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VICTORIAN CURRICULUM VERSION 2.0

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SECOND EDITION

OXFORD MATHS

10/10A

Series Consultant: Thomas Christiansen
Alexander Blanksby Morgan Levick
Daniel Mansfield Echo Gu Jennifer Nolan
Melanie Koetsveld Joe Marsiglio
Lyn Elms Dina Antoniou

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oxforddigital **This module is available on Oxford Digital.**

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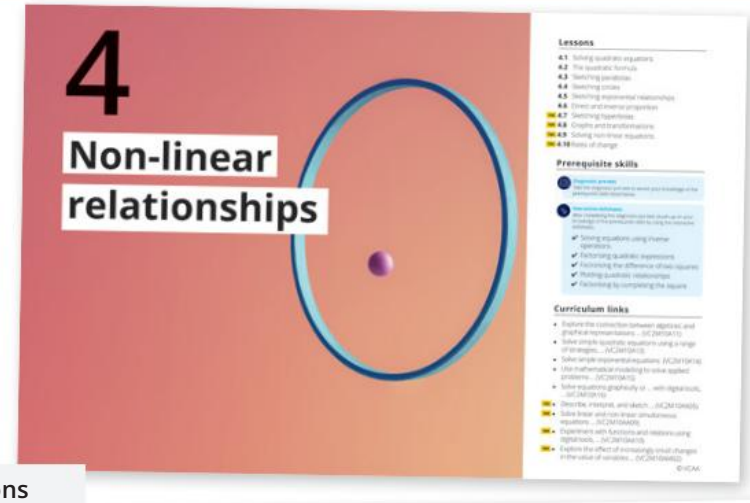
Oxford Maths 7–10 Victorian Curriculum utilises an innovative suite of print and digital resources to guide students on a focused mathematics journey. The series makes maths accessible to students with differing levels of understanding, increasing engagement by giving learners the opportunity to achieve success at their own skill level while also providing comprehensive syllabus coverage.

Key features of Student Books

- Complete access to all digital resources.
- Australian Maths Trust (AMT) spreads offer unique questions designed to challenge students and build engagement.
- STEAM projects encourage inter-disciplinary thinking.
- Semester reviews provide an opportunity to revise key concepts from each semester.
- An illustrated glossary of newly introduced mathematical terms.



Each module opens with:

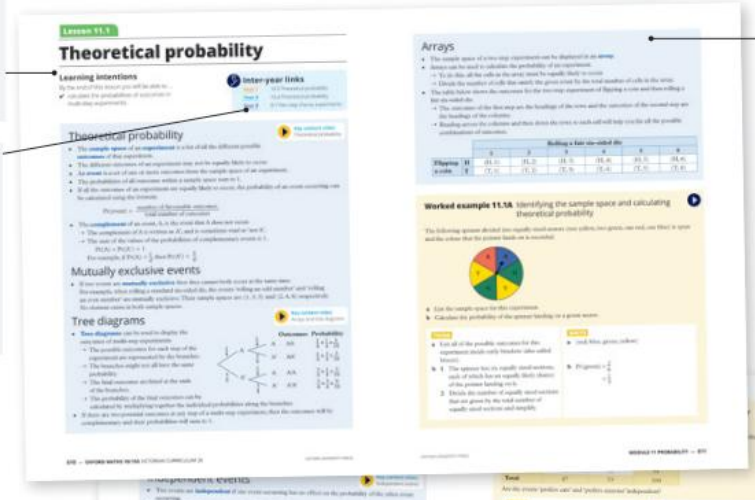
- **Prerequisite skills** with reference to an online diagnostic pre-test and interactive skillsheets.
- **Curriculum links** to all relevant content descriptions in the Victorian Curriculum.
- **Materials** used to complete the exercises.

Learning intentions

- Signpost the foundational skills being developed in each section.

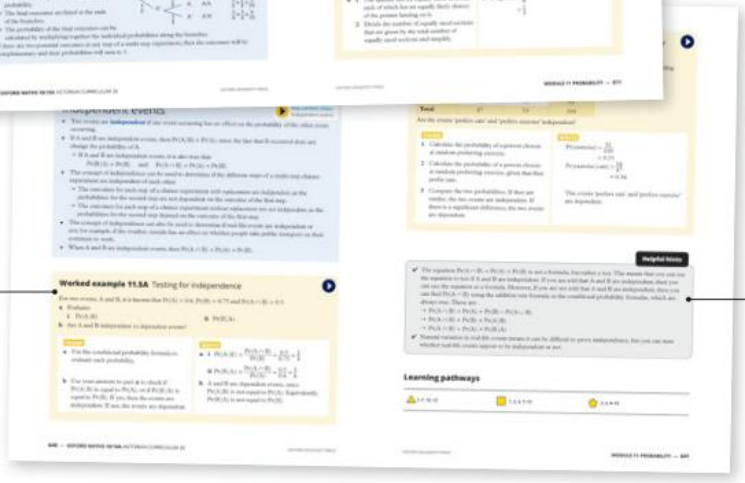
Inter-year links

- Provide easy access to support and extension material from each of the 7–10 Student Books as students build knowledge year on year.



New theory

- Backed by the latest pedagogical research to promote engagement with the material.
- Filled with precise diagrams that bring key concepts to life, and aid understanding.



Worked examples

- Outline a step-by-step thought process for solving essential questions with direct reference to the exercises.

Helpful hints

- Provide additional strategies for tackling problems.
- Highlight important elements of the theory.
- Point out common misconceptions.

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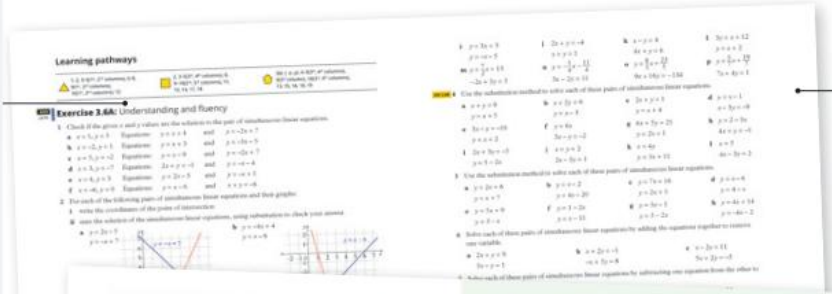
Differentiated learning pathways

- Each exercise is separated into three pathways, tailoring for students of all skill levels.
- Each pathway can be assigned based on results of the diagnostic pre-tests that are recommended at the beginning of every module.

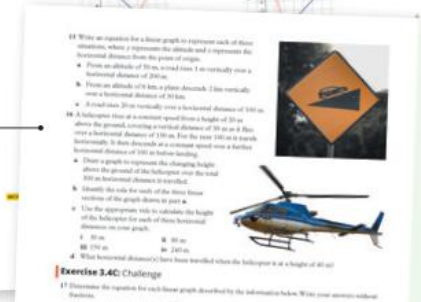


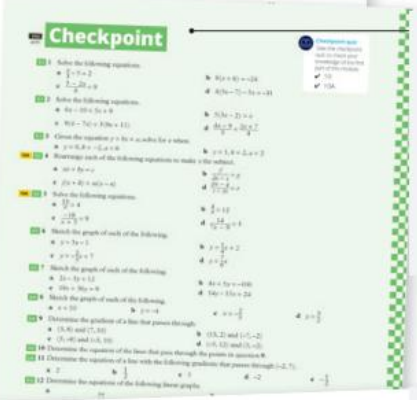
Understanding and fluency

- Basic exercises dedicated to practising key concepts.

Problem solving and reasoning

- Comprehensive exercises bring together new ideas and provide engaging contexts from real-world problems.



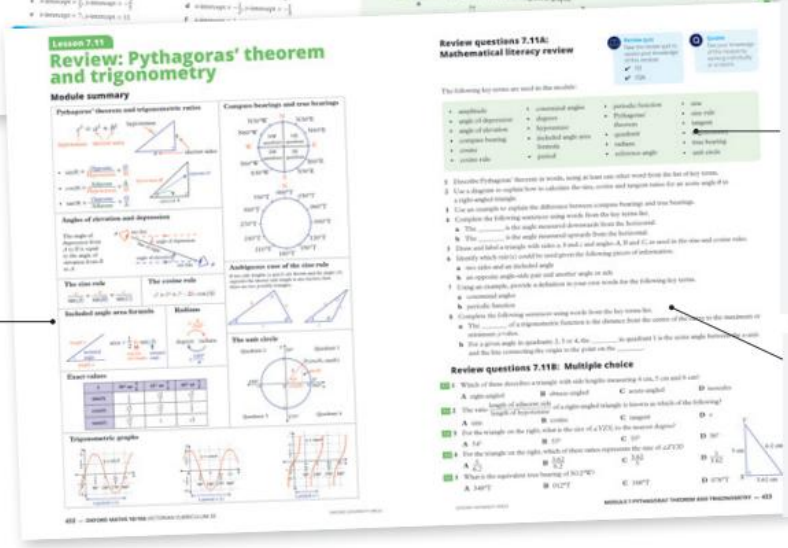


Checkpoint

- A section in the middle of each module dedicated to summarising key skills and encouraging memory retention.
- Reference to an online checkpoint quiz to gauge student progress.

Challenge

- Advanced exercises designed to build engagment and anticipate future learning outcomes.



Review questions

- Additional practice questions to further consolidate understanding at the end of each module.
- Reference to an online review quiz to track results.
- Reference to Quizlet test to revise new terminology.

Module summary

- Condenses all the theory from each section into one accessible revision page.

Mathematical literacy reviews

- A comprehensive literacy review assesses understanding of key terms.

Integrated STEAM projects

- Take the hard work out of cross-curricular learning with engaging STEAM projects. Two fully integrated projects are included at the end of each book in the series, and are scaffolded and mapped to the Science, Maths and Humanities curricula. The same projects also feature in the corresponding Oxford Humanities and Oxford Science series to assist cross-curricular learning.

Problem solving through design thinking

- Each STEAM project investigates a real-world problem that students are encouraged to problem-solve using design thinking.

Full digital support

- Each STEAM project is supported by a wealth of digital resources, including student booklets (to scaffold students through the design-thinking process of each project), videos to support key concepts and skills, and implementation and assessment advice for teachers.

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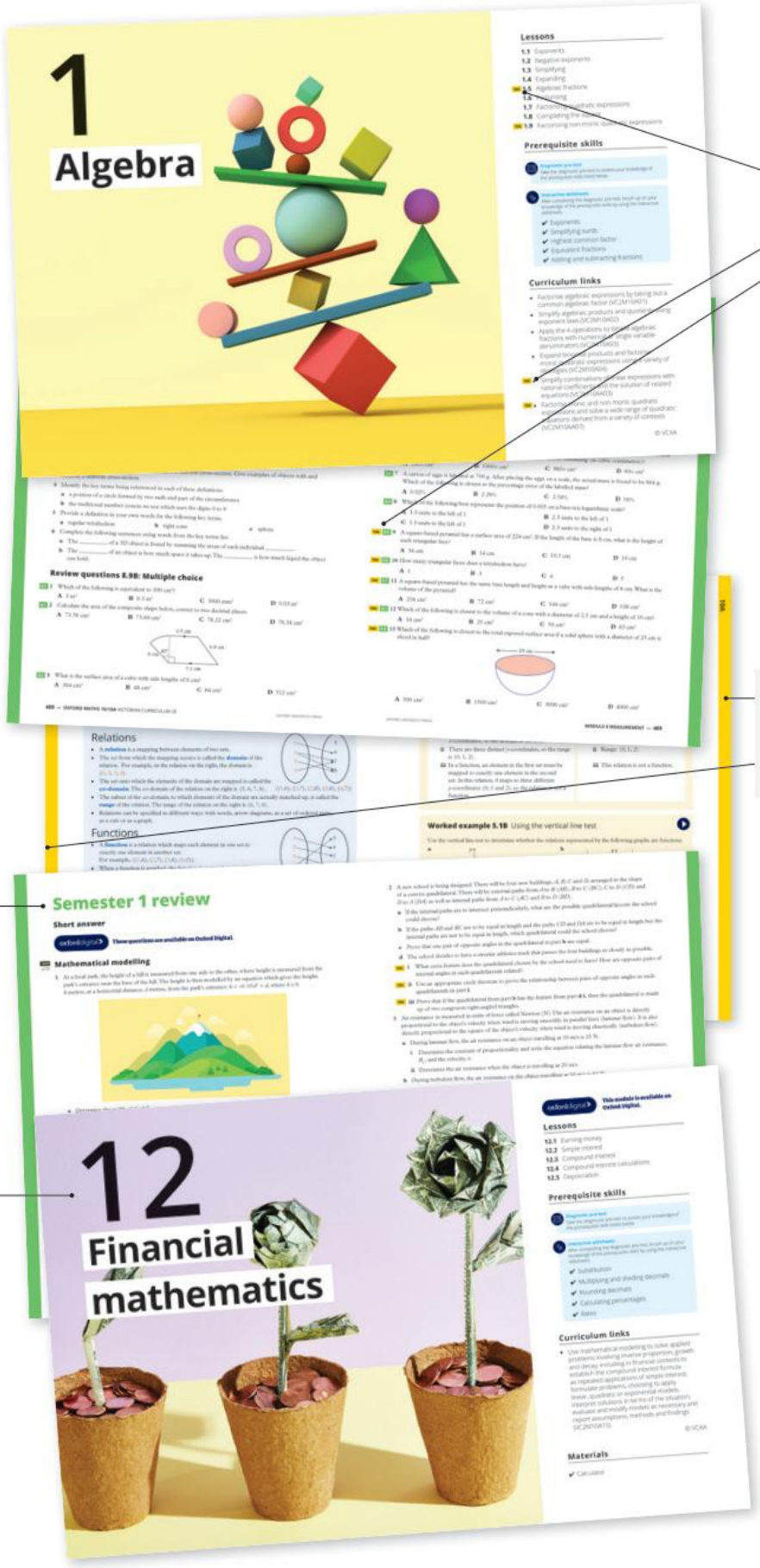

- Yellow lozenges indicate 10A lessons, questions and curriculum content descriptions.
- Yellow side banners indicate that the lesson (including theory, worked examples and exercise questions) is 10A content.
- Semester review short answer questions are available on Oxford Digital.
- Module 12 Financial mathematics is available on Oxford Digital.

- Additional answers for lessons in the Student Book are available on Oxford Digital.

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Introducing Oxford Maths 7–10 (Second edition)

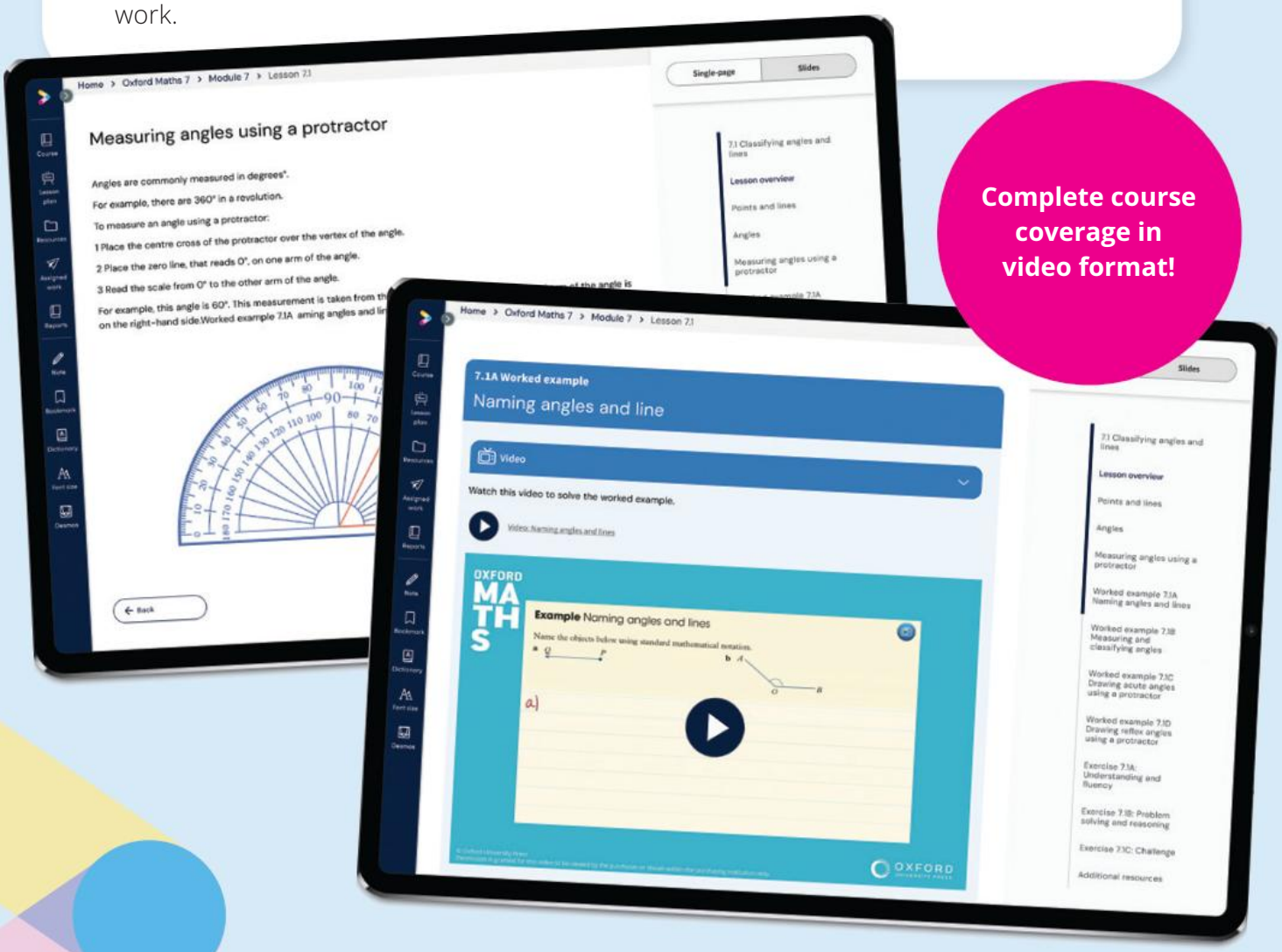
oxforddigital

This Student Book is supported by a range of digital resources and features for students and teachers on Oxford Digital.

Key features for students

Student digital access allows you to:

- **view** all the Student Book content in a fully accessible, reflowable format that's delivered in bite-sized chunks so you can work at you own pace
- **highlight**, **take notes**, **bookmark** pages, or define words with the built-in **Australian Oxford Dictionary**
- **watch** hundreds of short **content videos** and **worked example videos** to help you revise anything you don't understand, catch up on things you've missed or complete your homework
- **complete** thousands of interactive questions and quizzes as you work through the content and get the answers and results sent to you instantly
- **use** a suite of **Desmos calculators and graphing tools** to help you complete your work.



Complete course coverage in video format!

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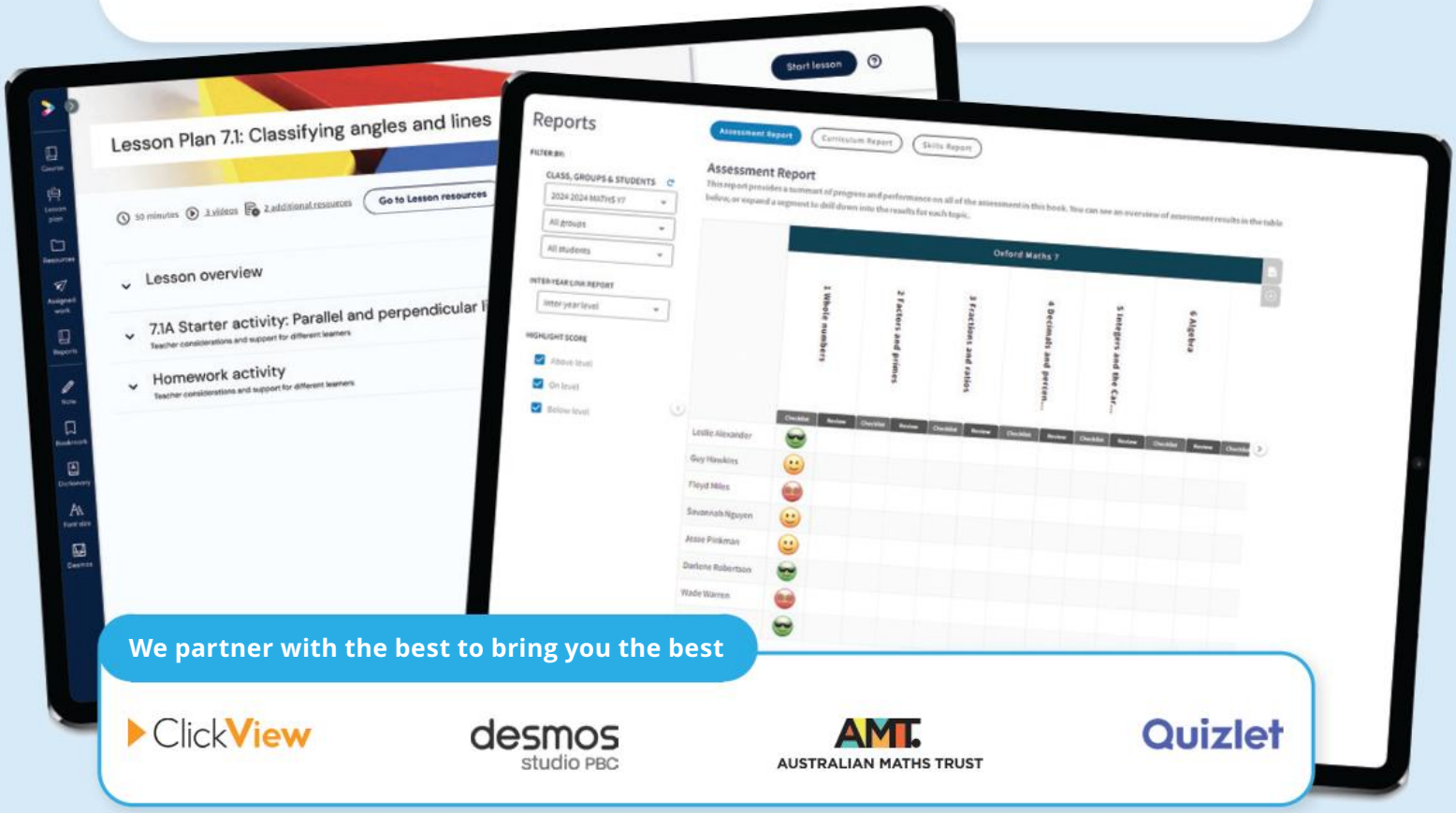
Key features for teachers

Teacher digital access allows you to:

- **elevate** your teaching and reduce planning and preparation time with **Live Lesson mode**. This Premium feature is an Australian first that lets you upgrade from traditional print-based lesson plans to **fully integrated, perfectly sequenced and timed interactive lessons complete with classroom activities** that drive engagement and participation
- **personalise** learning for every student and **differentiate** content based on student strengths and weaknesses. Assign support or extension resources to any student in any year level using **inter-year links**
- **identify gaps in understanding** with diagnostic quizzes that generate personalised learning pathway reports for every student based on ability. Learning pathway reports provide clear learning goals and exercises for every lesson including interactive resources designed to target misconceptions and give each student what they need to address them!
- **begin every lesson** with ready-made learning intentions and success criteria
- **revolutionise** your planning, marking and reporting with powerful analytics on student performance and progress:
  - **Assessment report** shows how students are performing in each online interactive assessment, providing instant feedback for teachers about areas of understanding
  - **Curriculum report** summarises student performance against specific curriculum content descriptors and curriculum codes
  - **Skills report** displays students' understanding of prerequisite skills in mathematics.



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1

Algebra

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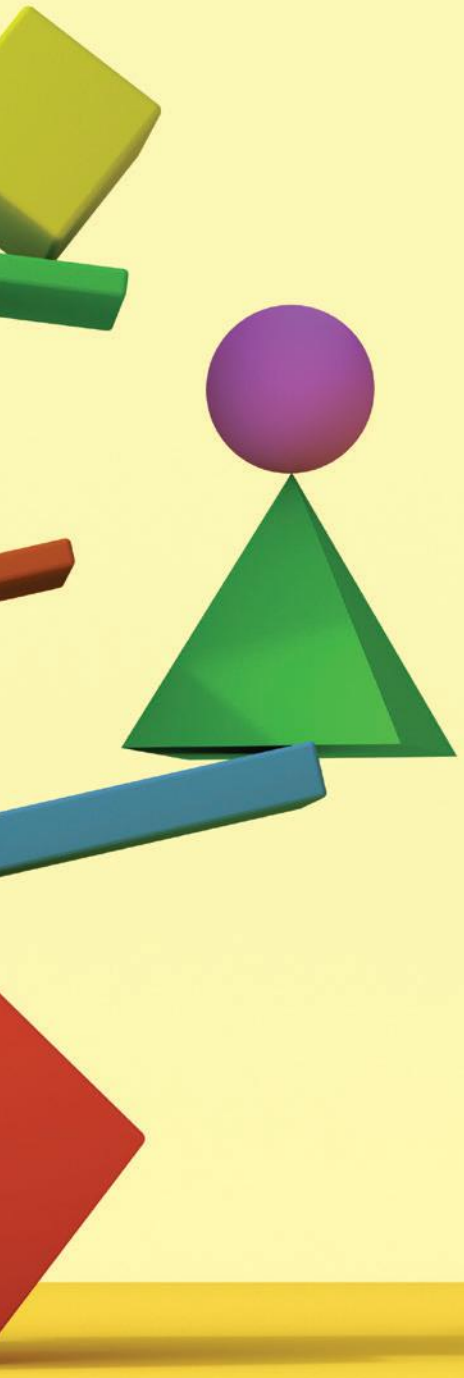

Lessons

1.1 Exponents
1.2 Negative exponents
1.3 Simplifying
1.4 Expanding
10A **1.5** Algebraic fractions
1.6 Factorising
1.7 Factorising quadratic expressions
1.8 Completing the square
10A **1.9** Factorising non-monic quadratic expressions

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ Exponents
- ✔ Simplifying surds
- ✔ Highest common factor
- ✔ Equivalent fractions
- ✔ Adding and subtracting fractions

Curriculum links

- Factorise algebraic expressions by taking out a common algebraic factor (VC2M10A01)
- Simplify algebraic products and quotients using exponent laws (VC2M10A02)
- Apply the 4 operations to simple algebraic fractions with numerical or single variable denominators (VC2M10A03)
- Expand binomial products and factorise monic quadratic expressions using a variety of strategies (VC2M10A04)
- 10A Simplify combinations of linear expressions with rational coefficients and the solution of related equations (VC2M10AA03)
- 10A Factorise monic and non-monic quadratic expressions and solve a wide range of quadratic equations derived from a variety of contexts (VC2M10AA07)

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Exponents

Learning intentions

By the end of this lesson you will be able to ...

✔ use the exponent laws to simplify algebraic expressions.

Inter-year links

Year 7	1.7 Exponents and square roots
Year 8	4.1 Exponents
Year 9	1.1 Exponents

Exponent form

Key content video
Exponents

- **Exponent form**, is used to represent repeated multiplication.
  - → The **base** is the number or variable that is multiplied repeatedly.
  - → The **exponent** (or **index**) indicates the number of times the base is multiplied.

exponent

base → a^3 = $a \times a \times a$

exponent form — expanded form

Exponent laws

Key content video
Products and quotients of powers

Key content video
Powers of powers and zero exponents

Key content video
Simplifying expressions

- The **exponent laws** are used for simplifying expressions that include terms written in exponent form.
- The exponent laws for multiplying or dividing terms in exponent form only apply when both terms have the same base.
  - → Any non-zero base raised to the power of zero is equal to one.

Exponent law	Algebraic form	Example
Product of powers law	$a^m \times a^n = a^{m+n}$	$2^3 \times 2^5 = 2^{3+5} = 2^8$
Quotient of powers law	$a^m \div a^n = a^{m-n}$	$2^5 \div 2^3 = 2^{5-3} = 2^2$
Power of a power law	$(a^m)^n = a^{m \times n}$	$(2^3)^5 = 2^{3 \times 5} = 2^{15}$
The zero exponent	$a^0 = 1$	$2^0 = 1$

- We can use a combination of exponent laws to simplify complex expressions.

$$(a \times b)^m = \underbrace{a \times b \times \ldots \times a \times b}_{m \text{ times}} = a^m \times b^m \qquad (2 \times 3)^5 = 2^5 \times 3^5$$

$$\left(\frac{a}{b}\right)^m = \underbrace{\frac{a}{b} \times \ldots \times \frac{a}{b}}_{m \text{ times}} = \frac{a^m}{b^m} \qquad \left(\frac{2}{3}\right)^5 = \frac{2^5}{3^5}$$

Worked example 1.1A Using the exponent laws

Use the appropriate exponent law to simplify each of these expressions.

a $(g^3)^4$ **b** $\left(\frac{w}{3}\right)^3$ **c** $3k^2 \times 6k^5$ **d** $8a^{12} \div 16a^7$

THINK

a Apply the power of a power law to expand the brackets by multiplying the exponents.

WRITE

a $(g^3)^4 = g^{3 \times 4}$
$= g^{12}$

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b **1** Expand the brackets by applying the exponent to the numerator and the denominator.

2 Simplify.

b $\left(\frac{w}{3}\right)^3 = \frac{w^3}{3^3}$

$= \frac{w^3}{27}$

c **1** Multiply the coefficients.

2 Since the exponents have the same base, apply the product of powers law by adding the exponents.

3 Simplify.

c $3k^2 \times 6k^5 = 18 \times k^2 \times k^5$

$= 18 \times k^{2+5}$

$= 18k^7$

d **1** Write the expression as a fraction.

2 Simplify the fraction by dividing the coefficients by their highest common factor (HCF).

3 Since the exponents have the same base, apply the quotient of powers law by subtracting the exponents.

4 Simplify.

d $8a^{12} \div 16a^7 = \frac{8a^{12}}{16a^7}$

$= \frac{\cancel{8}^{1}a^{12}}{\cancel{16}^{2}a^7}$

$= \frac{a^{12}}{2a^7}$

$= \frac{a^{12-7}}{2}$

$= \frac{a^5}{2}$

Worked example 1.1B Combining exponent laws to simplify algebraic expressions

Use the appropriate exponent laws to simplify these expressions.

a $3a^4 \times 10a \div 6a^2$ **b** $(2z^7)^3$ **c** $\frac{x^6 \times 6x^2}{8(x^4)^2}$

THINK

a **1** Write the expression as a fraction.

2 Simplify the numerator by multiplying the coefficients and using the product of powers law to add the exponents.

3 Simplify the fraction by dividing the coefficients by their HCF.

4 Apply the quotient of powers law to subtract the exponents.

b **1** Expand the brackets.

2 Apply the power of a power law to multiply the exponents and simplify.

c **1** Simplify the numerator by applying the product of powers law. Simplify the denominator by applying the power of a power law.

2 Simplify the fraction by dividing the coefficients by their HCF.

3 Divide the numerator and denominator by the common factor x^8. Alternatively, simplify using the quotient of powers law and the zero exponent.

WRITE

a $3a^4 \times 10a \div 6a^2 = \frac{3a^4 \times 10a}{6a^2}$

$= \frac{30a^5}{6a^2}$

$= \frac{\cancel{30}^{5}a^5}{\cancel{6}^{1}a^2}$

$= \frac{5a^5}{a^2}$

$= 5a^{5-2}$

$= 5a^3$

b $(2z^7)^3 = 2^3 \times (z^7)^3$

$= 8 \times z^{7 \times 3}$

$= 8z^{21}$

c $\frac{x^6 \times 6x^2}{8(x^4)^2} = \frac{6x^8}{8x^8}$

$= \frac{\cancel{6}^{3}x^8}{\cancel{8}^{4}x^8}$

$= \frac{3x^8}{4x^8}$

$= \frac{3\cancel{x^8}}{4\cancel{x^8}}$

$= \frac{3}{4}$

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Helpful hints

- ✔ Remember, the product and quotient of powers laws only apply if the expressions have the same base.
- ✔ Expanding brackets raised to an exponent only works if the brackets contain division and multiplication. You cannot apply the same method to expand brackets containing addition and subtraction!

  $(a + b)^m \neq a^m + b^m$ and $(a - b)^m \neq a^m - b^m$

  In order to expand these expressions, you will need to use the distributive law covered in **Lesson 1.4**.
- ✔ Avoid errors in your working out by applying the exponent laws one at a time.

Learning pathways

1–7, 8(a, b), 10(a–e), 11	1(e–h), 3(d, g–i), 4–6(2nd, 3rd columns), 7, 8(b, d), 9–10(a–e), 12, 17	6–7(f–i), 8, 9–10(f–i), 13–17

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Exercise 1.1A: Understanding and fluency

1 Use the appropriate exponent laws to simplify these expressions.

a $a^7 \div a^3$ **b** $(b^4)^5$ **c** $c^9 \times c^6$ **d** $d^3 \div d^2$

e $y^5 \times y^5$ **f** $(m^7)^2$ **g** $n^{15} \div n$ **h** $h \times h^6$

i $\frac{x^{11}}{x^5}$ **j** $(p^6)^3$ **k** $\frac{w^4}{w}$ **l** $a^2 \times a \times a^7$

2 Answer the following questions.

a What is the value of a non-zero term expressed in exponent form raised to the power of zero?

b Write the value of each of the following.

i 3^0 **ii** 100^0 **iii** m^0 **iv** $5^9 \div 5^9$

3 Use the property $a^0 = 1$ to simplify each of these expressions.

a $3a^0$ **b** $(3a)^0$ **c** $8m^0$ **d** $(8m)^0$

e $(-5x)^0$ **f** $a^0 + b^0$ **g** $j^0 - k^0$ **h** $p^0 + p^0$

i $2c^0 + 4g^0$ **j** $w^0 + x^0 - y^0$ **k** $(m + n)^0$ **l** x^0

4 Use the property $(a \times b)^m = a^m \times b^m$ to write each of these expressions without brackets.

a $(a \times b)^5$ **b** $(k \times p)^9$ **c** $(4 \times c)^3$ **d** $(7 \times y)^{13}$ **e** $(xy)^4$

f $(cd)^7$ **g** $(3w)^2$ **h** $(6g)^6$ **i** $(2p)^8$

5 Use $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ to write each of these expressions without brackets.

a $\left(\frac{a}{b}\right)^4$ **b** $\left(\frac{f}{g}\right)^7$ **c** $\left(\frac{k}{2}\right)^6$

d $\left(\frac{5}{x}\right)^6$ **e** $\left(\frac{d}{c}\right)^{-2}$ **f** $\left(\frac{m}{3}\right)^{-5}$

WE 1.1A **6** Use the appropriate exponent laws to simplify these expressions.

a $5x^2 \times x^3$ **b** $2a^2 \times 3a^5$ **c** $(3x)^2$

d $(2x^4)^3$ **e** $\frac{4a^5}{a^2}$ **f** $\frac{12x^8}{6x^3}$

g $\frac{4x^3}{2x^3}$ **h** $3x^6 \times 2x^2 \times 5x^4$ **i** $\frac{(6a^4)^2}{12a^5}$

WE 1.1B **7** Use the appropriate exponent laws to simplify these expressions.

a $\frac{4y^2 \times 12y^5}{6y^3}$ **b** $3(2x^3)^3$ **c** $(3k)^2 \times (2k)^3$

d $(3b^5)^2 \times 5b^2$ **e** $3a^3 \times (2a^4)^2 \times 5a^9$ **f** $15c^4 \times 6c^3 \div (3c)^2$

g $\frac{(2n^2)^2 \times 5n^3}{10n^5}$ **h** $\frac{3v^2}{4} \times \frac{8v^4}{6}$ **i** $\left(\frac{5m^5}{2}\right)^2 \times 4m^3$

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Exercise 1.1B: Problem solving and reasoning

8 Use the exponent laws to simplify each expression. Write your answers in exponent form.

a $\dfrac{p^4 \times (p^5)^3}{p^{10}}$ **b** $\dfrac{2(t^2)^8 \times (3t^5)^3}{3t^{11}}$ **c** $\dfrac{(5m^{11}n^{10})^8 \times (5mn^6)^2}{(5m^9n^7)^5 \times (5m^2n^3)^3}$ **d** $\dfrac{(8j^5p)^0 \times (j^0p^4)^3}{(jp^2)^6}$

9 Use the exponent laws to decide whether each of these statements is true or false. Explain your reasoning. For each false statement, change the right-hand side to make the statement true.

a $x^7 \times x \times x^7 = x^{14}$ **b** $(5a)^3 = 5 \times a^3$ **c** $-k^0 = -1$

d $a^2b^3 \times a^3b^4 = a^{12}b^{12}$ **e** $x^7y^4 \div y^6 = \dfrac{x^7}{y^2}$ **f** $\left(\dfrac{m}{n}\right)^5 = \dfrac{m^5}{n}$

g $\dfrac{w^6 \times w^6}{w^{12}} = 0$ **h** $\dfrac{(b^5)^4 \times b^2}{(b^3)^7} = b$ **i** $\dfrac{a^8b^5}{a^4b^3} \times \dfrac{a^2b^2}{a^9b^4} = a^3$

10 Find the value of x that makes the statement true.

a $a^x \times a^3 = a^{12}$ **b** $b^x = 1$ **c** $c^x \div c^6 = c^4$

d $(d^3)^x = d^{12}$ **e** $\left(\dfrac{m}{n}\right)^x = \dfrac{m^7}{n^7}$ **f** $\dfrac{w^x \times w^4}{w^7} = w^6$

g $(2a^x)^5 = 32a^{30}$ **h** $\dfrac{(n^3)^x \times n^4}{(n^2)^6} = n^7$ **i** $\dfrac{a^3b^6}{a^4b^x} \times \dfrac{a^7(b^2)^3}{a^5b^4} = a$

11 Write $\dfrac{8^5 \times 2^4}{4^7}$ in simplest exponent form with a base of 2.

12 Write $\dfrac{3^{4x} \times 9^{2x}}{27^x \times 3^{3x}}$ in simplest exponent form with a base of 3.

13 Use the appropriate exponent laws to simplify $\dfrac{25^{3x} \times 625^x}{125^{2x} \times 3^{3x}}$.

14 Find the value of x and y to make each of these statements true.

a $a^xb \times a^y(b^4)^y = a^{12}b^{33}$ **b** $\dfrac{a^xb^y}{a^2b^3} \times \dfrac{a^{x+y}b^y}{a^3b^7} = a^3b^{14}$

Exercise 1.1C: Challenge

15 A square number is a number that can be written in the form n^2, where n is a whole number.

a Show that $\dfrac{2^{13} \times 3^{11} \times 5^9}{30^7}$ is a square number.

b Find the possible values of a if $\dfrac{5^5 \times 7^7}{a}$ is a square number.

c Describe the prime factorisation of a square number.

16 Evaluate $\dfrac{6^5 \times 5^7}{2^4 \times 15^5}$ without a calculator.

17 Many students mistakenly believe that $(x + y)^n = x^n + y^n$ for non-zero integers x and y, and positive integer n. This false equation is sometimes called the 'Freshman's dream'.

a Expand the expression $(x + y)^2$ to show the 'Freshman's dream' is false for $n = 2$.

b Use a calculator to substitute the following values into the 'Freshman's dream' and show that the resulting equation is false.

i $x = 2, y = 3, n = 3$ **ii** $x = 4, y = 5, n = 4$ **iii** $x = 1, y = 5, n = 5$

c For a prime number p and non-zero integers x and y, the following equation is true.

$(x + y)^p = x^p + y^p + mp$

Here, m is an integer which depends on the values of x and y. For example, when $p = 2$, we get $m = xy$.

i Using the values in part **b i** check that this new equation holds and identify the corresponding values of m.

ii Use the identity $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ to show that the new equation holds in the case $p = 3$. What is m in terms of x and y?

iii Find values of non-zero integers x, y and n, where n is composite, such that $(x + y)^n \neq x^n + y^n + mn$ where m is an integer.

Online resources:

- **Interactive skillsheet** Exponent laws
- **CAS instructions** CAS basics
- **CAS instructions** Simplifying exponents
- **Quick quiz** 1.1

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Negative exponents

Learning intentions

By the end of this lesson you will be able to ...

✔ use negative exponents in algebraic expressions.

Inter-year links

Year 8	4.4 Combining the exponent laws
Year 9	1.4 Negative exponents

Negative exponents

Key content video: Negative exponents

Key content video: Simplifying expressions

- When a number is multiplied by its **reciprocal** (or multiplicative inverse), the result is 1.
  - → a and $\frac{1}{a}$ are reciprocals, as $a \times \frac{1}{a} = 1$.
  - → $\frac{a}{b}$ and $\frac{b}{a}$ are reciprocals, as $\frac{a}{b} \times \frac{b}{a} = 1$.
- A power with a **negative exponent** is the reciprocal of the base with the positive exponent.

$$\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^{m}$$

- The exponent laws apply to expressions containing terms with negative exponents.
- These properties allow for expressions containing negative exponents to be expressed with positive exponents. For example,

$$3hr^{-2} = \frac{3h}{r^2}.$$

- Fractions can also be written in exponent form using negative exponents. For example,

$$\frac{2b^3}{a} = 2a^{-1}b^3.$$

Algebraic property	Example
$a^{-1} = \frac{1}{a}$	$2^{-1} = \frac{1}{2}$
$\frac{1}{a^{-1}} = a$	$\frac{1}{3^{-1}} = 3$
$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a}$	$\left(\frac{2}{3}\right)^{-1} = \frac{3}{2}$
$a^{-m} = \frac{1}{a^m}$	$3^{-2} = \frac{1}{3^2}$
$\frac{1}{a^{-m}} = a^m$	$\frac{1}{4^{-3}} = 4^3$
$\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^{m}$	$\left(\frac{2}{3}\right)^{-4} = \left(\frac{3}{2}\right)^{4}$

Algebraic form	Example
$a^{-m} \times a^{-n} = a^{(-m)+(-n)} = a^{-m-n}$	$5^{-3} \times 5^{-2} = 5^{(-3)+(-2)} = 5^{-5}$
$a^{-m} \div a^{-n} = a^{(-m)-(-n)} = a^{-m+n}$	$5^{-3} \div 5^{-2} = 5^{(-3)-(-2)} = 5^{-1}$
$(a^{-m})^{-n} = a^{(-m)\times(-n)} = a^{mn}$	$(5^{-3})^{-2} = 5^{(-3)\times(-2)} = 5^{6}$

Worked example 1.2A Determining the reciprocal of a term

Determine the reciprocal of each of the following.

a $2x$ **b** $\frac{2m}{n}$

THINK

1. Write the base with a negative exponent as a fraction if it is not already.
2. Find the reciprocal of the fraction. Swap the numerator and denominator.
3. Simplify the result.

WRITE

a $(2x)^{-1} = \left(\frac{2x}{1}\right)^{-1}$

$= \frac{1}{2x}$

b $\left(\frac{2m}{n}\right)^{-1} = \frac{n}{2m}$

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Worked example 1.2B Using the exponent laws with negative exponents

Use the appropriate exponent laws to simplify each of these expressions.

a $(a^5)^{-2}$ **b** $3b^{-2} \times b^{-3}$ **c** $9c^{-3} \div 3c^5$

THINK

a Apply the power of a power law, multiply the exponent of a by the exponent outside the brackets.

b Apply the product of powers law, add the exponents of b and multiply the result by the coefficient 3.

c **1** Write the expression as a fraction.

2 Simplify the fraction by dividing both the numerator and denominator by their highest common factor (HCF). Apply the quotient of powers law to subtract the exponents of c.

3 Simplify.

WRITE

a $(a^5)^{-2} = a^{5 \times (-2)}$
$= a^{-10}$

b $3b^{-2} \times b^{-3} = 3 \times b^{-2 + (-3)}$
$= 3 \times b^{-2 - 3}$
$= 3 \times b^{-5}$
$= 3b^{-5}$

c $9c^{-3} \div 3c^5 = \dfrac{9c^{-3}}{3c^5}$
$= 3 \times c^{-3 - 5}$
$= 3c^{-8}$

Worked example 1.2C Writing terms with positive exponents

Write an equivalent term with only positive exponents for each of the following.

a x^{-3} **b** $\dfrac{1}{y^{-8}}$ **c** $x^{-5}y^3$ **d** $\left(\dfrac{z}{3}\right)^{-2}$

THINK

a Apply $a^{-m} = \dfrac{1}{a^m}$ to write the expression with a positive exponent.

b Apply $\dfrac{1}{a^{-m}} = a$ to write the expression with a positive exponent.

c **1** Write the term as a product of two factors.

2 Apply $a^{-m} = \dfrac{1}{a^m}$ to write x^{-5} with a positive exponent.

3 Simplify.

d **1** Apply $\left(\dfrac{a}{b}\right)^{-m} = \left(\dfrac{b}{a}\right)^m$ to write the expression with a positive exponent.

2 Remove the brackets by applying the exponent to the numerator and the denominator.

3 Simplify.

WRITE

a $x^{-3} = \dfrac{1}{x^3}$

b $\dfrac{1}{y^{-8}} = y^8$

c $x^{-5}y^3 = x^{-5} \times y^3$
$= \dfrac{1}{x^5} \times y^3$
$= \dfrac{y^3}{x^5}$

d $\left(\dfrac{z}{3}\right)^{-2} = \left(\dfrac{3}{z}\right)^2$
$= \dfrac{3^2}{z^2}$
$= \dfrac{9}{z^2}$

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Worked example 1.2D Combining exponent laws to simplify negative exponents

Use the appropriate exponent laws to simplify these expressions. Write your answers using only positive exponents.

a $\left(\frac{x^2}{x^{-1}}\right)^{-3}$ **b** $6g^{-4} \times 2g \div 3g^{-2}$ **c** $\frac{3}{m^{-4}} \times (m^2)^{-2}$ **d** $\left(\frac{y}{x}\right)^{-2} \times \left(\frac{x}{y^{-2}}\right)^2$

THINK

a 1 Apply the quotient of powers law to simplify the fraction inside the bracket.

2 Apply the power of a power law to remove the brackets.

3 Apply $a^{-m} = \frac{1}{a^m}$ to write the expression with a positive exponent.

b 1 Write the expression as a fraction.

2 Simplify the numerator by multiplying the coefficients, 6 and 2, in the numerator and applying the product of powers law to add the exponents.

3 Simplify the fraction by dividing the coefficients by their HCF.

4 Apply the quotient of powers law to subtract the exponents.

5 Apply $a^{-1} = \frac{1}{a}$ to write the number as a fraction with a positive exponent.

c 1 Separate the denominator from the numerator and write each factor with a positive exponent by applying:

$\frac{1}{a^{-m}} = a^m$ and $a^{-m} = \frac{1}{a^m}$.

2 Apply the power of a power law to remove the brackets, then the quotient of powers law to subtract the exponents.

3 Simplify. Remember that any non-zero base raised to a zero exponent is equal to one.

d 1 Apply $\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$ and $\frac{1}{a^{-m}} = a^m$ to write the expression using positive exponents.

2 Apply the associative property of multiplication and the power of a power law to remove the brackets and write the expression as one fraction.

3 Simplify the expression using the product and quotient of powers laws.

WRITE

a $\left(\frac{x^2}{x^{-1}}\right)^{-3} = (x^{2-(-1)})^{-3}$

$= (x^{2+1})^{-3}$

$= (x^3)^{-3}$

$= x^{3 \times (-3)}$

$= x^{-9}$

$= \frac{1}{x^9}$

b $6g^{-4} \times 2g \div 3g^{-2} = \frac{6g^{-4} \times 2g}{3g^{-2}}$

$= \frac{\cancel{12}^{4} g^{-4+1}}{\cancel{3}^{1} g^{-2}}$

$= \frac{4g^{-3}}{g^{-2}}$

$= 4g^{-3-(-2)}$

$= 4g^{-3+2}$

$= 4g^{-1}$

$= 4 \times \frac{1}{g}$

$= \frac{4}{g}$

c $\frac{3}{m^{-4}} \times (m^2)^{-2} = 3 \times \frac{1}{m^{-4}} \times (m^2)^{-2}$

$= 3 \times m^4 \times \frac{1}{(m^2)^2}$

$= 3 \times m^4 \times \frac{1}{m^4}$

$= 3 \times m^{4-4}$

$= 3m^0$

$= 3$

d $\left(\frac{y}{x}\right)^{-2} \times \left(\frac{x}{y^{-2}}\right)^2 = \left(\frac{x}{y}\right)^2 \times (xy^2)^2$

$= \frac{x^2 \times x^2y^{2\times 2}}{y^2}$

$= \frac{x^{2+2}y^4}{y^2}$

$= x^4y^{4-2}$

$= x^4y^2$

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Worked example 1.2E Writing fractions in exponent form

Write the following fractions in exponent form.

a $\dfrac{5y^{-4}}{x^{-7}}$ **b** $\dfrac{1}{(2x)^{-2}}$ **c** $\left(\dfrac{3x}{y}\right)^{-2}$

THINK

a 1 Apply $\dfrac{1}{a^{-m}} = a^m$ to write the denominator of the fraction with a positive exponent.

2 Simplify by removing the multiplication sign between variables and arrange variables in alphabetical order if not already.

b 1 Apply $\dfrac{1}{a^{-m}} = a^m$ to write the denominator of the fraction with a positive exponent.

2 Simplify by applying the exponent to both the coefficient and the variable.

c 1 Apply the exponent to both the numerator and the denominator of the fraction.

2 Remove the brackets in the numerator by applying the negative exponent to both the coefficient, 3, and the variable, x.

3 Rewrite the fraction as the product of the numerator and the reciprocal of the denominator.

4 Use $\dfrac{1}{a^{-m}} = a^m$ to write $\dfrac{1}{y^{-2}}$ in exponent form.

5 Use $a^{-m} = \dfrac{1}{a^m}$ to write the coefficient with a positive exponent. Calculate the value of the coefficient.

6 Simplify by removing the multiplication signs between variables and arrange variables in alphabetical order if not already.

WRITE

a $\dfrac{5y^{-4}}{x^{-7}} = 5y^{-4} \times x^7$

$= 5x^7y^{-4}$

b $\dfrac{1}{(2x)^{-2}} = (2x)^2$

$= 4x^2$

c $\left(\dfrac{3x}{y}\right)^{-2} = \dfrac{(3x)^{-2}}{y^{-2}}$

$= \dfrac{3^{-2} \times x^{-2}}{y^{-2}}$

$= 3^{-2} \times x^{-2} \times \dfrac{1}{y^{-2}}$

$= 3^{-2} \times x^{-2} \times y^2$

$= \dfrac{1}{3^2} \times x^{-2} \times y^2$

$= \dfrac{1}{9}x^{-2}y^2$

Helpful hints

✔ When converting the denominator of a fraction into exponent form, use brackets to make sure the exponent is applied to both the coefficient and all variables in the denominator.
For example:

$$\frac{1}{2xy^2z} = (2xy^2z)^{-1} \qquad \frac{1}{2xy^2z} \neq 2xy^2z^{-1}$$

✔ The sequence showing positive powers of a base can be extended to show negative exponents.

$$\ldots, 3^{-3} = \frac{1}{27}, 3^{-2} = \frac{1}{9}, 3^{-1} = \frac{1}{3}, 3^0 = 1, 3^1 = 3, 3^2 = 9, 3^3 = 27, \ldots$$

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Learning pathways

△ 1–9(1st, 2nd columns), 10(a–c), 12(a, b)	□ 1–9(3rd, 4th columns), 10(c, e, f), 11, 12, 15(a), 19	⬠ 5–9(4th column), 10(d, g, h), 12–20

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Exercise 1.2A: Understanding and fluency

WE 1.2A **1** Determine the reciprocal of each of the following.

a 1 **b** -2 **c** $\frac{1}{3}$ **d** $-5d$

e $2w$ **f** $\frac{1}{2n}$ **g** $\frac{z}{xy}$ **h** $\frac{b}{a^2}$

WE 1.2B **2** Simplify each of these expressions.

a $x^5 \times x^{-2}$ **b** $a^{-2} \times a^{-3}$ **c** $n^2 \times n^{-2}$ **d** $y^{-6} \times y^{10}$

e $a^{-4} \div a^{-9}$ **f** $m^4 \div m^{-3}$ **g** $n^{-3} \div n^{-2}$ **h** $n^0 \div n^{-5}$

i $(k^{-6})^{-2}$ **j** $(d^{-5})^3$ **k** $y^3 \times y^{-4} \times y^2$ **l** $n^{-7} \times n^{-3} \times n^{12}$

3 Find the reciprocal of each of your answers in question **2**.

4 Simplify each of these expressions.

a $a^{-1}b \times ab^{-2}$ **b** $mn \times nm^{-2}$ **c** $a^{-3}b^{-2} \times a^3b^2$ **d** $-x^3y^{-2} \times y^{-3}x^2$

e $a^{-1}bc \times cab^{-2}$ **f** $cab^{-2} \times a^{-1}bc \times cab^{-2}$ **g** $-x^{-1}y \times xy^{-2}$ **h** $x^{-1}yz \times xz^{-2}$

WE 1.2C **5** Write each of these terms using only positive exponents.

a x^{-5} **b** $c^{-9}d^4$ **c** j^5k^{-3} **d** $x^{-1}y^8$

e $2a^3b^{-6}$ **f** $3p^{-4}q^{-5}$ **g** $w^4x^{-5}y^7$ **h** $k^{-3}m^5n^{-8}$

i $6d^2e^{-7}f$ **j** $11a^{-3}b^{-5}c^{-9}$ **k** $\frac{1}{x^{-3}}$ **l** $\frac{b^4}{c^{-6}}$

m $\frac{4m^3}{p^{-7}}$ **n** $\frac{a^{-4}}{b^{-2}}$ **o** $\frac{8x^{-2}}{2y^{-5}}$ **p** $\frac{c^{-1}d^8}{e^{-4}}$

6 Write each of these expressions without brackets and using only positive exponents.

a $(x \times y)^{-4}$ **b** $(4 \times a)^{-6}$ **c** $(9 \times n)^{-1}$ **d** $(5p)^{-3}$

e $(3p)^{-2}$ **f** $(km)^{-5}$ **g** $\left(\frac{x}{4}\right)^{-3}$ **h** $\left(\frac{7}{d}\right)^{-1}$

WE 1.2D **7** Use the appropriate exponent laws to simplify each of these expressions. Write your answers using only positive exponents.

a $(x^3)^{-2}$ **b** $(2y^{-3})^2$ **c** $(w^{-3})^{-4}$ **d** $\left(\frac{2x^4}{x^{-3}}\right)^2$

e $\left(\frac{a^4}{3}\right)^{-2}$ **f** $(2m)^{-2} \times (2m)^3$ **g** $\left(\frac{h^{-4}}{2}\right)^{-3}$ **h** $3c^{-4} \times 5c^{-8} \times 2c^3$

i $(2d)^{-2} \times 8d^5$ **j** $(6t^2)^{-2} \times 72(t^4)^{-3}$ **k** $(4x^{-2})^2 \times (2x^{-3})^{-3} \div (2x^4)$ **l** $\left(\frac{a^2}{3}\right)^{-3} \times \left(\frac{2a^{-1}}{3}\right)^2$

8 Simplify each of these expressions. Write your answers using only positive exponents.

a $x^3 \times x^{-5}$ **b** $2x^{-4} \times x^{-1}$ **c** $7x^{-3} \times 3x^6 \times 2x^{-2}$ **d** $4x^{-8} \times 5x^3 \times x$

e $x^{-11} \div x^{-9}$ **f** $6x^4 \div (3x^{-5})$ **g** $3x^{-7} \div (12x^2)$ **h** $9x^6 \div (15x^{10})$

i $(x^{-3})^2 \times x^{-5}$ **j** $(x^{-4})^6 \times 10x^{18}$ **k** $3x^{-6} \times (x^{-1})^7 \times (x^3)^2$ **l** $(x^{-2})^7 \times (x^{-5})^{-1} \times (x^{-3})^4$

WE 1.2E **9** Write the following fractions in exponent form.

a $\frac{1}{x^2}$ **b** $\frac{1}{y^{-2}}$ **c** $\frac{a^{-4}}{b^{-2}}$ **d** $\frac{n^{-4}}{m^{-2}n^2}$

e $\frac{1}{(2x)^6}$ **f** $\frac{1}{(pq)^{-2}}$ **g** $\frac{123}{g^{-234}}$ **h** $\frac{t}{(2t)^{-2}}$

i $\left(\frac{u}{10}\right)^{-10}$ **j** $\left(\frac{3q^2}{p}\right)^{-1}$ **k** $\left(\frac{3y}{xy}\right)^{-2}$ **l** $3\left(\frac{2m}{3n}\right)^{-3}$

10 Use the exponent laws to help you write all fractions in exponent form.

a $(ab)^4 \times a^7b^3$ **b** $3(m^2n)^6$ **c** $\frac{d^5}{h^4} \times \left(\frac{d}{h}\right)^7$ **d** $\left(\frac{a^5}{b^{-2}}\right)^6$

e $\left(\frac{wx^5}{u^3}\right)^4 \times \left(\frac{u^5}{wx}\right)^3$ **f** $\left(\frac{c^6}{a^6n^2}\right)^2 \times \frac{(5a^4)^3}{c^8n^{-5}}$ **g** $\frac{(m^5n^4)^3 \times (m^6n)^4}{(mn^3)^5}$ **h** $\frac{(2x^2)^6(y^4)^2}{(4x^4y^5)^3}$

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Exercise 1.2B: Problem solving and reasoning

11 Evaluate each of the following.

a 5^{-2} **b** 3^{-3} **c** $\dfrac{5^{-2}}{10^{-3}}$

d $\left(\dfrac{3^{-1}}{2^2}\right)^{-2}$ **e** $4^{-2} + 2^{-4}$ **f** $3^{-2} + 3^{-1} + 3^0$

12 Use the exponent laws to decide whether each of the following statements is true or false. Explain your reasoning. For each false statement, change the right-hand side to make the statement true.

a $\dfrac{a^5}{a^9} = a^4$ **b** $\dfrac{x^3}{x^6} = x^{\frac{1}{2}}$ **c** $\dfrac{y^3}{y^9} = \dfrac{1}{y^6}$

d $\left(\dfrac{3a^2b^{-1}}{c^3}\right)^2 = \dfrac{6a^4}{b^2c^6}$ **e** $(5m^4n^{-3})^{-2} = \dfrac{n^6}{25m^8}$ **f** $\dfrac{2a^{-4}b^3}{5^{-1}ab^2} = \dfrac{10b}{a^5}$

g $\dfrac{3(x^{-2}y^1)^3}{x^3y^{-2}z} = \dfrac{3y^5}{x^5z}$ **h** $(6abc)^{-2} \times (3a^{-1}b)^2 \div (4a^4c^2)^{-1} = 1$

13 Find the value of x that makes each of these statements true.

a $\dfrac{a^x}{a^4} = \dfrac{1}{a}$ **b** $\dfrac{3}{a^x} = 3a^4$ **c** $\left(\dfrac{2a^{-2}b}{c^3}\right)^x = \dfrac{a^4c^6}{4b^2}$

d $\dfrac{(ab)^x}{a^2b^4} = \dfrac{a}{b}$ **e** $\dfrac{a^3b^x}{a^{3x}b^2} = \dfrac{1}{b}$ **f** $\dfrac{y^7 \times y^x}{y^{2x}} = y^2$

14 Simplify each of these expressions.

a $x^{2a}y^{b+3} \times x^5y^{3b-2}$ **b** $\dfrac{m^{3x+4}n^{x-2}}{m^xn^{x-5}}$

15 Simplify each of these expressions.

a $x^{2a+1} \times x^{a-5}$ **b** $(y^m)^2 \times y^{m+5}$ **c** $\dfrac{a^{3x+3}}{a^{2x}}$ **d** $\dfrac{b^{2x}}{b^y} \times b^3 \times \dfrac{b^{y+1}}{b^x}$

16 Use the exponent laws to simplify each of the following expressions. Write your answers with positive exponents only.

a $\dfrac{3a^2b^4c}{8a^5b^3} \times \dfrac{4a^5b^4}{b^3c^2}$ **b** $\dfrac{10ac^5}{5a^5b^9c^2} \times \dfrac{3a^4b^3c}{(2bc^3)^2}$

17 There is usually more than one method to simplify an expression.

a Simplify $\left(\dfrac{3a^2b^3}{2c^4}\right)^{-3}$ and write your answer using only positive exponents, by:

i raising each numeral and pronumeral to the power of −3 and then simplifying

ii flipping the fraction and then raising each numeral and pronumeral to the power of 3.

b Which method did you prefer and why?

Exercise 1.2C: Challenge

18 Determine which of the following expressions is not equal to $\left(\dfrac{c^7d^3}{4a^3b^{-6}}\right)^{-2} \times \left(\dfrac{2a^2b^{-4}}{c^5d^2}\right)^{-3}$.

A $\left(\dfrac{4a^3}{b^6c^7d^3}\right)^2 \times \left(\dfrac{b^4c^5d^2}{2a^2}\right)^3$ **B** $\dfrac{c^{14}d^6}{(2a^2b^{-4})^3} \times \dfrac{(4a^3b^{-6})^2}{c^{15}d^6}$ **C** $\dfrac{(4a^3b^{-6})^2}{(2a^2b^{-4})^3c^{14}d} \times \dfrac{d}{c^{-15}}$ **D** $\left(\dfrac{4a^3b^{-6}}{c^7d^3}\right)^2 \times \dfrac{c^{15}d^6}{(2a^2b^{-4})^3}$

19 Simplify the following expression. Write your answer with positive exponents only.

$$\frac{4(x^2y^{-1}z)^3}{10x^4y^2} \div \frac{2x^5z^2}{5x^3y^4z^{-1}}$$

20 The product of a number and its reciprocal is 1.

a Find all numbers which are equal to their reciprocal.

b Explain why 0 does not have a reciprocal.

Online resources:

Interactive skillsheet
Negative exponents

Quick quiz
1.2

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Lesson 1.3

Simplifying

Learning intentions

By the end of this lesson you will be able to ...

✔ simplify and evaluate algebraic expressions.

Inter-year links

Year 7	6.4 Substitution
Year 8	5.3 Adding and subtracting algebraic terms
Year 9	2.1 Simplifying

Evaluating algebraic expressions

Key content video
Evaluating algebraic expressions

- When pronumerals are given numerical values, an algebraic expression can be evaluated by **substitution**.

  For example, if $x = 3$ and $y = 2$:

$$\begin{aligned} x^2 + 5y - \frac{x}{3} &= 3^2 + 5(2) - \frac{3}{3} \\ &= 9 + 10 - 1 \\ &= 18 \end{aligned}$$

Adding and subtracting terms

Key content video
Adding and subtracting algebraic terms

- **Like terms** are the terms of an algebraic expression that contain the same pronumerals with the same exponents. The order of the pronumerals does not matter.

  For example, the following are sets of like terms:

  → x^2 and $3x^2$

  → 3, 16 and 100

  → $2y^2z$ and $8zy^2$

  → $\frac{4w^3h^4}{t}$, $\frac{2w^3h^4}{t}$ and $\frac{10h^4w^3}{t}$

- Like terms can be added or subtracted to simplify an algebraic expression.

  For example,

$$7x^2y + 2 + 3yx^2 - 8 = 10x^2y - 6$$

Multiplying and dividing terms

Key content video
Multiplying and dividing algebraic terms

- To multiply algebraic terms, multiply the coefficients and combine the pronumerals, using the exponent laws if necessary.

  For example,

$$\begin{aligned} 3a^3b \times (-2a^2b^2) &= 3 \times (-2) \times a^{3+2} \times b^{1+2} \\ &= -6a^5b^3 \end{aligned}$$

- To divide algebraic terms, divide the coefficients and combine the pronumerals, using the exponent laws if necessary.

  For example,

$$\begin{aligned} \frac{6a^5b^3}{3a^3b} &= \frac{{}^{2}\cancel{6} \times a^5 \times b^3}{{}_{1}\cancel{3} \times a^3 \times b} \\ &= 2 \times a^{5-3} \times b^{3-1} \\ &= 2a^2b^2 \end{aligned}$$

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Worked example 1.3A Using substitution to evaluate algebraic expressions

If $a = -2$, $b = 5$ and $c = -1$, evaluate:

a a^3bc^2 **b** $2b - ab + ac$ **c** $\frac{a^2}{4} + \frac{ab}{2} - c$

THINK

a Replace a with -2, b with 5 and c with -1. If the exponent of a negative number is odd, the power is negative.

b Replace a with -2, b with 5 and c with -1. Put negative numbers in brackets to avoid errors.

c Replace a with -2, b with 5 and c with -1. Remember the fraction line acts like a pair of brackets and groups the expression in the numerator together.

WRITE

a $a^3bc^2 = (-2)^3(5)(-1)^2$
$= -8 \times 5 \times 1$
$= -40$

b $2b - ab + ac = 2(5) - (-2)(5) + (-2)(-1)$
$= 10 + 10 + 2$
$= 22$

c $\frac{a^2}{4} + \frac{ab}{2} - c = \frac{(-2)^2}{4} + \frac{(-2) \times (5)}{2} - (-1)$
$= \frac{4}{4} + (-5) + 1$
$= 1 - 5 + 1$
$= -3$

Worked example 1.3B Adding and subtracting algebraic terms

Simplify each of these expressions by collecting like terms.

a $9a + 12b - 3 + 3a + 2a - 1$ **b** $2m^2n - 8n - 10n + n^{-2} + 4nm^2$

c $\frac{3a}{4} + \frac{2}{b} - \frac{a}{4} - \frac{4}{b}$ **d** $\frac{x}{2a} + \frac{2x}{3a}$

THINK

a 1 Rearrange the expression, grouping like terms together.
2 Simplify by collecting like terms.

b 1 Rearrange the expression by grouping like terms together.
2 Simplify by collecting like terms.

c 1 Rearrange the expression by grouping like terms together.
2 Simplify by collecting like terms.

d 1 Identify the lowest common denominator (LCD).
2 Rewrite the expression using equivalent fractions with the LCD.
3 Simplify the expression by adding or subtracting the numerators.

WRITE

a $9a + 12b - 3 + 3a + 2a - 1$
$= (9a + 3a + 2a) + (12b) + (-3 - 1)$
$= 14a + 12b - 4$

b $2m^2n - 8n - 10n + n^{-2} + 4nm^2$
$= (2m^2n + 4m^2n) + (n^{-2}) + (-8n - 10n)$
$= 6m^2n + n^{-2} - 18n$

c $\frac{3a}{4} + \frac{2}{b} - \frac{a}{4} - \frac{4}{b} = \left(\frac{3a}{4} - \frac{a}{4}\right) + \left(\frac{2}{b} - \frac{4}{b}\right)$
$= \frac{2a}{4} - \frac{2}{b}$

d LCD $= 6a$

$\frac{x}{2a} + \frac{2x}{3a} = \frac{x \times 3}{2a \times 3} + \frac{2x \times 2}{3a \times 2}$
$= \frac{3x}{6a} + \frac{4x}{6a}$
$= \frac{7x}{6a}$

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Worked example 1.3C Multiplying and dividing algebraic terms

Simplify these expressions by writing each one as a single term with positive exponents only.

a $2ab^2 \times 6a^3c$ **b** $12pqr^2 \div (4qr)$ **c** $\frac{ab}{12} \div \frac{3b}{5}$

THINK

a Multiply the coefficients and apply the product of powers law to multiply the terms with the same base by adding their exponents.

b 1 Write the expression as a fraction.

2 Simplify the coefficients by dividing them both by their HCF of 4.

3 Apply the quotient of powers law to divide the terms with the same base by subtracting their exponents.

4 Simplify. Remember that any non-zero base raised to a zero exponent is equal to 1.

c 1 Write the expression as the product of two fractions.

2 Write the expression as a single fraction by multiplying the numerators together and the denominators together.

3 Look for common factors. Divide both the numerator and the denominator by any common factors.

WRITE

a $2ab^2 \times 6a^3c = 12 \times a^{1+3}b^2c$

$= 12a^4b^2c$

b $12pqr^2 \div (4qr) = \frac{12pqr^2}{4qr}$

$= \frac{\cancel{12}^{3}pqr^2}{\cancel{4}^{1}qr}$

$= \frac{3pqr^2}{qr}$

$= 3pq^{1-1}r^{2-1}$

$= 3pq^0r^1$

$= 3pr$

c $\frac{ab}{12} \div \frac{3b}{5} = \frac{ab}{12} \times \frac{5}{3b}$

$= \frac{a\cancel{b} \times 5}{12 \times 3\cancel{b}}$

$= \frac{a \times 5}{12 \times 3}$

$= \frac{5a}{36}$

Helpful hints

- ✔ The order in which you write the coefficients and pronumerals in an algebraic term doesn't matter. But it is easier to identify like terms if you write the pronumerals in alphabetical order. For example,

$$acb + cb - ba + 12bca + 8ab - 6bac = abc + 12abc - 6abc + bc - ab + 8ab$$
$$= 7abc + 7ab + bc$$

- ✔ Fractions can be written in exponent form to identify like terms.
- ✔ Fractions which include pronumerals can be added, subtracted, multiplied and divided in the same way as numerical fractions.

  When adding and subtracting, identify the lowest common denominator of like terms, and then add or subtract the numerators.

  $\frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd}$ $\frac{a}{b} - \frac{c}{d} = \frac{ad - cb}{bd}$

  When multiplying and dividing, apply the usual rules for multiplying and dividing fractions.

  $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$ $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$

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Learning pathways

1, 2–4(1st column), 5, 6(a–f), 8–12, 14(1st column) | 2–4, 6, 7, 9, 11, 12, 14(a–d) | 4, 6–8, 13–15

ANS p679

Exercise 1.3A: Understanding and fluency

1 For each of the following terms:

i identify the coefficient **ii** write an example of a like term.

a $3abcd$ **b** $-4mn$ **c** xy^2 **d** $9k^2m^4p$

WE 1.3A **2** If $a = 1$, $b = 3$ and $c = -2$, evaluate:

a abc **b** $ab + bc$ **c** $ac - 2bc$ **d** a^4b^2

e $3ab^2c - ac$ **f** $7ac + 3bc - 4ac$ **g** $a^3b^2 + bc^3$ **h** $2a^2b + 3ac^2 - b^3$

i $\frac{a^2b}{4} + \frac{ab^2}{2}$ **j** $\frac{b}{6} + \frac{a}{3} + \frac{c}{2}$ **k** $\frac{a^2b}{4} + \frac{b^2c}{2} - \frac{ac^2}{2}$ **l** $\frac{abc - ab - ac - bc}{2}$

WE 1.3B **3** Simplify each of these expressions by collecting like terms.

a $8x - 5x + 6x$ **b** $3ab - 4ab - 9ab$ **c** $4y^2 + y^2 + 5y^2$

d $7m^2n + 2m^2n - m^2n$ **e** $4a + 6d + 8a + 3d$ **f** $k - 3m + 5m + 7k$

g $\frac{4}{a} + 8y - \frac{2}{a} - 11y$ **h** $\frac{a^2}{3} + \frac{6a}{b} - \frac{5a^2}{3} + \frac{a}{b}$ **i** $\frac{3xy}{7} + \frac{2x^2}{y} - \frac{xy}{7} + \frac{x^2}{y}$

j $\frac{7a}{8} - \frac{3ba}{4} + \frac{ab}{2} - \frac{a}{12}$ **k** $\frac{nm^2}{2} - \frac{mn^2}{3} + \frac{m^2n}{4} - \frac{n^2m}{5}$ **l** $xy - \frac{4xy}{5} + \frac{3x}{4} - \frac{2y}{3} + \frac{xy}{2} - x + y$

4 Simplify each of these expressions by collecting like terms.

a $5a + 2b - a + 3b + 7a - 5b$ **b** $6mn - 3n - n + 2n^2 + m - 7mn$

c $x + 4x^2 - 8x + 1 + 5x - x^2$ **d** $2x^2y - 5x^2y - 4xy + 3 + 7xy - 8$

e $\frac{9p^2}{2a} - \frac{3p}{a^2} + \frac{2p^3}{b^2} + \frac{6p^2}{a} - \frac{9}{b} + \frac{3p}{4a^2}$ **f** $\frac{a^2b}{c} + \frac{4a^2}{2c} + \frac{8ba^2}{3c} - \frac{b^2}{4c} + \frac{3ab^2}{5c} - \frac{5a^2}{6c}$

g $\frac{5ab}{c} + \frac{2ac}{b} + \frac{3bc}{a}$ **h** $\frac{3ab}{4c} + \frac{4ba}{3c}$

i $\frac{6x^2y}{z} - \frac{2xy}{3z} + \frac{5yx}{z} - \frac{yx^2}{3z}$ **j** $\frac{14map}{3b} + \frac{2amp}{5b} + \frac{pam}{6b} - \frac{mpa}{9b}$

5 Simplify each of the following expressions by writing each one as a single term with positive exponents.

a $5ab \times 2cd$ **b** $-7xy \times 3mn$ **c** $4kp \times k$ **d** $abc \div b$ **e** $6mnp \div mp$ **f** $8wx \div 4wy$

WE 1.3C **6** Simplify each of the following expressions by writing each one as a single term using only positive exponents.

a $4x^2y^2 \times x^3y$ **b** $3ab^2 \times 2a^5b$ **c** $4m^5n^{-3} \times 6m^2n$

d $\frac{5abc}{2} \div \frac{ab}{5}$ **e** $\frac{x^5y^3z^2}{2} \div \frac{2xyz^2}{5}$ **f** $\frac{m^5n^3}{2} \times \frac{2m^2n^2}{3}$

g $6x^2y \times (xy)^2$ **h** $\frac{4xy}{5} \times \frac{3x}{4} \times \frac{2y}{3}$ **i** $\frac{ab}{5} \times \left(\frac{3a}{4}\right)^2 \div \frac{2b}{3}$

7 Simplify each of the following expressions by writing each one as a single term using only positive exponents.

a $\frac{2xy}{z} \times \left(\frac{3xy^2}{z^2}\right)^2$ **b** $\frac{2xy}{z} \div \left(\frac{3xy^2}{z^2}\right)^2$ **c** $\left(\frac{2}{abc}\right)^3 \times \left(\frac{2}{a^2b}\right)^{-1}$

d $\left(\frac{2}{abc}\right)^3 \div \left(\frac{2}{a^2b}\right)^{-1}$ **e** $\frac{abc}{xyz} \times \left(\frac{xy^2z^3}{abc}\right)^{-2}$ **f** $\frac{abc}{xyz} \div \left(\frac{xy^2z^3}{abc}\right)^{-2}$

Exercise 1.3B: Problem solving and reasoning

8 The surface area of this box can be calculated using the expression $2lb + 2lh + 2bh$. Calculate the surface area of a box with each of the following sets of dimensions.

a $l = 25$ cm, $b = 15$ cm, $h = 10$ cm

b $l = 0.5$ m, $b = 0.3$ m, $h = 0.2$ m

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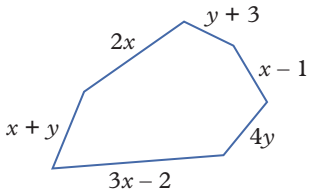
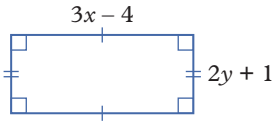
9 Write the perimeter of each of these shapes as an algebraic expression in simplest form.

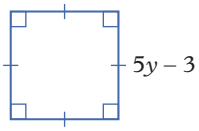
a



b



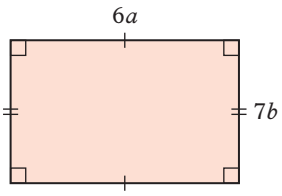
c



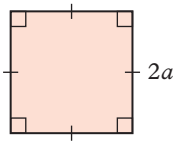
10 Calculate the perimeter of each shape in question **9** if $x = 7$ cm and $y = 2$ cm.

11 Write the area of each shaded shape below as an algebraic expression in simplest form. When required, leave your answer as an exact value.

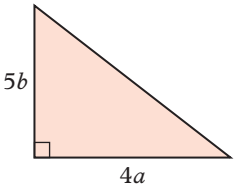
a



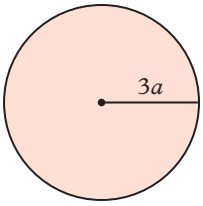
b



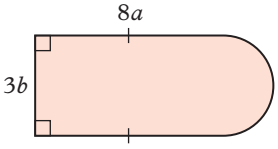
c



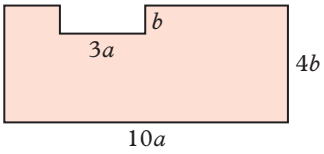
d



e

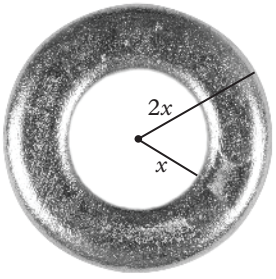


f



12 Calculate the area of each shape in question **11** if $a = 4$ m and $b = 3$ m. Leave your answers as exact values containing π.

13 Consider the top surface of a metal washer shaped like an annulus as shown on the right.



a Write an algebraic expression for the area of the top surface of one washer.

b List three possible values for x that will result in an area of between 400 mm^2 and 500 mm^2 for the surface area of the top of the washer.

c Write an algebraic expression for the total length of the inner and outer circular edges of one washer.

d Use the values you wrote for part **b** to calculate the total length of the inner and outer circular edges of one washer, correct to one decimal place.

14 Use the order of operations and exponent laws to simplify each of the following expressions.

a $2a \div 2 + 5a \times 3b - 4a$

b $19xy^2 + 8x \div 4 + 5x^2y^2 \div x - 3x$

c $3(2a)^2 + (5ab^3)^2 - 2(3a)^2$

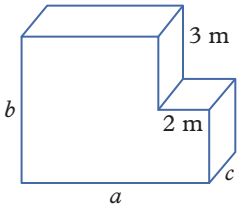
d $(-6xy)^2 + 4x^2y^2 \div x^2 + (3y)^2 - 5xy^2 \times 4x$

e $2a^2b \times 4b + 3ab^2 \times 5a$

f $\dfrac{6xy^2z}{2xz} + \dfrac{5xyz}{y} - \dfrac{7x^2y^2z}{14x^2z} + \dfrac{xz(2y)^2}{2y^2}$

Exercise 1.3C: Challenge

15 Consider the object on the right. Hint: All angles are right angles, meaning all opposite sides are parallel and of equal length.



a Write an expression for the volume of the shape in cubic metres (m^3) in terms of a, b and c in its simplest form.

b Write an expression for the surface area of the shape in square metres (m^2) in terms of a, b and c in its simplest form.

Online resources:

- **Interactive skillsheet** Like terms
- **Interactive skillsheet** Multiplying terms
- **Interactive skillsheet** Dividing terms
- **Interactive skillsheet** Simplifying expressions with algebraic fractions
- **Quick quiz** 1.3

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Lesson 1.4

Expanding

Learning intentions

By the end of this lesson you will be able to ...

✔ expand algebraic expressions.

Inter-year links

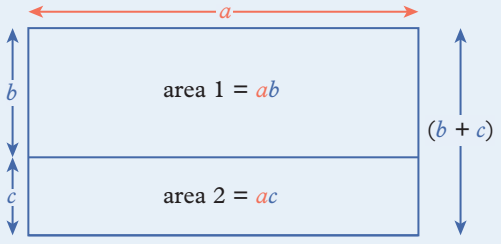
Year 7	6.3 Order of operations in algebra
Year 8	5.6 Expanding
Year 9	2.2 Expanding

Expanding one pair of brackets

Key content video
Expanding algebraic expressions

- The **distributive law** is used to distribute or **expand** products over addition and subtraction.

$$a(b + c) = ab + ac$$



total area = $a(b + c) = ab + ac$

Expanding binomial products

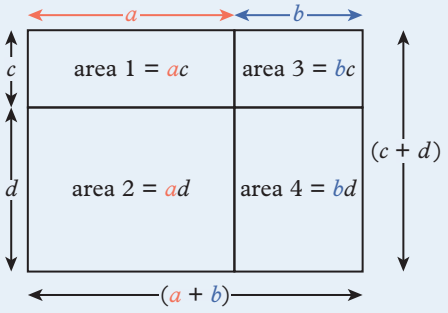
Key content video
Expanding binomial products

- A **binomial** is an algebraic expression that contains two terms. Examples are:

$$x + 1 \qquad a^3 - 3a^4 \qquad 21m^2 + n$$

- The distributive law can be used to expand the product of two binomials.

$$(a + b)(c + d) = ac + ad + bc + bd$$



total area = $(a + b)(c + d) = ac + ad + bc + bd$

- Expanding binomial products often requires simplification using the exponent laws and the identification of like terms.
- The **difference of two squares** is a specific form of expansion with the simplified rule:

$$(a + b)(a - b) = a^2 - b^2$$

- The **expansion of a perfect square** is a specific form of expansion with the simplified rule:

$$(a + b)^2 = a^2 + 2ab + b^2 \qquad (a - b)^2 = a^2 - 2ab + b^2$$

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Worked example 1.4A Expanding one pair of brackets

Expand each of these algebraic expressions to remove the brackets.

a $4(a + 11)$ **b** $5k(k - 6)$ **c** $6g^3(g^4 + 3g)$ **d** $cd(3d - c^2)$

THINK

a 1 Multiply each term inside the brackets by the term outside the brackets.
2 Simplify.

b 1 Multiply each term inside the brackets by the term outside the brackets.
2 Simplify. Remember to take care with the + and – signs when simplifying.

c 1 Multiply each term inside the brackets by the term outside the brackets.
2 Simplify. Apply the product of powers law, multiplying the terms with the same base by adding their exponents.

d 1 Multiply each term inside the brackets by the term outside the brackets.
2 Simplify. Apply the product of powers law, multiplying the terms with the same base by adding their exponents.

WRITE

a $4(a + 11) = 4 \times a + 4 \times 11$

$= 4a + 44$

b $5k(k - 6) = 5k \times k + 5k \times (-6)$

$= 5k^2 - 30k$

c $6g^3(g^4 + 3g) = 6g^3 \times g^4 + 6g^3 \times 3g$

$= 6g^7 + 18g^4$

d $cd(3d - c^2) = cd \times 3d + cd \times (-c^2)$

$= 3cd^2 - c^3d$

Worked example 1.4B Expanding a binomial product

Expand each of these algebraic expressions to remove the brackets.

a $(x + 4)(y + 9)$ **b** $(-f + 3)(4f - 6)$ **c** $(2z^2 + z)(z + 8)$ **d** $(h^2g - 2)(h + 4g^3)$

THINK

a 1 Multiply each term inside the second pair of brackets, y and 9, by the first term in the first pair of brackets, x. Then multiply them by the second term in the first pair of brackets, 4.
2 Simplify each term.

b 1 Multiply each term inside the second pair of brackets, $4f$ and -6, by the first term in the first pair of brackets, $-f$. Then multiply them by the second term in the first pair of brackets, 3.
2 Simplify each term.
3 Simplify any like terms.

WRITE

a $(x + 4)(y + 9)$

$= x \times y + x \times 9 + 4 \times y + 4 \times 9$

$= xy + 9x + 4y + 36$

b $(-f + 3)(4f - 6)$

$= (-f) \times 4f + (-f) \times (-6) + 3 \times 4f + 3 \times (-6)$

$= -4f^2 + 6f + 12f - 18$

$= -4f^2 + 18f - 18$

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c 1 Multiply each term inside the second pair of brackets, z and 8, by the first term in the first pair of brackets, $2z^2$. Then multiply them by the second term in the first pair of brackets, z.

2 Use the exponent laws to simplify each term.

3 Simplify any like terms.

d 1 Multiply each term inside the second pair of brackets, h and $4g^3$, by the first term in the first pair of brackets, h^2g. Then multiply them by the second term in the first pair of brackets, -2.

2 Use the exponent laws to simplify each term.

c

$(2z^2 + z)(z + 8)$

$= 2z^2 \times z + 2z^2 \times 8 + z \times z + z \times 8$

$= 2z^3 + 16z^2 + z^2 + 8z$

$= 2z^3 + 17z^2 + 8z$

d

$(h^2g - 2)(h + 4g^3)$

$= h^2g \times h + h^2g \times 4g^3 + (-2) \times h + (-2) \times 4g^3$

$= gh^3 + 4g^4h^2 - 2h - 8g^3$

Helpful hints

- ✔ It's important to be able to recognise when an expression is the difference of two squares, or a perfect square. These are both very useful in algebra!
- ✔ Remember to pay close attention to minus signs when you are expanding an expression! Using brackets around negative terms can make it clearer.
- ✔ To make sure you correctly expand the product of two binomials, it may help to include the following step:

$$(a + b)(c + d) = a(c + d) + b(c + d) = ac + ad + bc + bd.$$

Learning pathways

1–2(1st, 2nd columns), 3, 5–6(1st column), 7–9	1–2(3rd, 4th columns), 3, 4, 5–6(2nd column), 7–10, 12	4, 5–6(3rd column), 7–9, 11–14

ANS p679

Exercise 1.4A: Understanding and fluency

WE 1.4A **1** Expand each of these algebraic expressions to remove the brackets.

a $3(a + 7)$ **b** $5(b - 4)$ **c** $2(6 - c)$ **d** $4(1 + d)$

e $-6(e + 3)$ **f** $-7(f - 2)$ **g** $-3(g - 1)$ **h** $-9(5 - h)$

i $x(x + 2)$ **j** $2y(y - 4)$ **k** $-k(k + 5)$ **l** $-3p(p - 1)$

m $m(3n + k)$ **n** $4a(3b + 6c)$ **o** $-w(w - 8x)$ **p** $-2p(2 + 5p)$

2 Expand each of these algebraic expressions to remove the brackets.

a $3(a - 4)$ **b** $-5(b - 3)$ **c** $4(2x + 3)$ **d** $-7(2y + 4)$

e $5a(2 - a)$ **f** $2m(m^2 + 3)$ **g** $8ab(3a + 1)$ **h** $4w(2x - w)$

i $5k^2(3 - 2k^2)$ **j** $6x^2y(2x - y^3)$ **k** $5cd(3c - 2d)$ **l** $-2a^2c(4a - 3b)$

3 Expand and simplify each of these algebraic expressions.

a $2(x + 5) + 7x$ **b** $4(y - 7) + 15$ **c** $m(n + 6) - 3m$

d $-3x(2 - x) + 5x^2 + x$ **e** $7p^3 - 8p + 5p(1 - p^2)$ **f** $x^4(x^3 - 2x) - x^5$

4 Expand and simplify each of these algebraic expressions.

a $4(x - 2) + 2(x + 5)$ **b** $m(m + 8) + 6(m - 3)$ **c** $2a(3a + 1) - 3(2 - 4a)$

d $k^2(k^3 + 2) + 5(k^2 + 1)$ **e** $3p^4(p^2 - 2) + 2p(p^3 + 6)$ **f** $5y^3(2y^5 + 3) - 2y(7y^2 - 4)$

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WE 1.4B **5** Expand each of these algebraic expressions to remove the brackets.

a $(a + 4)(b + 2)$ **b** $(c + 3)(d + 6)$ **c** $(m + 7)(n + 5)$
d $(x + 6)(y + 8)$ **e** $(k + 5)(p - 1)$ **f** $(f - 2)(g + 3)$
g $(x - 9)(y - 4)$ **h** $(2m + 3)(n + 1)$ **i** $(4x + 5)(3y - 7)$
j $(3a - 2b)(5c - 4d)$ **k** $(y + 9)(y - 1)$ **l** $(5 - k)(6 - k)$
m $(6p + 5)(2p + 7)$ **n** $(2x + 3)(4x - 1)$ **o** $(5y - 2)(3y - 4)$

6 Expand each of these algebraic expressions to remove the brackets.

a $(2a - b)(a + 3)$ **b** $(4t + u)(t - u)$ **c** $(2x + 3y)(x + 4y)$
d $(5 - x)(x + 4)$ **e** $(6x^2 + y)(x + 3y)$ **f** $(ab^2 + 1)(b^2 - 2a)$
g $(a + b)(b + c)$ **h** $(b + 3c)(b + d)$ **i** $(a + 2b)(3 - c)$
j $(a^3 + b^2)(3ab + 2)$ **k** $(x^4 - 3z)(5x - 2y)$ **l** $(c^2 + 4a^3b)(c^2 - 4a^3b)$

Exercise 1.4B: Problem solving and reasoning

7 Some binomial products are in a recognisable form.

a Expand each of these binomial products.

i $(x + 2)(x - 2)$ **ii** $(k + 7)(k - 7)$ **iii** $(m + 8)(m - 8)$
iv $(w - 6)(w + 6)$ **v** $(y - 1)(y + 1)$ **vi** $(a + b)(a - b)$

b Describe any pattern or shortcut you can see in part **a**. What is special about the two binomial factors that were multiplied together in part **a**?

c The binomial products in this question are all examples of the 'difference of two squares'. Why do you think the rule is called the 'difference of two squares'?

d Does it matter whether the product is $(a + b)(a - b)$ or $(a - b)(a + b)$? Explain.

e Use the 'difference of two squares' rule to expand each of these binomial products.

i $(a - 3)(a + 3)$ **ii** $(x + 10)(x - 10)$ **iii** $(m + n)(m - n)$
iv $(3 + x)(3 - x)$ **v** $(1 + d)(1 - d)$ **vi** $(2x - 5)(2x + 5)$
vii $(2 - 3k)(2 + 3k)$ **viii** $(4g + h)(4g - h)$ **ix** $(5y + 2w)(5y - 2w)$

8 This question considers another recognisable form of binomial products.

a Expand each of these binomial products.

i $(x + 3)(x + 3)$ **ii** $(y + 8)(y + 8)$ **iii** $(a + b)(a + b)$
iv $(k - 5)(k - 5)$ **v** $(p - 6)(p - 6)$ **vi** $(x - y)(x - y)$

b Describe any pattern or shortcut you can see in part **a**. What is special about the two binomial factors that were multiplied together in part **a**?

c These are all examples of 'perfect squares'. Why do you think the rule is called the 'perfect square' rule?

d Use the 'perfect square' rule to expand each of these algebraic expressions.

i $(a + 2)^2$ **ii** $(x + 5)^2$ **iii** $(p + 4)^2$
iv $(y + 10)^2$ **v** $(k + 9)^2$ **vi** $(m + n)^2$
vii $(5 + x)^2$ **viii** $(1 + 2d)^2$ **ix** $(3w + 7)^2$

e Why is $(a - b)^2 = a^2 - 2ab + b^2$ also a perfect square?

f Use the 'perfect square' rule to expand each of these algebraic expressions.

i $(a - 3)^2$ **ii** $(b - 4)^2$ **iii** $(x - 9)^2$
iv $(n - 11)^2$ **v** $(2 - x)^2$ **vi** $(g - h)^2$
vii $(4w - 1)^2$ **viii** $(5 - 2y)^2$ **ix** $(3m - 2p)^2$

9 Use the 'difference of two squares' and 'perfect square' rules to expand and simplify each of these expressions.

a $(x^2 + 2)(x^2 - 2)$ **b** $(x^2 - 5)(x^2 + 5)$ **c** $(x^2 + y^2)(x^2 - y^2)$
d $(x^2 + 4)^2$ **e** $(x^2 - 3)^2$ **f** $(x^3 - 1)^2$

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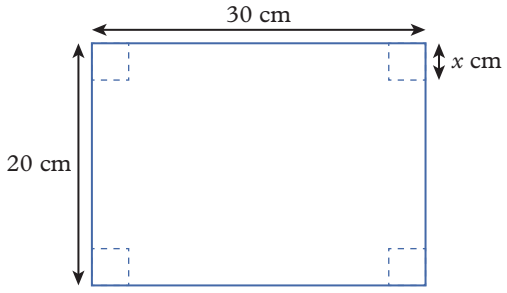
10 Expand each of these algebraic expressions to remove the brackets.

a $(2x + 3)(x^2 + 3x - 1)$ **b** $(x + 2)(2x^2 + x - 1)$ **c** $(2 - x)(1 - x^2 + 2x)$

d $(x^2 + 2)(2x^2 + x - 1)$ **e** $(2x^2 - 3)(x^2 - 3x + 1)$ **f** $(x^2 + 2x)(3x^2 - x - 3)$

11 A rectangular piece of cardboard is 30 cm long and has a width of 20 cm. A square of side length x cm is cut from each corner so that the cardboard can be folded to form an open box.



a Write an expression containing x for:

i the length of the box **ii** the width of the box **iii** the height of the box.

b Draw a labelled diagram of the open box.

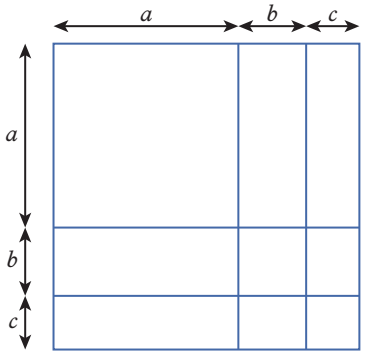
c Write expressions for the following, simplifying by expanding to remove any brackets.

i the volume of space contained in the box **ii** the inner surface area of the box

d If $x = 3$ cm, calculate:

i the volume of space contained in the box **ii** the inner surface area of the box.

Exercise 1.4C: Challenge



12 It is possible to expand brackets with three terms in each bracket.

a Use the diagram on the right to help you expand $(a + b + c)^2$.

b Expand $(x + 2y + 3)^2$.

c Expand $(2a - 3b + c)(2a + 3b - c)$.

13 Expand each of the following.

a $(x + 1)^3$ **b** $(x - 3)^3$

c $(x + 2)^4$ **d** $(x + y)^2(x - y)^2$

14 In the diagram on the right, the area of the shaded rectangle is $(x - 2)(x - 5)$, which can be expanded as $x^2 - 7x + 10$.

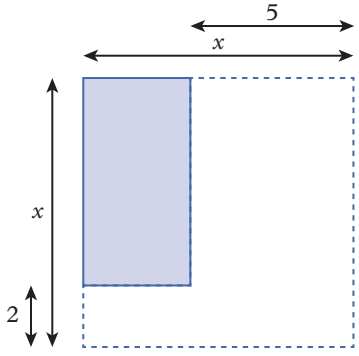
a Copy the diagram into your workbook and draw:

i the outline of a rectangle with area $2 \times x$ square units

ii the outline of a rectangle with area $5 \times x$ square units

iii the outline of a rectangle with area 10 square units.

b Explain why $+10$ appears in the expansion of $(x - 2)(x - 5)$, even though the rectangle with area 10 square units is not shaded.



Online resources:

- **Interactive skillsheet** Expanding over one pair of brackets
- **Interactive skillsheet** Expanding binomial products
- **Investigation** Binomial expansion and Pascal's Triangle
- **CAS instructions** Expanding
- **Quick quiz** 1.4

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Lesson 1.5

Algebraic fractions

Learning intentions

By the end of this lesson you will be able to ...

✔ simplify expressions including algebraic fractions.

Inter-year links

Year 7	3.2 Equivalent fractions
Year 8	5.5 Dividing algebraic terms
Year 9	2.1 Simplifying

Simplifying algebraic fractions

- **Algebraic fractions** are fractions that contain at least one pronumeral. They can be simplified, just like any other fractions, by identifying common factors of the numerator and the denominator.
- The fastest way to simplify a fraction is to divide the numerator and the denominator by their highest common factor (HCF).

Multiplying and dividing algebraic fractions

- The product of two algebraic fractions is the product of the numerators over the product of the denominators. For example,

$$\frac{3x}{2} \times \frac{x}{x+1} = \frac{3x \times x}{2 \times (x+1)}$$
$$= \frac{3x^2}{2(x+1)}$$

Key content video
Multiplying and dividing algebraic fractions

- The quotient of two algebraic fractions is found by multiplying the dividend (the fraction before the division sign) by the reciprocal of the divisor (the fraction after the division sign). For example,

$$\frac{2x}{y} \div \frac{5}{x} = \frac{2x}{y} \times \frac{x}{5}$$
$$= \frac{2x^2}{5y}$$

Adding and subtracting algebraic fractions

- Adding and subtracting algebraic fractions is no different from adding and subtracting any other fractions.
  1. Identify the lowest common denominator (LCD) by identifying the lowest common multiple (LCM) of the denominators.
  2. Write each fraction as an equivalent fraction with the LCD.
  3. The sum of the algebraic fractions with a common denominator is then found by adding the numerators. The difference of algebraic fractions with a common denominator is found by subtracting the numerators.

     For example, the LCD of $\frac{x+1}{4}$ and $\frac{x}{6}$ is 12, so:

Key content video
Adding and subtracting algebraic fractions

$$\frac{x+1}{4} + \frac{x}{6} = \frac{x+1}{4} \times \frac{3}{3} + \frac{x}{6} \times \frac{2}{2}$$
$$= \frac{3(x+1)}{12} + \frac{2x}{12}$$
$$= \frac{3x+3+2x}{12}$$
$$= \frac{5x+3}{12}$$

$$\frac{x+1}{4} - \frac{x}{6} = \frac{x+1}{4} \times \frac{3}{3} - \frac{x}{6} \times \frac{2}{2}$$
$$= \frac{3(x+1)}{12} - \frac{2x}{12}$$
$$= \frac{3x+3-2x}{12}$$
$$= \frac{x+3}{12}$$

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Worked example 1.5A Simplifying algebraic fractions

Simplify each of these algebraic fractions.

a $\dfrac{8(2x+1)}{16}$

b $\dfrac{(x+9)(3x+1)}{(x+2)(x+9)}$

THINK

a 1 Look for factors common to both the numerator and the denominator to determine the HCF. The HCF is 8.

2 Divide both the numerator and the denominator by 8.

b 1 Look for factors common to both the numerator and the denominator to determine the HCF. The HCF is $(x+9)$.

2 Divide both the numerator and the denominator by $(x+9)$.

WRITE

a $\dfrac{8(2x+1)}{16} = \dfrac{\cancel{8}^{1}(2x+1)}{\cancel{16}^{2}}$

$= \dfrac{2x+1}{2}$

b $\dfrac{(x+9)(3x+1)}{(x+2)(x+9)} = \dfrac{\cancel{(x+9)}^{1}(3x+1)}{(x+2)\cancel{(x+9)}^{1}}$

$= \dfrac{3x+1}{x+2}$

Worked example 1.5B Adding and subtracting algebraic fractions with binomial numerators

Write each of these expressions as one fraction in simplest form.

a $\dfrac{2x+3}{7} + \dfrac{x-1}{3}$

b $\dfrac{7x-1}{10} - \dfrac{3x+1}{6}$

THINK

a 1 The LCM of 7 and 3 is 21. So write each fraction as an equivalent fraction with an LCD of 21.

2 Write the expression as a single fraction by adding the numerators.

3 Expand the brackets.

4 Simplify the numerator by adding and subtracting like terms.

b 1 The LCM of 10 and 6 is 30. So write each fraction as an equivalent fraction with an LCD of 30.

2 Write the expression as a single fraction by subtracting the numerators.

3 Expand the brackets.

4 Simplify the numerator by adding and subtracting like terms.

5 Simplify by dividing the numerator and the denominator by their HCF of 2.

WRITE

a $\dfrac{2x+3}{7} + \dfrac{x-1}{3} = \dfrac{2x+3}{7} \times \dfrac{3}{3} + \dfrac{x-1}{3} \times \dfrac{7}{7}$

$= \dfrac{3(2x+3)}{21} + \dfrac{7(x-1)}{21}$

$= \dfrac{3(2x+3)+7(x-1)}{21}$

$= \dfrac{6x+9+7x-7}{21}$

$= \dfrac{13x+2}{21}$

b $\dfrac{7x-1}{10} - \dfrac{3x+1}{6} = \dfrac{7x-1}{10} \times \dfrac{3}{3} - \dfrac{3x+1}{6} \times \dfrac{5}{5}$

$= \dfrac{3(7x-1)}{30} - \dfrac{5(3x+1)}{30}$

$= \dfrac{3(7x-1)-5(3x+1)}{30}$

$= \dfrac{21x-3-15x-5}{30}$

$= \dfrac{6x-8}{30}$

$= \dfrac{3x-4}{15}$

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Worked example 1.5C Multiplying and dividing algebraic fractions

Simplify the following expressions.

a $\dfrac{x(x-2)}{(x+3)(2x+1)} \times \dfrac{(x+1)(2x+1)}{(x-2)(x+3)}$

b $\dfrac{x+4}{x^2} \div \dfrac{3(x+4)}{x-9}$

THINK

a 1 Write the expression as a single fraction by multiplying the numerators together and the denominators together.

2 Look for common factors. Divide both the numerator and the denominator by $(x-2)$ and $(2x+1)$.

3 Simplify.

b 1 Write the expression as the product of two fractions.

2 Write the expression as a single fraction by multiplying the numerators together and the denominators together.

3 Look for common factors. Divide both the numerator and the denominator by $(x+4)$.

WRITE

a $\dfrac{x(x-2)}{(x+3)(2x+1)} \times \dfrac{(x+1)(2x+1)}{(x-2)(x+3)}$

$= \dfrac{x(x-2) \times (x+1)(2x+1)}{(x+3)(2x+1) \times (x-2)(x+3)}$

$= \dfrac{x\cancel{(x-2)}^{1}(x+1)\cancel{(2x+1)}^{1}}{(x+3)\cancel{(2x+1)}^{1}\cancel{(x-2)}^{1}(x+3)}$

$= \dfrac{x(x+1)}{(x+3)(x+3)}$

$= \dfrac{x(x+1)}{(x+3)^2}$

b $\dfrac{x+4}{x^2} \div \dfrac{3(x+4)}{x-9} = \dfrac{x+4}{x^2} \times \dfrac{x-9}{3(x+4)}$

$= \dfrac{(x+4) \times (x-9)}{x^2 \times 3(x+4)}$

$= \dfrac{\cancel{(x+4)}^{1}(x-9)}{3x^2\cancel{(x+4)}^{1}}$

$= \dfrac{x-9}{3x^2}$

Worked example 1.5D Adding and subtracting algebraic fractions

Simplify the following expressions.

a $\dfrac{4}{x(x+1)} + \dfrac{3}{(x+1)(x-1)}$

b $\dfrac{7x}{x+1} - \dfrac{x-3}{x}$

THINK

a 1 The LCM of $x(x+1)$ and $(x+1)(x-1)$ is $x(x+1)(x-1)$. Write each fraction as an equivalent fraction with the denominator $x(x+1)(x-1)$.

2 Write the expression as a single fraction by subtracting the numerators.

3 Expand the brackets in the numerator.

4 Simplify the numerator by adding and subtracting like terms.

WRITE

a $\dfrac{4}{x(x+1)} + \dfrac{3}{(x+1)(x-1)}$

$= \dfrac{4}{x(x+1)} \times \dfrac{(x-1)}{(x-1)} + \dfrac{3}{(x+1)(x-1)} \times \dfrac{x}{x}$

$= \dfrac{4(x-1)}{x(x+1)(x-1)} + \dfrac{3x}{x(x+1)(x-1)}$

$= \dfrac{4(x-1)+3x}{x(x+1)(x-1)}$

$= \dfrac{4x-4+3x}{x(x+1)(x-1)}$

$= \dfrac{7x-4}{x(x+1)(x-1)}$

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b **1** The LCM of $(x + 1)$ and x is $x(x + 1)$. So write each fraction as an equivalent fraction with a denominator of $x(x + 1)$.

2 Write the expression as a single fraction by subtracting the numerators.

3 Expand the binomial product in the numerator.

4 Simplify the numerator by adding and subtracting like terms.

b
$$\frac{7x}{x+1} - \frac{x-3}{x} = \frac{7x}{x+1} \times \frac{x}{x} - \frac{x-3}{x} \times \frac{x+1}{x+1}$$
$$= \frac{7x \times x}{x(x+1)} - \frac{(x-3)(x+1)}{x(x+1)}$$
$$= \frac{7x^2 - (x-3)(x+1)}{x(x+1)}$$
$$= \frac{7x^2 - (x^2 - 3x + x - 3)}{x(x+1)}$$
$$= \frac{7x^2 - (x^2 - 2x - 3)}{x(x+1)}$$
$$= \frac{7x^2 - x^2 + 2x + 3}{x(x+1)}$$
$$= \frac{6x^2 + 2x + 3}{x(x+1)}$$

Helpful hints

✔ You don't need to keep the brackets around a binomial factor if the other factors have been removed and there is no exponent. For example,
$$\frac{2(x+3)}{4} = \frac{x+3}{2}$$

✔ Remember, if you want to turn an algebraic fraction into a sum of algebraic terms, you must divide *every* term in the numerator by the denominator.
$$\frac{x+9}{3} \neq x + 3$$

Learning pathways

1–5, 6(a, c, e, g, i, k), 7(a–d), 8, 9(1st, 2nd columns), 10, 12(a–c)	4–5(2nd column), 6(b, d, f, h, j, l), 7(e–h), 8–15, 16(a–c)	4–6(3rd column), 7(h–l), 8, 10, 12, 15–17

ANS p681

Exercise 1.5A: Understanding and fluency

1 Simplify each of these algebraic fractions.

a $\frac{3x}{6}$ **b** $\frac{9x}{9xy}$ **c** $\frac{8x}{2x}$ **d** $\frac{4xy}{6y}$

e $\frac{2(x+3)}{10}$ **f** $\frac{8(x-5)}{4}$ **g** $\frac{x(x-4)}{x}$ **h** $\frac{3x(x+1)}{3x}$

2 Simplify each of these algebraic fractions.

a $\frac{(x-2)(x+5)}{(x+1)(x-2)}$ **b** $\frac{(x-6)(x+4)}{(x-2)(x+4)}$ **c** $\frac{(x-1)(x+3)}{(x+3)(x+1)}$

d $\frac{(x-7)(x+2)}{(x-7)(x-8)}$ **e** $\frac{(x+4)(x-5)}{(x+4)(x-5)}$ **f** $\frac{(x-1)(x+1)}{(x+1)(x-1)}$

g $\frac{2(x-1)(x+11)}{(x+11)(x+3)}$ **h** $\frac{(x+8)(x-7)}{4(x+9)(x-7)}$ **i** $\frac{10(x-4)(x+4)}{30(x-12)(x-4)}$

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WE 1.5A **3** Simplify each of these algebraic fractions.

a $\frac{8x}{4xy}$ **b** $\frac{14abc}{7ac}$ **c** $\frac{5(a+1)}{a(a+1)}$

d $\frac{x(x+3)}{2(x+3)}$ **e** $\frac{5(m-2)}{15m(m-2)}$ **f** $\frac{(n+3)(n+7)}{2(n+3)}$

g $\frac{3y^2(y+x)}{y(x+y)}$ **h** $\frac{x^2(x-4)}{x(x+1)(x-4)}$ **i** $\frac{5(y-3)(y+4)}{(y+4)(y-4)}$

j $\frac{(t+2)(r+3)}{(r+3)(r+2)}$ **k** $\frac{(x+2)(x-1)(2x+7)}{(2x+7)(x-4)(x+2)}$ **l** $\frac{(p^2+2)(p+6)(p-2)}{3(p+6)(p+2)(p^2+2)}$

4 Simplify each of these expressions.

a $\frac{3x}{x+2} \times \frac{x+2}{6x}$ **b** $\frac{x-3}{x} \times \frac{5x}{x-3}$ **c** $\frac{x+4}{x+1} \times \frac{x+1}{x+4}$

d $\frac{6x}{x-5} \div \frac{2x}{x-5}$ **e** $\frac{x+7}{4x} \div \frac{x+7}{20x}$ **f** $\frac{8x}{2x+1} \div \frac{8}{2x+1}$

WE 1.5B **5** Write each of these expressions as a single fraction in simplest form.

a $\frac{c+1}{4} + \frac{c}{6}$ **b** $\frac{m-2}{3} + \frac{m+1}{4}$ **c** $\frac{x-2}{2} - \frac{x}{5}$ **d** $\frac{4k+1}{4} - \frac{2k}{5}$

e $\frac{5x+1}{8} + \frac{x-3}{10}$ **f** $\frac{x}{6} - \frac{x-1}{3}$ **g** $\frac{b+4}{8} - \frac{b+1}{12}$ **h** $\frac{2a-1}{4} - \frac{3a+1}{6}$

6 Write each of these expressions as a single fraction in simplest form.

a $\frac{x+1}{2} + \frac{x+5}{3}$ **b** $\frac{x+2}{5} + \frac{x-4}{6}$ **c** $\frac{x-3}{4} + \frac{x-1}{8}$ **d** $\frac{x+4}{3} - \frac{x+2}{4}$

e $\frac{x-6}{8} - \frac{x-3}{12}$ **f** $\frac{x+1}{2} - \frac{x+4}{10}$ **g** $\frac{2x+3}{3} + \frac{x-7}{5}$ **h** $\frac{x+5}{6} + \frac{2x+1}{9}$

i $\frac{4x+7}{3} + \frac{5x-3}{2}$ **j** $\frac{3x-1}{4} - \frac{2x-5}{5}$ **k** $\frac{4-x}{7} - \frac{x-4}{3}$ **l** $\frac{x^2+2}{5} - \frac{x^2+3}{8}$

WE 1.5C **7** Simplify the following expressions.

a $\frac{a+2}{a-4} \times \frac{2a+1}{a+2}$ **b** $\frac{x(x+1)}{x+5} \times \frac{x+5}{x(x-3)}$

c $\frac{t+3}{4(t+5)} \div \frac{t(t+3)}{8(t+5)}$ **d** $\frac{m(m+2)}{(m-3)(3m+1)} \times \frac{3m+1}{m(m+7)}$

e $\frac{(x+3)(x-2)}{x(x-2)} \times \frac{x(x+5)}{(x-3)}$ **f** $\frac{x+4}{(x-2)(x+4)} \div \frac{x+5}{x-2}$

g $\frac{2y+3}{y-4} \div \frac{y+3}{3(y-4)}$ **h** $\frac{(m+1)(m-2)}{(m+4)(m-3)} \times \frac{m(m+4)}{(m-1)(m-2)}$

i $\frac{(c^2+6)(c-4)}{(c+2)(c^2+3)} \times \frac{(2c+1)(c^2+3)}{(c-4)(2c+1)}$ **j** $\frac{n^2(n+p)}{n+2p} \times \frac{(n-p)(n+2p)}{n^3(n+p)}$

k $\frac{x^2y^3(x+2y)}{y(x-2y)(x+y)} \div \frac{x^4(x+2y)(x+1)}{(x+y)(x-2y)}$

l $\frac{5(2a+b)}{(4-3a)(b+2)} \times \frac{(a+b)(b+2)}{(a-b)(a+b)} \div \frac{b+2a}{2(4-3a)(a^2+b^2)}$

8 Simplify the following expressions.

a $\frac{6(x+5)(x-5)}{(x-2)(x+2)} \times \frac{(x-1)(x+2)}{3(x+5)(x-1)}$ **b** $\frac{(x+3)(x+8)}{4x(x+2)(x-3)} \times \frac{4x(x+2)(x-6)}{(x+3)(x+6)}$

c $\frac{(2x-1)(3x+2)}{(3x-2)(x+1)} \div \frac{(x+2)(2x-1)}{(5x+1)(3x-2)}$ **d** $\frac{5x(3x-5)}{(2x+5)(7x-4)} \div \frac{10(3x-5)(x+8)}{(7x-4)(2x+5)}$

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WE 1.5D **9** Simplify the following expressions.

a $\frac{2}{m+1}+\frac{3}{m}$ **b** $\frac{k}{k^2+1}+\frac{3}{k}$ **c** $\frac{3}{y+2}+\frac{1}{y-3}$ **d** $\frac{x}{x+3}+\frac{2}{x}$

e $\frac{t}{t+5}+\frac{2}{t-1}$ **f** $\frac{5}{x}-\frac{2}{2x+1}$ **g** $\frac{5}{x+2}-\frac{3}{x+1}$ **h** $\frac{4}{2h-1}-\frac{3}{h+2}$

i $\frac{m}{m-3}-\frac{1}{2m+1}$ **j** $\frac{a^2}{3a+1}+\frac{4}{a}$ **k** $\frac{x}{2}-\frac{2x+3}{x+1}$ **l** $\frac{x+1}{x}-\frac{x}{x+1}$

Exercise 1.5B: Problem solving and reasoning

10 Find the algebraic expression that is equal to a if $\frac{a}{2}+\frac{x+1}{4}=\frac{3x-5}{4}$.

11 Find the algebraic expression that is equal to b if $\frac{b}{5}-\frac{x-2}{2}=\frac{x+12}{10}$.

12 Write the following expressions as single fractions in simplest form.

a $\left(\frac{1}{x}+x\right)\left(\frac{1}{x}-x\right)$ **b** $\left(\frac{1}{x}+x\right)^2$

c $\frac{x}{x+1}\left(\frac{2}{x-1}-\frac{1}{x}\right)$ **d** $\left(\frac{2}{x+1}-\frac{1}{x}\right)\left(\frac{1}{x-1}-\frac{1}{2x}\right)$

13 Consider the expression $\frac{x}{x+1}-\frac{x-1}{x}$.

a Write this expression as a single fraction in simplest form.

b Use your answer to part **a** to evaluate the following.

i $\frac{6}{7}-\frac{5}{6}$ **ii** $\frac{9}{10}-\frac{8}{9}$ **iii** $\frac{99}{100}-\frac{98}{99}$ **iv** $\frac{1000}{1001}-\frac{999}{1000}$

14 Consider the expression $\frac{x}{y}-\frac{x}{y+1}$.

a Write this expression as a single fraction in simplest form.

b Use your answer to part **a** to evaluate the following.

i $\frac{5}{6}-\frac{5}{7}$ **ii** $\frac{7}{10}-\frac{7}{11}$ **iii** $\frac{67}{100}-\frac{67}{101}$ **iv** $\frac{11}{999}-\frac{11}{1000}$

15 Simplify the following expressions.

a $\frac{1}{(x+2)}+\frac{3}{(x+2)(x-2)}$ **b** $\frac{2}{(x+1)(x-1)}+\frac{2(x-1)}{(x+1)}$

c $\frac{1}{(x+1)}+\frac{2}{(x+2)(x+1)}$ **d** $\frac{4x}{(x-1)}+\frac{3}{x(x-1)(x-2)}$

e $\frac{4x+2}{x(x-2)}-\frac{3x}{(x+2)(x-2)}$ **f** $\frac{x}{x+1}+\frac{2x}{(x+1)(x+2)}-\frac{3x}{(x+1)(x+2)(x+3)}$

Exercise 1.5C: Challenge

16 Consider the following algebraic property: $x-y=-(y-x)$.

Use this property to simplify the following algebraic fractions.

a $\frac{a-b}{b-a}$ **b** $\frac{4(m-n)}{n-m}$ **c** $\frac{(x-1)(5-x)(x-3)}{(x+4)(1-x)(x-5)}$

d $\frac{(x+1)^2}{x-1}\times\frac{1-x}{x+1}$ **e** $\frac{x-5}{x+5}\div\frac{5-x}{x-5}$ **f** $\frac{2(2-x)}{x^2-4}\div\frac{x-2}{x+2}$

17 If $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$ and $a^2+b^2+c^2=1$, what is the value of $a+b+c$?

Online resources:

- **Interactive skillsheet** Multiplying and dividing algebraic fractions
- **Interactive skillsheet** Adding and subtracting algebraic fractions
- **Worksheet** Simplifying algebraic fractions
- **Investigation** The shoemaker's knife
- **CAS instructions** Simplifying algebraic fractions
- **Quick quiz** 1.5

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Checkpoint

Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this module.
✔ 10
✔ 10A

1.1 **1** Use the appropriate exponent laws to simplify each of these expressions.

a $x^2 \times x^7$ **b** $y^6 \div y^2$ **c** $\frac{2m^4}{m^3}$

d $(a^2)^5$ **e** $(3xy^2)^2$ **f** $\left(\frac{2a}{b^2}\right)^3$

1.1 **2** Use the property $a^0 = 1$ to simplify each of the following.

a x^0 **b** $2y^0$ **c** $(3a^2)^0$ **d** $x^0 + y^0 + z^0$

1.2 **3** Rewrite each of these terms with positive exponents.

a y^{-3} **b** $4x^{-2}$ **c** $\frac{a^{-7}}{6}$

d $\frac{2}{b^{-3}}$ **e** $\frac{a^{-2}}{2b^5}$ **f** $5c^{-3}d^4$

1.2 **4** Simplify the following expression, writing your answer using only positive exponents.

$$\frac{3a^2b^3}{a^3} \times \left(\frac{2a^{-3}}{b^2}\right)^2$$

1.3 **5** Simplify each of these expressions.

a $4x + 3x - x$ **b** $5m^2 + 3m^2$

c $3a + 5b + 4a$ **d** $c + 3d - 4c + 2d$

e $3a^2 - 4a + 2a^2 + 2a$ **f** $2x^2y + 3xy - 4x^2y + xy^2$

1.3 **6** Simplify the following expressions.

a $\frac{x}{3} - \frac{x}{2}$ **b** $\frac{x}{a} + \frac{xy^2}{b} - \frac{7x}{a} + \frac{4y^2x}{b}$

c $\frac{2x}{3y} + \frac{5x}{6y}$ **d** $\frac{2x}{3y} - \frac{5x}{6y}$

e $\frac{2x}{3y} \times \frac{5x}{6y}$ **f** $\frac{2x}{3y} \div \frac{5x}{6y}$

1.4 **7** Expand and simplify each of these algebraic expressions.

a $3(a + 7)$ **b** $3(2y + 1) + 4$

c $2(x - 5) + x + 2$ **d** $-3(x + 2) - 4(x - 4)$

1.4 **8** Expand and simplify each of the following.

a $(a + 1)(a + 2)$ **b** $(x - 4)(x + 6)$

c $(k - 5)(k + 5)$ **d** $(2x - 3)(x + 2)$

1.4 **9** Expand the expression $(x^2 - y)(x + 2y)$.

10A 1.5 **10** Write each of these expressions as a single fraction in simplest form.

a $\frac{x - 1}{2} + \frac{x}{3}$ **b** $\frac{y + 1}{3} + \frac{5y}{2}$

c $\frac{m^2 + 1}{4} - \frac{m}{5}$ **d** $\frac{2x + 1}{3} - \frac{x + 2}{7}$

10A 1.5 **11** Simplify each of these algebraic fractions.

a $\frac{a(a + 4)}{3(a + 4)}$ **b** $\frac{(x + 4)(x - 3)}{5(x - 3)}$

c $\frac{(x + 4)(x - 3)}{(x + 7)(2x - 3)} \div \frac{(x - 3)(x + 4)}{2x - 3}$ **d** $\frac{m^2(m - 1)(m + 3)}{(m + 3)(m + 1)} \times \frac{(m + 1)(m - 4)}{m^3(m - 1)}$

10A 1.5 **12** Simplify the expression $\frac{x}{x - 2} - \frac{3}{x + 1}$.

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Factorising

Learning intentions

By the end of this lesson you will be able to ...

✔ factorise algebraic expressions using a variety of techniques.

Inter-year links

Year 7	2.4 Factors and the highest common factor
Year 8	5.7 Factorising
Year 9	2.3 Factorising using the HCF

Factorising using the highest common factor

Key content video
Factorising using the HCF

- The distributive law can be used to **factorise** algebraic expressions by identifying the HCF of the terms in the expression.

$$\underset{\text{expanded form}}{ab + ac} = \underset{\text{factorised form}}{a(b + c)}$$

Factorising by grouping terms

Key content video
Factorising by grouping terms

- It is possible to find binomial factors of large algebraic expressions by grouping pairs of terms and factorising each pair using their HCF.

$$\begin{aligned} ac + ad + bc + bd &= a(c + d) + b(c + d) \\ &= (a + b)(c + d) \end{aligned}$$

For example, $x^2 + 2x - 3x - 6$ can be factorised into a binomial product by grouping the terms $x^2 + 2x$ and $-3x - 6$, and then factorising each of them with a common factor of $x + 2$:

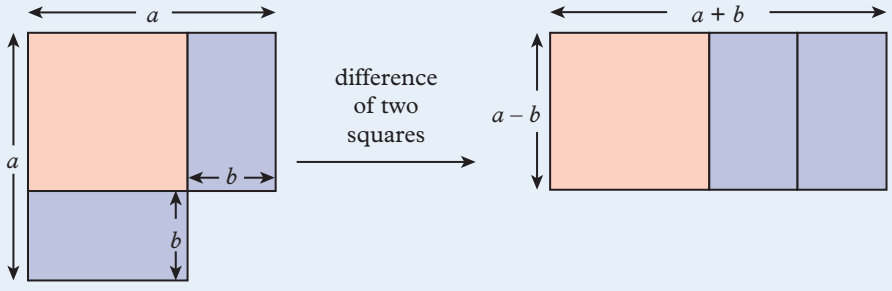
$$\begin{aligned} x^2 + 2x - 3x - 6 &= x(x + 2) - 3(x + 2) \\ &= (x + 2)(x - 3) \end{aligned}$$

Factorising using the 'difference of two squares' rule

Key content video
Factorising the difference of two squares

- Binomials that have the same terms but a different sign, such as $a + b$ and $a - b$, are conjugates of each other.
- If the expression is the difference of two squares, you can factorise it into the product of conjugate pairs using the 'difference of two squares' rule:

$$a^2 - b^2 = (a + b)(a - b)$$



- If the terms of a binomial expression have opposite signs, but both terms are not perfect squares, you can still factorise the expression using surds. For example,

$$x^2 - 3 = (x + \sqrt{3})(x - \sqrt{3})$$

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Factorising using the 'perfect square' rule

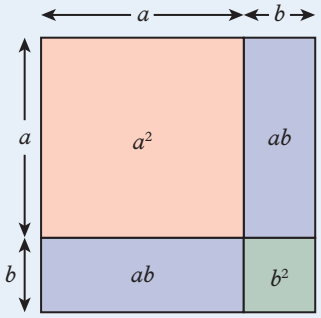
Key content video
Factorising using the perfect square rule

- If the expression is of the same form as a perfect square, then you can factorise it using the 'perfect square' rule.

$$a^2 + 2ab + b^2 = (a + b)^2$$

For example,

$$x^2 + 6x + 9 = x^2 + 2(x)(3) + 3^2$$
$$= (x + 3)^2.$$



Worked example 1.6A Factorising using the highest common factor

Factorise each of these expressions.

a $4n + 36$ **b** $t^2 - 7t$ **c** $5x^5y - 13x^2y^3$ **d** $3klm + 9km - 6k$

THINK	WRITE
a 1 Identify that the HCF is 4 and write each term as a product of the HCF and its other factor.	**a** $4n + 36 = 4 \times n + 4 \times 9$
2 Use the distributive law. Write the HCF before a pair of brackets containing the sum of the remaining factors.	$= 4(n + 9)$
b 1 Identify that the HCF is t and write each term as a product of the HCF and its other factor.	**b** $t^2 - 7t = t \times t + t \times (-7)$
2 Use the distributive law to factorise the expression.	$= t(t - 7)$
c 1 Identify that the HCF is x^2y and write each term as a product of the HCF and its other factor.	**c** $5x^5y - 13x^2y^3 = x^2y \times 5x^3 + x^2y \times (-13y^2)$
2 Use the distributive law to factorise the expression.	$= x^2y(5x^3 - 13y^2)$
d 1 Identify that the HCF is $3k$ and write each term as a product of the HCF and its other factor.	**d** $3klm + 9km - 6k = 3k \times lm + 3k \times 3m + 3k \times (-2)$
2 Use the distributive law. Write the HCF before a pair of brackets containing the sum of the remaining factors.	$= 3k(lm + 3m - 2)$

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Worked example 1.6B Factorising by grouping terms

Factorise $a^2 + 3a - 15b - 5ab$ by grouping terms.

THINK

1. Check for the HCF of all four terms. In this case there isn't one. Pair the terms in groups of two so that each pair has a common factor.
2. Use the distributive law to factorise each pair of terms.
3. Factorise using the common binomial factor, which is $(a + 3)$. Put the common binomial factor in front of a new pair of brackets containing the sum of the other factors.

WRITE

$a^2 + 3a - 15b - 5ab = (a^2 + 3a) + (-15b - 5ab)$

$= a(a + 3) - 5b(a + 3)$

$= (a + 3)(a - 5b)$

Worked example 1.6C Factorising using the 'difference of two squares' rule

Factorise each of these expressions.

a $16h^2 - 49$ **b** $12a^4 - 27b^2$ **c** $(r + 11)^2 - 1$

THINK

a 1. Check for the HCF of all the terms. In this case there isn't one. Write each term as a square in exponent form.
2. Factorise using the 'difference of two squares' rule.

b 1. Check for the HCF of all the terms. In this case, 3 is the HCF. Using the distributive law, factorise using the HCF.
2. Write each term in the binomial factor as a square in exponent form.
3. Factorise the binomial factor using the 'difference of two squares' rule.

c 1. Check for the HCF of all the terms. In this case, there isn't one. Write each term as a square in exponent form.
2. Factorise using the 'difference of two squares' rule.
3. Simplify each of the bracketed expressions.

WRITE

a $16h^2 - 49 = (4h)^2 - 7^2$

$= (4h + 7)(4h - 7)$

b $12a^4 - 27b^2 = 3(4a^4 - 9b^2)$

$= 3([2a^2]^2 - [3b]^2)$

$= 3(2a^2 + 3b)(2a^2 - 3b)$

c $(r + 11)^2 - 1 = (r + 11)^2 - 1^2$

$= (r + 11 + 1)(r + 11 - 1)$

$= (r + 12)(r + 10)$

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Worked example 1.6D Factorising using the 'difference of two squares' rule and surds

Factorise each of these expressions.

a $25x^2 - 7$

b $(y + 5)^2 - 12$

THINK

a 1 Check for the HCF of the two terms. In this case there isn't one. Write each term as a square in exponent form.

2 Factorise using the 'difference of two squares' rule.

b 1 Check for the HCF of the two terms. In this case there isn't one. Write each term as a square in exponent form.

2 Simplify the surd by factorising 12 so it is a product of 3 and 4, where 4 is a perfect square.

3 Factorise using the 'difference of two squares' rule.

WRITE

a $25x^2 - 7 = (5x)^2 - (\sqrt{7})^2$

$= (5x + \sqrt{7})(5x - \sqrt{7})$

b $(y + 5)^2 - 12 = (y + 5)^2 - (\sqrt{12})^2$

$= (y + 5)^2 - (\sqrt{4 \times 3})^2$

$= (y + 5)^2 - (\sqrt{4} \times \sqrt{3})^2$

$= (y + 5)^2 - (2\sqrt{3})^2$

$= (y + 5 + 2\sqrt{3})(y + 5 - 2\sqrt{3})$

Worked example 1.6E Factorising using the 'perfect square' rule

Factorise:

a $t^2 - 18t + 81$

b $16x^2 - 56x + 49$

THINK

a 1 Check for the HCF of all three terms. In this case there isn't one. Identify that the square root of t^2 is t, the negative square root of 81 is -9, and $-18t$ is twice the product of both these roots: $2 \times (-9) \times t$.

2 Factorise using the 'perfect square' rule.

b 1 Check for the HCF of all three terms. In this case there isn't one. Identify that the square root of $16x^2$ is $4x$, the negative square root of 49 is -7, and $-56x$ is twice the product of both these roots: $2 \times (-7) \times (4x)$.

2 Factorise using the 'perfect square' rule.

WRITE

a $t^2 - 18t + 81 = t^2 + 2(-9)t + (-9)^2$

$= (t - 9)^2$

b $16x^2 - 56x + 49 = (4x)^2 + 2 \times (-7) \times (4x) + (-7)^2$

$= (4x - 7)^2$

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Helpful hints

- ✔ It's important that you are able to recognise when an expression can be factorised by grouping terms or by identifying it as the difference of two squares, or as a perfect square.
- ✔ Factorising an expression is the reverse process of expanding an expression. You can check if your factorised solution is correct by expanding it to see if you get the same expression that you started with.

expanding →

$$7(a + 2) = 7a + 14$$

← factorising

- ✔ Always check to see if the terms of an expression have any common factors. It is always best to start with common factors before you move on with any other factorising methods.

  For example, it is easier to identify that $3x^2 + 18x + 27$ can be solved using the 'perfect square' rule once you take out 3 as a common factor:

  $$3(x^2 + 6x + 9) = 3(x + 3)^2$$

- ✔ Remember that the difference of two squares requires one term to be positive and the other to be negative. For example, the following are NOT the difference of two squares:

  $$x^2 + y^2 \quad \text{and} \quad -a^2 - 4$$

Learning pathways

1–2(1st, 2nd columns), 3, 4(a, c, e, g), 5, 6–7(1st column), 8, 9(1st, 2nd columns), 10(a, d, g), 11, 12, 13(1st, 2nd columns), 14, 15(a–d)	1–2(3rd, 4th columns), 4(b, e, g, h), 6–7(2nd column), 9(2nd, 3rd columns), 10, 13(3rd, 4th columns), 15(e–h), 17, 18	2(4th column), 4(f–h), 6–7(3rd column), 9(2nd, 4th columns), 10, 13(4th column), 15(i–l), 16, 17, 19–21

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Exercise 1.6A: Understanding and fluency

WE 1.6A

1 Factorise each of these expressions.

a $5a + 35$ **b** $30b + 6$ **c** $24 + 33c$ **d** $12d + 8$

e $8e - 18$ **f** $35f - 21$ **g** $24 + 16g$ **h** $10 - 40h$

i $9j + 9k$ **j** $16w - 28x$ **k** $m^2 + 2m$ **l** $n^2 - 6n$

m $5a + a^2$ **n** $9p - p^2$ **o** $3q^2 + 3q$ **p** $2r^2 - 8r$

2 Factorise each of these expressions.

a $3x - 12$ **b** $6y - 15$ **c** $8x^2 - 16x$ **d** $4k^2 + 6k$

e $5c + 11c^2$ **f** $8u - 4u^2$ **g** $6m^2 - 3m$ **h** $y^2 + 5y^3$

i $14t - 8t^2$ **j** $9p^3 + 3p^2$ **k** $4ab + 3a$ **l** $5t^2u - 3tu$

m $4m^2n + 2mn^2$ **n** $6x^2y + 3xy$ **o** $15c^2 + 21cd$ **p** $18a^2b^3c - 12a^4bc^2$

3 Complete the following factorisations so each expression is factorised using a negative HCF.

a $-2ab - 4a^2 = (-2a) \times \square + (-2a) \times 2a$
$= -2a(\square + 2a)$

b $-10x^3 + 35x = \square \times 2x^2 + \square \times (-7)$
$= \square(2x^2 - 7)$

4 Factorise each of these expressions using a negative HCF.

a $-6mn - 18$ **b** $-3ab - 3bc$ **c** $-y^2 - 10y$ **d** $-4k + 8k^2$

e $-20a^3 - 30a$ **f** $-12x^2y + 14xy$ **g** $-w^5 + w^7$ **h** $-4b^3 + 2b^2 - 6b^4$

5 Answer the following questions about binomial factors.

a Using the binomial factor $(x - 5)$ as the HCF, show that $y(x - 5) + 2(x - 5)$ is written in factorised form as $(x - 5)(y + 2)$.

b In the same way, show that $4a(3 + k^2) - 9(3 + k^2) = (3 + k^2)(4a - 9)$.

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6 Factorise each of these expressions using an HCF that is a binomial factor.

a $a(x + 3) + 5(x + 3)$ **b** $m(k - 2) + 4(k - 2)$ **c** $y(y + 5) + 2(y + 5)$
d $x(x - 1) + 9(x - 1)$ **e** $k(k + 6) - 3(k + 6)$ **f** $p(p - 9) - 6(p - 9)$
g $7(4 - a) + a(4 - a)$ **h** $3n(2n - 5) + 4(2n - 5)$ **i** $x(y^2 + 2) + 8(y^2 + 2)$
j $a^2(a - 4) + 3(a - 4)$ **k** $2x(y^2 + 1) - 5(y^2 + 1)$ **l** $5x(7 - x^3) + 2(7 - x^3)$

WE 1.6B **7** Factorise each of these expressions by grouping terms.

a $ab + 4b + 3a + 12$ **b** $x^2 - 7x + 2xy - 14y$ **c** $mn + 5m - 6n - 30$
d $a^2 + 2a + 5a + 10$ **e** $x^2 - x + 4x - 4$ **f** $c^2 + 8c - 3c - 24$
g $6y - 3y^2 + 8 - 4y$ **h** $p^3 + 2p^2 + 5p + 10$ **i** $x^4 + x^2 + 3x^2 + 3$
j $2m^3 + 4m - 3m^2 - 6$ **k** $4 + 4k^2 + k^5 + k^3$ **l** $xy - 15 + 5y - 3x$

8 Consider the 'difference of two squares' rule.

a Expand $(x + 5)(x - 5)$. **b** Explain how you can factorise $x^2 - 25$.

WE 1.6C **9** Factorise each of these expressions.

a $x^2 - 4$ **b** $a^2 - 36$ **c** $100 - y^2$ **d** $64b^2 - 9$
e $25 - 49p^2$ **f** $4a^2 - 81b^2$ **g** $3m^2 - 3$ **h** $8k^2 - 18$
i $(xy)^2 - 16w^2$ **j** $m^6 - n^2$ **k** $4a^{10} - 36b^6$ **l** $(h + 3)^2 - 25$
m $(2 - x)^2 - x^2$ **n** $1 - (a + b)^2$ **o** $4 - (y - 5)^2$ **p** $(x + 3)^2 - (x - 6)^2$

WE 1.6D **10** Factorise each of these expressions. Hint: Where needed, use a surd to write a term as a square.

a $x^2 - 7$ **b** $a^2 - 13$ **c** $19 - y^2$
d $16k^2 - 5$ **e** $(p + 8)^2 - 2$ **f** $(m - 4)^2 - 6$
g $(x + 7)^2 - 3$ **h** $(y - 1)^2 - 19$ **i** $(d - 17)^2 - 17$

11 Consider the 'perfect square' rule.

a Expand $(x + 5)^2$. **b** Explain how you can factorise $x^2 + 10x + 25$.

12 Consider again the 'perfect square' rule.

a Expand $(y - 8)^2$. **b** Explain how you can factorise $y^2 - 16y + 64$.

WE 1.6E **13** Factorise each of these expressions using what you know from the 'perfect square' rule.

a $x^2 + 12x + 36$ **b** $m^2 + 8m + 16$ **c** $p^2 + 20p + 100$ **d** $y^2 - 6y + 9$
e $x^2 - 14x + 49$ **f** $g^2 - 4g + 4$ **g** $1 + 2a + a^2$ **h** $25 - 10x + x^2$
i $81 - 18b + b^2$ **j** $w^2 + 2wx + x^2$ **k** $k^2 - 2km + m^2$ **l** $9x^2 + 12x + 4$

Exercise 1.6B: Problem solving and reasoning

14 Consider the rectangular items below.

a Write an expression for the missing side length of each rectangular item shown, given that:

i the area of the welcome mat is $(x^2 - 100)$ cm^2

ii the area of the stained glass panel is $(2x^2 + 3x)$ cm^2.

b Write an expression in factorised form for the perimeter of each of the items above.

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15 Simplify each of these algebraic fractions by factorising the numerator and denominator where possible.

a $\dfrac{a^2 + 2a}{ab + 2b}$ **b** $\dfrac{xy + 2x}{3y^2 + 6y}$ **c** $\dfrac{abc^2 - 4ab^2}{ab}$ **d** $\dfrac{2mn + m^2n^2}{mn + 2}$

e $\dfrac{x^2 - y^2}{2x + 2y}$ **f** $\dfrac{3x + 15}{x^2 + 5x}$ **g** $\dfrac{9x^2 - 15x}{12x - 20}$ **h** $\dfrac{m^2 - 49}{5m^2 - 35m}$

i $\dfrac{a^2 + 2ab + b^2}{a^2 - b^2}$ **j** $\dfrac{x^2 + 6x + 9}{x^2 - 9}$ **k** $\dfrac{16a^2 - 25}{8a^2 - 10a}$ **l** $\dfrac{4x^2 + 28x + 49}{49 - 4x^2}$

16 Simplify each of the following.

a $\dfrac{3x + 21}{2x^2 - 10x} \times \dfrac{4x^2 - 20x}{x^2 - 49}$ **b** $\dfrac{x^2 - 1}{10x^2 + 30x} \times \dfrac{15x + 45}{x^2 + 2x + 1} \times \dfrac{2x^2 + 2x}{3x - 3}$

17 Consider an odd number represented by the pronumeral n.

a Write expressions for the next two odd numbers.

b Write an expression for the sum of n and the next two odd numbers.

c Factorise your expression from part **b**. Explain how this relates to one of the three odd numbers.

d Use the shortcut you found in part **c** to find the sum of 423, 425 and 427 without adding.

e Investigate a shortcut for adding five consecutive odd numbers. Show your working.

f Use the shortcut you found in part **e** to find the sum of 2671, 2673, 2675, 2677 and 2679 without adding.

Exercise 1.6C: Challenge

18 In question **17**, you found expressions for the sum of three consecutive odd numbers and five consecutive odd numbers.

a Investigate shortcuts for finding the sum of an even number of consecutive odd terms. Start with two consecutive odd numbers, then investigate for four consecutive odd numbers, then six consecutive odd numbers, where the first term is n each time.

b Without adding, find the sum of the numbers 73, 75, 77, 79, 81, 83, 85, 87, 89, 91.

19 The difference between two numbers is 7 and the difference between the squares of the two numbers is 105. What is the sum of the two numbers?

20 Factorising using the 'difference of two squares' method can be useful for simplifying certain multiplications. Evaluate each of the following without a calculator by using the 'difference of two squares' factorisation method.

a 19×21 **b** 28×32 **c** 83×77 **d** 54×46

21 Legend has it that the famous mathematician Carl Friedrich Gauss was once punished for misbehaving in primary school by being told he had to find the sum of all the numbers from 1 to 100 before leaving class. Much to his teacher's surprise, he did so in seconds.

The shortcut Gauss probably used was to start by pairing the highest and lowest numbers together, like so:

$$(1 + 100) + (2 + 99) + (3 + 98) + (4 + 97) + \ldots$$

a What is the sum of each binomial pair in Gauss's method?

b How many binomial pairs will there be once all of the numbers from 1 to 100 are grouped together?

c Hence, write the sum of all the numbers from 1 to 100 as the product of two numbers. Evaluate this product to find Gauss's solution.

d This method can be generalised to write the sum of all the numbers from 1 to n, for any positive integer n. Write the sum of all the numbers from 1 to n as a factorised expression in terms of n.

Online resources:

- **Interactive skillsheet** Factorising using the HCF
- **Interactive skillsheet** Factorising by grouping terms
- **Interactive skillsheet** Factorising the difference of two squares
- **CAS instructions** Factorising
- **Quick quiz** 1.6

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Lesson 1.7

Factorising quadratic expressions

Learning intentions

By the end of this lesson you will be able to ...

✔ factorise monic quadratic trinomials.

Inter-year links

Year 7	2.4 Factors and the highest common factor
Year 8	5.7 Factorising
Year 9	2.5 Factorising quadratic expressions

Key content video
Factorising monic quadratic expressions

Quadratic trinomials

- A **quadratic** expression is an algebraic expression that contains a squared pronumeral, with no exponents greater than 2 in the expression.
  The following are all quadratic expressions:

  $6t^2$ $\quad$ $x^2 + 5$ $\quad$ $k^2 + 14k + 30$ $\quad$ $4b^2 - a^2$

- A quadratic **trinomial** is an algebraic expression of the form $ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$.

  $ax^2 + bx + c$

  → a is the **leading coefficient**, b is the coefficient of the linear term, and c is the constant term that can be viewed as the coefficient of x^0.

- A **monic** quadratic trinomial is a quadratic trinomial with leading coefficient equal to 1.

  $x^2 + bx + c$

Factorising monic quadratic trinomials

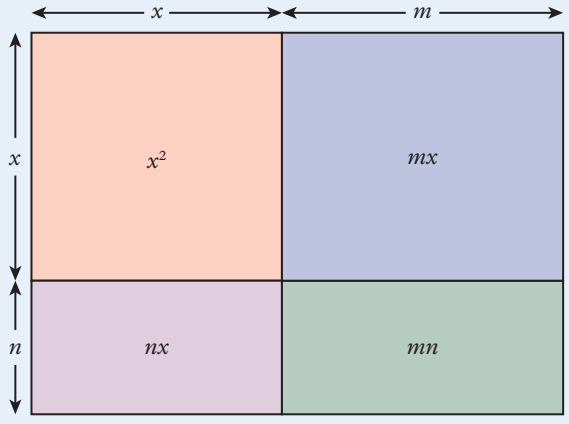
- Expanding a binomial product of the form $(x + m)(x + n)$ results in a monic quadratic trinomial:

$$\begin{aligned}(x + m)(x + n) &= x^2 + mx + nx + mn \\ &= x^2 + (m + n)x + mn \\ &= x^2 + bx + c\end{aligned}$$

  where $b = m + n$ and $c = m \times n$.

- The process can be reversed to factorise a monic quadratic trinomial, $x^2 + bx + c$, by finding two numbers (m and n) that add to give b and multiply to give c.
  For example, in the quadratic trinomial $x^2 + 4x + 3$, the linear coefficient is 4 (that is, $b = 4$) and the constant is 3 (that is, $c = 3$). So the values $m = 3$ and $n = 1$ satisfy the conditions $b = m + n$ and $c = m \times n$.

$$\begin{aligned}x^2 + 4x + 3 &= x^2 + (3 + 1)x + (3 \times 1) \\ &= (x + 3)(x + 1)\end{aligned}$$



expanding ←

$x^2 + 4x + 3 = (x + 1)(x + 3)$

quadratic trinomial (expanded form) $\quad$ binomial product (factorised form)

→ factorising

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Worked example 1.7A Factorising simple quadratic trinomials

Factorise each of these quadratic trinomials.

a $x^2 + 6x + 5$ **b** $x^2 + 9x + 18$

THINK

a **1** For $x^2 + 6x + 5$, list the factor pairs of the constant term, 5. Remember that, if the positive/negative sign is changed for both factors, the product will be the same. So there are two combinations of factor pairs with the same numerals.

2 Add the factor pairs together and identify which pair add to the linear coefficient, 6.

3 Write the expression in factorised form.

b **1** For $x^2 + 9x + 18$, list the factor pairs of the constant term, 18.

2 Add the factor pairs together and identify which pair add to the linear coefficient, 9.

3 Write the expression in factorised form.

WRITE

a
$1 \times 5 = 5 \rightarrow 1 + 5 = 6$
$(-1) \times (-5) = 5 \rightarrow -1 - 5 = -6$
$x^2 + 6x + 5 = (x + 1)(x + 5)$

b
$1 \times 18 = 18 \rightarrow 1 + 18 = 19$
$(-1) \times (-18) = 18 \rightarrow -1 - 18 = -19$
$2 \times 9 = 18 \rightarrow 2 + 9 = 11$
$(-2) \times (-9) = 18 \rightarrow -2 - 9 = -11$
$3 \times 6 = 18 \rightarrow 3 + 6 = 9$
$(-3) \times (-6) = 18 \rightarrow -3 - 6 = -9$
$x^2 + 9x + 18 = (x + 3)(x + 6)$

Worked example 1.7B Factorising more complex quadratic trinomials

Factorise each of these quadratic trinomials.

a $x^2 + 2x - 3$ **b** $x^2 - 7x - 8$ **c** $x^2 - 5x + 6$

THINK

1 List the factor pairs of the constant term. Remember that, if the positive/negative sign is changed for both factors, the product will be the same. So there are always two combinations of factor pairs with the same numerals.

2 Add the factor pairs together and identify which pair add to the linear coefficient.

3 Write the expression in factorised form.

WRITE

a
$1 \times (-3) = -3 \rightarrow 1 - 3 = -2$
$(-1) \times 3 = -3 \rightarrow -1 + 3 = 2$
$x^2 + 2x - 3 = (x + 3)(x - 1)$

b
$1 \times (-8) = -8 \rightarrow 1 - 8 = -7$
$(-1) \times 8 = -8 \rightarrow -1 + 8 = 7$
$2 \times (-4) = -8 \rightarrow 2 - 4 = -2$
$(-2) \times 4 = -8 \rightarrow -2 + 4 = 2$
$x^2 - 7x - 8 = (x - 8)(x + 1)$

c
$1 \times 6 = 6 \rightarrow 1 + 6 = 7$
$(-1) \times (-6) = 6 \rightarrow -1 - 6 = -7$
$2 \times 3 = 6 \rightarrow 2 + 3 = 5$
$(-2) \times (-3) = 6 \rightarrow -2 - 3 = -5$
$x^2 - 5x + 6 = (x - 3)(x - 2)$

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Worked example 1.7C Factorising quadratic trinomials by first taking out a common factor

Factorise these by first taking out a common factor.

a $2x^2 - 14x + 12$ **b** $-x^2 + 4x - 3$

THINK

1. If the coefficient of x^2 is not 1, look for a common factor of the three terms (–1 can be a common factor). Write the HCF before the brackets. Inside the brackets, write the results of dividing each term by the HCF.
2. List the factor pairs of the constant term each time. Remember that, if the positive/negative sign is changed for both factors, the product will be the same. So there are always two combinations of factor pairs with the same numerals.
3. Add the factor pairs together and identify which pair add to the linear coefficient.
4. Write the expression in factorised form.

WRITE

a $2x^2 - 14x + 12 = 2(x^2 - 7x + 6)$

$1 \times 6 = 6 \rightarrow \quad 1 + 6 = 7$

$(-1) \times (-6) = 6 \rightarrow -1 - 6 = -7$

$2 \times 3 = 6 \rightarrow \quad 2 + 3 = 5$

$(-2) \times (-3) = 6 \rightarrow -2 - 3 = -5$

$2x^2 - 14x + 12 = 2(x^2 - 7x + 6)$

$= 2(x - 6)(x - 1)$

b $-x^2 + 4x - 3 = -(x^2 - 4x + 3)$

$1 \times 3 = 3 \rightarrow \quad 1 + 3 = 4$

$(-1) \times (-3) = 3 \rightarrow -1 - 3 = -4$

$-x^2 + 4x - 3 = -(x^2 - 4x + 3)$

$= -(x - 3)(x - 1)$

Helpful hints

- ✔ You can always check your factorised quadratic by expanding your solution to see if you get the same expression that you started with.
- ✔ You don't always need to consider all the factor pairs of the constant term, c, to factorise a quadratic of the form $x^2 + bx + c$.
  If the constant term, c, is positive, then both factors have the same sign. So:
  → if b is also positive, both factors are positive
  → if b is negative, both factors are negative.
  If the constant term, c, is negative, then the factors have different signs.
- ✔ Remember, always look for common factors! You should start by checking to see if the leading coefficient is a common factor of all the terms.

Learning pathways

1, 2, 3(1st column), 4, 5, 6(1st column), 7(1st, 2nd columns), 8(a–d), 9(a, c), 10	3(2nd, 3rd columns), 6–7(2nd, 3rd columns), 8(e–h), 9(b, d), 10–13	3(3rd column), 6–7(2nd, 3rd columns), 8(e–i), 9(b, d), 11, 13–17

ANS p683

Exercise 1.7A: Understanding and fluency

1. For each of these pairs of numbers, identify the two numbers that multiply together to give the first number and add together to give the second number.

a 7, 8 **b** 16, 10 **c** 45, 14 **d** 60, 16 **e** 36, 13 **f** 24, 11

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2 Use your results from question **1** to factorise each of these quadratic trinomials.

a $x^2 + 8x + 7$ **b** $x^2 + 10x + 16$ **c** $x^2 + 14x + 45$

d $x^2 + 16x + 60$ **e** $x^2 + 13x + 36$ **f** $x^2 + 11x + 24$

WE 1.7A **3** Factorise each of these quadratic trinomials.

a $x^2 + 3x + 2$ **b** $x^2 + 5x + 6$ **c** $x^2 + 11x + 10$

d $x^2 + 12x + 27$ **e** $x^2 + 7x + 10$ **f** $x^2 + 9x + 14$

g $x^2 + 6x + 9$ **h** $x^2 + 12x + 11$ **i** $x^2 + 11x + 18$

j $x^2 + 12x + 32$ **k** $x^2 + 16x + 63$ **l** $x^2 + 14x + 40$

m $x^2 + 11x + 30$ **n** $x^2 + 13x + 22$ **o** $x^2 + 15x + 36$

4 For each of these pairs of numbers, identify the two numbers that multiply together to give the first number and add together to give the second number.

a $-8, 7$ **b** $6, -7$ **c** $-35, -2$ **d** $-15, 2$ **e** $-8, -2$ **f** $30, -11$

5 Use your results from question **4** to factorise each of these quadratic trinomials.

a $x^2 + 7x - 8$ **b** $x^2 - 7x + 6$ **c** $x^2 - 2x - 35$

d $x^2 + 2x - 15$ **e** $x^2 - 2x - 8$ **f** $x^2 - 11x + 30$

WE 1.7B **6** Factorise each of these quadratic trinomials.

a $x^2 + 4x - 12$ **b** $x^2 - 8x - 20$ **c** $x^2 - 5x + 6$

d $x^2 + 6x - 27$ **e** $x^2 - 13x + 40$ **f** $x^2 - x - 20$

g $x^2 + 4x - 21$ **h** $x^2 - 14x + 33$ **i** $x^2 - 3x - 28$

j $x^2 + x - 12$ **k** $x^2 - 7x - 18$ **l** $x^2 - 15x + 56$

WE 1.7C **7** Factorise each of these quadratic trinomials by first taking out a common factor.

a $2x^2 + 18x + 16$ **b** $3x^2 + 15x + 18$ **c** $7x^2 + 14x - 56$

d $-5x^2 - 35x - 60$ **e** $-4x^2 + 28x - 40$ **f** $6 - 5x - x^2$

g $2x^2 + 14x - 36$ **h** $19x^2 - 38x - 57$ **i** $-3x^2 + 33x + 78$

j $6x - 2x^2 - 4$ **k** $\frac{1}{2}x^2 + \frac{3}{2}x - 2$ **l** $\frac{1}{3}x^2 + x - 6$

8 Simplify each of the following algebraic fractions.

a $\dfrac{x^2 + 6x + 8}{x + 2}$ **b** $\dfrac{x + 2}{x^2 - 4}$ **c** $\dfrac{x^2 + 6x}{x^2 + 4x - 12}$

d $\dfrac{x^2 - 8x + 15}{x^2 - 9}$ **e** $\dfrac{2x^2 - 50}{x^2 + 7x + 10}$ **f** $\dfrac{11x^2 - 22x}{11x^2 - 44}$

g $\dfrac{x^2 + 2x + 1}{1 - x^2}$ **h** $\dfrac{4x^2 + 8x - 60}{2x^2 + 18x + 40}$ **i** $\dfrac{3x^2 - 9x - 30}{6x^2 + 42x + 60}$

9 Simplify each of these expressions using the most appropriate method.

a $\dfrac{x^2 + x - 20}{x^2 - 6x + 8} \times \dfrac{x^2 - 4}{x^2 - 25} \times \dfrac{8x^2 - 40x}{4x^2 + 8x}$ **b** $\dfrac{x}{x + 7} \times \dfrac{x^2 - 49}{3x} \div \dfrac{x^2 - 5x - 14}{x^2 - 4x - 12}$

c $\dfrac{3x^2 + 3x}{x - 3} \times \dfrac{x^2 - 7x + 12}{6x^2 + 6x} + \dfrac{x + 5}{3}$ **d** $\dfrac{x^2 + 9x + 18}{2x + 6} \div \dfrac{3x + 18}{5x - 1} + \dfrac{3x - 5}{8}$

10 Choose the most appropriate method to factorise each of these expressions. Then factorise them.

Exercise 1.7B: Problem solving and reasoning

a $x^2 + 15x + 44$ **b** $x^2 - 2x - 15$ **c** $x^2 + 4x + 4$

d $3x^2 - 18x + 24$ **e** $5x^2 - 45$ **f** $2x^2 - 8x$

11 Explain why each of these expressions cannot be factorised using the methods covered so far.

a $x^2 + 9$ **b** $(x - 3)^2 + 16$ **c** $(x + 1)^2 + 4$

d $x^2 - 4x + 5$ **e** $(x + 2)^2 - 4x$ **f** $x^2 + 12x + 40$

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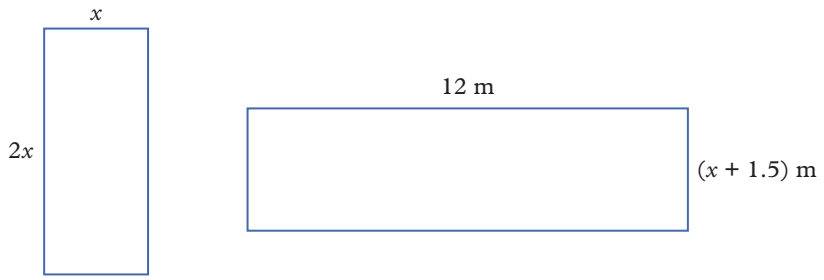
12 Consider the following two rectangles:

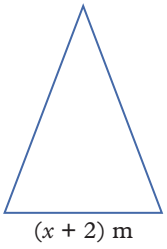


a Write an expression for the area of each rectangle in terms of x.

b Write an expression for the total area of the two rectangles in terms of x.

c The total area of the two rectangles can be combined and reshaped to form two equally sized squares. What is the side length of these squares, in terms of x?

13 The triangle below has a base of $(x + 2)$ metres and an area of $\left(\frac{1}{2}x^2 + 4x + 6\right)$ square metres. Find an expression, in terms of x, for the height of the triangle.



Exercise 1.7C: Challenge

14 It is possible for an expression to be a quadratic with two variables.

a Expand each of the following:

i $(x + y)(x + y)$ **ii** $(x + 2y)(x + 3y)$

b Factorise each of the following:

i $x^2 + 10x + 16$ **ii** $x^2 + 10xy + 16y^2$

c Factorise each of the following:

i $x^2 + 7x + 10$ **ii** $x^2 + 7xy + 10y^2$

d Factorise each of the following:

i $x^2 - 6xy + 8y^2$ **ii** $y^2 - 9xy + 14x^2$ **iii** $4x^2 + 5xy + y^2$

15 If possible, factorise each of the following quadratic expressions, using a method already covered in this module.

a $x^2 + 6x$ **b** $x^2 + 6x + 1$ **c** $x^2 + 6x + 2$ **d** $x^2 + 6x + 3$

e $x^2 + 6x + 4$ **f** $x^2 + 6x + 5$ **g** $x^2 + 6x + 6$ **h** $x^2 + 6x + 7$

i $x^2 + 6x + 8$ **j** $x^2 + 6x + 9$

16 What values can k have for each quadratic trinomial below to be able to be factorised using the methods already covered in this module?

a $x^2 - 6x + k$ **b** $x^2 + 20x + k$

17 Factorise the following expressions.

a $x^4 - 16$ **b** $64a^2 - 4b^2$ **c** $a^2 - 2ax + x^2 - y^2$

Online resources:

Interactive skillsheet Factorising quadratic expressions

Worksheet Factorising algebraic expressions

Quick quiz 1.7

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Lesson 1.8

Completing the square

Learning intentions

By the end of this lesson you will be able to ...

✔ complete the square of quadratic expressions.

Inter-year links

Year 8	5.7 Factorising
Year 9	2.4 Factorising the difference of two squares

Adding a term to make a perfect square

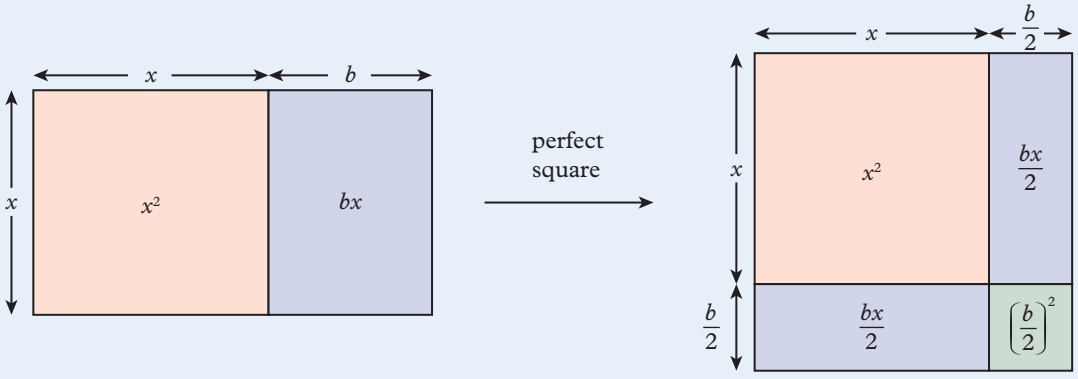
Key content video
Completing the square

- For any quadratic of the form $x^2 + bx$, a constant term of the form $\left(\frac{b}{2}\right)^2$ can be added to make it a perfect square.

$$x^2 + 2x\left(\frac{b}{2}\right) + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

'perfect square' rule

$$a^2 + 2ab + b^2 = (a + b)^2$$



For example, consider the expression $x^2 + 6x$. The coefficient of the linear term is 6, so we can add $\left(\frac{6}{2}\right)^2 = 9$ to the binomial to make it a perfect square:

$$x^2 + 6x + 9 = x^2 + 2(x)(3) + 3^2$$
$$= (x + 3)^2$$

Completing the square

- **Completing the square** is a process of changing a quadratic expression. It uses the 'perfect square' rule to convert expressions in the form $x^2 + bx + c$, to the form $(x - h)^2 + k$.
  - → Just as zero can be added to any number without changing its value, an expression equivalent to zero can be added to an expression without changing its value: $+\left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 = 0$.
  - → To complete the square, the term $\left(\frac{b}{2}\right)^2$ must be *added and subtracted* from the expression.

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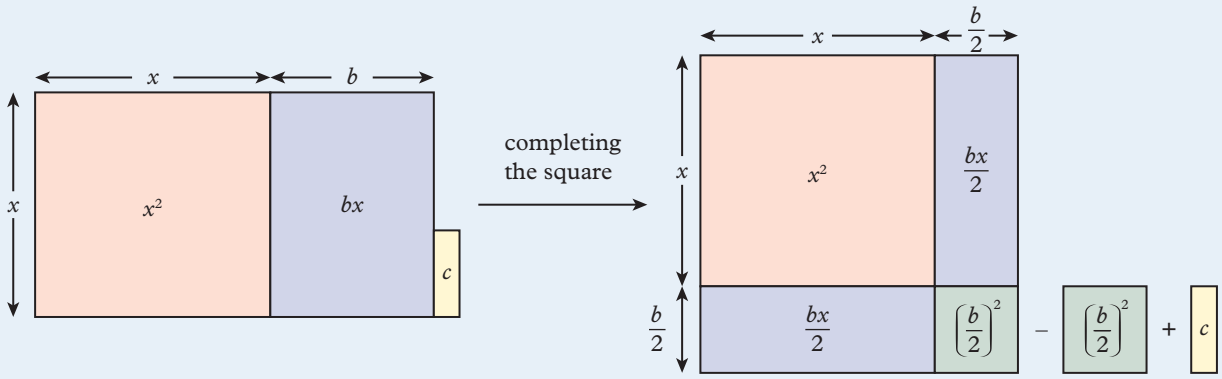

- Expressions in the form $x^2 + bx + c$ can be converted to the form $(x - h)^2 + k$ using the following process.

$$\begin{aligned} x^2 + bx + c &= x^2 + bx + 0 + c \\ &= x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \\ &= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \\ &= (x - h)^2 + k \end{aligned}$$

when $h = -\frac{b}{2}$ and $k = -\left(\frac{b}{2}\right)^2 + c$

For example, to complete the square for $x^2 + 8x + 1$:

1 find the constant term needed to make a perfect square

$$b = 8$$
$$\left(\frac{b}{2}\right)^2 = 16$$

2 add and subtract this value

$$\begin{aligned} x^2 + 8x + 1 &= x^2 + 8x + 0 + 1 \\ &= x^2 + 8x + 16 - 16 + 1 \end{aligned}$$

3 group the first three terms (the perfect square), and the last two terms can be simplified

$$\begin{aligned} &= (x^2 + 8x + 16) - 16 + 1 \\ &= (x^2 + 8x + 16) - 15 \end{aligned}$$

4 factorise the perfect square.

$$= (x + 4)^2 - 15$$

Factorising using the 'difference of two squares' rule

- Once a monic quadratic trinomial is written in the form $(x - h)^2 + k$, the value of k determines whether it can be factorised and written as the product of two linear factors.

  → If $k = 0$, then the expression is already factorised and is written as the square of a linear expression.

  → If $k > 0$, then the expression cannot be factorised using real numbers.

  → If $k < 0$, the expression can be factorised using the 'difference of two squares' rule.

- For example, to factorise the expression $(x + 4)^2 - 15$, from above:

$$a^2 - b^2 = (a + b)(a - b)$$

$$\begin{aligned} (x + 4)^2 - 15 &= (x + 4)^2 - (\sqrt{15})^2 \\ &= (x + 4 + \sqrt{15})\,(x + 4 - \sqrt{15}). \end{aligned}$$

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Worked example 1.8A Adding a term to make a perfect square

Add a term to each of these expressions to make a new expression that is a perfect square in expanded form.

a $x^2 + 10x$ **b** $x^2 - 8x$

THINK

a 1 For $x^2 + 10x$, substitute the coefficient of the linear term, 10, into the expression $\left(\frac{b}{2}\right)^2$ to find the correct term.

2 Add the term to make a perfect square.

b 1 For $x^2 - 8x$, substitute the coefficient of the linear term, -8, into the expression $\left(\frac{b}{2}\right)^2$ to find the correct term.

2 Add the term to make a perfect square.

WRITE

a $\left(\frac{b}{2}\right)^2 = \left(\frac{10}{2}\right)^2$

$= 5^2$

$= 25$

$x^2 + 10x + 25$

b $\left(\frac{b}{2}\right)^2 = \left(\frac{-8}{2}\right)^2$

$= (-4)^2$

$= 16$

$x^2 - 8x + 16$

Worked example 1.8B Factorising by completing the square

Factorise each of these quadratic trinomials by first completing the square.

a $x^2 + 12x + 15$ **b** $x^2 - 2x - 7$ **c** $x^2 + x - 3$

THINK

1 Substitute the coefficient of the linear term into the expression $\left(\frac{b}{2}\right)^2$ each time to find the term to make a perfect square.

2 Add and subtract the term from the quadratic so that the overall expression remains unchanged.

3 Factorise the perfect square expression using the 'perfect square' rule and simplify the constant terms at the end of the expression.

4 Write as the difference of two squares. Simplify the square root if possible.

5 Factorise using the 'difference of two squares' rule and simplify where possible.

WRITE

a The coefficient of the linear term in $x^2 + 12x + 15$ is 12.

$\left(\frac{12}{2}\right)^2 = 6^2$

$= 36$

$$\begin{aligned} x^2 + 12x + 15 &= x^2 + 12x + 36 - 36 + 15 \\ &= (x^2 + 12x + 36) - 36 + 15 \\ &= (x^2 + 2[6]x + 6^2) - 21 \\ &= (x + 6)^2 - 21 \\ &= (x + 6)^2 - (\sqrt{21})^2 \\ &= (x + 6 + \sqrt{21})(x + 6 - \sqrt{21}) \end{aligned}$$

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b The coefficient of the linear term in $x^2 - 2x - 7$ is -2.

$$\left(\frac{-2}{2}\right)^2 = (-1)^2$$
$$= 1$$
$$\begin{aligned} x^2 - 2x - 7 &= x^2 - 2x + 1 - 1 - 7 \\ &= (x^2 - 2x + 1) - 1 - 7 \\ &= (x^2 + 2[-1]x + [-1]^2) - 8 \\ &= (x - 1)^2 - 8 \\ &= (x - 1)^2 - (\sqrt{8})^2 \\ &= (x - 1)^2 - (2\sqrt{2})^2 \\ &= (x - 1 + 2\sqrt{2})(x - 1 - 2\sqrt{2}) \end{aligned}$$

c The coefficient of the linear term in $x^2 + x - 3$ is 1.

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$$
$$\begin{aligned} x^2 + x - 3 &= x^2 + x + \frac{1}{4} - \frac{1}{4} - 3 \\ &= \left(x^2 + x + \frac{1}{4}\right) - \frac{1}{4} - 3 \\ &= \left(x^2 + 2\left[\frac{1}{2}\right]x + \left[\frac{1}{2}\right]^2\right) - \frac{1}{4} - \frac{12}{4} \\ &= \left(x + \frac{1}{2}\right)^2 - \frac{13}{4} \\ &= \left(x + \frac{1}{2}\right)^2 - \left(\frac{\sqrt{13}}{2}\right)^2 \\ &= \left(x + \frac{1 + \sqrt{13}}{2}\right)\left(x + \frac{1 - \sqrt{13}}{2}\right) \end{aligned}$$

Helpful hints

- ✔ For expressions in the form $ax^2 + bx + c$, the coefficient a must be factorised out before completing the square can begin, resulting in $a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$.
- ✔ Factorising using the 'difference of two squares' rule only works if $k < 0$. If $k > 0$, then the expression cannot be factorised.

Learning pathways

1–5, 6(1st column), 7, 8, 11	3–5, 6(2nd column), 7, 9, 10, 12, 13	3, 4, 6(3rd column), 7, 10, 12, 14–16

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Exercise 1.8A: Understanding and fluency

1 Recall the 'perfect square' rule.

a **i** Expand $(x + 3)^2$. **ii** Use your answer for part **i** to factorise $x^2 + 6x + 9$.

b **i** Expand $(x - 9)^2$. **ii** Use your answer for part **i** to factorise $x^2 - 18x + 81$.

c Factorise each of these quadratic trinomials.

i $x^2 + 8x + 16$ **ii** $x^2 + 4x + 4$ **iii** $x^2 - 12x + 36$

iv $x^2 + 16x + 64$ **v** $x^2 - 10x + 25$ **vi** $x^2 - 20x + 100$

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2 Copy and complete each of these expressions to make a perfect square.

a $x^2 + 6x + __ = (x + 3)^2$ **b** $x^2 + 10x + __ = (x + 5)^2$

c $x^2 - 8x + __ = (x - 4)^2$ **d** $x^2 - 4x + __ = (x - 2)^2$

e $x^2 + 2x + __ = (x + 1)^2$ **f** $x^2 - 16x + __ = (x - 8)^2$

WE 1.8A **3** Add a term to each of these expressions to make a new perfect square in expanded form.

a $x^2 + 14x$ **b** $x^2 - 6x$ **c** $x^2 + 18x$

d $x^2 + 20x$ **e** $x^2 - 12x$ **f** $x^2 + 22x$

4 Factorise each perfect square expression you wrote for question **3**.

5 The quadratic trinomial $x^2 + 6x + 7$ can be written as $(x^2 + 6x + 9) - 2$. Explain how knowledge of the 'perfect square' rule and the 'difference of two squares' rule can be used to factorise it.

WE 1.8B **6** Factorise each of these quadratic trinomials by first completing the square.

a $x^2 + 8x + 9$ **b** $x^2 - 10x + 14$ **c** $x^2 + 4x - 6$ **d** $x^2 + 2x - 5$

e $x^2 - 6x - 4$ **f** $x^2 - 12x + 13$ **g** $x^2 + 10x - 4$ **h** $x^2 - 14x + 46$

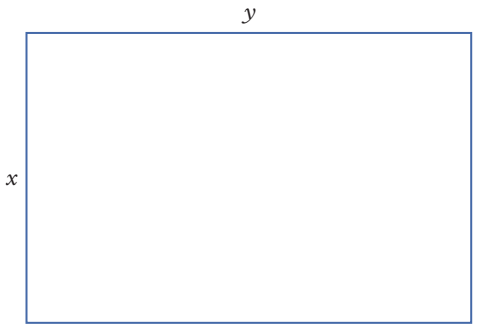
i $x^2 + 8x + 7$ **j** $x^2 + 3x + 1$ **k** $x^2 - 5x + 3$ **l** $x^2 + 11x + 9$

7 Factorise each of these quadratic trinomials by first taking out a common factor.

a $2x^2 + 8x + 2$ **b** $-x^2 - 6x + 2$ **c** $-5x^2 + 10x + 5$ **d** $3x^2 - 15x + 6$

Exercise 1.8B: Problem solving and reasoning

8 To create a square from the rectangle on the right, other rectangles can be cut out or attached. Assume $y > x$.



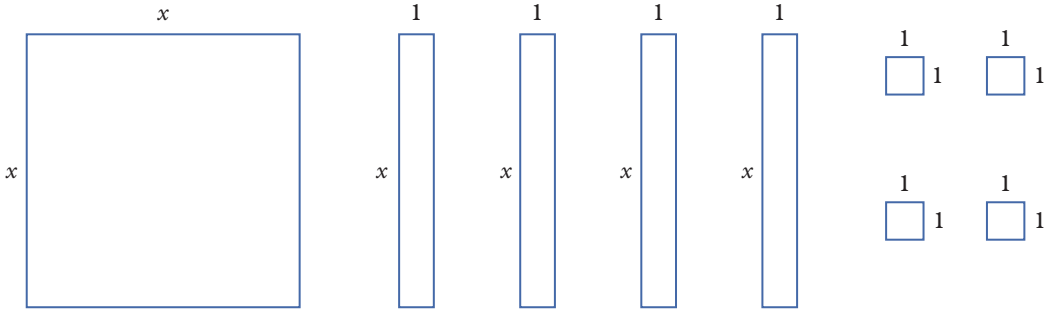
a What are the dimensions of the rectangle needing to be cut out to turn the rectangle into a square?

b If a rectangle is cut out to turn the rectangle into a square, write an expression for the area of the resulting square.

c What are the dimensions of the rectangle needing to be attached to turn the rectangle into a square?

d If a rectangle is attached to turn the rectangle into a square, write an expression for the area of the resulting square.

9 Consider the following shapes.



a Show how the shapes can be arranged to form a square.

b Express the area of the square you arranged in part **a** as an expanded quadratic expression.

c Express the area of the square you arranged in part **a** as a factorised quadratic expression.

10 In **Exercise 1.7C**, you were asked to factorise as many of the following quadratic expressions as possible using the methods that had been covered. Now that you know how to complete the square, factorise each of the following quadratic expressions using the most appropriate method.

a $x^2 + 6x$ **b** $x^2 + 6x + 1$ **c** $x^2 + 6x + 2$ **d** $x^2 + 6x + 3$

e $x^2 + 6x + 4$ **f** $x^2 + 6x + 5$ **g** $x^2 + 6x + 6$ **h** $x^2 + 6x + 7$

i $x^2 + 6x + 8$ **j** $x^2 + 6x + 9$

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11 It is important to consider the method used to factorise quadratic trinomials.

a Factorise $x^2 + 7x + 6$ by first completing the square.

b Look at your answer to part **a**. Is there an easier way of obtaining it? Explain.

12 Not all quadratic trinomials can be factorised using real numbers. If a quadratic cannot be factorised by completing the square, then it cannot be factorised at all.
For example, a quadratic like $x^2 + 2x + 5$ becomes $(x + 1)^2 + 3$ after completing the square, and this expression cannot be factorised using real numbers because it cannot be expressed as the difference of two squares.
Complete the square for each of the following expressions to find the specified values.

a Find the values of c for which $x^2 + 8x + c$ cannot be factorised using real numbers.

b Find the values of c for which $x^2 - 10x + c$ cannot be factorised using real numbers.

c Find the values of b for which $x^2 + bx + 1$ cannot be factorised using real numbers.

13 Explain why each of these expressions cannot be factorised using real numbers with the methods covered in this module so far.

a $x^2 + 6x + 10$ **b** $x^2 - 8x + 21$ **c** $3x^2 + 7x + 2$

d $12x^2 + 64x + 45$ **e** $2x^2 + 4x + 4$

Exercise 1.8C: Challenge

14 Factorise each of the following expressions using real numbers, if possible.

a $2x^2 + 6x - 56$ **b** $x^2 - 3x + 5$ **c** $x^3 + 3x^2 - 18x$

d $12 + x - x^2$ **e** $98 - 2x^2$ **f** $4x^2 + 25$

15 Non-monic quadratic trinomials can also be factorised by completing the square.
For example:

$$\begin{aligned}
2x^2 + 4x - 9 &= 2\left(x^2 + 2x - \frac{9}{2}\right) \\
&= 2\left[(x^2 + 2x + 1) - 1 - \frac{9}{2}\right] \\
&= 2\left[\left(x + 1\right)^2 - \frac{11}{2}\right] \\
&= 2\left(x + 1 - \sqrt{\frac{11}{2}}\right)\left(x + 1 + \sqrt{\frac{11}{2}}\right) \\
&= 2\left(x + 1 - \frac{\sqrt{22}}{2}\right)\left(x + 1 + \frac{\sqrt{22}}{2}\right)
\end{aligned}$$

Factorise each of the following quadratic trinomials by completing the square:

a $2x^2 + 12x + 5$ **b** $2x^2 + 7x + 1$ **c** $3x^2 + 11x - 1$

16 Completing the square of a general quadratic trinomial can be used to determine whether the expression can be factorised using real numbers.

a Complete the square to factorise the general monic quadratic trinomial $x^2 + bx + c$.

b Considering the fact that the square root of a negative number is not a real number, write an algebraic expression in terms of b and c that must be *greater than or equal to zero* for $x^2 + bx + c$ to be factorised using real numbers.

Online resources:

Interactive skillsheet Factorising by completing the square

CAS instructions Completing the square

Quick quiz 1.8

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Factorising non-monic quadratic expressions

Learning intentions

By the end of this lesson you will be able to ...

✔ factorise non-monic quadratic trinomials.

Inter-year links

Year 8	5.7 Factorising
Year 9	2.5 Factorising quadratic expressions

Non-monic quadratic trinomials

- A **non-monic** quadratic trinomial is a quadratic trinomial in which the leading coefficient is not equal to one. These trinomials are often more difficult to factorise.

  $ax^2 + bx + c$ when $a \neq 1$

- When a non-monic quadratic trinomial, $ax^2 + bx + c$, can be factorised, it has the general form $(mx + n)(px + q)$ where m, n, p and q are non-zero constants.

  $ax^2 + bx + c = (mx + n)(px + q)$

Factorising by splitting the middle term

Key content video
Factorising by splitting the middle term

- Expanding the general form of the factorised non-monic quadratic trinomial gives:

$$\begin{aligned}(mx + n)(px + q) &= mpx^2 + mqx + npx + nq \\ &= mpx^2 + (mq + np)x + nq \\ &= ax^2 + bx + c\end{aligned}$$

  So: $a = m \times p$ $\quad b = mq + np$ $\quad c = n \times q$

  This means there are two numbers, mq and np, which add to give b and multiply to give $a \times c$.

  $b = mq + np$ and $a \times c = (mq)(np)$

- If the middle term, bx, of a quadratic trinomial, $ax^2 + bx + c$, is split into the sum of two terms whose coefficients multiply to give $a \times c$, the quadratic can be factorised by grouping terms. This method of factorisation is called splitting the middle term.

  For example, $3x^2 + 10x + 8$ is in the form $ax^2 + bx + c$. We have $a = 3$, $b = 10$ and $c = 8$.
  To split the middle term, $10x$, we must find two numbers that add to give b (which is 10) and multiply to give $a \times c$ (which is $3 \times 8 = 24$).

  The numbers 4 and 6 satisfy these conditions:

  $4 + 6 = 10$ and $4 \times 6 = 24$

  So:
$$\begin{aligned}3x^2 + 10x + 8 &= 3x^2 + 4x + 6x + 8 \\ &= x(3x + 4) + 2(3x + 4) \\ &= (3x + 4)(x + 2)\end{aligned}$$

- If you choose the correct values of mq and np, the order you write the two new terms when you split the middle term does not matter. The resulting factorisation is the same.

  For example, switching the $4x$ and the $6x$ above gives:

$$\begin{aligned}3x^2 + 10x + 8 &= 3x^2 + 6x + 4x + 8 \\ &= 3x(x + 2) + 4(x + 2) \\ &= (x + 2)(3x + 4)\end{aligned}$$

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Factorising by completing the square

- A non-monic quadratic trinomial can also be factorised by completing the square. This is done by taking out the leading coefficient as a factor.

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

Then we can complete the square using the same method as before.

Worked example 1.9A Factorising by splitting the middle term

Factorise each of these quadratic trinomials by first splitting the middle term.

a $3x^2 + 8x + 4$ **b** $6x^2 + 7x - 3$ **c** $10x^2 - x - 2$

THINK

a 1 Identify the values of $a \times c$ and b in the general form of the trinomial, $ax^2 + bx + c$.

2 List the factor pairs of the product $a \times c = 12$. Remember to consider both positive and negative factors.

3 Add the factor pairs together and identify which pair add to the linear coefficient ($b = 8$).

4 Split the middle term of the trinomial into two terms using the pair of numbers from the previous step as the coefficients.

5 Factorise by grouping terms.

b 1 Identify the values of $a \times c$ and b in the general form of the trinomial, $ax^2 + bx + c$.

2 List the factor pairs of the product $a \times c = -18$. Remember to consider both positive and negative factors.

3 Add the factor pairs together and identify which pair add to the linear coefficient ($b = 7$).

4 Split the middle term of the trinomial into two terms using the pair of numbers from the previous step as the coefficients.

5 Factorise by grouping terms.

c 1 Identify the values of $a \times c$ and b in the general form of the trinomial, $ax^2 + bx + c$.

2 List the factor pairs of the product $a \times c = -20$. Remember to consider both positive and negative factors.

3 Add the factor pairs together and identify which pair add to the linear coefficient ($b = -1$).

4 Split the middle term of the trinomial into two terms using the pair of numbers from the previous step as the coefficients.

5 Factorise by grouping terms.

WRITE

a $3x^2 + 8x + 4$

$b = 8$ and $a \times c = 3 \times 4 = 12$

$1 \times 12 = 12 \rightarrow 1 + 12 = 13$

$(-1) \times (-12) = 12 \rightarrow -1 - 12 = -13$

$2 \times 6 = 12 \rightarrow 2 + 6 = 8$

$(-2) \times (-6) = 12 \rightarrow -2 - 6 = -8$

$3 \times 4 = 12 \rightarrow 3 + 4 = 7$

$(-3) \times (-4) = 12 \rightarrow -3 - 4 = -7$

$3x^2 + 8x + 4 = 3x^2 + 2x + 6x + 4$

$= x(3x + 2) + 2(3x + 2)$

$= (3x + 2)(x + 2)$

b $6x^2 + 7x - 3$

$b = 7$ and $a \times c = 6 \times (-3) = -18$

$1 \times (-18) = -18 \rightarrow 1 + -18 = -17$

$(-1) \times 18 = -18 \rightarrow -1 + 18 = 17$

$2 \times (-9) = -18 \rightarrow 2 - 9 = -7$

$(-2) \times 9 = -18 \rightarrow -2 + 9 = 7$

$3 \times (-6) = -18 \rightarrow 3 - 6 = -3$

$(-3) \times 6 = -18 \rightarrow -3 + 6 = 3$

$6x^2 + 7x - 3 = 6x^2 - 2x + 9x - 3$

$= 2x(3x - 1) + 3(3x - 1)$

$= (3x - 1)(2x + 3)$

c $10x^2 - x - 2$

$b = -1$ and $a \times c = 10 \times (-2) = -20$

$1 \times (-20) = -20 \rightarrow 1 - 20 = -19$

$(-1) \times 20 = -20 \rightarrow -1 + 20 = 19$

$2 \times (-10) = -20 \rightarrow 2 - 10 = -8$

$(-2) \times 10 = -20 \rightarrow -2 + 10 = 8$

$4 \times (-5) = -20 \rightarrow 4 - 5 = -1$

$(-4) \times 5 = -20 \rightarrow -4 + 5 = 1$

$10x^2 - x - 2 = 10x^2 + 4x - 5x - 2$

$= 2x(5x + 2) - 1(5x + 2)$

$= (5x + 2)(2x - 1)$

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Worked example 1.9B Factorising by completing the square

Factorise $5x^2 - 10x + 1$ by completing the square.

THINK	WRITE
1 Factorise the trinomial by writing the expression as a product of the leading coefficient, which is 5, and a monic quadratic.	$5x^2 - 10x + 1 = 5\left(x^2 - 2x + \frac{1}{5}\right)$
2 The linear term of the monic quadratic is -2, so add and subtract the term $\left(\frac{-2}{2}\right)^2 = 1$ from the quadratic so that the overall expression remains unchanged.	$= 5\left(x^2 - 2x + 1 - 1 + \frac{1}{5}\right)$ $= 5\left([x^2 - 2x + 1] - \frac{5}{5} + \frac{1}{5}\right)$
3 Factorise the perfect square expression using the 'perfect square' rule and simplify the constant terms at the end of the expression.	$= 5\left(\left[x - 1\right]^2 - \frac{4}{5}\right)$
4 Write as the difference of two squares. Simplify the square root by identifying any perfect square factors and rationalising the denominator.	$= 5\left(\left[x - 1\right]^2 - \left[\sqrt{\frac{4}{5}}\right]^2\right)$ $= 5\left(\left[x - 1\right]^2 - \left[\frac{2}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}\right]^2\right)$ $= 5\left(\left[x - 1\right]^2 - \left[\frac{2\sqrt{5}}{5}\right]^2\right)$
5 Factorise using the 'difference of two squares' rule.	$= 5\left(x - 1 + \frac{2\sqrt{5}}{5}\right)\left(x - 1 - \frac{2\sqrt{5}}{5}\right)$

Helpful hints

✔ Factorising non-monic quadratic trinomials by completing the square often results in an expression containing fractions with surds.

Learning pathways

1, 2, 3–5(1st, 2nd columns), 6(a, c, d), 7(a, b), 8(a, c, e, g), 12	3–6(2nd, 3rd columns), 7(c, d), 8(b, d, f, h), 9, 10, 13	4–6(2nd, 3rd columns), 7(e, f), 8(f–i), 11, 14–16

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Exercise 1.9A: Understanding and fluency

1 Each quadratic trinomial below is in the form $ax^2 + bx + c$. For each one:

i identify ac and b

ii find two numbers that multiply together to give ac and add together to give b

iii use the two numbers you found in part **ii** to split the middle term into two terms.

a $2x^2 + 5x + 2$ **b** $3x^2 + 11x + 6$ **c** $4x^2 + 17x + 4$

d $3x^2 + 10x - 8$ **e** $6x^2 - 7x - 3$ **f** $5x^2 - 13x + 6$

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2 Factorise each of these expressions by grouping terms.

a $3x^2 + 2x + 9x + 6$ **b** $4x^2 + x + 16x + 4$ **c** $2x^2 + x + 4x + 2$

d $6x^2 - 9x + 2x - 3$ **e** $3x^2 - 2x + 12x - 8$ **f** $5x^2 - 3x - 10x + 6$

WE 1.9A **3** Factorise each of these quadratic trinomials by first splitting the middle term.

a $2x^2 + 7x + 6$ **b** $3x^2 + 5x + 2$ **c** $4x^2 + 12x + 5$

d $7x^2 + 9x + 2$ **e** $5x^2 + 14x + 8$ **f** $2x^2 + 9x + 9$

g $6x^2 + 7x + 2$ **h** $10x^2 + 17x + 3$ **i** $8x^2 + 14x + 5$

4 Factorise each of these quadratic trinomials by first splitting the middle term.

a $3x^2 - 8x - 3$ **b** $4x^2 + 5x - 6$ **c** $2x^2 - 9x + 10$

d $5x^2 - x - 4$ **e** $8x^2 + 10x - 3$ **f** $4x^2 + 4x - 15$

g $7x^2 - 18x + 8$ **h** $12x^2 + x - 1$ **i** $3x^2 - 11x - 20$

5 Factorise each of these quadratic trinomials. Where appropriate, take out any common factors first. Hint: The common factor can be negative.

a $4x^2 + 10x + 6$ **b** $12x^2 + 27x + 6$ **c** $10x^2 + 38x - 8$

d $8x^2 + 22x + 15$ **e** $6x^2 + 3x - 30$ **f** $12x^2 - 8x - 4$

g $-4x^2 - 6x - 2$ **h** $-12x^2 - 21x + 6$ **i** $-6x^2 - 17x - 12$

6 Simplify each of these expressions. Remember to factorise first.

a $\dfrac{2x^2 + 15x + 18}{x + 6}$ **b** $\dfrac{3x + 4}{3x^2 + 13x + 12}$ **c** $\dfrac{4x^2 + 18x + 14}{2x + 2}$

d $\dfrac{6x^2 - 5x + 1}{3x^2 + 14x - 5}$ **e** $\dfrac{5x^2 - 28x - 12}{2x^2 - 15x + 18}$ **f** $\dfrac{-8x^2 + 4x + 40}{2x^2 - x - 10}$

7 Simplify each of these expressions.

a $\dfrac{5x^2 - 8x + 3}{x^2 - 1} \times \dfrac{2x^2 + x - 1}{2x^2 - 7x + 3}$ **b** $\dfrac{2x^2 + 13x + 21}{6x^2 - x - 2} \times \dfrac{3x^2 - 14x + 8}{x^2 - x - 12}$

c $\dfrac{4x^2 + 13x - 12}{2x^2 + 13x + 20} \div \dfrac{12x^2 - 5x - 3}{6x^2 + 17x + 5}$ **d** $\dfrac{4x^2 + 28x + 49}{4x^2 - 49} \div \dfrac{2x^2 - x - 28}{2x^2 + 15x + 28}$

e $\dfrac{3x^2 + 6x - 24}{3x^2 - 7x + 2} \times \dfrac{15x^2 + x - 2}{6x + 24}$ **f** $\dfrac{2x^3 - x^2 - 3x}{2x^2 + 9x - 18} \div \dfrac{x^3 - 9x}{x^2 + 9x + 18}$

WE 1.9B **8** Factorise each of these quadratic trinomials by completing the square.

a $3x^2 + 18x + 21$ **b** $2x^2 - 20x + 40$ **c** $5x^2 + 10x - 30$

d $3x^2 + 12x + 3$ **e** $2x^2 - 12x - 3$ **f** $4x^2 + 8x - 1$

g $6x^2 - 24x - 5$ **h** $3x^2 + 24x + 1$ **i** $9x^2 - 18x + 2$

9 Simplify the following expression by first factorising each fraction.

$$\frac{2x^2 - 3x - 20}{x^3 - 16x} \times \frac{3x^2 + 16x + 16}{4x^2 - 25} \div \frac{9x^2 + 24x + 16}{70x - 28x^2}$$

Exercise 1.9B: Problem solving and reasoning

10 When the leading coefficient of a quadratic trinomial is negative, the expression can be factorised to a product of –1 and a quadratic trinomial with a positive leading coefficient.

a Factorise $4x^2 + 7x - 2$.

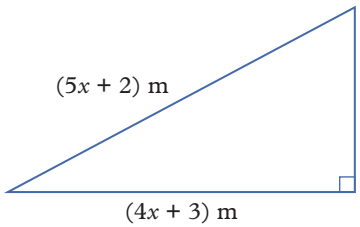
b Show that $-4x^2 - 7x + 2$ factorises to $(1 - 4x)(x + 2)$.

c Factorise each of these quadratic trinomials.

i $-2x^2 + x + 3$ **ii** $-3x^2 + 16x - 5$ **iii** $-5x^2 - 3x + 14$

iv $-4x^2 + 5x + 6$ **v** $-6x^2 - x + 2$ **vi** $-25x^2 + 10x - 1$

11 Consider the right-angled triangle shown on the right.



a Find an expression for the length of the missing side, in terms of x.

b The expression you wrote for part **a** should include a square root symbol. Factorise the expression appearing under that square root symbol.

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12 Factorise each of the following by first completing the square.

a $x^2 + 8x + 5$ **b** $2x^2 + 8x + 5$ **c** $3x^2 + 8x + 5$

13 Decide whether each of these expressions can be factorised using the methods covered in this module so far. Write those that can be factorised in factorised form.

a $2x^2 + 4x + 6$ **b** $15x^2 - 2x - 8$ **c** $3x^2 - 12x + 3$

d $4x^2 - 8x + 16$ **e** $-2x^2 + 12x - 22$ **f** $-6x^2 + 69x - 189$

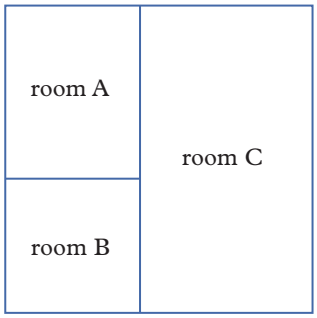
Exercise 1.9C: Challenge

14 A karate instructor bought a gym with three training rooms: A, B and C. To make each room a safe place for training, the instructor plans to set down some large square mats of length x metres.

The dimensions of all the rooms are a number of mats plus or minus an integer number of metres. Training room A has an area of $(8x^2 + 2x - 15)$ m^2. The length of the square floor of training room B is the same as the smaller floor dimension of training room A. Assume $x < 4$ m.

Answer the questions about the karate gym.

karate gym



a Write expressions for the dimensions of the floors of training rooms A and B in terms of the length of the mats.

b Write an expression for the total floor area of training rooms A and B combined, in terms of the length of the mats.

c If the shape of the karate gym is a square, what are the dimensions of the gym?

d Write an expression for the total area of training rooms A and C combined, as the difference of two squares.

e Factorise your answer for part **d**.

f Divide the total area of the gym by the area of one mat to determine an expression for the maximum number of mats that the instructor will need to cover the floor of the gym. Give your answer as a trinomial.

g If $x = 3$ m, what is the maximum number of mats that can fit on the floor of the gym?

15 A quartic is an expression of the form $ax^4 + bx^3 + cx^2 + dx + e$ where a, b, c, d and e are constants. If a quartic does not include any exponents that are odd numbers (so the b and d coefficients are zero), it can be factorised using the same methods for factorising quadratic trinomials.

For example, $x^4 + 5x^2 + 6$ can be factorised to $(x^2 + 3)(x^2 + 2)$.

a Factorise the following expressions.

i $x^4 + 7x^2 + 10$ **ii** $x^4 + 14x^2 + 24$ **iii** $x^4 + 4x^2 + 3$

b Write each of the following quartics as the product of one quadratic factor and two linear factors, by using the 'difference of two squares' method of factorisation.

i $x^4 - 1$ **ii** $x^4 - 16$ **iii** $x^4 - 81$

c Write each of the following quartics as the product of four linear factors.

i $x^4 - 13x^2 + 36$ **ii** $x^4 - 26x^2 + 25$ **iii** $x^4 - 11x^2 + 10$

16 Completing the square of a general quadratic trinomial can be used to determine whether the expression can be factorised using real numbers.

a Complete the square for the general quadratic trinomial $ax^2 + bx + c$.

b Use your answer for part **a** to factorise the general quadratic trinomial $ax^2 + bx + c$.

c Write a simplified expression in terms of a, b and c that *must be non-negative* for the quadratic to be factorised.

Online resources:

Interactive skillsheet Factorising non-monic quadratic expressions

Quick quiz 1.9

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Lesson 1.10

Review: Algebra

Module summary

Exponent laws and properties

$a^m \times a^n = a^{m+n}$	$a^m \div a^n = a^{m-n}$
$(a^m)^n = a^{m \times n}$	$(a \times b)^m = a^m \times b^m$
$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$	$\left(\frac{a}{b}\right)^{-m} = \left(\frac{b}{a}\right)^m$
$a^0 = 1$	$a^{-1} = \frac{1}{a}$

Simplifying terms and expressions

- Like terms contain the same pronumerals raised to the same exponents.
- Like terms can be added and subtracted by adding and subtracting their coefficients.

$$7x^2y + 2 + 3yx^2 - 8 = 10x^2y - 6$$

Expanding

Use the distributive law to expand brackets.

$$a(b + c) = ab + ac$$

$$(a + b)(c + d) = ac + ad + bc + bd$$

Factorising using the HCF

expanding

$$2x^2 - 6x = 2x(x - 3)$$

expanded form factorised form

factorising

Factorising by grouping terms

$$x^2 + 2x - 3x - 6 = x(x + 2) - 3(x + 2)$$
$$= (x + 2)(x - 3)$$

Factorising monic quadratic trinomials

$$x^2 + bx + c$$

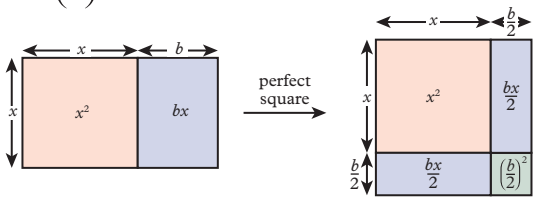
- Find numbers, m and n, that add to give b and multiply to give c.

$$x^2 + (m + n)x + mn = (x + m)(x + n)$$

- Substitute those two numbers into the factorised form (the binomial product).

Adding a term to make a perfect square

Add $\left(\frac{b}{2}\right)^2$ to $x^2 + bx$ to make a perfect square.



$$x^2 + 2x\left(\frac{b}{2}\right) + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

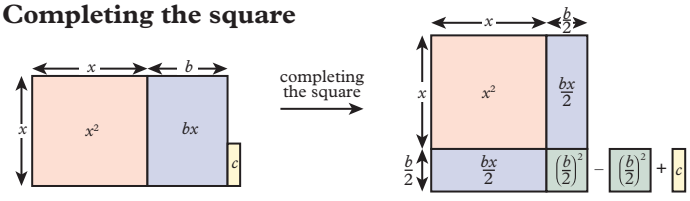
Factorising by splitting the middle term

$$ax^2 + bx + c = mpx^2 + mqx + npx + nq$$

- Find the unknowns that add to give b and multiply to give $a \times c$.
- Use these numbers to split the linear term.
- Factorise by grouping terms.

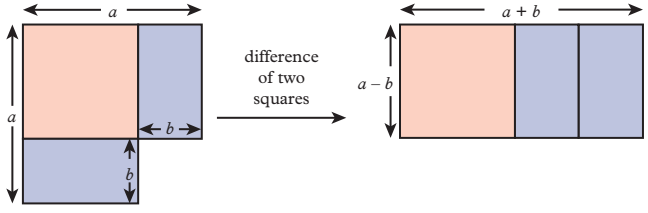
$$mpx^2 + mqx + npx + nq = (mx + n)(px + q)$$

Completing the square



$$x^2 + bx + c = x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$
$$= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c$$

Factorising using the difference of two squares



$$a^2 - b^2 = (a + b)(a - b)$$
$$(x - 1)^2 - 2 = (x - 1)^2 - (\sqrt{2})^2$$
$$= (x - 1 + \sqrt{2})(x - 1 - \sqrt{2})$$

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Review questions 1.10A: Mathematical literacy review

Review quiz
Take the review quiz to assess your knowledge of this module.
✔ 10
✔ 10A

Quizlet
Test your knowledge of this module by working individually or in teams.

The following key terms are used in this module:

- algebraic fraction
- base
- binomial
- coefficient
- completing the square
- constant term
- difference of two squares
- distributive law
- expand
- exponent
- exponent form
- exponent laws
- exponent notation
- factorise
- highest common factor (HCF)
- index
- leading coefficient
- like terms
- monic quadratic trinomial
- negative exponent
- non-monic
- perfect square
- pronumeral
- quadratic
- reciprocal
- simplify
- splitting the middle term
- substitution
- term
- trinomial
- variable

1 Identify the following from these given expressions.

$6x^3 - 2x + 9,\ x + 5,\ \dfrac{4}{x+1} - 7x$

a a binomial

b a trinomial

c a constant term

d an algebraic fraction

e a negative coefficient

f leading coefficient of the trinomial

g an exponent

2 Match the expressions in the left-hand column with a key term in the right-hand column.

a	$a^2 - b^2$	**I**	monic quadratic trinomial
b	$(a + b)^2$	**II**	splitting the middle term
c	$a(b + c) = ab + ac$	**III**	non-monic quadratic trinomial
d	$\dfrac{4a}{8(a+b)} = \dfrac{a}{2(a+b)}$	**IV**	completing the square
e	$x^2 + 2x + 1$	**V**	difference of two squares
f	$3x^2 - x + 5$	**VI**	simplifying
g	$x^2 + 6x + 8 = (x + 3)^2 - 1$	**VII**	perfect square
h	$2x^2 - 5x + 3 = 2x^2 - 2x - 3x + 3$	**VIII**	expanding

3 Complete the following sentence using words from the list of key terms.

a If a term in the denominator has a _____, this is the same as multiplying by the same term in the numerator but with a positive _____.

b Factorisation is the reverse process to _____ and the first step is to divide by the _____.

4 Select the best option(s) for this statement: A factor of a factorised quadratic expression could be _____.

A a constant

B a single term involving the pronumeral

C a binomial expression

D a quadratic expression

E all of the above

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Review questions 1.10B: Multiple choice

1.1 **1** What is the simplified form of $x^2 \times (x^3)^4$?

A x^9 **B** x^{24} **C** x^{20} **D** x^{14}

1.1 **2** Which of the following expressions is not equal to the others?

A $(x^3 \times x^3 \times x^3)^2$ **B** $(x^4)^3$ **C** $(x^3)^2 \times (x^3)^2$ **D** $x^2 \times (x^2)^5$

1.2 **3** What is the value of the missing exponent in $x^{-1}yz \times yz^{\square} = \frac{y^2}{xz}$?

A 2 **B** 1 **C** -1 **D** -2

1.2 **4** What is $\frac{36x^2y}{4x^3y^{-2}}$ when it is simplified?

A $\frac{9}{xy^3}$ **B** $\frac{9y^3}{x}$ **C** $9x^{-1}y^{-3}$ **D** $\frac{9}{xy}$

1.3 **5** Which of the following expressions is equivalent to $a^2b - 2ab - ab^2 + 5ab$?

A $7ab$ **B** $a^2b + 3ab + ab^2$
C $3ab$ **D** $a^2b + 3ab - ab^2$

1.4 **6** What is the expanded form of $6t^2w(4w^3 + 3t^4)$?

A $24t^2w^3 + 18t^6w$ **B** $24t^2w^4 + 18t^8w$
C $24t^2w^4 + 18t^6w$ **D** $24t^2w^3 + 18t^8w$

1.4 **7** What is the expanded form of $(x - 4)(x + 3)$?

A $x^2 + x + 12$ **B** $x^2 + x - 12$ **C** $x^2 - x + 12$ **D** $x^2 - x - 12$

10A 1.5 **8** What is the lowest common denominator of $\frac{2x}{(x+2)}$, $\frac{xy}{(x+1)(x+2)}$ and $\frac{2y}{x(x+2)}$?

A $x(x + 1)(x + 2)^3$ **B** $(x + 1)(x + 2)$
C $x(x + 1)(x + 2)$ **D** $(x + 2)$

10A 1.5 **9** Which of the following expressions is equal to $\frac{2x}{3} + \frac{x+1}{4}$?

A $\frac{3x+1}{7}$ **B** $\frac{11x+3}{12}$ **C** $\frac{3x+1}{12}$ **D** $\frac{11x+3}{7}$

1.6 **10** What is the fully factorised form of $8x^2y^3 + 12x^3y$?

A $8x^2y(y^2 + 4x)$ **B** $4x^2y(2y^2 + 3x)$
C $4xy(2xy^2 + 3x^2)$ **D** $4xy(2y^2 + 3x)$

1.6 **11** What is the factorised form of $3(a - 2) - b(a - 2)$?

A $(a - 2)(3 - b)$ **B** $(a - 2)(b - 3)$
C $(a - 2)(3 + b)$ **D** $(a + 2)(3 + b)$

1.7 **12** The quadratic trinomial $x^2 + 7x + c$ *cannot* be factorised for which one of the following values of c?

A 18 **B** -8 **C** 10 **D** 12

1.7 **13** Which of these is a quadratic trinomial?

A $5x^2 + 4$ **B** $2x^3 + 4x^2 - 6$
C $x + 5x - 3$ **D** $4x - 2 + 3x^2$

1.8 **14** Which of the following expressions is equal to $x^2 + 6x + 7$?

A $(x + 3)^2 - 2$ **B** $\left(x + \frac{3}{2}\right)^2 - 2$
C $(x + 6)^2 + 2$ **D** $(x + 3)^2 + 2$

10A 1.9 **15** What is the factorised form of $9x^2 + 21x + 10$?

A $(9x + 5)(x + 2)$ **B** $(9x + 10)(x + 1)$
C $(3x + 5)(3x + 2)$ **D** $(3x + 10)(3x + 1)$

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Review questions 1.10C: Short answer

1.1 **1** Simplify each of the following.

a $a^2 \times a^5$ **b** $(2x^2y)^3 \times 3xy^2$ **c** $3(x^2y)^0$ **d** $\dfrac{12ab^2c^5}{(2bc)^2}$

1.2 **2** Write each of these terms in simplified form with positive exponents.

a $\dfrac{xy^{-3}}{x^{-2}y}$ **b** $\dfrac{4a^2b^{-1}}{12a^{-2}b^4}$ **c** $\dfrac{18p^3q^{-1}r^5}{15p^{-1}q^4r^{-2}}$

1.3 **3** Simplify each of these expressions, writing the answers with positive exponents.

a $(x^2)^3 \times x^{-2} \div x^{-4}$ **b** $[(p^2q^{-3})^2 \div p^4q^{-3}] \times p^{-1}q$ **c** $(2a^2)^{-3} \times (3ab^{-1})^{-2} \div 12b^5$

1.3 **4** Simplify each of these expressions.

a $4xy - 10xy + x^2 - 4x^2 + 2x^2y - 5xy^2$ **b** $ab^2 + (ab)^2 - 3a^2b^2 + 2a^2b - 4b^2a$

c $pq - 5pq + p^2 - q^2 - (3p)^2 + (2q)^2$

1.3 **5** Name the like terms in each of these lists.

a $3abc^2,\ a^2b^2c^2,\ -2a^2b^2c,\ (2abc)^2$ **b** $5xyz^2,\ -2z^2xy,\ 3yz^2x,\ 4x^2y^2y^2$

c $2p^2q^2,\ -(pq)^2,\ p^2 \times q \times -q,\ (p \div q)^2$

1.3 **6** If $a = -2$, $b = -1$ and $c = 3$, evaluate each of these expressions.

a $a^2 + b^2 + c^2$ **b** $abc - ab^2c - c^3 \div b$ **c** $(a + b)^2 - (c - a)^2$

1.4 **7** Simplify each of the following expressions.

a $\dfrac{4ab}{3c} + \dfrac{6ba}{c}$ **b** $\dfrac{4ab}{3c} - \dfrac{6ba}{c}$ **c** $\dfrac{4ab}{3c} \times \dfrac{6ba}{c}$ **d** $\dfrac{4ab}{3c} \div \dfrac{6ba}{c}$

1.4 **8** Expand and simplify each of these expressions.

a $(x - 3)(x + 7)$ **b** $(x + 2)^2$ **c** $(x - 4)^2$

d $(2x + 1)(x - 4)$ **e** $(a - b)(a + b)$ **f** $(a^2 + b)(a + b^2)$

1.4 **9** Expand each of these expressions, giving your answers in simplest form.

a $(2x^2 - 4)(3x^2 + 5)$ **b** $(2x + 3y)(2x - 3y)$ **c** $(3x^2 - 4)^2$

1.4 **10** Expand and simplify each of these expressions.

a $3(x^2 - 2) - x(x + 5)$ **b** $-2x(3 - x) + 5x(2 - 3x)$ **c** $x(x + 1) - x(x - 1) + x(x - 3)$

10A 1.5 **11** Simplify each of these expressions.

a $\dfrac{3x - 1}{4} + \dfrac{4 - x}{3}$ **b** $\dfrac{(x - 1)(x - 2)}{(x + 2)(x - 3)} \times \dfrac{(x + 2)(x + 3)}{(x - 1)(x - 2)}$ **c** $\dfrac{5x + 2}{2y} - \dfrac{4 - x}{y}$

1.6 **12** Factorise each of these expressions by grouping terms.

a $x^2 - 3x - 4x + 12$ **b** $x^3 + x^2 - x - 1$ **c** $16x + 6x^2 - 32 - 12x$

1.6 **13** Use the 'difference of two squares' rule to factorise each of these expressions. Write your answers in simplest form.

a $18x^2 - 8y^2$ **b** $(x - 3)^2 - x^2$ **c** $(x + 2)^2 - (x - 2)^2$

1.6 **14** Factorise each of these expressions, including surds in your answer if necessary.

a $x^2y^2 - 13$ **b** $(x + 1)^2 - 3$ **c** $7 - (a - 1)^2$

1.7 **15** Factorise each of these expressions.

a $x^2 + 8x + 12$ **b** $x^2 - 2x - 63$ **c** $-x^2 + 6x - 8$ **d** $-6x^2 - 6x + 36$

1.7 **16** Simplify each of these expressions.

a $\dfrac{x^2 - 4}{x^2 - x - 6} \times \dfrac{x^2 + x - 12}{x^2 + x - 6}$ **b** $\dfrac{x^2 - 5x + 6}{x^2 - 16} \div \dfrac{x^2 + x - 6}{x^2 - 3x - 4}$

1.8 **17** Factorise each of these expressions by first completing the square.

a $x^2 + 10x + 20$ **b** $x^2 - 8x - 1$

10A 1.9 **18** Factorise each of these expressions.

a $3x^2 + 29x + 18$ **b** $8x^2 + 10x - 3$ **c** $2x^2 - 7x - 22$ **d** $12x^2 - 17x + 6$

10A 1.9 **19** Factorise each of these expressions, if possible.

a $2x^2 + 12x - 4$ **b** $5x^2 - 20x + 30$ **c** $6x^2 + 21x - 90$ **d** $-4x^2 + 12x - 2$

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Review questions 1.10D: Mathematical modelling

1 Sameera wants to know about factorising quadratic trinomials of the form $x^2 + bx + 12$, where b is an integer.

a Suppose $b = 0$. Explain why Sameera can't use the 'difference of two squares' rule.

b Is there a value of b that would allow the 'perfect square' rule to be used? If not, why not?

c Sameera wants to find all values of b that allow the quadratic expression to be factorised into a product of two linear factors in the form $(x + c)$ where c is a non-zero integer.

i Complete the square of $x^2 + bx + 12$ and use your answer to show that $b^2 \geq 49$.

ii Find all possible values of b by considering the factor pairs of 12 and the corresponding factorisations.

10A **d** Sameera next considers quadratic trinomials of the form $2x^2 + bx + 12$. Find all values of b for which the quadratic trinomial can be written in the form $(x + n)(px + q)$.

2 Consider the following quadratic equations.

a Find values of a and b such that $(ax - 3)(6x + 5) = 12x^2 - 8x + b$.

b Find values of a, b and c such that $(4x - b)(ax + 2) = 12x^2 + cx - 2$.

c Find values of a, b and c such that $(ax - 3)(bx + 5) = 12x^2 - 31x + c$.

d Find values of a, b and c such that $(ax + 1)(-2x + b) = 12x^2 + cx - 15$.

e How many possible sets of values of a, b, c and d are there, such that $(ax - 3)(bx + c) = 12x^2 + dx - 15$?

10A **f** For $(ax + b)(cx + d) = ex^2 + fx + g$:

i Determine the rule to calculate the value of f.

ii How many possible values of f are there, given e and g are prime integers?

iii How many possible values of f are there if e is a composite integer with 4 distinct factors and g is a prime integer? Verify your answer with an example.

iv How many possible values of f are there if both e and g are composite integers with 4 and 6 distinct factors, respectively?

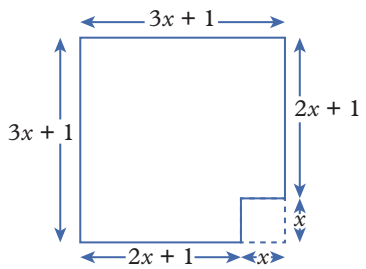
v How many possible values of f are there if both e and g are composite integers with m and n distinct factors, respectively?

vi If the equation was $(ax + b)(cx + d) = ex^2 + fx - g$, how would this change your answer to part **v**?

3 This question explores areas of composite shapes.

a For each of the following shapes, find an expression for the area as a difference of squares, and then in factorised form.

i



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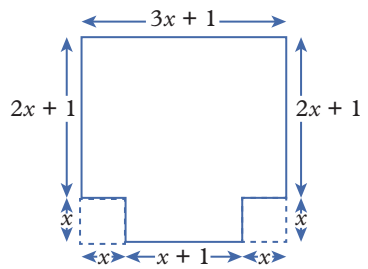

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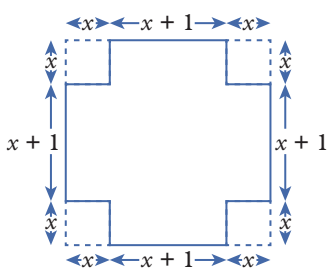

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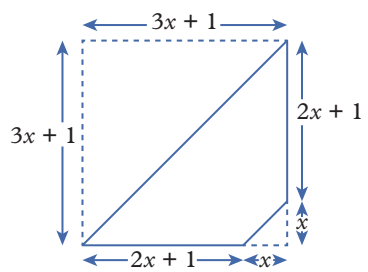
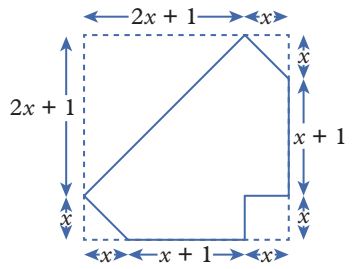


b The area of the shape on the right, in terms of x, can be expressed as $ax^2 + bx + c$. Find the values of a, b and c.



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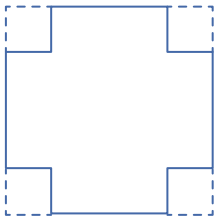
10A **c** Four identical small squares are cut off a larger square as shown below. The area of the resulting shape is $A = (9x + 3)(x + 3)$.



Let y be an expression in terms of x representing the side length of the larger square.
Let c be an expression in terms of x representing the side length of the four small squares.
Find the expressions for y and c assuming that $x > 0$.

Checklist

Now that you have completed this module, reflect on your ability to do the following.

I can do this	I need to review this
☐ Use the exponent laws to simplify algebraic expressions	☐ Go back to Lesson 1.1 Exponents
☐ Write algebraic terms with only positive exponents ☐ Use the exponent laws to simplify expressions with negative exponents	☐ Go back to Lesson 1.2 Negative exponents
☐ Evaluate algebraic expressions using substitution ☐ Identify like terms to simplify algebraic expressions ☐ Add, subtract, multiply and divide algebraic terms	☐ Go back to Lesson 1.3 Simplifying
☐ Expand algebraic expressions with one pair of brackets ☐ Expand binomial products	☐ Go back to Lesson 1.4 Expanding
☐ Simplify algebraic fractions by identifying common factors ☐ Multiply and divide algebraic fractions ☐ Add and subtract algebraic fractions	☐ Go back to Lesson 1.5 Algebraic fractions [10A]
☐ Factorise algebraic expressions by identifying the highest common factor ☐ Factorise algebraic expressions by grouping terms ☐ Factorise algebraic expressions using the 'difference of two squares' and the 'perfect square' rules	☐ Go back to Lesson 1.6 Factorising
☐ Factorise monic quadratic trinomials by using factor pairs of the constant	☐ Go back to Lesson 1.7 Factorising quadratic expressions
☐ Add a term to a quadratic binomial to make a perfect square ☐ Factorise monic quadratic trinomials by completing the square	☐ Go back to Lesson 1.8 Completing the square
☐ Factorise non-monic quadratic trinomials by splitting the middle term ☐ Factorise non-monic quadratic trinomials by completing the square	☐ Go back to Lesson 1.9 Factorising non-monic quadratic expressions [10A]

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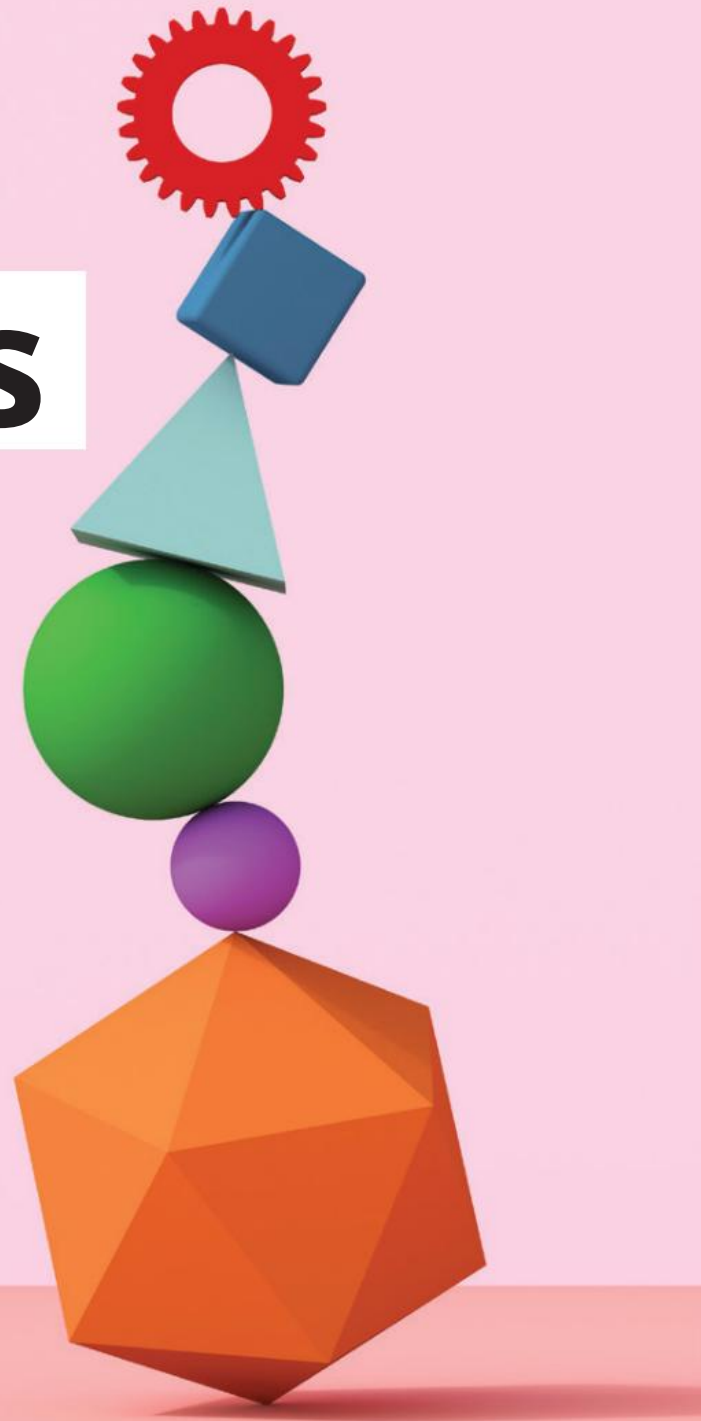
2

Real numbers



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Lessons

10A **2.1** Surds
10A **2.2** Multiplying and dividing surds
10A **2.3** Adding and subtracting surds
10A **2.4** Rationalising the denominator
10A **2.5** Fractional exponents
10A **2.6** Logarithms
10A **2.7** Logarithm laws
10A **2.8** Exponential equations

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ Exponents
- ✔ Roots
- ✔ Multiplying and dividing terms
- ✔ Like terms
- ✔ Equivalent fractions

Curriculum links

10A • Define rational and irrational numbers and perform operations with surds and fractional indices (VC2M10AN01)

10A • Perform operations on numbers involving fractional exponents and surds (VC2M10AN02)

10A • Use the definition of a logarithm to establish and apply the laws of logarithms and investigate logarithmic scales in measurement (VC2M10AN03)

10A • Explore the inverse relationship between exponential functions and logarithmic functions and the solution of related equations (VC2M10AA04)

© VCAA

Materials

- ✔ Calculator

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Lesson 2.1

Surds

Learning intentions

By the end of this lesson you will be able to ...

✔ identify and simplify surds.

Inter-year links

Year 7	1.7 Exponents and square roots
Year 8	4.5 Roots
Year 9	1.6 Rational and irrational numbers

Real numbers

- A **real number** is a number that can be represented by a point on a number line.
- The real number system is made up of **rational numbers** and **irrational numbers**.
- Rational numbers are numbers that can be written as a fraction where both the numerator and denominator are integers. The set of rational numbers includes integers, fractions, terminating decimals and recurring decimals.
- Irrational numbers are numbers that cannot be written as fractions with both the numerator and denominator being integers. This includes **surds** and other non-terminating, non-repeating decimals.

Surds

- A surd is an irrational expression of the form $\sqrt[n]{x}$, where x is a rational number and n is an integer greater than or equal to 2.
  - → Surds cannot be simplified to remove the root.
  - → Surds are exact values that can be rounded to a decimal approximation.
  - → Square roots, cube roots and fourth roots are all types of surds.
  - → When the value of n is not provided, it is assumed to be 2 (a square root).
- For a surd to be a real number, when the value of n is even the value of x must be positive ($x > 0$).
- When a number is written, full size, in front of a surd, the surd is being multiplied by that number. For example, $a\sqrt{b} = a \times \sqrt{b}$.
- A surd multiplied by a rational number is also considered to be a surd.

Square roots

- For all positive real numbers, finding the square root of a number and squaring a number are inverse operations.

  $(\sqrt{a})^2 = a = \sqrt{a^2}$
- $\sqrt{a}$ is the positive square root of x. For example, $\sqrt{9} = 3$.
  - → The negative square root is represented by putting a negative sign in front of the root symbol. For example, $-\sqrt{9} = -3$.
- The value $\sqrt{0}$ is 0.

Simplifying surds

Key content video
Simplifying surds

- The square root of a product can be written as the product of the square roots of its positive factors.

  $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
- A square root can be simplified if the number under the square root sign has a perfect square factor.

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- To simplify a square root (in this case, $\sqrt{12}$):
  1. Rewrite the number under the root sign as a product of its factors, with the first factor being the greatest possible perfect square.
  2. Write the square root as the product of its square root factors, using $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$.
  3. Simplify the square roots of any perfect squares.

$$\begin{aligned}\sqrt{12} &= \sqrt{4 \times 3}\\ &= \sqrt{4} \times \sqrt{3}\\ &= 2 \times \sqrt{3}\\ &= 2\sqrt{3}\end{aligned}$$

Worked example 2.1A Simplifying a surd in $\sqrt{a}$ form

Write $\sqrt{54}$ in its simplest form.

THINK

1. Rewrite the number under the root sign as a product of its factors, with the first factor being the greatest possible perfect square.
2. Write the square root as the product of its square root factors, using $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$.
3. Simplify the square roots of any perfect squares.

WRITE

$\sqrt{54} = \sqrt{9 \times 6}$

$= \sqrt{9} \times \sqrt{6}$

$= 3 \times \sqrt{6}$

$= 3\sqrt{6}$

Worked example 2.1B Simplifying a surd in $a\sqrt{b}$ form

Write $4\sqrt{72}$ in its simplest form.

THINK

1. Rewrite the number under the root sign as a product of its factors, with the first factor being the greatest possible perfect square.
2. Simplify the square roots of any perfect squares.
3. Multiply the coefficients and simplify the surd.

WRITE

$4\sqrt{72} = 4\sqrt{36 \times 2}$

$= 4 \times \sqrt{36} \times \sqrt{2}$

$= 4 \times 6 \times \sqrt{2}$

$= 24 \times \sqrt{2}$

$= 24\sqrt{2}$

Worked example 2.1C Identifying surds

Which of the following numbers are surds?

a $2\sqrt{8}$

b $24\sqrt{16}$

THINK

a $2\sqrt{8} = 2 \times 2\sqrt{2} = 4 \times \sqrt{2}$

$\sqrt{2} = 1.414\,213\,562\ldots$, so $2\sqrt{8}$ is a surd.

b $24\sqrt{16} = 24 \times 4 = 96$

So $24\sqrt{16}$ is not a surd.

WRITE

a $2\sqrt{8}$ is a surd.

b $24\sqrt{16}$ is not a surd.

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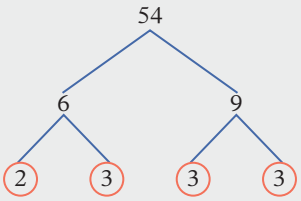
Helpful hints

- ✔ Factor trees can be helpful in determining perfect square factors. Prime factors that appear more than once in a prime factorisation can be combined to give a perfect square factor. For example, in the factor tree shown, $3 \times 3 = 3^2 = 9$ is a perfect square factor of 54.
- ✔ If there are no repeated prime factors of a root, then the root cannot be simplified.



Learning pathways

1, 2, 4–5(1st–3rd columns), 6, 7(1st–3rd columns), 8, 10, 11, 15, 16, 17(1st, 2nd columns)	3, 4–5(3rd–5th columns), 7(3rd–5th columns), 8, 9(a, c), 10, 11(b, d, j, l), 12, 13, 15, 16, 17(3rd, 4th columns), 20(a, b)	3, 4–5(4th, 5th columns), 7(4th, 5th columns), 9(b, d), 10, 11(i–l), 13–15, 17(5th, 6th columns), 18, 19, 20(c, d)

ANS p685

Exercise 2.1A: Understanding and fluency

1 Calculate the value of each of these.

a $\sqrt{16}$ **b** $\sqrt{81}$ **c** $\sqrt{49}$ **d** $\sqrt{4}$ **e** $\sqrt{100}$
f $\sqrt{25}$ **g** $3\sqrt{36}$ **h** $4\sqrt{121}$ **i** $\sqrt{1}$ **j** $\frac{1}{2}\sqrt{16}$

2 Simplify each of these expressions.

a $\sqrt{81} \times \sqrt{7}$ **b** $\sqrt{16} \times \sqrt{3}$ **c** $\sqrt{4} \times \sqrt{6}$
d $\sqrt{25} \times \sqrt{10}$ **e** $\sqrt{9} \times \sqrt{2}$ **f** $\sqrt{5} \times \sqrt{36}$
g $2 \times \sqrt{16} \times \sqrt{11}$ **h** $3 \times \sqrt{100} \times \sqrt{5}$ **i** $\frac{1}{2}\sqrt{144} \times \sqrt{10}$

3 Calculate the value of each of these without using a calculator.

a $\sqrt{121}$ **b** $5\sqrt{36}$ **c** $-4\sqrt{64}$
d $-\frac{\sqrt{16}}{4}$ **e** $4\sqrt{16} + \sqrt{64}$ **f** $\sqrt{36} + \sqrt{64}$
g $\sqrt{36 + 64}$ **h** $\sqrt{9 + 16}$ **i** $\sqrt{25 + 144}$
j $-\sqrt{9} - \frac{\sqrt{36}}{2}$ **k** $2\sqrt{100} - \frac{3\sqrt{144}}{4}$ **l** $\sqrt{9^2}$

WE 2.1A **4** Simplify the following square roots.

a $\sqrt{28}$ **b** $\sqrt{50}$ **c** $\sqrt{24}$ **d** $\sqrt{27}$ **e** $\sqrt{48}$
f $\sqrt{75}$ **g** $\sqrt{63}$ **h** $\sqrt{60}$ **i** $\sqrt{56}$ **j** $\sqrt{45}$
k $\sqrt{128}$ **l** $\sqrt{108}$ **m** $\sqrt{180}$ **n** $\sqrt{147}$ **o** $\sqrt{162}$
p $\sqrt{125}$ **q** $\sqrt{450}$ **r** $\sqrt{288}$ **s** $\sqrt{270}$ **t** $\sqrt{567}$

WE 2.1B **5** Write each of the following surds in simplest form.

a $5\sqrt{20}$ **b** $3\sqrt{8}$ **c** $6\sqrt{12}$ **d** $9\sqrt{18}$ **e** $4\sqrt{80}$
f $3\sqrt{90}$ **g** $12\sqrt{150}$ **h** $-3\sqrt{48}$ **i** $7\sqrt{96}$ **j** $-2\sqrt{245}$
k $8\sqrt{192}$ **l** $10\sqrt{252}$ **m** $-5\sqrt{112}$ **n** $7\sqrt{99}$ **o** $-12\sqrt{68}$
p $100\sqrt{300}$ **q** $\frac{1}{2}\sqrt{320}$ **r** $\frac{1}{2}\sqrt{216}$ **s** $8\sqrt{248}$ **t** $\frac{3}{4}\sqrt{176}$

6 Show that the surd $5\sqrt{27}$ simplifies to $15\sqrt{3}$. Note: 'Show that' means to show all the necessary calculations.

7 Write each of the following surds in simplest form.

a $\sqrt{44}$ **b** $\sqrt{125}$ **c** $\sqrt{120}$ **d** $\sqrt{72}$ **e** $\sqrt{150}$
f $-\sqrt{216}$ **g** $\sqrt{126}$ **h** $\sqrt{240}$ **i** $-3\sqrt{18}$ **j** $\sqrt{245}$
k $\frac{1}{2}\sqrt{192}$ **l** $\frac{\sqrt{108}}{3}$ **m** $-2\sqrt{48}$ **n** $15\sqrt{80}$ **o** $3\sqrt{84}$
p $\frac{3}{5}\sqrt{450}$ **q** $-8\sqrt{800}$ **r** $-\frac{2}{3}\sqrt{900}$ **s** $-\sqrt{1000}$ **t** $\frac{\sqrt{288}}{4}$

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Exercise 2.1B: Problem solving and reasoning

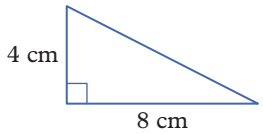
WE 2.1C **8** Which of the following numbers are surds?

a $-\sqrt{81}$ **b** $\sqrt{0.64}$ **c** $\sqrt{0.036}$ **d** $-3\sqrt{0.004}$

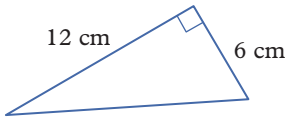
e $\sqrt{1.369}$ **f** $\sqrt{20.25}$ **g** $\frac{3}{4}\sqrt{100}$ **h** $100\sqrt{\frac{1}{4}}$

9 Use Pythagoras' theorem to calculate the exact length of the unknown side in each triangle below. Where appropriate, write the answer in simplest surd form.

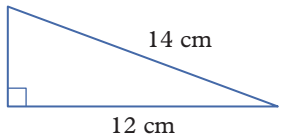
a



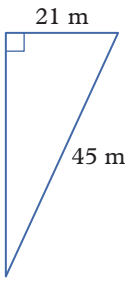
b



c



d



10 A surd in which the entire number is written beneath the root sign is called an entire surd. For example, $\sqrt{6}$ is an entire surd, whereas $3\sqrt{6}$ is not an entire surd.

a Fill in the boxes below to write $3\sqrt{6}$ as an entire surd.

$$\begin{aligned} 3\sqrt{6} &= \sqrt{3^2} \times \sqrt{6} \\ &= \sqrt{\square} \times \sqrt{6} \\ &= \sqrt{\square \times 6} \\ &= \sqrt{\square} \end{aligned}$$

b Consider the first line of working in part **a**. Why has the coefficient been written as a perfect square beneath the root sign?

c How are the surd we started with ($3\sqrt{6}$) and the answer (the entire surd) related to each other?

d Briefly explain how you could check that your answer is correct when you write a surd as an entire surd.

e Follow the working shown in part **a** to write $4\sqrt{3}$ as an entire surd.

11 Write each of the following as an entire surd.

a $3\sqrt{5}$ **b** $2\sqrt{3}$ **c** $4\sqrt{2}$ **d** $6\sqrt{7}$ **e** $4\sqrt{10}$ **f** $5\sqrt{11}$

g $8\sqrt{6}$ **h** $5\sqrt{5}$ **i** $4\sqrt{15}$ **j** $-8\sqrt{2}$ **k** $12\sqrt{7}$ **l** $-5\sqrt{3}$

12 Let a and b represent positive integers.

a Write $a\sqrt{b}$ as an entire surd.

b Use the general form of the entire surd you wrote in part **a** to check your answers to question **11**.

c In your own words, explain how a simplified surd can be expressed as an entire surd.

13 Write each of the following as an entire surd.

a $m\sqrt{n}$ **b** $7\sqrt{a}$ **c** $x\sqrt{5}$ **d** $10\sqrt{xy}$

e $d\sqrt{2c}$ **f** $3\sqrt{4abc}$ **g** $a\sqrt{ab^2}$ **h** $xy\sqrt{x^2yz^2}$

14 Write each of the following in simplest form.

a $\sqrt{x^2y}$ **b** $\sqrt{x^2y^2}$ **c** $\sqrt{ab^2}$ **d** $\sqrt{x^2yz^2}$

e $\sqrt{x^4}$ **f** $\sqrt{x^4y}$ **g** $\sqrt{x^4y^2}$ **h** $\sqrt{a^4b^2c}$

15 An alternative method for simplifying a surd involves writing the number appearing under the root sign as a product of its prime factors. Consider the surd $\sqrt{12}$.

a Write 12 as a product of its prime factors in exponent form.

b Write the product of the prime factors of 12 under the root sign and show that it simplifies to $2\sqrt{3}$.

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16 When using the method described in question **15**, you will notice that not all numbers under the root sign result in perfect squares. But, often, they can be written differently so that they do contain perfect squares. Consider $\sqrt{54}$.

a Write 54 as a product of its prime factors in exponent form.

b Use your answer to part **a** to show that $\sqrt{3^2 \times 3 \times 2}$ is equivalent to $\sqrt{54}$.

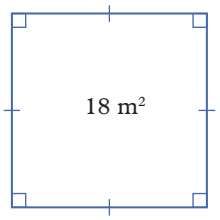
c Simplify the expression in part **b**.

17 Use the methods from questions **15** and **16** to fully simplify each of the following.

a $\sqrt{18}$	**b** $\sqrt{28}$	**c** $\sqrt{32}$	**d** $\sqrt{27}$	**e** $\sqrt{24}$	**f** $\sqrt{50}$
g $3\sqrt{20}$	**h** $2\sqrt{128}$	**i** $5\sqrt{63}$	**j** $8\sqrt{40}$	**k** $7\sqrt{48}$	**l** $6\sqrt{45}$
m $\sqrt{180}$	**n** $5\sqrt{147}$	**o** $6\sqrt{162}$	**p** $\sqrt{125}$	**q** $2\sqrt{450}$	**r** $\sqrt{288}$

Exercise 2.1C: Challenge

18 Consider the square on the right with an area of 18 m².



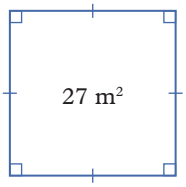
a Write the exact side length of the square as an entire surd and a simplified surd.

b The 18 m² square can be divided into nine 2 m² squares. Write the exact side length of each of the 2 m² squares.

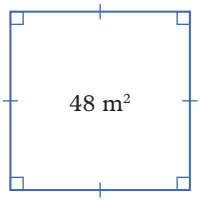
c Describe the relationship between the side lengths of the two different sized squares in parts **a** and **b** and the number of squares the 18 m² square is divided into.

d Determine the side length of each of the following blue squares. Note: In part **iii**, the area of each small square is 2 m²; in part **iv**, the area of each small square is 3 m². Write your answer as an entire surd and a simplified surd each time.

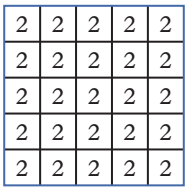
i



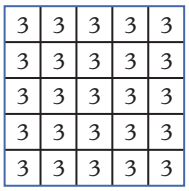
ii



iii



iv



19 Simplifying surds is important; it allows for like surds to be identified. (Like surds will be covered in **Lesson 2.3**.)

a Simplify each of the following surds.

i $\sqrt{8}$ **ii** $\sqrt{27}$ **iii** $\sqrt{125}$ **iv** $\sqrt{a^3}$, where $a > 0$

b Simplify each of the following surds.

i $\sqrt{32}$ **ii** $\sqrt{243}$ **iii** $\sqrt{3125}$ **iv** $\sqrt{a^5}$, where $a > 0$

c Simplify each of the following surds.

Hint: Look for perfect cube factors.

i $\sqrt[3]{16}$ **ii** $\sqrt[3]{81}$ **iii** $\sqrt[3]{625}$ **iv** $\sqrt[3]{a^4}$

d Simplify each of the following surds.

i $\sqrt[4]{32}$ **ii** $\sqrt[4]{243}$ **iii** $\sqrt[4]{3125}$ **iv** $\sqrt[4]{a^5}$, where $a > 0$

20 Arrange the surds in each of the following lists in ascending order.

a $\sqrt{46}, \sqrt{22}, \sqrt{42}, \sqrt{10}, \sqrt{27}, \sqrt{5}, \sqrt{21}, \sqrt{30}$

b $2\sqrt{3}, 3, 6, \sqrt{6}, \sqrt{39}, 2\sqrt{6}, \sqrt{11}, 4\sqrt{3}$

c $3\sqrt{3}, 2\sqrt{5}, 3\sqrt{6}, \sqrt{83}, \sqrt{22}, \sqrt{43}, 4, 2\sqrt{11}$

d $7\sqrt{10}, 12\sqrt{2}, 3\sqrt{43}, 2\sqrt{57}, 3\sqrt{14}, 20, 4\sqrt{5}, 2\sqrt{6}$

Online resources:

- **Interactive skillsheet** Simplifying surds
- **CAS instructions** Simplifying surds
- **Quick quiz** 2.1

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Multiplying and dividing surds

Learning intentions

By the end of this lesson you will be able to ...

✔ multiply and divide surds.

Inter-year links

Year 8	5.4 Multiplying algebraic terms
Year 9	2.1 Simplifying

Multiplying surds

Key content video
Multiplying and dividing surds

- The product of two surds written in the form $\sqrt{a}$ is equal to the square root of the product of the numbers under the root signs.

$$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$$

- To multiply surds written in the form $a\sqrt{b}$, multiply the surd terms and multiply the coefficients.

$$a\sqrt{b} \times c\sqrt{d} = a \times c \times \sqrt{b \times d}$$

Dividing surds

- A square root divided by a square root can be written as the square root of the quotient.

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

- To divide surds written in the form $a\sqrt{b}$, divide the surd terms and divide the coefficients.

$$a\sqrt{b} \div c\sqrt{d} = \frac{a\sqrt{b}}{c\sqrt{d}} = \frac{a}{c} \times \sqrt{\frac{b}{d}}$$

Worked example 2.2A Multiplying surds

Simplify the following expressions.

a $\sqrt{6} \times \sqrt{5}$ **b** $\sqrt{15} \times \sqrt{6}$ **c** $5\sqrt{3} \times 8\sqrt{11}$

THINK

a Multiply the square roots together. Remember: $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$.

b 1 Multiply the square roots together. Remember: $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$.

2 Simplify the square roots of any perfect squares.

c 1 Multiply the square roots together and multiply the coefficients together.

2 Write the product of the coefficient first. Then write the product of the surd terms under the root sign.

WRITE

a $\sqrt{6} \times \sqrt{5} = \sqrt{6 \times 5}$
$= \sqrt{30}$

b $\sqrt{15} \times \sqrt{6} = \sqrt{15 \times 6}$
$= \sqrt{90}$
$= \sqrt{9 \times 10}$
$= \sqrt{9} \times \sqrt{10}$
$= 3 \times \sqrt{10}$
$= 3\sqrt{10}$

c $5\sqrt{3} \times 8\sqrt{11} = 5 \times 8 \times \sqrt{3} \times \sqrt{11}$
$= 40 \times \sqrt{3 \times 11}$
$= 40 \times \sqrt{33}$
$= 40\sqrt{33}$

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Worked example 2.2B Dividing surds

Simplify the following expressions.

a $\sqrt{27} \div \sqrt{3}$ **b** $\dfrac{\sqrt{35}}{\sqrt{15}}$ **c** $\dfrac{36\sqrt{60}}{9\sqrt{12}}$

THINK

a **1** Divide one square root by the other.

Remember: $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$.

2 Simplify the fraction.

3 Simplify the square root.

b **1** Divide one square root by the other.

Remember: $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$.

2 Simplify the fraction.

3 Write in the form $\dfrac{\sqrt{a}}{\sqrt{b}}$.

$\sqrt{7}$ and $\sqrt{3}$ cannot be further simplified

c **1** Divide the surd terms and divide the coefficients.

2 Divide one square root by the other.

3 Simplify the fractions.

$\sqrt{5}$ cannot be simplified further.

WRITE

a $\sqrt{27} \div \sqrt{3} = \dfrac{\sqrt{27}}{\sqrt{3}}$

$= \sqrt{\dfrac{27}{3}}$

$= \sqrt{9}$

$= 3$

b $\dfrac{\sqrt{35}}{\sqrt{15}} = \sqrt{\dfrac{35}{15}}$

$= \sqrt{\dfrac{7}{3}}$

$= \dfrac{\sqrt{7}}{\sqrt{3}}$

c $\dfrac{36\sqrt{60}}{9\sqrt{12}} = \dfrac{36 \times \sqrt{60}}{9 \times \sqrt{12}}$

$= \dfrac{36}{9} \times \dfrac{\sqrt{60}}{\sqrt{12}}$

$= \dfrac{36}{9} \times \sqrt{\dfrac{60}{12}}$

$= 4 \times \sqrt{5}$

$= 4\sqrt{5}$

Helpful hints

- ✔ Make sure that the horizontal line that is part of the root symbol covers all relevant numbers or pronumerals.
- ✔ Your skills with fractions will be very useful to you in this module. Simplifying when dividing roots is similar to multiplying and dividing fractions.

Learning pathways

1(a–l), 2, 3, 4(a–l), 5(a–h), 6–9, 10(a, c, e, g), 11, 12(a, c, e, g), 13, 14, 16, 21(a, b)	1(i–p), 2(a, b, e, f), 3, 4(e–l), 5(i–l), 7(e–h), 8, 10(b, d, f, h), 11(f, h, i, k), 12(b, d, f, h), 17, 18(a–e), 21(a, d, e), 23	4(i–p), 5(i–l), 10(f–i), 11(h–l), 12(g–i), 15, 18(e–i), 19–22, 24

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ANS p686

Exercise 2.2A: Understanding and fluency

Do not use a calculator for the questions in this exercise.

WE 2.2A **1** Simplify the following expressions.

a $\sqrt{7} \times \sqrt{5}$ **b** $\sqrt{3} \times \sqrt{11}$ **c** $\sqrt{2} \times (-\sqrt{7})$ **d** $-\sqrt{15} \times \sqrt{7}$

e $\sqrt{13} \times \sqrt{5}$ **f** $-\sqrt{2} \times \sqrt{11}$ **g** $\sqrt{7} \times \sqrt{3}$ **h** $\sqrt{19} \times \sqrt{5}$

i $\sqrt{7} \times 3\sqrt{11}$ **j** $5\sqrt{3} \times 4\sqrt{2}$ **k** $-4\sqrt{2} \times 7\sqrt{11}$ **l** $12\sqrt{6} \times (-3\sqrt{5})$

m $11\sqrt{2} \times 6\sqrt{17}$ **n** $8\sqrt{6} \times 8\sqrt{7}$ **o** $4\sqrt{3} \times (-6\sqrt{11})$ **p** $12\sqrt{3} \times 11\sqrt{5}$

2 Simplify the following expressions.

a $\sqrt{7} \times \sqrt{7}$ **b** $\sqrt{3} \times \sqrt{3}$ **c** $\sqrt{2} \times \sqrt{2}$ **d** $\sqrt{6} \times \sqrt{6}$

e $\sqrt{5} \times \sqrt{5}$ **f** $-\sqrt{11} \times (-\sqrt{11})$ **g** $\sqrt{10} \times \sqrt{10}$ **h** $\sqrt{9} \times \sqrt{9}$

3 Consider your answers from question **2**.

a What do you notice about each result?

b Simplify $\sqrt{a} \times \sqrt{a}$, where a is a positive real number.

4 Simplify each of these expressions, writing your answers in surd form where appropriate.

a $3\sqrt{3} \times \sqrt{3}$ **b** $-\sqrt{11} \times 7\sqrt{11}$ **c** $4\sqrt{2} \times 5\sqrt{2}$ **d** $6\sqrt{5} \times 2\sqrt{5}$

e $\sqrt{5} \times 3\sqrt{2}$ **f** $15\sqrt{6} \times 2\sqrt{5}$ **g** $-4\sqrt{2} \times 8\sqrt{13}$ **h** $12\sqrt{3} \times \frac{1}{4}\sqrt{11}$

i $2\sqrt{12} \times 7\sqrt{3}$ **j** $\frac{1}{2}\sqrt{5} \times 3\sqrt{20}$ **k** $\frac{3}{4}\sqrt{13} \times -8\sqrt{2}$ **l** $\frac{3}{5}\sqrt{7} \times \frac{5}{3}\sqrt{15}$

m $-12\sqrt{6} \times (-6\sqrt{7})$ **n** $\frac{1}{2}\sqrt{15} \times \frac{3}{4}\sqrt{13}$ **o** $15\sqrt{10} \times \left(-\frac{1}{3}\sqrt{7}\right)$ **p** $-10\sqrt{17} \times 1.1\sqrt{5}$

WE 2.2B **5** Simplify the following expressions.

a $\sqrt{34} \div \sqrt{2}$ **b** $\sqrt{24} \div \sqrt{8}$ **c** $\sqrt{15} \div \sqrt{15}$ **d** $\sqrt{15} \div (-\sqrt{3})$

e $\dfrac{\sqrt{42}}{\sqrt{7}}$ **f** $\dfrac{\sqrt{56}}{\sqrt{8}}$ **g** $4\sqrt{18} \div \sqrt{3}$ **h** $22\sqrt{40} \div 11\sqrt{8}$

i $\dfrac{-24\sqrt{35}}{8\sqrt{7}}$ **j** $\dfrac{16\sqrt{50}}{8\sqrt{10}}$ **k** $-12\sqrt{60} \div 3\sqrt{6}$ **l** $8\sqrt{55} \div 16\sqrt{11}$

6 Determine the value of each of these expressions.

a $\sqrt{32} \div \sqrt{2}$ **b** $\sqrt{24} \div \sqrt{6}$ **c** $\sqrt{54} \div \sqrt{6}$ **d** $\sqrt{40} \div \sqrt{10}$

7 Simplify each of these expressions, writing your answers in surd form.

a $12\sqrt{15} \div (-\sqrt{3})$ **b** $35\sqrt{24} \div 7\sqrt{8}$ **c** $-12\sqrt{48} \div 4\sqrt{24}$ **d** $\dfrac{18\sqrt{35}}{9\sqrt{7}}$

e $\dfrac{-32\sqrt{42}}{-8\sqrt{6}}$ **f** $\dfrac{35\sqrt{5}}{-15}$ **g** $\dfrac{-24}{-8\sqrt{3}}$ **h** $\sqrt{240} \div 5\sqrt{48}$

8 Show that the result of the calculation $6\sqrt{3} \times 3\sqrt{2}$ is $18\sqrt{6}$.
Note: 'Show that' means to show all the necessary calculations.

9 Consider $\dfrac{24\sqrt{40}}{8\sqrt{20}}$. By dividing the coefficients and then dividing the surd terms, show that the result is $3\sqrt{2}$.

10 Simplify each of these expressions. Write your answers in fractional surd form where appropriate.

a $6\sqrt{15} \div \sqrt{15}$ (Remember: $6\sqrt{15} \div \sqrt{15}$ is equivalent to $\dfrac{6\sqrt{15}}{\sqrt{15}}$.)

b $\dfrac{12\sqrt{30}}{16\sqrt{6}}$ **c** $44\sqrt{34} \div 28\sqrt{42}$ **d** $\dfrac{-20\sqrt{75}}{25\sqrt{45}}$ **e** $\dfrac{36\sqrt{35}}{16\sqrt{7}}$

f $\dfrac{22\sqrt{5}}{-33\sqrt{5}}$ **g** $\dfrac{8\sqrt{3} \times 6}{-\sqrt{3}}$ **h** $\frac{1}{2}\sqrt{15} \div 4\sqrt{8}$ **i** $9\sqrt{32} \div 18\sqrt{8}$

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11 Simplify each of these expressions.

a $\frac{\sqrt{54}}{\sqrt{6}}$ **b** $\sqrt{\frac{81}{100}}$ **c** $\sqrt{32} \times 3\sqrt{2}$

d $\frac{\sqrt{3}}{\sqrt{27}}$ **e** $5\sqrt{12} \times -2\sqrt{3}$ **f** $\sqrt{\frac{36}{9}}$

g $7\sqrt{8} \times \sqrt{18}$ **h** $4\sqrt{6} \times \sqrt{8} \times 6\sqrt{12}$ **i** $8\sqrt{45} \div 5\sqrt{5}$

j $\frac{\sqrt{32}}{16\sqrt{8}}$ **k** $\frac{-4\sqrt{75}}{15\sqrt{3}}$ **l** $\frac{-5\sqrt{343}}{15\sqrt{7}}$

12 Simplify the following fractions.

a $\frac{3\sqrt{5} \times 4\sqrt{6}}{6\sqrt{3}}$ **b** $\frac{5\sqrt{7} \times \sqrt{6}}{25\sqrt{14}}$ **c** $\frac{12\sqrt{48}}{3\sqrt{2} \times 5\sqrt{6}}$

d $\frac{11\sqrt{132}}{4\sqrt{11} \times (-2\sqrt{6})}$ **e** $\frac{8\sqrt{3} \times 2\sqrt{8}}{6\sqrt{6} \times \sqrt{2}}$ **f** $\frac{4\sqrt{5} \times -3\sqrt{24}}{5\sqrt{10} \times 6\sqrt{2}}$

g $\frac{\sqrt{3} \times \sqrt{15} \times \sqrt{8}}{\sqrt{6} \times 9\sqrt{12}}$ **h** $\frac{4\sqrt{12} \times 3\sqrt{10} \times (-6\sqrt{7})}{5\sqrt{8} \times 8\sqrt{15}}$ **i** $\frac{-5\sqrt{6} \times 2\sqrt{28} \times (-\sqrt{44})}{3\sqrt{21} \times (-4\sqrt{18})}$

Exercise 2.2B: Problem solving and reasoning

13 To square a number, we multiply the number by itself. Find the value of each of these expressions.

a $(\sqrt{5})^2$ **b** $(\sqrt{8})^2$ **c** $(\sqrt{2})^2$

d $(\sqrt{7})^2$ **e** $(\sqrt{9})^2$ **f** $(4\sqrt{5})^2$

g $(3\sqrt{6})^2$ **h** $(8\sqrt{3})^2$ **i** $(9\sqrt{11})^2$

14 Consider your answer for each part of question **13**.

a Does the square of a surd expressed in the form $\sqrt{a}$ or $a\sqrt{b}$ result in a rational answer or an irrational answer?

b Show that your answer to part **a** will always be true.

c Write the surds in parts **f–i** as entire surds (a surd with only one number beneath one root sign).

d Describe the connection between the number beneath the square root of an entire surd and the square of that surd.

15 Simplify each of these expressions.

a $(\sqrt{x})^2$ **b** $(\sqrt{ab})^2$

c $\sqrt{6y} \times \sqrt{6y}$ **d** $\sqrt{x+y} \times \sqrt{x+y}$

16 Find the value of each of these expressions.

a $\left(\frac{5}{\sqrt{7}}\right)^2$ **b** $\left(\frac{1}{\sqrt{5}}\right)^2$ **c** $\left(-\frac{3}{\sqrt{6}}\right)^2$ **d** $\left(\frac{\sqrt{10}}{\sqrt{13}}\right)^2$

17 Consider $3\sqrt{392} \times 5\sqrt{192}$.

a Calculate the value of the expression using each of these methods:

Method 1: Multiply the coefficients, then multiply the surd terms and write the surd in simplest form.

Method 2: Simplify each surd first and then perform the multiplication.

b Do both methods from part **a** produce the same answer?

c When do you think it is best to use:

i method 1 **ii** method 2?

18 Simplify each of the following, writing each answer as a fraction in simplest surd form where appropriate.

a $3\sqrt{48} \div \sqrt{6}$ **b** $\frac{2\sqrt{54}}{6\sqrt{3}}$ **c** $15\sqrt{104} \div 25\sqrt{8}$

d $\frac{54\sqrt{75}}{63\sqrt{15}}$ **e** $\frac{-21\sqrt{84}}{6\sqrt{7}}$ **f** $\frac{4\sqrt{300}}{3\sqrt{25}}$

g $\frac{8\sqrt{960}}{-\sqrt{48}}$ **h** $\sqrt{384} \div 5\sqrt{64}$ **i** $\frac{-\sqrt{504}}{\sqrt{54}}$

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19 Simplify each of these expressions, writing your answer in simplest surd form.

a $\dfrac{22\sqrt{48}}{8\sqrt{2} \times 4\sqrt{3}}$ **b** $\dfrac{4\sqrt{10} \times 6\sqrt{24}}{2\sqrt{5} \times 3\sqrt{2}}$ **c** $\dfrac{3\sqrt{8} \times 7\sqrt{12}}{12\sqrt{6} \times (-8\sqrt{2})}$

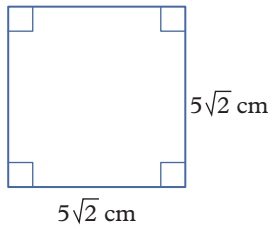
d $\dfrac{2\sqrt{15} \times 4\sqrt{12} \times \sqrt{10}}{\sqrt{2} \times 5\sqrt{12}}$ **e** $\dfrac{3\sqrt{14} \times (-5\sqrt{12})}{4\sqrt{2} \times \sqrt{7}}$ **f** $\dfrac{2\sqrt{35} \times 3\sqrt{2} \times 6\sqrt{8}}{5\sqrt{5} \times 12\sqrt{14}}$

g $\dfrac{-4\sqrt{8} \times 3\sqrt{7} \times 12\sqrt{45}}{2\sqrt{14} \times 6\sqrt{15} \times 3\sqrt{3}}$ **h** $\dfrac{-6\sqrt{26} \times 12\sqrt{2} \times 8\sqrt{12} \times 16\sqrt{33}}{2\sqrt{11} \times 4\sqrt{13} \times 18\sqrt{3} \times 22\sqrt{8}}$

20 Simplify $\dfrac{a\sqrt{bc^2} \times b^2\sqrt{ac}}{c\sqrt{ab^2}}$, where a, b and c are positive real numbers.

21 For each of these shapes, calculate: **i** the approximate area (correct to two decimal places), and **ii** the exact area.

a



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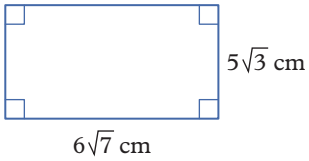

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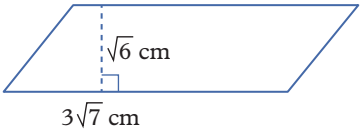

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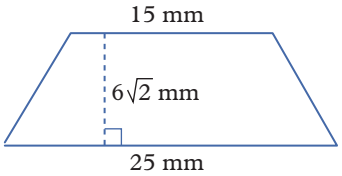

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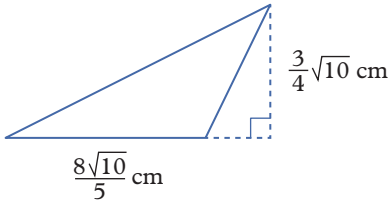

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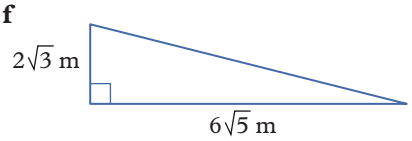

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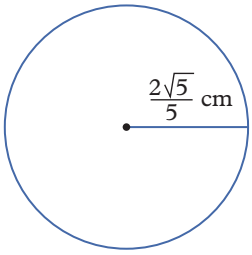

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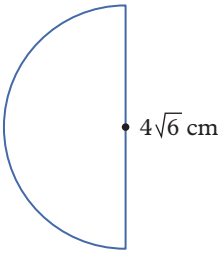
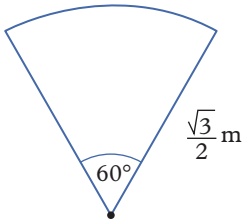


i



Exercise 2.2C: Challenge

22 Consider a square with side length $\sqrt{a}$, where $\sqrt{a}$ is a surd in its simplest form. Is it possible for the value representing the perimeter to be the same as the value representing the area of this square? Explain your answer.

23 Determine the exact (simplified) perimeter of a square with an area of:

a 5 m² **b** 17 m² **c** 18 m²

d 20 m² **e** $\frac{3}{4}$ m² **f** $\frac{175}{36}$ m²

24 For each of the following, determine the smallest surd, x, that will make the product an integer.

a $x \times \sqrt{180}$ **b** $x \times \sqrt{5400}$ **c** $x \times \sqrt{2835}$

d $x \times \sqrt{8064}$ **e** $x \times \sqrt{1911}$ **f** $x \times \sqrt{11\,154}$

Online resources:

Interactive skillsheet Multiplying and dividing surds

Worksheet Multiplying and dividing surds

Quick quiz 2.2

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Lesson 2.3

Adding and subtracting surds

Learning intentions

By the end of this lesson you will be able to ...

✔ add and subtract surds.

Inter-year links

Year 8	5.3 Adding and subtracting algebraic terms
Year 9	2.1 Simplifying

Adding and subtracting surds

Key content video
Adding and subtracting surds

- **Like surds** are surds that in simplest form contain exactly the same surd term.
  For example, $2\sqrt{2}$ and $3\sqrt{2}$ are like surds, whereas $2\sqrt{5}$ and $2\sqrt{3}$ are not like surds.
- To add or subtract like surds, add or subtract the coefficients.
  $$a\sqrt{b} + c\sqrt{b} = (a + c)\sqrt{b}$$
  For example, $2\sqrt{2} + 3\sqrt{2} = 5\sqrt{2}$.
- Sometimes, like surds need to be simplified to have the same surd term. Simplify each surd in an expression by rewriting the number under each root sign as a product of its factors, with the first factor being the greatest possible perfect square. Then, if the simplified surds are like surds, they can be added or subtracted.
  For example,
  $$\begin{aligned} 2\sqrt{27} - \sqrt{12} &= 2\sqrt{9 \times 3} - \sqrt{4 \times 3} \\ &= 2 \times \sqrt{9} \times \sqrt{3} - \sqrt{4} \times \sqrt{3} \\ &= 2 \times 3 \times \sqrt{3} - 2\sqrt{3} \\ &= 6\sqrt{3} - 2\sqrt{3} \\ &= 4\sqrt{3} \end{aligned}$$

Worked example 2.3A Identifying like surds

State whether the surds in each of these pairs are like surds or not.

a $2\sqrt{6}$ and $5\sqrt{6}$ **b** $7\sqrt{6}$ and $7\sqrt{3}$ **c** $5\sqrt{2}$ and $\sqrt{8}$

THINK	WRITE
a The surds are like surds because they have exactly the same surd term.	**a** Yes, they are like surds.
b The surds are not like surds because they do not have exactly the same surd term.	**b** No, they are not like surds.
c $\sqrt{8}$ can be simplified first. $\sqrt{8} = \sqrt{4} \times \sqrt{2}$ $= 2\sqrt{2}$ $\sqrt{8}$ and $2\sqrt{2}$ are like surds because $2\sqrt{2}$ and $5\sqrt{2}$ have exactly the same surd term.	**c** Yes, they are like surds.

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Worked example 2.3B Adding and subtracting surds

Simplify the following expressions.

a $7\sqrt{3} + 4\sqrt{3} - \sqrt{3}$

b $6\sqrt{5} - 3\sqrt{2} - 8\sqrt{2} + 5\sqrt{5}$

THINK

a 1 Group the like surds together. All the surds in this expression are like surds.

2 Simplify the expression by adding and subtracting the coefficients.

b 1 Group the like surds together. There are two groups of like surds in this expression: groups of $\sqrt{5}$ and groups of $\sqrt{2}$.

2 Add the like surds in the expression by adding and subtracting the coefficients.

WRITE

a $7\sqrt{3} + 4\sqrt{3} - \sqrt{3} = 7\sqrt{3} + 4\sqrt{3} - 1\sqrt{3}$

$= 10\sqrt{3}$

b $6\sqrt{5} - 3\sqrt{2} - 8\sqrt{2} + 5\sqrt{5} = 6\sqrt{5} + 5\sqrt{5} - 3\sqrt{2} - 8\sqrt{2}$

$= 11\sqrt{5} - 11\sqrt{2}$

Worked example 2.3C Adding and subtracting surds after first simplifying

Simplify the following expressions.

a $3\sqrt{2} + 4\sqrt{8}$

b $4\sqrt{48} - 2\sqrt{75}$

THINK

a 1 Simplify each surd in the expression. The first term, $3\sqrt{2}$, is already in its simplest form. Rewrite the number under the other root sign as a product of its factors, with the first factor being the greatest possible perfect square.

2 Simplify the like terms in the expression by adding the coefficients.

b 1 Simplify each surd in the expression, rewriting the number under each root sign as a product of its factors, with the first factor being the greatest possible perfect square.

2 Subtract the like surds in the expression by subtracting the coefficients.

WRITE

a $3\sqrt{2} + 4\sqrt{8} = 3\sqrt{2} + 4\sqrt{4 \times 2}$

$= 3\sqrt{2} + 4 \times \sqrt{4} \times \sqrt{2}$

$= 3\sqrt{2} + 4 \times 2 \times \sqrt{2}$

$= 3\sqrt{2} + 8\sqrt{2}$

$= 11\sqrt{2}$

b $4\sqrt{48} - 2\sqrt{75} = 4\sqrt{16 \times 3} - 2\sqrt{25 \times 3}$

$= 4 \times \sqrt{16} \times \sqrt{3} - 2 \times \sqrt{25} \times \sqrt{3}$

$= 4 \times 4 \times \sqrt{3} - 2 \times 5 \times \sqrt{3}$

$= 16\sqrt{3} - 10\sqrt{3}$

$= 6\sqrt{3}$

Helpful hints

- ✔ Pay attention to your surds. Do not attempt to simplify the sum or difference of unlike surds!
- ✔ Starting every question by asking yourself, 'Can I simplify these surds?', is a great way to avoid common mistakes.

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Learning pathways

1, 2(1st column), 3–5, 6(1st column), 7(1st column), 8(1st column), 9(a–f), 10(a–c), 11, 12, 15, 17(a, b)

2(2nd column), 4, 5, 6–10(1st column), 13, 14, 16(a, b), 17(b, c), 18

6–10(2nd column), 16, 17(c, d), 18–22

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Exercise 2.3A: Understanding and fluency

WE 2.3A **1** State whether the terms in each of these pairs are like surds or not.

a $3\sqrt{5}$ and $\sqrt{5}$ **b** $4\sqrt{2}$ and $2\sqrt{4}$ **c** 3 and $\sqrt{3}$ **d** $-3\sqrt{6}$ and $6\sqrt{6}$

e $5\sqrt{5}$ and $2\sqrt{2}$ **f** $3\sqrt{7}$ and $3\sqrt{10}$ **g** $2\sqrt{2}$ and $-\sqrt{2}$ **h** $2\sqrt{2}$ and $\sqrt{8}$

WE 2.3B **2** Simplify each expression by adding (or subtracting) the like terms, if possible. Hint: Look for like surds.

a $2\sqrt{3} + \sqrt{2} + 5\sqrt{3}$ **b** $4\sqrt{3} - 2\sqrt{2} + \sqrt{2}$

c $-3\sqrt{10} - 4\sqrt{7} - \sqrt{6} + 2\sqrt{3}$ **d** $6\sqrt{13} + 4\sqrt{39} - \sqrt{3} + 2\sqrt{13}$

e $9\sqrt{2} - \sqrt{2} - 6\sqrt{5} + 5\sqrt{2}$ **f** $3\sqrt{6} + 4\sqrt{5} - \sqrt{6} + 5 + 2\sqrt{6} - 4$

g $3\sqrt{11} + 4\sqrt{55} - \sqrt{5}$ **h** $7\sqrt{21} - 6\sqrt{3} + 5\sqrt{3} - 6\sqrt{21}$

i $6\sqrt{6} + 3\sqrt{5} - \sqrt{6} + 5 + 2\sqrt{6}$ **j** $-3\sqrt{15} - 2\sqrt{7} - \sqrt{7} - 4\sqrt{15}$

3 Write each surd in simplest form and then identify any like surds in each list.

a $2\sqrt{3}$ $\sqrt{2}$ $4\sqrt{2}$ $2\sqrt{5}$ $3\sqrt{2}$ **b** $3\sqrt{7}$ $14\sqrt{2}$ $7\sqrt{3}$ $2\sqrt{14}$

c $-3\sqrt{6}$ $3\sqrt{12}$ $\sqrt{3}$ $\sqrt{18}$ $2\sqrt{27}$ **d** $9\sqrt{32}$ $4\sqrt{8}$ $\sqrt{128}$ $5\sqrt{200}$

WE 2.3C **4** Simplify each of these expressions.

a $5\sqrt{2} + 4\sqrt{8}$ **b** $4\sqrt{36} - 2\sqrt{12}$ **c** $3\sqrt{24} + \sqrt{45}$

d $2\sqrt{28} + 8\sqrt{63}$ **e** $4\sqrt{72} + 5\sqrt{98}$ **f** $-2\sqrt{8} + 7\sqrt{16}$

5 Simplify each of these expressions.

a $5\sqrt{27} + \sqrt{12} - 8\sqrt{48}$ **b** $2\sqrt{20} - 4\sqrt{80} + 2\sqrt{45}$ **c** $\sqrt{28} - 3\sqrt{112} + 2\sqrt{63}$

d $5\sqrt{12} + \sqrt{27} - 8\sqrt{3}$ **e** $2\sqrt{8} + 4\sqrt{32} + 3\sqrt{2}$ **f** $\sqrt{7} - 3\sqrt{28} + 5\sqrt{7}$

6 Simplify each of these expressions.

a $3\sqrt{45} - \sqrt{5} - 2\sqrt{80} + 2\sqrt{20}$ **b** $\sqrt{108} - 4\sqrt{27} - 2\sqrt{12} + 2\sqrt{75}$

c $3\sqrt{24} - 4\sqrt{54} + 5\sqrt{24}$ **d** $3\sqrt{18} - 4\sqrt{50} - \sqrt{8} + 2\sqrt{98}$

e $3\sqrt{24} - 7\sqrt{54} - \sqrt{96} + 4\sqrt{20}$ **f** $4\sqrt{108} - 5\sqrt{75} - 6\sqrt{48} + \sqrt{8}$

g $\sqrt{128} - 5\sqrt{27} + 2\sqrt{32} - 3\sqrt{75}$ **h** $2\sqrt{75} - 6\sqrt{3} + 2\sqrt{48} - 6\sqrt{12}$

i $3\sqrt{12} - \sqrt{28} + 5\sqrt{8} - 2\sqrt{20}$ **j** $6\sqrt{54} + \sqrt{45} - 5\sqrt{96} - 2\sqrt{27}$

7 Simplify each of these expressions.

a $\sqrt{180} + 4\sqrt{125} - 3\sqrt{245}$ **b** $4\sqrt{243} + 3\sqrt{300} - 2\sqrt{432}$

c $3\sqrt{338} - 12\sqrt{147} - 2\sqrt{50} + \sqrt{48}$ **d** $3\sqrt{216} - 4\sqrt{125} - 2\sqrt{147} + 5\sqrt{45}$

e $\sqrt{63} + 7\sqrt{567} - 2\sqrt{96} + 8\sqrt{588}$ **f** $\sqrt{200} - 4\sqrt{180} + 5\sqrt{864} - 3\sqrt{160}$

g $-4\sqrt{320} - 25 + \sqrt{245} - 36 + \sqrt{27}$ **h** $12\sqrt{72} - 6\sqrt{192} + 2\sqrt{128} - 5\sqrt{507}$

Exercise 2.3B: Problem solving and reasoning

8 Simplify each of these surd expressions containing pronumerals.

For example: $\sqrt{27x} + \sqrt{12x} = \sqrt{9 \times 3 \times x} + \sqrt{4 \times 3 \times x}$
$= 3\sqrt{3x} + 2\sqrt{3x}$
$= 5\sqrt{3x}$

a $\sqrt{x} + \sqrt{16x} - \sqrt{25x}$ **b** $3\sqrt{4x} + 2\sqrt{x} - 5\sqrt{36x}$ **c** $\sqrt{9x} + \sqrt{12x} + \sqrt{16x} - 2\sqrt{48x}$

d $\sqrt{8xy} - 2\sqrt{32xy} + \frac{1}{3}\sqrt{18xy}$ **e** $\sqrt{150xy} + 2\sqrt{384xy} - \sqrt{6xy}$ **f** $\sqrt{24x^2y} + 3x\sqrt{96y} - 5\sqrt{54x^2y}$

g $\sqrt{3xy^2} - 5y\sqrt{12x} + 7\sqrt{27xy^2}$ **h** $3\sqrt{100x^2y} - \sqrt{147x^2y} + \frac{1}{2}\sqrt{48x^2y} + \sqrt{16x^2y}$

i $2x\sqrt{80xy} + 5\sqrt{12xy} - \sqrt{125x^2y} - 2\sqrt{432x^2y}$

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9 The distributive law is used to expand expressions, as shown below.

$$a(b + c) = ab + ac$$

The distributive law can be applied to expressions involving surds.

Expand each of the expressions below. Where necessary, write all surds in simplest form.

a $6(2 + \sqrt{3})$

b $8(\sqrt{5} - 4)$

c $5(4 - 6\sqrt{2})$

d $4(3\sqrt{2} + 2\sqrt{3})$

e $2(5\sqrt{6} - 3\sqrt{5})$

f $-5\sqrt{7}(2\sqrt{6} - 3\sqrt{10} + 2)$

g $3\sqrt{6}(5\sqrt{3} + 2\sqrt{10})$

h $5\sqrt{7}(2\sqrt{7} - 12)$

i $6\sqrt{11}(2\sqrt{3} - 5\sqrt{7} + \sqrt{11})$

j $(3 - 2\sqrt{5} + 5\sqrt{15})\sqrt{5}$

k $-3\sqrt{3}(4\sqrt{6} - 2\sqrt{10} + 9\sqrt{15})$

10 Binomial products can be expanded as shown below.

$$(a + b)(c + d) = ac + ad + bc + bd$$

Binomial products can also contain surds. Expand and simplify each of these expressions.

a $(5 + 3\sqrt{3})(2 + 4\sqrt{6})$

b $(3\sqrt{2} + 6)(5\sqrt{5} - 8)$

c $(7\sqrt{5} - 3\sqrt{2})(4\sqrt{2} + 8\sqrt{6})$

d $(\sqrt{6} + 2\sqrt{3})(6\sqrt{2} + 5\sqrt{3})$

e $(4\sqrt{8} - 5\sqrt{5})(3\sqrt{3} - 9\sqrt{8})$

f $(4\sqrt{10} + 3\sqrt{5})(7\sqrt{2} - 6\sqrt{12})$

11 Expand and simplify each of these perfect squares.

a $(5 + \sqrt{6})^2$

b $(\sqrt{7} - 3)^2$

c $(3\sqrt{5} + 2\sqrt{3})^2$

d $(4\sqrt{6} - 3\sqrt{7})^2$

12 The results of the expansions in question **11** followed a pattern that you've seen before; the 'perfect square' rule.

a Show that the expansion of $(a + b)^2$ will always produce the expression $a^2 + 2ab + b^2$.

b Show that the expansion of $(a - b)^2$ will always produce the expression $a^2 - 2ab + b^2$.

c Use these rules to help you check your answers to question **11**.

13 Expand and simplify each of these expressions.

a $(4 + \sqrt{6})(4 - \sqrt{6})$

b $(5\sqrt{5} - 3)(5\sqrt{5} + 3)$

c $(2\sqrt{3} + \sqrt{2})(2\sqrt{3} - \sqrt{2})$

d $(4\sqrt{5} + 7\sqrt{3})(4\sqrt{5} - 7\sqrt{3})$

14 Consider each expression in question **13**.

a What do you notice about the terms inside each pair of brackets?

b Are your answers to question **13** rational numbers or irrational numbers? Why do you think this is the case?

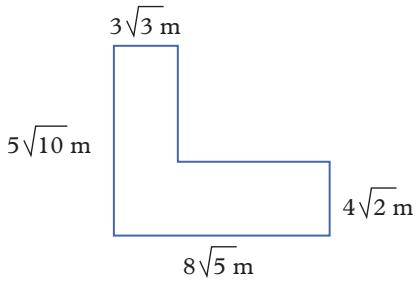
c The results of the expansions in question **13** follow a pattern that you've seen before; the 'difference of two squares' rule. Show that the expansion of binomial products in the form $(a + b)(a - b)$ or $(a - b)(a + b)$ will always be $a^2 - b^2$.

d Use this rule to check your answers for question **13**.

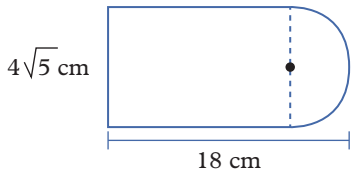
15 Use two non-zero numbers to demonstrate $\sqrt{a} + \sqrt{b} \neq \sqrt{a + b}$.

16 Calculate the exact area of each of the following composite shapes.

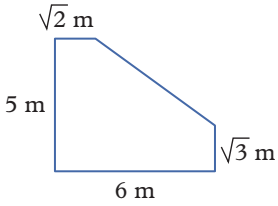
a



b



c



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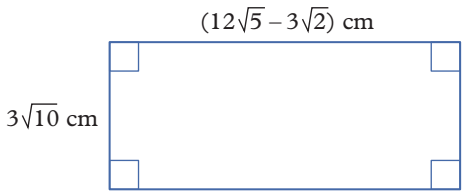
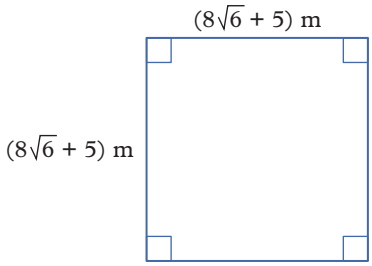
17 Calculate the exact area of each of these shapes. Write your answers in simplest surd form.

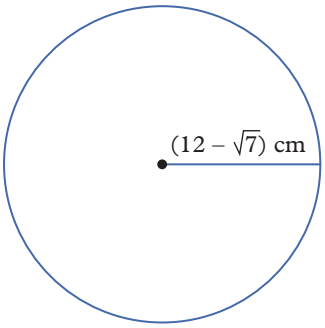
a



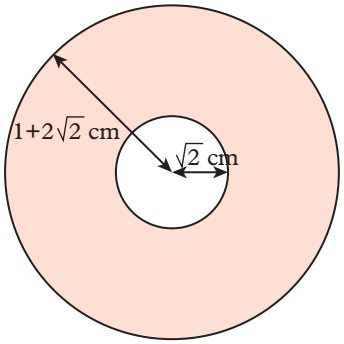
b



c



d



18 Calculate the exact perimeter of each shape in parts **a** and **b** of question **16**.

19 Evaluate each of the following.

a $\sqrt{3}(\sqrt{8} + \sqrt{72})$

b $-7\sqrt{15} + 5\sqrt{3} \times 2\sqrt{5}$

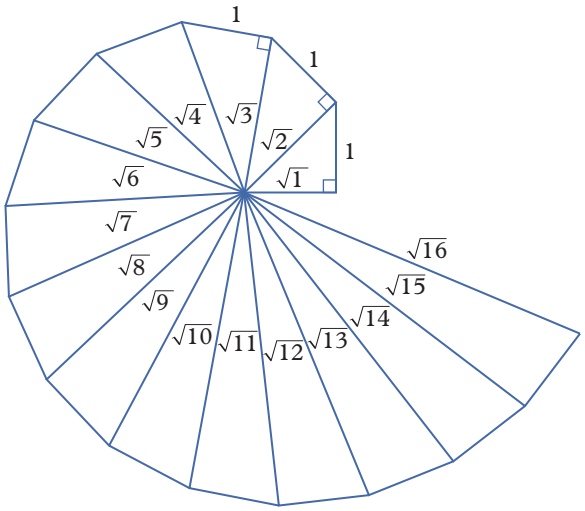
c $\dfrac{8\sqrt{30}}{3\sqrt{10}} + \dfrac{5\sqrt{42}}{6\sqrt{14}} + \dfrac{5\sqrt{105}}{9\sqrt{35}}$

d $\sqrt{50}\left(\dfrac{1}{\sqrt{10}} + \dfrac{1}{\sqrt{5}}\right) + \dfrac{\sqrt{10}}{2}(\sqrt{8} + \sqrt{18} + \sqrt{4})$

e $\dfrac{3\sqrt{55} - 5\sqrt{35}}{\sqrt{5}} + \dfrac{2\sqrt{77} - 2\sqrt{121}}{\sqrt{11}}$

f $\dfrac{6}{\sqrt{2}} \times \dfrac{6}{\sqrt{3}} + \left(\dfrac{4}{\sqrt{2}} + 2\sqrt{3}\right)\left(3\sqrt{2} - \dfrac{9}{\sqrt{3}}\right)$

20 The spiral of Theodorus is created by consecutively forming a right-angled triangle on the hypotenuse of the previous right-angled triangle, so that the length of one of the shorter sides is always 1. Determine the total length of line segments drawn for this section of the spiral of Theodorus. Write your answer as a simplified expression.



Exercise 2.3C: Challenge

21 Answer the following questions for each sequence shown below.

i $3\sqrt{5},\ 8\sqrt{5},\ 13\sqrt{5},\ 18\sqrt{5}, \ldots$

ii $2\sqrt{3},\ 3\sqrt{3} + 7,\ 4\sqrt{3} + 14,\ 5\sqrt{3} + 21, \ldots$

iii $3\sqrt{2},\ 6,\ 6\sqrt{2},\ 12, \ldots$

iv $\sqrt{5},\ 10,\ 20\sqrt{5},\ 200, \ldots$

a Describe the rule of the sequence.

b Continue the sequence to the tenth term.

c Evaluate the sum of the first 10 terms.

22 For each of the following lists, calculate: **i** the mean and **ii** the median.

a $\sqrt{2},\ \sqrt{4},\ \sqrt{8},\ \sqrt{16},\ \sqrt{32},\ \sqrt{64},\ \sqrt{128},\ \sqrt{256}$

b $\sqrt{5},\ \sqrt{3},\ 5\sqrt{2},\ 2\sqrt{5},\ 3\sqrt{3},\ \sqrt{2}$

Online resources:

- **Interactive skillsheet** Adding and subtracting surds
- **Worksheet** Adding and subtracting surds
- **Investigation** Tangrams and surds
- **Quick quiz** 2.3

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Rationalising the denominator

Learning intentions

By the end of this lesson you will be able to ...

✔ rationalise the denominator of a fraction containing surds.

Inter-year links

Year 8	5.4 Multiplying algebraic terms
Year 9	2.4 Factorising the difference of two squares

Rationalising the denominator of a simple fraction

Key content video
Rationalising the denominator

- Fractions that contain surds are conventionally written with a rational denominator. In most cases, this will be an integer denominator.
- If a fraction has a single surd as the denominator, steps can be taken to rationalise the denominator:

Step 1: Identify the surd in the denominator.

$$\frac{a}{\sqrt{b}}$$

single surd denominator

Step 2: Multiply both the numerator and the denominator by that surd.

$$= \frac{a}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}}$$

Step 3: Simplify the resulting fraction.

$$= \frac{a\sqrt{b}}{b}$$

The conjugate of a binomial expression

- The **conjugate** of a binomial expression with one surd term has the same two terms but the surd term has a different sign. For example, the conjugate of $a + \sqrt{b}$ is $a - \sqrt{b}$ and the conjugate of $a - \sqrt{b}$ is $a + \sqrt{b}$.
- The product of conjugate expressions is the difference of the squares of the two terms.

$$(a + b)(a - b) = a^2 - b^2$$
$$(a + \sqrt{b})(a - \sqrt{b}) = a^2 - b$$

conjugates rational number

For example, the conjugate of $2 + \sqrt{3}$ is $2 - \sqrt{3}$ and the product of the conjugate expression is:

$$\begin{aligned}(2 + \sqrt{3})(2 - \sqrt{3}) &= 2^2 + \sqrt{3} \times 2 - \sqrt{3} \times 2 - \sqrt{3}^2 \\ &= 2^2 - \sqrt{3}^2 \\ &= 4 - 3 \\ &= 1 \quad \text{which is a rational number.}\end{aligned}$$

Rationalising the denominator by using a conjugate expression

- When the denominator of a fraction is a binomial including one surd term and one integer term, the denominator can be rationalised by multiplying the numerator and denominator of the fraction by the conjugate of the denominator.

$$\frac{1}{a + \sqrt{b}} \times \frac{a - \sqrt{b}}{a - \sqrt{b}} = \frac{a - \sqrt{b}}{a^2 - b}$$

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Worked example 2.4A Rationalising the denominator of a fraction

Rewrite each of these fractions with a rational denominator.

a $\frac{7}{2\sqrt{5}}$ **b** $\frac{2\sqrt{3}}{\sqrt{2}}$ **c** $\frac{1-\sqrt{3}}{\sqrt{2}}$

THINK

a 1 Multiply the numerator and the denominator by the surd term of the denominator. It is not necessary to include the coefficient of the denominator as it is already rational.

2 Simplify the fraction.

b 1 Multiply the numerator and the denominator by the surd term of the denominator.

2 Simplify the fraction.

c 1 Multiply the numerator and the denominator by the surd term of the denominator.

2 Expand the expression using the distributive law.

3 Simplify the fraction.

WRITE

a $\frac{7}{2\sqrt{5}} = \frac{7}{2\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$

$= \frac{7\sqrt{5}}{2\sqrt{25}}$

$= \frac{7\sqrt{5}}{2 \times 5}$

$= \frac{7\sqrt{5}}{10}$

b $\frac{2\sqrt{3}}{\sqrt{2}} = \frac{2\sqrt{3}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{2\sqrt{6}}{\sqrt{4}}$

$= \frac{2^1\sqrt{6}}{2^1}$

$= \sqrt{6}$

c $\frac{1-\sqrt{3}}{\sqrt{2}} = \frac{1-\sqrt{3}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{(1-\sqrt{3})\sqrt{2}}{\sqrt{4}}$

$= \frac{\sqrt{2}-\sqrt{6}}{\sqrt{4}}$

$= \frac{\sqrt{2}-\sqrt{6}}{2}$

Worked example 2.4B Identifying the conjugate of an expression

Identify the conjugate for each of these expressions.

a $7 + \sqrt{2}$ **b** $1 - \sqrt{5}$

THINK

a To form a conjugate of a binomial expression with one surd term, change the sign of the surd term.
So $7 - \sqrt{2}$ is the conjugate of $7 + \sqrt{2}$.

b To form a conjugate of a binomial expression with one surd term, change the sign of the surd term.
So $1 + \sqrt{5}$ is the conjugate of $1 - \sqrt{5}$.

WRITE

a $7 - \sqrt{2}$

b $1 + \sqrt{5}$

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Worked example 2.4C Rationalising the denominator by using a conjugate expression

Write $\frac{2}{3+\sqrt{6}}$ with a rational denominator.

THINK

1 Multiply the numerator and the denominator by the conjugate of the denominator. The conjugate of $3+\sqrt{6}$ is $3-\sqrt{6}$.

2 Simplify the fraction by first expanding the expression in the numerator and then simplify the expression in the denominator using the 'difference of two squares' rule: $(a+b)(a-b)=a^2-b^2$.

3 Check that your answer has a rational denominator.

WRITE

$$\frac{2}{3+\sqrt{6}}=\frac{2}{3+\sqrt{6}}\times\frac{3-\sqrt{6}}{3-\sqrt{6}}$$

$$=\frac{2(3-\sqrt{6})}{(3+\sqrt{6})(3-\sqrt{6})}$$

$$=\frac{6-2\sqrt{6}}{9-6}$$

$$=\frac{6-2\sqrt{6}}{3}$$

Helpful hints

✔ It can be helpful to work out the binominal multiplication before you multiply the numerator. Then you can easily see if your multiplication will rationalise the denominator.
For example, $(4+\sqrt{3})(4-\sqrt{3})=16-3=13$

so $\frac{1}{4+\sqrt{3}}\times\frac{4-\sqrt{3}}{4-\sqrt{3}}=\frac{4-\sqrt{3}}{13}$

✔ To write a fraction containing a surd in the denominator in simplest form, you need to rationalise the denominator.

Learning pathways

1–3, 4–6(1st, 2nd columns), 7(a-c), 8(a), 9, 10(a, c), 13	3–6(3rd, 4th columns), 7(b–d), 8, 9, 10(b, d), 11, 12(a–c), 13(e, f), 14	5–6(3rd, 4th columns), 7(d, f), 8, 9, 10(e, f), 11, 12(d–f), 15

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Exercise 2.4A: Understanding and fluency

1 Review the skill of multiplying fractions.

a Determine the answer for each of these multiplications. Write your answers in simplest form.

i $\frac{1}{2}\times 1$ ii $\frac{3}{4}\times\frac{1}{1}$ iii $\frac{3}{4}\times\frac{4}{4}$ iv $\frac{2}{5}\times\frac{5}{5}$

b What did you notice about each of your results in part **a**?

c What is special about the second value in each of the multiplications in this question?

2 Continue reviewing the skill of multiplying fractions, this time with surds.

a Calculate the results for these expressions

i $\frac{2}{\sqrt{3}}\times\frac{1}{1}$ ii $\frac{2}{\sqrt{3}}\times\frac{3}{3}$ iii $\frac{2}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}$ iv $\frac{2}{\sqrt{2}}\times\frac{2}{2}$

b Which result from part **a** is a fraction with a rational denominator?

WE 2.4A 3 Rewrite each of these fractions with a rational denominator.

a $\frac{3}{\sqrt{6}}$ b $\frac{2}{\sqrt{7}}$ c $\frac{3}{\sqrt{3}}$ d $\frac{4\sqrt{5}}{\sqrt{7}}$ e $\frac{5}{3\sqrt{2}}$ f $\frac{7}{4\sqrt{5}}$

g $\frac{2\sqrt{3}}{\sqrt{5}}$ h $-\frac{5}{2\sqrt{5}}$ i $\frac{1}{7\sqrt{7}}$ j $\frac{4\sqrt{11}}{\sqrt{10}}$ k $\frac{8\sqrt{5}}{3\sqrt{2}}$ l $\frac{9\sqrt{6}}{4\sqrt{5}}$

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4 Rewrite each of these fractions with a rational denominator. Where appropriate, write the numerator in simplest surd form.

a $\frac{-\sqrt{6}}{\sqrt{2}}$ **b** $\frac{-\sqrt{15}}{\sqrt{6}}$ **c** $\frac{2\sqrt{8}}{3\sqrt{7}}$ **d** $\frac{2\sqrt{6}}{\sqrt{8}}$

e $\frac{6\sqrt{15}}{7\sqrt{3}}$ **f** $-\frac{3\sqrt{20}}{\sqrt{5}}$ **g** $\frac{5}{\sqrt{x}}$ **h** $\frac{x}{x\sqrt{y}}$

i $\frac{4x}{\sqrt{2y}}$ **j** $\frac{\sqrt{x}}{x\sqrt{y}}$ **k** $\frac{\sqrt{2y}}{2y\sqrt{x}}$ **l** $\frac{-3\sqrt{x}}{6x\sqrt{yz}}$

WE 2.4B **5** State the conjugate for each of these expressions.

a $3 + \sqrt{2}$ **b** $2 - \sqrt{3}$ **c** $-\sqrt{6} + 5$ **d** $-4 + 3\sqrt{5}$

e $3\sqrt{5} - 2$ **f** $-4\sqrt{7} - 1$ **g** $1 + 2\sqrt{3}$ **h** $\sqrt{2} + \sqrt{3}$

i $5 - \sqrt{3}$ **j** $-5\sqrt{6} + 2$ **k** $3\sqrt{7} - 2$ **l** $-5 - 2\sqrt{5}$

WE 2.4C **6** Rewrite each of these fractions with a rational denominator.

a $\frac{1}{2 + \sqrt{3}}$ **b** $\frac{3}{3 - \sqrt{2}}$ **c** $\frac{5}{6 + \sqrt{5}}$ **d** $\frac{3}{3 - 4\sqrt{5}}$

e $\frac{\sqrt{2}}{5 + 3\sqrt{2}}$ **f** $\frac{2\sqrt{3}}{1 + 2\sqrt{3}}$ **g** $\frac{3}{\sqrt{5} - 1}$ **h** $\frac{3\sqrt{7}}{2\sqrt{6} + 5}$

i $\frac{-4\sqrt{2}}{\sqrt{3} - 2}$ **j** $\frac{\sqrt{3}}{\sqrt{5} + 2}$ **k** $\frac{2\sqrt{5}}{3\sqrt{6} + 10}$ **l** $\frac{6\sqrt{10}}{5\sqrt{2} - 5}$

7 Rewrite each of these fractions with a rational denominator.

a $\frac{\sqrt{2} + 1}{2\sqrt{3} + 3}$ **b** $\frac{5 + 2\sqrt{3}}{4 + \sqrt{2}}$ **c** $\frac{1 + \sqrt{3}}{3 - \sqrt{5}}$ **d** $\frac{3 + \sqrt{5}}{\sqrt{6} + 1}$ **e** $\frac{2\sqrt{3} - 2}{3\sqrt{5} + 2}$ **f** $\frac{5\sqrt{2} - 1}{3 + 2\sqrt{5}}$

8 Write each of these fractions in simplest form, with a rational denominator.

a $\frac{x}{x - \sqrt{y}}$ **b** $\frac{\sqrt{x}}{\sqrt{x} - y}$ **c** $\frac{x + y}{x\sqrt{x} + y}$ **d** $\frac{x - \sqrt{y}}{x + \sqrt{y}}$

Exercise 2.4B: Problem solving and reasoning

9 You can perform additions and subtractions with fractions containing surds in a similar way as you do with regular fractions.

Consider $\frac{3}{\sqrt{2}} + \frac{1}{2}$.

a Rationalise the denominator of the first fraction.

b Write the result of the addition in simplest form.

10 Find the exact value of each of these expressions.

a $\frac{3}{\sqrt{5}} + \frac{1}{5}$ **b** $\frac{4}{\sqrt{3}} - \frac{1}{3}$ **c** $\frac{3\sqrt{2}}{2} + \frac{2}{\sqrt{3}}$

d $\frac{1}{4\sqrt{3}} + \frac{3}{2\sqrt{2}}$ **e** $\frac{5}{\sqrt{5}} - \frac{2}{3\sqrt{6}}$ **f** $\frac{2\sqrt{3}}{5\sqrt{2}} + \frac{\sqrt{5}}{4\sqrt{3}}$

11 Consider the expression $\frac{1}{\sqrt{5} - 1} + \frac{3}{\sqrt{5} + 1}$.

a Write each fraction in the expression as an equivalent fraction with a rational denominator.

b Add your answers from part **a** together, writing your answer in simplest form.

12 Simplify each of these expressions after first expressing each fraction with a rational denominator.

a $\frac{2}{\sqrt{6} - 2} + \frac{2}{\sqrt{6} + 2}$ **b** $\frac{2}{\sqrt{5} + 2} - \frac{2}{\sqrt{5} - 2}$ **c** $\frac{2}{\sqrt{6} - 1} + \frac{2}{\sqrt{3} + 1}$

d $\frac{5}{\sqrt{2} + 1} - \frac{4}{\sqrt{5} - 2}$ **e** $\frac{2\sqrt{3}}{2\sqrt{2} + 5} + \frac{\sqrt{3}}{3\sqrt{2} + 2}$ **f** $\frac{\sqrt{3} + 1}{\sqrt{3} + 4} - \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$

13 Write the reciprocal of each of the following. Write your answers with rational denominators and simplified surds.

a $\sqrt{2}$ **b** $5\sqrt{3}$ **c** $\frac{\sqrt{28}}{7}$ **d** $\frac{\sqrt{27}}{4}$ **e** $\sqrt{6} + 2$ **f** $\frac{3 + \sqrt{5}}{3 - \sqrt{5}}$

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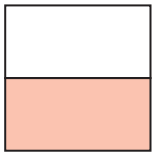
14 Consider the square on the right, which has been divided into congruent rectangles.



$\sqrt{2}$ cm

a State the area of the whole square.

b State the area of the shaded rectangle.

c The breadth of a rectangle can be determined by dividing its area by its length. Determine the breadth of the shaded rectangle. Leave your answer with an irrational denominator.

d The breadth of the shaded rectangle can also be determined by considering the fraction that the breadth of the shaded rectangle is of the square's side length. State what fraction the shaded rectangle's breadth is of the square's side length.

e State the breadth of the shaded rectangle in the form $\frac{a}{b}\sqrt{c}$, where a, b and c are integers.

f Describe the connection between the answers from parts **c** and **e** referring to 'rationalising the denominator'.

g Use a similar method to determine the breadth of the shaded region in the square on the right with a rational denominator.

$\sqrt{10}$ cm

Exercise 2.4C: Challenge

15 When an expression contains two surd terms, the expression has two conjugates resulting from changing the sign of one of the surd terms. For example, the conjugates of $\sqrt{a} + \sqrt{b}$ are $\sqrt{a} - \sqrt{b}$ and $-\sqrt{a} + \sqrt{b}$. The conjugates of $-\sqrt{6} - \sqrt{5}$ are $\sqrt{6} - \sqrt{5}$ and $-\sqrt{6} + \sqrt{5}$.

a Write all of the conjugates of the following expressions.

i $\sqrt{2} + \sqrt{3}$ **ii** $\sqrt{5} - \sqrt{3}$ **iii** $-2\sqrt{5} - 5\sqrt{6}$ **iv** $-2\sqrt{5} + 3\sqrt{7}$

b Evaluate the following products.

i $(\sqrt{2} + \sqrt{3})(\sqrt{2} - \sqrt{3})$ **ii** $(\sqrt{2} + \sqrt{3})(-\sqrt{2} + \sqrt{3})$ **iii** $(\sqrt{2} + \sqrt{3})(-\sqrt{2} - \sqrt{3})$

iv $(-\sqrt{2} + \sqrt{3})(\sqrt{2} - \sqrt{3})$ **v** $(\sqrt{2} - \sqrt{3})(-\sqrt{2} - \sqrt{3})$

c We can rationalise a denominator involving multiple surd terms by making use of the 'difference of two squares' rule. It is important that only the sign of one term is changed to create suitable conjugates.

Write the two conjugates which could be used to rationalise the denominator of each of the following.

i $\frac{1}{\sqrt{5} - \sqrt{3}}$ **ii** $\frac{1}{2\sqrt{2} + 5\sqrt{5}}$ **iii** $\frac{1}{-\sqrt{11} + 3\sqrt{2}}$ **iv** $\frac{1}{-5\sqrt{13} - 4\sqrt{3}}$

d Use the conjugates from part **c** to rationalise the denominator for each fraction in part **c**.

e Lucy was rationalising the denominator of $\frac{4}{-\sqrt{2} + \sqrt{3}}$. Her working is shown below right.

i Explain why Lucy did not get a rational denominator.

ii Explain what Lucy could do next to rationalise the denominator.

$$\frac{4}{-\sqrt{2} + \sqrt{3}} \times \frac{\sqrt{2} - \sqrt{3}}{\sqrt{2} - \sqrt{3}} = \frac{4(\sqrt{2} - \sqrt{3})}{-2 + \sqrt{6} + \sqrt{6} - 3}$$
$$= \frac{4\sqrt{2} - 4\sqrt{3}}{2\sqrt{6} - 5}$$

f Rationalise the denominator of each of the following fractions.

i $\frac{2}{\sqrt{3} + \sqrt{5}}$ **ii** $\frac{-4\sqrt{2}}{\sqrt{3} - \sqrt{2}}$ **iii** $\frac{\sqrt{3}}{\sqrt{5} + \sqrt{2}}$ **iv** $\frac{2\sqrt{5}}{3\sqrt{6} + 2\sqrt{10}}$

v $\frac{6\sqrt{10}}{5\sqrt{2} - 2\sqrt{5}}$ **vi** $\frac{5\sqrt{2} - 1}{3\sqrt{3} + 2\sqrt{5}}$ **vii** $\frac{\sqrt{7} - \sqrt{5}}{\sqrt{5} - \sqrt{7}}$ **viii** $\frac{x + y}{x\sqrt{x} + \sqrt{y}}$

g Evaluate $(\sqrt{2} + \sqrt{3} + \sqrt{5})(\sqrt{2} + \sqrt{3} - \sqrt{5})$.

h Write $\frac{4}{\sqrt{2} + \sqrt{3} + \sqrt{5}}$ with a rational denominator.

Online resources:

- **Interactive skillsheet** Rationalising the denominator
- **Interactive skillsheet** Rationalising the denominator using conjugates
- **Worksheet** Rationalising the denominator
- **Quick quiz** 2.4

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Checkpoint

Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this module.

10A 2.1 **1** State if the following roots are surds or not.

a $\sqrt{15}$ **b** $\sqrt{16}$ **c** $\sqrt{100}$ **d** $\sqrt{\frac{1}{16}}$

e $\sqrt{160}$ **f** $\sqrt{256}$ **g** $\sqrt{0.81}$ **h** $\sqrt{12.1}$

10A 2.1 **2** Simplify the following surds.

a $\sqrt{8}$ **b** $\sqrt{45}$ **c** $\sqrt{175}$ **d** $\sqrt{600}$

10A 2.1 **3** Simplify the following surds.

a $5\sqrt{12}$ **b** $7\sqrt{54}$ **c** $25\sqrt{72}$ **d** $5\sqrt{1280}$

10A 2.2 **4** Simplify the following products.

a $-\sqrt{5} \times (-\sqrt{7})$ **b** $\sqrt{14} \times (-\sqrt{22})$ **c** $-6\sqrt{3} \times 5\sqrt{6}$ **d** $\sqrt{18} \times \sqrt{112}$

10A 2.2 **5** Simplify the following quotients.

a $\frac{-\sqrt{60}}{-\sqrt{10}}$ **b** $\frac{-8\sqrt{34}}{12\sqrt{2}}$

c $\frac{\sqrt{864}}{-\sqrt{384}}$ **d** $\frac{9\sqrt{525}}{3\sqrt{252}}$

e $\frac{-5\sqrt{72}}{6\sqrt{15}}$ **f** $\frac{12\sqrt{338}}{13\sqrt{148}}$

10A 2.2 **6** Simplify the following fractions.

a $\frac{\sqrt{54} \times \sqrt{12}}{\sqrt{18}}$ **b** $\frac{8\sqrt{5} \times 5\sqrt{6}}{10\sqrt{3} \times 6\sqrt{15}}$

10A 2.3 **7** Simplify the following sums and differences.

a $3\sqrt{2} + 9\sqrt{2}$ **b** $3\sqrt{2} - 9\sqrt{2}$

c $5\sqrt{3} + 3\sqrt{5} + 15\sqrt{5} + 7\sqrt{3}$ **d** $21\sqrt{7} - 15\sqrt{11} - 23\sqrt{7} + 32\sqrt{11}$

10A 2.3 **8** Simplify the following sums and differences.

a $\sqrt{8} + \sqrt{128} + \sqrt{18}$ **b** $\sqrt{75} - \sqrt{27}$

c $\sqrt{150} + \sqrt{108} + \sqrt{54} + \sqrt{192}$ **d** $\sqrt{4000} - \sqrt{242} - \sqrt{90} - \sqrt{72}$

10A 2.3 **9** Simplify the following.

a $\frac{5\sqrt{6} + 13\sqrt{6}}{3\sqrt{2}}$ **b** $3\sqrt{3}(6\sqrt{2} - 2\sqrt{6})$

c $(\sqrt{5} - 2)(3 - \sqrt{7})$ **d** $(\sqrt{8} + \sqrt{27})(\sqrt{125} + \sqrt{48})$

e $(7\sqrt{12} + 6\sqrt{5})(5\sqrt{6} - 3\sqrt{40})$ **f** $(4\sqrt{32} + 2\sqrt{7})(5\sqrt{22} - \sqrt{72})$

10A 2.4 **10** Rationalise the denominators of the following fractions.

a $\frac{5}{\sqrt{2}}$ **b** $\frac{12}{\sqrt{3}}$ **c** $\frac{\sqrt{20}}{\sqrt{7}}$ **d** $\frac{\sqrt{55}}{\sqrt{30}}$

10A 2.4 **11** Rationalise the denominators of the following fractions.

a $\frac{5}{1 + \sqrt{2}}$ **b** $\frac{\sqrt{3}}{3 - \sqrt{3}}$

c $\frac{2\sqrt{5}}{3\sqrt{5} - 2}$ **d** $\frac{2 + \sqrt{7}}{-\sqrt{63} + 7}$

e $\frac{4 - \sqrt{5}}{-13 + \sqrt{5}}$ **f** $\frac{\sqrt{10} + 2\sqrt{3}}{2 - \sqrt{128}}$

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Fractional exponents

Learning intentions

By the end of this lesson you will be able to ...

✔ apply the exponent laws with fractional exponents.

Inter-year links

Year 8	4.2 Products and quotients of powers
Year 9	1.2 Products and quotients of powers

Fractional exponents

Key content video
Fractional exponents

- Fractional exponents can be used to express powers of roots.

$$\text{base} \longrightarrow a^{\frac{1}{3}} \longleftarrow \text{fractional exponent}$$

- Fractional exponents have an equivalent surd form:

$$a^{\frac{1}{3}} = \sqrt[3]{a} \qquad a^{\frac{3}{4}} = \sqrt[4]{a^3}$$

fractional exponent form — surd form fractional exponent form — surd form

→ The numerator of the fractional exponent is the number the base is raised to.

→ The denominator of the fractional exponent refers to the type of root applied to the base.

→ $\sqrt[4]{a^3}$ is equivalent to $(\sqrt[4]{a})^3$.

- The exponent laws also apply to fractional exponents.

	Integer exponents	Fractional exponents
Zero exponent law	$a^0 = 1$	$a^0 = 1$
Product of powers rule	$a^5 \times a^3 = a^{5+3}$	$a^{\frac{1}{5}} \times a^{\frac{1}{3}} = a^{\frac{1}{5}+\frac{1}{3}}$
Quotient of powers rule	$a^5 \div a^3 = a^{5-3}$	$a^{\frac{1}{5}} \div a^{\frac{1}{3}} = a^{\frac{1}{5}-\frac{1}{3}}$
Power of a power rule	$(a^5)^3 = a^{5\times 3}$	$\left(a^{\frac{1}{5}}\right)^{\frac{1}{3}} = a^{\frac{1}{5}\times\frac{1}{3}}$

→ Zero exponent law: Excluding 0, any number or variable with an exponent of 0 is equal to 1.

→ Product of powers rule: When multiplying numbers or variables in exponent form that have the same base, add the exponents.

→ Quotient of powers rule: When dividing numbers or variables in exponent form that have the same base, subtract the second exponent from the first.

→ Power of a power rule: When raising a number or variable to two exponents, multiply the exponents. Every base number or variable inside brackets should have its exponent multiplied by the exponent outside the brackets.

- Fractional exponents outside the brackets can also be applied to each factor inside the brackets like integer exponents.

Integer exponents	Fractional exponents
$(ab)^3 = a^3b^3$	$(ab)^{\frac{1}{3}} = a^{\frac{1}{3}}b^{\frac{1}{3}}$
$\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$	$\left(\frac{a}{b}\right)^{\frac{1}{3}} = \frac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}}$

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Worked example 2.5A Applying the exponent laws to fractional exponents

Simplify each of these expressions.

a $x^{\frac{1}{4}} \times x^{\frac{2}{3}}$ **b** $\dfrac{x^{\frac{3}{5}}}{x^{\frac{1}{3}}}$ **c** $\left(x^{\frac{1}{3}}\right)^{\frac{1}{2}}$

THINK

a Add the exponents of the common base, x. To add fractional exponents, convert them to equivalent fractions with a common denominator.

b Subtract the exponents of the common base, x. To subtract fractional exponents, convert them to equivalent fractions with a common denominator.

c Multiply the exponent of the base inside the brackets by the exponent outside the brackets.

WRITE

a $x^{\frac{1}{4}} \times x^{\frac{2}{3}} = x^{\frac{1}{4}+\frac{2}{3}}$
$= x^{\frac{3}{12}+\frac{8}{12}}$
$= x^{\frac{11}{12}}$

b $\dfrac{x^{\frac{3}{5}}}{x^{\frac{1}{3}}} = x^{\frac{3}{5}-\frac{1}{3}}$
$= x^{\frac{9}{15}-\frac{5}{15}}$
$= x^{\frac{4}{15}}$

c $\left(x^{\frac{1}{3}}\right)^{\frac{1}{2}} = x^{\frac{1}{3}\times\frac{1}{2}}$
$= x^{\frac{1}{6}}$

Worked example 2.5B Raising a term to a fractional exponent

Simplify $4\left(x^{\frac{2}{3}}y^{3}\right)^{\frac{5}{6}}$.

THINK

1. Raise each term inside the brackets to the exponent outside the brackets.
2. Multiply the exponent of every base inside the brackets by the exponent outside the brackets.
3. Simplify the fractional exponents.

WRITE

$4\left(x^{\frac{2}{3}}y^{3}\right)^{\frac{5}{6}} = 4\left(x^{\frac{2}{3}}\right)^{\frac{5}{6}}\left(y^{3}\right)^{\frac{5}{6}}$
$= 4 \times x^{\frac{2}{3}\times\frac{5}{6}} \times y^{3\times\frac{5}{6}}$
$= 4 \times x^{\frac{10}{18}} \times y^{\frac{15}{6}}$
$= 4x^{\frac{5}{9}}y^{\frac{5}{2}}$

Worked example 2.5C Multiplying terms with fractional exponents

Simplify $5x^{\frac{1}{2}}y^{\frac{1}{3}} \times 2x^{\frac{1}{5}}y^{\frac{3}{5}}$.

THINK

1. Multiply the coefficients and multiply the terms with the same bases. Add the exponents of the common bases, x and y.
2. Simplify the fractional exponents.

WRITE

$5x^{\frac{1}{2}}y^{\frac{1}{3}} \times 2x^{\frac{1}{5}}y^{\frac{3}{5}} = 5 \times 2 \times x^{\frac{1}{2}} \times x^{\frac{1}{5}} \times y^{\frac{1}{3}} \times y^{\frac{3}{5}}$
$= 10 \times x^{\frac{1}{2}+\frac{1}{5}} \times y^{\frac{1}{3}+\frac{3}{5}}$
$= 10 \times x^{\frac{7}{10}} \times y^{\frac{14}{15}}$
$= 10x^{\frac{7}{10}}y^{\frac{14}{15}}$

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Helpful hints

- ✔ Take care when writing fractional exponents. They should be smaller than the base and sit high up on the shoulder of the base to avoid confusion between the base and the exponent. For example, $x^{\frac{1}{3}}$ not $x\frac{1}{3}$.
- ✔ Take care not to mix up the exponent laws.
  - → When multiplying numbers with the same base, add the exponents.
  - → When dividing numbers with the same base, subtract the exponents.
  - → If a number raised to an exponent is in brackets and raised to another exponent, multiply the exponents.

Learning pathways

1–11, 13, 14, 15(a–f), 16	1–2(d–f), 3–6(e–h), 7–11, 13, 14, 15(g–l), 17, 20, 21, 23	3–6(4th column), 7, 8, 12, 13, 17–19, 21–24

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Exercise 2.5A: Understanding and fluency

1 Write each of these fractional exponents in the form $\sqrt{a}$, $\sqrt[n]{a}$ or $\sqrt[n]{a^m}$.

a $5^{\frac{1}{2}}$ **b** $12^{\frac{2}{3}}$ **c** $x^{\frac{3}{4}}$ **d** $32^{\frac{2}{3}}$ **e** $100^{\frac{1}{4}}$ **f** $x^{\frac{4}{5}}$

2 Write each of the surds in exponent form.

a $\sqrt{6}$ **b** $\sqrt[3]{x^2}$ **c** $\sqrt{5^3}$ **d** $\sqrt{x^3}$ **e** $\sqrt{x^5}$ **f** $\sqrt[4]{x^3}$

WE 2.5A **3** Use an appropriate exponent law to simplify each of these expressions.

a $x^{\frac{1}{5}} \times x^{\frac{2}{5}}$ **b** $x^{\frac{3}{10}} \times x^{\frac{1}{5}}$ **c** $x^{\frac{3}{4}} \times x^{\frac{1}{8}}$ **d** $x^{\frac{2}{5}} \times x^{\frac{1}{6}}$

e $x^{\frac{5}{6}} \times x^{\frac{1}{6}}$ **f** $x^{\frac{2}{3}} \times x^{\frac{1}{4}}$ **g** $x^{\frac{1}{4}} \times x^{\frac{1}{4}}$ **h** $x^{\frac{1}{3}} \times x^{\frac{1}{6}}$

4 Use an appropriate exponent law to simplify each of these expressions.

a $x^{\frac{13}{11}} \div x^{\frac{9}{11}}$ **b** $x^{\frac{3}{8}} \div x^{\frac{1}{8}}$ **c** $\dfrac{x^{\frac{1}{2}}}{x^{\frac{3}{7}}}$ **d** $\dfrac{x^{\frac{4}{5}}}{x^{\frac{3}{4}}}$

e $\dfrac{x^{\frac{1}{3}}}{x^{\frac{1}{6}}}$ **f** $x^{\frac{2}{5}} \div x^{\frac{3}{10}}$ **g** $\dfrac{x^{\frac{9}{4}}}{x^{\frac{1}{2}}}$ **h** $x^{\frac{6}{7}} \div x^{\frac{6}{7}}$

5 Use an appropriate exponent law to simplify each of these expressions.

a $\left(x^{\frac{3}{7}}\right)^{\frac{5}{4}}$ **b** $\left(x^{\frac{1}{3}}\right)^{\frac{3}{5}}$ **c** $\left(x^{\frac{3}{7}}\right)^{0}$ **d** $\left(x^{\frac{2}{5}}\right)^{\frac{3}{4}}$

e $\left(x^{\frac{1}{8}}\right)^{\frac{4}{9}}$ **f** $\left(x^{\frac{2}{3}}\right)^{\frac{3}{4}}$ **g** $\left(x^{\frac{27}{2}}\right)^{\frac{8}{9}}$ **h** $\left(x^{\frac{8}{45}}\right)^{\frac{9}{56}}$

6 Simplify each of these expressions.

a $5x^{\frac{1}{3}} \times 2x^{\frac{1}{4}}$ **b** $6x^{\frac{2}{5}} \times 5x^{\frac{3}{8}}$ **c** $9x^{\frac{3}{7}} \times -4x^{\frac{2}{9}}$ **d** $4x^{\frac{1}{4}} \div x^{\frac{1}{6}}$

e $12x^{\frac{3}{5}} \div 6x^{\frac{3}{8}}$ **f** $\dfrac{32x^{\frac{5}{6}}}{8x^{\frac{4}{5}}}$ **g** $3x^{\frac{1}{2}} \times 2x^{\frac{1}{3}}$ **h** $\dfrac{x^{\frac{3}{5}}}{4x^{\frac{1}{2}}}$

WE 2.5B **7** Simplify each of these expressions.

a $\left(5x^{\frac{1}{3}}\right)^{\frac{3}{5}}$ **b** $\left(16x^{\frac{2}{5}}\right)^{\frac{1}{4}}$ **c** $\left(27x^{\frac{3}{8}}\right)^{\frac{2}{3}}$ **d** $2\left(x^{\frac{2}{5}}y^{\frac{3}{4}}\right)^{\frac{1}{2}}$

e $4\left(x^{\frac{1}{5}}y^{\frac{3}{7}}\right)^{\frac{2}{3}}$ **f** $5\left(x^{\frac{2}{3}}y^{\frac{4}{5}}\right)^{\frac{3}{8}}$ **g** $5\left(x^{2}y^{\frac{3}{5}}\right)^{\frac{1}{2}}$ **h** $7\left(x^{\frac{1}{5}}y^{\frac{4}{11}}\right)^{\frac{5}{3}}$

WE 2.5C **8** Simplify each of these expressions.

a $4x^{\frac{1}{4}}y^{\frac{1}{5}} \times 7x^{\frac{1}{3}}y^{\frac{3}{4}}$ **b** $-6x^{\frac{2}{3}}y^{\frac{4}{7}} \times 12x^{\frac{5}{6}}y^{\frac{3}{7}}$ **c** $9x^{\frac{1}{6}}y^{\frac{2}{5}} \times 5x^{\frac{5}{8}}y$

d $25x^{\frac{3}{5}}y^{\frac{1}{4}} \div 5x^{\frac{1}{5}}y^{\frac{1}{12}}$ **e** $12x^{\frac{3}{8}}y^{\frac{5}{11}} \div 6x^{\frac{1}{5}}y^{\frac{1}{3}}$ **f** $\dfrac{7x^{\frac{1}{2}}y^{\frac{3}{7}}}{7x^{\frac{3}{7}}y^{\frac{2}{5}}}$

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Exercise 2.5B: Problem solving and reasoning

9 Consider the numerical expression $3^{\frac{1}{2}} \times 3^{\frac{1}{2}}$.

a Which of the exponent laws would you use to simplify it?

b Use the exponent law you chose to simplify the expression.

c Now consider the surd expression $\sqrt{3} \times \sqrt{3}$. Multiply the two surds and write your answer in simplest form.

d Compare your answers for parts **b** and **c**. What do you notice?

e Repeat parts **b–d** for each of these pairs of expressions.

i $2^{\frac{1}{2}} \times 2^{\frac{1}{2}}$ and $\sqrt{2} \times \sqrt{2}$ **ii** $5^{\frac{1}{2}} \times 5^{\frac{1}{2}}$ and $\sqrt{5} \times \sqrt{5}$

iii $10^{\frac{1}{2}} \times 10^{\frac{1}{2}}$ and $\sqrt{10} \times \sqrt{10}$ **iv** $x^{\frac{1}{2}} \times x^{\frac{1}{2}}$ and $\sqrt{x} \times \sqrt{x}$

f Using your results from part **e**, explain what you notice about the relationship between the fractional exponent of $\frac{1}{2}$ and the square root sign.

10 Now consider the numerical expression $3^{\frac{1}{3}} \times 3^{\frac{1}{3}} \times 3^{\frac{1}{3}}$.

a Use the appropriate exponent law to simplify the expression.

b Calculate $\sqrt[3]{3} \times \sqrt[3]{3} \times \sqrt[3]{3}$.

c Compare your answers for parts **a** and **b**. What do you notice?

d Repeat parts **a–c** for each of these pairs of expressions.

i $2^{\frac{1}{3}} \times 2^{\frac{1}{3}} \times 2^{\frac{1}{3}}$ and $\sqrt[3]{2} \times \sqrt[3]{2} \times \sqrt[3]{2}$ **ii** $5^{\frac{1}{3}} \times 5^{\frac{1}{3}} \times 5^{\frac{1}{3}}$ and $\sqrt[3]{5} \times \sqrt[3]{5} \times \sqrt[3]{5}$

iii $10^{\frac{1}{3}} \times 10^{\frac{1}{3}} \times 10^{\frac{1}{3}}$ and $\sqrt[3]{10} \times \sqrt[3]{10} \times \sqrt[3]{10}$ **iv** $x^{\frac{1}{3}} \times x^{\frac{1}{3}} \times x^{\frac{1}{3}}$ and $\sqrt[3]{x} \times \sqrt[3]{x} \times \sqrt[3]{x}$

e What are the differences between the surd expressions in question **10** part **b** and in question **9** part **c**?

f Use your results to explain what you notice about the relationship between the fractional exponent of $\frac{1}{3}$ and the cube root sign.

11 Consider the term $x^{\frac{2}{3}}$.

a Show that this term has the equivalent forms of $\left(x^2\right)^{\frac{1}{3}}$ and $\left(x^{\frac{1}{3}}\right)^2$.

b Show that both expressions in part **a** are equivalent to $\sqrt[3]{x^2}$ and $\left(\sqrt[3]{x}\right)^2$.

c Use a calculator to substitute each of the values below for x in $x^{\frac{2}{3}}$. Check that it produces the same results as substituting those values into each of the expressions from parts **a** and **b**.

i $x = 5$ **ii** $x = 2$ **iii** $x = 3$ **iv** $x = 10$ **v** $x = 12$ **vi** $x = 27$

12 Show that any fractional exponent of the form $\frac{m}{n}$ is equivalent to $\sqrt[n]{(\ )^m}$ or $\left(\sqrt[n]{\ }\right)^m$. That is, show that $x^{\frac{m}{n}}$ is equivalent to $\sqrt[n]{x^m}$ or $\left(\sqrt[n]{x}\right)^m$.

13 Calculate each of these values, without using a calculator.

a $25^{\frac{1}{2}}$ **b** $36^{\frac{1}{2}}$ **c** $1^{\frac{1}{2}}$ **d** $144^{\frac{1}{2}}$ **e** $100^{\frac{1}{2}}$ **f** $196^{\frac{1}{2}}$

g $8^{\frac{1}{3}}$ **h** $27^{\frac{1}{3}}$ **i** $125^{\frac{1}{3}}$ **j** $1^{\frac{1}{3}}$ **k** $1^{\frac{1}{5}}$ **l** $32^{\frac{1}{5}}$

14 Look at each part of question **13**.

i If each fractional exponent was negative, briefly explain how the answers would change.

ii What would the answer for each part of question **13** be if the exponents were negative?

iii Which parts can have a negative base? Explain why.

iv What would the value of each of the parts you identified in part **iii** be if the bases were negative?

15 Write each of these values in simplest surd form.

a $18^{\frac{1}{2}}$ **b** $8^{\frac{1}{2}}$ **c** $24^{\frac{1}{2}}$ **d** $108^{\frac{1}{2}}$ **e** $152^{\frac{1}{2}}$ **f** $54^{\frac{1}{3}}$

g $128^{\frac{1}{3}}$ **h** $320^{\frac{1}{2}}$ **i** $243^{\frac{1}{2}}$ **j** $432^{\frac{1}{2}}$ **k** $5^{\frac{3}{2}}$ **l** $6^{\frac{3}{2}}$

16 Write each of these expressions using fractional exponents.

a $\sqrt{x^3 y}$ **b** $\sqrt[3]{x^2 y}$ **c** $\sqrt{x^3 y^5}$ **d** $\sqrt{3xy^3}$ **e** $\sqrt{16x^{-1}y}$ **f** $\sqrt[3]{x^4 y^2}$

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17 Simplify each of these expressions, writing your answers with fractional exponents.

a $12x^{\frac{1}{4}}y^{\frac{1}{5}} \times 9x^{\frac{1}{5}}y^{\frac{2}{3}}$ **b** $\dfrac{x^{\frac{1}{3}}y^{\frac{3}{4}}}{5x^{\frac{1}{9}}y^{\frac{2}{5}}}$ **c** $\left(4x^{\frac{2}{5}}y^{\frac{1}{2}}\right)^{\frac{1}{2}}$ **d** $\sqrt{4x^3y} \times 3\sqrt{x^5y^3}$

e $3\sqrt{x^4y^3} \div \sqrt{9xy^2}$ **f** $(\sqrt{x^3y^5})^{\frac{1}{3}}$ **g** $\sqrt{8x^3yz^5} \times 2\sqrt{3xz^3}$ **h** $\left(\sqrt[3]{x^6y^{\frac{1}{3}}z}\right)^2$

18 Write each of these expressions in simplest form.

a $\sqrt{16x^2y}$ **b** $\sqrt{12x^{-3}y^7}$ **c** $\sqrt{\dfrac{72x^4y^3}{24xy}}$

d $\sqrt[3]{27xy^2}$ **e** $\sqrt[4]{16x^3y^2z}$ **f** $\sqrt{343x^{-1}y^5}$

19 Simplify each of these expressions, writing your answers with positive exponents where appropriate.

a $3x^{-\frac{1}{3}}y^{\frac{2}{3}} \times 6x^{\frac{3}{4}}y^{-\frac{1}{5}}$ **b** $12x^{\frac{1}{4}}y^{\frac{1}{3}} \times 6x^{-\frac{2}{3}}y^{-\frac{1}{4}}$ **c** $\dfrac{xy^2}{x^{\frac{1}{3}}y^{\frac{5}{2}}}$ **d** $\dfrac{3x^{\frac{1}{3}}y^{\frac{2}{5}}}{12x^{\frac{2}{5}}y^{\frac{3}{4}}}$

e $\left(\dfrac{x^{-\frac{1}{3}}}{y^3}\right)^{\frac{1}{4}}$ **f** $\left(\dfrac{2x^{-\frac{2}{5}}y^{\frac{3}{4}}}{y^2}\right)^4$ **g** $\dfrac{5x^{-\frac{1}{2}} \times 4x^{\frac{1}{3}}}{2x^{-\frac{2}{5}}}$ **h** $\sqrt{\dfrac{x^{-\frac{1}{3}}y^{\frac{2}{3}}}{3x^{-\frac{1}{4}}y^{-\frac{1}{5}}}}$

Exercise 2.5C: Challenge

20 Consider the expression $\dfrac{3^{\frac{2}{3}} \times 9^{-\frac{1}{2}}}{81^{\frac{1}{4}}}$.

a Use an appropriate combination of the exponent laws to simplify the expression.

b Use your answer to part **a** to help you solve this equation for x: $9^x = \dfrac{3^{\frac{2}{3}} \times 9^{-\frac{1}{2}}}{81^{\frac{1}{4}}}$

21 Solve each of these equations for x.

a $25^{x+2} = \dfrac{1}{5}$ **b** $27^{x-3} = 9^{x+1}$ **c** $8^x = \dfrac{16^{-\frac{1}{4}} \times 2^{\frac{1}{2}}}{64^{\frac{1}{6}}}$

22 Consider the diagrams on the right.

a The diagrams compare multiplying by a fraction and raising a number to a fractional exponent. Describe what is similar and what is different.

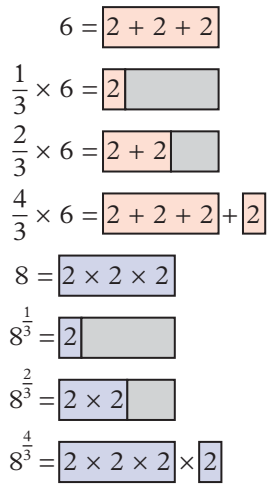
b Evaluate each of the following.

i $81^{\frac{1}{4}}$ **ii** $81^{\frac{3}{4}}$ **iii** $81^{\frac{5}{4}}$

iv $343^{\frac{7}{3}}$ **v** $81^{\frac{5}{2}}$ **vi** $256^{\frac{3}{8}}$

c Write the values appearing in the questions in part **b** in surd form.

d Describe how the denominator of an exponent and the root of a surd are related, as well as how the numerator of an exponent and the exponent of a surd are related. Your description should refer to the number of repeated factors a base has.



23 Use exponents for the following equations, where a and b are positive integers.

a $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$ **b** $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$

24 Write each of the following in the form $\sqrt[a]{x^b}$, where a and b are positive integers.

a $10^{2.5}$ **b** $2^{4.25}$ **c** $3^{0.16}$ **d** $5^{5.555}$

Online resources:

Interactive skillsheet — Fractional exponents

Worksheet — Fractional exponents

Quick quiz — 2.5

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Lesson 2.6

Logarithms

Learning intentions

By the end of this lesson you will be able to ...

✔ convert between exponential and logarithmic forms, evaluate logarithms and solve simple logarithmic equations.

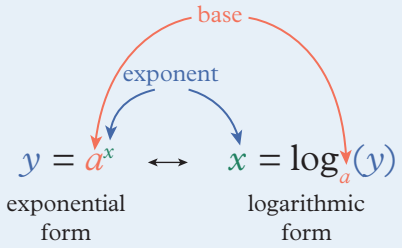
Inter-year links

Year 8	4.1 Exponents
Year 9	1.1 Exponents

Logarithms and exponentials

Key content video
Logarithmic form

- Any equation in which the pronumeral appears as an exponent is called an exponential equation. For example, $2^x = 8$ is an exponential equation with the solution $x = 3$.
- A **logarithm** asks, 'What exponent applied to base a gives y?' It is the number to which the base a is raised to produce a specific number y.
  For example, the equation $2^x = 8$ can be converted to $x = \log_2(8) = 3$. It is read as 'the logarithm, base 2, of 8 is equal to 3'.
  → The base of a logarithm must be a positive real number and cannot be equal to 1: $a > 0$, $a \neq 1$.
  → The diagram on the right shows the general relationship between exponential form and logarithmic form.



- To evaluate a logarithm, ask yourself, 'What exponent applied to a gives y?' In the diagram above, the answer is x.
- Logarithms and exponents are inverse operations of each other.
- Logarithmic identities include:

$$\log_a(a) = 1 \qquad \log_a(1) = 0 \qquad \log_a(a^x) = x \qquad a^{\log_a(x)} = x$$

Worked example 2.6A Converting between exponential form and logarithmic form

For each of these, write an equivalent equation in the form specified.

a $3^4 = 81$ (in logarithmic form)

b $\log_2(16) = 4$ (in exponential form)

THINK	WRITE
a Remember that $a^x = y$ is the same as $\log_a(y) = x$. Identify the value of each variable and rewrite the equation.	**a** $\log_3(81) = 4$
b Recall that $\log_a(y) = x$ is the same as $a^x = y$. Identify the value of each variable and rewrite the equation.	**b** $2^4 = 16$

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Worked example 2.6B Calculating the value of logarithms

Find the value of each of these logarithms.

a $\log_3(9)$ **b** $\log_2(64)$

THINK

a Determine the value of the logarithm. The term $\log_3(9)$ is asking, 'What exponent applied to 3 gives 9?' Since $3^2 = 9$, the value of the logarithm is 2.

b Determine the value of the logarithm. The term $\log_2(64)$ is asking, 'What exponent applied to 2 gives 64?'. Since $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6 = 64$, the value of the logarithm is 6.

WRITE

a $\log_3(9) = 2$

b $\log_2(64) = 6$

Worked example 2.6C Converting from exponential form with a negative exponent to logarithmic form

For each part:

i find the value of the term

ii write an equation in logarithmic form.

a 5^{-1} **b** $\left(\frac{1}{4}\right)^{-2}$ **c** 3^{-3}

THINK

a 1 Write an equivalent value with a positive exponent and simplify.

2 Use the answer to part **i** to write an equation in logarithmic form. (Remember that $a^x = y$ is the same as $\log_a(y) = x$.)

b 1 Write an equivalent value with a positive exponent and simplify.

2 Use the answer to part **i** to write an equation in logarithmic form. (Remember that $a^x = y$ is the same as $\log_a(y) = x$.)

c 1 Write an equivalent value with a positive exponent and simplify.

2 Use the answer to part **i** to write an equation in logarithmic form. (Remember that $a^x = y$ is the same as $\log_a(y) = x$.)

WRITE

a i $5^{-1} = \frac{1}{5^1}$

$= \frac{1}{5}$

ii $\log_5\left(\frac{1}{5}\right) = -1$

b i $\left(\frac{1}{4}\right)^{-2} = \left(\frac{4}{1}\right)^2$

$= 16$

ii $\log_{\frac{1}{4}}(16) = -2$

c i $3^{-3} = \frac{1}{3^3}$

$= \frac{1}{27}$

ii $\log_3\left(\frac{1}{27}\right) = -3$

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Helpful hints

- ✔ When the base of the logarithm and the number in the brackets are the same, the result is always 1. It doesn't matter how big or small the values are! For example, $\log_{238\,042}(238\,042) = 1$.
- ✔ Be careful when writing the base of the logarithms. The base needs to be written in a smaller size than the number in the brackets. For example, write $\log_2(3)$ rather than $\log 2(3)$.

Learning pathways

1–9	1–5(3rd, 4th columns), 6, 7, 10–13, 14(a, b), 17(a–d)	1–5(3rd, 4th columns), 6, 7, 12–17

ANS p690

Exercise 2.6A: Understanding and fluency

WE 2.6A **1** Write each of these equations in logarithmic form.

a $3^3 = 27$ **b** $4^1 = 4$ **c** $5^4 = 625$ **d** $10^3 = 1000$
e $6^2 = 36$ **f** $9^3 = 729$ **g** $3^4 = 81$ **h** $2^7 = 128$

2 Write each of these equations in exponential form.

a $\log_2(32) = 5$ **b** $\log_3(9) = 2$ **c** $\log_5(25) = 2$ **d** $\log_7(49) = 2$
e $\log_4(16) = 2$ **f** $\log_2(1024) = 10$ **g** $\log_6(216) = 3$ **h** $\log_{10}(100) = 2$

WE 2.6B **3** Calculate each of these values.

a $\log_2(16)$ **b** $\log_4(256)$ **c** $\log_5(125)$ **d** $\log_{10}(1000)$
e $\log_8(64)$ **f** $\log_3(27)$ **g** $\log_9(81)$ **h** $\log_2(32)$
i $\log_{10}(10\,000)$ **j** $\log_5(625)$ **k** $\log_6(36)$ **l** $\log_3(81)$
m $\log_2(8)$ **n** $\log_4(64)$ **o** $\log_7(49)$ **p** $\log_3(243)$

WE 2.6C **4** For each of the following: **i** evaluate, and **ii** write an equation in logarithmic form.

a 6^{-1} **b** 4^{-1} **c** 2^{-2} **d** 2^{-5}
e 5^{-2} **f** 10^{-1} **g** 10^{-3} **h** 3^{-3}
i $\left(\frac{1}{2}\right)^{-1}$ **j** $\left(\frac{1}{3}\right)^{-2}$ **k** $\left(\frac{1}{5}\right)^{-3}$ **l** $(0.01)^{-1}$

5 Evaluate the following.

a $\log_2\left(\frac{1}{8}\right)$ **b** $\log_3\left(\frac{1}{27}\right)$ **c** $\log_{10}(0.01)$ **d** $\log_4\left(\frac{1}{4}\right)$
e $\log_3\left(\frac{1}{81}\right)$ **f** $\log_2\left(\frac{1}{64}\right)$ **g** $\log_4\left(\frac{1}{64}\right)$ **h** $\log_8\left(\frac{1}{64}\right)$

6 This question investigates the logarithm of negative numbers.

a Evaluate: **i** 2^4, and **ii** 2^{-4}.

b Is there a real number such that 2 to the power of that number is -16? (What is the value of $\log_2(-16)$?)

c Repeat part **b** for each of the following values.

i $\log_5(-25)$ **ii** $\log_3(-27)$ **iii** $\log_{10}(-1000)$
iv $\log_6(-36)$ **v** $\log_4(-64)$ **vi** $\log_8(-8)$

d What can you conclude about the logarithm of any negative number to any base?

7 This question investigates logarithms to the base 1.

a Consider $\log_1(5)$. Is it possible for this to be written in an equivalent unique exponential form?

b Is it possible for any of the following to be written in an equivalent exponential form?

i $\log_1(3)$ **ii** $\log_1(10)$ **iii** $\log_1(6)$ **iv** $\log_1(4)$ **v** $\log_1(1)$

c What can you conclude about the logarithm of any number to the base 1?

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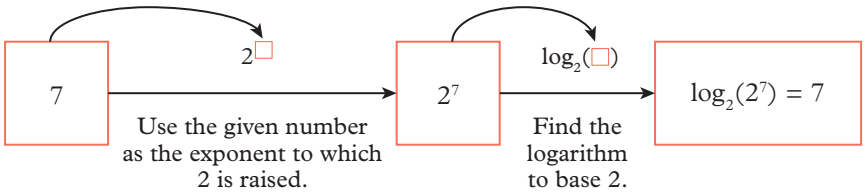
Exercise 2.6B: Problem solving and reasoning

8 Since exponents and logarithms are inverse operations, they 'undo' each other. If you raise a base to an exponent (which is called 'exponentiating'), then you find the logarithm to the same base of that result, you will get the original number you raised the base to.

a Describe the process shown below.



b Evaluate each of the following.

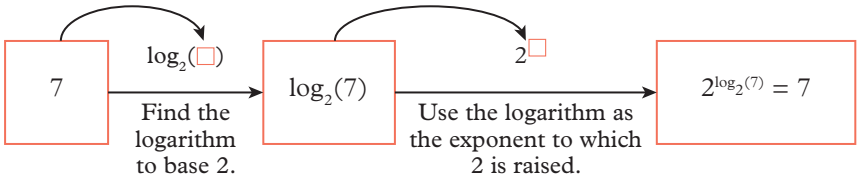
i $\log_5(x)$, where $x = 5^7$ **ii** $\log_7(x)$, where $x = 7^5$ **iii** $\log_{12}(x)$, where $x = 12^9$

iv $\log_2(x)$, where $x = 2^{140}$ **v** $\log_{19}(x)$, where $x = 19^{-6}$ **vi** $\log_{456}(x)$, where $x = 456^{123}$

c Describe, in terms of how we evaluate logarithms, why $\log_a(a^x) = x$ for $a > 0$.

9 If we find the logarithm of a number to a given base, then exponentiate that with the same base, we get the original number back. (See question **8** if you do not know what 'exponentiate' means.)

a Describe the process shown below.



b Evaluate each of the following.

i 5^x, where $x = \log_5(7)$

ii 7^x, where $x = \log_7(5)$

iii 21^x, where $x = \log_{21}(11)$

iv 2^x, where $x = \log_2(531)$

v 101^x, where $x = \log_{101}(8)$

vi 963^x, where $x = \log_{963}(147)$

c Describe, in terms of how we evaluate exponential expressions, why $a^{\log_a(x)} = x$, when $a > 0$ and $x > 0$.

10 A knowledge of the inverse relationship between logarithms and exponents is useful when solving logarithmic equations.

Consider the equation $\log_x(64) = 2$, when $x > 0$.

a Copy and complete the following, using both sides of the equation as exponents to which x is raised.

$\log_x(64) = 2$

$x^{\square} = x^{\square}$

$____ = x^{\square}$

$x = ____$

b What is the value of x? Explain why there is only one solution.

c Write the exponential equation in the last line of your answer for part **a** in the form $x^a = b^a$.

d Notice that, in the equation shown in part **c**, the two exponents are the same. If those exponents are the same, equating the bases will produce the solution. How does the answer found using this method compare with your answer to part **b**?

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11 Use the methods discussed in question **10** to solve each of these logarithmic equations for x.

a $\log_x(64) = 6$

b $\log_x(125) = 3$

c $\log_x(16) = 2$

d $\log_x(8) = 3$

e $\log_x(128) = 7$

f $\log_x(81) = 4$

g $\log_x\left(\frac{1}{4}\right) = -1$

h $\log_x\left(\frac{1}{8}\right) = -3$

12 Consider the equation $\log_4(x) = 3$.

a Copy and complete the following, using both sides of the equation as exponents to which 4 is raised.

$$\log_4(x) = 3$$
$$4^{\square} = 4^{\square}$$
$$____ = 4^{\square}$$

b Find the value of x.

c Briefly explain how using both sides of the equation as exponents to which the base, 4, is raised expresses the logarithmic equation in exponential form.

13 Use the method from question **12** to solve each of these logarithmic equations for x.

a $\log_3(x) = 3$

b $\log_5(x) = 2$

c $\log_6(x) = 1$

d $\log_{10}(x) = 4$

e $\log_2(x) = 5$

f $\log_4(x) = 3$

g $\log_3(x) = -3$

h $\log_2(x) = -2$

14 Review the power of a power rule.

a Show that $(a^2)^{\frac{1}{2}} = a$ and $(a^{\frac{1}{2}})^2 = a$.

b Use the rule in part **a** to complete the following:

$$12^{\square} = 144$$
$$12 = 144^{\square}$$
$$\log_{144}(12) = \log_{144}(144^{\square})$$
$$\log_{144}(12) = ____$$

c Simplify $\log_{a^x}(a)$.

Exercise 2.6C: Challenge

15 Use the method from question **14** to find the value of each of the following:

a $\log_{729}(9)$

b $\log_{729}(3)$

c $\log_{625}(5)$

d $\log_{512}(2)$

16 Use the findings from question **8** and question **14** to find the value of the following:

a $\log_6(36) \times \log_{36}(6)$

b $\log_4(256) \times \log_{256}(4)$

c $\log_{\frac{1}{2}}(8) \times \log_8\left(\frac{1}{2}\right)$

d $\log_a(b) \times \log_b(a)$

17 Evaluate the following.

a $\log_2(\sqrt{2})$

b $\log_{\sqrt{2}}(2)$

c $\log_2\left(\frac{1}{\sqrt{8}}\right)$

d $\log_2\left(\sqrt[3]{128}\right)$

e $\log_3(3\sqrt{3})$

f $\log_5\left(\sqrt[8]{125}\right)$

g $\sqrt{\log_3(81)}$

h $\log_{\sqrt{125}}\left(\frac{\sqrt{5}}{5}\right)$

Online resources:

Interactive skillsheet Converting between exponential form and logarithmic form

Interactive skillsheet Calculating the value of logarithms

Quick quiz 2.6

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Logarithm laws

Learning intentions

By the end of this lesson you will be able to ...

✔ simplify expressions using the logarithm laws.

Inter-year links

Year 8	4.4 Combining the exponent laws
Year 9	1.2 Products and quotients of powers
Year 9	1.3 Raising exponents and the zero exponent

Logarithm laws

Key content video
Logarithm laws

- The logarithm laws can be applied to expressions in logarithmic form:
  $y = \log_a(x)$, where $a > 0$, $a \neq 1$, $x > 0$.
- The logarithm laws include:
  - → The **logarithm of a product** is equal to the sum of the logarithms of the factors.
    $\log_a(mn) = \log_a(m) + \log_a(n)$
  - → The **logarithm of a quotient** is equal to the difference of the logarithm of the dividend and the logarithm of the divisor.
    $\log_a\left(\frac{m}{n}\right) = \log_a(m) - \log_a(n)$
  - → The **logarithm of a power** is equal to the product of the exponent of the power and the logarithm of the base.
    $\log_a(m^n) = n \times \log_a(m)$
- Logarithmic identities include:
  $\log_a(a) = 1$

  $\log_a(1) = 0$

Worked example 2.7A Calculating logarithmic expressions

Evaluate the following.

a $\log_3(1)$

b $\log_4(4)$

THINK	WRITE
a The logarithm of 1 is always equal to 0, no matter what the base is. $3^0 = 1$, so $\log_3(1) = 0$.	**a** $\log_3(1) = 0$
b The logarithm of a number to a base equal to the number itself is always 1. $4^1 = 4$, so $\log_4(4) = 1$.	**b** $\log_4(4) = 1$

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Worked example 2.7B Adding and subtracting logarithms

Use an appropriate logarithm law to evaluate:

a $\log_4(32) + \log_4(2)$

b $\log_5(50) - \log_5(2)$

THINK	WRITE
a 1 Both logarithms in the sum contain the same base, so use the 'logarithm of a product' law: $\log_a(x) + \log_a(y) = \log_a(xy)$	**a** $\log_4(32) + \log_4(2) = \log_4(32 \times 2)$
2 Write 64 as a power of 4.	$= \log_4(64)$ $= \log_4(4^3)$
3 Simplify the answer.	$= 3$
b 1 Both logarithms in the expression contain the same base, so use the 'logarithm of a quotient' law: $\log_a(x) - \log_a(y) = \log_a\left(\frac{x}{y}\right)$	**b** $\log_5(50) - \log_5(2) = \log_5\left(\frac{50}{2}\right)$
2 Write 25 as a power of 5.	$= \log_5(25)$ $= \log_5(5^2)$
3 Simplify the answer.	$= 2$

Worked example 2.7C Determining the logarithm of a number in exponential form

Use an appropriate logarithm law to calculate:

a $\log_2(4^3)$

b $\log_5\left(\frac{1}{5}\right)$

THINK	WRITE
a 1 Use the 'logarithm of a power' law to simplify the logarithm: $\log_a(x^n) = n \times \log_a(x)$	**a** $\log_2(4^3) = 3 \times \log_2(4)$
2 Calculate the logarithm and simplify.	$= 3 \times \log_2(2^2)$ $= 3 \times 2$ $= 6$
b 1 Write the fraction as an integer with a negative exponent.	**b** $\log_5\left(\frac{1}{5}\right) = \log_5(5^{-1})$
2 Use the 'logarithm of a power' law to simplify the logarithm: $\log_a(x^n) = n \times \log_a(x)$	$= -1 \times \log_5(5)$
3 Use the logarithm identity $\log_a(a) = 1$ to simplify the expression.	$= -1 \times 1$ $= -1$

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Worked example 2.7D Simplifying a logarithmic expression

Calculate $\log_6(48) + \log_6(3) - 2\log_6(2)$.

THINK	WRITE
1 Use the 'logarithm of a power' law to rewrite the third term.	$\log_6(48) + \log_6(3) - 2\log_6(2)$
2 Use the 'logarithm of a product' law to simplify the first two terms.	$= \log_6(48) + \log_6(3) - \log_6(2^2)$ $= \log_6(48 \times 3) - \log_6(2^2)$
3 Simplify the product.	$= \log_6(144) - \log_6(4)$
4 Use the 'logarithm of a quotient' law to simplify the expression to one term.	$= \log_6\left(\frac{144}{4}\right)$ $= \log_6(36)$
5 Work out the value of the logarithm. (Remember: $6^2 = 36$.)	$= \log_6(6^2)$ $= 2$

Helpful hints

- ✔ The order of operations is important when solving logarithms. Recall your knowledge of BIDMAS. The I stands for indices, which can also be called exponents. Logarithms and exponential equations are in the same category as indices. After brackets, you'll need to simplify or apply logarithms and indices.
- ✔ Remember that a logarithm can only be applied to positive numbers. This is important when solving logarithmic equations for which there might be two solutions.

  For example, $2\log_2(x) = \log_2(9)$

  $\log_2(x^2) = \log_2(9)$

  $x^2 = 9$

  $x = \pm 3$

  Except, x cannot be negative, so $x = 3$ only.

Learning pathways

1–5, 6(a–f), 7–11, 12–13(1st column), 14(1st, 2nd columns), 15, 16	3, 5, 6(g–l), 9–11, 12–13(2nd column), 14(3rd, 4th columns), 15, 17(a, c, e), 18, 19(a–c), 20, 22	6(g–l), 9, 11, 13(2nd column), 14(3rd, 4th columns), 17, 18, 19(d–f), 20, 21, 23

ANS p691

Exercise 2.7A: Understanding and fluency

WE 2.7A **1** Evaluate each of the following.

a $\log_5(5)$ **b** $\log_3(3)$ **c** $\log_{10}(10)$ **d** $4\log_5(5)$

e $3\log_3(3)$ **f** $\log_3(1)$ **g** $\log_2(1)$ **h** $4\log_5(1)$

2 Use the 'logarithm of a product' law, $\log_a(mn) = \log_a(m) + \log_a(n)$, to write each of the following expressions as the sum of two logarithms.

a $\log_{10}(3 \times 5)$ **b** $\log_2(6 \times 3)$ **c** $\log_5(2.5 \times 10)$ **d** $\log_2(xy)$

WE 2.7B **3** Write each of these expressions as the logarithm of a product.

a $\log_3(6) + \log_3(2)$ **b** $\log_{10}(5) + \log_{10}(3)$ **c** $\log_2(10) + \log_2(5)$

d $\log_4(6) + \log_4(12)$ **e** $\log_a(5) + \log_a(3)$ **f** $\log_2(x) + \log_2(y)$

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4 Use the 'logarithm of a quotient' law, $\log_a\left(\frac{m}{n}\right) = \log_a(m) - \log_a(n)$, to write each of the following expressions as the difference of two logarithms.

a $\log_{10}\left(\frac{5}{7}\right)$ **b** $\log_2\left(\frac{5}{3}\right)$ **c** $\log_5\left(\frac{11}{20}\right)$ **d** $\log_2\left(\frac{a}{b}\right)$

5 Write each of these expressions as the logarithm of a quotient.

a $\log_3(12) - \log_3(5)$ **b** $\log_{10}(50) - \log_{10}(15)$ **c** $\log_2(25) - \log_2(10)$

d $\log_4(36) - \log_4(9)$ **e** $\log_a(3) - \log_a(15)$ **f** $\log_2(x) - \log_2(y)$

6 Use an appropriate logarithm law to evaluate each of the following.

a $\log_3(9) + \log_3(9)$ **b** $\log_6(108) - \log_6(3)$ **c** $\log_2(8) + \log_2(4)$

d $\log_{10}(500) + \log_{10}(20)$ **e** $\log_4(256) - \log_4(16)$ **f** $\log_3(243) - \log_3(9)$

g $\log_6(9) + \log_6(4)$ **h** $\log_5(175) - \log_5(7)$ **i** $\log_2(32) + \log_2(16)$

j $\log_{10}(25) + \log_{10}(4)$ **k** $\log_3(54) - \log_3(6)$ **l** $\log_4(768) - \log_4(3)$

7 Use the logarithm laws to calculate these values.

a $\log_6(27) + \log_6(4) - \log_6(3)$ **b** $\log_{10}(60) - \log_{10}\left(\frac{3}{5}\right)$ **c** $\log_3\left(\frac{3}{2}\right) + \log_3(18)$

8 Use the 'logarithm of a power' law, $\log_a(x^n) = n \times \log_a(x)$, to rewrite each of the following.

a $\log_3(7^2)$ **b** $\log_5(2^9)$ **c** $\log_4(8^3)$ **d** $\log_9(11^4)$

WE 2.7C **9** Use an appropriate logarithm law to evaluate each of the following.

a $\log_3(3^5)$ **b** $\log_4(4^2)$ **c** $\log_6(6^5)$ **d** $\log_3(9^4)$

e $3\log_3(27)$ **f** $4\log_4(64)$ **g** $5\log_5(5^2)$ **h** $\frac{1}{2}\log_4(4^4)$

10 Use an appropriate logarithm law to evaluate each of the following.

a $\log_{15}\left(\frac{1}{15^2}\right)$ **b** $\log_{12}\left(\frac{1}{12}\right)$ **c** $\log_{19}\left(\left[\frac{1}{19}\right]^2\right)$ **d** $\log_{103}\left(\frac{1}{103^{14}}\right)$

e $5\log_2\left(\frac{1}{64}\right)$ **f** $7\log_6\left(\frac{1}{216}\right)$ **g** $\frac{1}{9}\log_3\left(\frac{4}{108}\right)$ **h** $\frac{5}{2}\log_9\left(\frac{3}{243}\right)$

11 Write each of these expressions in a simpler form and evaluate.

a $2\log_9(3)$ **b** $8\log_4(2)$ **c** $2\log_3(9)$

d $\log_a(\sqrt{a})$ **e** $\log_3(5) - \log_3\left(\frac{5}{3}\right)$ **f** $2\log_x(\sqrt{xy}) - \log_x(y)$

WE 2.7D **12** Use the logarithm laws to evaluate each of the following.

a $\log_6(12) + \log_6(9) + \log_6(2)$ **b** $\log_2(24) + \log_2(6) - \log_2(36)$

c $\log_5(40) + \log_5(10) - \log_5(16)$ **d** $\log_3(15) - \log_3(45)$

e $\log_4(2) - \log_4(32)$ **f** $\log_{10}(20) + \log_{10}(45) + \log_{10}\left(\frac{10}{9}\right)$

13 Evaluate each of the following.

a $3\log_4(2) + \log_4(8)$ **b** $\log_5(225) - 2\log_5(3)$

c $2\log_3(9) + 2\log_3(3)$ **d** $4\log_{10}(5) + 4\log_{10}(2)$

e $\log_2(72) - 2\log_2(6)$ **f** $5\log_2(4) - \log_2(2^3)$

g $2\log_2(4) + \log_2(14) - \log_2(7)$ **h** $4\log_3(2) - \log_3(32) + \log_3(54)$

i $\log_5(15) + 2\log_5(5) + 3\log_5(2) - \log_5(24)$ **j** $3\log_{10}(5) + 2\log_{10}(3) + \log_{10}\left(\frac{8}{9}\right)$

14 Solve each of these logarithmic equations for x.

a $\log_x(27) = 3$ **b** $\log_4(64) = x$ **c** $\log_3(x) = 2$ **d** $\log_7(2401) = x$

e $\log_6(x) = 3$ **f** $\log_2(x) = 6$ **g** $\log_5\left(\frac{1}{125}\right) = x$ **h** $\log_x(6) = \frac{1}{2}$

i $\log_3(x) = 0$ **j** $\log_x\left(\frac{1}{16}\right) = 4$ **k** $\log_2(\sqrt{2}) = x$ **l** $\log_x\left(\frac{1}{16}\right) = -2$

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Exercise 2.7B: Problem solving and reasoning

15 Fractions containing logarithmic expressions in the numerator and the denominator can be simplified in a similar way to common fractions; that is, by cancelling any common factors in the numerator and denominator. Consider $\frac{\log_{10}(3)}{\log_{10}(9)}$.

a Use a calculator to find the value of this fraction.

b Show that an equivalent logarithmic expression for the denominator of the fraction is $2\log_{10}(3)$.

c Show that the fraction simplifies to the same answer you obtained in part **a**.

16 Three students attempt to simplify the fraction $\frac{\log_6(8)}{\log_6(2)}$.

Hailey says the correct answer is 3, Danielle thinks the answer is 4 and Joyce is certain that it must be $\log_6\left(\frac{8}{2}\right)$ or $\log_6(4)$. Which student do you think is correct and why?

17 Simplify each of these expressions.

a $\frac{\log_{10}(16)}{\log_{10}(2)}$ **b** $\frac{\log_6(216)}{\log_6(36)}$ **c** $\frac{\log_3(8)}{\log_3(2)}$

d $\frac{\log_4(64)}{\log_4(256)}$ **e** $\frac{\log_{10}(16)}{\log_{10}(64)}$ **f** $\frac{\log_5(125)}{\log_5(5)}$

18 Consider the equation $\log_5(2x) + \log_5(3) = \log_5(12)$. The following parts of this question highlight two techniques that can be used to solve equations like this. Work through both parts to solve the equation in two different ways.

a Part 1:

i Subtract $\log_5(12)$ from both sides of the equation and show that the use of the logarithm laws can help you to simplify the equation to $\log_5\left(\frac{x}{2}\right) = 0$.

ii Write the simplified equation from part **i** in exponential form and solve it for x.

b Part 2:

i Look at the original equation. Apply an appropriate law to the left-hand side of the equation to show that it simplifies to $\log_5(6x) = \log_5(12)$.

ii If the same base appears on both sides, the number parts of each logarithm can be equated; that is, $6x = 12$. Show that this method produces the same value for x as you found in part **a**.

19 Use either method from question **18** to solve each of these logarithmic equations for x.

a $\log_3(x) + \log_3(5) = \log_3(15)$ **b** $\log_4(2x) + \log_4(8) = 3$ **c** $\log_5(5x) - \log_5(4) = 2$

d $\log_5(10) - \log_5(x) = \log_5(15)$ **e** $\log_2(2x + 1) - \log_2(x) = 4$ **f** $\log_{10}(2x) + \log_{10}(3) = 4 + \log_{10}(5)$

20 Apply the laws of logarithms to evaluate the following expressions given $\log_a 3 = 0.683$ and $\log_a 4 = 0.862$.

a $\log_a 12$ **b** $\log_a 2$ **c** $\log_a 6$

Exercise 2.7C: Challenge

21 Solve this logarithmic equation for x: $\log_5(x^2 - 9x + 21) = 0$.

22 Show that the solution to $\log_{10}(2^{2x+1}) = \log_{10}(4)$ is $\frac{1}{2}$.

23 If $2\log_x\left(\frac{7}{5}\right) + \log_x\left(\frac{704}{35}\right) - 3\log_x\left(\frac{4}{5}\right) = \log_x(y)$, find the value of y.

Online resources:

- **Interactive skillsheet** Logarithm of a product
- **Interactive skillsheet** Logarithm of a power
- **Worksheet** Logarithms
- **Quick quiz** 2.7
- **Interactive skillsheet** Logarithm of a quotient
- **Interactive skillsheet** Simplifying a logarithmic expression
- **Investigation** Before calculators ... there were slide rules!

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Lesson 2.8

Exponential equations

Learning intentions

By the end of this lesson you will be able to ...

✔ solve simple exponential equations, and problems related to exponential growth and decay.

Inter-year links

Year 8	6.2 Solving linear equations
Year 9	3.1 Solving linear equations

Solving exponential equations

Key content video
Solving exponential equations

- To solve exponential equations of the form $a^x = y$ for x, use one of the two methods below.

  Method 1: If y can be written in exponential form as a term with the base a, you can equate the exponents. That is, if $a^x = a^m$ then $x = m$.

  For example, $2^x = 8$ can be written as $2^x = 2^3$, so $x = 3$.

  In some cases, you might need to rewrite both sides of the equation so they have the same base.

  For example, $8^x = 4$ can be written as $2^{3x} = 2^2$, so $3x = 2$.

  Simplifying shows that $x = \frac{2}{3}$.

  Method 2: Take the logarithm with the same base of both sides and simplify.

  For example, $2^x = 8$ can be written as:

$$\begin{aligned} 2^x &= 8 \\ \log_2(2^x) &= \log_2(8) \\ \log_2(2^x) &= \log_2(2^3) \\ x &= 3 \end{aligned}$$

- Exponential relationships can be used to model many real situations such as the growth and decay of population or investments.

Worked example 2.8A Solving exponential equations by writing with the same base

Solve each of these equations.

a $2^x = 32$ **b** $9^x = 27$

THINK	WRITE
a 1 Write the equation so that both sides have the same base (in this case, 2).	**a** $2^x = 32$
2 Solve the equation by equating the exponents. The bases are the same and it is an equation, so the exponents are equal.	$2^x = 2^5$ $x = 5$
b 1 Write the equation so that both sides have the same base (in this case, 3).	**b** $9^x = 27$
2 Equate the exponents because it is an equation and the bases are the same.	$(3^2)^x = 3^3$ $3^{2x} = 3^3$ $2x = 3$
3 Solve the linear equation for x.	$x = \frac{3}{2}$

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Worked example 2.8B Solving exponential equations using logarithmic form

Solve each of these equations. Give your answer as an exact value each time.

a $2^x = 5$ **b** $4 \times 3^x = 2$ **c** $5^{x-4} = 125$

THINK

a Write the equation in logarithmic form. The logarithm cannot be simplified so x is already in exact form.

b **1** Simplify by dividing both sides of the equation by 4.

2 Write the equation in logarithmic form.

3 Simplify the logarithm. The fraction can be simplified using negative exponents.

c **1** Write the equation in logarithmic form.

2 Solve for x by adding 4 to both sides.

3 Which exponent of base 5 will give an answer of 125? The answer is 3, then simplify.

WRITE

a $2^x = 5$

$x = \log_2(5)$

b $4 \times 3^x = 2$

$3^x = \frac{1}{2}$

$x = \log_3\left(\frac{1}{2}\right)$

$= \log_3(2^{-1})$

$= -\log_3(2)$

c $5^{x-4} = 125$

$x - 4 = \log_5(125)$

$x = \log_5(125) + 4$

$= 3 + 4$

$= 7$

Helpful hints

- ✔ Remember that logarithms 'undo' exponentials, and exponentials 'undo' logarithms. These are inverse operations.
- ✔ You will need expert skills in negative exponents, factors and surds to effectively solve exponential equations!

Learning pathways

1–6, 7(a, c, e, g), 9, 11, 12	3–7, 10(a–c), 13, 15, 16	3–4, 6(c, e, f), 7(b, d, f, h), 8, 10(c–e), 14, 15, 17, 18

ANS p691

Exercise 2.8A: Understanding and fluency

WE 2.8A **1** Find x by equating the exponents to solve each of these equations.

a $2^x = 2^9$ **b** $3^x = 3^6$ **c** $4^x = 4^{-3}$

d $8^x = 8$ **e** $5^{x-2} = 5^6$ **f** $2^{x+4} = 2^8$

g $10^{2-x} = 10^{-5}$ **h** $2^{-x-7} = 2^3$ **i** $9^{2x} = 9^6$

j $6^{3x} = 6^{15}$ **k** $7^{2x-1} = 7^{11}$ **l** $3^{4x+5} = 3^5$

2 Solve each of these equations.

a $2^x = 8$ **b** $3^x = 9$

c $4^x = 1024$ **d** $10^x = 10\,000$

e $6^{x-5} = 36$ **f** $2^{x+3} = 64$

g $9^{2x-1} = 729$ **h** $3^{-x+4} = 243$

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WE 2.8B **3** Solve each of these equations. Give your answer as an exact value each time.

a $2^x = 11$ **b** $5^x = 92$ **c** $92^x = 5$

d $14^x = \frac{1}{4}$ **e** $6 \times 3^x = 3$ **f** $2 \times 7^x = 53$

g $5^{2x} = 17$ **h** $10^{3x} = 75$ **i** $8^x + 3 = 36$

j $11^{6x} - 7 = 48$ **k** $3 \times 17^x + 5 = 41$ **l** $9 \times 12^{-4x} - 10 = 36$

4 Solve each of these equations. Give your answer as an exact value in the form $a\log_b(c) + d$, where a, b, c and d are rational numbers.

a $8^{x-1} = 10$ **b** $3^{x+1} = 12$ **c** $5^{x+3} = 2$

d $2^{x-5} = 6$ **e** $2^{3-x} = 48$ **f** $9^{-x+2} = 20$

g $6^{3-x} = 4$ **h** $2^{6-x} = 30$ **i** $4^{2x+3} = 24$

j $3^{4x+1} = 87$ **k** $8 \times 7^{5x+2} = 24$ **l** $6 \times 10^{4x+3} + 11 = 20$

5 Solve the following equations.

a $2^{x^3} = 256$ **b** $2^{x^2} = 512$ **c** $3^{x^2} = 81$

6 Solve the following equations.

a $(3^x - 9)(2^x - 8) = 0$ **b** $(5^x - 125)(10^x - 10\,000) = 0$

c $(2^x - 64)(2^x - 16) = 0$ **d** $(3^x - 1)(3^x - 3)(3^x - 9) = 0$

e $\left(2^x - \frac{1}{32}\right)\left(4^x - \frac{1}{64}\right) = 0$ **f** $(4 \times 7^x - 28)\left(2^{3x-5} - 1\right) = 0$

Exercise 2.8B: Problem solving and reasoning

7 The value of x in each of the exponential equations below is not a whole number. Between which two whole numbers will the value of x be for each of the equations? For example, $2^x = 10$ has a solution for x that is between 3 and 4, because $2^3 = 8$ and $2^4 = 16$.

a $2^x = 5$ **b** $3^x = 290$

c $5^x = 17$ **d** $10^x = 6850$

e $3^{2x} = 14$ **f** $7^{2x} = 3000$

g $4^{x+1} = 38$ **h** $6^{x-2} = 210$

8 Consider the exponential equation $9 \times 2^{6x-5} = 27$.

a Use exponent laws to write the equation in the form $b \times a^x = c$, where a, b and c are rational numbers.

b Solve the equation. Give your answer as an exact value.

9 Explain why the following equations have no real solutions.

a $3^x = -1$ **b** $3^x = 0$

10 Solve the following equations by first factorising the expression.

a $(2^x)^2 - 8 \times 2^x = 0$

b $(5^2)^x - 5 \times 5^x = 0$

c $10^{2x} - 15 \times 10^x = 0$

d $4^x - 6 \times 2^x + 8 = 0$

e $9^x - 5 \times 3^x - 14 = 0$

11 After the water boils in an electric kettle, the temperature of the water T (in °C) after m minutes follows the relationship $T = 20 + 80 \times 0.8^m$ as it cools.

a What was the temperature of the boiled water at the start? Hint: $m = 0$.

b What is the temperature of the water, to the nearest degree, after 3 minutes?

c How many minutes, correct to one decimal place, does it take for the temperature to be half of what it was at the start?

d What is the lowest temperature that this kettle of water can reach?

e Is the relationship discussed here showing growth or decay?

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12 A population of rabbits can be modelled by the relationship $n = 9^t + 1$, where n is the number of rabbits after t months.

a How many rabbits were there to start with?

b How many rabbits are there after 2 months?

c How many months, correct to two decimal places, does it take for there to be 1000 rabbits?

d Is the model discussed here showing population growth or population decay?

13 The human population can be modelled using different exponential functions. One model uses $P = 2.6 \times 1.017^t$, where P is the population in billions and t is the number of years since 1950.

a Use this model to find a population figure for:

i 2020 **ii** 2040.

b In what year does this model predict that the population will reach 15 billion?

c The population in 2010 was estimated to be 6.9 billion. How did this compare to the population calculated using this model? Give reasons why you think the population numbers might be different.

14 Bacteria grow by doubling every hour such that the number, N, of bacteria on a petri dish after t hours is given by $N = 45 \times 2^t$.

a State the number of bacteria on the petri dish initially.

b Determine the number of hours it would take for the number of bacteria to grow to 45 000:

i as an exact value

ii correct to the nearest hour.

c Initially, the petri dish was left for a day. Determine the number of bacteria on the petri dish after that day, correct to two significant figures.

d A virus is introduced to the petri dish that kills the bacteria. The number, N, of bacteria alive t hours after the virus is introduced is given by the equation $N = \dfrac{100}{1 - 0.99^{t+1}}$.

Determine the number of bacteria still alive one hour after the virus is introduced, correct to the nearest bacterium.

e Fill in the following table of values to determine the number of bacteria alive after 66, 67, 68, 69, 70 hours, correct to the nearest bacterium.

t	66	67	68	69	70
N					

f After how many hours will the number of bacteria reduce to 200?

g Determine the number of hours it will take for the number of bacteria to reduce to 200, as an exact value.

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15 Tristan fills a glass with tap water at room temperature, which is 25°C. He wants to cool it by placing it in the refrigerator which has a constant internal temperature of 4°C. The temperature, T°C, of the water in the glass t minutes after the glass is placed in the refrigerator is given by the equation $T = 21 \times 2.718^{-kt} + 4$, where k is a positive number.

a Tristan checks the temperature of the glass of water after 5 minutes to find it is 15°C. Determine the value of k, correct to two decimal places.

b Determine how long the water will take to cool to 9°C, correct to the nearest minute.

c Unhappy with how long the water will take to cool to 9°C, Tristan puts the glass in the freezer which has a constant internal temperature of −18°C. The temperature, T°C, of the water in the glass t minutes after the glass is placed in the freezer is given by the equation $T = 33 \times 2.718^{-mt} - 18$, where m is a positive number.

i Tristan assumes that $m = k$ while the glass is in the freezer. Determine how long, to the nearest 30 seconds, Tristan expects the glass to take to cool to 9°C.

ii Tristan checks the temperature of the glass of water after 1 minute to find it is 7°C. Determine the actual value of m, correct to two decimal places.

Exercise 2.8C: Challenge

16 Solve $\left(x^2 + x - 11\right)^{x^2 - 5x - 14} = 1$ for x.

17 Scientific calculators often have buttons for two types of logarithms; the common (base 10) logarithm, usually labelled 'log', and the natural logarithm with a special base, e, that is approximately 2.71, labelled 'ln'. To evaluate logarithms with other bases on these calculators, we need to change the base of the logarithm we are evaluating to 10 (or to the special base, e). For example, to change $\log_2(54)$ to a base 10 logarithm, we change the base to 10, then divide by the base 10 logarithm of 2. That is, $\log_2(54) = \dfrac{\log_{10}(54)}{\log_{10}(2)}$.

a Convert the following logarithms to base 10.

i $\log_3(7)$ **ii** $\log_7(3)$ **iii** $\log_{11}(45)$ **iv** $\log_{101}(12)$ **v** $\log_{19}(24)$ **vi** $\log_a(b)$

b Write the following quotients as a single logarithm, for which the base is the value appearing inside the brackets of the logarithm in the denominator.

i $\dfrac{\log_{10}(8)}{\log_{10}(9)}$ **ii** $\dfrac{\log_{10}(9)}{\log_{10}(8)}$ **iii** $\dfrac{\log_2(18)}{\log_2(5)}$ **iv** $\dfrac{\log_e(6)}{\log_e(15)}$ **v** $\dfrac{\log_e(75)}{\log_e(2)}$ **vi** $\dfrac{\log_x(v)}{\log_x(t)}$

c Write $x = \log_2(54)$ in exponential form.

d Solve the exponential equation from part **c** using a base 10 logarithm to show that $x = \dfrac{\log_{10}(54)}{\log_{10}(2)}$.

e Use a similar method to show that $\log_b(a) = \dfrac{\log_x(a)}{\log_x(b)}$.

18 Solve each of these equations. Give your answers correct to three significant figures.

a $9^x = 10$ **b** $5^x = 30$ **c** $\frac{1}{4} \times 2^x = 2000$

d $3^{6x} = 120$ **e** $10^{\frac{1}{8}x + 5} = 40\,000$ **f** $7 \times 16^x + 9 = 130$

g $25^{3x} - 10 = 28$ **h** $6 \times 7^{2x - 7} + 1 = 1258$ **i** $\dfrac{100}{1 - 5^{2x}} = 200$

Online resources:

Interactive skillsheet Solving exponential equations

Investigation The exponential growth of one grain of rice

Quick quiz 2.8

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Lesson 2.9

Review: Real numbers

Module summary

Surds

$\sqrt[n]{x}$ x is rational, n is an integer, $n \geq 2$

- Surds cannot be simplified to remove the root
- Surds are exact values
- When n is even, x must be positive

$a\sqrt{b} = a \times \sqrt{b}$

Square roots

- $(\sqrt{a})^2 = a$
- $\sqrt{a^2} = a$

Multiplying surds

- $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$
- $a\sqrt{b} \times c\sqrt{d} = a \times c \times \sqrt{b \times d}$

Dividing surds

- $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$
- $a\sqrt{b} \div c\sqrt{d} = \dfrac{a\sqrt{b}}{c\sqrt{d}} = \dfrac{a}{c} \times \sqrt{\dfrac{b}{d}}$

Adding and subtracting surds

- $a\sqrt{b} + c\sqrt{b} = (a + c)\sqrt{b}$
- $6\sqrt{5} - 3\sqrt{2} - 8\sqrt{2} + 5\sqrt{5} = 11\sqrt{5} - 11\sqrt{2}$

Rationalising the denominator

$(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$

conjugates rational number

- $\dfrac{a}{\sqrt{b}} = \dfrac{a}{\sqrt{b}} \times \dfrac{\sqrt{b}}{\sqrt{b}} = \dfrac{a\sqrt{b}}{b}$
- $\dfrac{1}{a + \sqrt{b}} = \dfrac{1}{a + \sqrt{b}} \times \dfrac{a - \sqrt{b}}{a - \sqrt{b}} = \dfrac{a - \sqrt{b}}{a^2 - b}$

Fractional exponents

base $a^{\frac{1}{3}}$ fractional exponent

exponent, root

$a^{\frac{2}{3}} = (\sqrt[3]{a})^2$

fractional exponential form surd form

$a^{\frac{1}{3}} = \sqrt[3]{a}$ $a^{\frac{3}{4}} = \sqrt[4]{a^3} = (\sqrt[4]{a})^3$

	Integer exponents	Fractional exponents
Zero exponent law	$a^0 = 1$	$a^0 = 1$
Product of powers rule	$a^5 \times a^3 = a^{5+3}$	$a^{\frac{1}{5}} \times a^{\frac{1}{3}} = a^{\frac{1}{5}+\frac{1}{3}}$
Quotient of powers rule	$a^5 \div a^3 = a^{5-3}$	$a^{\frac{1}{5}} \div a^{\frac{1}{3}} = a^{\frac{1}{5}-\frac{1}{3}}$
Power of a power rule	$(a^5)^3 = a^{5\times 3}$	$(a^{\frac{1}{5}})^{\frac{1}{3}} = a^{\frac{1}{5}\times\frac{1}{3}}$
	$(ab)^3 = a^3b^3$	$(ab)^{\frac{1}{3}} = a^{\frac{1}{3}}b^{\frac{1}{3}}$
	$\left(\dfrac{a}{b}\right)^3 = \dfrac{a^3}{b^3}$	$\left(\dfrac{a}{b}\right)^{\frac{1}{3}} = \dfrac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}}$

Logarithms and exponentials

- $\log_a(1) = 0$
- $\log_a(a) = 1$
- $\log_a(mn) = \log_a(m) + \log_a(n)$
- $\log_a(m^n) = n \times \log_a(m)$
- $\log_a\left(\dfrac{m}{n}\right) = \log_a(m) - \log_a(n)$

base, exponent

$a^y = x \leftrightarrow y = \log_a(x)$

exponential form logarithmic form

Solving exponential equations

Method 1:
Write both sides of the equation with the same base.

$8^x = 16$
$2^{3x} = 2^4$
$3x = 4$
$x = \dfrac{4}{3}$

Method 2:
Take the logarithm with the same base on both sides

$8^x = 16$
$\log_2(8^x) = \log_2(16)$
$\log_2(2^{3x}) = 4$
$3x = 4$
$x = \dfrac{4}{3}$

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Review questions 2.9A: Mathematical literacy review

Review quiz
Take the review quiz to assess your knowledge of this module.

Quizlet
Test your knowledge of this module by working individually or in teams.

The following key terms are used in this module:

- base
- base-ten
- conjugate
- cube root
- exponent
- exponential form
- irrational number
- like surds
- logarithm
- logarithmic form
- order of magnitude
- power
- product
- quotient
- rationalise
- rational number
- real number
- recurring decimal
- square root
- surds

1 State whether the following statements are true or false. If false, write a corresponding true statement and give an example.

a All fractions with the denominator and numerator being integers are rational numbers.
b All recurring decimals are irrational numbers.
c All terminating decimals are irrational numbers.
d All surds are irrational numbers.
e A number with a fractional exponent will always be irrational.
f A square root is always a rational number.
g A cube root can be a rational number.

2 Select the best option to complete this sentence: The logarithm of a number is …

A the quotient of the number and the base.
B the number raised to the power of the base.
C the number multiplied by the base.
D the power of the base needed to make the number.

3 Using an example, provide a definition in your own words for the following key terms.

a rational number
b like surds

4 Complete the following sentences using words from the list of key terms.

a $9 = 3^2$ is written in _____, and the corresponding $\log_3(9) = 2$ is written in _____.
b $2 + \sqrt{3}$ is the _____ of $2 - \sqrt{3}$ and is used to _____ the denominator of $\frac{5}{2 - \sqrt{3}}$.

Review questions 2.9B: Multiple choice

10A 2.1 **1** Which of these numbers is irrational?

A $\sqrt{120}$ **B** $-\sqrt{16}$ **C** $3\sqrt{9}$ **D** $\sqrt[3]{8}$

10A 2.1 **2** Which of these numbers is the simplest form of $4\sqrt{32}$?

A $8\sqrt{8}$ **B** $4\sqrt{2}$ **C** $8\sqrt{2}$ **D** $16\sqrt{2}$

10A 2.2 **3** Which of these surd expressions is equivalent to $\sqrt{35}$?

A $\sqrt{30} \times \sqrt{5}$ **B** $\sqrt{7} \times 5$ **C** $\sqrt{7} \times \sqrt{5}$ **D** $\sqrt{30} + \sqrt{5}$

10A 2.3 **4** Which of these surd expressions does $2\sqrt{50} - 6\sqrt{2} + 4\sqrt{18}$ simplify to?

A $8\sqrt{2}$ **B** $2\sqrt{50} - 6\sqrt{2} + 4\sqrt{18}$ **C** $16\sqrt{2}$ **D** $28\sqrt{2}$

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10A 2.3 **5** After expanding, what does $2\sqrt{6}(4\sqrt{3} + 5\sqrt{10})$ simplify to?

A $8\sqrt{18} + 10\sqrt{60}$ **B** $24\sqrt{2} + 20\sqrt{15}$ **C** $24\sqrt{2} + 5\sqrt{10}$ **D** $72\sqrt{2} + 40\sqrt{15}$

10A 2.4 **6** Which of these is equivalent to $\frac{3\sqrt{5}}{2\sqrt{2}}$ written with a rational denominator?

A $\frac{3\sqrt{10}}{2}$ **B** $\frac{3\sqrt{10}}{4}$ **C** $\frac{3\sqrt{10}}{8}$ **D** $\frac{15}{2\sqrt{10}}$

10A 2.5 **7** Which of these expressions is *not* equivalent to $x^{\frac{3}{4}}$?

A $\sqrt[4]{x^3}$ **B** $(\sqrt[4]{x})^3$ **C** $(x^3)^{\frac{1}{4}}$ **D** $(\sqrt[3]{x})^4$

10A 2.6 **8** What is $x^y = z$ equivalent to in logarithmic form?

A $\log_x(z) = y$ **B** $\log_x(y) = z$ **C** $\log_z(x) = y$ **D** $\log_z(y) = x$

10A 2.7 **9** Which of these is an equivalent logarithmic expression for $\log_{10}(42)$?

A $\log_{10}(20) + \log_{10}(22)$ **B** $\log_{10}(6) + \log_{10}(7)$ **C** $\log_{10}(6) \times \log_{10}(7)$ **D** $\frac{\log_{10}(84)}{\log_{10}(2)}$

10A 2.8 **10** If $8^x = 32$, which of these is the solution to the equation?

A $x = \frac{1}{4}$ **B** $x = 4$ **C** $x = \frac{3}{5}$ **D** $x = \frac{5}{3}$

Review questions 2.9C: Short answer

10A 2.1 **1** Determine which of these numbers are surds.

a $3\sqrt{8}$ **b** $\frac{1}{\sqrt{5}}$ **c** $6\sqrt{4}$ **d** $\sqrt{6} - 1$ **e** $\sqrt{\frac{9}{25}}$ **f** $\frac{\sqrt{6}}{4}$

10A 2.1 **2** Use a calculator to find the value of each of these expressions, correct to three decimal places.

a $2\sqrt{6}$ **b** $\frac{\sqrt{5} - 2}{4}$

10A 2.1 **3** Show that each of these numbers is rational.

a 0.65 **b** 0.2175 **c** $0.5\dot{2}$ **d** $1.\overline{23}$

10A 2.1 **4** Write each of these terms in simplest form.

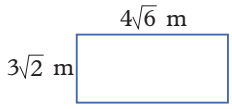
a $2\sqrt{24}$ **b** $3\sqrt{125}$ **c** $\frac{1}{2}\sqrt{108}$

10A 2.2 **5** Calculate the value of each of these expressions, writing your answers in simplest form.

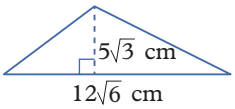
a $3\sqrt{12} \times 4\sqrt{6}$ **b** $-\sqrt{20} \times 5\sqrt{45}$

10A 2.5 **6** Calculate the exact area of each of these shapes, giving your answers in simplest surd form.

a



b



10A 2.2 **7** Calculate the value of each of these expressions.

a $\sqrt{11} \times \sqrt{7}$ **b** $3\sqrt{6} \times 9\sqrt{5}$

10A 2.2 **8** Calculate the value of each of these expressions.

a $\sqrt{24} \div \sqrt{8}$ **b** $\frac{4\sqrt{45}}{\sqrt{15}}$

10A 2.3 **9** Simplify each of these expressions.

a $3\sqrt{2} - 6\sqrt{2} + \sqrt{2}$ **b** $3\sqrt{7} + 2\sqrt{5} - \sqrt{7} + 3\sqrt{5}$

c $2\sqrt{10} - \sqrt{2} + 4\sqrt{5} + \sqrt{2}$ **d** $5\sqrt{6} - 2\sqrt{6} + 5 + 3\sqrt{5} - 3$

10A 2.3 **10** Simplify each of these expressions.

a $2\sqrt{24} - 2\sqrt{54} - 5\sqrt{96} + \sqrt{20}$ **b** $\sqrt{108} - 2\sqrt{75} + 2\sqrt{48} + 3\sqrt{8}$

c $4\sqrt{180} - 6\sqrt{125} + 3\sqrt{245}$ **d** $3\sqrt{200} + 4\sqrt{180} + \sqrt{160} - 3\sqrt{128}$

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10A 2.3 **11** Expand and simplify each of these expressions.

a $(2 + 4\sqrt{3})(3 - 2\sqrt{8})$ **b** $(2\sqrt{3} + 4\sqrt{5})^2$

10A 2.4 **12** Write each of these fractions with a rational denominator, writing the numerator in simplest surd form where appropriate.

a $\frac{5}{2\sqrt{6}}$ **b** $\frac{3\sqrt{15}}{2\sqrt{5}}$ **c** $\frac{4\sqrt{5}}{\sqrt{3} + 2\sqrt{6}}$ **d** $\frac{2 + \sqrt{6}}{3\sqrt{7} - 5\sqrt{3}}$

10A 2.5 **13** Use an appropriate exponent law to simplify each of these expressions. Write each answer in:

i fractional exponent form, and **ii** an appropriate surd form.

a $\left(x^{\frac{3}{5}}\right)^{\frac{5}{2}}$ **b** $x^{\frac{1}{2}} \times x^{\frac{1}{3}}$ **c** $\frac{x^{\frac{2}{3}}}{9x^{\frac{1}{4}}}$

d $\sqrt{8x^3y^2} \times \sqrt{3xy}$ **e** $\left(9x^{\frac{1}{4}}y^{\frac{2}{5}}\right)^{\frac{1}{2}}$ **f** $\frac{\sqrt{3x^5y^2z}}{5\sqrt{8x^2yz^2}}$

10A 2.6 **14** Write each of these equations in logarithmic form.

a $3^4 = 81$ **b** $6^3 = 216$ **c** $2^5 = 32$ **d** $x^y = w$

10A 2.6 **15** Write each of these equations in exponential form.

a $\log_2(8) = 3$ **b** $\log_4(64) = 3$ **c** $\log_{10}\left(\frac{1}{10\,000}\right) = -4$ **d** $\log_x(z) = b$

10A 2.7 **16** Use the logarithm laws to calculate the value of each of these expressions.

a $\log_5(5^3)$ **b** $\log_4(96) - \log_4(6)$

c $3\log_9(3) + \log_9(27)$ **d** $2\log_3(4) - \log_3(32) + \log_3(54)$

10A 2.7 **17** Solve each of the following logarithmic equations for x.

a $\log_x(32) = 5$ **b** $\log_5(x) = -3$

c $\log_4(256) = x$ **d** $\log_x\left(\frac{1}{8}\right) = 3$

e $\log_2(x) + \log_2(5) = \log_2(12)$ **f** $\log_5(4x) - \log_5(5) = 3$

10A 2.8 **18** Solve each of the following exponential equations for x.

a $4^{x+2} = 16$ **b** $81^{x+1} = 27$ **c** $5^{3x} = 15$ **d** $6^{2x+1} = 42$

Review questions 2.9D: Mathematical modelling

10A **1** A storage chest is in the shape of a cube with edge lengths each measuring $7\sqrt{10} + 11\sqrt{7}$

a Draw a labelled diagram to represent the chest.

b Calculate the exact area of one of the faces of the chest.

c Is the area you calculated in part **b** a rational number or an irrational number?

d All the external faces of the chest need to be covered with a special coating to protect it against wear and tear. What is the total surface area that needs to be covered?

e If the special coating is sold for \$12.80 per square metre, how much will it cost to cover the chest?

f An alternative protective paint offers the best protection when three coats are applied. It is advertised that 1 L of this protective paint will cover an area of 5 m². Show that a 1 L can of the protective paint is enough to provide the best protection for the chest.

g Calculate the percentage of the paint in the 1 L can from part **f** that will be used to cover the chest with three coats.

h A narrow 80 cm long pole needs to fit inside the chest. Show that it is not possible for the pole to lie along the bottom of the chest.

i Is there any way that the pole from part **h** can fit inside the chest? Use appropriate mathematics to show your working and to support your answer.

j What assumptions did you need to consider for the calculations you have performed in this task?

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10A **2** Toby has begun beekeeping. He started with 10 000 bees in his four-frame hive. One month later, there are 20 000 bees in the hive. The number of bees, B, in Toby's hive after t months can be modelled using the equation $B = a \times 10^{bt}$.

a State the value of a.

b Determine the value of b:

i as an exact value **ii** as a decimal, correct to two decimal places.

c Write the equation modelling the number of bees in the form $B = a \times d^t$.

d Determine the expected number of bees after 6 months.

e Toby's four-frame hive can support up to 1 000 000 bees. After how many months does Toby need to expand his hive to allow for the population of bees to continue growing? Write your answer:

i as an exact value **ii** correct to the nearest month.

f After some years, Toby's population of bees became relatively constant at 10 million. Toby began to notice his population of bees declining in number. After a year, there were approximately 9.1 million bees. Determine the equation for the number of bees, B, n years after Toby noticed the population was declining, in the form $B = p \times q^n$.

g If Toby does not make any changes, how many years will it take for the number of bees to drop below his original 10 000 bees?

Checklist

Now that you have completed this module, reflect on your ability to do the following.

I can do this	I need to review this
☐ Simplify surds in $\sqrt{a}$ form ☐ Simplify surds in $a\sqrt{b}$ form ☐ Determine whether a root is a surd or not	☐ Go back to Lesson 2.1 Surds [10A]
☐ Multiply two or more surds together ☐ Divide a surd by another surd	☐ Go back to Lesson 2.2 Multiplying and dividing surds [10A]
☐ Identify like surds ☐ Add and subtract like surds	☐ Go back to Lesson 2.3 Adding and subtracting surds [10A]
☐ Rationalise the denominator of simple fractions ☐ Identify the conjugate of a binomial expression with a surd ☐ Use a conjugate to rationalise the denominator of fractions	☐ Go back to Lesson 2.4 Rationalising the denominator [10A]
☐ Apply the exponent laws to fractional exponents	☐ Go back to Lesson 2.5 Fractional exponents [10A]
☐ Convert between exponential form and logarithmic form ☐ Evaluate logarithms ☐ Solve simple logarithmic equations	☐ Go back to Lesson 2.6 Logarithms [10A]
☐ Simplify logarithmic expressions involving addition and subtraction ☐ Simplify logarithmic expressions involving exponents ☐ Simplify expressions using the logarithm laws	☐ Go back to Lesson 2.7 Logarithm laws [10A]
☐ Solve simple exponential equations ☐ Use exponential equations to solve questions involving exponential growth or decay	☐ Go back to Lesson 2.8 Exponential equations [10A]

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3 Linear relationships

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Lessons

3.1 Solving linear equations
10A **3.2** Solving linear equations with algebraic fractions
3.3 Sketching linear graphs
3.4 Determining linear equations
3.5 Parallel and perpendicular lines
3.6 Simultaneous linear equations
3.7 Solving linear inequalities

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ The Cartesian plane
- ✔ Plotting linear relationships
- ✔ Intercepts
- ✔ Representing simple linear inequalities on the Cartesian plane

Curriculum links

- Substitute values into formulas to determine an unknown and rearrange formulas to solve for a particular term (VC2M10A05)
- Solve problems involving linear equations, including those derived from formulas (VC2M10A07)
- Solve linear inequalities and graph their solutions on a number line (VC2M10A08)
- Solve simultaneous linear equations, using algebraic and graphical techniques including using digital tools (VC2M10A09)
- Solve problems involving gradients of parallel and perpendicular lines (VC2M10A10)
- Solve linear equations involving simple algebraic fractions (VC2M10A12)
- 10A Simplify combinations of linear expressions with rational coefficients and the solution of related equations (VC2M10AA03)

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Lesson 3.1

Solving linear equations

Learning intentions

By the end of this lesson you will be able to ...

✔ rearrange and solve linear equations.

Inter-year links

Year 7	6.6 Solving equations using inverse operations
Year 8	6.3 Solving equations with the unknown on both sides
Year 9	3.1 Solving linear equations

Linear equations

Key content video
Solving linear equations

- A **linear equation** is an equation featuring pronumerals that are only raised to the power of 1. These pronumerals represent variable or unknown values.

Linear equations	Non-linear equations
$4x - 1 = 6$	$2x^2 = 32$
$y = x + 3$	$y = \frac{1}{x}$
$2a - \frac{3}{5}b = 10$	$a^3 + 6b = 0$

- If a linear equation has only one variable, algebraic methods can be used to find a solution for the unknown value.
- If a linear equation has more than one variable then it has more than one solution for each variable. It cannot be solved for numerical solutions by itself and can only be rearranged.

Solving linear equations using inverse operations

- If the same operation is performed on both sides of an equation, the result is an equation equivalent to the original equation. For example, by adding 4 to both sides, the equation $5x - 4 = 16$ is equivalent to $5x = 20$.
- Operations performed on an unknown in an equation can be undone by performing the appropriate inverse operation to both sides of the equation.

Operation	Inverse operation
$+ 3$	$- 3$
$- 3$	$+ 3$
$\times 3$	$\div 3$
$\div 3$	$\times 3$

For example,
$$\frac{5x - 4}{2} = 8 \quad (\times 2)$$
$$5x - 4 = 16 \quad (+ 4)$$
$$5x = 20 \quad (\div 5)$$
$$x = 4$$

Changing the subject of an equation

Key content video
Changing the subject of an equation

- Equations often contain several variables. As one or more of the variables change, the other variables must also change to maintain the equality of the relationship. For example, we can write the equation *distance* = *speed* × *time* as $d = st$.
- An unknown in an equation with several variables can be determined using substitution if values for all other variables are known.
- A pronumeral is the **subject** of an equation if it is by itself on one side of the equation and does not appear on the other side of the equation. For example, d is the subject in the equation $d = st$.
- The subject of an equation is written 'in terms of' the variables on the other side of the equation.
- Inverse operations can be used to rearrange an equation to make any variable the subject.

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Worked example 3.1A Solving linear equations with the unknown on both sides

Solve each of these equations by finding the value of x.

a $9x + 2 = 7x - 8$

b $5 - 2x = 6x - 1$

THINK

a **1** Remove the pronumeral term from one side of the equation by subtracting the pronumeral that has the smaller coefficient ($7x$) from both sides of the equation.

2 Solve the equation using inverse operations.

b **1** Remove the negative pronumeral term by adding $2x$ to both sides of the equation.

2 Solve the equation using inverse operations.

3 Simplify the solution and write x on the left-hand side.

WRITE

a $9x + 2 = 7x - 8 \quad (-7x)$

$2x + 2 = -8 \quad (-2)$

$2x = -10 \quad (\div 2)$

$x = -5$

b $5 - 2x = 6x - 1 \quad (+2x)$

$5 = 8x - 1 \quad (+1)$

$6 = 8x \quad (\div 8)$

$\frac{6}{8} = x$

$x = \frac{3}{4}$

Worked example 3.1B Solving linear equations involving brackets

Solve the following linear equations by first expanding to remove the brackets.

a $2(x - 4) = 6$

b $6(x - 1) = 2(x + 3)$

THINK

a **1** Expand the brackets.

2 Identify and apply the correct inverse operations to both sides of the equation.

3 Write the solution to the equation with x on the left-hand side.

b **1** Expand the brackets on both sides of the equation.

2 Identify and apply the correct inverse operations on both sides of the equation to move the pronumeral terms to one side of the equation.

3 Simplify both sides of the equation by collecting like terms.

4 Solve the linear equation and write x on the left-hand side.

WRITE

a $2(x - 4) = 6$

$2x - 8 = 6$

$2x = 14$

$x = 7$

b $6(x - 1) = 2(x + 3)$

$6x - 6 = 2x + 6$

$6x - 2x = 6 + 6$

$4x = 12$

$x = 3$

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Worked example 3.1C Solving for an unknown with substitution

Determine the value of x in the following equations using the given values of the other pronumerals.

a $y = mx$, where $y = 10$ and $m = 2$

b $y = abx + \frac{c}{d}$, where $y = 10$, $a = 2$, $b = 3$, $c = -32$ and $d = 4$

THINK

a 1 Substitute the known values into the equation.

2 Solve the equation using inverse operations.

b 1 Substitute the known values into the equation.

2 Simplify the equation by evaluating any calculations.

3 Solve the equation using inverse operations.

WRITE

a $y = mx$

$10 = 2x \quad (\div 2)$

$x = 5$

b $y = abx + \frac{c}{d}$

$10 = 2(3)x + \frac{-32}{4}$

$10 = 6x - 8 \quad (+ 8)$

$18 = 6x \quad (\div 6)$

$x = 3$

Worked example 3.1D Rearranging equations

Rearrange each equation to make x the subject on the left-hand side.

a $y = mx$

b $y = mx + c$

THINK

a 1 Identify the correct inverse operation to make x the subject. Divide both sides by m.

2 Rewrite the equation with x on the left-hand side.

b 1 Identify the correct inverse operations to make x the subject. Subtract c from both sides.

2 Divide both sides by m and rewrite the equation with x on the left-hand side.

WRITE

a $y = mx \quad (\div m)$

$\frac{y}{m} = x$

$x = \frac{y}{m}$

b $y = mx + c \quad (- c)$

$y - c = mx \quad (\div m)$

$x = \frac{y - c}{m}$

Helpful hints

✔ Writing the inverse operation beside the appropriate line of working is a great way to keep track of your calculations.

$\frac{x}{3} - 5 = 7 \quad (+ 5)$

✔ Writing the pronumeral on the left-hand side of your solution is only a convention. Once you have isolated the unknown on one side of the equation, you have the solution.

For example, $3 = x$ is the same as $x = 3$.

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✔ When given a problem in words, define your pronumerals before writing the equation to represent the problem.
For example, if you are told that a community garden grows twice as many zucchinis as tomatoes, you could define the pronumerals like this: t = the number of tomatoes, z = the number of zucchinis, and the formula can be written as: $2t = z$.

✔ When solving linear equations with the unknown on both sides, the side you choose to remove the pronumeral from does not affect the result. However, it is usually easier to perform operations on both sides of the equation which results in the coefficient of the pronumeral being positive.

Learning pathways

1–6 (1st column), 7(a, c), 8(a, e, i), 9, 10(a, b), 11, 13	1–6 (2nd column), 7(b, d), 8(2nd, 3rd columns), 10(c, d), 13–15	1–6 (3rd column), 7(e, f), 8(d, h, l), 12, 14–17

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Exercise 3.1A: Understanding and fluency

1 Solve each of these equations.

a $3x - 4 = 15$ **b** $\frac{x}{5} + 6 = -2$ **c** $3 = 5 - 7x$

d $8 = \frac{5x}{3}$ **e** $\frac{x+3}{10} = -5$ **f** $6 = 3(x + 5)$

g $3(x - 2) = -11$ **h** $1 - \frac{x}{2} = 5$ **i** $-4(5 - 3x) = 9$

j $\frac{2x-5}{8} = -3$ **k** $\frac{7-6x}{5} = 10$ **l** $6 = \frac{5(x-4)}{7}$

m $-4 = \frac{9x}{4} + 4$ **n** $12 - \frac{8x}{15} = 8$ **o** $8 = \frac{4(11-3x)}{9}$

2 Solve each of these equations by finding the value of x.
Hint: You will need to perform operations that involve fractions and decimals.

a $x - \frac{1}{2} = -9$ **b** $2x + 3.6 = 7$ **c** $8.5x = 34$

d $\frac{x}{0.28} + 3.2 = -5.6$ **e** $\frac{3-0.5x}{4} = 0.45$ **f** $\frac{11}{3}x = \frac{5}{7}$

g $\frac{4}{3}x + \frac{5}{6} = \frac{3}{2}$ **h** $\frac{3}{4}\left(x + \frac{9}{5}\right) = \frac{11}{10}$ **i** $\frac{8}{7}\left(\frac{21}{2}x + \frac{56}{5}\right) = \frac{48}{35}$

WE 3.1A **3** Solve each of these equations.

a $5x + 3 = 2x + 9$ **b** $6x - 7 = 4x + 3$ **c** $3x + 4 = x - 2$

d $4x - 5 = 2x - 3$ **e** $3x + 8 = 2x + 1$ **f** $5x + 6 = 2 + 3x$

g $4x - 5 = 13 - 2x$ **h** $2x - 1 = 11 - x$ **i** $2x + 3 = 6 - 5x$

j $3x - 5 = 7x + 11$ **k** $2 - x = 4x - 8$ **l** $8x + 5 = 11x + 20$

WE 3.1B **4** Solve each of these equations.

a $2(x + 4) - 5 = -7$ **b** $3(x - 1) + 2 = 8$ **c** $4(x + 3) + 2x = 0$

d $9(x - 5) - x = 3$ **e** $2(3x + 4) + 7 = 13$ **f** $7(2x - 11) - 4x = -7$

g $5(2 - x) - 1 = -6$ **h** $4(1 - 3x) + 3x = 3$ **i** $-3(4 - x) - 2x + 5 = -1$

5 Solve each of these equations by first expanding to remove the brackets.

a $3(x + 2) = 2x + 11$ **b** $8x - 9 = 5(x - 3)$ **c** $7x + 3 = 6(x + 1)$

d $6(x + 5) = 4(5 - x)$ **e** $4(x - 7) = 2 - x$ **f** $7(x - 12) = -3(x - 2)$

g $3(x + 6) = 2(x + 7)$ **h** $9(x - 3) = 5(x + 1)$ **i** $4(x - 1) = 2(x + 2)$

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6 Solve each of these equations by first expanding to remove the brackets.

a $5(2x + 1) = 7x + 2$ **b** $x - 4 = 2(3 - 2x)$ **c** $2(3x - 4) = 11(x - 3)$

d $4(1 - 2x) = 6(2x - 1)$ **e** $5(1 - 3x) = 3(5x - 1)$ **f** $5(2x - 2) = 6(3x + 1)$

WE 3.1C **7** Determine the value of x in the following equations using given values of the other pronumerals.

a $y = mx$, where $y = 12$ and $m = 3$ **b** $y = mx$, where $y = 3$ and $m = 12$

c $y = a + x$, where $y = 9$ and $a = 4$ **d** $y = a + x$, where $y = 4$ and $a = 9$

e $y = abx + \frac{c}{d}$, where $y = 40$, $a = -2$, $b = 6$, $c = 20$, and $d = 5$

f $y - k = m(x - h)$, where $y = 15$, $k = -3$, $m = 3$, $h = -5$

WE 3.1D **8** Rearrange each of these equations to make x the subject on the left-hand side.

a $x + a = b$ **b** $k = x - p$ **c** $cx = d$ **d** $g + 3x = h$

e $y = 4x + 5$ **f** $\frac{a + x}{m} = n$ **g** $7x - 2w = 5$ **h** $b - \frac{x}{a} = c$

i $x + 3d = 5c - 7x$ **j** $6x - e = 4x + f$ **k** $2(x + y) = v$ **l** $\frac{x}{p} + q = w$

Exercise 3.1B: Problem solving and reasoning

9 Liam is saving to buy a tennis racquet that costs \$219.
He is able to save \$24 per week and currently has \$105.

a Choose a pronumeral to represent the unknown quantity in the problem.

b Write an equation to represent the problem using your chosen pronumeral for the unknown value.

c Solve the equation using inverse operations.

d Write your answer to the problem in weeks.

10 For each of these problems, set up an equation and solve it using inverse operations.

a Edith bought four makeup kits online for a total cost of \$387, which included the delivery charge of \$29. What was the cost of each makeup kit?

b The perimeter of a rectangular car park is 88 m. If the length of the car park is 12 m more than its breadth, what are the car park's dimensions?

c Lisa, Sara and Bindi scored a combined total of 29 goals in a netball match. Lisa scored five more goals than Sara, and Bindi scored three goals less than Sara. How many goals did each girl score?

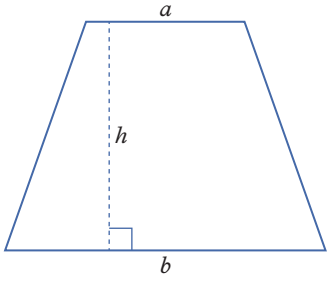
d Oliver has a budget of \$1800 for his birthday party. The cost of hiring a local hall is \$400 and catering is \$27 per person. What is the maximum number of people that Oliver can afford to attend his party?

11 Shana and Paulo have the same amount of money. Shana buys six dumplings and has \$3.80 left over. Paulo buys four dumplings and has \$9.20 left over.

a Write an equation to represent this situation.

b Solve the equation to find the cost of one dumpling.

12 The formula $A = \frac{h}{2}(a + b)$ links the area, A, of a trapezium with its three length measurements, a, b and h, as shown in the diagram on the right.



a Calculate the area, A, of a trapezium for which $a = 4$ cm, $b = 6$ cm and $h = 10$ cm.

b Calculate the height, h, of a trapezium for which $a = 15$ mm, $b = 19$ mm and $A = 119$ mm^2.

c Calculate a when $b = 5$ m, $h = 3$ m and $A = 18$ m^2.

d Rearrange the formula to make a the subject.

e Calculate a when:

i $b = 37$ cm, $h = 24$ cm and $A = 1068$ cm^2 **ii** $b = 2.4$ m, $h = 6.1$ m and $A = 30.5$ m^2.

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13 Equations in the form $y = kx$ are very common, as they describe a multiplicative relationship between the variables x, y, and k. Some examples are: rectangle area = width × length, distance = speed × time, mass = density × volume, momentum = mass × velocity, force = mass × acceleration, voltage = resistance × current, and power = current × voltage.

a Rearrange $y = kx$ to make x the subject.

b Rearrange $y = kx$ to make k the subject.

c Write down 4 pairs of values for x and y such that $k = 4$.

d Write down 4 pairs of values for x and k such that $y = 36$.

14 Temperatures are commonly written in one of three different units:

- degrees Celsius, °C, calibrated originally on a 100 degree scale, from water freezing at 0°C to water boiling at 100°C at sea level
- degrees Fahrenheit, °F, calibrated originally on a 180 degree scale, from water freezing at 32°F to water boiling at 212°F at sea level
- kelvin, K, calibrated from absolute zero, 0 K, and such that a one kelvin increase in temperature is equivalent to a 1°C increase in temperature.

With multiple units, we need to be able to convert between the three units. Converting between degrees Celsius and kelvin is the simplest, as absolute zero, 0 K, equals −273.15°C, and both units increase at the same rate.

a Write an equation that converts the temperature in degrees Celsius, C, to the temperature in kelvin, K.

b Convert the following temperatures to kelvin.

i 0°C **ii** 100°C **iii** 26.85°C **iv** 180°C

c Rearrange this equation to make C the subject.

d Convert the following temperatures to degrees Celsius.

i 500 K **ii** 1000 K **iii** 20 K **iv** 180 K

15 To convert a temperature in degrees Fahrenheit, F, to the temperature in degrees Celsius, C, we use the equation $C = \frac{5}{9}(F - 32)$.

a Convert the following temperatures to degrees Celsius. Round your answers to two decimal places where necessary.

i 0°F **ii** 100°F **iii** 50°F **iv** 86°F

b Rearrange this equation to make F the subject.

c Convert the following temperatures to degrees Fahrenheit.

i 30°C **ii** −10°C **iii** 20°C **iv** 180°C

16 The equation $E = mgh + \frac{1}{2}mv^2$ gives the total energy of a body as the sum of the potential energy, mgh, and the kinetic energy, $\frac{1}{2}mv^2$.

a Determine the value of h, where $E = 1500$, $m = 20$, $g = 10$ and $v = 9$.

b Rearrange the equation to make h the subject. Ensure that any fractions are simplified.

c Determine the value of m, where $E = 8000$, $h = 12$, $g = 10$ and $v = 5$. Write your answer correct to two decimal places.

d Rearrange the equation to make m the subject. Ensure that any fractions are simplified.

Exercise 3.1C: Challenge

17 The sum of five consecutive odd numbers is 50 more than triple the smallest number. Write an equation you can solve to find the five numbers. Hint: Let the smallest number be x.

Online resources:

- **Interactive skillsheet** Solving equations using inverse operations
- **Interactive skillsheet** Solving equations with the unknown on both sides
- **Interactive skillsheet** Changing the subject of a formula
- **Investigation** Calendar capers
- **CAS instructions** Solving equations
- **Quick quiz** 3.1

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Lesson 3.2

Solving linear equations with algebraic fractions

Learning intentions

By the end of this lesson you will be able to ...

✔ rearrange and solve linear equations with algebraic fractions.

Inter-year links

Year 7	6.6 Solving equations using inverse operations
Year 8	6.3 Solving equations with the unknown on both sides
Year 9	3.1 Solving linear equations

Linear equations with algebraic fractions

Key content video
Solving linear equations with algebraic fractions

- Algebraic fractions are fractions that contain at least one pronumeral.
  → When the pronumeral is in the denominator, the denominator of the fraction can't be zero. For example, in $\frac{2}{x-2}$, $x - 2$ can't be 0, therefore x can't be 2.
- Linear equations involving algebraic fractions can be solved just like any other linear equations by performing the appropriate inverse operations on both sides of the equation until the pronumeral is isolated on one side of the equation.
- When solving linear equations involving multiple algebraic fractions, the fractions can be removed by multiplying by the lowest common denominator (LCD) on both sides of the equation.
  For example, the LCD of the three fractions in the linear equation $\frac{x-2}{7} + \frac{x+2}{5} = \frac{x}{2}$ is 70.

$$\frac{x-2}{7} + \frac{x+2}{5} = \frac{x}{2}$$
$$70 \times \frac{x-2}{7} + 70 \times \frac{x+2}{5} = 70 \times \frac{x}{2}$$
$$10(x-2) + 14(x+2) = 35x$$

Worked example 3.2A Solving linear equations with an unknown in the denominator

Solve $\frac{24}{x+2} - 1 = 5$ by finding the value of x.

THINK	WRITE
1 Use inverse operations so that the fraction is by itself on one side.	$\frac{24}{x+2} - 1 = 5$ $(+ 1)$
2 Multiply both sides of the equation by the denominator of the fraction.	$\frac{24}{x+2} = 6$ $(\times (x + 2))$
3 Identify and apply the appropriate inverse operations to both sides of the equation.	$24 = 6(x + 2)$ $(\div 6)$ $4 = x + 2$ $(- 2)$ $2 = x$
4 Write the solution to the equation with x on the left-hand side.	$x = 2$

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Worked example 3.2B Solving linear equations that involve two or more fractions

Solve the following linear equations.

a $\frac{2x-5}{3}+\frac{x-7}{5}=2$

b $\frac{2x-5}{3}-\frac{x}{5}=\frac{2x}{30}$

THINK

a 1 Determine the LCD of the two fractions as 15. Multiply both sides of the equation by the LCD.

2 Cancel the numbers in the denominators and expand the brackets.

3 Simplify by combining like terms.

4 Apply the correct inverse operations to solve the linear equation.

b 1 Determine the LCD of the three fractions as 30. Multiply both sides of the equation by the LCD.

2 Cancel the numbers in the denominators and expand the brackets.

3 Rearrange the equation to have terms with the unknown on one side of the equation.

4 Simplify by combining like terms.

5 Apply the correct inverse operation to solve the linear equation.

6 Simplify the result.

WRITE

a

$$\frac{2x-5}{3}+\frac{x-7}{5}=2 \qquad (\times 15)$$

$$15\times\frac{2x-5}{3}+15\times\frac{x-7}{5}=2\times 15$$

$$5(2x-5)+3(x-7)=30$$

$$10x-25+3x-21=30$$

$$13x-46=30 \qquad (+46)$$

$$13x=76 \qquad (\div 13)$$

$$x=\frac{76}{13}$$

b

$$\frac{2x-5}{3}-\frac{x}{5}=\frac{2x}{30} \qquad (\times 30)$$

$$30\times\frac{2x-5}{3}-30\times\frac{x}{5}=\frac{2x}{30}\times 30$$

$$10(2x-5)-6x=2x$$

$$20x-50-6x=2x$$

$$20x-6x-2x=50$$

$$12x=50 \qquad (\div 12)$$

$$x=\frac{50}{12}$$

$$x=\frac{25}{6}$$

Worked example 3.2C Changing the subject of an equation

Rearrange $\frac{h+1}{x}=\frac{a}{b}$ to make x the subject on the left-hand side.

THINK

1 Multiply both sides of the equation by the denominator containing x.

2 Apply the appropriate inverse operation to both sides of the equation until x is isolated on one side of the equation.

3 Rewrite the equation with x on the left-hand side.

WRITE

$$\frac{h+1}{x}=\frac{a}{b} \qquad (\times x)$$

$$h+1=\frac{ax}{b} \qquad (\times b)$$

$$b(h+1)=ax \qquad (\div a)$$

$$\frac{b(h+1)}{a}=x$$

$$x=\frac{b(h+1)}{a}$$

or, in expanded form:

$$x=\frac{bh+b}{a}$$

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Helpful hints

✔ When working with algebraic fractions, remember that the terms in the numerator are grouped together. When multiplying the numerator by a coefficient, write all the terms in the numerator inside a pair of brackets then expand to remove the brackets. For example, $a \times \frac{b+c}{d} = \frac{a(b+c)}{d}$.

Learning pathways

- 1–6 (1st column), 7, 11(a), 12(a–c), 15(a), 16(a)
- 1–6 (2nd column), 8, 9, 11(a, b), 12(a–d), 14, 15(a)
- 1–6 (3rd column), 10, 11(b, c), 12, 13, 15–17

ANS p693

Exercise 3.2A: Understanding and fluency

1 Solve each of these equations. Start by multiplying both sides of the equation by the common denominator.

a $\frac{2x+1}{5} = \frac{x-7}{5}$ **b** $\frac{4x-3}{3} = \frac{2x+5}{3}$ **c** $\frac{7x-2}{4} = \frac{3x-10}{4}$

d $\frac{x+5}{6} = \frac{11-2x}{6}$ **e** $\frac{9-2x}{11} = \frac{4-7x}{11}$ **f** $\frac{3(x-4)}{2} = \frac{4-x}{2}$

g $\frac{5(x+1)}{2} = -\frac{3(x+27)}{4}$ **h** $\frac{2(x+11)}{7} = \frac{4(x+2)}{5}$ **i** $\frac{9(21-x)}{4} = \frac{4(x+22)}{3}$

2 Solve each of these equations. Start by writing each fraction as an equivalent fraction with the same denominator as the other. Hint: Some solutions may not be integers.

a $\frac{x-2}{5} = \frac{x+4}{2}$ **b** $\frac{x-4}{3} = \frac{x+5}{4}$ **c** $\frac{x+2}{2} = \frac{x+7}{3}$

d $\frac{x+3}{4} = \frac{x+6}{7}$ **e** $\frac{x-8}{3} = \frac{x+1}{6}$ **f** $\frac{3x+5}{4} = \frac{x+9}{2}$

g $\frac{2x+1}{5} = \frac{x-2}{4}$ **h** $\frac{x-7}{3} = \frac{5-x}{4}$ **i** $\frac{2x-1}{6} = \frac{3x+2}{5}$

j $\frac{5x+1}{3} = \frac{3x+2}{2}$ **k** $\frac{8-2x}{9} = \frac{5-7x}{20}$ **l** $\frac{10-x}{3} = \frac{2x-1}{4}$

WE 3.2A **3** Solve each of these equations.

a $\frac{8}{x} = 4$ **b** $\frac{45}{x} = -9$ **c** $\frac{6}{x} = -1$

d $\frac{-12}{x} = 3$ **e** $\frac{-20}{x} = -5$ **f** $\frac{15}{x} = 2$

g $\frac{10}{3x} = 1$ **h** $\frac{-2}{5x} = -7$ **i** $\frac{6}{x} + 5 = 8$

j $\frac{11}{x} + 7 = -4$ **k** $\frac{7}{2x} - 1 = 13$ **l** $\frac{-24}{5x} - 2 = 0$

m $\frac{5}{x+2} = 8$ **n** $2 = \frac{21}{x-7}$ **o** $\frac{10}{2x-5} = -3$

4 Solve each of these equations.

a $\frac{5x}{6} + \frac{4x}{3} = \frac{13}{2}$ **b** $\frac{3x}{4} - \frac{4x}{3} = \frac{21}{12}$

c $\frac{7x}{8} = \frac{1}{2} + \frac{5x}{6}$ **d** $\frac{4x}{9} + \frac{7}{3} = \frac{7}{4} - \frac{11x}{12}$

e $\frac{2}{5}\left(\frac{3x}{4} + \frac{7}{2}\right) = \frac{9x}{5}$ **f** $\frac{55}{6}\left(\frac{14x}{3} - \frac{2}{5}\right) = \frac{22}{15}\left(\frac{5}{6} - \frac{15x}{2}\right)$

g $\frac{5x}{4} + \frac{1}{10} = \frac{7x}{12} + \frac{1}{2}$ **h** $\frac{5}{3}\left(\frac{1}{2} + \frac{4x}{5}\right) = \frac{1}{30} + \frac{2x}{9}$

i $\frac{14}{3}\left(\frac{3}{10} - \frac{15x}{56}\right) = \frac{1}{10}\left(57x - \frac{111}{2}\right)$

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WE 3.2B **5** Solve each of these equations.

a $\frac{3x}{4} + \frac{2x+1}{3} = 1$

b $\frac{1-3x}{4} + \frac{2x}{3} = 3$

c $\frac{2x-3}{5} + \frac{x+7}{3} = 1$

d $\frac{x+5}{2} - \frac{x-1}{6} = 8$

e $\frac{3x}{4} + \frac{1-x}{3} = \frac{5x}{2}$

f $\frac{2x+3}{3} + \frac{1-2x}{6} = \frac{5x-2}{2}$

6 Solve each of these equations.

a $\frac{3(4-2x)}{4} + \frac{2(3x+9)}{3} = 1$

b $\frac{3(-4x-4)}{4} + \frac{2x+10}{2} = 3$

c $\frac{5x+6}{5} + \frac{1-2x}{3} = 3x+4$

d $\frac{3(2-x)}{4} + \frac{3(x-2)}{2} = \frac{x-1}{8}$

e $\frac{1-3x}{4} + \frac{1-x}{3} = 2(x-1)$

f $2x + \frac{1-2x}{6} = \frac{5x}{2} + 3$

7 Determine the value of x in the following equations using given values of the other pronumerals.

a $y = \frac{k}{x}$ where $y = 8$ and $k = 80$

b $y = \frac{k}{x}$ where $y = 80$ and $k = 8$

WE 3.2C **8** Rearrange each equation to make x the subject on the left-hand side.

a $y = \frac{x}{k}$

b $y = \frac{k}{x}$

c $\frac{x+b}{a} = c$

9 Rearrange each equation from question **8** to make pronumerals other than x the subject. Write a separate equation for each such variable.

Exercise 3.2B: Problem solving and reasoning

10 Consider the equation $\frac{2t+1}{g-x} = \frac{t}{x}$.

a Show that the equation can be rearranged to $x = \frac{tg}{3t+1}$.

b Use your equation from part **a** to evaluate x when $g = 10$ and $t = 5$.

c Solve the original equation, $\frac{2t+1}{g-x} = \frac{t}{x}$, for x when $g = 10$ and $t = 5$. Verify that the answer is the same as part **b**.

11 Rearrange each of the equations below to make: **i** x the subject, **ii** k the subject, and **iii** m the subject.

a $y = mkx$

b $y = \frac{kx}{m}$

c $y = kx + m$

12 The equation of a line can be written in many ways. Two common forms are $y = mx + c$ and $Ax + By = D$.

a Rearrange $Ax + By = D$ to make y the subject.

b By comparing the coefficients of your answer in part **a** with $y = mx + c$, write two equations: one where m is in terms of A, B and D and another where c is in terms of A, B and D.

c Rearrange your equations from part **b** to make A and D the subjects.

d The form $Ax + By = D$ is more commonly used to avoid writing fractions when y is the subject. Write $y = \frac{p}{q}x + c$ in the form $Ax + By = D$, where A, B and D are integers.

e Another form of the equation of a line is $m = \frac{y-k}{x-h}$. Rearrange $m = \frac{y-k}{x-h}$ to make y the subject. Fully expand your answer.

13 The optic equation, $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$, equates the sum of the reciprocals of two numbers with the reciprocal of a third.

The optic equation is used for dealing with many systems, such as:

- the distances between an object and its image through a thin lens,

$$\frac{1}{\text{distance from object to lens}} + \frac{1}{\text{distance from image to lens}} = \frac{1}{\text{focal length}}$$

- the resistances in a parallel circuit,

$$\frac{1}{\text{resistance}_1} + \frac{1}{\text{resistance}_2} = \frac{1}{\text{total resistance}}$$

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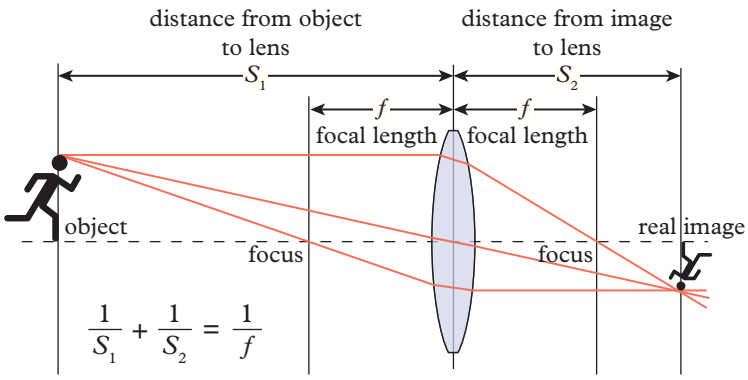

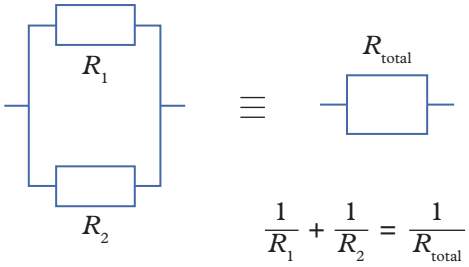

a Show that, for the optic equation, $c = \frac{ab}{a + b}$.

b Calculate:

i the focal length when the distance from the object to the lens is 4 cm and the distance from the image to the lens is 12 cm

ii the total resistance in a parallel circuit when the two resistances are 500 ohms and 2000 ohms.

c Rearrange the optic equation to make a the subject.

d Determine:

i the distance from the lens to the object when the distance from the lens to the image is 10 cm and the focal length is 8 cm

ii the resistance of a resistor that has been connected in parallel with a 16 ohm resistor, where the total resistance is 8 ohms.

Exercise 3.2C: Challenge

14 Two tanks are being filled with water at different rates, m L/min and n L/min. The two tanks will have the same amount of water in their tanks after t minutes.

a If the first tank starts with a L and the second tank with b L, write an equation in terms of a, b, m, n and t that would give the time taken for the tanks to fill to the same amount.

b Determine the time it would take if the first tank starts with 50 L and fills at 5 L/min and the second tank starts with 35 L and fills at 8 L/min.

c Rearrange the equation from part **a** to make t the subject.

d Verify your answer to part **b** by substituting it into your equation from part **c**.

15 Consider the equation $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$.

a Rearrange the equation to make c the subject.

b Rearrange the equation to make a the subject.

16 Solve each of these equations.

a $\frac{2x + 5}{x - 4} = 7$ **b** $\frac{9 - 4x}{3x + 5} = 10$ **c** $\frac{5}{6} = \frac{7x - 3}{3 - 2x}$

17 Make x the subject for each of the following.

a $\frac{mx + p}{nx + q} = r$ **b** $\frac{a + by}{b + ax} = \frac{b - ay}{a - bx}$ **c** $\frac{a + by}{b - ax} = \frac{b + ay}{a - bx}$

Online resources:

Interactive worksheet Solving equations with the unknown in the denominator

Quick quiz 3.2

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Lesson 3.3

Sketching linear graphs

Learning intentions

By the end of this lesson you will be able to ...

✔ sketch linear graphs using a variety of methods.

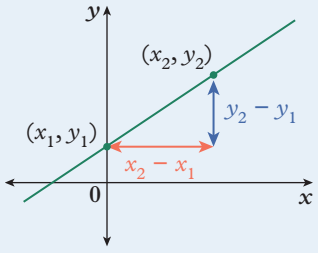
Inter-year links

Support	The Cartesian plane
Year 7	5.4 The Cartesian plane
Year 8	6.5 Plotting linear relationships
Year 9	3.5 Sketching linear graphs

Key content video
Sketching linear graphs

Linear graphs

- A linear equation with two pronumerals describes a **linear relationship** between two variables.
- A linear relationship can be represented on the Cartesian plane as a straight line called a **linear graph**.
- An intercept is the point where a graph crosses an axis.
- The **gradient** is a numerical measure of the slope or steepness of a graph. It can be calculated as the ratio of the change in the y-values (the rise) to the change in the x-values (the run).



$$\text{gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$

- Unless a graph is a vertical or a horizontal line, two points must be plotted in order to sketch the graph.

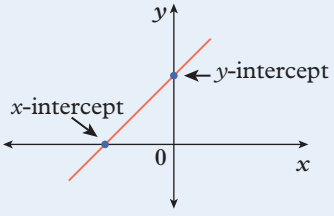
Gradient–intercept form

- The equations for all linear relationships can be expressed in **gradient–intercept form** shown on the right, where:
  - → m is the gradient of the line
  - → c is the y-coordinate of the y-intercept (the point where the line crosses the y-axis).
- Gradient–intercept form can be used to identify the gradient and y-intercept, enabling us to sketch the graph.

$$y = \underbrace{m}_{\text{gradient}}x + \underbrace{c}_{y\text{-intercept}}$$

The x-and-y-intercept method

- The x-intercept is the point at which the graph crosses the x-axis, that is, where $y = 0$.
- The y-intercept is the point at which the graph crosses the y-axis, that is, where $x = 0$.
- The values $x = 0$ and $y = 0$ can be substituted into a linear equation in order to find the line's x- and y-intercepts and sketch the graph.



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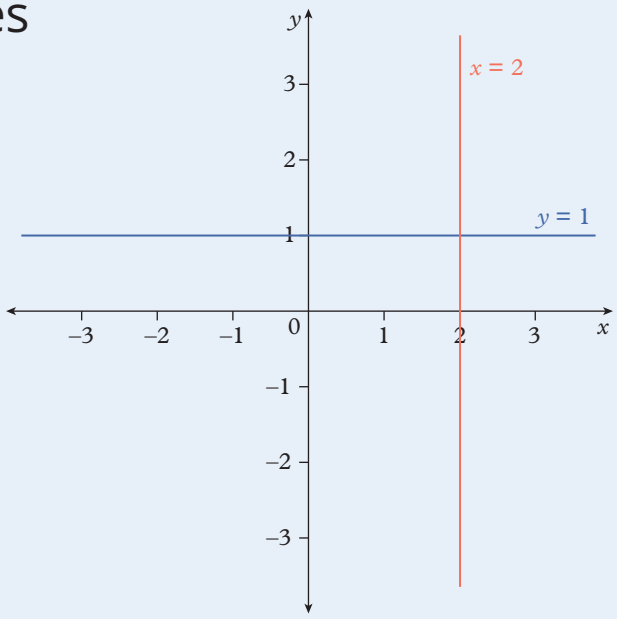
Horizontal and vertical lines

- A horizontal line has a gradient of zero. Horizontal lines have equations of the form $y = c$, where c is any real number.
  For example, the blue line on the right shows the horizontal line $y = 1$.
- A vertical line has an undefined gradient. Vertical lines have equations of the form $x = a$, where a is any real number.
  For example, the red line on the right shows the vertical line $x = 2$.

Horizontal line	Vertical line
$y = c$	$x = a$



Worked example 3.3A The gradient–intercept method

Sketch the graph of $y = -\frac{1}{3}x - 1$.

THINK

1. Identify the gradient, m, and the y-intercept, c, by comparing the equation to the gradient–intercept form $y = mx + c$.
2. Compare the gradient with the gradient formula. Identify values of $y_2 - y_1$ and $x_2 - x_1$ that result in the gradient. Make sure $x_2 - x_1$ is a positive value.
3. Plot a point at the y-intercept and label the coordinates $(0, -1)$.
4. Identify a second point. Since $y_2 - y_1 = -1$, move down by 1 unit. Since $x_2 - x_1 = 3$, move to the right by 3 units.
5. Rule a straight line through the points. Label the graph with its equation.

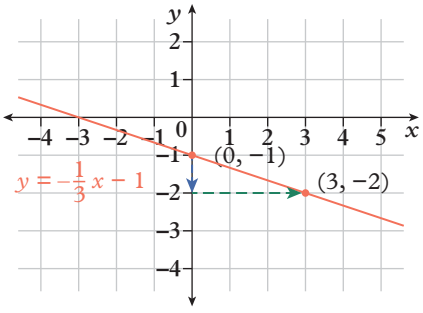
WRITE

$y = -\frac{1}{3}x - 1$

$m = -\frac{1}{3}$, $c = -1$

$m = \frac{y_2 - y_1}{x_2 - x_1} = -\frac{1}{3}$

Let $y_2 - y_1 = -1$ and $x_2 - x_1 = 3$



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Worked example 3.3B The *x*-and-*y*-intercept method

Sketch the graph of $4x - 3y = 12$ using the x-and-y-intercept method.

THINK

1 Determine the x-intercept by substituting $y = 0$ into the equation and solving for x.

2 Determine the y-intercept by substituting $x = 0$ into the equation and solving for y.

3 Plot and label the x- and y-intercepts on the Cartesian plane.

4 Rule a straight line through the points. Label the graph with its equation.

WRITE

For the x-intercept, $y = 0$:

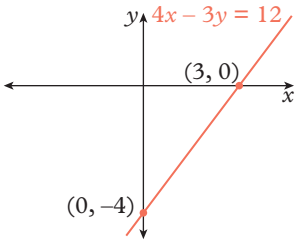
$$\begin{aligned} 4x - 3y &= 12 \\ 4x - 3(0) &= 12 \\ 4x &= 12 \\ x &= 3 \end{aligned}$$

x-intercept is at $(3, 0)$

For the y-intercept, $x = 0$:

$$\begin{aligned} 4x - 3y &= 12 \\ 4(0) - 3y &= 12 \\ -3y &= 12 \\ y &= -4 \end{aligned}$$

y-intercept is at $(0, -4)$



Worked example 3.3C Horizontal and vertical lines

For each linear equation below, **i** list the coordinates of any intercepts and **ii** sketch the graph.

a $y = 3$ **b** $x = -1$

THINK

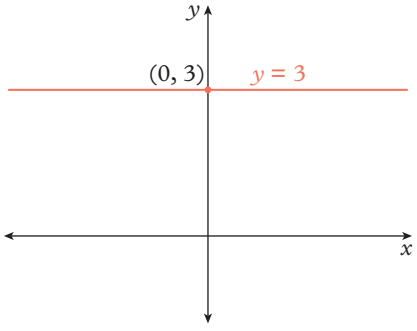
a i Look at the form of the equation. The equation tells us that y is 3 for any value of x, so the graph is a horizontal line passing through a y-intercept at $(0, 3)$.

ii Rule a straight horizontal line through $(0, 3)$. Label the graph with its equation.

WRITE

a i y-intercept at $(0, 3)$

ii



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THINK

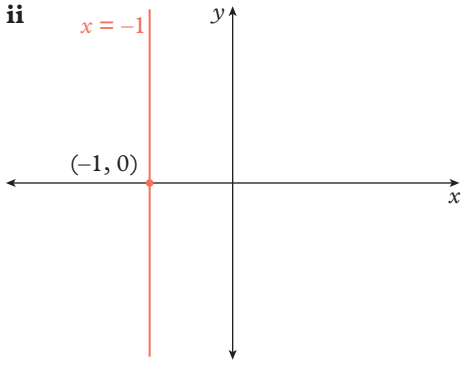
b i Look at the form of the equation. The equation tells us that x is -1 for any value of y, so the graph is a vertical line passing through an x-intercept at $(-1, 0)$.

ii Rule a straight vertical line through $(-1, 0)$. Label the graph with its equation.

WRITE

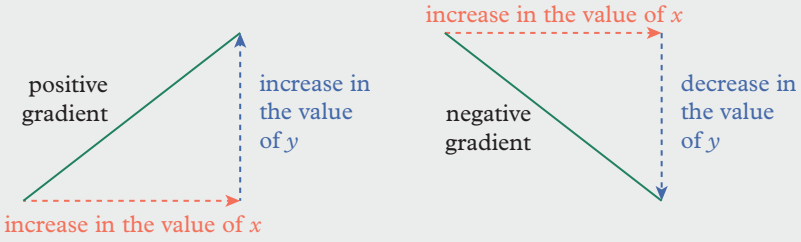
b i x-intercept at $(-1, 0)$

ii



Helpful hints

- ✔ Check that the slope of your graph matches the sign of the gradient. A positive gradient slopes upwards to the right and a negative gradient slopes downwards to the right.



- ✔ If you used the gradient–intercept method to sketch your graph, check the x- and y-intercepts of your graph by substituting $y = 0$ and $x = 0$ into the equation.
- ✔ If you used the x-and-y-intercept method to sketch your graph, rearrange the equation to the form $y = mx + c$ and check to see if the gradient, m, matches the gradient of your graph.
- ✔ The different ways you can sketch a linear graph may seem overwhelming. In order to sketch a linear graph, identify and plot any two points on the graph and then rule a straight line between them.
- ✔ Remember to always label a graph with its equation as well as any x- and y-intercepts.

Learning pathways

1–2(1st, 2nd columns), 3, 5, 7–9(1st, 2nd columns), 12–13(1st, 2nd columns), 14, 15, 16(1st column), 19(a–d)	1–2(2nd, 3rd columns), 4(a, b), 6(a, b), 7–9(2nd, 3rd columns), 12–13(2nd, 3rd columns), 15, 16–17(2nd column), 19, 21	1(3rd column), 2(3rd, 4th columns), 4(c, d), 6(c, d), 7–9(3rd column), 10, 11, 13(3rd column), 15, 17(2nd column), 18, 20, 22–24

Exercise 3.3A: Understanding and fluency

WE 3.3A **1** Sketch the graph of each of these linear relationships using the gradient–intercept method.

a $y = 2x - 4$ **b** $y = \frac{2}{3}x + 1$ **c** $y = -3x + 5$

d $y = \frac{5}{4}x - 2$ **e** $y = -5x - 1$ **f** $y = -\frac{1}{2}x + 3$

g $y = 3x - 4$ **h** $y = x + 6$ **i** $y = -x - 3$

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2 Rearrange each of these equations to be in the form $y = mx + c$, and then **i** identify the gradient, **ii** identify the y-intercept, and **iii** sketch the graph.

a $y = 1 + 2x$ **b** $y = 4 - 3x$ **c** $y = -3 - \frac{3}{4}x$ **d** $y = 3(x - 2)$

e $y = -2(x - 3)$ **f** $y - 4 = -(5x + 6)$ **g** $3x + y = 2$ **h** $-4x + y = -5$

i $6x + 3y = 9$ **j** $-4x + 5y = 15$ **k** $3x - 2y = -4$ **l** $x + 3y + 6 = 0$

3 Consider the linear relationship $y = 3x$.

a Compared to the gradient–intercept form, $y = mx + c$, identify the value of:

i m **ii** c.

b Sketch the graph of $y = 3x$ using the gradient–intercept method.

4 Sketch each of the following linear relationships using the gradient–intercept method.

a $y = 2x$ **b** $y = -5x$ **c** $y = -x$ **d** $y = \frac{2}{5}x$

5 Consider the linear relationship $y = 4$.

a Compared to the gradient–intercept form, $y = mx + c$, identify the value of:

i m **ii** c.

b Sketch the graph of $y = 4$ using the gradient–intercept method.

6 Sketch each of these linear relationships using the gradient–intercept method.

a $y = 6$ **b** $y = -2$ **c** $y = \frac{1}{3}$ **d** $y = -\frac{7}{2}$

WE 3.3B **7** Sketch the graph of each of these linear relationships using the x-and-y-intercept method.

a $x + 2y = 2$ **b** $3x + y = -6$ **c** $2x + 4y = 8$ **d** $5x + 10y = 20$

e $4x + 3y = -12$ **f** $-6x + 2y = 6$ **g** $8x - y = -8$ **h** $-x + 4y = 2$

i $9x - 3y = -18$ **j** $-12x + 9y = 36$ **k** $7x + 4y = 21$ **l** $2x - y = 5$

8 For each of the linear relationships, **i** find the x-intercept, **ii** find the y-intercept and **iii** sketch the graph.

a $y = x + 5$ **b** $y = x - 4$ **c** $y = 3x + 6$

d $y = 2x - 8$ **e** $y = 2 - x$ **f** $y = -6x - 3$

9 Sketch the graph of each of these using the x-and-y-intercept method.

a $2y = 3x + 6$ **b** $4y = 4x - 8$ **c** $x + 3y - 3 = 0$

d $3x + 6y + 18 = 0$ **e** $5x - 2y - 10 = 0$ **f** $2x - y - 7 = 0$

10 Consider the linear equation $y = 2x$.

a Find:

i the x-intercept **ii** the y-intercept.

b Do you have enough information from part **a** to sketch the graph? Explain.

c What other information do you need? Discuss this with a classmate.

d Another way of identifying a second point to plot is to find the matching y-value for a chosen x-value. Any x-value can be used, but we often choose $x = 1$.

i Find the value of y in $y = 2x$ when x is 1. Hint: Substitute $x = 1$ into the equation.

ii List the coordinates of the two points you can now use to sketch the graph of $y = 2x$.

iii Plot the two points and rule a straight line through them to produce the graph of $y = 2x$. Label your graph with its equation.

11 Use the approach used in question **10** to sketch the graphs for the following linear equations.

a $y = 4x$ **b** $y = -3x$ **c** $y = x$ **d** $y = -\frac{1}{2}x$

WE 3.3C **12** For each of the linear equations, **i** list the coordinates of any intercepts and **ii** sketch the graph.

a $y = 8$ **b** $x = 3$ **c** $y = -5$ **d** $x = -1$

e $x = 7$ **f** $y = 0$ **g** $x = 0$ **h** $x = -4.5$

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13 How many axis intercepts does the graph of each of these linear equations have? List them.

a $y = 2x - 4$ **b** $y = -7$ **c** $4x + 6y = 12$

d $y = 9x$ **e** $y = 5(x - 10)$ **f** $3x - 5y + 15 = 0$

g $x = 28$ **h** $x = 3y + 6$ **i** $y = 5.6$

Exercise 3.3B: Problem solving and reasoning

14 Consider the linear relationship $x = py + q$.

a Write this equation in the form:

i $y = a(x - b)$ **ii** $y = mx + c$.

b What do the values of p and q correspond with on the graph of $x = py + q$?

c Write $x = 3$ in the form $x = py + q$.

d From your answer to part **c**, identify the values of:

i p **ii** q.

e Write your answer from part **c** in the forms $y = a(x - b)$ and $x = py + q$.

f Use your answers from the previous parts to explain why the gradient of the line $x = 3$ (and in general, all lines of the form $x = q$) is undefined.

15 Describe which method is best for sketching the following graphs.

a linear graphs with equations in the general form $y = mx + c$ (for example, $y = 3x + 2$)

b linear graphs with equations in the general form $ax + by = d$ (for example, $4x + 2y = 8$)

c linear graphs that pass through the origin (for example, $y = 3x$)

d linear graphs that are horizontal lines (for example, $y = 7$)

e linear graphs that are vertical lines (for example, $x = -2$)

16 Use the most appropriate method to sketch the following equations.

a $5x - 2y = -10$ **b** $y = 2x + 4$

c $x + y = 5$ **d** $y = 2x$

e $y = 5$ **f** $y = 8 - 4x$

g $x = -9$ **h** $y = 3(x - 2)$

i $2x + 4y - 8 = 0$ **j** $y = -3x$

k $y = -2x + 7$ **l** $y = 3 - 6x$

17 Sketch the following lines for the given x- or y-values. Label any endpoints with their coordinates.

a $y = 5x$, for $x \geq 0$ **b** $y = 2x - 4$, for $x \leq 3$

c $y = 7$, for $-2 \leq x \leq 4$ **d** $x = -3$, for $-1 \leq y \leq 1$

e $2x + 3y = 12$, for $0 \leq x \leq 6$ **f** $y = x$, for $x > 2$

g $y = 3 - x$, for $x < 3$ **h** $y = -3$, for $-1 < x < 4$

i $x - 4y = 2$, for $-2 \leq x < 6$ **j** $x = 6$, for $y < 3$

18 For a line segment between the points (x_1, y_1) and (x_2, y_2), the length of the line segment is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

a Sketch each of these linear relationships on the same Cartesian plane.

i $x = -6$, for $3 \leq y \leq 13$ **ii** $y = 3$, for $-6 \leq x \leq 2$

iii $y = -\frac{5}{4}x + \frac{11}{2}$, for $-6 \leq x \leq 2$

b Calculate the perimeter of the shape formed, correct to two decimal places.

c Calculate the area of the shape formed.

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19 A racing car accelerates from a constant speed of 20 m/s. Its speed, s, after t seconds of accelerating can be calculated using the formula $s = 20 + 5t$.

a Draw the graph of this linear relationship for $0 \le t \le 10$ (that is, t values from 0 to 10 seconds).

b What does the s-intercept represent on this graph?

c What does the gradient represent on this graph?

d From the graph, what is the speed of the racing car after:

i 3 seconds

ii 5 seconds

iii 7.5 seconds?

e Convert each speed in part **d** to kilometres per hour.

f From the graph, how long does it take the racing car to reach speeds of:

i 30 m/s

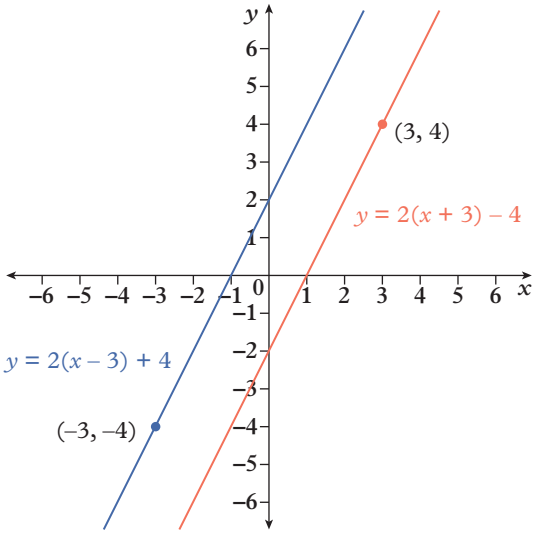
ii 55 m/s

iii 65 m/s?

g Using the graph, how long does it take the racing car to reach a speed of 216 km/h?

h Using the graph, what is the highest speed, in kilometres per hour, reached by the racing car? How long does it take to reach this speed?

20 The gradient–intercept form $y = mx + c$ is a specific case of the gradient–point form $y = m(x - h) + k$, where (h, k) is a point on the line. For example, $y = 2(x - 3) + 4$ passes through the point $(3, 4)$ and $y = 2(x + 3) - 4$ passes through the point $(-3, -4)$.



a Substitute $(h, k) = (0, c)$ and show that the equation $y = m(x - h) + k$ becomes $y = mx + c$.

b Sketch the following lines by plotting the point (h, k) and then use the gradient to plot another point in the same way you would by using the gradient–intercept method.

i $y = -3(x - 2) + 6$

ii $y = 3(x - 2) + 6$

iii $y = \frac{5}{2}(x + 3) - 4$

iv $y = -\frac{3}{4}(x + 7) + 10$

c Determine the coordinates for the x- and y-intercepts of $y = m(x - h) + k$ in terms of m, h and k.

d Once again, consider the gradient–point form equation $y = m(x - h) + k$.

i Substitute $(h, k) = (b, 0)$ to determine a new general form for writing a linear relationship in terms of the x-intercept.

ii Explain what you can do to the equation $y = mx + c$ to write it in this form.

iii Explain how you can use this form to determine the x-intercept.

e Use your method from part **d** to write the following equations in the new general form and determine their x-intercepts.

i $y = 3x - 12$

ii $y = 2x - 5$

iii $y = -3x - 12$

iv $y = 2x + 5$

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Exercise 3.3C: Challenge

21 The edges of shapes on the Cartesian plane can be defined by line segments.

a Sketch these linear relationships on the same Cartesian plane:

- $2x + 4y = 8$, for $4 \le x \le 8$
- $x = -2$, for $-2 \le y \le 2$
- $y = 0$, for $0 \le x \le 4$.
- $y = -2$, for $-2 \le x \le 8$
- $y = -x$, for $-2 \le x \le 0$

b Use Pythagoras' theorem to calculate the perimeter of the shape formed by the graph lines.

c Calculate the area of the shape formed.

22 The edges of shapes on the Cartesian plane can be defined by line segments.

a Sketch these linear relationships on the same Cartesian plane.

- $y = -4x + 10$, for $0 \le x \le 2$
- $y = 4x + 10$, for $-2 \le x \le 0$
- $x - 4y = -10$, for $-10 \le x \le -2$
- $4y = x - 10$, for $2 \le x \le 10$
- $y = 4x - 10$, for $0 \le x \le 2$
- $y = -4x - 10$, for $-2 \le x \le 0$
- $x + 4y = -10$, for $-10 \le x \le -2$
- $x + 4y = 10$, for $2 \le x \le 10$

b Calculate the perimeter of the shape formed by the graph lines.

c Calculate the area of the shape formed.

d Join the points $(-2, -2)$ and $(2, 2)$ to form a line segment. Write a rule to represent the line segment.

e Join the points $(-2, 2)$ and $(2, -2)$ to form a line segment. Write a rule to represent the line segment.

23 When an equation is written in the form $ax + by = c$, we can determine both of the intercepts simultaneously.

a Determine the coordinates of the x- and y-intercepts of $ax + by = c$ in terms of a, b and c.

b Divide both sides of the equation $ax + by = c$ by c.

c Explain the relationship between the coefficients of x and y in your answer for part **b** and the coordinates in part **a**.

d Use the method in part **b** to help you determine the intercepts of the following lines.

i $2x + 3y = 5$

ii $4x - 7y = 11$

iii $2x - 3y = 12$

iv $9y - 2x = 36$

v $8x - 4y = -32$

vi $20x + 50y = 10$

24 We can compare graphs by sketching them on the same Cartesian plane.

a Sketch each pair of graphs on the same Cartesian plane.

i $2x + 3y = 12$ and $2x - 3y = 12$

ii $2x + 3y = 12$ and $\frac{1}{2}x - \frac{1}{3}y = 12$

iii $2x + 3y = 12$ and $2x - 3y = \frac{1}{12}$

iv $5x + 2y = 20$ and $2x - 5y = 20$

v $5x + 2y = 20$ and $-2x + 5y = 20$

vi $5x + 2y = 20$ and $2x + 5y = -20$

b Determine which pairs of graphs are perpendicular.

c Write the equation of a line perpendicular to $ax + by = 1$ in the same form and in terms of a and b.

Online resources:

- **Interactive skillsheet** Sketching linear relationships using intercepts
- **Interactive skillsheet** Sketching linear relationships using the gradient–intercept method
- **Worksheet** Linear graphs
- **CAS instructions** Graphing functions
- **Quick quiz** 3.3

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Lesson 3.4

Determining linear equations

Learning intentions

By the end of this lesson you will be able to ...

✔ determine the equation of a linear relationship from a graph, the gradient and a point, or two points.

Inter-year links

Support	The Cartesian plane
Year 7	5.4 The Cartesian plane
Year 8	6.8 Finding linear equations
Year 9	3.6 Determining linear equations

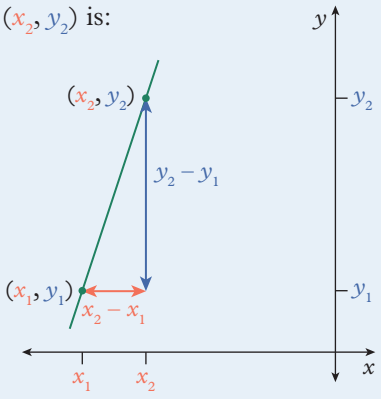
Determining the gradient

- The formula for the gradient, m, between any two points (x_1, y_1) and (x_2, y_2) is:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

For example, the gradient of a line that passes through $(2, -1)$ and $(1, 3)$ can be calculated as:

$$m = \frac{3 - (-1)}{1 - 2}$$
$$= \frac{4}{-1}$$
$$= -4$$



Determining linear equations

Key content video
Determining linear equations

- To determine the equation of a line in the form $y = mx + c$:
  1. determine the gradient, m
  2. identify the coordinates of a point on the graph
  3. substitute the gradient and the coordinates into the formula $y = mx + c$ to determine c.
- If the y-coordinate of the y-intercept can be read from the graph, then steps **2–3** are not necessary; the value of c is already given.

Worked example 3.4A Determining the gradient

Determine the gradient of a line that passes through $(3, -8)$ and $(-7, -10)$.

THINK

1. Substitute the given x- and y-coordinates into the gradient formula.
2. Calculate and simplify the gradient.

WRITE

Let $(x_1, y_1) = (3, -8)$ and $(x_2, y_2) = (-7, -10)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{(-10) - (-8)}{(-7) - 3}$$
$$= \frac{-10 + 8}{-7 - 3}$$
$$= \frac{-2}{-10}$$
$$= \frac{1}{5}$$

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Worked example 3.4B Determining the equation given the gradient and a point

Determine the equation for a line with a gradient of 4 that passes through (–2, 7).

THINK

1 Substitute the values for the gradient and the coordinates into the gradient–intercept equation: $y = mx + c$.

2 Solve the equation for c and write the final equation in gradient–intercept form.

WRITE

From the question:

$m = 4 \qquad x = -2 \qquad y = 7$

$y = mx + c$

$7 = 4 \times (-2) + c$

$7 = -8 + c \qquad (+8)$

$c = 15$

So, $y = 4x + 15$.

Worked example 3.4C Determining the equation given two points

Determine the equations for the lines that pass through these given pairs of points.

a (2, 6) and (–4, 3) **b** (0, 2) and (3, –7)

THINK

a 1 Substitute the x- and y-coordinates into the gradient formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$.

2 Substitute the values for the gradient and one of the sets of coordinates into the gradient–intercept equation: $y = mx + c$.

3 Solve for c and write the final equation in gradient–intercept form.

b 1 Substitute the x- and y-coordinates into the gradient formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$.

2 Check if any of the coordinates given are on the x- or y-axis. If one of the coordinates has an x value of 0, the point is a y-intercept, and the corresponding y value is the value of c in $y = mx + c$.

3 Write the final equation in gradient–intercept form.

WRITE

a Let $(x_1, y_1) = (2, 6)$ and $(x_2, y_2) = (-4, 3)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 6}{-4 - 2} = \frac{-3}{-6} = \frac{1}{2}$$

Use: $x = 2 \qquad y = 6$

$y = mx + c$

$6 = \frac{1}{2} \times 2 + c$

$6 = 1 + c \qquad (-1)$

$c = 5$

So, $y = \frac{1}{2}x + 5$.

b Let $(x_1, y_1) = (0, 2)$ and $(x_2, y_2) = (3, -7)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-7 - 2}{3 - 0} = \frac{-9}{3} = -3$$

(0, 2) is the y-intercept.

So, $c = 2$.

$y = -3x + 2$.

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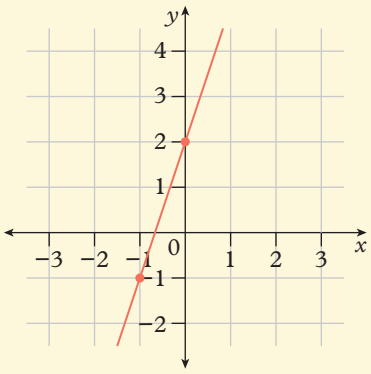
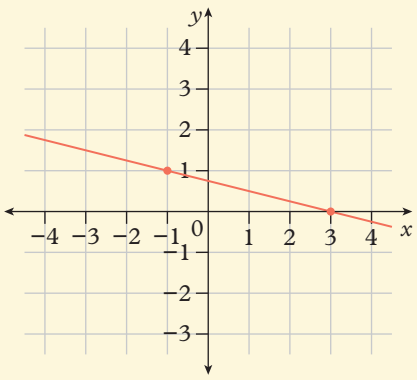
Worked example 3.4D Determining the equation given the graph

Determine the equation of each of these linear graphs.

a



b



THINK

a **1** Identify two coordinates on the graph.

2 Determine the value of the gradient, m, using the chosen coordinates.

3 Determine the value of c by identifying the y-coordinate of the y-intercept.

4 Write the equation of the linear graph by substituting m and c into the equation.

b **1** Identify two integer (whole number) coordinates on the graph.

2 Determine the value of the gradient, m, using the chosen coordinates.

3 Check if the value of c can be read from the graph. The y-coordinate of the y-intercept is unclear on the graph, so choose a clearer coordinate to substitute into the gradient–intercept equation: $y = mx + c$.

4 Solve the equation for c and write the final equation in gradient–intercept form.

WRITE

a Let $(x_1, y_1) = (-1, -1)$ and $(x_2, y_2) = (0, 2)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{2 - (-1)}{0 - (-1)}$$
$$= 3$$
$$c = 2$$

$$y = mx + c$$
$$y = 3x + 2$$

b Let $(x_1, y_1) = (-1, 1)$ and $(x_2, y_2) = (3, 0)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{0 - 1}{3 - (-1)}$$
$$= -\frac{1}{4}$$

Use: $x = 3 \quad y = 0$

$$y = mx + c$$
$$0 = -\frac{1}{4} \times 3 + c$$
$$0 = -\frac{3}{4} + c \qquad \left(+\frac{3}{4}\right)$$
$$c = \frac{3}{4}$$
$$y = -\frac{1}{4}x + \frac{3}{4}$$

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Helpful hints

✔ The order in which points are substituted into the gradient formula does not matter. Just always make sure that the coordinates of a point match up vertically in the equation.

point 2 point 1 point 1 point 2

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{or} \quad m = \frac{y_1 - y_2}{x_1 - x_2}$$

✔ Remember, if you know the y-intercept, $(0, c)$, all you need to do is determine the gradient, m, and substitute both values into the gradient–intercept form of the equation: $y = mx + c$.

Learning pathways

1–3, 4–5(a–e), 6, 7, 8(1st column), 9, 10, 15, 17	1, 3, 4–5(e–i), 8(2nd column), 11–13, 15–17	3, 4(a, d, g, j), 5(e–i), 8(2nd column), 14–16, 18, 19

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Exercise 3.4A: Understanding and fluency

WE 3.4A **1** Determine the gradient of the line passing through each of these pairs of points.

a (2, 3) and (3, 7)
b (5, 4) and (10, –6)
c (4, 2) and (6, 3)
d (0, 5) and (4, 1)
e (–3, –2) and (–1, 8)
f (–1, –1) and (2, –10)

2 For each line passing through the given pair of points, determine the **i** gradient and **ii** equation of the line by substituting the correct values into $y = mx + c$.

a (0, 2) and (2, 10)
b (0, –7) and (3, –4)
c (0, 5) and (4, 7)
d (–1, –4) and (0, 6)
e (–2, 1) and (0, –5)
f (–8, 6) and (0, 4)

3 Use the x- and y-intercepts given below to determine the equation of the linear graph on which they appear.

a x-intercept: (2, 0), y-intercept: (0, 4)
b x-intercept: (–3, 0), y-intercept: (0, 3)
c x-intercept: (3, 0), y-intercept: (0, –6)
d x-intercept: (1, 0), y-intercept: (0, 5)
e x-intercept: (5, 0), no y-intercept
f no x-intercept, y-intercept: (0, –2)

WE 3.4B **4** Use the given information to determine the equation for each of these lines.

a gradient of 3; passes through (1, 4)
b gradient of 2; passes through (3, 9)
c gradient of –4; passes through (–1, 2)
d gradient of –1; passes through (2, 7)
e gradient of 5; passes through (–3, –6)
f gradient of –3; passes through (–4, 11)
g gradient of $\frac{6}{5}$; passes through (15, 4)
h gradient of $-\frac{3}{4}$; passes through (8, –5)
i gradient of $\frac{5}{2}$; passes through (–3, –6)
j gradient of $-\frac{7}{9}$; passes through (–6, 4)

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WE 3.4C **5** Determine the equation for the line that passes through the given pair of points.

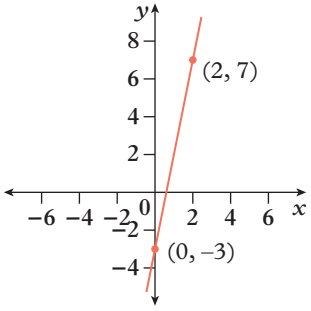
a (1, 3) and (3, 7) **b** (5, 2) and (6, 8) **c** (2, 4) and (4, –2)

d (3, –2) and (6, –8) **e** (–2, –3) and (2, 1) **f** (–4, –2) and (–1, –5)

g (1, –3) and (4, 4) **h** (–7, 5) and (–2, –2) **i** (–5, 3) and (1, 4)

6 Use $y = mx + c$ to help determine the equation of the line passing through the points shown.

a



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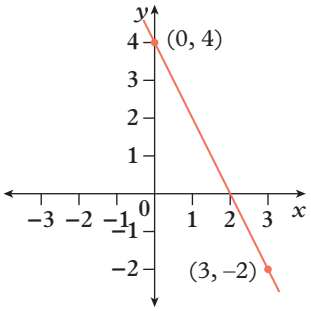

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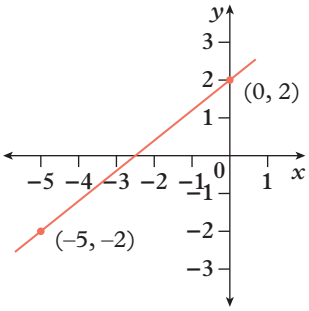
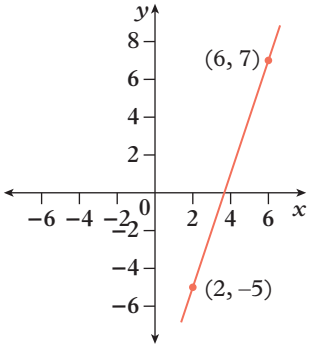


7 Determine the equation of the line passing through the points shown.

a



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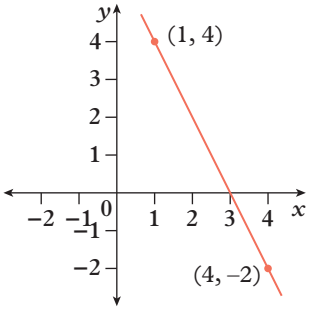

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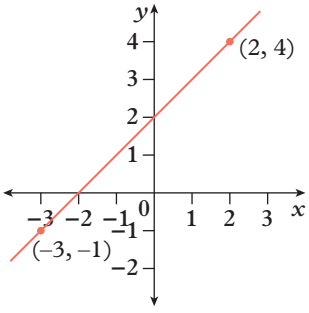
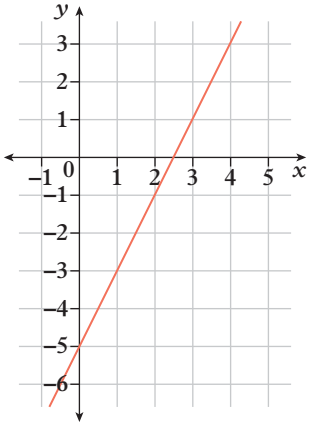


WE 3.4D **8** Determine the equation of each of these linear graphs.

a



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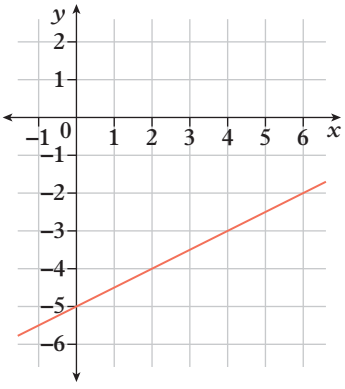

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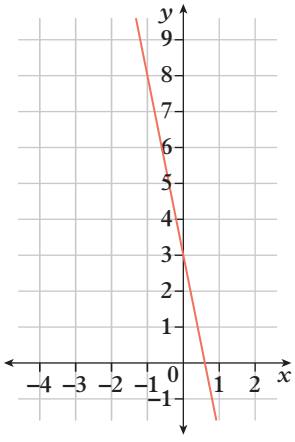

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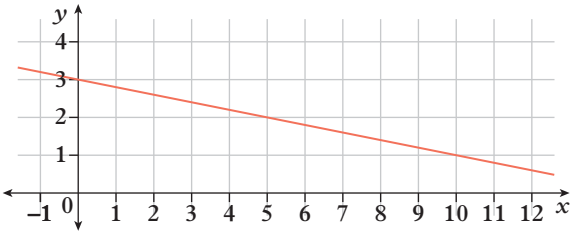

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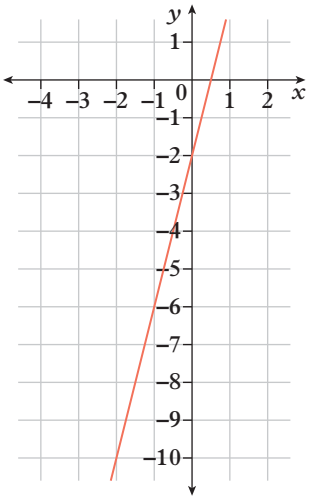

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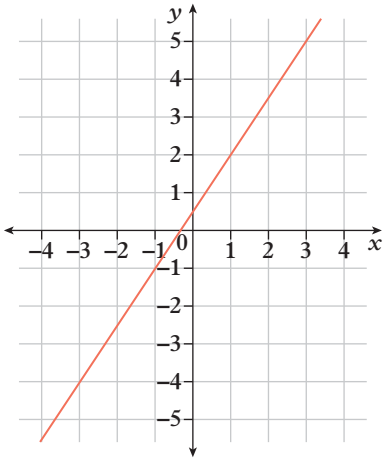

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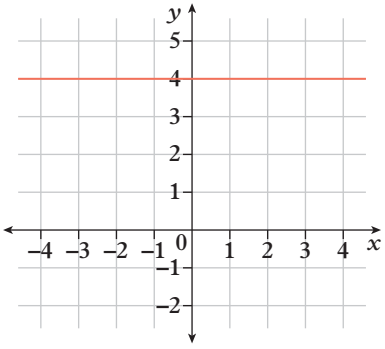

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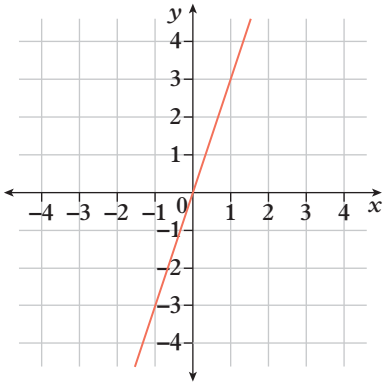

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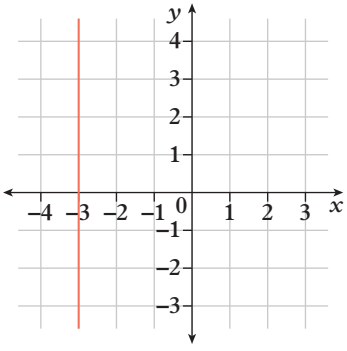

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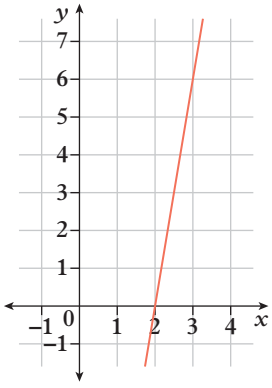

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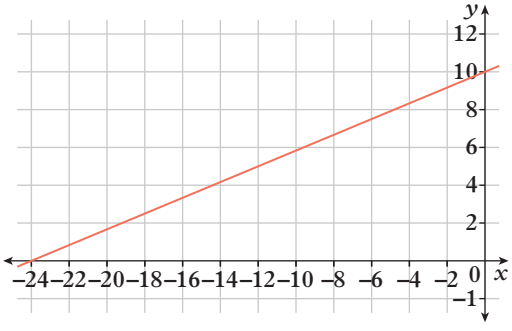

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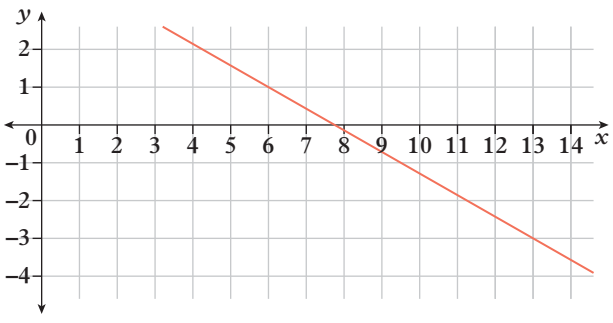

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Exercise 3.4B: Problem solving and reasoning

9 Liza and Tate are finding the gradient of a line that passes through (–2, 2) and (4, 5). Liza writes $m = \frac{5-2}{4-(-2)}$, while Tate writes $m = \frac{2-5}{-2-4}$.

a Show that Liza and Tate both find the gradient to be $m = \frac{1}{2}$.

b Juan attempts to find the gradient of the same line and writes $m = \frac{5-2}{-2-4}$. Explain what Juan has done wrong and how his answer will compare to the correct gradient found by Liza and Tate.

c Show that $m = \frac{y_1 - y_2}{x_1 - x_2}$ is equivalent to $m = \frac{y_2 - y_1}{x_2 - x_1}$.

10 Liza and Tate are now finding the equation of the line that passes through (–2, 2) and (4, 5). To find the y-intercept, c, Liza writes the equation $2 = \frac{1}{2}(-2) + c$ and Tate writes $5 = \frac{1}{2}(4) + c$.

a Show that Liza and Tate will get the same value for c.

b Juan attempts to find the equation of the same line and says it is $y = 4x + 5$. Explain what Juan has done wrong.

11 Consider the equation $y = mx + c$.

a For a line passing through the point (h, k), the coordinates can be substituted into the equation $y = mx + c$ to produce $k = mh + c$. Rearrange this equation to make c the subject.

b Substitute the equation you wrote for c into $y = mx + c$.

c Rearrange your answer to part **b** and show that the equation can be expressed in gradient–point form: $y = m(x - h) + k$.

12 Determine the equations of the lines with the features described in **a**–**h** below, expressed in the gradient–point form $y = m(x - h) + k$ derived in question **11**, (where m is the gradient and (h, k) is a point on the line).

a gradient of $\frac{3}{4}$, passes through (8, 2)

b gradient of $-\frac{7}{3}$, passes through (15, –6)

c (–3, 5) and (1, 6) lie on the line

d (–6, 5) and (–1, –3) lie on the line

e (–4, 2) and (2, –1) lie on the line

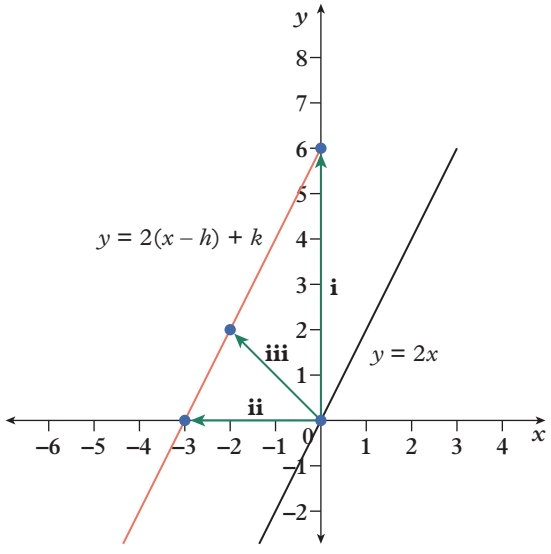
f (–7, 3) and (4, 3) lie on the line

g x-intercept = 4, y-intercept = 3

h x-intercept = –5, y-intercept = 2

13 Use your answers in question **12** to express the equations of the lines in the gradient–intercept form, $y = mx + c$.

14 Consider the linear graphs $y = 2x$ and $y = 2(x - h) + k$ shown on the right. The line $y = 2(x - h) + k$ can be considered as a translation of the line $y = 2x$. Three possible translations are shown as arrows on the diagram.



a State the translation each arrow describes.

b Write the equation of the translated line in the form $y = 2(x - h) + k$ using the three points at the end of the arrows.

c For each of the following, describe the translation required for the graph of the first equation to transform into the graph of the second equation.

i $y = 3x \rightarrow y = 3(x + 1) - 2$

ii $y = 3x \rightarrow y = 3(x - 1) + 2$

iii $y = -3x \rightarrow y = -3(x - 1) + 2$

iv $y = \frac{5}{2}x \rightarrow y = \frac{5}{2}(x + 4) + 6$

v $y = \frac{2}{5}x \rightarrow y = \frac{2}{5}\left(x - \frac{3}{2}\right) + \frac{4}{3}$

vi $y = 3x + 6 \rightarrow y = 3(x - 4) + 8$

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15 Write an equation for a linear graph to represent each of these situations, where y represents the altitude and x represents the horizontal distance from the point of origin.

a From an altitude of 50 m, a road rises 1 m vertically over a horizontal distance of 200 m.

b From an altitude of 6 km, a plane descends 2 km vertically over a horizontal distance of 30 km.

c A road rises 20 m vertically over a horizontal distance of 100 m.

16 A helicopter rises at a constant speed from a height of 20 m above the ground, covering a vertical distance of 30 m as it flies over a horizontal distance of 100 m. For the next 100 m it travels horizontally. It then descends at a constant speed over a further horizontal distance of 100 m before landing.

a Draw a graph to represent the changing height above the ground of the helicopter over the total 300 m horizontal distance it travelled.

b Identify the rule for each of the three linear sections of the graph drawn in part **a**.

c Use the appropriate rule to calculate the height of the helicopter for each of these horizontal distances on your graph.

i 30 m **ii** 80 m

iii 150 m **iv** 240 m

d What horizontal distance(s) have been travelled when the helicopter is at a height of 40 m?

Exercise 3.4C: Challenge

17 Determine the equation for each linear graph described by the information below. Write your answers without fractions.

a x-intercept of 4, passes through (20, 8)

b gradient of 0, passes through (–15, –23)

c passes through (–25, 10) and (11, –14)

d passes through the origin and (–17, –12)

18 Sketch the linear graph for each situation in question **15**. What values of x would be appropriate to use? Rewrite each equation you wrote in your answers to question **15** with the appropriate restriction of x-values written as an inequality statement.

19 The x-intercept of $ax + by = 1$ is given by the reciprocal of a and the y-intercept is given by the reciprocal of b. Use this fact to:

i write the equations of the following lines in the form $ax + by = 1$, where a and b are real numbers

ii write the equations in the form $px + qy = r$, where p, q and r are integers.

a x-intercept $= \frac{2}{3}$, y-intercept $= \frac{2}{7}$

b x-intercept $= -\frac{3}{5}$, y-intercept $= \frac{3}{2}$

c x-intercept $= \frac{3}{5}$, y-intercept $= -\frac{2}{5}$

d x-intercept $= -\frac{1}{2}$, y-intercept $= -\frac{1}{3}$

e x-intercept $= 7$, y-intercept $= 11$

f x-intercept $= 1$, y-intercept $= -1$

Online resources:

Interactive skillsheet Gradient

Interactive skillsheet Determining linear equations

Quick quiz 3.4

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Checkpoint

Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this module.
✔ 10
✔ 10A

3.1 **1** Solve the following equations.

a $\frac{x}{3} - 5 = 2$ **b** $8(x + 6) = -24$

c $\frac{5 - 2x}{4} = 9$ **d** $4(3x - 7) - 5x = -31$

3.1 **2** Solve the following equations.

a $6x - 10 = 5x + 9$ **b** $5(3x - 2) = x$

c $9(4 - 7x) = 3(9x + 11)$ **d** $\frac{4x - 9}{3} = \frac{2x + 7}{4}$

3.1 **3** Given the equation $y = kx + a$, solve for x when:

a $y = 0, k = -2, a = 6$ **b** $y = 1, k = 2, a = 2$

10A 3.2 **4** Rearrange each of the following equations to make x the subject.

a $ax + by = c$ **b** $\frac{f}{dx - e} = g$

c $j(x + k) = m(x - n)$ **d** $\frac{px - q}{s - tx} = r$

10A 3.2 **5** Solve the following equations.

a $\frac{10}{x} = 4$ **b** $\frac{4}{x} = 12$

c $\frac{-18}{x + 5} = 9$ **d** $\frac{14}{7x - 9} = 3$

3.3 **6** Sketch the graph of each of the following.

a $y = 3x - 1$ **b** $y = \frac{1}{4}x + 2$

c $y = -\frac{2}{5}x + 7$ **d** $y = \frac{7}{6}x$

3.3 **7** Sketch the graph of each of the following.

a $2x - 3y = 12$ **b** $4x + 5y = -100$

c $18x + 36y = 9$ **d** $14y - 15x = 24$

3.4 **8** Sketch the graph of each of the following.

a $x = 10$ **b** $y = -4$ **c** $x = -\frac{2}{3}$ **d** $y = \frac{3}{2}$

3.4 **9** Determine the gradient of a line that passes through:

a (3, 8) and (7, 10) **b** (13, 2) and (–7, –2)

c (5, –8) and (–3, 10) **d** (–5, 12) and (3, –2).

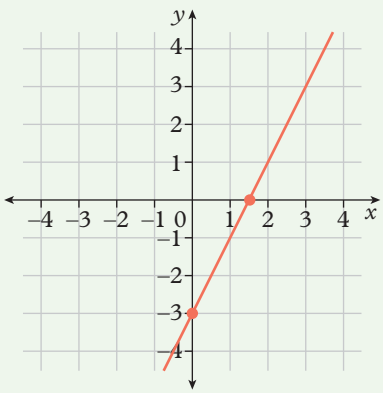
3.4 **10** Determine the equation of the lines that pass through the points in question **9**.

3.4 **11** Determine the equation of a line with the following gradients that passes through (–2, 7).

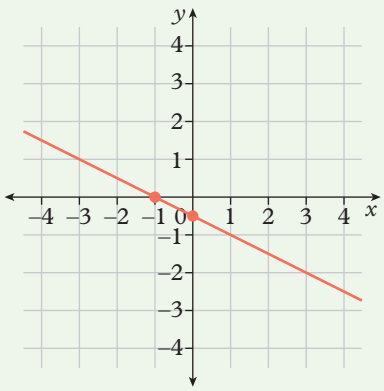
a 2 **b** $\frac{1}{2}$ **c** 1 **d** –2 **e** $-\frac{1}{2}$

3.4 **12** Determine the equations of the following linear graphs.

a



b



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Lesson 3.5

Parallel and perpendicular lines

Learning intentions

By the end of this lesson you will be able to ...

✔ use the gradients of parallel or perpendicular lines to solve problems.

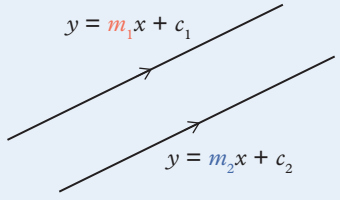
Inter-year links

Year 8	6.8 Finding linear equations
Year 9	3.4 Gradients and intercepts

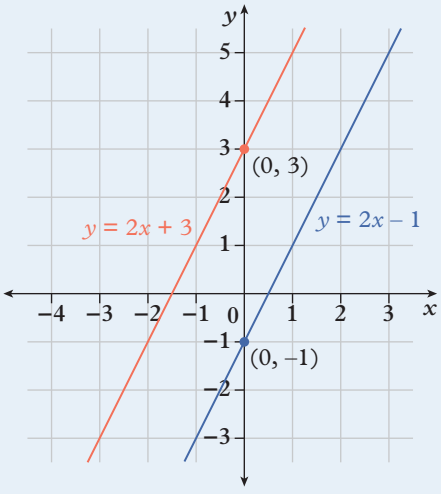
Key content video
Parallel and perpendicular lines

Parallel lines

- **Parallel lines** are two or more straight lines that never meet.
- Two lines are parallel if they have the same gradient.
  If the equations for two lines are $y = m_1x + c_1$ and $y = m_2x + c_2$, then the lines are parallel if $m_1 = m_2$.
  For example, the lines $y = 2x + 3$ and $y = 2x - 1$ shown on the right are parallel.

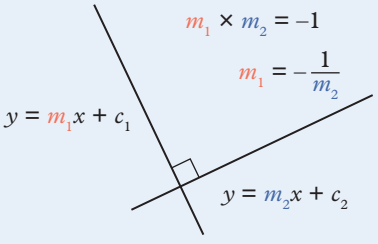


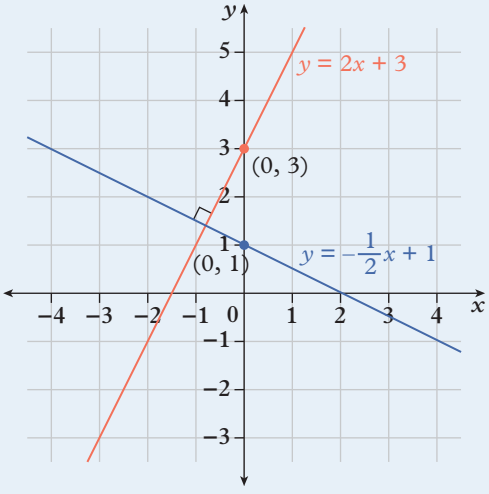
$m_1 = m_2$



Perpendicular lines

- **Perpendicular lines** are straight lines that meet at right angles.
- Two lines are perpendicular if the product of their gradients is –1, or expressed another way, if one gradient is the negative reciprocal of the other.
  If the equations of two lines are $y = m_1x + c_1$ and $y = m_2x + c_2$, then the lines are perpendicular if $m_1 \times m_2 = -1$, or expressed another way, if $m_2 = -\frac{1}{m_1}$.
  For example, the lines $y = 2x + 3$ and $y = -\frac{1}{2}x + 1$ are perpendicular because $2 \times \left(-\frac{1}{2}\right) = -1$.





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Equations of parallel and perpendicular lines

- Follow these steps to determine the equation of a parallel or perpendicular line in the form $y = mx + c$.
  1. Determine the gradient of the given line, m_2.
  2. Determine the gradient of the new line, m_1.
     - → If lines are parallel, then their gradients are equal: $m_1 = m_2$.
     - → If lines are perpendicular, then their gradients are the negative reciprocal of each other: $m_2 = -\frac{1}{m_1}$.
  3. Identify the x- and y-coordinates of a point on the new line.
  4. Substitute into the formula $y = mx + c$ to determine c.

Worked example 3.5A Determining whether two lines are parallel or perpendicular

For each pair of equations, decide whether the lines are parallel, perpendicular or neither. Give reasons.

a $y - 2 = 5x$
$y - 5x = 9$

b $1 - 4y = 2x + 3$
$1 + 4x = 2y$

c $y = 3x - 2$
$3x + y = -2$

THINK

a 1 Rearrange $y - 2 = 5x$ to make y the subject and identify the gradient.

2 Rearrange $y - 5x = 9$ to make y the subject and identify the gradient.

3 State whether the lines are parallel, perpendicular or neither by comparing the two gradients.

b 1 Rearrange $1 - 4y = 2x + 3$ to make y the subject and identify the gradient.

2 Rearrange $1 + 4x = 2y$ to make y the subject and identify the gradient.

3 State whether the lines are parallel, perpendicular or neither by comparing the two gradients.

c 1 Identify the gradient of $y = 3x - 2$ by looking at the coefficient of x.

2 Rearrange $3x + y = -2$ to make y the subject and identify the gradient.

3 State whether the lines are parallel, perpendicular or neither by comparing the two gradients.

WRITE

a $y - 2 = 5x$
$y = 5x + 2$
So $m_1 = 5$.
$y - 5x = 9$
$y = 5x + 9$
So $m_2 = 5$.
Since $m_1 = m_2$, the lines have the same gradient and are parallel.

b $1 - 4y = 2x + 3$
$-4y = 2x + 2$
$y = -\frac{1}{2}x - \frac{1}{2}$
So $m_1 = -\frac{1}{2}$.
$1 + 4x = 2y$
$2y = 1 + 4x$
$y = 2x + \frac{1}{2}$
So $m_2 = 2$.
Since $m_1 \times m_2 = -1$, the lines are perpendicular.

c The gradient of $y = 3x - 2$ is 3.
So $m_1 = 3$.

$3x + y = -2$
$y = -3x - 2$
So $m_2 = -3$.
The lines are not parallel, since $m_1 \neq m_2$. The lines are also not perpendicular, since $m_1 \times m_2 \neq -1$. Therefore, the lines are neither parallel nor perpendicular.

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Worked example 3.5B Determining parallel lines

Determine the equation for a line that is parallel to the graph of $y = -7x - 5$ and passes through $(2, 6)$.

THINK	WRITE
1 Identify the gradient of the graph of $y = -7x - 5$ and use this to define the gradient of the parallel line. Parallel lines have equal gradients.	The gradient of $y = -7x - 5$ is -7.
2 Substitute the values for the gradient and the coordinates into the gradient–intercept equation: $y = mx + c$.	For the parallel line, use $m = -7$, with $x = 2$ and $y = 6$. $y = mx + c$ $6 = -7 \times 2 + c$
3 Solve the equation for c and write the final equation in gradient–intercept form.	$6 = -14 + c$ $c = 20$ The equation is $y = -7x + 20$.

Worked example 3.5C Determining perpendicular lines

Determine the equation for a line that is perpendicular to the graph of $y = \frac{2}{3}x + 11$ and passes through $(10, -8)$.

THINK	WRITE
1 Identify the gradient for the graph of $y = \frac{2}{3}x + 11$ and use this to define a gradient for the perpendicular line. Perpendicular lines have gradients that are the negative reciprocal of each other, $m_2 = -\frac{1}{m_1}$.	The gradient of $y = \frac{2}{3}x + 11$ is $\frac{2}{3}$. So $m_1 = \frac{2}{3}$. $m_2 = -\frac{1}{m_1}$ $m_2 = \frac{-1}{\left(\frac{2}{3}\right)}$ $m_2 = -\frac{3}{2}$
2 Substitute the values for the gradient and the coordinates into the gradient–intercept equation: $y = mx + c$.	Use $x = 10$ and $y = -8$. $y = mx + c$ $-8 = -\frac{3}{2} \times 10 + c$
3 Solve the equation for c and write the final equation in gradient–intercept form.	$-8 = -15 + c$ $c = 7$ The equation is $y = -\frac{3}{2}x + 7$.

Helpful hints

- ✔ If y is already the subject of the equation, you do not need to rearrange the equation to the form $y = mx + c$ to identify the gradient, m. The coefficient of x is the gradient.
  For example, the gradient of $y = 1 - 4x$ is -4 and the gradient of $y = \frac{1 + 2x}{3}$ is $\frac{2}{3}$.
- ✔ Remember, if you know the y-intercept, $(0, c)$, all you need to do is determine the gradient, m, and substitute both values into the gradient–intercept equation $y = mx + c$.

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Learning pathways

1–3, 4(a–e), 5, 6(a, c, e, g), 7, 8(a, c, e, g), 9–11, 13, 16	4(e–i), 5, 6(a, c, e, g), 7, 8(a, c, e, g), 9, 12, 17, 19, 21	5, 6(b, d, f, h), 7, 8(b, d, f, h), 9, 14, 15, 18, 20, 22–26

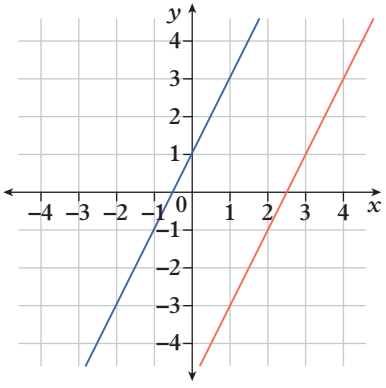
ANS p697

Exercise 3.5A: Understanding and fluency

1 Consider the following pairs of lines on the Cartesian plane.

a Calculate the gradient for each line.

i



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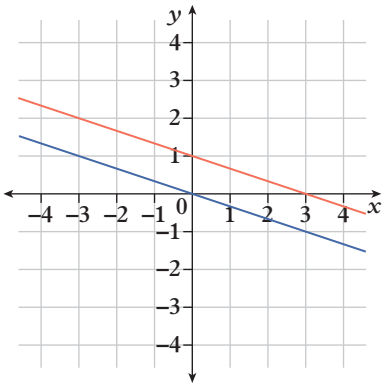


b What do you notice about both pairs of lines?

c What can you say about the gradients of parallel lines?

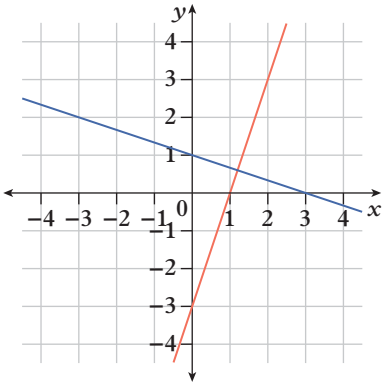
d Decide whether each equation below describes a linear graph parallel to those in part **a i**.

i $y = 2x - 9$ **ii** $y = 5 + 2x$ **iii** $y = 3x + 2$

2 Consider the following pairs of lines on the Cartesian plane.

a Calculate the gradient for each line.

i



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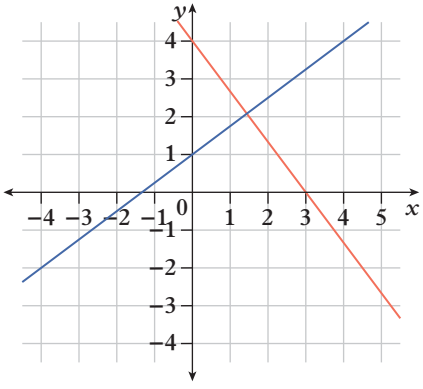

b Find the product of the gradients for each pair of lines in part **a**.

c What do you notice about both pairs of lines?

d What can you say about the product of the gradients of perpendicular lines?

e Write an equation for determining the gradient m_2 of a line that is perpendicular to another line with gradient m_1.

f Decide whether each of the following equations describes a linear graph that is perpendicular to those in part **a i**.

i $y = 3x - 4$ **ii** $y = \frac{1}{3}x + 4$ **iii** $y = 9 - \frac{1}{3}x$

3 If each of the numbers below is the gradient of a line, then find the gradient of corresponding perpendicular lines.

a 5 **b** -8 **c** -1 **d** $\frac{1}{6}$ **e** $\frac{4}{5}$ **f** $-\frac{3}{2}$

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4 For each linear equation below:

i rearrange the equation so that it is in gradient–intercept form: $y = mx + c$

ii write the gradient of a line parallel to the line described by the given equation

iii write the gradient of a line perpendicular to the line described by the given equation.

a $3x + y = 11$ **b** $-4x + y = 7$ **c** $-\frac{5}{3}x + y = \frac{2}{3}$

d $3x + 2y = 8$ **e** $5x - 2y = 4$ **f** $8x + 5y = -11$

g $9y - 3x = 15$ **h** $x = -5y - 3$ **i** $\frac{y-2}{x+5} = 1$

WE 3.5A **5** Decide whether each of these pairs of equations describe lines that are parallel, perpendicular or neither. Give a reason for your answer.

a $y = 4x - 3$ and $4x - y = 5$ **b** $3y - 2x = 12$ and $3x + 2y = 8$

c $y = \frac{3}{8}x$ and $3y = 4 - 8x$ **d** $2x + y = 3$ and $y = 2x + 6$

e $5x + y + 4 = 0$ and $x + 5y = 7$ **f** $2x - 7y = 11$ and $14y = 4x + 1$

g $8x - 12y = 7$ and $18y + 12x = -11$ **h** $6x + 3y = 7$ and $20y - 10x = 4$

WE 3.5B **6** Determine the equation for the line that is:

a parallel to the graph of $y = 4x + 3$ and passes through $(2, -5)$

b parallel to the graph of $y = -3x + 7$ and passes through $(-1, -3)$

c parallel to the graph of $y = -x - 6$ and passes through $(-4, 2)$

d parallel to the graph of $y = \frac{1}{2}x + 9$ and passes through $(6, 0)$

e parallel to the graph of $y = -\frac{1}{3}x - 8$ and passes through $(15, 0)$

f parallel to the graph of $y = \frac{2}{7}x + 5$ and passes through $(0, 11)$

g parallel to the graph of $y = \frac{3}{2} - \frac{5}{4}x$ and passes through $(0, -5)$

h parallel to the graph of $y = \frac{5}{6}x - \frac{9}{4}$ and passes through $(-9, 12)$.

7 Rearrange each of these equations to help you determine the equation for a line that is:

a parallel to the graph of $8x + 2y = 10$ and passes through $(5, -2)$

b parallel to the graph of $5x + 10y = 30$ and passes through $(-8, 1)$

c parallel to the graph of $7x - 3y = 5$ and passes through $(0, 6)$

d parallel to the graph of $40y - 24x = 6$ and passes through $(5, 0)$.

WE 3.5C **8** Determine the equation for the line that is:

a perpendicular to the graph of $y = 5x + 9$ and passes through $(5, -8)$

b perpendicular to the graph of $y = -7x - 3$ and passes through $(0, 3)$

c perpendicular to the graph of $y = x + 4$ and passes through $(-1, -2)$

d perpendicular to the graph of $y = -\frac{2}{3}x + 1$ and passes through $(-4, 4)$

e perpendicular to the graph of $y = \frac{4}{9}x - 7$ and passes through $(0, 8)$

f perpendicular to the graph of $y = -\frac{1}{3}x + 2$ and passes through $(-10, 0)$

g perpendicular to the graph of $y = \frac{14}{3} - \frac{5}{6}x$ and passes through $(9, 0)$

h perpendicular to the graph of $y = \frac{4}{7}x - \frac{8}{21}$ and passes through $(-28, 3)$.

9 Rearrange each of these equations to help you determine the equation for a line that is:

a perpendicular to the graph of $15x + 3y = 12$ and passes through $(-10, 7)$

b perpendicular to the graph of $12x - 36y = 72$ and passes through $(-9, -6)$

c perpendicular to the graph of $9x + 7y = 13$ and passes through $(0, 47)$

d perpendicular to the graph of $45y - 18x = 63$ and passes through $(-40, 0)$.

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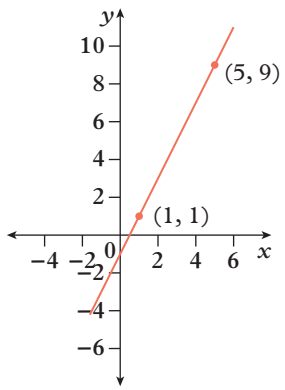
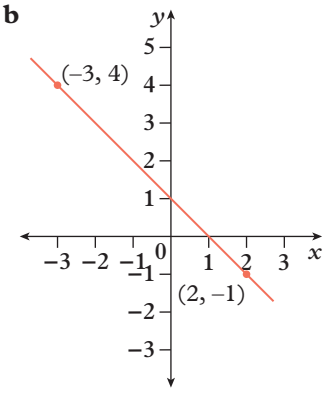
Exercise 3.5B: Problem solving and reasoning

10 Write the equation for a line that is perpendicular to each line shown and has a y-intercept of 3.

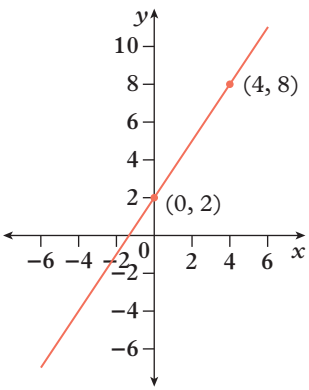
a



b



c



11 For each line shown in question **10**, write the equation for a line that has a y-intercept of –4 and is:

i parallel to the given line

ii perpendicular to the given line.

12 Consider the following pairs of linear equations.

i $2x + 3y = 12$ and $2x - 3y = 12$

ii $2x + 3y = 12$ and $\frac{1}{2}x - \frac{1}{3}y = 12$

iii $2x + 3y = 12$ and $2x - 3y = \frac{1}{12}$

iv $5x + 2y = 20$ and $2x - 5y = 20$

v $5x + 2y = 20$ and $-2x + 5y = 20$

vi $5x + 2y = 20$ and $2x + 5y = -20$

a Sketch each pair of graphs on the same Cartesian plane.

b Determine which pairs of graphs are perpendicular.

c Write the equation of a line perpendicular to $ax + by = 1$ in the same form and in terms of a and b.

13 Consider the linear equations $x = 2$ and $y = 4$.

a Sketch the graphs of $x = 2$ and $y = 4$ on the same Cartesian plane.

b Explain why each of the following student's explanations is either correct or incorrect.

i Aamira says the graphs are perpendicular because they meet at a right angle.

ii Jack says the graphs are not perpendicular because $\frac{0}{1} \times \frac{1}{0} = 1 \neq -1$.

c Complete each of the following sentences.

i Vertical lines are perpendicular to horizontal lines because they meet at a _____ degree angle.

ii Vertical lines are perpendicular to horizontal lines, but we cannot use the formula ___ × ___ = ___ because the gradient of a vertical line is _____.

iii Vertical lines are perpendicular to horizontal lines, so the equations _____ and _____ are perpendicular.

14 Show that the line joining (2, –3) and (4, 5) on the Cartesian plane is parallel to the graph of $y = 4x - 7$.

15 Show that the line joining (–11, –7) and (–1, –2) on the Cartesian plane is perpendicular to the graph of $y = -2x + 5$.

16 A line passes through the points (–2, 3) and (7, 9).

a Find the gradient of the line.

b Write the equation for a linear graph that is:

i parallel to the line and passes through (6, –1)

ii perpendicular to the line and passes through (–4, 8).

17 A line segment joins (–6, 7) and (0, –11). Write the equation for a linear graph that is:

a parallel to this line segment and passes through the origin

b perpendicular to this line segment and passes through (–6, 7)

c perpendicular to this line segment and passes through its midpoint.

Hint: Use $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ to find the coordinates of the midpoint.

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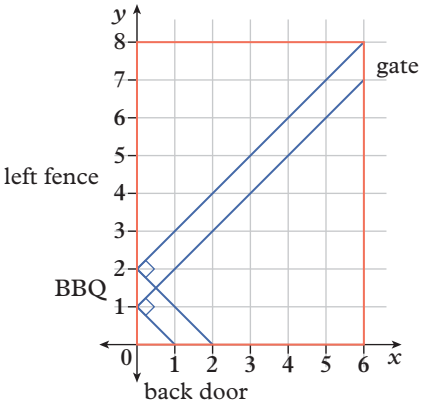
18 In his backyard, Husayn is planning to add a concrete path that goes from the back door to the barbecue on the left fence, then continues to the gate in the right corner. He wants the path to make a right turn at the barbecue. The path is planned to be 1 m wide and the back door is 1 m away from the left fence. Determine the equations of the four lines that mark the edge of the proposed path.



19 Four points are plotted on a Cartesian plane: $A(-2, 1)$, $B(5, 4)$, $C(-6, -4)$ and $D(8, 2)$.

a Is line segment AB parallel to line segment CD?

b Is line segment AC perpendicular to line segment BD?

20 The following three points are plotted on a Cartesian plane: $A(-2, 3)$, $B(1, 12)$ and $C(7, 10)$.

a Show that line segments drawn between these points form a right-angled triangle with the points A, B and C as vertices.

b Is the triangle ABC an isosceles right-angled triangle? Show your reasoning.

c Calculate the perimeter of the triangle, correct to one decimal place.

d Calculate the area of the triangle, correct to one decimal place.

Hint: Use Pythagoras' theorem to find the length between two points $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

21 Use your knowledge of parallel lines and Pythagoras' theorem to:

a show that quadrilateral $ABCD$ with vertices at $A(1, 4)$, $B(2, 6)$, $C(4, 7)$ and $D(3, 5)$ is a rhombus

b show that quadrilateral $EFGH$ with vertices at $E(1, 3)$, $F(8, 6)$, $G(4, -2)$ and $H(-3, -5)$ is a parallelogram.

Exercise 3.5C: Challenge

22 Prove that quadrilateral $KLMN$ with vertices at $K(0, 0)$, $L(2, 4)$, $M(6, 2)$ and $N(4, -2)$ is a square. Then find its perimeter and area, correct to one decimal place.

23 Prove that quadrilateral $PQRS$ with vertices at $P(-7, 5)$, $Q(-3, 1)$, $R(0, 4)$ and $S(-4, 8)$ is a rectangle. Then find its perimeter and area, correct to one decimal place.

24 A perpendicular bisector is the line that passes through the midpoint of a line segment between two points at a right angle. Write the equation for the perpendicular bisector of the line segment joining $(-2, 3)$ and $(8, -7)$.

25 Prove that quadrilateral $WXYZ$ with vertices at $W(-9, -11)$, $X(-1, 8)$, $Y(6, 9)$ and $Z(7, 2)$ is a kite.
Hint: Also consider the diagonals of the quadrilateral.

26 Without plotting points on a Cartesian plane, what type of quadrilateral is $ABCD$ if its vertices are $A(2, 1)$, $B(8, 4)$, $C(9, 1)$ and $D(-3, -5)$? Show your reasoning, using your knowledge of parallel and perpendicular lines.

Online resources:

Interactive skillsheet Parallel and perpendicular lines

Worksheet Working with linear relationships

Quick quiz 3.5

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Lesson 3.6

Simultaneous linear equations

Learning intentions

By the end of this lesson you will be able to ...

✔ solve simultaneous linear equations using an appropriate method.

Inter-year links

Year 8	6.6 Solving linear equations using graphs
Year 8	6.7 Intersecting lines
Year 9	3.5 Sketching linear graphs

Simultaneous linear equations

- **Simultaneous linear equations** are a set of two or more equations using two or more variables, whose values can simultaneously satisfy all the equations in the set.
- The solutions to a pair of simultaneous linear equations are the values for the variables that are solutions to both equations.

  For example, to check whether $x = 2$ and $y = 7$ are solutions to $y = 5x - 3$ and $y = -x + 9$, substitute the value $x = 2$ into both equations.

$$\begin{aligned} y &= 5x - 3 \\ y &= 5(2) - 3 \\ y &= 10 - 3 \\ y &= 7 \end{aligned} \qquad \begin{aligned} y &= -x + 9 \\ y &= -1(2) + 9 \\ y &= -2 + 9 \\ y &= 7 \end{aligned}$$

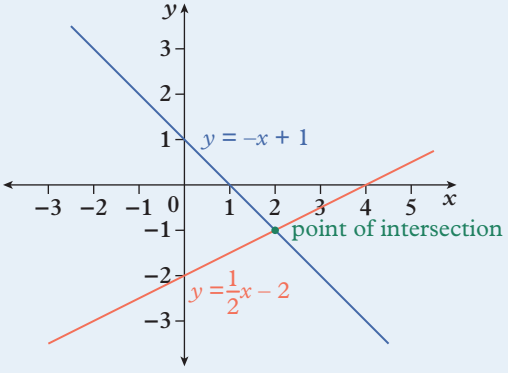
  When $x = 2$, the result of both equations is $y = 7$, showing that the solution to the pair of simultaneous linear equations is $x = 2$ and $y = 7$.

Solving simultaneous linear equations graphically

Key content video
Solving simultaneous linear equations graphically

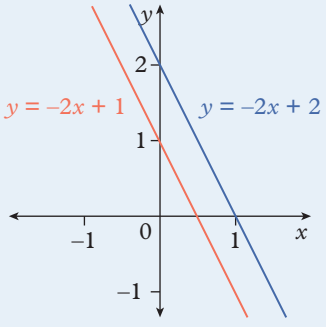
- If two linear graphs are not parallel, then they have a **point of intersection**. The x- and y-values at the point of intersection are the solution to the set of simultaneous equations consisting of the equations of the two lines.

  For example, the pair of simultaneous linear equations $y = -x + 1$ and $y = \frac{1}{2}x - 2$ can be represented by the pair of linear graphs shown on the right. Since their point of intersection occurs at $(2, -1)$, then the solution to the simultaneous equations is $x = 2$ and $y = -1$.



- If the graphs are overlapping, there are infinitely many solutions. All points on the graph are solutions to the pair of equations.
- If the linear graphs are distinct and parallel, the simultaneous linear equations have no solution. The graphs do not intersect.

  For example, the pair of simultaneous linear equations $y = -2x + 2$ and $y = -2x + 1$ can be represented by the pair of parallel linear graphs shown on the right.



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Solving simultaneous linear equations by substitution

Key content video
Solving simultaneous linear equations by substitution

- The **substitution method** involves substituting an expression from one equation into another equation.
- When at least one of the equations already has one variable as the subject, using the substitution method is usually the simplest.

  For example, if $y = 4 - 2x$ and $y + 11 = 3x$ are being solved simultaneously, substituting y from the first equation into the second equation will give us an equation in terms of only x that can be solved to determine the value for x.

  Substitute $y = 4 - 2x$ into $y + 11 = 3x$:

$$\begin{aligned} (4 - 2x) + 11 &= 3x \\ 4 - 2x + 11 &= 3x && (+\ 2x) \\ 4 + 11 &= 5x \\ 15 &= 5x && (\div\ 5) \\ x &= 3 \end{aligned}$$

  Substituting the value $x = 3$ into either equation gives the result $y = -2$:

$$\begin{aligned} y &= 4 - 2x & y + 11 &= 3x \\ y &= 4 - 2(3) & y + 11 &= 3(3) \\ y &= 4 - 6 & y + 11 &= 9 && (-\ 11) \\ y &= -2 & y &= -2 \end{aligned}$$

  So, the solution to the pair of simultaneous equations is $x = 3$ and $y = -2$.

Solving simultaneous linear equations by elimination

Key content video
Solving simultaneous linear equations by elimination

- The left-hand side (LHS) and right-hand side (RHS) of two equations can be added or subtracted from each other to create a new equation.

  For example, the LHS and RHS of $6 = 1 + 5$ and $3 + 4 = 7$ can be added to get $6 + (3 + 4) = (1 + 5) + 7$.
- The **elimination method** uses this idea to 'add' or 'subtract' simultaneous linear equations to create a new equation in which one of the variables is eliminated.

  For example, the pair of simultaneous linear equations $x + y = 4$ and $-3x + y = -8$ can be solved by elimination when the second equation is 'subtracted' from the first.

$$\begin{aligned} x + y &= 4 && ① \\ -3x + y &= -8 && ② \\ ① - ②:\quad (x + y) - (-3x + y) &= (4) - (-8) \\ x + 3x + \cancel{y} - \cancel{y} &= 4 + 8 \\ 4x &= 12 && (\div\ 4) \\ x &= 3 \end{aligned}$$

the variable y is eliminated

  Substituting this value for x into either equation gives the result $y = 1$. The pair of simultaneous equations has been solved.
- One or both of the equations might need to be multiplied by a constant so that the coefficient of one of the variables becomes zero when the equations are added/subtracted.

  In the example below, the first equation is multiplied by 2. This results in the coefficient of x being 2 in both equations.

$$\begin{aligned} x + y &= 1 && ① \quad \text{multiply ① by 2} \quad && 2x + 2y = 2 \\ 2x + 3y &= 7 && ② \quad \longrightarrow && 2x + 3y = 7 \end{aligned}$$

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Worked example 3.6A Solving simultaneous linear equations graphically

Solve this pair of simultaneous linear equations using a graph.

$y = x + 2$ $\qquad$ $y = 3x - 2$

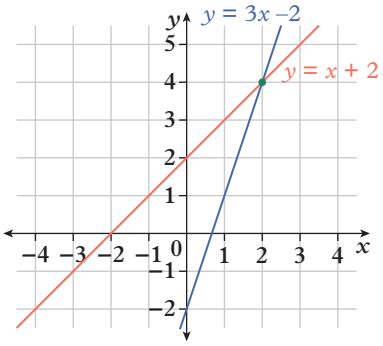
THINK

1 Sketch the graphs of $y = x + 2$ and $y = 3x - 2$ on the same Cartesian plane, using graph paper. Use a ruler to extend both lines until they intersect.

2 Use the coordinates of the point of intersection to determine a solution to the simultaneous linear equations.

3 Check the solution by substituting the value for x into both equations. If both equations result in the same y value, the solution is correct.

WRITE



The point of intersection is at (2, 4), so the solution to the simultaneous linear equations is $x = 2, y = 4$.

Check: $y = x + 2$ $\qquad$ $y = 3x - 2$

$y = (2) + 2$ $\qquad$ $y = 3(2) - 2$

$y = 4$ $\qquad$ $y = 6 - 2$

$y = 4$

Worked example 3.6B Solving simultaneous linear equations by substitution

Solve this pair of simultaneous linear equations using the substitution method.

$5x + 2y = 2$ $\qquad$ $y = 7x + 20$

THINK

1 Number each equation for reference.

2 Substitute the expression for y from equation ② into equation ①. Use brackets when substituting the expression.

3 Solve for x.

4 Substitute the solution for x into either equation and solve it for y.

WRITE

$5x + 2y = 2$ ①

$y = 7x + 20$ ②

Substituting $y = 7x + 20$ into ①:

$5x + 2y = 2$

$5x + 2(7x + 20) = 2$

$5x + 14x + 40 = 2$

$19x + 40 = 2$ $\quad (-40)$

$19x = -38$ $\quad (\div 19)$

$x = -2$

Substitute $x = -2$ into ②:

$y = 7x + 20$

$y = 7(-2) + 20$

$y = 6$

The solution is $x = -2, y = 6$.

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Worked example 3.6C Solving simultaneous linear equations by elimination

Solve this pair of simultaneous linear equations using the elimination method.

$y - 3x = -10 \qquad 3x + 2y = 16$

THINK

1 Number each equation for reference.

2 Determine which variable can be easily eliminated. The coefficient for x in equation ① is the negative value of the coefficient for x in equation ②. So, the equations can be added to remove x.

3 Solve the resulting equation for y.

4 Substitute the solution for y into either equation to solve it for x.

WRITE

$y - 3x = -10$ ①

$3x + 2y = 16$ ②

① + ②:

$(y - 3x) + (3x + 2y) = (-10) + (16)$

$y - \cancel{3x} + \cancel{3x} + 2y = -10 + 16$

$3y = 6 \qquad (\div 3)$

$y = 2$

Substitute $y = 2$ into ①:

$y - 3x = -10$

$(2) - 3x = -10 \qquad (-2)$

$-3x = -12 \qquad (\div (-3))$

$x = 4$

The solution is $x = 4$, $y = 2$.

Worked example 3.6D Solving simultaneous linear equations by elimination (multiplying by a constant)

Solve each of these pairs of simultaneous linear equations using the elimination method.

a $y - x = 11$
$3y - 2x = 27$

b $4x - 3y = -5$
$2y - 5x = 1$

THINK

a 1 Number each equation for reference.

2 Determine which variable can be easily eliminated. The coefficient of x in equation ① is -1 and the coefficient of x in equation ② is -2. Equation ② can stay the same, multiply equation ① by 2.

3 Equation '2 × ①' and equation ② have the same coefficients of x, so subtract the equations to remove x.

4 Solve the equation for y.

WRITE

a $y - x = 11$ ①

$3y - 2x = 27$ ②

2 × ①:

$y - x = 11 \quad (\times 2)$

$2y - 2x = 22$

(2 × ①) − ②:

$(2y - 2x) - (3y - 2x) = (22) - (27)$

$2y - \cancel{2x} - 3y + \cancel{2x} = 22 - 27$

$-y = -5$

$y = 5$

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5 Substitute the solution for y into either equation and solve it for x.

Substituting $y = 5$ into ①:

$y - x = 11$

$(5) - x = 11 \quad (-5)$

$-x = 6$

$x = -6$

The solution is $x = -6$, $y = 5$.

b 1 Number each equation for reference.

$4x - 3y = -5$ ①

$2y - 5x = 1$ ②

2 The coefficient of y in equation ① is -3 and the coefficient of y in equation ② is 2. Multiply equation ① by 2 and equation ② by 3.

$2 \times$ ①:

$4x - 3y = -5 \quad (\times 2)$

$8x - 6y = -10$

$3 \times$ ②:

$2y - 5x = 1 \quad (\times 3)$

$6y - 15x = 3$

3 The coefficients of y in equation '$2 \times$ ①' and equation '$3 \times$ ②' can be added to remove y.

$(2 \times$ ①$) + (3 \times$ ②$)$:

$(8x - 6y) + (6y - 15x) = (-10) + (3)$

$8x - \cancel{6y} + \cancel{6y} - 15x = -10 + 3$

$-7x = -7$

4 Solve for x.

$x = 1$

Substituting $x = 1$ into ①:

$4x - 3y = -5$

$4(1) - 3y = -5$

$4 - 3y = -5 \quad (-4)$

$-3y = -9 \quad (\div (-3))$

$y = 3$

5 Substitute the solution for x into either equation to solve it for y.

The solution is $x = 1$, $y = 3$.

Helpful hints

- ✔ Always number your equations for easy reference.
- ✔ Remember, you can always check your solution to a pair of simultaneous equations by substituting one of the values back into each equation to see if the result matches your solution for the other variable.
- ✔ When solving simultaneous linear equations graphically, make sure to choose a suitable size for the units of your x- and y-axes and use graph paper so that you can identify the intersection point with accuracy.
- ✔ While parallel lines have no intersection, that does not mean that only perpendicular lines do. Any pair of non-parallel lines on the Cartesian plane meet at a point of intersection.
- ✔ When solving simultaneous linear equations using the substitution method, always make sure you substitute algebraic expressions enclosed in brackets, then remove the brackets by expanding in the next step.
- ✔ Remember, you can't eliminate a variable by adding or subtracting equations unless the magnitude of the coefficients are the same.

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Learning pathways

1, 2, 3–5(1st, 2nd columns), 6–8, 9(1st, 2nd columns), 10(1st, 2nd columns), 12	2, 3–5(3rd, 4th columns), 8, 9–10(2nd, 3rd columns), 11, 13, 14, 17, 18	3(k, l, o, p), 4–5(3rd, 4th columns), 9(3rd column), 10(3rd, 4th columns), 13, 15, 16, 18, 19

ANS p699

Exercise 3.6A: Understanding and fluency

1 Check if the given x and y values are the solution to the pair of simultaneous linear equations.

a $x = 1, y = 5$ Equations: $y = x + 4$ and $y = -2x + 7$

b $x = -2, y = 1$ Equations: $y = x + 3$ and $y = -3x - 5$

c $x = 5, y = -2$ Equations: $y = x - 9$ and $y = -2x + 7$

d $x = 3, y = -7$ Equations: $2x + y = -1$ and $y = -x - 4$

e $x = 4, y = 3$ Equations: $y = 2x - 5$ and $y = -x + 1$

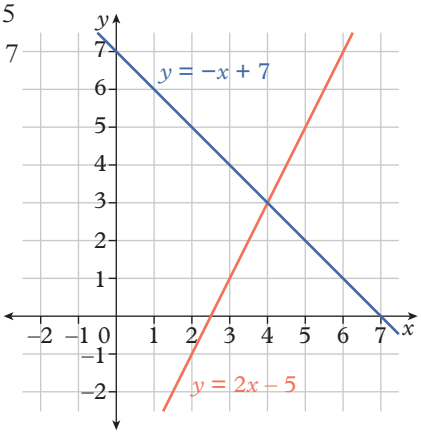
f $x = -6, y = 0$ Equations: $y = x - 6$ and $x + y = -6$

2 For each of the following pairs of simultaneous linear equations and their graphs:

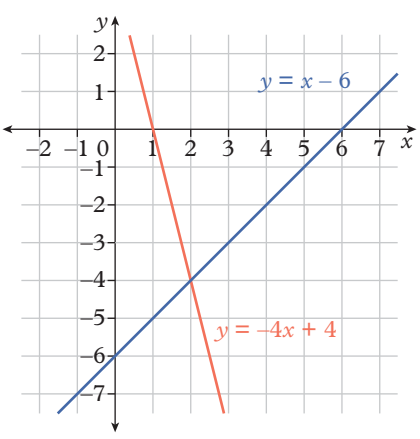
i write the coordinates of the point of intersection

ii state the solution of the simultaneous linear equations, using substitution to check your answer.

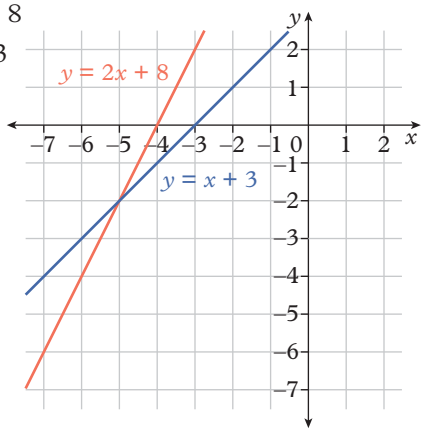
a $y = 2x - 5$
$y = -x + 7$



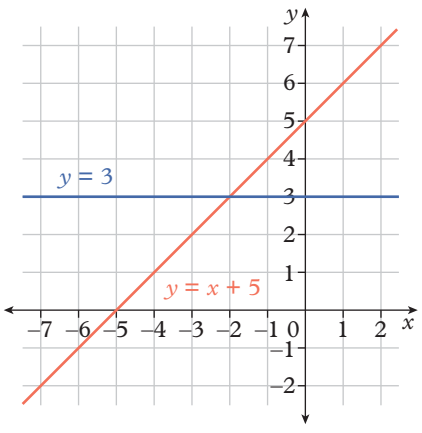
b $y = -4x + 4$
$y = x - 6$



c $y = 2x + 8$
$y = x + 3$



d $y = x + 5$
$y = 3$



WE 3.6A **3** Use graphs to help you solve these pairs of simultaneous linear equations (**a–p**). Use substitution to check your solutions.

a $y = 2x - 6$
$y = x - 1$

b $y = 3x$
$y = x + 4$

c $y = x - 2$
$y = 10 - x$

d $y = x + 7$
$y = -3x - 1$

e $y = 3x + 6$
$2x + y = 1$

f $y = -2x - 5$
$y = 2x + 7$

g $y = 2x - 4$
$y = x - 1$

h $2x + y = 6$
$y = x - 6$

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i $y = 3x + 3$
$y = -x - 5$

j $2x + y = -4$
$x + y = 1$

k $x - y = 4$
$4x + y = 6$

l $3y = x + 12$
$y = x + 2$

m $y = \frac{7}{2}x + 13$
$-2x + 3y = 5$

n $y = -\frac{5}{4}x - \frac{11}{4}$
$3x - 2y = 11$

o $y = \frac{8}{5}x + \frac{23}{5}$
$9x + 16y = -134$

p $y = \frac{5}{7}x + \frac{19}{7}$
$7x + 4y = 1$

WE 3.6B **4** Use the substitution method to solve each of these pairs of simultaneous linear equations.

a $x + y = 9$
$y = x + 5$

b $x + 2y = 6$
$y = x - 3$

c $2x + y = 1$
$y = x + 4$

d $y = x - 1$
$x - 3y = -9$

e $3x - y = -10$
$y = x + 2$

f $y = 6x$
$5x - y = -2$

g $6x + 5y = 21$
$y = 2x + 1$

h $y = 2 - 3x$
$4x + y = -1$

i $2x + 3y = -5$
$y = 5 - 2x$

j $x = y + 2$
$2x - 3y = 1$

k $x = 4y$
$y = 3x + 11$

l $x = 5$
$4x - 3y = 2$

5 Use the substitution method to solve each of these pairs of simultaneous linear equations.

a $y = 2x + 6$
$y = x + 7$

b $y = x - 2$
$y = 4x - 20$

c $y = 7x + 16$
$y = 2x + 1$

d $y = x - 6$
$y = 4 - x$

e $y = 5x + 9$
$y = 3 - x$

f $y = 1 - 2x$
$y = x - 11$

g $y = 3x - 5$
$y = 5 - 2x$

h $y = 4x + 14$
$y = -4x - 2$

6 Solve each of these pairs of simultaneous linear equations by adding the equations together to remove one variable.

a $2x + y = 9$
$3x - y = 1$

b $x + 2y = -1$
$-x + 5y = 8$

c $x - 2y = 11$
$5x + 2y = -5$

7 Solve each of these pairs of simultaneous linear equations by subtracting one equation from the other to remove one variable.

a $x + 4y = 17$
$x + 2y = 11$

b $3x + y = 3$
$x + y = 7$

c $4x - 3y = 33$
$4x + y = 5$

WE 3.6C **8** Use the elimination method to solve each of these pairs of simultaneous linear equations.

a $-3x + y = 10$
$3x - 4y = -4$

b $x - 2y = 2$
$3x - 2y = 14$

c $6x - y = -8$
$-6x + 5y = 16$

d $6x + 5y = 19$
$4x + 5y = 21$

e $4x + 3y = 29$
$2x - 3y = 1$

f $-5x + y = 11$
$-5x - 2y = 23$

WE 3.6D **9** Use the elimination method to solve each of these pairs of simultaneous linear equations.

a $5x + 3y = 21$
$2x - y = 4$

b $x + 4y = -2$
$4x - 3y = 11$

c $3x + y = -5$
$7x + 2y = -14$

d $6x - 7y = 9$
$-2x + 5y = 5$

e $4x + 3y = -24$
$5x - 12y = 33$

f $2x + 5y = 26$
$10x - 3y = -38$

g $4x + 3y = 16$
$3x - 2y = -5$

h $3x + 7y = -2$
$5x + 11y = -4$

i $-8x + 5y = -21$
$5x + 2y = 8$

j $3x - 10y = -32$
$-7x + 3y = 34$

k $-9x + 2y = -35$
$8x - 9y = -5$

l $-4x - 5y = 22$
$-3x + 7y = -5$

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Exercise 3.6B: Problem solving and reasoning

10 Use the most appropriate method to solve each of these pairs of simultaneous equations.
Hint: Not all solutions are integers.

a $y = 3x - 6$
$4x + 11y = 8$

b $y = 6x - 5$
$y = 3x - 4$

c $4x - 5y = 4$
$2x + 5y = 5$

d $y = -7$
$y = 4x + 29$

e $x = 5y - 6$
$x - y = 2$

f $2y = x - 4$
$3x - 2y = 0$

g $2x - y = -4$
$4x + y = 22$

h $x + 5y = -5$
$x - 2y = 9$

i $-x - 7y = 5$
$-2x - 7y = 3$

j $y = 2.5x + 4.3$
$y = 7.5x + 8.3$

k $4x = y - 6$
$8x + 3y = 13$

l $6x - 7y = -119$
$8x - 5y = 19$

11 Explain why there is no solution to the simultaneous linear equations $y = 2x + 5$ and $2x - y = 1$.

12 Sophie has decided to make hand-painted cards. She invests \$60 in a set of paintbrushes and enough paint to make 40 cards. The cost of each plain white card is \$2 and she plans to sell the painted cards for \$6 each.

a Explain how the amount of money, a, to produce n cards (where $0 \le n \le 40$) can be written as $a = 2n + 60$.

b Explain how the amount of money, a, received for selling n cards can be written as $a = 6n$.

c Draw graphs of the two equations on the same Cartesian plane. Label the horizontal axis as n and the vertical axis as a.

d What are the coordinates of the point of intersection of the two graphs? What does this tell you?

e How many cards should Sophie produce and then sell to break even (recover her costs but nothing extra)?

f How many cards should Sophie produce and sell to make a profit?

13 The perimeter around the top of a rectangular billiard table is 810 cm.

a Use l and w to write a linear equation for the perimeter of the table.

b Use l and w to write a linear equation describing the relationship between the length and width of the top of the table if the length is twice the width.

c Solve the simultaneous linear equations you wrote for parts **a** and **b** and find the dimensions of the top of the billiard table.

14 Consider this pair of simultaneous equations.

$y = 4 - 3x$
$6x + 2y = 8$

a Determine if the following coordinate pairs are solutions of both equations.

i $(1, 1)$ **ii** $(2, -2)$ **iii** $(3, -5)$

b Explain why the pair of simultaneous linear equations has an infinite number of solutions.

c If a pair of simultaneous linear equations has an infinite number of solutions, does that mean any pair of coordinates (x, y) is a solution? Use an example to explain why or why not.

15 Consider this pair of simultaneous equations:

$7x - 6y = -14$ ①
$5x - 3y = -1$ ②

a Demonstrate that the same solution is obtained using each of the strategies described below.

Strategy A: Multiply equation ② by 2 to form equation ③ and then subtract equation ③ from equation ①.

Strategy B: Multiply equation ② by –2 to form equation ④ and then add equations ① and ④.

Strategy C: Multiply equation ① by 5 to form equation ⑤ and multiply equation ② by 7 to form equation ⑥. Subtract equation ⑥ from equation ⑤.

b Are there other strategies you could use? Explain.

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16 Chloe is planning a party and obtains prices from two catering companies. Each caterer has a fixed price for delivery and set-up and a cost per person for the food.

Angie's Catering: fixed price \$200, \$28 per person

Cool Food Club: fixed price \$100, \$32 per person

Chloe is unsure how many people she will invite.

a Write a linear equation to represent the cost of hiring Angie's Catering. Remember to define the two variables you are using.

b Write a linear equation to represent the cost of hiring Cool Food Club.

c Show the graphs of both linear equations on the same Cartesian plane.

d Which company is cheaper if catering for 18 people? Explain how you can see this from your graph.

e Which company is cheaper if catering for 36 people?

f How many people can be catered for so that the cost of using either company is the same? What is this cost?

g Write a summary advising Chloe on what her options are.

Exercise 3.6C: Challenge

17 A linear graph has a y-intercept of -5. Another linear graph has a gradient of -3. The two linear graphs intersect at the point $(4, -6)$.

a Show the two graphs on the same Cartesian plane.

b Find the equation for each linear graph.

c Write the solution to the simultaneous linear equations.

d Use substitution to check that the solution satisfies both equations.

18 There can be more than one solution to a pair of simultaneous equations.

a Show that $x = 15$, $y = -5$ and $z = 5$ is a solution to the simultaneous equations $x + y = 10$ and $x + z = 20$.

b Show that $x = 22$, $y = -12$ and $z = -2$ is a solution to the simultaneous equations $x + y = 10$ and $x + z = 20$.

c Explain why we cannot solve the simultaneous equations $x + y = 10$ and $x + z = 20$ for one set of values of x, y and z.

19 We can find general solutions to a pair of simultaneous equations.

a Show that the solution to the simultaneous equations $ax + by = c$ and $ax + qy = r$ is $x = \dfrac{br - cq}{ab - aq}$ and $y = \dfrac{c - r}{b - q}$.

b Solve the simultaneous equations $3x + 4y = 9$ and $3x + 6y = 7$ by substituting the values of a, b, c, q and r into the solution from part **a**.

c Show that the solution to the simultaneous equations $ax + by = c$ and $px + qy = r$ is $x = \dfrac{br - cq}{bp - aq}$ and $y = \dfrac{cp - ar}{bp - aq}$.

d Solve the simultaneous equations $3x + 4y = 9$ and $5x + 6y = 7$ by substituting the values of a, b, c, p, q and r into the solution from part **c**.

Online resources:

- **Interactive skillsheet** Solving simultaneous linear equations graphically
- **Interactive skillsheet** Solving simultaneous linear equations by elimination
- **Worksheet** Solving simultaneous linear equations
- **Quick quiz** 3.6
- **Interactive skillsheet** Solving simultaneous linear equations by substitution
- **Investigation** Drawing a circumcircle around a triangle
- **CAS instructions** Solving simultaneous equations

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Lesson 3.7

Solving linear inequalities

Learning intentions

By the end of this lesson you will be able to ...

✔ solve linear inequalities and graph linear inequalities on the Cartesian plane.

Inter-year links

Year 7	6.6 Solving equations using inverse operations
Year 8	6.9 Solving linear inequalities
Year 9	3.2 Solving linear inequalities

Solving linear inequalities

Key content video
Solving linear inequalities

- A **linear inequality** is an inequality containing pronumerals only raised to the power of 1 and no two pronumerals are multiplied together.
- If the same operation (addition or subtraction, multiplication or division of positive numbers) is performed on both sides of an inequality sign, this results in an equivalent inequality of a different form. The possible values of the unknown remains the same.

  For example, $3x - 8 > 10 \quad (+ 8)$
  $3x > 18 \quad (\div 3)$
  $x > 6$
- Inverse operations can be used to solve linear inequalities. However, the inequality sign needs to be reversed when both sides of the inequality are multiplied or divided by a negative number.

  For example, $5 - 2x \leq 1 \quad (- 5)$
  $-2x \leq -4 \quad (\div (-2))$
  $x \geq 2$
- A linear inequality can also have two variables. For example, $y > 5 - 2x$ and $3x + 4y < 5$.
- Linear inequalities with two variables have infinitely many solutions that can be written as ordered pairs in the form (x, y).
- Each ordered pair produces a true statement when the values of x and y are substituted into the inequality.

  For example, $(0, 0)$ is one of the solutions to $3x + 4y < 5$.

  This is because $3 \times 0 + 4 \times 0 = 0$, and $0 < 5$ is a true statement.

Representing linear inequalities on the Cartesian plane

Key content video
Representing linear inequalities on the Cartesian plane

- Linear inequalities can be represented on the Cartesian plane using a boundary line and shading.
  - → A *solid* boundary line indicates that the points on the boundary line are included, meaning one side of the inequality is either 'less than or equal to' (≤) or 'greater than or equal to' (≥) the other side of the inequality. All the points on the boundary line and in the shaded area are solutions to the inequality.
  - → A *dashed* line indicates that one side of the inequality is either 'less than' (<) or 'greater than' (>) the other side of the inequality. All the points in the shaded area are solutions to the linear inequality.
- The boundary line of linear inequalities on the Cartesian plane can be found by changing the inequality sign in the inequality to an equals sign.

  For example:
  - → The boundary line of $2x + 3 < 1$ can be found by solving $2x + 3 = 1$
  - → The boundary line of $2x + 3 < y$ is $2x + 3 = y$.
- The boundary line divides the Cartesian plane into two halves. Points on either side of the boundary line can be used as test points to determine which half should be shaded.

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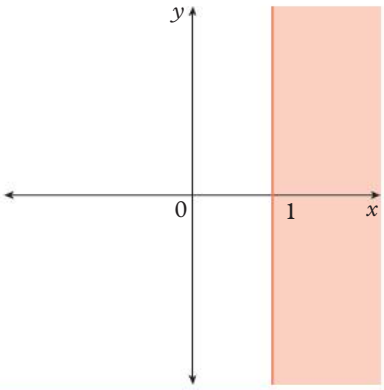
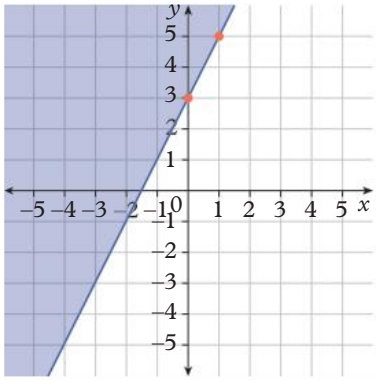
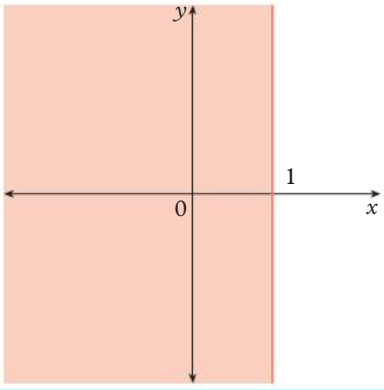
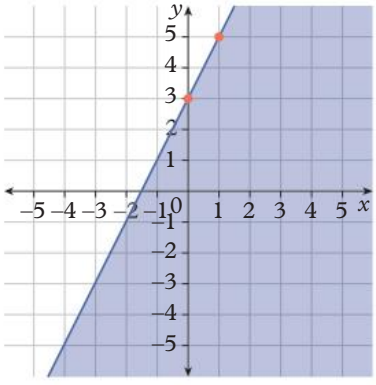
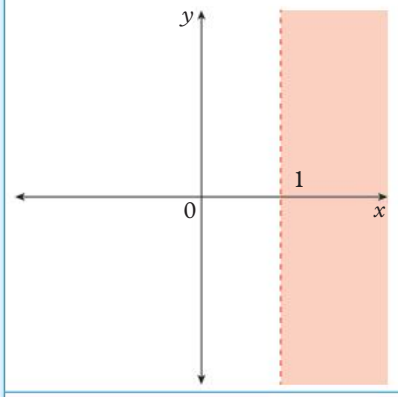
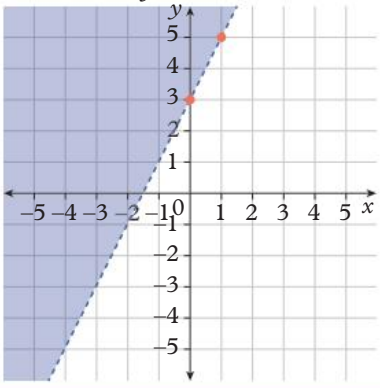
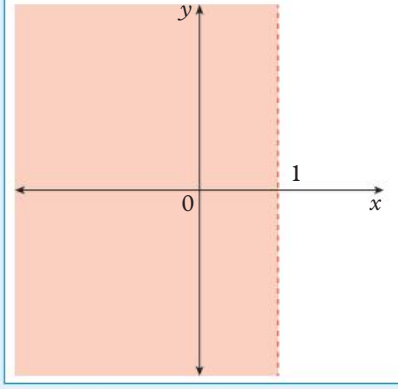
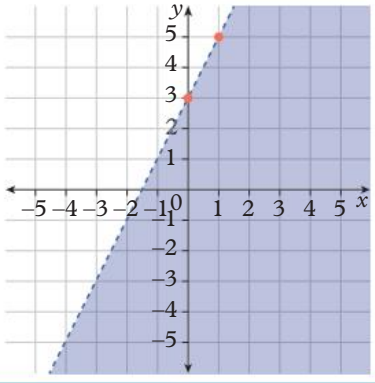
- One-variable inequalities can be represented on the Cartesian plane in a similar way. Since $x = k$ is a vertical line on the Cartesian plane, draw a vertical boundary line through $(k, 0)$ on the x-axis and shade the corresponding half.

Inequality with one variable	Inequality with two variables
$x \geq 1$ 	$y \geq 2x + 3$ 
$x \leq 1$ 	$y \leq 2x + 3$ 
$x > 1$ 	$y > 2x + 3$ 
$x < 1$ 	$y < 2x + 3$ 

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Worked example 3.7A Solving linear inequalities

Solve each of these inequalities.

a $5x + 3 < -2$ **b** $3 > -1 - \frac{x}{7}$ **c** $7x - 2 \geq 3x - 10$

THINK	WRITE
a 1 Apply the inverse operations to both sides in the correct order.	**a** $5x + 3 < -2 \quad (-3)$ $5x < -5 \quad (\div 5)$ $x < -1$
2 Write the solution with x on the LHS.	
3 Check your solution. Substitute $x = -1$ into the LHS of the inequality. The result should equal the RHS. Then substitute any value for x that is less than -1 into the original inequality to check the inequality sign. The result should be less than the RHS.	Check by substituting $x = -1$ into $5x + 3 < -2$. LHS $= 5(-1) + 3$ $= -2$ $=$ RHS (For $x = -1$, LHS = RHS as required.) Check by substituting a value less than -1 (for example, $x = -2$). LHS $= 5(-2) + 3$ $= -7$ $-7 < -2$ (For $x < -1$, LHS < RHS as required.)
b 1 Apply the inverse operations to both sides in the correct order. Reverse the inequality sign when both sides of the inequality are multiplied or divided by a negative number.	**b** $3 > -1 - \frac{x}{7} \quad (+1)$ $4 > -\frac{x}{7} \quad (\times (-7))$ $-28 < x$
2 Write the solution with x on the LHS, reverse the inequality sign if needed.	$x > -28$
3 Check your solution. Substitute $x = -28$ into the RHS of the inequality. The result should equal the LHS. Then substitute any value for x that is greater than -28 into the original inequality to check the inequality sign. The result should be less than the LHS.	Check by substituting $x = -28$ into $3 > -1 - \frac{x}{7}$. RHS $= -1 - \frac{-28}{7}$ $= -1 + 4$ $= 3$ $=$ LHS (For $x = -28$, LHS = RHS as required.) Check by substituting a value greater than -28 (for example, $x = 0$). RHS $= -1 - \frac{0}{7}$ $= -1$ $-1 < 3$ (For $x > -28$, LHS > RHS as required.)
c 1 Use inverse operations to remove the pronumeral on the side where the coefficient is less than the coefficient on the other side.	**c** $7x - 2 \geq 3x - 10 \quad (-3x)$
2 Identify and apply the correct inverse operations to both sides of the inequality in the correct order.	$4x - 2 \geq -10 \quad (+2)$ $4x \geq -8 \quad (\div 4)$ $x \geq -2$
3 Check your solution by substituting appropriate values for x into the original inequality to see if the result is a true statement.	Check by substituting $x = -2$ into $7x - 2 \geq 3x - 10$. LHS $= 7(-2) - 2$ $= -16$ RHS $= 3(-2) - 10$ $= -16$ (For $x = -2$, LHS = RHS as required.)

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Check by substituting a value greater than -2 (for example, $x = 0$).

LHS $= 7(0) - 2$
$= -2$

RHS $= 3(0) - 10$
$= -10$ (For $x > -2$, LHS > RHS as required.)

Worked example 3.7B Determining if a point is a solution to a linear inequality

Determine if the following number pairs are solutions of $3x + 2y \geq 23$.

a (0, 0) **b** (5, 4) **c** (4, 5) **d** (4, 6)

THINK	WRITE
a 1 Substitute $x = 0$ and $y = 0$ into the LHS of the linear inequality.	**a** $3x + 2y = 3(0) + 2(0)$
2 Calculate the LHS of the linear inequality.	LHS $= 0$
3 Compare the LHS and RHS of the linear inequality. LHS ≥ RHS is a false statement, so the number pair is not a solution to the given linear inequality.	$0 \geq 23$ is a false statement (0, 0) is not a solution of $3x + 2y \geq 23$.
b 1 Substitute $x = 5$ and $y = 4$ into the LHS of the linear inequality.	**b** $3x + 2y = 3(5) + 2(4)$
2 Calculate the LHS of the linear inequality.	LHS $= 15 + 8$ $= 23$
3 Compare the LHS and RHS of the linear inequality. The LHS = RHS, it satisfies the inequality LHS ≥ RHS so the number pair is a solution to the given linear inequality.	$23 \geq 23$ is a true statement (5, 4) is a solution of $3x + 2y \geq 23$.
c 1 Substitute $x = 4$ and $y = 5$ into the LHS of the linear inequality.	**c** $3x + 2y = 3(4) + 2(5)$
2 Calculate the LHS of the linear inequality.	LHS $= 12 + 10$ $= 22$
3 Compare the LHS and RHS of the linear inequality. LHS ≥ RHS is a false statement, so the number pair is not a solution to the given linear inequality.	$22 \geq 23$ is a false statement (4, 5) is not a solution of $3x + 2y \geq 23$.
d 1 Substitute $x = 4$ and $y = 6$ into the LHS of the linear inequality.	**d** $3x + 2y = 3(4) + 2(6)$
2 Calculate the LHS of the linear inequality.	LHS $= 12 + 12$ $= 24$
3 Compare the LHS and RHS of the linear inequality. The LHS > RHS, it satisfies the inequality LHS ≥ RHS so the number pair is a solution to the given linear inequality.	$24 \geq 23$ is a true statement (4, 6) is a solution of $3x + 2y \geq 23$.

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Worked example 3.7C Representing linear inequalities on the Cartesian plane

Represent the following linear inequalities on the Cartesian plane.

a $y > 2x + 4$ **b** $2y + x \leq 2$

THINK

a 1 Sketch $y = 2x + 4$ on the Cartesian plane by finding the x- and y-intercepts.

2 Connect the intercepts with a dashed line as the inequality sign is > not ≥.

3 Use (0, 0) as a test point to find out which side should be shaded.

4 Determine which side of the Cartesian plane should be shaded. As (0, 0) is not a solution to $y > 2x + 4$, shade the area above the boundary line.

b 1 Sketch $2y + x = 2$ on the Cartesian plane by finding the x- and y-intercepts.

WRITE

a $y = 2x + 4$

When $x = 0$,

$y = 2(0) + 4$

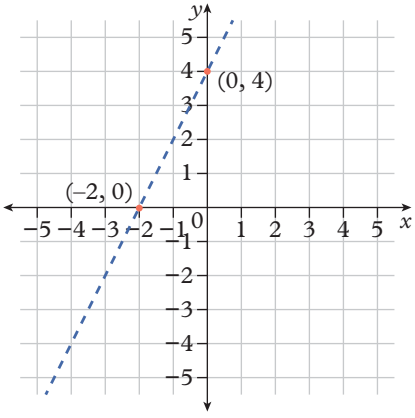
$= 4$

When $y = 0$,

$0 = 2x + 4$

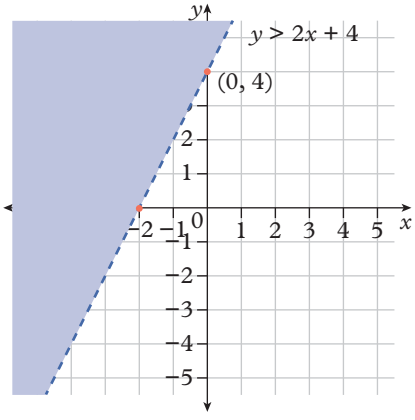
$x = -2$

The x- and y-intercepts are (0, 4) and (–2, 0).



Substitute (0, 0) into the inequality.

$0 > 2(0) + 4$ is a false statement. (0, 0) is not a solution of $y > 2x + 4$.



b $2y + x = 2$

When $x = 0$,

$2y + 0 = 2$

$y = 1$

When $y = 0$,

$0 + x = 2$

$x = 2$

The x- and y-intercepts are (0, 1) and (2, 0).

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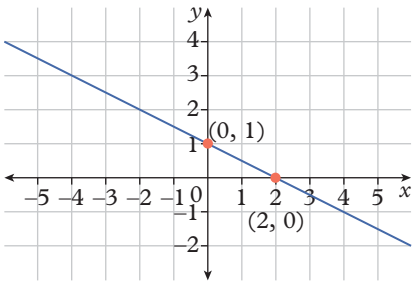
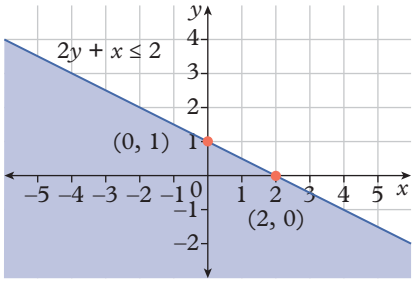
2 Connect the intercepts with a solid line.



3 Use (0, 0) as a test point to find out which side should be shaded.

Substitute (0, 0) into the inequality.
$2(0) + 0 \leq 2$ is a true statement. (0, 0) is a solution of $2y + x \leq 2$.

4 As (0, 0) is a solution to $2y + x \leq 2$, shade the area below the boundary line.



Helpful hints

- ✔ Be careful with the direction of the inequality sign when multiplying and dividing inequalities by negative numbers.
- ✔ If you need to swap the LHS and RHS of an inequality, remember to reverse the inequality sign so that it is still pointing towards the correct direction.
- ✔ When sketching the boundary line of a linear inequality, you can use either two intercepts or one intercept and the gradient.

Learning pathways

1, 2–4(1st, 2nd columns), 6–8, 13, 14	4(2nd column), 6-11, 15	4(3rd column), 7, 8, 12, 16

Exercise 3.7A: Understanding and fluency

1 Determine which of the following are linear inequalities.

a $x = 1$ **b** $y = 2x + 1$ **c** $y + 2 < 2x + 1$ **d** $x < 2x$
e $x + y + x \geq 0$ **f** $y < x^2 + 1$ **g** $x^2 < 1$ **h** $y > \frac{1}{2}$
i $y < \frac{1}{x}$ **j** $y \leq 2x$ **k** $xy > 2$ **l** $y < \frac{x}{2}$

WE 3.7A **2** Solve the following linear inequalities.

a $x + 5 < 9$ **b** $2 \geq 2x + 1$ **c** $12 < 2x - 4$ **d** $2x < -20$
e $x + 12 \geq 0$ **f** $23 < x + 1$ **g** $11(x + 1) \leq 121$ **h** $\frac{(x-3)}{2} > 72$
i $12(2x + 2) \leq 144$ **j** $100 \leq \frac{3}{5}x$ **k** $\frac{2x}{5} - 10 \geq 0$ **l** $21 < \frac{x}{7} + 7$

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3 Solve the following linear inequalities.

a $-x + 3 < 9$ **b** $2 \geq -2x + 1$ **c** $-12x - 10 < 110$ **d** $-2x < -20$

e $-2x + 12 \geq 0$ **f** $23 < -x + 1$ **g** $11(2 - x) \leq 121$ **h** $\frac{(3 - x)}{2} > 72$

i $12(2 - 2x) \leq 144$ **j** $12 \leq 10 - \frac{3}{5}x$ **k** $7 - \frac{2x}{5} \geq 3$ **l** $21 < 7 - \frac{x}{7}$

4 Solve the following linear inequalities.

a $x + 3 < 9x - 2$ **b** $2x \geq 5x + 1$ **c** $12x - 10 < 2x + 110$ **d** $2 - 2x < x - 20$

e $-3x + 12 \geq 2x$ **f** $x + 23 < -x + 1$ **g** $2(2 - x) \leq 12x$ **h** $\frac{(3 - x)}{2} > 7$

i $12(2 - 2x) \leq 2x + 5$ **j** $12x \leq 10 - \frac{3}{5}x$ **k** $7 - \frac{2x}{5} \geq 3x$ **l** $2x - 1 < 7 - \frac{x}{7}$

5 Solve the following linear inequalities.

a $\frac{x + 3}{12} < \frac{9}{12}$ **b** $\frac{x + 1}{3} < \frac{2}{3}$ **c** $\frac{3 - x}{2} < \frac{9 - x}{12}$

d $\frac{1 - x}{6} > \frac{x - 2}{12}$ **e** $\frac{x - 3}{21} < \frac{9 - x}{7}$ **f** $\frac{x + 3}{5} \leq \frac{9 - x}{7}$

WE 3.7B **6** Determine if the following points are solutions of the given inequalities.

i (0, 0) **ii** (1, 1) **iii** (3, 5)

a $2x + 1 < y$ **b** $y \geq 3 - x$ **c** $x - 5 \leq y$ **d** $y < 8 + 2x$

7 Determine if the following points are solutions of the given inequalities.

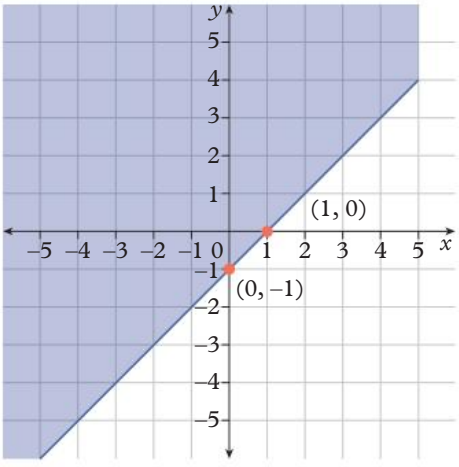
i (0, 0) **ii** (2, 3) **iii** (1, 9)

a $2x + 1 \leq 2y + 3$ **b** $3x + 2y \geq 3$ **c** $x + 2y \leq 8$ **d** $x + y \leq 3 + 2x$

8 Use your answer to question **7** to match the inequalities with the graphs below.

a $2x + 1 \leq 2y + 3$ **b** $3x + 2y \geq 3$ **c** $x + 2y \leq 8$ **d** $x + y \leq 3 + 2x$

A



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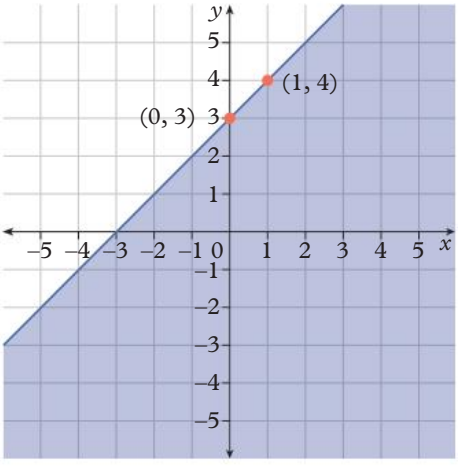
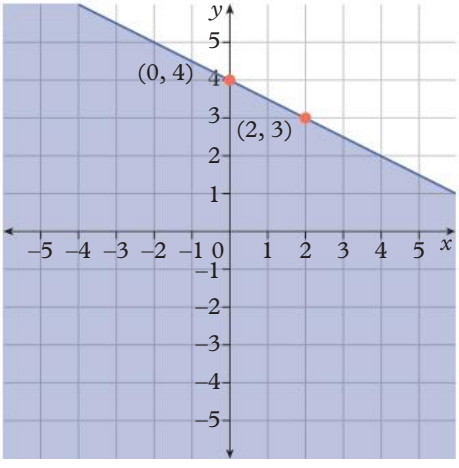
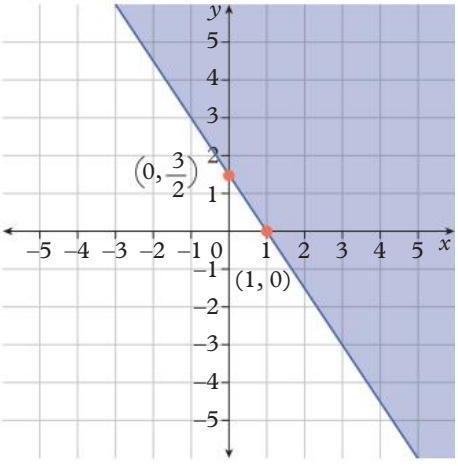


C



D



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WE 3.7C **9** Represent the linear inequalities in question **6** on the Cartesian plane.

10 Represent the following linear inequalities on the Cartesian plane.

a $y + 2x \leq 5$ **b** $2y + 4x > 2$ **c** $3y + 2x < 2$

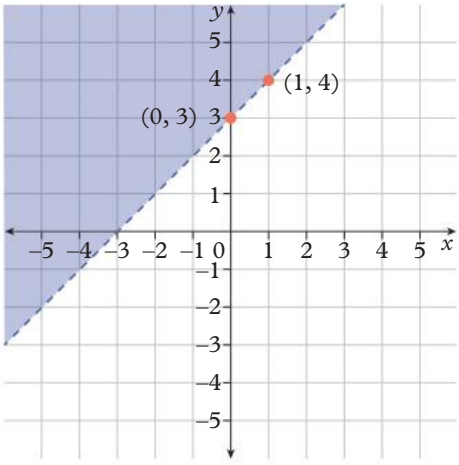
11 Represent the following linear inequalities on the Cartesian plane.

a $y + 2x \leq 5 - x$ **b** $y + 4x > 2 + 2y$ **c** $x + y + 1 \leq 2x$

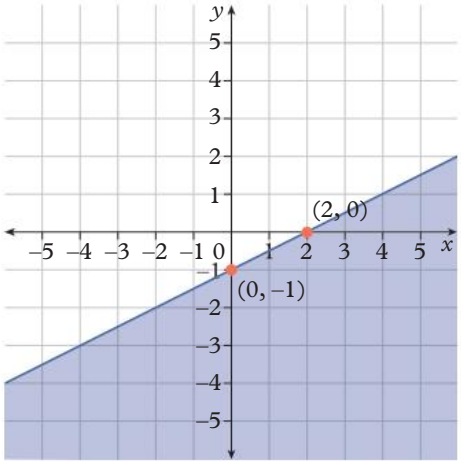
d $\frac{1 - x}{6} < \frac{-2y}{12}$ **e** $\frac{x - 3}{21} \geq \frac{9 - y}{7}$ **f** $\frac{y + 3}{5} > \frac{9 - x}{7}$

12 Write the inequalities represented by the following graphs.

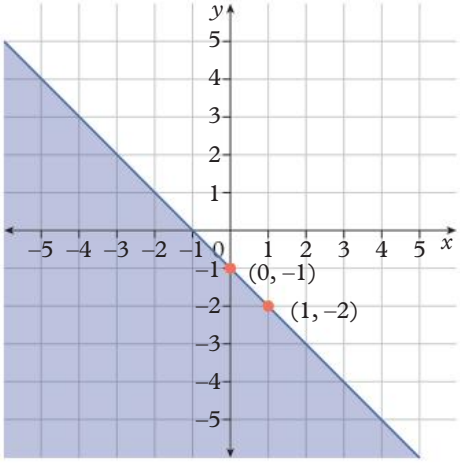
a



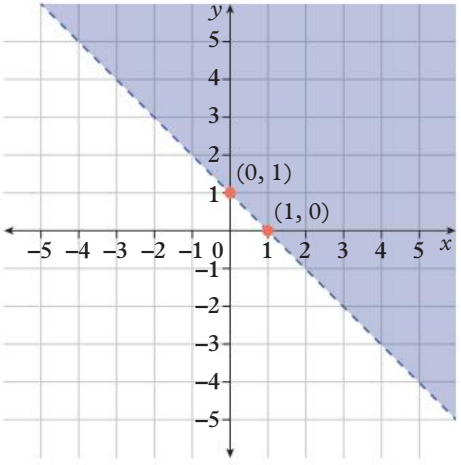
b



c



d



Exercise 3.7B: Problem solving and reasoning

13 Write a linear inequality to represent each of the following situations.

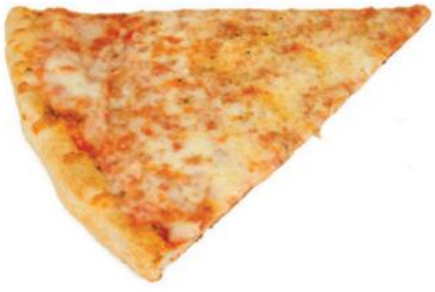
a The width and length of a rectangle are w and l, and the perimeter of the rectangle is greater than 20 cm.

b Eric has \$100 to spend on fertilisers and pesticides. Each bag of fertiliser costs \$22 and each bag of pesticide costs \$6.

c The capacity of a water tank is 2145 L. Each shower uses 55 L of water, and each bath uses 80 L of water.

d Emily has decided to buy p pizzas for g guests. Each pizza is cut into 8 pieces. Emily wants to eat 4 pieces and ensure each guest can have at least 3 pieces.

e Mark is making plans for a triathlon. He plans to swim for s hours and bike for b hours. The fastest he can complete the running course is in 3 hours 45 minutes and he aims to finish the whole race in 12 hours.



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14 Ken runs a small business selling products A and B. He buys product A for \$10 each and product B for \$20 each and resells them later at a higher price.

a Write an inequality to represent the relationship between the numbers of product A, x, and product B, y, Ken can stock up on if he has \$400.

b Ken prices product A and B at \$15 and \$25, respectively. Ken wants to make at least \$100 in profit. Ken wrote the inequality $15x + 25y \geq 400 + 100$ to represent the amount of money paid by customers versus his costs and desired profit. Explain why this is incorrect.

c Write the correct inequality to represent Ken's profit.

d What is Ken's profit if he only stocks up on product B and manages to sell all the stock?

e Ken is confident that he can sell all the products in stock. How many of product A and product B should Ken buy if he wants to maximise his profit?

15 Ada is buying souvenirs for her friends in the airport. Each snow globe costs \$10 and weighs 1 kg and each hoodie costs \$20 and weighs 0.7 kg.

a Write an inequality to represent the relationship between the number of snow globes, x, and the number of hoodies, y, Ada could afford if she has \$50.

b Graph the inequality on the Cartesian plane.

c Explain why you only need to include the first quadrant in the graph for part **b**.

d Use your graph to find out if Ada can buy 2 snow globes and 2 hoodies.

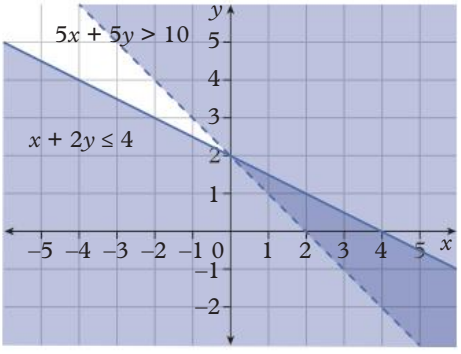
e List all the combinations of snow globes and hoodies Ada could buy with \$50.

f The weight limit of Ada's carry-on luggage is 7 kg. Her laptop weighs 3 kg. Write an inequality to represent the relationship between the number of snow globes, x, and the number of hoodies, y, if Ada wants to fit all the souvenirs and her laptop into the luggage.

g Which combination(s) in part **e** satisfy the inequality in part **f**?

Exercise 3.7C: Challenge

16 When two or more inequalities are sketched on the same set of axes, the shaded areas overlap and form an intersecting region. Points in the intersecting region satisfy both inequalities. For example, the graph on the right shows $x + 2y \leq 4$ and $5x + 5y > 10$ on the same set of axes.



a Sketch the following inequalities on the same set of axes.

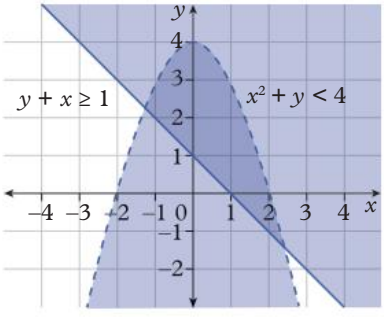
i $y \geq 2x + 4$ and $y < -x$

ii $y + 2x \geq 4$ and $y + x < 1$

iii $2x - y \leq 2$ and $y < 3 - x$

iv $y + 2x < 2$ and $y + x < 2$

b Use the graph on the right to find out if the given value pairs satisfy $x^2 + y < 4$ and $y + x \geq 1$



i $x = 1, y = 1$ **ii** $x = 2, y = 1$

iii $x = 2, y = -2$ **iv** $x = -1, y = 2$

v $x = 0, y = 4$ **vi** $x = -2, y = 3$

Online resources:

- **Interactive skillsheet** Representing linear inequalities on the Cartesian plane
- **Interactive skillsheet** Solving linear inequalities
- **Worksheet** Graphing linear inequalities
- **Quick quiz** 3.7

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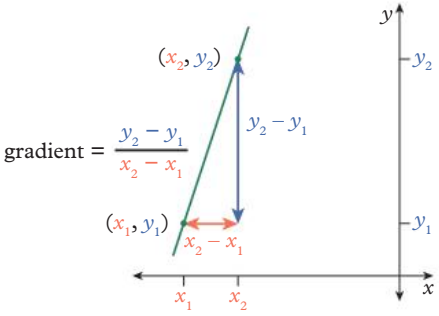
Lesson 3.8

Review: Linear relationships

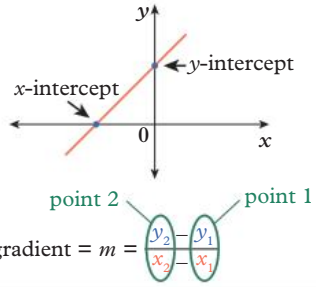
Module summary

Gradient and intercepts



$y = mx + c$

gradient y-intercept

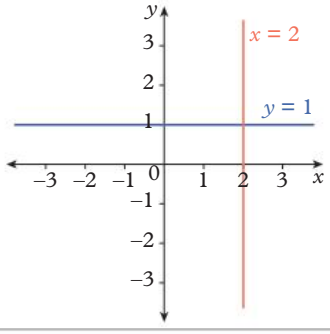


Equations and formulas

- A solution to an equation makes the equation a true statement.
- An unknown value in an equation can be determined by substitution if the values of all other variables are known.
- Inverse operations can be used to change the subject of a formula.

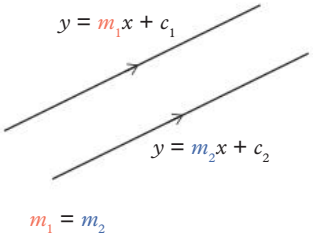
Horizontal and vertical lines

- Horizontal: $y = c$, gradient = 0
- Vertical: $x = c$, gradient is undefined



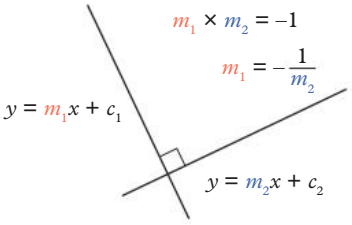
Parallel lines

Gradients are equal.



Perpendicular lines

The product of the gradients is –1.



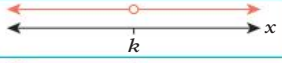
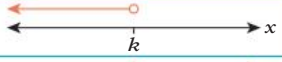
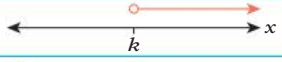
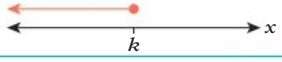
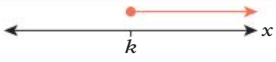
Solving simultaneous linear equations

Solving graphically
The coordinates for the point of intersection of the two graphs give the solution to the simultaneous linear equations.

Solving by substitution
Substitute an expression from one equation into the other equation to create a new equation containing only one variable.

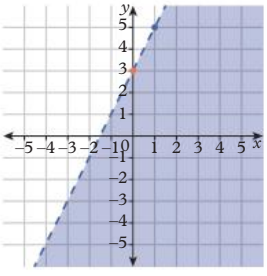
Solving by elimination
Add or subtract one equation (or the multiple of one equation) from the other equation to eliminate one of the variables.

Linear inequalities with one variable

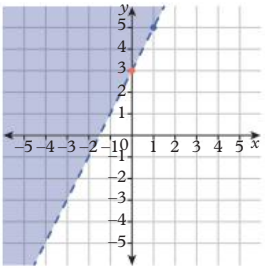
Inequality	Number line representation
$x \neq k$	
$x < k$	
$x > k$	
$x \leq k$	
$x \geq k$	
$x \geq k$	

Linear inequalities with two variables

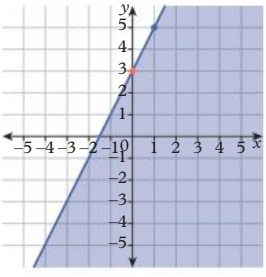
$y < 2x + 3$



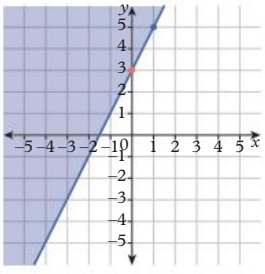
$y > 2x + 3$



$y \leq 2x + 3$



$y \geq 2x + 3$



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Review questions 3.8A: Mathematical literacy review

Review quiz
Take the review quiz to assess your knowledge of this module.
✔ 10
✔ 10A

Quizlet
Test your knowledge of this module by working individually or in teams.

The following key terms are used in this module:

- boundary line
- elimination method
- formula
- gradient
- gradient-intercept form
- horizontal
- intercept
- linear equation
- linear graph
- linear inequality
- linear relationship
- parallel
- perpendicular
- point of intersection
- simultaneous equations
- subject of an equation
- substitution method
- vertical

1 Given two linear equations in gradient–intercept form, $y = mx + c$, what do the two values of m in each equation tell us? State True or False for the following statements.

a They indicate if the lines are parallel to each other.
b They indicate if the lines are perpendicular.
c They indicate what the y-intercepts are.
d They indicate what the x-intercepts are.
e They indicate if the line is horizontal.

2 Complete the following sentences using words from the list of key terms.

a If we are trying to find a common solution(s) to a group of equations, they are referred to as _____.
b If we are trying to solve this pair of simultaneous equations, $y = 2x + 1$ and $3x - y = 10$, we could use the _____ because one of the equations already has one variable on its own as the _____.
c If we are trying to solve this pair of simultaneous equations, $2x + y = 5$ and $3x - y = 10$, we could use the _____ because each equation has the same or opposite coefficient for one of the variables.
d A _____ with two variables has an infinite number of points that satisfy the equation, and they lie on a straight line. A _____ has an infinite number of solutions that lie in a region. The _____ is either solid or dashed.

Review questions 3.8B: Multiple choice

3.1 **1** The solution to $\frac{x-4}{2} = -3$ is:
A $x = 2$ **B** $x = -2$ **C** $x = 10$ **D** $x = -10$

10A 3.2 **2** Which of these equations has the solution $x = -5$?
A $\frac{12}{x} = -2.4$ **B** $\frac{2x+3}{13} = 1$ **C** $3x - 7 = 8$ **D** $3x + 1 = 8(x + 6)$

3.3 **3** The gradient of the line $12x + 4y = 8$ is:
A 3 **B** −3 **C** 12 **D** −12

3.3 **4** Which one of the following equations is not equivalent to the others?
A $y = 3x - 6$ **B** $-3x + 6 = y$ **C** $2 - x = -\frac{y}{3}$ **D** $12 + 2y - 6x = 0$

3.4 **5** The equation for the linear graph with a gradient of $\frac{3}{4}$ and a y-intercept of $\frac{5}{8}$ is:
A $8x - 6y + 5 = 0$ **B** $6x - 5y + 8 = 0$ **C** $6x - 8y + 5 = 0$ **D** $6y = 8x + 5$

3.5 **6** The gradient of the line passing through the points $(1, 8)$ and $(-3, -6)$ is:
A −1 **B** $\frac{2}{7}$ **C** $\frac{7}{2}$ **D** $-\frac{7}{2}$

3.5 **7** A line parallel to the line $3x + 2y = 6$ would have a gradient of:
A $\frac{2}{3}$ **B** $-\frac{3}{2}$ **C** $\frac{3}{2}$ **D** $-\frac{2}{3}$

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3.5 **8** A line has a gradient of $-\frac{3}{4}$. A line perpendicular to this line would have a gradient of:

A $\frac{4}{3}$ **B** $-\frac{3}{4}$ **C** $\frac{3}{4}$ **D** $-\frac{4}{3}$

3.6 **9** The pair of linear equations $y = x + 3$ and $y = -2x + 6$ intersect at which of these points?

A $(-3, 0)$ **B** $(3, 6)$ **C** $(1, 8)$ **D** $(1, 4)$

3.6 **10** The pair of linear equations $y = 3x + 1$ and $6x - 2y + 2 = 0$ have how many simultaneous solutions?

A none **B** one **C** two **D** many

3.6 **11** When $y = 3x - 4$ is substituted into the equation $x - 2y = 3$, the resulting equation is:

A $-5x - 8 = 3$ **B** $-5x + 8 = 3$ **C** $5x + 8 = 3$ **D** $5x - 8 = 3$

3.6 **12** Consider the two equations:

$2x - y + 3 = 0$ ①

$4x - y + 5 = 0$ ②

If equation ② is subtracted from equation ①, the resulting equation is:

A $-2x - 2 = 0$ **B** $-2x - 8 = 0$ **C** $2x + 2 = 0$ **D** $2x + 8 = 0$

3.7 **13** If $3x + 2 > 5x - 6$, which of the following could be a value for x?

A 8 **B** 6 **C** 4 **D** 2

3.7 **14** Which of the following value pairs could not be a solution to $3x + 2y < 11$?

A $(0, 5)$ **B** $(1, 2)$ **C** $(2, 3)$ **D** $(-1, 6)$

Review questions 3.8C: Short answer

3.1 **1** Solve each of these equations. Leave your answer as a fraction where appropriate.

a $3 - 2(4 - 5x) = -20$ **b** $4(3x + 5) = 3(5x - 3)$

10A 3.2 **2** Solve each of these equations, leaving your answer as a fraction if appropriate.

a $\frac{4 + x}{3} = \frac{2x - 5}{3}$ **b** $\frac{5 - 3x}{2} = \frac{2x + 5}{3}$

10A 3.2 **3** Rearrange each of these equations so that the pronumeral in the brackets is the subject.

a $A = \frac{h(a + b)}{2}$ (b) **b** $A = 2(l + b)h$ (h)

3.3 **4** Rearrange the following linear relationships into gradient–intercept form.

a $6x + y = 0$ **b** $2y + 3 = x$

3.3 **5** For each of the following relationships:

i find the coordinates of the intercepts

ii sketch the relationship on the Cartesian plane.

a $4x + 5y = 20$ **b** $16x + 36y = 4$

3.4 **6** Determine the equation of each linear graph described by the information below. Write your answer in gradient–intercept form.

a gradient of 2, passes through $(-2, -1)$

b gradient of -3, passes through $(-2, 3)$

3.5 **7** Write an equation for a line that is:

a parallel to the graph of $y = -4x + 5$, with y-intercept of -3

b perpendicular to the graph of $y = 2x - 7$, with y-intercept of -4.

3.5 **8** Write an equation for a line that is:

a parallel to the graph of $y = 3x + 2$ and passes through $(3, 2)$

b perpendicular to the graph of $y = 3x + 2$ and passes through $(-3, -2)$.

3.6 **9** Use a graph to help you solve these pairs of simultaneous equations.

a $y = 4x - 2$
$y = -3x + 5$

b $2x - y + 5 = 0$
$3x - y = 1$

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3.6 **10** Use the substitution method to solve each of these pairs of simultaneous equations.

a $3x + 4y = -1$
$y = x - 2$

b $y = 4x + 3$
$x - 2y = 8$

c $7x - y = 4$
$y = 4x + 2$

3.6 **11** Use the elimination method to solve each of these pairs of simultaneous equations.

a $2x + y = 8$
$4x - y = 4$

b $x + 5y = 13$
$x + 2y = 4$

c $2x - y = 7$
$3x + 5y = 4$

3.7 **12** Solve these inequalities.

a $5 - 2x < 7$

b $\frac{2 - 3x}{5} > 2$

3.7 **13** Represent the following linear inequalities on the Cartesian plane with boundary lines and shading.

a $y + x \leq 3$

b $3y + 6x > 3$

c $x + 2y \leq 5 - y$

d $x + 7y \leq y + 2x$

Review questions 3.8D: Mathematical modelling

1 Jack purchases and then sells muffins and croissants. The muffins cost \$1.60 each and the croissants cost \$0.80 each. Jack sells the muffins for \$3.30 each and the croissants for \$2.50 each. Jack has a budget of \$100 per day. Let m be the number of muffins Jack buys and c be the number of croissants Jack buys.

a Write an inequality that describes how Jack can spend his budget each day.

b If Jack buys 20 muffins, write an inequality that describes the number of croissants he can buy.

c If Jack buys 30 croissants, write an inequality that describes the number of muffins he can buy.

d Assume Jack is able to sell all the muffins and croissants he buys on a particular day. Let P be the profit Jack will make that day from muffins and croissants. Write an equation that describes the profit Jack will make.

e On another day, Jack made \$170 profit.

i Sketch the graph of the number of muffins against the number of croissants Jack could have bought and sold. Hint: Put muffins on the vertical axis.

ii On the day Jack made \$170 profit, the price he paid for the muffins and croissants was \$99.20. Determine the number of muffins and croissants Jack bought and sold that day.

2 Consider the graphs of the linear equations $y = 2x - 1$ and $y = x + 5$.

a Find the point of intersection of the two graphs.

b Sketch the following linear inequalities on the same set of axes: $y \geq 2x - 1$ and $y < x + 5$. Shade the region where the points satisfy both inequalities.

c Does the point $(0, -1)$ lie in the region? Explain.

d Does the point $(0, 5)$ lie in the region? Explain.

e Does your solution to part **a** lie in the region? Explain.

f Does the point $(0, 0)$ lie in this region?

g Find the values of a and b such that:

i the point $(0, 0)$ lies in the region where $y \geq 2x - a$ and $y < x + b$

ii the point $(0, 0)$ lies in the region where $y \leq 2x - a$ and $y \geq x + b$

iii the point $(2, 3)$ lies in the region where $y \geq 2x - a$ and $y < x + b$.

3 Consider a triangle with vertices $X(4, -4)$, $Y(-2, 2)$ and $Z(1, 5)$ on the Cartesian plane.

a Draw this triangle on a Cartesian plane.

b Write an equation for each of the line segments XY, YZ and ZX. Limit the values of x for each equation with an inequality statement.

c Let the length of the side opposite vertex X be x, the side opposite vertex Y be y, and the side opposite vertex Z be z.

i Calculate the length of the sides x, y and z. Leave your answers as exact values.

ii Show that the triangle XYZ is right-angled.

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d Let M be the midpoint of the line segment ZX.

i Find the coordinates of M. Join M to vertex Y.

ii Calculate the length of MY.

iii The line segment MY divides $\triangle XYZ$ into two smaller triangles. Classify these triangles.

e The perpendicular height of the triangle from vertex Y to the base ZX. Let HY be this perpendicular height of the triangle, where H is a point on the line segment XZ.

i Find the coordinates of H.

ii Calculate the length of YH.

f Calculate the area of $\triangle XYZ$.

g Show that the length of YH is $\frac{xz}{y}$.

h Explain how a semicircle can be drawn to enclose $\triangle XYZ$. What does this tell us about the angle in a semicircle?

Checklist

Now that you have completed this module, reflect on your ability to do the following.

I can do this	I need to review this
☐ Solve linear equations with the unknown on both sides ☐ Solve linear equations involving brackets ☐ Solve for an unknown with substitution ☐ Change the subject of an equation	☐ Go back to Lesson 3.1 Solving linear equations
☐ Solve linear equations with an unknown in the denominator ☐ Solve linear equations involving two or more fractions ☐ Change the subject of an equation involving fractions	☐ Go back to Lesson 3.2 Solving linear equations with algebraic fractions [10A]
☐ Sketch a linear graph using the gradient–intercept method ☐ Sketch a linear graph using the x- and y-intercepts ☐ Sketch horizontal and vertical lines	☐ Go back to Lesson 3.3 Sketching linear graphs
☐ Determine the gradient of a line given two points ☐ Determine the equation of a linear relationship given the gradient and a point ☐ Determine the equation of a linear relationship given two points ☐ Determine the equation of a linear relationship given the graph	☐ Go back to Lesson 3.4 Determining linear equations
☐ Determine if two lines are parallel or perpendicular from their equations ☐ Determine linear equations for parallel lines ☐ Determine linear equations for perpendicular lines	☐ Go back to Lesson 3.5 Parallel and perpendicular lines
☐ Solve simultaneous linear equations graphically ☐ Solve simultaneous linear equations using substitution ☐ Solve simultaneous linear equations using elimination	☐ Go back to Lesson 3.6 Simultaneous linear equations
☐ Solve one-variable linear inequalities ☐ Verify if a point on the Cartesian plane is a solution to a linear inequality ☐ Represent inequalities with two variables on the Cartesian plane	☐ Go back to Lesson 3.7 Solving linear inequalities

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4

Non-linear relationships

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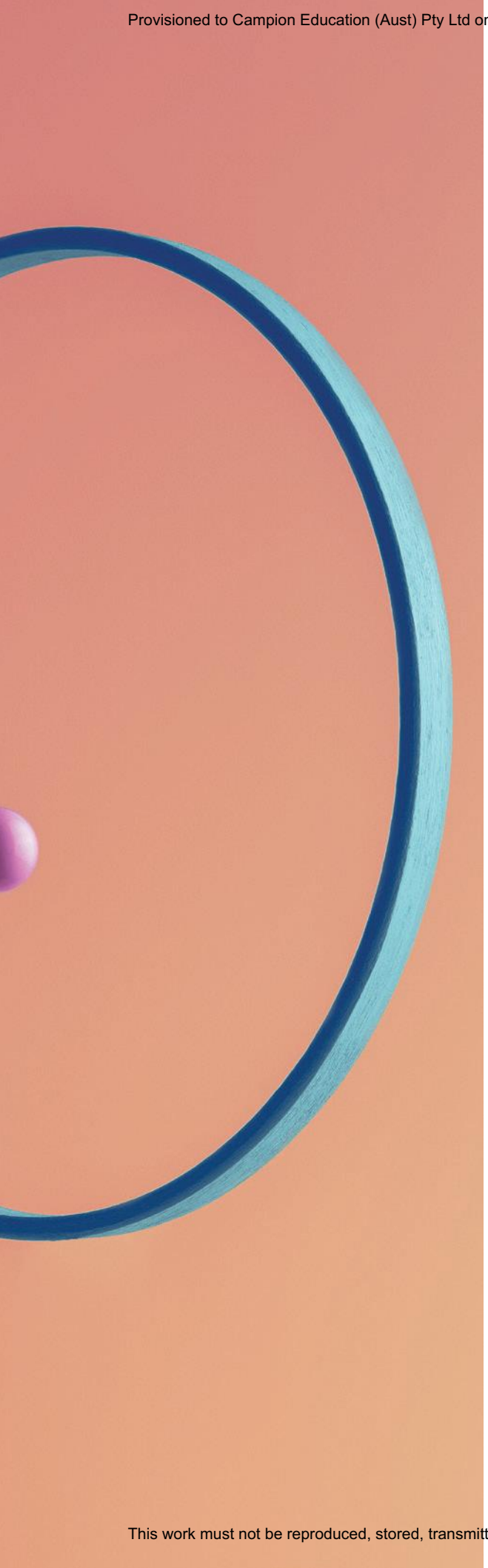

Lessons

4.1 Solving quadratic equations
4.2 The quadratic formula
4.3 Sketching parabolas
4.4 Sketching circles
4.5 Sketching exponential relationships
4.6 Direct and inverse proportion
10A **4.7** Sketching hyperbolas
10A **4.8** Graphs and transformations
10A **4.9** Solving non-linear equations
10A **4.10** Rates of change

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ Solving equations using inverse operations
- ✔ Factorising quadratic expressions
- ✔ Factorising the difference of two squares
- ✔ Plotting quadratic relationships
- ✔ Factorising by completing the square

Curriculum links

- Explore the connection between algebraic and graphical representations ... (VC2M10A11)
- Solve simple quadratic equations using a range of strategies, ... (VC2M10A13)
- Solve simple exponential equations (VC2M10A14)
- Use mathematical modelling to solve applied problems ... (VC2M10A15)
- Solve equations graphically or ... with digital tools, ... (VC2M10A16)
- 10A Describe, interpret, and sketch ... (VC2M10AA05)
- 10A Solve linear and non-linear simultaneous equations ... (VC2M10AA09)
- 10A Experiment with functions and relations using digital tools, ... (VC2M10AA10)
- 10A Explore the effect of increasingly small changes in the value of variables ... (VC2M10AM02)

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Lesson 4.1

Solving quadratic equations

Learning intentions

By the end of this lesson you will be able to ...

✔ solve quadratic equations through factorisation.

Inter-year links

Year 7	6.2 Terms, expressions and equations
Year 8	6.4 Solving simple quadratic equations
Year 9	4.1 Solving quadratic equations

Quadratic equations

Key content video
Solving quadratic equations

- The general form of a quadratic equation is: $ax^2 + bx + c = 0$, where a, b and c are constants and $a \neq 0$.
  → When $a = 1$, the equation is called monic; otherwise, the equation is non-monic.
- Non-monic quadratic equations in the form $ax^2 + bx + c = 0$ can be turned into a monic quadratic equation by dividing both sides of the equation by the coefficient of the **leading term**, a.

The Null Factor Law

- The **Null Factor Law** states that, if the product of factors is 0, one or more of the factors must be 0; that is:
  if $a \times b = 0$, then $a = 0$ or $b = 0$, or both $a = 0$ and $b = 0$.
- Some quadratic equations can be solved by making one side equal to zero and factorising the non-zero expression on the other side.
- When we apply the Null Factor Law to quadratic equations in factorised form, $(x + m)(x + n) = 0$, we get:
  $x + m = 0$ or $x + n = 0$, therefore $x = -m$ or $x = -n$.
- Some monic quadratic equations in the form $x^2 + bx + c = 0$ can be factorised to $(x + m)(x + n) = 0$, where $b = m + n$ and $c = m \times n$.
- Solutions can be written in a shorthand way using a ± ('plus or minus') sign.
  For example:
  $x = -2$ or $x = 2$ can be written as $x = \pm 2$.
- The expansion of a perfect square, $(a \pm b)^2 = a^2 \pm 2ab + b^2$, and the difference of two squares rule, $(a + b)(a - b) = a^2 - b^2$, can also be used to factorise quadratic equations.

Solving quadratic equations by completing the square

Key content video
Solving quadratic equations by completing the square

- For any quadratic equation of the form $ax^2 + bx + c = 0$, the same operations can be performed on both sides of the equation to create a perfect square on one side of the equation.

$$ax^2 + bx + c = 0$$
$$ax^2 + bx = -c$$
$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$
$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$
$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

- If the RHS of the equation is positive or zero after completing the square, you can take the square root of both sides of the equation to get the possible values of $x + \frac{b}{2a}$, then use inverse operations to find the value of x.
- There are two solutions to the equation $x^2 = c$ when $c > 0$. After completing the square, if the RHS is positive, remember to consider both the positive and negative values for $x + \frac{b}{2a}$.

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Worked example 4.1A Solving quadratic equations by factorising

Solve each of these equations.

a $(x + 5)(x - 4) = 0$ **b** $x^2 - 4x - 12 = 0$ **c** $x^2 - 16 = 0$ **d** $5x^2 = -35x$

THINK

a 1 Check that the quadratic equation is in factorised form and one side of the equation is equal to 0.

2 Apply the Null Factor Law by equating each factor to 0.

3 Solve each linear equation.

b 1 Check that one side of the quadratic equation is equal to 0.

2 Factorise the quadratic equation.

3 Apply the Null Factor Law.

4 Solve each linear equation.

c 1 Check that one side of the equation is equal to 0.

2 Factorise the quadratic expression using the 'difference of two squares' rule.

3 Apply the Null Factor Law.

4 Solve each linear equation. Use ± to write the solutions.

d 1 Rearrange the equation so one side of the equation is equal to 0.

2 Factorise the quadratic expression by taking out a common factor.

3 Apply the Null Factor Law.

4 Solve each linear equation.

WRITE

a $(x + 5)(x - 4) = 0$

$x + 5 = 0$ or $x - 4 = 0$

$x = -5$ or $x = 4$

b $x^2 - 4x - 12 = 0$

$(x - 6)(x + 2) = 0$

$x - 6 = 0$ or $x + 2 = 0$

$x = 6$ or $x = -2$

c $x^2 - 16 = 0$

$x^2 - 4^2 = 0$

$(x + 4)(x - 4) = 0$

$x + 4 = 0$ or $x - 4 = 0$

$x = -4$ or $x = 4$

$x = \pm 4$

d $5x^2 = -35x$

$5x^2 + 35x = 0$

$5x(x + 7) = 0$

$5x = 0$ or $x + 7 = 0$

$x = 0$ or $x = -7$

Worked example 4.1B Solving quadratic equations by completing the square

Solve each of these equations. Write the solutions as exact values.

a $4(x - 2)^2 - 1 = 0$ **b** $x^2 - 12x - 3 = 0$ **c** $-2x^2 + 8x + 8 = 0$

THINK

a 1 Use inverse operations to rearrange the equation, and leave the LHS as a perfect square.

2 Take the square root of both sides of the equation, and use ± to show the two roots.

3 Use inverse operations on both sides of the equation to solve for x.

WRITE

a $4(x - 2)^2 - 1 = 0$ $(+ 1)$

$4(x - 2)^2 = 1$ $(\div 4)$

$(x - 2)^2 = \frac{1}{4}$

$x - 2 = \pm\frac{1}{2}$ $(+ 2)$

$x = 2 \pm \frac{1}{2}$

$x = \frac{3}{2}$ or $x = \frac{5}{2}$

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b **1** Use inverse operations to move the constant term to the RHS of the equation.

2 Make the LHS of the equation a perfect square by adding $\left(\frac{-12}{2}\right)^2 = 36$ to both sides of the equation.

3 Write the LHS as a perfect square; simplify the RHS.

4 Take the square root of both sides of the equation, and use ± to show the two roots.

5 Use inverse operations on both sides of the equation to solve for x.

c **1** Divide all terms by the coefficient of the x^2 term to turn the quadratic equation into a monic quadratic equation.

2 Use inverse operations to move the constant term to the RHS of the equation.

3 Make the LHS of the equation a perfect square by adding $\left(\frac{-4}{2}\right)^2 = 4$ to both sides of the equation.

4 Write the LHS as a perfect square; simplify the RHS.

5 Take the square root of both sides of the equation, and use ± to show the two roots. Simplify the surd.

6 Use inverse operations on both sides of the equation to solve for x.

b $x^2 - 12x - 3 = 0$ $(+3)$

$x^2 - 12x = 3$ $(+36)$

$x^2 - 12x + 36 = 3 + 36$

$(x - 6)^2 = 39$

$x - 6 = \pm\sqrt{39}$ $(+6)$

$x = 6 \pm \sqrt{39}$

c $-2x^2 + 8x + 8 = 0$ $(\div -2)$

$x^2 - 4x - 4 = 0$ $(+4)$

$x^2 - 4x = 4$ $(+4)$

$x^2 - 4x + 4 = 4 + 4$

$(x - 2)^2 = 8$

$x - 2 = \pm\sqrt{8}$

$x - 2 = \pm 2\sqrt{2}$ $(+2)$

$x = 2 \pm 2\sqrt{2}$

Helpful hints

✔ Watch for changes in sign when solving quadratic equations for x.

$$(x + 6)(x - 2) = 0$$
$$x + 6 = 0 \text{ or } x - 2 = 0$$
$$x = -6 \text{ or } x = +2$$

✔ Although it is possible to add, subtract and multiply by x, you must not divide both sides of an equation by x, in the case of $x = 0$.

$$\frac{1}{0} = \text{undefined}$$

✔ The Null Factor Law can only be used to solve equations if one side of the equation is equal to zero. You cannot use the rule if the equation is equal to any other number.

For example, if $x(x - 2) = 1$, it does not imply that $x = 1$ or $x - 2 = 1$.

But, if $x(x - 2) = 0$, two possible solutions are $x = 0$ or $x - 2 = 0$.

✔ Being able to confidently factorise quadratic expressions is an important skill to have when solving quadratic equations. Go back to **Lessons 1.6** to **1.9** to revise this skill.

✔ When solving quadratic equations, look out for familiar rules like the difference of two squares and the expansion of a perfect square.

$$(a + b)(a - b) = a^2 - b^2$$
$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$

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Learning pathways

1–3(1st, 2nd column), 4, 5–9(1st, 2nd column), 10, 12, 15

1–3(2nd column), 5–9(2nd column), 11–13, 15, 16

1–3(3rd column), 5–9(3rd column), 12–18

ANS p701

Exercise 4.1A: Understanding and fluency

WE 4.1A **1** Solve each of the following equations. Show all steps of your working.

a $(x + 2)(x - 4) = 0$ **b** $(x + 7)(x + 1) = 0$ **c** $(x - 3)(x - 6) = 0$

d $(x + 9)(x - 9) = 0$ **e** $x(x - 5) = 0$ **f** $x(2x - 1) = 0$

g $2x(x + 1) = 0$ **h** $7x(2x + 4) = 0$ **i** $\frac{1}{2}x(4x - 3) = 0$

j $(x + \sqrt{2})(x - \sqrt{2}) = 0$ **k** $(x - \sqrt{5})(x + \sqrt{5}) = 0$ **l** $(x + 4\sqrt{3})(x - 4\sqrt{3}) = 0$

m $(2x + 3)(x - 4) = 0$ **n** $(3x - 1)(3x + 1) = 0$ **o** $(7x + 2)(4x - 5) = 0$

2 Solve each of these equations.

a $x^2 + 7x + 12 = 0$ **b** $x^2 + 3x - 10 = 0$ **c** $x^2 - 9x + 8 = 0$

d $x^2 - 25 = 0$ **e** $x^2 - 81 = 0$ **f** $4 - x^2 = 0$

g $x^2 - 3x = 0$ **h** $x^2 + x = 0$ **i** $2x^2 + 8x = 0$

j $10x^2 - 4x = 0$ **k** $3x^2 = 18x$ **l** $7x^2 = -70x$

3 Solve each of these quadratic equations, and leaving your answers as exact values.
Hint: First rearrange each equation into the form $ax^2 + bx + c = 0$.

a $x^2 + 4x = 12$ **b** $x^2 + 4 = 5x$ **c** $6x^2 = 54$

d $x^2 - 3x - 20 = 20$ **e** $x^2 - 6x = 3x$ **f** $x^2 + 1 = 4x$

g $5x - x^2 = x$ **h** $x^2 + 21 = 11x + 3$ **i** $2x^2 - 3x = x^2 + 18$

j $x(x + 4) = 21$ **k** $(x - 5)^2 = 1$ **l** $x^2 + 3x + 10 = 1 - 3x$

4 Consider $3(x + 6)(x - 8) = 0$.

a Why is this equivalent to $(x + 6)(x - 8) = 0$?

b Solve $(x + 6)(x - 8) = 0$ and then write the solutions for $3(x + 6)(x - 8) = 0$.

c Use substitution to check that you have the correct solutions.

5 Solve each of these quadratic equations.

a $2(x + 5)(x - 6) = 0$ **b** $7(x - 2)(x - 4) = 0$ **c** $-5(x + 8)(x + 1) = 0$

d $-4(x + 9)(x - 3) = 0$ **e** $-11(x - 4)(x + 4) = 0$ **f** $-x(x + 7) = 0$

g $-6x(x - 2) = 0$ **h** $-2x(3x + 5) = 0$ **i** $-8(2x - 3)(x + 10) = 0$

6 Solve each of these quadratic equations.
Hint: Take out a common factor first.

a $2x^2 + 6x + 4 = 0$ **b** $3x^2 + 6x - 45 = 0$ **c** $6x^2 - 30x + 24 = 0$

d $-5x^2 - 5x + 30 = 0$ **e** $-4x^2 + 16x + 48 = 0$ **f** $-x^2 - x + 56 = 0$

g $11x^2 - 44 = 0$ **h** $-7x^2 + 63 = 0$ **i** $-16x^2 + 8x = 0$

7 Solve each of these equations. Write the solutions as exact values.

a $4(x - 2)^2 = 0$ **b** $3(x - 3)^2 - 9 = 0$ **c** $(2 - 2x)^2 - 7 = 9$

d $x^2 - 12x = 4$ **e** $x^2 - 2x - 1 = 0$ **f** $-x^2 + 12x - 3 = 0$

WE 4.1B **8** Solve each of these quadratic equations by completing the square.

a $x^2 + 2x - 6 = 0$ **b** $x^2 - 4x + 2 = 0$ **c** $x^2 + 8x + 11 = 0$

d $x^2 - 12x + 28 = 0$ **e** $x^2 + 20x + 25 = 0$ **f** $x^2 - x - 1 = 0$

g $3x^2 + 18x + 21 = 0$ **h** $4x^2 + 8x - 16 = 0$ **i** $-x^2 + 4x - 1 = 0$

j $-6x^2 - 48x - 36 = 0$ **k** $2x^2 - 12x - 3 = 0$ **l** $5x^2 + 5x - 2 = 0$

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9 Solve each of these quadratic equations.

a $2x^2 + 13x + 15 = 0$ **b** $5x^2 - 22x + 8 = 0$ **c** $6x^2 + x - 2 = 0$

d $4x^2 + 25x - 21 = 0$ **e** $3x^2 + 2x = 1$ **f** $7x^2 = 11x + 6$

Exercise 4.1B: Problem solving and reasoning

10 Explain why you can divide both sides of $3x(x - 2) = 0$ by 3, but you should not divide both sides of the equation by $3x$.

11 How many real solutions does each of these quadratic equations have?

a $(x - 3)(x - 5) = 0$ **b** $(x + 2)(x + 2) = 0$ **c** $(x - 4)(x + 4) = 0$

d $(x - 7)^2 = 0$ **e** $x(x - 6) = 0$ **f** $x^2 + 9 = 0$

12 Find solution/s for each of these equations, if real solutions exist. Hint: Some of these equations cannot be solved using real numbers.

a $x^2 - 9 = 0$ **b** $x^2 + 4x - 77 = 0$ **c** $x^2 - 8x + 16 = 0$

d $x^2 + 4 = 0$ **e** $x^2 + 4x = 0$ **f** $x^2 - 6x - 1 = 0$

13 Nutsa dives with her arms outstretched from a platform into a diving pool. The height of her fingertips above the pool surface can be represented by the quadratic relationship $h = -2t^2 + 4t + 6$, where h is the height of her fingertips above the water, in metres, after t seconds.

a What is the height of Nutsa's fingertips above the water after:

i 1 second **ii** 2 seconds?

b What is the height of Nutsa's fingertips when she starts the dive?

c How long does the dive take? Hint: How long does it take for Nutsa's fingertips to hit the surface of the water?

d Explain why there is only one time value for your answer to part **c**, even though you have solved a quadratic equation that has two solutions.

14 Oliver buys rice to serve with stir-fried meals over the coming weeks. As time passes, the amount of rice Oliver has left can be represented by the quadratic relationship $y = x^2 - 10x + 25$, where y is the amount left, in kilograms, after x weeks.

a How much rice did Oliver buy?

b How much rice is left after 2 weeks?

c How long does it take for all of the rice to be used?

d Did Oliver use the same amount of rice each week? Explain.

15 The area of a rectangular vegetable plot is 70 m^2. The length is 3 m longer than the breadth.

a Write a quadratic equation to represent this. Hint: Let x represent the width of the vegetable plot, in metres.

b Solve the quadratic equation.

c Write the dimensions of the vegetable plot.

16 The width of a mobile phone screen is 5 cm less than its length. Write an equation to represent the situation if the area of the screen is 66 cm^2, and then solve your equation to find the dimensions of the screen.

Exercise 4.1C: Challenge

17 Solve each of the following equations to find the value(s) of x by substituting the relation given in the brackets into the equation.

a $x^4 - 12x^2 + 36 = 0$ $(p = x^2)$ **b** $\frac{1}{x^2} + \frac{4}{x} + 4 = 0$ $\left(q = \frac{1}{x}\right)$

18 If $x - \frac{1}{x} = 3$, find the value of $x^2 + \frac{1}{x^2}$.

Online resources:

Interactive skillsheet Solving quadratic equations

Investigation Beware of irreversible operations

Quick quiz 4.1

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Lesson 4.2

The quadratic formula

Learning intentions

By the end of this lesson you will be able to ...

✔ solve quadratic equations using the quadratic formula.

Inter-year links

Year 7	6.4 Substitution
Year 8	5.2 Substitution

The quadratic formula

Key content video
The quadratic formula

- The **quadratic formula** can be used to solve equations of the general form $ax^2 + bx + c = 0$, where a, b and c are constants and $a \neq 0$.
- The solutions for $ax^2 + bx + c = 0$ are:

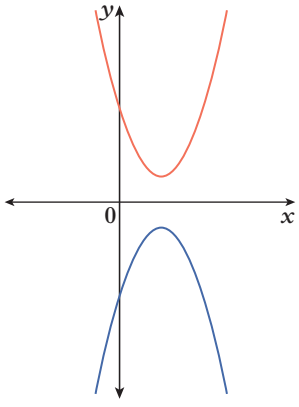
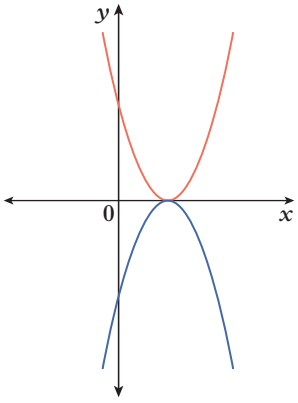
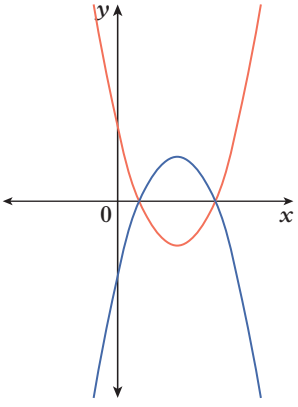
$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

This can be written more concisely as:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The discriminant

- The **discriminant** is $\Delta = b^2 - 4ac$. This is the expression under the square root in the quadratic formula. Evaluating this expression after substituting values for a, b and c allows us to identify the number of real solutions for the quadratic equation.
  - → If $\Delta < 0$, there are no real solutions as the square root of a negative number is not a real number.
  - → If $\Delta = 0$, there is one real solution. It is $x = \frac{-b}{2a}$ and the quadratic expression is a perfect square.
  - → If $\Delta > 0$, there are two real solutions. They are $x = \frac{-b \pm \sqrt{\Delta}}{2a}$.

Discriminant	$\Delta < 0$ (negative)	$\Delta = 0$ (zero)	$\Delta > 0$ (positive)
Number of solutions	No real solutions	One real solution	Two real solutions
Graph			

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Worked example 4.2A Identifying the number of solutions to a quadratic equation

Using the discriminant, identify the number of solutions to each of these quadratic equations.

a $3x^2 + 7x + 2 = 0$ **b** $x^2 + 16 = 8x$ **c** $-2x^2 + 3x - 5 = 0$

THINK

1. Rearrange each equation so that the right-hand side is 0.
2. Identify a, b and c.
3. Write the discriminant formula and substitute values for a, b and c. Simplify the result.
4. Identify the number of real solutions.

WRITE

a $3x^2 + 7x + 2 = 0$

$a = 3, b = 7, c = 2$

$\Delta = b^2 - 4ac$

$= 7^2 - 4 \times 3 \times 2$

$= 49 - 24$

$= 25$

Since $\Delta > 0$, the equation has two real solutions.

b $x^2 + 16 = 8x$

$x^2 - 8x + 16 = 0$

$a = 1, b = -8, c = 16$

$\Delta = b^2 - 4ac$

$= (-8)^2 - 4 \times 1 \times 16$

$= 64 - 64$

$= 0$

Since $\Delta = 0$, the equation has one real solution.

c $-2x^2 + 3x - 5 = 0$

$a = -2, b = 3, c = -5$

$\Delta = b^2 - 4ac$

$= 3^2 - 4 \times (-2) \times (-5)$

$= 9 - 40$

$= -31$

Since $\Delta < 0$, the equation has no real solutions.

Worked example 4.2B Using the quadratic formula to solve quadratic equations

Use the quadratic formula to solve each of these equations. If solutions exist, write them as **i** exact values, and **ii** approximate values, correct to two decimal places.

a $2x^2 - 5x + 1 = 0$ **b** $9x^2 + 12x + 4 = 0$ **c** $4x^2 = 3x - 1$

THINK

1. If necessary, rearrange the equation so that one side of the equation equals 0.
2. Identify a, b and c.
3. Determine the discriminant and identify the number of solutions.
4. Write the quadratic formula and substitute in values for a, b and c. Simplify the result.
5. Write the solutions as exact values.
6. Use a calculator to help you rewrite the values, correct to two decimal places.

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WRITE

a $2x^2 - 5x + 1 = 0$

$a = 2, b = -5, c = 1$

$\Delta = b^2 - 4ac$

$= (-5)^2 - 4 \times 2 \times 1$

$= 25 - 8$

$= 17$

$\Delta > 0$. The equation has two real solutions.

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{-(-5) \pm \sqrt{17}}{2(2)}$

$= \dfrac{5 \pm \sqrt{17}}{4}$

i $x = \dfrac{5 - \sqrt{17}}{4}$ or $\dfrac{5 + \sqrt{17}}{4}$

ii $x \approx 0.22$ or 2.28

b $9x^2 + 12x + 4 = 0$

$a = 9, b = 12, c = 4$

$\Delta = b^2 - 4ac$

$= (12)^2 - 4 \times 9 \times 4$

$= 144 - 144$

$= 0$

$\Delta = 0$. The equation has one real solution.

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{-12 \pm \sqrt{0}}{2(9)}$

$= \dfrac{-12}{18}$

$= -\dfrac{2}{3}$

i $x = -\dfrac{2}{3}$

ii $x \approx -0.67$

c $4x^2 = 3x - 1$

$4x^2 - 3x + 1 = 0$

$a = 4, b = -3, c = 1$

$\Delta = b^2 - 4ac$

$= (-3)^2 - 4 \times 4 \times 1$

$= 9 - 16$

$= -7$

$\Delta < 0$. The equation has no real solutions.

Helpful hints

- ✔ If the discriminant is positive, don't forget to evaluate both possible solutions using the quadratic formula.

  Remember that the quadratic formula is equal to $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, so:

  $$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ or } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

- ✔ Make sure one side of the equation equals zero before identifying a, b and c.
- ✔ When substituting the values of a, b and c into the quadratic formula, include the sign of each number.

  For example, if $-2x^2 - 6x - 4 = 0$, then $a = -2$, $b = -6$ and $c = -4$.

  So, the solutions are $x = \dfrac{-(-6) \pm \sqrt{(-6)^2 - 4(-2)(-4)}}{2(-2)}$, which simplifies to $x = -2$ or $x = -1$.

- ✔ If you have already calculated the discriminant of a quadratic equation, you can substitute it into the quadratic formula.

  For example, $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-b \pm \sqrt{\Delta}}{2a}$.

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Learning pathways

1, 2, 3–4(1st, 2nd columns), 5, 6, 8, 11, 13

2, 3–4(2nd column), 5–10, 14, 15, 18

2–4(3rd column), 5–7, 11, 14–18

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Exercise 4.2A: Understanding and fluency

WE 4.2A **1** Using the discriminant, identify the number of real solutions to each of these quadratic equations.

a $x^2 + 5x + 7 = 0$ **b** $3x^2 - 8x - 4 = 0$ **c** $x^2 - 9 = 0$

d $2x^2 + 3x + 4 = 0$ **e** $12x - x^2 = 36$ **f** $6x^2 - 5 = 7x$

2 For each of the quadratic equations below:

i calculate the discriminant

ii identify the number of real solutions

iii solve by first factorising and then using the Null Factor Law

iv check whether your answers for part **iii** confirms your result from part **ii**.

a $x^2 - 2x - 15 = 0$ **b** $x^2 + 4x + 4 = 0$ **c** $x^2 - 49 = 0$

d $3x^2 - 18x = 0$ **e** $x^2 - 6x + 7 = 0$ **f** $x^2 + 2x + 3 = 0$

WE 4.2B **3** Use the quadratic formula to solve each of these equations. Write any real solution(s) as: **i** exact values and **ii** approximate values, correct to two decimal places.

a $x^2 - 7x + 5 = 0$ **b** $3x^2 - x - 1 = 0$ **c** $2x^2 + x - 4 = 0$

d $-3x^2 - x + 9 = 0$ **e** $-6x^2 - 5x + 3 = 0$ **f** $x^2 - 2x - 1 = 0$

g $x^2 + 8x + 4 = 0$ **h** $4x^2 - 12x + 1 = 0$ **i** $-x^2 + 4x + 2 = 0$

j $5x^2 + 10x - 3 = 0$ **k** $2x^2 = 11 - 3x$ **l** $x^2 + 2x = 7$

4 Use the quadratic formula to find any real solutions to each of the following quadratic equations. If solutions exist, write them as exact values.

a $x^2 + 3x = 1$ **b** $x^2 + 6x + 12 = 0$ **c** $\frac{1}{2}x^2 + 3x + 2 = 0$

d $x^2 + 3x + 5 = 0$ **e** $2x^2 + 5x + 1 = 0$ **f** $x^2 + 16 = 8x$

g $x^2 + 2x = 4$ **h** $x^2 = 3 - 7x$ **i** $10 = x^2 + 8x$

j $x^2 - 5 = x$ **k** $x^2 + 5 = x$ **l** $x^2 + \sqrt{5}x = 1$

5 Use the quadratic formula to find the solutions to each of the following quadratic equations.

a $m^2 + 3m - 7 = 0$ **b** $u^2 + 4u = 1$ **c** $r^2 + 9r + 1 = 0$

d $y^2 - 3 = 2y$ **e** $3h^2 + 7h = 1$ **f** $w^2 = 5 + 6w$

6 Solve $\frac{1}{x} = x - 3$ by first multiplying both sides by x to create a quadratic equation.

Exercise 4.2B: Problem solving and reasoning

7 Consider the quadratic equation $x^2 - 5x + 6 = 0$.

a Solve the equation by factorising the left-hand side and using the Null Factor Law.

b Now solve the equation again, using the quadratic formula.

c Which method was quicker or easier to use?

d The solutions to any quadratic equation can be found using the quadratic formula. Why might you try using the method described in part **a** first?

8 When answering each of these, consider $\sqrt{b^2 - 4ac}$ in the quadratic formula.

a Explain why there are no real solutions to a quadratic equation when the discriminant is negative.

b Explain why there is only one real solution when the discriminant is zero.

c Explain why there are two real solutions when the discriminant is positive.

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9 Copy and complete the lines of working below to derive the quadratic formula. You will complete the square to factorise the left-hand side and then use the Null Factor Law to find the solutions for $ax^2 + bx + c = 0$ (where $a \neq 0$).

THINK	**WRITE**
1 Write the quadratic equation.	$ax^2 + bx + c = 0$
2 Factorise by taking out the coefficient of x^2.	$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0$
3 Divide both sides by a.	$x^2 + \frac{b}{a}x + \frac{\square}{\square} = 0$
4 Add a new third term to complete the square and compensate by subtracting the same term.	$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{\square} = 0$
5 Form a perfect square using the first three terms and simplify the fourth term.	$\left(x + \frac{b}{2a}\right)^2 - \frac{\square}{4a^2} + \frac{c}{a} = 0$
6 Group the last two terms together.	$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2}{4a^2} - \frac{c}{a}\right) = 0$
7 Combine the last two terms using a common denominator.	$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\square - 4ac}{4a^2}\right) = 0$
8 Write the left-hand side as the difference of two squares.	$\left(x + \frac{b}{2a}\right)^2 - \left(\sqrt{\frac{b^2 - 4ac}{4a^2}}\right)^2 = 0$
9 Simplify the second square.	$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2 = 0$
10 Factorise the left-hand side using the 'difference of two squares' rule.	$\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} - \square\right) = 0$
11 Use the Null Factor Law.	$x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} = 0$ or $x + \frac{b}{2a} - \square = 0$
12 Solve each linear equation.	$x = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$ or $x = -\frac{b}{2a} + \square$
13 Simplify each solution.	$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ or $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$

The quadratic formula for finding the solutions to $ax^2 + bx + c = 0$ is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

10 Quadratic equations can be solved in different ways depending on the form of the equation.

a Copy and complete the lines below to derive the solution of a quadratic equation in the form $a(x - h)^2 + k = 0$.

$a(x - h)^2 + k = 0$

$a(x - h)^2 =$ _____

$(x - h)^2 =$ _____

$x - h =$ _____

$x =$ _____

b How can you use the value of a, h and k to determine if a quadratic equation in the form $a(x - h)^2 + k = 0$ would have zero, one or two real solutions?

c Determine how many real solutions each of the following quadratic equations have, using your conclusion in part **b**.

i $4(x - 5)^2 - 1 = 0$ **ii** $5(x + 4)^2 = 0$ **iii** $-3(x - 1)^2 - 2 = 0$

d Find the solutions of the quadratic equations in part **c** using the formula in part **a** if real solutions exist.

11 The profit, p (in dollars), made from selling n soft toys is given by the relationship $p = -0.1n^2 + 17n + 200$.

a What is the profit when 30 toys are sold?

b What is the smallest number of toys that need to be sold to make a profit of at least \$850?

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12 A rectangular paddock has an area of 5000 m^2; the length is 20 m greater than the width. Solve a quadratic equation to find the dimensions of the paddock, correct to one decimal place.

13 The perimeter of a framed painting is 100 cm. Within the 2 cm frame, the area of the painting that can be viewed is 300 cm^2. Find the dimensions of the framed painting, correct to the nearest millimetre.

14 The discriminant of a quadratic equation can be used to find the number of solutions, but it can also be used to find the nature of the solutions, if they exist. If a quadratic equation has integer coefficients and the discriminant is a square number, then the solutions are rational (that is, integers or fractions).

a Consider each of the following equations with integer coefficients. Find the discriminant and use it to determine whether the solutions are rational or irrational.

i $0 = x^2 + 5x - 2$

ii $0 = 2x^2 + 3x - 2$

iii $0 = x^2 - 7x + 10$

iv $0 = 5x^2 + 11x + 4$

v $0 = 3x^2 - x - 4$

b Explain with reference to the quadratic formula why, if the discriminant is a square number, the solutions are guaranteed to be rational, assuming all coefficients are integers.

15 The quadratic formula can be used to identify the x-intercepts of a quadratic relationship.

a Find the coordinates of the x-intercepts, if they exist, of the graphs of each of the following quadratic relationships. You do not need to sketch the graph.

i $y = x^2 + 7x + 10$

ii $y = x^2 - 6x + 8$

iii $y = x^2 + 3x + 5$

iv $y = x^2 - 8x + 16$

v $y = -2x^2 + 6x - 7$

b Explain how the discriminant of a quadratic relationship relates to the number of x-intercepts of its graph.

Exercise 4.2C: Challenge

16 Consider the equation $kx^2 + 6x + 3 = 0$.

a Find the value of k so that the equation has only one real solution.

b For what values of k does the equation have:

i two real solutions

ii no real solutions?

17 For each of these equations, find the value(s) of k that give the number of real solutions shown in square brackets.

a $x^2 + 2x + k = 0$ [one real solution]

b $2x^2 - 4x + k = 0$ [two real solutions]

c $kx^2 + 6x - 1 = 0$ [no real solutions]

d $kx^2 - 3x + 1 = 0$ [one real solution]

e $3x^2 + kx + 3 = 0$ [two real solutions]

f $x^2 + kx + 25 = 0$ [no real solutions]

18 A quadratic equation is in the form $ax^2 + bx + 5 = 0$.

a Find the discriminant of the equation in terms of a and b.

b If the equation has one and only one solution, write the solution in terms of a and b.

c If $x = -1$ is the only solution to the equation, use your answers to parts **a** and **b** to find the values of a and b.

Online resources:

Interactive skillsheet The quadratic formula

Worksheet Solving quadratic equations

Quick quiz 4.2

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Lesson 4.3

Sketching parabolas

Learning intentions

By the end of this lesson you will be able to ...

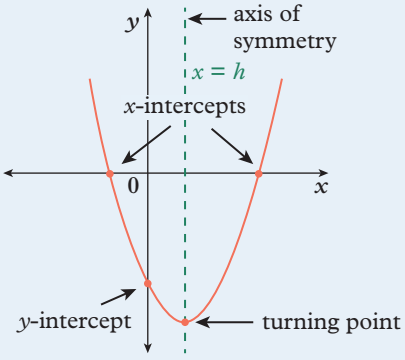
✔ sketch parabolas using intercepts or transformations.

Inter-year links

Year 7	8.3 Translations
Year 8	9.1 Transformations
Year 9	4.3 Sketching parabolas using intercepts
Year 9	4.4 Sketching parabolas using transformations

Parabolas

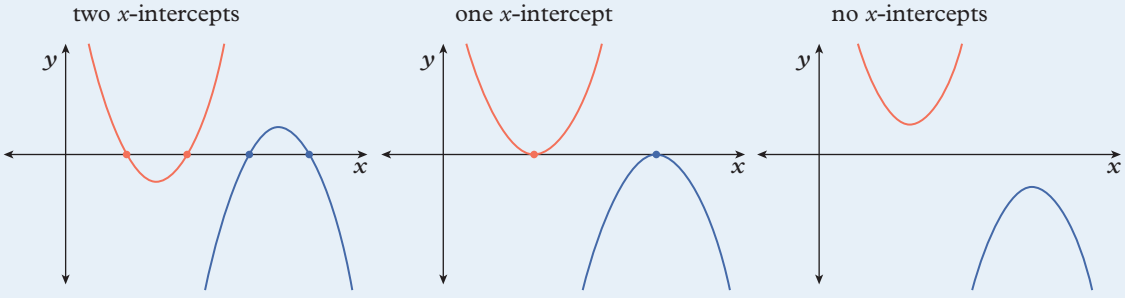
- The graph of a quadratic relationship is called a **parabola**.
- The key features of a parabola include:
  - → x- and y-intercepts
  - → a **turning point** (or **vertex**)
  - → an **axis of symmetry**, $x = h$, where h is the x-coordinate of the turning point.
- Quadratic relationships are written in **general form** when they are written as $y = ax^2 + bx + c$.



Sketching parabolas using intercepts

Key content video
Sketching parabolas using intercepts

- One way of sketching a parabola is to use the x- and y-intercepts.
  - → The y-intercept is found by substituting $x = 0$ into the equation and simplifying.
  - → The x-intercept(s) are found by substituting $y = 0$ into the equation and solving for x. A parabola can have two, one or no x-intercepts.



- Quadratic relationships with two x-intercepts $(x_1, 0)$ and $(x_2, 0)$ can be written in **intercept form**, as shown below.

$$y = a(x - x_1)(x - x_2)$$

Leading coefficient
If $a > 0$, upright
If $a < 0$, inverted

x-intercepts
$(x_1, 0)$, $(x_2, 0)$

- A quadratic relationship in general form can be written in intercept form by factorising the right-hand side (RHS).

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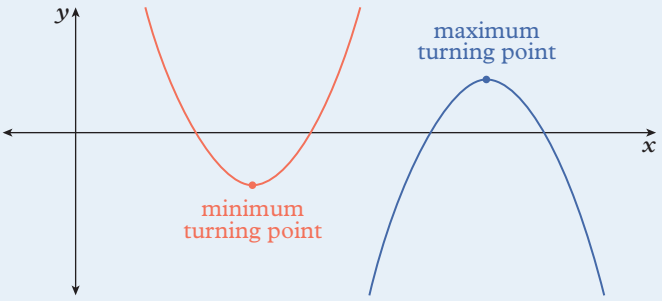
- A **minimum turning point** is the turning point of a graph that opens upwards or is 'upright'.
- A **maximum turning point** is the turning point of a graph that opens downwards or is 'inverted'.
- Turning points can be identified using the axis of symmetry or the midpoint between the x-intercepts (or any two x values with the same y value).



Sketching parabolas using transformations

Key content video
Sketching parabolas using transformations

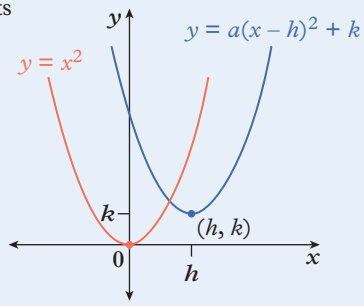
- Quadratic relationships can be written in **turning point form**, as shown below, where the coordinates of the turning point are (h, k):

$$y = a(x - h)^2 + k$$

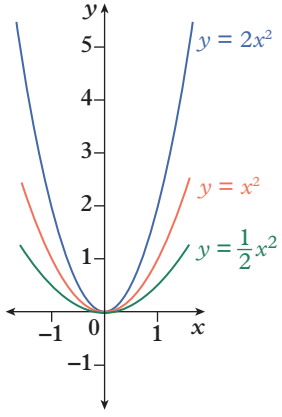
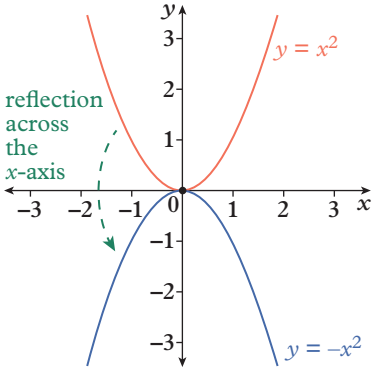
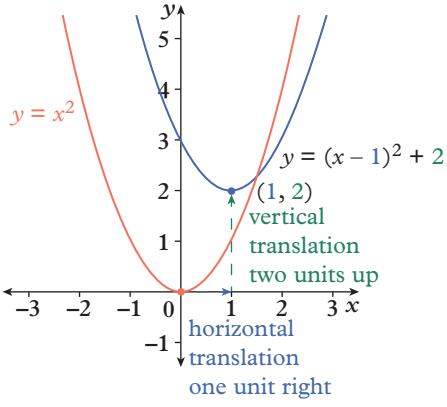
Vertical translation of k units
If $k > 0$, up
If $k < 0$, down

Dilate in the y-direction
If $a > 0$, upright
If $a < 0$, inverted

Horizontal translation of h units
If $h > 0$, right
If $h < 0$, left



- The following **transformations** can be performed on the graph of $y = x^2$ to produce the graph of $y = a(x - h)^2 + k$:
  - → a **dilation** of a factor of a in the y-direction. If a is greater than 1 or less than –1, the parabola becomes narrower. If a is between 1 and –1, the parabola becomes wider.
  - → a **reflection** across the x-axis if $a < 0$
  - → a horizontal **translation** of h units right (or left if $h < 0$)
  - → a vertical translation of k units up (or down if $k < 0$).

Dilation	Reflection	Translation
		

- Transformations on the graph of $y = x^2$ should be described in the order: Dilations, Reflections, Translations.

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Worked example 4.3A Sketching a parabola using intercepts

Sketch the graph of $y = -x^2 + 4x + 5$ by first finding the x- and y-intercepts. Label the turning point with its coordinates.

THINK

1 Find the x-intercepts by substituting $y = 0$ into the relationship and solving it for x. Factorise so that the Null Factor Law can be used.

2 Find the y-intercept by substituting $x = 0$ into the relationship and simplifying.

3 Find the coordinates of the turning point. First, calculate the x-coordinate by finding the midpoint of the two x-intercepts, then substitute that value into the equation to find the corresponding y value. Use both values to write down the coordinates of the turning point.

4 Plot the intercepts and the turning point on the Cartesian plane.

5 Draw an inverted parabola through the points and label the graph with the equation.

WRITE

For the x-intercepts, $y = 0$:

$$-x^2 + 4x + 5 = 0$$
$$-(x^2 - 4x - 5) = 0$$
$$x^2 - 4x - 5 = 0$$
$$(x + 1)(x - 5) = 0$$
$$x + 1 = 0 \quad \text{or } x - 5 = 0$$
$$x = -1 \text{ or } \quad x = 5$$

The x-intercepts are $(-1, 0)$ and $(5, 0)$.

For the y-intercept, $x = 0$:

$$y = -(0)^2 + 4(0) + 5$$
$$= 5$$

The y-intercept is $(0, 5)$.

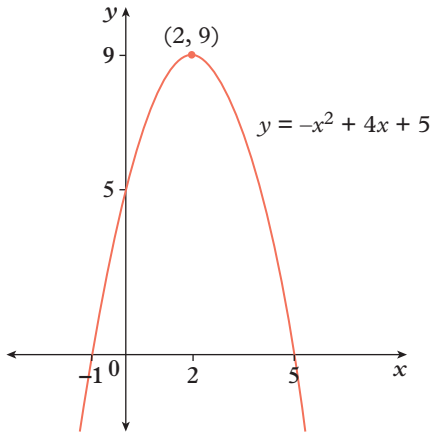
For the x-coordinate of turning point:

$$x = \frac{-1 + 5}{2}$$
$$= \frac{4}{2}$$
$$= 2$$

When $x = 2$:

$$y = -(2)^2 + 4(2) + 5$$
$$= -4 + 8 + 5$$
$$= 9$$

The turning point is $(2, 9)$.



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Worked example 4.3B Sketching a parabola using two transformations

Sketch the graph of each of the following parabolas by performing transformations on $y = x^2$. Label the y-intercept and the turning point each time.

a $y = -(x + 3)^2$

b $y = (x - 2)^2 - 1$

THINK

a **1** Identify the transformations. The equation has the form $a(x - h)^2$ where $a = -1$, so the graph of $y = x^2$ is reflected in the x-axis. Because $h = -3$, the graph of $y = x^2$ also undergoes a horizontal translation of three units to the left, resulting in a turning point of $(h, k) = (-3, 0)$.

2 Find the y-intercept by substituting $x = 0$ into the equation and simplifying.

3 Sketch the graphs of $y = x^2$ and $y = -(x + 3)^2$, and label their turning points and intercepts.

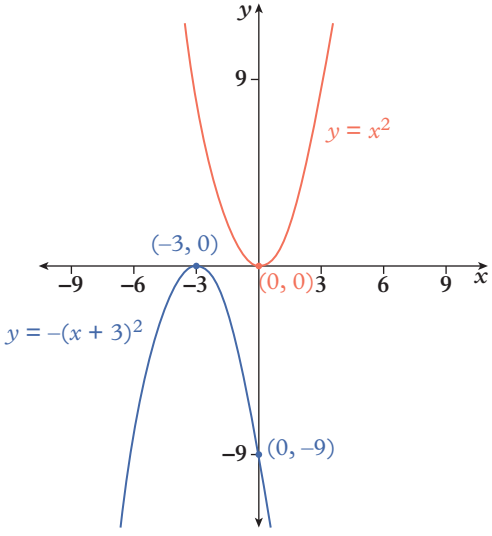
WRITE

a The turning point is $(-3, 0)$.

For the y-intercept, $x = 0$:

$y = -(x + 3)^2 = -(0 + 3)^2 = -9$

The y-intercept is $(0, -9)$.



b **1** Identify the transformations. The equation has the form $(x - h)^2 + k$, where $h = 2$ and $k = -1$, so the graph of $y = x^2$ undergoes a horizontal translation of two units to the right and a vertical translation of one unit down, giving a turning point of $(h, k) = (2, -1)$.

2 Find the y-intercept by substituting $x = 0$ into the equation and simplifying.

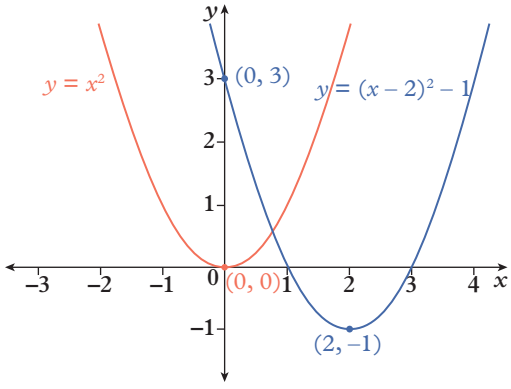
3 Sketch the graphs of $y = x^2$ and $y = (x - 2)^2 - 1$, and label their turning points and intercepts.

b The turning point is $(2, -1)$.

For the y-intercept, $x = 0$:

$y = (x - 2)^2 - 1 = (0 - 2)^2 - 1 = 3$

The y-intercept is $(0, 3)$.



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Worked example 4.3C Sketching a parabola using multiple transformations

Sketch the graph of $y = x^2$ on a Cartesian plane, then perform transformations on $y = x^2$ to sketch the graph of $y = -3(x - 1)^2 + 2$. For both graphs, label the y-intercept and the turning point.

THINK

1 Identify the transformations. The equation has the form $a(x - h)^2 + k$ where $a = -3$, $h = 1$ and $k = 2$, so the graph of $y = x^2$ is:
- reflected in the x-axis
- dilated in the y-direction by a factor of 3
- translated horizontally 1 unit to the right
- translated vertically 2 units up

resulting in a turning point of $(h, k) = (1, 2)$.

2 Find the y-intercept by substituting $x = 0$ into the equation and simplifying.

3 Sketch the graphs of $y = x^2$ and $y = -3(x - 1)^2 + 2$, and label their turning points and intercepts.

WRITE

$$y = -3(x - 1)^2 + 2$$

Reflection in the x-axis and dilated in the y-direction by a factor of 3

Horizontal translation of one unit to the right

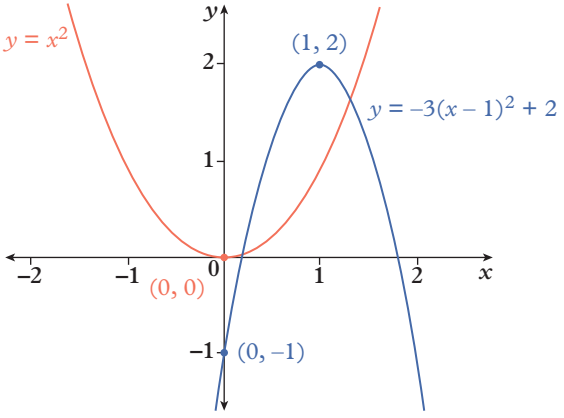
Vertical translation of two units up

The turning point is $(1, 2)$.

For the y-intercept, $x = 0$:

$$\begin{aligned} y &= -3(x - 1)^2 + 2 \\ &= -3(0 - 1)^2 + 2 \\ &= -1 \end{aligned}$$

The y-intercept is $(0, -1)$.



Helpful hints

- ✔ The solutions to the equation $0 = ax^2 + bx + c$ are the x-intercepts of the graph of $y = ax^2 + bx + c$.
- ✔ In turning point form:
  - → the size and direction of vertical translations are determined by the value of k
  - → the size and direction of horizontal translations are determined by the value of h.
- ✔ Always label key features when sketching a parabola. These include any axis intercepts and the turning point.

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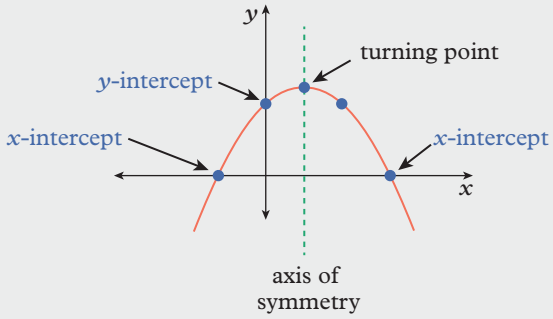
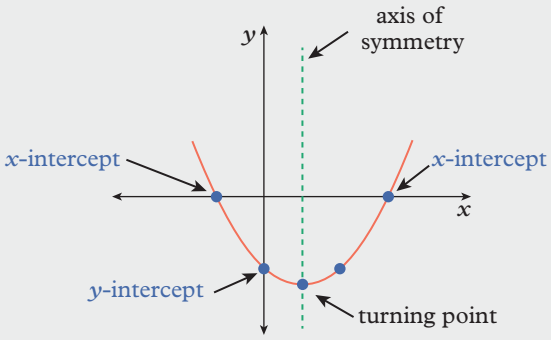
✔ If you need help drawing a symmetrical curve, draw a dashed line of symmetry and reflect points drawn on one side of the line to the other side.





✔ For $y = a(x - h)^2 + k$, if $-1 < a < 1$, the graph of $y = x^2$ has undergone a 'dilation' in the direction of the y-axis which compresses the graph.

For example, if $y = x^2$ becomes $y = \frac{1}{2}x^2$, then it has been 'dilated' by a factor of $\frac{1}{2}$ in the direction of the y-axis, which really means it has been compressed by a factor of 2 in the direction of the y-axis and has become wider.

Learning pathways

1–3, 4(a–c), 5–8, 9(a, b, d, e), 10–12, 14, 15, 17, 18, 20	1–2(a–c), 3(b), 4(d–f), 5, 6–7(d–f), 8, 9(a, c, d, f), 10–16, 18, 19(a, b), 20	1–2(d–f), 3(b), 4(g–i), 5, 8(d–f), 9, 10(d–f), 11–15, 17, 19–21

Exercise 4.3A: Understanding and fluency

1 For each quadratic relationship below:

i find the coordinates of any x-intercepts

ii find the coordinates of the y-intercept

iii state whether its graph is upright or inverted

iv determine the axis of symmetry

v find the coordinates of the turning point.

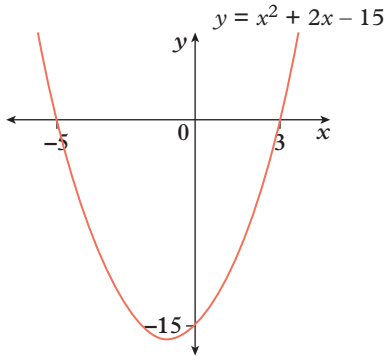
a $y = x^2 + 6x$ **b** $y = -x^2 + 10x$ **c** $y = x^2 - 6x + 5$

d $y = x^2 - 1$ **e** $y = -x^2 - 4x + 12$ **f** $y = 9 - x^2$

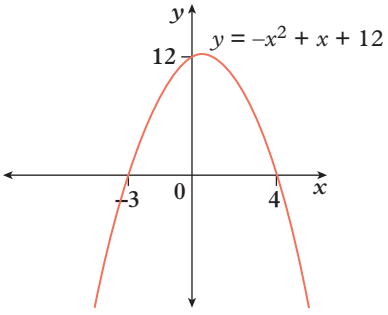
2 Sketch the graph of each quadratic relationship in question **1**, using your answers from parts **i-v** to help you.

3 Find the coordinates of the turning point for each of these parabolas.

a



b



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WE 4.3A **4** Use intercepts to help you sketch the graph of each quadratic relationship below. Label each turning point with its coordinates.

a $y = x^2 - 6x + 8$ **b** $y = -x^2 + 10x - 16$ **c** $y = x^2 + 4x - 12$

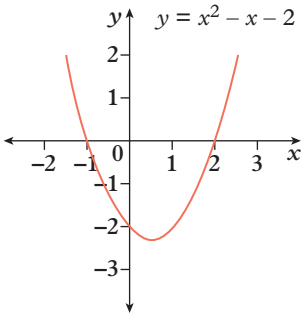
d $y = 4 - x^2$ **e** $y = x^2 + 2x$ **f** $y = -x^2 + 10x - 25$

g $y = x^2 - 4x - 21$ **h** $y = x^2 - x - 6$ **i** $y = x^2 + 7x + 10$

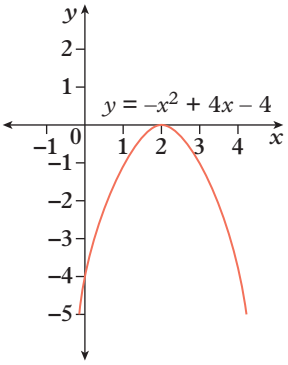
5 Consider the graphs of the three quadratic relationships shown below.

a How many x-intercepts does each parabola have?

i



ii



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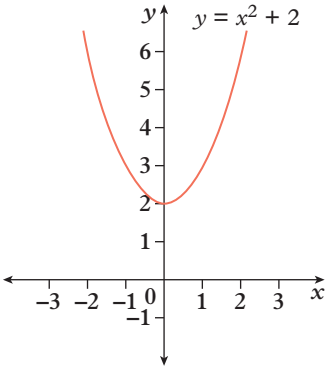

b Use the graphs in part **a** to help you solve the following:

i $x^2 - x - 2 = 0$ **ii** $-x^2 + 4x - 4 = 0$ **iii** $x^2 + 2 = 0$.

6 Perform transformations on the graph of $y = x^2$ to help you sketch the graph of each quadratic equation below. Clearly show the coordinates of the turning point each time.

a $y = x^2 + 3$ **b** $y = x^2 + 2$ **c** $y = x^2 - 5$

d $y = x^2 + 6$ **e** $y = x^2 - 4$ **f** $y = x^2 - 7.5$

7 Perform a transformation on the graph of $y = x^2$ to help you sketch the graph of each quadratic equation below. Clearly show the y-intercept and the coordinates of the turning point each time.

a $y = (x - 4)^2$ **b** $y = (x + 3)^2$ **c** $y = (x - 5)^2$

d $y = (x + 2)^2$ **e** $y = (x - 8)^2$ **f** $y = (x + 1.5)^2$

WE 4.3B **8** Sketch the graph of each of the following quadratic relationships by performing transformations on $y = x^2$. Label the y-intercept and the turning point each time.

a $y = (x - 1)^2 + 2$ **b** $y = (x + 5)^2 - 3$ **c** $y = -(x - 2)^2 + 7$

d $y = -(x + 4)^2 - 6$ **e** $y = (x + 7)^2 + 4$ **f** $y = -(x - 6)^2 - 1$

WE 4.3C **9** For each quadratic relationship:

i identify whether its graph is narrower or wider than the graph of $y = x^2$

ii identify whether the parabola is upright or inverted

iii identify the coordinates of the turning point

iv sketch the graph.

a $y = 2(x - 4)^2 - 8$ **b** $y = -3(x + 1)^2 + 12$ **c** $y = 5(x - 3)^2$

d $y = -4x^2 + 1$ **e** $y = \frac{1}{3}(x + 6)^2 - 3$ **f** $y = -\frac{1}{2}(x - 2)^2 - 9$

10 Write the rule for the parabola that would be produced by performing each transformation (or sequence of transformations) below on the graph of $y = x^2$.

a dilated in the y-direction by a factor of 6

b reflected in the x-axis

c translated 5 units right

d translated 2 units down

e dilated in the y-direction by a factor of $\frac{1}{4}$ and reflected in the x-axis

f reflected in the x-axis and translated 3 units up

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Exercise 4.3B: Problem solving and reasoning

11 Identify intervals on the real number line over which the quadratic relationships in question **8** are positive.

12 Write the equation for the parabola produced after performing each sequence of transformations below on the graph of $y = x^2$.

a Dilated in the y-direction by a factor of 2 then translated 5 units right.

b Reflected in the x-axis then translated 3 units down and 4 units left.

c Dilated in the y-direction by a factor of $\frac{1}{3}$ and reflected in the x-axis.

d Dilated in the y-direction by a factor of 5, reflected in the x-axis then translated 7 units up.

13 The solutions of a quadratic equation in general form are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

a Write the coordinates of the x-intercepts of $ax^2 + bx + c = 0$ if $\sqrt{b^2 - 4ac} \geq 0$.

b Use your answer to part **a** to write a formula for the axis of symmetry.

c Substitute the values from the equations in question **5** to verify if your formula for the axis of symmetry applies to quadratic equations with two, one and no x-intercepts.

14 Consider the graphs of $y = (x + 5)(x - 1)$, $y = 2(x + 5)(x - 1)$ and $y = -(x + 5)(x - 1)$.

a Find the coordinates of the x-intercepts for each graph.

b Explain why the graphs of these equations are different even though all parabolas have the same x-intercepts.

c Find the y-intercept for each parabola.

d Find the coordinates of the turning point for each parabola.

e Sketch all three parabolas on the same Cartesian plane.

15 Thomas throws a discus. The position of the discus can be represented by $h = -0.01(d + 4)(d - 72)$, where h is the height of the discus above the ground, in metres, and d is the horizontal distance from where the discus was thrown, in metres.

a Sketch the graph of this relationship by finding the intercepts.

b Thomas is standing on a platform. At what height off the ground was the discus thrown?

c What horizontal distance did the discus travel before hitting the ground?

d What is the greatest height that the discus reached?

16 A ball was kicked off the ground. Its path can be represented by $y = -0.05x^2 + 1.6x$, where x is the horizontal distance (in metres) and y is the vertical distance (in metres).

a Sketch the graph of this relationship.

b What was the maximum height the ball reached? What horizontal distance had it travelled when it reached its maximum height?

c What horizontal distance did the ball travel before hitting the ground?

d If the ball was caught at a height of 1.8 m above the ground as it was moving downwards, what horizontal distance did the ball travel?

17 Riley tries to kick a goal during a game of football but accidentally hits the post, scoring a behind. The height of the ball above the ground can be represented by the relationship $h = -4(t - 1)^2 + 5$, where h is the height, in metres, after t seconds.

a What are the coordinates of the turning point of the graph of this relationship?

b Sketch the graph, from $t = 0$ to $t = 1.5$.

c Use the graph to find:

i the height at which the ball made contact with Riley's boot

ii the maximum height of the ball during Riley's kick

iii the height of the football when it hits the post, 1.5 seconds after it was kicked.

d If the ball had not hit the post, how long would it have taken for the ball to hit the ground after Riley kicked it? Give your answer correct to one decimal place.

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18 A soccer ball is kicked from the ground into the air. The path of the ball follows the relationship $y = -\frac{1}{40}(x - 20)^2 + 10$, where y is the height above the ground of the ball for a horizontal distance, x, from where the ball was kicked. Both x and y are in metres.

a Sketch the graph of this relationship from when the ball was kicked to when it landed.

b What was the maximum height of the soccer ball?

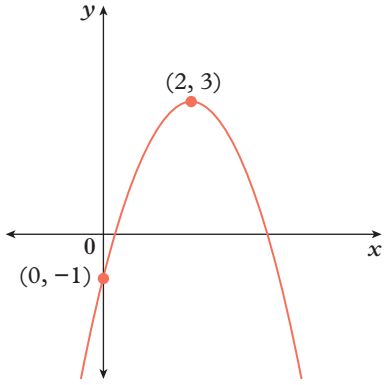
c What horizontal distance did the ball travel before it hit the ground?

Exercise 4.3C: Challenge

19 Given a graph of a quadratic relationship, we need three pieces of information about the graph to work out its equation.

a For each of the following graphs, write down the three pieces of information provided. Hint: How much information does knowing the turning point give you?

i



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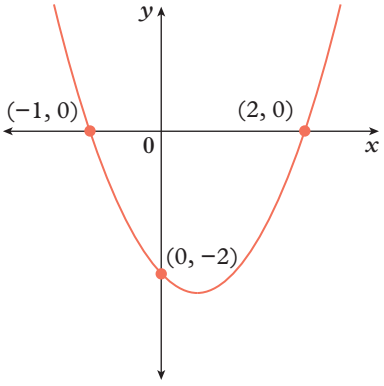

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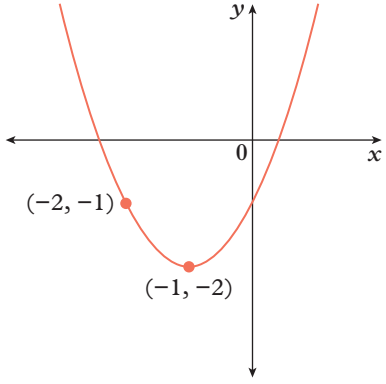

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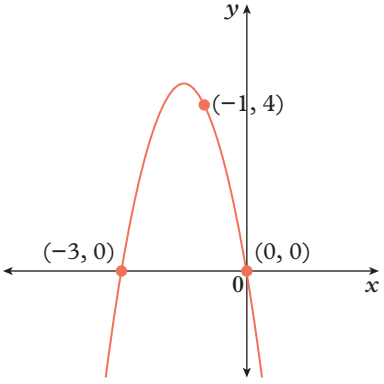
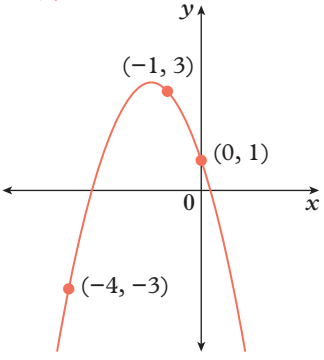


b For each of the graphs in part **a**, identify the form of quadratic relationship which best corresponds with the information provided, and then determine the equation of the graph.

c Describe when you should use turning point form and when you should use intercept form when finding the equation of a parabola.

20 The x-intercepts of a parabola are $(1, 0)$ and $(-2, 0)$. Do you have enough information to determine the equation of the graph? Explain.

21 Find the equation of the quadratic relationship on the right by substituting the three points into the equation $y = ax^2 + bx + c$ and solving the resulting system of simultaneous equations.



Online resources:

- **Interactive skillsheet** Sketching parabolas using intercepts
- **Interactive skillsheet** Sketching parabolas using transformations
- **Investigation** The connection between linear and quadratic graphs
- **Quick quiz** 4.3

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Lesson 4.4

Sketching circles

Learning intentions

By the end of this lesson you will be able to ...

✔ sketch circles of different radii, after horizontal and/or vertical translations.

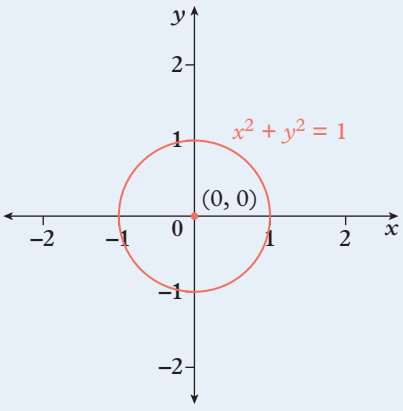
Inter-year links

Year 7	8.3 Translations
Year 8	9.1 Transformations
Year 9	4.5 Other non-linear relationships

Circles

Key content video
Graphing circles

- The graph of $x^2 + y^2 = 1$ is a circle with a centre at (0, 0) and a radius of $r = 1$ unit.
- The general equation of a circle graph is: $(x - h)^2 + (y - k)^2 = r^2$ where:
  - → the circle has a radius of r units
  - → the coordinates of the centre of the circle are (h, k).
- The following transformations can be performed on the graph of $x^2 + y^2 = r^2$ to produce the graph of $(x - h)^2 + (y - k)^2 = r^2$:
  - → a dilation in all directions from (h, k) as r increases
  - → a horizontal translation of h units right (or left if $h < 0$)
  - → a vertical translation of k units up (or down if $k < 0$).

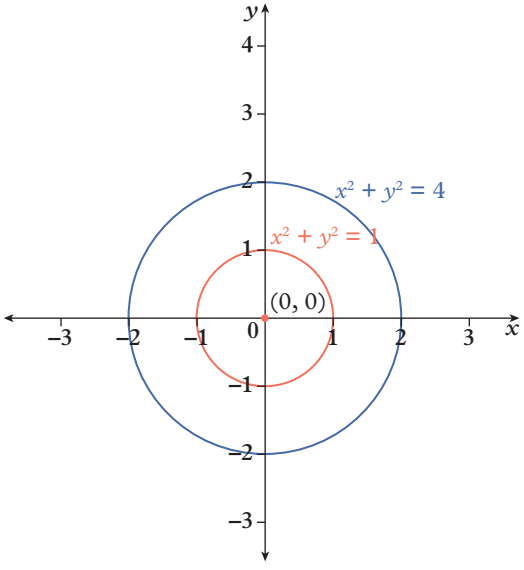
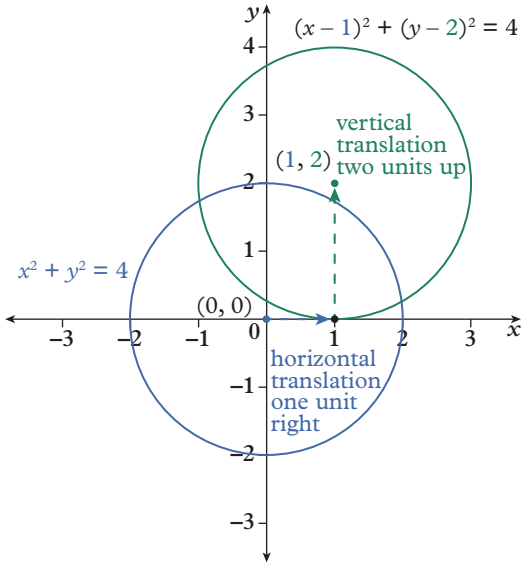


$$(x - h)^2 + (y - k)^2 = r^2$$

Horizontal translation of h units
If $h > 0$, right
If $h < 0$, left

Vertical translation of k units
If $k > 0$, up
If $k < 0$, down

Radius of r units

Dilation	Translation
	

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Worked example 4.4A Sketching simple circles

Sketch the graph of $x^2 + y^2 = 36$ by first finding the horizontal and vertical extremes of the circle.

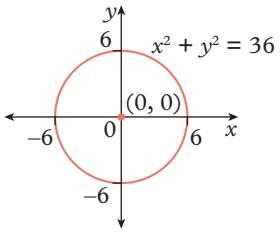
THINK

1. Identify the centre of the circle. The equation has the form $x^2 + y^2 = r^2$, so $h = 0$ and $k = 0$, and the centre of the circle is at the origin, $(0, 0)$.
2. Identify the radius. The equation has the form $x^2 + y^2 = r^2$ so $r^2 = 36$. A radius cannot be negative, so reject the negative value.
3. Identifying the horizontal and vertical extremes that are 6 units left, right, below and above the centre $(0, 0)$.
4. Sketch the graph. Label the graph with its equation.

WRITE

Centre: $(0, 0)$

$r^2 = 36$
$r = \pm\sqrt{36}$
$= \pm 6$
The radius is 6 units.
Horizontal extremes: $(-6, 0)$ and $(6, 0)$
Vertical extremes: $(0, -6)$ and $(0, 6)$



Worked example 4.4B Sketching circles using translations

Sketch the graph of $(x + 2)^2 + (y - 4)^2 = 9$ using translations.

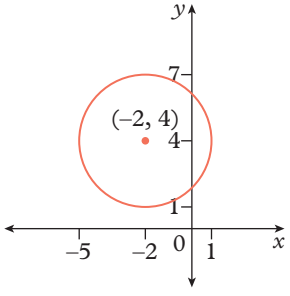
THINK

1. Identify the translations. The equation has the form $y = (x - h)^2 + (y - k)^2 = r^2$ where $h = -2$ and $k = 4$, so the coordinates of the centre are $(h, k) = (-2, 4)$.
2. Identify the radius. $r^2 = 9$. A radius cannot be negative, so reject the negative value.
3. Sketch the graph by marking the centre of the circle at $(-2, 4)$, then identifying the horizontal and vertical extremes that are 3 units left, right, below and above the centre: $(-5, 4)$, $(1, 4)$, $(-2, 1)$ and $(-2, 7)$.

WRITE

Centre: $(-2, 4)$

$r^2 = 9$
$r = \pm\sqrt{9}$
$= \pm 3$
The radius is 3 units.



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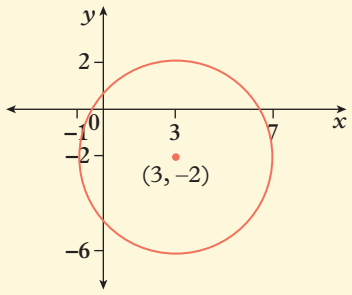
Worked example 4.4C Determining the equation of a circle

Determine the equation of the following circles.

a a circle with a radius of 6 units and a centre at (–2, 5)

b



THINK

The general equation of a circle is $y = (x - h)^2 + (y - k)^2 = r^2$. Identify the values of h, k and r, substitute them into the equation and simplify.

WRITE

a The centre is $(-2, 5) = (h, k)$.

$h = -2$

$k = 5$

Radius: $r = 6$

$(x - h)^2 + (y - k)^2 = r^2$

$(x - [-2])^2 + (y - 5)^2 = 6^2$

$(x + 2)^2 + (y - 5)^2 = 36$

b The centre is $(3, -2) = (h, k)$

$h = 3$

$k = -2$

Radius: $r = 4$

$(x - h)^2 + (y - k)^2 = r^2$

$(x - 3)^2 + (y - [-2])^2 = 4^2$

$(x - 3)^2 + (y + 2)^2 = 16$

Helpful hints

- ✔ Remember that the values of h and k have the opposite sign to the one appearing with them in the equation. For example, in the equation $(x + 2)^2 + (y - 4)^2 = 9$, $h = -2$ and $k = +4$
- ✔ Make sure you sketch your circles on a set of axes drawn to the same scale, because scales that don't match will make your circle look like an oval or an ellipse.
- ✔ Remember that the value of r is the positive square root of the right-hand side of the equation. For example, the radius of $(x - 1)^2 + (y - 3)^2 = 9$ is 3, not 9 (or –3).
- ✔ Make sure that you mark and label the centre of your circles!

Learning pathways

1–9	1(e, f), 3–10	1(e, f), 3(b, d, f, h), 5(b, d, f, h), 6–11

Exercise 4.4A: Understanding and fluency

WE 4.4A **1** Sketch the graph of each of these equations.

a $x^2 + y^2 = 9$ **b** $x^2 + y^2 = 1$ **c** $x^2 + y^2 = 64$

d $4x^2 + 4y^2 = 100$ **e** $15x^2 + 15y^2 = 60$ **f** $\frac{x^2}{16} + \frac{y^2}{16} = 1$

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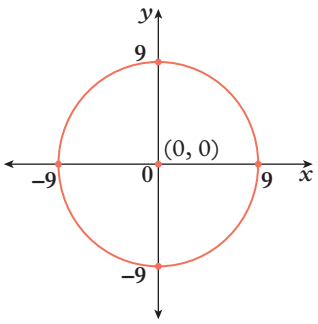
2 Write the equation for each of these circles.

a



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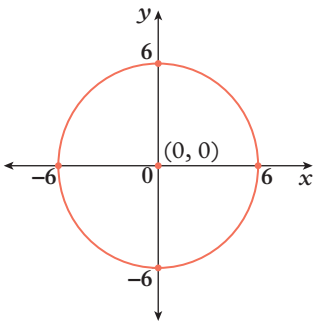

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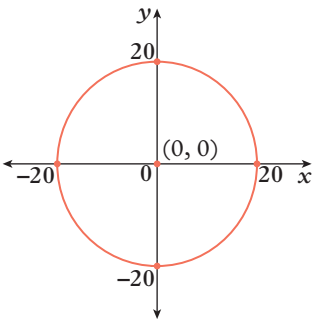


WE 4.4B **3** Sketch the graph of each of these equations. List the centre and radius of each circle.

a $(x-3)^2 + (y-2)^2 = 9$

b $(x-5)^2 + (y-7)^2 = 25$

c $(x+1)^2 + (y-3)^2 = 64$

d $(x-6)^2 + (y+6)^2 = 36$

e $(x-2)^2 + y^2 = 16$

f $x^2 + (y+5)^2 = 4$

g $(x+3)^2 + (y+5)^2 = 49$

h $(x+4)^2 + (y+3)^2 = 81$

4 Use digital technology to reproduce the first four graphs from question **3**. Compare those graphs to the ones you sketched.

WE 4.4C **5** Use the information provided to write the equation for each of these circles.

a a circle with radius 12 units and centre at (0, 0)

b a circle with radius 15 units and centre at (0, 0)

c a circle with radius 1 unit and centre at (4, –2)

d a circle with radius 7 units and centre at (–5, –6)

e a circle with radius 3 units and centre at (4, 4)

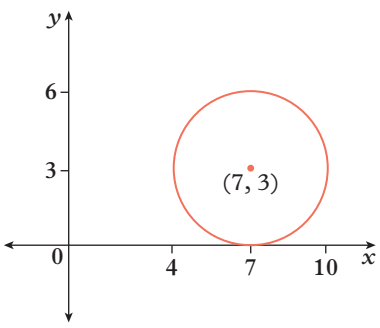
f a circle with radius 6 units and centre at (0, –5)

g a circle with radius 1.5 units and centre at (–1, 2.5)

h a circle with radius $\sqrt{5}$ units and centre at (2, –3)

6 Write the equation for each of these circles.

a



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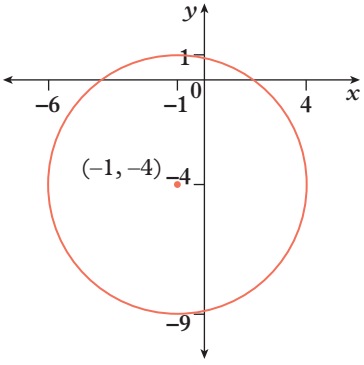

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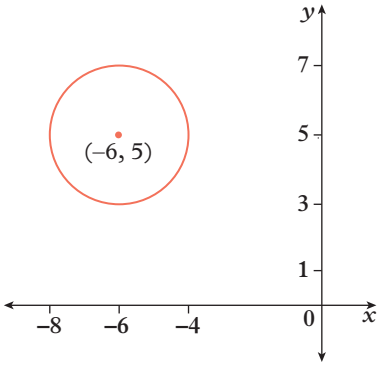

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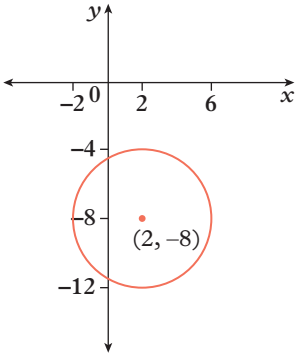

7 Circles on the Cartesian plane can be described in terms of transformations performed on the graph of $x^2 + y^2 = 1$.

a The circle with equation $x^2 + y^2 = 1$ is dilated in all directions from the origin by a factor of 3, then translated 2 units right and 1 unit down. What is the equation of the transformed circle?

b The circle with equation $x^2 + y^2 = 1$ is transformed to become a circle with equation $(x+7)^2 + (y-4)^2 = 25$. Describe the transformations that have occurred. Hint: Don't forget to describe the centre of the dilation.

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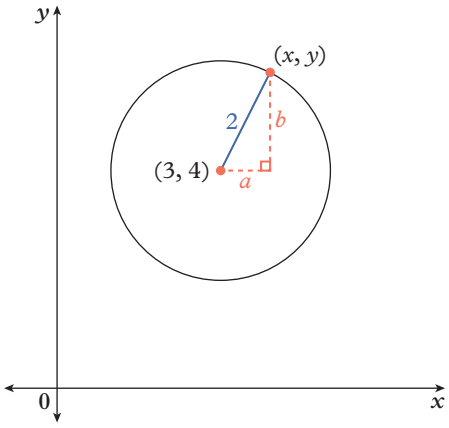
Exercise 4.4B: Problem solving and reasoning

8 A circle is a set of points that are equidistant from a single point. This geometric description of a circle, along with Pythagoras' theorem, explains the Cartesian equation of a circle.

Consider the diagram on the right, where (3, 4) is the centre and (x, y) is generalised to be any point on the circle.



a Explain why $a = x - 3$ and $b = y - 4$.

b Use Pythagoras' theorem to explain why the Cartesian equation of this circle is $(x - 3)^2 + (y - 4)^2 = 2^2$.

9 An arch is modelled using the relationship $(x - 3)^2 + y^2 = 9$, where x is the horizontal distance from the place the left-hand side of the arch meets the ground and y is the corresponding height at that distance. Both x and y are measured in metres, $x \geq 0$ and $y \geq 0$.

a Sketch the graph of this relationship.

b What is the horizontal distance along the ground from one side of the arch to the other?

c What is the maximum height of the arch?

d Describe the position on the ground below a point where the arch is at half its maximum height.

10 The outline of a circular spa is mapped onto a Cartesian plane, representing a 15 m long and 10 m wide outdoor area of a seaside resort, using the relationship $(x - 6.2)^2 + (y - 4.8)^2 = 2.25$.

a Sketch the graph of this relationship.

b What is the minimum distance from the water to the edge of the outdoor area?

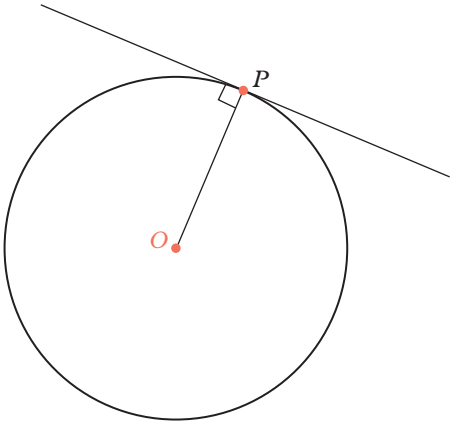
c Calculate the area of the spa to the nearest square metre.

d What percentage of the outdoor area is taken up by the spa?

Exercise 4.4C: Challenge

11 A tangent to a circle is a straight line which just touches the circle once, at one point on its circumference. The radius of the circle at the point of tangency is perpendicular to the tangent line.

Consider the circle with equation $(x - 1)^2 + (y - 2)^2 = 5$.



a What is the centre of this circle?

b Show that (3, 3) is a point on the circle.

c Find the gradient of the radius that connects the centre of the circle to the point (3, 3).

d Find the gradient of the line perpendicular to the radius, referred to in part **c**.

e Use the gradient you found in part **d** to find the equation of the tangent to the circle at (3, 3). Sketch the graphs of the circle and your line on a graphing calculator to confirm that the straight line is a tangent to the circle.

Online resources:

- **Interactive skillsheet** Circles
- **Worksheet** Circles on the Cartesian plane
- **CAS instructions** Graphing circles
- **Quick quiz** 4.4

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Lesson 4.5

Sketching exponential relationships

Learning intentions

By the end of this lesson you will be able to ...

✔ sketch exponential relationships by finding asymptotes and considering transformations.

Inter-year links

Year 7	8.3 Translations
Year 8	9.1 Transformations
Year 9	4.5 Other non-linear relationships

Exponential relationships

Key content video
Graphing exponential relationships

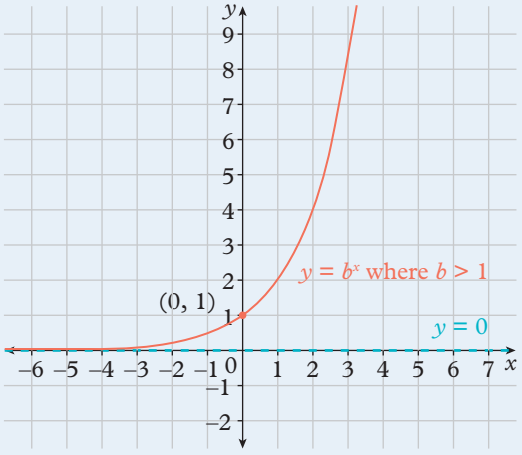
- Exponential relationships can be used to model many real situations involving multiplicative growth and decay; for example, situations involving population growth or investments.
- The basic form of an **exponential relationship** has the following equation:

$$y = b^x$$

exponent

base

where $b > 1$.



- The graph has:
  - → a y-intercept of 1
  - → a horizontal asymptote along the x-axis.
- An **asymptote** is a boundary or line that a curve approaches as the value of x or y tends towards positive or negative infinity ($\pm\infty$). The asymptote for $y = b^x$ is the horizontal line with equation $y = 0$.

Sketching exponential equations using transformations

- The general equation of an exponential relationship is: $y = a \times b^{nx} + k$

  where the graph has:
  - → an asymptote at $y = k$
  - → a y-intercept at $(0, a + k)$.
- The following transformations can be performed on the graph of $y = b^x$ to produce a sketch of $y = a \times b^{nx} + k$:
  - → a dilation in the y-direction as a varies
  - → a vertical reflection across the x-axis if $a < 0$
  - → a dilation in the x-direction as n varies
  - → a horizontal reflection across the y-axis if $n < 0$
  - → a vertical translation of k units up (or down if $k < 0$).

Dilation in the x-direction
If $n > 0$, upright
If $n < 0$, horizontal reflection

$$y = a \times b^{nx} + k$$

Dilation in the y-direction
If $a > 0$, upright
If $a < 0$, vertical reflection

Vertical translation of k units
If $k > 0$, up
If $k < 0$, down

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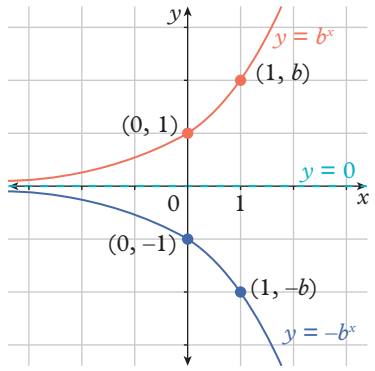
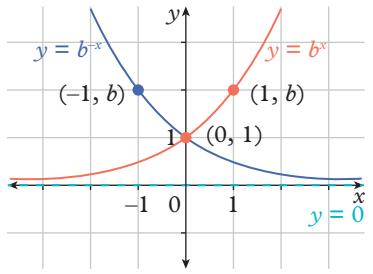
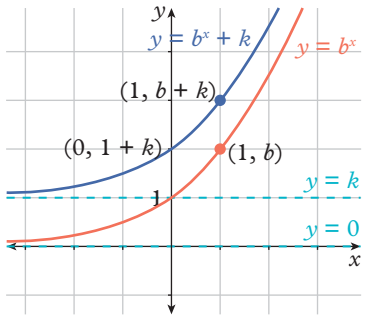
Vertical reflection across x-axis	Horizontal reflection across y-axis	Vertical translation
		

- Transformations can be applied to individual points to help sketch a graph. A point (x, y) on $y = b^x$ is transformed to the point $\left(\frac{x}{n}, ay + k\right)$ on $y = a \times b^{nx} + k$.
- Graphs of exponential relationships can be used to solve exponential equations.

Worked example 4.5A Sketching the graph of an exponential relationship by performing reflections

On the same Cartesian plane, sketch the graphs of the following exponential relationships by reflecting the graph of $y = 2^x$. For each graph, label the y-intercept and asymptote.

a $y = -2^x$ **b** $y = 2^{-x}$

THINK

1 Compare both equations with $y = 2^x$ to identify the transformations. Both equations are in the form $y = a \times 2^{nx} + k$.

In $y = -2^x$, $a = -1$, $n = 1$, $k = 0$.

In $y = 2^{-x}$, $a = 1$, $n = -1$, $k = 0$.

When $a = -1$, the graph is being reflected across the x-axis and when $n = -1$, the graph is being reflected across the y-axis. In both graphs, the asymptote remains $y = 0$ because $k = 0$.

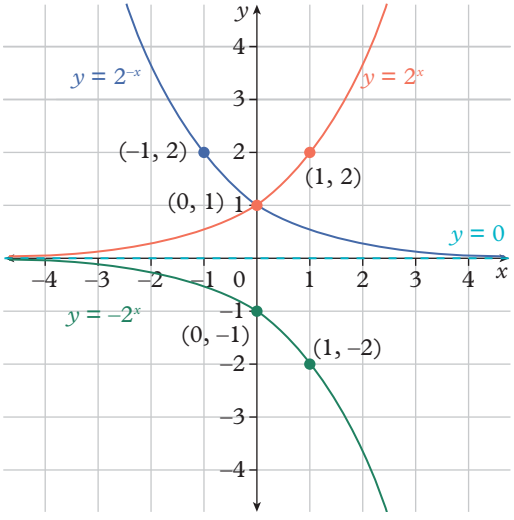
2 Transform the point $(1, 2)$ and the y-intercept $(0, 1)$. A point (x, y) on $y = 2^x$ becomes $(x, -y)$ after being reflected in the x-axis, and $(-x, y)$ after being reflected in the y-axis.

3 Perform the reflection, sketch the curve through the y-intercept and reference point, using the horizontal asymptote as a boundary. Label the y-intercept and asymptote.

WRITE

a Asymptote: $y = 0$, y-intercept: $(0, -1)$, curve passes through $(1, -2)$.

b Asymptote: $y = 0$, y-intercept: $(0, 1)$, curve passes through $(-1, 2)$.



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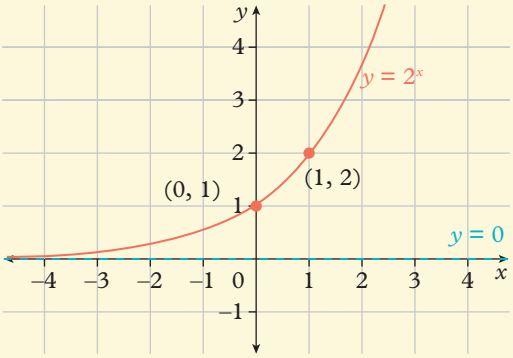
Worked example 4.5B Sketching the graph of an exponential relationship by performing vertical translations

On the same Cartesian plane, sketch the graph of the following exponential relationships by transforming the graph of $y = 2^x$. For each graph, label the y-intercept and asymptote.

a $y = 2^x + 1$

b $y = 2^x - 3$



THINK

1 Compare both equations with $y = 2^x$ to identify the transformations. Both equations are in the form $y = 2^x + k$.

In **a**, $k = 1$, so the asymptote of the graph is $y = 1$. Translate the graph of $y = 2^x$ one unit up.

In **b**, $k = -3$, so the asymptote of the graph is $y = -3$. Translate the graph of $y = 2^x$ three units down.

2 Transform the point $(1, 2)$ and the y-intercept $(0, 1)$. A point (x, y) on $y = 2^x$ becomes $(x, y + 1)$ after being translated up one unit, and $(x, y - 3)$ after being translated down three units.

In **a**:

$(1, 2) \mapsto (1, 2 + 1) = (1, 3)$ and $(0, 1) \mapsto (0, 1 + 1) = (0, 2)$

In **b**:

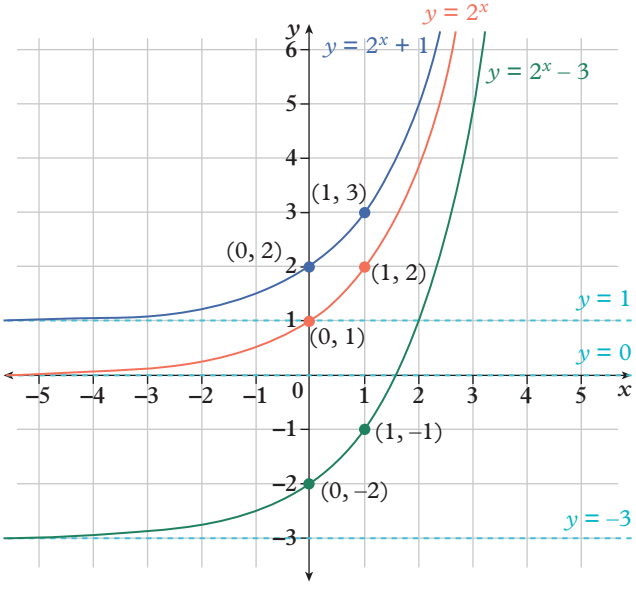
$(1, 2) \mapsto (1, 2 - 3) = (1, -1)$ and $(0, 1) \mapsto (0, 1 - 3) = (0, -2)$.

3 Perform the translation and sketch the curve through the y-intercept and reference point, using the horizontal asymptote as a boundary. Label the y-intercept and asymptote.

WRITE

a Asymptote: $y = 1$, y-intercept: $(0, 2)$, curve passes through $(1, 3)$.

b Asymptote: $y = -3$, y-intercept: $(0, -2)$, curve passes through $(1, -1)$.



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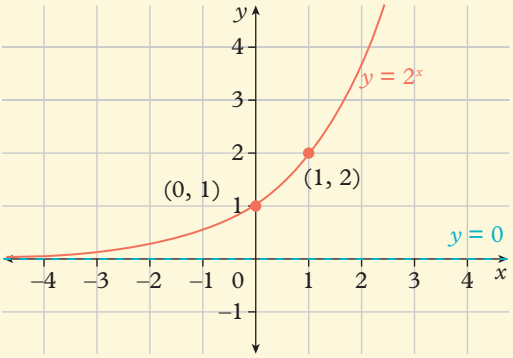
Worked example 4.5C Sketching the graph of an exponential relationship using vertical dilations

On the same Cartesian plane, sketch the graph of $y = 2 \times 2^x$ by transforming the graph of $y = 2^x$. Label the y-intercept and asymptote.



THINK

1 Compare the equation with $y = 2^x$ to identify the transformations. The equation is in the form $y = a \times 2^{nx} + k$, where $k = 0$. The asymptote for the graph is $y = 0$.

2 $a = 2$, $n = 1$, so dilate $y = 2^x$ in the y-direction. A point (x, y) on $y = 2^x$ becomes (x, ay) after the dilation. Since $a = 2$, the y-intercept changes from $(0, 1)$ to $(0, 2)$:

$(0, 1) \mapsto (0, a \times 1) = (0, 2 \times 1) = (0, 2)$.

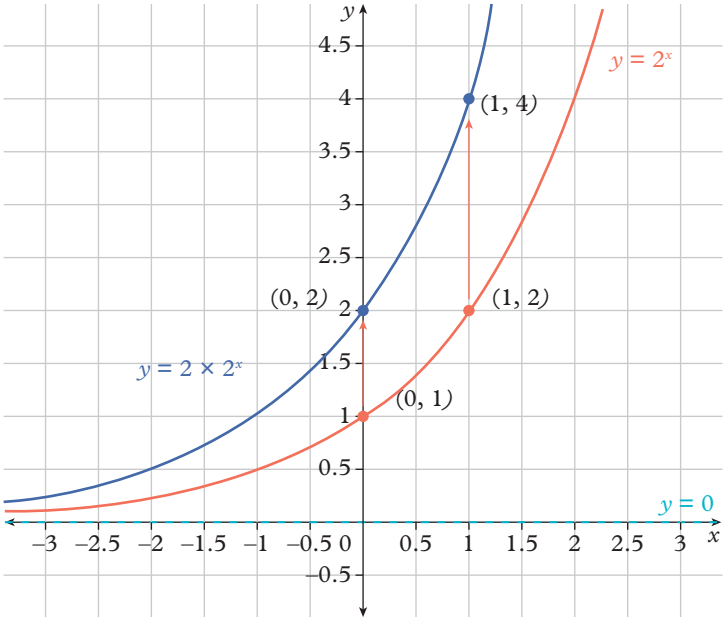
The reference point changes from $(1, 2)$ to $(1, 4)$:

$(1, 2) \mapsto (1, a \times 2) = (1, 2 \times 2) = (1, 4)$.

3 Sketch the curve for an exponential relationship through the y-intercept and the reference point, using the horizontal asymptote as a boundary. Label the y-intercept and asymptote.

WRITE

Asymptote: $y = 0$, y-intercept: $(0, 2)$, curve passes through $(1, 4)$.



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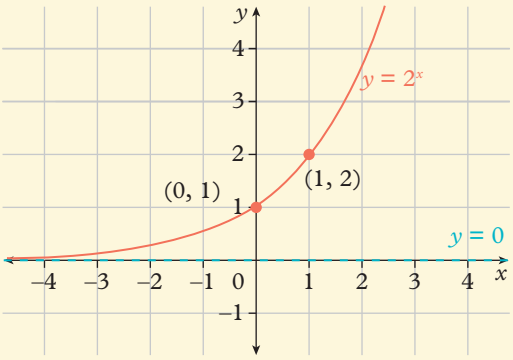
Worked example 4.5D Sketching the graph of an exponential relationship using horizontal dilations

On the same Cartesian plane, sketch the graph of $y = 2^{2x}$ by transforming the graph of $y = 2^x$. Label the y-intercept and asymptote.



THINK

1 Compare the equations with $y = 2^x$ to identify the transformations. The equation is in the form $y = a \times 2^{nx} + k$, where $k = 0$. The asymptote for the graph is $y = 0$.

2 $a = 1$, $n = 2$, so dilate $y = 2^x$ in the x-direction. A point (x, y) on $y = 2^x$ becomes $\left(\frac{x}{n}, y\right)$ after the dilation. Since $n = 2$, the y-intercept remains the same at $(0, 1)$:

$(0, 1) \mapsto \left(\frac{0}{n}, 1\right) = \left(\frac{0}{2}, 1\right) = (0, 1)$.

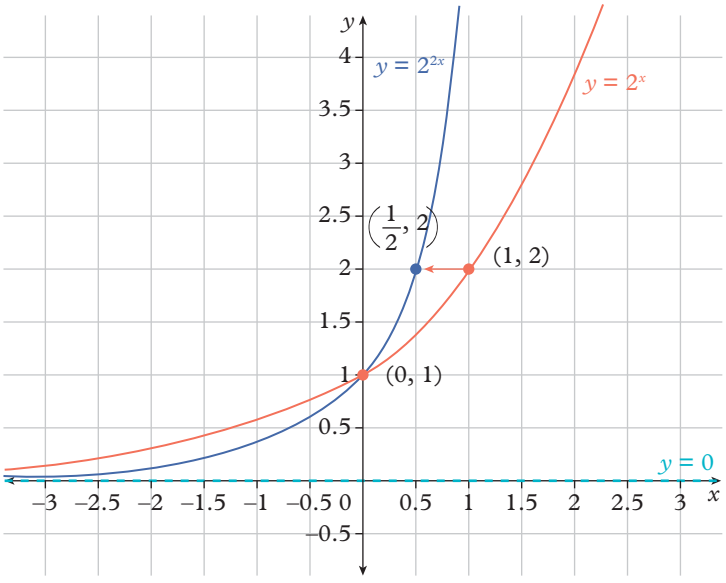
The reference point changes from $(1, 2)$ to $\left(\frac{1}{2}, 2\right)$:

$(1, 2) \mapsto \left(\frac{1}{n}, 2\right) = \left(\frac{1}{2}, 2\right) = \left(\frac{1}{2}, 2\right)$.

3 Sketch the curve for an exponential relationship through the y-intercept and the reference point, using the horizontal asymptote as a boundary. Label the y-intercept and asymptote.

WRITE

Asymptote $y = 0$, y-intercept: $(0, 1)$, curve passes through $\left(\frac{1}{2}, 2\right)$.



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Helpful hints

- ✔ Using your graphing calculator to alter the exponential function will help you to better understand the transformations. You can start by moving a negative sign to different places in the same relationship to increase or decrease values, or you can compare the graphs for the following equations:
  - $y = -2^x$
  - $y = 2^{-x}$
  - $y = -2^{-x}$
  - $y = 2^x + 1$
  - $y = 2^x - 1$
- ✔ The graph of an exponential relationship gets increasingly closer to the asymptote but will never cross or touch it.
- ✔ Remember that every exponential graph has an asymptote and you must always include it, even when it is $y = 0$ (which is the x-axis).

Learning pathways

1–11, 13, 15	2–7, 9–12, 14, 16, 18	1(e, f), 3–7(d–f), 9, 10, 14, 17, 18

Exercise 4.5A: Understanding and fluency

1 Plot the graph of each of these exponential relationships by first completing a table of values for integer values of x from –2 to 2. Is there a pattern in the y value?

a $y = 3 \times 2^{-x}$ **b** $y = \frac{3}{5} \times 3^x$ **c** $y = 2 \times 4^x$

2 Consider the exponential relationships $y = 2^x$ and $y = 3^x$.

a Plot both graphs for x-values from –3 to 3 on the same Cartesian plane. You can plot the graphs using a table of values or digital technology.

b Compare the two curves. How are they different?

c Which features are the same?

d Write the y-intercept for each graph.

e Write the rule for the asymptote for each graph.

f Predict how the graph of $y = 5^x$ compares to the two curves you have drawn.

3 Consider the exponential relationships $y = \left(\frac{1}{2}\right)^x$ and $y = \left(\frac{1}{3}\right)^x$.

a Plot both graphs for x-values from –3 to 3 on the same Cartesian plane. You can plot the graphs using a table of values or digital technology.

b Compare the two curves. How are they different?

c Which features are the same?

d Write the y-intercept for each graph.

e Write the equation for the asymptote of each graph.

f Predict how the graph of $y = \left(\frac{1}{5}\right)^x$ compares to the two curves you have drawn.

WE 4.5A **4** Sketch the graph of each exponential relationship by performing reflections.

a $y = -3^x$ **b** $y = 3^{-x}$ **c** $y = -3^{-x}$

d $y = \left(\frac{1}{2}\right)^{-x}$ **e** $y = -\left(\frac{1}{2}\right)^x$ **f** $y = -\left(\frac{1}{2}\right)^{-x}$

WE 4.5B **5** Sketch the graph of each exponential relationship by performing a vertical translation.

a $y = 2^x + 2$ **b** $y = 2^x - 2$ **c** $y = 2^x + 1$

d $y = \left(\frac{1}{3}\right)^x + 1$ **e** $y = \left(\frac{1}{3}\right)^x + 2$ **f** $y = \left(\frac{1}{3}\right)^x - 2$

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WE 4.5C **6** Sketch the graph of each exponential relationship by performing a vertical dilation.

a $y = 2 \times 3^x$ **b** $y = 3 \times 3^x$ **c** $y = 0.5 \times 3^x$

d $y = 2 \times \left(\frac{1}{2}\right)^x$ **e** $y = 3 \times \left(\frac{1}{2}\right)^x$ **f** $y = 0.5 \times \left(\frac{1}{2}\right)^x$

WE 4.5D **7** Sketch the graph of each exponential relationship by performing a horizontal dilation.

a $y = 3^{2x}$ **b** $y = 3^{\frac{x}{2}}$ **c** $y = 3^{\frac{x}{3}}$

d $y = \left(\frac{1}{2}\right)^{2x}$ **e** $y = \left(\frac{1}{2}\right)^{\frac{x}{2}}$ **f** $y = \left(\frac{1}{2}\right)^{\frac{x}{3}}$

8 What is the rule obtained when the graph of $y = b^x$ is reflected in:

a the x-axis **b** the y-axis **c** the x-axis and the y-axis?

9 Add a row to the table of values you created in question **1** to find the ratio between consecutive values of the dependent variable, y. What do you notice? Use the exponent laws to explain what you find.

Exercise 4.5B: Problem solving and reasoning

10 Sienna invests \$2000 at 10% p.a. interest, compounded annually. The amount, A, (in dollars) of the investment after n years can be represented by $A = 2000 \times (1.1)^n$.

a What will the graph of this relationship look like? Sketch the graph showing the key features.

b Describe what happens to the value of A as n increases.

c Produce the graph of $A = 2000 \times (1.1)^n$ for $0 \le n \le 10$, using a table of values or digital technology to help you. Use the graph to find:

i the amount of the investment after 4 years

ii the number of years it takes for the investment amount to be more than \$4000.

11 The value of Gabriel's work tools depreciates at the rate of 20% p.a. compounded annually. The asset value, V, (in dollars) after n years can be represented by $V = 3500 \times (1.25)^{-n}$.

a What will the graph of this relationship look like? Sketch the graph showing the key features.

b What was the initial value of Gabriel's tools?

c Describe what happens to the value of V as n increases.

d Produce the graph of $V = 3500 \times (1.25)^{-n}$ for $0 \le n \le 10$, using a table of values or digital technology to help you. Use the graph to find the number of years it takes for the asset value of the tools to be less than \$1000.

12 A common mistake with exponential relationships is writing $a \times b^x$ as $(ab)^x$. For example, incorrectly writing 3×2^x as 6^x.

a Show that 3×2^x is not equivalent to 6^x by substituting $x = 2$ into each expression.

b Find, by inspection or by trial and error, the value of x for which 3×2^x is equal to 6^x.

c Considering your answers from parts **a** and **b**, explain the difference between the descriptions 'equal to' and 'equivalent to' in this context.

d Considering your answers from parts **a** and **b**, explain why checking the value of one variable is not enough to decide whether two expressions are equivalent.

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13 For each of the following exponential relationships:

i use the exponent laws to rewrite the relationship in the form $y = a \times 2^x + k$

ii identify the corresponding transformations on $y = 2^x$

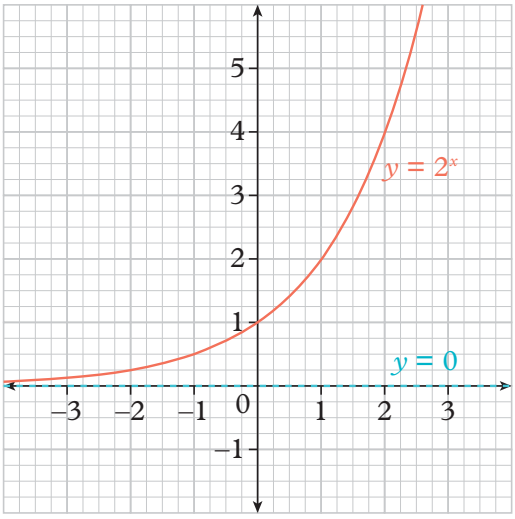
iii sketch the graph.

a $y = 2^{x+3}$ **b** $y = 2^{x+2}$ **c** $y = 2^{x+1}$

d $y = 2^{x-1}$ **e** $y = 2^{x-2}$ **f** $y = 2^{x-3}$

14 Use your answers to question **13** to explain why exponential relationships can always be graphed without considering horizontal translations.

15 Graphs of exponential relationships can be used to solve simple exponential equations. Use the graph of $y = 2^x$ provided to identify the solutions to the following equations.



a $2^x = 2$ **b** $2^x = 4$ **c** $2^x = \frac{1}{2}$

d $2^x = 1$ **e** $2^x = \frac{1}{4}$ **f** $2^x = 0$

16 Solve the following equations by first sketching the exponential relationship $y = 2^{-x}$.

a $2^{-x} = 1$ **b** $2^{-x} = 8$ **c** $2^{-x} = 4$

d $2^{-x} = \frac{1}{2}$ **e** $2^{-x} = \frac{1}{8}$ **f** $2^{-x} = 16$

17 For each of the following equations:

i identify a suitable exponential relationship to help you solve the equation

ii sketch the graph of the exponential relationship using a digital tool

iii solve the equation, correct to two decimal places, using the digital tool.

a $3^x = 6$ **b** $4^x = 5$ **c** $2^x = \frac{3}{2}$ **d** $3^{-x} = 8$ **e** $2^{-x} = 3$ **f** $4^{-x} = 7$

Exercise 4.5C: Challenge

18 Describe the transformations performed on the graph of $y = 2^x$ to produce the graph of $y = -6 \times 2^{2x} - 2$. Sketch the graph.

Online resources:

Interactive skillsheet Exponential relationships

Quick quiz 4.5

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ANS p703

Checkpoint

Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this module.

4.1 **1** Use the Null Factor Law to solve each of the following equations.

a $x(x-2)=0$ **b** $(x+3)(x-4)=0$
c $(x+1)(x-1)=0$ **d** $(2x+3)(x-5)=0$

4.1 **2** Solve each of the following quadratic equations by first factorising and then applying the Null Factor Law.

a $x^2+5x=0$ **b** $2x^2-12x=0$
c $x^2+6x+8=0$ **d** $x^2-22x+40=0$
e $x^2+16x-17=0$ **f** $x^2+x-6=0$

4.1 **3** Solve each of the following equations and give your answers as exact values.

a $x^2-4=0$ **b** $x^2-7=0$
c $x^2=22$ **d** $5x^2=45$
e $3x^2-15=0$ **f** $-2x^2+12=0$

4.2 **4** Use the discriminant to identify the number of solutions for each of the following quadratic equations.

a $x^2-5x+3=0$ **b** $x^2-6x=-9$
c $2x^2+7x=10$ **d** $2x^2+5x+4=0$

4.2 **5** Use the quadratic formula to solve the following quadratic equations, leaving your answers as exact values.

a $x^2+3x+1=0$ **b** $x^2-2x+1=0$
c $3x+1=x^2$ **d** $5x^2+x=2$
e $4x^2+12x+9=0$ **f** $\frac{1}{2}x^2+5x=3$

4.3 **6** Sketch the graph of $y=(x-1)(x-5)$, labelling all axis intercepts and the turning point.

4.3 **7** Sketch the graph of $y=-2(x+1)(x-2)$, labelling all axis intercepts and the turning point.

4.3 **8** Sketch the graphs of each of the following quadratic relationships using transformations, labelling all relevant points.

a $y=x^2+1$ **b** $y=x^2-4$
c $y=(x-2)^2$ **d** $y=(x+3)^2$

4.3 **9** Sketch the graphs of each of the following quadratic relationships using transformations, labelling all relevant points with exact values.

a $y=(x-3)^2+1$ **b** $y=2(x+1)^2+2$
c $y=-x^2+1$ **d** $y=-(x+1)^2-1$
e $y=-(x-2)^2+1$ **f** $y=(x-3)^2-7$

4.3 **10** Write the rule for each quadratic relationship resulting from performing each of these sequences of transformations on the graph of $y=x^2$.

a a reflection in the x-axis, then a translation 3 units left and 1 unit up

b a dilation in the y-direction by a factor of 3 and a reflection in the x-axis, then a translation 4 units right and 2 units down

4.4 **11** Sketch the graphs of the following circles. Label the centre and the horizontal and vertical extremes of each graph.

a $x^2+y^2=9$ **b** $(x-1)^2+y^2=4$
c $x^2+(y+2)^2=1$ **d** $(x+1)^2+(y-2)^2=16$

4.5 **12** Sketch the graphs of the following exponential relationships using transformations. Label the asymptote and y-intercept of each graph.

a $y=2^{-x}$ **b** $y=3\times 2^x$ **c** $y=-2^{-x}$

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Lesson 4.6

Direct and inverse proportion

Learning intentions

By the end of this lesson you will be able to ...

✔ solve problems involving direct and inverse proportion.

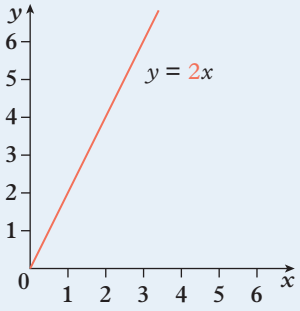
Inter-year links

Year 9 3.8 Direct proportion

Direct proportion

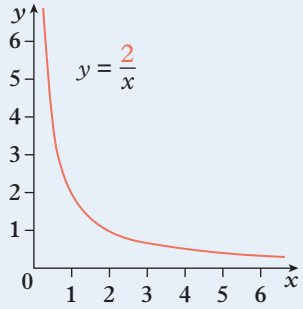
Key content video
Direct and inverse proportion

- y is said to be **directly proportional** to x if $y = 0$ when $x = 0$ and the rate of change of y with respect to x is constant.
- Proportionality is denoted using the symbol '$\propto$'.
- If $y \propto x$, the equation for the relationship between x and y is $y = kx$, where k, the **constant of proportionality**, is equal to the rate of change of y with respect to x (the gradient).



Inverse proportion

- y is said to be **inversely proportional** to x if $y \propto \frac{1}{x}$.
  → This is equivalent to the product of x and y being constant.
- If $y \propto \frac{1}{x}$, the equation for the inverse relationship between x and y is $y = \frac{k}{x}$, where k is the constant of proportionality.
- In many applications, only positive values of x, y and k are considered.
- The graph of $y = \frac{k}{x}$ has:
  → no x- or y-intercepts
  → two asymptotes: $x = 0$ and $y = 0$.
- The graph of $y = \frac{k}{x}$ is called a rectangular hyperbola, which is discussed in more detail in **Lesson 4.7**.



Worked example 4.6A Finding the constant of proportionality

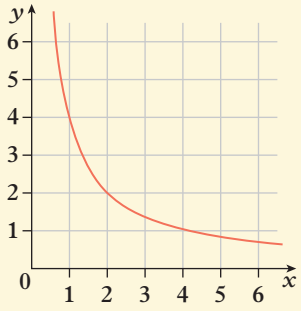
Find the constant of proportionality, k, using the information given about each relationship below.

a $y \propto x$,

x	0	2	4	6	8
y	0	24	48	72	96

b $y \propto \frac{1}{x}$ and when $x = 4$, $y = 3$.

c $y \propto \frac{1}{x}$,



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THINK

a If $y \propto x$, then $y = kx$. Select any pair of values in the table and substitute the corresponding values of x and y into $y = kx$, then solve the equation for k.

b If $y \propto \frac{1}{x}$, then $y = \frac{k}{x}$. Substitute the given values into $y = \frac{k}{x}$ and solve the equation for k.

c To find the constant, k, select any point on the graph, substitute the corresponding values of x and y into $y = \frac{k}{x}$, then solve the equation for k.

WRITE

a

x	0	2	4	6	8
y	0	24	48	72	96

$y = kx$

$24 = k \times 2$

$k = 12$

b $y = \frac{k}{x}$

$y = \frac{k}{4}$

$3 = \frac{k}{4}$

$k = 12$

c

$y = \frac{k}{x}$

$2 = \frac{k}{2}$

$k = 4$

Worked example 4.6B Plotting inversely proportional relationships

Plot the following table of values on the Cartesian plane, showing only positive x- and y-values. Join the points together with a smooth curve.

x	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
y	4	2	$\frac{4}{3}$	1	$\frac{4}{5}$	$\frac{2}{3}$

THINK

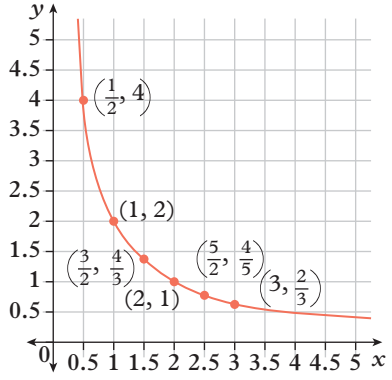
a Write out the coordinates of each point listed in the table.

b Draw a set of axes to show only the region of the Cartesian plane where x and y are both positive.

c Plot the points on the Cartesian plane and join the points with a smooth curve.

WRITE

$\left(\frac{1}{2}, 4\right)$, $(1, 2)$, $\left(\frac{3}{2}, \frac{4}{3}\right)$, $(2, 1)$, $\left(\frac{5}{2}, \frac{4}{5}\right)$, $\left(3, \frac{2}{3}\right)$



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Helpful hints

- ✔ Direct proportion is an example of a linear relationship. Remember that for the relationship $y = kx$:
  k = constant of proportionality = rate of change = gradient
- ✔ For a relationship to be directly or inversely proportional, the constant of proportionality, k, must be a constant, but it can still be fractional or irrational.

Learning pathways

1(a–c), 2, 3(a–c), 4–7, 8(i), 10, 12	1–5(a–b), 6–8, 9, 13, 14	1–7(a–b), 11, 13, 15–17

ANS p705

Exercise 4.6A: Understanding and fluency

WE 4.6A **1** If $y \propto x$, find the constant of proportionality for each of these relationships, using the information given.

a $y = 12$ when $x = 3$

b $y = 9$ when $x = 6$

c $y = 10$ when $x = 20$

d

x	0.5	1	1.5
y	2.5	5	7.5

e

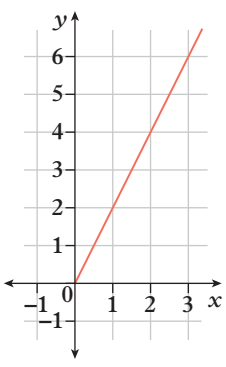
x	1	2	3
y	9	18	27

f

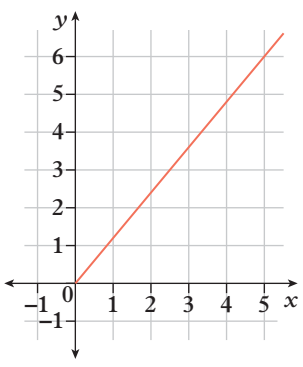
x	5	10	15
y	2	4	6

2 For each of the following graphs, $y \propto x$. Find the constant of proportionality.

a



b



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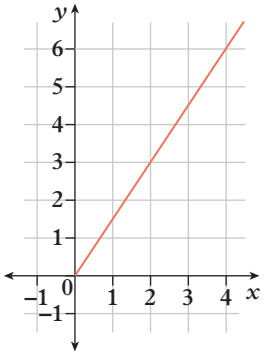
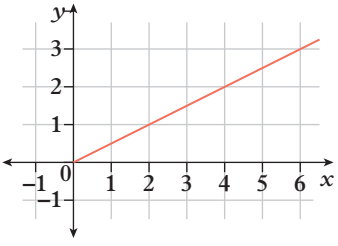


d



3 If $y \propto \frac{1}{x}$, use the information given to find the constant of proportionality for each of the following relationships.

a $y = 1$ when $x = 6$

b $y = 0.4$ when $x = 10$

c $y = 3$ when $x = 5$

d

x	2	4	6
y	6	3	2

e

x	2	3	4
y	3	2	1.5

f

x	2	5	10
y	0.5	0.2	0.1

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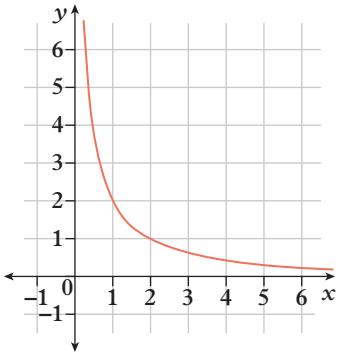
4 For each of the following graphs, $y \propto \frac{1}{x}$. Find the constant of proportionality.

a



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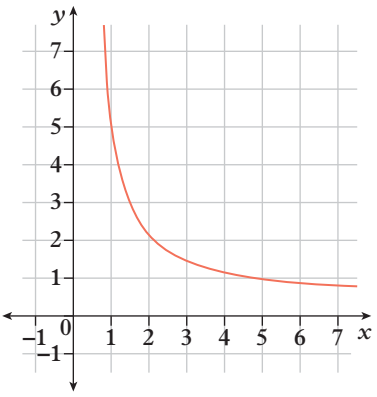

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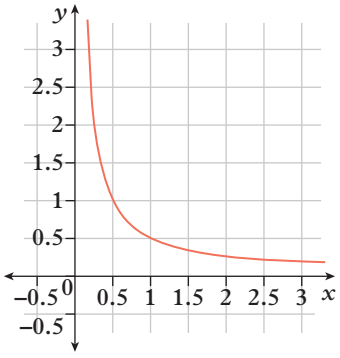

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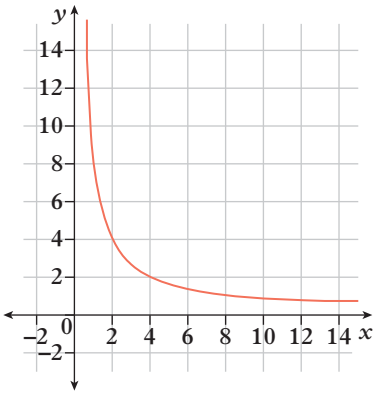

WE 4.6B **5** Plot the following tables of values on the Cartesian plane, showing only positive x- and y-values. Join the points together with a smooth curve.

a

x	1	2	3	4	6
y	12	6	4	3	2

b

x	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
y	2	1	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{3}$

c

x	5	10	20	30
y	12	6	3	2

6 Determine the constant of proportionality for all relationships in question **5**.

7 Decide whether the variables in each of the following relationships are directly or inversely proportional, or neither.

a The height of a tree and the length of its shadow

b The cost of hiring a plumber who charges \$70 per hour plus a \$90 call out fee

c The time taken to travel a fixed distance and the speed of travel (assuming the speed is constant)

d The area of a square and its side length

e The height of water in a cylindrical glass and the volume of the water

f The number of people paying a bill and the cost per person

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8 Consider the incomplete table below, which shows a proportional relationship.

x	0.5	1	2	4	8
y	8	4	2		

a Is y directly proportional or inversely proportional to x?

b Find the constant of proportionality.

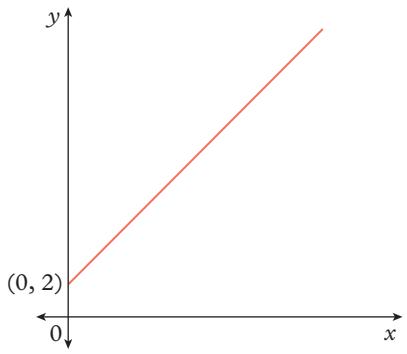
c Calculate the two y values missing from the table.

d Plot the points from the table on a graph and draw a smooth curve to join the points.

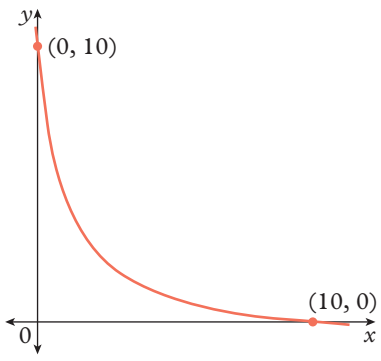
Exercise 4.6B: Problem solving and reasoning

9 For each of the following graphs, determine if the relationship is directly proportional or inversely proportional, or explain why it is neither.

a



b



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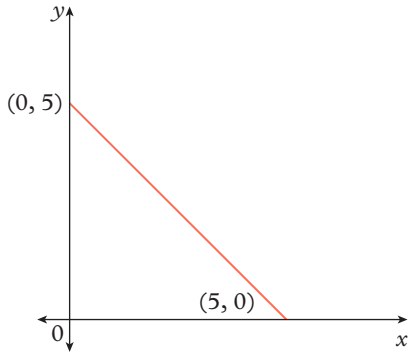

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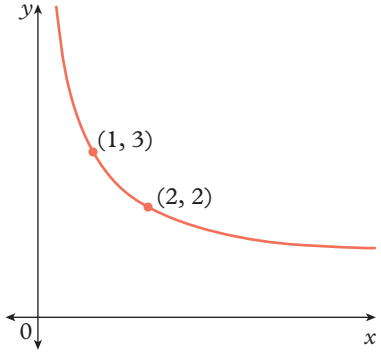

10 The number of days it takes to complete a construction job is inversely proportional to the number of people working on the job. If it would take 12 days for 5 workers to complete the job, then how many days would it take for 8 workers to complete the job?

11 Consider the following relationship between x and y.

x	0.5	1	1.5	2
y	12	6	4	3

a If $y \propto \frac{1}{x}$, find the constant of proportionality.

b If $x \propto \frac{1}{y}$, find the constant of proportionality.

c Your answers for parts **a** and **b** should be the same. Is this always the case for inversely proportional relationships, or were these answers just a coincidence?

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12 To attend a formal, a group of Year 10 students are considering hiring a limousine, which costs a fixed amount regardless of how many passengers ride in the limousine. The rental company tells the students that, if 5 of them hire the limousine, it will cost \$34.20 per person.

a What is the total cost of 5 people hiring the limousine for the night?

b What is the constant of proportionality for $C \propto \frac{1}{n}$, where C is the cost per person and n is the number of people?

c Explain why the total cost and the constant of proportionality for $C \propto \frac{1}{n}$ will always be the same for situations like this.

13 The equation $y = \frac{k}{x}$ can be rearranged to $xy = k$. This means that if $y \propto \frac{k}{x}$, then x and y multiply to a constant value. Verify, for each relationship in question **5**, that the product of each pair of values of x and y is equal to the constant of proportionality found in question **6**.

14 Use the formula $speed = \frac{distance}{time}$ to explain why time and speed are inversely proportional for a fixed distance.

Exercise 4.6C: Challenge

15 The circumference of a circle is directly proportional to the diameter of the circle. Malachi tries to find the constant of proportionality for $C \propto d$ by measuring, as best he can, the circumferences and diameters of several circles. His measurements, in centimetres, are shown in the table below.

d	1	2	3	4	5	6	7
C	3.2	6.2	9.3	13	16.2	18.9	21.7

a For each circle, find the constant of proportionality for $C \propto d$ as a decimal.

b Since there are several different values, Malachi estimates the constant of proportionality by finding the average of the seven ratios. What is his estimation of the constant? Round your answer to two decimal places.

c What famous constant is Malachi trying to approximate? Is he close?

16 The brightness of light decreases inversely proportionally to the square of the distance from the light source. This means that $I \propto \frac{1}{d^2}$ where I is a measure of light intensity (in watts per square metre) and d is the distance (in metres) from the light source at which the measurement is taken.

Consider a lamp with a light intensity of 1.2 W/m², measured at a distance of 2 metres from the lamp.

a Find the constant of proportionality.

b Find:

i the light intensity at a distance of 5 metres from the lamp

ii the distance from the lamp, to the nearest centimetre, if the light intensity is 100 W/m².

c By what percentage does the intensity of the light on an object decrease when the object is moved twice as far from the lamp?

17 Write a short proof of each of the following statements.

a If $x \propto y$ and $y \propto z$, then $x \propto z$.

b If $a \propto \frac{1}{b}$ and $b \propto \frac{1}{c}$, then $a \propto c$.

Online resources:

- **Interactive skillsheet** Direct proportion
- **Interactive skillsheet** Inverse proportion
- **Investigation** Modelling problems involving inverse proportion
- **Quick quiz** 4.6

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Computational Thinking

Curriculum link

Solve equations graphically or using systematic numerical guess-check-and-refine with digital tools, with consideration of whether all solutions have been found (VC2M10A16)

Inter-year links

- Year 7 Classifying polygons
- Year 8 Binary numbers
- Year 8 Geometric proofs
- Year 9 Triangles

Points of intersection

A point of intersection of two graphs is a point which lies on both graphs. Finding these points is not always straightforward.

If both graphs are linear, we can apply the methods discussed in **Lesson 3.6**. Some intersections of linear and non-linear, or two non-linear graphs, are able to be solved algebraically, or by sketching accurate graphs. Further details on these methods are discussed in **Lesson 4.9**.

Sometimes, we are unable to identify the exact points of intersection of two graphs. Digital tools can help to accurately sketch graphs, identify (exact or approximate) points of intersection, and determine whether two graphs intersect.

Check your understanding

1 Answer the following questions.

a Attempt to solve the equation $x^2 - 4x + 5 = 2^x - 1$ using methods discussed in this book. What problem do you encounter when you try to solve the equation?

b Consider the non-linear relationships $y = x^2 - 4x + 5$ and $y = 2^x - 1$.

i Use a digital tool to graph both relationships on the same set of axes.

ii Use a digital tool to find all points of intersection of the two graphs, correct to two decimal places.

iii Explain how your answer to part **ii** relates to the equation in part **a**.

c Use a similar method to solve the equation $x^2 + 3x - 5 = 2^x + 1$, correct to two decimal places.

d Determine the number of solutions to the equation $x^2 + 3x - 5 = 3^x + 1$.

Applying the bisection method using digital tools

There are many numerical methods to solve equations which do not rely on digital graphing tools. One such process is called the 'bisection method', an algorithm that finds a numerical approximation of the solutions to an equation in terms of x, where one side of the equation is equal to zero.

The steps of the algorithm can be summarised as follows.

1 Identify two x-values, a and b, between which a solution must lie.

2 Find the midpoint, c, of a and b.

3 Determine whether the solution must lie between a and c or c and b; take these to be the new values of a and b.

4 Repeat steps **2** and **3** until $a = b$, correct to two decimal places. More iterations can be performed if a greater level of accuracy is required.

We can apply the bisection method to solve the equation $x^2 - 4x - 2^x + 6 = 0$ using the spreadsheet program Microsoft Excel. This equation has one solution which must lie between $x = 0$ and $x = 2$.

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Instructions	Explanation
Open a Microsoft Excel document. Complete row 1 as shown below.	Define the variables to be used in the algorithm. Subscripts are represented with underscores (i.e. a_n is a_n).
In cell A2, type '1'. In cell B2, type '0'. In cell D2, type '2'. In cell F2, type '=(B2+D2)/2'. Press Enter.	Setting $a_1 = 0$ and $b_1 = 2$ is the first step in the algorithm. Set c_1 as the midpoint of a_1 and b_1.
In cell C2, type '=B2^2-4*B2-2^B2+6'. In cell E2, type '=D2^2-4*D2-2^D2+6'. In cell G2, type '=F2^2-4*F2-2^F2+6'. Press Enter after filling in each cell.	In cells C2, E2 and G2 we find the value of the expression $x^2 - 4x - 2^x + 6$ at $x = a_1$, $x = b_1$ and $x = c_1$, respectively.
Highlight cells A2 to H2 and drag down to fill row 3. Delete the entries in cells B3 and D3. In cell B3, type '=IF(G2>0,F2,B2)'. In cell D3, type '=IF(G2<0,F2,D2)'.	Using the spreadsheet program to copy values saves time. The entries in B3 and D3 need to be updated so that the solution lies between a_2 and b_2.
Highlight cells A3 to G3 and drag down to show values up to $n = 9$. Highlight all cells in both rows 2 to 10 and columns B to G and and change the format to show three decimal places.	We can see that $a_9 = 1.305$ and $b_9 = 1.313$, so $a_9 = b_9 = 1.31$, correct to two decimal places. This gives the solution to the equation as $x = 1.31$, correct to two decimal places.

	A	B	C	D	E	F	G
1	n	a_n	y_a	b_n	y_b	c_n	y_c
2	1	0.000	5.000	2.000	-2.000	1.000	1.000
3	2	1.000	1.000	2.000	-2.000	1.500	-0.578
4	3	1.000	1.000	1.500	-0.578	1.250	0.184
5	4	1.250	0.184	1.500	-0.578	1.375	-0.203
6	5	1.250	0.184	1.375	-0.203	1.313	-0.011
7	6	1.250	0.184	1.313	-0.011	1.281	0.086
8	7	1.281	0.086	1.313	-0.011	1.297	0.037
9	8	1.297	0.037	1.313	-0.011	1.305	0.013
10	9	1.305	0.013	1.313	-0.011	1.309	0.001

Check your understanding

2 Answer the following questions.

a Copy the example of the bisection method above into a Microsoft Excel document (or another similar spreadsheet program). Did you find the same solution?

b Explain why using a spreadsheet program like Microsoft Excel is helpful when applying the bisection method to solve an equation.

Task

Use Microsoft Excel (or another similar spreadsheet program) to solve the following equations using the bisection method. Give solutions correct to two decimal places and explain any decisions you make.

1 $x^2 - 2 - 2^x = 0$

2 $3^{-x} - 2x^2 = 0$

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Lesson 4.7

Sketching hyperbolas

Learning intentions

By the end of this lesson you will be able to ...

✔ sketch rectangular hyperbolas by finding asymptotes and considering transformations.

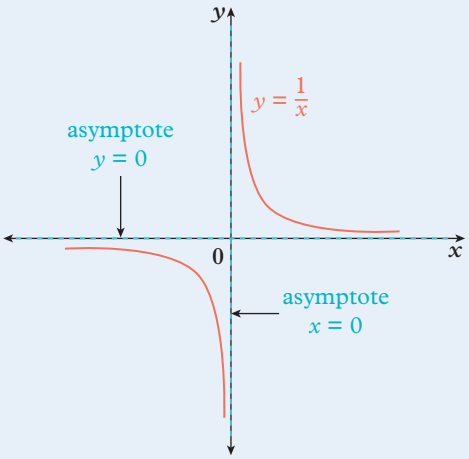
Inter-year links

Year 7	8.3 Translations
Year 8	9.1 Transformations
Year 9	4.5 Other non-linear relationships

Hyperbolas

Key content video
Graphing hyperbolas

- The basic form of a simple **rectangular hyperbola** is the graph of $y = \frac{1}{x}$. The graph has two asymptotes and no x- or y-intercepts.
- The asymptotes for $y = \frac{1}{x}$ lie along the x-axis and along the y-axis and have equations $y = 0$ and $x = 0$.
- The general equation of a hyperbola is: $y = \frac{a}{x-h} + k$ where the graph has asymptotes at $x = h$ and $y = k$.
- The following transformations can be performed on the graph of $y = \frac{1}{x}$ to produce the graph of $y = \frac{a}{x-h} + k$:
  - → a dilation in the y-direction as a varies
  - → a reflection in the x-axis if $a < 0$
  - → a horizontal translation of h units right (or left if $k < 0$).
  - → a vertical translation of k units up (or down if $k < 0$).

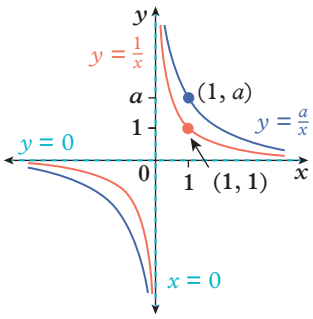
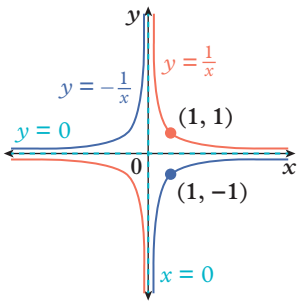
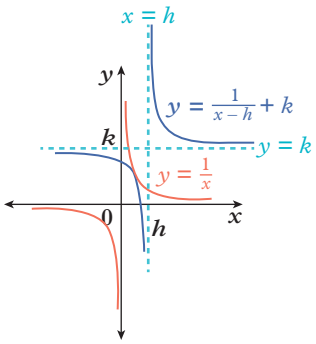


Dilation in the y-direction
If $a > 0$, upright
If $a < 0$, inverted

$$y = \frac{a}{x-h} + k$$

Vertical translation of k units
If $k > 0$, up
If $k < 0$, down

Horizontal translation of h units
If $h > 0$, right
If $h < 0$, left

Dilation	Reflection	Translation
		

- When sketching a hyperbola, always label the horizontal and vertical asymptotes and any axis intercepts. If there are no axis intercepts, label at least one other point (usually $x = 1$).

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Worked example 4.7A Sketching the graph of a hyperbola using dilations

Perform transformations to sketch the graph of each hyperbola below on the same Cartesian plane. Label the asymptotes.

a $y = \frac{1}{x}$ **b** $y = \frac{3}{x}$ **c** $y = \frac{6}{x}$

THINK

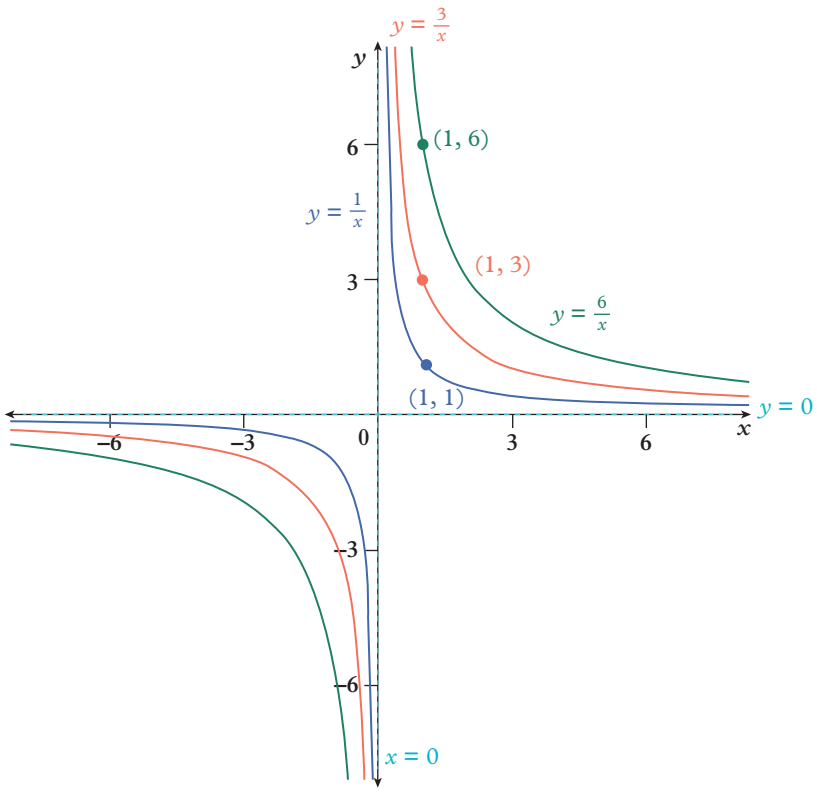
1. Identify the asymptotes. The asymptotes for $y = \frac{a}{x}$ are the horizontal line with equation $y = 0$ and the vertical line with equation $x = 0$.
2. Identify any intercepts.
3. To find the scale of the dilation, label another point on the curve with its coordinates. Find the value of y when $x = 1$.
4. Sketch the basic curve for a rectangular hyperbola using the asymptotes as boundaries. Label a known point on the curve to indicate that a dilation has occurred.

WRITE

a Horizontal asymptote: $y = 0$
Vertical asymptote: $x = 0$
No x- or y-intercepts
When $x = 1$, the point on the curve is $(1, 1)$.

b Horizontal asymptote: $y = 0$
Vertical asymptote: $x = 0$
No x- or y-intercepts
When $x = 1$, the point on the curve is $(1, 3)$.

c Horizontal asymptote: $y = 0$
Vertical asymptote: $x = 0$
No x- or y-intercepts
When $x = 1$, the point on the curve is $(1, 6)$.



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Worked example 4.7B Sketching the graph of a hyperbola using translations

Sketch the graph of $y = \frac{1}{x+2} - 4$. Label the asymptotes and intercepts.

THINK

1 Identify the translations that need to be performed on $y = \frac{1}{x}$. Compare to $y = \frac{a}{x-h} + k$ to identify the translations: $h = -2$ (move 2 units left) and $k = -4$ (move 4 units down).

WRITE

$$y = \frac{1}{x+2} - 4$$

Horizontal translation of 2 units left

Vertical translation of 4 units down

2 Identify the rule for each asymptote: $x = h$ and $y = k$.

The vertical asymptote is $x = -2$.
The horizontal asymptote is $y = -4$.

3 Find the x-intercept.

When $y = 0$:

$$0 = \frac{1}{x+2} - 4$$
$$\frac{1}{x+2} = 4$$
$$x + 2 = \frac{1}{4}$$
$$x = \frac{1}{4} - 2$$
$$= -\frac{7}{4}$$

The x-intercept is $\left(-\frac{7}{4}, 0\right)$.

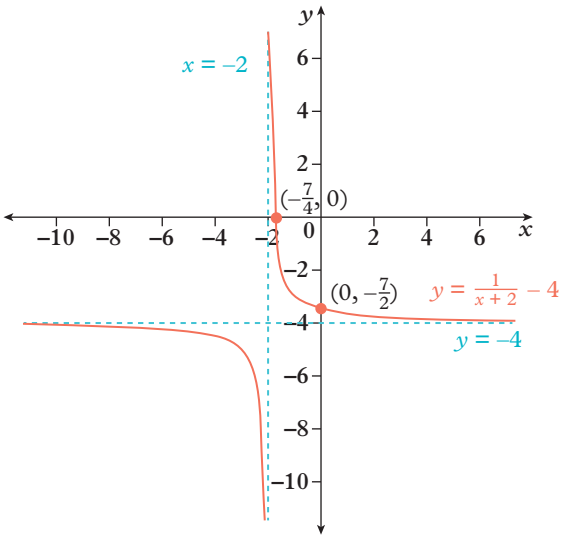
4 Find the y-intercept.

When $x = 0$:

$$y = \frac{1}{0+2} - 4$$
$$= \frac{1}{2} - 4$$
$$= -\frac{7}{2}$$

The y-intercept is $\left(0, -\frac{7}{2}\right)$.

5 Sketch the basic curves for a hyperbola using the asymptotes as boundaries. Label the intercepts and asymptotes.



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Worked example 4.7C Sketching the graph of a hyperbola using reflections

Sketch the graph of $y = -\frac{4}{x} + 1$. Label the asymptotes and intercepts.

THINK

1 Identify the three translations that need to be performed on $y = \frac{1}{x}$: reflection in the x-axis; $k = 1$ (move 1 unit up) and a dilation in the y-direction by a factor of 4.

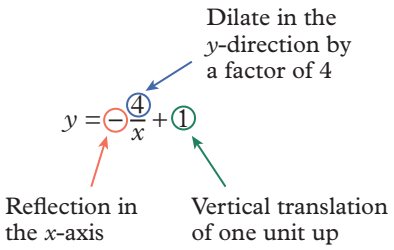
2 Identify the rule for each asymptote: $x = 0$ and $y = 1$.

3 Find the x-intercept.

4 Find the y-intercept.

5 Sketch the basic curves for a hyperbola using the asymptotes as boundaries. Label the intercepts and asymptotes.

WRITE



The vertical asymptote is $x = 0$.

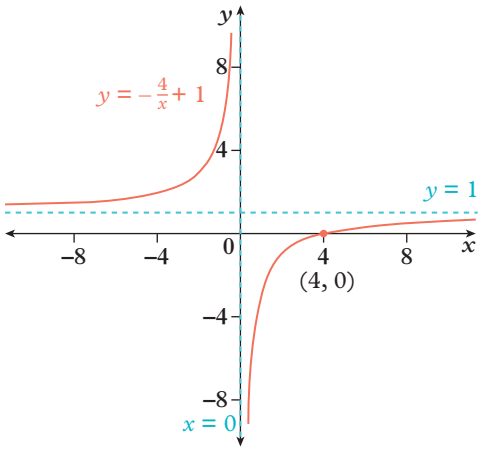
The horizontal asymptote is $y = 1$.

For the x-intercept, $y = 0$:

$$y = -\frac{4}{x} + 1$$
$$0 = -\frac{4}{x} + 1$$
$$\frac{4}{x} = 1$$
$$x = 4$$

The x-intercept is $(4, 0)$.

There is no y-intercept.



Helpful hints

- ✔ Be careful not to interchange h and k. The h values refers to the horizontal translation, and the k value refers to the vertical translation. Think 'h for horizontal' to help you remember.
- ✔ Don't forget to sketch the x- and y-asymptotes, even when they are on an axis.
- ✔ The coordinates (h, k) give the point of intersection for the two asymptotes of the hyperbola.
- ✔ When sketching a hyperbola, start by drawing the asymptotes.

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Learning pathways

1–9, 11, 12 | 1–9, 11, 13, 14(a, b) | 1(d, e), 2, 4–11, 13–15

Exercise 4.7A: Understanding and fluency

WE 4.7A **1** Sketch the graph of each of these hyperbolas. Label the asymptotes.

a $y = \frac{4}{x}$ **b** $y = \frac{7}{x}$ **c** $y = \frac{2}{x}$ **d** $y = \frac{5}{x}$ **e** $y = \frac{10}{x}$

2 Sketch the graph of each of these relationships by performing a vertical translation. Clearly show the asymptotes and intercepts.

a $y = \frac{1}{x} + 4$ **b** $y = \frac{1}{x} - 7$ **c** $y = \frac{1}{x} + 5$ **d** $y = \frac{1}{x} - 1$

3 Sketch the graph of each of these relationships by performing a horizontal translation. Clearly show the asymptotes and intercepts.

a $y = \frac{1}{x-5}$ **b** $y = \frac{1}{x+4}$ **c** $y = \frac{1}{x-7}$ **d** $y = \frac{1}{x+1}$

WE 4.7B **4** Sketch the graph of each of these equations. Clearly show the asymptotes and intercepts.

a $y = \frac{1}{x-3} - 2$ **b** $y = \frac{1}{x+5} + 3$ **c** $y = \frac{1}{x-2} + 4$ **d** $y = \frac{1}{x+4} - 5$ **e** $y = \frac{1}{x-1} - 8$

5 Use digital technology to produce the first four graphs from question **4**. Compare them to your sketches.

WE 4.7C **6** Sketch the graph for each equation below, clearly showing the asymptotes and intercepts.

a $y = -\frac{1}{x} + 3$ **b** $y = -\frac{2}{x}$ **c** $y = -\frac{1}{x-2}$ **d** $y = -\frac{5}{x}$ **e** $y = -\frac{1}{x-2} + 3$

7 Use transformations to help you sketch the graph of each of these equations. Clearly show the asymptotes and intercepts.

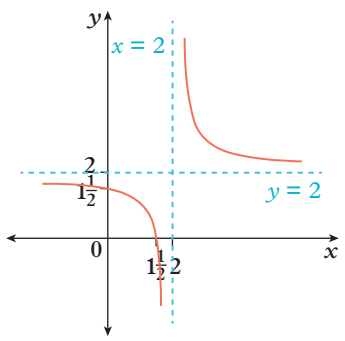
a $y = -\frac{1}{x+4}$ **b** $y = -\frac{1}{x} - 2$ **c** $y = -\frac{1}{x-3} + 5$ **d** $y = -\frac{1}{x+5} - 1$ **e** $y = -\frac{2}{x-1} - 4$

8 Use digital technology to produce the first four graphs from question **7**. Compare them to the ones you sketched.

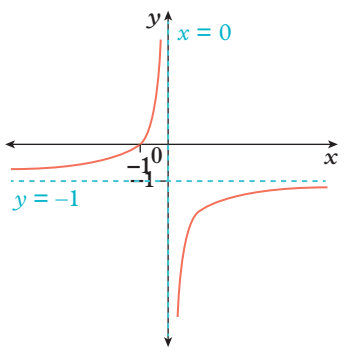
Exercise 4.7B: Problem solving and reasoning

9 Write the equation for each hyperbola shown below. Each hyperbola has a dilation factor of $a = \pm 1$.

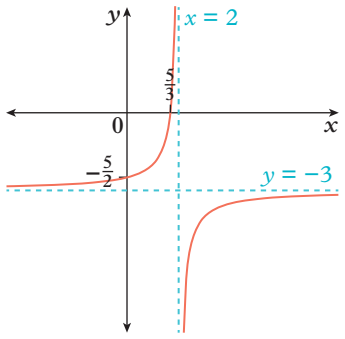
a



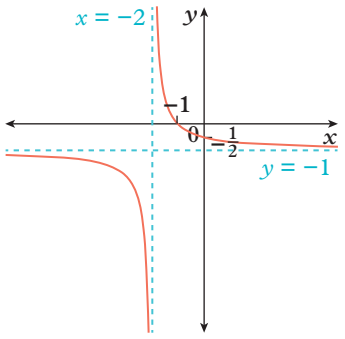
b



c



d



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10 Why does reflecting the graph of $y = \frac{1}{x}$ in the x-axis produce the same graph as reflecting $y = \frac{1}{x}$ in the y-axis?

11 One way to describe the graph of $y = \frac{1}{x}$ is to describe the behaviour of the curve as x approaches very large or very small values. Complete each statement below using a term from this list:

- positive infinity
- zero
- negative infinity

a When x approaches infinity, y approaches _____.

b When x approaches zero from the right, y approaches _____.

c When x approaches zero from the left, y approaches _____.

d When x approaches negative infinity, y approaches _____.

12 At a bank, the average waiting time for customers in a queue is related to the number of bank tellers on duty and can be modelled by the relationship $W_n = \frac{12}{n}$, where W_n is the average waiting time, in minutes, when there are n bank tellers on duty.

a Sketch a graph of this relationship.

b On average, how long does a customer wait when the number of bank tellers on duty is:

i 2 **ii** 3 **iii** 4

iv 6 **v** 12 **vi** 24?

c What is the maximum waiting time for a customer? When does this occur?

d According to this model, will there be occasions when the waiting time is zero? Explain.

13 The graph of a rectangular hyperbola always has a horizontal asymptote and a vertical asymptote. Consider the hyperbola $y = \frac{1}{x-2} + 3$, which has asymptotes at $x = 2$ and $y = 3$.

a Substitute $x = 2$ into the equation and explain why this causes the y-value to be undefined.

b Explain why the error from part **a** will not occur for any other value of x that is not 2.

c Substitute $y = 3$ into the equation and attempt to solve it. What goes wrong when you attempt to solve it for x?

d Explain why the error from part **c** will not occur for any other value of y that is not 3.

Exercise 4.7C: Challenge

14 The equation of a hyperbola can be determined if the number of unknowns matches the number of pieces of information given.

a Find the rule for a hyperbola of the form $y = \frac{8}{x-h}$ that passes through (6, 2).

b Find the rule for a hyperbola of the form $y = \frac{3}{x-h} + 2$ that passes through (–2, 1).

c Find the rule for a hyperbola of the form $y = \frac{a}{x-h}$ that passes through (–7, –1) and (1, 1).

15 Sketch the graphs of each of the following. Remember that the horizontal asymptote occurs for the value of x that would result in the denominator being equal to zero.

a $y = \frac{1}{2x-1}$ **b** $y = \frac{1}{1-x} + 2$ **c** $y = \frac{-2}{1-2x} - 1$

Online resources:

Interactive skillsheet
Hyperbolas

Quick quiz
4.7

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Lesson 4.8

Graphs and transformations

Learning intentions

By the end of this lesson you will be able to ...

✔ use algebraic and graphical techniques to relate the equation and graph of a non-linear relationship.

Inter-year links

Year 7	8.5 Combined transformations
Year 8	9.1 Transformations
Year 9	4.5 Other non-linear relationships

Transforming non-linear relationships

Key content video
Graphs and transformations

- The equations of **non-linear relationships** are often written in ways that emphasise the link between the algebraic and graphical forms of the relationship. These are usually related to transformations or axis intercepts.
- The different forms of some non-linear relationships are summarised in the following table.

Basic form	Transformation form
$y = x^2$	$y = a(x - h)^2 + k$
$y = \frac{1}{x}$	$y = \frac{a}{x - h} + k$
$y = b^x$	$y = a \times b^{nx} + k$
$x^2 + y^2 = 1$	$(x - h)^2 + (y - k)^2 = r^2$

- A quadratic relationship can also be written in intercept form as $y = a(x - x_1)(x - x_2)$, or in general form as $y = ax^2 + bx + c$.
- Complete the follow steps to determine the equation of a non-linear relationship from its graph using transformations.

1. Determine the basic form of the equation from the shape of the graph.
2. Identify any transformations performed on the basic graph to create the transformed graph.
3. Identify the unknown constants in the transformation form of the equation.
4. Write the equation of the graph in transformation form.

Transforming points

- When sketching non-linear relationships using transformations, it is often helpful to first transform individual reference points on the basic graph to their location on the transformed graph.
- The following transformations can be applied to a non-linear relationship.
  - → dilation by a factor of a in the y-direction
  - → dilation by a factor of n in the x-direction
  - → reflection in the x-axis ($a < 0$)
  - → reflection in the y-axis ($n < 0$)
  - → translation h units right (or left if $h < 0$)
  - → translation k units up (or down if $k < 0$)
- When these transformations are applied to a point (x, y) on the basic graph of a non-linear relationship, the resulting point can be found using the formula:

$$(x, y) \mapsto \left(\frac{x}{n} + h, ay + k\right).$$

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Completing the square

- Completing the square is a useful technique to help graph parabolas and circles.
- The 'transformation form' of a quadratic relationship is usually called 'turning point form' and can be found by completing the square of an equation in the form $y = ax^2 + bx + c$.
- The equation of a circle is sometimes written in expanded form. Complete the square twice, once in terms of x, and again in terms of y, and then simplify to write in transformation form.

  For example:

$$x^2 + 2x + y^2 + 4y = 4$$
$$(x + 1)^2 - 1 + (y + 2)^2 - 4 = 4$$
$$(x + 1)^2 + (y + 2)^2 = 9$$

Worked example 4.8A Identifying the rule for a non-linear relationship

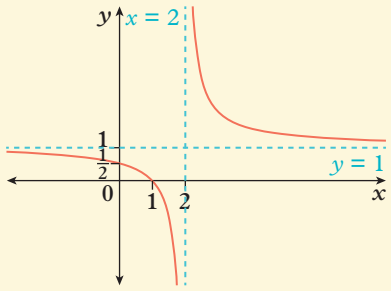
For the graph shown on the right:

a describe the relationship

b identify the basic graph used to produce the graph of this relationship

c identify the transformations that have been performed on the basic graph

d write the equation for the relationship shown.



THINK	WRITE
a Identify the shape. This graph is a rectangular hyperbola.	**a** The graph is a rectangular hyperbola.
b Use the shape of the graph to identify the basic equation form of the type of relationship it represents.	**b** $y = \frac{1}{x}$
c **1** Check if a reflection has been performed.	**c** There has been no reflection, because the curves are in the same quadrants as the basic graph.
2 Identify the translations performed.	There is an asymptote at $y = 1$ so there has been a translation of 1 unit up. There is an asymptote at $x = 2$ so there has been a translation of 2 units right.
3 Check for dilations by substituting known values of x and y.	The rule is of the form: $y = \frac{a}{x-2} + 1$ For $x = 1, y = 0$: $0 = \frac{a}{1-2} + 1$ $\frac{a}{-1} = -1$ $a = 1$ There has been no dilation.
d Use your answers for parts **b** and **c** to help you write the equation of the graph.	**d** $y = \frac{1}{x-2} + 1$

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Worked example 4.8B Transforming points

Consider the non-linear relationship $y = -5 \times 2^{3x} + 4$.

a Identify the values of the variables a, n, h and k and substitute them into the point transformation formula.

b The following are points on the graph of $y = 2^x$. Use your formula from part **a** to find their images on the graph of $y = -5 \times 2^{3x} + 4$.

i $(0, 1)$ **ii** $(1, 2)$ **iii** $(2, 4)$ **iv** $\left(-1, \frac{1}{2}\right)$

THINK

a Identify the related transformation form of the non-linear relationship. Compare this with the given non-linear relationship and identify the value of each of the variables.

b Substitute the values of x and y into the formula from part **a** and simplify.

WRITE

a $y = a \times 2^{nx} + k$

$y = -5 \times 2^{3x} + 4$

$a = -5, n = 3, h = 0, k = 4$

$(x, y) \mapsto \left(\frac{x}{n} + h, ay + k\right)$

$(x, y) \mapsto \left(\frac{x}{3}, -5y + 4\right)$

b **i** $(0, 1) \mapsto \left(\frac{0}{3}, -5 \times 1 + 4\right) = (0, -1)$

ii $(1, 2) \mapsto \left(\frac{1}{3}, -5 \times 2 + 4\right) = \left(\frac{1}{3}, -6\right)$

iii $(2, 4) \mapsto \left(\frac{2}{3}, -5 \times 4 + 4\right) = \left(\frac{2}{3}, -16\right)$

iv $\left(-1, \frac{1}{2}\right) \mapsto \left(-\frac{1}{3}, -5 \times \frac{1}{2} + 4\right) = \left(-\frac{1}{3}, \frac{3}{2}\right)$

Worked example 4.8C Completing the square to find the equation of a circle

Complete the square twice to write the following equation in the form $(x - h)^2 + (y - k)^2 = r^2$.

$x^2 + 6x + y^2 - 2y + 1 = 0$

THINK

1 Arrange the equation with the constant on the RHS.

2 Identify quadratic expressions in terms of x and y on the LHS.

3 Complete the square of both quadratic expressions separately.

4 Substitute the completed square expressions back into the equation from step **1**.

5 Simplify the equation by taking any constants to the RHS.

WRITE

$x^2 + 6x + y^2 - 2y + 1 = 0$

$x^2 + 6x + y^2 - 2y = -1$

$x^2 + 6x$ and $y^2 - 2y$

$x^2 + 6x = \left(x + \frac{6}{2}\right)^2 - \left(\frac{6}{2}\right)^2$

$= (x + 3)^2 - 9$

$y^2 - 2y = \left(y - \frac{2}{2}\right)^2 - \left(\frac{2}{2}\right)^2$

$= (y - 1)^2 - 1$

$x^2 + 6x + y^2 - 2y = -1$

$(x + 3)^2 - 9 + (y - 1)^2 - 1 = -1$

$(x + 3)^2 + (y - 1)^2 = 9$

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Helpful hints

- ✔ You may have noticed that the constant k, which relates to vertical translations, is *subtracted* in the rule for a circle but *added* in the rules for other graphs. In all cases, a vertical translation of k units is achieved by substituting $y - k$ for y. The only difference is how the equations are rearranged.
- ✔ To determine an equation with n unknowns, you need n pieces of information. This information could include points on the graph or any transformations.
- ✔ Always choose the best form of an equation to match the information you are given. For example, if you know the x-intercepts of a parabola, choose intercept form instead of turning point form.

Learning pathways

1–4, 5(a, c, e), 6–8, 10, 11(a, b), 12–15, 17–19	1,2(d–f), 3, 4, 6, 7–9(1st column), 10, 11(c, d), 12–15, 17, 19, 20	1(e–h), 3, 4, 7–9(2nd column), 12–14, 16,18–23

Exercise 4.8A: Understanding and fluency

WE 4.8A **1** For each of the following graphs:

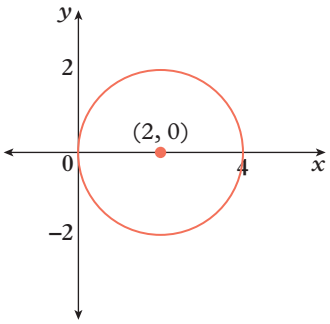
i describe the relationship

ii identify the basic graph used to produce the graph of the relationship

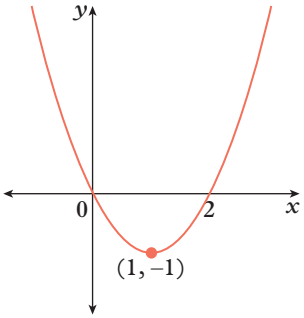
iii identify the transformations that have been performed on the basic graph to produce the graph shown

iv write the rule for the relationship.

a



b



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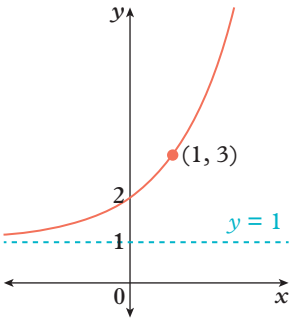

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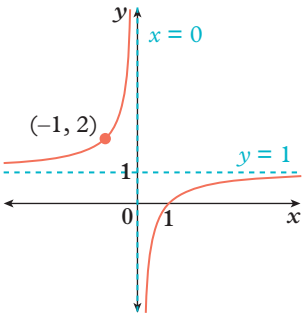

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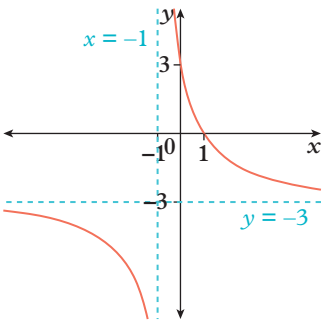

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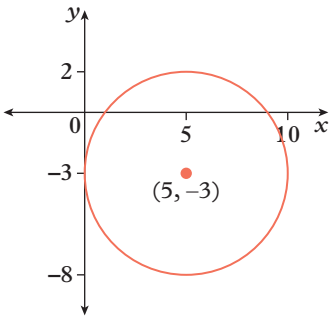

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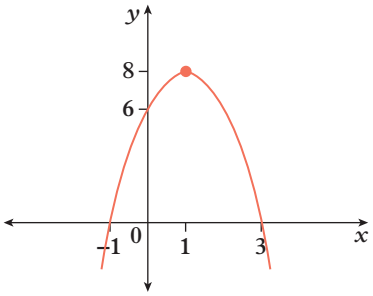

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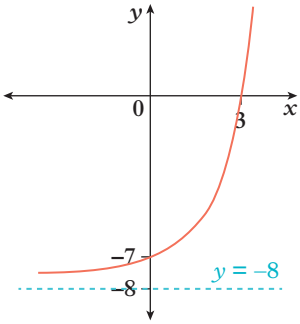

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2 For each equation below:

i describe the type of non-linear relationship it represents

ii identify the transformations to be performed on the basic graph of the relationship

iii list any key features, including axis intercepts

iv use transformations to sketch the graph, showing all key features.

a $y = -(x + 3)^2 - 5$ **b** $y = 3^{x+2} - 1$ **c** $(x - 4)^2 + (y + 2)^2 = 9$

d $y = \frac{1}{x - 6} + 2$ **e** $y = 3(x - 2)^2 - 12$ **f** $y = -2^x + 1$

WE 4.8B **3** Consider the non-linear relationship $y = 3 \times 2^{2x} - 1$.

a Identify the values of the variables a, n, h and k and substitute them into the point transformation formula.

b The following are points on the graph of $y = 2^x$. Use your formula from part **a** to find their images on the graph of $y = 3 \times 2^{2x} - 1$.

i $(0, 1)$ **ii** $(1, 2)$ **iii** $(2, 4)$ **iv** $\left(-1, \frac{1}{2}\right)$

4 Consider the non-linear relationship $y = -\frac{2}{x + 1} - 1$.

a Identify the values of the variables a, h and k and substitute them into the point transformation formula. Hint: Use $n = 1$ when transforming points on hyperbolas.

b The following are points on the graph of $y = \frac{1}{x}$. Use your formula from part **a** to find their images on the graph of $y = -\frac{2}{x + 1} - 1$.

i $(1, 1)$ **ii** $(-1, -1)$ **iii** $\left(\frac{1}{2}, 2\right)$ **iv** $\left(-\frac{1}{2}, -2\right)$

5 Complete the square of the following quadratic relationships to write them in turning point form.

a $y = x^2 - 4x + 6$ **b** $y = x^2 + 6x - 11$

c $y = x^2 - 5x + 10$ **d** $y = x^2 + 9x - 25$

e $y = 2x^2 - 10x - 12$ **f** $y = 2x^2 + 5x - 6$

6 Consider the quadratic trinomials $x^2 + 6x + 2$ and $y^2 - 4y + 7$.

a Complete the square of each quadratic trinomial.

b Sketch the circle with equation $x^2 + 6x + 2 + y^2 - 4y + 7 = 0$.

WE 4.8C **7** Complete the square twice to write the following equations in the form $(x - h)^2 + (y - k)^2 = r^2$.

a $x^2 + 2x + y^2 + 2y + 1 = 0$ **b** $x^2 + 6x + y^2 - 4y + 9 = 0$

c $x^2 - 6x + y^2 - 10y + 25 = 0$ **d** $x^2 - 8x + y^2 - 6y + 9 = 0$

e $x^2 + 20x + y^2 + 14y + 124 = 0$ **f** $x^2 - 18x + y^2 - 12y + 108 = 0$

8 Use your answers to sketch the circles in question **7**.

9 Rearrange the following equations and then complete the square twice to write them in the form $(x - h)^2 + (y - k)^2 = r^2$.

a $x^2 + y^2 - 2x - 2y - 1 = 0$ **b** $x^2 + y^2 - 6x + 4y + 9 = 0$

c $x^2 + y^2 + 32 = 8y - 10x$ **d** $x^2 + y^2 - 69 = 14x - 12y$

Exercise 4.8B: Problem solving and reasoning

10 The equation $H = -x^2 + 6x + 1$ models the path of a poorly hit baseball, where H represents the height above the ground of the ball (in metres) and x represents the horizontal distance travelled by the ball (in metres).

a Substitute $x = 0$ into the equation to find the height above the ground of the ball when it was hit by the batter.

b Complete the square for $-x^2 + 6x + 1$.

c Find the maximum height reached by the ball.

d Find the horizontal distance the ball travelled before hitting the ground. Give your answer in metres, rounded to two decimal places.

e Use the information found in parts **a–d** to help you sketch the graph of H versus x, labelling all relevant points.

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11 Sketch each of the following parabolas by first completing the square. Label all relevant points as exact values.

a $y = x^2 + 6x + 9$ **b** $y = x^2 - 2x + 3$ **c** $y = x^2 - 4x + 1$ **d** $y = 3x^2 - 6x + 5$

12 From the given information, write an equation for each of these quadratic relationships in turning point form.

a The graph is an upright parabola, with the turning point at (2, 6) and a y-intercept of 10.

b The graph is an upright parabola, with the turning point at (–4, –12) and a y-intercept of 20.

c The graph is an inverted parabola, with the turning point at (–1, 7) and a y-intercept of 4.

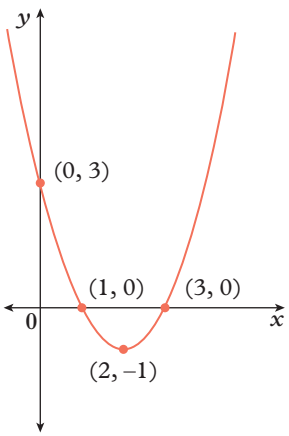
d The graph is an inverted parabola, with the turning point at (3, –5) and a y-intercept of –8.

13 When determining the equation of a graph, consider the information provided in the graph when choosing which form of equation to use. For each of the parabolas below:

i determine the equation in intercept form

ii determine the equation in turning point form.

a



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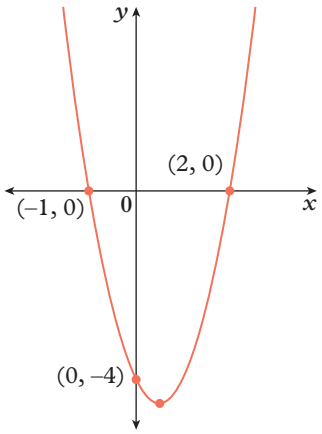
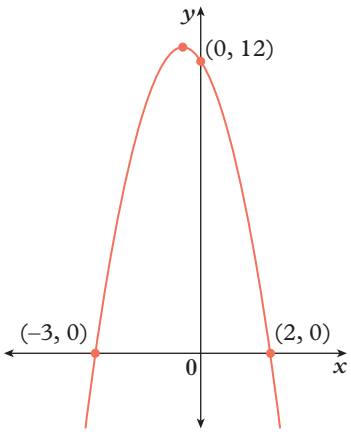


c



14 When determining the equation of a parabola, when should you find the equation in turning point form and when is it better to use intercept form?

15 The area of grass watered by a sprinkler forms a circle of radius 4 m.

a Write an equation for the relationship that describes this circle if the sprinkler is located at the origin of a Cartesian plane.

b If the sprinkler is moved 5 m north (up) and 7 m west (left) of its original position on the Cartesian plane, write a new equation for the circle that encloses the area being watered.

c Does the new area overlap with the original area? Explain.

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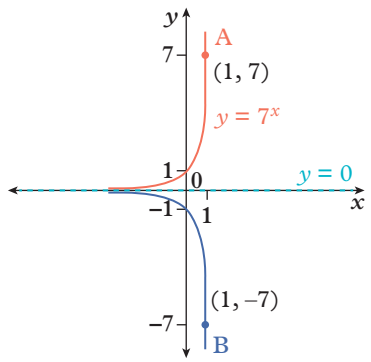
16 Find the equation of the circle that passes through the points (–10, 5), (–4, 11), (2, 5) and (–4, –1).

17 For each pair of graphs shown below:

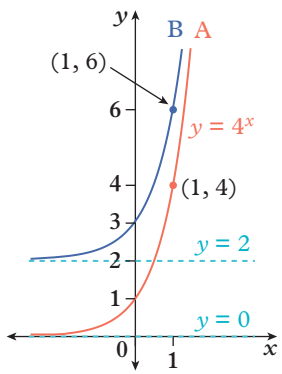
i identify the transformation performed on graph A to produce graph B

ii write the rule for graph B.

a



b

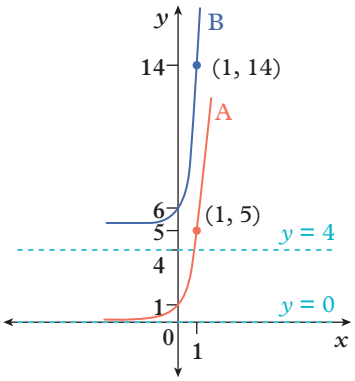


18 For each pair of graphs shown below:

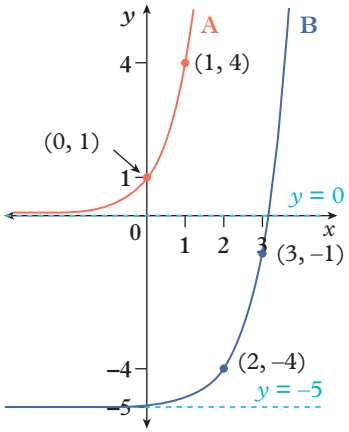
i identify the transformations performed on graph A to produce graph B

ii write the rules for graphs A and B.

a



b



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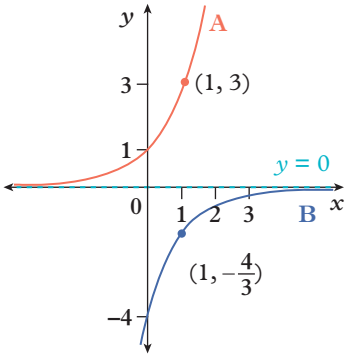
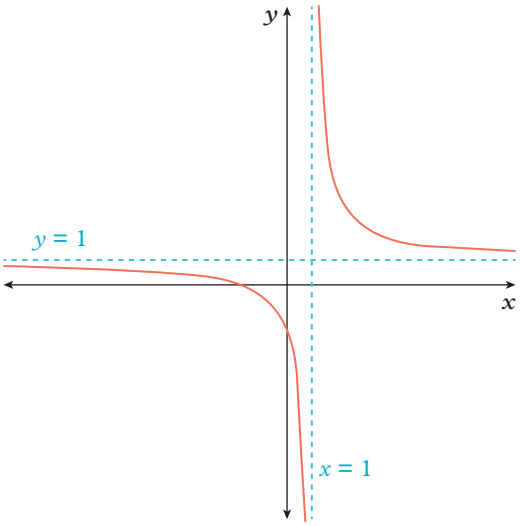


19 Consider the graph of the hyperbola below. Do you have enough information to determine the equation of the graph? Explain why or why not.



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20 For each non-linear relationship below, determine whether the information provided is always sufficient to determine the equation of a graph. When the information is insufficient, provide an example of two non-linear relationships which show this is not enough information.

a quadratic relationship: turning point and y-intercept

b circle: the centre and any point on the circle

c exponential relationship: asymptote and y-intercept

d rectangular hyperbola: both asymptotes and one point on the graph

Exercise 4.8C: Challenge

21 Given a point transformation formula, we can identify a sequence of transformations to identify and map a transformed point back to its original location. (x, y) is used to identify a point on the original graph and (x', y') represents a point on the transformed graph.

Consider the point transformation given by $(x, y) \mapsto (2x, -3y - 1)$.

a Solve the equation $x' = 2x$ for x.

b Solve the equation $y' = -3y - 1$ for y.

c Write a point transformation $(x', y') \mapsto (x, y)$ to determine the original locations of points on the transformed graph.

d The point $(2, 6)$ lies on the transformed graph. Find its original location.

e Show that first applying the point transformation $(x, y) \mapsto (2x, -3y - 1)$ and then applying your transformation from part **c** results in the original point.

22 When multiple transformations are performed on a single graph, the order is very important. For example, dilating the graph of $y = x^2$ by a factor of 2 in the y-direction and then translating it 1 unit up results in the graph of $y = 2x^2 + 1$. However, translating the graph of $y = x^2$ up by 1 unit and then dilating it by a factor of 2 in the y-direction results in the graph of $y = 2x^2 + 2$.

a Find the equation of the graph that results from applying the following transformations to the graph of $y = x^2$, in the given order.

i dilate by a factor of $\frac{1}{2}$ in the y-direction, then translate up by 2 units

ii translate down by 3 units, then dilate by a factor of 2 in the y-direction

b Find the equation of the graph that results from applying the following transformations, in order, to the graph of $y = x^2$.

i reflect in the x-axis, then translate up by 3 units

ii translate up by 3 units, then reflect in the x-axis

23 Determine the equations of the following non-linear relationships.

a Quadratic relationship passing through $(0, 2)$, $(1, 5)$ and $(-2, 2)$

b Quadratic relationship passing through $(3, 2)$, $(-2, 7)$ and $(2, -1)$

c Circle passing through $(0, -1 + \sqrt{3})$, $(0, -1 - \sqrt{3})$ and $(3, -1)$

d Rectangular hyperbola passing through $(-1, 0)$, $(-2, 1)$ and $(-5, -2)$

e Exponential relationship with $b = 2$ passing through $(-2, -5)$, $(-1, -1)$ and $(0, 1)$

Online resources:

Interactive skillsheet Determining non-linear relationships

Quick quiz 4.8

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Lesson 4.9

Solving non-linear equations

Learning intentions

By the end of this lesson you will be able to ...

✔ solve non-linear equations.

Inter-year links

Year 7 6.6 Solving equations using inverse operations

Year 8 6.7 Intersecting lines

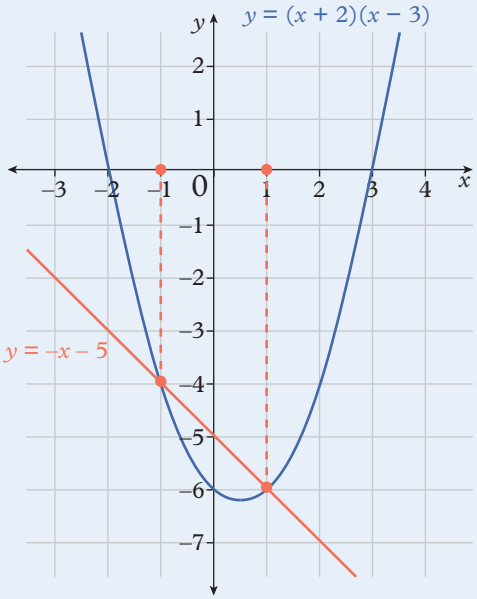
Year 9 4.1 Solving quadratic equations

Solving non-linear equations graphically

Key content video
Solving non-linear equations

- Equations with one unknown can be solved graphically by plotting the graphs of y = [LHS] of the equation and y = [RHS] of the equation on the same Cartesian plane, and determining the x-coordinates of any points of intersection. These x-coordinates are the solutions to the original equation.

  For example, the solutions to the equation $(x + 2)(x - 3) = -x - 5$ are the x-coordinates of the points of intersection of the graphs $y = (x + 2)(x - 3)$ and $y = -x - 5$; that is $x = \pm 1$.



Solving simultaneous equations

- A system of simultaneous equations can include linear and non-linear relationships. The solutions to a system of simultaneous equations are the points which satisfy all relationships.

  For example, the solutions to the system of equations $y = (x + 2)(x - 3)$ and $y = -x - 5$ are the points $(-1, -4)$ or $(1, -6)$.
- To solve a system of simultaneous equations, reduce the system to one equation with one unknown, solve for that unknown, and then substitute the value(s) back into the system to determine the corresponding value(s) of the other unknown. For non-linear equations, the substitution method (see **Lesson 3.6**) is usually preferred, although the elimination method can sometimes be used.

The bisection method

Key content video
The bisection method

- A **zero** of an expression in terms of x is a value of x which, when substituted, results in the expression being equal to 0.

  For example, -2 and 3 are the zeros of the expression $(x + 2)(x - 3)$.
- An **interval** is a set of real numbers lying between two endpoints, which may or may not be included in the interval. The endpoints are called the upper and lower bounds of the interval.

  For example, $0 \le x < 2$ is an interval with a lower bound of 0 and an upper bound of 2. In this interval, 0 is included, and 2 is not included.

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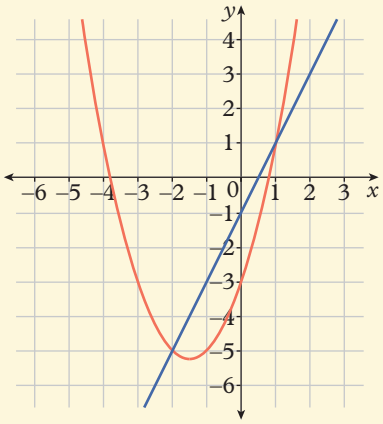
- The **bisection method** is an algorithm that approximates a zero of an expression numerically (as opposed to solving algebraically).
- To apply the bisection method to an expression in terms of x:
  1. Find an interval $a < x < b$ where the expression is positive for $x = a$ and negative for $x = b$ (or negative for $x = a$ and positive for $x = b$).
  2. Find the midpoint of the interval: $c = \frac{a + b}{2}$.
  3. Choose a new interval $a < x < c$ or $c < x < b$ such that when the endpoints of the new interval are substituted into the expression, there is exactly one positive value and one negative value.
  4. Repeat steps **2** and **3** until $a = b$, correct to a desired number of decimal places. In this lesson, we will look for $a = b$ correct to two decimal places.

Worked example 4.9A Solving non-linear equations graphically

Determine the solution(s) to $2x - 1 = x^2 + 3x - 3$ by identifying the x-coordinate(s) of the point(s) of intersection of the graphs of $y = 2x$ and $y = x^2 + 3x - 3$.

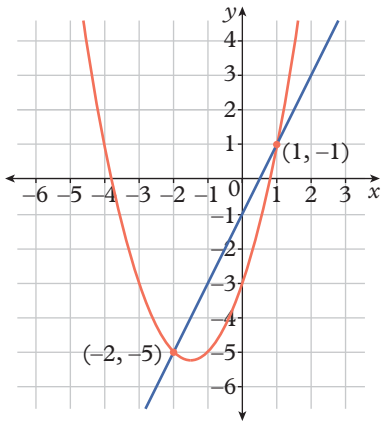


THINK

1. Identify the point(s) of intersection of the graphs.

2. Write the x-coordinate(s) of the point(s) of intersection as the solution(s) to the equation.

WRITE



$x = -2$ or $x = 1$

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Worked example 4.9B Solving non-linear equations algebraically

Solve the following equations for x algebraically.

a $2x + 3 = x^2 - 3x - 3$

b $2x + 3 = \dfrac{3}{x + 4}$

THINK

1. Rearrange the equation so that it is the form of a quadratic expression equal to 0.
2. Solve the quadratic equation.

WRITE

a
$$\begin{aligned} 2x + 3 &= x^2 - 3x - 3 \\ 0 &= x^2 - 5x - 6 \\ 0 &= (x - 6)(x + 1) \\ x - 6 = 0 &\text{ or } x + 1 = 0 \\ x = 6 &\text{ or } \quad x = -1 \end{aligned}$$

b
$$\begin{aligned} 2x + 3 &= \frac{3}{x + 4} \\ (2x + 3)(x + 4) &= 3 \\ 2x^2 + 11x + 12 &= 3 \\ 2x^2 + 11x + 9 &= 0 \\ (2x + 9)(x + 1) &= 0 \\ 2x + 9 = 0 &\text{ or } x + 1 = 0 \\ x = -\frac{9}{2} &\text{ or } \quad x = -1 \end{aligned}$$

Worked example 4.9C Solving simultaneous non-linear equations

Solve the following systems of simultaneous equations algebraically. Give your answers as pairs of coordinates in the form (x, y).

a $x^2 + y^2 = 289$ and $y = -x - 7$

b $(x - 1)^2 + (y - 4)^2 = 16$ and $y = x + 7$

THINK

1. Number each equation for reference and substitute the expression for y from equation ② into equation ①. Use brackets when substituting the expression.
2. Solve the resulting equation for x.
3. Substitute the solutions for x into equation ② and solve for y.

WRITE

a $x^2 + y^2 = 289$ ①

$y = -x - 7$ ②

Substitute $y = -x - 7$ into ①:
$$\begin{aligned} x^2 + y^2 &= 289 \\ x^2 + (-x - 7)^2 &= 289 \\ x^2 + x^2 + 14x + 49 &= 289 \\ 2x^2 + 14x - 240 &= 0 \\ x^2 + 7x - 120 &= 0 \\ (x - 8)(x + 15) &= 0 \\ x = 8 &\text{ or } x = -15 \end{aligned}$$

Substitute $x = 8$ and $x = -15$ into ②:

$x = 8$: $y = -(8) - 7 = -15$

$x = -15$: $y = -(-15) - 7 = 8$

The solutions are $(8, -15)$ or $(-15, 8)$.

b $(x - 1)^2 + (y - 4)^2 = 16$ ①

$y = x + 7$ ②

Substitute $y = x + 7$ into ①:
$$\begin{aligned} (x - 1)^2 + (y - 4)^2 &= 16 \\ (x - 1)^2 + ((x + 7) - 4)^2 &= 16 \\ (x - 1)^2 + (x + 3)^2 &= 16 \\ x^2 - 2x + 1 + x^2 + 6x + 9 &= 16 \\ 2x^2 + 4x - 6 &= 0 \\ x^2 + 2x - 3 &= 0 \\ (x + 3)(x - 1) &= 0 \\ x = -3 &\text{ or } x = 1 \end{aligned}$$

Substitute $x = -3$ and $x = 1$ into ②:

$x = -3$: $y = (-3) + 7 = 4$

$x = 1$: $y = (1) + 7 = 8$

The solutions are $(-3, 4)$ or $(1, 8)$.

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Worked example 4.9D Using the bisection method

Consider the equation $2x^2 - 3x - 7 = 0$ on the interval $0 < x < 4$.

a Substitute the upper and lower bounds of the interval into the LHS of the equation and show that one bound evaluates to a positive value and the other bound evaluates to a negative value.

b Use the bisection method to determine the solution in the given interval, correct to two decimal places.

THINK

a 1 Substitute the upper and lower bounds into the LHS of the equation.

WRITE

a $x = 0$: $2(0)^2 - 3(0) - 7$
$= 0 - 0 - 7$
$= -7$
$x = 4$: $2(4)^2 - 3(4) - 7$
$= 32 - 12 - 7$
$= 13$

2 Show that one value is positive and the other value is negative by writing two inequalities.

$-7 < 0$ and $13 > 0$

b 1 Set $a_1 = 0$ and $b_1 = 4$.

2 Determine the midpoint of a_1 and b_1 as $c_1 = \frac{a_1 + b_1}{2}$.

b $a_1 = 0$, $b_1 = 4$

$c_1 = \frac{0 + 4}{2}$
$= \frac{4}{2}$
$= 2$

3 Substitute c_1 into the LHS of the equation, rounding correct to three decimal places where necessary. If $a_n = b_n$ correct to two places, then c_n is the desired approximate solution. Otherwise, continue to the next step.

$x = c_1 = 2$: $2(2)^2 - 3(2) - 7 = 8 - 6 - 7$
$= -5$

$a_2 = c_1 = 2$
$b_2 = b_1 = 4$

4 Set $a_2 = c_1$ and $b_2 = b_1$ or $a_2 = a_1$ and $b_2 = c_1$ so that substituting $x = a_2$ and $x = b_2$ into the LHS of the equation results in one positive value and one negative value.

5 Repeat steps 2–4 and continue replacing the values of a_n, b_n and c_n until the approximate solution is determined. For each iteration, write the values of a_n, b_n and c_n, as well as the values of the LHS expression, in a table.

n	a_n	y_a	b_n	y_b	c_n	y_c
1	0	–7	4	13	2	–5
2	2	–5	4	13	3	2
3	2	–5	3	2	2.5	–2
4	2.5	–2	3	2	2.75	–0.125
5	2.75	–0.125	3	2	2.875	0.906…
6	2.75	–0.125	2.875	0.906…	2.812…	0.382…
7	2.75	–0.125	2.8125	0.382…	2.781…	0.126…
8	2.75	–0.125	2.7812…	0.126…	2.765…	0.000…
9	2.75	–0.125	2.765…	0.000…	2.757…	–0.06…
10	2.757…	–0.062…	2.765…	0.000…	2.761…	–0.030…
11	2.761…	–0.030…	2.765…	0.000…	2.763…	–0.015…
12	2.763…	–0.015…	2.765…	0.000…	2.764…	–0.007…
13	2.764…	–0.007…	2.765…	0.000…	2.765…	–0.003…
14	2.765…	–0.003…	2.765…	0.000…	2.765…	–0.001…

6 Write the solution correct to two decimal places.

Therefore, $x = 2.77$, correct to two decimal places.

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Helpful hints

- ✔ When using a graphical method, make sure to check whether the question is asking for the solution to an equation with one unknown or the point of intersection of two graphs. For one unknown, the solution should be x = [solution]. For a point of intersection, the answer should be a pair of coordinates (x, y).
- ✔ Graphical methods rely on graphs being drawn with a high level of accuracy, which is harder to do for non-linear graphs than linear graphs. Graphical calculators and online graphing software can sketch the graphs with an incredibly high level of precision, from which the intersection points can be determined by inspection, or by using the software to identify the solution.
- ✔ The lower bound, a_n, is the lower x-value but not necessarily the x-value that produces a negative value. Likewise, the upper bound, b_n, is the greater x-value but not necessarily the x-value that produces a positive value.
- ✔ Use a table to keep track of the values being used at each iteration of the bisection method.

Learning pathways

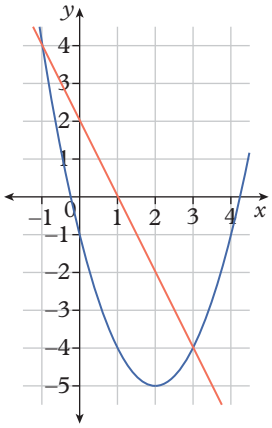
1, 2, 4(a–c), 5(a, c, e), 6(a–c), 8, 9(a–d), 11, 13(a–c), 14	2, 3, 4(d–f), 5(b, d, f), 6(d–f), 8(a–c), 9(b–e), 10, 11, 13, 14, 16(a–c)	2, 4(d–f), 5(d–f), 6–8(d–f), 9(e, f), 12(a, c), 13, 15–17

ANS p707

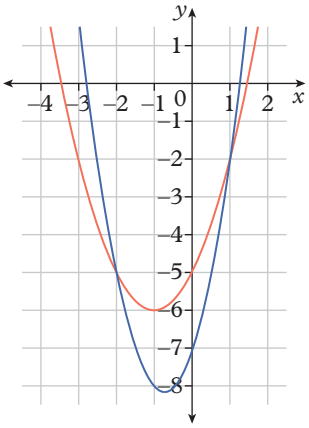
Exercise 4.9A: Understanding and fluency

WE 4.9A **1** Determine the solution(s) to the following equations by identifying the x-coordinate(s) of the point(s) of intersection of the given graphs, representing y = [LHS] and y = [RHS].

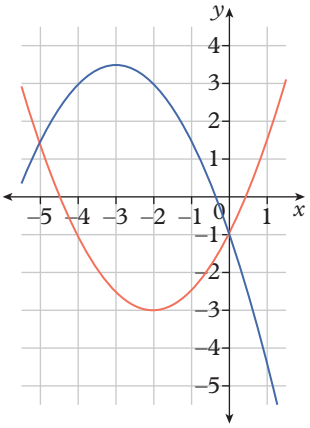
a $-2x + 2 = x^2 - 4x - 1$



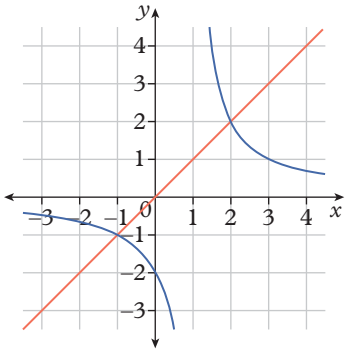
b $x^2 + 2x - 5 = 2x^2 + 3x - 7$



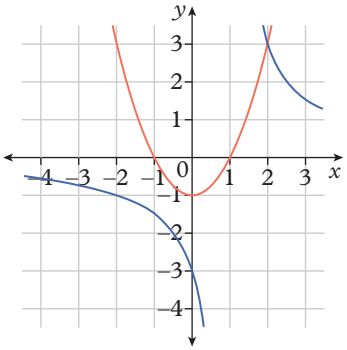
c $\frac{1}{2}x^2 + 2x - 1 = -\frac{1}{2}x^2 - 3x - 1$



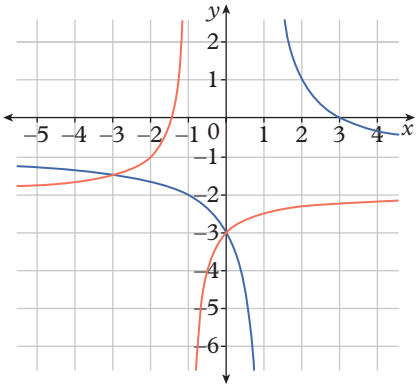
d $x = \frac{2}{x - 1}$



e $x^2 - 1 = \frac{3}{x - 1}$



f $\frac{-1}{x + 1} - 2 = \frac{2}{x - 1} - 1$



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2 For each of the following equations:

i determine which pair of graphs (**A–F**) are the graphs of y =[LHS] and y =[RHS]

ii determine the solution(s) to the equation graphically.

a $2x + 1 = x^2 + 6x + 5$

b $2x + 1 = x^2 - 3x + 5$

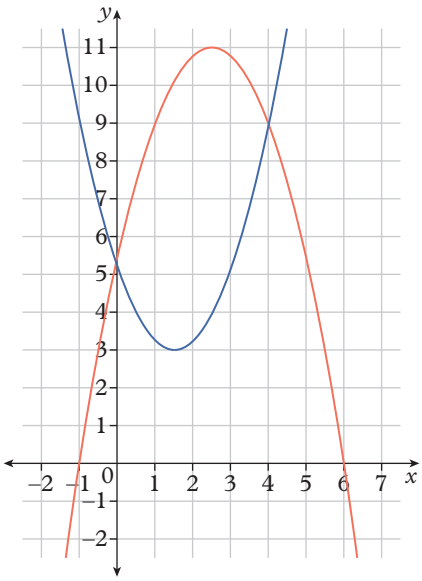
c $x + 5 = x^2 + 6x + 5$

d $x + 5 = \dfrac{4}{x + 1} + 1$

e $x + 5 = \dfrac{-4}{x - 1} + 1$

f $x^2 - 3x + 5 = -x^2 + 5x + 5$

A



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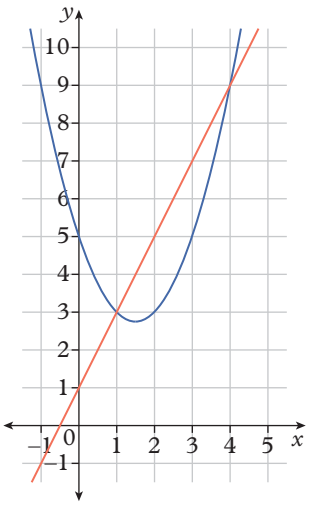

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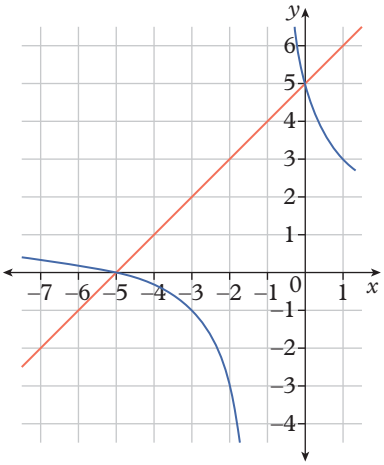
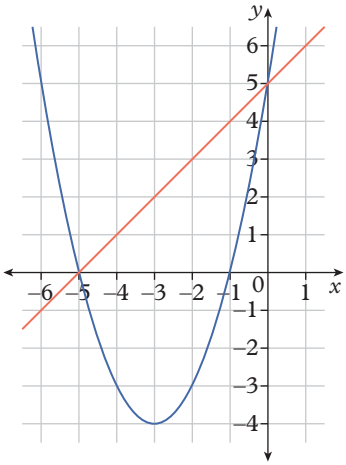
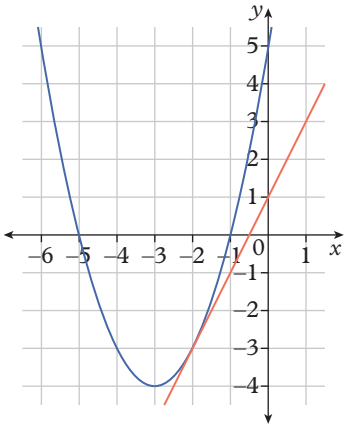


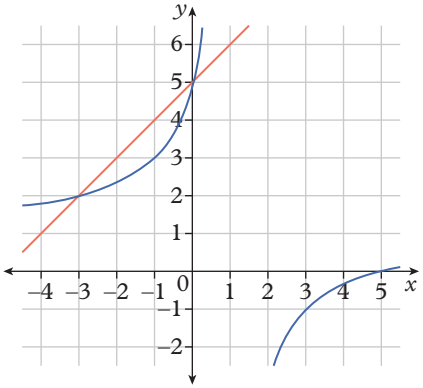
D



E



F



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3 Sketch the graphs of y = LHS and y = RHS of the following equations, and hence state the expected number of solutions of the equation. Do not determine the solutions to the equations.

a $2x - 4 = x^2 - 5$ **b** $x^2 - 5 = -x^2 + 1$ **c** $-x^2 + 1 = \frac{1}{x}$

d $\frac{1}{x} = 3 - x$ **e** $3 - x = -x^2 + 2$ **f** $-x^2 + 2 = 2^x$

WE 4.9B **4** Solve the following equations for x algebraically.

a $5x + 10 = x^2 + 6x - 20$ **b** $3x^2 - 5x + 1 = x^2 - 6x + 5$ **c** $\frac{6}{x+2} = x + 3$

d $-\frac{1}{3x+1} = x - 3$ **e** $\frac{1}{x-4} = -\frac{1}{x}$ **f** $2x + 5 = \frac{5x+1}{3x-4}$

WE 4.9C **5** Solve the following systems of simultaneous equations algebraically. Give your answers as pairs of coordinates in the form (x, y).

a $x^2 + y^2 = 25$ and $y = x + 7$ **b** $x^2 + y^2 = 9$ and $y = x - 3$

c $x^2 + y^2 = 100$ and $y = \frac{3}{4}x + \frac{25}{2}$ **d** $(x + 3)^2 + (y - 8)^2 = 9$ and $y = x - 3$

e $(x - 5)^2 + (y - 10)^2 = 225$ and $y = x - 16$ **f** $(x - 2)^2 + (y + 1)^2 = 25$ and $y = -\frac{4}{3}x - \frac{20}{3}$

WE 4.9D **6** For each of the following:

i substitute the upper and lower bounds of the interval into the LHS of the equation and show that one bound evaluates to a positive value and the other bound evaluates to a negative value.

ii use the bisection method to determine the solution in the interval, correct to two decimal places.

a $x^2 - 5x - 7 = 0,\ -2 < x < 0$ **b** $x^2 - 3x - 3 = 0, 3 < x < 4$

c $x^2 + 2x - 9 = 0, 2 < x < 3$ **d** $2x^2 + 3x - 3 = 0, 0 < x < 1$

e $-3x^2 + 10x + 7 = 0, 3 < x < 4$ **f** $10x^2 - 52x - 9 = 0, 5 < x < 6$

7 Use the bisection method to determine the other solution for each of the equations in question **6**, correct to two decimal places. Hint: First find two x-values, that the solution from question **6** does not lie in between, such that one evaluates to a positive value and the other evaluates to a negative value.

Exercise 4.9B: Problem solving and reasoning

8 Determine the point(s) of intersection between the graphs with the following equations algebraically. Hint: Use a graphical method to confirm the number of points of intersection.

a $y = (x - 3)^2$ and $y = 2x - 3$ **b** $y = (x - 3)^2 + 2$ and $y = 6 - (x - 1)^2$

c $y = (x - 3)^2 + 4$ and $y = (x - 1)^2 + 4$ **d** $y = (x - 3)(x - 2)$ and $y = (x - 3)(2 - x)$

e $y = \frac{-1}{x+3} + 1$ and $y = \frac{2}{x+1} + 2$ **f** $y = \frac{32}{x+1} + 4$ and $y = \frac{5}{x-2} + 7$

9 Determine the values of a and b (where they appear) in each of the following equations, such that the equation has the solution(s) in square brackets.

a $ax^2 + x + 1 = x^2 - 3x$ $[x = -1]$

b $2x + 1 = \frac{4}{x-a}$ $[x = -1]$

c $ax^2 - 4x + a = ax - a$ $[x = 2]$

d $x^2 + ax + b = 5x - 6$ $[x = -5, x = 3]$

e $3x + a = \frac{7}{bx-5}$ $\left[x = -1, x = \frac{13}{6}\right]$

f $x^2 + ax + b = bx + a$ $[x = -1 + \sqrt{3}, x = -1 - \sqrt{3}]$

10 Consider the equation $5x^2 + 12x + 3 = 0$.

a Solve the equation for x using the quadratic formula.

b Solve the equation for x using the bisection method, correct to two decimal places.

c Determine the percentage error of the approximate solutions, correct to three significant figures, using the formula:

$$\text{percentage error} = \frac{|\text{exact value} - \text{approximated value}|}{\text{exact value}} \times 100\%.$$

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11 For each of the following equations:

i rearrange the equation to write it as a quadratic expression equal to 0

ii sketch the graph of the quadratic expression, plotting the turning point and y-intercept without determining the x-intercepts

iii state the pair of consecutive integers each x-intercept lies between

iv use the bisection method to determine the solution(s) to the equation, correct to two decimal places.

a $x^2 + 3x + 4 = -7x - 15$

b $4(x + 1)(x + 2) = (x + 3)^2 + 10$

c $4x - 20 = \dfrac{14}{x - 1}$

12 Use the bisection method with an appropriate equation to approximate the values of the following, correct to two decimal places.

a $\sqrt{2}$ **b** $\sqrt{3}$ **c** $\sqrt{5}$ **d** $\sqrt{7}$

13 We can determine the points of intersection between parabolas and circles algebraically. First, make x^2 the subject of the parabola's equation, substitute that expression for x^2, and then solve for y. Finally, substitute the value(s) of y into the parabola's equation to determine the corresponding x-values. Determine the exact point(s) of intersection of the following pairs of graphs where they exist.

a $x^2 + y^2 = 1$ and $y = x^2$

b $x^2 + y^2 = 4$ and $y = x^2 - 3$

c $x^2 + y^2 = 4$ and $y = x^2 - 5$

d $x^2 + (y - 1)^2 = 9$ and $y = 4 - x^2$

e $x^2 + (y - 2)^2 = 16$ and $y = 2x^2 - 6$

f $(x + 1)^2 + (y - 2)^2 = 25$ and $y = (x + 1)^2 + 2$

14 Consider the equation $x^2 - 3x + 2.25 = 0$.

a Solve the equation algebraically.

b Explain why you can't directly apply the bisection method as discussed in this lesson to solve the equation.

Exercise 4.9C: Challenge

15 Determine the point(s) of intersection of the following pairs of equations graphically.

a $y = \dfrac{60}{x}$ and $x^2 + y^2 = 169$

b $y = -x^2 + 13$ and $x^2 + y^2 = 25$

16 The x-coordinate(s) of the point(s) of intersection of a line with an exponential relationship can be determined numerically using the bisection method. Determine the point(s) of intersection between the following pairs of graphs where they exist, correct to two decimal places. Hint: Sketch the two graphs to narrow down the intervals(s) that any intersections occur in, or to identify if there is no intersection.

a $y = 2^x$ and $y = 5$

b $y = 2^x$ and $y = x + 3$

c $y = 2^x$ and $y = -x + 2$

d $y = 2^x$ and $y = x$

e $y = 3^x$ and $y = 2x + 3$

f $y = 3^{-x}$ and $y = 2x + 3$

17 The equation $x^2 = 2^x$ has three solutions.

a Complete the following table to find two of the solutions.

x	−2	−1	0	1	2	3	4	5
x^2								
2^x								

b Use the table to explain why the third solution must be between −1 and 0.

c Rearrange the equation so that it is an expression equal to 0, and hence use the bisection method to determine the third solution, correct to two decimal places.

Online resources:

Interactive skillsheet Solving non-linear equations

Interactive skillsheet The bisection method

Investigation Solving simultaneous equations using cobweb diagrams

Quick quiz 4.9

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Lesson 4.10

Rates of change

Learning intentions

By the end of this lesson you will be able to ...

✔ determine average rate of change and approximate instantaneous rate of change.

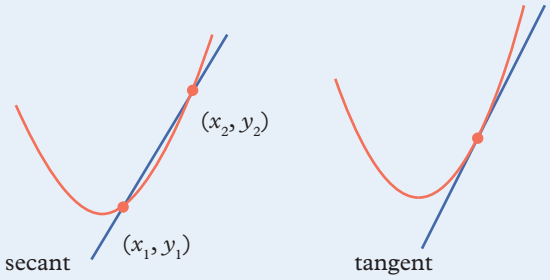
Inter-year links

Year 8	6.8 Finding linear equations
Year 9	3.4 Gradient and intercepts

Key content video
Rates of change

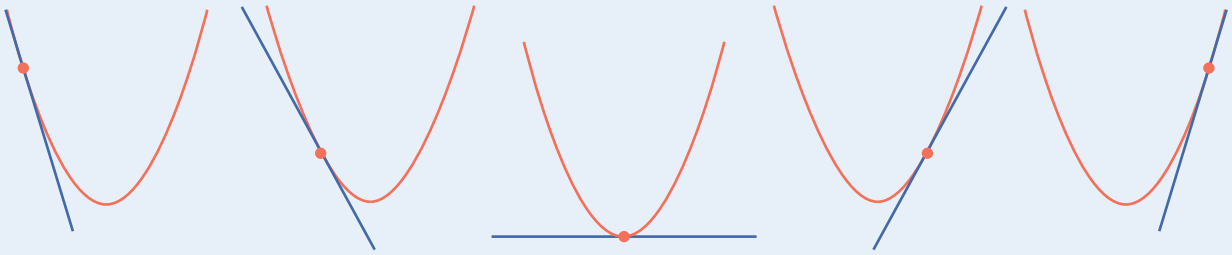
Secants and tangents

- A line that passes through at least two points of a non-linear graph is called a **secant**.
- The **tangent** of a curve at a point is the straight line that just touches, but does not cross, the curve at the point.

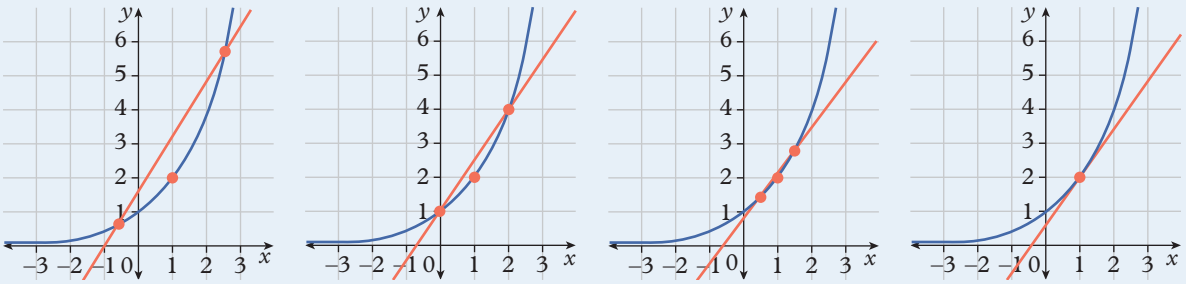


Gradient

- The gradient, m, of a linear relationship $y = mx + c$ is a constant, where $m = \frac{y_2 - y_1}{x_2 - x_1}$.
- The gradient of a non-linear relationship at a point is equal to the gradient of the tangent line at the point. The gradient of a non-linear relation depends on the value of x.



- The gradient of the tangent can be approximated by calculating the gradient of nearby secants that have increasingly smaller intervals between the two points that close in on the point of interest. The gradients of the secants approach the gradient of the tangent.



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Rates of change

- The **instantaneous rate of change** at a point, m_T, is the gradient at that point.
- The **average rate of change** over an interval is the gradient of the line connecting the two endpoints of the interval.
  - → For linear relationships: the average rate of change is equal to the gradient of the line, which is constant.
  - → For non-linear relationships: the average rate of change from point A to point B is the gradient of the secant that passes through A and B.

Worked example 4.10A Calculating the average rate of change

Calculate the average rate of change of $y = x^2$ from $x = 3$ to $x = 4$.

THINK

1 Determine the corresponding y-coordinates of the two points by substituting the x-coordinates into the equation.

2 Write the coordinates of the two points.

3 The average rate of change is the gradient of the secant between the two points.

WRITE

For $x_1 = 3$,

$y_1 = 3^2$

$= 9$

For $x_2 = 4$,

$y_2 = 4^2$

$= 16$

$(3, 9), (4, 16)$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{16 - 9}{4 - 3}$$
$$= \frac{7}{1}$$
$$= 7$$

Worked example 4.10B Approximating the instantaneous rate of change

Approximate the instantaneous rate of change of $y = x^2$ at $x = 3$ by evaluating the gradient of the secants between $x = 3$ and $x = 3.5$, $x = 3$ and $x = 3.1$, and $x = 3$ and $x = 3.01$.

THINK

1 Calculate the corresponding y-coordinates for each of the x-coordinates.

2 Calculate the gradient of the secants between the given points.

3 Approximate the instantaneous rate of change by estimating what value the gradients of the secants are approaching.

WRITE

x	3	3.5	3.1	3.01
y	9	12.25	9.61	9.0601

$$m_1 = \frac{12.25 - 9}{3.5 - 3} = 6.5$$
$$m_2 = \frac{9.61 - 9}{3.1 - 3} = 6.1$$
$$m_3 = \frac{9.0601 - 9}{3.01 - 3} = 6.01$$

The gradient of the secants is approaching 6, therefore, $m_T \approx 6$.

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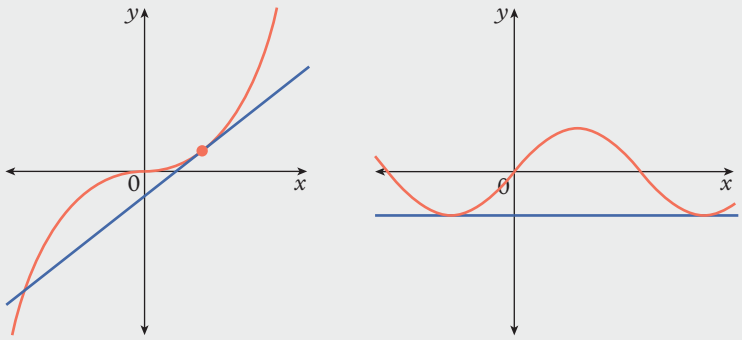
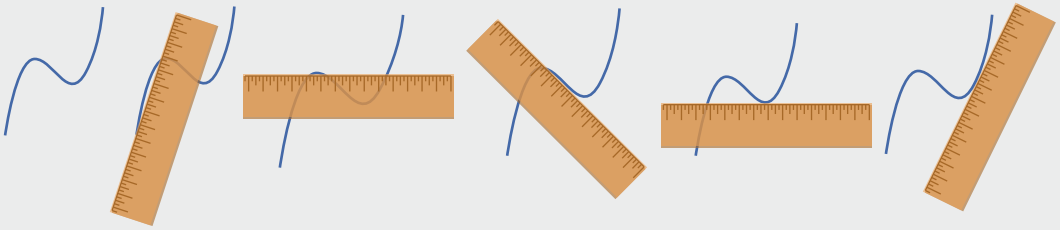
Helpful hints

✔ The tangent at a point does not cross the curve in the nearby vicinity of the point but can touch or cross the graph again elsewhere. For example:



✔ The tangent can be visualised by sliding a ruler along the graph of a non-linear relationship, following the graph's shape.



Learning pathways

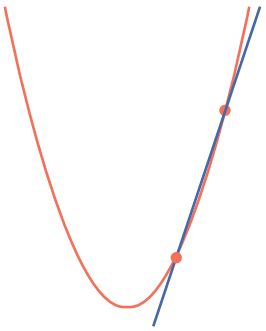
1–4, 6, 8, 9	3–5, 7, 8, 10, 11	3–4, 7, 10–13

ANS p709

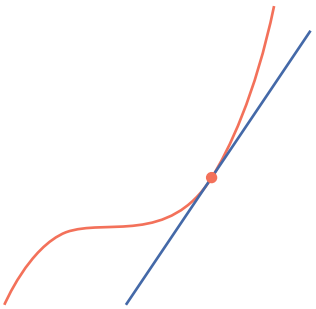
Exercise 4.10A: Understanding and fluency

1 Determine which of the following blue lines are tangents.

a



b



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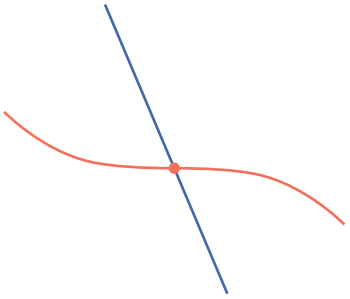

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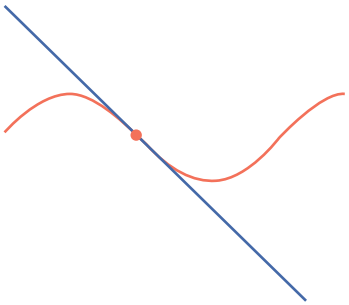

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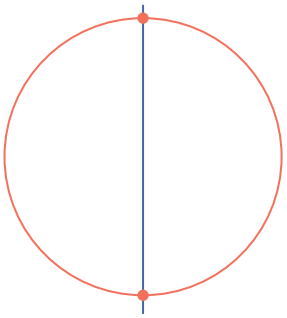

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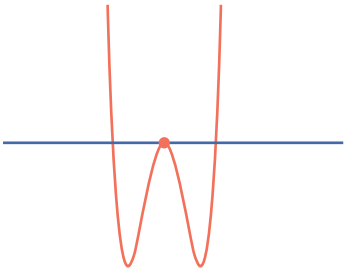

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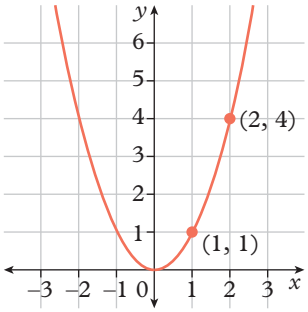
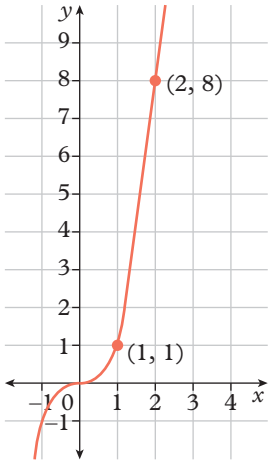
2 Calculate the average rate of change between the given points on the following graphs.

a



b



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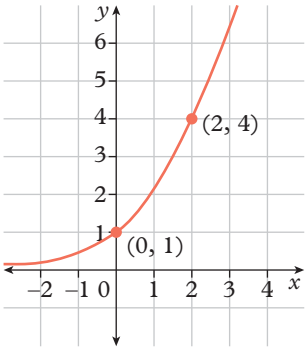

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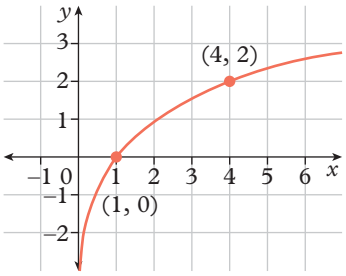

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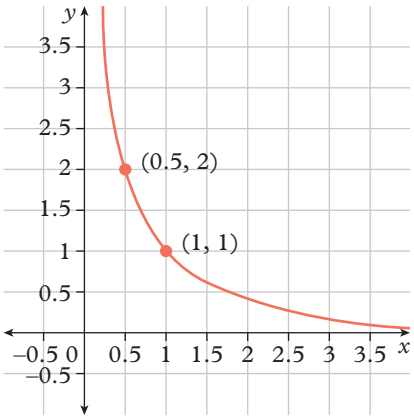

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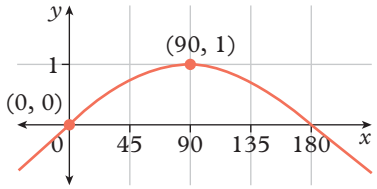

WE 4.10A **3** Calculate the average rate of change between the given x values of the following relationships:

a $y = x^2$ from $x = -3$ to $x = -2$

b $y = x^4$ from $x = 2$ to $x = 2.5$

c $y = 10^x$ from $x = -1$ to $x = 1$

d $y = x^3 - 3x^2$ from $x = 1$ to $x = 2$.

WE 4.10B **4** Approximate the instantaneous rate of change at the given x-coordinate by evaluating the gradients of the secants in the following intervals:

a $y = x^2$ at $x = -3$ between $x = -3$ to $x = -2.9$, $x = -3$ to $x = -2.99$, and $x = -3$ to $x = -2.999$

b $y = x^4$ at $x = 2$ between $x = 2$ to $x = 2.1$, $x = 2$ to $x = 2.01$, and $x = 2$ to $x = 2.001$

c $y = 10^x$ at $x = 0$ between $x = -1$ to $x = 1$, $x = -0.5$ to $x = 0.5$, and $x = -0.1$ to $x = 0.1$

d $y = x^3 - 3x^2$ at $x = 1$ between $x = 1$ to $x = 2$, $x = 1$ to $x = 1.1$, and $x = 1$ to $x = 1.01$.

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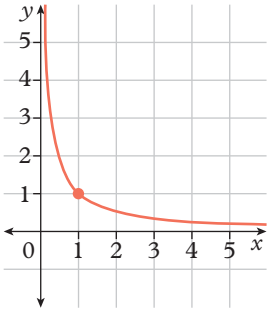
5 For each of the following graphs:

i use a ruler or straight edge to visualise the tangent at the point shown, and write down the y-intercept of that tangent, if it exists, to the nearest integer

ii calculate the gradient of the tangent using your answer to part **i** and the point shown.

a



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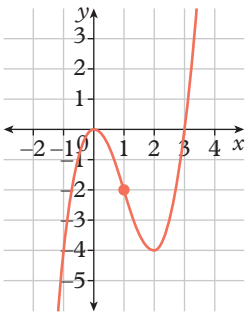

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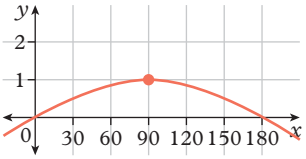

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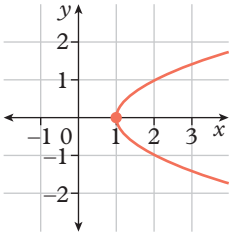
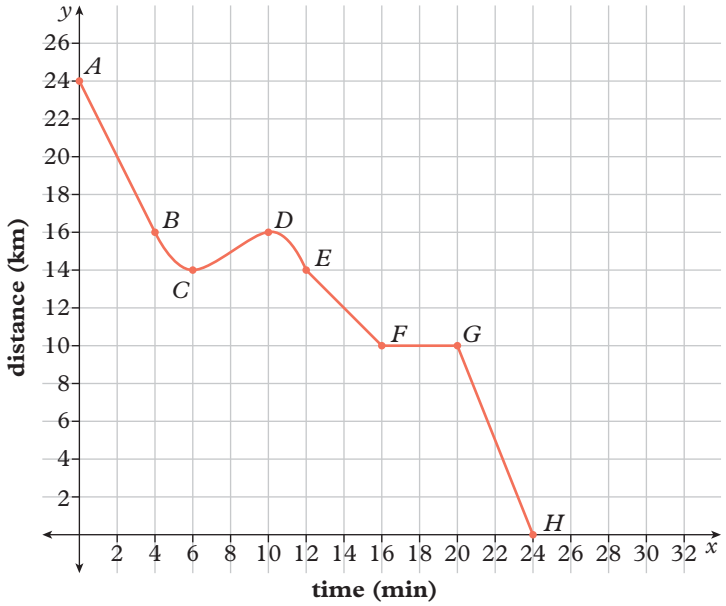


6 Consider the travel graph below.



Determine the average velocity in km/min:

a from A to B **b** from B to C **c** from C to D **d** from D to E **e** from A to H.

7 Consider the quadratic relationship $y = x^2 - 5x + 6$.

a Approximate the instantaneous rate of change at $x = 1$ by evaluating the gradient of the secants between $x = 1$ and $x = 1.5$, $x = 1$ and $x = 1.1$, and $x = 1$ and $x = 1.01$.

b Sketch the graph and the tangent to the graph at $x = 1$.

Exercise 4.10B: Problem solving and reasoning

8 Joseph measured the instantaneous rate of change of a graph at three places, and in each case, the result was 5. Is the graph linear? Explain.

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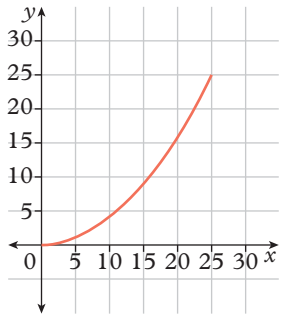
9 Describe how the gradient changes as x increases from 0 to 25.

a



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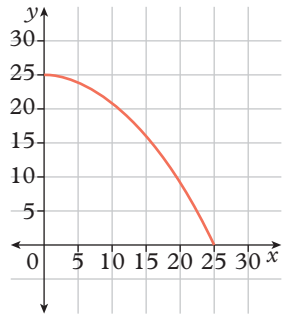

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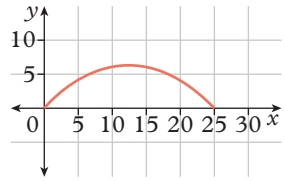

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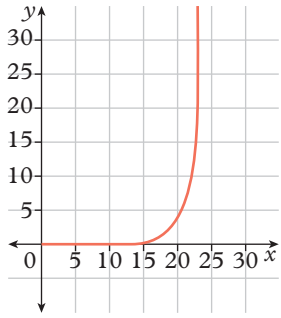

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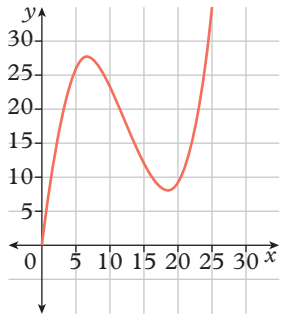

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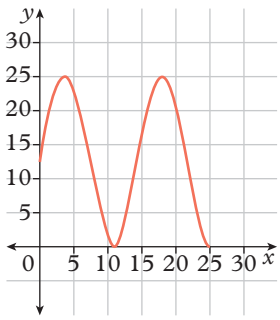
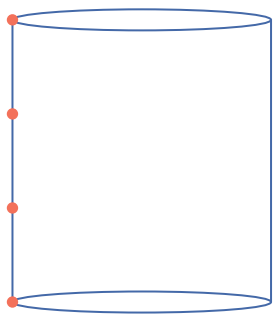


10 Water is being poured into each of the following vases at a constant rate. Sketch a graph with the height of the water on the vertical axis and time on the horizontal axis.
Note: Each red dot on the vase marks an increase of 1 unit in height.

a



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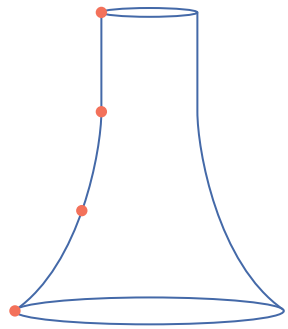

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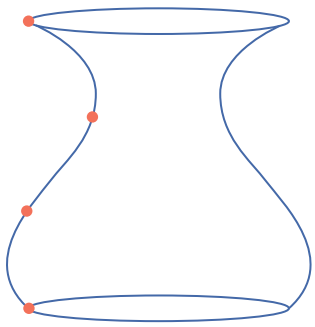

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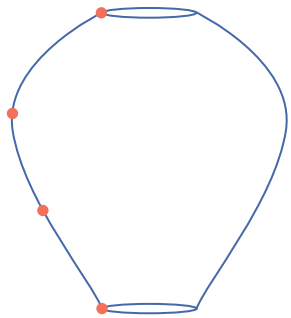

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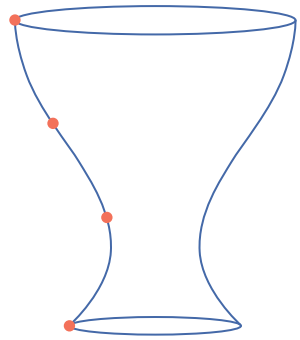

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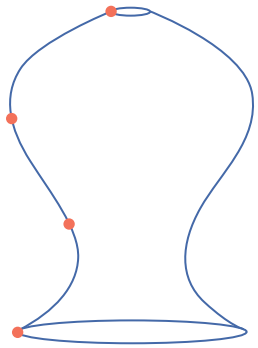

Exercise 4.10C: Challenge

11 When approximating the gradient of a curve at $x = a$, we calculate the gradient of the secant connecting $x = a$ and $x = a + h$ for decreasing values of h. Write an expression for the gradient of the following relationships at the given point using $x = a$ and $x = a + h$.

a $y = x^2$ at $x = 3$ **b** $y = x^3$ at $x = 1$ **c** $y = 3x^2 + 5x + 1$ at $x = 0$

12 Alternatively, we can use two points on either side of $x = a$: $x = a - h$ and $x = a + h$, for decreasing values of h. Write an expression for the gradient of the relationships in question **11** using $x = a - h$ and $x = a + h$.

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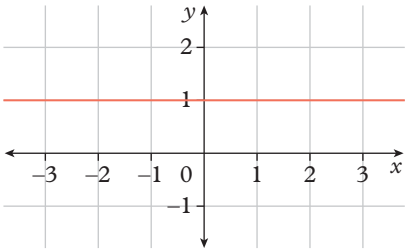
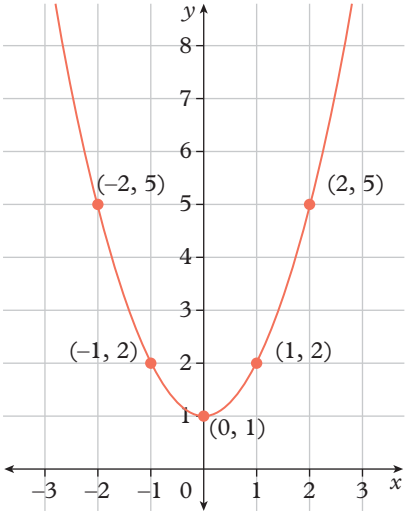
13 The graph on the right shows the gradient of $y = x + 1$. The equation of the graph is $y = 1$ because the gradient of $y = x + 1$ is always equal to 1.



a Sketch a graph to show the gradient of $y = x + 2$.

b If the graph of the gradient of an equation is a horizontal line, is the equation necessarily unique? Explain.

c Determine the gradient of the secants connecting adjacent labelled points in the following graph.



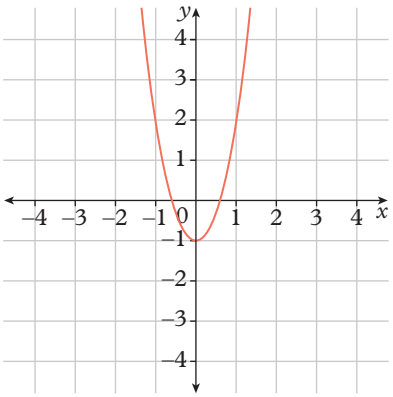
d Use your answers to part **c** to approximate the gradient of $y = x^2 + 1$ at the following points

x	−1.5	−0.5	0	0.5	1.5
Gradient					

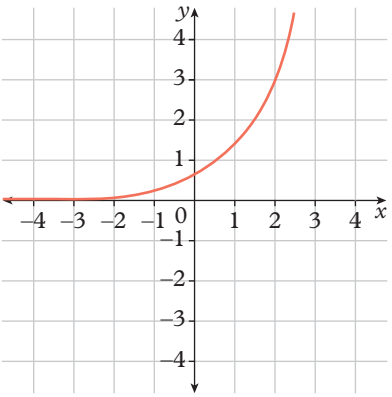
e Use the table of values from part **d** to sketch the graph of the gradient of $y = x^2 + 1$.

f Which of the following graphs shows the gradient of $y = x^2 - 1$?

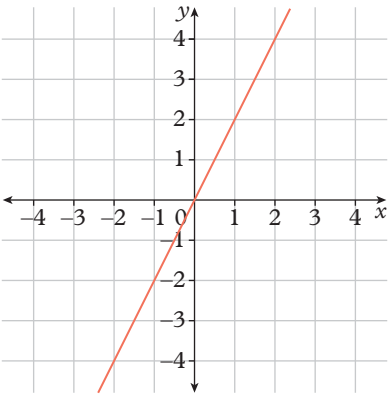
A



B



C



g If the graph of the gradient of an equation is a straight line, is the original equation necessarily unique? Explain.

Online resources:

Interactive skillsheet Rates of change

Investigation Rates of change in quadratic relationships

Quick quiz 4.10

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Lesson 4.11

Review: Non-linear relationships

Module summary

The Null Factor Law

If $(x + m)(x + n) = 0$, then $x + m = 0$ or $x + n = 0$.

Quadratic formula

$ax^2 + bx + c = 0$ $\quad x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The discriminant

$\Delta = b^2 - 4ac$

- $\Delta < 0$: no real solutions
- $\Delta = 0$: one real solution
- $\Delta > 0$: two real solutions

Direct proportion

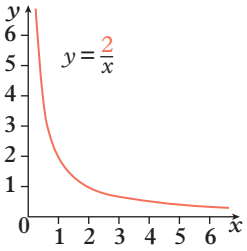
$y = kx$

constant of proportionality

k = rate of change = gradient

Inverse proportion

$y = \dfrac{k}{x}$ ← constant of proportionality



Quadratic relationships

Horizontal translation of h units
If $h > 0$, right
If $h < 0$, left

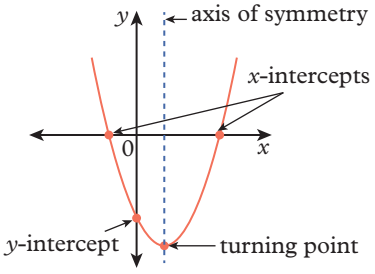
$y = a(x - h)^2 + k$

Dilation as a varies
Reflection if $a < 0$

Vertical translation of k units
If $k > 0$, up
If $k < 0$, down

x-intercepts

$y = a(x - x_1)(x - x_2)$



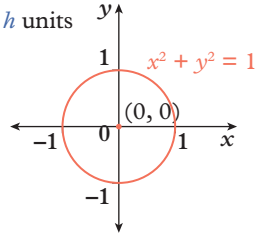
Circles

$(x - h)^2 + (y - k)^2 = r^2$

Radius of r units

Vertical translation of k units
If $k > 0$, up
If $k < 0$, down

Horizontal translation of h units
If $h > 0$, right
If $h < 0$, left



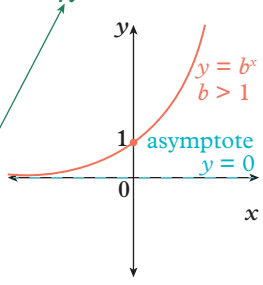
Exponential relationships

Dilation in the x-direction
If $n > 0$, upright
If $n < 0$, horizontal reflection

$y = a \times b^{nx} + k$

Dilation in the y-direction
If $a > 0$, upright
If $a < 0$, vertical reflection

Vertical translation of k units
If $k > 0$, up
If $k < 0$, down



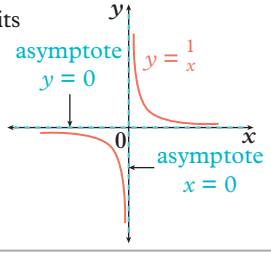
Hyperbolas

Dilation in the y-direction
If $a > 0$, upright
If $a < 0$, inverted

$y = \dfrac{a}{x - h} + k$

Vertical translation of k units
If $k > 0$, up
If $k < 0$, down

Horizontal translation of h units
If $h > 0$, right
If $h < 0$, left



Solving non-linear equations

- Points of intersection of two graphs represent solutions to non-linear equations.
- Equations can also be solved algebraically, or numerically using the bisection method.

Rate of change

- The average rate of change of an interval is the gradient of the line connecting the two endpoints of the interval.
- The instantaneous rate of change at a point, m_T, is the gradient at that point, and can be estimated by finding the gradient of a sequence of secants which approach the point.

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Review questions 4.11A: Mathematical literacy review

Review quiz
Take the review quiz to assess your knowledge of this module.
✔ 10
✔ 10A

Quizlet
Test your knowledge of this module by working individually or in teams.

The following key terms are used in this module:

- asymptote
- average rate of change
- axis of symmetry
- basic form
- bisection method
- centre
- circle
- constant of proportionality
- dilation
- direct proportion
- discriminant
- exponential relationship
- general form
- gradient
- instantaneous rate of change
- intercept form
- interval
- inverse proportion
- leading term
- maximum turning point
- minimum turning point
- non-linear relationship
- Null Factor Law
- parabola
- quadratic equation
- quadratic formula
- quadratic relationship
- radius
- rectangular hyperbola
- reflection
- secant
- solution
- tangent
- transformation
- transformation form
- translation
- turning point
- turning point form
- vertex
- x-intercept
- y-intercept
- zero (of an expression)

1 Match the relationships in the left-hand column with a key term in the right-hand column.

a	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	**A**	Circle
b	$y = \dfrac{6}{x} + 3$	**B**	General quadratic relationship
c	$y = a(x - h)^2 + k$	**C**	Quadratic formula
d	$(x - 1)^2 + y^2 = 4$	**D**	Exponential relationship
e	$y = ax^2 + bx + c$	**E**	Rectangular hyperbola
f	$y = 2^x$	**F**	Turning point form
g	$\Delta = b^2 - 4ac$	**G**	Inverse proportion
h	$y \propto x$	**H**	Direct proportion
i	$y \propto \dfrac{1}{x}$	**I**	Discriminant

2 Which descriptions best explain the transformations of the graph of $y = x^2$ to produce the graph of $y = a(x - h)^2 + k$?

A There is a dilation of a in the y-direction. This has no impact on any x-intercepts.
B There is a dilation of a in the y-direction. This has no impact on any y-intercepts.
C If $a < 0$, there is a reflection across the y-axis.
D If $a < 0$, there is a reflection across the x-axis.
E There is a horizontal translation k units and a vertical translation of h units.
F There is a horizontal translation h units and a vertical translation of k units.
G Any translations affect the shape but not the vertex.
H Any translations affect the shape and the vertex.
I Any translations affect the vertex but not the shape.
J Any translations do not affect the shape or the vertex.

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3 Complete the following sentences using words from the list of key terms.

a The _____ is the line that the exponential graph $y = 3^x$ approaches for large negative values of x.

b The gradient of a _____ is needed to calculate the average rate of change. The gradient of the _____ is needed to calculate the instantaneous rate of change.

c When two variables are _____ proportional, the gradient of the straight line is the same as the _____.

Review questions 4.11B: Multiple choice

4.1 **1** Which of these is a quadratic equation?

A $y - 2 = x^2 + 4$ **B** $x^2 + y^2 = 9$ **C** $y = \frac{5}{x}$ **D** $3^x = 9$

4.2 **2** For the quadratic equation $2x - 4x^2 = 3$, which of these is the value of the discriminant?

A 52 **B** –8 **C** –44 **D** 44

4.3 **3** The coordinates of the x-intercepts of the graph of $y = x^2 - 6x + 5$ are:

A (1, 0) and (5, 0) **B** (0, 1) and (0, 5) **C** (–3, 0) and (2, 0) **D** (–1, 0) and (–5, 0)

4.3 **4** Which of the following statements about the graph of $y = x^2 + 2x - 3$ is *incorrect*?

A It is upright. **B** The graph has two x-intercepts.

C The turning point has coordinates (–4, –1). **D** The y-intercept is (0, –3).

4.3 **5** Compared with the graph of $y = x^2$, the graph of $y = (x - 2)^2$ is:

A dilated by a factor of 2 in the y-direction. **B** translated 2 units left.

C translated 2 units up. **D** translated 2 units right.

4.3 **6** The coordinates of the turning point of the graph of $y = (x - 2)^2 - 5$ are:

A (–2, –5) **B** (2, –5) **C** (–2, 5) **D** (2, 5)

4.4 **7** The coordinates of the centre of the circle $(x - 2)^2 + (y + 3)^2 = 16$ are:

A (2, –3) **B** (–2, 3) **C** (–2, –3) **D** (2, 3)

4.5 **8** Which of these equations represents an exponential relationship?

A $y = 3x^2 + 4$ **B** $x^2 + y^2 = 4$ **C** $y = 3^x + 4$ **D** $y = \frac{1}{x}$

4.5 **9** What is the rule obtained when the graph of $y = 6 \times 2^{3x} + 3$ is reflected across the x-axis?

A $y = 6 \times 2^{-3x} + 3$ **B** $y = -6 \times 2^{3x} + 3$ **C** $y = -6 \times 2^{-3x} - 3$ **D** $y = 6 \times 2^{\frac{x}{3}} + 3$

4.6 **10** Which of the following represents an example of a directly proportional relationship?

A

x	0	1	2	3
y	2	7	12	17

B

x	0	1	2	3
y	0	6	12	18

C

x	1	2	3	4
y	48	24	16	12

D

x	5	10	15	20
y	12	22	32	42

10A 4.7 **11** What are the equations of the asymptotes of the graph of $y = \frac{1}{x + 2} + 5$?

A $x = 2, y = 5$ **B** $x = -2, y = 5$ **C** $x = 2, y = -5$ **D** $x = -2, y = -5$

10A 4.8 **12** The equation of a circle is $x^2 + 10x + y^2 - 8y = 32$. What translations need to be performed on the graph of $x^2 + y^2 = 9$ to graph this circle?

A 10 units left and 8 units up **B** 5 units right and 4 units down

C 10 units right and 8 units down **D** 5 units left and 4 units up

10A 4.9 **13** The solutions to the equation $\frac{2}{x} + 1 = x + 2$ are:

A $x = 0, x = 3$ **B** $x = -3, x = 0$ **C** $x = 2, x = -1$ **D** $x = -2, x = 1$

10A 4.10 **14** What is the average rate of change of $y = x^3 - 1$ between $x = -1$ and $x = 1$?

A 0 **B** 1 **C** 2 **D** –1

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Review questions 4.11C: Short answer

4.1 **1** Solve each of these equations. Write your solutions as exact values.

a $(x-2)^2 = 0$ **b** $(4x-3)(3x+4) = 0$ **c** $(\sqrt{2}-x)(\sqrt{2}+x) = 0$

d $3x^2 - 6x - 24 = 0$ **e** $5x^2 - 7 = 0$ **f** $3x^2 - 12x + 12 = 0$

4.1 **2** Solve these equations by completing the square. Write your solutions as exact values.

a $x^2 + 6x + 4 = 0$ **b** $x^2 - 3x - 3 = 0$

4.2 **3** Use the discriminant to identify the number of real solutions for each of these quadratic equations.

a $x^2 - 4x + 4 = 0$ **b** $3x^2 = 2x - 4$ **c** $5x - 4x^2 = 2$

d $4 - 3x^2 = 8x$ **e** $1 - x - x^2 = 0$ **f** $6x^2 + 4x = -1$

4.2 **4** Use the quadratic formula to solve each of these equations. Write your solutions as: **i** exact values and **ii** approximate values, correct to two decimal places.

a $x^2 + 3x = 10$ **b** $5x^2 = 3x + 1$ **c** $x^2 = 5x - 2$

d $x^2 + 5x + 2 = 0$ **e** $2x^2 = x + 2$ **f** $3x^2 + 7x + 3 = 0$

4.3 **5** For each of these quadratic relationships, find the coordinates of the: **i** x-intercept(s) and **ii** y-intercept.

a $y = -x^2 + 25$ **b** $y = x^2 - 3x - 4$ **c** $y = 2x^2 - 8x + 8$ **d** $y = -x^2 - x + 6$

4.3 **6** Use intercepts to help you sketch the graphs of the relationships below. Label the turning point with its coordinates each time.

a $y = x^2 - 4x + 3$ **b** $y = -x^2 - 4x + 5$

4.3 **7** For each of the quadratic relationships below:

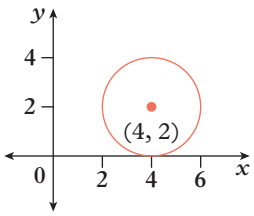
i describe the translations performed on $y = x^2$

ii write the coordinates of the turning point.

a $y = (x-5)^2 + 2$ **b** $y = (x+6)^2 - 4$

c $y = (x-1)^2 - 1$ **d** $y = (x+3)^2 + 5$

e $y = (x-4)^2 + 2$ **f** $y = 3(x+7)^2 - 3$



4.4 **8** Write the equation for the circle on the right.

4.5 **9** Consider the exponential relationship $y = 5^x$.

a Sketch the graph of $y = 5^x$.

b Describe the transformations performed on the graph of $y = 5^x$ to produce the graph of:

i $y = 3 \times 5^x$ **ii** $y = 5^x + 3$

c Sketch all three graphs from parts **a** and **b** on the same Cartesian plane.

4.6 **10** At a craft store, the cost of fabric is directly proportional to the number of square metres purchased. Mary purchased 7.5 square metres of fabric for \$93.75.

a Find the constant of proportionality.

b How much would it cost Mary for 4 square metres of fabric?

c If Mary has \$200, how many square metres of fabric can she purchase?

4.6 **11** Consider the following table which shows an inversely proportional relationship between F and a.

a	2	4	5	10
F	10	5	4	2

a Find the constant of proportionality for $F \propto \frac{1}{a}$.

b Find the constant of proportionality for $a \propto \frac{1}{F}$.

c Find the value of F if $a = 2.5$.

10A 4.7 **12** Describe the transformations performed on the graph of $y = \frac{1}{x}$ to produce the graph of each of these equations:

a $y = \frac{1}{x} - 5$ **b** $y = -\frac{1}{x}$ **c** $y = \frac{4}{x}$ **d** $y = \frac{1}{x+5} + 6$

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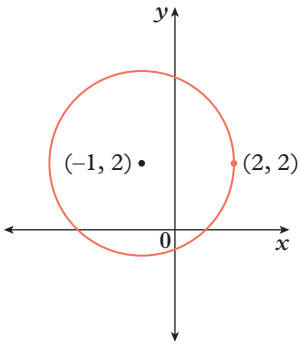
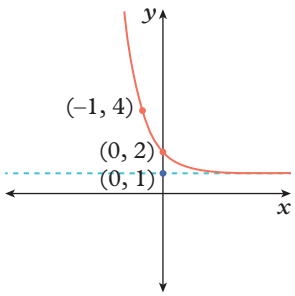
10A 4.8 **13** Determine the equations of the following non-linear relationships in the given forms.

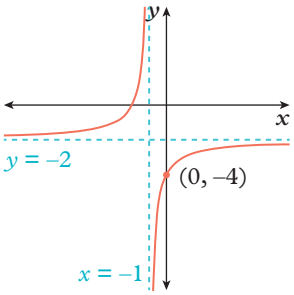
a $(x-h)^2 + (y-k)^2 = r^2$



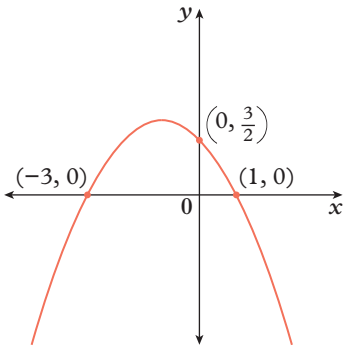
b $y = A \times 3^{nx} + k$



c $y = \dfrac{a}{x-h} + k$



d $y = a(x-h)^2 + k$



10A 4.8 **14** Sketch the graph of the circle with equation $x^2 + y^2 - 6x + 4y + 9 = 0$. Label the centre and the horizontal and vertical extremes.

10A 4.8 **15** Consider the non-linear relationship $y = \dfrac{2}{x-1} - 3$.

a Identify the values of the variables a, h and k, and substitute them into the point transformation formula.

b The following are points on the graph of $y = \dfrac{1}{x}$. Use your formula from part **a** to find their images on the graph of $y = \dfrac{2}{x-1} - 3$.

i $(1, 1)$ **ii** $\left(2, \frac{1}{2}\right)$ **iii** $(-1, -1)$ **iv** $\left(-2, -\frac{1}{2}\right)$

c Verify that the transformed points lie on the graph of $y = \dfrac{2}{x-1} - 3$.

10A 4.9 **16** Solve the following systems of simultaneous equations: **i** graphically, and **ii** algebraically.

a $y = x - 1$ and $y = x^2 + 2x - 3$

b $y = x + 1$ and $(x-2)^2 + (y-1)^2 = 4$

c $y = -\dfrac{2}{x+1}$ and $y = -\dfrac{1}{2}x - \dfrac{1}{2}$

10A 4.9 **17** Consider the equation $x^2 - x - 1 = 0$ on the interval $0 < x < 2$.

a Substitute the upper and lower bounds of the interval into the left-hand side of the equation, and show that one evaluates to a positive value and the other to a negative value.

b Use the bisection method to determine the solution to the equation in the given interval, correct to two decimal places.

c Identify a suitable interval and apply the bisection method to determine the second solution to the equation, correct to two decimal places.

d Use the quadratic formula to determine the exact solutions to the equation.

10A 4.10 **18** Approximate the instantaneous rate of change at the given x-coordinate by evaluating the gradients of the secant between the following intervals:

a $y = x^3$ at $x = -3$ between $x = -3$ to $x = -2.9$, $x = -3$ to $x = -2.99$, and $x = -3$ to $x = -2.999$

b $y = x^2 - x$ at $x = 2$ between $x = 2$ to $x = 2.1$, $x = 2$ to $x = 2.01$, and $x = 2$ to $x = 2.001$.

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Review questions 4.11D: Mathematical modelling

1 The parabolas you have looked at so far are vertical parabolas, because they each have a vertical axis of symmetry. There are also horizontal parabolas, for which the axis of symmetry is horizontal. Similar to the way we have used $y = x^2$ as the basic rule for a vertical parabola, we can use $x = y^2$ as the basic rule for a horizontal parabola.

A horizontal parabola can be represented by the general equation $x - h = a(y - k)^2$, where (h, k) is the turning point of the parabola, and a is a constant.

a Consider the graph of $x + 4 = (y + 1)^2$.

i Provide values for a, h and k.

ii What are the coordinates of the turning point?

iii There is one x-intercept. Calculate its coordinates.

iv There are two y-intercepts. Calculate the coordinates of both points.

v Use the coordinates of the turning point and the intercepts to sketch the graph on a Cartesian plane.

vi Describe the transformations that need to be performed on the graph of $x = y^2$ to produce the same graph you sketched in part **v**.

b Follow the procedure in part **a** to sketch the graph of $x - 4 = -(y + 1)^2$ on the same Cartesian plane as the previous graph.

c Compare the two graphs from parts **a** and **b**. What is the effect of the negative value of a?

2 By now, you are familiar with the quadratic formula expressed in the form $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

An alternative form of the quadratic formula is $x = \dfrac{2c}{-b \pm \sqrt{b^2 - 4ac}}$.

a Show that both quadratic formulas give the same solution for the quadratic equation $x^2 - x - 3 = 0$. Give your solutions correct to one decimal place.

10A **b** Start with the first formula given above. Show how algebraic methods can be used to transform the standard quadratic formula into the second formula. Hint: Take the positive value of the square root, then rationalise the numerator.

10A **3** The graph of a hyperbola consists of two branches, which are rotationally symmetrical around a centre point. That is, there is a point in the centre of the hyperbola, and when the graph is rotated 180° about this point, the result is the same graph.

A hyperbola with equation $y = \dfrac{a}{x - h} + k$ has its centre at (h, k).

The shortest distance between the two branches of a hyperbola can be found by drawing a straight line, angled at 45° (that is, with a gradient of 1 or −1) that passes through the centre of the hyperbola.

Consider the hyperbola described by the equation $y = \dfrac{2}{x - 1} + 2$.

a What is the centre of this hyperbola?

b Sketch the graph of $y = \dfrac{2}{x - 1} + 2$.

c Find, using trial and error, the equation for the circle with the same centre as the hyperbola, that just touches each branch of the hyperbola.

d Use the equation of the circle you found in part **c** to find the shortest distance between the two branches of $y = \dfrac{2}{x - 1} + 2$.

e Consider the hyperbola $y = \frac{a}{x}$, where $a > 0$, and the circle $x^2 + y^2 = r^2$.

Find the relationship between a and r, such that the circle just touches each branch of the hyperbola. It is possible to find this relationship algebraically, but it might be easier to find it with trial and error, using technology, by first sketching the circle for specific values of r and then finding corresponding values of a.

f Find the shortest distance between the two branches of the general hyperbola: $y = \dfrac{a}{x - h} + k$, where $a > 0$.

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Checklist

Now that you have completed this module, reflect on your ability to do the following.

I can do this	I need to review this
☐ Solve quadratic equations by factorisation ☐ Solve quadratic equations by completing the square	☐ Go back to Lesson 4.1 Solving quadratic equations
☐ Identify the number of solutions to a quadratic equation using the discriminant ☐ Solve quadratic equations using the quadratic formula	☐ Go back to Lesson 4.2 The quadratic formula
☐ Sketch parabolas using the intercepts and turning point ☐ Sketch parabolas in turning point form using transformations	☐ Go back to Lesson 4.3 Sketching parabolas
☐ Sketch circles of the form $x^2 + y^2 = r^2$ ☐ Sketch circles of the form $(x - h)^2 + (y - k)^2 = r^2$ using translations ☐ Determine the equation of a circle from a description or graph	☐ Go back to Lesson 4.4 Sketching circles
☐ Identify the key features of the graph of an exponential relationship ☐ Sketch exponential relationships using transformations	☐ Go back to Lesson 4.5 Sketching exponential relationships
☐ Identify the constant of proportionality in directly and inversely proportional relationships ☐ Solve modelling problems involving direct and inverse proportion	☐ Go back to Lesson 4.6 Direct and inverse proportion
☐ Sketch hyperbolas of the form $y = \frac{a}{x}$ ☐ Sketch rectangular hyperbolas using reflections and dilations ☐ Sketch rectangular hyperbolas using translations	☐ Go back to Lesson 4.7 Sketching hyperbolas [10A]
☐ Determine the equation of the graph of a non-linear relationship ☐ Complete the square of non-linear relationships	☐ Go back to Lesson 4.8 Graphs and transformations [10A]
☐ Solve non-linear equations graphically ☐ Solve non-linear equations algebraically ☐ Solve non-linear equations numerically using the bisection algorithm	☐ Go back to Lesson 4.9 Solving non-linear equations [10A]
☐ Find the average rate of change and the gradient of the secant between two points ☐ Approximate the instantaneous rate of change of the gradient of the tangent at a point	☐ Go back to Lesson 4.10 Rates of change [10A]

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5

Polynomials

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Lessons

10A **5.1** Functions and relations
10A **5.2** Polynomials
10A **5.3** Dividing polynomials
10A **5.4** Remainder and factor theorems
10A **5.5** Solving polynomial equations
10A **5.6** Sketching graphs of polynomials using intercepts
10A **5.7** Sketching graphs of polynomials using transformations

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ Expanding binomial products
- ✔ Factorising quadratic expressions
- ✔ Solving quadratic equations
- ✔ Sketching non-linear relationships
- ✔ Sketching parabolas using transformations

Curriculum links

10A
- Investigate the concept of a polynomial and apply the factor and remainder theorems to solve problems (VC2M10AA01)

10A
- Apply understanding of polynomials to sketch a range of curves and describe the features of these curves from their equation (VC2M10AA06)

10A
- Use function notation to describe the relationship between dependent and independent variables in modelling contexts (VC2M10AA08)

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Lesson 5.1

Functions and relations

Learning intentions

By the end of this lesson you will be able to ...

✔ find the domain and range of a relation and determine if a relation is a function or not.

Inter-year links

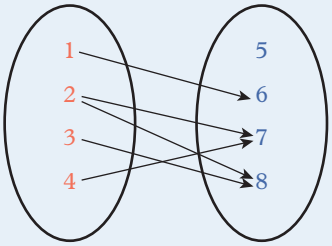
Year 7	6.4 Substitution
Year 8	5.2 Substitution
Year 9	4.4 Sketching parabolas using transformations

Key content video
Functions and relations

Sets

- A **set** is a collection of objects. Objects within a set are called **elements**.
- Sets can be defined by listing all the elements in a set between brackets, as in $\{1, 2, 3, 4\}$.
- The set of real numbers is written as $\mathbb{R}$, the set of integers as $\mathbb{Z}$ and the set of rational numbers as $\mathbb{Q}$.

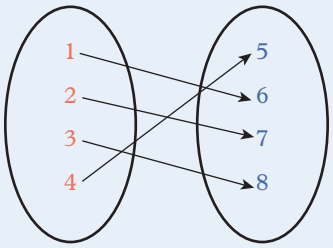
Relations



$\{(1,6), (2,7), (2,8), (3,8), (4,7)\}$

- A **relation** is a mapping between elements of two sets.
- The set from which the mapping occurs is called the **domain** of the relation. For example, in the relation on the right, the domain is $\{1, 2, 3, 4\}$.
- The set onto which the elements of the domain are mapped is called the **co-domain**. The co-domain of the relation on the right is $\{5, 6, 7, 8\}$.
- The subset of the co-domain, to which elements of the domain are actually matched up, is called the **range** of the relation. The range of the relation on the right is $\{6, 7, 8\}$.
- Relations can be specified in different ways: with words, arrow diagrams, as a set of ordered pairs, as a rule or as a graph.

Functions



$\{(1,6), (2,7), (3,8), (4,5)\}$

- A **function** is a relation which maps each element in one set to exactly one element in another set.
  For example, $\{(1,6), (2,7), (3,8), (4,5)\}$.
- When a function is graphed, the function is a map from its domain on the x-axis to its range on the y-axis.
  For example, the graph of $y = x^2$ has a domain of all real x-values ($x \in \mathbb{R}$) and a range of real y-values greater than or equal to zero ($y \geq 0$).
- The **vertical line test** is a visual way to identify whether a graph represents a function.
  - → If any vertical line intersects or touches the graph more than once, then there is at least one x-value with more than one corresponding y-value; therefore the graph fails the vertical line test and is not a function.
  - → If all vertical lines have at most one intersection with the graph, then for all x-values, there is at most one corresponding y-value; therefore the graph passes the vertical line test and is a function.
- Functions with a rule can be written using **function notation** in the form $f(x) = [\text{rule}]$.
  For example, $f(x) = x^2$ (read as 'f of x equals x squared') is the function which squares a number represented by the variable x.
- Function notation can be used to substitute values into an expression.
  For example, if $f(x) = x + 2$, then $f(3) = 3 + 2 = 5$.

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Worked example 5.1A Determining if a relation is a function

For each of the following relations represented by sets of ordered pairs, determine

i the domain of the relation

ii the range of the relation

iii whether the relations are functions.

a $\{(0, 0), (1, 1), (2, 2), (3, 3)\}$

b $\{(0, 0), (0, 1), (0, 2), (1, 0), (2, 0)\}$

THINK

a i In this relation, there are four distinct x-coordinates, so the domain is $\{0, 1, 2, 3\}$.

ii There are four distinct y-coordinates, so the range is $\{0, 1, 2, 3\}$.

iii Since each x-coordinate maps to exactly one y-coordinate, this relation is a function.

b i In this relation, there are three distinct x-coordinates, so the domain is $\{0, 1, 2\}$.

ii There are three distinct y-coordinates, so the range is $\{0, 1, 2\}$.

iii In a function, an element in the first set must be mapped to exactly one element in the second set. In this relation, 0 maps to three different y-coordinates (0, 1 and 2), so the relation is not a function.

WRITE

a i Domain: $\{0, 1, 2, 3\}$.

ii Range: $\{0, 1, 2, 3\}$.

iii This relation is a function.

b i Domain: $\{0, 1, 2\}$

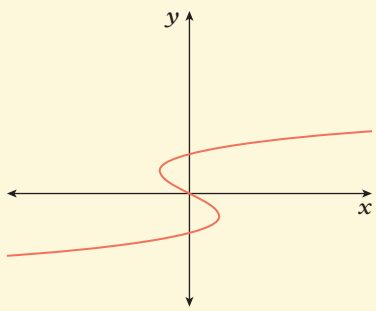
ii Range: $\{0, 1, 2\}$

iii This relation is not a function.

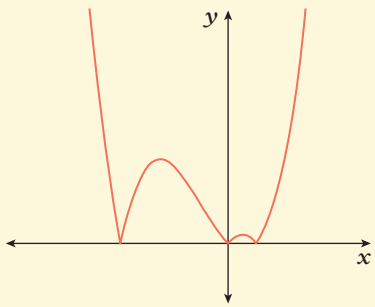
Worked example 5.1B Using the vertical line test

Use the vertical line test to determine whether the relations represented by the following graphs are functions.

a



b

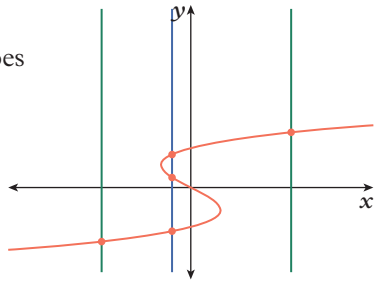


THINK

a Draw several vertical lines in the Cartesian plane and determine if there is a vertical line that intersects or touches the graph more than once. The blue vertical line intersects the graph three times; therefore, the graph fails the vertical line test and is not a function.

WRITE

a This graph does not represent a function.



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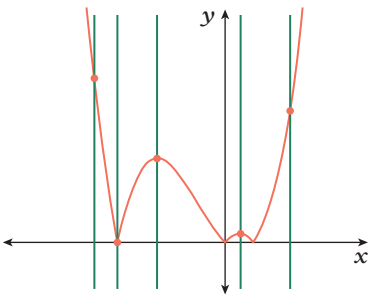
b Draw several vertical lines in the Cartesian plane and determine if there is a vertical line that intersects or touches the graph more than once. All possible vertical lines drawn on this graph intersect the graph exactly once; therefore, the graph passes the vertical line test and is a function.

b This graph represents a function.



Worked example 5.1C Using function notation

For each of the following functions, evaluate $f(0)$, $f(1)$ and $f(-3)$.

a $f(x) = x^3 + 1$

b $f(x) = 2^x - x$

THINK

Substitute the values 0, 1 and –3 into each function in the place of x and evaluate the expression.

WRITE

a $f(x) = x^3 + 1$

$f(0) = 0^3 + 1 = 0 + 1 = 1$

$f(1) = 1^3 + 1 = 1 + 1 = 2$

$f(-3) = (-3)^3 + 1 = -27 + 1 = -26$

b $f(x) = 2^x - x$

$f(0) = 2^0 - 0 = 1 - 0 = 1$

$f(1) = 2^1 - 1 = 2 - 1 = 1$

$f(-3) = 2^{-3} - (-3) = \frac{1}{2^3} + 3 = \frac{1}{8} + 3 = \frac{25}{8}$

Helpful hints

- ✔ All graphs (and their corresponding rules) that you have studied, such as linear graphs, parabolas, exponential graphs, hyperbolas and circles, are examples of relations. However, not all relations are functions. For example, circles are not functions.
- ✔ Remember that in a function, each element in the first set maps to exactly one element in the second set. In a relation, an element may map to more than one element in the second set.
- ✔ When applying the vertical line test, remember that a relation is not a function if any vertical line touches or intersects the graph more than once, and the relation is a function if no such vertical line can be drawn.
- ✔ In function notation, the variable inside the brackets should match the variable used in the rule of the function.

Learning pathways

1–5, 6(1st, 2nd columns), 7(a, c, e), 8–12, 14, 15, 18, 21(a, b)	4, 5, 6(2nd, 3rd columns), 7(b, d, f), 13, 15, 16(a,b), 18, 19, 21(c, d)	3–5, 6(3rd, 4th columns), 7(c, f), 15–20, 22–24

Exercise 5.1A: Understanding and fluency

1 Write each of the following sets.

a the set of whole numbers from 1 to 10

b the set of integers greater than –5 and less than 4

c the set of positive prime numbers less than 20

d the set of the first five letters of the alphabet

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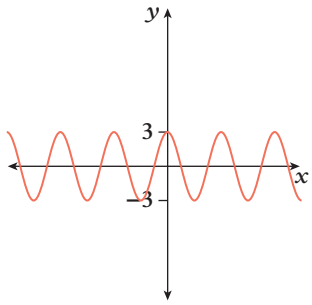
WE 5.1A **2** For each of the following relations, represented by sets of ordered pairs, determine:

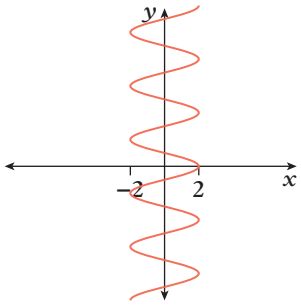
i the domain of the relation
ii the range of the relation
iii whether the relations are functions.

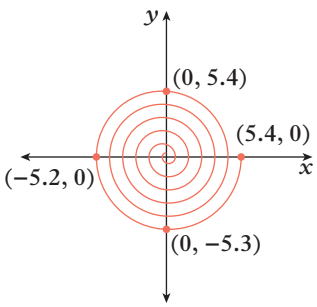
a {(0, 3), (1, 4), (2, 7), (3, 10)}
b {(1, 5), (5, 1), (2, 6), (6, 2), (3, 7), (7, 3)}
c {(1, 9), (5, 8), (–2, 7), (5, 11), (3, 1), (20, 5)}
d {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)}
e {(12, –5), (12, 4), (12, 0), (12, 1)}
f {(–5, 12), (4, 12), (0, 12), (1, 12)}

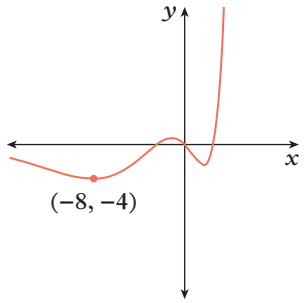
3 For each of the relations in question **2** which are not functions, identify two ordered pairs which can be used to show that the relation is not a function.

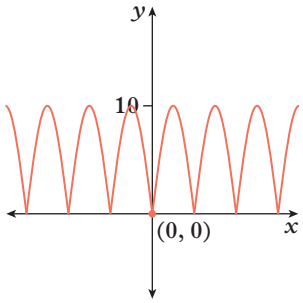
WE 5.1B **4** Use the vertical line test to determine whether the relations represented by the following graphs are functions.

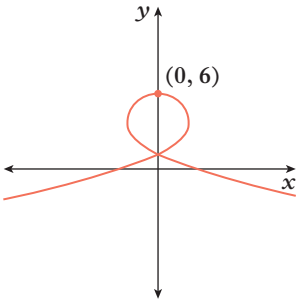
a


b


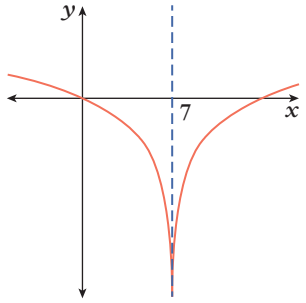
c


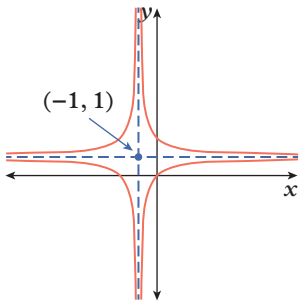
d


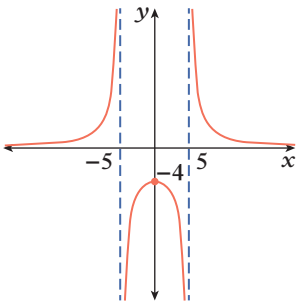
e


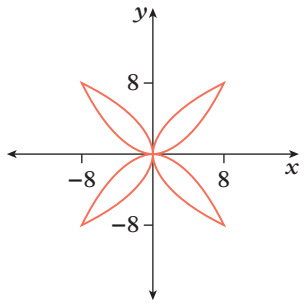
f


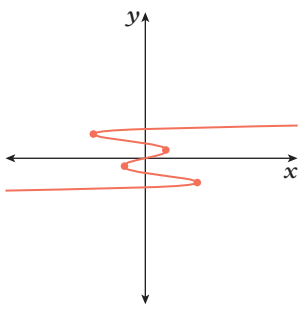
5 Determine whether the relations represented by the following graphs are functions. In this question, asymptotes are indicated by dashed blue lines.

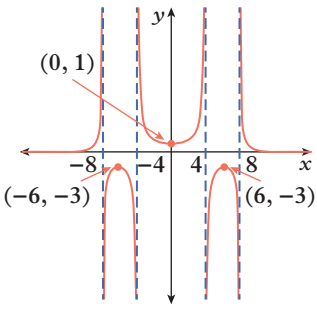
a


b


c


d


e


f


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WE 5.1C **6** For each of the functions below, evaluate: **i** $f(0)$, **ii** $f(1)$, and **iii** $f(-1)$.

a $f(x) = x + 1$ **b** $f(x) = 2x^2 + x$ **c** $f(x) = 3^x + 1$ **d** $f(x) = 3 \times 2^{-x}$

e $f(x) = \dfrac{3}{x+2}$ **f** $f(x) = \dfrac{1}{2x-3} + 1$ **g** $f(x) = \log_2(x+3)$ **h** $f(x) = 3\log_2(5-2x)$

7 For each of the functions below, evaluate: **i** $f(2)$, **ii** $f(-2)$, and **iii** $f(5)$.

a $f(x) = x + 1 + 2^x$ **b** $f(x) = x^2 - 1 + 3^x$ **c** $f(x) = 2x^2 + x + 4 \times 2^{-x}$

d $f(x) = 3^x + \dfrac{3}{x+5}$ **e** $f(x) = 2^x + \dfrac{1}{4x-4}$ **f** $f(x) = -3 \times 2^{-x} - \dfrac{2}{x-3}$

Exercise 5.1B: Problem solving and reasoning

8 This question investigates linear relationships.

a Sketch the following linear graphs.

i $y = x$ **ii** $y = 2x + 3$ **iii** $y = -3x - 1$ **iv** $y = 4$ **v** $x = -2$

b Determine which of the graphs in part **a** are functions.

c Give a reason why vertical lines are not functions.

d Explain why all linear graphs which are not vertical lines are functions.

9 This question investigates quadratic relationships.

a Sketch the following parabolas.

i $y = x^2$ **ii** $y = 2x^2$ **iii** $y = -(x-2)^2 + 1$ **iv** $y = (x+2)^2 + 4$

b Explain why all quadratic relationships of the form $y = ax^2 + bx + c$ are functions.

10 This question investigates exponential relationships.

a Sketch the following exponential graphs.

i $y = 2^x$ **ii** $y = 3 \times 2^{-x}$ **iii** $y = -2^x$ **iv** $y = -2^{-x}$ **v** $y = 2^x + 1$

b Explain why all exponential relationships of the form $y = a \times b^{nx} + k$ are functions.

11 This question investigates graphs of hyperbolas.

a Sketch the following hyperbolas.

i $y = \dfrac{1}{x}$ **ii** $y = \dfrac{-2}{x}$ **iii** $y = \dfrac{1}{x-2}$ **iv** $y = \dfrac{-3}{x} + 1$

b Explain why all hyperbolic relationships of the form $y = \dfrac{a}{x-h} + k$ are functions.

12 This question investigates graphs of circles.

a Sketch the following circles.

i $x^2 + y^2 = 1$ **ii** $(x-1)^2 + y^2 = 4$ **iii** $x^2 + (y+2)^2 = 9$ **iv** $(x+1)^2 + (y-3)^2 = 1$

b Explain why all circles are not functions.

13 A rule gives a function if each possible input, x, gives exactly one output, y. Use this condition to explain why the following statements are true.

a The rule $y = x^3$ gives a function. **b** The rule $x = -2$ does not give a function.

14 Two functions are equal if they map the elements of one set, X, to another set, Y, in the same way. Consider the functions given by $f(x) = x$ and the infinite set of ordered pairs $\{(1, 1), (2, 2), (3, 3), \ldots, (n, n), \ldots\}$ where n is a positive integer.

a Describe each function in words.

b The first function, $f(x)$, maps from $\mathbb{R}$ to $\mathbb{R}$. Identify two sets that the second function maps between.

c Graph both functions on the same Cartesian plane. Show eight of the ordered pairs from the second function on your graph.

d Use your answers from parts **a** to **c** to determine whether the two functions are equal.

15 The *implied domain* of a function is the largest possible domain of the function.

a Determine the implied domain of the following functions.

i $f(x) = x^2$ **ii** $f(x) = 2x + 3$ **iii** $f(x) = 2^x$

b Sometimes it is necessary to exclude elements of a set to write the implied domain. The exclusion of the element a from the set X is written as $X \backslash \{a\}$.

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Determine the implied domain of the following functions. Hint: The denominator of a fraction can't be zero.

i $f(x) = \frac{1}{x}$ **ii** $f(x) = \frac{2}{x-2}$ **iii** $f(x) = \frac{1}{x+3} + 3$

c It is possible to consider the domain of relations which are not functions.

Determine the implied domain of the following circles. Write your answers in the form $a \le x \le b$.
Hint: Refer to your answers to question **12**.

i $x^2 + y^2 = 1$ **ii** $(x-1)^2 + y^2 = 4$ **iii** $x^2 + (y+2)^2 = 9$ **iv** $(x+1)^2 + (y-3)^2 = 1$

16 State the domain (the set of x-values) of each of the graphs in question **4**.

17 State the domain of each of the graphs in question **5**.

18 Recall that the range of a function is the set of the output values of the function.

a Sketch each of the following functions.

i $f(x) = x + 1$ **ii** $f(x) = 3x - 1$ **iii** $f(x) = x^2$ **iv** $f(x) = x^2 + 1$

v $f(x) = 3^x$ **vi** $f(x) = 2^x - 1$ **vii** $f(x) = \frac{2}{x+2}$ **viii** $f(x) = \frac{1}{x-3} + 1$

b Determine the range of each of the functions in part **a**. Use appropriate set notation or write an inequality using y where required.

19 State the range of each of the graphs in question **4**.

20 State the range of each of the graphs in question **5**.

21 Function notation can be used to substitute any expression into a function. For example, if $f(x) = x^2$, then $f(4x) = (4x)^2 = 16x^2$. For each of the functions below, find:

i $f(3x)$ **ii** $f(x+1)$ **iii** $f(2x-1)$.

a $f(x) = x + 2$ **b** $f(x) = 2x^2$ **c** $f(x) = 2^x$ **d** $f(x) = -x^2 + x$

22 Function notation can be used to describe transformations on a function.

- A dilation by a factor of a in the direction of the y-axis is written in the form $a \times f(x)$.
- A dilation by a factor of $\frac{1}{n}$ in the direction of the x-axis is written in the form $f(nx)$.
- A reflection in the y-axis is written as $f(-x)$.
- A reflection in the x-axis is written as $-f(x)$.
- A translation k units up is written in the form $f(x) + k$.
- A translation h units to the right is written as $f(x - h)$.

a For the functions below, describe the transformation performed on the function $f(x)$ in words.

i $f(x) + 2$ **ii** $f(x) - 3$ **iii** $f(x + 5)$ **iv** $f(x - 4)$

v $-f(x)$ **vi** $f(-x)$ **vii** $2f(x)$ **viii** $f(2x)$

b Apply each of the transformations in part **a** to the function $f(x) = x^2$ to find the rule of the transformed function in the form $y = a(x-h)^2 + k$ respectively.

Exercise 5.1C: Challenge

23 For the functions $f(x) = x + 1$, $g(x) = x^2 + x$ and $h(x) = 2^x$, find:

a $f(g(x))$ **b** $g(f(x))$ **c** $f(h(x))$ **d** $h(f(x))$ **e** $g(h(x))$ **f** $h(g(x))$.

24 A function is called injective (or one-to-one) if no two elements in the domain map to the same element in the range. If a horizontal line drawn through any part of a graph intersects or touches the graph only once, then the graph represents an injective function. This is known as the horizontal line test.

a Sketch the following functions.

i $f(x) = 2x + 1$ **ii** $f(x) = -x^2$ **iii** $f(x) = 2^{-x} + 1$ **iv** $f(x) = \frac{1}{x+2} - 1$

b Use the horizontal line test to determine which of the functions in part **a** are injective.

c Give a reason why no quadratic function, $y = ax^2 + bx + c$, on its implied domain is injective.

Online resources:

Interactive skillsheet
Functions and relations

Quick quiz
5.1

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Lesson 5.2

Polynomials

Learning intentions

By the end of this lesson you will be able to ...

✔ operate on, evaluate and identify the features of polynomials.

Inter-year links

Year 7	6.2 Terms, expressions and equations
Year 8	5.6 Expanding
Year 9	2.2 Expanding

Polynomials

Key content video
Polynomials

- A **polynomial** is an expression that contains at least one variable, such as x.
  Each term containing that variable is raised to a non-negative integer value (0, 1, 2, 3, …).
  For example, $x^3 + 7x^2 + 3x + 2$ and $3x^2 + 2xy - y^2$ are examples of polynomials, because the exponent of each variable is a positive integer value.
  → The expressions $3x^{\frac{1}{3}} - 9x$ and $\frac{2}{x} + x^2$ are not polynomials because the variables are raised to negative exponents or fractional exponents.
- A single-variable polynomial function can be written as:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x^1 + a_0$$

  where n is the highest exponent, and $a_n, a_{n-1}, a_{n-2}, \ldots a_2, a_1, a_0$ are coefficients.
- $P(x)$ is read as 'P of x', meaning the polynomial P is in terms of the variable x.
- The **degree** of a polynomial (n) is the largest exponent of the variable. Polynomials are named according to their degree. For example, $P(x) = x^2 - 2x + 1$ is a degree 2 polynomial called a quadratic.
  → Some common polynomials are listed in the table on the next page.
- The leading term (a_nx^n) includes the variable with the degree of the polynomial and is usually written first.
  → The coefficient of the leading term (a_n) is called the leading coefficient.
- A polynomial is said to be monic if its leading coefficient is 1. All other polynomials are non-monic.
- The **constant term** (a_0) of a polynomial is the term with no variable.
- Polynomials are generally written with the variables in descending order of their exponents.
- All single-variable polynomials are functions. Function notation can be used to indicate the value of a polynomial, $P(x)$, at $x = c$ by writing $P(c)$.

Operations on polynomials

- Polynomials can be added and subtracted by collecting like terms. For example,

$$\begin{aligned}&(3x^5 + 4x^3 + 2x + 1) + (4x^5 + 2x^3 - 3x - 10)\\ &= (3 + 4)x^5 + (4 + 2)x^3 + (2 - 3)x + (1 - 10)\\ &= 7x^5 + 6x^3 - x - 9\end{aligned}$$

- The distributive law states that each term in one set of brackets is to be multiplied by each term in another set of brackets.
  → The distributive law can be applied to polynomials.

$$(x^2 + x + 1)(x^2 - 2x - 3) = x^2(x^2 - 2x - 3) + x(x^2 - 2x - 3) + 1(x^2 - 2x - 3)$$

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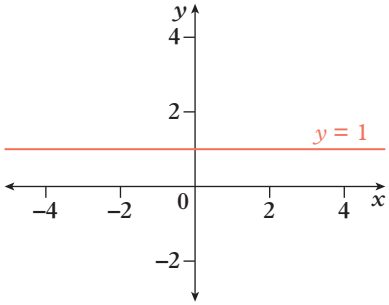
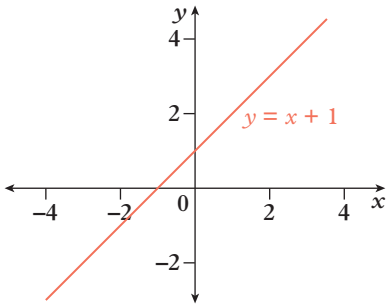
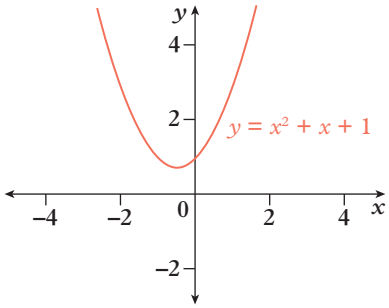
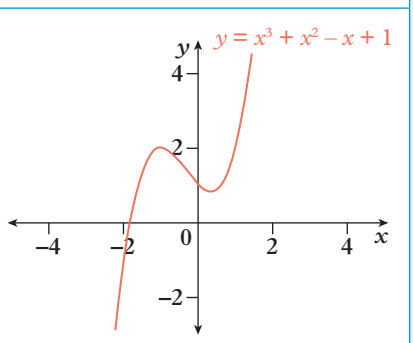
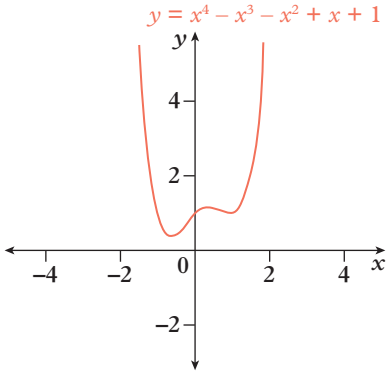
Types of polynomials			
degree	**name**	**example**	**graph**
0	constant	1 (or $1x^0$)	
1	linear	$x + 1$ (or $x^1 + 1$)	
2	quadratic	$x^2 + x + 1$	
3	cubic	$x^3 + x^2 - x + 1$	
4	quartic	$x^4 - x^3 - x^2 + x + 1$	

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Worked example 5.2A Identifying features of a polynomial

For the polynomial $2x^3 - x^2 + 7x + 4$, identify:

a the number of terms
b the degree of the polynomial
c the constant term
d the leading term
e the leading coefficient
f the coefficient of the x^2 term
g whether the polynomial is monic or non-monic.

THINK	WRITE
a Terms are separated by + and – signs.	**a** There are four terms.
b Look for the highest power of x.	**b** The degree is 3.
c Look for a term without a variable (or a term for which the power of the variable is 0).	**c** The constant term is 4.
d Look for the term that has the highest power of x.	**d** The leading term is $2x^3$.
e Write the coefficient of the leading term.	**e** The leading coefficient is 2.
f Write the coefficient of the term containing x^2. The sign shown between it and the term before belongs to the x^2 term. ($-x^2$ means $-1x^2$.)	**f** The coefficient of x^2 is -1.
g The leading coefficient is not 1 so the polynomial is non-monic.	**g** The polynomial is non-monic.

Worked example 5.2B Evaluating a polynomial

If $P(x) = 2x^3 - x^2 + 7x + 4$, evaluate:

a $P(3)$
b $P(0)$
c $P(-1)$.

THINK	WRITE
a Substitute $x = 3$ into the expression and evaluate.	**a** $P(x) = 2x^3 - x^2 + 7x + 4$ $P(3) = 2(3)^3 - (3)^2 + 7(3) + 4$ $= 54 - 9 + 21 + 4$ $= 70$
b Substitute $x = 0$ into the expression and evaluate.	**b** $P(x) = 2x^3 - x^2 + 7x + 4$ $P(0) = 2(0)^3 - (0)^2 + 7(0) + 4$ $= 0 - 0 + 0 + 4$ $= 4$
c Substitute $x = -1$ into the expression and evaluate.	**c** $P(x) = 2x^3 - x^2 + 7x + 4$ $P(-1) = 2(-1)^3 - (-1)^2 + 7(-1) + 4$ $= -2 - 1 - 7 + 4$ $= -6$

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Worked example 5.2C Expanding the product of two polynomials

Expand and simplify each of these products.

a $2x^3(x^2 - 3x + 4)$

b $(x^2 - 2)(x^3 - x + 5)$

c $(2x^4 - x^2 + 3)(x^3 + 4x - 1)$

THINK

a Apply the distributive law. Multiply each term inside the pair of brackets by the term outside the brackets.

b Apply the distributive law. Multiply each term in the second pair of brackets by each term in the first pair of brackets. Simplify like terms.

c Apply the distributive law. Multiply each term in the second pair of brackets by each term in the first pair of brackets. Simplify like terms.

WRITE

a

$$2x^3(x^2 - 3x + 4) = 2x^5 - 6x^4 + 8x^3$$

b

$$\begin{aligned}(x^2 - 2)(x^3 - x + 5) &= x^2(x^3 - x + 5) - 2(x^3 - x + 5)\\ &= x^5 - x^3 + 5x^2 - 2x^3 + 2x - 10\\ &= x^5 - 3x^3 + 5x^2 + 2x - 10\end{aligned}$$

c

$$\begin{aligned}(2x^4 - x^2 + 3)(x^3 + 4x - 1) &= 2x^4(x^3 + 4x - 1) - x^2(x^3 + 4x - 1) + 3(x^3 + 4x - 1)\\ &= 2x^7 + 8x^5 - 2x^4 - x^5 - 4x^3 + x^2 + 3x^3 + 12x - 3\\ &= 2x^7 + 7x^5 - 2x^4 - x^3 + x^2 + 12x - 3\end{aligned}$$

Worked example 5.2D Expanding the product of three polynomials

Expand and simplify $(x + 2)(x - 3)(x + 4)$.

THINK

1. Multiply two of the factors and simplify. The second and third factors can be multiplied.
2. Multiply the linear factor by the quadratic factor.
3. Simplify like terms.

WRITE

$$\begin{aligned}(x + 2)(x - 3)(x + 4) &= (x + 2)(x^2 + 4x - 3x - 12)\\ &= (x + 2)(x^2 + x - 12)\\ &= x(x^2 + x - 12) + 2(x^2 + x - 12)\\ &= x^3 + x^2 - 12x + 2x^2 + 2x - 24\\ &= x^3 + 3x^2 - 10x - 24\end{aligned}$$

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Helpful hints

- ✔ When multiplying through by a negative term, use brackets for clarity. For example,
  $-7p(3q + 4) = -7p \times 3q + (-7p \times 4)$
- ✔ Show all your working to avoid making errors in arithmetic and errors related to the signs.
- ✔ After expanding expressions containing brackets, always simplify your results by looking for like terms. Simplify, simplify, simplify!

Learning pathways

1–4, 6–8, 9(a, c, e, g), 10, 11, 13	1, 3–8, 9(b, c, f, h), 10, 12, 14, 16(a–d), 18(a, b)	1, 5, 6(g–l), 8, 9(d, f, h), 10, 12, 15–17, 18(c), 19

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Exercise 5.2A: Understanding and fluency

WE 5.2A **1** For each polynomial below, identify:

i the number of terms
ii the degree of the polynomial
iii the constant term
iv the leading term
v the leading coefficient
vi the coefficient of the x^2 term
vii whether the polynomial is monic or non-monic.

a $2x^3 + 3x^2 + 4x + 5$
b $4x^5 + x^4 + 7x^3 - 2x^2 + 9x - 3$
c $-5x^4 - 2x^3 + 5x^2 + 1$
d $x^7 + x^6 - x^5 + x^4 + x^3 - x^2 + x$
e $9 - 3x - 6x^3 + 2x^2 - 7x^6$
f $3 - 11x^{10} + 5x^8$

2 Simplify each of these polynomials by collecting like terms.

a $4x^3 + 2x^2 - 4x - 1 + x^3 - 5x^2 + 3$
b $x^5 - 3x^2 + x^3 - 2x^4 - x^3 + 7x^2$
c $x^4 + 2 + 4x^2 + x^4 - 2x^2 - 5 + 6x^3$
d $(2x^2 - 5x + 1) + (3x^2 - 6x + 8)$
e $(5x^3 - 2x + 1) - (2x^3 - 7x + 4)$
f $(3x^2 + x + x^3) - (4x^4 - 3x^2 + 5x^3)$

WE 5.2B **3** If $P(x) = x^3 - 2x^2 + 5x - 8$, evaluate:

a $P(2)$ **b** $P(0)$ **c** $P(3)$ **d** $P(-3)$.

4 If $P(x) = 3x^4 - 4x^3 + x^2 - 6x + 1$, evaluate:

a $P(3)$ **b** $P(1)$ **c** $P(-1)$ **d** $P(-2)$.

5 If $P(x) = 2x^3 + 3x^2 - 5x + 1$, evaluate:

a $P\left(\frac{1}{2}\right)$ **b** $P\left(-\frac{2}{3}\right)$ **c** $P(\sqrt{2})$ **d** $P\left(-\frac{1}{\sqrt{3}}\right)$.

6 If $P(x) = 2x^3 - x^2 + 3x - 6$ and $Q(x) = 3x^2 - 7x + 2$, find:

a $P(x) + Q(x)$ **b** $P(x) - Q(x)$ **c** $Q(x) - P(x)$ **d** $3P(x)$
e $-2Q(x)$ **f** $2P(x) + Q(x)$ **g** $P(x) - 4Q(x)$ **h** $3P(x) - 2Q(x)$
i $P(x) \times Q(x)$ **j** $-3Q(2)$ **k** $P(3) + Q(3)$ **l** $5P(0) - 4Q(0)$.

WE 5.2C **7** Expand and simplify the following expressions.

a $x^3(x^2 + 2x + 7)$ **b** $4x^2(x^2 - 5x + 2)$ **c** $-6x^3(x^4 - 4x^2 + 1)$
d $(x + 3)(x^2 - 2x + 4)$ **e** $(x^2 - 1)(x^2 + 3x - 7)$ **f** $(x^4 - 3x + 2)(x^3 + 4x^2 - 1)$

WE 5.2D **8** Expand and simplify the following expressions.

a $3x(x + 2)(x + 3)$ **b** $-7x^2(x + 5)(x - 2)$ **c** $(x + 3)(x + 1)(x + 6)$
d $(x - 3)(x - 4)(x + 1)$ **e** $(3x - 2)(x - 5)(x - 2)$ **f** $(4x + 1)(2x + 7)(x - 4)$

9 Expand and simplify each of these expressions.

a $(x^3 + 4)^2$ **b** $(x^2 + 3x)^2$ **c** $2x(x^2 - 2)^2$ **d** $(x^3 - 5x^2)^2$
e $(x + 2)^3$ **f** $(2x - 3)^3$ **g** $(x + 3)^4$ **h** $(1 - x)^4$

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Exercise 5.2B: Problem solving and reasoning

10 State whether or not each of the following is a polynomial.

a $2x^3 - 3x^2 + 5x$ **b** $2x^{-3} - 3x^{-2} + 5x^{-1}$ **c** $2x^{\frac{3}{7}} - 3x^{\frac{2}{7}} + 5x^{\frac{1}{7}}$ **d** $\frac{2}{x^3} - \frac{3}{x^2} + \frac{5}{x}$

e $\frac{2}{7}x^3 - \frac{3}{7}x^2 + \frac{5}{7}x$ **f** $2x^3 - 3x^2 + 5x + \frac{8}{x+1}$ **g** $\sqrt{2}x^3 - \sqrt{3}x^2 + \sqrt{5}x$ **h** $2\sqrt{x^3} - 3x + 5\sqrt{x}$

11 What is the maximum number of terms in a single-variable polynomial of degree 5?

12 Consider two polynomials, $P(x)$ and $Q(x)$, where $P(x)$ is of degree n, and $Q(x)$ is of degree m.

a State:

i the maximum number of terms in $P(x)$ **ii** the minimum number of terms in $P(x)$

iii the degree of $2P(x)$ **iv** the degree of $[P(x)]^2$.

b If $n > m$, state the degree of:

i $P(x) + Q(x)$ **ii** $P(x) - Q(x)$ **iii** $P(x)Q(x)$.

13 If $P(2) = 2$, find the value of k in $P(x) = x^3 - 2x^2 + 3x + k$.

14 If $P(-1) = 7$, find the value of k in $P(x) = 3x^4 + 2x^3 - kx - 5$.

15 If $P(3) = 31$ and $P(-2) = -19$, find the values of a and b in $P(x) = x^3 + ax^2 + bx + 1$.

16 If $P(x) = x^3 - 3x^2 + 2x + 1$, write simplified expressions for:

a $P(a)$ **b** $P(-3a)$ **c** $P(a^2)$ **d** $P(-a^2)$ **e** $P(a + 2)$ **f** $P(2a - 1)$.

17 Show that $(2x - 1)^4 = 16x^4 - 32x^3 + 24x^2 - 8x + 1$.

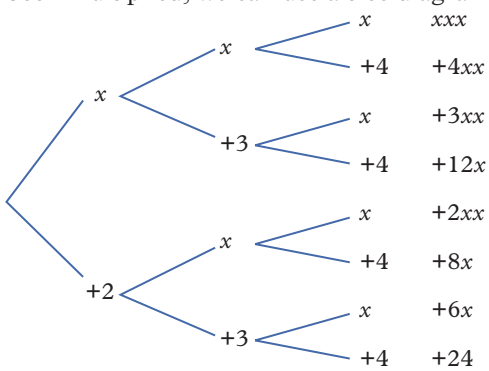
Exercise 5.2C: Challenge

18 Expand each of these expressions.

a $(x^2 - 2)^3$ **b** $(x^3 + x - 1)^2$ **c** $(2x^3 + 1)^4$

19 Consider $(x + 2)(x + 3)(x + 4)$. We can expand the expression by adding the products of every combination of one term from each binomial factor.

To ensure all combinations have been multiplied, we can use a tree diagram, as shown below.



The tree diagram shows all the products that need to be added together. The path along each sequence of connected branches on the tree diagram is represented by a set of brackets in this calculation:

$$\begin{aligned}(x + 2)(x + 3)(x + 4) &= (x \times x \times x) + (x \times x \times 4) + (x \times 3 \times x) + (x \times 3 \times 4) + (2 \times x \times x) + (2 \times x \times 4) \\ &\quad + (2 \times 3 \times x) + (2 \times 3 \times 4) \\ &= x^3 + 4x^2 + 3x^2 + 12x + 2x^2 + 8x + 6x + 24 \\ &= x^3 + 9x^2 + 26x + 24\end{aligned}$$

a Use a tree diagram to help you expand $(x + 5)(x - 3)(x - 8)$.

b Use a tree diagram to help you expand $(x - 2)(x + 5)(x - 3)(x - 8)$.

Online resources:

- **Interactive skillsheet** Polynomials
- **Interactive skillsheet** Multiplying polynomials
- **Worksheet** Working with polynomials
- **CAS instructions** Expanding
- **Quick quiz** 5.2

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Lesson 5.3

Dividing polynomials

Learning intentions

By the end of this lesson you will be able to ...

✔ divide polynomials by linear expressions.

Inter-year links

Year 7	1.6 Dividing whole numbers
Year 8	5.5 Dividing algebraic terms
Year 9	2.5 Factorising quadratic expressions

Polynomial division

Key content video
Dividing polynomials

- A polynomial can be divided by another polynomial using long division.
- The **dividend** is the expression which is being divided by another expression.
- The **divisor** is the expression that the dividend is being divided by.
- The **quotient** is the resulting expression after the division has occurred.
- The **remainder** is the part of the dividend that is left over when the expression is not a multiple of the divisor.
  → The degree of the remainder is always less than the degree of the divisor.

$$\begin{array}{rl} & x + 2 \leftarrow \text{quotient} \\ \text{divisor} \rightarrow x + 1 & \overline{)\,x^2 + 3x + 4} \leftarrow \text{dividend} \\ & -(x^2 + x) \\ \hline & \phantom{)\,x^2 +{}} 2x + 4 \\ & \phantom{)\,x^2 +{}} -(2x + 2) \\ \hline & \phantom{)\,x^2 + 2x +{}} 2 \leftarrow \text{remainder} \end{array}$$

- There are four main steps to remember:
  1. Divide the leading term of the dividend by the leading term of the divisor. The result will be the first term of the quotient.
  2. Multiply the resulting term by the divisor and simplify. Write the new expression below the dividend, aligning the like terms.
  3. Subtract the new expression from the dividend, and write the result below.
  4. Repeat steps **1** to **3**, treating the result from step **3** as the new dividend, until an expression with a degree less than that of the divisor is reached. This expression is the remainder.
- The quotient and remainder are important elements to identify. In the example above, the quotient is $x + 2$ and the remainder is 2.
  We can write this as:

$$\underbrace{(x^2 + 3x + 4)}_{\text{dividend}} \div \underbrace{(x + 1)}_{\text{divisor}} = \underbrace{x + 2}_{\text{quotient}} \quad \underbrace{\text{remainder } 2}_{\text{remainder}}$$

  or as:

$$\frac{\overbrace{x^2 + 3x + 4}^{\text{dividend}}}{\underbrace{x + 1}_{\text{divisor}}} = \overbrace{x + 2}^{\text{quotient}} + \frac{\overbrace{2}^{\text{remainder}}}{\underbrace{x + 1}_{\text{divisor}}}$$

- Alternatively, the polynomial can be written as:

  dividend = divisor × quotient + remainder

  $x^2 + 3x + 4 = (x + 1)(x + 2) + 2$
- In general, if $P(x)$ is divided by $D(x)$ to produce the quotient $Q(x)$ and the remainder $R(x)$, then $P(x) = D(x)Q(x) + R(x)$.

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Worked example 5.3A Dividing a polynomial by a linear expression

Use long division to find the quotient and remainder for $(x^3 + 2x^2 - 9x - 3) \div (x - 2)$.

THINK

1. Divide the dividend's leading term, x^3, by the divisor's leading term, x, and write the result (x^2) in the quotient line.
2. Expand $x^2(x - 2)$ to give $x^3 - 2x^2$ and write the result aligned under the dividend. Subtract like terms.
3. Divide $4x^2$ by x and write the result ($4x$) in the quotient line.
4. Expand $4x(x - 2)$ to give $4x^2 - 8x$ and write the result aligned under the dividend. Subtract like terms.
5. Divide $-x$ by x and write the result (-1) in the quotient line.
6. Expand $-1(x - 2)$ to give $-x + 2$ and write the result aligned under the dividend. Subtract like terms.
7. Identify the quotient and the remainder.

WRITE

$$
\begin{array}{rl}
 & \phantom{x^3 + {}} x^2 + 4x - 1 \\
 & x - 2 \overline{\big) x^3 + 2x^2 - 9x - 3} \\
x^2(x-2) \rightarrow & \phantom{x^3 + {}} -(x^3 - 2x^2) \\
\hline
 & \phantom{x^3 + {}} 4x^2 - 9x - 3 \\
4x(x-2) \rightarrow & \phantom{x^3 + {}} -(4x^2 - 8x) \\
\hline
 & \phantom{x^3 + 4x^2 + {}} -x - 3 \\
-1(x-2) \rightarrow & \phantom{x^3 + 4x^2 + {}} -(-x + 2) \\
\hline
 & \phantom{x^3 + 4x^2 + -x + {}} -5
\end{array}
$$

The quotient is $x^2 + 4x - 1$.
The remainder is -5.

Worked example 5.3B Dividing a polynomial that has coefficients of zero by a linear expression

Use long division to find the quotient and remainder for $(3x^4 + 2x^2 - 5) \div (x + 1)$.

THINK

1. Write the 'missing' terms in the dividend, with zero as the coefficient, to keep like terms aligned.
2. Divide $3x^4$ by x and write $3x^3$ in the quotient line.
3. Expand $3x^3(x + 1)$ to be $3x^4 + 3x^3$. Subtract like terms.
4. Divide $-3x^3$ by x and write $-3x^2$ in the quotient line.
5. Expand $-3x^2(x + 1)$ to be $-3x^3 - 3x^2$. Subtract like terms.
6. Divide $5x^2$ by x and write $5x$ in the quotient line.
7. Expand $5x(x + 1)$ to be $5x^2 + 5x$. Subtract like terms.
8. Divide $-5x$ by x and write -5 in the quotient line.
9. Expand $-5(x + 1)$ to be $-5x - 5$. Subtract like terms.
10. Identify the quotient and the remainder.

WRITE

$$
\begin{array}{rl}
 & \phantom{x^4 + {}} 3x^3 - 3x^2 + 5x - 5 \\
 & x + 1 \overline{\big) 3x^4 + 0x^3 + 2x^2 + 0x - 5} \\
3x^2(x+1) \rightarrow & \phantom{x^4 + {}} -(3x^4 + 3x^3) \\
\hline
 & \phantom{x^4 + {}} -3x^3 + 2x^2 + 0x - 5 \\
-3x^2(x+1) \rightarrow & \phantom{x^4 + {}} -(-3x^3 - 3x^2) \\
\hline
 & \phantom{x^4 + -3x^3 + {}} 5x^2 + 0x - 5 \\
5x(x+1) \rightarrow & \phantom{x^4 + -3x^3 + {}} -(5x^2 + 5x) \\
\hline
 & \phantom{x^4 + -3x^3 + 5x^2 + {}} -5x - 5 \\
-5(x+1) \rightarrow & \phantom{x^4 + -3x^3 + 5x^2 + {}} -(-5x - 5) \\
\hline
 & \phantom{x^4 + -3x^3 + 5x^2 + -5x + {}} 0
\end{array}
$$

The quotient is $3x^3 - 3x^2 + 5x - 5$.
The remainder is 0.

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Helpful hints

- ✔ Polynomial expressions need to be written in descending order, according to the exponents on the variables, for long division to work. For example, the expression $5 - 3x + x^2$ needs to be rearranged to $x^2 - 3x + 5$ before it can be divided using long division.
- ✔ Before dividing a polynomial, remember to insert the 'missing' terms with zero as the coefficient. It can be easy to make mistakes if you skip this step.
- ✔ The ultimate purpose of dividing polynomials is so that you can fully factorise cubics, quartics and other polynomials of higher degree. It is best to keep this skill at the front of your mind!

Learning pathways

1–5, 8, 9(a, c), 10(a, c), 14	1(e–h), 2(a, d, e, g, i, j), 3, 6, 8, 9, 10(b, d), 12, 15	1(g, h), 2(g–j), 3, 6, 7, 9, 10(b, d), 11, 13, 15, 16

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Exercise 5.3A: Understanding and fluency

1 Use long division to find the quotient and remainder for each of these division problems.

a $(x^2 + 10x + 26) \div (x + 7)$
b $(x^2 - 12x + 26) \div (x - 4)$
c $(5x^2 - 4x - 2) \div (x - 2)$
d $(9x^2 - 37x + 22) \div (x - 4)$
e $(14x^2 + 53x - 6) \div (2x + 7)$
f $(60x^2 - 52x + 79) \div (10x + 3)$
g $(-2x^2 - 113x - 28) \div (2x + 7)$
h $(36x^2 - 25x + 6) \div (1 - 4x)$

WE 5.3A **2** Use long division to find the quotient and remainder for each of these division problems.

a $(x^3 - 2x^2 - 5x + 7) \div (x - 3)$
b $(x^3 + 3x^2 + 7x - 1) \div (x + 1)$
c $(x^3 + 6x^2 - 4x - 15) \div (x - 2)$
d $(x^3 - 3x^2 - 20x + 25) \div (x + 4)$
e $(2x^3 + 3x^2 - 5x - 4) \div (x + 2)$
f $(3x^3 - 2x^2 - 7x + 2) \div (x - 1)$
g $(5x^3 - 12x^2 - 6x - 15) \div (x - 3)$
h $(4x^3 + 9x^2 - 8x - 1) \div (x + 1)$
i $(x^4 + 2x^3 - 4x^2 - 2x + 8) \div (x + 1)$
j $(x^4 - 5x^3 + 2x^2 - x - 3) \div (x - 2)$

WE 5.3B **3** Use long division to find the quotient and remainder for each of these division problems.

a $(x^3 + x + 21) \div (x + 3)$
b $(2x^3 - 3x^2 - 6) \div (x - 2)$
c $(4x^4 - 3x^2 - 5) \div (x - 1)$
d $(3x^4 - 41) \div (x + 2)$

4 For each of the parts **a**–**f** in question **1**:

i Write the answers for the division problems in the form: quotient $+ \dfrac{\text{remainder}}{\text{divisor}}$.
ii Write the dividends in the form: divisor × quotient + remainder.

5 For each of the parts **a**–**f** in question **2**:

i Write the answers for the division problems in the form: quotient $+ \dfrac{\text{remainder}}{\text{divisor}}$.
ii Write the dividends in the form: divisor × quotient + remainder.

6 Use long division to find the quotient and remainder for each of these division problems.

a $(6x^4 + x^3 + 4x^2 - 11x + 2) \div (2x - 1)$
b $(36x^4 + 6x^3 + 4x^2 - 10x + 2) \div (3x - 2)$
c $(-4x^4 + x^3 + 4x^2 - 9x + 6) \div (1 - 4x)$
d $(-4x^4 + 8x^3 + 33x^2 - 10x + 6) \div (5 - 2x)$
e $(100x^4 + 8x^3 + 34x^2 - 10x + 6) \div (2x + 6)$
f $(72x^4 - 6x^3 + 3x^2 - 201x + 6) \div (3x + 6)$

7 Use long division to find the quotient and remainder for each of these division problems.

a $(2x^3 - 3x - 5) \div (2x + 2)$
b $(9x^3 + 6x^2) \div (3x - 1)$
c $(8x^4 + 6x^2 - 12) \div (2x - 1)$
d $(-8x^4 + 6x^2 - 12) \div (1 - 2x)$
e $(9x^4 + 6x^2 - 12) \div (3 - 3x)$
f $(6x^4 + 4x^2 - 12x) \div (2x - 4)$

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8 Perform the following divisions and write your answers in the form $Q(x) + \frac{R(x)}{D(x)}$, where Q, R and D are polynomial expressions and the degree of R is less than the degree of D.

a $\frac{3x^3 + 6x^2 + 7x + 20}{x + 2}$

b $\frac{2x^3 - 11x^2 + 15x + 9}{x - 3}$

c $\frac{x^4 + x^3 - 30x^2 - 8x + 2}{x + 6}$

d $\frac{x^4 - 7x^3 + 12x - 34}{x - 7}$

9 Write the following polynomials in the form $(x - 3)Q(x) + R(x)$, where Q and R are polynomial expressions. Hint: Divide each polynomial by $(x - 3)$ to find the quotient and remainder.

a $x^2 + 2x - 9$

b $x^3 - 7x^2 + 21x - 21$

c $x^3 - 9x^2 + 20x - 17$

d $x^4 + 5x^3 - 15x^2 - 23x - 9$

10 Perform the following divisions and write your answer in the form $Q(x) + \frac{R(x)}{D(x)}$, where Q, R and D are polynomial expressions and the degree of R is less than the degree of D.

a $(4x + 3) \div (x + 2)$

b $(9x - 4) \div (x - 3)$

c $\frac{x - 8}{x - 2}$

d $\frac{10x + 4}{2x - 5}$

Exercise 5.3B: Problem solving and reasoning

11 Let $P(x)$ represent a polynomial with a degree of n, where n is a positive integer greater than 2.

a Consider the polynomial division $P(x) \div (x - a)$. Determine the degree of the quotient.

b Consider the polynomial division $P(x) \div D(x)$, where the degree of D is m, m is a positive integer, and $n > m$. Determine the degree of the quotient.

c Explain why, when $P(x)$ is divided by $x^2 - 3$, the remainder must be of the form $ax + b$, for some constants a and b.

12 Consider the division $\frac{x^3 + 9x^2 + 23x + 17}{x + 4}$.

a Rewrite the division with the numerator written in the form $(x + 4)Q(x) + R(x)$, where Q and R are polynomial expressions.

b Write your answer from part **a** in the form $\frac{(x + 4)Q(x)}{x + 4} + \frac{R(x)}{x + 4}$ and explain how $\frac{x^3 + 9x^2 + 23x + 17}{x + 4} = Q(x) + \frac{R(x)}{x + 4}$ for $x \neq -4$.

13 Consider the polynomial $P(x) = x^3 - 3x^2 - 10x + k$, where k is a real number.

a For what value of k does $P(x) \div (x - 2)$ give a remainder of zero?

b From your answer in part **a**, identify the linear factor and the quadratic factor of $P(x)$. Hence, show that you can write $P(x)$ as the product of three linear factors.

14 Consider $P(x) = (x + 2)x^2 + 5x(x + 2) - 36(x + 2)$.

a Expand and simplify $P(x)$.

b Factorise $P(x)$.

15 Use long division to find the quotient and remainder for each of these division problems.

a $(x^3 - 3x^2 + 2x - 4) \div (x^2 + 1)$

b $(x^4 + x^3 - 7x^2 + 3x + 5) \div (x^2 - 3)$

c $(2x^4 + 6x^2 - 1) \div (x^2 + 2)$

Exercise 5.3C: Challenge

16 Suppose $f(x) = \frac{P(x)}{D(x)}$ where P and D are polynomial expressions.

a If $P(x) = (x - a)Q(x)$ and $D(x) = x - a + b$, where $b \neq 0$, write an expression for $f(a)$.

b If $P(x) = (x - a)Q(x) + b$ and $D(x) = x - a$, where $b \neq 0$, explain why $f(a)$ is undefined.

c If $P(x) = (x - a)Q(x)$ and $D(x) = x - a$, write an expression for $f(x)$, where $x \neq a$.

Online resources:

- **Interactive skillsheet** Dividing polynomials
- **Investigation** Synthetic division
- **CAS instructions** Dividing polynomials
- **Quick quiz** 5.3

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Lesson 5.4

Remainder and factor theorems

Learning intentions

By the end of this lesson you will be able to ...

✔ use the remainder and factor theorems to factorise polynomial expressions.

Inter-year links

Year 7	6.4 Substitution
Year 8	5.2 Substitution
Year 9	2.5 Factorising quadratic expressions

The remainder theorem

Key content video
The remainder and factor theorems

- The **remainder theorem** states that when a polynomial $P(x)$ is divided by $(x - a)$, the remainder is $P(a)$.
  For example, the remainder of $P(x) \div (x - 2)$ is $P(2)$, and the remainder of $P(x) \div (x + 3)$ is $P(-3)$.

The factor theorem

- A zero of a polynomial function $P(x)$ is a value a such that $P(a) = 0$.
- The **factor theorem** states that if a is a zero of $P(x)$, then $(x - a)$ is a factor of $P(x)$.
  For example, for $P(x) = x^3 + x^2 - 10x + 8$, since $P(1) = 0$, $(x - 1)$ must be a factor of $P(x)$.
- Once a linear factor of a cubic polynomial is identified, long division can be used to find the corresponding quadratic factor.
  For example, the quotient when dividing $P(x) = x^3 + x^2 - 10x + 8$ by $(x - 1)$ is $x^2 + 2x - 8$.
- Factorising the quotient allows us to write the polynomial as a product of its linear factors.
  For example, $P(x) = x^3 + x^2 - 10x + 8$
  $= (x - 1)(x^2 + 2x - 8)$
  $= (x - 1)(x - 2)(x + 4)$
- In general, if $P(x) \div (x - a) = Q(x)$, then $P(x) = (x - a)Q(x)$.

Worked example 5.4A Using the remainder theorem

Find the remainder when $x^3 + 4x^2 + x - 6$ is divided by:

a $x - 2$ **b** $x + 3$.

THINK

First, name the polynomial.

a For a divisor of $(x - 2)$, calculate $P(2)$ and write the remainder.

b For a divisor of $(x + 3)$, calculate $P(-3)$ and write the remainder.

WRITE

Let $P(x)$ be $x^3 + 4x^2 + x - 6$.

a $P(2) = (2)^3 + 4(2)^2 + (2) - 6$
$= 8 + 16 + 2 - 6$
$= 20$
The remainder is 20.

b $P(-3) = (-3)^3 + 4(-3)^2 + (-3) - 6$
$= -27 + 36 - 3 - 6$
$= 0$
The remainder is 0.

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Worked example 5.4B Deciding whether a divisor is a factor of a polynomial

Decide whether each of these divisors is a factor of $x^3 + 2x^2 - x - 2$.

a $x - 3$ **b** $x + 1$

THINK

First, name the polynomial.

a For $(x - 3)$ to be a factor, the remainder should be equal to zero. Check if $P(3) = 0$.

b For $(x + 1)$ to be a factor, the remainder should be equal to zero. Check if $P(-1) = 0$.

WRITE

Let $P(x)$ be $x^3 + 2x^2 - x - 2$.

a $P(3) = 27 + 18 - 3 - 2$
$= 40$
$(x - 3)$ is not a factor of $P(x)$.

b $P(-1) = -1 + 2 + 1 - 2$
$= 0$
$(x + 1)$ is a factor of $P(x)$.

Worked example 5.4C Using the factor theorem to find a linear factor of a polynomial

Use the factor theorem to find a linear factor for each of the following polynomials.

a $P(x) = x^3 + 5x^2 - 2x - 24$ **b** $P(x) = x^3 - 3x^2 - 9x + 27$

THINK

a 1 Write the polynomial.

2 Determine a value of x where $P(x) = 0$. Try $P(1)$, $P(-1)$, $P(2)$, etc. That is, substitute x-values that are positive or negative factors of the constant term in $P(x)$, which is -24.

3 Use the factor theorem to identify a linear factor of $P(x)$.

b 1 Write the polynomial.

2 Substitute x-values that are positive or negative factors of the constant term in $P(x)$, which is 27. Try $P(1)$, $P(-1)$, $P(3)$, etc.

3 Use the factor theorem to identify a linear factor of $P(x)$.

WRITE

a $P(x) = x^3 + 5x^2 - 2x - 24$
$P(1) = 1 + 5 - 2 - 24$
$= -20$
$\neq 0$
$P(-1) = -1 + 5 + 2 - 24$
$= -18$
$\neq 0$
$P(2) = 8 + 20 - 4 - 24$
$= 0$
So $(x - 2)$ is a factor of $P(x)$.

b $P(x) = x^3 - 3x^2 - 9x + 27$
$P(1) = 1 - 3 - 9 + 27$
$= 16$
$\neq 0$
$P(-1) = -1 - 3 + 9 + 27$
$= 32$
$\neq 0$
$P(3) = 27 - 27 - 27 + 27$
$= 0$
So $(x - 3)$ is a factor of $P(x)$.

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Worked example 5.4D Factorising a cubic polynomial

Factorise $x^3 - 7x^2 + 7x + 15$.

THINK	WRITE
1 Name the polynomial.	Let $P(x)$ be $x^3 - 7x^2 + 7x + 15$.
2 Determine a value of x for which $P(x) = 0$. Try $P(1)$ and $P(-1)$.	$P(1) = 1 - 7 + 7 + 15$ $= 16$ $\neq 0$ $P(-1) = -1 - 7 - 7 + 15$ $= 0$
3 Identify a linear factor of $P(x)$.	So $(x + 1)$ is a factor of $P(x)$.
4 Use long division to find the quotient when $P(x)$ is divided by $(x + 1)$. The remainder should be zero if the division has been performed correctly.	$\begin{array}{r} x^2 - 8x + 15 \\ x+1 \overline{\big) x^3 - 7x^2 + 7x + 15} \\ \underline{x^3 + x^2 } \\ -8x^2 + 7x + 15 \\ \underline{-8x^2 - 8x } \\ 15x + 15 \\ \underline{15x + 15} \\ 0 \end{array}$
5 Write $P(x)$ as the product of the divisor and quotient.	$P(x) = (x + 1)(x^2 - 8x + 15)$
6 Factorise the quadratic factor.	$= (x + 1)(x - 3)(x - 5)$

Helpful hints

- ✔ Remember that if $P(a) = 0$, then $(x - a)$ is a linear factor of $P(x)$, not $(x + a)$.
- ✔ When factorising, try the positive and negative factors of the constant term in the polynomial.
  For example, when determining a linear factor of $x^2 + 2x + 6$, try substituting the positive and negative factors of the constant term, 6, which are: 1, –1, 2, –2, 3, –3, 6 and –6.
- ✔ When a linear factor has been established and you are performing a polynomial division, you will know you have performed the division correctly if the remainder is zero.

Learning pathways

1–7, 8(1st column), 9–11, 14, 15	4–6, 7–8(2nd column), 10, 12, 13, 16, 18, 19, 21(a)	4, 6(e–h), 7(2nd column), 8(3rd column), 12, 16–18, 20–22

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Exercise 5.4A: Understanding and fluency

1 Consider the polynomial $P(x) = x^3 + x^2 - 10x + 8$.

a Use long division to find the remainder when $P(x)$ is divided by $(x - 3)$.

b Evaluate $P(3)$. What do you notice?

c Use long division to find the remainder when $P(x)$ is divided by $(x + 2)$.

d Evaluate $P(-2)$. What do you notice?

e Without using long division, use the pattern you noticed in parts **a–d** to show how to work out the remainder when $P(x)$ is divided by $(x - 4)$.

f Use long division to verify your answer to part **e**.

WE 5.4A **2** Use the remainder theorem to find the remainder when $x^3 - 5x^2 - 8x + 12$ is divided by:

a $x - 1$ **b** $x + 1$ **c** $x - 2$ **d** $x + 2$.

3 Use the remainder theorem to find the remainder when $2x^3 + 3x^2 - 8x + 3$ is divided by:

a $x + 1$ **b** $x - 2$ **c** $x + 2$ **d** $x + 3$.

4 Use the remainder theorem to find the remainder when $x^4 - 7x^3 + 5x^2 + 31x - 30$ is divided by:

a $x - 2$ **b** $x + 1$ **c** $x - 3$ **d** $x + 4$.

WE 5.4B **5** Decide whether each of these divisors is a factor of $x^3 + 3x^2 - 6x - 8$.

a $x - 1$ **b** $x + 1$ **c** $x - 3$ **d** $x + 3$

e $x - 4$ **f** $x + 4$ **g** $x + 2$ **h** $x - 2$

6 Decide whether each of these divisors is a factor of $x^3 - 4x^2 - 7x + 10$.

a $x - 2$ **b** $x - 3$ **c** $x - 1$ **d** $x + 3$

e $x + 1$ **f** $x + 2$ **g** $x - 5$ **h** $x - 4$

WE 5.4C **7** Use the factor theorem to find a linear factor for each of the following polynomials.

a $P(x) = x^3 + 8x^2 + 9x - 18$ **b** $P(x) = x^3 - x^2 - 14x + 24$

c $P(x) = x^3 - 4x^2 - 9x + 36$ **d** $P(x) = x^3 - 19x - 30$

e $P(x) = 3x^3 - 10x^2 + x + 6$ **f** $P(x) = 2x^3 - 5x^2 - 14x + 8$

g $P(x) = 8x^3 - 26x^2 + 17x + 6$ **h** $P(x) = 4x^3 + 4x^2 - 21x + 9$

WE 5.4D **8** Factorise each of the following polynomials.

a $x^3 + 6x^2 + 3x - 10$ **b** $x^3 + 4x^2 - 19x + 14$ **c** $x^3 + 5x^2 - 4x - 20$

d $x^3 - 5x^2 - 8x + 48$ **e** $x^3 - 2x^2 - 9x + 18$ **f** $2x^3 + x^2 - 8x - 4$

g $4x^3 - 9x^2 - 19x + 30$ **h** $6x^3 - 23x^2 - 6x + 8$ **i** $12x^3 - 17x^2 + 2x + 3$

9 Factorise each of the polynomials in question **7**.

Exercise 5.4B: Problem solving and reasoning

10 Let $P(x) = x^3 + 9x^2 + 23x + 15$.

a For what three values of k does $P(x) \div (x + k)$ result in a remainder of zero?

b Write $P(x)$ as the product of three linear factors.

11 What is the maximum number of linear factors $P(x)$ can have if $P(x)$ is:

a a quadratic polynomial **b** a cubic polynomial **c** a quartic polynomial

d a polynomial of degree 7 **e** a polynomial of degree 12 **f** a polynomial of degree n?

12 a How many linear factors does the polynomial $P(x)$ have if $P(x) = (x + 2)(x - 3)(x + 4)$?

b Without expanding completely, what is the constant term of $P(x)$? Explain.

c To find linear factors using the factor theorem, we look for values of x for which $P(x) = 0$. What are these x-values and how do they relate to the constant term?

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13 Consider the polynomial $P(x) = x^3 + 2x^2 - 5x - 6$.

a Use the factor theorem to find a linear factor of $P(x)$.

b Use the factor theorem to find another linear factor of $P(x)$.

c Use the factor theorem to find a third linear factor of $P(x)$.

d Write $P(x)$ as a product of three factors.

e Check your answer to part **d** by expanding and simplifying the product.

14 Consider your answers from question **5**.

a Identify three factors of the polynomial $P(x) = x^3 + 3x^2 - 6x - 8$.

b Write $P(x)$ as a product of three factors.

c Check your answer to part **b** by expanding and simplifying the product.

15 Consider your answers from question **6**.

a Identify three factors of the polynomial $P(x) = x^3 - 4x^2 - 7x + 10$.

b Write $P(x)$ as a product of three factors.

16 Using only the factor theorem, find three factors of $(x^3 + 2x^2 - 11x - 12)$. Hence, write the polynomial as a product of three factors.

17 A polynomial, P, can be written in the form $P(x) = (x - a)Q(x) + r$, where Q is a polynomial.

a Solve $x - a = 0$ for x.

b Substitute the solution from part **a** into $P(x)$.

c Explain why your answer from part **b** is equal to the remainder of the polynomial division $P(x) \div (x - a)$.

d Another polynomial, F, can be written in the form $F(x) = (bx - a)G(x) + h$, where G is a polynomial.

i Solve $bx - a = 0$ for x.

ii Substitute your solution from part **i** into $F(x)$.

iii Explain why your answer from part **ii** is equal to the remainder of the polynomial division $F(x) \div (bx - a)$.

18 Solve the following problems using the remainder and/or factor theorems.

a When $(x^3 + 2x^2 + ax + 5)$ is divided by $(x - 3)$ the remainder is 5. Find the value of a.

b When $(x^3 + bx^2 - 6x - 7)$ is divided by $(x + 2)$ the remainder is 17. Find the value of b.

c If $(x + 4)$ is a factor of $(x^3 + 13x^2 + cx - 20)$, find the value of c.

d When $(x^3 + mx^2 + nx + 5)$ is divided by $(x + 1)$ the remainder is 13, and when divided by $(x - 2)$ the remainder is 31. Find the values of m and n.

e Find the values of a and b if the polynomial $P(x) = 3x^6 - 5x^3 + ax^2 + bx + 10$ is divisible by $(x + 1)$ and $(x - 2)$.

Exercise 5.4C: Challenge

19 Explain why the strategy used in question **16** is not suitable for factorising each of these polynomials.

a $6x^3 + 13x^2 + 4x - 3$ **b** $x^3 + 5x^2 + 10x + 8$

20 Find a monic polynomial of degree 3 that has a remainder of 5 when divided by $(x - 2)$.

21 Factorise each of these quartic polynomials.

a $x^4 - x^3 - 7x^2 + x + 6$ **b** $x^4 + 8x^3 + 17x^2 - 2x - 24$ **c** $2x^4 + 13x^3 + 21x^2 + 2x - 8$

22 Let $P(x) = x^4 + 13x^2 + 36$.

a Show that $(x^2 + 4)$ is a factor of $P(x)$.

b Factorise $P(x)$.

Online resources:

- **Interactive skillsheet** Remainder and factor theorems
- **Worksheet** Dividing and factorising polynomials
- **Investigation** Rational roots of polynomials
- **CAS instructions** Factorising
- **Quick quiz** 5.4

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Checkpoint

Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this module.

10A 5.1 **1** Determine which of the following relations are functions.

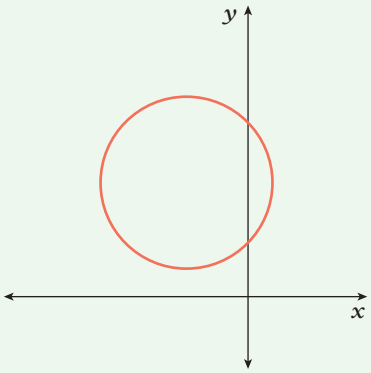
a $\{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1)\}$

b $\{(-5, -5), (-3, 2), (0, 2), (1, 3), (1, 4), (2, 5)\}$

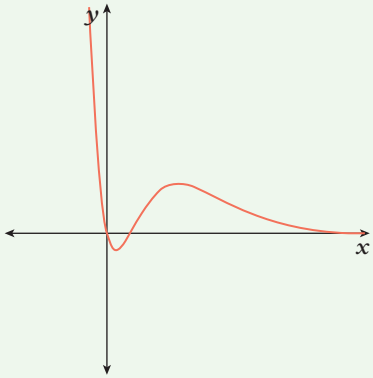
c $y = x^2 + 2x$

d $x^2 + (y - 3)^2 = 9$

e



f



10A 5.2 **2** State whether or not each of the following is a polynomial. If it is not a polynomial, state the term(s) in the expression that cannot be included in a polynomial. If it is a polynomial, state its:

i degree **ii** leading term **iii** leading coefficient.

a $5x^2 - 3x^5 + 4x^7$

b $9x^{100} - 9x^{-100}$

c $\dfrac{5y^{81}}{9} - \sqrt{3}y^{42} + \dfrac{2}{3}y^{27} - 1.2$

d $81y^5 - 42\sqrt{y^3} + 27y^{\frac{2}{3}} - 1.2$

10A 5.2 **3** Suppose $P(x) = 5x^3 - 3x^2 + 2x - 1$ and $Q(x) = 2x^2 - 8x + 7$. Write an expression for each of the following.

a $P(x) + Q(x)$ **b** $P(x) - Q(x)$ **c** $6P(x)$ **d** $P(x) \times Q(x)$

10A 5.3 **4** Determine the quotient and remainder for each of the following polynomial divisions.

a $(x^2 + 2x - 29) \div (x - 4)$

b $(x^3 - 3x^2 + 2x + 33) \div (x + 2)$

c $\dfrac{2x^2 - 8}{x + 3}$

d $\dfrac{6x^3 + 19x^2 + 4x - 8}{3x + 2}$

10A 5.3 **5** Write the following results in the form $Q(x) + \dfrac{R(x)}{D(x)}$, where $D(x)$, $Q(x)$ and $R(x)$ are polynomial expressions, and the degree of R is less than the degree of D.

a $(x^2 + 16x - 9) \div (x + 4)$; quotient: $x + 12$, remainder: -57

b $(5x^3 - 9x^2 + 2x + 1) \div (x - 6)$; quotient: $5x^2 + 21x + 128$, remainder: 769

c $\dfrac{5x^3 - 6x^2 + 7}{x^2 + 4}$; quotient: $5x - 6$, remainder: $-20x + 31$

10A 5.4 **6** Determine the remainder when $P(x) = 3x^3 + 4x^2 - 13x + 6$ is divided by:

a $x + 1$ **b** $x - 1$ **c** $x - 3$ **d** $3x - 2$.

10A 5.4 **7** For each of the following, determine whether the linear expression is a factor of the non-linear polynomial.

a $x - 2$ and $x^3 - 5x^2 + 7x + 4$

b $x + 1$ and $5x^3 + 13x^2 + 5x - 3$

c $x^3 - 2$ and $x^2 - 11x + 13$

d $4x - 3$ and $12x^3 + 11x^2 - 23x + 6$

10A 5.4 **8** Factorise each of the following polynomials.

a $x^3 - 4x^2 - 25x + 28$

b $x^3 - 7x - 6$

10A 5.4 **9** Factorise each of the following polynomials.

a $5x^3 - 15x^2 - 170x + 600$

b $x^4 - x^3 - 43x^2 - 23x + 210$

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Lesson 5.5

Solving polynomial equations

Learning intentions

By the end of this lesson you will be able to ...

✔ solve polynomial equations using the Null Factor Law.

Inter-year links

Year 8	6.2 Solving linear equations
Year 9	4.1 Solving quadratic equations

Polynomials and the Null Factor Law

Key content video
Solving polynomial equations

- The Null Factor Law states that, for real numbers a and b, if $a \times b = 0$, then $a = 0$ or $b = 0$, or both a and b are 0.
- The polynomial equation $P(x) = 0$ can be solved by applying the Null Factor Law when $P(x)$ is in factor form.
  → The values of x that make $P(x) = 0$ are called **roots**.
- If $P(x) = (x - a)(x - b)(x - c)$, then the solutions to $P(x) = 0$ are $x = a$, $x = b$ and $x = c$.

$$\overbrace{P(x) = \underbrace{\underbrace{(x - a)(x - b)(x - c)}_{\text{expression}}}_{\text{equation}}}^{\text{function}} = 0$$

$$x - a = 0 \Rightarrow x = a$$
$$x - b = 0 \Rightarrow x = b$$
$$x - c = 0 \Rightarrow x = c$$

- A polynomial, $P(x)$, of degree n can have a maximum of n linear factors. Therefore, the number of solutions to the equation $P(x) = 0$ is at most n.
- If a polynomial, $P(x)$, of degree n has repeated factors, then the number of unique solutions to $P(x) = 0$ is less than n.
  For example, the factor $(x + 1)$ is repeated twice in $P(x) = x(x + 1)^2$, so even though the degree of the polynomial is 3, the equation $P(x) = 0$ only has two solutions.

$$P(x) = x(x + 1)^2 = 0$$

$(x + 1)$ is a repeated factor

$$x = 0 \Rightarrow x = 0$$
$$x + 1 = 0 \Rightarrow x = -1$$

Factorising polynomials

- To factorise $P(x)$ so that it is a product of linear factors:
  1. apply the factor theorem to find a linear factor
  2. divide $P(x)$ by the linear factor to obtain the quotient
  3. factorise the quotient.
- If $P(x)$ is a cubic polynomial, the quotient is a quadratic factor that *may* be further factorised into two linear factors (making a total of three linear factors).
  For example, $P(x) = (x - 1)(x^2 + 2x - 8)$
  $= (x - 1)(x - 2)(x + 4)$.

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Worked example 5.5A Solving polynomial equations in factor form

Solve each of these equations.

a $(x-1)(x+3)(x-2)=0$

b $(2x+1)(x-5)(x-4)^2=0$

THINK	WRITE
a 1 The left-hand side is in factor form and the right-hand side is equal to 0, so we apply the Null Factor Law.	**a** $(x-1)(x+3)(x-2)=0$
2 Solve each linear equation.	$x-1=0$ or $x+3=0$ or $x-2=0$ $x=1$ or $x=-3$ or $x=2$
3 Write the solutions in ascending order.	$x=-3, 1$ or 2
b 1 Apply the Null Factor Law.	**b** $(2x+1)(x-5)(x-4)^2=0$
2 Solve each linear equation.	$2x+1=0$ or $x-5=0$ or $x-4=0$ $x=-\frac{1}{2}$ or $x=5$ or $x=4$
3 Write the solutions in ascending order.	$x=-\frac{1}{2}, 4$ or 5

Worked example 5.5B Solving a cubic polynomial equation

Solve $x^3-5x^2+2x+8=0$.

THINK	WRITE
1 Name the polynomial.	Let $P(x)$ be x^3-5x^2+2x+8.
2 Use the factor theorem to identify a linear factor of the polynomial.	$P(1)=1-5+2+8$ $=6$ $\neq 0$ $P(-1)=-1-5-2+8$ $=0$ So $(x+1)$ is a factor of $P(x)$.
3 Use long division to find the quotient when $P(x)$ is divided by $(x+1)$.	x^2-6x+8 $x+1\overline{)x^3-5x^2+2x+8}$ x^3+x^2 $-6x^2+2x+8$ $-6x^2-6x$ $8x+8$ $8x+8$ 0
4 Write $P(x)$ as the product of the divisor and quotient.	$P(x)=(x+1)(x^2-6x+8)$
5 Factorise the quadratic factor.	$=(x+1)(x-2)(x-4)$
6 Solve $P(x)=0$ using the Null Factor Law.	For $P(x)=0$: $(x+1)(x-2)(x-4)=0$ $x=-1, 2$ or 4

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Helpful hints

- ✔ Ensure that one side of the equation is equal to zero before applying the Null Factor Law. Remember that 'null' means zero. For example, if $(x + 2)(x + 3)(x + 4) = 1$, you *do not* solve the equations $x + 2 = 1$, $x + 3 = 1$ or $x + 4 = 1$. You need to expand the polynomial, make the equation equal to zero, factorise again, and then apply the Null Factor Law.
- ✔ Remember, sometimes a quadratic equation has irrational solutions, or no real solutions at all. If you factorise a polynomial, $P(x)$, into the product of linear and quadratic factors, then you might need to use the quadratic formula $\left(x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$ to determine any irrational solutions, if they exist.

  For example, $P(x) = x^3 - 2x^2 - 6x + 4$

  $P(x) = (x + 2)(x^2 - 4x + 2)$

  $P(x) = (x + 2)(x - 2 + \sqrt{2})(x - 2 - \sqrt{2})$
- ✔ The discriminant ($\Delta = b^2 - 4ac$) can be used to check whether a quadratic factor can be factorised further. Remember that when $\Delta > 0$ there are two real solutions, when $\Delta = 0$ there is one real solution, and when $\Delta < 0$ there are no real solutions and the expression cannot be further factorised.
- ✔ The zeros of a polynomial function $P(x)$ are the roots of the equation $P(x) = 0$.

Learning pathways

1, 2, 3–4(1st, 2nd columns), 5–8, 9(a, b), 10–12, 14	3–4(2nd, 3rd columns), 5–8, 9(c, d), 12, 13, 15, 17	4(2nd, 3rd columns), 5–7, 9(e, f), 13, 15, 16, 18, 19

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Exercise 5.5A: Understanding and fluency

WE 5.5A **1** Solve each of these equations.

a $(x + 2)(x + 5)(x - 4) = 0$ **b** $(x + 1)(x - 3)(x + 4) = 0$ **c** $(x - 6)(x - 2)(x + 3) = 0$

d $x(x + 2)(x - 9) = 0$ **e** $(2x - 1)(x + 1)(x - 1) = 0$ **f** $(3x - 2)(2x + 7)(x + 5) = 0$

2 Solve each of these equations.

a $(x + 3)(x - 4)(x + 7)(x - 1) = 0$ **b** $(x - 2)(x - 5)(x - 3)(x + 6) = 0$ **c** $(x + 4)(x + 1)(x + 2)^2 = 0$

d $x^2(x + 5)(x - 6) = 0$ **e** $5x(3x + 1)(x + 4)(4 - x) = 0$ **f** $(4x - 3)(2x + 5)(x^2 + 1) = 0$

3 Fully factorise the left-hand side of each of these equations and then solve each equation.

a $(x + 1)(x^2 + 8x + 12) = 0$ **b** $(x - 3)(x^2 - x - 20) = 0$ **c** $(x - 2)(x^2 - 6x + 8) = 0$

d $(x + 4)(x^2 + 6x - 7) = 0$ **e** $(x + 3)(x^2 - 16) = 0$ **f** $x(x - 1)(x^2 - 1) = 0$

g $(x - 5)(x^2 - 10x + 25) = 0$ **h** $(x + 2)(x^2 + 6x + 9) = 0$ **i** $(x - 4)(x^2 - x - 6) = 0$

j $(x + 1)(x^2 + 2x + 1) = 0$ **k** $3(x + 3)(5x^2 + 30x) = 0$ **l** $(x - 2)(2x^2 + 4x - 6) = 0$

WE 5.5B **4** Solve each of these equations.

a $x^3 + 6x^2 + 5x - 12 = 0$ **b** $x^3 - 5x^2 - 4x + 20 = 0$ **c** $x^3 + x^2 - 36x - 36 = 0$

d $x^3 + 10x^2 + 21x = 0$ **e** $x^3 - 7x - 6 = 0$ **f** $x^3 + 2x^2 + 5x + 10 = 0$

g $x^3 + 3x^2 - 9x - 27 = 0$ **h** $x^3 + 9x^2 + 24x + 16 = 0$ **i** $x^3 - 3x^2 - 3x - 4 = 0$

5 Solve each of these equations by first taking out a common factor.

a $2x^3 - 2x^2 - 20x - 16 = 0$ **b** $3x^3 - 15x^2 - 3x + 15 = 0$

c $5x^3 + 10x^2 - 20x - 40 = 0$ **d** $4x^3 + 24x^2 + 44x + 24 = 0$

6 Solve each of these equations by first taking out a negative common factor.

a $-x^3 - 2x^2 + 9x + 18 = 0$ **b** $-x^3 + 4x^2 + 17x - 60 = 0$

c $-2x^3 + 8x^2 + 2x - 8 = 0$ **d** $-3x^3 - 3x^2 + 24x + 36 = 0$

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7 Solve each of these equations using the quadratic formula after a linear factor has been taken out.

a $x^3 + x^2 - 3x + 1 = 0$ **b** $x^3 + 6x^2 + 9x + 2 = 0$ **c** $x^3 + x^2 - 10x - 12 = 0$ **d** $x^3 + 4x^2 - 27x - 20 = 0$

8 Work through the following steps to solve $P(x) = 0$, where $P(x) = x^4 + x^3 - 7x^2 - x + 6$.

a Identify one of the linear factors of $P(x)$.

b Use long division to find the quotient when $P(x)$ is divided by the linear factor identified in part **a**. Write $P(x)$ as the product of the divisor and the quotient, $Q(x)$.

c Identify a linear factor of $Q(x)$ and use long division to find its quadratic factor.

d Factorise the quadratic factor you found in part **c**.

e Write $P(x)$ as a product of four linear factors, and hence solve $P(x) = 0$ using the Null Factor Law.

9 Solve each of these equations.

a $x^4 - 5x^3 + 5x^2 + 5x - 6 = 0$ **b** $x^4 - 4x^3 - 7x^2 + 22x + 24 = 0$ **c** $x^4 + 7x^3 + 8x^2 - 28x - 48 = 0$

d $x^4 - x^3 - 19x^2 - 11x + 30 = 0$ **e** $x^4 - 5x^3 + 20x - 16 = 0$ **f** $x^4 - 13x^2 + 36 = 0$

Exercise 5.5B: Problem solving and reasoning

10 What is the maximum number of real solutions for $P(x) = 0$, if $P(x)$ is:

a a linear polynomial **b** a quadratic polynomial **c** a cubic polynomial

d a quartic polynomial **e** a polynomial of degree 6 **f** a polynomial of degree n?

11 Consider $P(x) = x^3 - 2x^2 - 5x + 6$.

a Using only the factor theorem, find three linear factors of $P(x)$.

b Solve $P(x) = 0$.

12 Explain why $x^3 + x^2 - 2x - 8 = 0$ has only one real solution.

13 Explain why $x^3 - 5x^2 + 3x + 9 = 0$ has only two real solutions.

14 The volume of a lamington is 192 cm^3. Its length is twice its height, and its width is 2 cm more than its height.

a Write a polynomial to represent the volume of the slice.

b Solve a polynomial equation to find the dimensions of the slice.

15 The volume of a Toblerone chocolate box is 450 cm^3. The height of the box's triangular face is 1 cm less than the length of the base of that face. The perpendicular length of the box is five times the length of the base of the triangular face. Find the dimensions of the box.

16 A polynomial equation has the solutions $x = 1$, $x = -1$, $x = 2$ and $x = -3$.

a Write all possible equations in factorised form, if the polynomial:

i is of degree 4 and its leading coefficient is 1 **ii** is of degree 5 and its leading coefficient is 1

iii is of degree 5 and its leading coefficient is 2 **iv** is of degree 6 and its leading coefficient is 5.

b If the polynomial is of degree n and its leading coefficient is a, where n is a non-zero positive integer and a is a real number, then write a generalised equation for the factorised form.
Hint: Let p, q and r be non-zero positive integers for three of the four possible exponents.

Exercise 5.5C: Challenge

17 $P(x) = x^4 - 2x^3 - 13x^2 + 14x + 24$ has the quadratic factor $(x^2 - x - 2)$. Factorise $P(x)$ and solve $P(x) = 0$.

18 Solve each of these equations.

a $x^5 - x^4 - 17x^3 - 19x^2 + 16x + 20 = 0$ **b** $x^5 - 2x^4 - 15x^3 + 20x^2 + 44x - 48 = 0$

19 Solve the equation $(x + 2)(x - 3)(x + 4) = (x + 2)(x - 10)$.

Online resources:

Interactive skillsheet Solving polynomial equations

CAS instructions Solving equations

Quick quiz 5.5

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Lesson 5.6

Sketching graphs of polynomials using intercepts

Learning intentions

By the end of this lesson you will be able to ...

✔ sketch the graph of cubic and quartic relationships.

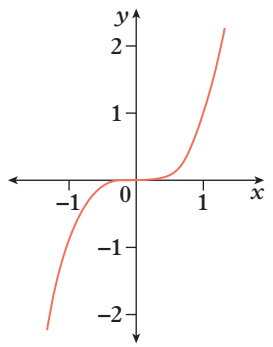
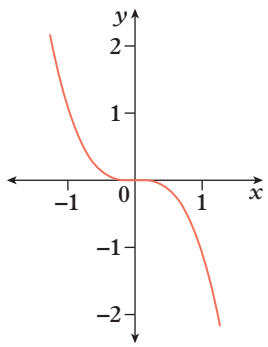
Inter-year links

Year 9 — 4.3 Sketching parabolas using intercepts

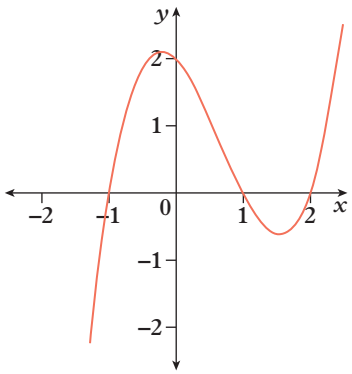
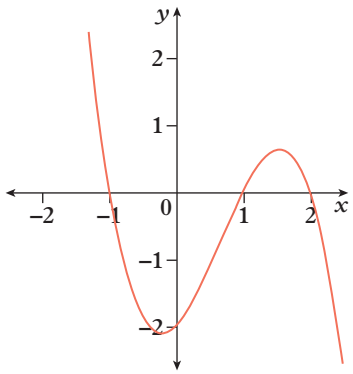
Sketching graphs of cubic relationships

Key content video: Sketching graphs of polynomials using intercepts

- When the leading coefficient of a cubic function is *positive*, the curve starts from the *bottom-left* of the Cartesian plane.
- When the leading coefficient of a cubic function is *negative*, the graph is reflected in the x-axis, and the curve starts from the *top-left* of the Cartesian plane.

$y = x^3$	$y = -x^3$
	

- The graph of the function $P(x) = (x - a)(x - b)(x - c)$ has x-intercepts at $(a, 0)$, $(b, 0)$ and $(c, 0)$, and a y-intercept at $(0, -abc)$.
- The graph of the function $P(x) = -(x - a)(x - b)(x - c)$ is the reflection in the x-axis of $P(x) = (x - a)(x - b)(x - c)$ and has x-intercepts at $(a, 0)$, $(b, 0)$ and $(c, 0)$, and a y-intercept at $(0, abc)$.

$y = (x - 2)(x - 1)(x + 1)$	$y = -(x - 2)(x - 1)(x + 1)$
	

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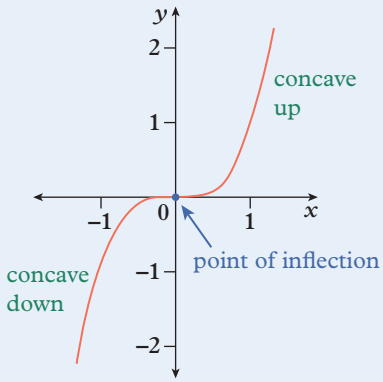
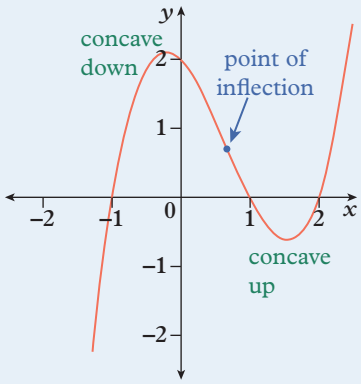
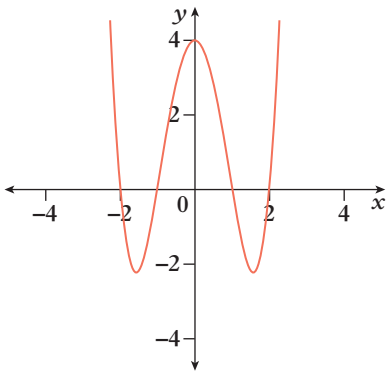
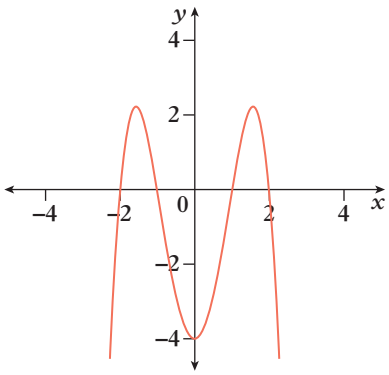
- A **point of inflection** is the point where a curve changes between being 'concave up' and 'concave down'. For example, the point of inflection is shown on each of the curves below.





Sketching graphs of quartic relationships

- The graph of the function $P(x) = (x - a)(x - b)(x - c)(x - d)$ has x-intercepts at $(a, 0)$, $(b, 0)$, $(c, 0)$ and $(d, 0)$, and a y-intercept at $(0, abcd)$. The curve starts from the *top-left* of the Cartesian plane.
- The graph of the function $P(x) = -(x - a)(x - b)(x - c)(x - d)$ is the reflection in the x-axis of $P(x) = (x - a)(x - b)(x - c)(x - d)$. The curve starts from the *bottom-left* of the Cartesian plane.

$y = (x - 2)(x - 1)(x + 1)(x + 2)$	$y = -(x - 2)(x - 1)(x + 1)(x + 2)$
	

- To sketch the graph of a polynomial relationship:
  1. write the polynomial in factor form
  2. determine the orientation of the graph from the sign of the leading coefficient
  3. identify the x-intercepts (find x when $y = 0$)
  4. identify the y-intercept (find y when $x = 0$)
  5. draw a smooth curve through the known points
  6. if necessary, find the coordinates of another point to confirm the orientation of the graph.

Repeated factors

- When a linear factor, $(x - a)$, is repeated in a polynomial, the factor is raised to the power of an exponent greater than 1. For example, in the cubic function $P(x) = (x + 1)(x - 2)^2$, the linear factor $(x - 2)$ is repeated twice. In the quartic function $P(x) = (x - 3)(x + 5)^3$, the linear factor $(x + 5)$ is repeated three times.
- When a linear factor is repeated in $P(x)$, the number of solutions to the equation $P(x) = 0$ is less than the degree of $P(x)$. This reduces the number of x-intercepts in the graph of $y = P(x)$.
- The **multiplicity** of a root, a, in a polynomial, $P(x)$, is the number of times $(x - a)$ is repeated as a factor of $P(x)$. The sum of the multiplicities of the linear factors of a fully factorised polynomial is equal to the degree of the polynomial.

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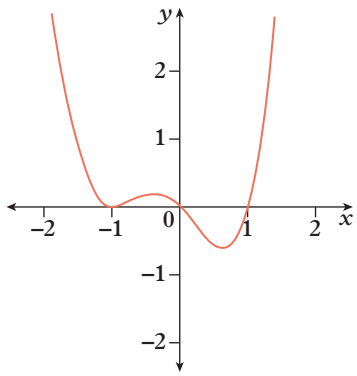
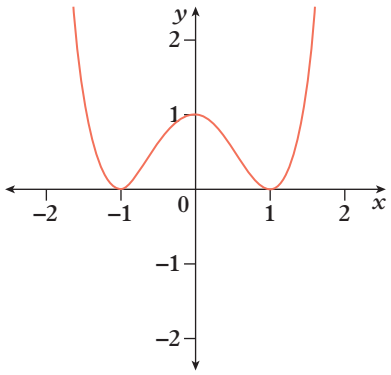
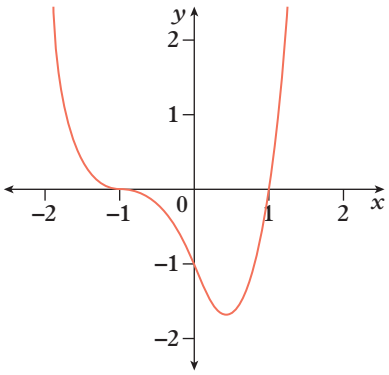
- When a linear factor, $(x - a)$, is repeated in a polynomial, it means that the x-intercept at a has a zero gradient as it passes through the x-axis. For a non-constant polynomial, points with zero gradients correspond to points of inflection and turning points.

  For example, consider how repeated factors affect the shape of the graphs of the quartic relationships in the table below. Look at the x-intercepts corresponding to each repeated factor.

One factor with multiplicity of 2 (one double root)	Two factors with multiplicity of 2 (two double roots)	One factor with multiplicity of 3 (one triple root)
$y = x(x - 1)(x + 1)^2$	$y = (x - 1)^2(x + 1)^2$	$y = (x - 1)(x + 1)^3$
		

Worked example 5.6A Sketching the graph of a cubic relationship

Sketch the graph of each of these cubic relationships.

a $y = (x + 2)(x + 4)(x - 3)$ **b** $y = -x(x - 1)(x + 2)$

THINK

a 1 Substitute $y = 0$ and use the Null Factor Law to find the x-intercepts.

2 Substitute $x = 0$ to find the y-intercept.

3 Mark the four intercepts on a Cartesian plane and draw a smooth curve through them. The leading coefficient is positive, so the curve starts from the bottom-left of the Cartesian plane. Label the graph with its equation.

WRITE

When $y = 0$:

$(x + 2)(x + 4)(x - 3) = 0$

$x + 2 = 0$ or $x + 4 = 0$ or $x - 3 = 0$

$x = -2, -4$ or 3

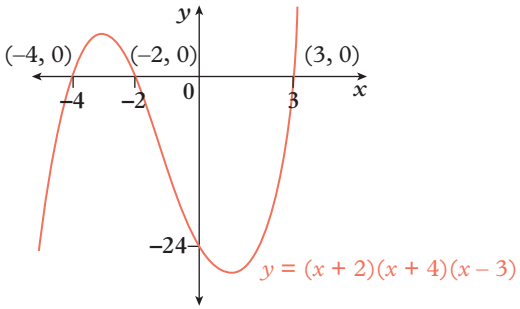
The x-intercepts are at $(-4, 0)$, $(-2, 0)$ and $(3, 0)$.

When $x = 0$:

$y = (2)(4)(-3)$

$= -24$

The y-intercept is at $(0, -24)$.



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b **1** Substitute $y = 0$ and use the Null Factor Law to find the x-intercepts.

When $y = 0$:

$-x(x-1)(x+2) = 0$

$-x = 0$ or $x - 1 = 0$ or $x + 2 = 0$

$x = -2, 0$ or 1

The x-intercepts are at $(-2, 0)$, $(0, 0)$ and $(1, 0)$.

2 Substitute $x = 0$ to find the y-intercept.

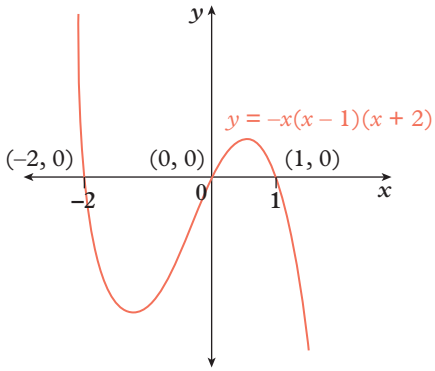
When $x = 0$:

$y = -(0)(-1)(2)$

$= 0$

The y-intercept is at $(0, 0)$.

3 Mark the three intercepts on a Cartesian plane and draw a smooth curve through them. The leading coefficient is negative, so the curve starts from the top-left of the Cartesian plane due to the negative sign. Label the graph with its equation.



Worked example 5.6B Sketching the graph of a quartic relationship

Sketch the graph of $y = (x-2)(x+3)(x+1)(x-4)$.

THINK

1 Substitute $y = 0$ and use the Null Factor Law to find the x-intercepts.

WRITE

When $y = 0$:

$(x-2)(x+3)(x+1)(x-4) = 0$

$x - 2 = 0$ or $x + 3 = 0$ or $x + 1 = 0$ or $x - 4 = 0$

$x = -3, -1, 2$ or 4

The x-intercepts are at $(-3, 0)$, $(-1, 0)$, $(2, 0)$ and $(4, 0)$.

2 Substitute $x = 0$ to find the y-intercept.

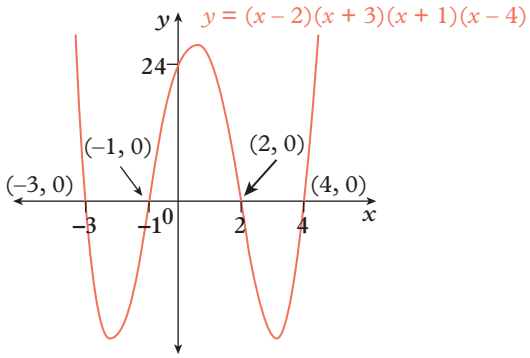
When $x = 0$:

$y = (-2)(3)(1)(-4)$

$= 24$

The y-intercept is at $(0, 24)$.

3 Mark the five intercepts on a Cartesian plane and draw a smooth curve through them. The leading coefficient is positive, so the curve starts from the top-left of the Cartesian plane. Label the graph with its equation.



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Helpful hints

- ✔ In this section, you don't need to plot the graph, only to sketch it. The only points to plot are the intercepts. Ask questions like: 'What happens when x is very large or very small?' and 'Which way does the graph need to turn to pass through all of the intercepts without touching an axis in any other place?'
- ✔ When sketching, remember to use an approximate scale to ensure accuracy.
- ✔ Plot all the intercepts before attempting to sketch the line.

Learning pathways

1–5, 6(a, b, e, g), 7–9, 10(a, b), 13	3–5, 6(c, d, f, h), 8, 9(b, d), 10(c–f), 12, 14	4, 5(d–f), 6(e–h), 9(b, d), 10(b, d, f), 11, 12, 15–17

Exercise 5.6A: Understanding and fluency

WE 5.6A **1** Sketch the graph of each of these cubic relationships.

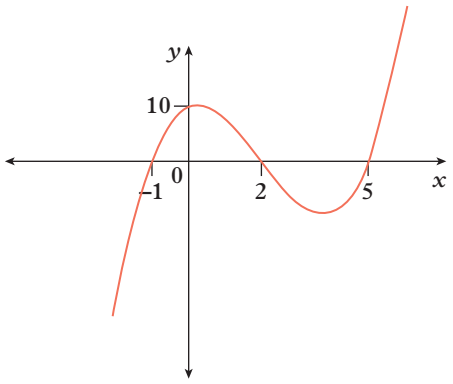
a $y = (x + 1)(x + 3)(x - 1)$ **b** $y = (x - 2)(x - 5)(x + 1)$ **c** $y = (x - 4)(x + 2)(x + 3)$

d $y = (x - 3)(x - 1)(x - 5)$ **e** $y = x(x + 3)(x - 3)$ **f** $y = 2x(x + 4)(x - 1)$

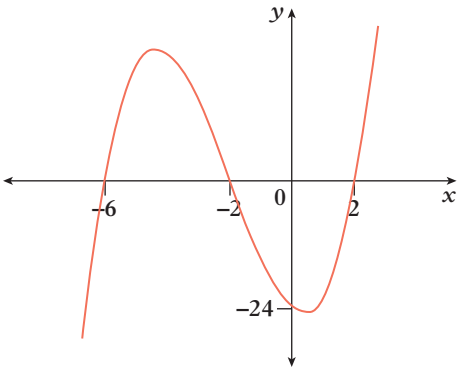
2 For each of the cubic relationships below:

i identify the x-intercepts **ii** identify the y-intercept **iii** write the rule in factorised form.

a



b



3 **a** Find the x- and y-intercepts for:

i $y = (x + 7)(x - 2)(x - 3)$ **ii** $y = -(x + 7)(x - 2)(x - 3)$.

b How is the graph of $y = -(x + 7)(x - 2)(x - 3)$ different from the graph of $y = (x + 7)(x - 2)(x - 3)$?

4 Sketch the graph of each of these cubic relationships.

a $y = -(x + 5)(x - 1)(x - 3)$ **b** $y = -(x - 6)(x + 1)(x - 2)$

c $y = -(x + 2)(x + 3)(x - 5)$ **d** $y = -x(x + 7)(x - 4)$

WE 5.6B **5** Sketch the graph of each of these quartic relationships.

a $y = (x - 1)(x + 2)(x + 3)(x - 3)$ **b** $y = (x + 4)(x + 1)(x - 1)(x + 3)$

c $y = x(x - 2)(x - 6)(x + 5)$ **d** $y = -(x - 3)(x + 3)(x + 1)(x - 1)$

e $y = (2x - 3)(x + 2)(x + 1)(x - 4)$ **f** $y = -(3x - 1)(x + 3)(x + 1)(x + 2)$

6 Sketch the graph of each of these polynomial relationships.
Hint: You will need to first factorise the polynomial.

a $y = x^3 - 3x^2 - 13x + 15$ **b** $y = -x^3 - 2x^2 + 16x + 32$

c $y = x^3 - 2x^2 - 3x$ **d** $y = -2x^3 + 8x^2 - 2x - 12$

e $y = x^4 - 15x^2 - 10x + 24$ **f** $y = x^4 - 9x^3 + 6x^2 + 56x$

g $y = -x^4 + 10x^2 - 9$ **h** $y = -2x^4 - 9x^3 + 18x^2 + 71x + 30$

7 Use a graphing application to check your graphs from question **6**.

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Exercise 5.6B: Problem solving and reasoning

8 Consider the relationship $y = (x + 2)(x - 3)^2$.

a What type of polynomial relationship is this?

b Find the x- and y-intercepts.

c As the leading coefficient of the polynomial is positive, should the graph start from the top-left or bottom-left of the Cartesian plane?

d Sketch the graph. Use a graphing application to verify your answer.

e What effect does the repeated factor of $(x - 3)$ have on the graph?

9 Sketch the graph of each of these cubic relationships.

a $y = (x + 4)(x - 1)^2$

b $y = (x + 2)^2(x - 5)$

c $y = -(x - 2)(x - 7)(x - 2)$

d $y = -(2x + 1)(x - 4)^2$

10 Sketch the graph of each of these quartic relationships.

a $y = (x - 3)(x + 2)(x + 1)^2$

b $y = -(x - 2)^2(x - 4)(x + 2)$

c $y = -x(x - 2)(x + 3)^2$

d $y = x^2(x + 5)(x - 3)$

e $y = (x - 1)^2(x + 2)^2$

f $y = -x^2(x - 4)^2$

11 A squared factor in a polynomial relationship means that an x-intercept is also a turning point. Investigate the effect of having a cubed factor in a quartic relationship. That is, find what happens at $x = a$ for a graph of the form $y = (x - a)^3(x - b)$. Use a graphing application to try different values for a and b.

12 Equations are used to model situations in real life. The subject of the equation is called the 'dependent variable', and the variable that the equation is in terms of is called the 'independent variable'. When graphing the equation, the independent variable is represented on the horizontal axis, and the dependent variable is represented on the vertical axis.

Consider the following modelling question.

The motion of a person on a particular water slide can be approximated using the cubic relationship $h(t) = -\frac{1}{5}(t^3 - 11t^2 + 39t - 45)$, where h is the height above the ground, in metres, and t is the time, in seconds, since the start of the slide.

a Which variable (h or t) is the independent variable, and which is the dependent variable? Explain your reasoning.

b Sketch a graph of this relationship.

c At what height above the ground does a person start the slide?

d How high above the ground is a person after 1 second?

e The ride descends to its lowest point. How long does this take?

f How long does it take for the slide to ascend from its lowest point and then descend again to its lowest point?

13 Amelia monitors the change in the value of shares during the month of June. She finds that the change in value, V, (in dollars) after x days can be approximated using a cubic relationship, where y increases from \$0 to \$21 after 5 days and is zero again after 12 days and 20 days.

a Sketch a graph of this cubic relationship.

b Find the equation for this relationship.

c Use this relationship to estimate the change in the value of the shares by the end of June.

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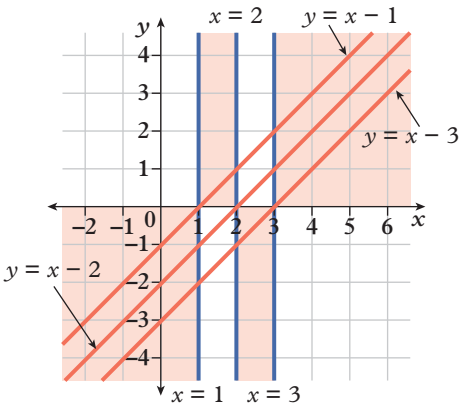
14 Shown on the right are the graphs of the lines $y = x - 1$, $y = x - 2$ and $y = x - 3$, as well as the vertical lines $x = 1$, $x = 2$ and $x = 3$. The shaded regions show where the function of the product of the lines, $y = (x - 1)(x - 2)(x - 3)$, will lie.



a The table provided below shows if a line will have positive or negative y-values for the given x-values. Copy the table and complete it by writing 'positive' or 'negative' in each space.

	$x < 1$	$1 < x < 2$	$2 < x < 3$	$x > 3$
$y = x - 1$	negative			
$y = x - 2$				
$y = x - 3$				
$y = (x - 1)(x - 2)(x - 3)$				

b Describe why regions of the graph are shaded.

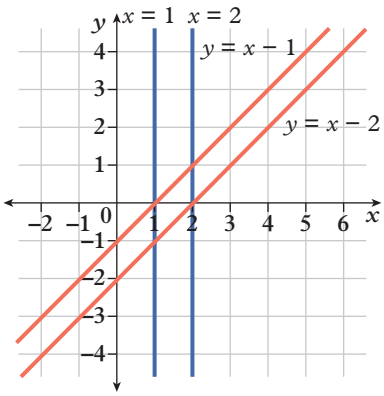
c The table lists inequalities that do not include the values for $x = 1$, $x = 2$, and $x = 3$.

i What is the value of the product $y = (x - 1)(x - 2)(x - 3)$ for these x-values?

ii What feature(s) of the graph of $y = (x - 1)(x - 2)(x - 3)$ do these x-values correspond with?

d Use the information you found in part **c** to help you sketch the graph of $y = (x - 1)(x - 2)(x - 3)$. Label all axis intercepts.

e Shown below are the graphs of the lines $y = x - 1$ and $y = x - 2$, as well as the vertical lines $x = 1$ and $x = 2$.



Copy the table below and complete it by writing 'positive' or 'negative' in each space.

	$x < 1$	$1 < x < 2$	$x > 2$
$y = x - 1$			
$y = x - 2$			
$y = x - 2$			
$y = (x - 1)(x - 2)^2$			

f Copy the graphs on the previous page in part **e** and shade the regions in which the graph of $y = (x - 1)(x - 2)^2$ will lie. Explain your reasoning.

g Sketch the graph of $y = (x - 1)(x - 2)^2$. Label all axis intercepts.

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15 This question explores higher-order polynomials.

a Consider graphs of the form $y = x^n$ and $y = -x^n$, where n is an odd positive integer.

i Sketch $y = x$, $y = x^3$ and $y = x^5$ on the same set of axes.

ii Sketch $y = -x$, $y = -x^3$ and $y = -x^5$ on the same set of axes.

iii By considering the extreme ends of the graphs from parts **i** and **ii**, describe the rough shape of the graphs of polynomials of the form $y = x^n$ and $y = -x^n$ when n is odd.

b Consider graphs of the form $y = x^n$ and $y = -x^n$, where n is an even positive integer.

i Sketch $y = x^2$, $y = x^4$ and $y = x^6$ on the same set of axes.

ii Sketch $y = -x^2$, $y = -x^4$ and $y = -x^6$ on the same set of axes.

iii By considering the extreme ends of the graphs from parts **i** and **ii**, describe the rough shape of the graphs of polynomials of the form $y = x^n$ and $y = -x^n$ when n is even.

c Use a graphing application to sketch the following graphs. Do *not* find or plot any intercepts.

i $y = x^2 + x$

ii $y = x^3 + x^2 + x$

iii $y = x^4 + x^3 + x^2 + x$

iv $y = x^5 + x^4 + x^3 + x^2 + x$

v $y = x^6 + x^5 + x^4 + x^3 + x^2 + x$

vi $y = x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x$

d Describe the rough shape of polynomials with a leading term of x^n and $-x^n$, when n is odd or even.

Exercise 5.6C: Challenge

16 The first step for factorising polynomials requires the use of the factor theorem.

a Zaya wishes to factorise $P(x) = x^3 - 10x^2 - 11x + 180$ so she can sketch $y = P(x)$. So far, she has found the following: $P(-6) = -330$, $P(-2) = 154$, $P(2) = 126$, $P(6) = -30$ and $P(10) = 70$.

i How many x-intercepts does the graph of $y = P(x)$ have?

ii If Zaya knows that the x-intercepts have integer x-values, then list a possible set of solutions for $P(x) = 0$.

iii Which number(s) can be ruled out from the set? Explain why.

b Sara wishes to factorise $Q(x) = x^4 - 6x^3 - x^2 + 54x - 72$ so she can sketch $y = Q(x)$. So far, she has found the following: $Q(1) = -24$, $Q(-1) = -120$, $Q(2) = 0$, $Q(-2) = -120$, $Q(3) = 0$, $Q(-3) = 0$, $Q(4) = 0$ and $Q(-4) = 336$.

i State the coordinates of the x-intercepts of $y = Q(x)$.

ii Explain how Sara could use the remaining substitutions to sketch the graph of $y = Q(x)$ more accurately.

17 This question explores the factor form of higher-order polynomials with repeated factors.

a Sketch a rough graph of $y = x(x - 6)(x - 3)(x + 2)(x + 4)(x + 6)$.
You do not need to label any intercepts.

b Sketch a rough graph of $y = (x - 8)(x - 5)^3(x + 1)^2(x + 5)^3(x + 10)^2$.
You do not need to label any intercepts.

Online resources:

- **Interactive skillsheet** Sketching graphs of cubic relationships
- **Worksheet** Solving polynomial equations and graphing polynomial relationships
- **Investigation** Maximising the volume of a box
- **Quick quiz** 5.6
- **Interactive skillsheet** Sketching graphs of quartic relationships
- **Desmos activity** Sketching graphs of polynomials using intercepts
- **Investigation** Exploring repeated roots
- **CAS instructions** Graphing functions

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Lesson 5.7

Sketching graphs of polynomials using transformations

Learning intentions

By the end of this lesson you will be able to ...

✔ perform transformations on the graph of a polynomial relationship.

Inter-year links

Year 9 — 4.4 Sketching parabolas using transformations

Transformations of a polynomial relationship

Key content video: Sketching graphs of polynomials using transformations

- Transformations such as dilations, reflections and translations can be performed on the graph of $y = P(x)$.
  - → A dilation by a factor of a in the direction of the y-axis produces $y = aP(x)$.
  - → A reflection in the x-axis produces $y = -P(x)$.
  - → A reflection in the y-axis produces $y = P(-x)$.
  - → A vertical translation of k units produces $y = P(x) + k$.
  - → A horizontal translation of h units produces $y = P(x - h)$.
- A combination of transformations can be performed on the graph of $y = P(x)$ to produce the graph of $y = aP(n(x - h)) + k$.

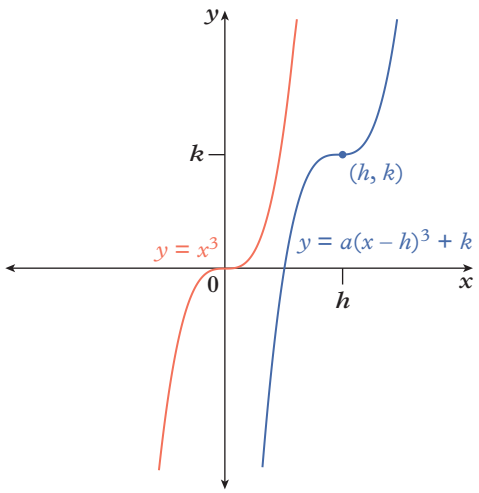
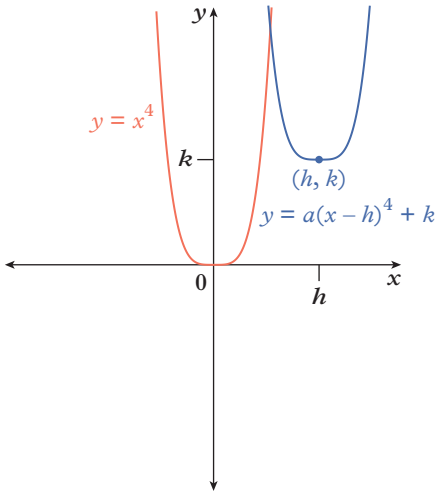
$$y = aP(n(x - h)) + k$$

Vertical translation of k units
For $k > 0$, move up
For $k < 0$, move down

Dilation by factor of a in the y-direction
For $a < 0$, reflect in the x-direction

For $n = -1$, reflect in the y-axis

Horizontal translation of h units
For $h > 0$, move right
For $h < 0$, move left

Cubic relationships	Quartic relationships
	

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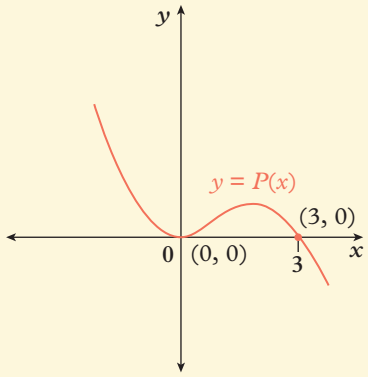
Worked example 5.7A Performing transformations on the graph of a polynomial relationship

Perform a transformation on the graph of $y = P(x)$ (shown on the right) to produce the graph of:

a $y = P(x) + 2$

b $y = P(x + 2)$

c $y = -P(x)$.



THINK

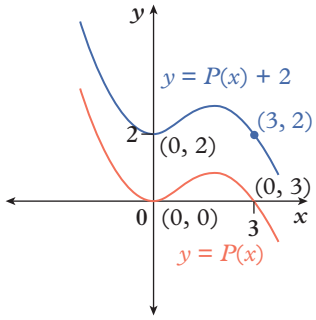
a 1 Identify the transformation. The original graph needs to be translated 2 units up.

2 Determine the impact of the transformation on the intercepts: (0, 0) moves to (0, 2) and (3, 0) moves to (3, 2).

3 Draw $y = P(x)$ and $y = P(x) + 2$ on the same Cartesian plane.

WRITE

a The graph of $y = P(x)$ will be translated 2 units up.

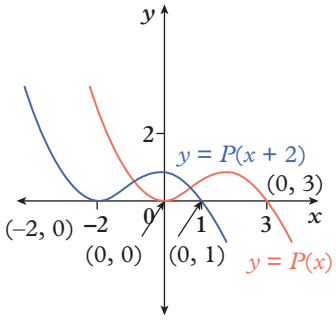


b 1 Identify the transformation. The original graph needs to be translated 2 units left.

2 Determine the impact of the transformation on the intercepts. (0, 0) moves to (–2, 0) and (3, 0) moves to (1, 0).

3 Draw $y = P(x)$ and $y = P(x + 2)$ on the same Cartesian plane.

b The graph of $y = P(x)$ will be translated 2 units left.

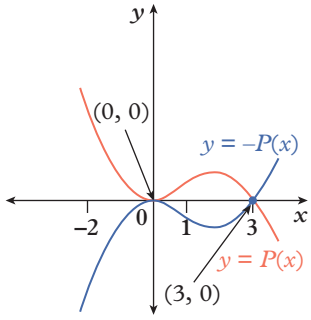


c 1 Identify the transformation. The original graph needs to be reflected in the x-axis.

2 Determine the impact of the transformation on the intercepts: (0, 0) and (3, 0) remain the same.

3 Draw $y = P(x)$ and $y = -P(x)$ on the same Cartesian plane.

c The graph of $y = P(x)$ will be reflected in the x-axis.



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Helpful hints

- ✔ You will only become familiar with the shape and turns of a polynomial by sketching them manually. Practice makes permanent!
- ✔ It can be easy to miss the negative sign at the front of a function. Be careful to pay attention, because a negative factor has a large impact on the shape of the graph. It completely flips it!

Learning pathways

1–7, 9, 10, 12	2–8, 11, 12, 13(a)	2, 5–8, 11, 13–15

Exercise 5.7A: Understanding and fluency

WE 5.7A **1** Perform a transformation on the graph of $y = P(x)$ (shown below) to produce the graph of:

a $y = P(x) + 1$

b $y = P(x) - 2$

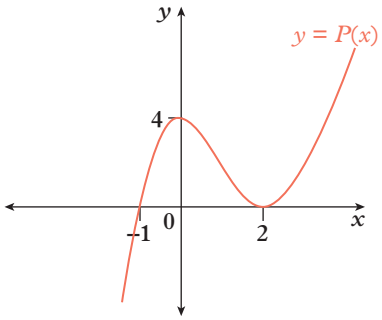
c $y = P(x - 1)$

d $y = P(x + 3)$

e $y = -P(x)$

f $y = 2P(x)$

g $y = P(-x)$.



2 Perform a transformation on the graph of $y = P(x)$ (shown below) to produce the graph of:

a $y = P(x) + 3$

b $y = P(x) - 1$

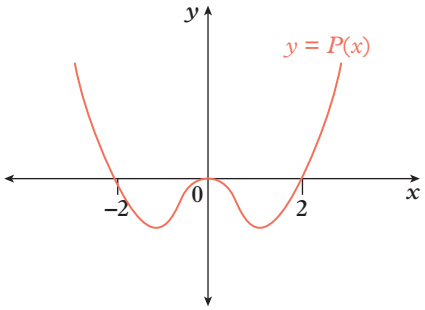
c $y = P(x - 3)$

d $y = P(x + 4)$

e $y = -P(x)$

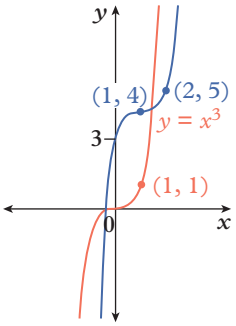
f $y = 2P(x)$

g $y = P(-x)$.

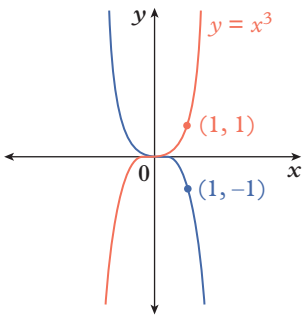


3 Identify the transformations performed on the graph of $y = x^3$ to produce each graph shown in blue below.

a



b



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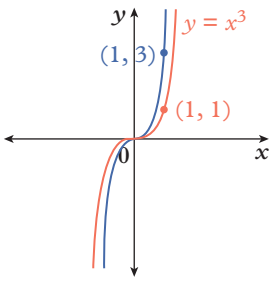
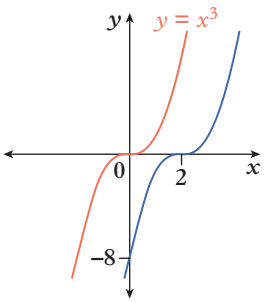
c



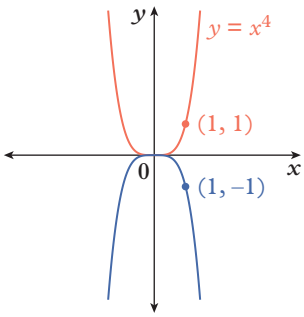
d



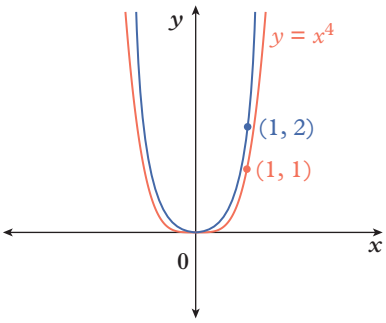
4 Write the equation for each of the transformed graphs in question **3**.

5 Identify the transformations performed on the graph of $y = x^4$ to produce each graph shown in blue below.

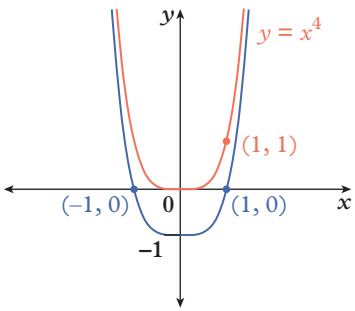
a



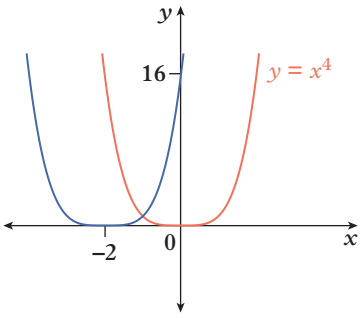
b



c



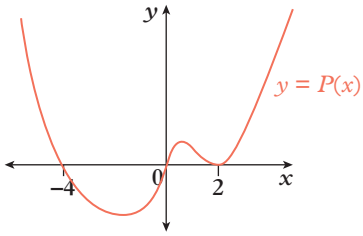
d



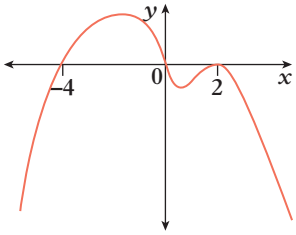
6 Write the equation for each of the transformed graphs in question **5**.

Exercise 5.7B: Problem solving and reasoning

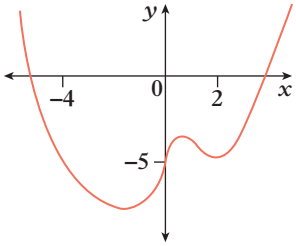
7 Consider the graph of $y = P(x)$ shown on the right. Identify the equation for each of the graphs below, in terms of $P(x)$.



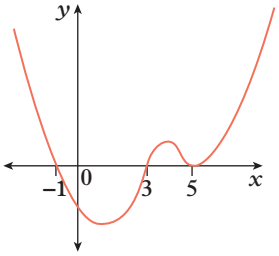
a



b



c



8 Compare the graph produced after reflecting $y = x^3$ in the x-axis with the graph produced after reflecting $y = x^3$ in the y-axis. What is the equation for each of these transformed graphs?

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9 For each cubic relationship below:

i describe the transformations that need to be performed on $y = x^3$ to produce the graph of the relationship

ii identify the coordinates of the point of inflection

iii find the x- and y-intercepts

iv sketch the graph.

a $y = 8x^3 + 1$ **b** $y = \frac{1}{2}x^3 - 4$ **c** $y = \frac{1}{2}(x - 3)^3 + 4$ **d** $y = -2(x + 1)^3 - 2$

10 Identify the coordinates of the point of inflection and the x-intercept for cubic relationships of the form $y = a(x - h)^3 + k$.

11 Dilations can also occur in the direction of the x-axis by multiplying the x-coordinates instead of the y-coordinates. Consider the table of values for a polynomial shown on the right.

x	0	1	2	3
$y = P(x)$	5	0	−5	2

When dilating in the direction of the x-axis, we replace x with a new expression, then work backwards to find the x-values for the transformed graph. This way, the y-coordinates corresponding to the original x-values now correspond with the transformed x-values.

a Complete each table of values below by performing the appropriate operations to the x-coordinates.

i

$\frac{x}{2}$	0	1	2	3
x				
$y = P\left(\frac{x}{2}\right)$	5	0	−5	2

ii

$2x$	0	1	2	3
x				
$y = P(2x)$	5	0	−5	2

iii

$\frac{x}{5}$	0	1	2	3
x				
$y = P\left(\frac{x}{5}\right)$	5	0	−5	2

iv

$5x$	0	1	2	3
x				
$y = P(5x)$	5	0	−5	2

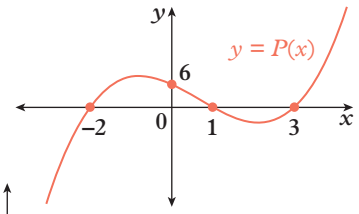
v

$-\frac{x}{2}$	0	1	2	3
x				
$y = P\left(-\frac{x}{2}\right)$	5	0	−5	2

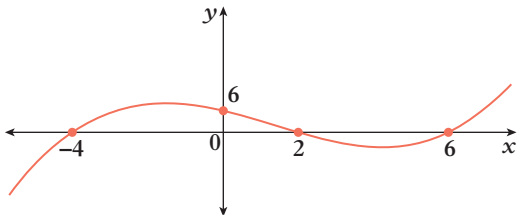
vi

$-2x$	0	1	2	3
x				
$y = P(-2x)$	5	0	−5	2

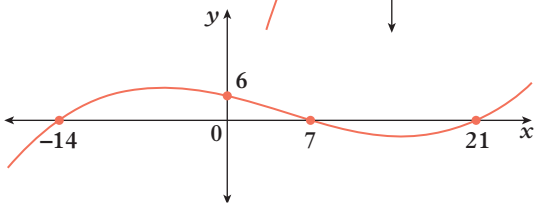
b Using the graph of $y = P(x)$ shown on the right, identify the rule for each of the graphs below in terms of $P(x)$.



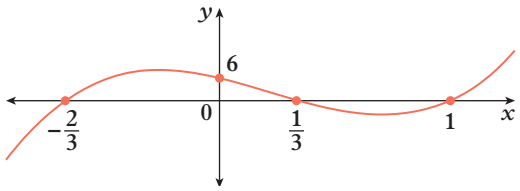
i



ii



iii



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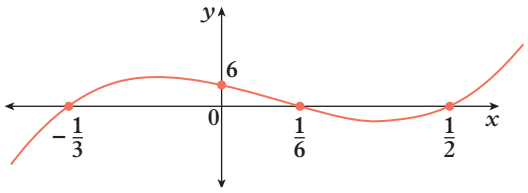

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12 Dylan is sketching transformations of the graph of the polynomial $y = P(x)$. He has written the transformation he will apply for each of the following. State whether his descriptions are correct or incorrect. For any incorrect description, provide the correct transformation.

a $y = P(x) + 6$; translate 6 units up

b $y = 4P(x)$; dilate by a factor of 4 in the direction of the x-axis

c $y = P(x - 5)$; translate 5 units left

d $y = P(3x)$; dilate by a factor of $\frac{1}{3}$ in the direction of the x-axis

13 A transformation of a polynomial, $P(x)$, involving horizontal dilations can be written in the form:

$$y = aP(n(x - h)) + k.$$

The following sequence of transformations was performed on the polynomial $P(x)$.

- dilation by a factor of a in the direction of the y-axis
- dilation by a factor of $\frac{1}{n}$ in the direction of the x-axis
- reflection in the x-axis if $a < 0$
- reflection in the y-axis if $n < 0$
- translation h units to the right and k units up

a For each of the following polynomials, write a sequence of transformations on the polynomial $P(x)$ which would result in the given polynomial.

i $y = 2P(3x) + 324$ **ii** $y = \frac{1}{3}P\left(\frac{x}{2}\right)$ **iii** $y = 5P(-x) - 15$ **iv** $y = -2P(-4x) + 12$

b The polynomial $y = P(x)$ is transformed according to the following sequence of transformations. Identify the equation of the transformed polynomial in terms of $P(x)$.

- dilation by a factor of 3 in the direction of the x-axis
- reflection in the x-axis
- translation 2 units to the left and 4 units down

c Consider the transformed polynomial $y = -P\left(\frac{1}{3}x + 2\right) - 4$, which should look similar to your answer to part **b**. Factorise the expression inside the brackets in the form $n(x - h)$, and then write a sequence of transformations that could be applied to $P(x)$ to obtain the transformed polynomial.

d Use parts **b** and **c** to explain why it is helpful to write transformations in function notation with the expression inside the brackets factorised.

Exercise 5.7C: Challenge

14 Consider the function $f(x) = (x - 2)^3 + 27$.

a Expand the rule for the function f.

b Using your answer from part **a**, sketch the following graphs.

i $y = 4x^3 - 24x^2 + 48x + 76$ **ii** $y = (x - 2)^3 - 6(x - 2)^2 + 12(x - 2) + 21$

iii $y = -x^3 - 6x^2 - 12x + 19$

15 For polynomials of the form $y = x^n$, a dilation in the x-axis is equivalent to a different dilation in the y-axis. This equivalence can be determined algebraically by expanding or factorising the expression.
For example, $y = (2x)^3$ is a dilation of $y = x^3$ by a factor of $\frac{1}{2}$ in the direction of the x-axis.
When expanded, $y = (2x)^3 = 8x^3$ is a dilation of $y = x^3$ by a factor of 8 in the direction of the y-axis.
For each of the following functions:

i $y = (3x)^2$ **ii** $y = \left(\frac{x}{2}\right)^4$ **iii** $y = 25x^2$ **iv** $y = \frac{625}{81}x^4$

a describe the equivalent dilations in both axes of the graph of $y = x^n$ for it to be transformed to the graph of each of the above polynomials

b describe the equivalent dilations in both axes of the graph of each of the polynomials above for it to be transformed to the graph of $y = x^n$.

Online resources:

Interactive skillsheet Sketching polynomials using transformations

Investigation Transforming polynomials

Quick quiz 5.7

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Lesson 5.8

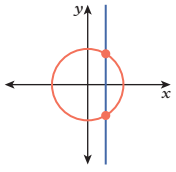
Review: Polynomials

Module summary

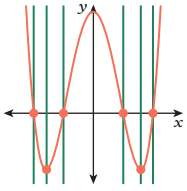
Functions and relations

- A relation is a map between two sets.
- A function is a relation where each element in the domain is mapped to exactly one element in the range.
- Graphs of functions will pass the vertical line test.

→ This graph fails the vertical line test; it is only a relation.



→ This graph passes the vertical line test; it is a function (as well as a relation).



The distributive law

$$(x^2 + x + 1)(x^2 - 2x - 3) = x^2(x^2 - 2x - 3) + x(x^2 - 2x - 3) + 1(x^2 - 2x - 3)$$

Division of polynomials

$$\begin{array}{r} x + 2 \\ \text{divisor} \rightarrow x + 1 \overline{\big) x^2 + 3x + 4} \\ -(x^2 + x) \\ \hline 2x + 4 \\ -(2x + 2) \\ \hline 2 \end{array}$$

← quotient

← dividend

← remainder

$$\frac{\overbrace{x^2 + 3x + 4}^{\text{dividend}}}{\underbrace{x + 1}_{\text{divisor}}} = \overbrace{x + 2}^{\text{quotient}} + \frac{\overbrace{2}^{\text{remainder}}}{\underbrace{x + 1}_{\text{divisor}}}$$

The remainder theorem

When a polynomial, $P(x)$, is divided by $(x - a)$, the remainder is $P(a)$.

The factor theorem

When a polynomial, $P(x)$, is divided by $(x - a)$ and the remainder $P(a)$ is zero, then $(x - a)$ is a factor of $P(x)$.

The Null Factor Law

If $P(x) = (x - a)(x - b)(x - c)$, then the solutions for $P(x) = 0$ are $x = a, b, c$.

$$(x - a)(x - b)(x - c) = 0$$

$$x - a = 0 \quad \Rightarrow \quad x = a$$

$$x - b = 0 \quad \Rightarrow \quad x = b$$

$$x - c = 0 \quad \Rightarrow \quad x = c$$

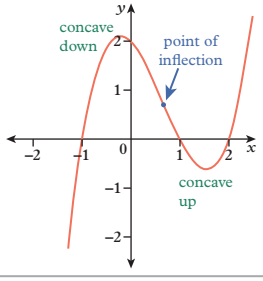
Polynomials

$$\overbrace{P(x) = \underbrace{\underbrace{x^2 + 2x + 1}_{\text{expression}} = 0}_{\text{equation}}}^{\text{function}}$$

- 1 is a degree 0 polynomial.
- x is a degree 1 polynomial.
- x^2 is a degree 2 polynomial.
- x^3 is a degree 3 polynomial.

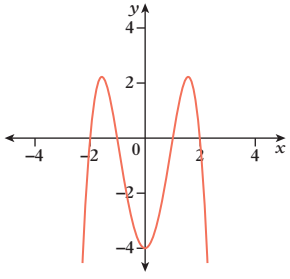
Cubic polynomials

$$P(x) = (x - 2)(x - 1)(x + 1)$$



Quartic polynomials

$$P(x) = -(x - 2)(x - 1)(x + 1)(x + 2)$$



Sketching polynomials using intercepts

1. Write the polynomial in factorised form.
2. Determine the orientation of the graph.
3. Identify the x- and y-intercepts.
4. Draw a smooth curve through the known points.
5. If necessary, find the coordinates of another point to confirm the orientation of the graph.
6. Check the multiplicities of linear factors in the polynomial are correctly represented in the graph.

Sketching polynomials using transformations

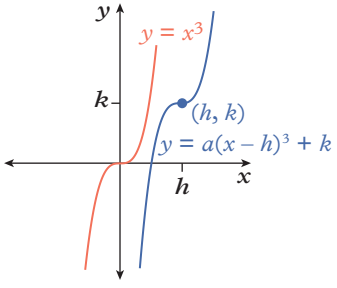
$$y = aP(n(x - h)) + k$$

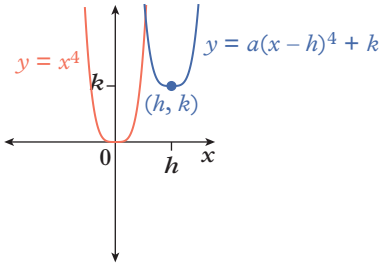
Vertical translation of k units
For $k > 0$, move up
For $k < 0$, move down

Dilation by factor of a in the y-axis
For $a < 0$, reflect in the x-axis

Horizontal translation of h units
For $h > 0$, move right
For $h < 0$, move left

For $n = -1$, reflect in the y-axis





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Review questions 5.8A: Mathematical literacy review

Review quiz
Take the review quiz to assess your knowledge of this module.

Quizlet
Test your knowledge of this lesson by working individually or in teams.

The following key terms are used in this module:

- co-domain
- coefficient
- constant term
- degree (of a polynomial)
- dilation
- dividend
- divisor
- domain
- element
- factor theorem
- function
- function notation
- leading coefficient
- leading term
- monic polynomial
- multiplicity
- non-monic polynomial
- Null Factor Law
- point of inflection
- polynomial
- quotient
- range
- reflection
- relation
- remainder
- remainder theorem
- repeated factor
- root
- set
- term
- transformation
- translation
- vertical line test
- zero (of a polynomial)

1 Which of the following expressions are polynomials?

a $x^2 + 2x + 1$ **b** $x^{-2} - 2$ **c** $\dfrac{1}{(x^3 + 3x)}$

d $\log_2(x + 1)$ **e** 2 **f** $y^3 - 2xy + y^2 - x$

2 For the polynomial $3x^6 - 4x^3 + 2x^2 - 1$, identify:

a the number of terms **b** the degree of the polynomial

c the constant term **d** the leading term

e the leading coefficient **f** the coefficient of the x^3 term.

3 It is important to be able to identify relations as functions.

a Write the definition of a function. **b** Give an example of a relation which is not a function.

4 How does a relation fail the vertical line test?

5 Complete the following sentences using words from the list of key terms.

a When dividing one polynomial by another polynomial, the _____ is the expression being divided by the _____ and the _____ is the resulting expression.

b The _____ of a _____ polynomial is equal to 1 and the _____ of a _____ polynomial is not equal to 1.

c A _____ of a polynomial, $P(x)$, is a _____ of the equation $P(x) = 0$.

d The _____ of a polynomial is the exponent of the variable in the _____ term.

Review questions 5.8B: Multiple choice

10A 5.1 **1** Which of these relations is not a function?

A $y = (x - 2)^2 + 2$ **B** $y = \dfrac{2}{x - 2} + 2$ **C** $y = x^6 + x^5 + x^4 + x^3$ **D** $x^2 + y^2 = 1$

10A 5.2 **2** Which of these expressions is a polynomial?

A $x^2 + \sqrt{x}$ **B** $\dfrac{3x}{x^2 + 1}$ **C** $x^3 - 2x$ **D** $x^2 + x^{-1}$

10A 5.2 **3** The degree of the polynomial $4x^2 - 3x^4 + x^3 - 2x$ is:

A 1 **B** 2 **C** 3 **D** 4

10A 5.2 **4** The coefficient of the leading term in the expression $3 - 2x^2 + 5x^3 + 7x$ is:

A 5 **B** −2 **C** 3 **D** 7

10A 5.3 **5** When $x^2 + 6x - 3$ is divided by $x - 2$, the remainder is:

A 13 **B** −19 **C** 5 **D** −11

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10A 5.3 **6** Which of these is correct?

A $(3x^2 - 2x + 1) \div (x - 1) = 3x - 5$, remainder -9

B $(4x^2 - x + 8) \div (x - 2) = 4x - 9$, remainder -10

C $(2x^2 - x + 1) \div (x - 5) = 2x + 9$, remainder 37

D $(6x^2 - 13x + 15) \div (x - 3) = 6x + 5$, remainder 30

10A 5.4 **7** When $P(x) = x^3 - 2x^2$ is divided by $(x - 1)$, the remainder is:

A $P(0)$ **B** $P(1)$ **C** $P(2)$ **D** $P(-1)$

10A 5.4 **8** Which of these expressions is a factor of $x^3 - 4x^2 + x + 6$?

A $x - 1$ **B** $x - 4$ **C** $x - 2$ **D** $x + 6$

10A 5.5 **9** The leading term in a polynomial is raised to the power n. The maximum number of solutions it could have is:

A $n - 1$ **B** $n + 1$ **C** n **D** $\frac{n}{2}$

10A 5.5 **10** How many different solutions does $(x + 1)(x - 1)(x + 2)(x + 1) = 0$ have?

A 1 **B** 2 **C** 3 **D** 4

10A 5.6 **11** The graph of $y = (x - 3)(x + 2)(x - 5)$ has a y-intercept of:

A 30 **B** -2 **C** 3 **D** -30

10A 5.7 **12** The graph of $y = (x - 1)^3$ has a point of inflection at:

A $x = -1$ **B** $x = 1$ **C** $x = 0$ **D** $y = 1$

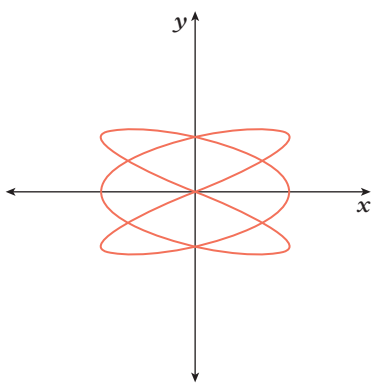
10A 5.7 **13** The transformation that needs to be performed on the graph of $y = x^3$ to produce the graph of $y = (x + 5)^3$ is a translation of:

A 5 units right **B** 5 units left **C** 5 units up **D** 5 units down

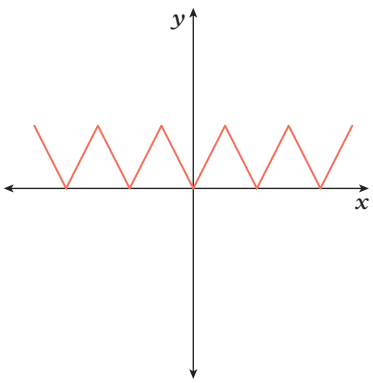
Review questions 5.8C: Short answer

10A 5.1 **1** Determine which of the following relations are functions.

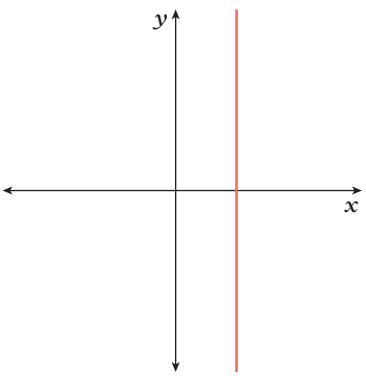
a



b



c



10A 5.1 **2** For each of the following functions, evaluate: **i** $f(0)$, **ii** $f(5)$ and **iii** $f(-2)$.

a $f(x) = x^2 + 1$ **b** $f(x) = x^3 - x + 2^x$ **c** $f(x) = -x^3 + \frac{1}{x + 1}$

10A 5.2 **3** Decide whether each of these expressions is a polynomial. For those that are polynomials, correctly identify the type of polynomial using its descriptive name: linear, quadratic, cubic or quartic.

a $5 - 3x^2 + 4x - 6x^3$ **b** $1 - 3x$

c $\frac{4x}{5}$ **d** $\frac{6}{7x}$

e $6x^2 - 5x + 12$ **f** $\frac{1}{2}x - 3x^4$

10A 5.2 **4** For the polynomial $3x^4 + 2x^6 - 4x^5 + x - 8x^3 - 7x^2 - 5x^8$, identify:

a the number of terms it has **b** its degree

c the constant term **d** the leading term

e the leading coefficient **f** the coefficient of the x^2 term.

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10A 5.3 **5** Use long division to find the quotient and remainder for each of the following.

a $(x^2 + 5x - 2) \div (x - 2)$ **b** $(3x^2 - x + 4) \div (x + 1)$

c $(x^2 - x + 8) \div (x - 4)$ **d** $(x^3 + 2x^2 - x + 3) \div (x^2 + 1)$

10A 5.3 **6** Expand and simplify the right-hand side to verify whether each of these statements is true.

a $2x^2 + 5x - 1 = (2x + 3)(x + 1) - 4$ **b** $x^2 - 3x + 5 = (x - 1)(x - 2) + 7$

10A 5.4 **7** Use the factor theorem to find a linear factor of each of the following polynomials, $P(x)$.

a $P(x) = x^3 - x^2 - 5x - 3$ **b** $P(x) = x^3 + 4x^2 + 7x + 12$

c $P(x) = x^3 + 3x^2 - 6x + 2$ **d** $P(x) = x^3 + x^2 + 4$

10A 5.4 **8** Fully factorise each of the following polynomials.

a $x^3 - 7x^2 + 14x - 8$ **b** $x^3 - 7x - 6$

10A 5.4 **9** Two factors of the polynomial $x^3 - 39x - 70$ are $(x - 7)$ and $(x + 5)$. What is the third factor?

10A 5.5 **10** Solve each of the following.

a $(x - 5)(x + 2)(x - 1)(x + 1) = 0$ **b** $x(x + 3)(x - 3)(x - 2) = 0$

c $(x + 3)^2(x - 5)^2 = 0$ **d** $x^2(x + 4)(x - 4) = 0$

10A 5.5 **11** Fully factorise and solve each of the following.

a $x(x - 2)(x^2 - 4) = 0$ **b** $(x - 4)(x^2 - 8x + 16) = 0$

10A 5.6 **12** Sketch the following polynomials.

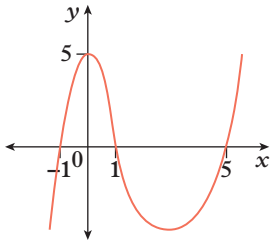
a $y = (x - 3)(x + 1)(x - 2)$ **b** $y = -(x + 2)^2(x - 4)$

10A 5.6 **13** For the cubic graph shown on the right:

a identify the x-intercepts

b identify the y-intercept

c write the equation as a product of factors.



10A 5.6 **14** Consider $y = x^4 - 5x^2 + 4$.

a Write this relationship in terms of its factors.

b Identify the x- and y-intercepts.

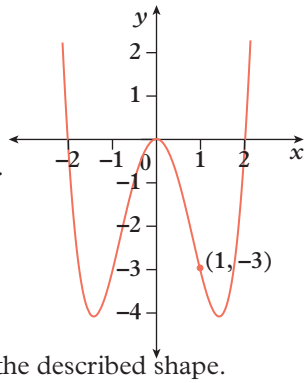
c Sketch the graph.

10A 5.6 **15** A quartic graph is shown on the right.

a Write its equation in:

i factor form **ii** expanded form.

b Determine the multiplicity of each root of the quartic relationship.



10A 5.6 **16** Consider the graph of $y = x(x - 1)^3$.

a Identify the x- and y-intercepts.

b Determine the multiplicity of each root of $y = x(x - 1)^3$.

c Sketch the graph.

10A 5.6 **17** For each of the following, write the equation of a graph that satisfies the described shape.

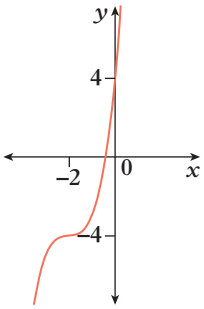
a a cubic polynomial with one double root which passes through the origin

b a quartic polynomial with two double roots whose range is $y \leq 0$

c a cubic polynomial with one triple root which passes through the point $(3, 0)$

d a quartic polynomial with one double root which passes through the origin and the point $(-1, 0)$.

10A 5.7 **18** Describe the transformation(s) performed on the graph of $y = x^3$ to produce the graph shown on the right.



10A 5.7 **19** Let $P(x)$ be the quartic polynomial relationship in question **15**. Sketch the graphs of the following transformations of $P(x)$.

a $P(x + 1)$

b $P(-x)$

c $-P(x)$

d $-2P(x)$

e $P(x) + 4$

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Review questions 5.8D: Mathematical modelling

10A **1** You should be familiar with the infinity symbol. It looks like the number 8 on its side. Its shape is similar to the graph of a cubic function with three x-intercepts, combined with its reflection in the x-axis.

a It is possible to model the infinity sign using two cubic relationships sketched over the same set of x-values.

i Copy and complete the table below for $y = x^3 - x$.

x	y
–1	
–0.6	
–0.5	
0	
0.5	
0.6	
1	

ii Repeat part **i** for $y = -x^3 + x$.

iii On a Cartesian plane, plot the points from the tables in parts **i** and **ii**. Join the points with a smooth curve.

iv Describe the domain of the two graphs.

v Do the turning points for the two relationships occur at $x = -0.5$ and $x = 0.5$? Explain.

vi Use a graphing application to sketch the graphs of these two relationships for the set of x-values described in part **iv**. Find the coordinates of the turning points.

b It is also possible to model the infinity symbol by tracing the path a point takes as it moves from an angle of 0° with the x-axis to an angle of 360° with the x-axis.

The coordinates of all points (x, y) on the infinity symbol can be represented by the values $\left(\cos[\theta], \frac{\sin[2\theta]}{2}\right)$, where θ is the angle made with the x-axis by the line joining a point with the origin.

i Construct a table like the one below for x and y, using angles of θ increasing by 15° between 0° and 360°. (This number of points is necessary for an accurate representation of the graph.)

ii Complete the table, giving your answers correct to two decimal places if necessary.
(You might find some of the trigonometric values to be negative as the angle increases beyond 90°. You will understand the reason for this when you study the trigonometric ratios for angles greater than 90° in Module 7.)

Angle (θ)	**x ($\cos[\theta]$)**	**y $\left(\frac{\sin[2\theta]}{2}\right)$**
0°		
15°		
30°		
45°		
...		
315°		
330°		
345°		
360°		

iii Plot the points (x, y) on a Cartesian plane. Join them with a smooth curve.
(Alternatively, you could use a spreadsheet or graphing application to plot the graph.)

iv Describe the shape of the graph.

v Describe the domain and range of the graph.

c Compare the two models for infinity symbols from parts **a** and **b**. Do you consider one to be a better model of the infinity symbol than the other? Explain.

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10A **2** For a cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$ with three (not necessarily distinct) roots, r_1, r_2 and r_3, the following three equations hold:

- $r_1 + r_2 + r_3 = -\frac{b}{a}$
- $r_1 r_2 + r_1 r_3 + r_2 r_3 = \frac{c}{a}$
- $r_1 r_2 r_3 = -\frac{d}{a}$

These formulas are known as Vieta's formulas for cubic polynomials.

a Verify Vieta's formulas for the following cubic polynomials.

i $P(x) = 3(x + 1)(x - 3)(x + 4)$ **ii** $P(x) = -2(x - 1)^2(x + 10)$

b Now you will derive the formulas above.
Let $P(x) = ax^3 + bx^2 + cx + d$ be a cubic polynomial with three roots r_1, r_2 and r_3.

i Write $P(x)$ in factorised form. Hint: Don't forget to consider the leading coefficient of $P(x)$.

ii Expand your answer from part **b** and collect like terms.

iii By equating the coefficients in your answer to part **c** with those in $P(x) = ax^3 + bx^2 + cx + d$, write three equations which relate r_1, r_2 and r_3 to a, b, c and d.

iv Rearrange your answers from part **d** to complete the derivation of Vieta's formulas.

v Use a similar technique to derive Vieta's formulas for quartic polynomials.

Checklist

Now that you have completed this module, reflect on your ability to do the following.

I can do this	I need to review this
☐ Explain the difference between functions and relations ☐ Determine whether a graph represents a function ☐ Use function notation	☐ Go back to Lesson 5.1 Functions and relations [10A]
☐ Identify the features of a polynomial ☐ Evaluate a polynomial using substitution ☐ Expand and simplify a product of polynomials	☐ Go back to Lesson 5.2 Polynomials [10A]
☐ Divide a polynomial by a linear expression ☐ Divide a polynomial that has coefficients of zero by a linear expression	☐ Go back to Lesson 5.3 Dividing polynomials [10A]
☐ Use the remainder theorem to find the remainder of a polynomial divided by a linear expression ☐ Use the factor theorem to find a linear factor of a polynomial ☐ Factorise cubic polynomials	☐ Go back to Lesson 5.4 Remainder and factor theorems [10A]
☐ Solve polynomial equations in factorised form using the Null Factor Law ☐ Solve polynomial equations in expanded form	☐ Go back to Lesson 5.5 Solving polynomial equations [10A]
☐ Sketch the graph of a cubic relationship ☐ Sketch the graph of a quartic relationship ☐ Sketch the graph of cubic and quartic relationships with repeated factors	☐ Go back to Lesson 5.6 Sketching graphs of polynomials using intercepts [10A]
☐ Perform transformations on the graph of a polynomial relationship ☐ Identify the transformations performed on a graph	☐ Go back to Lesson 5.7 Sketching graphs of polynomials using transformations [10A]

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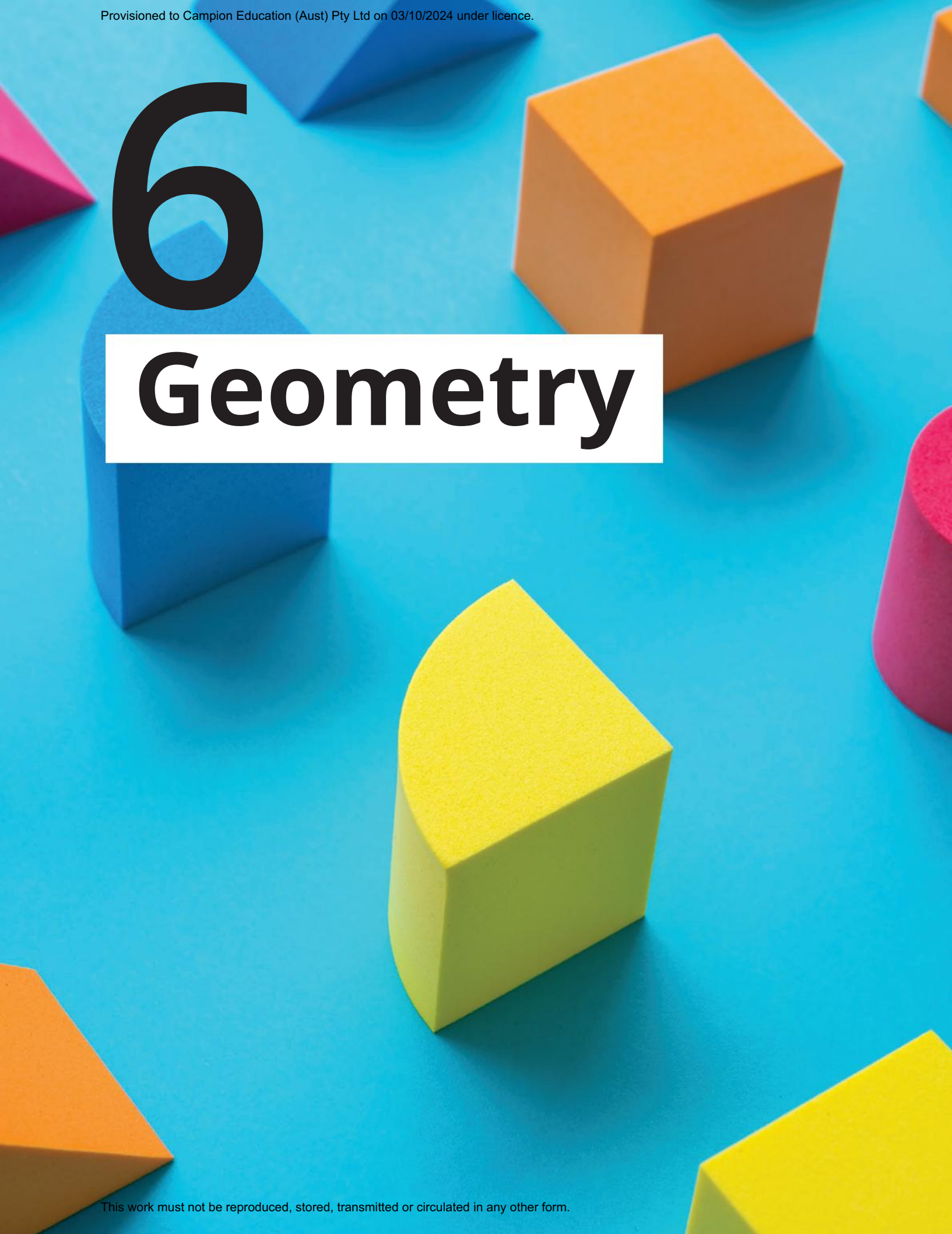
6

Geometry



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Lessons

6.1 Geometric properties
6.2 Geometric proofs
6.3 Congruence and similarity
6.4 Proofs and quadrilaterals
10A **6.5** Circle geometry: angles
10A **6.6** Circle geometry: chords
10A **6.7** Circle geometry: tangents and secants

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ Triangle properties
- ✔ Quadrilateral properties

Curriculum links

- Apply deductive reasoning to formulate proofs involving shapes in the plane and use theorems to solve spatial problems (VC2M10SP01)
- 10A Prove and apply relationships between angles and various lines associated with circles (radii, diameters, chords, tangents) (VC2M10ASP01)

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Lesson 6.1

Geometric properties

Learning intentions

By the end of this lesson you will be able to ...

✔ use geometric properties to solve problems involving angles.

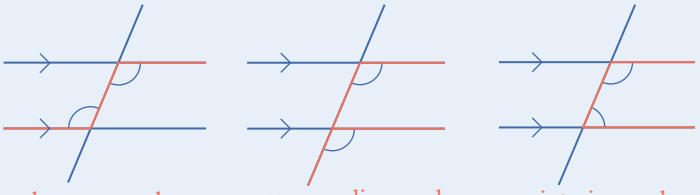
Inter-year links

Year 7	7.2 Angles at a point
Year 8	7.1 Angles

Key content video
Calculating unknown angles

Angles

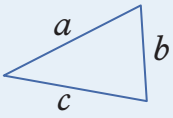
- **Complementary angles** sum to 90°.
- **Supplementary angles** sum to 180°.
- Angles at a point sum to 360°.
- **Vertically opposite angles** are equal.
- When parallel lines are crossed by a **transversal**, a number of equal and supplementary angles are formed.



alternate angles (equal) | corresponding angles (equal) | co-interior angles (supplementary)

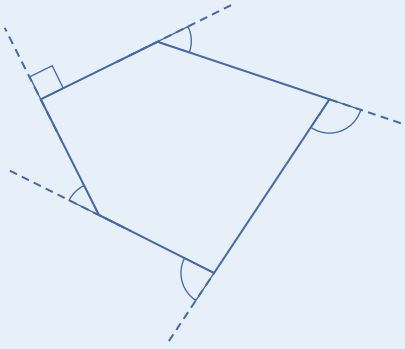
Polygons

- A **polygon** is a closed shape with straight sides.
  → A **regular polygon** has all sides equal in length and all **interior angles** equal in size.
- A **triangle** is a three-sided polygon with an interior angle sum of 180°.
- The **angle–side relationship** states that the longest side of a triangle is always opposite the largest angle in a triangle.
- The **triangle inequality** states that the sum of the lengths of any two sides of a triangle is always greater than the length of the third side.



$$a + b > c$$
$$a + c > b$$
$$b + c > a$$

- A **quadrilateral** is a four-sided polygon with an interior angle sum of 360°.
- The interior angle sum of any polygon is $(n - 2) \times 180°$, where n is the number of sides of the polygon.
- The **exterior angles** of a convex polygon can be identified by extending the sides of the polygon beyond the vertices of the polygon.



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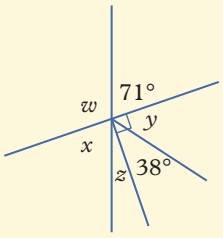
Worked example 6.1A Finding unknown angles using angle properties

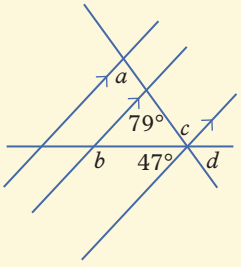
Find the size of the labelled unknown angles in this diagram and state your reasoning.



THINK	WRITE
1 Angle w is supplementary to the 71° angle.	$w = 180° - 71°$ (supplementary angles) $= 109°$
2 Angle x is vertically opposite the 71° angle.	$x = 71°$ (vertically opposite angles)
3 Angle y is complementary to the 38° angle.	$y = 90° - 38°$ (complementary angles) $= 52°$
4 Angle z is supplementary to the angles 71°, y and 38°.	$z = 180° - 71° - 52° - 38°$ (supplementary angles) $= 19°$

Worked example 6.1B Finding the size of an angle using angle relationships with parallel lines

Find the values of the pronumerals in this diagram and state your reasoning.



THINK	WRITE
1 Angle a is a corresponding angle to the 79° angle. Corresponding angles on parallel lines are equal.	$a = 79°$ (corresponding angles)
2 Angle b is co-interior with the 47° angle. Co-interior angles on parallel lines are supplementary.	$b = 180° - 47°$ (supplementary angles) $= 133°$
3 Angle c is alternate to the 79° angle. Alternate angles on parallel lines are equal.	$c = 79°$ (alternate angles)
4 Angle d is supplementary to angle c and the unlabelled angle beside it, which is vertically opposite, and therefore, equal to 47°.	$d = 180° - 79° - 47°$ (supplementary angles) $= 54°$

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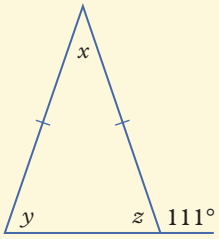
Worked example 6.1C Using triangle properties to find the the size of an angle

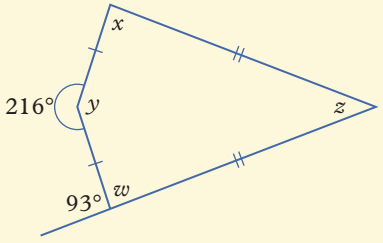
Find the value of each pronumeral in this diagram.



THINK	WRITE
1 Angle z is supplementary to the 111° angle.	$z = 180° - 111°$ $= 69°$
2 Angle y is equal to angle z because the triangle is isosceles. Base angles of an isosceles triangle are equal.	$y = z = 69°$
3 Angle x is supplementary to angles y and z because the interior angle sum of a triangle is 180°.	$x = 180° - 69° - 69°$ $= 42°$

Worked example 6.1D Using quadrilateral properties to find the size of an angle

Find the value of each pronumeral in the kite on the right.



THINK	WRITE
1 Angle w is supplementary to 93°.	$w = 180° - 93°$ $= 87°$
2 Angle x is equal to angle w because the opposite angles formed from the unequal sides of a kite are equal.	$x = w = 87°$
3 Angle y and the angle labelled 216° meet at a point, therefore they add to 360°.	$y = 360° - 216°$ $= 144°$
4 Angles w, x, y and z add to 360° because the interior angle sum of a quadrilateral is 360°.	$z = 360° - 144° - 87° - 87°$ $= 42°$

Helpful hints

✔ When calculating unknown angles, you might not be given the required information to determine all the unknown angles (at first). You may need to use the angle sizes you can calculate to help you find the angle sizes you are unable to determine.

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Learning pathways

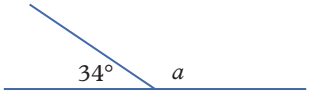
1–7, 9(a, c), 11, 13, 14	1, 3–5, 8, 9(b, d, e), 10, 12–13, 15(a, b), 16(b, d)	3, 5, 6, 8, 9(d–f), 10, 12–14, 15(c), 16, 17

ANS p715

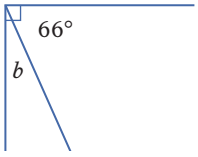
Exercise 6.1A: Understanding and fluency

WE 6.1A **1** Find the sizes of the labelled unknown angles in these diagrams and state your reasoning.

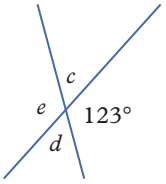
a



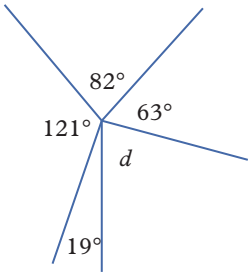
b



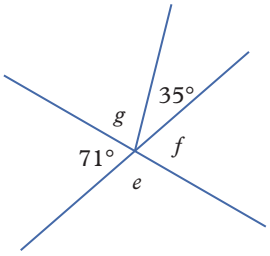
c



d



e



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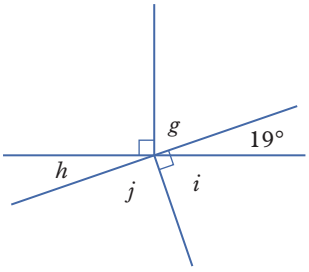
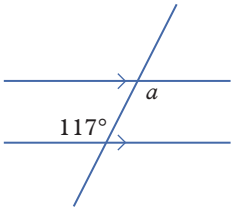


WE 6.1B **2** Find the value of each pronumeral in these diagrams and state your reasoning.

a



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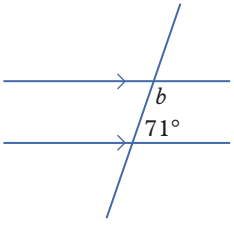

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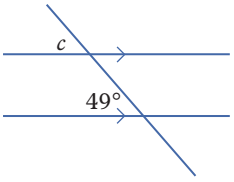
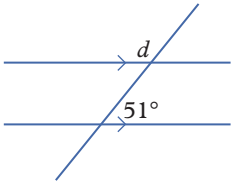
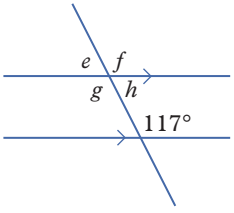


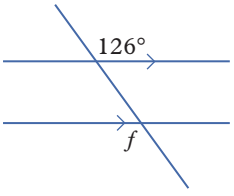
d



e

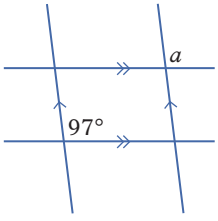


f

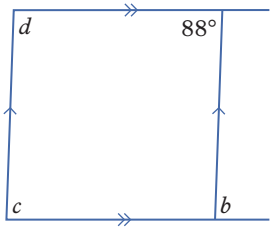


3 Find the value of each pronumeral in these diagrams.

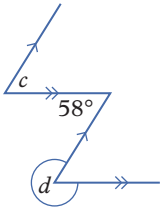
a



b



c



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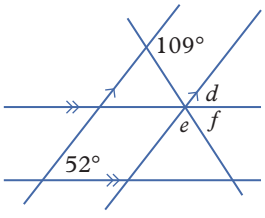

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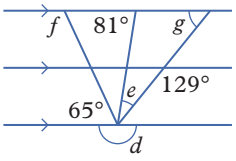

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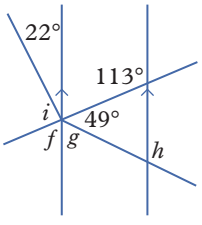

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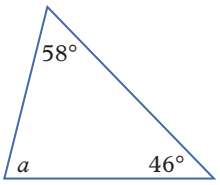
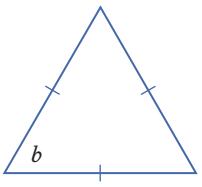
WE 6.1C **4** Find the value of each pronumeral in these diagrams.

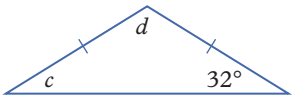
a



b



c



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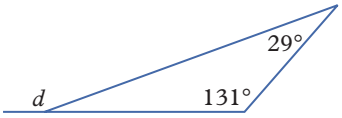
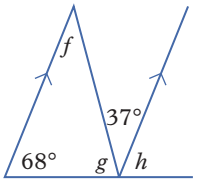


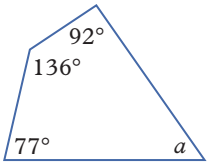
e

f

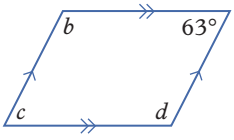


WE 6.1D **5** Find the value of each pronumeral in these diagrams.

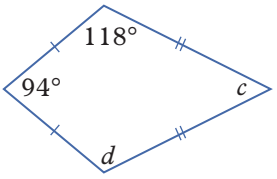
a



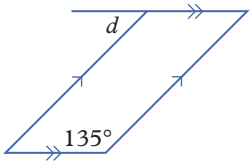
b



c



d



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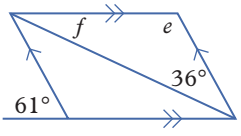

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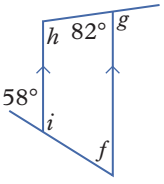
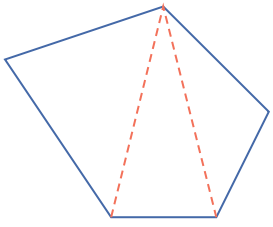
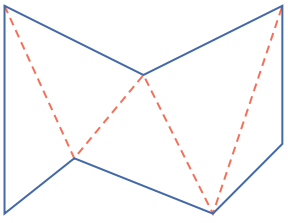


6 Use the expression $(n - 2) \times 180°$ to find the internal angle sum of a polygon with:

a 20 sides **b** 50 sides **c** 100 sides.

7 Every polygon can be dissected into triangles by connecting pairs of vertices, as shown in the diagrams below.





a How many triangles can a polygon with n sides be dissected into?

b How does the result from part **a** explain the formula for the sum of the interior angles of a polygon?

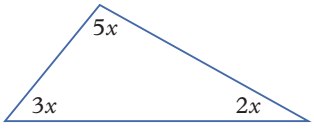
Exercise 6.1B: Problem solving and reasoning

8 Find the number of sides of a regular polygon that has each of its internal angles equal to:

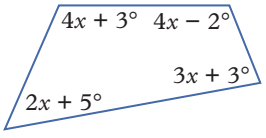
a 160° **b** 144° **c** 170° **d** 179.64°.

9 Find the value of x in each of these diagrams. Round your answers to two decimal places where necessary.

a



b



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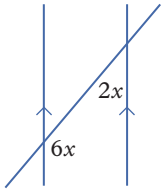
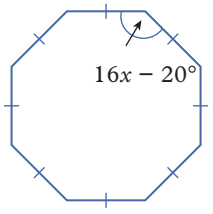
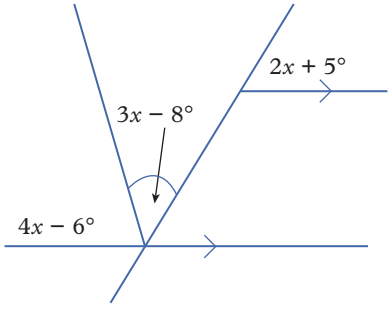
c



d



e



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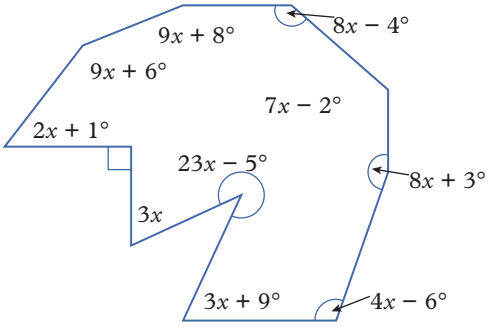
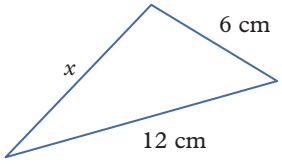


10 Explain why the triangle inequality is true for all triangles.

11 Use the triangle inequality to give a range of possible values for the unknown side lengths in these triangles.

a



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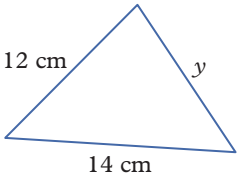

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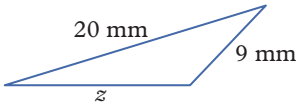
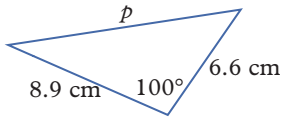


12 Use the angle–side relationship and the triangle inequality to give a range of possible values for the unknown side lengths in these triangles.

a



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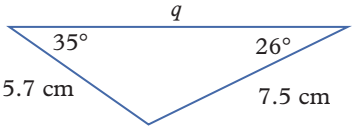

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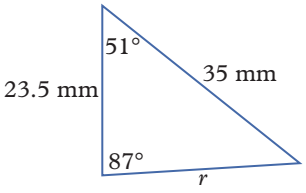

13 It is possible to extend the sides of any polygon to create exterior angles.

a Draw an equilateral triangle and extend each of the sides to create the exterior angles of the triangle. What is the size of each exterior angle of an equilateral triangle?

b What is the sum of the exterior angles of an equilateral triangle?

c Draw a square and extend each of the sides to create the exterior angles of the square. What is the size of the exterior angle of a square?

d What is the sum of the exterior angles of a square?

e Use your diagrams and answers to parts **a–d** to explain why the sum of the exterior angles of a convex polygon is equal to 360°.

14 Decide whether each of these statements is true or false. If false, rewrite it as a true statement.

a Angles at a point add to 180°.

b Corresponding angles on parallel lines are supplementary.

c Complementary angles add to 180°.

d A quadrilateral can have a maximum of one concave angle.

e The sum of an interior angle and the adjacent exterior angle is 360°.

f The number of sides of a polygon determines the interior angle sum of that polygon.

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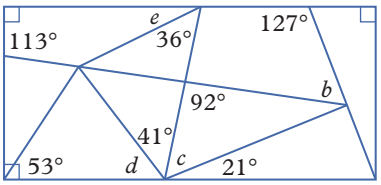
15 Find the value of each pronumeral in these diagrams.

a



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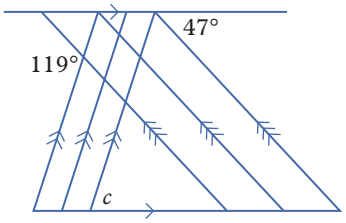

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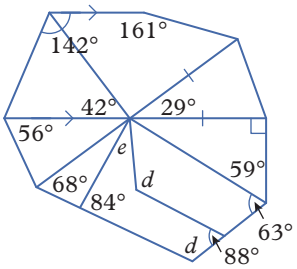
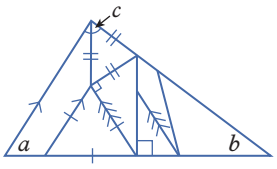


Exercise 6.1C: Challenge

16 Find the value of each pronumeral in these diagrams.

a



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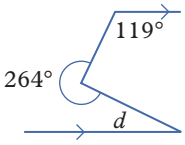

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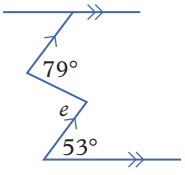
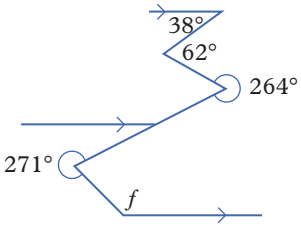
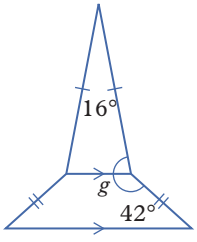


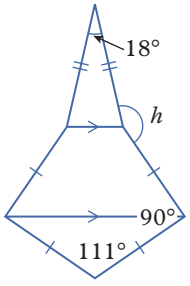
d



e



f



17 Explementary angles are a pair of angles that form a revolution; that is, they add to 360°. Determine the size of each of the following angles:

a the explement of 30°

b the explement of 175°

c the explement of the supplement of 45°

d the explement of the complement of 81°

e the complement of the explement of 324°

f the supplement of the explement of 267°

g the complement of the supplement of the explement of 200°

h the explement of the angle alternate to 64° on parallel lines

i the explement of the angle co-interior to 145° on parallel lines

j the explement of the exterior angle of a regular hexagon.

Online resources:

Interactive skillsheet Angles at a point

Interactive skillsheet Angles and parallel lines

Quick quiz 6.1

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Geometric proofs

Learning intentions

By the end of this lesson you will be able to ...

✔ understand and write mathematical proofs in geometry.

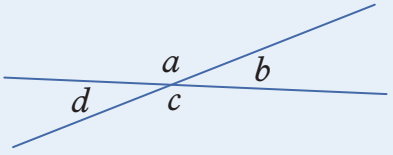
Inter-year links

Year 7	7.3 Angles and parallel lines
Year 8	7.1 Angles

Mathematical proofs

Key content video
Geometric proofs

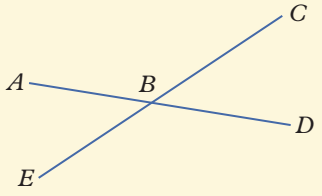
- In mathematics, a practical demonstration can be used to show something is true. For example, we can show that vertically opposite angles are equal by cutting out one of the angles and placing it on top of the other angle.



- On the other hand, a mathematical **proof** explains why something is true through a series of logically connected statements. Mathematical proofs are constructed to show that a theory is true in all cases.
- Mathematical proofs begin with self-evident or given statements. For example, you can use the fact that there are 180° on a straight line as a given statement to prove that vertically opposite angles are equal (as shown in **Worked example 6.2A**).
- The following list of given statements can be used to create the proofs in this section:
  - → Angles at a right angle add to 90°.
  - → Angles on a straight line add to 180°.
  - → Angles at a point add to 360°.
  - → If point B lies in the interior of $\angle AOC$, then $\angle AOB + \angle BOC = \angle AOC$ (angle addition postulate).
  - → The interior angle sum of a triangle is 180° (as shown in **Lesson 6.2**, question **4**).
  - → If $a = b$ and $a = c$, then $b = c$ (transitive law).
  - → The same number can be added or subtracted from both sides of an equation to create an equivalent equation (addition and subtraction properties of equality).
  - → Any other information as indicated by the question (e.g. angle and parallel line properties).
- The symbol $\because$ is shorthand for 'because' and the symbol $\therefore$ is shorthand for 'therefore'.
- The letters 'QED' or the 'tombstone', ■ or □, are often written at the end of a proof. The acronym QED stands for the Latin phrase 'quod erat demonstrandum', which means 'which was to be demonstrated'.
- One way to structure mathematical proofs is to lay them out in two columns, with mathematical statements given in the left-hand column and reasons why each statement is true given in the right-hand column.

Worked example 6.2A Proving that vertically opposite angles are equal

For the diagram below, given AD and CE intersect at B, prove that $\angle ABC = \angle DBE$.



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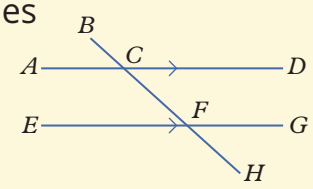
THINK

1. Set up a two-column table with 'Statements' as the heading of the left-hand column and 'Reasons' as the heading of the right-hand column.
2. Use the fact that angles on a straight line add to 180° to help you write two equations, making sure to include a common angle in both equations (for example, $\angle CBD$).
3. The right-hand sides of the equations from step **2** are the same, so the left-hand sides of the equations must also be equal (according to the transitive law). Express this in a new equation.
4. Simplify the new equation by subtracting $\angle CBD$ from both sides of the equation.
5. Include a symbol (□ or ■) or write 'QED' to show that your proof is complete.

WRITE

Statements	Reasons
$\angle ABC + \angle CBD = 180°$	Angles on a straight line add to 180°
$\angle DBE + \angle CBD = 180°$	Angles on a straight line add to 180°
$\angle ABC + \angle CBD = \angle DBE + \angle CBD$	Transitive law
$\therefore \angle ABC = \angle DBE$ ■	Subtraction property of equality

Worked example 6.2B Proving that alternate angles on parallel lines are equal



For the diagram on the right, prove that $\angle ACF = \angle CFG$, given that $\angle BCD = \angle CFG$.

THINK

1. Set up a two-column table for the proof and write the given information in the first row.
2. Use the fact that angles on a straight line add to 180° to write two equations, making sure to include a common angle in both equations (for example, $\angle DCF$).
3. Equate the left-hand sides of the equations (because the right-hand sides are equal).
4. Simplify the equation by subtracting $\angle DCF$ from both sides of the equation.
5. Because $\angle BCD = \angle CFG$, substitute $\angle CFG$ for $\angle BCD$ in the equation (using the transitive law).

WRITE

Statements	Reasons
$\angle BCD = \angle CFG$	Given
$\angle ACF + \angle DCF = 180°$	Angles on a straight line add to 180°
$\angle BCD + \angle DCF = 180°$	Angles on a straight line add to 180°
$\angle ACF + \angle DCF = \angle BCD + \angle DCF$	Transitive law
$\angle ACF = \angle BCD$	Subtraction property of equality
$\therefore \angle ACF = \angle CFG$ ■	Transitive law

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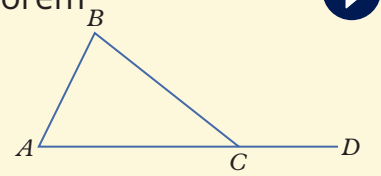
Worked example 6.2C Proving the exterior angle theorem

From the diagram on the right, prove that $\angle BCD = \angle ABC + \angle CAB$.



THINK

1. Set up a two-column table for the proof.
2. Use the fact that angles on a straight line sum to 180° to write an equation linking the exterior angle of the triangle to the adjacent interior angle of the triangle.
3. Write an equation linking the interior angles of the triangle.
4. Equate the left-hand sides of the two equations (because the right-hand sides are equal).
5. Simplify by subtracting $\angle BCA$ from both sides.

WRITE

Statements	Reasons
$\angle BCD + \angle BCA = 180°$	Angles on a straight line add to 180°
$\angle ABC + \angle CAB + \angle BCA = 180°$	Interior angle sum of a triangle is 180°
$\angle BCD + \angle BCA = \angle ABC + \angle CAB + \angle BCA$	Transitive law
$\therefore \angle BCD = \angle ABC + \angle CAB$ ■	Subtraction property of equality

Helpful hints

✔ Be patient when constructing mathematical proofs. Start with the information that you are given and build on that information. You may not see the whole proof at first, but by taking it one step at a time, you will work to the required conclusion.

Learning pathways

1–6, 10 | 2, 4–8, 10 | 4–10

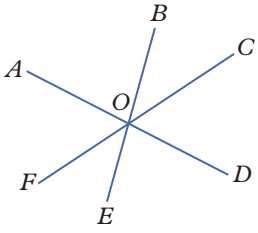
ANS p716

Exercise 6.2A: Understanding and fluency

WE 6.2A **1** For the diagram on the right, given AD, BE and CF intersect at O, prove that $\angle EOF = \angle BOC$. Complete the proof twice, starting with each of the following opening statements:

a $\angle EOF + \angle FOB = 180°$

b $\angle EOF + \angle AOF + \angle AOB = 180°$.



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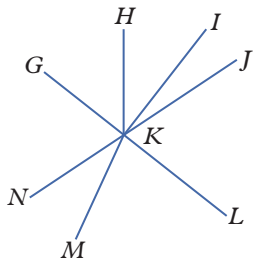
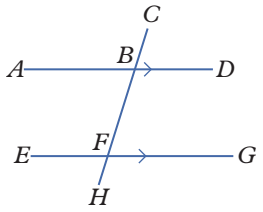
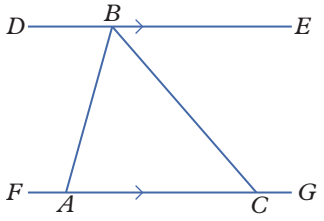
2 For the diagram below, prove that $\angle GKJ = \angle NKL$.



WE 6.2B **3** For the diagram below, prove that $\angle DBF$ is supplementary to $\angle BFG$, given that $\angle ABC = \angle EFB$.

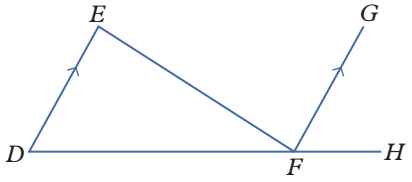


WE 6.2C **4** Use your knowledge of angles and parallel lines, as well as the diagram below, to prove that the internal angle sum of a triangle is 180°. You may use the conclusions from the worked examples in this lesson.

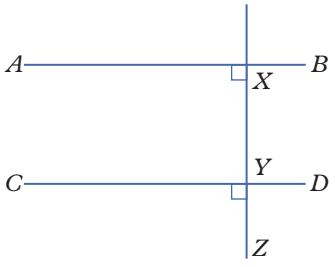


Exercise 6.2B: Problem solving and reasoning

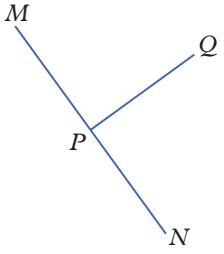
5 For the diagram below, use the fact that DE and FG are parallel to prove that $\angle EFH = \angle DEF + \angle FDE$.



6 Given that X lies on AB and Y lies on CD, prove that AB is parallel to CD.
Note: 'If co-interior angles are supplementary, they lie on parallel lines' can be used as a reason.



7 Given that $\angle MPQ$ and $\angle QPN$ are supplementary, prove that MN is a straight line.
Note: 'Supplementary angles sum to 180°' can be used as a reason.



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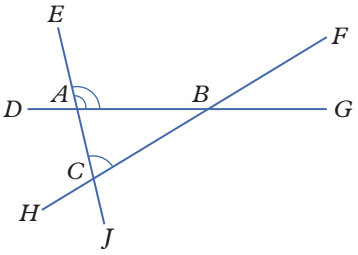
Exercise 6.2C: Challenge

8 There are different forms of mathematical proofs. The proofs you have seen in this section so far are called 'direct proofs', where a statement is established from known facts. Another form of proof is 'proof by contradiction', where we assume that what we want to prove is not true, and then we demonstrate that this assumption is false.

We can use proof by contradiction to prove that, if corresponding angles are equal, the lines being considered are parallel. Start with the statement, 'If two corresponding angles are equal, the lines being considered are not parallel'.



a Why does the lines not being parallel mean that the lines would intersect?

b Consider the diagram on the right, showing intersecting lines DG and HF, cut by the transversal EJ.

Name the two angles in the diagram that you are assuming to be equal.

c The other assumption that you must make is that the length $AC > 0$. Why do you need to make this assumption?

d If $\angle EAB = \angle ACB = x$, show that $\angle BAC = 180° - x$.

e If $\angle ABC$ can be represented by y, write a statement adding together all the interior angles of $\triangle ABC$.

f Show that simplifying this statement leaves you with $y = 0°$.

g How does this imply that AC does equal 0?

h Explain how this is 'proof by contradiction' that, if corresponding angles are equal, the lines must be parallel.

i Write the proof in full.

9 Another form of proof is 'proof by contrapositive'. The contrapositive of the statement 'if X, then Y' is 'if not Y, then not X'. These two statements are logically equivalent. That is, if you are able to prove the contrapositive, the original statement is also proved to be true.

a Write the contrapositive of the statement, 'When two lines are cut by a transversal, if the corresponding angles are equal then the lines are parallel'.

b In the diagram from question **8**, let EJ be the transversal. Prove the contrapositive of the statement from part **a** using the diagram and the fact that AB is not parallel to CB ($AB \nparallel CB$).

Note: The following additional reasons can be used:

- Three points that do not form a line form a triangle.
- The exterior angle of a triangle is equal to the sum of the two non-adjacent interior angles.
- If an angle is 0°, the two arms of the angle are overlapping each other.

10 It is important not to make errors when completing proofs because, if you are not careful, you may think you have proven something that is actually false. The following example appears to 'prove' that $2 = 1$. Find the error.

$a = b$, where a and b are not equal to zero.

So: $a^2 = ab$

and: $a^2 - b^2 = ab - b^2$

Factorising gives: $(a + b)(a - b) = b(a - b)$

Dividing by $(a - b)$ on both sides gives: $a + b = b$

Remember that $a = b$.

So: $b + b = b$

$2b = b$

Dividing by b gives: $2 = 1$

Online resources:

Investigation Euclid's *Elements*

Quick quiz 6.2

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Lesson 6.3

Congruence and similarity

Learning intentions

By the end of this lesson you will be able to ...

✔ apply congruence and similarity tests to triangles in proofs.

Inter-year links

Year 7	7.4 Triangles
Year 8	9.3 Congruent triangles
Year 9	6.2 Similar figures

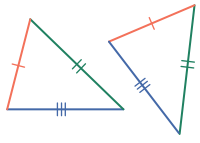
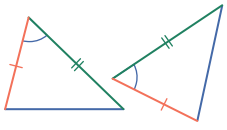
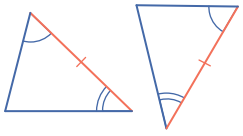
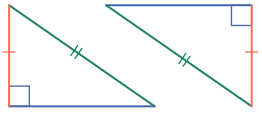
Congruent figures

- **Congruent** figures are identical in shape and size but can be in any position or orientation.
- Two figures are congruent if their corresponding sides are the same length and their corresponding angles are the same size.
  → The symbol for congruence is ≅ ('is congruent to').

Congruent triangles

Key content video
Congruent triangles

- If two triangles meet any of the four conditions shown in the table below, they are congruent.

SSS (side–side–side)	SAS (side–angle–side)	AAS (angle–angle–side)	RHS (right angle–hypotenuse–side)
All three side lengths are equal. 	Two of the side lengths are equal and the included angle (between those sides) is equal. 	Two of the angles and a corresponding pair of side lengths are equal. 	For right-angled triangles, the hypotenuses and another pair of side lengths are equal. 

- If $\triangle ABC \cong \triangle DEF$, then vertex A corresponds to vertex D, vertex B corresponds to vertex E, and vertex C corresponds to vertex F.

Similar figures

- **Similar** figures are identical in shape but can be different in size.
- For two figures to be similar, all angles must be equal in size and the lengths of all corresponding sides must be in the same ratio.
- If two figures are similar, an unknown side length or angle can be found if the scale factor is known.
  → The symbol for similarity is ~ ('is similar to').
- The scale factor between similar figures can be calculated using the formula:

$$\text{scale factor} = \frac{\text{image length}}{\text{original length}}$$

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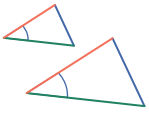
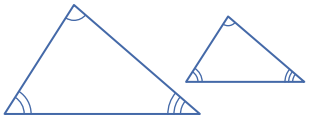
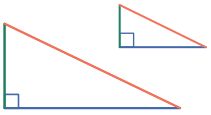
Similar triangles

Key content video
Similar triangles

- For two triangles to be similar, they must meet one of four conditions of similarity.

SSS (side–side–side)	**SAS (side–angle–side)**	**AAA (angle–angle–angle)**	**RHS (right angle–hypotenuse–side)**
The lengths of all three pairs of corresponding sides are in the same ratio.	The lengths of two pairs of corresponding sides are in the same ratio and the included angles are equal. 	Each of the three angles in one triangle is equal to the corresponding angle in the other triangle. 	In two right-angled triangles, the lengths of their hypotenuses are in the same ratio as the lengths of another pair of corresponding sides. 

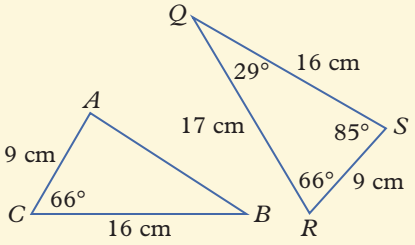
Proving triangles are congruent or similar

Key content video
Proving triangles are similar or congruent

- In addition to the given statements in **Lesson 6.2**, the following statements can be used to create the proofs in this section:
  - → Vertically opposite angles are equal.
  - → Alternate angles on parallel lines are equal.
  - → Corresponding angles on parallel lines are equal.
  - → Co-interior angles on parallel lines are supplementary.
- Properties of different types of triangles can also be used.

Worked example 6.3A Identifying whether two triangles are congruent

Decide whether these triangles are congruent, giving a reason for your answer.



THINK

1 Look at the given information to determine which congruence condition to use. We are given the sizes of two sides and an included angle for $\triangle ABC$. This fits the SAS condition for congruence.

WRITE

$\triangle ABC$ has two sides and an included angle labelled, which may fit the SAS condition for congruence.

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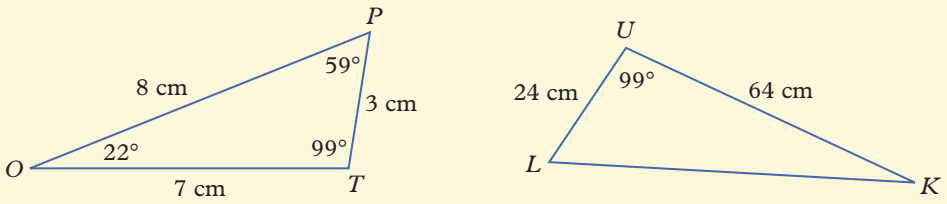
2 Identify the corresponding sides in the second triangle.	$RS = AC = 9$ cm $SQ = CB = 16$ cm
3 Compare the sizes of the included angles between the matching sides in each triangle.	$\angle ACB = 66° \neq \angle RSQ$.
4 Write a conclusion.	The two triangles are not congruent because they fail the SAS condition for congruence.

Worked example 6.3B Identifying whether two triangles are similar

Decide whether these triangles are similar, giving a reason for your answer.



THINK	WRITE
1 Look at the given information to determine which similarity condition to use. Since $\triangle LUK$ has less information to work with, start with the conditions on $\triangle LUK$. $\triangle LUK$ has two sides and an included angle labelled, which fits the SAS condition for similarity.	$\triangle LUK$ has two sides and an included angle labelled, which may fit the SAS condition for similarity.
2 Use the given angle in $\triangle LUK$ to find the corresponding sides. Observe that $PT < OT$ and $LU < UK$. Identify the corresponding pairs as the two shorter sides and the two longer sides.	$\angle PTO = \angle LUK = 99°$ PT corresponds to LU. TO corresponds to UK. OP corresponds to KL.
3 Determine the scale factor used for one of the pairs of matching sides (LU and PT).	scale factor $= \frac{LU}{PT}$ $= \frac{24}{3}$ $= 8$
4 Find the scale factor used for the other pair of matching sides (TO and UK).	scale factor $= \frac{UK}{TO}$ $= \frac{64}{7}$ ≈ 9.14
5 Write a conclusion.	The two triangles are not similar because these two scale factors are not the same, meaning their corresponding side lengths are not in the same ratio. They fail the SAS condition for similarity.

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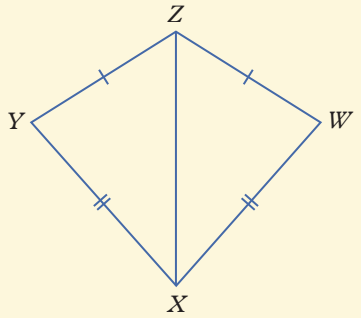
Worked example 6.3C Proving that two triangles are congruent

For the diagram on the right, prove that $\triangle XYZ \cong \triangle XWZ$ using a congruence condition.



THINK

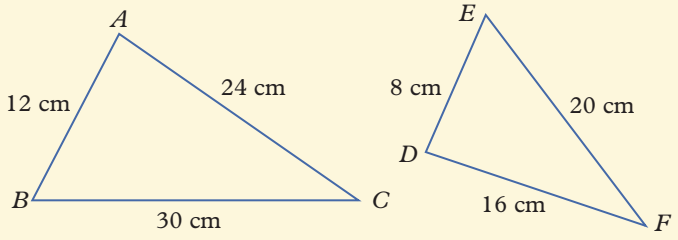
1. Set up a two-column table for your proof.
2. List the given information.
3. Determine whether the given information meets any of the conditions for congruence.
4. Complete the proof.

WRITE

Statements	Reasons
$XY = XW$	Given
$ZY = ZW$	Given
XZ is common to both triangles	Common side
$\therefore \triangle XYZ \cong \triangle XWZ$ ■	SSS (congruent)

Worked example 6.3D Proving that two triangles are similar

Prove that $\triangle ABC$ is similar to $\triangle DEF$ using one of the conditions for similarity.



THINK

1. Identify the corresponding side for the two triangles. For example, AB is the shortest side in $\triangle ABC$ and DE is the shortest side in $\triangle DEF$.
2. Calculate the scale factor for each pair of corresponding sides.
3. Determine whether the given information meets any of the conditions for similarity.
4. Complete the proof.

WRITE

Statements	Reasons
$\frac{DE}{AB} = \frac{8}{12} = \frac{2}{3}$	Scale factor between corresponding sides
$\frac{EF}{BC} = \frac{20}{30} = \frac{2}{3}$	Scale factor between corresponding sides
$\frac{DF}{AC} = \frac{16}{24} = \frac{2}{3}$	Scale factor between corresponding sides
$\therefore \triangle ABC \sim \triangle DEF$ ■	SSS (similar)

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Helpful hints

- ✔ If two triangles meet one of the conditions for congruence, they meet all the conditions for congruence. Likewise, if two triangles fail one of the conditions for congruence, they fail all the conditions for congruence.
- ✔ When writing a congruence or similarity statement, the corresponding vertices in each triangle must be written in the same order.

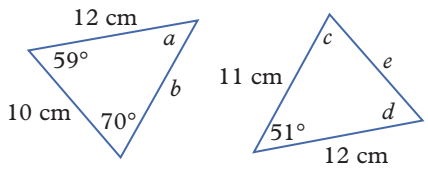
Learning pathways

1–5, 6(a, c, d), 7, 8(a), 9(a), 10(a, b), 12, 15, 19	1(c, d), 2(d–f), 3, 4, 5(d–f), 6, 8, 9(b), 10(c,d), 11, 13, 16, 17, 18(a) ,19–21	3, 4, 6(d–f), 7–9, 11, 14, 16, 18(b), 19–24

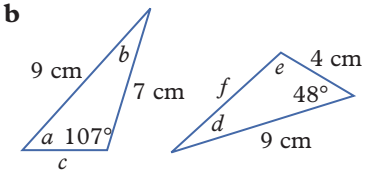
Exercise 6.3A: Understanding and fluency

1 Find the unknown side lengths and angles in each of these pairs of congruent triangles.

a



b



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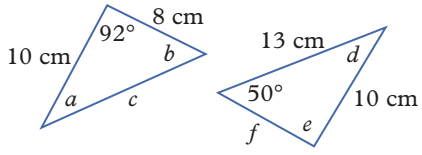

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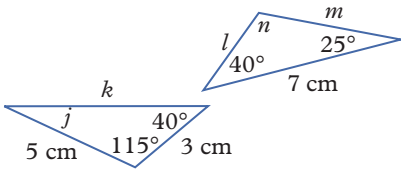
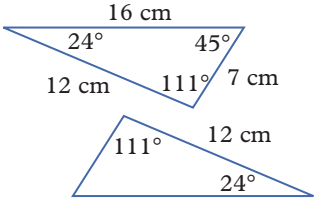


WE 6.3A **2** Decide whether the triangles in each of these pairs are congruent and state the congruence condition used.

a



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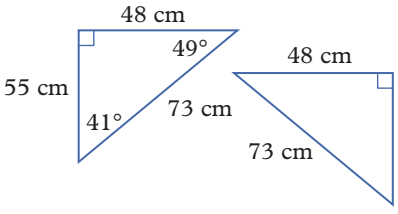

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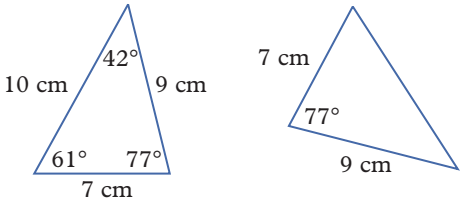

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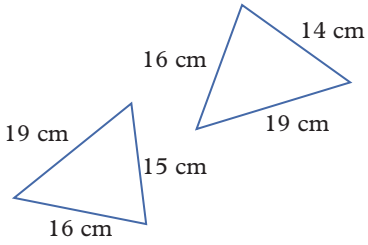

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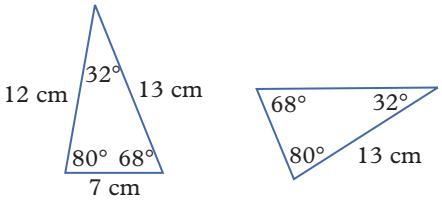

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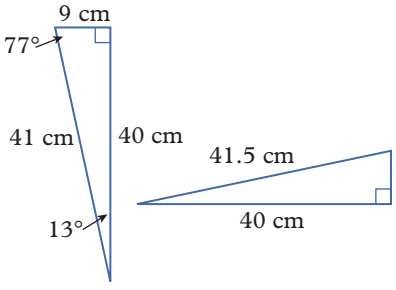

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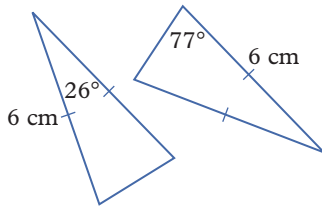
3 Use your understanding of triangle properties to decide whether the triangles in each of these pairs are congruent.

a



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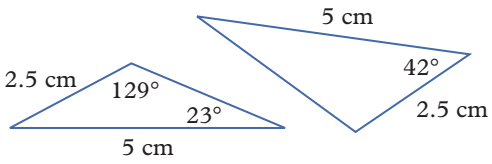

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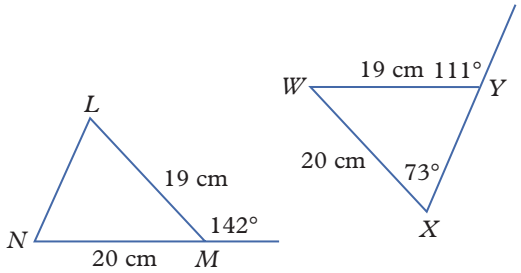

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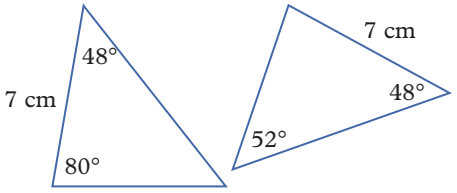
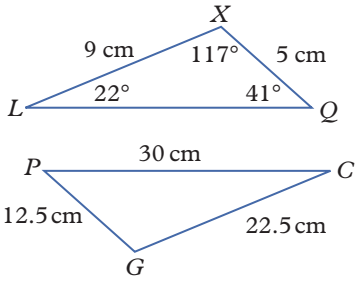


4 Find the scale factor between the triangles in each of these pairs of similar triangles. Assume that the first triangle in each pair is the original.

a



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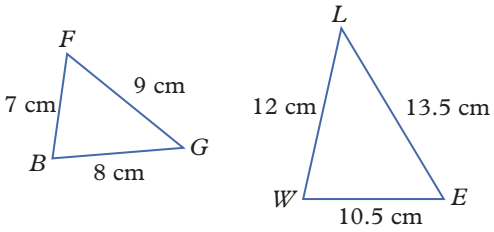

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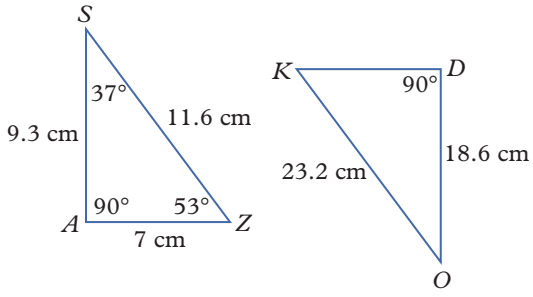

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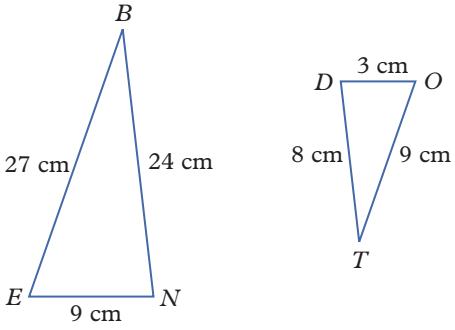
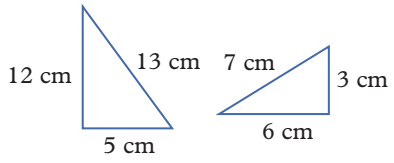


WE 6.3B **5** Decide whether the triangles in each of these pairs are similar and state the similarity condition used.

a



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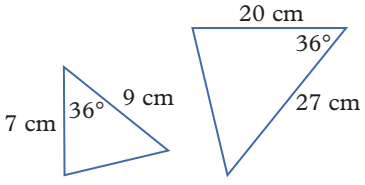

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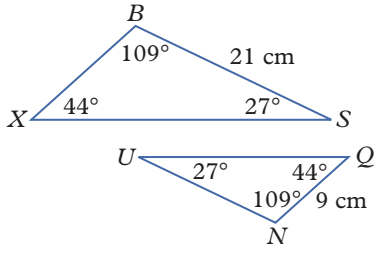

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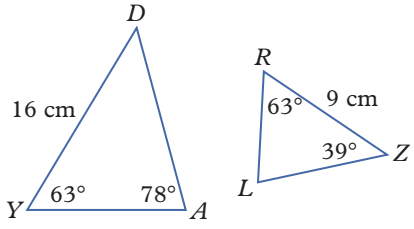

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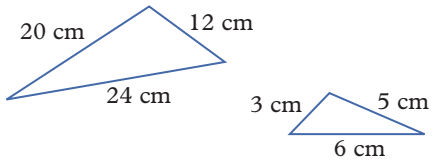

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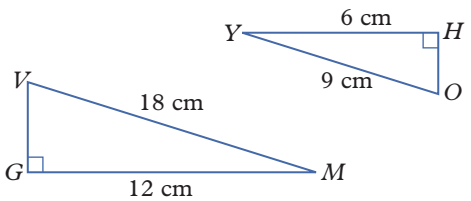
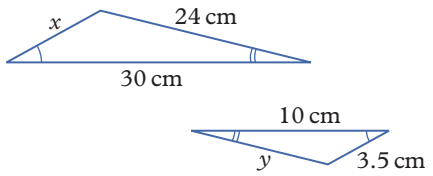


6 Find the unknown side lengths if the triangles in each pair below are similar.

a



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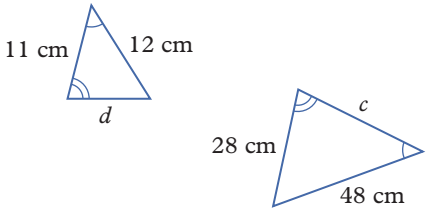

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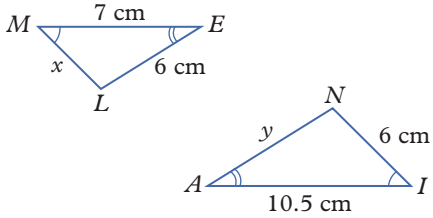

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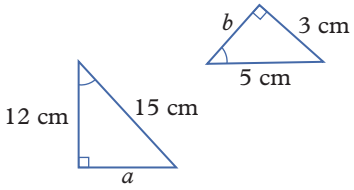

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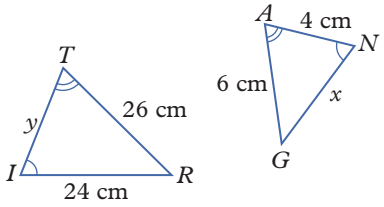

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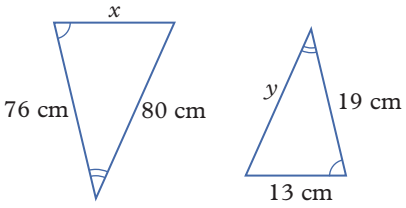
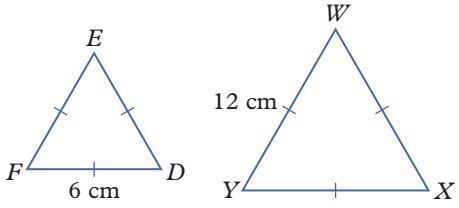


7 Use your understanding of triangle properties to decide whether the triangles in each of these pairs are similar.

a



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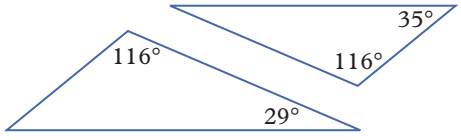

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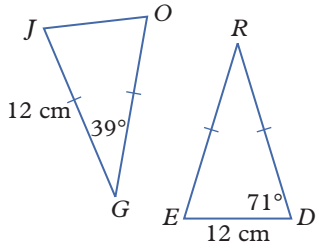

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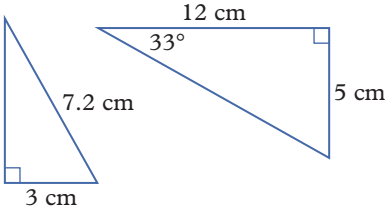
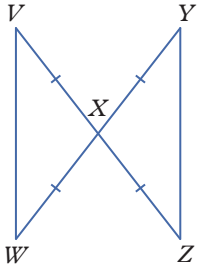
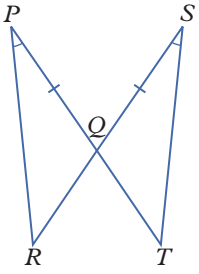


WE 6.3C **8** Consider the following diagrams.

a For the diagram below, prove that $\triangle VWX \cong \triangle YZX$.



b For the diagram below, prove that $\triangle PQR \cong \triangle SQT$.



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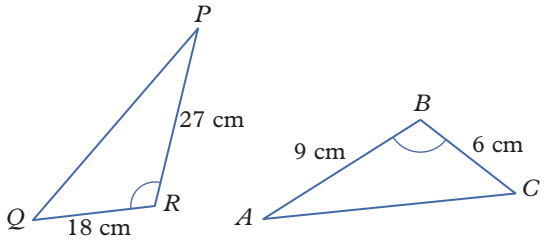
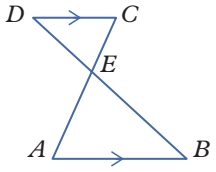
WE 6.3D **9** Consider the following diagrams.

a Prove that $\triangle ABC \sim \triangle PRQ$.



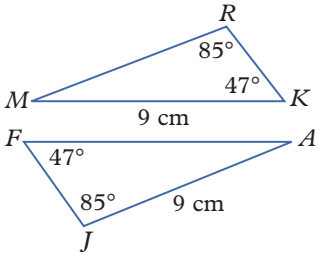
b Prove that $\triangle ABE \sim \triangle CDE$.



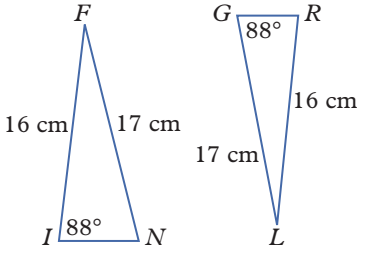
Exercise 6.3B: Problem solving and reasoning

10 Lachlan said that the information given for the pairs of triangles below proves that they are congruent triangles. Explain why he is wrong in each case.

a



b



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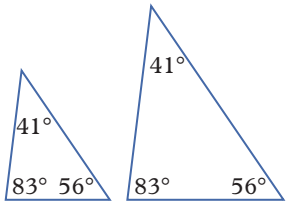

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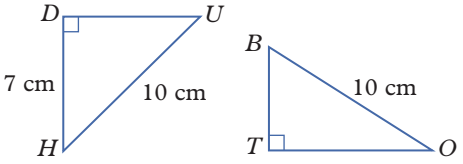


11 Decide whether each of these statements is true or false, giving reasons for your answers.

a If two triangles meet the condition AAA, they are not congruent.

b If two triangles have two corresponding sides of the same length and one corresponding angle of the same size, the triangles are congruent.

c A pair of triangles can meet one congruence condition but fail another.

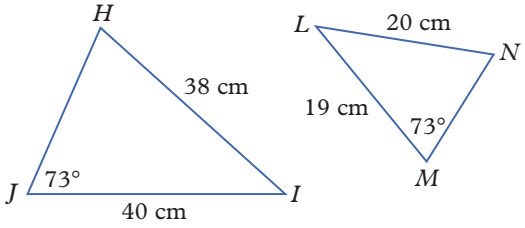
d If $\triangle ABC$ is congruent to $\triangle DEF$, and $\triangle DEF$ is congruent to $\triangle GHI$, then $\triangle GHI$ is congruent to $\triangle ABC$.

e If two quadrilaterals have all matching sides equal in length, then they are congruent.

f All equilateral triangles are congruent.

12 Jessica said that the triangles in each pair below could be proven to be similar. Explain why she is wrong in each case.

a



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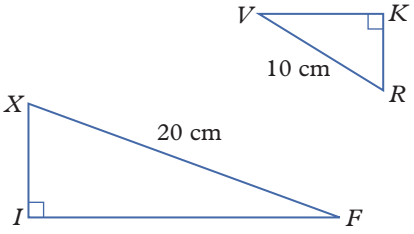
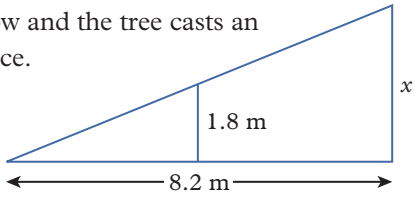


13 A 1 m ruler is placed upright next to a tree. If the ruler casts a 2.3 m shadow and the tree casts an 11.2 m shadow, how tall is the tree? Round your answer to one decimal place.

14 The horizontal length of a ski jump is 8.2 m long. It has 1.8 m tall vertical supports at the halfway mark. How tall is the ski jump at its highest point?



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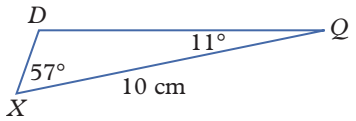
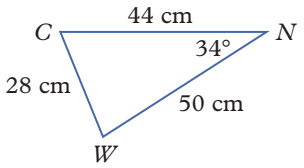
15 Select two pairs of similar triangles from the options below. Provide reasons for your selection.

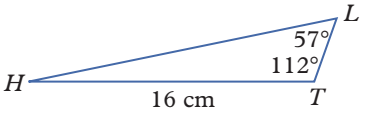
A



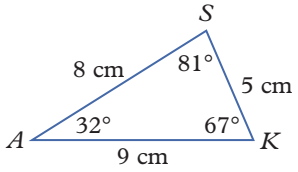
B



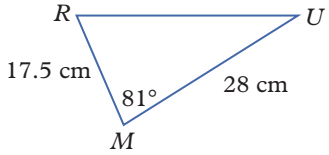
C



D



E



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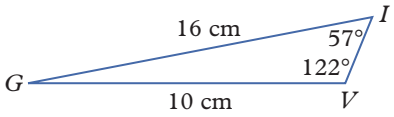


16 Decide whether each of these statements is true or false. Give reasons for your answers.

a If two triangles meet the condition AAA, they are not necessarily similar.

b If $\triangle ABC$ is similar to $\triangle DEF$, and $\triangle DEF$ is similar to $\triangle GHI$, then $\triangle GHI$ is similar to $\triangle ABC$.

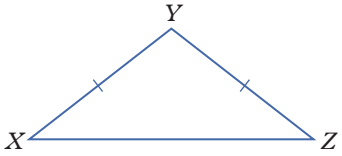
c All isosceles triangles are similar.

d All squares are similar.

e If two triangles fail the condition SAS, they are not necessarily similar.

f If two quadrilaterals have all sides in the same ratio, they are not necessarily similar.

17 If a triangle is isosceles, then the angles opposite those sides are equal. You can prove this fact using congruent triangles. To help you, consider $\triangle XYZ$ on the right.



a Draw a line from Y to the midpoint of XZ, and name this point W.

b Name the two triangles you have now formed.

c Which pairs of sides do you know are equal in these two triangles?

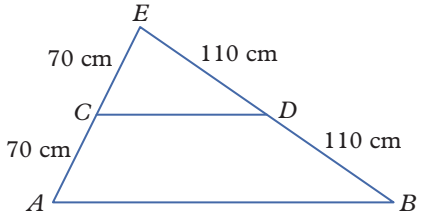
d Name the side common to both triangles.

e Use a congruence condition to prove that $\triangle WXY \cong \triangle WZY$.

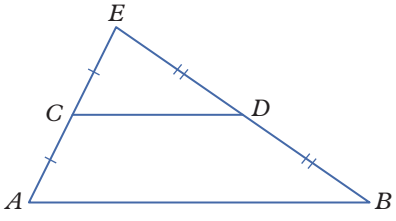
f Explain why you now know that $\angle WXY = \angle WZY$.

18 Consider the following diagrams.

a Prove that $\triangle ABE \sim \triangle CDE$.

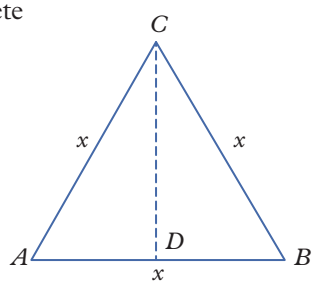


b Prove that $\triangle ABE \sim \triangle CDE$.



Exercise 6.3C: Challenge

19 Consider the equilateral triangle on the right, where CD is a bisector of AB. Complete the following parts to show that the interior angles of an equilateral triangle are 60°.



a What are the lengths of AD and BD?

b Use trigonometry to calculate $\angle CAD$ and $\angle CBD$.

c Use the property that the sum of the interior angles of a triangle is 180° to calculate $\angle ACB$.

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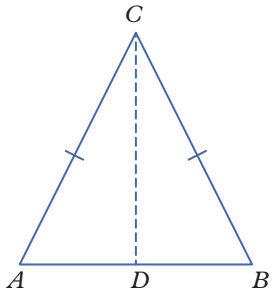
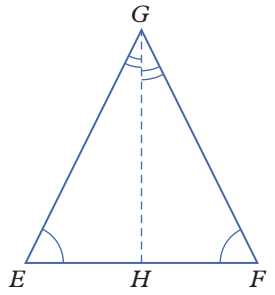
20 Use the diagram below, where CD is a bisector of AB, to prove that if two sides of a triangle are equal, then the angles opposite those sides are equal.



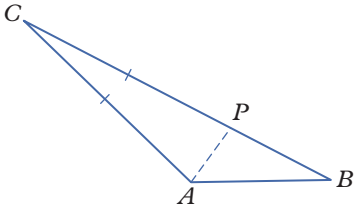
21 Use the diagram below, where GH is a bisector of $\angle EGF$, to prove that if two angles of a triangle are equal, then the sides opposite those angles are equal.



22 The angle–side relationship states that the longest side of a triangle is always opposite the largest angle in a triangle. This can be expanded to state that in $\triangle ABC$, if $BC \geq AC \geq AB$, then $\angle BAC \geq \angle ABC \geq \angle ACB$.

a Explain why the angle–side relationship holds for all equilateral triangles.

b Use the diagram on the right to prove that if $BC > AC$, then $\angle BAC > \angle CBA$. Hint: Start by defining point P.



c For an isosceles triangle with side lengths satisfying $BC \geq AC \geq AB$, explain why $BC > AC = AB$ or $BC = AB > AC$.

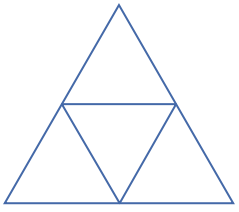
d The result in part **b** means that if one side of a triangle is longer than another side, then the angle opposite the longer side is larger than the angle opposite the shorter side.

i Use parts **b** and **c** to explain why the angle–side relationship holds for all isosceles triangles.

ii Use part **b** to explain why the angle–side relationship holds for all scalene triangles.

23 A flagpole has a guide wire attached at its midpoint and tethered to the ground 2.7 m away. A metre ruler, standing vertically, touches the guide wire when placed 75 cm away from where it is tethered to the ground. How tall is the flagpole?

24 Prove that drawing a triangle whose vertices are the midpoints of the sides of an equilateral triangle splits that equilateral triangle into four smaller congruent equilateral triangles.



Online resources:

- **Interactive skillsheet** Congruent triangles
- **Interactive skillsheet** Similar triangles
- **Worksheet** Congruent and similar triangles
- **Quick quiz** 6.3

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Lesson 6.4

Proofs and quadrilaterals

Learning intentions

By the end of this lesson you will be able to ...

✔ prove properties of quadrilaterals.

Inter-year links

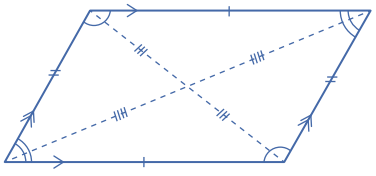
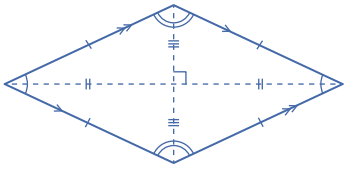
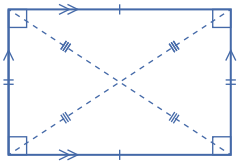
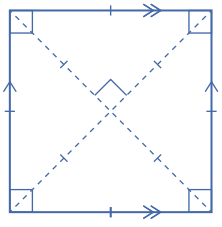
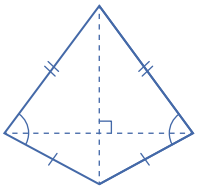
Year 7	7.5 Quadrilaterals
Year 8	9.6 Quadrilaterals

Geometry terminology

Key content video
Proving quadrilateral properties

- 'Bisect' means to cut in half.
- The symbol for 'is parallel to' is ∥.
- The symbol for 'is perpendicular to' is ⊥.

Quadrilateral properties

Definition	Properties	Example
Parallelogram A quadrilateral with two pairs of parallel sides.	Opposite sides are equal in length. Opposite angles are equal. Diagonals bisect each other.	
Rhombus A parallelogram with two adjacent sides equal in length.	All parallelogram properties. All sides are equal in length. Diagonals bisect each other at right angles. Diagonals bisect the interior angles.	
Rectangle A parallelogram with a right angle.	All parallelogram properties. Diagonals are equal in length. All interior angles are right angles.	
Square A rectangle with two adjacent sides which are equal in length.	All parallelogram properties. All rhombus properties. All rectangle properties.	
Kite A quadrilateral with two pairs of adjacent sides which are equal in length.	One pair of opposite angles are equal. Diagonals are perpendicular. One diagonal bisects the other diagonal. One diagonal bisects a pair of interior angles.	

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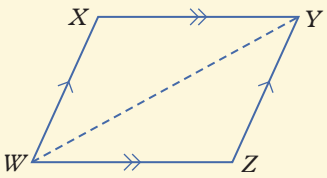
Worked example 6.4A Proving opposite sides of a parallelogram are equal



Use the diagram on the right to prove that the opposite sides of a parallelogram are equal in length.

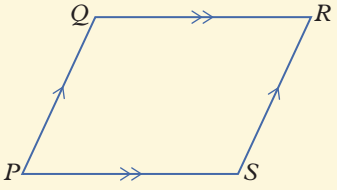
THINK

1. Set up a two-column table for your proof.
2. Use the fact that the diagonal of the parallelogram is common to both $\triangle WXY$ and $\triangle YZW$.
3. Use your knowledge of angles and parallel lines to identify equal angles.
4. Use a congruence condition to show that the triangles are congruent.
5. Show that the opposite sides of the parallelogram are equal in length (as corresponding sides of the congruent triangles).
6. Complete the proof with a closing statement.

WRITE

Statements	Reasons
YW is common to both $\triangle WXY$ and $\triangle YZW$	Common side
$\angle XYW = \angle ZWY$	Alternate angles on parallel lines are equal
$\angle YWX = \angle WYZ$	Alternate angles on parallel lines are equal
$\therefore \triangle WXY \cong \triangle YZW$	Meets the AAS condition of congruence
$WX = YZ$	Corresponding sides of congruent triangles are equal
$WZ = YX$	Corresponding sides of congruent triangles are equal
$\therefore$ Opposite sides of a parallelogram are equal in length ■	

Worked example 6.4B Proving opposite angles in a parallelogram are equal



Use the diagram on the right to prove that the opposite angles of a parallelogram are equal in size.

THINK

1. Use your understanding of angles and parallel lines to write two equations about co-interior angles, making sure the same angle appears in both equations.
2. Equate the left-hand sides of the equations you wrote (because the right-hand sides are equal).
3. Simplify by subtracting the same angle from both sides.
4. Repeat steps **1–3** for the other pair of co-interior angles.
5. Complete the proof with a closing statement.

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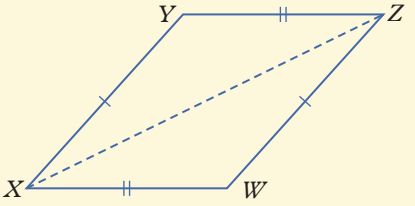
WRITE

Statements	Reasons
$PQ \parallel SR$	Given
$\angle QPS + \angle PSR = 180°$	Co-interior angles on parallel lines are supplementary
$PS \parallel QR$	Given
$\angle QPS + \angle PQR = 180°$	Co-interior angles on parallel lines are supplementary
$\angle PSR + \angle QRS = 180°$	Co-interior angles on parallel lines are supplementary
$\angle QPS + \angle PSR = \angle QPS + \angle PQR$	Transitive law
$\angle QPS + \angle PSR = \angle PSR + \angle QRS$	Transitive law
$\angle PSR = \angle PQR, \angle QRS = \angle QPS$	Subtraction property of equality
$\therefore$ Both pairs of opposite angles of the parallelogram are equal in size ■	

Worked example 6.4C Proving a quadrilateral is a parallelogram

Use the diagram on the right to show that a quadrilateral with two pairs of opposite sides equal in length is a parallelogram.



THINK

1. Show that the two triangles formed by the diagonal are congruent.
2. Identify the corresponding angles in these congruent triangles.
3. If alternate angles are equal, then the lines must be parallel. Note: $\parallel$ means 'is parallel to'.
4. Use the definition of a parallelogram to prove that $WXYZ$ is a parallelogram.

WRITE

Statements	Reasons
$XY = ZW$	Given
$YZ = WX$	Given
XZ is common to both $\triangle XWZ$ and $\triangle ZYX$	Common side
$\therefore \triangle XWZ \cong \triangle ZYX$	SSS condition (congruence)
$\angle WXZ = \angle YZX$	Corresponding angles of congruent triangles are equal.
$\angle XZW = \angle ZXY$	Corresponding angles of congruent triangles are equal.
$YZ \parallel XW$	$\angle WXZ$ and $\angle YZX$ are alternate angles. If alternate angles are equal, they lie on parallel lines.
$XY \parallel WZ$	$\angle XZW$ and $\angle ZXY$ are alternate angles. If alternate angles are equal, they lie on parallel lines.
$\therefore WXYZ$ is a parallelogram ■	A quadrilateral with two pairs of parallel lines is a parallelogram.

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Helpful hints

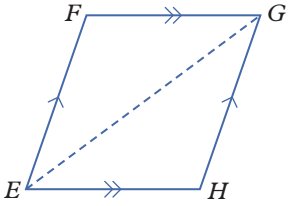
- ✔ If you are unsure where to begin when constructing a mathematical proof, start by writing all the information you are given (using mathematical terminology) as well as visualising what you are trying to prove.
- ✔ Quadrilaterals and other polygons are labelled in a cyclic manner around the shape in order. That is, pairs of adjacent letters name the edges of the quadrilateral.

Learning pathways

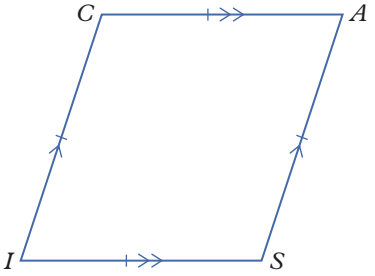
1–4, 6, 8–10	1–5, 7, 10, 11, 13	3, 4, 7, 9, 11, 12, 14–16

Exercise 6.4A: Understanding and fluency

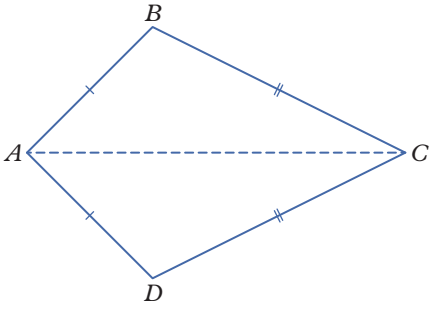
WE 6.4A **1** Given that $FG = EF$, prove that all four sides of the rhombus on the right are equal in length.



WE 6.4B **2** Use the diagram on the right to prove that the opposite angles of the rhombus are equal.



WE 6.4C **3** Use the diagram below to prove that a kite has one pair of opposite angles equal in size.



Exercise 6.4B: Problem solving and reasoning

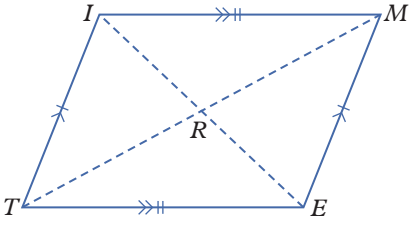
4 Consider the parallelogram on the right.

a Which angle is equal to $\angle IMR$?

b Which angle is equal to $\angle MIR$?

c Prove that $\triangle MIR \cong \triangle TER$.

d Explain how this shows that the diagonals of a parallelogram bisect each other.



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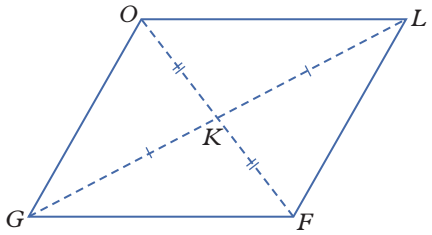
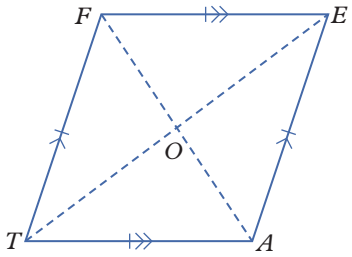
5 Prove that the shape below is a parallelogram.



6 Consider the rhombus on the right.



a Follow similar steps to those given in question **4** to prove that the diagonals of the rhombus bisect each other.

b Show that the four triangles formed by the diagonals are congruent to one another.

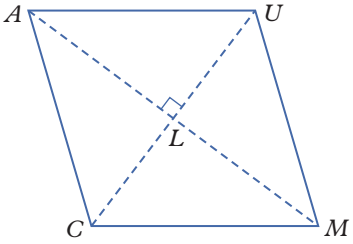
c Explain why $\angle FOE = \angle AOE$.

d Use your answer from part **c** to help you prove that the diagonals of a rhombus bisect each other at right angles.

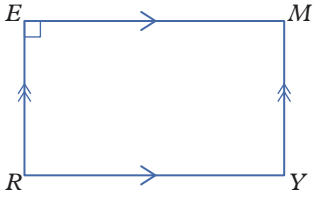
7 Prove that the shape on the right is a rhombus if:

a $AL = ML$ and $CL = UL$

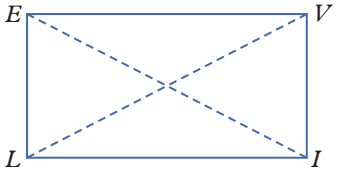
b $AU \parallel CM$ and $AC \parallel UM$.



8 Use the diagram below to show that all four interior angles of a rectangle are 90°.



9 Consider figure $LEVI$ below, where $VE = IL$, $VE \parallel IL$ and $\angle EVI = 90°$.

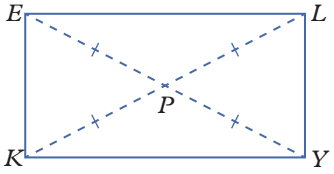


a Prove that $\angle VIL$, $\angle ILE$ and $\angle LEV$ are all 90°.

b Prove that $EL = IV$ and $EL \parallel IV$.

c Prove that $\triangle LEV \cong \triangle ELI$.

10 Prove that the shape on the right is a rectangle by following these steps.



a Use what you know about isosceles triangles and about equal alternate angles to prove that $ELYK$ is a parallelogram.

b Use $\angle LYE + \angle YEL + \angle ELK + \angle KLY$ to prove that $\angle ELY = 90°$.

c Explain why this shows $ELYK$ is a rectangle.

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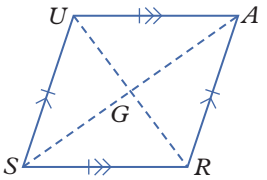
11 Consider the rhombus.

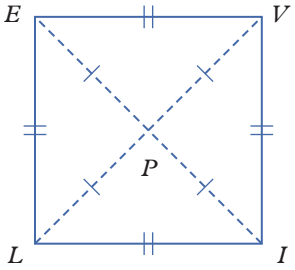


a Prove that $UG = RG$ and that $AG = SG$ (in other words, that the diagonals bisect each other).

b Prove that $\triangle UAG \cong \triangle RAG$.

c Explain why this method shows that the diagonals of a rhombus bisect its interior angles.

12 For the diagram below, prove that $\angle EVL = 45°$.



13 Use mathematical reasoning to demonstrate all of the statements below.

a Use a square and a rhombus to show that having all corresponding sides the same length is not enough to prove congruence for shapes other than triangles.

b Use a parallelogram and a rhombus to show that meeting the SAS condition of congruence does not prove congruence for shapes other than triangles.

c Use a square and a rectangle to show that meeting the AAS condition of congruence does not prove congruence for shapes other than triangles.

14 Use mathematical reasoning to demonstrate all of the statements below.

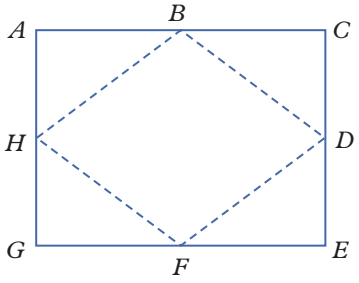
a Use a rectangle and a parallelogram to show that having all corresponding sides in the same ratio is not enough to prove similarity for shapes other than triangles.

b Use a parallelogram and a rhombus to show that having all corresponding angles the same size is not enough to prove similarity for shapes other than triangles.

Exercise 6.4C: Challenge

15 A regular hexagon is split into six triangles by drawing three diagonal lines joining opposite vertices. Draw the shape as described and show that the triangles contained within are congruent.

16 $ACEG$ is a rectangle. If the vertices of $HBDF$ touch the midpoints of each of the rectangle's sides, prove that $HBDF$ is a rhombus and that the opposite angles of $HBDF$ are equal.



Online resources:

Investigation Finding the shortest path around a rectangle

Quick quiz 6.4

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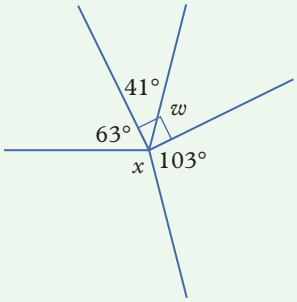
ANS p717

Checkpoint

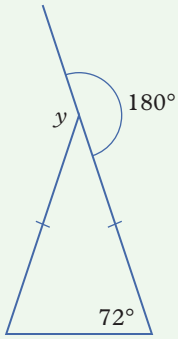
Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this module.

6.1 **1** Determine the values of the pronumerals.

a

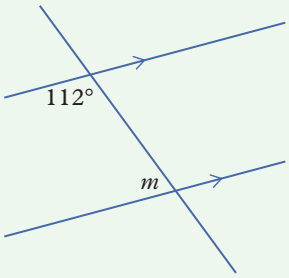


b



6.1 **2** Determine the values of the pronumerals.

a



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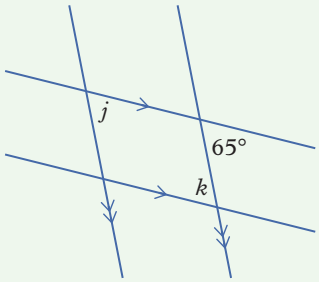
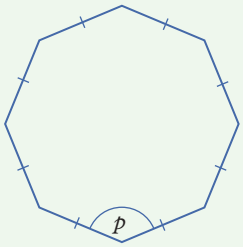


6.1 **3** Determine the values of the pronumerals.

a



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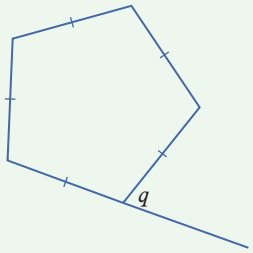
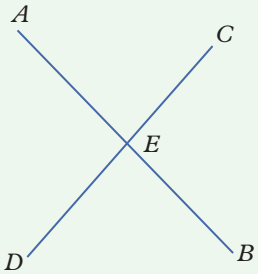
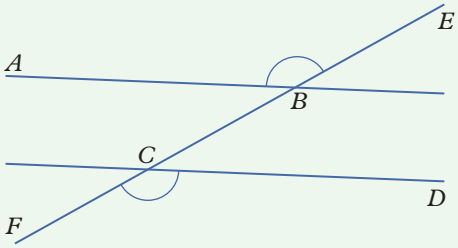


6.2 **4** Given that AB and CD in the diagram on the right are line segments, prove that $\angle AEC = \angle DEB$.



6.2 **5** Given that $\angle ABE = \angle DCF$, prove that $AB \parallel CD$.



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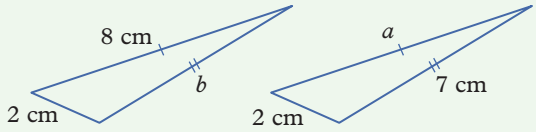
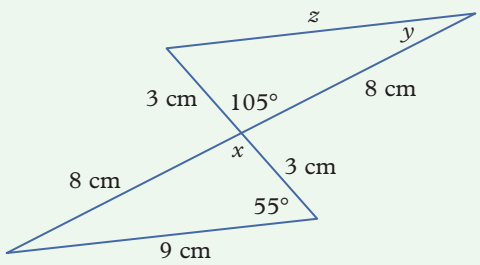
6.3 **6** Determine the values of the pronumerals.

a

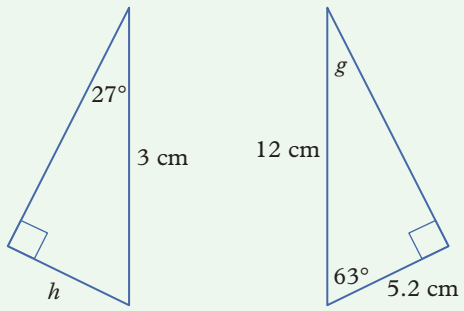


b

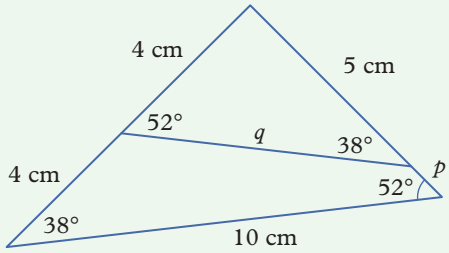


6.3 **7** Determine the values of the pronumerals.

a

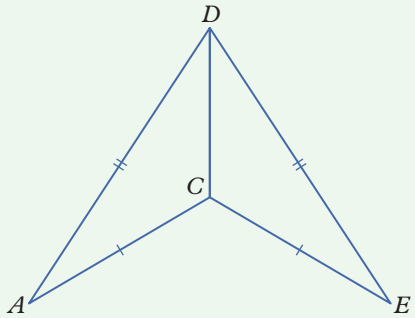


b

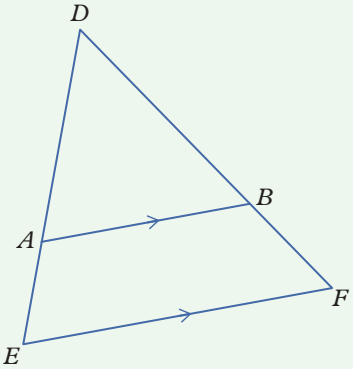


6.3 **8** Consider the following diagrams.

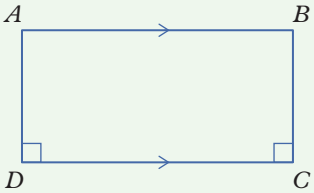
a Prove that $\triangle ADC \cong \triangle EDC$, and therefore, that $\angle ACD = \angle ECD$.



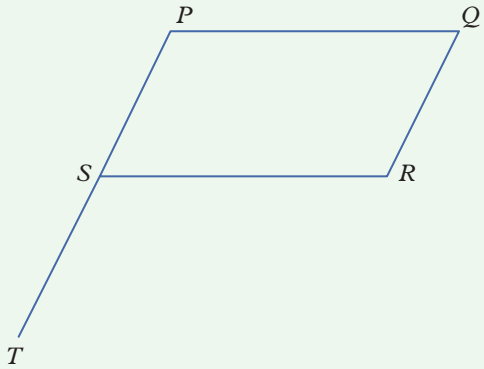
b Prove that $\triangle ADB \sim \triangle EDF$.



6.4 **9** Given that $AB \parallel DC$, $BC \perp CD$ and $AD \perp CD$, prove that $ABCD$ on the right is a rectangle, and therefore, that $AC = BD$.



6.4 **10** Given that $\angle SPQ = \angle RST = \angle SRQ$, prove that $PQRS$ is a parallelogram.



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Lesson 6.5

Circle geometry: angles

Learning intentions

By the end of this lesson you will be able to ...

✔ understand and apply circle theorems related to angles.

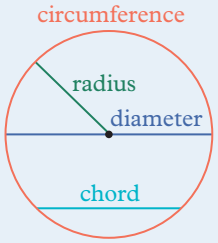
Inter-year links

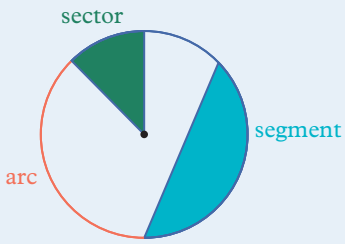
Year 8 7.2 Triangles

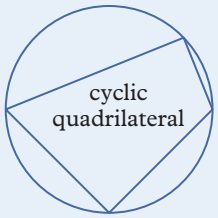
Circle terminology

Key content video
Circle geometry: angles

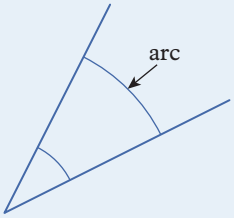
- A circle is a set of points equidistant from a central point.
- **Arc**: A part of the **circumference** of a circle.
  → A minor arc measures less than a semicircle (180°). A major arc measures more than a semicircle.
- **Chord**: A line segment connecting two points on a circle's circumference.
- **Sector**: A portion of a circle enclosed by two radii and an arc on the circumference.
- **Segment**: A region of a circle enclosed by a chord and an arc on the circumference.
- A cyclic quadrilateral is a quadrilateral whose vertices all lie on the circumference of a single circle.

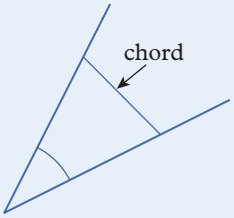


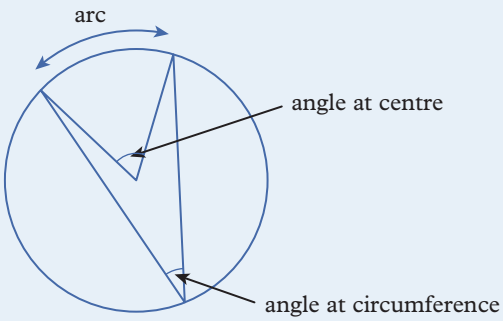




- An angle whose rays meet the endpoints of an arc or chord is said to be 'subtended by' the arc or chord.







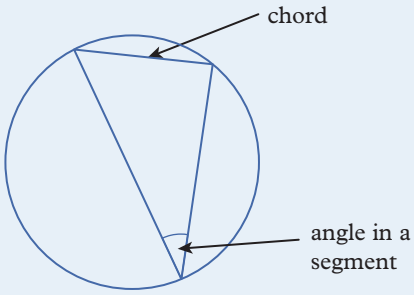
→ The angle at the centre refers to an angle which is subtended by an arc or chord at the centre of a circle.

→ The angle at the circumference refers to an angle which is subtended by an arc or chord at the circumference of a circle.

→ The arc on which an angle at the centre or circumference stands is the arc which subtends that angle.

→ Two angles which are both subtended by the same arc are said to be 'standing on the same arc'.

→ An angle in a segment is an angle subtended by the endpoints of the chord that forms that segment to a point on the circumference.



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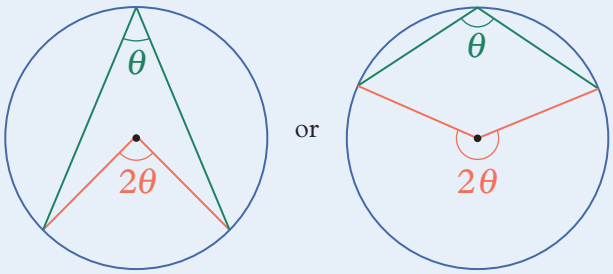
Circle theorems 1–5

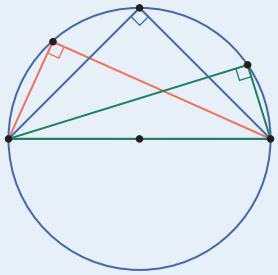
- **Circle theorem 1:** The angle subtended by an arc at the centre of a circle is twice the size of the angle subtended by the same arc at the circumference.
  This is also known as the 'angle at the centre' theorem.

Circle theorem 1



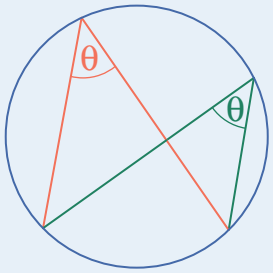
- **Circle theorem 2:** Any angle subtended at the circumference of a circle by the ends of a diameter of the circle is a right angle.
  This is also known as the 'angle in a semi-circle' theorem or Thales' theorem.

Circle theorem 2



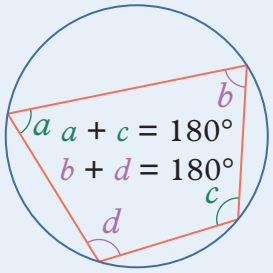
- **Circle theorem 3:** All angles subtended at the circumference by the same arc are equal in size.
  This is also known as the 'angles in the same segment' theorem.

Circle theorem 3



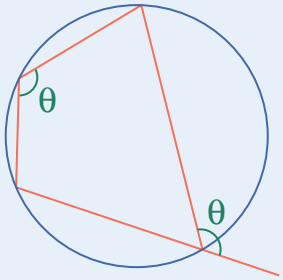
- **Circle theorem 4:** The opposite angles of a cyclic quadrilateral are supplementary.
  This is also known as the 'cyclic quadrilateral' theorem.

Circle theorem 4



- **Circle theorem 5:** Any exterior angle of a cyclic quadrilateral is equal to the opposite interior angle.
  This is an extension of the 'cyclic quadrilateral' theorem.

Circle theorem 5



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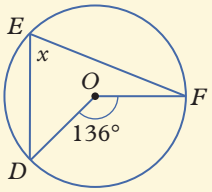
Worked example 6.5A Using circle theorem 1

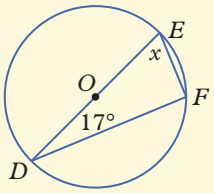
Find the value of x in the circle on the right.



THINK	WRITE
1 The angle subtended by an arc at the centre of a circle is twice the size of the angle subtended at the circumference. So $\angle DOF = 2\angle DEF$.	$136° = 2x$
2 Write your answer.	$x = 68°$

Worked example 6.5B Using circle theorem 2

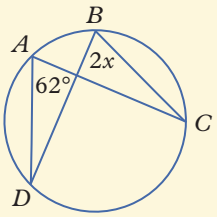
Find the value of x in the circle on the right.



THINK	WRITE
1 Any angle subtended at the circumference by the diameter of a circle is a right angle. So $\angle DFE$ is a right angle.	$\angle DFE = 90°$
2 Use the angle sum of a triangle to help you write an equation to find the value of x.	$x + 17° + 90° = 180°$
3 Solve the equation and write your answer.	$x = 180° - 17° - 90°$ $x = 73°$

Worked example 6.5C Using circle theorem 3

Find the value of x in the circle on the right.



THINK	WRITE
1 All angles subtended by the same arc at the circumference are equal in size, therefore $\angle DBC$ and $\angle DAC$ are equal in size.	$\angle DBC = \angle DAC$
2 Substitute values and solve for x.	$2x = 62°$ $x = 31°$

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Helpful hints

- ✔ If you can't remember the circle theorems at first, carefully inspecting the diagrams can be helpful. The given diagram might help you remember the circle theorem or theorems you need to use!
- ✔ Don't attempt to memorise the exact phrasing of each circle theorem. Applying the theorems in different scenarios will help you to remember them.

Learning pathways

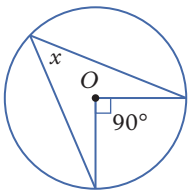
1–4, 5(a–c), 6–8	1–5, 9, 10, 12(a)	4, 5, 11, 12

ANS p718

Exercise 6.5A: Understanding and fluency

WE 6.5A **1** Find the value of x in each of these circles.

a



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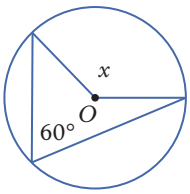

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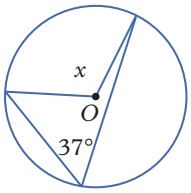
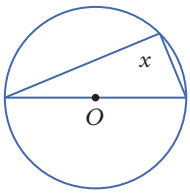


WE 6.5B **2** Find the value of x in each of these circles.

a



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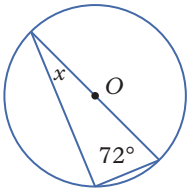

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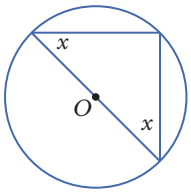
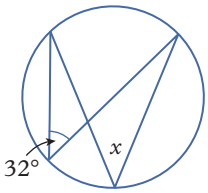
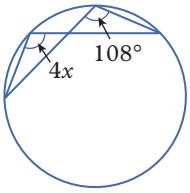


WE 6.5C **3** Find the value of each pronumeral in these circles.

a



b



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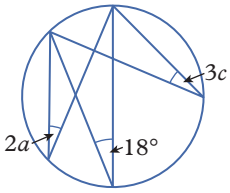
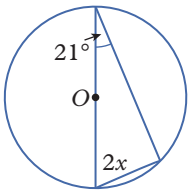


4 Find the value of each pronumeral in these circles, and state which circle theorem is used.

a



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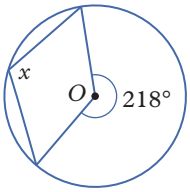
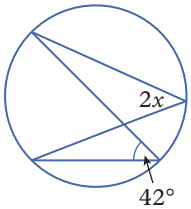
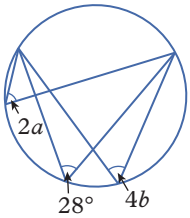


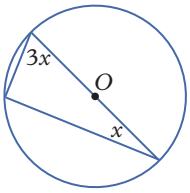
c



d



e



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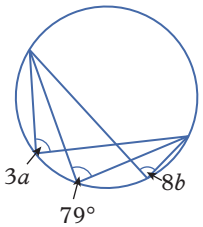

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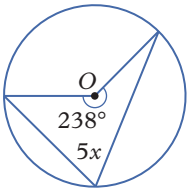
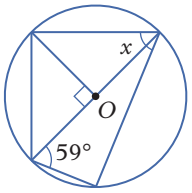
Exercise 6.5B: Problem solving and reasoning

5 Use your understanding of triangle and circle properties and theorems to find the value of x in each of these diagrams.

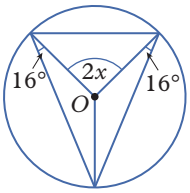
a



b



c



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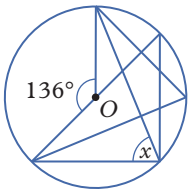
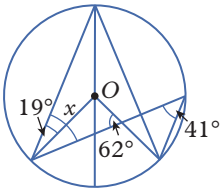


e



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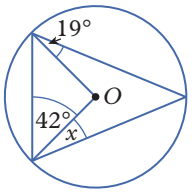
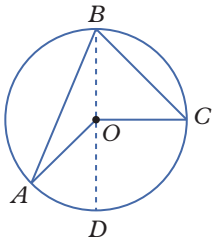


6 This question proves circle theorem 1. Consider the diagram on the right.



a How do you know that $AO = BO = CO$?

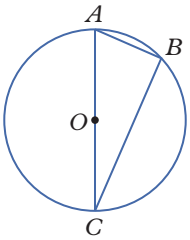
b How do you know that $\angle ABO = \angle OAB$?

c Use the theorem for the exterior angle of a triangle to show that $\angle AOD = 2\angle ABO$.

d Use your answers for parts **b** and **c** to show that $\angle DOC = 2\angle OBC$.

e Explain how this shows that $\angle AOC = 2\angle ABC$.

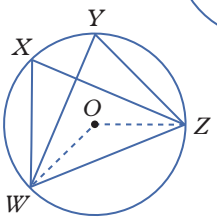
7 This question proves circle theorem 2. Consider the diagram on the right.



a What is the angle at the centre of the circle?

b Use circle theorem 1 to show that $\angle ABC = 90°$.

8 This question proves circle theorem 3. Consider the diagram on the right.

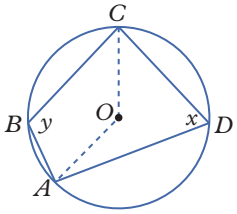


a Use circle theorem 1 to explain why $\angle WOZ = 2\angle WXZ$.

b Similarly, show that $\angle WOZ = 2\angle WYZ$.

c What can you say about $\angle WYZ$ and $\angle WXZ$?

9 This question proves circle theorem 4. Consider the diagram on the right, showing a cyclic quadrilateral.



a Explain why the obtuse angle $\angle AOC = 2x$ and the reflex angle $\angle AOC = 2y$.

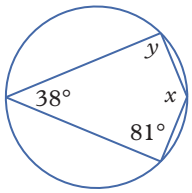
b Use what you know about angles at a point to explain why reflex $\angle AOC = 360° - 2x$.

c Use your answers from parts **a** and **b** to show that $y = 180° - x$.

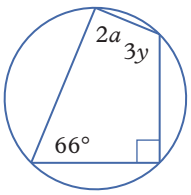
d Copy the diagram, replacing radii OC and OA with OB and OD, then use this to show that $\angle BAD = 180° - \angle BCD$.

e Find the value of each pronumeral in the circles below.

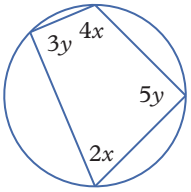
i



ii



iii



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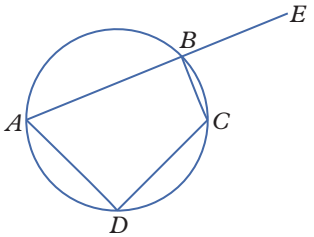
10 This question proves circle theorem 5. Consider the diagram below, showing another cyclic quadrilateral.



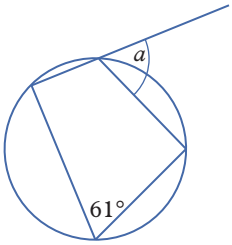
a Use circle theorem 4 to explain why $\angle ADC = 180° - \angle ABC$.

b Why does $\angle EBC = 180° - \angle ABC$?

c Explain why this shows that $\angle ADC = \angle EBC$.

d Find the value of each pronumeral in the diagrams below.

i



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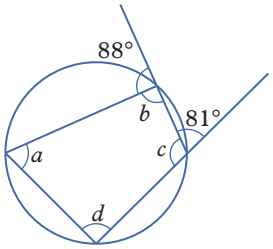
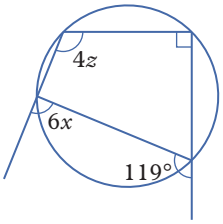


iii

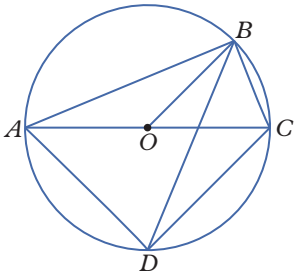


11 Using the diagrams from questions **6–10**, write a complete proof for each of the circle theorems 1–5.

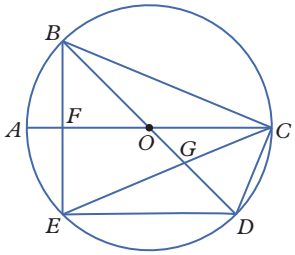
Exercise 6.5C: Challenge

12 Use the given information to answer these questions.

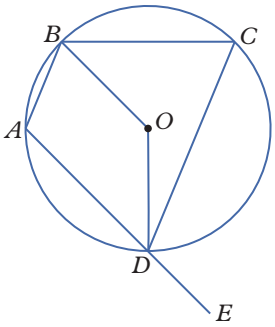
a Find $\angle DBC$, given that $\angle ADB = 68°$, $OA = OB = OC$ and $\angle BCD = 92°$.



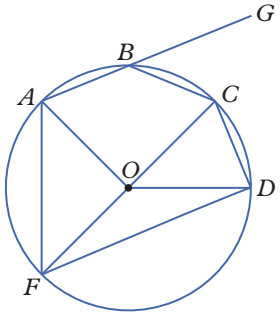
b Find $\angle DEG$, given that $BE \perp AC$, $\angle OCG = 19°$ and $\angle FOG = 142°$.



c Find $\angle ODC$, given that $\angle BAD = 96°$ and $\angle CBO = 46°$.



d Find $\angle OAB$, given that $\angle GBC = 46°$, $\angle BCD = 138°$ and $\angle AFD = 70°$.



Online resources:

Interactive skillsheet Circle geometry: angles

Investigation Clock angles

Quick quiz 6.5

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Lesson 6.6

Circle geometry: chords

Learning intentions

By the end of this lesson you will be able to ...

✔ understand and apply circle theorems related to chords.

Inter-year links

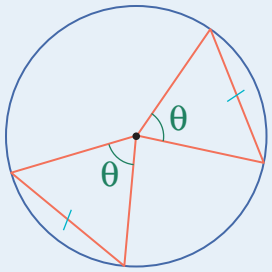
Year 8 7.2 Triangles

Circle theorems 6–9

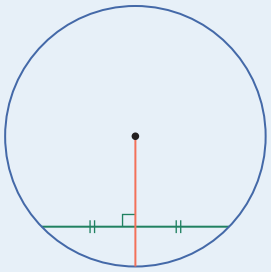
Key content video
Circle geometry: chords

- **Circle theorem 6:** Chords of equal length subtend equal angles at the centre of the circle.
  This is also known as the 'equal chords, equal angles' theorem.
- **Circle theorem 7:** If a radius and a chord intersect perpendicularly, then the radius bisects the chord.
- **Circle theorem 8:** Chords that are equal in length are **equidistant** (at equal distances) from the centre of the circle.
- **Circle theorem 9:** When two chords intersect inside a circle, this divides each chord into two line segments so that the product of the lengths of the line segments for both chords is the same.

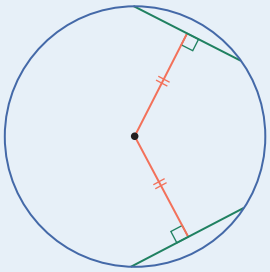
Circle theorem 6



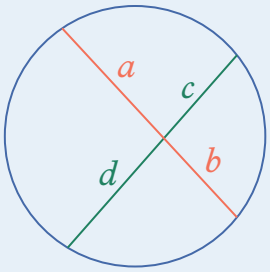
Circle theorem 7



Circle theorem 8



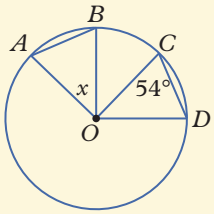
Circle theorem 9



$ab = cd$

Worked example 6.6A Using circle theorem 6

Given $AB = CD$, find the value of x in the circle on the right.



THINK	WRITE
1 Since OC and OD are radii, they are of equal length. This means that $\triangle COD$ is an isosceles triangle.	$OC = OD$
2 Find the two unknown interior angles of $\triangle COD$.	$\triangle COD$ is isosceles, so $\angle OCD = \angle ODC = 54°$. $\angle COD = 180° - 54° - 54°$ $= 72°$
3 Chords that are equal in length subtend equal angles at the centre of the circle. So $\angle AOB = \angle COD$.	$\angle AOB = \angle COD = 72°$
4 Write your answer.	$x = 72°$

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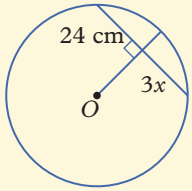
Worked example 6.6B Using circle theorem 7

Find the value of x in the circle on the right.



THINK

1. If a radius and a chord intersect perpendicularly, then the radius bisects the chord. So $3x$ is equal to 24 cm.
2. Solve for x.

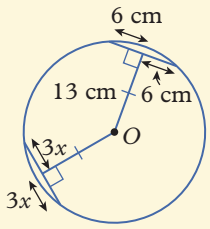
WRITE

$3x = 24$ cm

$x = 8$ cm

Worked example 6.6C Using circle theorem 8

Find the value of x in the circle on the right.



THINK

1. Chords that are equal in length are equidistant from the centre of the circle. So the two chords must be equal in length.
2. Solve for x, including units in your answer.

WRITE

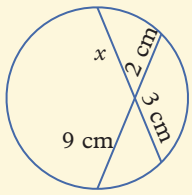
$3x + 3x = 6 + 6$

$6x = 12$

$x = 2$ cm

Worked example 6.6D Using circle theorem 9

Find the value of x in the circle on the right.



THINK

1. When two chords intersect, the products of the line segments they are divided into are equal. Write the product of the lengths of the line segments for each chord.
2. Simplify the equation.
3. Solve for x, including units in your answer.

WRITE

Chord 1: $x \times 3$ cm

Chord 2: 2 cm $\times$ 9 cm

$x \times 3 = 2 \times 9$

$3x = 18$

$x = 6$ cm

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Helpful hints

- ✔ All radii of the same circle are equal in size.
- ✔ When chords intersect, each of the equal products results from multiplying the lengths of the two parts of the same line segment.

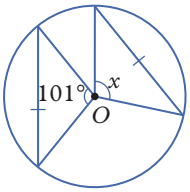
Learning pathways

1–9 ,12, 14	1–3, 4(b, c), 5–7, 10–12, 14, 15	5(d–i), 6(d–f), 7(d–f), 13–18

Exercise 6.6A: Understanding and fluency

WE 6.6A **1** Find the value of x in each of these circles.

a



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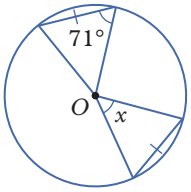

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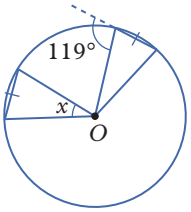
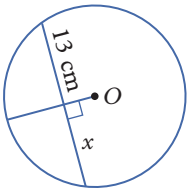


WE 6.6B **2** Find the value of x in each of these circles.

a



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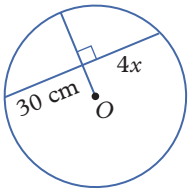

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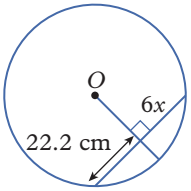
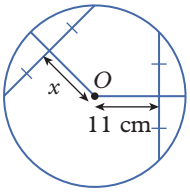


WE 6.6C **3** Find the value of each pronumeral in these circles.

a



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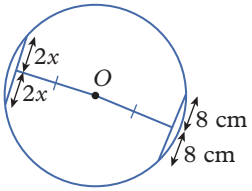

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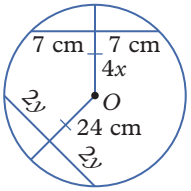
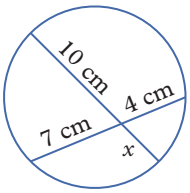


WE 6.6D **4** Find the value of x in each of these circles, correct to one decimal place.

a



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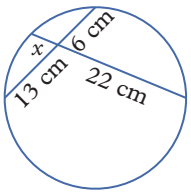

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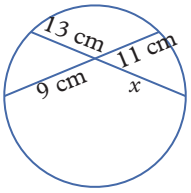
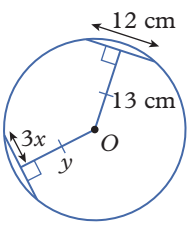


5 Find the value of each pronumeral in these circles. Round your answer to two decimal places where necessary.

a



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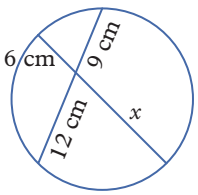

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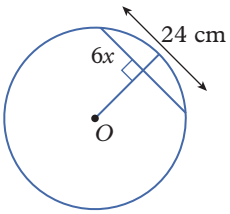

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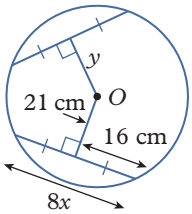

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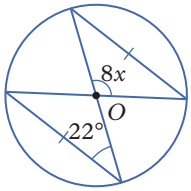

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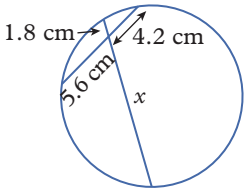

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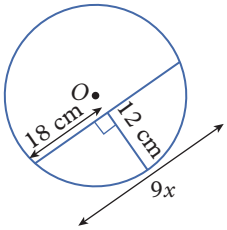

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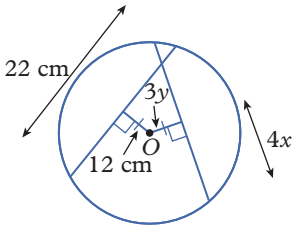

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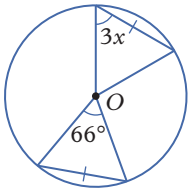
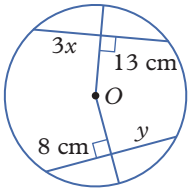


6 Use your understanding of circle properties and theorems and any other necessary geometric properties to find the value of each pronumeral in these circles. Round your answer to two decimal places where necessary.

a



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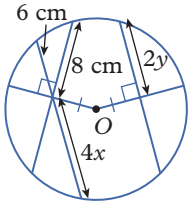

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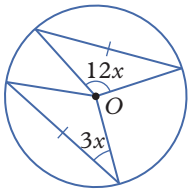

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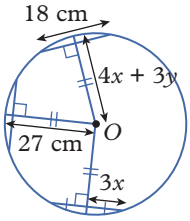

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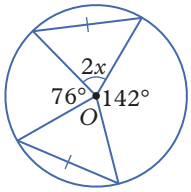

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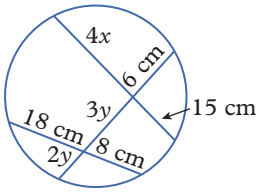
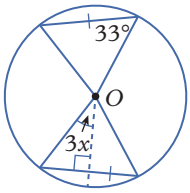


7 Find the value of each pronumeral in these circles. Round your answer to two decimal places where necessary.

a



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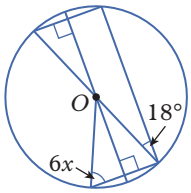

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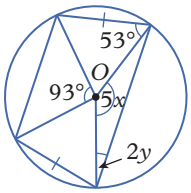

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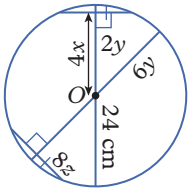

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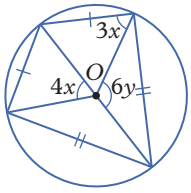

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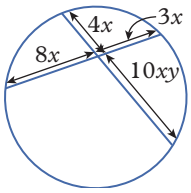
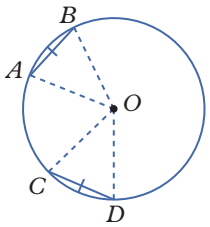


Exercise 6.6B: Problem solving and reasoning

8 This question proves circle theorem 6.
Consider the diagram on the right, in which $AB = CD$.



a Given that OA, OB, OC and OD are radii, what can you say about them?

b What type of triangles does this mean $\triangle OAB$ and $\triangle OCD$ are?

c Explain why $\triangle OAB \cong \triangle OCD$.

d Explain why $\angle COD = \angle AOB$.

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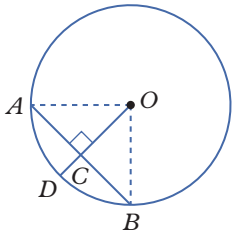
9 This question proves circle theorem 7. Consider the following diagram.



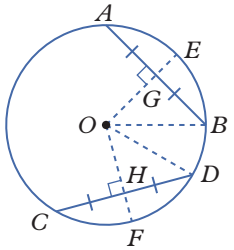
a Explain why $OA = OB$.

b Explain why $\angle OCA = \angle OCB$.

c Use one of the conditions for the congruence of triangles to explain why $\triangle ACO \equiv \triangle BCO$.

d Explain why $AC = BC$.

10 This question proves circle theorem 8. Consider the following diagram in which $AB = CD$.



a Explain why $OE = OB = OD = OF$.

b Use circle theorem 7 to explain why $AG = GB$ and $CH = HD$.

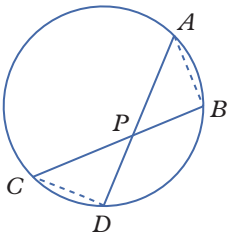
c Likewise, use circle theorem 7 and the fact that $AB = CD$ to explain why $GB = HD$.

d Use one of the conditions for the congruence of triangles to explain why $\triangle OHD \cong \triangle OGB$.

e Explain why $OH = OG$.

f Use your understanding of Pythagoras' theorem to explain why the shortest distance from the centre of a circle to a chord is perpendicular to that chord.

11 This question proves circle theorem 9. Consider the diagram on the right.



a Explain why $\angle CPD = \angle APB$.

b Use circle theorem 3 to explain why $\angle DCB = \angle BAD$.

c Explain why $\angle PDC = \angle PBA$.

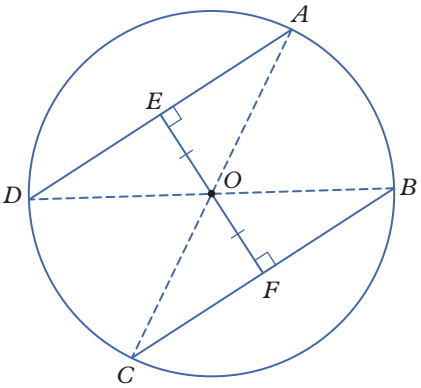
d Explain how you can tell that $\triangle CPD$ is similar to $\triangle APB$.

e Explain why you can state that $\frac{AP}{BP} = \frac{CP}{DP}$, and hence, that $AP \times DP = CP \times BP$.

12 Prove that, if a cyclic quadrilateral is a parallelogram, then it must be a rectangle.

13 Using the diagrams from questions **8–11**, write a complete proof for each of the circle theorems 6–9.

14 Use the diagram below to prove that if two chords are equidistant from the centre of a circle, then the chords are equal in length.



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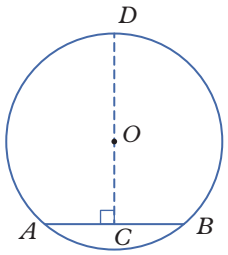
15 Another form of proof is 'proof by contradiction', where we assume that what we want to prove is not true, and then we demonstrate that this assumption is false.

We can use a proof by contradiction to show that the perpendicular bisector of a chord passes through the centre of a circle.

Consider the following diagram, where CD is a perpendicular bisector of AB.



Start with the assumption that the perpendicular bisector of a chord does not pass through the centre. That is, that O does not lie on the line segment CD.

a What is the size of $\angle ACD$?

b Now consider the radius of the circle that perpendicularly intersects AB.

i Explain how you know that this radius must go through C.

ii Explain how this demonstrates that O lies on the line segment CD.

iii Explain where the contradiction arises and what this demonstrates.

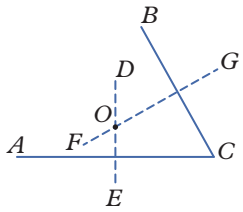
Exercise 6.6C: Challenge

16 Non-collinear points are points that do not lie on the same straight line.

a Use the diagram, below right, where DE is a perpendicular bisector of AC and FG is a perpendicular bisector of BC, to prove that there is only one circle that passes through any three non-collinear points.

The following properties can be used in your proof:

- Every point on a perpendicular bisector of a line segment is equidistant from the endpoints of the line segment.
- A point which is equidistant from two distinct points lies on the perpendicular bisector of the two points.
- Two non-parallel straight lines only intersect at a unique point.
- The perpendicular bisectors of two non-parallel lines are not parallel.

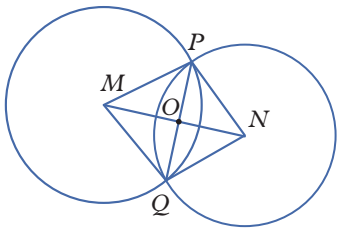


b Explain how the result from part **a** proves that given any three non-collinear points, the point of intersection of the perpendicular bisectors of any two sides of the triangle formed by the three points is the centre of the circle which passes through all three points.

17 A circle passes through the points (1, 1), (5, 1) and (1, 6). State the coordinates of the centre of the circle.

18 Consider the diagram on the right, which shows two intersecting circles. Prove that the line joining the centre of the circles (MN) bisects the chord common to both circles (PQ) at right angles. You may use the following properties in your proof:

- The perpendicular bisector of a chord passes through the centre of a circle.
- There is a unique straight line passing through any two points.



Online resources:

Interactive skillsheet
Circle geometry: chords

Quick quiz
6.6

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Lesson 6.7

Circle geometry: tangents and secants

Learning intentions

By the end of this lesson you will be able to ...

✔ understand and apply circle theorems related to tangents and secants.

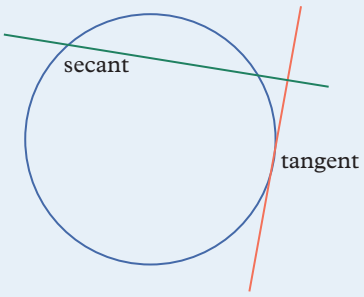
Inter-year links

Year 8 7.2 triangles

Tangents and secants

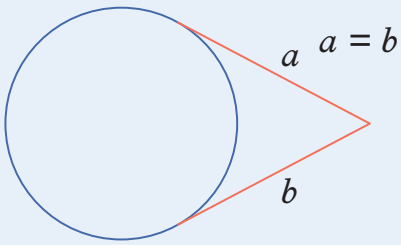
Key content video
Circle geometry: tangents and secants

- A **tangent** is a line that touches the circumference of a circle at one point only.
  → The angle between a radius of a circle and a tangent to a circle is 90°.
- A **secant** is a line that intersects a circle twice.

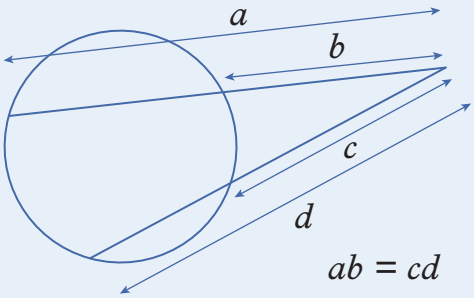


- **Circle theorem 10:** If two tangents intersect outside a circle, the distances along the tangent from the intersection to the circumference of the circle are equal.
- **Circle theorem 11:** If two secants are drawn from a circle to an intersecting point outside the circle, the product of the entire secant length by the external secant length is the same for each secant.
- **Circle theorem 12:** If a tangent and a secant intersect outside a circle, the product of the entire secant length by the external secant length is equal to the square of the tangent length.
- **Circle theorem 13:** The angle that lies between a tangent and a chord is equal to the angle subtended by the same chord in the alternate segment. This is also known as the 'alternate chord theorem'.

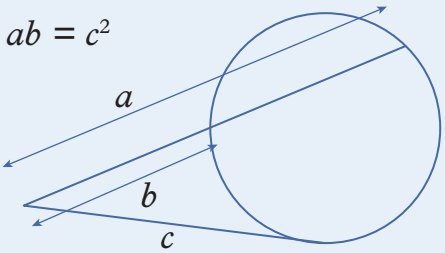
Circle theorem 10



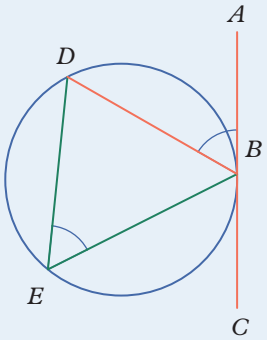
Circle theorem 11



Circle theorem 12



Circle theorem 13
$\angle ABD = \angle DEB$



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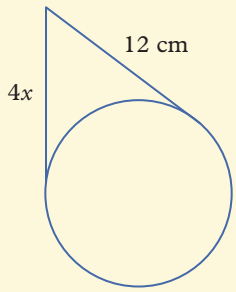
Worked example 6.7A Using circle theorem 10

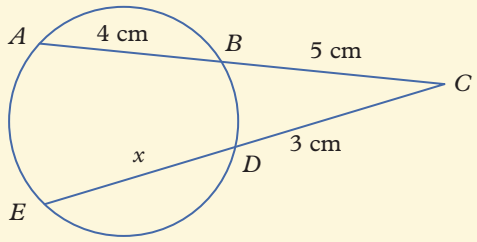
Find the value of x in the diagram on the right.



THINK	WRITE
1 The distances from the intersection of two tangents to the circumference of a circle are equal.	$4x = 12$
2 Solve for x, including units in your answer.	$x = 3$ cm

Worked example 6.7B Using circle theorem 11

Find the value of x in the following circle.



THINK	WRITE
1 When two secants are drawn from a circle to an intersecting point outside the circle, the product of the entire secant length by the external secant length is the same for each secant.	$AC \times BC = CE \times CD$
2 Write the length of each line segment.	$AC = 4 + 5$ $= 9$ cm $BC = 5$ cm
3 Substitute these lengths into the relationship and simplify.	$CE = (3 + x)$ cm $CD = 3$ cm $9 \times 5 = (3 + x) \times 3$ $45 = 3(3 + x)$ $15 = 3 + x$
4 Solve for x, including units in your answer.	$x = 12$ cm

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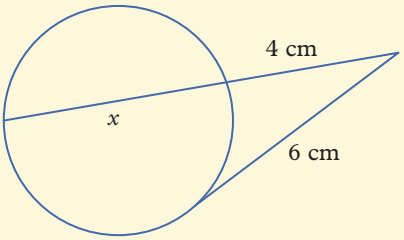
Worked example 6.7C Using circle theorem 12

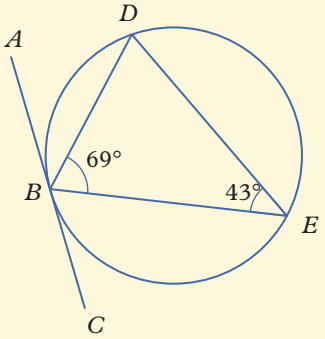
Find the value of x in the circle on the right.



THINK	WRITE
1 When a tangent and a secant intersect outside a circle, the product of the entire secant length by the external secant length is equal to the square of the tangent length.	$a \times b = c^2$
2 Write the length for each line segment.	$a = (x + 4)$ cm $b = 4$ cm $c = 6$ cm
3 Substitute into the formula and simplify.	$(x + 4) \times 4 = 6^2$ $4\,(x + 4) = 36$ $x + 4 = 9$
4 Solve for x, including units in your answer.	$x = 5$ cm

Worked example 6.7D Using circle theorem 13

Find the value of $\angle CBE$ in the circle on the right.



THINK	WRITE
1 Write the known information.	$\angle BED = 43^\circ$ $\angle DBE = 69^\circ$
2 The angle that lies between a tangent and a chord is equal to the angle subtended by the same chord in the alternate segment.	$\angle ABD = \angle BED$ $= 43^\circ$
3 Angles on a straight line sum to 180°.	$\angle ABD + \angle DBE + \angle CBE = 180^\circ$ $43^\circ + 69^\circ + \angle CBE = 180^\circ$
4 Solve for $\angle CBE$.	$\angle CBE = 68^\circ$

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Helpful hints

✔ When applying the alternate chord theorem, it is important to identify the chord and the correct angle subtended by this chord.

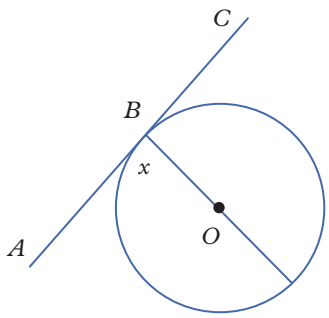
Learning pathways

1–5, 6(a–c), 7–9	2–5, 6(d–f), 8–12	6, 7, 12–15

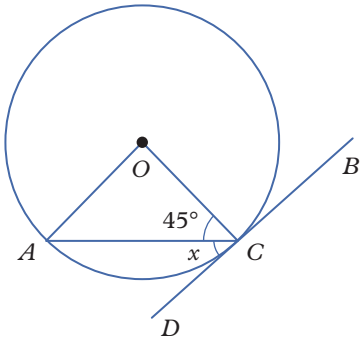
Exercise 6.7A: Understanding and fluency

1 Use the property that the angle between a radius of a circle and a tangent to a circle is 90° to find the value of x in each of these diagrams.

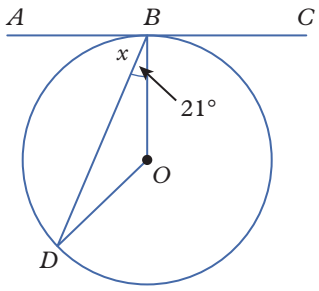
a



b

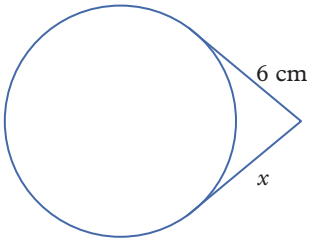


c



WE 6.7A **2** Find the value of the unknown pronumerals in each of these diagrams.

a



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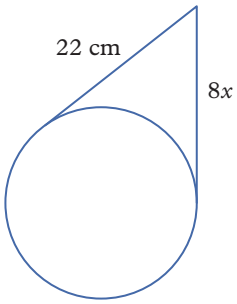

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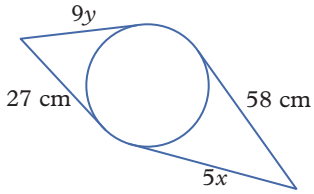
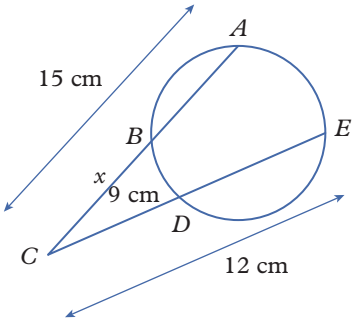


WE 6.7B **3** Find the value of x in each of these diagrams.

a



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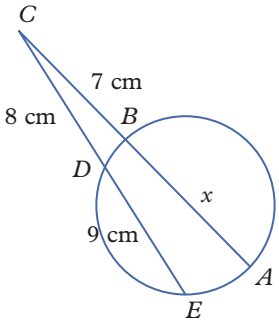

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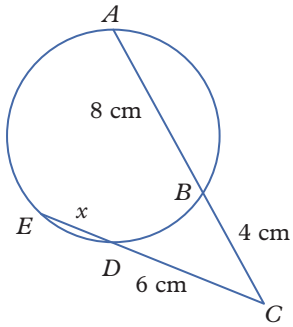

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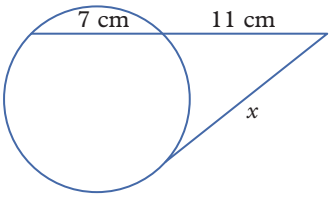
WE 6.7C **4** Find the value of x, correct to one decimal place, in each of these diagrams.

a



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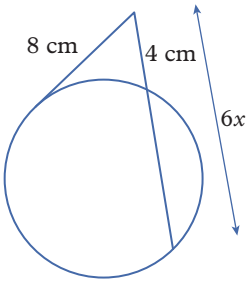

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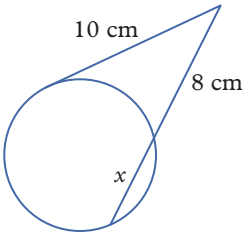
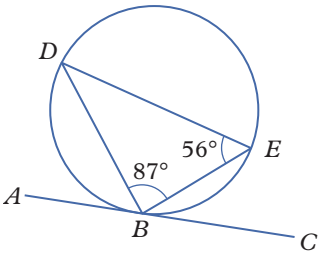


WE 6.7D **5** Find the value of $\angle CBE$ in each of these diagrams.

a



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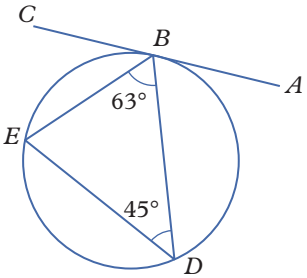

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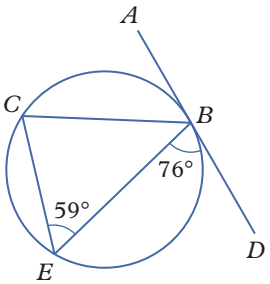
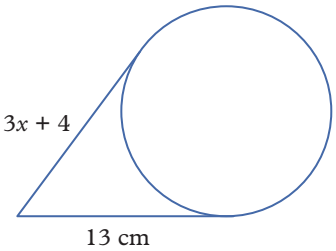


Exercise 6.7B: Problem solving and reasoning

6 Find the value of each pronumeral, giving reasons with your answers.

a



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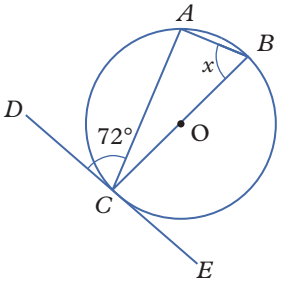

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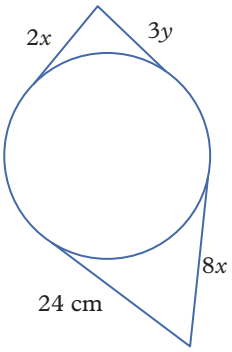

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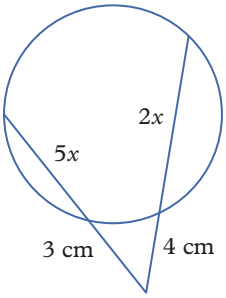

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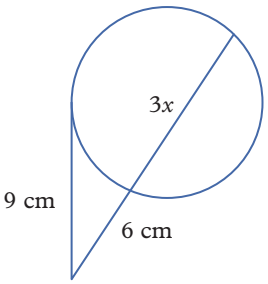

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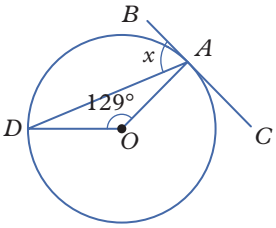
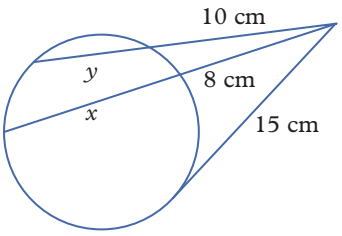


7 Use your understanding of circle theorems and geometric properties to find the value of each pronumeral. Give reasons with your answers.

a



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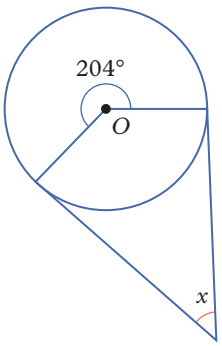

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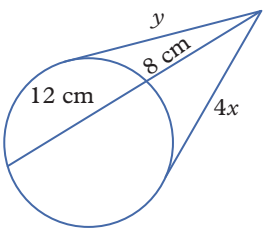

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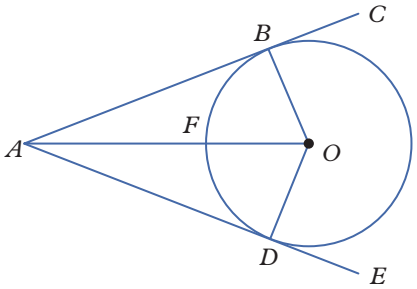
8 This question proves circle theorem 10. Consider the following diagram.



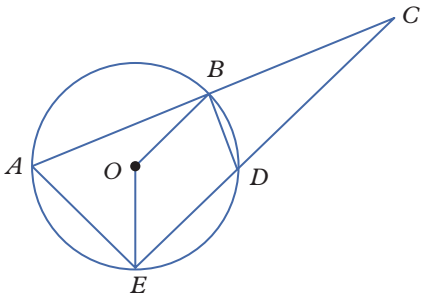
a How do you know that $\angle ABO = \angle ADO = 90^\circ$?

b Explain why $OB = OD$.

c Explain why $\triangle ABO \cong \triangle ADO$.

d Explain why this means that $AB = AD$.

9 This question proves circle theorem 11. Consider the following diagram.



a Explain why $\angle BOE = 2\angle BAE$.

b Explain why the reflex angle $\angle BOE = 360^\circ - 2\angle BAE$.

c Explain how you know $\angle BDE = 180^\circ - \angle BAE$.

d Hence explain why $\angle BDC = \angle BAE$.

e Now consider $\triangle DCB$ and $\triangle ACE$.

i What angle do they have in common?

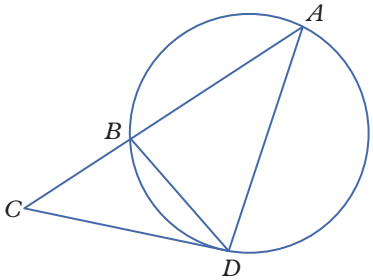
ii What pair of angles have you already shown are equal?

iii Explain why these triangles must therefore be similar.

iv Show that $\frac{CA}{CD} = \frac{EC}{BC}$.

v Explain why this can also be written as $AC \times BC = CE \times CD$.

10 This question proves circle theorem 12. Consider the following diagram.



a Which angle do $\triangle BCD$ and $\triangle DAC$ have in common?

b Construct OD and OB, where O is the centre of the circle, and explain why $\angle CAD = \angle CDB$.

c Explain why $\triangle BCD$ and $\triangle DCA$ are similar.

d Explain how you know that $\frac{CB}{CD} = \frac{CD}{CA}$.

e Explain why this can be written as $CA \times CB = (CD)^2$.

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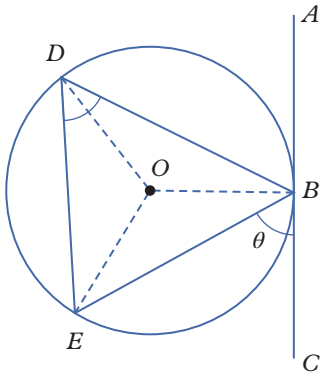
11 This question proves circle theorem 13. Consider the diagram on the right.



a Explain why $\angle OBE$ and $\angle OEB$ are equal.

b Write an equation that links $\angle EOB$ and $\angle OBE$ and explain why this equation is true.

c Explain why $\angle OBC$ is a right angle.

d Write an equation that links $\angle OBE$ and θ, with θ as the subject of the equation.

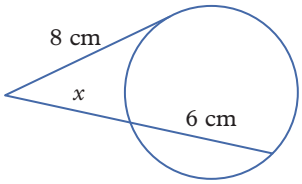
e Write an equation that links $\angle EOB$ and θ, showing how this equation was formed.

f Explain why $\angle EOB = 2 \times \angle EDB$.

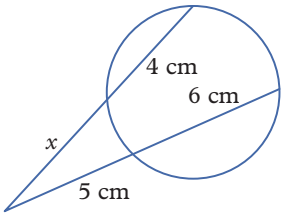
g Show that $\angle EDB = \theta$.

12 Find the value of each pronumeral, correct to one decimal place, in these diagrams.

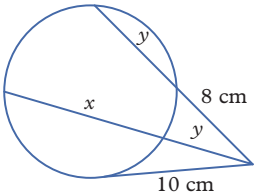
a



b



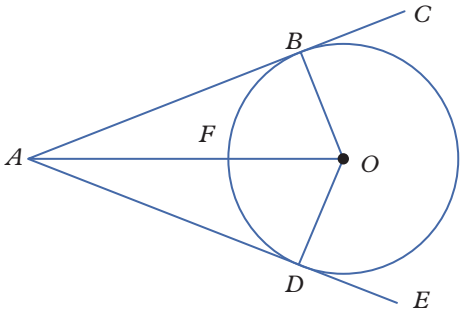
c



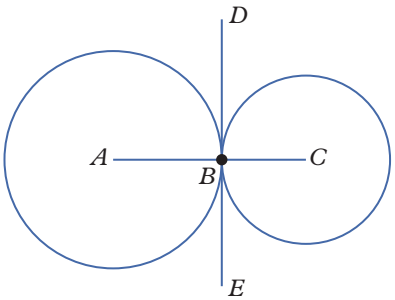
13 Using the diagrams from questions **8–11**, write a complete proof for each of the circle theorems 10–13.

Exercise 6.7C: Challenge

14 Use the diagram from question **8** (reproduced below) to prove that a straight line drawn from the centre of the circle to the intersection point of two tangents to the circle bisects the angle formed by the two tangents.



15 Use the following diagram, where DE is a tangent common to both circles, to prove that when two circles touch, the point of contact and the centres of the circles are collinear (i.e. they lie on a straight line).



Online resources:

Interactive skillsheet Circle geometry: tangents and secants

Worksheet Circle geometry

Quick quiz 6.7

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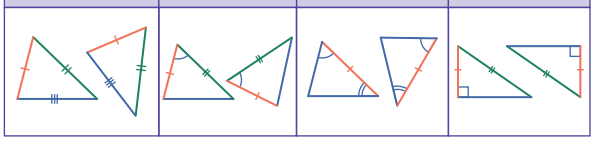
Lesson 6.8

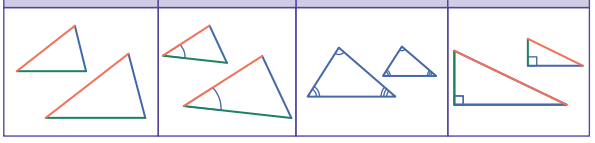
Review: Geometry

Module summary

Congruent triangles

SSS	SAS	AAS	RHS
			

Similar triangles

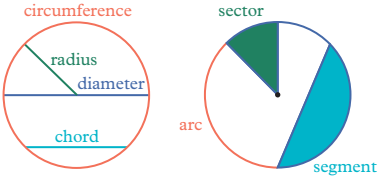
SSS	SAS	AAA	RHS
			

Triangle and quadrilateral properties

Definition	Properties
Triangle A three-sided polygon	The sum of the interior angles is 180°. The exterior angle is equal to the sum of the two interior opposite angles.
Equilateral triangle A triangle with all sides equal in length	Each interior angle is 60°.
Isosceles triangle A triangle with exactly two sides equal in length	The angles opposite the equal side lengths are equal.
Scalene triangle A triangle with no two sides equal in length	No angles are equal.
Quadrilateral A four-sided polygon	The sum of the interior angles is 360°. The exterior angle is equal to the sum of the two interior opposite angles.
Trapezium A quadrilateral with at least one pair of parallel sides	Diagonals of regular trapeziums are equal in length.
Parallelogram A quadrilateral with two pairs of parallel sides	Opposite sides are equal in length. Opposite angles are equal. Diagonals bisect each other.
Rhombus A parallelogram with two adjacent sides equal in length	All parallelogram properties. All sides are equal in length. Diagonals bisect each other at right angles. Diagonals bisect the interior angles.
Rectangle A parallelogram with all right angles	All parallelogram properties. Diagonals are equal in length.
Square A rectangle with two adjacent sides which are equal in length	All parallelogram properties. All rhombus properties. All rectangle properties.
Kite A quadrilateral with two pairs of adjacent sides which are equal in length	One pair of opposite angles are equal. Diagonals are perpendicular. One diagonal bisects the other diagonal. One diagonal bisects a pair of interior angles.

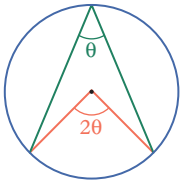
Terminology

- 'Bisect' means to cut in half.
- The symbol for 'is parallel to' is ||.
- The symbol for 'is perpendicular to' is ⊥.

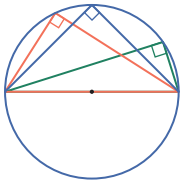


Circle theorems

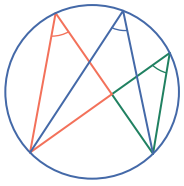
Circle theorem 1



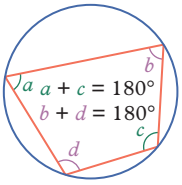
Circle theorem 2



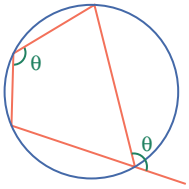
Circle theorem 3



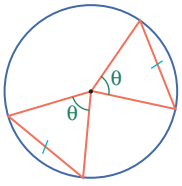
Circle theorem 4



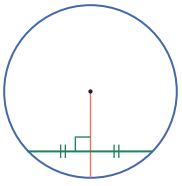
Circle theorem 5



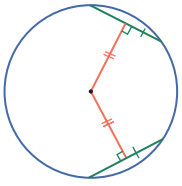
Circle theorem 6



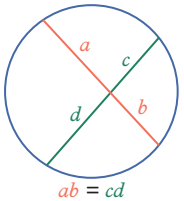
Circle theorem 7



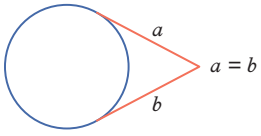
Circle theorem 8



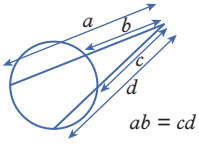
Circle theorem 9



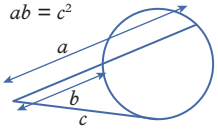
Circle theorem 10



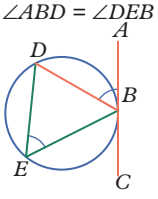
Circle theorem 11



Circle theorem 12



Circle theorem 13



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Review questions 6.8A: Mathematical literacy review

Review quiz
Take the review quiz to assess your knowledge of this module.
✔ 10
✔ 10A

Quizlet
Test your knowledge of this module by working individually or in teams.

The following key terms are used in this module:

- angle–side relationship
- arc
- bisect
- chord
- circumference
- complementary angles
- congruent
- convex polygon
- cyclic quadrilateral
- definition
- equidistant
- equilateral triangle
- exterior angle
- hypotenuse
- interior angle
- isosceles triangle
- kite
- parallel
- parallelogram
- perpendicular
- polygon
- proof
- property
- quadrilateral
- rectangle
- regular polygon
- rhombus
- scalene triangle
- secant
- sector
- segment
- similar
- square
- subtend
- supplementary angles
- tangent
- test
- theorem
- transversal
- trapezium
- triangle
- triangle inequality
- vertically opposite angles

1 What makes a polygon a regular polygon?

2 Explain how the angle–side relationship relates to the hypotenuse.

3 Explain the difference between a definition and a property of a plane shape.

4 Identify the key terms being referenced in each of these definitions.

a a polygon in which all interior angles are less than 180°

b a line segment connecting two points on a circle's circumference

5 Using an example, provide a definition in your own words for the following key terms.

a triangle inequality **b** tangent

6 Complete the following sentences using words from the key terms list.

a The opposite angles of a cyclic quadrilateral are _____.

b A _____ is a statement or result that can be demonstrated to be true based on known facts.

Review questions 6.8B: Multiple choice

6.1 **1** Which of these angles is supplementary to 25°?

A 75° **B** 335° **C** 155° **D** 65°

6.1 **2** What is the size of an individual internal angle in a regular decagon?

A 36° **B** 72° **C** 120° **D** 144°

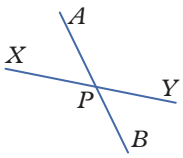
6.2 **3** Straight lines AB and XY intersect at P. Given that $\angle APY = 2\angle APX$, which of the following statements is true?

A $\angle BPX = 2\angle APY$

B $\angle XPB = 2\angle APX$

C $\angle YPB = 2\angle APX$

D $\angle APX = 2\angle BPY$



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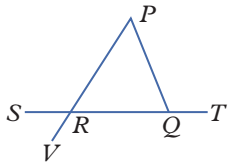
6.2 **4** Which of the following statements is true for the diagram below?

A $\angle RPQ + \angle PQR = \angle SRP$

B $\angle RPQ = \angle RQP$

C $\angle PRS = \angle PQT$

D $\angle TQP = \angle PRQ + \angle RQP$



6.3 **5** Which of these is *not* a test for congruent triangles?

A AAA **B** SSS **C** SAS **D** AAS

6.3 **6** Which of these is *not* a test for similar triangles?

A AAA **B** SSS **C** SAS **D** AAS

6.4 **7** Which of the following is not a property of all rhombuses?

A Opposite sides are equal in length

B Diagonals bisect each other at right angles

C Opposite angles are equal

D All interior angles are right angles

10A 6.5 **8** For an arc in a circle, the angle subtended at the centre of the circle and the angle subtended at the circumference are complementary angles. What are the values of the two angles?

A 60° and 30° **B** 45° and 45° **C** 90° and 45° **D** 120° and 60°

10A 6.6 **9** A circle contains two chords of the same length. Which of these statements is *not* true?

A The arcs on which the two chords stand are the same length.

B The chords are the same perpendicular distance from the centre.

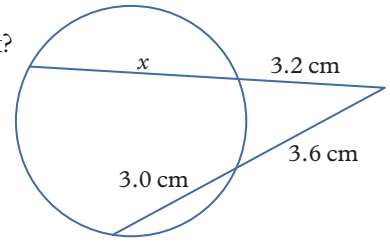
C The chords cannot intersect.

D The chords subtend equal angles at the centre of the circle.

10A 6.7 **10** Which of the following best represents the value of x in the diagram on the right?

A 2.8 cm **B** 3.4 cm

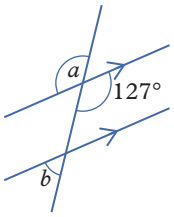
C 4 cm **D** 4.2 cm



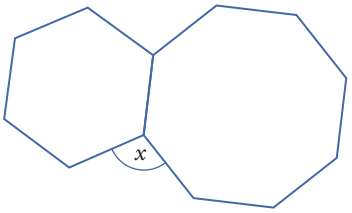
Review questions 6.8C: Short answer

6.1 **1** Determine the values of the pronumerals in these diagrams.

a



b



Note: Both polygons are regular.

6.2 **2** $\triangle ABC$ has an internal line drawn from vertex B, meeting the opposite side AC at point D. Also, $\angle ABC$ is a right angle.

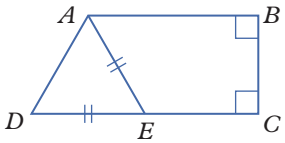
a Draw a sketch of the triangle.

b Use only the given information to prove that $\angle ADB = \angle DBC + \angle BCD$.

6.2 **3** Refer to the diagram on the right and write a proof to show that:

a $\angle AEC = 2\angle ADE$

b $2\angle ADE + \angle BAE = 180°$.



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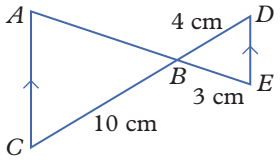
6.3 **4** Consider the diagram on the right.



a Provide a reason why $\triangle ABC$ is similar to $\triangle EBD$.

b If $\triangle EBD$ is the reduced image of $\triangle ABC$, what is the scale factor?

c What is the length of AB?

6.4 **5** Rectangle $ABCD$ has side lengths of 4 cm and 3 cm. Its diagonals intersect at point P.

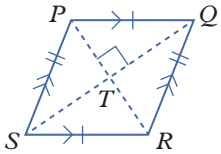
a Draw a diagram to represent this rectangle.

b Explain why $\angle ABP$ is equal to $\angle CDP$, and $\angle PAB$ is equal to $\angle PCD$.

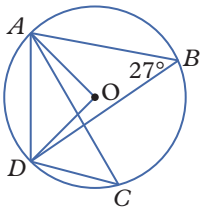
c Explain why $\triangle PAB \equiv \triangle PCD$.

d If $\angle BDC = 37°$, find the size of $\angle CBD$.

6.4 **6** Use the information in the following diagram to prove that figure $PQRS$ is a rhombus.

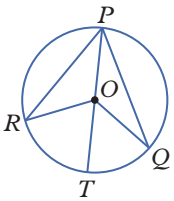


10A 6.5 **7** Determine the size of these angles in the diagram on the right.

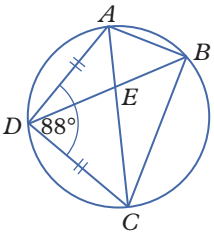


a $\angle AOD$ **b** $\angle ACD$ **c** $\angle OAD$

10A 6.5 **8** For the following diagram, prove that
$2\angle RPO + 2\angle QPO + \angle POR + \angle POQ = 360°$.



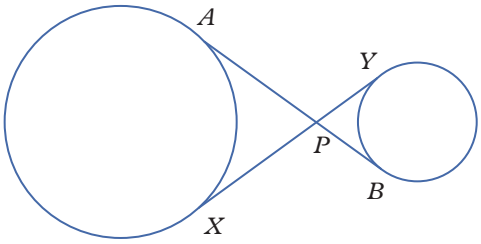
10A 6.6 **9** Consider the following diagram.



a Determine the size of $\angle DBC$.

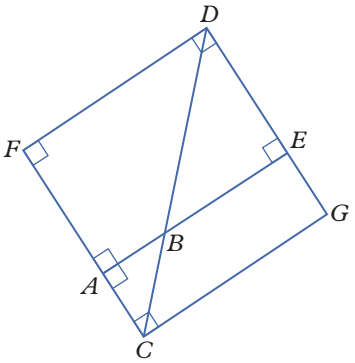
b Prove that $\angle DBC = \angle ABD$.

10A 6.7 **10** In the following diagram, AB and XY are tangents to the two circles. Prove that $AB = XY$.



Review questions 6.8D: Mathematical modelling

1 Consider the quadrilateral $FDGC$ on the right. Any facts proven in parts **a–c** can be used as given facts in parts **d–f**.



a Prove that $\triangle CFD \cong \triangle CGD$.

b Prove that $\triangle CAB \sim \triangle CFD$.

c Prove that $\triangle CAB \sim \triangle DEB$.

d If $AC = DE$, prove that B is the midpoint of CD.

e If $\frac{FC}{AC} = 3$, prove that $\frac{DE}{AC} = 2$.

f If $\angle DBE = 45°$, prove that $FDGC$ is a square.

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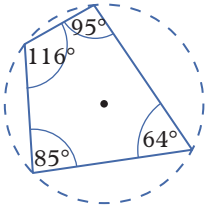
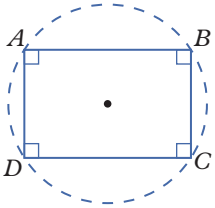
10A **2** All the edges of a cyclic quadrilateral are chords of the same circle. We can consider these chords and the chords that are the diagonals of the cyclic quadrilateral in terms of the circle theorems.

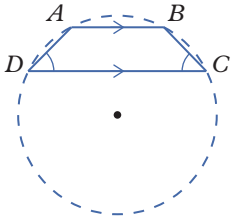
a Explain why this quadrilateral is cyclic.



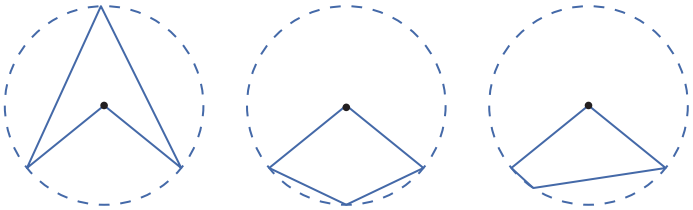
b Prove that all rectangles are cyclic, using *ABCD* shown below.

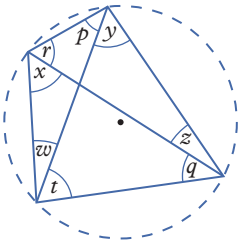


c An isosceles trapezium has equal base angles. Prove that all isosceles trapezia are cyclic, using *ABCD* shown below.



d Explain why quadrilaterals formed using circle theorem 1 (the 'angle at the centre' theorem) are not cyclic.





e Consider the diagram on the right, where diagonals have been added to the cyclic quadrilateral from part **a**.

i Explain why $p = q$, $r = t$, $x = y$ and $w = z$.

ii If $x = 54°$, determine the values of p, q, r, t, w, y and z.

Checklist

Now that you have completed this module, reflect on your ability to do the following.

I can do this	I need to review this
☐ Use geometric properties to find unknown angles	☐ Go back to Lesson 6.1 Geometric properties
☐ Understand what constitutes a mathematical proof ☐ Write a mathematical proof consisting of a sequence of logically connected steps	☐ Go back to Lesson 6.2 Geometric proofs
☐ Find unknown side lengths and angles in congruent and similar triangles ☐ Identify whether a pair of triangles are congruent or similar ☐ Prove that a pair of triangles are congruent or similar	☐ Go back to Lesson 6.3 Congruence and similarity
☐ Write proofs of the properties of quadrilaterals	☐ Go back to Lesson 6.4 Proof and quadrilaterals
☐ Understand and apply circle theorems related to angles (Circle theorems 1–5)	☐ Go back to Lesson 6.5 Circle geometry: angles [10A]
☐ Understand and apply circle theorems related to chords (Circle theorems 6–9)	☐ Go back to Lesson 6.6 Circle geometry: chords [10A]
☐ Understand and apply circle theorems related to tangents and secants (Circle theorems 10–13)	☐ Go back to Lesson 6.7 Circle geometry: tangents and secants [10A]

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Semester 1 review

Short answer

oxforddigital These questions are available on Oxford Digital.

ANS p720

Mathematical modelling

1 At a local park, the height of a hill is measured from one side to the other, where height is measured from the park's entrance near the base of the hill. The height is then modelled by an equation which gives the height, h metres, at a horizontal distance, d metres, from the park's entrance: $h = -0.05d^2 + d$, where $h \geq 0$.

a Determine the width of the hill.

b Determine the maximum height of the hill and its horizontal distance from the entrance.

c Sketch the graph of the equation for an appropriate domain.

d Determine the horizontal distances to the points on the hill that are 4 m above ground level. Write your answers:

i as exact values

ii correct to two decimal places.

10A **e** The highest point of the hill does not actually lie halfway between the edges of the base of the hill. To better model the hill, a polynomial model can be used. The polynomial also models the depth of a pond on the other side of the hill, for which the depth is the negative value of the height. The pond is 8 m wide.
Using the width of the hill from part **a**, write the equation of the polynomial in the form $h = \frac{1}{380}d(d - a)(d - b)$, where a and b are integers.

10A **f** Write the polynomial in part **e** in the form $h = \frac{1}{380}(d^3 + pd^2 + pd)$, where p and q are integers.

10A **g** Sketch the graph of the polynomial over an appropriate domain.

10A **h** Using the polynomial model, use technology to determine, to two decimal places:

i the maximum height of the hill and the horizontal distance from the entrance to the highest point

ii the maximum depth of the pond and the horizontal distance from the entrance to the deepest point.

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2 A new school is being designed. There will be four new buildings, A, B, C and D, arranged in the shape of a convex quadrilateral. There will be external paths from A to B (AB), B to C (BC), C to D (CD) and D to A (DA) as well as internal paths from A to C (AC) and B to D (BD).

a If the internal paths are to intersect perpendicularly, what are the possible quadrilateral layouts the school could choose?

b If the paths AB and BC are to be equal in length and the paths CD and DA are to be equal in length but the internal paths are not to be equal in length, which quadrilateral could the school choose?

c Prove that one pair of opposite angles in the quadrilateral in part **b** are equal.

d The school decides to have a circular athletics track that passes the four buildings as closely as possible.

10A **i** What extra feature does the quadrilateral chosen by the school need to have? How are opposite pairs of internal angles in such quadrilaterals related?

10A **ii** Use an appropriate circle theorem to prove the relationship between pairs of opposite angles in such quadrilaterals in part **i**.

10A **iii** Prove that if the quadrilateral from part **b** has the feature from part **d i**, then the quadrilateral is made up of two congruent right-angled triangles.

3 Air resistance in measured in units of force called Newton (N). The air resistance on an object is directly proportional to the object's velocity when wind is moving smoothly in parallel lines (laminar flow). It is also directly proportional to the square of the object's velocity when wind is moving chaotically (turbulent flow).

a During laminar flow, the air resistance on an object travelling at 10 m/s is 25 N.

i Determine the constant of proportionality and write the equation relating the laminar flow air resistance, R_L, and the velocity, v.

ii Determine the air resistance when the object is travelling at 20 m/s.

b During turbulent flow, the air resistance on the object travelling at 10 m/s is 60 N.

i Determine the constant of proportionality and write the equation relating the turbulent flow air resistance, R_T, and the velocity, v.

ii Determine the air resistance when the object is travelling at 20 m/s.

c Sketch the graphs of the two air resistance equations on the same pair of axes.

d Determine the speeds at which the air resistance during laminar and turbulent flow are the same.

e The time it takes for the object to travel a set distance is inversely proportional to its velocity. If an object moves at 8 m/s and takes 12.5 seconds to travel a set distance, then determine the constant of proportionality and write the equation for the velocity, v, of the object in terms of the time, t, it takes to travel the set distance.

f What is the set distance that is used to determine the relationship between velocity and time in part **e**?

10A **g** Sketch the graph of the velocity versus the time it takes to travel the set distance.

10A **h** Can the time it takes to travel the set distance ever equal zero? Explain.

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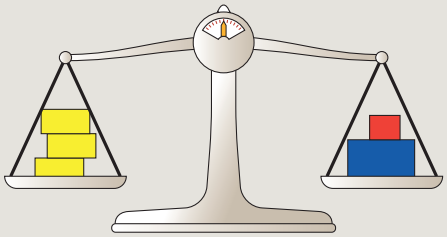
ANS p721

EXPLORATIONS 1

1 Balancing bricks

Indy has a collection of coloured plastic bricks and a balance scale. All bricks of the same colour weigh the same, but bricks of different colours weigh different amounts.



a Indy discovers the following about red, blue, yellow and green bricks:

- 1 blue brick weighs the same as 1 yellow and 1 green brick
- 3 yellow bricks weigh the same as 1 blue and 1 red brick
- 1 blue brick weighs more than 1 green and 3 red bricks
- 14 red bricks weigh more than 1 blue brick
- at least one combination of the other bricks weighs the same as 1 yellow brick.

Find all such combinations.

b Indy discovers the following about black, white, orange and purple bricks:

- 1 orange brick weighs the same as 1 white and 1 black brick
- 3 white bricks weigh the same as 1 orange and 1 purple brick
- 2 white bricks weigh less than 5 purple bricks
- only one combination of the other bricks weighs the same as 1 black brick.

Find that combination.

2 Times tables

In each of the following partially completed multiplication tables, the letters A, B, C, … represent different whole numbers, subject to some extra conditions. Find the value of each letter and complete the table.

a Letters A to J represent the numbers 1 to 10, in some order. The sum of numbers A to E is equal to the sum of numbers F to I.

×	A	B	C	D	E
F	18				
G				20	
H					
I					24
J			10		

b Letters A to H represent the even numbers from 2 to 16, in some order. The sum of E, G and H is twice the sum of A, B, C and D.

×	A	B	C	D
E		40		
F			24	
G				112
H				

c Letters A to I represent the numbers 2 to 10, in some order. The sum of the squares of A, B and C is equal to the sum of the squares of D, E, G, H and I.

×	A	B	C	D	E
F			30		
G		28			
H				27	
I					

3 How irrational!

a Rationalise the denominator to simplify $\dfrac{1}{\sqrt{2}+\sqrt{3}}$.

b Let a and b be positive integers and consider the fraction $\dfrac{1}{\sqrt{a}+\sqrt{b}}$.

i Find, in terms of a and b, an alternative expression for this fraction in which the denominator is an integer.

ii Does your answer always make sense? If not, describe when it fails and find the correct answer in this case.

c Rationalise the denominator twice to simplify $\dfrac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$.

d Let a, b and c be positive integers with $c = a + b$ and consider the fraction $\dfrac{1}{\sqrt{a}+\sqrt{b}+\sqrt{c}}$.

i Find, in terms of a, b and c, an alternative expression for this fraction in which the denominator is an integer.

ii Use your rule to simplify $\dfrac{1}{2\sqrt{2}+\sqrt{3}+\sqrt{5}}$.

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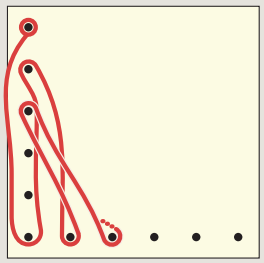
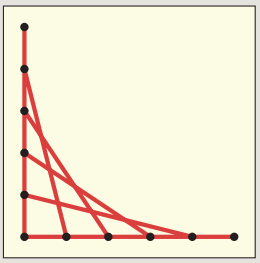
4 String art





If you have access to a pegboard and string, try the following hands-on activity first. Alternatively, you can use free online geometry software, such as GeoGebra *or* Desmos. *For best results, use as many pegs (or virtual pegs) as possible.*

On a pegboard, vertical and horizontal rows of equally spaced pegs are arranged in an L-shape. A piece of string is looped from the first (top) peg of the vertical row to the first (left-most) peg of the horizontal row, then to the second vertical peg, second horizontal peg, third vertical peg and so on – see the first diagram. When the string is pulled tight, it forms the design in the second diagram. The larger the number of pegs used, the smoother the curve formed by the boundary of the intertwined string. What is this curve?

To model this mathematically, it is best to rotate the board 45° anti-clockwise.

a Follow these instructions to make your own 'string art' on a sheet of grid paper:

- Draw x- and y-axes with the origin in the middle of the page.
- Plot the lines $y = x - 10$ and $y = -x - 10$, extending them to the edge of the page. (There is nothing special about the number 10. Later we will replace it with an arbitrary number, d.)
- Pick any point on one of the lines (for example, (13, 3) on $y = x - 10$). Find the point on the other line whose y-coordinate is the negative of your first point's y-coordinate (for example, (−7, −3) on $y = -x - 10$). Draw a line through your pair of points, extending it to the edge of the page.
- Repeat the previous step several times. As more and more lines are added, a familiar shape should emerge!

b Let's prove that the curve produced by the lines in part **a** is the parabola $y = \frac{1}{40}x^2$. To do so, it is enough to show that each such line is a tangent to this parabola. To warm up, we start with a concrete example with two given points. Consider the points (13, 3) and (−7, −3).

i Find, in gradient–intercept form, the equation of the line that passes through the points.

ii Verify that this line intersects the parabola $y = \frac{1}{40}x^2$ exactly once.

c Now try the same strategy in more generality.

Let $P(k + 10, k)$ be a point on the line $y = x - 10$, where k is a fixed but arbitrary number.

i Find the point Q on the line $y = -x - 10$ whose y-coordinate is $-k$.

ii Find, in gradient–intercept form, the equation of the line through P and Q.

iii Verify that this line intersects the parabola $y = \frac{1}{40}x^2$ exactly once.

d Finally, let's see how the parabola changes when the original 'lines of pegs' are changed.

For some non-zero constant d, consider the lines $y = x - d$ and $y = -x - d$ with points $P(k + d, k)$ and $Q(k - d, -k)$, respectively. There is some parabola of the form $y = ax^2$ which, for all values of k, intersects the line through P and Q exactly once. Verify this claim by finding an expression for a in terms of d.

5 Fifth degree

This question investigates integers raised to an exponent.

a Calculate the first five fifth powers: 1^5, 2^5, 3^5, 4^5 and 5^5. What do you notice about their last digits? Does this always happen?

b Fully factorise the expression $n^5 - n$. Hence show that $n^5 - n$ is divisible by 30 for all integers n.
Hint: If n, $n - 1$ and $n + 1$ are not divisible by 5, then n is of the form $5k \pm 2$ for some integer k.

c How does the result of part **b** answer the final question of part **a**?

d Show that $n^5 - n$ is divisible by 240 for all odd integers n.

e For what values of n is $n^5 - n$ divisible by 9?

f Deduce from part **b** that every integer has the same final digit as its ninth power. For what other powers is this true?

Explorations inspired by the Australian Maths Trust's competitions and programs: www.amt.edu.au

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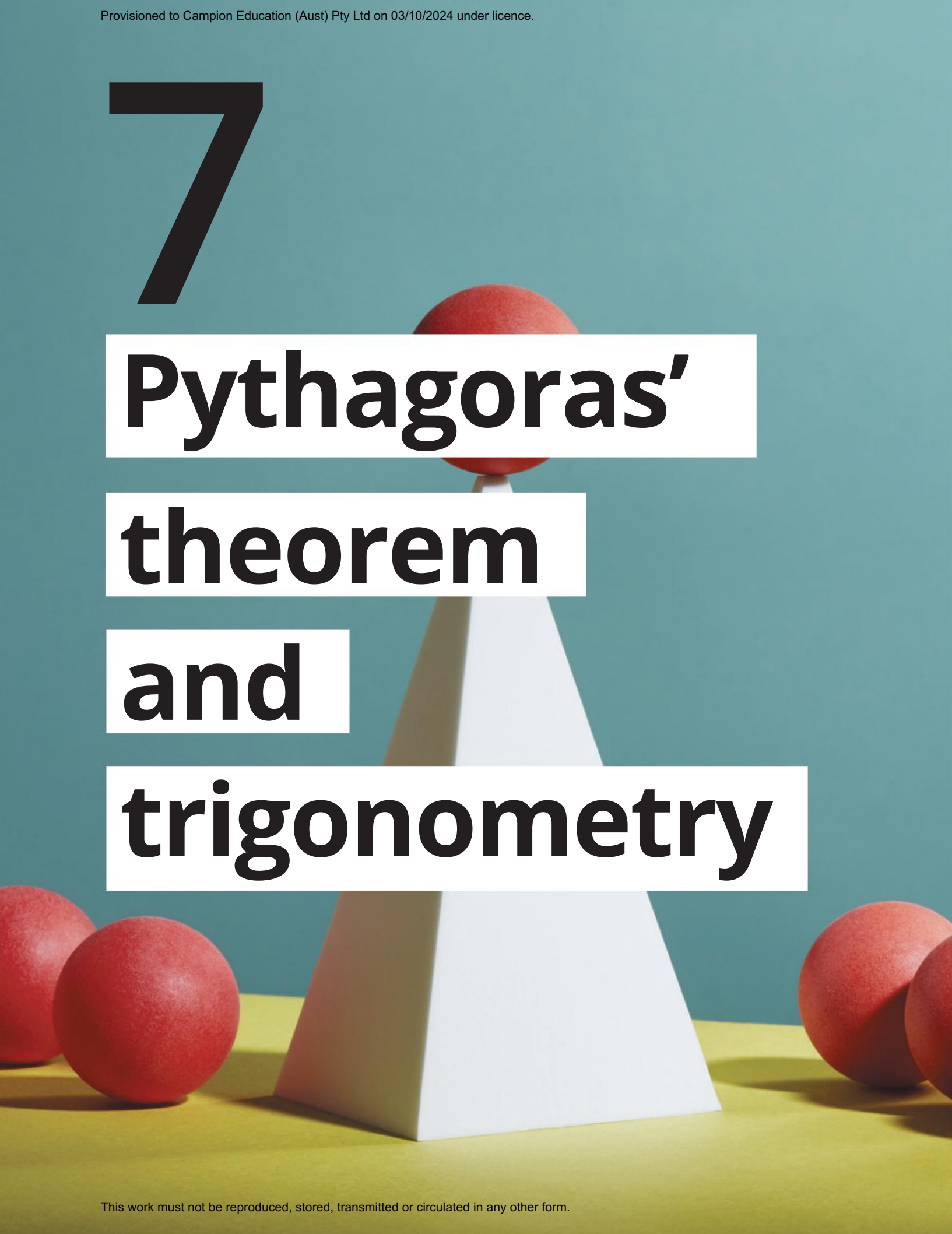
7

Pythagoras' theorem and trigonometry



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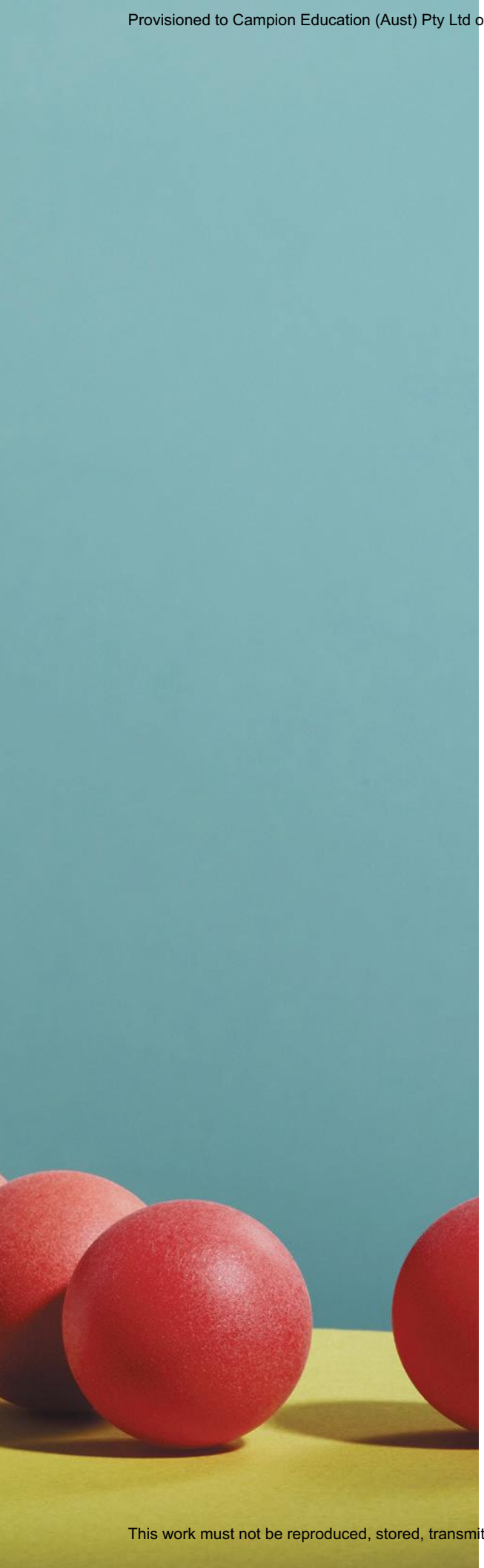

Lessons

7.1 Pythagoras' theorem
7.2 Trigonometry
7.3 Applications of trigonometry
10A **7.4** Three-dimensional problems
10A **7.5** The sine and area rules
10A **7.6** The cosine rule
10A **7.7** The unit circle
10A **7.8** Exact values
10A **7.9** Trigonometric graphs
10A **7.10** Solving trigonometric equations

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ Angles and parallel lines
- ✔ Naming 3D objects
- ✔ Rounding decimals
- ✔ Simplifying surds

Curriculum links

- Solve practical problems by applying Pythagoras' theorem and trigonometry to right-angled triangles, including problems involving direction and angles of elevation and depression (VC2M10M03)
- 10A Establish the sine, cosine and area rules for any triangle and solve related problems (VC2M10ASP02)
- 10A Use the unit circle to define the simple trigonometric functions of $y = \sin(x)$, $y = \cos(x)$, and $y = \tan(x)$ as functions of a real variable, and graph them with and without the use of digital tools (VC2M10ASP03)
- 10A Solve simple trigonometric equations (VC2M10ASP04)
- 10A Apply Pythagoras' theorem and trigonometry to solving three-dimensional problems in right-angled triangles (VC2M10ASP05)

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Materials

- ✔ Protractor
- ✔ Calculator
- ✔ Grid paper

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Lesson 7.1

Pythagoras' theorem

Learning intentions

By the end of this lesson you will be able to ...

✔ use Pythagoras' theorem to solve problems involving right-angled triangles.

Inter-year links

Year 7	1.7 Exponents and square roots
Year 8	7.3 Pythagoras' theorem
Year 9	5.1 Pythagoras' theorem

Pythagoras' theorem

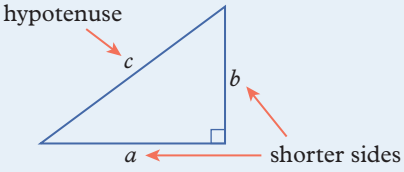
Key content video: Pythagoras' theorem

Key content video: Finding the length of a shorter side

- **Pythagoras' theorem** states the relationship between the three side lengths of a right-angled triangle: the sum of the squares of the shorter sides that meet at a right angle is equal to the square of the **hypotenuse**.

$$c^2 = a^2 + b^2$$

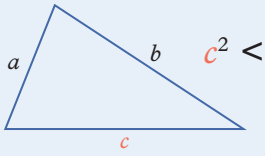
hypotenuse shorter sides



Inequalities for acute and obtuse triangles

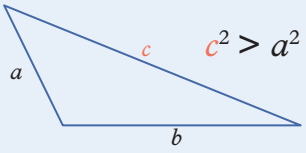
Key content video: Triangle inequalities

- In an *acute* triangle, the square of the longest side is *less than* the sum of the squares of the other two sides.



$c^2 < a^2 + b^2$

- In an *obtuse* triangle, the square of the longest side is *greater than* the sum of the squares of the other two sides.



$c^2 > a^2 + b^2$

Worked example 7.1A Determining whether a triangle is acute, right-angled or obtuse

Determine whether a triangle with side lengths of 24 cm, 20 cm and 30 cm is acute, right-angled or obtuse.

THINK	WRITE
1 Identify the length of the longest side and call it c.	If $c = 30$ cm:
2 Calculate the value of c^2.	$c^2 = 30^2$ $= 900$

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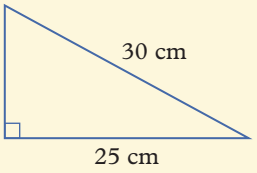
3 Identify the lengths of the two shorter sides and call them a and b.	If $a = 20$ cm and $b = 24$ cm:
4 Calculate the value of $a^2 + b^2$.	$a^2 + b^2 = 20^2 + 24^2$ $= 400 + 576$ $= 976$
5 Compare c^2 to $a^2 + b^2$. If these two values are equal, the triangle is right-angled. If c^2 is greater than $a^2 + b^2$, the triangle is obtuse. If c^2 is less than $a^2 + b^2$, the triangle is acute.	$c^2 < a^2 + b^2$ The triangle is acute.

Worked example 7.1B Using Pythagoras' theorem to calculate an unknown side

Calculate the length of the unknown side of this triangle, giving your answer as an exact value.



THINK	WRITE
1 Since the triangle is right-angled, state Pythagoras' theorem and define a, b and c. Remember that c is the hypotenuse and use b for the unknown side.	$c^2 = a^2 + b^2$, where $c = 30$ cm and $a = 25$ cm:
2 Substitute the values for a and c into the equation for Pythagoras' theorem and simplify the equation.	$30^2 = 25^2 + b^2$ $900 = 625 + b^2$
3 Solve the equation for b using inverse operations, and include units in your answer.	$b^2 = 900 - 625$ $= 275$ $b = \sqrt{275}$ $= 5\sqrt{11}$ cm

Helpful hints

- ✔ Make sure you correctly identify the hypotenuse. It is always the longest side and is opposite the right angle on the triangle. It may not be obvious, so always look at the numbers carefully.
- ✔ It doesn't matter which of the shorter sides you assign to be a or b, but the hypotenuse must always be c.
- ✔ Remember to always give your answer with correct units.
- ✔ Any set of three whole numbers that satisfies Pythagoras' theorem is called a Pythagorean triad or Pythagorean triple. For example, 3, 4, 5 and 5, 12, 13 are both Pythagorean triads.

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Learning pathways

1–4, 7(a, c, f), 8, 10	1(a, c, e), 2(b, d, f, h), 5–9, 11, 12	1(a, c, e), 2(e, g, i), 3(b, d), 7(c, d, f), 11, 13

ANS p721

Exercise 7.1A: Understanding and fluency

WE 7.1A **1** Determine whether each set of side lengths form an acute, right-angled or obtuse triangle.

a 1 cm, 2.4 cm, 2.6 cm

b 7.5 cm, 4.2 cm, 8.5 cm

c 5 cm, 7 cm, 6 cm

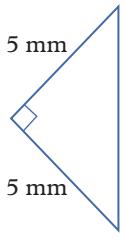
d 26 cm, 24 cm, 0.1 m

e 1 m, 96 cm, 28 cm

f 3.5 cm, 85 mm, 93 mm

WE 7.1B **2** Calculate the length of the unknown side in each of these triangles, and give your answer as an exact value in simplest form. Leave your answer in centimetres for parts **g–i**.

a



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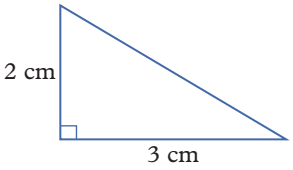
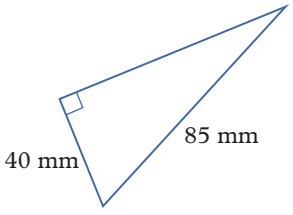


c



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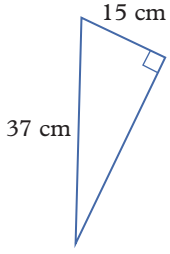

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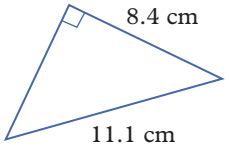

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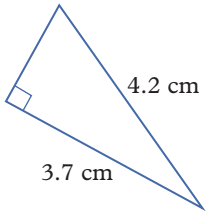

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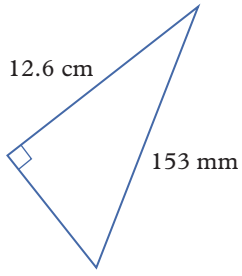

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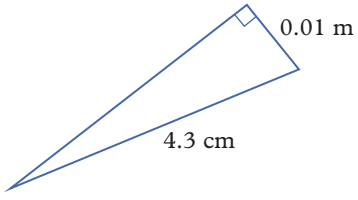

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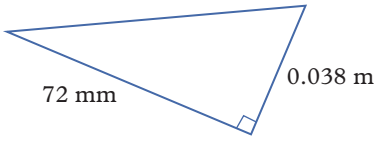

3 Calculate the length of the diagonal of a square with each of these side lengths. Give your answer as an exact value in simplest form.

a 2 cm **b** 30 mm **c** 13 cm **d** 35 cm

4 Calculate the length of the diagonal of a square with side lengths of x cm.

5 An equilateral triangle has side lengths of 10 cm.

a Draw a sketch of the triangle. From the top vertex, draw a line that is perpendicular to the base. Label the lengths of all sides of the figure.

b Use Pythagoras' theorem to calculate the height of the triangle. Give your answer as an exact value.

c Calculate the area of the triangle.

6 Calculate the area of an equilateral triangle with side lengths of x cm. Give your answer as an exact value.

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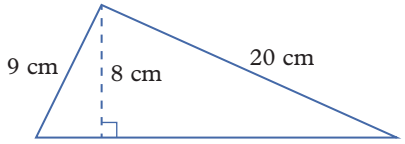
7 Calculate the perimeter of the following triangles by first finding the length of the unknown side, using the given height of each triangle to help you. Give your answers correct to two decimal places.

a



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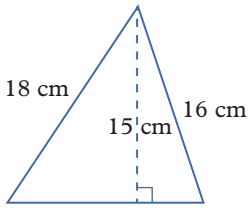

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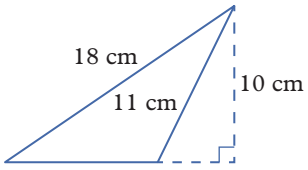

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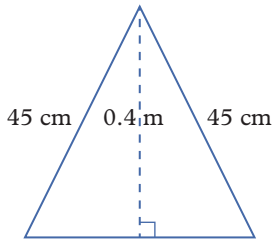

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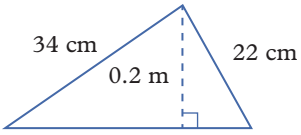

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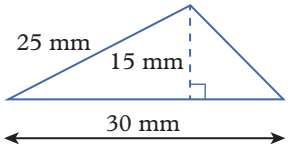

8 Determine whether the triangles in question **7** are acute, right-angled or obtuse.

Exercise 7.1B: Problem solving and reasoning

9 The hypotenuse of a right-angled triangle is 25 cm long. Find the length of the other two sides if:

- **a** one of those sides is twice as long as the other
- **b** one of those sides is three times as long as the other
- **c** one of those sides is five times as long as the other.

10 A rectangular running field measures 50 m by 100 m. How much shorter is it to run directly from one corner to the diagonally opposite corner than it is to travel between these two points around the outside of the field?

11 The flagpole which sits on Parliament House in Canberra stands 81 m high. The Australian flag flying from the top section of the pole is said to be the largest Australian flag flying in Australia: it's about the size of the side of a double-decker bus! It measures 12.8 m wide, with a diagonal of 14.3 m.

- **a** What is the height of the flag?
- **b** Calculate the perimeter of the flag.
- **c** What is the distance from the base of the flagpole to the lowest point where the flag is attached to the pole?
- **d** Assuming that the flag is flying flat and is perpendicular to the pole, calculate the distance from the base of the flagpole to the:
  - **i** lower unattached corner of the flag
  - **ii** higher unattached corner of the flag.

12 A ladder rests against a vertical wall, with the top of the ladder 7 m above the ground. When the bottom of the ladder is moved 1 m further away from the wall, the ladder is flat on the ground with one end against the wall.

- **a** Draw a diagram to illustrate this situation.
- **b** Calculate the length of the ladder.

Exercise 7.1C: Challenge

13 The orthocentre of a triangle is the intersection point of the perpendicular heights of each side of the triangle. Consider $\triangle ABC$ which has the coordinates $A(-5, 0)$, $B(-1, 8)$ and $C(2, -1)$.

- **a** Determine the equation of the lines of all three perpendicular heights.
- **b** Find the intersection point of the perpendicular heights.
- **c** Find the exact distance each vertex of $\triangle ABC$ is from the orthocentre.

Online resources:

- **Interactive skillsheet** Pythagoras' theorem
- **Interactive skillsheet** Finding the length of the hypotenuse
- **Interactive skillsheet** Finding the length of a shorter side
- **Quick quiz** 7.1

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Lesson 7.2

Trigonometry

Learning intentions

By the end of this lesson you will be able to ...

✔ use trigonometric ratios to solve problems involving right-angled triangles.

Inter-year links

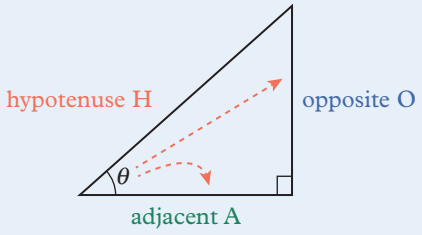
Year 8	9.5 Similar triangles
Year 9	6.4 Trigonometric ratios

Using trigonometry to find side lengths

Key content video: Trigonometric ratios

Key content video: Using trigonometry to find side lengths

- **Trigonometry** describes the relationship between the angles and side lengths of a triangle.
- The Greek letter θ (theta) is often used as a pronumeral for angles.
- The sides of a right-angled triangle can be labelled as:
  - → the hypotenuse: the longest side of the right-angled triangle opposite the right angle
  - → the opposite side: the side opposite an interior reference angle
  - → the adjacent side: the side adjacent to (or next to) an interior reference angle that is not the hypotenuse.



- The trigonometric ratios are known as the **sine** ratio, the **cosine** ratio and the **tangent** ratio. They are the ratios between two side lengths of a right-angled triangle with respect to a given interior reference angle.
- For right-angled triangles with a reference angle of θ:
  - → $\sin(\theta) = \dfrac{\text{Opposite}}{\text{Hypotenuse}} = \dfrac{\text{O}}{\text{H}}$
  - → $\tan(\theta) = \dfrac{\text{Opposite}}{\text{Adjacent}} = \dfrac{\text{O}}{\text{A}}$
  - → $\cos(\theta) = \dfrac{\text{Adjacent}}{\text{Hypotenuse}} = \dfrac{\text{A}}{\text{H}}$
- The equations for the sine, cosine and tangent ratios can be rearranged to determine the side lengths of a right-angled triangle given an angle and one side length.
- A calculator can be used to determine the sine, cosine or tangent values of any angle.

Using trigonometry to find angles

Key content video: Using trigonometry to find angles

- The non-right angles in a right-angled triangle can be determined using the inverse sine ($\sin^{-1}$), inverse cosine ($\cos^{-1}$) or inverse tangent ($\tan^{-1}$) of a ratio as they are the inverse operations of the sine, cosine and tangent ratios.
  - → $\theta = \sin^{-1}\left(\dfrac{\text{Opposite}}{\text{Hypotenuse}}\right)$
  - → $\theta = \tan^{-1}\left(\dfrac{\text{Opposite}}{\text{Adjacent}}\right)$
  - → $\theta = \cos^{-1}\left(\dfrac{\text{Adjacent}}{\text{Hypotenuse}}\right)$
- A calculator can be used to determine the unknown angle given the sine, cosine or tangent values.

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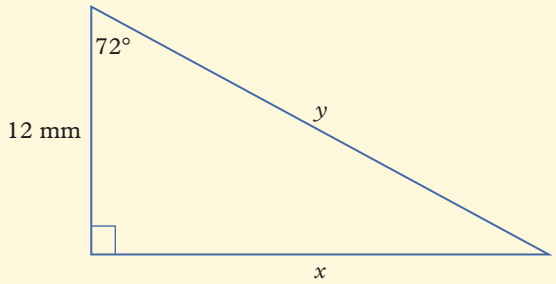
Worked example 7.2A Using trigonometry to determine unknown side lengths

Find the length of each unknown side in this triangle, correct to one decimal place.



THINK	WRITE
1 Identify the hypotenuse (H), opposite side (O) and adjacent side (A) with respect to the reference angle 72°.	H = y, O = x, A = 12 mm
2 Decide which trigonometric ratio to use to find the length of x. Since this involves the sides opposite and adjacent to the reference angle (O and A), use tangent.	$\tan(\theta) = \frac{\text{O}}{\text{A}}$
3 Substitute values for θ, O and A into $\tan(\theta) = \frac{\text{O}}{\text{A}}$.	$\tan(72°) = \frac{x}{12}$
4 Solve the equation for x. Use a calculator to multiply 12 by tan(72°). Round the value you find for x to one decimal place and include units.	$x = 12 \times \tan(72°)$ $= 36.9322\ldots$ ≈ 36.9 mm
5 Decide which trigonometric ratio to use to find the length of y. Since this involves the side adjacent to the reference angle and the hypotenuse (A and H), use cosine.	$\cos(\theta) = \frac{\text{A}}{\text{H}}$
6 Substitute values for θ, A and H into $\cos(\theta) = \frac{\text{A}}{\text{H}}$.	$\cos(72°) = \frac{12}{y}$
7 Solve the equation for y. Use a calculator to divide 12 by cos(72°). Round the value you find for y to one decimal place and include units.	$y \times \cos(72°) = 12$ $y = \frac{12}{\cos(72°)}$ $= 38.8328\ldots$ ≈ 38.8 mm

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Worked example 7.2B Using the inverse of the trigonometric ratios

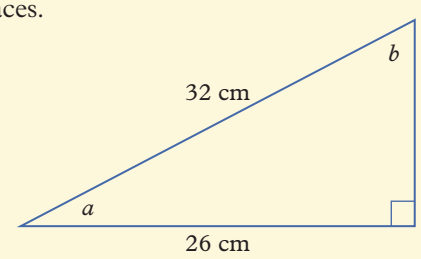
Use a calculator to find the size of θ for these trigonometric ratios, to the nearest degree.

a $\sin(\theta) = 0.439$ **b** $\tan(\theta) = \frac{17}{5}$

THINK	WRITE
a 1 Rearrange to make θ the subject of the equation using the inverse of the sine ratio.	**a** $\sin(\theta) = 0.439$ $\theta = \sin^{-1}(0.439)$
2 Use the $\sin^{-1}$ key on your calculator to find the inverse sine of 0.439.	$\theta = 26.0400\ldots$
3 Round the angle to the nearest degree.	$\theta \approx 26°$
b 1 Rearrange to make θ the subject of the equation using the inverse of the tangent ratio.	**b** $\tan(\theta) = \frac{17}{5}$ $\theta = \tan^{-1}\left(\frac{17}{5}\right)$
2 Use the $\tan^{-1}$ key on your calculator to find the inverse tangent of $\frac{17}{5}$.	$\theta = 73.6104\ldots$
3 Round the angle to the nearest degree.	$\theta \approx 74°$

Worked example 7.2C Using trigonometry to determine unknown angles

Find the size of each of the two unknown angles in this triangle. Give your answers correct to two decimal places.



THINK	WRITE
1 Decide which angle to calculate first, and identify the known side lengths with respect to that chosen angle.	$\theta = a$, H = 32 cm, A = 26 cm
2 Decide which trigonometric ratio to use. Because the adjacent side (A) and hypotenuse (H) are involved, use cosine.	$\cos(\theta) = \frac{A}{H}$
3 Substitute values for θ, A and H into $\cos(\theta) = \frac{A}{H}$.	$\cos(a) = \frac{26}{32}$
4 Rearrange to make a the subject of the equation using the inverse of the cosine ratio.	$a = \cos^{-1}\left(\frac{26}{32}\right)$
5 Use the $\cos^{-1}$ key on your calculator to find the inverse cosine of $\frac{26}{32}$.	$= 35.659\ldots$
6 Round your answer correct to two decimal places.	$\approx 35.66°$
7 Use the value of a and the right angle to find the value of b. Remember that the sum of the interior angles of a triangle is equal to 180°.	$b = 180° - 90° - a$ $= 90° - 35.659\ldots°$ $\approx 54.34°$

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Helpful hints

- ✔ When finding an unknown side length, label the triangle to identify which side length you know (O, A or H) and which side length you want to find (O, A or H). Then choose the trigonometric ratio whose formula contains both.
- ✔ When finding an unknown angle, label the triangle to identify which two side lengths you know (O, A or H). Then choose the trigonometric ratio whose formula contains both, and find the unknown angle by using the corresponding inverse trigonometric operation.
- ✔ Remember that $\sin^{-1}$ does *not* mean sine to the power of –1 (the reciprocal of sine). Instead it is the opposite (inverse) operation of the sine operation.
- ✔ Remember to write $\theta = \sin^{-1}\left(\frac{4}{5}\right)$, not $\sin^{-1}(\theta) = \frac{4}{5}$.
- ✔ It helps to draw diagrams roughly to scale. Remember, longer sides are opposite larger angles and shorter sides are opposite smaller angles.
- ✔ Ensure that your calculator is in 'D' or 'DEG' (degrees) mode.

Learning pathways

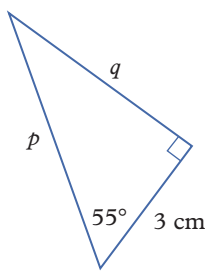
1–5, 7–9	1–4, 6, 10, 12(a, b), 14, 16–18	1, 3(d–f), 6, 10–13, 15–18

ANS p722

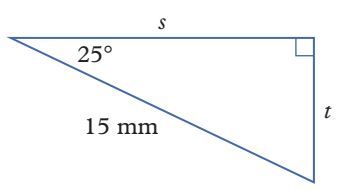
Exercise 7.2A: Understanding and fluency

WE 7.2A **1** Find the lengths of the unknown sides in each of these triangles, correct to one decimal place.

a



b



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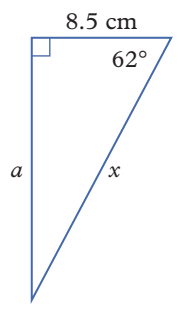

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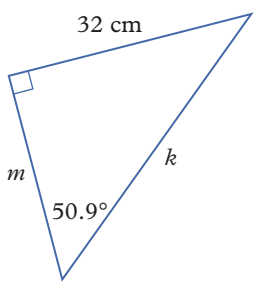

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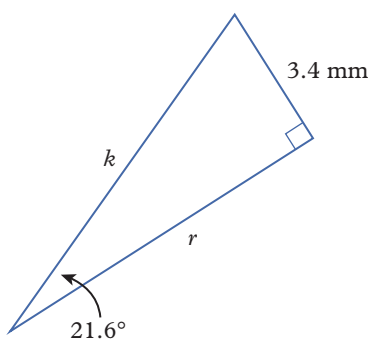

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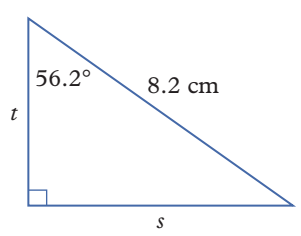

WE 7.2B **2** Use a calculator to find the size of θ for the trigonometric ratios below. Write your answers to the nearest degree.

a $\sin(\theta) = 0.378$ **b** $\cos(\theta) = 0.845$ **c** $\tan(\theta) = 2.376$

d $\sin(\theta) = \frac{10}{13}$ **e** $\cos(\theta) = \frac{44}{99}$ **f** $\tan(\theta) = \frac{1}{100}$

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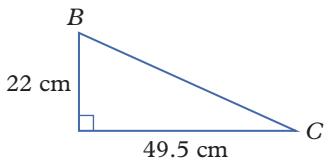
WE 7.2C **3** Find the size of each of the two unknown angles in each of these triangles. Give your answers to the nearest degree.

a



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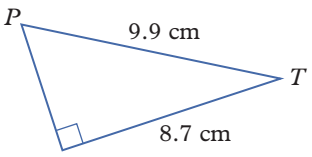

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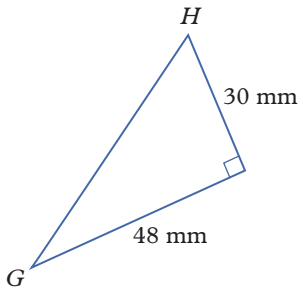

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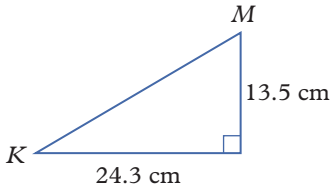

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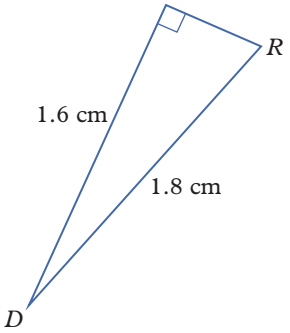

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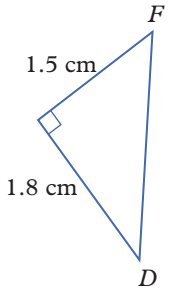
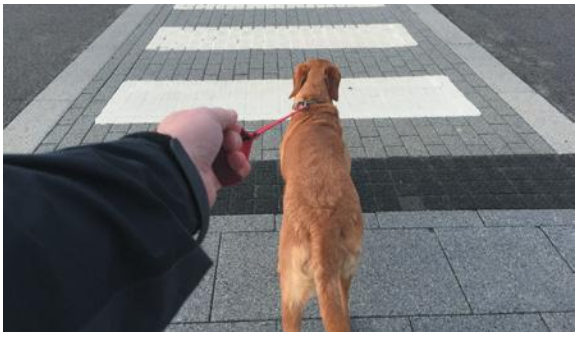


4 Shun is walking his dog on a leash. The leash is 2.5 m long, and his dog is 55 cm tall. Shun's dog is pulling him along so that the taut leash makes an angle of 85° with the vertical.



a Draw a diagram to show this information.

b How far in front of Shun is his dog, correct to the nearest centimetre?

c What is the vertical height of the end of the leash in Shun's hand, to the nearest centimetre?

5 The front face of a brick has a diagonal length of 242 mm. This diagonal forms an angle of 18.3° with the longer side of the face. What are the dimensions of the face of the brick, correct to the nearest millimetre?

6 The Leaning Tower of Pisa is a freestanding bell tower next to the cathedral in the Italian city of Pisa. The top of the tower is displaced 3.9 m horizontally from where it would be if it were standing vertically. The distance from the centre of the base to the centre of the top is 56.28 m.

a Draw a labelled sketch of the tower using this distance and its horizontal displacement. Mark the angle of 'lean' of the tower.

b Describe which trigonometric ratio you would use to calculate the angle of the tower's lean, marked in part **a**.

c Calculate the angle of lean of the tower. Give your answer to the nearest degree.

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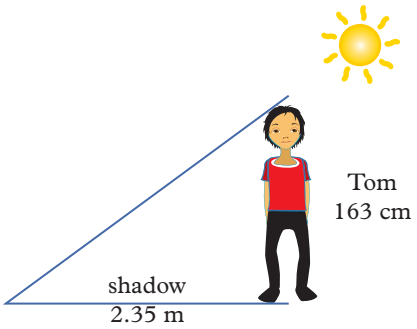
7 Tom and his brother Sam were standing in the sunshine measuring the length of each other's shadow. They each knew their height, so they collated the measurements (see the table on the right).

	Height	Length of shadow
Tom	163 cm	2.35 m
Sam	141 cm	2.04 m

a Tom drew the diagram on the right to represent his shadow and height. Draw a similar diagram to display Sam's measurements.



b Calculate the base acute angle for the triangles in each diagram. (Remember to take into account the difference in units.) Give your answers in degrees, correct to one decimal place. What does this base angle represent?

c Based on your answers from parts **a** and **b**:

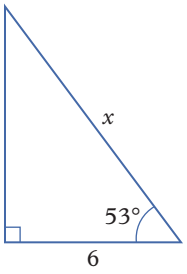
i explain why the two triangles formed by the boys' heights and their shadows must be similar shapes

ii draw a diagram combining the two triangles from the diagram above and your own diagram from part **a**.

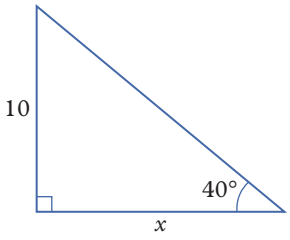
Exercise 7.2B: Problem solving and reasoning

8 Ji-ho has attempted to find the value of x in the following triangles.

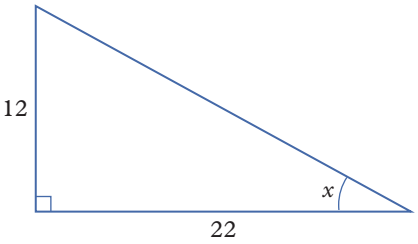
a Explain the mistake Ji-ho made in all three questions.



$\cos(53°) = \frac{x}{6}$
$x = 6\cos(53°)$
$x = 3.61089...$
$x \approx 3.6$



$\sin(40°) = \frac{x}{10}$
$x = 10\sin(40°)$
$x = 6.42787...$
$x \approx 6.4$



$\tan(x) = \frac{22}{12}$
$x = \tan^{-1}\left(\frac{22}{12}\right)$
$x = 61.3895...°$
$x \approx 61.4°$

b Explain how you can ensure you don't make the mistakes in part **a**.

9 Parliament House in Canberra is 300 m long and 300 m wide, being one of the largest buildings in the southern hemisphere.

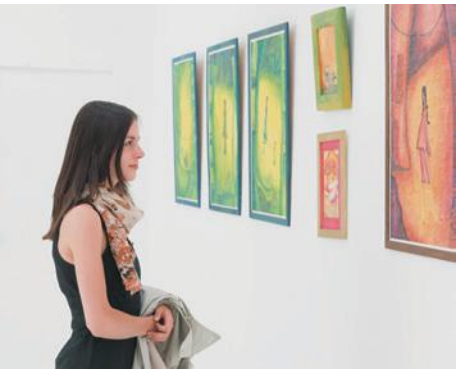
a Draw a labelled diagram to show the shape and size of the building from above.

b Use Pythagoras' theorem to calculate the diagonal length of the building.

c Use trigonometry to calculate the diagonal length of the building. Hint: Identify the size of the angle you can use.

d Comment on your answers for parts **b** and **c**.

10 Sandy is standing 2 m in front of a picture hanging on a wall. The picture is 100 cm wide and is hung so the bottom of the frame is 1.5 m from the floor. If she lowers her eyes 5.7° from the horizontal, the bottom of the frame of the painting is directly in her line of sight. If she raises her eyes 16.7° from the horizontal, the top of the frame is directly in her line of sight.



a At what height are Sandy's eyes?

b What are the dimensions of the painting?

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11 Some road signs display the gradient of a road, particularly when the road is approaching a hill or mountain climb. The gradient can be written in different forms, commonly as a ratio, a percentage or a fraction. For example, 1 : 10, 10% and $\frac{1}{10}$ can all be used to represent the same gradient.

Patricia is a long-distance truck driver delivering goods between various locations. She generally plans her route to avoid steep hill climbs, if possible. Her next delivery takes her via a mountainous stretch, where she has a choice of three possible routes, described below.

Route 1: Over a horizontal distance of 12 km, with the rise being 6 km

Route 2: Over a horizontal distance of 18 km, with the rise being 7.5 km

Route 3: Over a horizontal distance of 15 km, with the rise being 6.5 km

a Write the gradient of each route, correct to one decimal place, as:

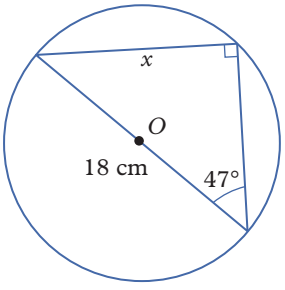
i a ratio **ii** a percentage.

b What is the angle of the rise of each road over the distances given, correct to one decimal place?

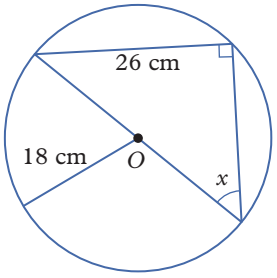
c Rank the three routes in order of increasing steepness.

12 Determine the value of x in each of the following diagrams, and correct your answer to two decimal places.

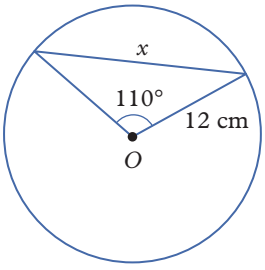
a



b



c



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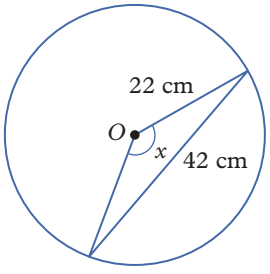
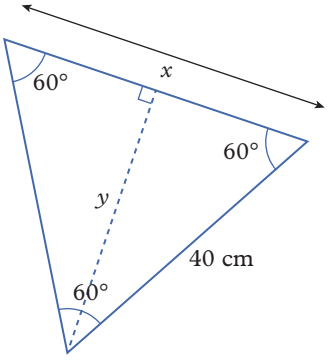


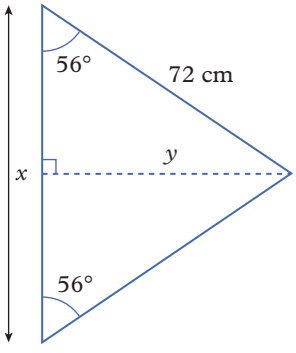
13 A hexagon is a six-sided polygon. Consider a regular hexagon with side lengths of 12 cm. Calculate the distance between its parallel sides, correct to two decimal places.

14 Consider the following triangles.

i



ii



a Determine the value of x and y in the following triangles, correct to two decimal places where required.

b Determine the area of each triangle, correct to one decimal place.

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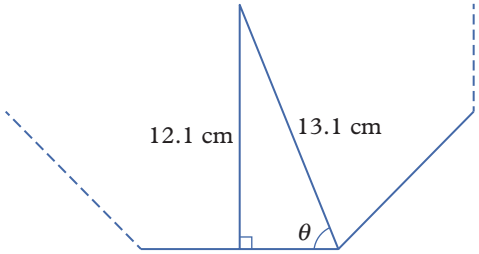
Exercise 7.2C: Challenge

15 The vertex of a regular polygon where line segments have been drawn from the polygon's centre to the vertex and the midpoint of the adjacent edge is shown below.

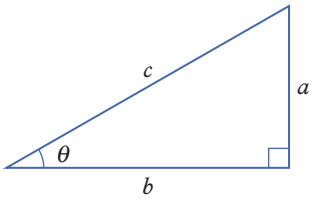


a Determine the value of θ, correct to one decimal place.

b Use the angle from part **a** to determine what type of regular polygon the vertex is from.

c Determine the perimeter of the regular polygon, correct to the nearest centimetre.

16 Investigate what happens to the value of the sine ratio as the angle increases from 0° to 90° in a right-angled triangle. Consider the triangle in the diagram shown on the right.



a The sine of θ is the ratio of the length of the opposite side to the length of the hypotenuse; that is, $\frac{a}{c}$. When θ is almost 0°:

i What is the length of a?

ii What does this mean for the value of the ratio $a : c$?

b Now consider the situation when θ becomes almost 90°.

i How does the value of a compare with the value of c?

ii What does this mean for the value of the ratio $\frac{a}{c}$?

c Use your answers from parts **a** to **b** to describe what happens to the value of the sine ratio as the angle increases from 0° to 90°.

d Repeat parts **a–c** for the cosine ratio, considering the relevant line segments for that ratio.

e Repeat parts **a–c** for the tangent ratio, considering the relevant line segments for that ratio.

17 From your investigations in question **16**, you can determine minimum and maximum values for each of the three trigonometric ratios for angles in the range 0° to 90°. Copy and complete the following to summarise your findings.

The sine ratio has a minimum value of _____ and a maximum value of _____.

The cosine ratio has a minimum value of _____ and a maximum value of _____.

The tangent ratio has a minimum value of _____ with _____ maximum value.

18 Check your answers for question **17** with a calculator. Provide some values to support your conclusions.

Online resources:

- **Interactive skillsheet** Trigonometry
- **Interactive skillsheet** Finding angles by trigonometry
- **Quick quiz** 7.2
- **Interactive skillsheet** Finding lengths by trigonometry
- **Worksheet** Pythagoras' theorem and trigonometry
- **CAS instructions** Trigonometry

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Lesson 7.3

Applications of trigonometry

Learning intentions

By the end of this lesson you will be able to ...

✔ use angles of elevation, angles of depression, and bearings to solve worded problems.

Inter-year links

Year 7	7.3 Angles and parallel lines
Year 8	6.2 Solving linear equations
Year 9	6.5 Using trigonometry to find side lengths

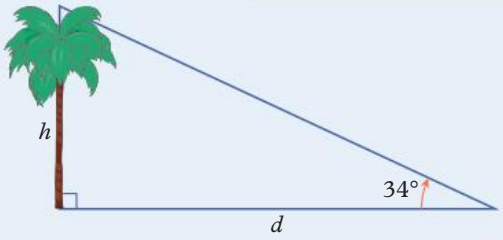
Angles of elevation and depression

Key content video
Angles of elevation and depression

- The **angle of elevation** is the angle measured upward from the horizontal.

  For example, in the diagram on the right, from a point on the beach that is a distance, d, away from the base of the tree there is a 34° angle of elevation to the top of a palm tree.

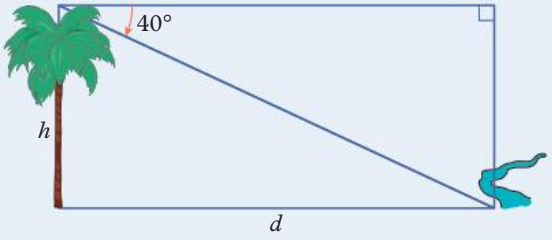
  The height of the palm tree is given by $h = d\tan(34°)$.



- The **angle of depression** is the angle measured down from the horizontal.

  For example, if someone climbs to the top of the same palm tree and measures a 40° angle of depression from the top of the tree to a nearby ocean inlet.

  The distance from the base of the tree to the inlet would be given by $d = \dfrac{h}{\tan(40°)}$.



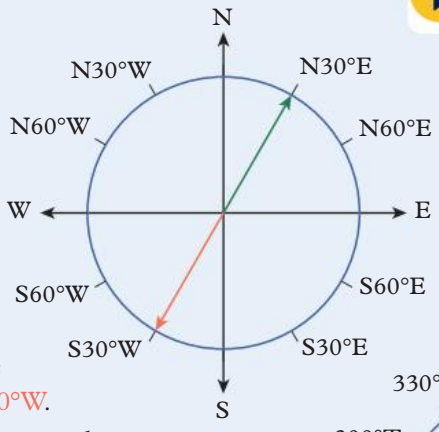
Bearings

Key content video
Bearings

- The four cardinal directions are north, south, east and west.
- **Compass bearings** describe direction using angles between 0° and 90°.

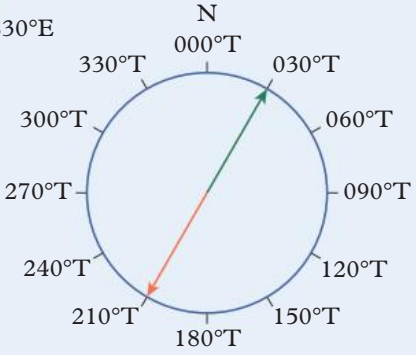
  The bearings are written with a capital letter for the north–south direction, followed by the angle, followed by a capital letter for the east–west direction.

  For example, the angle 30° clockwise of north would be written as N30°E, and the opposite direction would be written as S30°W.



- **True bearings** describe a direction using an angle between 0° and 360° clockwise from north. They are written with three digits for the angle, followed by a capital T.

  For example, the angle 30° clockwise of north would be written as 030°T, and the opposite direction would be written as 210°T.



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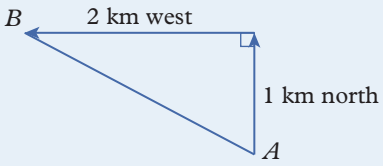
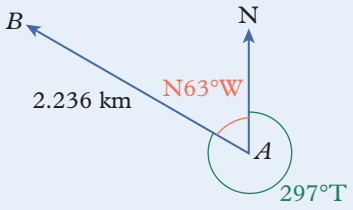
- It is often useful to know how far a position is in terms of distance along the cardinal directions. For example, if point B is 1 km north and 2 km west of point A, then Pythagoras' theorem can be used to find the total distance and bearings from point A to point B to be approximately 2.236 km N63°W or 297°T.





Worked example 7.3A Calculating distances using angles of elevation and depression

Cape Pillar in the Tasman National Park is home to the tallest sea cliffs in the southern hemisphere.

a If a tourist measures her kayak to be 200 m from the base of the closest cliff face and the angle of elevation to the tip of the cliff is 56°, how tall is the cliff? Round your answer to the nearest metre.

b The next day, the same tourist hikes to the top of the cliff and spots a boat near the horizon at an angle of depression of 2°. How far away is the boat from the tourist? Round your answer to the nearest metre.

THINK

a 1 Draw a diagram to represent the situation. It does not need to be to scale.

2 Substitute the known values into the equation for the tangent ratio.

3 Solve for the height of the cliff and round to the nearest metre.

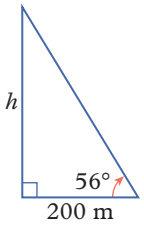
b 1 Draw a diagram to represent the situation. It does not need to be to scale. Assign the angle of depression to the corresponding angle of elevation, as they are alternate angles.

2 Substitute the known values into the equation for the sine ratio.

3 Solve for the distance from the tourist to the boat and round to the nearest metre.

WRITE

a



$\theta = 56°$, O $= h$, A $= 200$ m,

$\tan(\theta) = \dfrac{\text{O}}{\text{A}}$

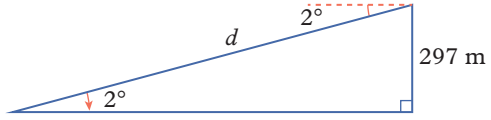
$\tan(56°) = \dfrac{h}{200}$

$h = 200\tan(56°)$

$= 296.5121\ldots$

≈ 297 m

b



$\theta = 2°$, O $= 297$ m, H $= d$

$\sin(\theta) = \dfrac{\text{O}}{\text{H}}$

$\sin(2°) = \dfrac{297}{d}$

$d = \dfrac{297}{\sin(2°)}$

$= 8510.1514\ldots$

≈ 8510 m

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Worked example 7.3B Calculating distances using bearings

For ships travelling the distances and directions below:

i Draw a labelled diagram to display the distance travelled along the relevant cardinal directions after departing from the port.

ii Use trigonometry to calculate the distances travelled along the relevant cardinal directions, to the nearest kilometre.

a 180 km on a bearing of S59°E

b 85 km on a bearing of 224°T

THINK

a i Draw a diagram to represent the situation. Start with the port and draw an upwards arrow to represent north, then draw a line in the direction towards the ship. Label the distance traveled on the diagram. Label the distances along the east–west and north–south axes of your diagram with pronumerals x and y.

ii 1 Use the sine ratio to calculate the distance east.

2 Use the cosine ratio to calculate the distance south.

b i 1 A true bearing of between 180°T and 270°T is in the SW quadrant; calculate the angle measured clockwise from south.

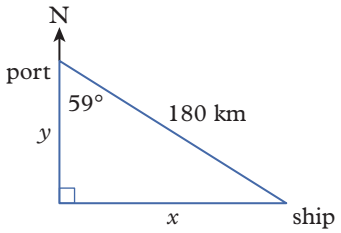
2 Draw a diagram to represent the situation. Start with the port and draw an upwards arrow to represent north, then a line in the correct direction towards the ship. Label the distance travelled. Label the distance along the east–west and north–south axes of your diagram with pronumerals x and y.

ii 1 Use the sine ratio to calculate the distance west.

2 Use the cosine ratio to calculate the distance south.

WRITE

a i



ii $\sin(\theta) = \frac{O}{H}$

$\sin(59°) = \frac{x}{180}$

$x = 180 \times \sin(59°)$

≈ 154 km

The ship is 154 km east of port.

$\cos(\theta) = \frac{A}{H}$

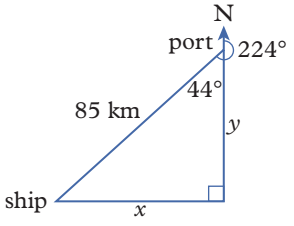
$\cos(59°) = \frac{y}{180}$

$y = 180 \times \cos(59°)$

≈ 93 km

The ship is 93 km south of port.

b i $224° - 180° = 44°$



ii $\sin(\theta) = \frac{O}{H}$

$\sin(44°) = \frac{x}{85}$

$x = 85 \times \sin(44°)$

≈ 59 km

The ship is 59 km west of port.

$\cos(\theta) = \frac{A}{H}$

$\cos(44°) = \frac{y}{85}$

$y = 85 \times \cos(44°)$

≈ 61 km

The ship is 61 km south of port.

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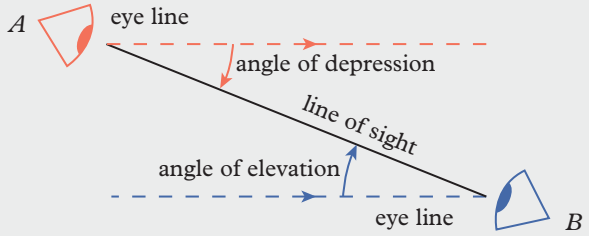
Helpful hints

- ✔ It always helps to draw a diagram and label the sides and angles with the known and unknown values. It does not need to be exactly to scale.
- ✔ The angle of depression from point A to point B is equal to the angle of elevation from point B to point A. This is because they form alternate angles on parallel eye lines.



- ✔ One way of remembering the cardinal directions is the phrase 'Never Eat Soggy Weet-Bix'. The first letter of each word matches the cardinal directions, read clockwise from north.

Learning pathways

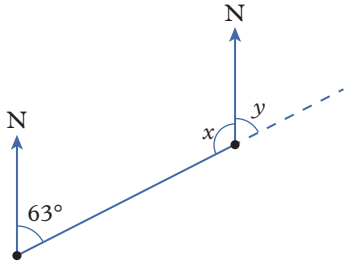
1–6, 8, 11, 12, 14, 15	1, 2, 5–7, 9, 10, 13, 16, 17, 19	1(d–f), 5, 6, 10, 14, 16–18, 20

ANS p723

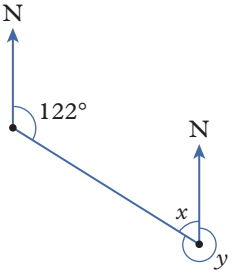
Exercise 7.3A: Understanding and fluency

1 Determine the value of the pronumerals in each of the following.

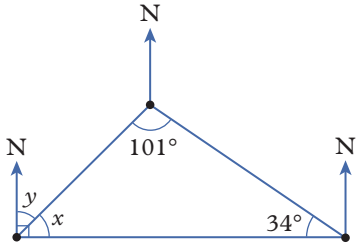
a



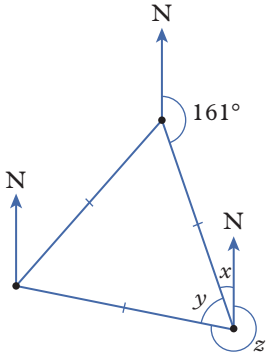
b



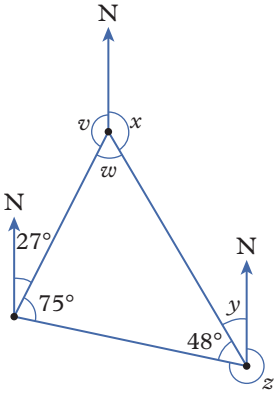
c



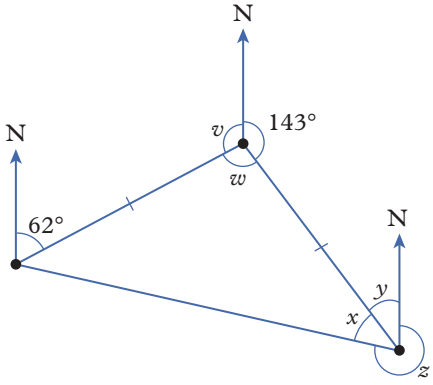
d



e



f



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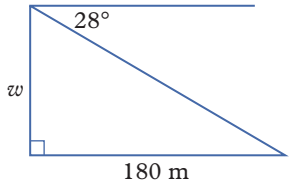
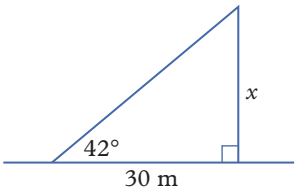
2 Use the angle of elevation or depression in each diagram to calculate the side length marked with the pronumeral. Give your answers correct to one decimal place.

a



b



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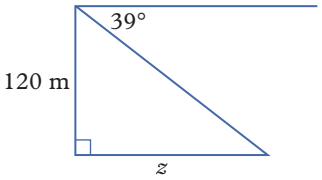

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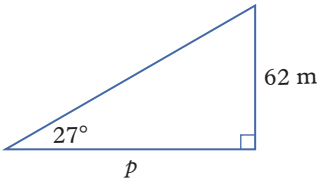

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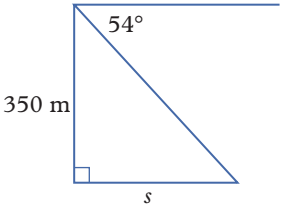

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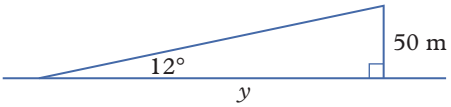


WE 7.3A **3** A lighthouse stands on a cliff above the sea. The angle of elevation to the top of the lighthouse from a point on level ground 100 m from its base is 14°.

a Find the height of the lighthouse, to the nearest metre.

b The angle of depression from the light at the top of the lighthouse to a ship at sea 2.5 km from the base of the cliff is 3.3°. Find the height of the light above the sea, to the nearest metre.

4 Convert the following bearings:

a from compass bearings to true bearings

i S30°E **ii** N52°W **iii** N5°E

b from true bearings to compass bearings.

i 200°T **ii** 100°T **iii** 300°T

WE 7.3B **5** For ships travelling each of the distances and directions below:

i draw a labelled diagram to display the distance travelled along the relevant cardinal directions after departing from the port

ii use trigonometry to calculate these distances travelled along the relevant cardinal directions, to the nearest kilometre.

a 250 km on a bearing of N30°E **b** 100 km on a bearing of 300°T

c 75 km on a bearing of S45°W **d** 210 km on a bearing of 090°T

6 If each ship in question **5** turns around and returns to its original port, write a description of the ship's path with respect to its turn-around position using:

i compass bearings and distance in kilometres

ii true bearings and distance in kilometres.

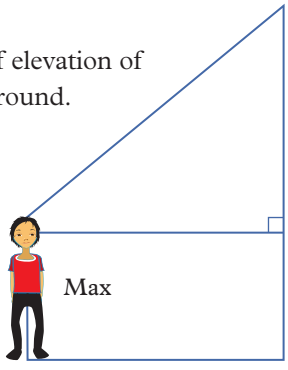
7 Stavros is descending an escalator at an angle of depression of 30°.

a If the escalator is 6 m high at the top, correct to two decimal places, what horizontal distance does it cover?

b If it takes 30 seconds for Stavros to descend the escalator, at what speed is the escalator moving? Give your answer correct to one decimal place, in m/s.

Exercise 7.3B: Problem solving and reasoning

8 Max throws a ball into the air. He views the ball at its highest point to be at an angle of elevation of 40° from him and 3 m horizontally away from him. Max's eye level is 172 cm above ground.



a Copy the diagram on the right and label it with the known values.

b What height does the ball reach above the level of Max's eyes? Give your answer to the nearest centimetre.

c What is the maximum height the ball reaches above the ground? Give your answer to the nearest centimetre.

d Why is it important to consider the height of a person's eyes in a situation like this?

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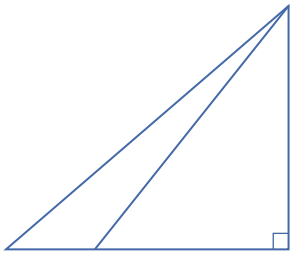
9 Can an angle of elevation or angle of depression be an obtuse angle? Explain.

10 Toshiko is standing at the edge of a 10 m cliff and sees a boat near the base of the cliff. Toshiko measures the angle of depression to the point where the bow of the boat (the front of the boat) touches the water as 24°.

a Correct to two decimal places, how far from the base of the cliff is the boat if Toshiko's eye height is 152 cm?

b Toshiko spots a shark directly below the boat and measures the angle of depression to the nose of the shark as 25°. If the nose of the shark is directly below where the bow of the boat touches the water, correct to two decimal places, how deep is the shark swimming in the water?

11 Ann is standing 8 m in front of a tree. She observes the angle of elevation to the top of the tree to be 52°. Beth is standing further from the tree than Ann, but in the same line. She observes the angle of elevation to be 40°. (Ignore the heights of the two girls in this problem.)



a Copy the diagram on the right and label it with the known values.

b Find the height of the tree calculated using Ann's observations, correct to one decimal place.

c Use the tree's height, together with Beth's angle of elevation, to calculate Beth's distance from the tree, correct to one decimal place.

d What is the distance between Ann and Beth? Correct your answer to one decimal place.

e Suppose Beth had been standing the same distance from the tree, but on the opposite side of the tree from Ann (with Beth and Ann on the same line).

i Draw a diagram to show this situation.

ii What would have been your answer to part **d** in this case?

12 A lookout tower is built on top of a cliff. At a point 50 m from the base of the cliff, the angle of elevation to the base of the tower is 58°, while the elevation to the top of the tower is 64°.

a Draw a diagram to display this information.

b Find the height of the tower, correct to one decimal place.

13 Two bushwalkers start walking from their campsite. Roger walks due north, while Ben walks N22.5°E. They both walk 25 km.

a Display the information in a diagram.

b How far north of his starting point is Ben when he completes his walk? Correct your answer to one decimal place.

c How far east of his starting point is Ben at this stage? Correct your answer to one decimal place.

d If Ben were to walk directly to meet Roger at his finishing point, how far would he have to walk? Correct your answer to one decimal place.

14 Jake is riding his bike from home to his friend Luke's house. He rides 1.2 km directly north, then makes a right-angled turn left and continues for 3.2 km west to Luke's house.

a Draw and label a scale drawing of Jake's route.

b Calculate the direct distance between Jake's house and Luke's house, correct to one decimal place.

c Use trigonometry to find the true bearing from Jake's house to Luke's house. (Remember that bearings are measured from the north in a clockwise direction.)

d What is the true bearing from Luke's house to Jake's house?

e The bearing in part **d** is known as the back-bearing of the bearing in part **c** (and vice versa). What do you notice about the relationship between these two bearings?

f Use your scale drawing from part **a** to:

i find the distance between the two houses, correct to one decimal place.

ii measure the bearings in both directions.

g Comment on any differences between your previous answers and those found using your scale drawing.

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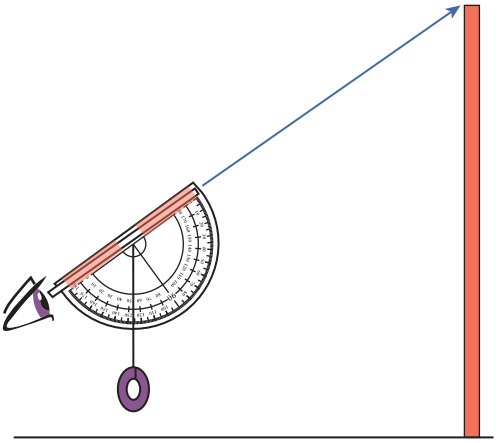
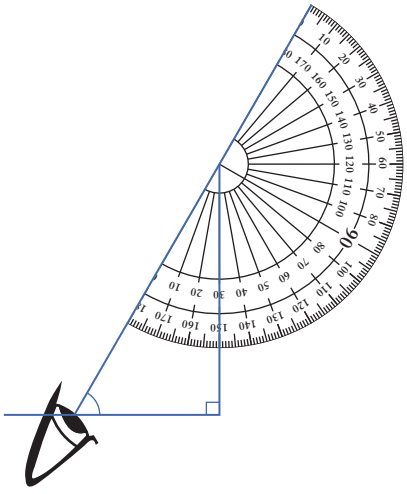
15 A clinometer (or inclinometer) is a tool used to measure angles of inclination. That is, angles of elevation or depression.

A makeshift clinometer can be constructed using a protractor, a straw, some string and a weight. You can then measure angles of elevation and depression by looking at the object through the straw, reading the angle the string makes with the protractor, then using geometry to determine the angle of elevation or depression.

Determine the angle of elevation or depression each of the following clinometers is measuring.



a



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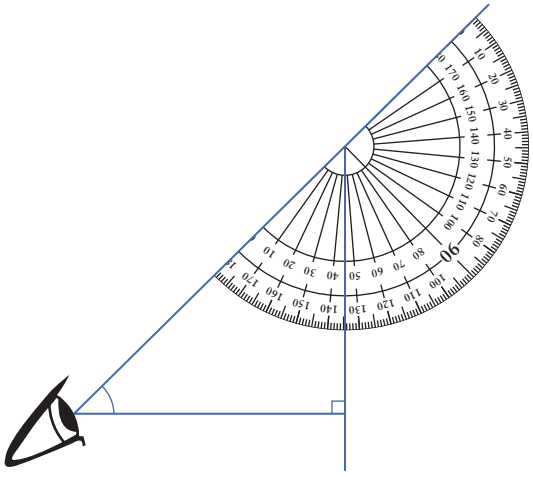

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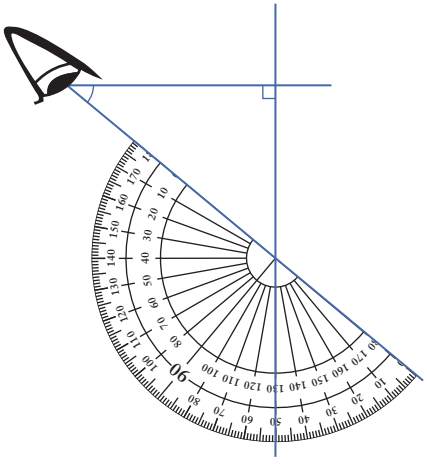

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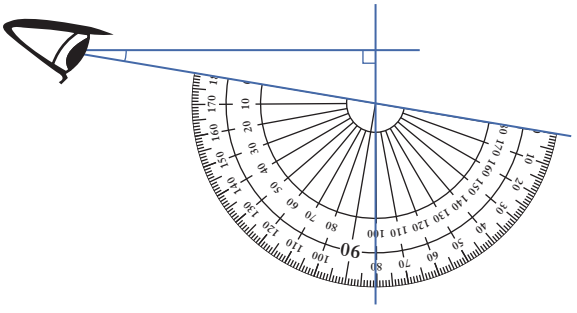

e Describe how you can use the measured angle on the clinometer to determine:

i the angle of elevation **ii** the angle of depression.

16 The members of a bushwalking group have been given instructions for their next hike.

- Start from a meeting place, labelled A, and walk 35 km at a bearing of 055°T to B.
- From B, walk 25 km at a bearing of 015°T to finish the walk at C.

a Draw a scale drawing of the bushwalkers' route.

b Correct to one decimal place, find how far north:

i B is from A **ii** C is from B **iii** C is from A.

c Correct to one decimal place, find how far east:

i B is from A **ii** C is from B **iii** C is from A.

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d Consider the right-angled triangle in which AC is the hypotenuse. Calculate:

i the distance from A to C correct to one decimal place

ii the true bearing from A to C

iii the true bearing from C to A.

e Using your scale drawing from part **a** to measure the distance and bearings from part **d**.

f Compare your measured answers with your answers from part **d**.

17 A new hike for bushwalkers took them along a different route.

- From the starting point, walk 18 km at a bearing of 300°T.
- Change direction and walk 23 km at a bearing of 200°T to reach your endpoint.

a Find the direct distance from the starting point to the endpoint, correct to one decimal place.

b What is the bearing of the starting point from the endpoint?

18 Jan is standing 2 m in front of a painting on a wall. The bottom of the painting's frame is 1.4 m from the floor, and Jan's eyes are 165 cm from the floor. The painting, including the frame, has a height of 70 cm.

a Draw a diagram to show this information.

b Find the angle of elevation from Jan's eyes to the top of the painting's frame.

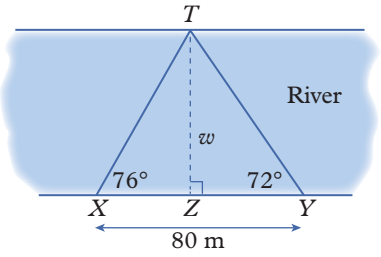
c What is the angle of depression from Jan's eyes to the bottom of the painting's frame?

d If Jan stood 1 m further away from the painting, through what angle would she have to sweep her eyes to look directly at the top of its frame, and then directly at the bottom of its frame?

Exercise 7.3C: Challenge

19 Council surveyors have been set the task of measuring the width of a river along a straight stretch in preparation for building a bridge across it. They have located a tree on the opposite side of the river and have taken angle measurements from two points, X and Y, which are 80 m apart, as shown in the diagram on the right.

Calculate the width of the river, correct to the nearest metre. Hint: Let the distance XZ be represented by a pronumeral.



20 A pilot was flying a routine route from A, directly east to B, a distance of 3500 km. She didn't know her compass was not operating correctly and she was actually travelling 10° south of her true direction. When she didn't arrive at her destination after travelling 3500 km, the pilot realised she was off course.

a How much further does she need to travel to fly directly to her destination from this point? Give your answer to the nearest kilometre.

b The pilot only has enough fuel for another 60 km, so she decides to land on a nearby runway 29 km away. The pilot measures the angle of depression from the plane to the runway to be 6° at an altitude of 3 km. The runway is a straight line from north to south and so the pilot must approach the runway in that direction. The bearing from the plane to the runway is 115°T. The pilot wants to travel directly east and then make a sharp right-angled turn to land the plane.

i What is the horizonal distance east before the pilot must make the turn? Give your answer to the nearest kilometre.

ii What is the horizontal distance south from the turn to the runway? Give your answer to the nearest kilometre.

c If the pilot had become aware of the error when she was halfway into the flight and adjusted her direction from that point, would she have made it to her destination?

Online resources:

- **Interactive skillsheet** Angles of elevation and depression
- **Interactive skillsheet** Compass bearings and true bearings
- **Worksheet** Applying Pythagoras' theorem and trigonometry
- **Investigation** How do you get to school?
- **Quick quiz** 7.3

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Lesson 7.4

Three-dimensional problems

Learning intentions

By the end of this lesson you will be able to ...

✔ solve problems involving 3D objects.

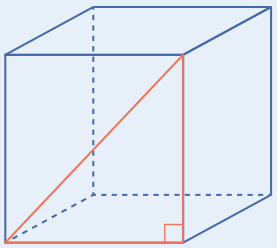
Inter-year links

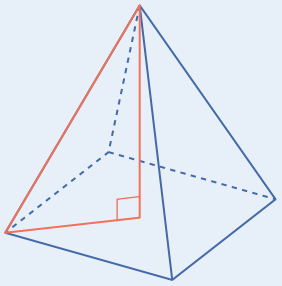
Year 8	8.6 Three-dimensional space
Year 9	6.6 Using trigonometry to find angles

Three-dimensional problems

Key content video
Finding lengths and angles within 3D objects

- Just as 2D shapes can contain many right-angled triangles, there are also many right-angled triangles to be found in 3D objects.
  - → Some right-angled triangles can be found on the surface of 3D objects, such as the triangle that has as its hypotenuse the diagonal joining vertex to vertex across the face of a cube.
  - → Other right-angled triangles can be found within a shape, such as the triangle shown in the diagram of the pyramid on the right. That triangle is formed by a slanting edge of the pyramid (the hypotenuse), a perpendicular line from the base of the square pyramid to the apex of the pyramid, and a line on the base joining the perpendicular to the slanting edge.





- Pythagoras' theorem can be used to find an unknown length, either on the surface of a 3D object or within a 3D object.

$$c^2 = a^2 + b^2$$

- Trigonometry can be used to find an unknown angle or length, either on the surface of a 3D object or within a 3D object.

$$\sin(\theta) = \frac{O}{H} \qquad \cos(\theta) = \frac{A}{H} \qquad \tan(\theta) = \frac{O}{A}$$

Worked example 7.4A Three-dimensional problems

A 20 cm long box has two square faces with 12 cm sides.

a Calculate the length of the diagonal across each of the square faces of the box, to the nearest centimetre.

b Find the length of the diagonal within the box, to the nearest centimetre.

c Find the angle the internal diagonal from part **b** makes with the diagonal across the square face, to the nearest degree.

THINK

a Substitute the side lengths of the square into Pythagoras' theorem to find the length of the diagonal across the square face.

WRITE

a
$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= 12^2 + 12^2 \\ &= 144 + 144 \\ &= 288 \\ c &= \sqrt{288} \\ &= 12\sqrt{2} \\ &\approx 17 \text{ cm} \end{aligned}$$

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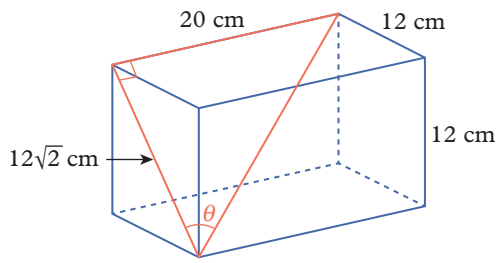
b **1** Draw a labelled diagram of the box. Identify the right-angled triangle formed by the two diagonals.

2 Substitute the side lengths of the two short sides of the right-angled triangle into Pythagoras' theorem to find the length of the diagonal within the box (the hypotenuse).

c **1** Choose a trigonometric ratio to use for calculating the angle. Note that all the sides are now known, so any trigonometric ratio can be used.

2 Solve for the angle using the inverse of the tangent ratio.

b



$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= (12\sqrt{2})^2 + (20)^2 \\ &= 288 + 400 \\ &= 688 \\ c &= \sqrt{688} \\ &\approx 26 \text{ cm} \end{aligned}$$

c $\tan(\theta) = \dfrac{\text{O}}{\text{A}}$

$\tan(\theta) = \dfrac{20}{12\sqrt{2}}$

$\theta = \tan^{-1}\left(\dfrac{20}{12\sqrt{2}}\right)$

$\approx 50°$

Helpful hints

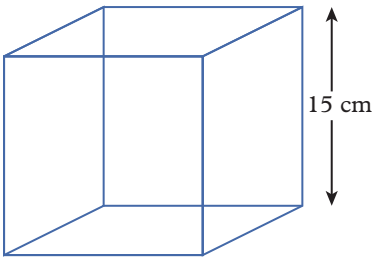
- ✔ It always helps to draw a diagram and label the sides and angles with the known and unknown values. It does not need to be exactly drawn to scale.
- ✔ Remember, the trick is to find right-angled triangles! This can often be done by:
  - → drawing a line across the diagonal of a rectangle
  - → drawing a line through any triangle from one vertex to the opposite side, to meet that side at a right angle.
- ✔ When being asked to round to a given number of decimal places, the rounding should only take place at the final stage of your work and not in any intermediate steps. For example, rounding the size of angles in the middle of your work can lead to considerable differences in the values of trigonometric ratios.

Learning pathways

1–5, 7, 8, 10, 13	1–4, 6, 9–11, 14, 15, 17	3, 6, 9–12, 14–16, 18, 19

ANS p725

Exercise 7.4A: Understanding and fluency



WE 7.4A **1** The side lengths of a cube are 15 cm.

a Calculate the length of the diagonal of the base of the cube, correct to two decimal places.

b Find the length of the diagonal within the cube, correct to two decimal places.

c Find the angle the internal diagonal makes with the base diagonal, correct to two decimal places.

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2 A box has a base 20 cm by 15 cm and is 10 cm tall.

a Provide working to show whether a 30 cm rod will fit in the box.

b If a rod is in contact with both a bottom corner and a diagonally opposite top corner of the box, what angle would it make with the base diagonal of the box? Give your answer correct to two decimal places.

3 A solid cube of side length 5 cm is sliced along its base diagonal.

a What is the shape of the exposed face? Draw a diagram with its dimensions labelled.

b Compare the size of the exposed face with the size of one of the faces of the cube.

4 A piece of cheese has the shape of a triangular prism. Its base is an isosceles triangle with side lengths of 10 cm, 10 cm and 4 cm, and the prism's height is 5 cm.

a Draw a labelled diagram of the triangular prism, showing all dimensions.

b Draw a labelled diagram of:

i the front view of the largest side

ii the top view (or base view).

c Draw a diagonal on your diagram of the front view.

i Find the length of the diagonal you drew, correct to one decimal place.

ii What angle does the diagonal make with the longer side, correct to two decimal places?

iii Without using trigonometry, explain how you could find the angle the diagonal makes with the shorter side.

d Calculate, correct to one decimal place:

i the perpendicular height of the base triangle

ii the area of the base.

e The cheese is sliced along the height of the base triangle.

i Draw the shape of one of the exposed faces. Label its dimensions.

ii Find the area exposed on both sides of the slice.

iii What is the diagonal length of the exposed face, correct to the nearest centimetre?

Exercise 7.4B: Problem solving and reasoning

5 Jack was given a tent for Christmas. He wants to make a cover the size as the tent to protect it when it rains. The front of the tent for equilateral triangle, with a height of 1.5 m.

a Sketch a diagram of the front of the tent. Label the height 1.5 m, and let the side lengths be x m. Highlight a right-angled triangle, and write the length of its base in terms of x.

b Find the value of x, correct to one decimal place.

c Jack has bought a rectangular tarpaulin sheet with one side long enough to fit the length of the tent from front to back. If the other side length is 3.5 m, explain whether this is sufficient to reach the ground from one side of the tent to the other.

6 Ethan is standing 70 m in front of a tree when he notices a much taller tree 37° to his left.

a Draw a sketch showing the relative position of the two trees described. Label the known values.

b Find the distance along the ground from Ethan to the base of the taller, tree correct to one decimal place.

c He measures the angle of elevation to the top of the taller tree as 13°. Draw a sketch showing the elevation from Ethan to the top of the tall tree. Label the known values.

d Use this diagram to find the height of the tall tree, correct to one decimal place.

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7 The Pyramid of Giza is the largest stone monument in the world. At one stage, it was also the tallest.

The base of the pyramid is a square with side lengths of 230 m. It stands 140 m tall.

a Draw a labelled sketch of the pyramid.

b Calculate the diagonal length of the base, correct to two decimal places.

c The apex of the pyramid lies directly over the midpoint of the base diagonal. Use the vertical right-angled triangle to calculate the angle that the sloping edge makes with the base diagonal, correct to two decimal places.

d A different right-angled triangle can be used to find the angle the sloping sides make with the base by measuring from the midpoint of one of the base sides.

i Draw a labelled diagram to display the dimensions of this triangle.

ii Calculate the angle the sloping side makes with the base, correct to two decimal places.

8 Cooks sometimes find that if they leave their stirring spoon unattended, it falls into the saucepan.

This saucepan is in the shape of a cylinder open at one end, with a circumference of 60 cm and a height of 12 cm.

a Find the diameter of the saucepan correct to one decimal place.

b A stirring spoon is 20 cm in length. Explain whether the entire spoon could fall below the rim of the saucepan.

9 An ice cream cone has a height of 10 cm. The circumference of the circular top of the cone is 15.5 cm.

a Find the radius of the circular top of the cone, correct to one decimal place.

b Draw a labelled two-dimensional sketch of the outline of the cone. Define any right angles within the shape.

c Find the slant length of the cone, correct to one decimal place.

10 The branches of a Christmas tree form a cone which is 3 m tall with a base diameter of 2 m. Sandy wants to run streamers from the top of the tree to the base of the branches, along the outer surface of the tree.

She has one 30 m long roll of streamer paper that she can cut up to do this. How many streamers the full length of the slant of the tree is she able to attach?

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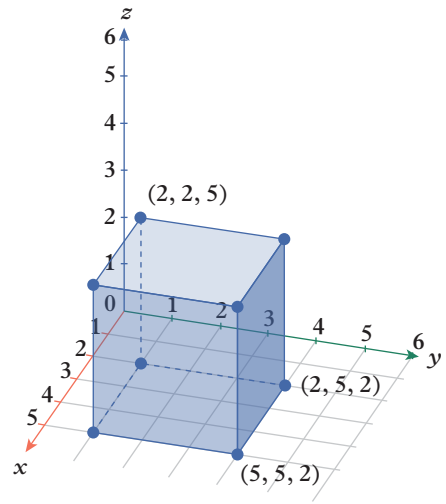
11 Consider the following cube in three-dimensional space.



a The coordinates of three of the vertices of the cube are given. State the coordinates of the other five vertices.

b Determine the length of the internal diagonal of the cube, between (2, 2, 5) and (5, 5, 2).

12 Derive a formula to calculate the length of a line segment in three-dimensional space between the coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2).

13 A regular tetrahedron is a three-dimensional object with four equilateral triangular faces.

a Identify the tetrahedron in the photo below and draw a sketch of it.

b Suppose that the length of each side of the tetrahedron you drew is 10 cm and its height from the base to the upper apex is 8.16 cm. Add these measurements to your sketch from part **a**.

c **i** Draw a diagram of a right-angled triangle with corresponding measurements from the tetrahedron. Label one side of the triangle with the height of the tetrahedron from part **a** and label the hypotenuse with the length of one edge of the tetrahedron.

ii Use your diagram to help you calculate the angle between the edge and the base of the tetrahedron, correct to one decimal place.

d Draw the shape of a face of the tetrahedron. Draw the perpendicular height, and label all known values.

i What is the perpendicular height of the face of the tetrahedron, correct to two decimal places?

ii Draw a right-angled triangle showing the perpendicular height of the face and the height of the tetrahedron. Use your diagram to calculate the angle between the base and a face of the tetrahedron, correct to one decimal place.

14 A tower stands at point T on flat ground. From point X on the ground, the angle of elevation to the top of the tower, point A, is 54°. At point Y on the ground, in the same straight line as XT, and 25 m beyond X, the angle of elevation to the top of the tower is 28°.

a Draw a labelled diagram to display the given information. Label AT, the height of the tower, as h, and the distance XT along the ground as d.

b Use a trigonometric ratio with $\triangle AXT$ to find an equation for h in terms of d.

c Use a trigonometric ratio with $\triangle AYT$ to find an equation for h in terms of d. Note that the length of YT is $(25 + d)$ m.

d Equate your two equations from parts **b** and **c**. Solve to find a value for d, correct to one decimal place.

e Use the value you found for d in part **d** with $\triangle AXT$ to find the height of the tower, correct to one decimal place.

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15 A rectangular prism has width, w, length, l, and height, h.

a Write expressions for the lengths of the diagonals along each of the faces.

b Write an expression for the length of the diagonal of the rectangular prism from one vertex to the vertex it is furthest away from.

16 A rectangular prism has a base with length, l, twice its width, w. The height of the prism, h, is three times its width.

a Write the length and height of the prism in terms of its width, w.

b Draw a labelled diagram of the prism.

c Find the length of the base diagonal in terms of w.

d Calculate the length of the internal diagonal within the prism in terms of w.

e Find the angle that the internal diagonal of the prism makes with the diagonal of the base. Give your answer to the nearest degree.

Exercise 7.4C: Challenge

17 The length of the diagonal within a cube is 6.2 cm. What is the side length of the cube?

18 Consider a cube of side length x cm.

a Draw a diagram of the cube.

b An ant crawls from the top front left-hand corner to the bottom back right-hand corner along the outer surface of the cube.

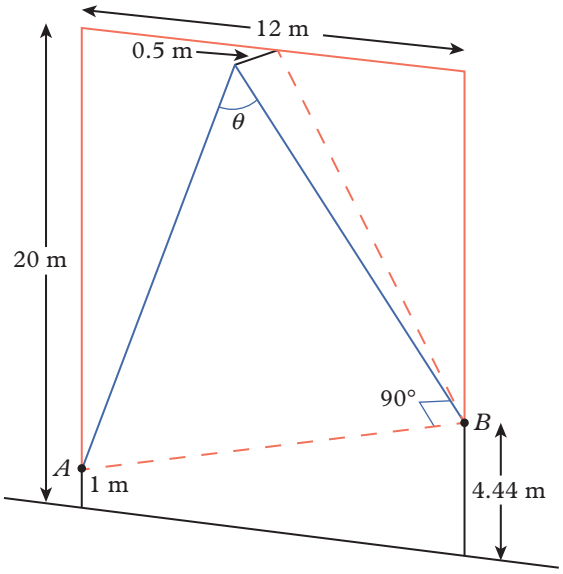
i Draw a labelled net of the cube and label the ant's starting and finishing points. Mark the shortest path the ant could take.

ii Find the length of the shortest path from your answer to part **i**. Give your answer in terms of the side length x.

19 Yolandi goes rock-climbing at an indoor rock-climbing gym. She starts by climbing a rectangular practice wall that is 12 m wide and 20 m high. A belay rope is suspended from the ceiling by a pulley and attached to Yolandi's waist to stop her from falling too far. The pulley is positioned at the centre, 0.5 m away from the wall. Yolandi starts from the left side of the wall with the rope tied to her waist at point A, 1 m above the floor. She then climbs directly to point B, 4.44 m high on the right side of the wall. She then continues directly to the top of the wall at its centre. Given that the angle the rope makes with the line from point A to point B is a right angle, correct to one decimal place, determine:

a the distance Yolandi climbs in total

b the angle, θ, that the rope turns from when Yolandi is on the left side of the wall to when Yolandi is at the right side of the wall.



Online resources:

Interactive skillsheet — Finding lengths within 3D objects

Interactive skillsheet — Finding angles within 3D objects

Quick quiz — 7.4

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Lesson 7.5

The sine and area rules

Learning intentions

By the end of this lesson you will be able to ...

✔ use the sine rule and the included angle area formula.

Inter-year links

Year 7	6.6 Solving equations using inverse operations
Year 8	6.2 Solving linear equations
Year 9	6.6 Using trigonometry to find angles

Key content video
The sine and area rules

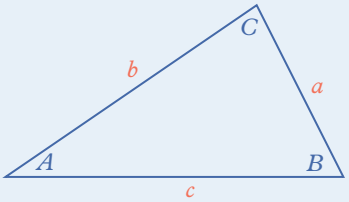
The sine of angles from 0° to 180°

- So far, sine has been defined in terms of the side lengths of a right-angled triangle with reference to an interior angle: $\sin(\theta) = \frac{O}{H}$. However, the interior angles of a right-angled triangle are limited to acute and right angles, $0° \leq \theta \leq 90°$.
- The interior angles of a general triangle can be acute and obtuse, so sine must be defined for $0 \leq \theta \leq 180°$.
- The sine values of supplementary angles are equal, $\sin(180° - \theta) = \sin(\theta)$. For example, $\sin(150°) = \sin(30°)$.

The sine rule

- The sides of a non-right-angled triangle are often labelled using lowercase letters a, b and c. The interior angles opposite those side lengths are then labelled using the capital letters: A, B and C.
- The side lengths and interior angles of any triangle, ABC, are related to each other by the **sine rule**:

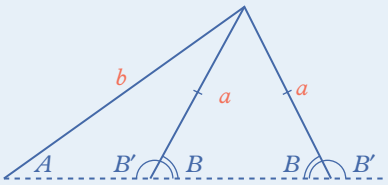
$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$



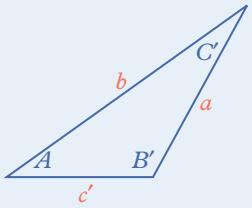
- The sine rule can be used to find missing side lengths or angles in cases where three of the values are already known and two of the known values are an opposite angle-side pair. That is, a pair of $a - A$, $b - B$ or $c - C$.

Ambiguous case of the sine rule

- There are two possible triangles that can be drawn when two side lengths (a and b) of a non-right-angled triangle are known, and the angle A opposite the shorter side length is also known.
- The important difference between the two possible triangles is that, for one of them, angle B (opposite side b) is acute while, for the other, angle B is obtuse.
  - → The acute angle for B can be determined using the sine rule with a calculator.
  - → The obtuse angle for B (which can be labelled as B') is determined using the fact that the two possible values for B are supplementary: $B + B' = 180°$.



- Once B and B' are both known, they can be used with the sine rule to determine the remaining unknown values for the first triangle (side length c and angle C), as well as for the second triangle (side length c' and angle C').



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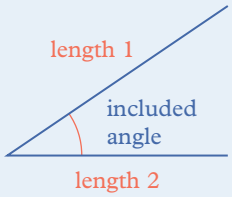
Included angle area formula

- An included angle is the interior angle made by two lines that meet at a common vertex.
- The area of any triangle can be determined using the included angle of two known side lengths. This relationship is known as the **included angle area formula**:

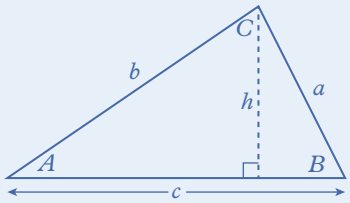


$$\text{area} = \frac{1}{2}\,bc\,\sin(A)$$

adjacent side lengths — included angle

- The included angle area formula, area $= \frac{1}{2}\,bc\,\sin(A)$, can be demonstrated by using the following diagram, where $\sin(A) = \frac{h}{b}$. Multiplying both sides by b, we get $h = b \times \sin(A)$.

$$\begin{aligned}\text{Area of } \triangle ABC &= \frac{1}{2} \times \text{base} \times \text{height}\\ &= \frac{1}{2} \times c \times h\\ &= \frac{1}{2} \times c \times b \times \sin(A)\\ &= \frac{1}{2}\,bc\,\sin(A)\end{aligned}$$



Worked example 7.5A The sine rule

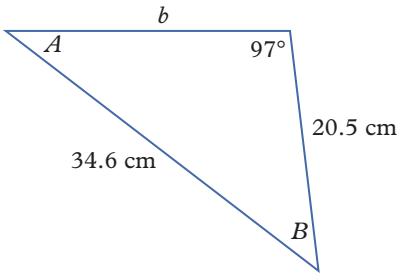
For $\triangle ABC$, find the unknown angles and side lengths, correct to one decimal place, using this information:

$a = 20.5$ cm, $c = 34.6$ cm, $C = 97°$

THINK

1 Draw a labelled diagram, roughly to scale.

WRITE



2 To solve for A, substitute the known values into an equation for the sine rule containing a and c.

$$\frac{a}{\sin(A)} = \frac{c}{\sin(C)}$$
$$\frac{20.5}{\sin(A)} = \frac{34.6}{\sin(97°)}$$
$$\frac{\sin(A)}{20.5} = \frac{\sin(97°)}{34.6}$$
$$\sin(A) = \frac{20.5 \times \sin(97°)}{34.6}$$
$$= 0.5881\ldots$$
$$A = \sin^{-1}(0.5881\ldots)$$
$$\approx 36.0°$$

3 The size of angle B can now be calculated, because the interior angles of a triangle add to 180°.

$$A + B + C = 180°$$
$$B \approx 180° - 97° - 36.0°$$
$$\approx 47.0°$$

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4 To solve for b to find the unknown side length, substitute the known values into any equation for the sine rule containing b.

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$
$$\frac{b}{\sin(47°)} = \frac{34.6}{\sin(97°)}$$
$$b = \frac{34.6 \times \sin(47°)}{\sin(97°)}$$
$$b \approx 25.5 \text{ cm}$$

Worked example 7.5B Ambiguous case of the sine rule

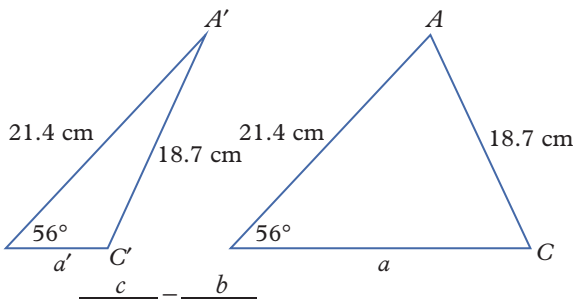
For $\triangle ABC$, find the two possible sets of unknown angles and side lengths, correct to one decimal place, using this information:

$b = 18.7$ m, $c = 21.4$ m, $B = 56°$

THINK

1 Draw a labelled diagram roughly to scale. Because this is an ambiguous case of the sine rule, draw both possible triangles and use the convention for differentiating both sets of unknown values from each other (C and C').

WRITE



2 To solve for C, substitute the known values into the equation for the sine rule containing b and c.

$$\frac{c}{\sin(C)} = \frac{b}{\sin(B)}$$
$$\frac{21.4}{\sin(C)} = \frac{18.7}{\sin(56°)}$$
$$\sin(C) = \frac{21.4 \times \sin(56°)}{18.7}$$
$$= 0.9487\ldots$$
$$C = \sin^{-1}(0.9487\ldots)$$
$$\approx 71.6°$$

3 The size of angle A can now be calculated, because the interior angles of a triangle add to 180°.

$$A + B + C = 180°$$
$$A \approx 180° - 56° - 71.6°$$
$$\approx 52.4°$$

4 To solve for a, substitute the known values into any equation for the sine rule containing a.

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$$
$$\frac{a}{\sin(52.4°)} = \frac{18.7}{\sin(56°)}$$
$$a = \frac{18.7 \times \sin(52.4°)}{\sin(56°)}$$
$$a \approx 17.9 \text{ m}$$

5 Calculate C' using the fact that C' is supplementary to C.

$$C + C' = 180°$$
$$C' \approx 180° - 71.6°$$
$$C' \approx 108.4°$$

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6 A' can now be calculated by subtracting the known angles from 180°.

$$A' + B' + C' = 180°$$
$$A' \approx 180° - 56° - 108.4°$$
$$\approx 15.6°$$

7 To solve for a', substitute the known values into any equation for the sine rule containing a'.

$$\frac{a'}{\sin(A')} = \frac{b}{\sin(B)}$$
$$\frac{a'}{\sin(15.6°)} = \frac{18.7}{\sin(56°)}$$
$$a' = \frac{18.7 \times \sin(15.6°)}{\sin(56°)}$$
$$a' \approx 6.1 \text{ m}$$

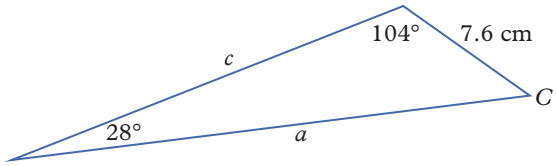
Worked example 7.5C The area rule

Find the area of $\triangle ABC$, correct to one decimal place, given the following information:
$b = 7.6$ cm, $A = 104°$, $B = 28°$

THINK

1 Draw a labelled diagram. Note that at least two side lengths must be known to calculate the area using the area rule.

WRITE



2 To solve for a, substitute the known values into an equation for the sine rule containing a and b.

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)}$$
$$\frac{a}{\sin(104°)} = \frac{7.6}{\sin(28°)}$$
$$a = \frac{7.6 \times \sin(104°)}{\sin(28°)}$$
$$a \approx 15.7 \text{ cm}$$

3 Calculate the included angle between a and b using the fact that the interior angles of a triangle add to 180°.

$$C + A + B = 180°$$
$$C = 180° - 104° - 28°$$
$$= 48°$$

4 Substitute the relevant known values into the area rule.

$$\text{area} = \frac{1}{2}ab\sin(C)$$
$$= \frac{1}{2} \times 15.7 \times 7.6 \times \sin(48°)$$
$$\approx 44.3 \text{ cm}^2$$

Helpful hints

- ✔ When using the sine rule, it helps to label the diagram with side lengths a, b and c, and their opposite angles A, B and C, respectively.
- ✔ To determine whether the given information for any triangle represents an ambiguous case of the sine rule, run through this short checklist:
  - → Are there two known side lengths and one known angle?
  - → Is the known angle opposite the shorter known side?

  If the answer is yes to both questions, it is an ambiguous case.

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Learning pathways

1–7, 8(a–c), 12, 13, 15, 16, 19 | 3–7, 8(b, d, f), 9, 11, 12, 14, 17, 19, 20, 22 | 4, 5, 7, 8(d–f), 10, 14, 16–19, 21–23

ANS p727

Exercise 7.5A: Understanding and fluency

1 Use a calculator to evaluate each of the following, correct to four decimal places where necessary.

a sin(30°) **b** sin(150°) **c** sin(40°) **d** sin(140°)

e sin(80°) **f** sin(100°) **g** sin(22°) **h** sin(158°)

2 Use the sine rule to find the unknown value for each of the following triangles, ABC.

a If $A = 45°$, $B = 72°$ and $a = 10$ mm, calculate b correct to one decimal place.

b If $C = 65°$, $a = 12$ cm and $c = 13.4$ cm, calculate A correct to one decimal place.

c If $B = 37°$, $c = 8$ cm and $b = 15$ cm, calculate A correct to one decimal place.

WE 7.5A **3** For each of these triangles, ABC, use the given information to find the unknown angles and side lengths, correct to one decimal place. Draw a labelled sketch of the triangle.

a $a = 8.5$ cm, $B = 81°$, $A = 37°$ **b** $c = 37.3$ mm, $b = 33.7$ mm, $C = 85°$

c $a = 22.5$ cm, $c = 28.3$ cm, $A = 49°$ **d** $b = 5.8$ cm, $B = 26°$, $A = 67°$

WE 7.5B **4** Use the given information to find the two possible sets of unknown angles and side lengths for each of these, correct to one decimal place. Draw a labelled sketch of both possible triangles.

a $c = 33.7$ mm, $b = 37.3$ mm, $C = 40°$ **b** $a = 14.9$ mm, $b = 26$ mm, $A = 32°$

c $b = 12.3$ cm, $c = 32.8$ cm, $B = 16°$ **d** $c = 22.1$ mm, $a = 23.8$ mm, $C = 67°$

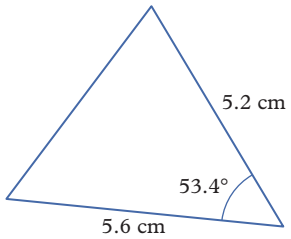
5 Find the area of each of these triangles, ABC, correct to one decimal place.

a $a = 9.5$ cm, $b = 7.6$ cm, $C = 28°$ **b** $b = 2.8$ mm, $c = 8.9$ mm, $A = 87°$

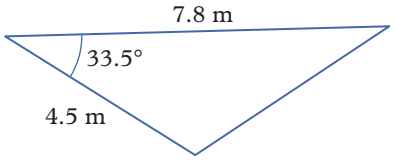
c $a = 4.7$ cm, $c = 9.2$ cm, $B = 57°$ **d** $c = 12.5$ cm, $b = 9.2$ cm, $A = 43°$

WE 7.5C **6** Find the area of each of these triangles, correct to one decimal place.

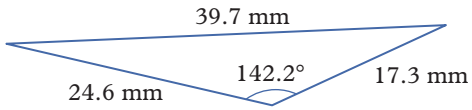
a



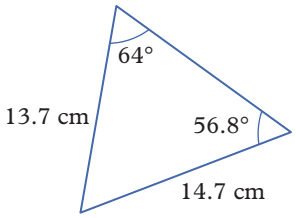
b



c



d



7 For each set of known values for the triangle ABC, explain why or why not:

i the sine rule can be used and

ii the set implies an ambiguous case of the sine rule.

a $c = 65$ cm, $b = 70$ cm, $C = 64°$ **b** $a = 60$ cm, $b = 70$ cm, $C = 64°$

c $b = 60$ cm, $B = 70°$, $A = 64°$ **d** $c = 65$ cm, $b = 70$ cm, $B = 64°$

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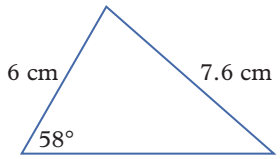
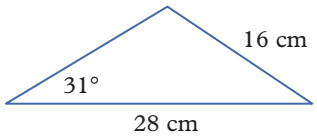
8 Find the perimeter of each triangle. Hint: Check which triangles result in an ambiguous case of the sine rule and decide which of the two possible triangles match the diagram.

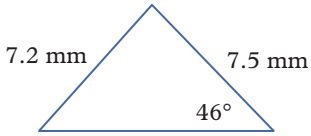
a



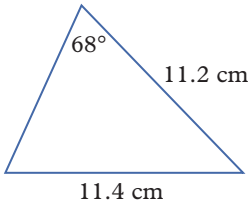
b



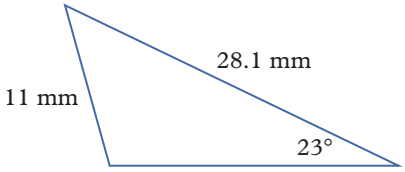
c



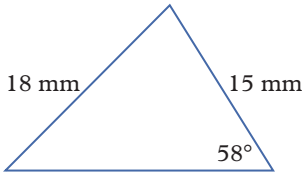
d



e

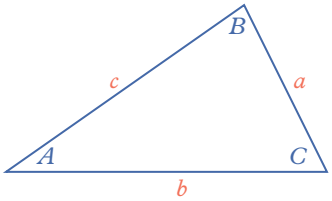


f



Exercise 7.5B: Problem solving and reasoning

9 Copy the triangle on the right into your workbook.



a Draw a perpendicular from vertex B to base b. Label the point where the perpendicular meets the base, D. Label the length of the perpendicular h.

b The trigonometric sine ratio is $\sin(\theta) = \frac{\text{O}}{\text{H}}$.

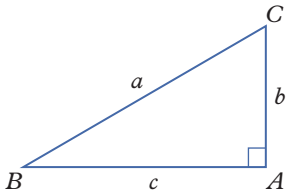
i Use the sine ratio for angle A to write a relationship between h and c. Make h the subject of the formula.

ii Use the sine ratio for angle C to write a relationship between h and a. Make h the subject of the formula.

iii Equate your two answers for parts **i** and **ii** and show how this can be written as $\frac{a}{\sin(A)} = \frac{c}{\sin(C)}$.

c Show that $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$ by repeating the same reasoning outlined in parts **a**, and **b** to find a similar relationship involving angle B.

10 Consider the right-angled triangle, $\triangle ABC$.



a Evaluate $\sin(A)$.

b Use the triangle to help you write expressions for $\sin(B)$ and $\sin(C)$.

c Substitute your expressions from parts **a** and **b** into the sine rule. Simplify.

d Explain your answer. Does the sine rule hold for right-angled triangles?

e What would have been your answer for part **c** if the triangle had been right-angled at $\angle ABC$ or $\angle BCA$?

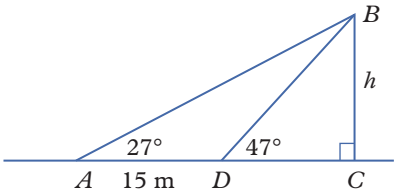
11 Refer to the diagram in question **10**. The area of the triangle can be calculated using the formula: area $= \frac{1}{2}ab\sin(C)$.

a Use the diagram to help you obtain an expression for $\sin(C)$.

b Substitute the expression you found in part **a** into the area formula and simplify.

c Explain the resulting formula.

12 Chloe measures the angle of elevation to the top of a building as 27°. When she walks 15 m closer to the building, she measures the angle of elevation to be 47°.



a Find the size of:

i $\angle ADB$ **ii** $\angle ABD$.

b Use the sine rule in $\triangle ABD$ to calculate the length of BD, correct to one decimal place.

c Use the sine ratio in $\triangle BCD$ to find the height of the building.

d Use the cosine ratio in $\triangle BCD$ to find how far Chloe was from the building when she took her second reading.

e How far was Chloe from the building initially?

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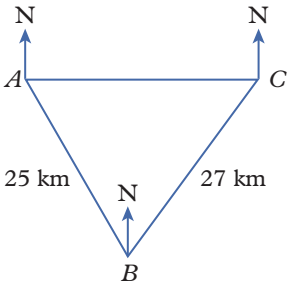
13 A group of bushwalkers started from point A, travelled 25 km to point B at a bearing of S30°E, then continued for 27 km to point C, which lies due east of their starting point.



a What is the size of $\angle BAC$?

b Find the size of $\angle ACB$ to the nearest degree.

c What is the size of $\angle ABC$?

d Find the distance the bushwalkers would need to travel from their finishing point to return directly to their starting point. Give your answer to one decimal place.

e Give the bearing:

i of C from B

ii of B from C

iii the bushwalkers would need to take to return directly to their starting point.

f Calculate the area enclosed by the bushwalkers' hike.

14 The banks of a creek run directly east–west. From where Kim stands on the banks of the creek, the bearing to a tree on the opposite side of the creek is 032°T. If she moves 20 m due east on the bank of the creek, the bearing to the same tree is 310°T.

a Draw a labelled diagram displaying as much information as possible.

b Find the distance (in metres, to one decimal place) of the tree from:

i Kim's first viewing point

ii Kim's second viewing point.

c Calculate the width of the creek.

15 A give-way sign is in the shape of an equilateral triangle, with side lengths of 40 cm. Calculate the area of the sign, to the nearest square centimetre.

16 A lighthouse stands 50 m tall on top of a cliff. From a ship at sea, the angle of elevation to the base of the lighthouse is 3°, while the angle of elevation to the top of the lighthouse is 4.2°.

a Draw a labelled diagram to display this information.

b Find the straight-line distance from the ship to:

i the base of the lighthouse **ii** the top of the lighthouse.

c Calculate the distance in kilometres (to one decimal place) from the ship to the base of the cliff.

d Find the height of the cliff, to the nearest metre.

17 A ship departs from Ashy Bay on a bearing of 295°T to Basalt Cove, 23 km away. From Basalt Cove, the ship needs to travel 12.4 km to Coral Quay, but the inexperienced crew have forgotten which bearing they need to travel along. The crew knows that Coral Quay is due west from Ashy Bay but so is Deserted Depths, which is also 12.4 km from Basalt Cove.

a Determine the bearings of Coral Quay and Deserted Depths from Basalt Cove, correct to the nearest degree.

b Determine the distances from Ashy Bay to Coral Quay and Deserted Depths, correct to one decimal place.

c One of the crew members incorrectly recalls that Coral Quay is closer to Ashy Bay than Deserted Depths is. The ship travels to the location closer to Ashy Bay but arrives at Deserted Depths instead. The ship now must travel from Deserted Depths to Coral Quay. Calculate the minimum total distance the ship will have travelled when it finally reaches Coral Quay, correct to one decimal place.

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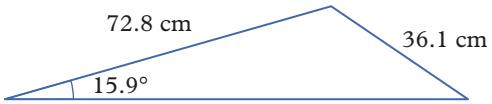
18 Consider the triangle below.

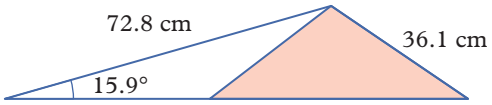


a Explain why the triangle is ambiguous.

b Correct to one decimal place, determine the two possible lengths of the third side.

c Correct to one decimal place, calculate the area of the two possible triangles.

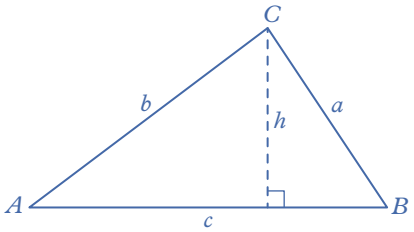
d When the two triangles are overlapped at the 15.9° angle, the excess is an isosceles triangle.



i Determine the area of the isosceles triangle using your answers to part **c**, correct to one decimal place.

ii Write an expression for the area of the isosceles triangle, using the angle between the equal side lengths.

19 Consider the $\triangle ABC$ on the right with height h and base c.



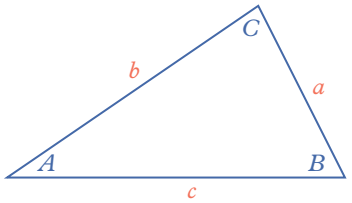
a Write a formula for the area of $\triangle ABC$ in terms of h and c.

b Write h in terms of A and b, and use your answer to write a formula for the area of $\triangle ABC$ in terms of A, b and c.

c Write h in terms of B and a, and use your answer to write a formula for the area of $\triangle ABC$ in terms of B, a and c.

d Compare your answers to parts **b** and **c** to the included angle area formula.

20 Consider the triangle ABC.



a Write three different expressions for the area of the triangle using the included angle area formula.

b Equate the expressions from part **a** and show that the resulting equation is equivalent to the sine rule.

c All interior angles of an equilateral triangle are 60°.

i Use a calculator to determine the value of $(\sin(60°))^2$ as a fraction.

ii Hence, write the exact value of $\sin(60°)$.

iii Hence, write a rule for the area of an equilateral triangle with a side length of l.

21 Regular n-sided polygons can be broken up into n congruent isosceles triangles by joining the centre of the polygon to each vertex. For each of the following regular polygons shown, determine:

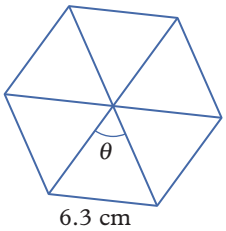
i the size of the angle θ at the centre

ii the distance from the centre to each vertex, correct to one decimal place

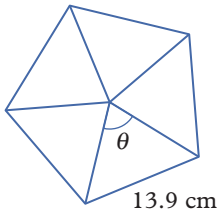
iii the area of each isosceles triangle, correct to one decimal place

iv the area of the regular polygon, correct to one decimal place.

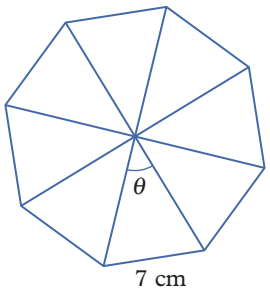
a



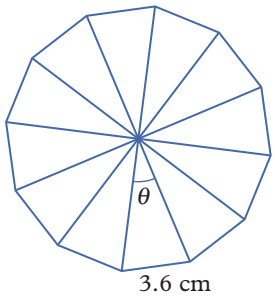
b



c



d



e Write a rule for the area of an n-sided regular polygon with side length l.

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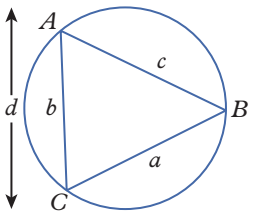
Exercise 7.5C: Challenge

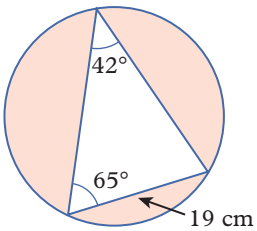


22 The circumcircle of a triangle is the circle that passes through the vertices of the triangle. The ratio of a side to the sine of the opposite angle is equal to the diameter of the circumcircle.

That is, $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)} = d$.

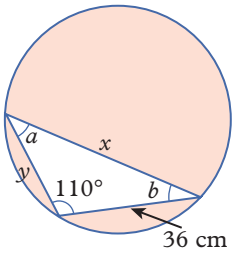
Answer the following questions, correct to two decimal places.

a Determine the diameter of the circumcircle of the triangle in the circle below.



b Using your answer from part **a**, find the area of:

i the triangle **ii** the circumcircle **iii** the shaded area.

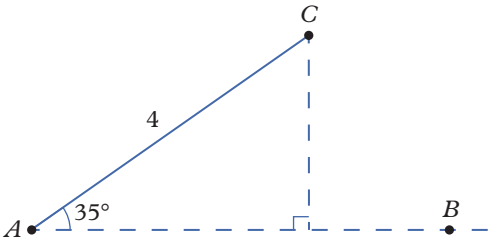


c The diameter of the circle on the right is 56 cm.

i Determine the value of each of the pronumerals, correct to two decimal places.

ii Determine the shaded area, correct to two decimal places.

23 So far, when dealing with the ambiguous case of the sine rule, we have assumed a triangle (or triangles) can be formed. However, it is possible for the side opposite the known angle to be too short to form a triangle. Consider the diagram below.

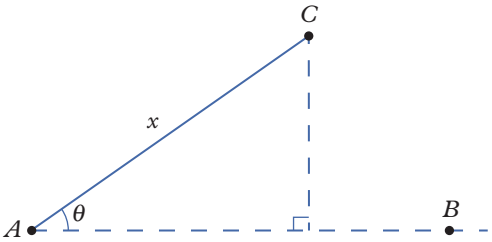


a Determine the perpendicular distance from C to the ray AB, correct to two decimal places.

b What is the minimum length of the line segment BC such that ABC is a triangle? Explain.

c What is the maximum length of the line segment BC such that there are two possible triangles ABC? Explain.

d For the diagram below, write an inequality for the length of line segment BC such that ABC is a triangle and also ambiguous.



Online resources:

- **Interactive skillsheet** The sine rule
- **Interactive skillsheet** The area rule
- **Investigation** Proof of Heron's formula
- **Quick quiz** 7.5

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Checkpoint

Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this module.
✔ 10
✔ 10A

7.1 **1** Determine whether the following lengths can form a right-angled triangle.

a 9 cm, 40 cm, 41 cm

b 39 cm, 89 cm, 80 cm

c 137 cm, 104 cm, 89 cm

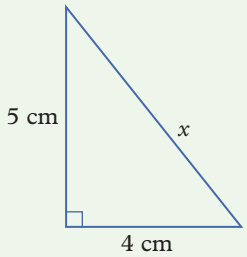
d 21 cm, 221 cm, 220 cm

7.1 **2** Determine the value of x in each of the following:

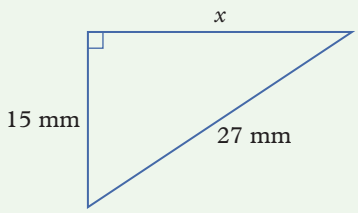
i as an exact value

ii correct to two decimal places.

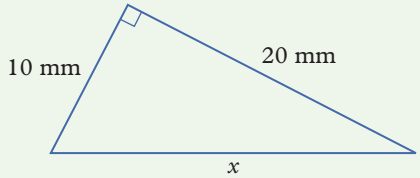
a



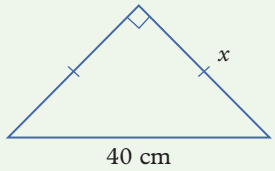
b



c

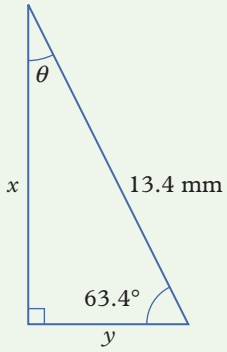


d

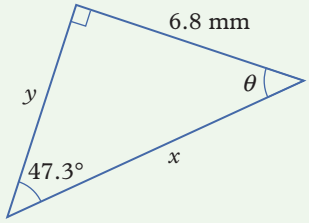


7.2 **3** Determine the value of each of the pronumerals, correct to one decimal place.

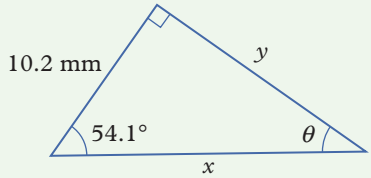
a



b

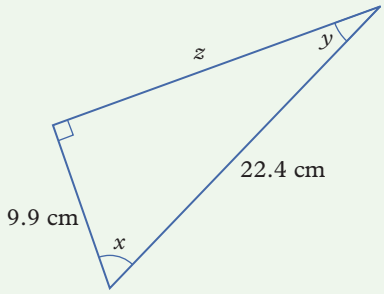


c

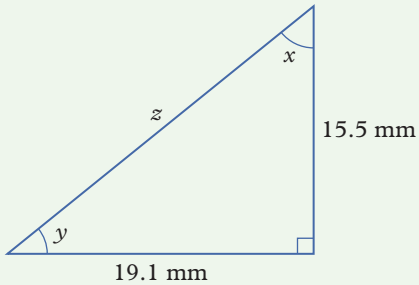


7.2 **4** Determine the value of each of the pronumerals, correct to one decimal place.

a



b



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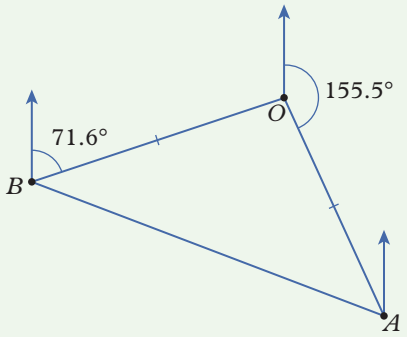
7.3 **5** Determine the true bearing of B from A in each of the following.

a The bearing of B from A is S85°E.

b The bearing of A from B is N35°W.

c



7.3 **6** Calculate each of the following, correct to one decimal place.

a The angle of elevation of the top of a large tree from a point on the ground 6 m away is 78°. Determine the height of the tree.

b A building is 120 m tall. Another building is 105 m tall. Determine the angle of depression from the top of the taller building to the top of the shorter building if the tops of the buildings are 30 m apart.

c The angle of depression of an overboard passenger from the deck of a ship is 24°. A 12 m rope pulled taut just reaches from the deck of the ship to the overboard passenger in the water. Determine how far, in total, the overboard passenger needs to be pulled towards the boat through the water and then up the side of the ship to get them back onto the deck of the boat.

d A child holds the 8 m string of a kite 130 cm from the ground. If the angle of elevation of the kite is 65°, how high above the ground is the kite flying?

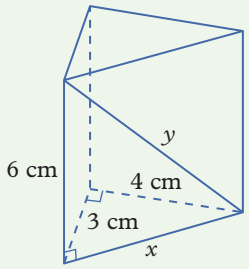
7.3 **7** Calculate each of the following distances, correct to one decimal place, and bearings correct to the nearest degree.

a A snorkeller swims to a point 230 m away at a bearing of 160° from their boat. Determine how far south and east they are from the boat.

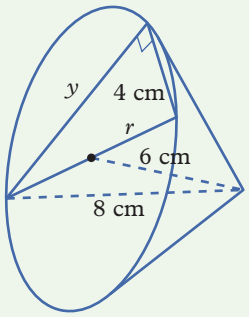
b A farmer walks 1.2 km east and then 800 m north along two sides of a paddock. Determine the true bearing of the location the farmer ended at from their starting point.

10A 7.4 **8** Determine the value of each of the pronumerals, correct to one decimal place.

a

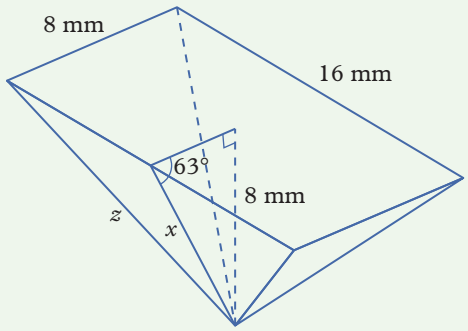


b

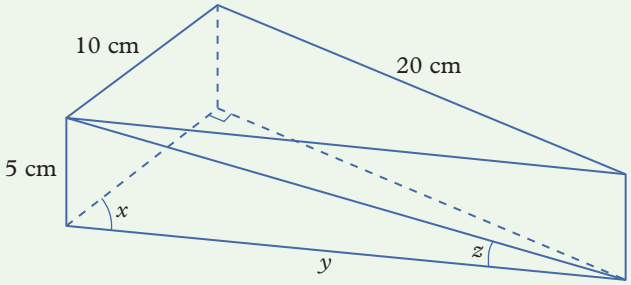


10A 7.4 **9** Determine the value of each of the pronumerals, correct to one decimal place.

a



b



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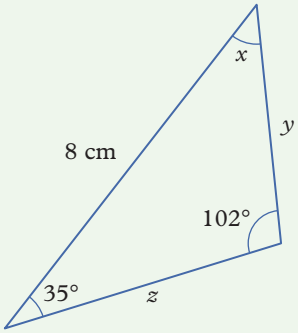
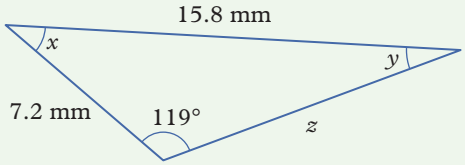
10A 7.5 **10** Determine the value of each of the pronumerals, correct to one decimal place.

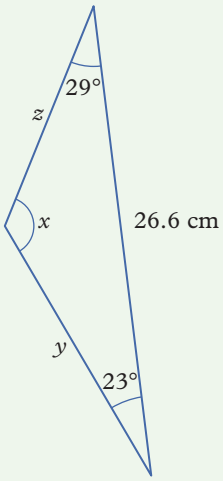
a



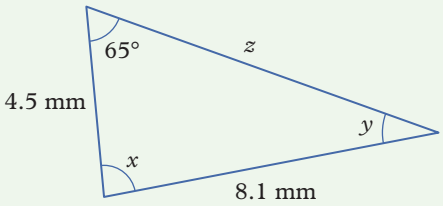
b



c

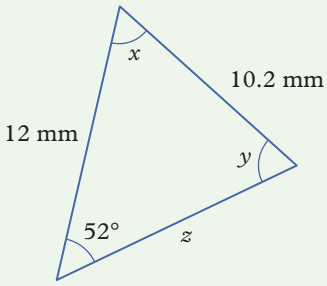


d

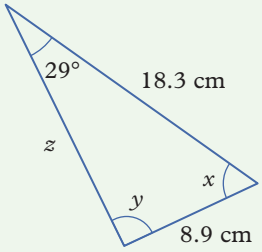


10A 7.5 **11** Determine all possible values of each of the pronumerals, correct to one decimal place.

a

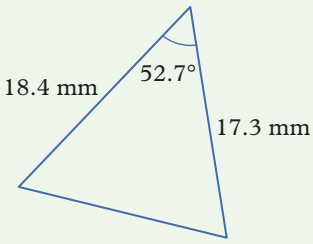


b

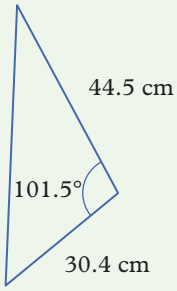


10A 7.5 **12** Determine the area of the following triangles, correct to two decimal places.

a



b



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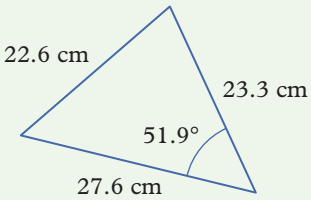

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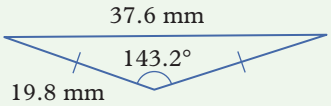

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Lesson 7.6

The cosine rule

Learning intentions

By the end of this lesson you will be able to ...

✔ use the cosine rule to link the three side lengths and one interior angle of a triangle.

Inter-year links

Year 7	6.6 Solving equations using inverse operations
Year 8	6.2 Solving linear equations
Year 9	6.6 Using trigonometry to find angles

The cosine of angles from 0° to 180°

Key content video
The cosine rule

- So far, cosine has been defined in terms of the side lengths of a right-angled triangle with reference to an interior angle: $\cos(\theta) = \frac{A}{H}$. However, the interior angles of a right-angled triangle are limited to acute and right angles, $0° \leq \theta \leq 90°$.
- The interior angles of a general triangle can be acute and obtuse, so cosine must be defined for $0 \leq \theta \leq 180°$. The cosine values of supplementary angles have opposite signs, $\cos(180° - \theta) = -\cos(\theta)$. For example, $\cos(120°) = -\cos(60°)$.
- In addition, $\cos(0°) = 1$ and $\cos(90°) = 0$.

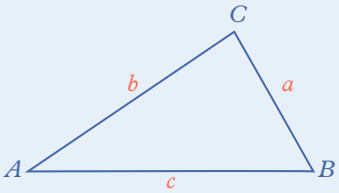
Using the cosine rule

- The side lengths and interior angles of any triangle, $\triangle ABC$, are related to each other by the **cosine rule**:

  $c^2 = a^2 + b^2 - 2ab\cos(C)$

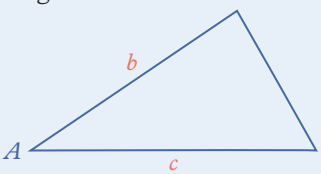
  → a, b and c are the three side lengths of a triangle

  → C is the included angle between a and b.

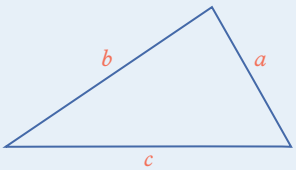


- The cosine rule is particularly useful for determining angles and side lengths in two situations when the sine rule is not suitable for finding the unknowns:

  Situation 1: The only known angle is the included angle between the only known side lengths.



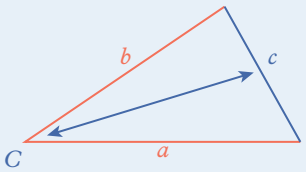
  Situation 2: All the side lengths are known, and all the angles are unknown.



- To help remember the cosine rule, remember that the side length and the angle at either end of the equation form an opposite side-angle pair, and the side lengths in the middle are the adjacent side lengths to the included angle.

opposite side-angle pair

$$c^2 = \underbrace{a^2 + b^2 - 2ab}_{\text{adjacent side lengths where } C \text{ is included}}\cos(C)$$



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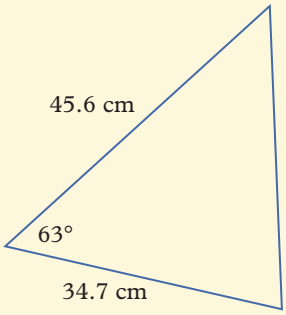
Worked example 7.6A Using the cosine rule to find a side length

Find the unknown side length in this triangle, correct to one decimal place.



THINK	WRITE
1 Identify the known values.	The included angle is 63° between the adjacent side lengths 45.6 cm and 34.7 cm.
2 Substitute the known values into the cosine rule.	$c^2 = a^2 + b^2 - 2ab\cos(C)$ $= (45.6)^2 + (34.7)^2 - 2 \times (45.6) \times (34.7) \times \cos(63°)$ $= 1846.7335\ldots$
3 Find the square root and round the answer to one decimal place. Write the units.	$c = \sqrt{1846.7335\ldots}$ ≈ 43.0 cm

Worked example 7.6B Using the cosine rule to find an unknown angle

The following lengths are given for the sides of $\triangle ABC$:

$a = 2.8$ m, $b = 5.3$ m, $c = 7.1$ m

Determine the size of the included angle between a and b, correct to one decimal place.

THINK	WRITE
1 Substitute the side lengths into the cosine rule.	$c^2 = a^2 + b^2 - 2ab\cos(C)$ $(7.1)^2 = (2.8)^2 + (5.3)^2 - 2 \times (2.8) \times (5.3) \times \cos(C)$
2 Solve for $\cos(C)$.	$50.41 = 35.93 - 29.68\cos(C)$ $14.48 = -29.68\cos(C)$ $\cos(C) = -0.4878\ldots$
3 Use the inverse cosine to calculate C.	$C = \cos^{-1}(-0.4879\ldots)$ $\approx 119.2°$

Helpful hints

- ✔ When you are calculating an angle using the cosine rule, always check that the value you have calculated for the cosine value is between –1 and 1.
- ✔ If you are calculating all the interior angles of a triangle using the cosine rule, check your calculations by making sure the three angles add to 180°.

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Learning pathways

1, 2(a–e), 3, 4, 5(a–c), 6–10, 13, 15	2(f–i), 3, 5–8, 11–13, 17, 18, 20	3(c, d), 5, 8, 11, 12, 14, 16, 17, 19, 21–23

ANS p727

Exercise 7.6A: Understanding and fluency

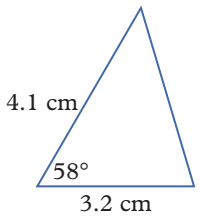
1 Use a calculator to evaluate each of the following, correct to four decimal places where necessary.

a cos(60°) **b** cos(120°) **c** cos(40°) **d** cos(140°)

e cos(80°) **f** cos(100°) **g** cos(22°) **h** cos(158°)

WE 7.6A **2** Find the unknown side length in each of these triangles, correct to one decimal place.

a



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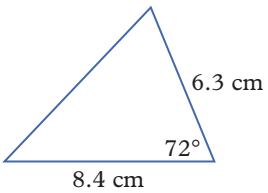

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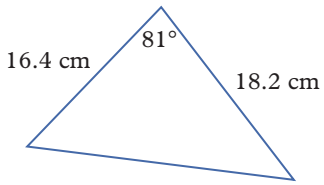

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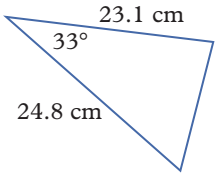

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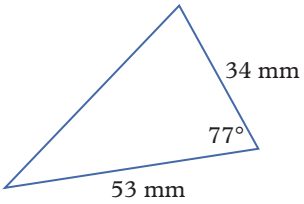

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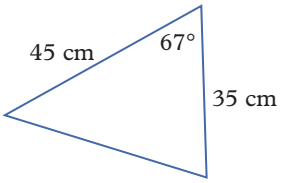

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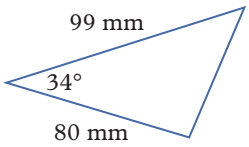

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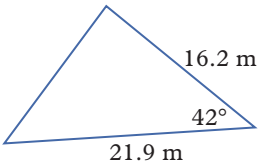

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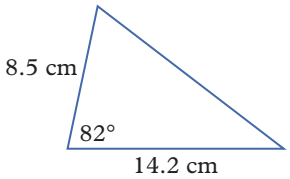


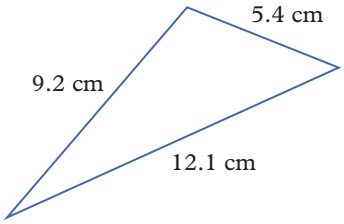
WE 7.6B **3** Each of the following sets of side lengths are for a different $\triangle ABC$. Find the sizes of all the angles of each triangle, correct to one decimal place.

a $a = 4.5$ cm, $b = 6$ cm and $c = 5$ cm **b** $a = 13$ mm, $b = 12$ mm and $c = 7$ mm

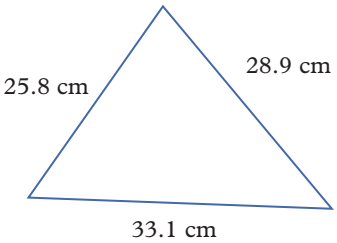
c $a = 1.5$ m, $b = 2.3$ m and $c = 1.8$ m **d** $a = 16$ cm, $b = 19$ cm and $c = 12$ cm

4 Find the sizes of the angles of each of these triangles, correct to one decimal place.

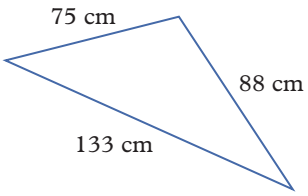
a Find the size of the largest interior angle.



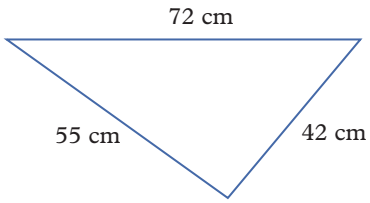
b Find the size of the smallest interior angle.



c Find the size of the median interior angle.



d Find the size of the largest exterior angle.



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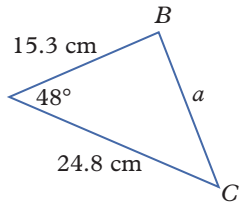
5 Find the unknown side lengths and angles of each of these triangles.

a



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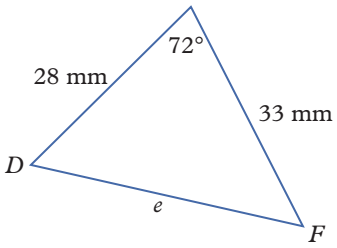

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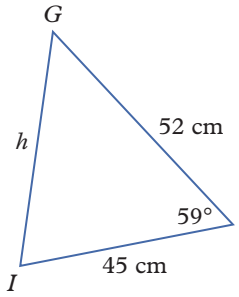

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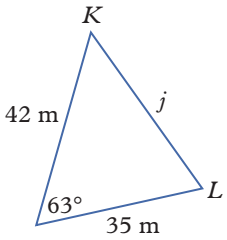

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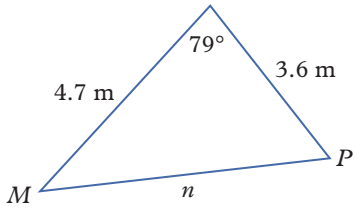

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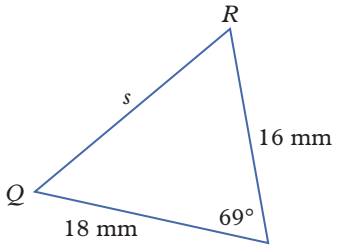
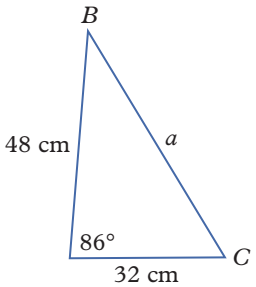


6 Sometimes there is a choice as to whether the cosine rule or the sine rule can be used to solve a problem. Consider the triangle on the right.



a Explain why it is not possible to use the sine rule to calculate the length of a.

b Use the cosine rule to find the length of a.

c Consider angle B and angle C.

i Explain how you could now use the sine rule to find the sizes of angles B and C.

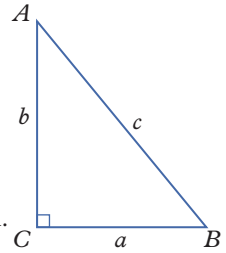
ii Calculate the sizes of those two angles.

d Angles B and C from part **c** could also have been calculated using the cosine rule. Find their sizes using this rule.

e Comment on your answers to parts **c ii** and **d**.

Exercise 7.6B: Problem solving and reasoning

7 Consider the right-angled triangle ABC with the right angle at C.



a Copy and complete the cosine rule for triangle ABC: $c^2 =$ _____.

b Evaluate $\cos(90°)$.

c Substitute the value from part **b** into your equation in part **a** and simplify. Describe the result.

d The cosine rule has the adjustment term $-2ab\cos(C)$ compared to Pythagoras' theorem.

i Suppose C is an acute angle. Use the adjustment term to explain why the side opposite angle C is shorter than the hypotenuse in the case where C is a right angle.

ii Suppose C is an obtuse angle. Use the adjustment term to explain why the side opposite angle C is longer than the hypotenuse in the case where C is a right angle.

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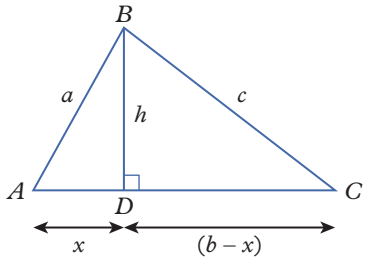
8 Consider $\triangle ABC$, with the perpendicular h from B dividing the base b into two sections of length x and $(b - x)$.



a Use $\triangle CDB$ and Pythagoras' theorem to complete the relationship $c^2 = h^2 +$ _____.

b Use $\triangle ADB$ and Pythagoras' theorem to complete the relationship $h^2 =$ _____.

c Substitute the value of h^2 you wrote for part **b** into your equation in part **a**.
Simplify the resulting equation and confirm that it can be written as $c^2 = a^2 + b^2 - 2bx$.

d Use $\triangle ADB$ and the cosine ratio to write a relationship with x as the subject of the equation.

e Substitute your value for x into the equation you wrote for part **c**. Your resulting equation should be:
$c^2 = a^2 + b^2 - 2ab\cos(C)$.

f Use a similar method to that used for writing an equation for c^2 in part **e** to write equations for b^2 and a^2.

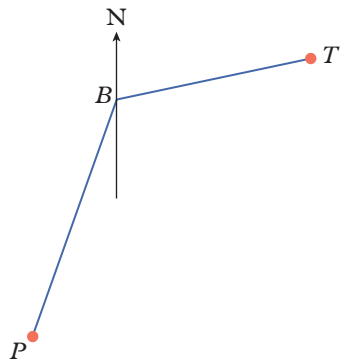
g The cosine rule can be rearranged to find the size of an angle of a triangle.
Write three versions of the cosine rule with the following cosine values as the subject.

i $\cos(A)$ **ii** $\cos(B)$ **iii** $\cos(C)$

h Using the cosine rule, what information would you need to calculate:

i a side length **ii** the size of an angle?

9 From a buoy at sea, Todd rows 2.5 km at a bearing of N80°E, while Pete rows 3.5 km at a bearing of S20°W.



a Copy the diagram and label all known sides and angles.

b What is the size of $\angle TBP$?

c Join TP. This represents the distance between the two rowers.
Use the cosine rule with $\triangle BTP$ to calculate this distance.

d Use the cosine rule with $\triangle BTP$ to calculate the size of $\angle BPT$.

e Use your answer for part **c** to find the bearing of Todd from Pete.

f Calculate the size of $\angle BTP$.

g Use your answer for part **f** to find the bearing of Pete from Todd.

10 A triangular garden bed has side lengths 5.3 m, 4.8 m and 4.5 m.

a Draw a labelled sketch of the garden bed.

b Identify the vertex with the largest angle. Hint: The largest angle lies opposite the longest side.

c Use the cosine rule to find the size of the angle you identified in part **b**, to the nearest degree.

d Use your answer to part **c** to calculate the area of the bed.

11 A deep-sea diver is attached to two safety ropes from two boats on the surface of the ocean. The boats are 50 m apart. One rope is 110 m long and the other is 100 m long.

a Draw a labelled diagram to describe the situation when the diver is at the maximum depth that is possible while attached to the two ropes.

b Use the cosine rule to find the angles the two ropes make with the surface of the water.

c On your diagram from part **a,** construct a right-angled triangle that will enable you to find the maximum depth to which the diver can descend.

d Calculate the maximum depth to which the diver can descend while attached to the two safety ropes.

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12 Emily is practising shooting hockey goals. The hockey goal is 3 m wide. She stands 5 m directly in front of one of the posts.

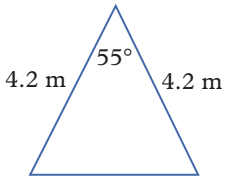
a Draw a diagram to display the situation.

b Use Pythagoras' theorem to calculate the distance from Emily to the other goalpost.

c Use the cosine rule to find the angle within which Emily must shoot to score a goal.

d Emily moves her position, so she is 5 m from each of the goal posts. Explain whether she has a greater chance of scoring a goal from this position.

13 The frame formed by the end support poles of a swing forms a triangular shape with the ground. The poles are each 4.2 m long with an angle of 55° between them. Write your answers correct to one decimal place.



a How far apart are the ends of the poles on the ground?

b Find the vertical height of the top of the frame above the ground.

14 The frame for a bridge consists of a series of triangular shapes connected together as shown in the photo on the right.

The sloping steel parts are 3.9 m long and the horizontal steel parts are 2.8 m long.

a Find the angle between any two sloping pieces of steel.

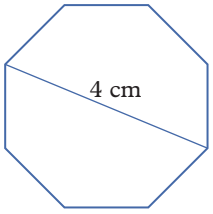
b Draw a sketch of several sections of the bridge, showing the angles and lengths of the connecting pieces.

c Find the height of the frame of the bridge in metres, correct to one decimal point.

15 The diagonal length of an octagon, between vertices that are four edges apart, measures 4 cm.

a Draw a sketch of the octagon. Add the three remaining diagonals that join vertices that are four edges apart.

b Consider one of the triangles within your sketch.



i What is the size of the angle at the centre of the octagon?

ii Find the sizes of the other two angles of the triangle.

c Calculate the length of the outer edge (the perimeter) of the whole octagon.

16 A light aircraft is set to fly on a course N78°E for a distance of 350 km. A strong wind moving from east to west blows the plane off course by 6°.

a Draw a labelled diagram to show this situation.

b If the pilot flies the planned distance on this incorrect path, how far off course will he be at the end of his journey?

17 A cube has a side length of 25 cm.

a Draw a labelled sketch of the cube.

b Use the cosine rule to find the length of the diagonal of the base. Leave your answer in a simplified exact form.

c Use the cosine rule again to find the length of the diagonal of the cube. Leave your answer in a simplified exact form.

18 The angles of a triangle are in the ratio 3 : 5 : 7.

a Find the size of all the angles.

b The two sides adjacent to the largest angle measure 2.58 cm and 4.30 cm.

i Correct to two decimal places, calculate the length of the third side.

ii Draw a labelled diagram to display the measurements of all the angles and sides.

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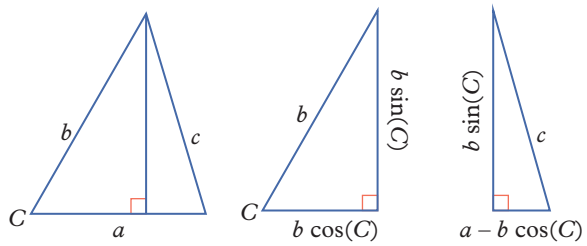
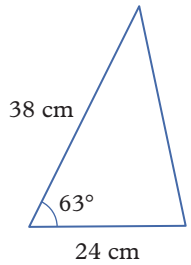
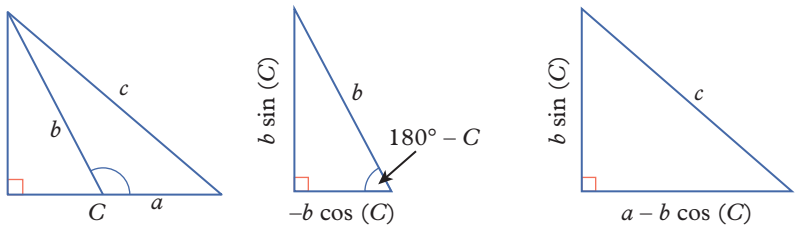
19 Consider the diagrams below. Using Pythagoras' theorem on the rightmost triangle, we can write the following equation: $(b\sin(C))^2 + (a - b\cos(C))^2 = c^2$.



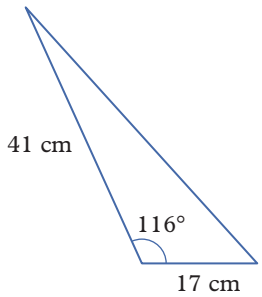
a Substitute the values from the triangle shown below into the equation above and hence determine the unknown side length, correct to one decimal place.



b The equation also holds true if C is an obtuse angle as $\sin(180° - C) = \sin(C)$, and $\cos(C)$ will be negative and so adds the extra length on.

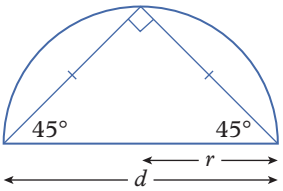


Substitute the values from the triangle shown below into the equation above and hence determine the unknown side length, correct to one decimal place.



c Use the fact that $(b\sin(C))^2 + (b\cos(C))^2 = b^2$ to show that $(b\sin(C))^2 + (a - b\cos(C))^2 = c^2$ is equivalent to $a^2 + b^2 - 2ab\cos(C) = c^2$.

20 A right-angled isosceles triangle is inscribed in a semicircle of radius r.



a Find the lengths of the triangle's two equal sides, in terms of r, using:

i Pythagoras' theorem

ii the cosine rule.

b Comment on your answers to part **a**.

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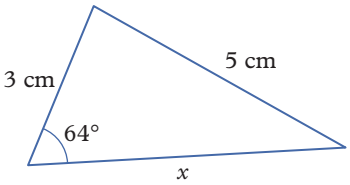
Exercise 7.6C: Challenge

21 Azar has written the equation $5^2 = 3^2 + x^2 - 2(3)x\cos(64°)$ for the triangle on the right.

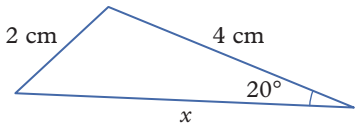


a Write the equation in the form $x^2 + bx + c = 0$, where b and c are real numbers.

b Solve the equation in part **a**. Give your answers correct to one decimal place.

c Are both solutions to the equation in part **b** possible side lengths of the triangle? Explain, stating the length of the unknown side length.

d Azar has written the equation $2^2 = 4^2 + x^2 - 2(4)x\cos(20°)$ for the triangle below, which is not drawn to scale.

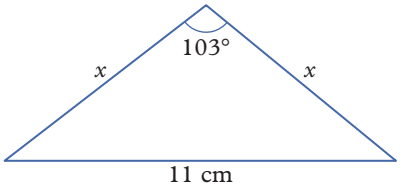


i Solve this equation. Give your answers correct to one decimal place.

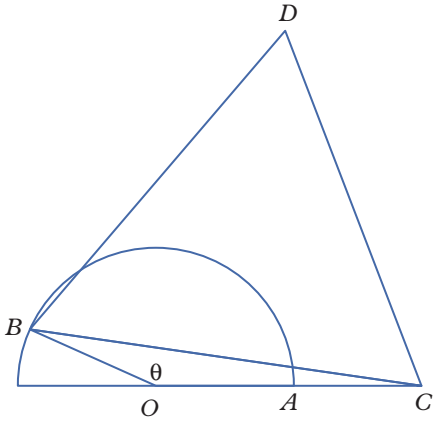
ii Are both solutions to the equation in part **d** possible side lengths of the triangle? Explain by referring to ambiguous triangles.

iii Compare using the sine and cosine rules to find the third side of a possibly ambiguous triangle.

e Consider the isosceles triangle shown below. Write and solve an equation for the value of the side lengths, x, correct to one decimal place.



22 Point B moves along the arc of semicircle O. The angle formed between OB and OA is θ. The radius of the semicircle is 1. Point C is on the same line as OA and $OA = AC$, connecting BC and creating an equilateral triangle BCD. As Point B moves along the arc of the semicircle, the area of $\triangle BCD$ also changes. Find the area of $\triangle BCD$ in terms of $\cos(\theta)$.



23 The triangle inequality states that a triangle with side lengths $a \geq b \geq c > 0$ exists precisely when $a < b + c$.

a Explain how the inequality $a \geq b + c$ relates to the triangle inequality.

b Use part **a**, the cosine rule and the fact that $-1 \leq \cos(\theta) \leq 1$ for all $\theta \in \mathbb{R}$ to prove the triangle inequality by contradiction.

Online resources:

Interactive skillsheet
The cosine rule

Quick quiz
7.6

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Lesson 7.7

The unit circle

Learning intentions

By the end of this lesson you will be able to ...

✔ solve problems involving the unit circle.

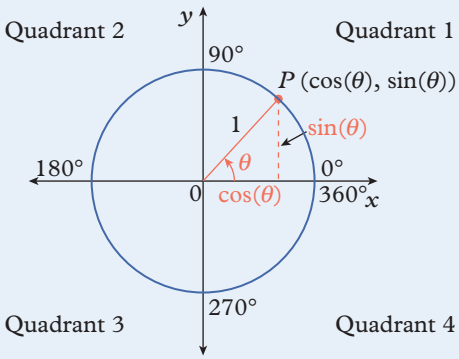
Inter-year links

Year 9 6.4 Trigonometric ratios

The unit circle

Key content video
The unit circle

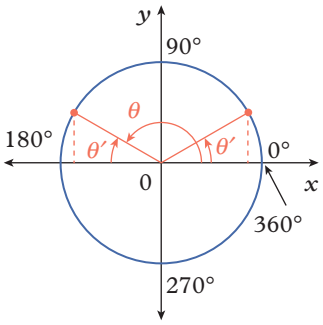
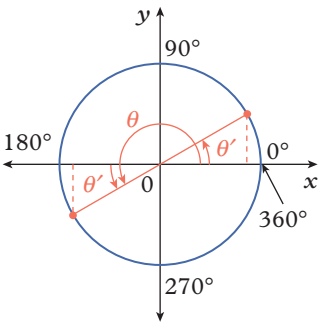
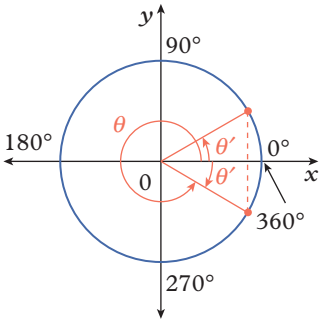
- The **unit circle** is a circle on the Cartesian plane, having its centre at the origin and a radius of 1 unit.
- The **quadrants** on the Cartesian plane are labelled in an anti-clockwise direction from the positive x-axis.
- If θ is the angle measured anti-clockwise from the positive x-axis to the radius that meets the unit circle at point P, then:
  - → The x-coordinate of P is the cosine of θ.
  - → The y-coordinate of P is the sine of θ.
  - → The gradient of the radius from the origin to P is the tangent of θ, as it is equal to $\frac{\sin(\theta)}{\cos(\theta)}$.



- **Coterminal angles** are angles that differ by a multiple of 360°.
  - → Two coterminal angles have the same sine, cosine and tangent ratios. For example, $\sin(10°) = \sin(370°)$.
  - → It can be helpful to imagine a point travelling around the unit circle in an anti-clockwise direction. As the angle defining the point goes beyond 360°, the point will continue around the circle.

The reference angle

- All angles on the unit circle in quadrants 2, 3 and 4 are related to a **reference angle** in quadrant 1. The reference angle is the acute angle between the x-axis and the line connecting the origin to the point on the unit circle.

	Quadrant 2	Quadrant 3	Quadrant 4
Angle (θ)	$90° < \theta < 180°$	$180° < \theta < 270°$	$270° < \theta < 360°$
Reference angle (θ')	$\theta' = 180° - \theta$ 	$\theta' = \theta - 180°$ 	$\theta' = 360° - \theta$ 

- Since, for any given angle, the x-coordinate is equal to the cosine ratio and the y-coordinate is equal to the sine ratio, we can relate the cosine and sine ratios for angles in quadrants 2, 3 and 4 to the cosine and sine ratios for angles in quadrant 1.
  - → The sine ratio is positive when the y-coordinate of the point on the unit circle is positive (quadrants 1 and 2), and negative where the y-coordinate is negative (quadrants 3 and 4).

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→ The cosine ratio is positive where the x-coordinate of the point on the unit circle is positive (quadrants 1 and 4), and negative where the x-coordinate is negative (quadrants 2 and 3).

→ The tangent ratio is positive where the gradient from the origin to the point on the unit circle is positive (quadrants 1 and 3), and negative where the gradient is negative (quadrants 2 and 4).

Worked example 7.7A Determining the quadrant in which an angle lies

Identify the quadrant in which each of these angles lies.

a 34° **b** 122° **c** 325°

THINK

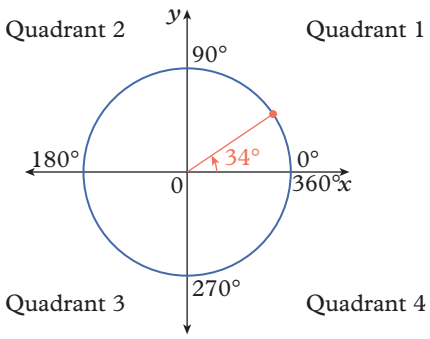
a Sketch the unit circle and mark the angle 34°. Because the size of the angle is between 0° and 90°, the angle is in quadrant 1.

b Sketch the unit circle and mark the angle 122°. Because the size of the angle is between 90° and 180°, the angle is in quadrant 2.

c Sketch the unit circle and mark the angle 325°. Because the size of the angle is between 270° and 360°, the angle is in quadrant 4.

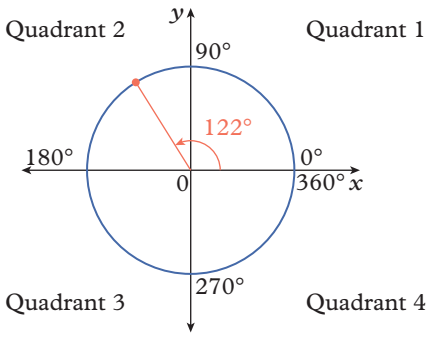
WRITE

a



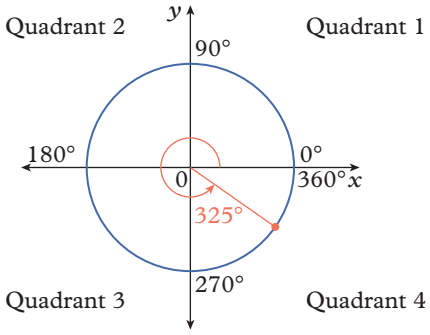
34° is in quadrant 1.

b



122° is in quadrant 2.

c



325° is in quadrant 4.

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Worked example 7.7B Determining the sign of a trigonometric expression

Identify the sign (positive or negative) of each of these trigonometric expressions.

a cos(55°) **b** tan(103°) **c** sin(230°) **d** tan(349°)

THINK

a Sketch the unit circle and mark the angle 55°. The angle is in quadrant 1, and the cosine value is positive in quadrant 1 because the x-coordinate of a point in quadrant 1 is positive.

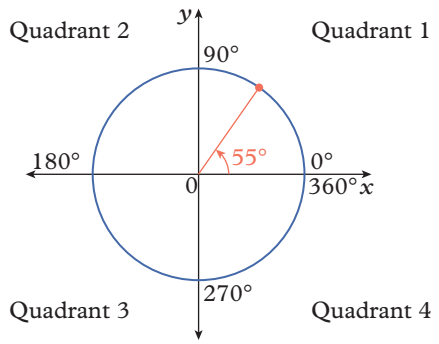
b Sketch the unit circle and mark the angle 103°. The angle is in quadrant 2, and the tangent value is negative in quadrant 2 because the gradient of the line from the origin to a point in quadrant 2 is negative.

c Sketch the unit circle and mark the angle 230°. The angle is in quadrant 3, and the sine value is negative in quadrant 3 because the y-coordinate of a point in quadrant 3 is negative.

d Sketch the unit circle and mark the angle 349°. The angle is in quadrant 4, and the tangent value is negative in quadrant 4 because the gradient of a line from the origin to a point in quadrant 4 is negative.

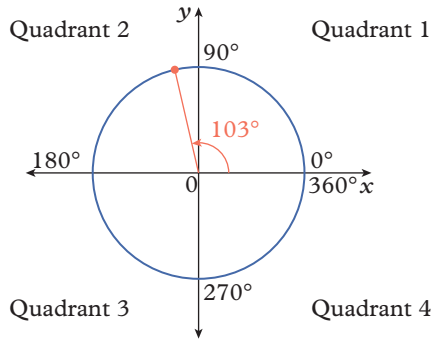
WRITE

a



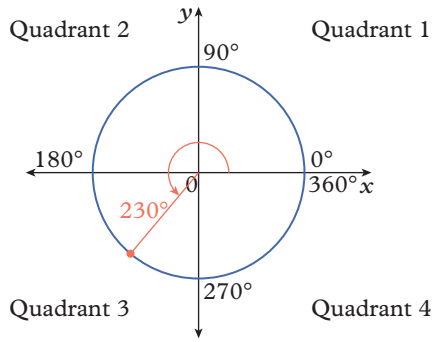
cos(55°) is positive.

b



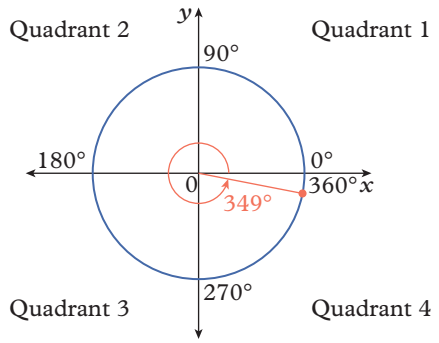
tan(103°) is negative.

c



sin(230°) is negative.

d



tan(349°) is negative.

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Worked example 7.7C Determining the reference angle

Write each trigonometric expression in terms of the reference angle using the correct sign.

a sin(255°) **b** cos(317°) **c** tan(145°)

THINK

a 1 Sketch the unit circle and mark the angle 255°.

2 The angle is in quadrant 3, so the reference angle is $\theta - 180°$.

3 sin(255°) is negative because the y-coordinate of a point in quadrant 3 is negative.

b 1 Sketch the unit circle and mark the angle 317°.

2 The angle is in quadrant 4, so the reference angle is $360° - \theta$.

3 cos(317°) is positive because the x-coordinate of a point in quadrant 4 is positive.

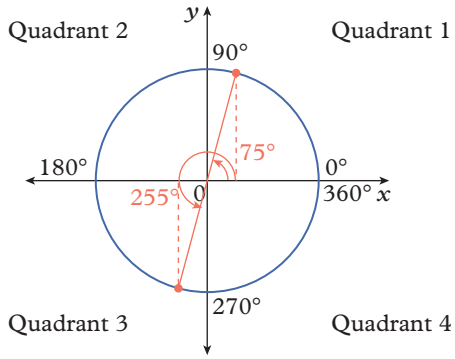
c 1 Sketch the unit circle and mark the angle 145°.

2 The angle is in quadrant 2, so the reference angle is $180° - \theta$.

3 tan(145°) is negative because the gradient of a line from the origin to any point in quadrant 2 is negative.

WRITE

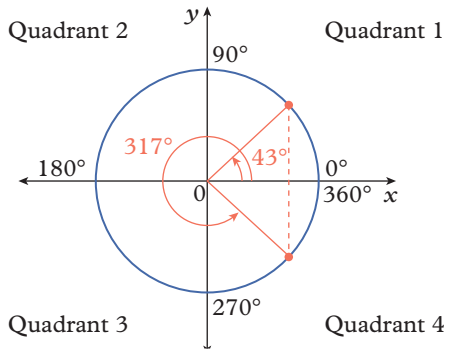
a



$255° - 180° = 75°$

$\sin(255°) = -\sin(75°)$

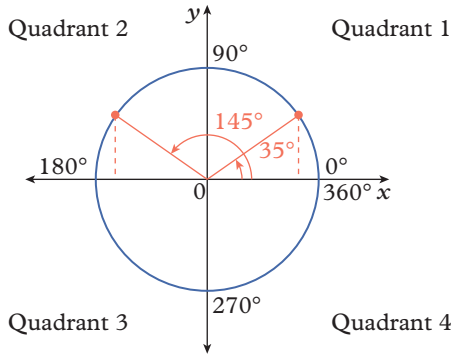
b



$360° - 317° = 43°$

$\cos(317°) = \cos(43°)$

c



$180° - 145° = 35°$

$\tan(145°) = -\tan(35°)$

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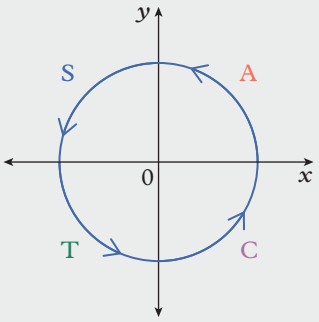
Helpful hints

✔ The mnemonic '**A**ll **S**tations **T**o **C**entral' can be used to help remember the signs of trigonometric ratios in different quadrants. All ratios are positive in quadrant 1, sin is positive in quadrant 2, tan is positive in quadrant 3, and cos is positive in quadrant 4.



All Stations To Central
L I A O
L N N S

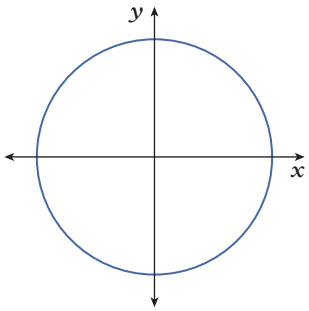
Learning pathways

1–3, 4(a–j), 5, 6, 8, 10	2, 3, 4(f–o), 5, 6, 8, 9, 11, 13	2, 3, 5–9, 12–14

ANS p729

Exercise 7.7A: Understanding and fluency

1 For each of the following, shade the appropriate regions of a unit circle diagram.



a Shade the regions where the sine of angles in that region is positive.
b Shade the regions where the cosine of angles in that region is positive.
c Shade the regions where the tangent of angles in that region is positive.
d Shade the regions where the sine of angles in that region is negative.
e Shade the regions where the cosine of angles in that region is negative.
f Shade the regions where the tangent of angles in that region is negative.

WE 7.7A **2** Identify the quadrant in which each of these angles lies.

a 100° **b** 200° **c** 140° **d** 295° **e** 260°
f 340° **g** 94° **h** 130° **i** 190° **j** 350°

WE 7.7B **3** Identify the sign (positive or negative) of each of these trigonometric expressions.
Hint: The answers from question **2** are will be useful.

a sin(100°) **b** cos(200°) **c** tan(140°) **d** cos(295°) **e** tan(260°)
f sin(340°) **g** cos(94°) **h** sin(130°) **i** cos(190°) **j** tan(350°)

4 Write the reference angle for each of the following.

a 140° **b** 92° **c** 200° **d** 190° **e** 302°
f 315° **g** 260° **h** 100° **i** 280° **j** 150°
k 345° **l** 170° **m** 220° **n** 255° **o** 299°

WE 7.7C **5** Express each of the following trigonometric expressions in terms of an appropriate reference angle.
Hint: Remember to consider whether the sign of each expression is positive or negative.

a sin(241°) **b** cos(304°) **c** tan(95°) **d** sin(164°) **e** cos(110°)
f cos(200°) **g** tan(200°) **h** tan(358°) **i** sin(285°) **j** sin(175°)

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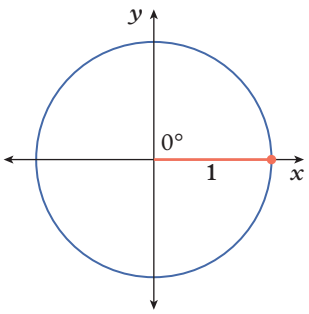
Exercise 7.7B: Problem solving and reasoning

6 The unit circle can be used to evaluate the sine and cosine of angles that are multiples of 90° (including 0°) by considering the coordinates of points on the unit circle. For each of the following identify:

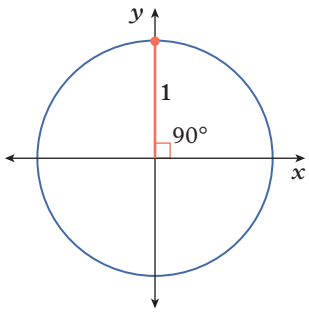
i the cosine of the marked angle (the x-coordinate of the point shown)

ii the sine of the marked angle (the y-coordinate of the point shown).

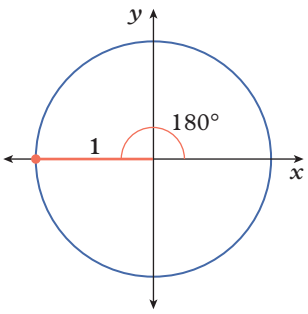
a



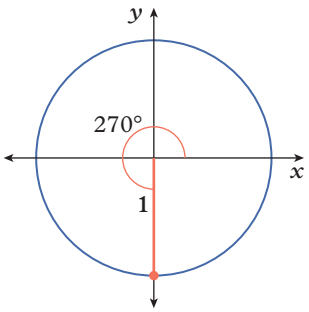
b



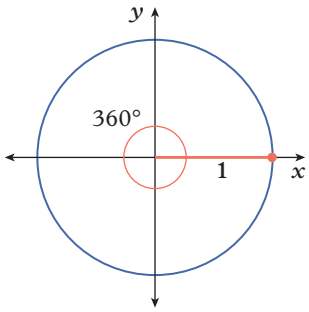
c



d

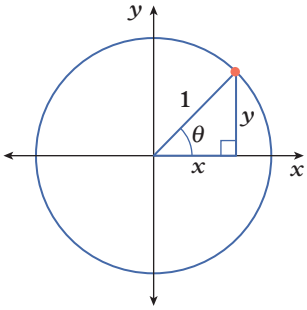


e



7 Every point on the unit circle forms a right-angled triangle, with the radius being the hypotenuse of the triangle.

a State the tangent of θ in the following right-angled triangle.



b Explain how the definition from part **a** can be described as 'the tangent of the angle is equal to the gradient of the radius from the origin to the point on the unit circle'.

c The lengths x and y also refer to the coordinate of the point where the radius touches the circle. Write the expression of the tangent of θ from part **a** in terms of sine and cosine.

d Use your expression from part **c** to write the following in terms of the tangent ratio.

i $\dfrac{\sin(20°)}{\cos(20°)}$ **ii** $\dfrac{\sin(152°)}{\cos(152°)}$ **iii** $\dfrac{\cos(218°)}{\sin(218°)}$ **iv** $\dfrac{\sin(347°)}{\cos(347°)}$

e The unit circle can also be used to evaluate the tangent of angles that are multiples of 90° (including 0°) by considering the gradients of a line from the origin to the point on the unit circle. Since the gradient is either horizontal or vertical at these angles, the tangent value is either zero or undefined, respectively. For each of the following, state the tangent of the marked angle using the gradient of the radius shown in question **6**.

i $\tan(0°)$ **ii** $\tan(90°)$ **iii** $\tan(180°)$ **iv** $\tan(270°)$ **v** $\tan(360°)$

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8 Angles greater than 360° are coterminal with angles between 0° and 360°, as they terminate or end at the same point on the unit circle. That is, every angle greater than 360° corresponds with an angle between 0° and 360°, as they point in the same direction. To determine a coterminal angle, add or subtract 360° from the given angle. Write each trigonometric expression in terms of:

i the coterminal angle between 0° and 360°

ii the reference angle, using the correct sign, or its value if it is a multiple of 90°.

a $\sin(400°)$ **b** $\cos(530°)$ **c** $\tan(800°)$ **d** $\sin(630°)$

e $\cos(777°)$ **f** $\cos(1024°)$ **g** $\tan(2000°)$ **h** $\cos(900°)$

9 Using the unit circle, we can also consider negative angles as angles measured clockwise from the positive x-axis. Negative angles are also coterminal with angles between 0° and 360°.
Write each trigonometric expression in terms of:

i the coterminal angle between 0° and 360°

ii the reference angle, using the correct sign, or its value if it is a multiple of 90°.

a $\sin(-45°)$ **b** $\cos(-225°)$ **c** $\tan(-135°)$ **d** $\sin(-315°)$

e $\cos(-508°)$ **f** $\tan(-436°)$ **g** $\cos(-630°)$ **h** $\sin(-1000°)$

10 Complete the following with different angles between 0° and 360°.

a $\sin(158°) = \sin(___°) = -\sin(___°) = -\sin(___°)$

b $\cos(213°) = \cos(___°) = -\cos(___°) = -\cos(___°)$

c $\tan(285°) = \tan(___°) = -\tan(___°) = -\tan(___°)$

11 Consider the following three equalities.

$\sin(234°) = -\sin(126°)$ $\cos(234°) = \cos(126°)$ $\tan(234°) = -\tan(126°)$

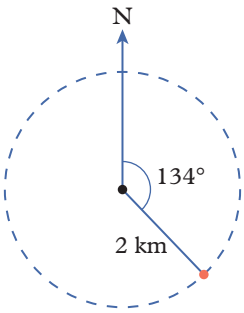
a Use a calculator to confirm that the pairs of trigonometric expressions are equal.

b Evaluate 234° + 126°.

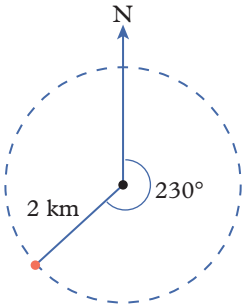
c Explain how these equalities relate to the fourth quadrant.

12 True bearings measure angles from the positive y-axis (north) clockwise, rather than from the positive x-axis, like angles anti-clockwise on the unit circle. The distance travelled along the bearing will scale the coordinates on the circle where the distance is the scale factor. If the centre point of the circle is the origin, state the coordinates of the points shown in terms of trigonometric expressions containing a reference angle in quadrant 1.

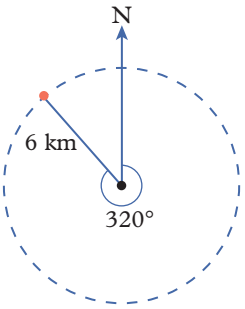
a



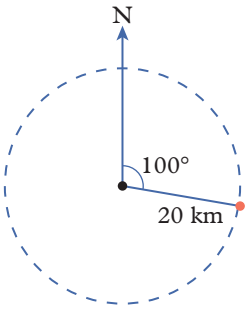
b



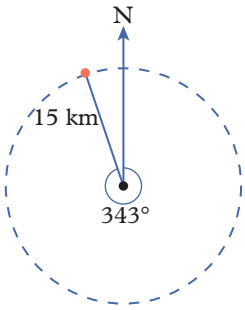
c



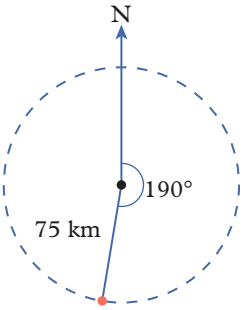
d



e



f



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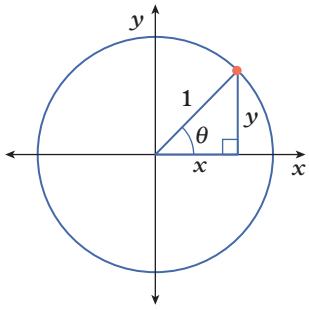
Exercise 7.7C: Challenge

13 Using the diagram shown, we can write the equation $x^2 + y^2 = 1$ using Pythagoras' theorem.



a Write the equation in terms of sine and cosine of θ. This is known as the Pythagorean identity.

b Rearrange the equation in part **a** to make:

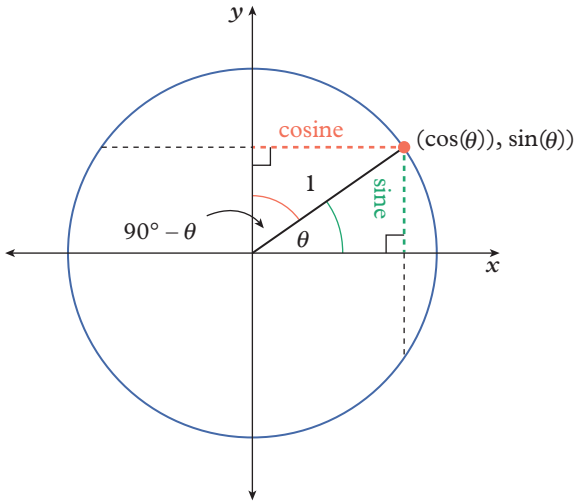
i $\sin(\theta)$ the subject **ii** $\cos(\theta)$ the subject.

c We can use the equation in part **a** (and its rearrangements in part **b**) to determine the cosine value of an angle if we know the sine of the angle and vice versa. The sign of the solution can be determined by considering which quadrant the angle lies in. Use the sine and cosine values provided to determine the value of the trigonometric expression in brackets, correct to four decimal places.

i $\sin(50°) = 0.7660$ $(\cos(50°))$ **ii** $\cos(116°) = -0.4384$ $(\sin(116°))$

iii $\sin(177°) = 0.0523$ $(\cos(177°))$ **iv** $\cos(240°) = -0.5$ $(\sin(240°))$

14 The name 'cosine' comes from an abbreviation of the phrase 'the sine of the complementary angle'. That is, $\sin(90° - \theta) = \cos(\theta)$. We can use this definition to turn sine expressions into cosine expressions of the complementary angle and vice versa.



This is true even when θ is not an acute angle. Write the following sine expressions using cosine, and cosine expressions using sine, using:

i the angle from the subtraction $90° - \theta$

ii the coterminal angle between 0° and 360° if applicable and

iii the reference angle using the correct sign or its value if it is a multiple of 90°.

a $\sin(30°)$ **b** $\sin(52°)$ **c** $\cos(24°)$ **d** $\cos(45°)$ **e** $\sin(158°)$

f $\cos(351°)$ **g** $\sin(211°)$ **h** $\sin(-15°)$ **i** $\cos(482°)$ **j** $\cos(-555°)$

Online resources:

Interactive skillsheet The sign of trigonometric expressions

Interactive skillsheet Reference angles

Quick quiz 7.7

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Exact values

Learning intentions

By the end of this lesson you will be able to ...

✔ determine exact values of trigonometric ratios in degrees and radians.

Inter-year links

Year 9 6.4 Trigonometric ratios

Degrees and radians

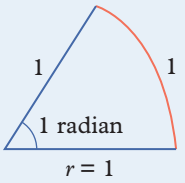
Key content video
Degrees and radians

- A **radian** is a unit for measuring angles that is related to circles. Radians are the ratio between the arc length and the radius of a circle such that 1 radian is the angle subtended by the arc length that is one radius long.
- There are 2π radians in one revolution of a circle. When written in terms of π, radians describe angles as fractions of a full turn, 2π.

  For example, $\frac{2\pi}{5}$ is a fifth of a full turn and $\frac{3\pi}{4} = \frac{3(2\pi)}{8}$ is three-eighths of a full turn.
- 180° is equivalent to π radians and 1 radian is exactly $\frac{180°}{\pi}$, or approximately 57.3°.

  → To convert from degrees to radians, multiply by $\frac{\pi}{180°}$.

  → To convert from radians to degrees, multiply by $\frac{180°}{\pi}$.

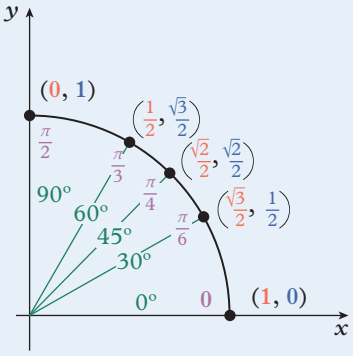


Exact values of trigonometric ratios

Key content video
Exact values of trigonometric ratios

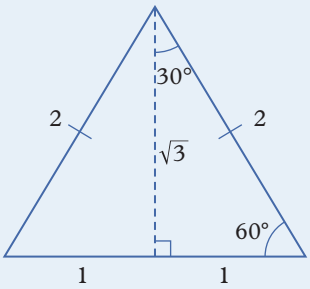
- The table below shows the exact values of the trigonometric ratios for five special angles in quadrant 1 of the unit circle.

θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
0° or 0	0	1	0
30° or $\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
45° or $\frac{\pi}{4}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	1
60° or $\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90° or $\frac{\pi}{2}$	1	0	undefined

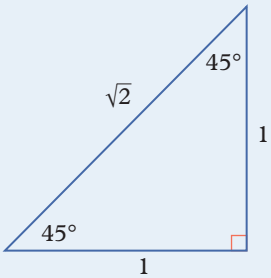


- There are two special triangles whose side lengths provide the exact values above. We can use Pythagoras' theorem to determine all angles and lengths of the two triangles.

half-equilateral triangle



half-square triangle



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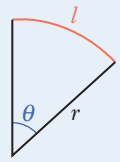
Arc length

- The circumference of the unit circle is 2π.
- The length of an arc, l, can be calculated by multiplying the angle θ (in radians) it subtends at the centre of the circle by the circle's radius, r.
- Since the unit circle has a radius of 1, any arc length on the circumference of the unit circle is equal to the radian measure of the angle θ subtending the arc.



$$l = r\theta$$

arc length radius angle in radians

Worked example 7.8A Converting between radians and degrees

Convert each of the following to the unit indicated in the brackets.

a 110° (convert to radians)

b $\frac{4\pi}{3}$ radians (convert to degrees)

THINK

a To convert from degrees to radians, multiply by $\frac{\pi}{180°}$.

b To convert from radians to degrees, multiply by $\frac{180°}{\pi}$.

WRITE

a $110° \times \frac{\pi}{180°} = \cancel{110°}^{11} \times \frac{\pi}{\cancel{180°}^{18}}$

$= \frac{11\pi}{18}$ radians

b $\frac{4\pi}{3} \times \frac{180°}{\pi} = \frac{\cancel{4\pi}^{4}}{\cancel{3}^{1}} \times \frac{\cancel{180°}^{60°}}{\cancel{\pi}^{1}}$

$= \frac{4 \times 60°}{1}$

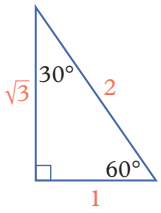
$= 240°$

Worked example 7.8B Finding the exact values of a trigonometric ratio

Use the unit circle and exact values for the trigonometric ratios to find: sin(150°), cos(150°) and tan(150°).

THINK

1 Sketch the unit circle and mark the angle 150°. It is in quadrant 2, so the reference angle is 180° – 150° = 30°.

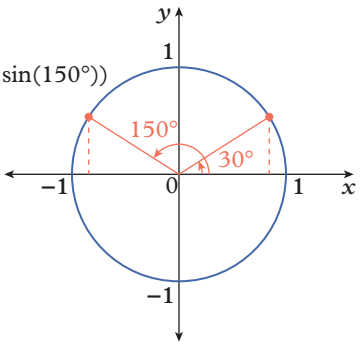


2 Calculate sin(30°), cos(30°) and tan(30°) using the side lengths of the special triangle above.

3 Apply the correct signs to the trigonometric ratios for quadrant 2.

WRITE

(cos(150°), sin(150°))



$\sin(30°) = \frac{1}{2}$, $\cos(30°) = \frac{\sqrt{3}}{2}$, $\tan(30°) = \frac{\sqrt{3}}{3}$

In quadrant 2, cosine and tangent are both negative, but sine is positive.

$\sin(150°) = \frac{1}{2}$, $\cos(150°) = -\frac{\sqrt{3}}{2}$, $\tan(150°) = -\frac{\sqrt{3}}{3}$

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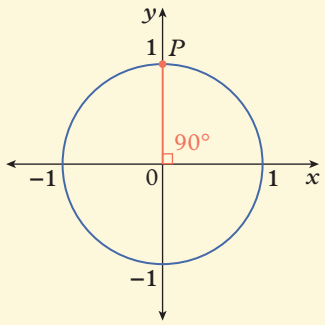
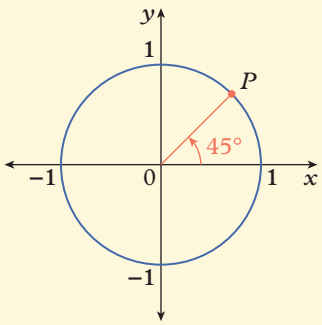
Worked example 7.8C Identifying points on the unit circle

Identify the coordinates of the point P in each case below.

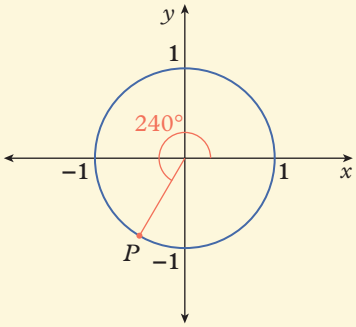
a

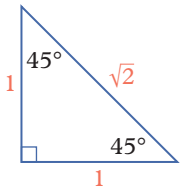
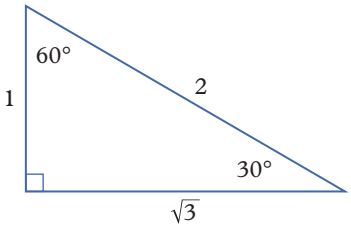


b



c



THINK	WRITE
a **1** Any point on the unit circle has an x-coordinate of $\cos(\theta)$ and a y-coordinate of $\sin(\theta)$. Write P in coordinate form using $\theta = 90°$.	**a** $P = (\cos(90°), \sin(90°))$
2 Identify the exact values of the trigonometric ratios. Remember that the unit circle has a radius of 1.	$= (0, 1)$
b **1** Write P in coordinate form using $\theta = 45°$.	**b** $P = (\cos(45°), \sin(45°))$
2 Identify the exact values of the trigonometric ratios. Use the half-square triangle to determine the values. 	$\cos(45°) = \frac{\sqrt{2}}{2}$ $\sin(45°) = \frac{\sqrt{2}}{2}$ $P = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
c **1** Write P in coordinate form using $\theta = 240°$.	**c** $P = (\cos(240°), \sin(240°))$
2 θ is in quadrant 3, so the reference angle is $240° - 180° = 60°$ Use the half-equilateral triangle to determine $\sin(60°)$ and $\cos(60°)$ 	$\sin(60°) = \frac{\sqrt{3}}{2}, \cos(60°) = \frac{1}{2}$
3 Apply the correct signs to the trigonometric ratios for quadrant 3.	$\sin(240°) = -\frac{\sqrt{3}}{2}, \cos(240°) = -\frac{1}{2}$ $P = (-\frac{\sqrt{3}}{2}, -\frac{1}{2})$

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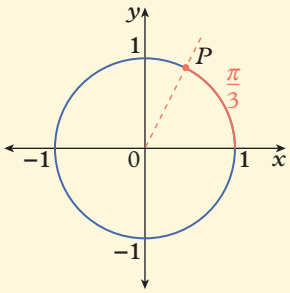
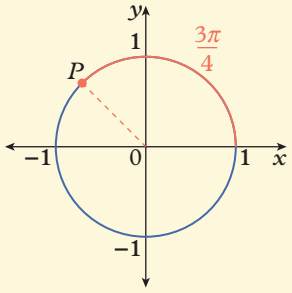
Worked example 7.8D Identifying points on the unit circle from an arc length

Identify the coordinates of point P from the given arc length in each case below.

a



b



THINK

a **1** Given the arc length, determine the angle that subtends the arc in the unit circle in radians.

2 Write P in coordinate form.

3 Identify the exact values of the trigonometric ratios. Use the half-equilateral triangle to determine the values.

WRITE

a $l = r\theta$

$\frac{\pi}{3} = 1 \times \theta$

$\theta = \frac{\pi}{3}$ radians

$P = \left(\cos\left(\frac{\pi}{3}\right), \sin\left(\frac{\pi}{3}\right)\right)$

$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

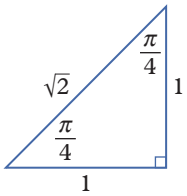
$P = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

b **1** Given the arc length, determine the angle that subtends the arc in the unit circle in radians.

2 Write P in coordinate form.

3 The angle is in quadrant 2, so the reference angle is $\pi - \frac{3\pi}{4} = \frac{\pi}{4}$.

4 Identify the exact values of the trigonometric ratios. Use the half-square triangle to determine the values; determine the sign in quadrant 2.

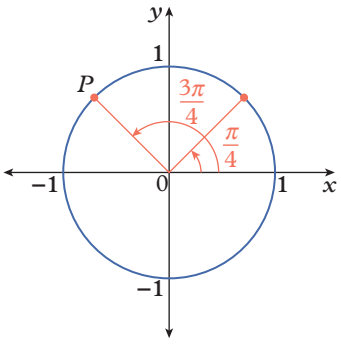


b $l = r\theta$

$\frac{3\pi}{4} = 1 \times \theta$

$\theta = \frac{3\pi}{4}$ radians

$P = \left(\cos\left(\frac{3\pi}{4}\right), \sin\left(\frac{3\pi}{4}\right)\right)$



$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}, \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$\cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}, \sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$P = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

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Helpful hints

- ✔ When converting from radians to degrees, remember $\pi = 180°$; substitute π with 180° and simplify. For example $\frac{2\pi}{5} = 2 \times \frac{180°}{5} = 72°$.
- ✔ Remember when converting from radians to degrees, you will need to get rid of π, so the converting factor is $\frac{180°}{\pi}$ (where you divide by π). When you need to convert from degrees to radians, you want π in your answer, so the converting factor is $\frac{\pi}{180°}$ (where you multiply by π).
- ✔ It is helpful to memorize the special trigonometric ratios for 30°, 60° and 45°, and remember $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ and $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

Learning pathways

1-10, 12, 13, 15	4, 6, 9, 11, 12, 14(a-c), 15, 16, 17(a, b)	4, 7, 9, 12, 14, 17(c, d), 18

Exercise 7.8A: Understanding and fluency

WE 7.8A **1** Convert each of the following to the unit given in brackets.

a 120° (radians)
b 90° (radians)
c 135° (radians)
d 3π radians (degrees)
e $\frac{2\pi}{3}$ radians (degrees)
f $\frac{5\pi}{4}$ radians (degrees)
g 190° (radians)
h 156° (radians)
i $\frac{2\pi}{5}$ radians (degrees)
j $\frac{17\pi}{12}$ radians (degrees)
k 136° (radians)
l $\frac{143\pi}{90}$ radians (degrees)

2 Determine the reference angle for each of the following angles (in degrees).

a 210°
b 120°
c 330°
d 135°
e 240°
f 315°

3 Determine the reference angle for each of the following angles (in radians).

a $\frac{3\pi}{4}$
b $\frac{5\pi}{3}$
c $\frac{11\pi}{6}$
d $\frac{7\pi}{6}$
e $\frac{2\pi}{3}$
f $\frac{5\pi}{4}$

WE 7.8B **4** Use the unit circle and exact values for the trigonometric ratios to evaluate the sine, cosine and tangent of each of these angles.

a 240°
b 300°
c 120°
d 315°
e 135°
f 150°

5 For each of the following angles:

i determine the reference angle in radians
ii evaluate the sine, cosine and tangent of the reference angle
iii use the unit circle to determine the sine, cosine and tangent of the original angle.

a $\frac{7\pi}{6}$
b $\frac{3\pi}{4}$
c $\frac{4\pi}{3}$

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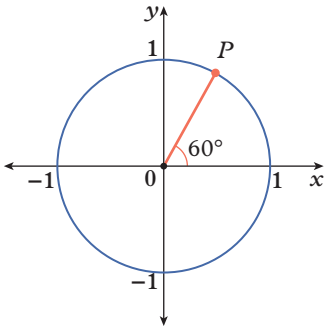
6 Use the unit circle and exact values for the trigonometric ratios to evaluate the sine, cosine and tangent of each angle.

a $\frac{\pi}{6}$ **b** $\frac{5\pi}{6}$ **c** $\frac{5\pi}{3}$

d $\frac{5\pi}{4}$ **e** $\frac{2\pi}{3}$ **f** $\frac{7\pi}{4}$

WE 7.8C **7** Identify the coordinates of the point P in each of these cases.

a



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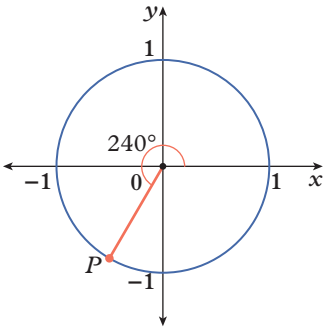

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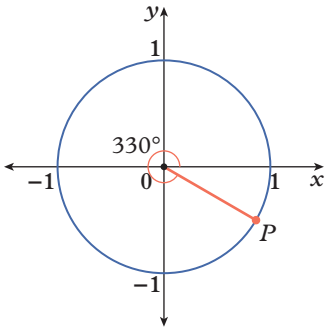

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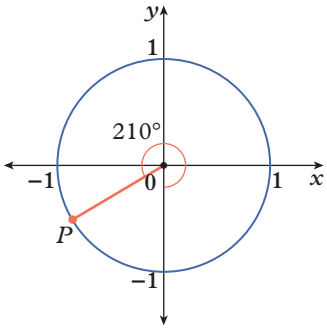

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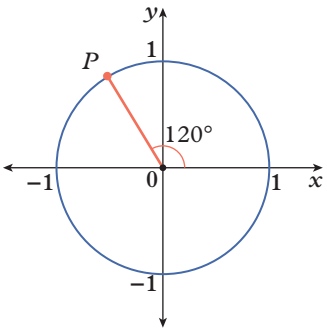
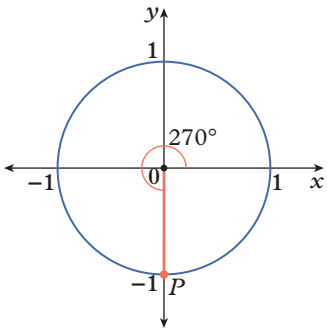


f



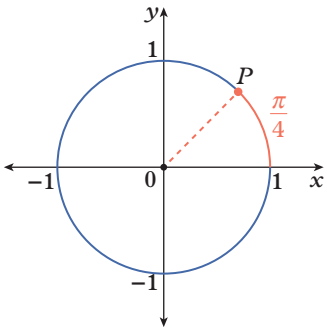
8 Give the exact coordinates of the point on the unit circle when the radial arm is rotated anti-clockwise by each of these angles from the positive x-axis.

a 225° **b** 300 **c** 150°

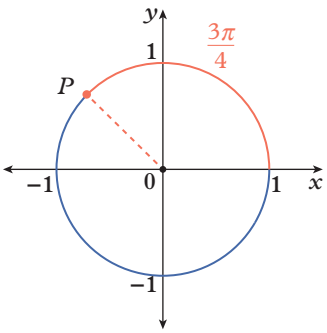
d 135° **e** 315° **f** 180°

WE 7.8D **9** Identify the coordinates of the point P in each of the following diagrams.

a



b



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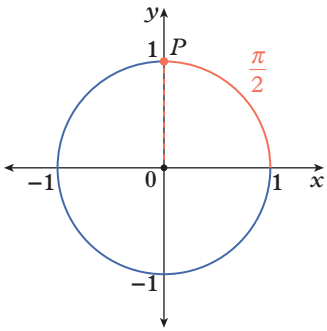

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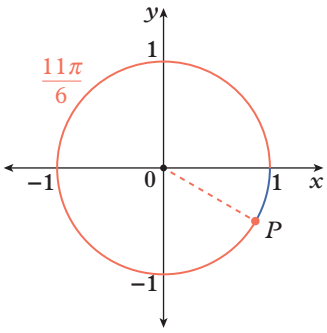

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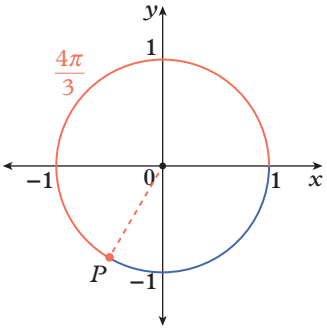
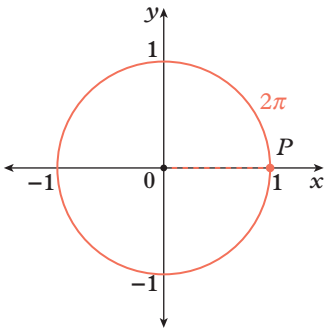


f



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Exercise 7.8B: Problem solving and reasoning

10 Answer each of the following in radians using your knowledge of geometry and without directly converting the angles to or from degrees.

a State the angle of:

i a revolution **ii** a straight angle **iii** a right angle.

b Write an inequality for the angle θ such that it is:

i acute **ii** obtuse **iii** reflex.

c Complete the following sentences.

i Complementary angles (two angles that form a _____) add to _____ radians.

ii Supplementary angles (two angles that form a _____) add to _____ radians.

d Determine the complement of the following angles.

i $\frac{\pi}{3}$ **ii** $\frac{3\pi}{8}$ **iii** $\frac{\pi}{4}$ **iv** $\frac{\pi}{10}$

e Determine the supplement of the following angles.

i $\frac{\pi}{3}$ **ii** $\frac{3\pi}{4}$ **iii** $\frac{\pi}{6}$ **iv** $\frac{5\pi}{12}$

f Determine the angle that would form a revolution with each of the following angles:

i $\frac{\pi}{3}$ **ii** $\frac{11\pi}{6}$ **iii** $\frac{5\pi}{4}$ **iv** $\frac{7\pi}{5}$

g State the interior angle sum of:

i a triangle **ii** a quadrilateral

iii a hexagon **iv** an n-sided polygon.

h State the size of each interior angle in:

i an equilateral triangle **ii** a square

iii a regular hexagon **iv** a regular n-sided polygon.

i Angles are said to be coterminal if the radial arm terminates or ends at the same point on the unit circle. State the coterminal angle that is one revolution greater than each of the following:

i $\frac{\pi}{3}$ **ii** $\frac{5\pi}{4}$ **iii** $\frac{11\pi}{6}$ **iv** $\frac{\pi}{2}$ **v** π

j State the coterminal angle that is one revolution less than each of the following:

i $\frac{\pi}{3}$ **ii** $\frac{5\pi}{4}$ **iii** $\frac{11\pi}{6}$ **iv** $\frac{\pi}{2}$ **v** π

11 Answer each of the following using your knowledge of geometry, and without directly converting the angles to degrees.

a For each angle, state the fraction of a full turn that it represents. Mark each angle on a unit circle.

i $\frac{2\pi}{3}$ **ii** $\frac{\pi}{4}$ **iii** $\frac{11\pi}{6}$ **iv** $\frac{3\pi}{2}$

v $\frac{8\pi}{7}$ **vi** $\frac{4\pi}{5}$ **vii** π **viii** $\frac{\pi}{180}$

b For each fraction of a full turn, state the angle in radians. Mark each angle on a unit circle.

i $\frac{5}{6}$ **ii** $\frac{3}{5}$ **iii** $\frac{7}{8}$ **iv** $\frac{1}{12}$

v $\frac{6}{11}$ **vi** $\frac{9}{10}$ **vii** $\frac{8}{15}$ **viii** $\frac{5}{16}$

12 The constructions of the half-square and half-equilateral triangles are helpful to learn in case you forget any of the side lengths or angles, as you can then reconstruct the triangle from the name.

a Draw a square, mark the side lengths 1 unit, and add one of the diagonals.

b Use your knowledge of geometry and Pythagoras' theorem to label all angles in degrees and lengths with their size.

c Repeat parts **a** and **b** using radians.

d Draw an equilateral triangle with a horizontal base, mark the side lengths 2 units, and add the perpendicular height to the base such that it bisects the top angle and the base length.

e Use your knowledge of geometry and Pythagoras' theorem to label all angles in degrees and lengths with their size.

f Repeat parts **d** and **e** using radians.

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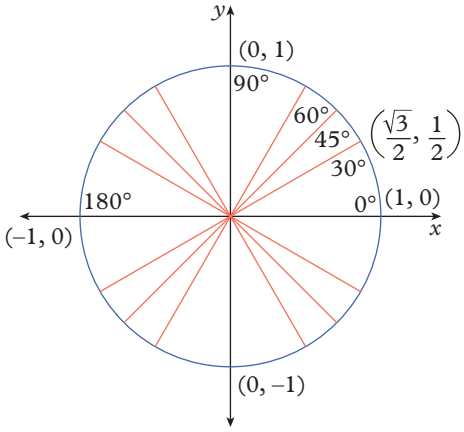
13 The exact values of sine and cosine for the angles 30°, 45° and 60°, together with their multiples, can be recorded on the unit circle.

a Draw a unit circle. Mark the coordinates of the points where the circle cuts the horizontal and vertical axes. Also, mark points on the circle representing angles of 30°, 45° and 60°, together with their multiples up to 360°.

b Label each of the points shown on the unit circle with its exact cosine and sine value as coordinates.

For example, the point representing 30° would be labelled as $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Take care to use the correct signs.

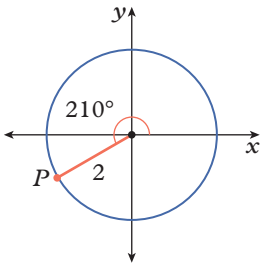


c Give the exact coordinates of each of these angles on the unit circle.

i 225° **ii** 300° **iii** 150° **iv** 135° **v** 315° **vi** 90°

14 Identify the coordinates of the point P in each of the following diagrams.

a



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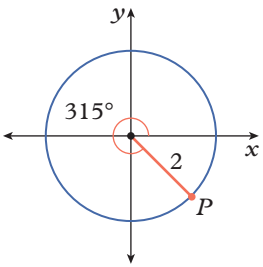

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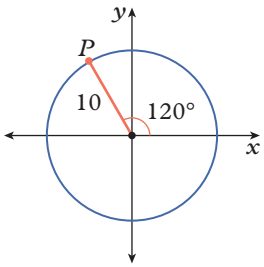

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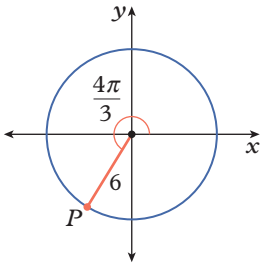

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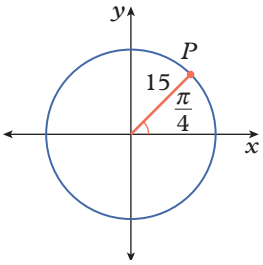

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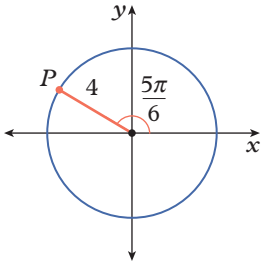
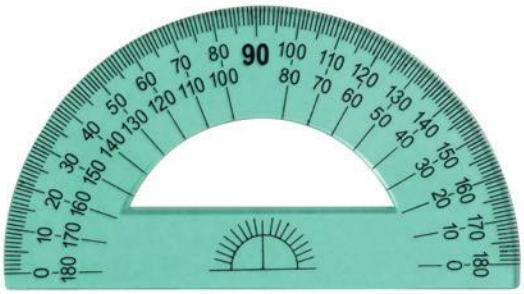


15 Complete the table.

	Radius	Angle subtending arc	Arc length
a	3 cm	$\frac{5\pi}{6}$	
b	6 cm	$\frac{2\pi}{3}$	
c		$\frac{3\pi}{4}$	
d		$\frac{5\pi}{12}$	$\frac{4\pi}{3}$ cm
e	4 cm		$\frac{9\pi}{2}$ cm
f			$\frac{2\pi}{7}$ cm



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16 There is a simple method for determining the reference angle and the quadrant in which an angle given in radians lies.

a For each of the following, state the reference angle in radians.

i $\frac{2\pi}{3}$ **ii** $\frac{4\pi}{3}$ **iii** $\frac{5\pi}{3}$ **iv** $\frac{5\pi}{6}$ **v** $\frac{5\pi}{4}$ **vi** $\frac{7\pi}{4}$ **vii** $\frac{7\pi}{6}$ **viii** $\frac{11\pi}{6}$ **ix** $\frac{3\pi}{4}$

b Explain how you can determine the reference angle for an angle in radians.

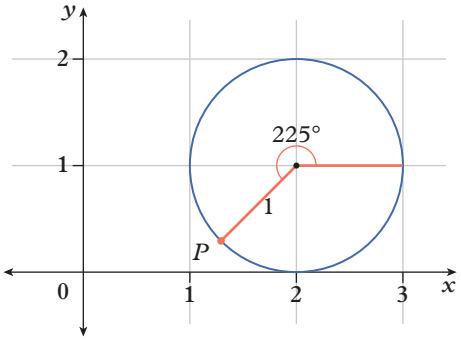
c Determine the quadrant each angle in part **a** is in.

d Explain how you can determine the quadrant for an angle in radians.

Exercise 7.8C: Challenge

17 Identify the coordinates of the point P in each case.

a



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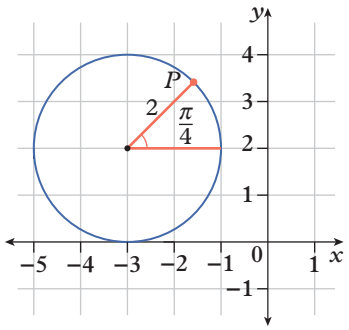

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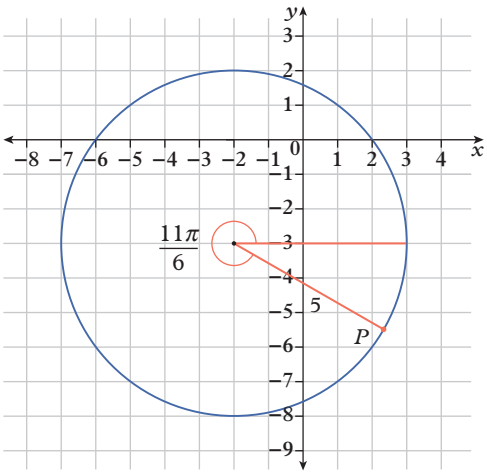

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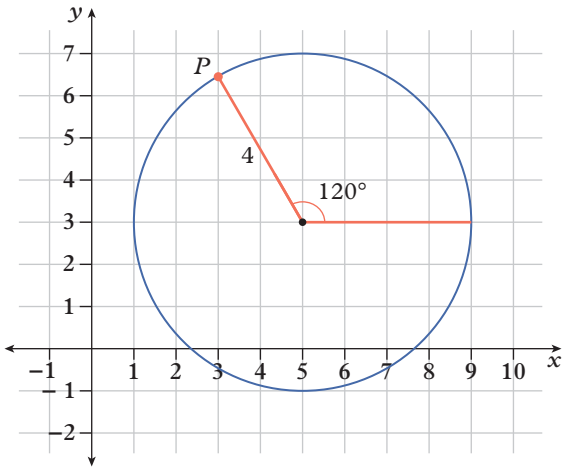
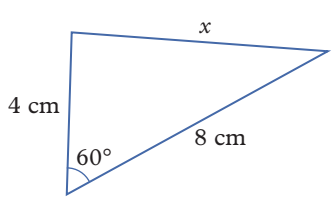


18 For each triangle below, find the exact values of:

i the length x in cm, and **ii** the area in cm^2.

a



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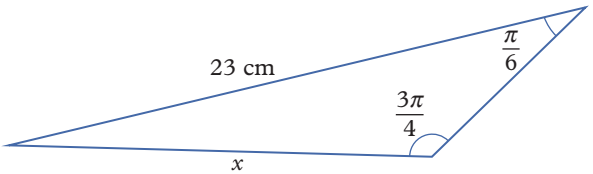

Online resources:

- **Interactive skillsheet** Converting between degrees and radians
- **Interactive skillsheet** Exact values of trigonometric ratios
- **Interactive skillsheet** Coordinates of points on the unit circle
- **Investigation** Remembering the exact values
- **Quick quiz** 7.8

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Trigonometric graphs

Learning intentions

By the end of this lesson you will be able to ...

✔ understand graphs of simple trigonometric functions, and use trigonometric identities to find exact values.

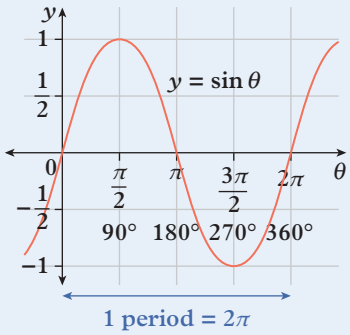
Inter-year links

Year 9 — 4.2 Plotting quadratic relationships

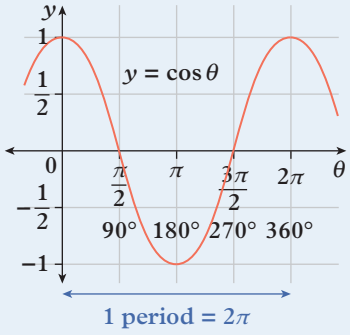
Trigonometric graphs

Key content video
Trigonometric graphs

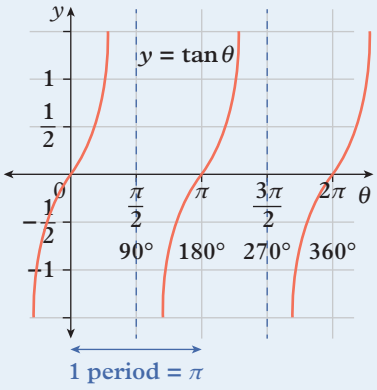
- The unit circle can be used to draw graphs of the simple trigonometric functions $y = \sin(\theta)$, $y = \cos(\theta)$ and $y = \tan(\theta)$.
- Trigonometric graphs can either be drawn by using digital tools or sketched by creating a table of values and joining the points with a smooth curve.
- In graphs of trigonometric functions, the horizontal axis represents the angle (θ), and the vertical axis represents the corresponding trigonometric ratio for this angle.
- Values beyond 360° are coterminal angles with angles in the range $0 \le \theta \le 360°$, therefore graphs of trigonometric functions form repeating patterns.
  - → As the output of trigonometric functions form a repeating pattern, we call them **periodic functions**.
  - → The **period** of a periodic function is the length of the interval along the horizontal axis that is required for the function to repeat itself.



- The y-values on the graph of $y = \sin(\theta)$ are equal to the corresponding y-coordinates on the unit circle for angle θ.
- The y-values on the graph of $y = \cos(\theta)$ are equal to the corresponding x-coordinates on the unit circle for angle θ.
- The shapes of the graphs of $y = \sin(\theta)$ and $y = \cos(\theta)$ are identical; if the graph of $y = \sin(\theta)$ is translated by 90° to the left, the graph of $y = \cos(\theta)$ is produced.
  - → The period of both $y = \sin(\theta)$ and $y = \cos(\theta)$ is 360°.
- The **amplitude** of the graphs of $y = \sin(\theta)$ and $y = \cos(\theta)$ is 1. This represents the vertical distance from the mean value of 0 (at the x-axis) to either the maximum or the minimum turning points.



- The y-values on the graph of $y = \tan(\theta)$ are equal to the values of the y-coordinates divided by the values of the x-coordinates on the unit circle for angle θ.
  - → The graph of $y = \tan(\theta)$ has asymptotes at every 180° on the θ-axis.
  - → The graph of $y = \tan(\theta)$ has x-intercepts at every 180° on the θ-axis.
  - → The period of $y = \tan(\theta)$ is 180°.



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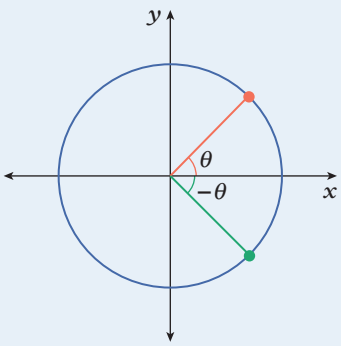
Trigonometric identities



- Negative angles on the unit circle are measured clockwise from the positive x-axis.
  → The sine, cosine and tangent of negative angles can be identified using the unit circle and reference angles.

- Trigonometric identities are equalities that are true for all values of θ. The following table provides some useful trigonometric identities.

Complementary angles	Supplementary angles	Negative angles
$\sin(90° - \theta) = \cos(\theta)$	$\sin(180° - \theta) = \sin(\theta)$	$\sin(-\theta) = -\sin(\theta)$
$\cos(90° - \theta) = \sin(\theta)$	$\cos(180° - \theta) = -\cos(\theta)$	$\cos(-\theta) = \cos(\theta)$
$\tan(90° - \theta) = \dfrac{1}{\tan(\theta)}$	$\tan(180° - \theta) = -\tan(\theta)$	$\tan(-\theta) = -\tan(\theta)$

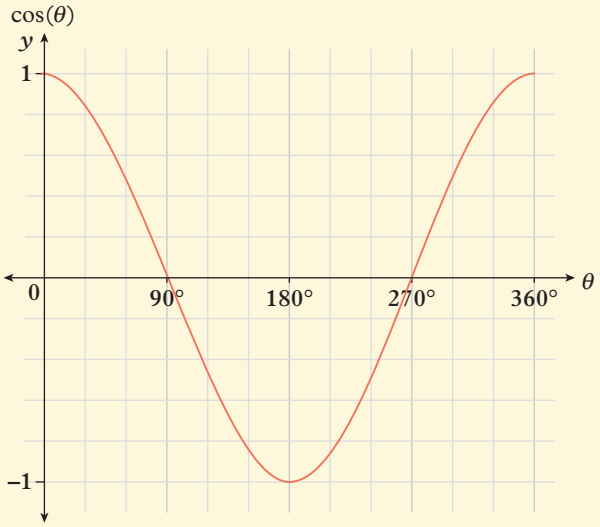
→ Note that the tangent ratios in this table can be determined by dividing the sine ratios by the corresponding cosine ratios.

Worked example 7.9A Reading a trigonometric ratio from a graph

Consider the graph of the cosine function below. Use the graph to read the cosine value for each of these angles. Give your answers as approximate values, if necessary.

a 360°

b 80°



THINK

a There is a maximum on the graph at $\theta = 360°$, so the vertical coordinate is exactly 1.

b Identify an approximate coordinate for the graph just to the left of (90°, 0).

WRITE

a $(360°, \cos(360°)) = (360°, 1)$
$\cos(360°) = 1$

b $(80°, \cos(80°)) \approx (80°, 0.2)$
$\cos(80°) \approx 0.2$

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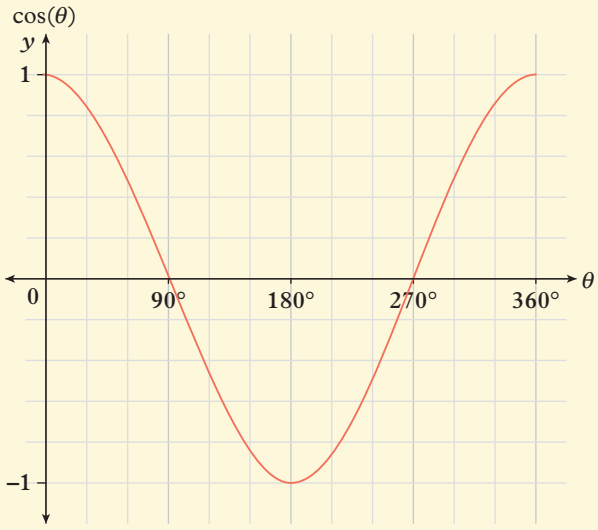
Worked example 7.9B Reading an angle from a trigonometric graph

Consider the following graph of the cosine function. Use the graph to find the angles from 0° to 360° that have the following cosine values. Give your answers to the nearest 10°.

a 0.6

b −0.2



THINK

a **1** There are two coordinates on the graph where $\cos(\theta) = 0.6$: one near 50° and one near 310°.

2 Check the answers by comparing the reference angles and the sign of the trigonometric expression.

b **1** There are two coordinates on the graph where $\cos(\theta) = -0.2$: one near 100° and one near 260°.

2 Check the answers by comparing the reference angles and the sign of the trigonometric expression.

WRITE

a $(\theta, 0.6) \approx (50°, 0.6)$ or $(310°, 0.6)$

1st quadrant: $\theta' = \theta$
$= 50°$

4th quadrant: $\theta' = 360° - \theta$
$= 360° - 310°$
$= 50°$

Cosine is positive in the first and fourth quadrants, so $\cos(50°) = \cos(310°)$.

$\theta \approx 50°$ or $310°$

b $(\theta, -0.2) \approx (100°, -0.2)$ or $(260°, -0.2)$

2nd quadrant: $\theta' = 180° - \theta$
$= 180° - 100°$
$= 80°$

3rd quadrant: $\theta' = \theta - 180°$
$= 260° - 180°$
$= 80°$

Cosine is negative in the second and third quadrants, so $\cos(100°) = \cos(260°)$.

$\theta \approx 100°$ or $260°$

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Worked example 7.9C Using trigonometric identities

Use appropriate trigonometric identities to write the following in terms of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$.

a $\cos(-\theta)$ **b** $\tan(360° - \theta)$ **c** $\sin(90° + \theta)$

THINK	WRITE
a Apply the negative angle trigonometric identity for cosine.	**a** $\cos(-\theta) = \cos(\theta)$
b 1 Identify that, on the unit circle, 360° and 0° are coterminal angles, so 360° can be replaced by 0°. Subtracting from 0° is equivalent to taking the negative angle.	**b** $\tan(360° - \theta) = \tan(0° - \theta)$ $= \tan(-\theta)$
2 Apply the negative angle trigonometric identity for tangent.	$= -\tan(\theta)$
c 1 Replace 90° with (180° – 90°). Group the resulting expression to be in the form of the supplementary angle trigonometric identity for sine.	**c** $\sin(90° + \theta) = \sin((180° - 90°) + \theta)$ $= \sin(180° - (90° - \theta))$
2 Apply the supplementary angle trigonometric identity for sine. The expression in brackets takes the place of θ in the identity.	$= \sin(90° - \theta)$
3 Apply the complementary angle trigonometric identity for sine.	$= \cos(\theta)$

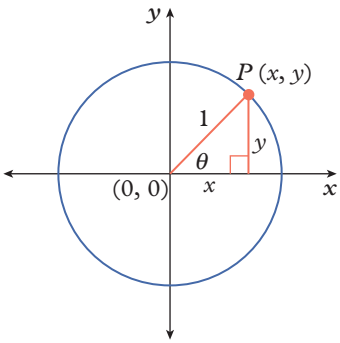
Helpful hints

- ✔ Remember to consider all 4 quadrants of the unit circle and convert to the correct reference angle when trying to determine the angles for a given y-value on a trigonometric graph. There are usually 2 angles with the same y-value per period.
- ✔ Keep in mind that $\sin(30°) = \frac{1}{2}$ and $\cos(60°) = \frac{1}{2}$. That is, the graphs do not equal half the amplitude at the halfway point between the x-axis and the maximum/minimum.

Learning pathways

1–11	4–10, 12–14, 18	5, 7, 8, 10, 12, 14–20

Exercise 7.9A: Understanding and fluency



1 Trace the point P around the unit circle in an anti-clockwise direction from its starting point on the positive x-axis back to its starting point. As you trace, consider the size of angle θ as P moves. Also consider what happens to the x-coordinate for $P(x, y)$ as it moves along the x-axis.

a As you traced anti-clockwise around the unit circle, you will have noticed that the size of angle θ increased from 0° to 360°, and that the x-coordinate of point P decreased as it moved from 0° to 90°. Describe what happened to the x-coordinate of P when the size of the angle moved between:

i 90° and 180° **ii** 180° and 270° **iii** 270° and 360°.

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b The change in the x-coordinates from part **a** is the change in the cosine of angle θ as it increases from 0° to 360°. Complete this table to obtain a clearer picture of the values for cosine as the angle increases from 0° to 360°. Write your answers correct to one decimal place.

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
cos(θ)													

c Plot the points from the table on grid or graph paper, with the size of angle θ along the horizontal axis scale and the cosine value, $\cos(\theta)$, up the vertical axis scale. Join your points with a smooth curve.

d Sketch on graph paper how this curve would look if the angles were extended to 720°.

e What is the value of $\cos(\theta)$ at its first and second maximum? What is the period of the cosine graph?

f What is the y-intercept of the cosine graph?

2 Consider the change in the y-coordinates as point $P(x, y)$ moves around the unit circle and angle θ increases from 0° to 360°. This represents the change in the sine value of the angle.

a Repeat the procedure from question **1** part **b**, constructing a table for the sine values, to obtain a clearer picture of the values for sine as the angle sweeps from 0° to 360°.

b Sketch the sine graph from 0° to 360°.

c What is the period of the graph?

d What is the y-intercept of the sine graph?

3 The change in the value of $\frac{\sin(\theta)}{\cos(\theta)}$ as point $P(x, y)$ moves around the unit circle and angle θ increases in size represents the change in the tangent value of the angle.

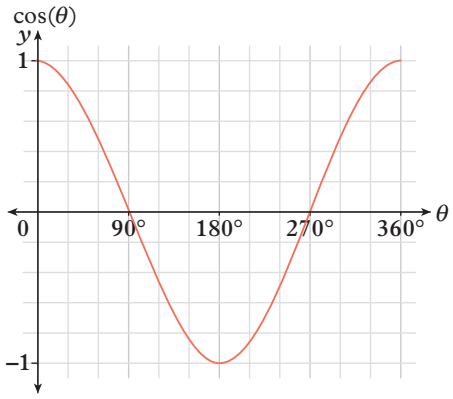
a Construct a table for the tangent values from 0° to 360° in intervals of 15°.

b Sketch the tangent graph from 0° to 360°.

c What is the period of the graph?

d Explain why the graph of $y = \tan(\theta)$ has a y-intercept of 0.

WE 7.9A **4** Consider the graph on the right of the cosine function.



a From the graph, write the cosine value for each of these angles.

i 0° **ii** 90°

iii 270° **iv** 180°

b Use the graph on the right to help you write the approximate cosine value (correct to one decimal place) for each of these angles.

i 40° **ii** 100°

iii 340° **iv** 200°

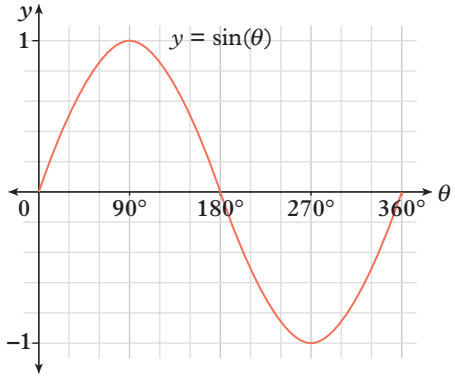
v 280° **vi** 140°

WE 7.9B **5** Use the graph from question **4** to help you find the angles from 0° to 360° with the following cosine values. Give your answers as approximate values, if necessary.

a 1 **b** –1 **c** 0

d 0.4 **e** –0.6 **f** 0.8

6 Consider the graph of the sine function on the right.



a From the graph, write the sine value for each of these angles.

i 360° **ii** 90°

iii 270° **iv** 180°

b Use the graph to help you write the approximate sine value (correct to one decimal place) for each of these angles.

i 40° **ii** 130°

iii 350° **iv** 190°

v 230° **vi** 140°

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7 Use the graph from question **6** to help you find the angles from 0° to 360° with the following sine values. Give your answers as approximate values, if necessary.

a 1 **b** –1 **c** 0

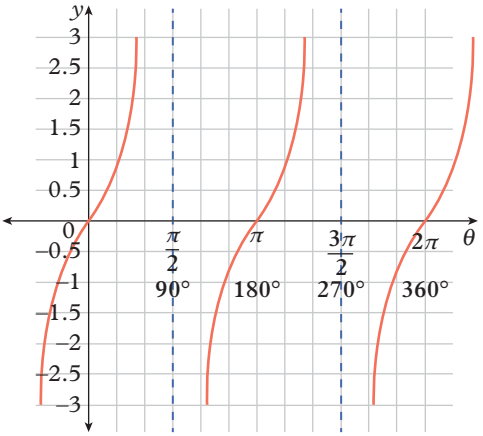
d 0.4 **e** –0.6 **f** 0.8

8 Consider the graph of the tan function on the right. Use the graph to help you write the approximate tan value (correct to one decimal place) for each of these angles.

a 30° **b** 150°

c 45° **d** 110°

e 210° **f** 250°



WE 7.9C **9** Use appropriate trigonometric identities to write the following in terms of $\sin(\theta)$, $\cos(\theta)$ or $\tan(\theta)$.

a $\sin(-\theta)$ **b** $\tan(-\theta)$

c $\cos(360° - \theta)$ **d** $\sin(360° - \theta)$

e $\cos(90° + \theta)$ **f** $\tan(90° + \theta)$

10 Use appropriate trigonometric identities to write the following in terms of $\sin(\theta)$, $\cos(\theta)$ or $\tan(\theta)$.

a $\sin(360° + \theta)$ **b** $\cos(540° - \theta)$ **c** $\tan(-90° + \theta)$

d $\cos(-90° - \theta)$ **e** $\sin(270° + \theta)$ **f** $\tan(270° - \theta)$

g $\sin(2\pi + \theta)$ **h** $\tan(3\pi + \theta)$ **i** $\cos(\frac{3\pi}{2} - \theta)$

Exercise 7.9B: Problem solving and reasoning

11 The graphs of the sine function and the cosine function are quite similar. If you were presented with a graph of $y = \sin(\theta)$ and $y = \cos(\theta)$, and the functions weren't labelled, how could you tell which graph was which?

12 Since the trigonometric functions are periodic, they can be graphed on any domain by first graphing one period, and then copying the graph to the left or right as required.

a To graph $y = \sin(\theta)$ on the domain $-2\pi \le \theta \le 2\pi$, complete the following steps.

i Graph $y = \sin(\theta)$ on the domain $0° \le \theta \le 2\pi$.

ii Extend your graph to the left, keeping the same shape, amplitude and period, until the graph extends from $\theta = -2\pi$ to $\theta = 2\pi$.

iii Verify that the points at $\theta = -2\pi$ and $\theta = 2\pi$ are correctly located on your graph.

b To graph $y = \cos(\theta)$ on the domain $-\frac{5\pi}{2} \le \theta \le -\frac{\pi}{2}$, complete the following steps.

i In pencil, graph $y = \cos(\theta)$ on the domain $0° \le \theta \le 2\pi$.

ii In pencil, extend your graph to the left, keeping the same shape, amplitude and period, until the graph extends from $\theta = -\frac{5\pi}{2}$ to $\theta = 2\pi$.

iii Erase the parts of your graph outside the domain $-\frac{5\pi}{2} \le \theta \le -\frac{\pi}{2}$.

iv Verify that the points at $\theta = -\frac{5\pi}{2}$ and $\theta = -\frac{\pi}{2}$ are correctly located on your graph.

c Graph $y = \sin(\theta)$ and $y = \cos(\theta)$ on the same set of axes on the following domains.

i $-\frac{3\pi}{2} \le \theta \le \frac{\pi}{2}$ **ii** $-4\pi \le \theta \le 0°$

13 Use trigonometric identities to explain why the graph of $y = \cos(\theta)$ is a horizontal translation of the graph of $y = \sin(\theta)$.

14 A trigonometric ratio involving an angle in quadrants 3 or 4 is often easier to evaluate using negative angles. For each of the trigonometric ratios below:

i rewrite the expression using a negative angle

ii simplify the expression to one with an acute or right angle using an appropriate trigonometric identity

iii evaluate the trigonometric ratio.

a $\sin(300°)$ **b** $\cos(315°)$ **c** $\tan(330°)$

d $\cos(270°)$ **e** $\sin(330°)$ **f** $\tan(300°)$

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15 There are many more trigonometric identities. The double angle trigonometric identity for sine is:

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta).$$

Verify the double angle formula for the following values of θ.

a $30°$ **b** $45°$ **c** $120°$ **d** $\frac{\pi}{3}$ **e** $\frac{\pi}{2}$ **f** π

16 The angle addition trigonometric identity for sine is:

$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y).$$

Verify the angle addition identity for the following values of x and y.

a $x = 30°, y = 30°$ **b** $x = 45°, y = 45°$ **c** $x = 60°, y = 30°$

d $x = 2\pi, y = \pi$ **e** $x = \frac{\pi}{2}, y = \frac{\pi}{3}$ **f** $x = \frac{\pi}{2}, y = \frac{\pi}{4}$

17 Exact values of some other angles can be determined using the identity from question **16**.

a Find the exact value of:

i $\sin(105°) = \sin(60° + 45°)$ **ii** $\sin(75°) = \sin(30° + 45°)$.

b Explain why your answers from part **a** are equal.

c Write an angle subtraction trigonometric identity for sine.
Hint: Write $\sin(x - y) = \sin(x + (-y))$ and apply the angle addition identity.

d Use your answer to part **c** to find the exact value of:

i $\sin(15°) = \sin(45° - 30°)$ **ii** $\sin(75°) = \sin(90° - 15°)$.

18 When graphing one period of a trigonometric function, it is common to choose a complete period, which includes the point $\theta = 0°$. The tangent function passes through the origin, has a vertical asymptote at $90°$ and has a period of $180°$.

a Determine the equation of the first vertical asymptote to the left of the one at $\theta = 90°$.

b Draw the asymptote from part **a** and the one at $\theta = 90°$ on the same set of axes.

c Graph one period of $y = \tan(\theta)$ between the two asymptotes.

d What is the equation of the next vertical asymptote to the left of the ones already drawn on your graph?

e Find a rule you can use to locate all asymptotes of $y = \tan(\theta)$ on any domain.

Exercise 7.9C: Challenge

19 Like other graphs, the graphs of trigonometric functions can be translated vertically and horizontally.

a Complete the table of values below for $y = \sin(\theta - 30°) + 4$.

$\theta - 30°$	$0°$	$90°$	$180°$	$270°$	$360°$
θ	$30°$	$120°$			
$\sin(\theta - 30°)$	0	1			
$\sin(\theta - 30°) + 4$	4	5			

b Sketch the graph of $y = \sin(\theta - 30°) + 4$ for $30° \le \theta \le 390°$.

c Sketch the graph of $y = \cos(\theta - 45°) - 3$ for $45° \le \theta \le 405°$.

20 Trigonometric graphs can also be dilated in the direction of the x- or the y-axis. Use a graphing application to answer the following questions.

a What transformation is required to produce the graph of $y = 2\sin(x)$ from the graph of $y = \sin(x)$? Is there a change in period or amplitude?

b What transformation is required to produce the graph of $y = \sin(2x)$ from the graph of $y = \sin(x)$? Is there a change in period or amplitude?

Online resources:

- **Interactive skillsheet** Sketching trigonometric graphs
- **Interactive skillsheet** Period and amplitude of trigonometric graphs
- **Interactive skillsheet** Trigonometric identities
- **Quick quiz** 7.9

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Lesson 7.10

Solving trigonometric equations

Learning intentions

By the end of this lesson you will be able to ...

✔ solve simple trigonometric equations.

Inter-year links

Year 9 — 6.6 Using trigonometry to find angles

Trigonometric equations

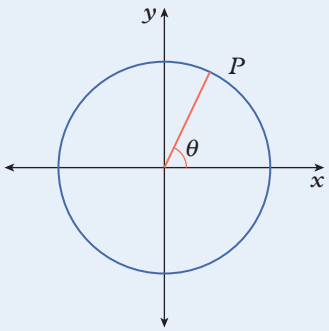
Key content video: Solving trigonometric equations

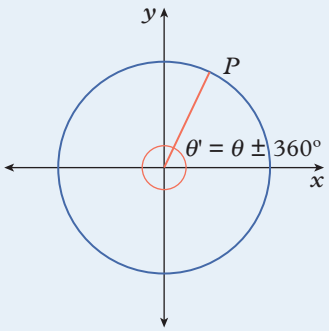
- The trigonometric functions $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ are all periodic, as their values repeat with every revolution of the angle θ.
- If θ is the angle measured anti-clockwise from the positive x-axis to the radius that meets the unit circle at point P, then:
  - → the x-coordinate of the point on the unit circle is $\cos(\theta)$
  - → the y-coordinate of the point on the unit circle is $\sin(\theta)$
  - → the gradient of the radius from the origin to the point P on the unit circle is $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$.
- When adding or subtracting 360° or 2π to θ, the new angle, θ', meets the unit circle at point P, therefore:

  $\sin(\theta) = \sin(\theta \pm 2\pi)$

  $\cos(\theta) = \cos(\theta \pm 2\pi)$

  $\tan(\theta) = \tan(\theta \pm 2\pi)$





- Equations involving trigonometric functions have infinitely many solutions, as trigonometric functions are periodic.

  For example, $\sin(\theta) = 0$ for $\theta = 0, \pi, 2\pi, 3\pi, 4\pi, \ldots$ or $\theta = 0, 180°, 360°, 540°, 720°, \ldots$
- The solutions to a trigonometric equation are often restricted to a domain given by an interval, such as $0 \le \theta \le 2\pi$ or $0° \le \theta \le 360°$.
- Most calculators only give one of the solutions for $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$ as the domains are restricted to $-90° \le \theta \le 90°$.

  To solve a trigonometric equation, e.g. $\sin(\theta) = a$, within a given domain:

  1. use the special triangles (or the exact values table in **Lesson 7.8**) to determine the reference angle
  2. identify the quadrants in which the ratio is positive or negative (depending on the sign of a) using ASTC
  3. find the possible values for θ; use $180 - \theta$ for quadrant 2, $180 + \theta$ for quadrant 3, and $360 - \theta$ for quadrant 4
  4. check the given domain; reject solutions that lie outside the domain, or find coterminal solutions that lie within the domain
  5. write all of the solutions to the equation.

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Worked example 7.10A Solving a trigonometric equation within a given domain

Solve the equation $\sin(\theta) = -\frac{1}{2}$, $0° \le \theta \le 720°$.

THINK

1 Use the special triangles to determine the angle in the first quadrant that has a sine value of $\frac{1}{2}$.

2 Identify the quadrants in which the sine ratio is negative.

3 Write possible values for θ using symmetry properties of trigonometric ratios in the unit circle.

4 Add or subtract multiples of 360° to find other solutions in the given domain.

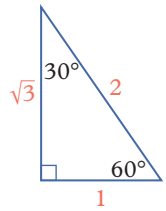
5 Write the solution to the equation.

WRITE

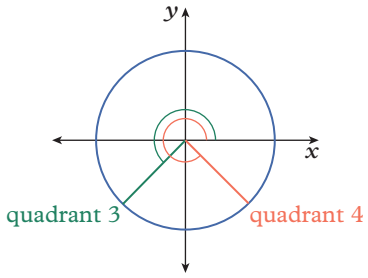
$\sin(\theta) = \frac{1}{2}$

$\theta = \sin^{-1}\left(\frac{1}{2}\right)$

$= 30°$



We were given that $\sin(\theta) = -\frac{1}{2}$ and sine is negative in the third quadrant and the fourth quadrant.



So: $\theta = 180° + 30°$

$= 210°$

or: $\theta = 360° - 30°$

$= 330°$

$210° + 360° = 570°$

$330° + 360° = 690°$

The solutions are 210°, 330°, 570° and 690°.

Helpful hints

✔ The mnemonic ASTC ('**A**ll **S**tations **T**o **C**entral') is a reminder that: **all** ratios are positive in quadrant 1, only **sin** is positive in quadrant 2, only **tan** is positive in quadrant 3, and only **cos** is positive in quadrant 4.

Learning pathways

1–3, 5–7, 9, 11, 15, 17	1, 4–6, 8, 10(a–e), 11, 12, 15–17(a, c, e), 18	4–6, 10(e–i), 11–14, 15–17(b, d, f), 19–21

ANS p730

Exercise 7.10A: Understanding and fluency

1 Consider the equation $\sin(\theta) = 1$, $0° \le \theta \le 360°$.

a Which angle(s) in the given domain have a sine value of 1? Hint: Refer to the unit circle.

b Write a list of all the solutions for $\sin(\theta) = 1$ in the domain $0° \le \theta \le 720°$.

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2 Consider the equation $\cos(\theta) = -1$.

a What solution(s) can be found in the domain $0° \le \theta \le 360°$?

b What are the solutions for $\cos(\theta) = -1$ in the domain $0° \le \theta \le 720°$?

3 Consider the equation $\sin(\theta) = 0$, $0° \le \theta \le 360°$.

a Which angle(s) in the given domain have a sine value of 0? Hint: Refer to the unit circle.

b Write a list of all the solutions for $\sin(\theta) = 0$ in the domain $0° \le \theta \le 720°$.

4 Solve each equation for the domain $0° \le \theta \le 360°$.

a $\cos(\theta) = 0$ **b** $\tan(\theta) = 0$ **c** $\sin(\theta) = -1$ **d** $\cos(\theta) = 1$

WE 7.10A **5** Solve each equation for the domain $0° \le \theta \le 720°$.

a $\cos(\theta) = \frac{\sqrt{3}}{2}$ **b** $\sin(\theta) = -\frac{1}{\sqrt{2}}$ **c** $\tan(\theta) = -\frac{1}{\sqrt{3}}$

d $\sin(\theta) = -\frac{\sqrt{3}}{2}$ **e** $\cos(\theta) = \frac{1}{\sqrt{2}}$ **f** $\tan(\theta) = 1$

6 Using a calculator to determine the reference angle, solve each equation for the domain $0° \le \theta \le 360°$, correct to the nearest degree.

a $\cos(\theta) = -0.4$ **b** $\sin(\theta) = -0.3$ **c** $\tan(\theta) = 0.9$

d $\sin(\theta) = 0.6$ **e** $\cos(\theta) = 0.6$ **f** $\tan(\theta) = -7$

Exercise 7.10B: Problem solving and reasoning

7 For each of the following equations, what are the smallest positive values of a and b such that there are two solutions in the domain $a \le \theta \le b$?

a $\cos(\theta) = 0$ **b** $\cos(\theta) = \frac{1}{2}$ **c** $\sin(\theta) = \frac{\sqrt{3}}{2}$

d $\sin(\theta) = \frac{1}{\sqrt{2}}$ **e** $\cos(\theta) = -\frac{\sqrt{3}}{2}$ **f** $\sin(\theta) = -1$

8 For each of the following equations, what are the smallest positive values of a and b such that there are four solutions in the domain $a \le \theta \le b$?

a $\sin(\theta) = -\frac{1}{2}$ **b** $\cos(\theta) = -\frac{1}{2}$ **c** $\sin(\theta) = -\frac{1}{\sqrt{2}}$ **d** $\cos(\theta) = -\frac{1}{\sqrt{2}}$

9 It is possible that a trigonometric equation has no real solution. Explain why each of these equations has no solution within any real domain.

a $3 - \sin(\theta) = 1$ **b** $2\cos(\theta) = 3$ **c** $\frac{1}{4}\sin(\theta) = 1$

10 Indicate whether each equation has a real solution within the domain $0° \le \theta \le 360°$. For those without a real solution, explain why this is so. There is no need to solve the equation.

a $\tan(\theta) = 4$ **b** $1 - \sin(\theta) = 1$ **c** $2\cos(\theta) = 2$

d $\frac{1}{2}\sin(\theta) = -\frac{1}{2}$ **e** $1 + \cos(\theta) = -1$ **f** $4\sin(\theta) = -\frac{1}{4}$

g $3 - \cos(\theta) = \frac{1}{3}$ **h** $\frac{1}{2}\cos(\theta) = -\frac{1}{4}$ **i** $\frac{1}{4} + \sin(\theta) = -1$

11 Take care when you consider the domain of a trigonometric equation. Don't always assume that a trigonometric equation has two or more solutions. Consider the following equations with the domains indicated. Write the number of solutions for each. (Don't write the values of the solutions.)

a $\sin(\theta) = \frac{1}{2}$, $0° \le \theta \le 90°$ **b** $\sin(\theta) = -\frac{1}{2}$, $0° \le \theta \le 90°$

c $\cos(\theta) = -\frac{1}{2}$, $0° \le \theta \le 180°$ **d** $\tan(\theta) = 0.3$, $0° \le \theta \le 180°$

e $\sin(\theta) = -0.4$, $0° \le \theta \le 270°$ **f** $\cos(\theta) = 0.8$, $0° \le \theta \le 270°$

12 Solve each equation for the domain indicated.

a $2\sin(\theta) - 3 = \sin(\theta) - 2$, $0° \le \theta \le 180°$ **b** $4\cos(\theta) + 5 = 4 + 5\cos(\theta)$, $0° \le \theta \le 180°$

c $\sqrt{2}\cos(\theta) + 1 = 0$, $0° \le \theta \le 270°$ **d** $2\sin(\theta) + \sqrt{3} = 0$, $0° \le \theta \le 270°$

e $4 - 4\tan(\theta) = 0$, $90° \le \theta \le 270°$ **f** $\sqrt{48}\tan(\theta) + 2 = -3 - \sqrt{3}\tan(\theta)$, $90° \le \theta \le 270°$

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13 If the pronumeral inside the brackets of the trigonometric function is multiplied by a coefficient, then start by redefining the specified domain in terms of the expression inside the brackets. For example, follow these steps to solve $\cos(3\theta) = \frac{1}{2}$, $0° \le \theta \le 180°$.

1. Write the equation and the domain of θ. $\cos(3\theta) = \frac{1}{2}$, $0° \le \theta \le 180°$
2. Multiply the domain by the coefficient of θ. $\cos(3\theta) = \frac{1}{2}$, $0° \le 3\theta \le 540°$
3. Solve the equation normally for all solutions in the new domain. $3\theta = 60°, 300°, 420°$
4. Divide all solutions by the coefficient of θ to find the solutions to the original equation. $\theta = 20°, 100°, 140°$

Solve each of the following equations for the domain $0° \le \theta \le 180°$.

a $\sin(2\theta) = \frac{1}{\sqrt{2}}$ **b** $\cos(4\theta) = -1$ **c** $\tan(3\theta) = -1$

14 Solve each equation for the domain $0° \le \theta \le 1080°$.

a $\sin\left(\frac{\theta}{2}\right) = -\frac{1}{2}$ **b** $\cos\left(\frac{\theta}{3}\right) = \frac{1}{\sqrt{2}}$ **c** $\tan\left(\frac{2\theta}{5}\right) = -\sqrt{3}$

15 Determine the sum of the solutions of the following equations for the domain $0° \le \theta \le 360°$.

a $\sin(\theta) = \frac{1}{2}$ **b** $\cos(\theta) = \frac{1}{2}$ **c** $\tan(\theta) = \frac{1}{\sqrt{3}}$ **d** $\sin(\theta) = -\frac{1}{2}$ **e** $\cos(\theta) = -\frac{1}{2}$ **f** $\tan(\theta) = -\frac{1}{\sqrt{3}}$

16 Solve each equation for the domain $-360° \le \theta \le 0°$.

a $\sin(\theta) = \frac{\sqrt{3}}{2}$ **b** $\cos(\theta) = -\frac{1}{\sqrt{2}}$ **c** $\tan(\theta) = \sqrt{3}$ **d** $\sin(\theta) = -1$ **e** $\cos(\theta) = 0$ **f** $\tan(\theta) = 0$

17 Solve the following equations for the domain $0 \le \theta \le 2\pi$.

a $\sin(\theta) = 0$ **b** $\cos(\theta) = \frac{\sqrt{3}}{2}$ **c** $\tan(\theta) = -1$ **d** $\sin(\theta) = \frac{1}{\sqrt{2}}$ **e** $\cos(\theta) = -1$ **f** $\tan(\theta) = \frac{1}{\sqrt{3}}$

18 The equation $\sin(\theta) = \cos(\theta)$ can be solved by using the identity $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$.

a Show that $\sin(\theta) = \cos(\theta)$ is equivalent to $\tan(\theta) = 1$.

b Solve the following equations for $0° \le \theta \le 360°$.

i $\sin(\theta) = \cos(\theta)$ **ii** $\sin(\theta) = \sqrt{3}\cos(\theta)$

iii $\sqrt{3}\sin(\theta) + \cos(\theta) = 0$ **iv** $3\sin(\theta) - \sqrt{3}\cos(\theta) = 0$

Exercise 7.10C: Challenge

19 Solve the following equations.

a $2\sin(2\theta) - 1 = 0$, $0° \le \theta \le 360°$ **b** $4\cos\left(\frac{\theta}{3}\right) + 2 = 0$, $180° \le \theta \le 540°$

c $10\sin(2\theta - 30°) + 5 = 0$, $180° \le \theta \le 540°$ **d** $12\cos\left(\frac{\theta}{4} + \pi\right) - \sqrt{72} = 0$, $-5\pi \le \theta \le 12\pi$

20 Consider the quadratic equation $2x^2 - 3x - 2 = 0$.

a Solve the quadratic equation.

b Hence, solve the equation $2(\sin(\theta))^2 - 3\sin(\theta) - 2 = 0$, $0° \le \theta \le 360°$.

21 Consider the quadratic equation $6x^2 - 17x + 12 = 0$.

a Solve the quadratic equation.

b Why are there no solutions to the equation $6(\sin(\theta))2 - 17\sin(\theta) + 12 = 0$?

c Why are there no solutions to the equation $6(\cos(\theta))2 - 17\cos(\theta) + 12 = 0$?

d How many solutions are there to the equation $6(\tan(\theta))2 - 17\tan(\theta) + 12 = 0$?

Online resources:

- **Interactive skillsheet** Solving trigonometric equations
- **Worksheet** Trigonometric rules and equations
- **CAS instructions** Solving trigonometric equations
- **Quick quiz** 7.10

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Lesson 7.11

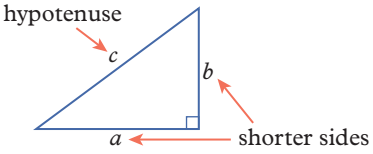
Review: Pythagoras' theorem and trigonometry

Module summary

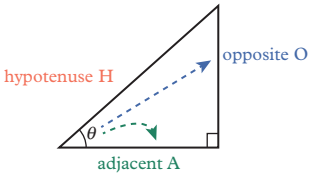
Pythagoras' theorem and trigonometric ratios

$$c^2 = a^2 + b^2$$

hypotenuse shorter sides

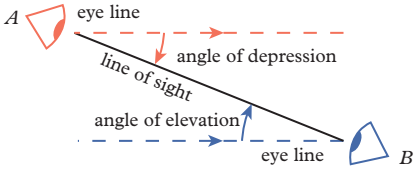


- $\sin(\theta) = \dfrac{\text{Opposite}}{\text{Hypotenuse}} = \dfrac{\text{O}}{\text{H}}$
- $\cos(\theta) = \dfrac{\text{Adjacent}}{\text{Hypotenuse}} = \dfrac{\text{A}}{\text{H}}$
- $\tan(\theta) = \dfrac{\text{Opposite}}{\text{Adjacent}} = \dfrac{\text{O}}{\text{A}}$



Angles of elevation and depression

The angle of depression from A to B is equal to the angle of elevation from B to A.



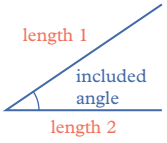
The sine rule

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

The cosine rule

$$a^2 = b^2 + c^2 - 2bc\cos(A)$$

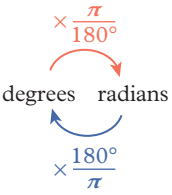
Included angle area formula



$$\text{area} = \frac{1}{2}bc\sin(A)$$

adjacent side lengths included angle

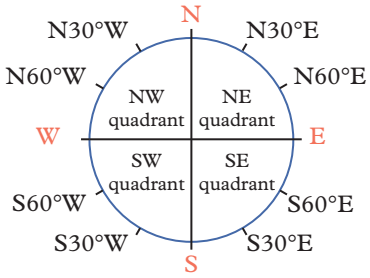
Radians

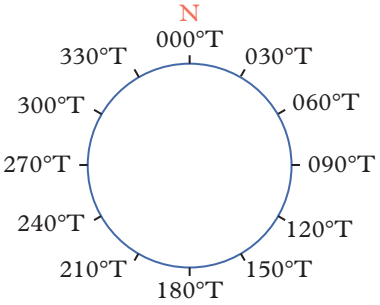


Exact values

θ	30° or $\frac{\pi}{6}$	45° or $\frac{\pi}{4}$	60° or $\frac{\pi}{3}$
$\sin(\theta)$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos(\theta)$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan(\theta)$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

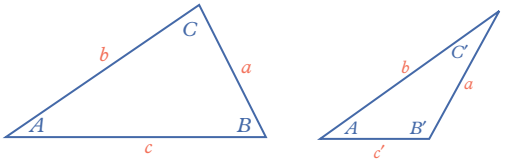
Compass bearings and true bearings



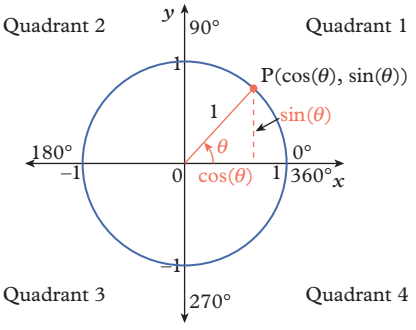


Ambiguous case of the sine rule

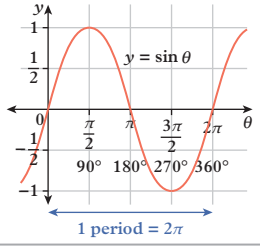
If two side lengths (a and b) are known and the angle (A) opposite the shorter side length is also known, then there are two possible triangles.

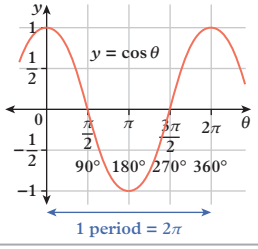


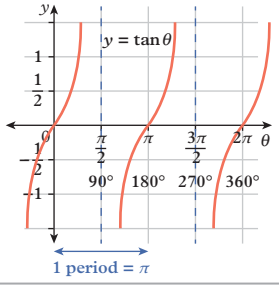
The unit circle



Trigonometric graphs







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Review questions 7.11A: Mathematical literacy review

Review quiz
Take the review quiz to assess your knowledge of this module.
✔ 10
✔ 10A

Quizlet
Test your knowledge of this module by working individually or in teams.

The following key terms are used in this module:

- amplitude
- angle of depression
- angle of elevation
- compass bearing
- cosine
- cosine rule
- coterminal angles
- degrees
- hypotenuse
- included angle area formula
- period
- periodic function
- Pythagoras' theorem
- quadrant
- radians
- reference angle
- sine
- sine rule
- tangent
- trigonometry
- true bearing
- unit circle

1 Describe Pythagoras' theorem in words, using at least one other word from the list of key terms.

2 Use a diagram to explain how to calculate the sine, cosine and tangent ratios for an acute angle θ in a right-angled triangle.

3 Use an example to explain the difference between compass bearings and true bearings.

4 Complete the following sentences using words from the key terms list.

a The _____ is the angle measured downwards from the horizontal.

b The _____ is the angle measured upwards from the horizontal.

5 Draw and label a triangle with sides a, b and c and angles A, B and C, as used in the sine and cosine rules.

6 Identify which rule(s) could be used given the following pieces of information.

a two sides and an included angle

b an opposite angle–side pair and another angle or side

7 Using an example, provide a definition in your own words for the following key terms.

a coterminal angles

b periodic function

8 Complete the following sentences using words from the key terms list.

a The _____ of a trigonometric function is the distance from the centre of the curve to the maximum or minimum y-value.

b For a given angle in quadrants 2, 3 or 4, the _____ in quadrant 1 is the acute angle between the x-axis and the line connecting the origin to the point on the _____.

Review questions 7.11B: Multiple choice

7.1 **1** Which of these describes a triangle with side lengths measuring 4 cm, 5 cm and 6 cm?

A right-angled **B** obtuse-angled **C** acute-angled **D** isosceles

7.1 **2** The ratio $\frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$ of a right-angled triangle is known as which of the following?

A sine **B** cosine **C** tangent **D** π

7.2 **3** For the triangle on the right, what is the size of $\angle YZX$, to the nearest degree?

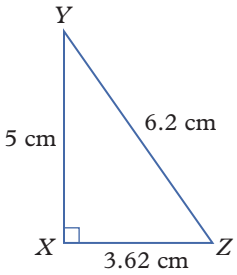
A 54° **B** 55° **C** 35° **D** 36°

7.2 **4** For the triangle on the right, which of these ratios represents the sine of $\angle ZYX$?

A $\frac{5}{6.2}$ **B** $\frac{3.62}{6.2}$ **C** $\frac{3.62}{5}$ **D** $\frac{5}{3.62}$

7.3 **5** What is the equivalent true bearing of N12°W?

A 348°T **B** 012°T **C** 168°T **D** 078°T



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10A 7.4 **6** Which of these values is closest to the diagonal length within a cube of side length 5 cm?

A 7 cm **B** 8 cm **C** 9 cm **D** 10 cm

10A 7.5 **7** What is the size of acute angle A in $\frac{10}{\sin(80°)} = \frac{9}{\sin(A)}$ closest to?

A 55° **B** 60° **C** 65° **D** 115°

10A 7.6 **8** In $\triangle ABC$, if you know the size of angle B, what do you also need to know to apply the cosine rule to find b?

A a and b **B** a and c **C** c and b **D** angle A

10A 7.7 **9** If a point, P, on the unit circle in quadrant 2 is given in terms of an acute reference angle, θ, then which of these coordinates for P are correct?

A $(-\cos(\theta), \sin(\theta))$ **B** $(\cos(\theta), -\sin(\theta))$ **C** $(-\cos(\theta), -\sin(\theta))$ **D** $(\cos(\theta), \sin(\theta))$

10A 7.8 **10** Which of these is the exact sine value for 30°?

A $\frac{1}{2}$ **B** $\frac{\sqrt{3}}{2}$ **C** $\frac{1}{\sqrt{3}}$ **D** $\frac{1}{\sqrt{2}}$

10A 7.9 **11** What is the period of $y = \sin(\theta)$?

A $\frac{\pi}{2}$ **B** π **C** 2π **D** 4π

10A 7.10 **12** Which of the following is not a solution to $2\cos(2\theta) = 1$?

A 30° **B** 60° **C** 150° **D** 210°

Review questions 7.11C: Short answer

For these questions, calculate each angle, correct to the nearest degree, and each length, correct to two decimal places (unless stated otherwise).

7.1 **1** An equilateral triangle has side lengths of 5 cm. Find the height of the triangle as:

a an exact value **b** an approximate value, correct to one decimal place.

7.2 **2** A rectangle has a diagonal length of 15.8 cm. The diagonal forms an angle of 31.3° with the longer side. What are the dimensions of the rectangle?

7.3 **3** Two buildings stand 20 m apart. One is taller than the other. From the top of the taller building to the base of the shorter building, the angle of depression is 75°. From the top of the shorter building to the top of the taller building, the angle of elevation is 60°. What is the height of the shorter building?

10A 7.4 **4** Find the length of the longest diagonal within each of the following 3D objects:

a a cylinder with base circumference of 20 cm and height of 15 cm

b a box 10 cm long, 8 cm wide and 15 cm tall

c a cube of side length 25 cm.

10A 7.4 **5** Calculate the angle each diagonal you calculated in question **4** makes with the height of the object, correct to the nearest degree.

10A 7.5 **6** Standing in a park, Anabel takes a bearing of 065°T to a tree. She walks 15 m due east and finds the bearing to the same tree to be 295°T.

a Draw a labelled diagram to represent this situation.

b Find the distance from the tree to Anabel, in both positions.

10A 7.5 **7** In $\triangle ABC$, $a = 5$ cm, $b = 6.1$ cm and $A = 39°$.

a Explain why two distinct triangles can be drawn with these measurements. Draw them.

b Find the dimensions of all the angles and sides of the two triangles drawn in part **a**.

10A 7.6 **8** A triangular garden bed in a park has sides of lengths 50 m, 40 m and 20 m.

a Draw a sketch of the shape of the garden bed.

b Find the size of the angle at each vertex of the triangular shape and the area of the triangle.

c What is the shortest distance from the vertex opposite the longest side to the longest side?

10A 7.7 **9** Express each of the following trigonometric expressions in terms of an appropriate reference angle.

a $\sin(120°)$ **b** $\cos(210°)$ **c** $\tan(300°)$ **d** $\cos(315°)$

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10A 7.8 **10** Angles can be measured in radians or in degrees.

a Convert the following angles to radians.

i 120° **ii** 315° **iii** 130°

b Convert the following angles to degrees.

i 3π **ii** 83π **iii** 98π

10A 7.8 **11** Use the unit circle to find each of the following values.

a sin(300°) **b** cos(225°) **c** sin(135°) **d** cos(315°)

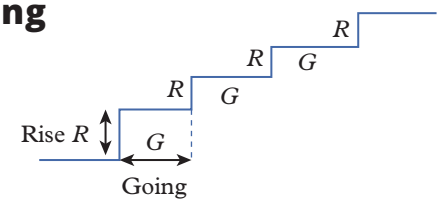
10A 7.9 **12** Use appropriate trigonometric identities to write the following in terms of $\sin(\theta)$, $\cos(\theta)$ and $\tan(\theta)$.

a $\sin(90° - \theta)$ **b** $\cos(180° - \theta)$ **c** $\tan(-\theta)$ **d** $\cos(360° - \theta)$

10A 7.10 **13** Solve each of these equations for the domain $0° \le \theta \le 360°$.

a $\cos(\theta) = -1$ **b** $\sin(\theta) = 0$ **c** $\cos(\theta) = \frac{\sqrt{3}}{2}$ **d** $\sin(\theta) = -\frac{1}{2}$

Review questions 7.11D: Mathematical modelling



1 There are building code regulations for the construction of staircases. For safety purposes, the tread cannot be too narrow, nor the staircase too steep. There are particular terms for staircases, as listed and shown in the diagram below.

- The rise (R) must have a minimum height of 115 mm and a maximum height of 190 mm.
- The going (G) must have a length within the range of 250–355 mm.
- The regulation also states that $2R + G$ must lie within the range of 550–700 mm.

a Consider a staircase with the minimum rise measurement of 115 mm and the minimum going measurement of 250 mm.

i Explain whether this staircase would be within the building regulation guidelines.

ii What is the slope of the staircase described? Give your answer to the nearest degree.

b Repeat part **a** using the maximum values allowed for the rise and going measurements.

c If the going measurement is the maximum value allowed, 355 mm, and the value of $2R + G$ falls within the required limits:

i find a range of acceptable values for the rise **ii** calculate the range of acceptable slope values.

d The number of steps is counted by the number of goings from the bottom to the top, starting with a going and ending with a going. Draw a set of five stairs with a going measurement of 300 mm and a rise of 150 mm.

i What is the total horizontal distance of the five stairs?

ii Find the total vertical height of the five stairs.

iii Calculate the direct distance between the base of the first step and the total vertical height calculated in part **ii**.

iv What is the slope of the staircase? Give your answer to the nearest degree.

e What do you consider would be ideal measurements for the going and rise of a comfortable staircase? Draw a sketch of your design, show that it falls within the regulation guidelines, and provide its slope.

f There are different building code regulations for wheelchair ramps. The slope of the ramp is of particular importance, because a wheelchair may tip backwards or roll down if the ramp is too steep. The code specifies that an internal ramp should be no steeper than 1 to 14, while an external ramp should have a gentler slope of 1 to 20.

i Draw diagrams of 20 m long internal and external ramps with these slopes.

ii For each ramp drawn in part **i**, find the going and rise measurements, rounding to three decimal places. Mark these measurements on your diagrams.

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10A **2** Emma rides her bicycle at a constant speed with a tag attached to a spoke at the rim of the wheel.

Tom used a video to record the height of the tag with respect to its starting position. He also made a record of the results in the table below.

Time (s)	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
Height (cm)	20	30	20	0	–20	–30	–20	0	20	30

a Graph the data from the table, showing time on the horizontal axis.

b Suggest whether the graph models a sine curve or a cosine curve.

c How long does it take for the wheel to rotate once?

d What was the starting position of the tag on the wheel? Explain how you deduced this.

e What is the radius of the wheel?

f Calculate the circumference of the wheel.

g At what speed does Emma ride? Give your answer in metres per second.

h What is the position of the tag after 10 seconds of riding at this constant speed? Explain how you arrived at this answer.

10A **3** Triangulation is a method to determine unknown distances and angles using trigonometry. In this question, we consider a simplified example using the diagram below, which shows some known distances and angles between seven points. A dashed line between two points means that a direct measurement could not be taken, resulting in the need to use triangles and trigonometry.

a Use the sine and cosine rules and your knowledge of triangles to find the following angles and side lengths, correct to one decimal place. In your working, use the values correct to one decimal place.

i $\angle ABF$ **ii** AB **iii** BF **iv** BC

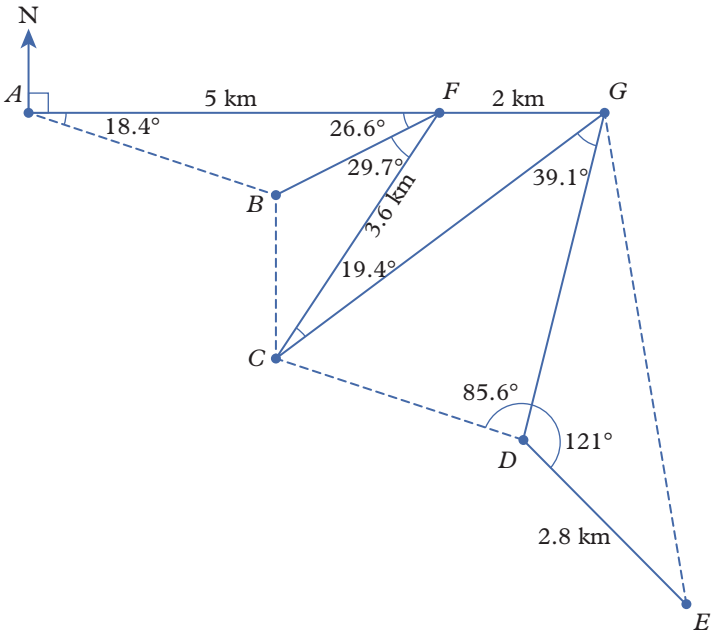
v $\angle CBF$ **vi** $\angle BCF$ **vii** $\angle CGF$ **viii** $\angle CFG$

ix CG **x** $\angle DCG$ **xi** CD **xii** DG

xiii EG **xiv** $\angle DEG$ **xv** $\angle DGE$

b Use your answers to part **a** to determine the distance between points A and E, correct to one decimal place, and the true bearing from A to E, correct to the nearest degree.

c How could you make your answers to part **b** more accurate?



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Checklist

Now that you have completed this module, reflect on your ability to do the following.

I can do this	I need to review this
☐ Determine whether a triangle is acute, right-angled or obtuse given the side lengths ☐ Use Pythagoras' theorem to determine unknown side lengths	☐ Go back to Lesson 7.1 Pythagoras' theorem
☐ Use trigonometry to find unknown side lengths ☐ Use trigonometry to find unknown angles	☐ Go back to Lesson 7.2 Trigonometry
☐ Use angles of elevation and depression to calculate distances ☐ Use compass bearings and true bearings in trigonometry problems	☐ Go back to Lesson 7.3 Applications of trigonometry
☐ Find lengths and angles within 3D objects	☐ Go back to Lesson 7.4 Three-dimensional problems [10A]
☐ Determine unknown side lengths and angles using the sine rule ☐ Determine the two possible solutions for an ambiguous case of the sine rule ☐ Determine the area of a triangle using the included angle area formula	☐ Go back to Lesson 7.5 The sine and area rules [10A]
☐ Determine unknown side lengths of a triangle using the cosine rule ☐ Determine unknown angles of a triangle using the cosine rule	☐ Go back to Lesson 7.6 The cosine rule [10A]
☐ Determine the unit circle quadrant in which an angle lies ☐ Determine the sign (positive or negative) of a trigonometric expression given an angle ☐ Find the reference angle in the first quadrant of any angle using symmetry properties	☐ Go back to Lesson 7.7 The unit circle [10A]
☐ Convert between degrees and radians ☐ Determine the exact values of the sine, cosine and tangent functions ☐ Identify points on the unit circle via angles and arc lengths in radians	☐ Go back to Lesson 7.8 Exact values [10A]
☐ Read a point on a trigonometric graph ☐ Sketch graphs of simple trigonometric functions ☐ Apply trigonometric identities involving complementary, supplementary and negative angles	☐ Go back to Lesson 7.9 Trigonometric graphs [10A]
☐ Solve simple trigonometric equations within a given domain	☐ Go back to Lesson 7.10 Solving trigonometric equations [10A]

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8

Measurement

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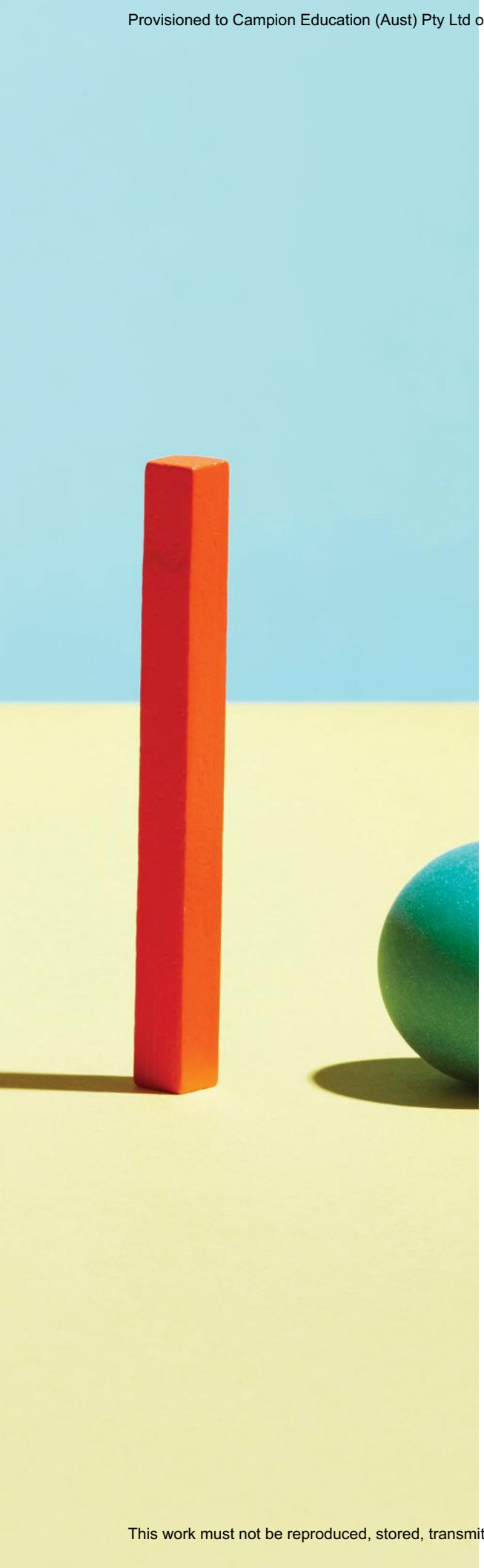

Lessons

8.1 Area review
8.2 Surface area review
8.3 Volume review
8.4 Compounding errors
8.5 Logarithmic scales
10A **8.6** Surface area of pyramids and cones
10A **8.7** Volume of pyramids and cones
10A **8.8** Surface area and volume of spheres

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ Using Pythagoras' theorem
- ✔ Area of simple shapes
- ✔ Area of a circle
- ✔ Scientific notation

Curriculum links

- Solve problems involving the surface area and volume of composite objects using appropriate units (VC2M10M01)
- Interpret and use logarithmic scales in applied contexts involving small and large quantities and change (VC2M10M02)
- Use mathematical modelling to solve practical problems involving direct and inverse proportion and scaling of objects; formulate problems and interpret solutions in terms of the situation, including the impact of measurement errors on the accuracy of results; evaluate and modify models as necessary, and report assumptions, methods and findings (VC2M10M04)
- 10A Solve problems involving surface area and volume of right pyramids, right cones, spheres and related composite solids (VC2M10AM01)

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Materials

- ✔ Calculator

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Lesson 8.1

Area review

Learning intentions

By the end of this lesson you will be able to ...

✔ convert units of measurement and calculate the area of composite shapes and sectors.

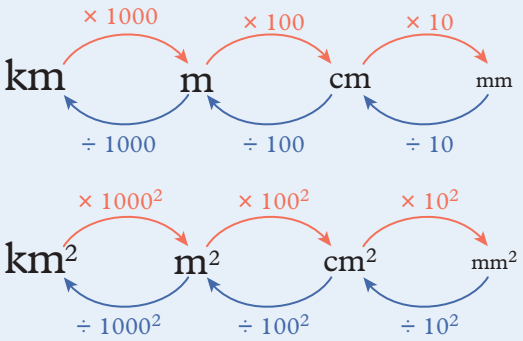
Inter-year links

Support	Understanding area
Year 7	9.5 Area of a triangle
Year 8	8.4 Area of a circle
Year 9	5.2 Area of composite shapes

Units of measurement

Key content video
Converting between units of area

- To convert between units of length, multiply or divide by the appropriate **conversion factor**.
- A unit of **area** can be converted to another unit of area by multiplying or dividing by the appropriate conversion factor.
- 1 ha (hectare) = 10 000 m^2



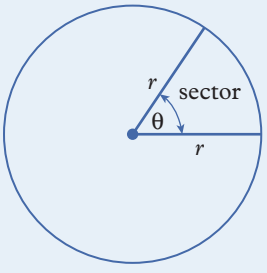
Composite shapes

- To find the area of a **composite shape**:
  1. Split the shape into individual parts
  2. Calculate any missing dimensions
  3. Calculate the areas of the individual parts
  4. Add or subtract the areas to find the total area.

Key content video
Area of composite shapes

Area of a sector

- The area of a sector of a circle is $\frac{\theta}{360°} \times \pi r^2$, where θ is the angle in degrees, between the two radii forming the sector.



Worked example 8.1A Converting units of length and area

Convert each length or area measurement below to the unit shown in brackets.

a 120 mm (m) **b** 1.2 km^2 (m^2) **c** 320 mm^2 (cm^2)

THINK

a Identify the correct conversion factor. To convert from millimetres (mm) to metres (m), first divide by 10 (to convert to centimetres), then divide by 100 (to convert to metres).

b Identify the correct conversion factor. To convert from square kilometres (km^2) to square metres (m^2) multiply by 1000^2.

c Identify the correct conversion factor. To convert from square millimetres (mm^2) to square centimetres (cm^2), divide by 10^2.

WRITE

a 120 mm = (120 ÷ 10 ÷ 100) m
= 0.12 m

b 1.2 km^2 = (1.2 × 1000^2) m^2
= 1 200 000 m^2

c 320 mm^2 = (320 ÷ 10^2) cm^2
= 3.2 cm^2

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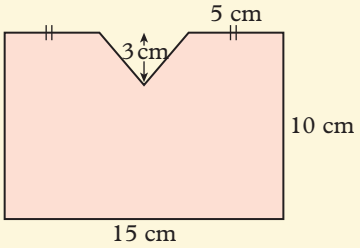
Worked example 8.1B Calculating the area of a composite shape

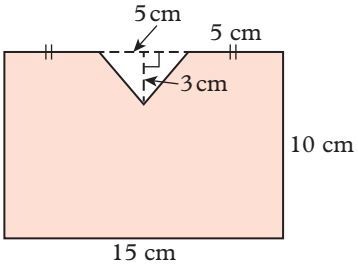
Calculate the area of this composite shape.



THINK

1. Redraw the shape with dashed lines, splitting it into individual shapes for which you can calculate the areas.
2. Label any unlabelled dimensions for each individual shape. The total length of the rectangle is 15 cm and the length either side of the triangle is 5 cm. The cut-out triangle must have a base of 5 cm (because $15 - 2 \times 5 = 5$).
3. Calculate the area of each individual shape.
4. Calculate the required area by subtracting the area of the triangle from the area of the rectangle.

WRITE



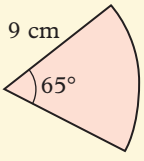
rectangle: $A = lw$
$= 10 \times 15$
$= 150 \text{ cm}^2$

triangle: $A = \frac{1}{2}bh$
$= \frac{1}{2} \times 5 \times 3$
$= 7.5 \text{ cm}^2$

total area $= 150 - 7.5$
$= 142.5 \text{ cm}^2$

Worked example 8.1C Calculating the area of a sector

Calculate the area of this sector, correct to one decimal place.



THINK

1. Identify the values of θ and r.
2. Substitute the values into the formula for the area of a sector.
3. Calculate the area and round to one decimal place, including the appropriate unit.

WRITE

$\theta = 65°$ $\quad r = 9 \text{ cm}$

$A = \frac{\theta}{360°} \times \pi r^2$
$= \frac{65°}{360°} \times \pi \times 9^2$
$= 45.945...$
$\approx 45.9 \text{ cm}^2$

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Helpful hints

- ✔ There can be more than one way to split a composite shape into individual component shapes.
- ✔ The following area formulas for simple shapes will be useful for the exercise.

  Square $= l^2$ rectangle $= lw$ triangle $= \frac{1}{2}bh$ trapezium $= \frac{1}{2}(a + b)h$ circle $= \pi r^2$

Learning pathways

1–3, 5(a–c), 6, 8–9(a, b)	1(a–f), 2–3(a, c), 4, 5(a, c, e), 7, 8(a, b), 9–11	1(g–i), 2(b, d), 3(a, c), 4, 5(b, d, f), 7, 8(c), 9–12

ANS p731

Exercise 8.1A: Understanding and fluency

For questions involving π, use the π key on your calculator.

WE 8.1A **1** Convert each length or area measurement below to the unit shown in brackets.

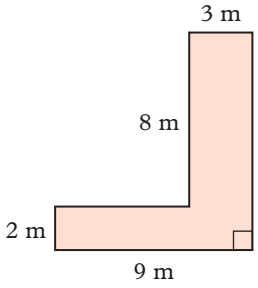
a 7 m (cm) **b** 4500 m (km) **c** 8.4 km^2 (m^2)

d 29 mm^2 (cm^2) **e** 24 000 m^2 (ha) **f** 0.88 m (mm)

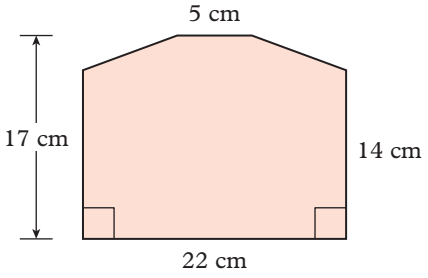
g 34 200 000 cm^2 (km^2) **h** 10.22 km (cm) **i** 3 km^2 (ha)

WE 8.1B **2** Calculate the shaded area of each of these composite shapes. Give your answers to one decimal place where necessary.

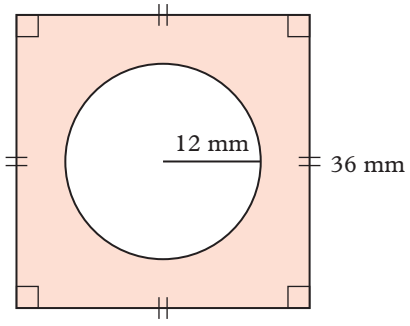
a



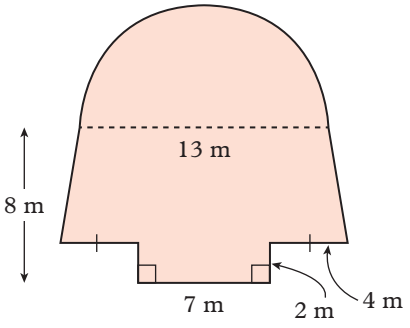
b



c

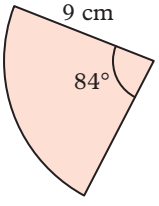


d

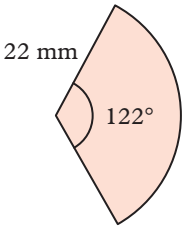


WE 8.1C **3** Calculate the area of each of these sectors, correct to one decimal place.

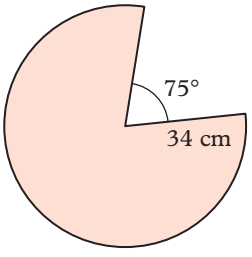
a



b



c



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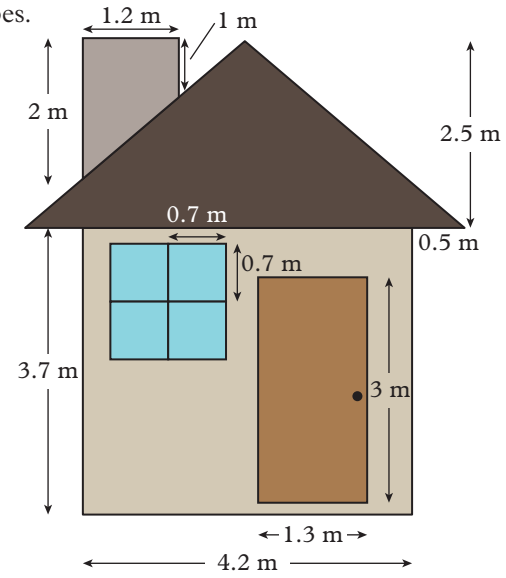
4 The front view of this house is made up of a number of basic shapes.



a Identify each shape.

b Calculate the area of each shape identified in part **a**.

c There are four window panes of equal size. Explain how you can calculate the area of all four window panes if you know the area of only one window pane.

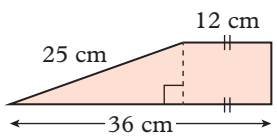
d The owners wish to paint the front of the house (including the door, roof and chimney).

i Use your answers for parts **b** and **c** to help you find the area of the house that will be painted.

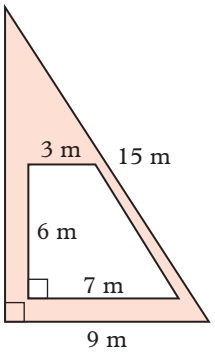
ii If 1 L of paint covers 15 m^2, how many litres of paint is needed? Give your answer to the nearest half litre.

5 Use your knowledge of Pythagoras' theorem to find any unknown lengths before calculating the shaded areas. Give your answers to one decimal place where necessary.

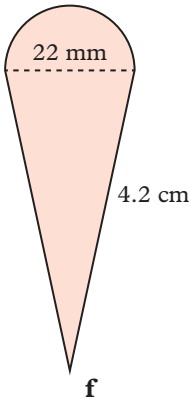
a



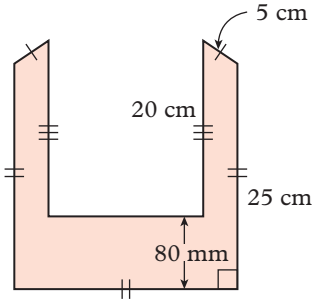
b



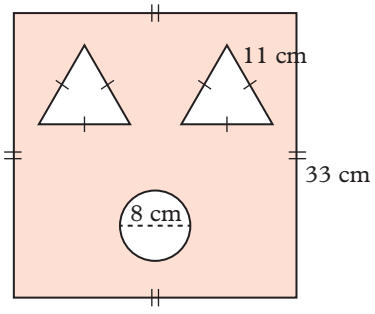
c



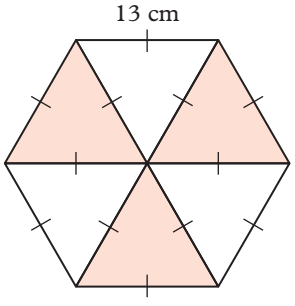
d



e



f



Exercise 8.1B: Problem solving and reasoning

6 A farm is in the shape of a trapezium. The length of the northern boundary of the farm is 265 m and the length of the southern boundary is 180 m. The perpendicular distance from the northern boundary to the southern boundary is 90 m.

a Draw a diagram of the farm and find its area.

b Write your answer to part **a** to the nearest hectare.

7 A design based on the ancient yin-yang symbol is shown on the right.

a Explain why the area of the yin-yang symbol which is shaded blue is equal to the area of the white section.

b If the radius of the full circle is 24 cm, find the area shaded blue, correct to one decimal place.

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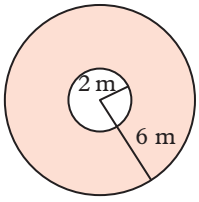
8 An annulus is the region between two circles of different sizes which share the same centre.

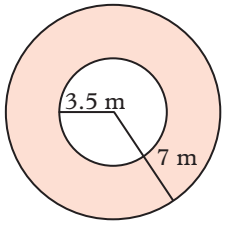
a Explain how you can find the area of the annulus below.



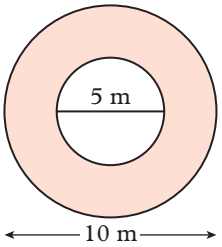
b Find the area of the annulus, correct to one decimal place.

c Find the area of the following annuli, correct to one decimal place.

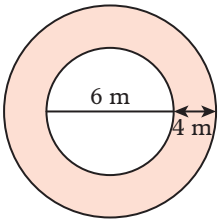
i



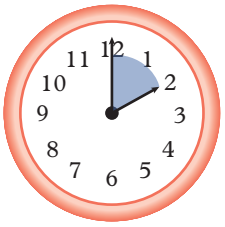
ii



iii



9 A clock has a minute hand that is 7 cm long and an hour hand that is 5 cm long.



a What is the difference in cm^2, correct to one decimal place, between the areas covered by the hands when they each complete a full revolution?

b The time shown on the clock face on the right is 2 o'clock. Calculate the angle between the minute hand and the hour hand.

c Find the difference in the areas that would be covered by the hands if they both moved clockwise from the position labeled 12 to the position labeled 2 on the clock face. Give your answer correct to one decimal place.

d Repeat part **c** if the hands both move clockwise from the position labeled 12 to the position labeled 7 on the clock face.

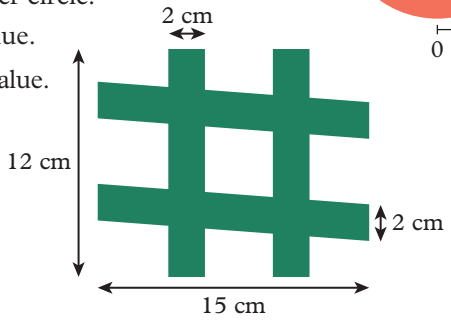
10 Consider the target on the right. The radius of the inner circle is 2 cm and the edge of each ring is 2 cm further from the centre of the inner circle.

a Find the shaded area, giving your answer as an exact value.

b Find the unshaded area, giving your answer as an exact value.

c Find the ratio of the shaded area to the unshaded area.

11 The area of a parallelogram is $A = bh$, where b is the base length and h is the perpendicular height. Find the total shaded area of this hash symbol.



Exercise 8.1C: Challenge

12 A regular hexagon can divided into six congruent equilateral triangles. Give your answers to the following questions as exact values.

a Find the height of an equilateral triangle with side lengths 10 cm.

b Find the area of an equilateral triangle with side lengths 10 cm.

c Find the area of a regular hexagon with side lengths 10 cm.

d Find the area of a regular hexagon with side lengths of a.

Online resources:

- **Interactive skillsheet** Converting between units of area
- **Interactive skillsheet** Area of composite shapes
- **Quick quiz** 8.1

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Lesson 8.2

Surface area review

Learning intentions

By the end of this lesson you will be able to ...

✔ calculate the surface area of prisms, cylinders and composite solids.

Inter-year links

Year 7	8.1 3D objects
Year 8	8.4 Area of a circle
Year 9	5.3 Surface area of prisms

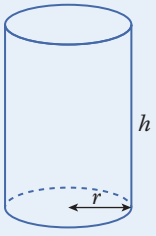
Surface area

Key content video
Surface area of prisms

Key content video
Surface area of cylinders

- The **surface area** (SA) of a 3D object is the total area of its outer surface. This is the sum of the areas of the faces (or surfaces) of that object.
- **Prisms** are 3D objects that have a uniform **cross-section** that is a polygon.
  → All faces of a prism are flat.
  → Cubes, rectangular prisms and triangular prisms are all common examples of prisms.
- **Cylinders** are 3D objects with a uniform circular cross-section and a curved face. The surface area of the curved face of a cylinder is equal to the circumference ($2\pi r$) of either end multiplied by the height (h):

  Area of curved face of a cylinder = $2\pi rh$

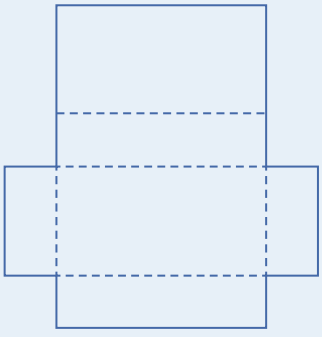


- The surface area of a cylinder (the area of the two circular ends and the curved surface) can be found using the formula:

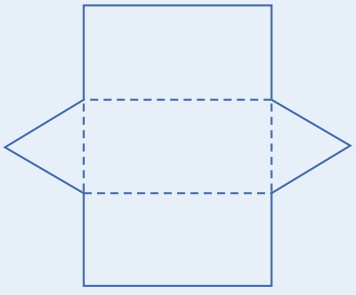
  $SA = 2\pi r^2 + 2\pi rh$

- A **net** is a 2D plan that can be folded to form a 3D object. Nets can be used to show all the faces of 3D objects and hence to help calculate the surface areas of those objects.

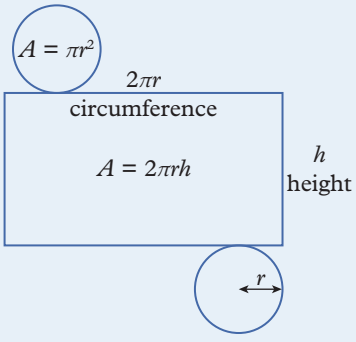
Net of a rectangular prism



Net of a triangular prism



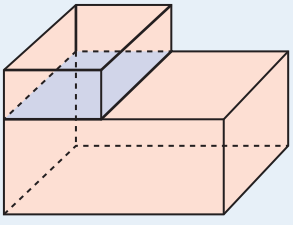
Net of a cylinder



Surface area of composite solids

Key content video
Surface area of composite objects

- A composite solid (or a composite object) consists of two or more simple solids.
- To calculate the surface area of a composite solid, identify all faces of the composite solid, and add the area of each face to find the surface area of the composite solid.
- The area where two solids join should not be included in the calculation of surface area.



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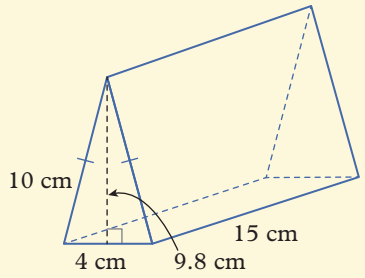
Worked example 8.2A Calculating the surface area of a prism

Calculate the surface area of this triangular prism.



THINK

1 Determine the shape of the faces of the triangular prism.

2 Calculate the area of each face. The triangular faces are isosceles triangles, meaning the two rectangles sharing common sides with them are identical.

3 Add the areas and include the appropriate unit.

WRITE

The prism has five faces: two identical rectangular faces, two identical triangular faces and a rectangular base.

rectangular faces: $A = 10 \times 15$
$= 150 \text{ cm}^2$ each

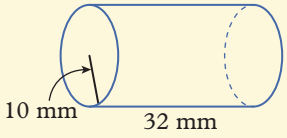
rectangular base: $A = 4 \times 15$
$= 60 \text{ cm}^2$

triangular faces: $A = \frac{1}{2} \times 4 \times 9.8$
$= 19.6 \text{ cm}^2$ each

$\text{SA} = 2 \times 150 + 60 + 2 \times 19.6$
$= 399.2 \text{ cm}^2$

Worked example 8.2B Calculating the surface area of a cylinder

Calculate the surface area of this cylinder, correct to one decimal place.



THINK

1 Identify the values of r and h for this cylinder.

2 Substitute the values you identified into the formula for the surface area of a cylinder and simplify.

3 Calculate the result using the π key on your calculator. Write the answer correct to one decimal place and include the appropriate unit.

WRITE

$r = 10 \text{ mm}$ $\quad h = 32 \text{ mm}$

$\text{SA} = 2\pi r^2 + 2\pi rh$
$= 2 \times \pi \times 10^2 + 2 \times \pi \times 10 \times 32$

$= 200\pi + 640\pi$
$= 840\pi$
$= 2638.937...$
$\approx 2638.9 \text{ mm}^2$

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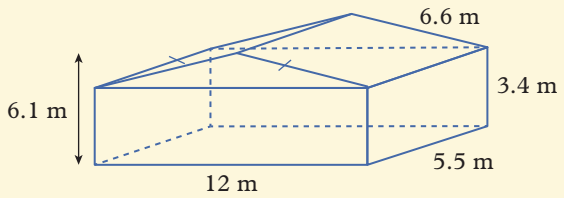
Worked example 8.2C Calculating the surface area of a composite solid

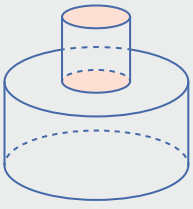
Calculate the surface area of this composite object.



THINK	WRITE
1 Determine the shape of the faces of the solid.	The solid has nine faces: three pairs of identical rectangular faces (including the pair of sloping faces), a pair of identical triangular faces and the rectangular base.
2 Calculate the area of each face.	rectangular faces (front and back): $A = 3.4 \times 12$ $= 40.8\ \text{m}^2$ each rectangular faces (sides): $A = 3.4 \times 5.5$ $= 18.7\ \text{m}^2$ each rectangular faces (sloping faces): $A = 6.6 \times 5.5$ $= 36.3\ \text{m}^2$ each rectangular base: $A = 12 \times 5.5$ $= 66\ \text{m}^2$ triangular faces: $A = \frac{1}{2} \times (6.1 - 3.4) \times 12$ $= \frac{1}{2} \times 2.7 \times 12$ $= 16.2\ \text{m}^2$ each
3 Add the areas and include the appropriate unit.	$\text{SA} = 2 \times 40.8 + 2 \times 18.7 + 2 \times 36.3 + 66 + 2 \times 16.2$ $= 81.6 + 37.4 + 72.6 + 66 + 32.4$ $= 290\ \text{m}^2$

Helpful hints

- ✔ If you can't remember the formula to calculate the surface area of a prism or cylinder, consider drawing a net of the solid and calculating the area of the shapes making up the net.
- ✔ In some instances, the area where the faces of solids join in a composite solid is the same as the area of the top face of the combined solid. Identifying this can help with your calculations.
  For example, in the diagram on the right, the surface area of the composite solid is the surface area of the bottom cylinder plus just the curved surface area of the top cylinder because its exposed circular end is the same as the joined surface calculated as part of the surface area of the bottom cylinder.



- ✔ The surface area of a prism can also be found by using the formula:
  SA = 2 × area of base + perimeter of base × height

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Learning pathways

1(1st, 2nd columns), 2–7, 9, 11(a)

1(2nd, 3rd columns), 3–9, 11, 12(a), 13, 14

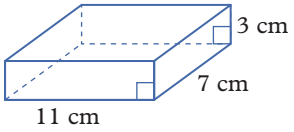
1(c, f, h, i), 2, 6–8, 10, 11–12(b), 13–15

ANS p732

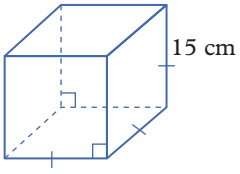
Exercise 8.2A: Understanding and fluency

WE 8.2A **1** Calculate the surface area of each of these prisms.

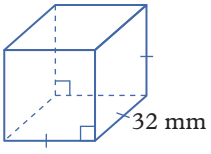
a



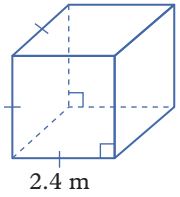
b



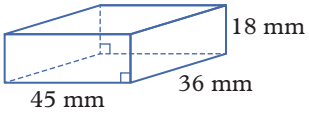
c



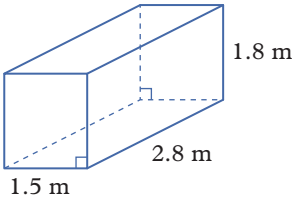
d



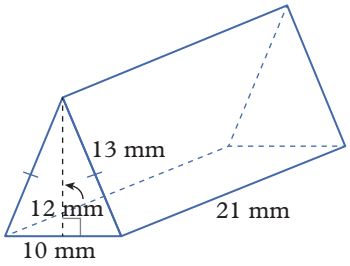
e



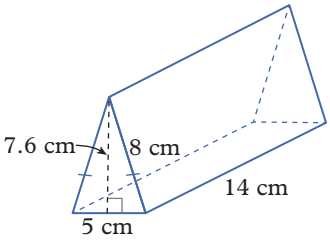
f



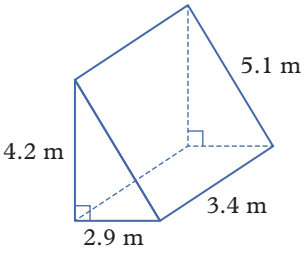
g



h

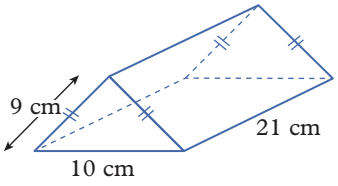


i

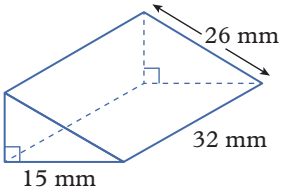


2 Calculate the surface area of each of these triangular prisms, correct to one decimal place.
Hint: Use Pythagoras' theorem.

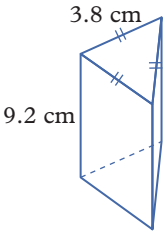
a



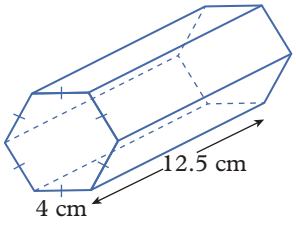
b



c

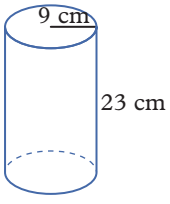


3 If the base of this hexagonal prism has an area of 41.52 cm^2, calculate the surface area.

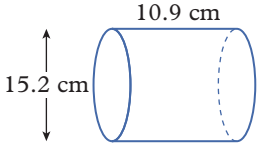


WE 8.2B **4** Calculate the surface area of each of these cylinders, correct to one decimal place.

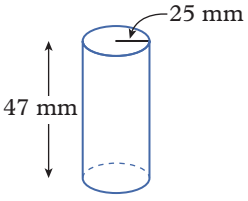
a



b



c



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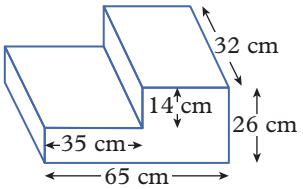
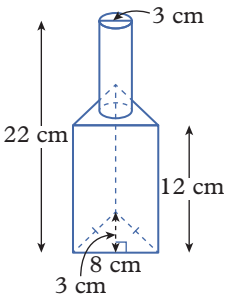
WE 8.2C **5** Calculate the surface area of each of these composite solids. Give your answers correct to one decimal place where appropriate.

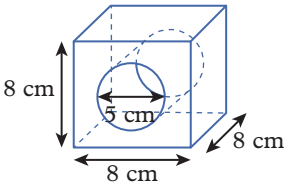
a



b



c



Exercise 8.2B: Problem solving and reasoning

6 Each die in this stack is the shape of a cube with side lengths of 15 mm.

a Calculate the surface area of one die.

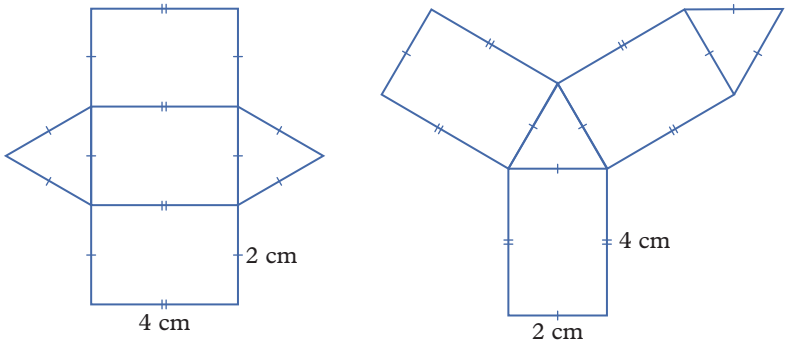
b Use your answer for part **a** to calculate the surface area of nine separate dice.

c When stacked, the nine dice form a rectangular prism. Using the information you have been given about each die, draw the rectangular prism in your books and label its dimensions.

d Calculate the surface area of the rectangular prism made of nine dice.

e Find the difference between your answers to parts **b** and **d**. Explain why these answers differ. Support your explanation with calculations.

7 The net of a 3D object is not unique. Here are two nets of the same triangular prism.



a Draw the triangular prism corresponding to the two nets above.

b Draw a third net of the triangular prism, which is different to the two given above.

8 An open pipe has a length of 33 cm and a diameter of 55 mm.

a Calculate the surface area of the outer surface of the pipe, correct to the nearest cm^2.

b To find your answer for part **a**, did you need to use the entire formula for the surface area of a cylinder: $SA = 2\pi rh + 2\pi r^2$? Explain.

9 Find the surface area of the outside of an open cylinder that has a radius of 4.5 cm and a height of 9.9 cm. Give your answer correct to one decimal place.

10 The teachers at a kindergarten want to add two cylindrical poles to the playground. One pole is to be painted yellow and the other red. The vertical poles have the same diameter of 125 mm, but they vary in height.

a The yellow pole is 1.8 m tall. Calculate the area needing to be painted yellow, correct to one decimal place. Assume that the top end and the side of the pole needs painting but the end in the ground does not.

b The red pole is double the height of the yellow pole. Compared to the amount of yellow paint needed, do the teachers need to purchase double the amount of red paint in order to paint this second pole? Why or why not?

c Calculate the area of the pole needing to be painted red, in square centimetres, correct to one decimal place. Does your result support your answer to part **b**?

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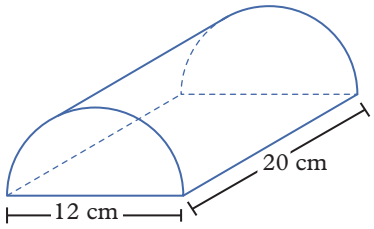
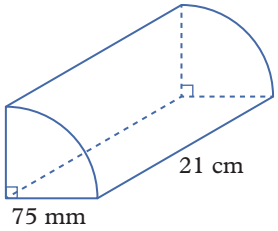
11 Find the surface area of the following cut cylinders. Give your answers in cm^2, correct to one decimal place.

a

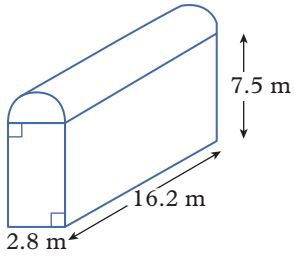


b

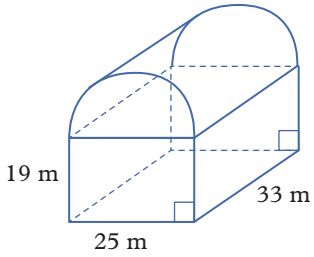


12 Calculate the surface area of each of these composite solids, correct to one decimal place.

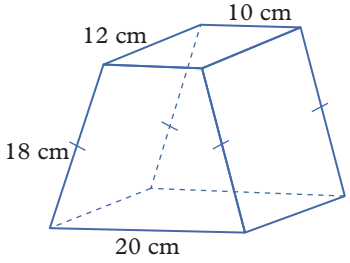
a



b

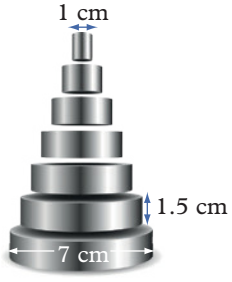


13 Calculate the surface area of this prism.



Exercise 8.2C: Challenge

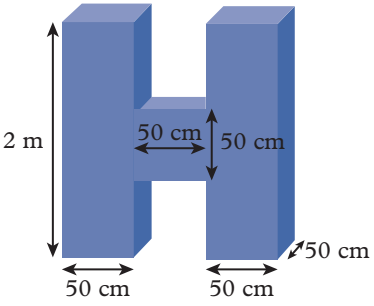
14 Seven cylinders with the same height but different diameters are stacked together as shown on the right. The top cylinder has a diameter of 1 cm and the diameter of each cylinder beneath it increases by 1 cm each time. The bottom cylinder has a diameter of 7 cm. If each cylinder is the same height, 1.5 cm, calculate the exposed surface area of the structure when the cylinders are sitting one on top of the other, as shown. Give your answer correct to one decimal place.



15 A company commissions an artist to make a giant version of their logo, an H, for the foyer of their office building. The specifications are in the diagram on the right. The H will be made of wood and then every face of the H will be painted, except the two square faces on the ground.

a Calculate the surface area of the H, not including the two rectangles touching the ground. Answer in square metres.

b The H will be painted blue, as that is the company's colour. The company plans to use a premium blue paint which is sold in 2 L cans. Each litre of this paint is enough to paint one coat on 4 square metres. How many cans of paint must they buy if they paint the H with a triple coat?



Online resources:

- **Interactive skillsheet** Surface area of prisms
- **Interactive skillsheet** Surface area of cylinders
- **Investigation** Net or not?
- **Quick quiz** 8.2

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Lesson 8.3

Volume review

Learning intentions

By the end of this lesson you will be able to ...

✔ calculate the volume of prisms, cylinders and composite solids.

Inter-year links

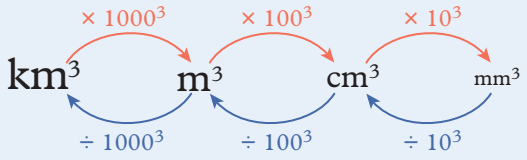
Year 7	9.6 Volume of right prisms
Year 8	8.5 Volume and capacity
Year 9	5.5 Volume of prisms and cylinders

Volume

Key content video: Volume and capacity

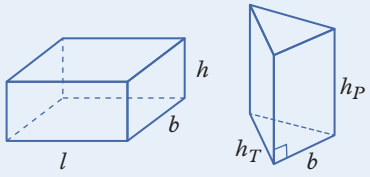
Key content video: Volume of cylinders

- **Volume** is measured in cubic units, such as mm^3, cm^3 and m^3.
- A unit of volume can be converted to another unit of volume by multiplying or dividing by the appropriate conversion factor.

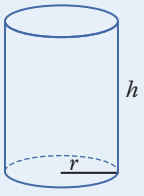


- The volume of any prism can be found using the formula $V = Ah$, where A is the area of the base and h is the height of the prism.
  - → The height is perpendicular to the base.
  - → When dealing with a triangular prism, subscripts can be used to represent the two heights. For example, in the diagram on the right, h_T represents the height of the triangular base while h_P represents the height of the prism.
- The volume of a cylinder can be found by multiplying the area of the base (πr^2) by the height (h).

  $V = \pi r^2 h$



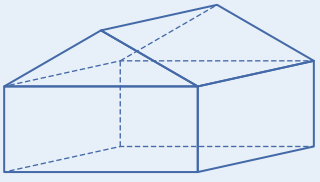
$V = lbh$ $\qquad$ $V = \frac{1}{2}bh_Th_P$



Volume of composite solids

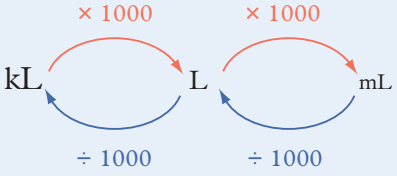
Key content video: Volume of composite solids

- The volume of a composite solid can be found by calculating the volumes of the individual component solids and adding them together. For example, the composite solid on the right consists of a triangular prism and a rectangular prism.



Capacity

- The **capacity** of a 3D object is a measure of how much the object can hold. A container with an inside volume of:
  - → 1 cm^3 holds 1 mL of liquid
  - → 1000 cm^3 holds 1 L of liquid
  - → 1 m^3 holds 1 kL of liquid.



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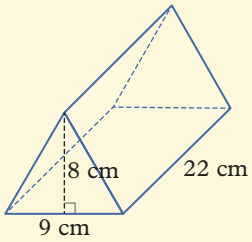
Worked example 8.3A Calculating the volume of a prism

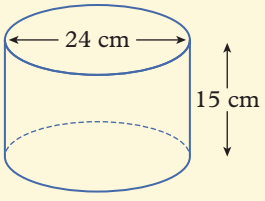
Calculate the volume of this triangular prism.



THINK	WRITE
1 Identify the base of the triangular prism and the perpendicular height.	The base is a triangle with base length 9 cm and height 8 cm. The height (h) of the prism is 22 cm.
2 Calculate the area of the base.	$A = \frac{1}{2}bh$ $= \frac{1}{2} \times 9 \times 8$ $= 36 \text{ cm}^2$ $h = 22 \text{ cm}$
3 Substitute the value of the area of the base and the perpendicular height into the formula for the volume of a prism.	$V = bh$ $= 36 \times 22$
4 Include the appropriate unit in your answer.	$= 792 \text{ cm}^3$

Worked example 8.3B Calculating the volume of a cylinder

Calculate the volume of this cylinder. Give your answer to one decimal place.



THINK	WRITE
1 Calculate the radius of the circular base.	$r = D \div 2$ $= 24 \div 2$ $= 12 \text{ cm}$ $h = 15 \text{ cm}$
2 Substitute the values of r and h into the formula to find the volume of the cylinder.	$V = \pi r^2 h$ $= \pi \times 12^2 \times 15$ $= 2160\pi$
3 Calculate using the π key on your calculator, round to one decimal place, and include the appropriate unit in your answer.	$= 6785.840...$ $\approx 6785.8 \text{ cm}^3$

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Helpful hints

✔ If the dimensions of a 3D object are given using different units, convert all lengths to the required unit specified in the question before you do anything else.

Learning pathways

1, 2(a–h), 3, 4, 5(a, b), 6(a–d), 7, 8, 9(b), 12, 15	1(b, d, f, h, j), 2(d–j), 3(c–e), 5(c), 6(b–e), 9(a, b), 10, 11, 17	1(g–j), 2(i–l), 6(d–f), 9(b, c), 13, 14, 16, 18–20

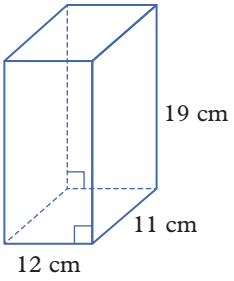
ANS p732

Exercise 8.3A: Understanding and fluency

1 Convert each of these volume measurements to the unit shown in brackets.

a 12.5 m^3 (cm^3)
b 240 000 mm^3 (cm^3)
c 34 200 000 cm^3 (m^3)
d 0.55 m^3 (mm^3)
e 67 200 mm^3 (m^3)
f 0.000 009 m^3 (mm^3)
g 7520 mm^3 (cm^3)
h 8.74 m^3 (cm^3)
i 142 900 cm^3 (m^3)
j 73 000 000 mm^3 (m^3)

WE 8.3A **2** Calculate the volume of each of these prisms.

a


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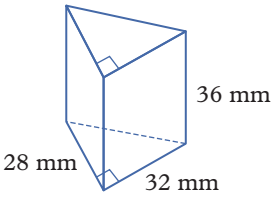

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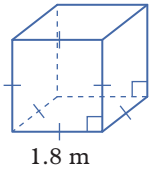

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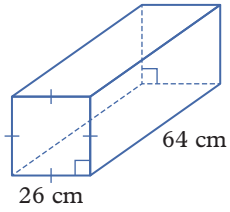

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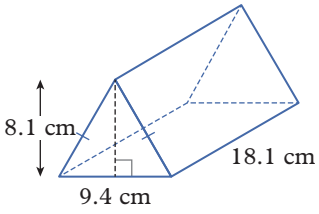

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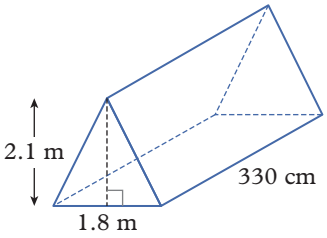

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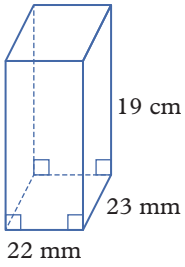

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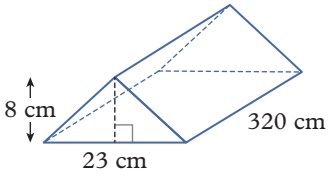
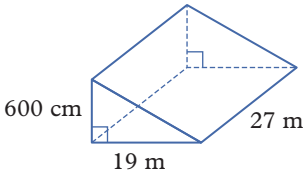


i


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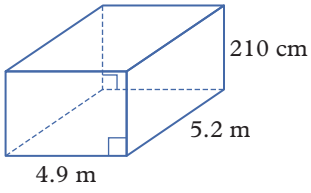

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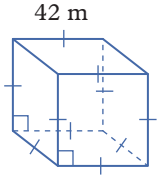

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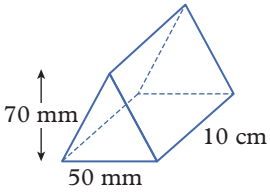

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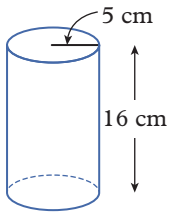
WE 8.3B **3** Calculate the volume of each of these cylinders, giving your answers correct to one decimal place. Where mixed units are provided, give your answers in cm^3.

a



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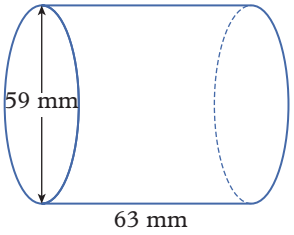

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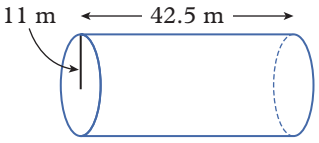

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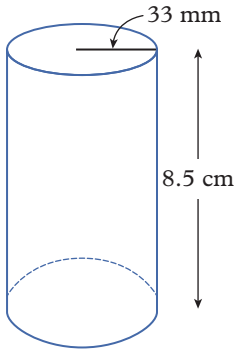

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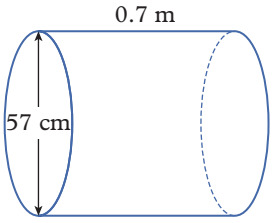

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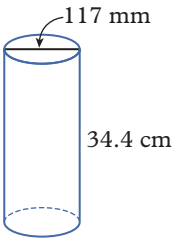


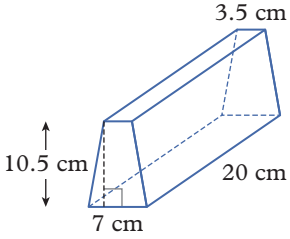
4 Paula is making a cube-shaped cushion with side lengths of 0.4 m.

a What is the internal volume of the cushion?

b Convert your answer to part **a** to cubic centimetres.

5 Calculate the volume of each of these prisms.

a



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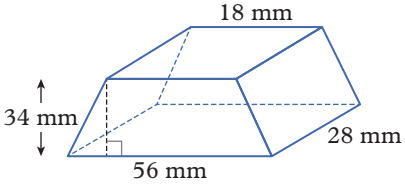

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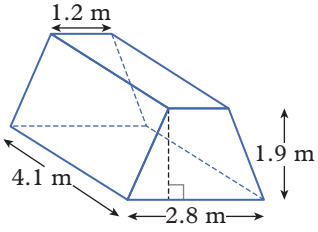
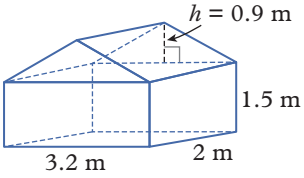


6 Calculate the volume of each of these composite solids. Where appropriate, give your answers correct to one decimal place.

Hint: Identify the individual solids first.

a



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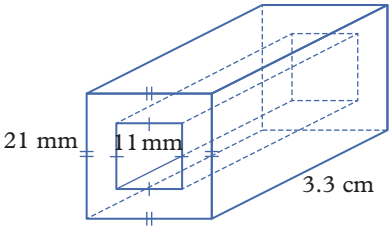

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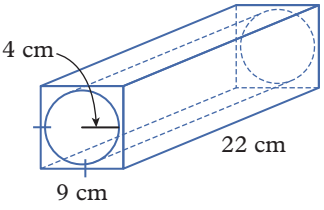

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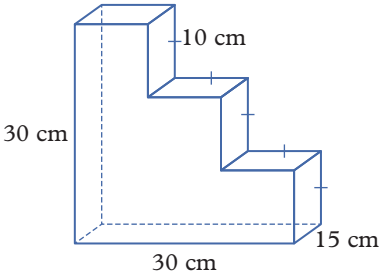

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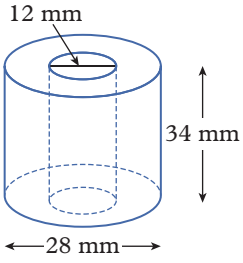

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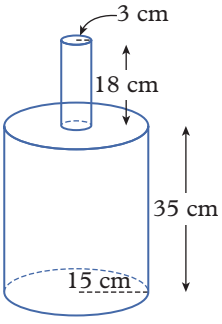

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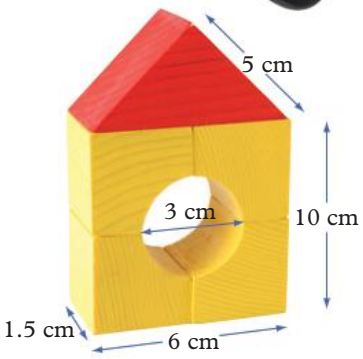
Exercise 8.3B: Problem solving and reasoning

7 A round hatbox has a diameter of 23 cm and a height of 12.5 cm. Calculate its volume, correct to one decimal place.



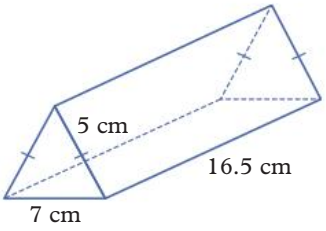
8 In the toy house on the right, constructed from different shaped blocks, three common solids can be identified.

a Identify the three solids.

b Find the volume of wood used to make this toy house, correct to the nearest cm^3.

9 Calculate the volume of each of these prisms. Give your answers correct to one decimal place where appropriate.

a



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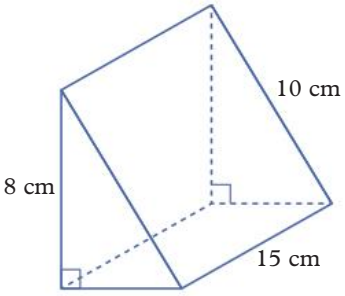

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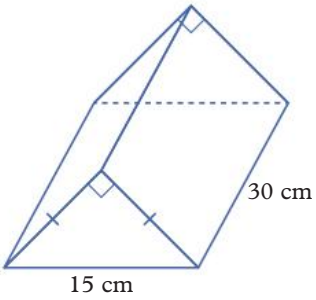
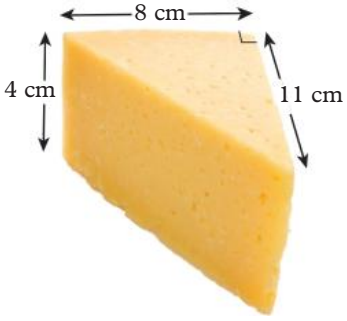


10 A piece of cheese is in the shape of a right-angled triangular prism.



a Calculate its volume.

b If 1 cm^3 of cheese has a mass of 0.001 kg, find the mass of the cheese, in kilograms, correct to four decimal places.

c The price of cheese is \$21 per kilogram. How much would the cheese on the right cost? Give your answer correct to the nearest cent.

11 A rectangular fish tank is 45 cm long, 20 cm wide and 50 cm tall.

a Calculate the volume of the fish tank.

b The fish tank should only be filled to a level 6 cm from the top. How many litres of water will be required to fill the tank to that height?

c A rock with a volume of 360 cm^3 is placed in the fish tank. By how much will the water level rise?

12 The volume of a particular prism is 2744 cm^3. If the prism is a cube, what is its side length?

13 The formula for the volume of a cylinder contains variables for its radius and height.

a If a cylinder has a volume of 1649.34 cm^3 and a radius of 5 cm, what is the height of the cylinder, to the nearest centimetre?

b If a cylinder has a volume of 207 cm^3 and a height of 5.7 cm, find the radius of its circular base, to the nearest millimetre.

14 Jack installs two cylindrical water tanks. The first tank is 2 m tall and the diameter of its base is 1.5 m. The second tank is 1.5 m tall and the diameter of its base is 2 m.

a Which tank has the larger capacity?

b What is the difference in the capacity of the two tanks? Give your answer to the nearest litre.

15 Consider a cylinder with a radius of r and a height of h.

a Calculate the volume of the cylinder.

b If the radius of the original cylinder is halved but the height remains the same, describe how the volume of the new cylinder compares to the original cylinder.

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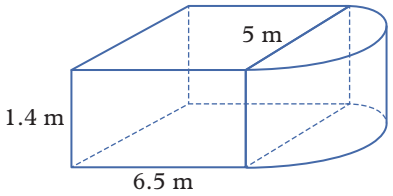
c If the radius of the original cylinder is doubled but the height remains the same, describe how the volume of the new cylinder compares to the original cylinder.

d If the radius of the original cylinder remains the same but the height is halved, describe how the volume of the new cylinder compares to the original cylinder.

e If the radius of the original cylinder remains the same but the height is doubled, describe how the volume of the new cylinder compares to the original cylinder.

16 A swimming pool is in the shape of a rectangular prism. A semicircular spa is built onto one end of the pool, as shown in the diagram on the right.

To make the pool and spa functional, the water levels must be below the top edge. If the pool and spa are filled to 70% of their depth, how much water is required, to the nearest litre?

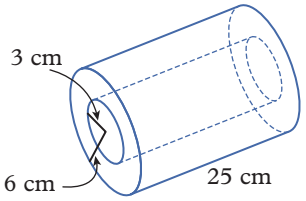


17 The plastic pipe shown on the right is hollow.

a What shape is the base of the pipe? Calculate its area correct to one decimal place.

b Use the area of the base to calculate the volume of plastic used for this pipe.

c Can you think of a different method for finding the volume? Explain your answer.



18 The roof of a house is in the shape of a triangular prism, so that all the rainfall that falls on the roof is collected in the gutters that surround the roof and filter into a water tank.

a The base of the roof is the shape of a rectangle with a length of 12.36 m and a width of 9.48 m. What is the area of the base of the roof?

b 15 mm of rain falls on the roof over a 24-hour period, with all the rain draining into the water tank. How much water is collected, correct to the nearest litre?

c The water tank is a cylinder with an internal radius of 75 cm. By how much does the level of the water in the tank rise over this 24-hour period, if no water from the tank is used? Give your answer in cm, correct to one decimal place.

Exercise 8.3C: Challenge

19 Nicko wishes to make an open box from a 30 cm by 30 cm piece of red cardboard. He decides to cut out a square of length 2 cm from each corner and fold up the sides to form the box. The edges of the box will be joined by sticky tape, so no tabs are needed to create the box.

a Draw a diagram showing the piece of cardboard with the corners cut out. Label the dimensions.

b Determine the length, width and height of the box once Nicko has folded up the sides.

c Calculate the volume of the box.

d Would the volume of the box increase or decrease if he cut out 3 cm square corners from the piece of cardboard before forming the rectangular prism? Explain your answer.

20 Consider these two nuts. One is circular and the other is in the shape of a regular hexagon. The inner diameter of each nut is the same, they have the same height, and both have a depth of 6.2 mm and a uniform cross-section.

a How much steel is required to manufacture each nut? Give your answers to one decimal place.

b Which nut requires more steel to make?

Online resources:

- **Interactive skillsheet** Volume of prisms
- **Interactive skillsheet** Volume of cylinders
- **Worksheet** Surface area and volume of prisms and cylinders
- **Quick quiz** 8.3

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Lesson 8.4

Compounding errors

Learning intentions

By the end of this lesson, you will be able to ...

✔ understand how repetitively using approximations in calculations can lead to compounding errors.

Inter-year links

Year 9 5.6 Errors in measurement

Errors in measurement

Key content video
Errors in measurement

- An **error** is the difference between the exact value and the estimated value of a quantity.

  error = exact value − estimated value
- The absolute value of a number is its distance from zero. Absolute values are always non-negative real numbers.

  → Straight line brackets are used to indicate an absolute value. For example, $|-3.4| = 3.4$.
- An **absolute error** is the absolute value of an error.

  absolute error = |exact value − estimated value|
- A **relative error** is the ratio between the absolute error and the exact value of a quantity.
- A **percentage error** is the absolute error as a percentage of the exact value of a quantity.

$$\text{relative error} = \frac{\text{absolute error}}{\text{exact value}} \qquad \text{percentage error} = \frac{\text{absolute error}}{\text{exact value}} \times 100\%$$

- All measurements are estimates of exact values. The accuracy of a measurement depends on the instrument being used.
- Exact values should be used in calculations wherever possible.
- Using approximated values repetitively in calculations leads to compounding errors, where the initial error is compounded in further calculations.

Worked example 8.4A Errors in measurement

Jonny buys mushrooms at the supermarket. The cashier's weighing scale measures the mushrooms to be 480 grams.

Jonny goes home and uses his own weighing scale, which measures the mushrooms to be exactly 478.81 grams. Correct to three significant figures, calculate the relative error in the cashier's measurement.

THINK	WRITE
1 If the actual exact weight is 478.81 grams, the cashier's measurement of 480 grams is the estimated value.	exact value = 478.81 grams estimated value = 480 grams
2 The absolute error is the absolute value of the difference between the exact and estimated values.	absolute error = $\lvert 478.81 - 480 \rvert$ = 1.19 grams
3 The relative error is found by dividing the absolute error by the exact value.	relative error = $\frac{1.19}{478.81}$ $\approx 0.002\,49$

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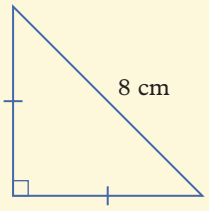
Worked example 8.4B Compounding errors



The hypotenuse of an isosceles right-angled triangle is 8 cm.

a Find the exact length of the two shorter sides.
b Find the length of the two shorter sides, correct to one decimal place.
c Calculate the percentage error of the approximate side lengths found in part **b**.
d Find the exact area of this triangle.
e Calculate the area of this triangle using your result from part **b**.
f Calculate the percentage error of the approximate area, correct to three decimal places.
g Explain why the percentage error from part **f** is larger than the percentage error from part **c**.

THINK

a In an isosceles right-angled triangle, the two shorter sides are equal in length. Pythagoras' theorem can be used to find unknown side lengths in right-angled triangles.

b Convert the result in part **a** to a decimal value, correct to one decimal place.

c The exact value is the result from part **a** and the estimated value is the result from part **b**. The absolute error is the difference between the two values. Calculate the percentage error by dividing the absolute error by the exact value and multiplying by 100%.

d Both base and height are equal to the side length found in part **a**. Substitute the values into the formula Area $= \frac{1}{2} \times$ base $\times$ height.

e Use the approximate result found in part **b** to calculate the area.

f 1 Find the absolute error as the difference between the exact area found in part **d** and the approximate area found in part **e**.

2 Calculate the percentage error by dividing the absolute error by the exact value and multiplying by 100%.

g Look at how the two estimated values were calculated.

WRITE

a Suppose the length of its shorter side is x cm.

$$x^2 + x^2 = 8^2$$
$$2x^2 = 64$$
$$x^2 = 32$$
$$x = \sqrt{32}$$
$$x = 4\sqrt{2}$$

The exact length of the two shorter sides is $4\sqrt{2}$ cm.

b $4\sqrt{2}\text{ cm} \approx 5.7\text{ cm}$

c absolute error $= |\text{exact value} - \text{estimated value}|$
$= |4\sqrt{2} - 5.7|$

percentage error $= \dfrac{\text{absolute error}}{\text{exact value}} \times 100\%$
$= \dfrac{|4\sqrt{2} - 5.7|}{4\sqrt{2}} \times 100\%$
$\approx 0.76\%$

d Area $= \frac{1}{2} \times 4\sqrt{2} \times 4\sqrt{2}$
$= 16\text{ cm}^2$

e Area $= \frac{1}{2} \times 5.7 \times 5.7$
$= 16.245\text{ cm}^2$

f Absolute error $= |16 - 16.245|$

Percentage error $= \dfrac{|16 - 16.245|}{16} \times 100\%$
$\approx 1.531\%$

g The approximate value for the side length is used twice in the calculation for the area, so the error for this value is compounded in the area calculation, making the error in the area calculation larger than in the side length calculation.

Helpful hints

✔ Make sure quantities are expressed in the same unit before calculating errors.

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Learning pathways

1–4, 6, 9 | 1, 3, 5–8, 10 | 3, 5–8, 11

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Exercise 8.4A: Understanding and fluency

WE 8.4A **1** A carpenter measures the length of a table using a long ruler. The measurement from the side of the ruler that uses the metric system is 120 cm, and the measurement using the imperial system is 4 feet. She finds that 120 cm converts to 3.937 feet, and 4 feet converts to 121.92 cm.

- **a** If the actual length of the table is 120 cm, what is the absolute error in her measurement using the imperial system?
- **b** If the actual length of the table is 4 feet, what is the relative error in her measurement using the metric system? Give your answer to four significant figures.

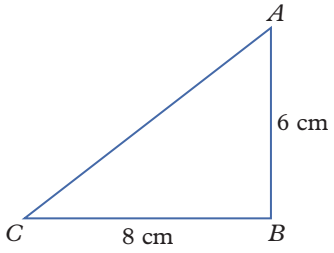
2 A chef weighs some butter using a scale. The measurement from the scale using the metric system is 185 g, and the measurement using the imperial system is 6.5 oz (ounces). He finds that 185 g converts to 6.5257 oz, and 6.5 oz converts to 184.27 g.

- **a** If the actual mass of the butter is 185 g, what is the percentage error in the measurement using the imperial system? Give your answer to four decimal places.
- **b** If the actual mass of the butter is 6.5 oz, what is the percentage error in the measurement using the metric system? Give your answer to four decimal places.

WE 8.4B **3** The circumference of a circle is 40 cm.

- **a** Determine the exact radius of the circle.
- **b** Find the approximate radius of the circle correct to one decimal place.
- **c** Determine the exact area of this circle.
- **d** Calculate the area of this circle using your result from part **b**.
- **e** Find the percentage error of the approximate area, correct to two decimal places.

Exercise 8.4B: Problem solving and reasoning



4 In the right-angled triangle ABC, $\angle ABC = 90°$, $AB = 6$ cm and $BC = 8$ cm.

- **a** Using a protractor, measure the size of $\angle ACB$.
- **b** Find the exact value of $\angle ACB$ using trigonometry.
- **c** Find the percentage error for your measurement for $\angle ACB$. Give your answer to two decimal places.

5 A cylinder has a radius of 2 cm and a height of 4 cm.

- **a** Calculate the area of the base of the cylinder, correct to one decimal place.
- **b** Calculate the percentage error for the base area calculated in part **a** compared to the exact base area.
- **c** Calculate the curved surface area of a cylinder, correct to one decimal place.
- **d** Determine the percentage error for the curved surface area calculated in part **c** compared to the exact area. Give your answer to four decimal places.
- **e** Using the result from parts **a** and **c**, find the surface area of the cylinder, correct to one decimal place.
- **f** Determine the percentage error of the surface area calculated in part **e** compared to the exact surface area. Give your answer to four decimal places.

6 Sally measures the diameter of a round coaster using a ruler as 8.4 cm.

- **a** Find the area of the coaster, correct to two decimal places.
- **b** Sally measures the coaster using a vernier calliper and it reads 8.352 cm. Determine the percentage error of the diameter measured using the ruler compared to the more accurate vernier calliper, correct to four decimal places.

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c What is the percentage error of the area calculated in part **a**, correct to four decimal places? Explain why the percentage error is greater for the area than for the diameter.

7 Pre-election polls are published with a stated margin of error. For example, before an election you may read something along the lines of, 'Labor has a lead over the Coalition of 45% to 43% with a 3% margin of error.'

a Why do you think the margin of error is stated when publishing a pre-election poll?

b What could be done to decrease the margin of error in a pre-election poll?

8 James rides his bike from home to his friend Karen's house which is 5 km away. It takes James 15 minutes to complete the journey. If Karen rides from her house to James' place, it will take 25 minutes.

a Find their speeds in km/h.

b Convert the speeds found in part **a** to m/s, giving your answers correct to one decimal place.

c If James and Karen both leave their house at the same time and ride towards each other, how long will it take for them to meet using:

i the speeds calculated in part **a**

ii the speeds calculated in part **b**.

Give your answer in minutes, correct to three decimal places.

d Calculate the percentage error of the time found in part **c ii**.

9 Jemima plans to make 25 gift bags for her upcoming birthday party. She can make them in 45 minutes. If her sister makes the gift bags instead, they can be made in 30 minutes.

a Find, correct to one decimal place, the number of gift bags Jemima and her sister can each make in one minute?

b If they work together, use the results from part **a** to determine approximately how long it will take to make all the gift bags. Give your answer in minutes, correct to one decimal place.

c If they work together, find the exact time taken to make all the gift bags.

d Calculate the absolute error of your answer in part **b**.

10 A homebuyer took out a loan from the bank. The monthly repayment is \$1317.172 242 per month for 30 years. The bank system only allows transaction amounts to be accurate to the nearest cent.

a Find the absolute error for each monthly payment.

b Find the absolute error for the total loan repayment, correct to the nearest cent.

Exercise 8.4C: Challenge

11 A supermarket sells packs of almonds priced at \$19.95. The weekly promotion marks the packs at 25% off. The cashier only accepts cash payment correct to the nearest 5 cents.

a Find the money paid by a customer purchasing one pack of almonds during this promotion. Is this an overpayment or underpayment?

b Find the money paid by a customer purchasing two, three, and four packs of almonds during the promotion and calculate the error of each of these cash payments by completing the following table.

Number of packs	Exact price (\$)	Payment amount (\$)	Error (\$)
1			
2			
3			
4			

c 52 customers visited the shop during the promotion. 24 of the customers bought one pack of almonds each, 16 customers bought two packs each, 7 customers bought three packs each and the rest bought four packs each. Determine the total error from the sale of almonds.

Online resources:

Interactive skillsheet
Measurement errors

Quick quiz
8.4

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Checkpoint

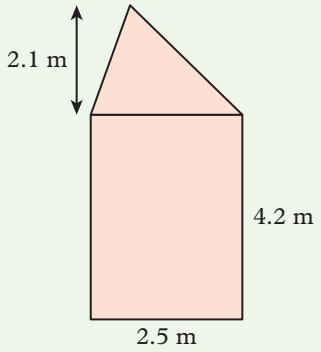
Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this module.

8.1 **1** Convert each length and area measurement to the unit shown in brackets.

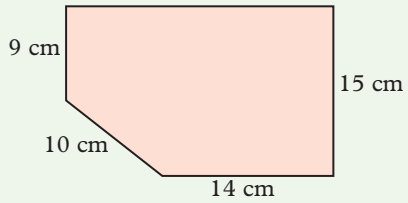
a 45 mm (cm) **b** 400 cm (m) **c** 4.53 m (mm)
d 4000 mm^2 (cm^2) **e** 5.2 km^2 (m^2) **f** 68 000 cm^2 (m^2)

8.1 **2** Find the area of the following composite shapes.

a

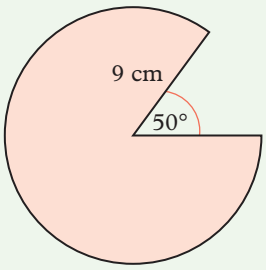


b

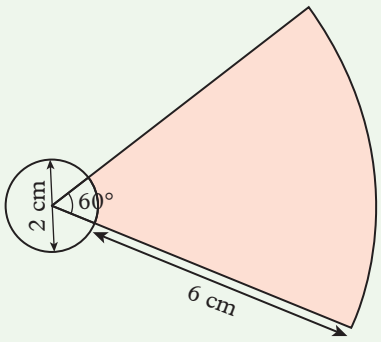


8.1 **3** Find the area of the following composite shapes. Give your answers correct to the nearest whole number.

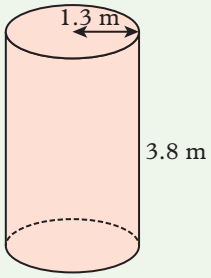
a



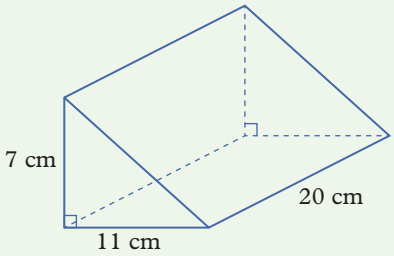
b



8.2 **4** Find the surface area of the following cylinder, correct to one decimal place.



8.2 **5** Use Pythagoras' theorem to help find the surface area of the following prism, correct to one decimal place.



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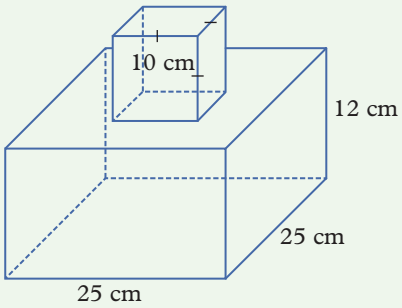
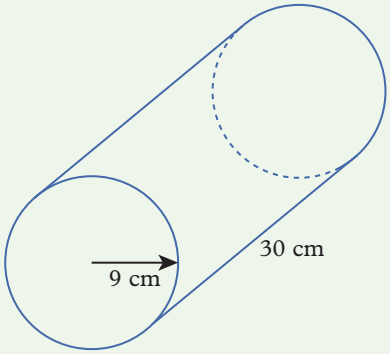
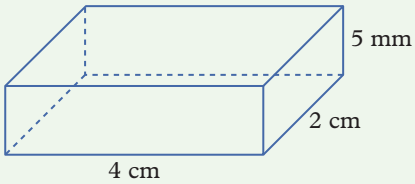
8.2 **6** A cube with side lengths of 10 cm sits on top of a square-based rectangular prism, as shown. Find the surface area of the composite solid.



8.3 **7** Calculate the volume of the following cylinder, correct to one decimal place.



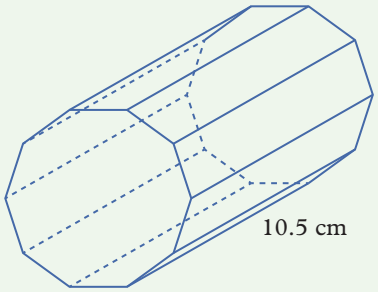
8.3 **8** Find the volume of this rectangular prism in:



a mm^3

b cm^3.

8.3 **9** The following prism has a cross-sectional area of 123.11 cm^2. Find the volume of the prism.



8.4 **10** A bar of chocolate is labelled as 150 g on the packaging. After placing the chocolate on a scale, the actual mass is found to be 156 g.

a Calculate the relative error of the labelled mass, correct to four significant figures.

b Calculate the percentage error of the labelled mass, correct to four significant figures.

8.4 **11** The radius of Earth at sea level is approximately 6378.137 km. A formula sheet for students states that the radius of Earth is 6.37×10^6 m.

a Calculate the absolute error of the radius of Earth given in the formula sheet in metres.

b Calculate the percentage error, correct to four decimal places.

c Discuss whether the error is significant or not.

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Lesson 8.5

Logarithmic scales

Learning intentions

By the end of this lesson you will be able to ...

✔ understand and use logarithmic scales.

Inter-year links

Year 8	1.1 Rounding and estimating
Year 9	1.5 Scientific notation

Scientific notation and orders of magnitude

Key content video
Scientific notation

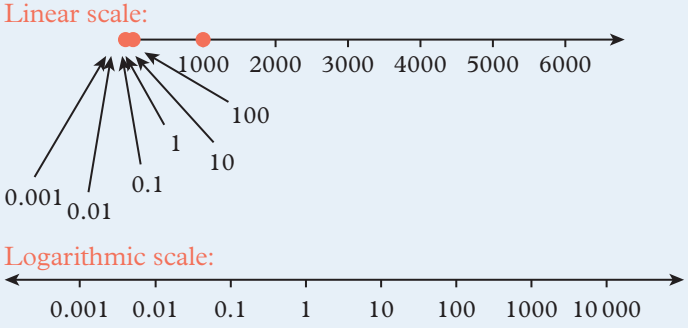
- **Base-ten** refers to the number system we use, where each digit in a number is from 0 to 9.
- **Scientific notation** is a simple way of writing very large or very small numbers in base-ten. A number written in scientific notation is in the form $a \times 10^m$, where $1 \le a < 10$ or $-10 < a \le -1$, and m is an integer.
- The **order of magnitude** of a number in base-ten is equal to the value of m when the number is written in scientific notation.

Base-ten number	0.001	0.01	0.1	1	10	100	1000
Scientific notation	1×10^{-3}	1×10^{-2}	1×10^{-1}	1×10^{0}	1×10^{1}	1×10^{2}	1×10^{3}
Order of magnitude	−3	−2	−1	0	1	2	3

Logarithmic scales

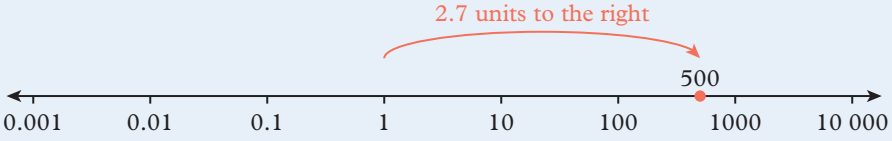
Key content video
Logarithmic scales

- **Logarithmic scales** can be used to display numerical data, which covers a wide range of values of differing orders of magnitude.
- A logarithmic scale is non-linear, which means the distance between consecutive numbers on the scale is not equal. Instead, in a logarithmic scale the distance between the orders of magnitude of the numbers is equal.
- To determine the position of a value on a logarithmic scale, the logarithm of the value should be determined using a calculator.



Use 1 as the reference point and move the specified number of units right or left on the scale.

For example, to find the position of 500 on the above scale, $\log_{10}(500) \approx 2.7$, so 500 is located 2.7 units to the right of 1 on the scale.



- If the range of values in a numerical data set are all within one order of magnitude of each other, it is more appropriate to use a linear scale than a logarithmic scale.
- Logarithmic scales should not be used if any of the quantities do not have a real-valued logarithm. Since the logarithm of zero is undefined and the logarithm of a negative value is not a real number, they cannot be displayed on a logarithmic scale.

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Worked example 8.5A Determining the order of magnitude of base-ten numbers

Determine the order of magnitude of the following base-ten numbers.

a 6.29×10^8 **b** 0.001 24

THINK

a For numbers written in scientific notation, the order of magnitude is the index of 10 in the scientific notation.

b **1** Write the number in scientific notation.

2 The order of magnitude is the index of 10 in the scientific notation.

WRITE

a The order of magnitude is 8.

b $0.001\,24 = 1.24 \times 10^{-3}$

The order of magnitude is -3.

Worked example 8.5B Determining the position of values on a logarithmic scale

Determine the position of the following values on a base-ten logarithmic scale, relative to 1.

a 1 000 000 **b** 0.0423 (correct to two decimal places)

THINK

a **1** Evaluate the base-ten logarithm of the number.

2 If the result if positive, the value lies this many units to the right of 1. If the result is negative, the value lies this many units to the left of 1.

b **1** Evaluate the base-ten logarithm of the number to two decimal places.

2 If the result if positive, the value lies this many units to the right of 1. If the result is negative, the value lies this many units to the left of 1.

WRITE

a $\log_{10}(1\,000\,000) = 6$

1 000 000 lies 6 units to the right of 1.

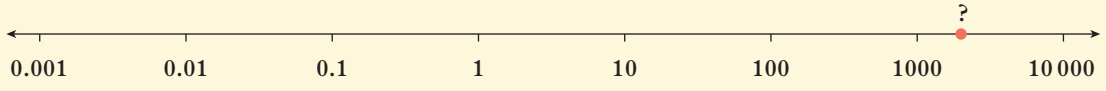
b $\log_{10}(0.0423) = -1.373...$

≈ -1.37

0.0423 lies 1.37 units to the left of 1.

Worked example 8.5C Determining values on a logarithmic scale

Determine the value of the quantity which is 3.3 units to the right of 1 on a base-ten logarithmic scale. Give your answer correct to the nearest whole number.



THINK

1 Evaluate 10 to the power of the given (logarithmic) value.

2 Round your answer to the nearest whole number.

WRITE

$10^{3.3} = 1995.2...$

≈ 1995

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Helpful hints

- ✔ An increase of 1 unit on a base-ten logarithmic scale corresponds to a quantity that is 1 order of magnitude ($10^1 = 10$ times) greater. Therefore, an increase of 2 units corresponds to 2 orders of magnitude ($10^2 = 100$ times) greater.
- ✔ A decrease of 1 unit on a base-ten logarithmic scale corresponds to a quantity that is 1 order of magnitude ($10^1 = 10$ times) smaller. Therefore, a decrease of 2 units corresponds to 2 orders of magnitude ($10^2 = 100$ times) smaller.
- ✔ A logarithmic scale is not appropriate if the range of quantities do not span multiple orders of magnitude or if any of the values do not have a real-valued logarithm.

Learning pathways

1–3, 4–6(a, b, e), 9, 11, 13(a–c), 14	2, 3, 4–7(c, d, f), 8, 9(a, c, e, g), 11, 12, 13(d–f), 16	2, 3, 7, 8, 9(b, d, f, h), 10, 11, 13(d–f), 14–18

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Exercise 8.5A: Understanding and fluency

WE 8.5A **1** Determine the order of magnitude of the following base-ten numbers.

a 8.2751×10^{12} **b** 2.1582×10^{-5} **c** 6.014×10^{3}

d 190 124 098 **e** 0.000 000 281 9 **f** 5217.241

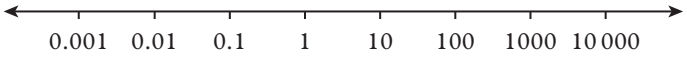
WE 8.5B **2** Determine the position of the following values on a base-ten logarithmic scale, relative to 1. Round your answers correct to two decimal places where necessary.

a 1000 **b** 0.000 001

c 10 **d** 200

e 0.02 **f** 0.0008

WE 8.5C **3** Determine the value of the quantities in the following positions on a base-ten logarithmic scale. Give your answers correct to three significant figures where necessary.



a 5 units to the right of 1 **b** 1 unit to the left of 1

c 3.4 units to the right of 1 **d** 2.7 units to the left of 1

e 0.1 units to the right of 1 **f** 4.6 units to the left of 1

4 Increase the following values by the specified order of magnitude.

a 4.215×10^3 by 1 order of magnitude **b** 4.215×10^3 by 2 orders of magnitude

c 4.215×10^3 by 4 orders of magnitude **d** 0.0729 by 1 order of magnitude

e 0.0729 by 2 orders of magnitude **f** 0.0729 by 4 orders of magnitude

5 Decrease the following values by the specified order of magnitude.

a 4.215×10^3 by 1 order of magnitude **b** 4.215×10^3 by 2 orders of magnitude

c 4.215×10^3 by 5 orders of magnitude **d** 0.0729 by 1 order of magnitude

e 0.0729 by 2 orders of magnitude **f** 0.0729 by 5 orders of magnitude

6 For each pair of numbers, determine how many orders of magnitude higher or lower the first number is than the second.

a 4.12×10^8 and 4.12×10^2 **b** 9.0210×10^{-4} and 9.0210×10^{-1}

c 3.14×10^{-8} and 3.14×10^7 **d** 42 000 and 420

e 0.000 88 and 0.88 **f** 72 400 000 and 0.000 072 4

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Exercise 8.5B: Problem solving and reasoning

7 What number would you need to multiply or divide by to move the following number of units on a base-ten logarithmic scale? Give your answers correct to the nearest integer where appropriate.

a 3 units to the left **b** 4 units to the right **c** 11 units to the right

d 6.3 units to the right **e** 2.1 units to the left **f** 11.1 units to the left

8 Consider the following values.

i 7293.85 **ii** 105 235.79 **iii** 0.002 135

iv 50 123 **v** 61.29 **vi** 0.000 003 279 3

a Write each of the numbers in scientific notation.

b State the order of magnitude of each number.

c Determine the base-ten logarithm of each number, correct to three decimal places.

9 Complete the following sentences. Write the missing values, correct to three significant figures.

a Increasing a value's base-ten logarithm by 0.5 corresponds to the value becoming _____ times greater.

b Increasing a value's base-ten logarithm by 1.5 corresponds to the value becoming _____ times greater.

c Decreasing a value's base-ten logarithm by 2.5 corresponds to the value becoming _____ times smaller.

d Decreasing a value's base-ten logarithm by 3.5 corresponds to the value becoming _____ times smaller.

e Multiplying a value by 2 increases its base-ten logarithm by _____.

f Multiplying a value by 4 increases its base-ten logarithm by _____.

g Dividing a value by 8 decreases its base-ten logarithm by _____.

h Dividing a value by 16 decreases its base-ten logarithm by _____.

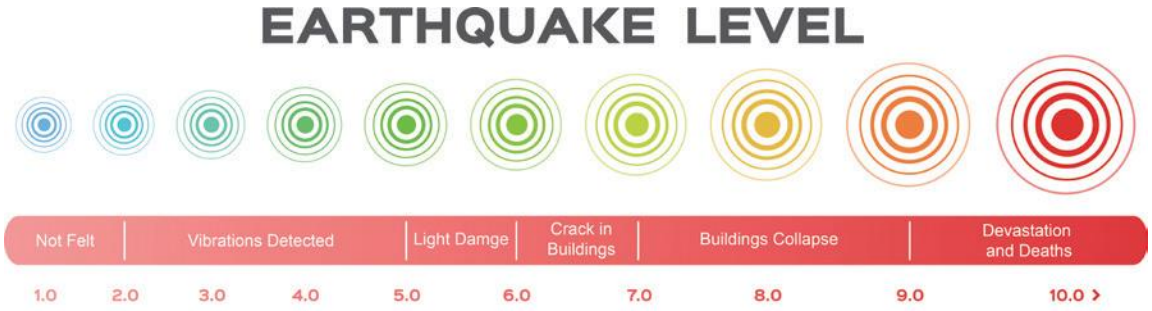
10 Use the approximate logarithmic values provided to determine the positions of the following quantities on a base-ten logarithmic scale, relative to 1.

a 5.291×10^4, given that $\log_{10}(5.291) \approx 0.724$ **b** 5.291×10^{-7}, given that $\log_{10}(5.291) \approx 0.724$

c 2.11×10^{-7}, given that $\log_{10}(2.11) \approx 0.324$ **d** 9.78×10^8, given that $\log_{10}(9.78) \approx 0.990$

e 7.02×10^{-6}, given that $\log_{10}(7.02) \approx 0.846$ **f** 1.05×10^{-1}, given that $\log_{10}(1.05) \approx 0.021$

11 The Richter scale measures the magnitude (size) of an earthquake by taking the base-ten logarithm of the amplitude of the largest seismic wave measured in micrometres (μm). Note that 1 μm = 1×10^{-6} m.



a Determine the magnitude of earthquakes with the following amplitudes of their largest seismic waves. Give your answers correct to one decimal place.

i 320 000 μm **ii** 1200 μm **iii** 75 600 μm

iv 600 μm **v** 21 460 mm **vi** 146 m

b Determine the amplitudes, in micrometres, of the largest seismic waves of earthquakes with the following magnitudes. Give your answers correct to the nearest micrometre.

i 5.0 **ii** 6.0 **iii** 6.5 **iv** 1.5 **v** 7.8 **vi** 4.3

c Each increase of one unit on the Richter scale corresponds to an earthquake that is 10 times stronger. For each pair of earthquake magnitudes, determine how many times stronger the larger magnitude is than the smaller magnitude. Give your answers correct to four significant figures where necessary.

i 4.2 and 5.2 **ii** 4.2 and 4.7 **iii** 4.2 and 5.7

iv 6.1 and 7.9 **v** 1.2 and 7.3 **vi** 5.1 and 5.2

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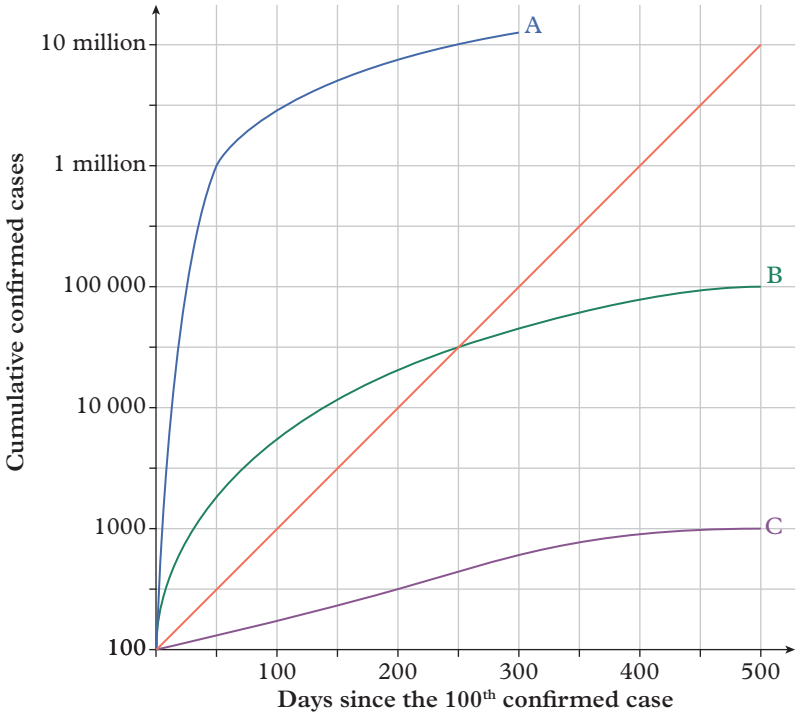
12 When change occurs at an exponential rate, the change is often displayed on a logarithmic scale. The number of people infected by a virus in three countries has been plotted against the number of days since the 100th infection on the following base-ten logarithmic scale.



a Explain why it would not be appropriate to use a linear scale to display this data.

b An orange line has been drawn at a 45° angle onto the graph so that it passes through the points (100, 1000), (200, 10 000), (300, 100 000). Describe what this line represents.

c Use the graph to determine:

i the number of days it took for Country A to go from its 100th infection to its one millionth infection

ii the number of days it took for Country A to go from its 1 millionth infection to its ten millionth infection

iii the number of days it took for Country B to go from its 100th infection to its hundred thousandth infection

iv correct to the nearest integer, the number of cumulative confirmed cases in Country B after 250 days

v the number of cumulative confirmed cases in Country C after 500 days.

d Describe the importance of the point where country B crosses the orange 45° line.

13 For each set of quantities, state if using a linear or base-ten logarithmic scale would be more appropriate to display the values.

a 790, 392, 597, 598, 800, 748, 828, 831, 814, 967

b 194 112, 42 461, 697 139, 17 186, 5599, 680 398, 67 184, 7065, 365 973

c 0.08, 0.8, 0.06, 0.09, 0.1, 0.05, 0.03, 0.7, 0.006

d 0.514, 0.64, 0.39, 0.421, 0.717, 0.49, 0.851, 0.874, 0.525, 0.33, 0.832

e 5.042, 0.118, 3.386, 4.4847, 0.096, 1.373, 7.717, 2.644, 5.005, 7.117, 0.476

f 0.000 173 4, 0.000 623 15, 0.000 513 54, 0.000 417 34, 0.0006, 0.000 961, 0.000 615 23, 0.000 252 31, 0.000 551 32

14 For each of the following sets of values, give a reason why a logarithmic scale would not be appropriate to display the data.

a –8.2, 62, 0.09, 1.2, –57, 0.1, –0.57, –6.23, 600, –7.92, –9.7, 4.31

b 2.4, 8.6, 6.5, 6.7, 3.2, 0, 0, 3.3, 4.4, 8.3, 7.3, 2.3

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15 Logarithmic scales are also useful for plotting data in histograms that involve very small and very large values, where otherwise the smaller values would all be grouped in the same class interval. The average masses of 47 species of animals, in kilograms, are listed below.

1390	1100	725	475	386	190	180	154.25	117.5	100	90	64.475
60	44.984	28.25	26.625	25.5	20	16.5	13.25	9.3	8.6	7.875	6.81
6	5.765	4.25	3.1	2.25	2.06	1.25	0.425	0.397	0.336	0.15	0.29
0.2	0.186	0.173	0.115	0.1	0.07	0.06	0.055	0.05	0.028	0.018	

a Complete the grouped data tables using the average masses listed above.

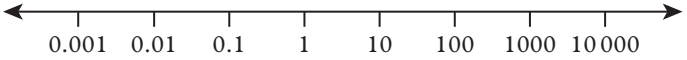
i

Class interval	Frequency
0–<200	
200–<400	
400–<600	
600–<800	
800–<1000	
1000–<1200	
1200–<1400	

ii

Class interval	Frequency
0.01–<0.1	
0.1–<1	
1–<10	
10–<100	
100–<1000	
1000–<10 000	

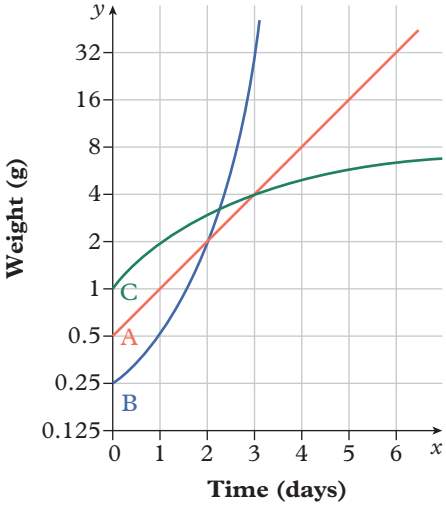
b Draw a histogram for each grouped data table in part **a**. Use a horizontal, base-ten logarithmic scale for the table in part **ii**, as shown.



c Describe the shape of the two histograms as either symmetric, positively skewed or negatively skewed.

d Explain why using a logarithmic scale helps to better understand the distribution of average masses than a linear scale.

16 A base-two logarithmic scale can be used when we want to consider the time it takes for a quantity to double in value. In a base-two logarithmic scale, the quantities double in value for each unit on the scale.
Consider the following graph.



a How long does it take for A to double in weight the first time?

b How long does it take for A to double in weight the second time?

c For the data shown, is the doubling time for A constant, increasing or decreasing?

d For the data shown, is the doubling time for B constant, increasing or decreasing?

e For the data shown, is the doubling time for C constant, increasing or decreasing?

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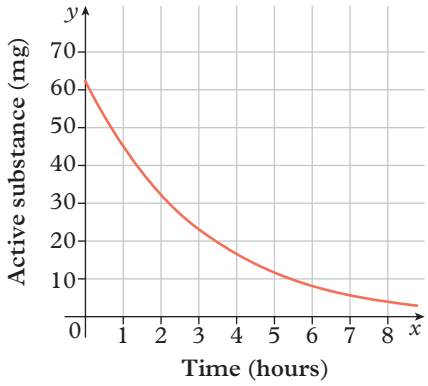
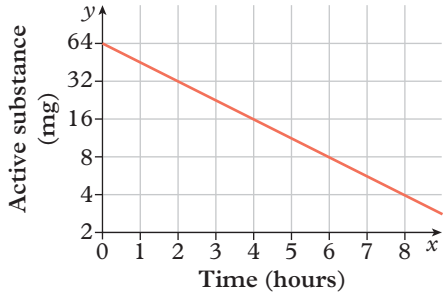
Exercise 8.5C: Challenge

17 The half-life of a drug is the time it takes for the amount of active substance of the drug in a body to reduce by 50%. The two graphs below show the amount of active amount of drug remaining using two different scales.





a What is the half-life of the drug?

b Discuss the advantages and disadvantages of using a base-two logarithmic scale instead of a linear scale to display this data.

18 Bels (B) are a unit of measurement for loud noise, named in honour of Alexander Graham Bell, the inventor of the telephone. Bels are measured on a logarithmic scale, on which an increase in 1 B is an increase of 10 times in loudness. More frequently, decibels (dB) are used where 10 dB = 1 B.

a Determine how many times louder the larger sound is in each of these pairs, correct to four significant figures.

i 15 dB and 25 dB
ii 60 dB and 70 dB
iii 50 dB and 90 dB
iv 95 dB and 145 dB
v 10 dB and 100 dB
vi 99 dB and 100 dB

b The gain, G, in decibels is 10 times the base-ten logarithm of the ratio, $\frac{P_{out}}{P_{in}}$, of the power in watts of the output power level, P_{out}, and the input power level, P_{in}.

That is, $G = 10\log_{10}\left(\frac{P_{out}}{P_{in}}\right)$.

Determine the gain, correct to the nearest decibel, given the following input and output power levels.

i output = 5 W, input = 50 W
ii output = 50 W, input = 5 W
iii output = 50 W, input = 5000 W
iv output = 2 W, input = 8 W
v output = 240 W, input = 240 W
vi output = 100 W, input = 1 W

c Determine the output power level in watts, correct to one decimal place, given the following input power levels and gains.

i input = 5 W, gain = 30 dB
ii input = 5 W, gain = 3 dB
iii input = 50 W, gain = 3 dB
iv input = 500 W, gain = 30 dB
v input = 240 W, gain = 60 dB
vi input = 240 W, gain = 20 dB

Online resources:

Interactive skillsheet — Logarithmic scales
CAS instructions — Scientific notation
Quick quiz — 8.5

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Lesson 8.6

Surface area of pyramids and cones

Learning intentions

By the end of this lesson you will be able to ...

✔ calculate the surface area of right pyramids and right cones.

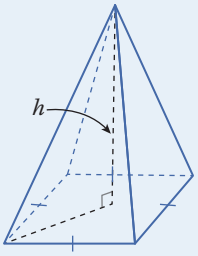
Inter-year links

Year 7	8.1 3D objects
Year 8	8.2 Area of triangles and rectangles
Year 9	5.1 Pythagoras' theorem

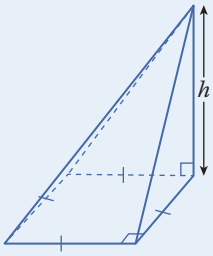
Key content video
Surface area of pyramids and cones

Pyramids

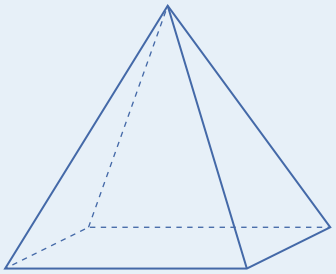
- A **pyramid** is a 3D object with a polygon as its base. All other faces of a pyramid are triangles that meet at the apex (highest point) of the pyramid.
  - → Pyramids are named according to the shape of their base.
  - → A **right pyramid** has an apex directly above the centre of its base.
  - → An **oblique pyramid** does not have its apex over the centre of its base.
- The surface area of a pyramid is the sum of the areas of all of its faces.

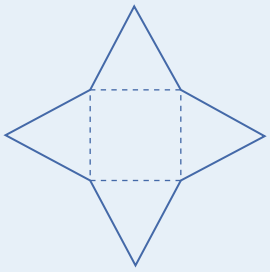


right square pyramid



oblique square pyramid



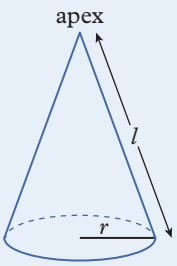


- A **tetrahedron** is a pyramid with a triangular base.
  A **regular tetrahedron** has four triangular faces that are congruent equilateral triangles.
- If the length of the base and the slant edge of a pyramid are known, Pythagoras' theorem can be used to find the height of a triangular face.

Cones

- A **cone** is a 3D object with a circular base and a curved surface that tapers from the circular base to the apex.
- A **right cone** has its apex directly above the centre of its base.
- The surface area of a right cone with radius r and slant height l is given by the formula: $SA = \pi r^2 + \pi rl$.
  - → πr^2 represents the circle at the bottom of the cone and πrl represents the curved surface (imagine the slant height rotating around the circumference of the base).
- The net of a cone consists of a circle and a sector of a circle.

Note: In this lesson, only right pyramids and right cones are considered.



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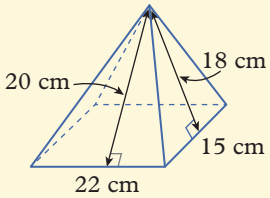
Worked example 8.6A Calculating the surface area of a pyramid

Calculate the surface area of the right rectangular pyramid.



THINK

1 Identify the number and shape of the faces of the pyramid. Since it is a right rectangular pyramid, the triangular faces opposite each other are congruent.

2 Calculate the area of each face.

3 Add the areas of the faces together and include the appropriate unit.

WRITE

The pyramid has five faces: one rectangular face and two pairs of congruent triangular faces.

rectangular face: $A = 22 \times 15$
$= 330 \text{ cm}^2$

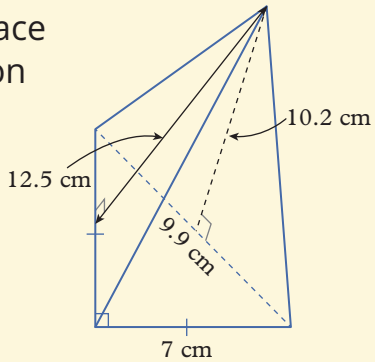
front triangular face: $A = \frac{1}{2} \times 22 \times 20$
$= 220 \text{ cm}^2$

side triangular face: $A = \frac{1}{2} \times 15 \times 18$
$= 135 \text{ cm}^2$

SA $= 330 + 2 \times 220 + 2 \times 135$
$= 1040 \text{ cm}^2$

Worked example 8.6B Calculating the surface area of a tetrahedron

Calculate the surface area of the tetrahedron on the right. The apex of the tetrahedron is directly above the centre of the 9.9 cm edge.



THINK

1 Identify the number and shapes of the faces of the tetrahedron.

2 Calculate the area of each face.

3 Add the areas of the faces together and include the appropriate unit.

WRITE

The base is an isosceles right-angled triangle:
$b = 7$ cm, $h = 7$ cm

There are two identical triangular faces:
$b = 7$ cm, $h = 12.5$ cm

There is one other triangular face:
$b = 9.9$ cm, $h = 10.2$ cm

base: $A = \frac{1}{2} \times 7 \times 7$
$= 24.5 \text{ cm}^2$

triangular face 1: $A = \frac{1}{2} \times 7 \times 12.5$
$= 43.75 \text{ cm}^2$ each

triangular face 2: $A = \frac{1}{2} \times 9.9 \times 10.2$
$= 50.49 \text{ cm}^2$

SA $= 24.5 + 2 \times 43.75 + 50.49$
$= 162.49 \text{ cm}^2$

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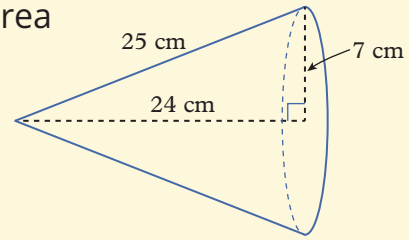
Worked example 8.6C Calculating the surface area of a cone

Calculate the surface area of this cone, correct to one decimal place.



THINK

1. Determine the radius and the slant height of the cone.
2. Write the formula for the surface area of a cone. Substitute values for r and l and simplify.
3. Calculate the result using the π key on your calculator. Write the answer correct to one decimal place and include the appropriate unit.

WRITE

$r = 7$ cm, $l = 25$ cm

$$\begin{aligned} SA &= \pi r^2 + \pi r l \\ &= \pi \times 7^2 + \pi \times 7 \times 25 \\ &= 49\pi + 175\pi \\ &= 224\pi \\ &= 703.716... \\ &\approx 703.7 \text{ cm}^2 \end{aligned}$$

Helpful hints

- ✔ If a tetrahedron is not regular, you'll need to work out the area of more than one triangular face to calculate the surface area.
- ✔ If the base of a right pyramid is a regular polygon, then the surface area can also be found using the formula:
  $SA = \text{base area} + \frac{1}{2} \times \text{perimeter of base} \times \text{slant height}$.

Learning pathways

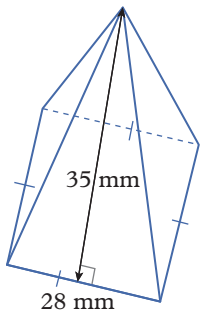
1–4, 5(a, d, e), 6, 7, 10	1(c, e, f), 2–4, 5(b, c, f), 6–8, 11, 12(b), 13	1(e, f), 2, 5(d–f), 6, 9, 11–15

ANS p735

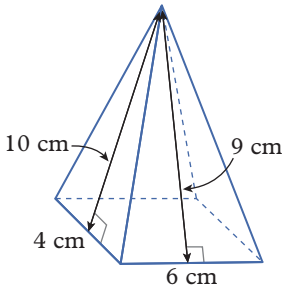
Exercise 8.6A: Understanding and fluency

WE 8.6A **1** Calculate the surface area of each of these right pyramids.

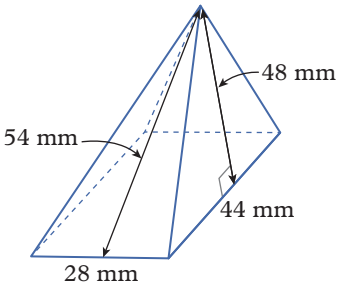
a



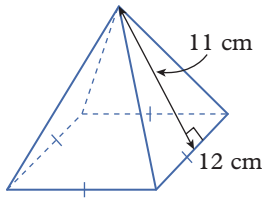
b



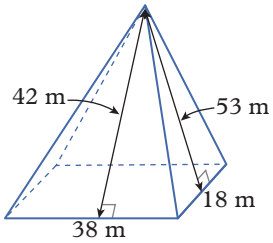
c



d



e



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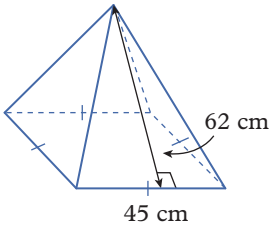

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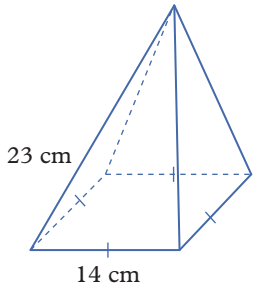
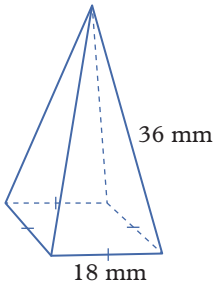
2 To calculate the surface area of each of these right pyramids, first find the height of a triangular face. Give your answers to one decimal place where necessary.

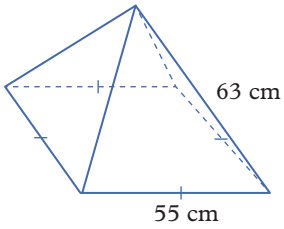
a



b



c



3 At the main entrance to the Louvre Museum in Paris, France, a pyramid has been constructed with a square base of length 35 m. Each identical sloping triangular face has a height of 27 m and is made of glass segments. Find the surface area of the glass walls of the pyramid.

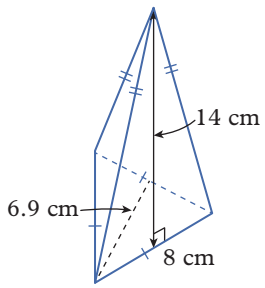
4 The Red Pyramid in Dahshur, Egypt, was originally covered in white limestone. It has a slant edge of length 187 m and a square base of side length 220 m.

a Draw a diagram of this pyramid and label its dimensions.

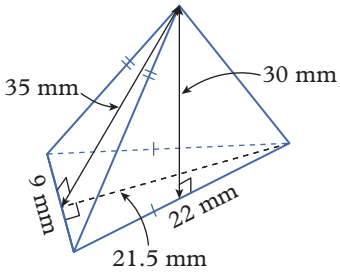
b Find the surface area of the pyramid that was covered with limestone.
Hint: Do not include the base in your calculations.

WE 8.6B **5** Calculate the surface area of each of these tetrahedrons, correct to one decimal place.
Hint: Use Pythagoras' theorem to find any missing heights.

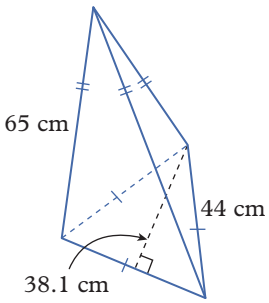
a



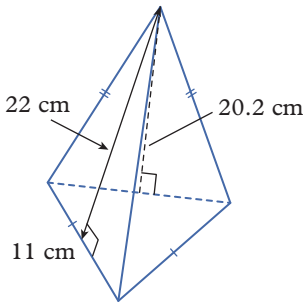
b



c



d



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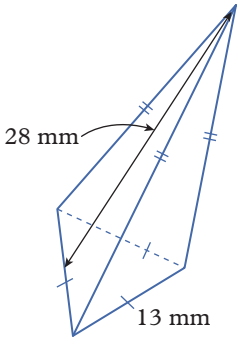

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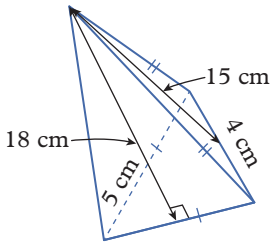

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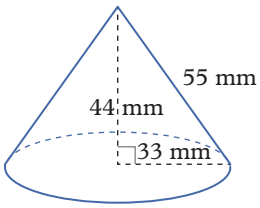
WE 8.6C **6** Calculate the surface area of each of these cones, correct to one decimal place.

a



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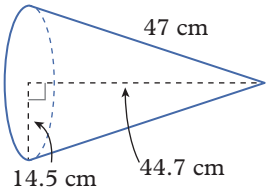

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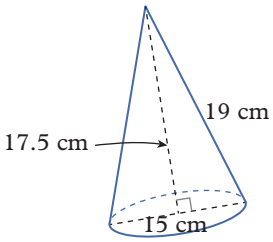

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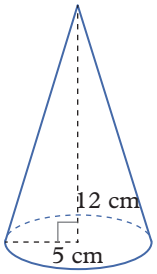

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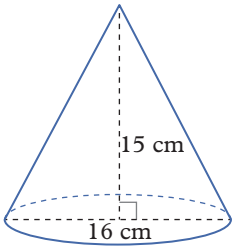

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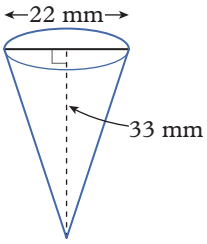


Exercise 8.6B: Problem solving and reasoning

7 A square-based right pyramid has a surface area of 561 cm^2. If the length of the base is 11 cm, find the height of all the triangular faces.

8 A cone has a radius of 15.5 cm and a height of 9.5 cm. Use Pythagoras' theorem to find the slant height of the cone, correct to one decimal place, and its surface area.

9 Consider the shape of a regular tetrahedron.

a What is the surface area of a regular tetrahedron with a side length of 4 cm?

b Develop a formula to determine the surface area of a regular tetrahedron with a side length of x cm.

10 A party hat is in the shape of a cone. The diameter of the base is 12 cm and the slant height is 18 cm. Find how much cardboard is needed to make this party hat assuming there is no overlap. Give your answer correct to one decimal place.
Hint: Do you need to use all parts of the formula for the surface area of a cone?

11 An eight-sided die is in the shape of an octahedron.

a How many equilateral triangles form the octahedron?

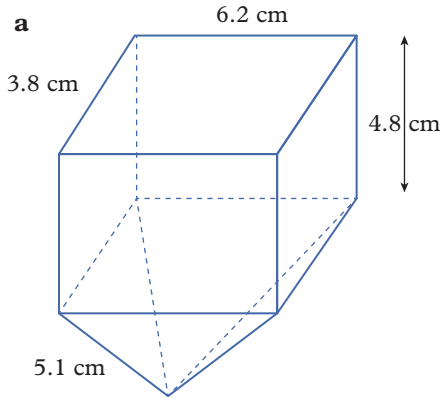
b If the side length of the octahedron is 14 mm, find the surface area of the octahedron, correct to one decimal place.

c The octahedron resembles two identical square pyramids sitting base to base. Calculate the surface area of one pyramid and double your answer.

d Are your answers for parts **b** and **c** equal? Why or why not?

12 Calculate the surface area of each of these composite solids. Give your answers correct to one decimal place.

a



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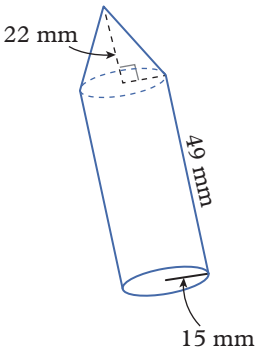

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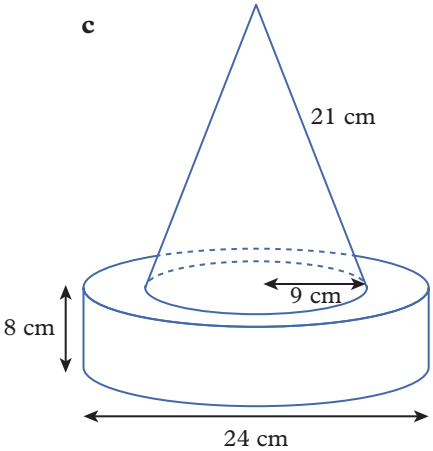

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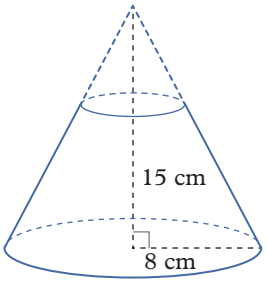
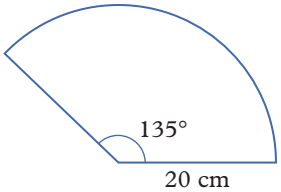
Exercise 8.6C: Challenge

13 The tip of a hollow cone is sliced through and removed. The remaining object is called a truncated cone. If the height of the removed tip was one-third of the height of the whole cone, find the outer surface area of the truncated cone, correct to one decimal place. (Assume the cone includes a base.)



14 An open cone can be made from a circle sector by curving the shape to join the two straight edges. The centre of the circle becomes the apex of the cone and the radius of the circle sector becomes the slant length of the open cone.

a Consider the sector on the right.



i Find the area of this circle sector (and therefore the surface area of the open cone made from this sector). Give your answer to one decimal place.

ii Hence, find the radius of the open cone made from this sector, correct to one decimal place.

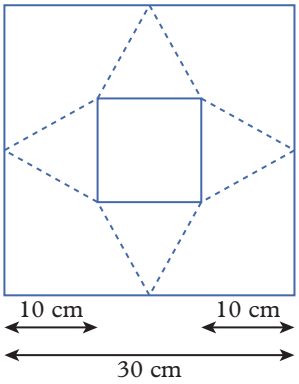
b A circle sector with radius r and interior angle θ is made into an open cone.

i Find the value of θ if the radius of the cone is half the radius of the sector.

ii Find the value of θ if the radius of the cone is one third the radius of the sector.

iii Find the radius of the cone in terms of r and θ.

15 Rhonda is creating a square-based right pyramid with a 30 cm by 30 cm square sheet of paper. She draws the square and dotted lines as shown in the image on the right.



a Find the surface area of the pyramid created by Rhonda when she cuts along the dotted lines and folds the triangles up.

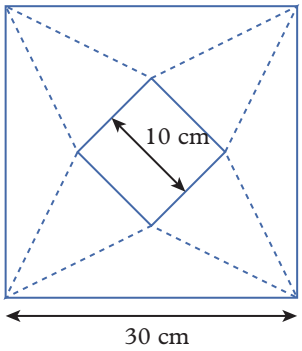
b Find the fraction of the square paper that goes to waste.

c Rhonda's friend Danielle suggests a different way to cut the paper in order to create a taller right pyramid with the same square base. She draws her design on another piece of paper, as shown below.

i Find the height of the triangles that become the slanted faces of the pyramid. Give your answer correct to one decimal place.

ii Hence, find the surface area of the pyramid that is created by cutting the dotted lines this way. Give your answer correct to one decimal place.

iii Find the percentage of the square paper that goes to waste with this design. Give your answer correct to one decimal place.



Online resources:

Interactive skillsheet Surface area of pyramids and cones

Quick quiz 8.6

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Lesson 8.7

Volume of pyramids and cones

Learning intentions

By the end of this lesson you will be able to ...

✔ calculate the volume of pyramids and cones.

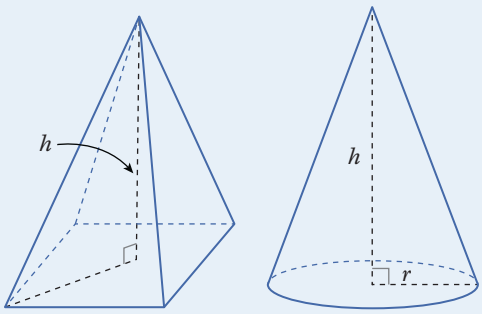
Inter-year links

Year 7	8.1 3D objects
Year 8	8.5 Volume and capacity
Year 9	5.5 Volume of prisms and cylinders

Volume of pyramids and cones

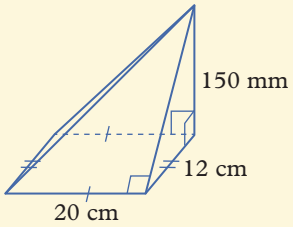
Key content video
Volume of pyramids and cones

- The volume of any pyramid (oblique or right) can be found using the formula $V = \frac{1}{3}Ah$, where A is the area of the base and h is the perpendicular height of the pyramid.
- The volume of a cone (oblique or right) can be found using $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the perpendicular height of the cone.



Worked example 8.7A Calculating the volume of a pyramid

Find the volume of this pyramid.



THINK	WRITE
1 Identify the shape of the base of the pyramid and calculate the area of the base.	The pyramid has a rectangular base. $l = 20$ cm $\quad b = 12$ cm $A = lb$ $= 20 \times 12$ $= 240$ cm^2
2 Convert the measurement given for the perpendicular height to the same units as the sides of the base.	$h = 150$ mm $= 15$ cm
3 Substitute the values of the perpendicular height and the area of the base into the formula for the volume of a pyramid.	$V = \frac{1}{3}Ah$ $= \frac{1}{3} \times 240 \times 15$
4 Calculate the result and include the appropriate unit.	$= 1200$ cm^3

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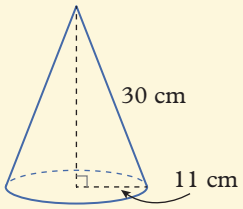
Worked example 8.7B Calculating the volume of a cone

Calculate the volume of this cone, correct to one decimal place.



THINK	WRITE
1 Calculate the value of the cone's perpendicular height by using Pythagoras' theorem.	$h^2 + 11^2 = 30^2$ $h^2 = 900 - 121$ $h^2 = 779$ $h = \sqrt{779}$
2 Substitute the values of the radius of the base and the perpendicular height into the formula for the volume of a cone.	$r = 11$ cm $V = \frac{1}{3}\pi r^2 h$ $= \frac{1}{3} \times \pi \times 11^2 \times \sqrt{779}$ $= 3536.573...$
3 Write the answer, correct to one decimal place, and include the appropriate unit.	≈ 3536.6 cm^3

Helpful hints

- ✔ If the perpendicular height of a pyramid or cone is not given, you may need to calculate it using Pythagoras' theorem.
- ✔ The formulas for the volume of a pyramid or a cone are essentially the same, with πr^2 representing the area of the base of a cone.

Learning pathways

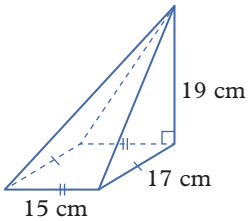
1–5, 8, 10, 11	1(d–i), 2, 4, 6, 9, 12, 15, 16	1(e, h, i), 2, 4(e, f), 6, 7, 9, 13–18

ANS p735

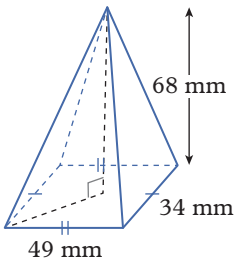
Exercise 8.7A: Understanding and fluency

WE 8.7A **1** Calculate the volume of each of these pyramids. Give your answers correct to one decimal place where appropriate.

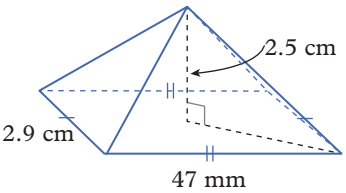
a



b



c



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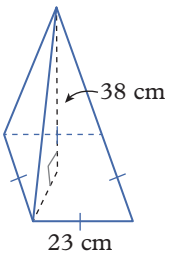
d



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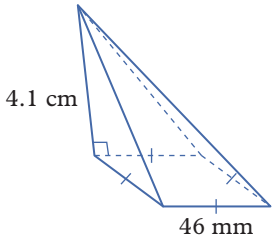

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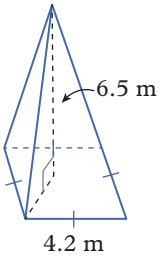

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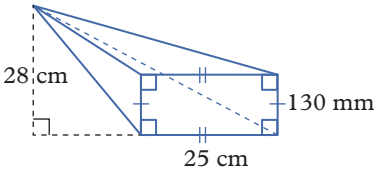

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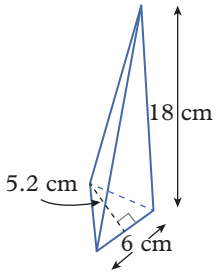
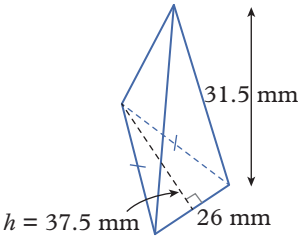
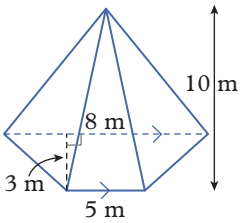


i

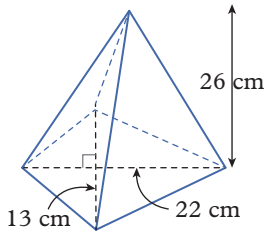


2 Find the volume of each of these pyramids by first calculating the area of the base. Give your answers correct to one decimal place where appropriate.

a



b



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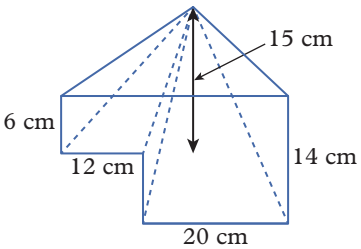
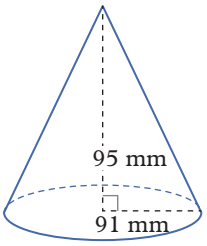


3 The Great Pyramid of Cheops in Giza, Egypt, can be approximated by a square-based pyramid. Its perpendicular height is approximately 147 m and the length of its base is approximately 230 m. Calculate its volume.

WE 8.7B **4** Find the volume of each of these cones. Give your answers correct to one decimal place.

a



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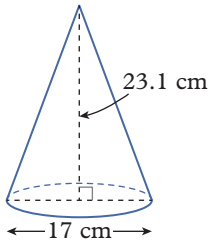

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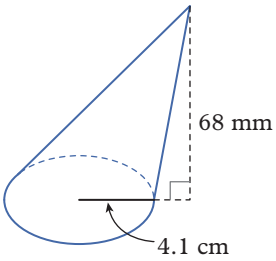

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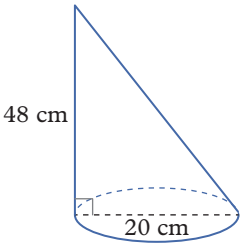

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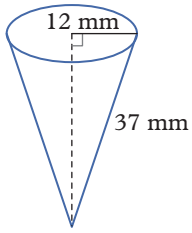

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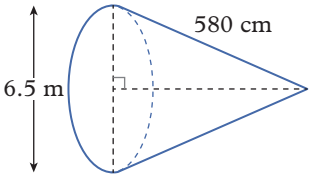

5 A traffic cone has a base with a diameter of 270 mm and it is 850 mm tall. Find the volume of the cone, giving your answer in cubic centimetres correct to one decimal place.

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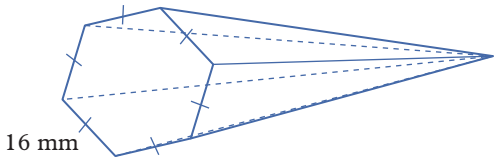
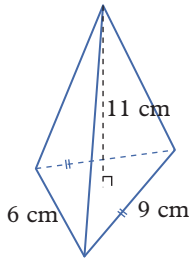
Exercise 8.7B: Problem solving and reasoning

6 Calculate the volume of this hexagonal pyramid of height 70 mm, correct to one decimal place.



7 Consider the tetrahedron on the right.



a Use Pythagoras' theorem to find the height of the triangular base.
Then find the area of the base. Give your answers correct to one decimal place.

b Find the volume of the tetrahedron. Give your answer in:

i cubic centimetres **ii** cubic millimetres.

c Write a formula for finding the volume of a tetrahedron.

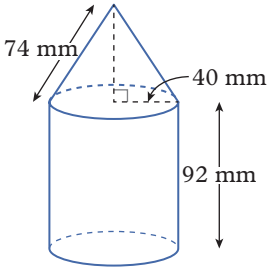
8 Use the appropriate measurement formulas to answer the following questions.

a If a cone has a volume of 89.8 cm^3 and the radius of its base is 3.5 cm, find the height of the cone, correct to one decimal place.

b If a second cone has the same volume as the cone from part **a**, but has a height of 10 cm, find the radius of this cone, correct to one decimal place.

9 A vase has the shape of an open inverted cone sitting in a frame. The circular top of the vase has a radius of 12 cm and the height of the vase is 46 cm. If the vase is filled to three-quarters of its capacity with water, find how much water, in litres, that the vase contains. Give your answer correct to one decimal place.

10 Consider the following composite solid.



a Calculate the surface area of this composite solid, correct to one decimal place.

b Which unknown dimension is also needed to calculate the volume of this solid?

c Use Pythagoras' theorem to find the unknown length. Calculate the volume of the solid. Give your answers correct to one decimal place.

11 A cone has a base with a radius of 3 cm. The cone is 5 cm tall.

a Calculate the volume of the cone, correct to one decimal place.

b Double the radius of the cone, keeping the height the same. Calculate the volume of the new cone, correct to one decimal place.

c Compare your answers to parts **a** and **b**. Describe what happens to the volume of a cone if the radius is doubled and the height remains the same.

d Now, double the height of the original cone and leave the radius unchanged. Calculate the volume of the new cone, correct to one decimal place.

e Write a sentence describing the effect on the volume of a cone when the height is doubled but the radius remains the same.

f Why is the volume increased by a different factor when the radius is doubled compared to when the height is doubled?

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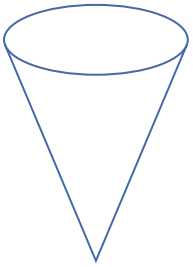
12 The pyramid framework on the right is 8 cm tall, has a rectangular base with a length of 9 cm and a breadth of 5 cm. A coin of diameter 3 cm and thickness 0.2 cm sits in the centre of the pyramid. What percentage of the space within the pyramid does the coin occupy? Give your answer correct to one decimal place.

13 A cone of height 40 cm and radius 12 cm is placed inside a square-based pyramid. The pyramid has a height of 40 cm and the edges of its square are 24 cm. This means that the pyramid is just big enough to contain the cone.

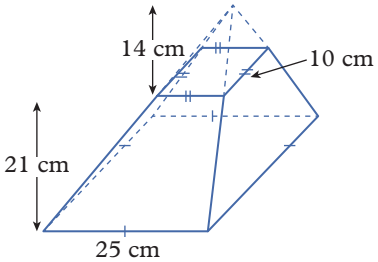
- **a** Find the volume of the cone, correct to one decimal place.
- **b** Find the volume of the pyramid, correct to one decimal place.
- **c** Hence, find the volume of empty space between the pyramid and the cone.
- **d** Find the exact ratio of the space in the pyramid taken up by the cone to the space in the pyramid that is empty.



14 A paper cup is in the shape of an open cone with a radius of 5 cm and a height of 10 cm.

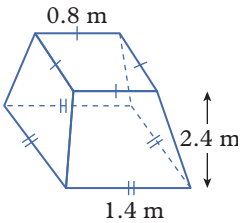
- **a** Find the volume of the cone, correct to one decimal place.
- **b** Find the height of the water in the cone when the cup is half full. Give your answer correct to one decimal place.

Exercise 8.7C: Challenge



15 The tip or apex of a square pyramid is cut off parallel to the base and is removed. The object remaining is called a truncated pyramid.

- **a** What shapes are the faces of the truncated pyramid?
- **b** Calculate the volume of the pyramid before the tip has been removed.
- **c** Calculate the volume of the smaller pyramid that has been cut off, correct to one decimal place.
- **d** Use your answers to parts **b** and **c** to calculate the volume of the truncated pyramid shown.



16 The base of a statue was cut from a concrete block formed as a square-based pyramid. If the height of the original block was 5.6 m before it was truncated, how much concrete was needed to construct this base? Give your answer in cubic metres, correct to one decimal place.

17 A drinking glass is 120 mm tall and is in the shape of a truncated cone. The base of the glass has a diameter of 55 mm and the top of the glass has a diameter of 75 mm.

- **a** If the height of the conical piece of glass used to construct the drinking glass was 192 mm, calculate the volume of the drinking glass, correct to one decimal place.
- **b** How many millilitres of iced tea can this glass hold if it is filled to its brim? Give your answer correct to the nearest whole number.

18 It is possible to write a formula for finding the volume of a truncated square-based pyramid and a truncated cone. Use your answers to questions **15–17** to assist you to find these formulas.

Online resources:

- **Interactive skillsheet** Volume of pyramids and cones
- **Investigation** Ice cream cone designs
- **Quick quiz** 8.7

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Surface area and volume of spheres

Learning intentions

By the end of this lesson you will be able to ...

✔ calculate the surface area and volume of spheres.

Inter-year links

Year 7	9.2 Circumference of a circle
Year 8	8.4 Area of a circle
Year 9	5.4 Surface area of cylinders

Spheres

Key content video
Surface area and volume of spheres

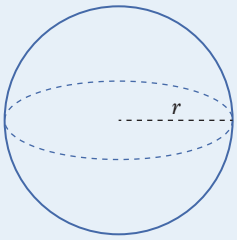
- A **sphere** is a round 3D object, with every point on its surface equidistant from its centre.
- The size of a sphere is determined by its radius (or diameter).
- The formula for the surface area of a sphere is:

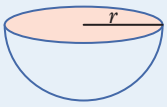
$$SA = 4\pi r^2$$

- The formula for the volume of a sphere is:

$$V = \frac{4}{3}\pi r^3$$

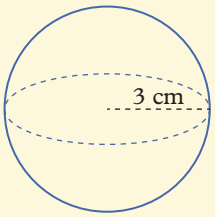
- A hemisphere is half of a sphere.
  In this book, closed hemispheres are drawn with a shaded circular face.





Worked example 8.8A Calculating the surface area of a sphere

Calculate the surface area of a sphere with a radius of 3 cm, correct to one decimal place.



THINK	WRITE
1 Substitute the length of the radius into the formula for the surface area of a sphere.	$r = 3$ cm $SA = 4\pi r^2$
2 Calculate the result, using the π key on your calculator.	$= 4 \times \pi \times 3^2$ $= 4 \times \pi \times 9$ $= 36\pi$ $= 113.097...$
3 Write your answer correct to one decimal place and include the appropriate unit.	≈ 113.1 cm^2

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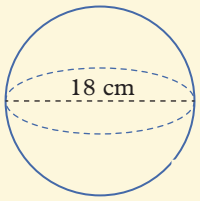
Worked example 8.8B Calculating the volume of a sphere

Calculate the volume of this sphere, correct to one decimal place.



THINK

1. Calculate the length of the radius of the sphere.
2. Substitute the length of the radius into the formula for the volume of a sphere.
3. Calculate the result using the π key on your calculator.
4. Write the answer correct to one decimal place and include the appropriate unit.

WRITE

$r = 18 \div 2$
$= 9$ cm

$V = \frac{4}{3}\pi r^3$
$= \frac{4}{3} \times \pi \times 9^3$
$= 972\pi$
$= 3053.628...$
≈ 3053.6 cm^3

Worked example 8.8C Calculating the radius of a sphere, given the volume

Calculate the radius of a sphere that has a volume of 500 cm^3. Give your answer correct to one decimal place.

THINK

1. Substitute the given volume into the formula for the volume of a sphere.
2. Rearrange the equation so r^3 is alone on one side of the equation.
3. Use a calculator to find r.
4. Write the answer correct to one decimal place and include the appropriate unit.

WRITE

$V = \frac{4}{3}\pi r^3$
$500 = \frac{4}{3}\pi \times r^3$
$500 \div \frac{4}{3}\pi = r^3$
$119.366... = r^3$
$r = \sqrt[3]{119.366...}$
$r = 4.923...$
$r \approx 4.9$ cm

Helpful hint

- ✔ When working with surface areas and volumes, always remember to include units with your final answers.
- ✔ When calculating the surface area of hemispheres, always check whether the flat circular face is included.

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Learning pathways

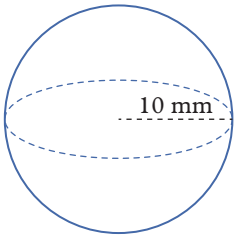
1–9, 11, 15, 16(a, b)	1–4, 6, 8, 12, 14, 16(b, c), 18	1(c, d), 2(c, d), 3, 4, 8(b, e, f), 10, 13, 14, 17, 19, 20

ANS p736

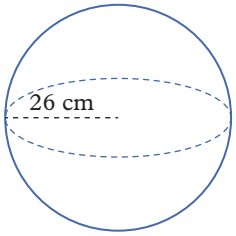
Exercise 8.8A: Understanding and fluency

WE 8.8A **1** Calculate the surface area of each of these spheres, correct to one decimal place.

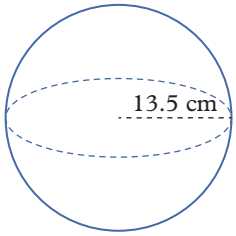
a



b



c



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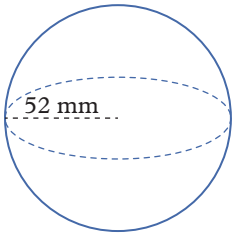
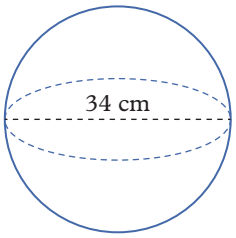


2 Calculate the surface area of each of these spheres by first calculating the radius. Give your answers correct to one decimal place.

a



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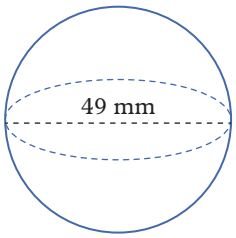

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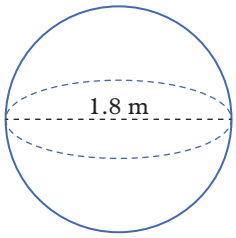

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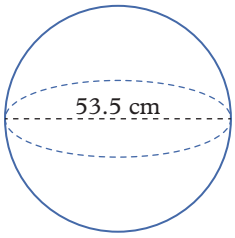


WE 8.8B **3** Calculate the volume of each sphere in question **1**, correct to one decimal place.

4 Calculate the volume of each sphere in question **2**, correct to one decimal place.

WE 8.8C **5** Calculate the radius of a sphere that has a volume of 5575 cm^3. Give your answer correct to the nearest cm.

6 If the surface area or the volume of a sphere (or a hemisphere) is given, equations can be formed and solved to find the dimensions (radius or diameter) of the solid.

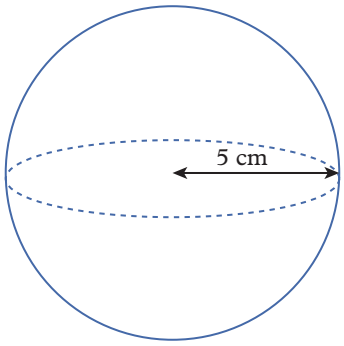
a Find the radius of a sphere that has a surface area of 144π cm^2.

b Find the radius of a sphere that has a volume of 972π cm^3.

c Find the radius of a sphere that has a surface area of 2124 cm^2.

d Find the diameter of a hemisphere that has a volume of 1072 cm^3.

7 Consider the solid sphere on the right.



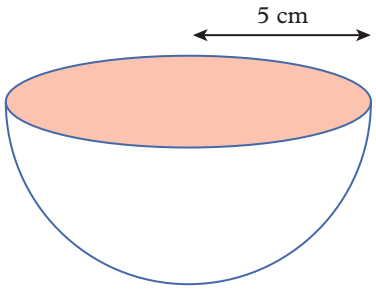
a Find the surface area of the sphere, as an exact value.

b The sphere is cut exactly in half, as shown below.

i Find the area of the circular face, as an exact value.

ii Find the area of the curved surface, as an exact value.

iii Hence, find the total surface area of the solid hemisphere, as an exact value.



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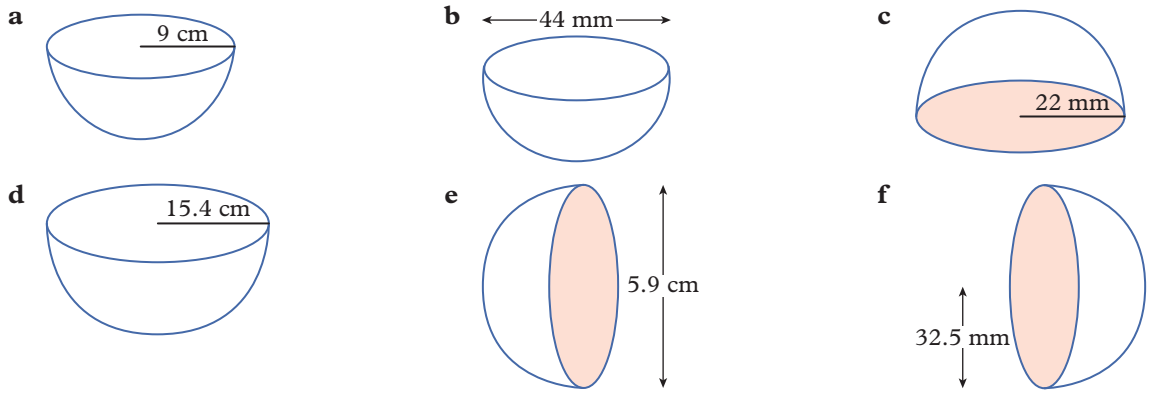
8 For each hemisphere below, calculate:

i its outer surface area, correct to one decimal place

ii its volume, correct to one decimal place.

Hint: First determine whether each hemisphere is open or closed. In this book, closed hemispheres are drawn with a shaded circular face.



Exercise 8.8B: Problem solving and reasoning

9 When inflated, the beach ball below has a diameter of 76 cm.

a Calculate its surface area, correct to one decimal place.

b Find the volume of air that it can hold, correct to one decimal place.

10 The radius of a sphere can be found when the volume or surface area of the sphere is known.

a The volume of a fully inflated volleyball is 0.15 m^3. Find the radius of the volleyball, to the nearest centimetre.

b The surface area of a tennis ball is approximately 564 cm^2. Find the radius of the tennis ball, correct to one decimal place.

11 The approximate diameters of the planets in our solar system are listed in the table below.

Planet	Diameter (km)	Surface area	Volume
Mercury	4878		
Venus	12 104		
Earth	12 756		
Mars	6787		
Jupiter	142 800		
Saturn	120 000		
Uranus	51 118		
Neptune	49 528		

a Copy and complete the table. Write the values in scientific notation, correct to two significant figures ($a \times 10^m$ where $1 \le a < 10$ and m is an integer).

b Approximately how many times greater is Earth's volume compared to the volume of Mercury?

c Approximately how many times greater is Jupiter's surface area compared to Earth's surface area?

d Approximately 71% of Earth's surface is covered by water. To the nearest square kilometre, how much of Earth's surface area is covered by water?

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12 The rind of a piece of fruit, like an orange, is the outer layer, which is usually not eaten and can be peeled off to reveal the edible inner part of the fruit. The surface area of the outer rind of this half blood orange is 116.9 cm^2.

a What is the diameter of the orange, correct to one decimal place?

b The thickness of the rind is approximately 0.5 cm. What volume does the edible part of the half orange occupy, to the nearest cubic centimetre?

13 A balloon is inflated to become a sphere with a radius of 9.5 cm.

a Two more balloons are inflated to exactly the same size as the first balloon. What is the total volume of the three balloons, correct to one decimal place?

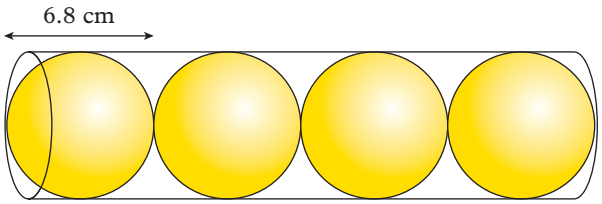
b When objects are filled with liquid or gas, you can refer to their capacity rather than volume.

What is the capacity of the three balloons, in litres? Give your answer correct to one decimal place.

c If the rate that the balloons can be inflated is 2.5 L/s, how long will it take to inflate all three balloons? Give your answer correct to one decimal place.

d What assumptions do you need to make when answering part **c**?

14 Tennis balls are sold in canisters. Each canister holds four tennis balls.



a If each tennis ball has a diameter of 6.8 cm, what is the combined volume of all four tennis balls, correct to one decimal place?

b Determine the dimensions of the canister and find its volume, correct to one decimal place. Assume that the tennis balls fit perfectly into the canister.

c How much space in the canister is not taken up by tennis balls? Give your answer correct to one decimal place.

d If the tennis balls were to be sold in a container with the same dimensions but in the shape of a rectangular prism instead of a cylinder, without any further calculations, would there be more or less unused space?

15 Rebecca buys an ice-cream cone. As well as the ice cream on top of the cone, the cone is completely filled with ice cream. Calculate how much ice cream Rebecca has in total by approximating the top as a perfect hemisphere. Give your answer to the nearest millilitre.

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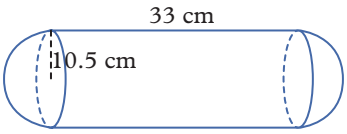
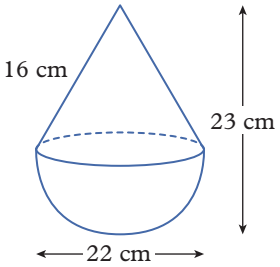
16 Calculate the volume of each of these composite solids, correct to one decimal place.

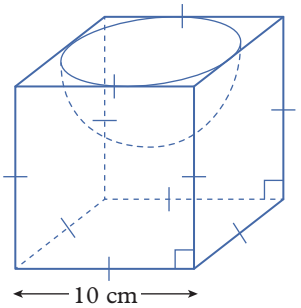
a



b



c



17 Consider the following objects.

- A cylinder with a height equal to its radius
- A cone with a height equal to its radius
- A sphere
- A hemisphere

If all four objects have the same radius, what is the ratio of the volumes of the four objects? Put your ratio in the following order.

cylinder : cone : sphere : hemisphere

18 A ping pong ball has a diameter of 40 mm.

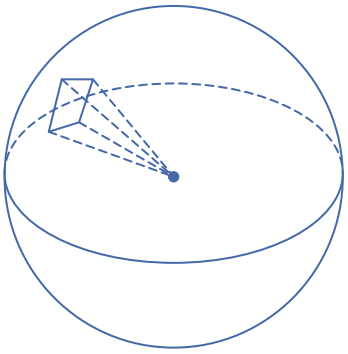
a Find the volume of the ping pong ball, correct to one decimal place.

b Density is defined as the mass per unit volume and can be found using the formula:

$$\text{density} = \frac{\text{mass}}{\text{volume}}.$$

If the mass of the ping pong ball is 2.7 g, calculate its density in g/cm^3. Give your answer correct to two decimal places.

Exercise 8.8C: Challenge



19 A sphere can be approximated using many right square pyramids, all with equal base lengths and slant heights equal to the radius of the sphere. The accuracy of the approximation increases as the number of pyramids increases. In this question, you will derive the formula for the surface area of a sphere.

a Suppose your approximation uses n pyramids with base area A. Write an expression for the volume of the sphere in terms of A, n and r, where r is the radius of the sphere.

b Equate your expression from part **a** with the usual formula for the volume of a sphere and simplify to find an expression equal to nA.

c Why does your answer to part **b** give the surface area of the sphere?

d Why doesn't this method work for small values of n (like $n = 6$)?

20 The following questions involve the volumes of solids.

a A solid sphere of radius 10 cm is placed inside the smallest cube that can contain it.

i Find the side length of the cube.

ii Find the volume of empty space in the cube, correct to one decimal place.

b A solid cube of side length 10 cm is enclosed in the smallest sphere that can contain it.

i Find the radius of the sphere, correct to one decimal place.

ii Find the volume of empty space in the sphere, correct to one decimal place.

Online resources:

- **Interactive skillsheet** Surface area of spheres
- **Interactive skillsheet** Volume of spheres
- **Worksheet** Surface area and volume of pyramids, cones and spheres
- **Quick quiz** 8.8

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Lesson 8.9

Review: Measurement

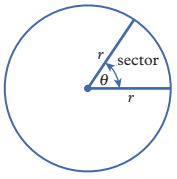
Module summary

Area of a composite shape

1. Split the shape into individual parts.
2. Calculate any missing dimensions.
3. Calculate the areas of the individual parts.
4. Add or subtract the areas to find the total area required.

Area of a sector

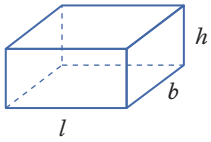
- Area $= \dfrac{\theta}{360°} \times \pi r^2$



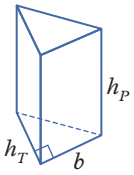
Volume of a prism

For any prism, $V = Ah$, where A is the area of the base and h is the height of the prism.

$V = lbh$

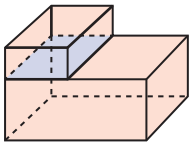


$V = bh_Th_P$



Surface area

The surface area (SA) of a 3D object is the total area of its outer surface. This is the sum of the areas of the faces (or surfaces) of that object.

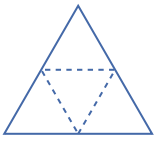


Scientific notation & orders of magnitude

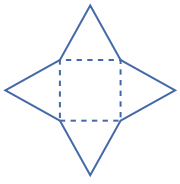
- $a \times 10^m$, where $1 \le a < 10$ or $-10 < a \le -1$ and m is an integer
- Order of magnitude in base-ten $= m$
- To increase a number by 1 order of magnitude: $\times$ 10
- To decrease a number by 1 order of magnitude: $\div$ 10

Pyramids

The surface area is the sum of the areas of all the faces.



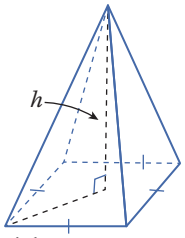
net of a regular tetrahedron



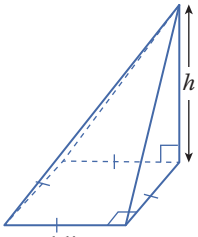
net of a square pyramid

$V = \dfrac{1}{3}Ah$

where A is the area of the base and h is the perpendicular height of the pyramid.

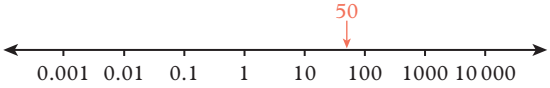


right square pyramid



oblique square pyramid

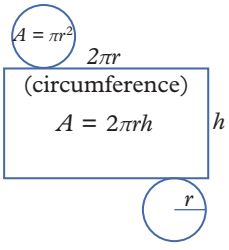
Logarithmic scales (base-ten)



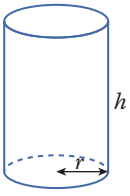
- The position of the value on the scale relative to 1 is the logarithm of the number. For example, log10(50) ≈ 1.7 so 50 will sit 1.7 units to the right of the number 1.
- Logarithmic scales should not be used if the data set has a zero or any negative values.

Cylinders

$SA = 2\pi r^2 + 2\pi rh$

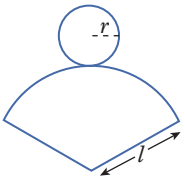


$V = \pi r^2 h$

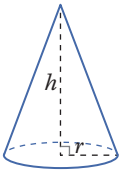


Cones

$SA = \pi r^2 + \pi rl$



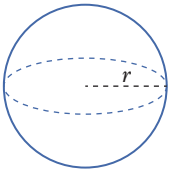
$V = \dfrac{1}{3}\pi r^2 h$



Spheres

$SA = 4\pi r^2$

$V = \dfrac{4}{3}\pi r^3$



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Review questions 8.9A: Mathematical literacy review

Review quiz
Take the review quiz to assess your knowledge of this module.
✔ 10
✔ 10A

Quizlet
Test your knowledge of this module by working individually or in teams.

The following key terms are used in this module:

- absolute error
- area
- base-ten
- capacity
- circumference
- composite shape
- cone
- conversion factor
- cross-section
- cylinder
- error
- face
- hemisphere
- logarithmic scale
- net
- oblique pyramid
- order of magnitude
- percentage error
- perpendicular
- prism
- pyramid
- regular tetrahedron
- relative error
- right cone
- right pyramid
- scientific notation
- sector
- sphere
- surface area
- tetrahedron
- uniform
- volume

1 Why is a cylinder not a prism?

2 What is the difference between an oblique pyramid and a right pyramid?

3 Explain what it means for a 3D object to have a uniform cross-section. Give examples of objects with and without a uniform cross-section.

4 Identify the key terms being referenced in each of these definitions.

a a portion of a circle formed by two radii and part of the circumference

b the traditional number system we use which uses the digits 0 to 9

5 Provide a definition in your own words for the following key terms.

a regular tetrahedron **b** right cone **c** sphere

6 Complete the following sentences using words from the key terms list.

a The _____ of a 3D object is found by summing the areas of each individual _____.

b The _____ of an object is how much space it takes up. The _____ is how much liquid the object can hold.

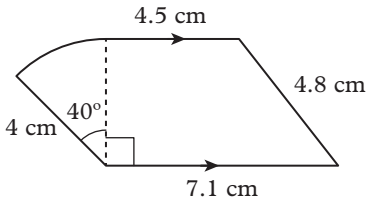
Review questions 8.9B: Multiple choice

8.1 **1** Which of the following is equivalent to 300 cm²?

A 3 m² **B** 0.3 m² **C** 3000 mm² **D** 0.03 m²

8.1 **2** Calculate the area of the composite shape below, correct to two decimal places.

A 73.58 cm² **B** 73.69 cm² **C** 78.22 cm² **D** 78.34 cm²



8.2 **3** What is the surface area of a cube with side lengths of 8 cm?

A 384 cm² **B** 48 cm² **C** 64 cm² **D** 512 cm²

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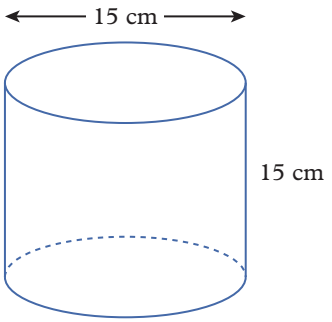
8.2 **4** A closed cylinder has a base diameter of 15 cm and a height of 15 cm. Which of the following is closest to the surface area of the cylinder?



A 225 cm^2 **B** 700 cm^2 **C** 750 cm^2 **D** 1060 cm^2

8.3 **5** A cylinder has a base diameter of 6 cm and a height of 20 cm. Which of the following is closest to the volume of the cylinder?

A 120 cm^3 **B** 188 cm^3 **C** 377 cm^3 **D** 565 cm^3

8.3 **6** A solid wooden cylinder has a base with radius 10 cm and is 10 cm tall. It has a cylinder of radius 2 cm drilled through its centre, from top to base. What is the volume of timber remaining (in cubic centimetres)?

A 640π cm^3 **B** 1000π cm^3 **C** 960π cm^3 **D** 80π cm^3

8.5 **7** A carton of eggs is labelled as 700 g. After placing the eggs on a scale, the actual mass is found to be 684 g. Which of the following is closest to the percentage error of the labelled mass?

A 0.02% **B** 2.29% **C** 2.34% **D** 16%

8.4 **8** Which of the following best represents the position of 0.005 on a base-ten logarithmic scale?

A 3.3 units to the left of 1 **B** 2.3 units to the left of 1

C 1.3 units to the left of 1 **D** 2.3 units to the right of 1

10A 8.6 **9** A square-based pyramid has a surface area of 224 cm^2. If the length of the base is 8 cm, what is the height of each triangular face?

A 56 cm **B** 14 cm **C** 10.5 cm **D** 10 cm

10A 8.6 **10** How many triangular faces does a tetrahedron have?

A 1 **B** 3 **C** 4 **D** 5

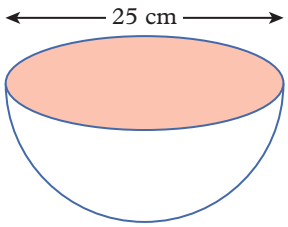
10A 8.7 **11** A square-based pyramid has the same base length and height as a cube with side lengths of 6 cm. What is the volume of the pyramid?

A 216 cm^3 **B** 72 cm^3 **C** 144 cm^3 **D** 108 cm^3

10A 8.7 **12** Which of the following is closest to the volume of a cone with a diameter of 2.5 cm and a height of 10 cm?

A 16 cm^3 **B** 25 cm^3 **C** 50 cm^3 **D** 65 cm^3

10A 8.8 **13** Which of the following is closest to the total exposed surface area if a solid sphere with a diameter of 25 cm is sliced in half?



A 500 cm^2 **B** 1500 cm^2 **C** 3000 cm^2 **D** 4000 cm^2

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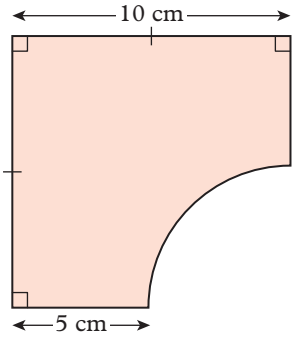
Review questions 8.9C: Short answer

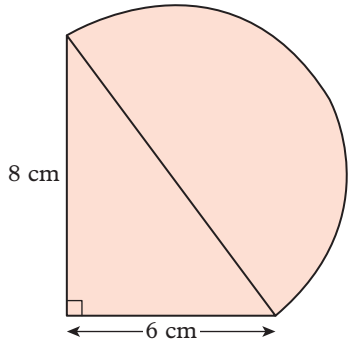
8.1 **1** A flat metal washer is in the form of an annulus. The radius of the inner ring of the washer is 1 cm, while the outer ring has a radius of 1.5 cm. What is the area of the annulus, correct to one decimal place?

8.1 **2** Determine the area of each of these composite shapes. Give your answers correct to one decimal place.

a



b



8.2 **3** The base of a triangular prism is a right-angled triangle with a base length of 25 mm and a height of 18 mm. The height of the prism is 30 mm. What is the SA of the prism?

8.2 **4** A set of three wooden cubes was made for a playground. The smallest cube had a side length of 40 cm. The next sized cube had side lengths twice as long as the smallest cube, while the side lengths of the largest cube were twice as long as those of the middle cube.

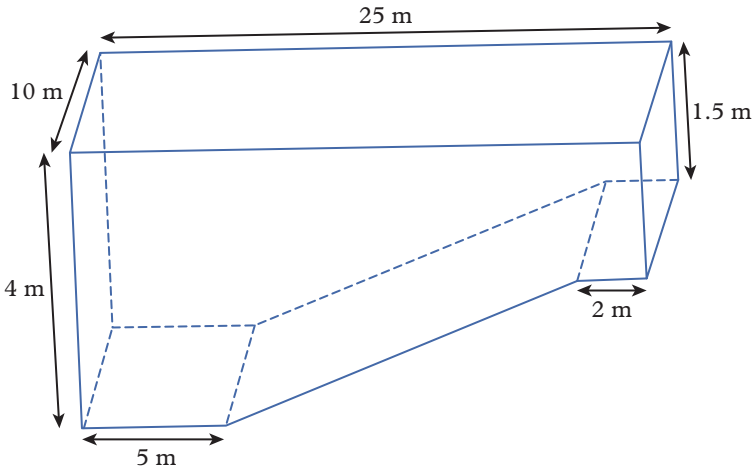
a Calculate the surface area of each cube.

b If 1 L of paint covers an area of 15 m^2, and the paint is only available in 5 L cans, how many cans of paint would be required to paint all three cubes?

8.3 **5** A can of baked beans has a cylindrical shape with a base diameter of 7 cm and a height of 10 cm. What is the capacity of the can, correct to the nearest millilitre?

8.3 **6** Calculate the volume of each cube in question **4**.

8.3 **7** An inground pool in the shape of the right prism shown is to be installed. The prism is not drawn to scale. Calculate the volume of soil which needs to be excavated so that the pool fits exactly into the hole.



8.4 **8** The diameter of a circle is 14 mm.

a Find the exact area of the circle.

b Find the approximate area of the circle, correct to one decimal place.

c Calculate the relative error of your approximation in part **b**, correct to 3 significant figures.

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8.5 **9** Increase or decrease the following numbers by the orders of magnitude provided.

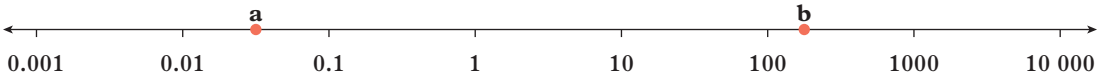
a Increase 2.65×10^2 by 2 orders of magnitude.

b Increase 0.084 by 4 orders of magnitude.

c Decrease 7.51×10^4 by 4 orders of magnitude.

d Decrease 9091 by 5 orders of magnitude.

8.5 **10** Consider the following base-ten logarithmic scale.



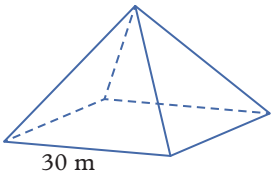
a A quantity is placed halfway between 0.01 and 0.1. What is its value, correct to three significant figures?

b A different quantity is placed one quarter of the way between 100 and 1000. What is its value, correct to the nearest integer?

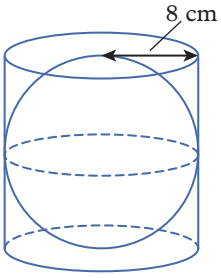
10A 8.6 **11** A square-based pyramid has a base with sides 5 cm in length. The height of each triangular face is 20 cm. Calculate the surface area of the pyramid.

10A 8.6 **12** The diameter of a closed cone is 10 cm and the cone's height is 16 cm. Calculate the surface area of the cone, correct to one decimal place.

10A 8.7 **13** A square-based pyramid has a base with sides of length 30 cm. The triangular faces are equilateral. Find the volume of the pyramid, correct to two decimal places.



10A 8.7 **14** A sphere fits neatly inside a cylinder with a base radius of 8 cm. Find the volume of the cylinder and the sphere in terms of π.



10A 8.8 **15** The following questions relate to calculating surface area of solid shapes.

a Calculate the surface area, correct to one decimal place where appropriate, of:

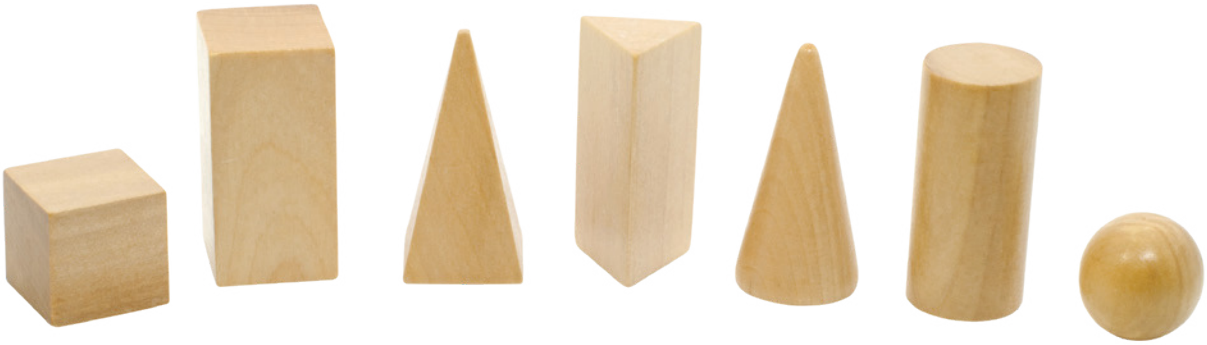
i a solid hemisphere with a diameter of 40 cm

ii a cube with sides 40 cm in length.

b The hemisphere is placed on top of the cube. What is the total surface area of this object?

Review questions 8.9D: Mathematical modelling

1 Ethan bought this set of wooden blocks for his young cousins.



a Name the solids in order from left to right.

b The cube has side lengths of 3 cm. Calculate its:

i surface area

ii volume.

c The rectangular prism has a base the same size as the cube, and is twice the height of the cube.

i Compare the surface area of the rectangular prism with that of the cube.

ii Compare the volume of the rectangular prism with that of the cube.

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d The triangular prism is the same shape as the rectangular prism cut vertically along the diagonal of its base.

i Draw a labelled diagram showing the shape of the triangular prism's base.

ii Calculate the surface area of the triangular prism, correct to one decimal place.

iii Is the surface area of the triangular prism half that of the rectangular prism that you calculated in part **c**? Explain.

iv Compare the volumes of the triangular prism and the rectangular prism.

e The base of the cylinder has a diameter the same length as the side of the cube, and the cylinder is the same height as the rectangular prism. Calculate, correct to one decimal place, the cylinder's:

i surface area **ii** volume.

f Create two tables listing the solids in parts **b–e** in increasing order of:

i surface area **ii** volume.

10A **g** The square base of the pyramid is the same shape as that of the base of the cube. Its height is the same as that of the rectangular prism. Calculate, correct to one decimal place, the pyramid's:

i surface area **ii** volume.

10A **h** The cone has the same base diameter and height as the cylinder. Calculate, correct to one decimal place, the cone's:

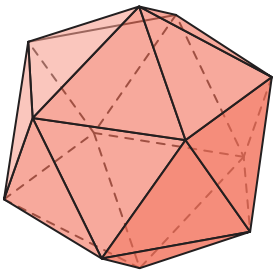
i surface area **ii** volume.

10A **i** The sphere has the same diameter as the cone. Calculate, correct to one decimal place, the sphere's:

i surface area **ii** volume.

10A **j** Using your answers for parts **g**, **h** and **i**, update your tables from part **f**.

10A **2** A regular icosahedron is a polyhedron (a 3D object with polygonal faces) with 20 equilateral triangle faces.



a Find the area of an equilateral triangle with side length a.

b Find the surface area of a regular icosahedron where each equilateral triangle face has side length a.

c A solid regular icosahedron can be divided into 20 congruent tetrahedrons. The base of each tetrahedron is one of the faces of the icosahedron, and the height of each tetrahedron is $\frac{\sqrt{3}(3+\sqrt{5})}{12}a$.

Determine the volume of a regular icosahedron in terms of a in exact form and as an approximation, correct to three decimal places.

d The circumscribed sphere of a regular icosahedron is the sphere which contains the icosahedron and touches all vertices of the icosahedron. The radius of the circumscribed sphere is $r = \frac{a}{4}\sqrt{10+2\sqrt{5}}$.

i Determine the surface area and volume of the circumscribed sphere of a regular icosahedron with side length a, correct to three decimal places.

ii For the case $a = 1$, calculate the percentage of the circumscribed sphere which is filled by the icosahedron, correct to two decimal places.

e A truncated icosahedron is obtained by cutting off each vertex of a regular icosahedron so that each resulting edge is one-third of the length of the edges in the original icosahedron. There are 12 regular pentagonal faces and 20 regular hexagonal faces in a truncated icosahedron. Soccer balls are based on the shape of a truncated icosahedron, where the pentagons are usually coloured black and the hexagons are usually white. The area of a regular pentagon with side length a is approximately $A = 1.720a^2$. The area of a regular hexagon with side length a is $A = \frac{3\sqrt{3}}{2}a^2$.

Determine the surface area of a truncated icosahedron with side length $\frac{a}{3}$ in terms of a, correct to three decimal places.

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f The formula for the volume of a truncated icosahedron with side length a is $V = \frac{125 + 43\sqrt{5}}{4}a^3$.

i Explain why the volume of a truncated icosahedron is less than the volume of the icosahedron it is constructed from.

ii Determine the volume of a truncated icosahedron with side length $\frac{a}{3}$ in terms of a, correct to three decimal places.

g The radius of the circumscribed sphere of a truncated icosahedron with side length a is $r = \frac{a}{4}\sqrt{58 + 18\sqrt{5}}$.

i Determine the surface area and volume of the circumscribed sphere of a truncated icosahedron with side length $\frac{a}{3}$, correct to three decimal places.

ii For the case $a = 1$, use your answers to parts **f ii** and **g i** to calculate the percentage of the circumscribed sphere which is filled by the truncated icosahedron, correct to two decimal places.

iii Suggest one reason why soccer balls are in the shape of a truncated icosahedron instead of a regular icosahedron.

Checklist

Now that you have completed this module, reflect on your ability to do the following.

I can do this	I need to review this
☐ Convert units of length, area, and volume ☐ Calculate the area of composite shapes ☐ Calculate the area of sectors	☐ Go back to Lesson 8.1 Area review
☐ Calculate the surface area of prisms ☐ Calculate the surface area of cylinders ☐ Calculate the surface area of composite solids consisting of prisms and cylinders	☐ Go back to Lesson 8.2 Surface area review
☐ Calculate the volume of prisms ☐ Calculate the volume of cylinder ☐ Calculate the volume of composite solids consisting of prisms and cylinders	☐ Go back to Lesson 8.3 Volume review
☐ Calculate the absolute error of a measurement ☐ Calculate the relative error of a measurement ☐ Calculate the percentage error of a measurement ☐ Explain how using approximations in calculations leads to compounding errors	☐ Go back to Lesson 8.4 Compounding errors
☐ Determine the order of magnitude of base-ten numbers ☐ Determine the position of values on a base-ten logarithmic scale ☐ Interpret values displayed on a logarithmic scale	☐ Go back to Lesson 8.5 Logarithmic scales
☐ Calculate the surface area of right pyramids ☐ Calculate the surface area of right cones	☐ Go back to Lesson 8.6 Surface area of pyramids and cones [10A]
☐ Calculate the volume of pyramids ☐ Calculate the volume of cones	☐ Go back to Lesson 8.7 Volume of pyramids and cones [10A]
☐ Calculate the surface area of spheres ☐ Calculate the volume of spheres	☐ Go back to Lesson 8.8 Surface area and volume of spheres [10A]

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9 Networks

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Lessons

9.1 Introduction to networks
9.2 Planar graphs
9.3 Polyhedrons
9.4 Special types of walks
9.5 Applications of networks

Prerequisite skills

Before starting work on this module, complete the investigation below: Colouring in the plane.

Investigation
Colouring in the plane

Diagnostic pre-test
Take the diagnostic pre-test to check your understanding of the investigation.

Curriculum links

- Interpret networks and network diagrams used to represent relationships in practical situations and describe connectedness (VC2M10SP02)

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Lesson 9.1

Introduction to networks

Learning intentions

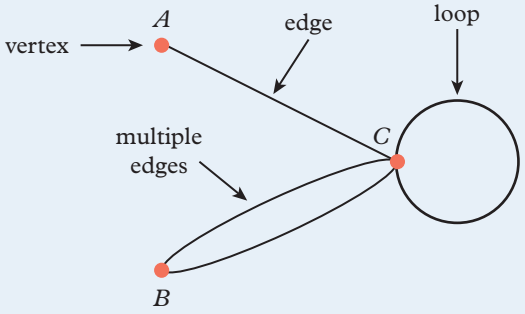
By the end of this lesson, you will be able to ...

✔ understand and use networks.

Networks

Key content video
Introduction to networks

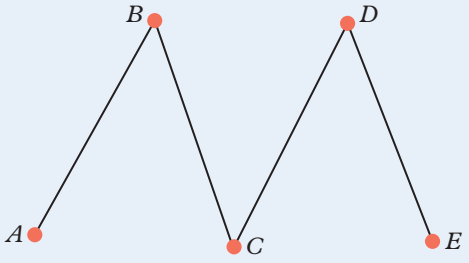
- A **network** (or **graph**) is a collection of objects and the relations between them. An object in a network is called a **vertex** (or node) and a relation between two vertices is called an **edge** (or line).
  - → Vertices and edges often have physical meanings. For example, vertices may correspond to things such as people, towns or electronic devices. Edges may correspond to connections such as friendships, roads or power cables.
  - → For convenience, we assume all graphs have at least one vertex.
- Edges connect exactly two vertices. Vertices on either side of an edge are said to be **adjacent**.
  - → Vertices are labelled with italic capital letters. For example, A, B and C.
  - → Edges can be drawn with straight or curved lines and are named using the vertices they connect. In the diagram shown, the edge AC is the same as the edge CA.
- The **degree** of a vertex is the number of edges connected to that vertex. For example, in the diagram on the right, vertex A has a degree of 1 and vertex B has a degree of 2.
  - → If the same pair of vertices have more than one edge between them, then the edges are called **multiple edges**. For example, in the diagram, vertices B and C are connected by multiple edges.
- A **loop** is an edge with the same vertex at both ends.
  - → A loop contributes two to the degree of that vertex. For example, in the diagram, vertex C has a degree of 5.



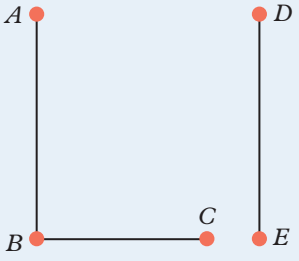
- The **handshaking lemma** states that in every graph, the sum of all vertex degrees is twice the total number of edges. For example, in the diagram above, the sum of degrees is $1 + 2 + 5 = 8$, and there are 4 edges in total.

Walks

- A **walk** is a sequence of vertices visited and the edges between them, starting and ending with a vertex. For example, the walk A-B-C is valid for both graphs below.
- A graph is said to be **connected** if there is a walk between every pair of vertices, and **disconnected** if there exists a pair of vertices with no walk between them.



Connected graph



Disconnected graph

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Worked example 9.1A Determining the degree of vertices in a graph

State the degree for each vertex in this graph.

THINK

1. Count the number of edges connected to each vertex.
2. Remember that a loop adds two to the degree of a vertex.

WRITE

Vertex A has a degree of 2.

Vertex B has a degree of 4.

Worked example 9.1B Drawing a network

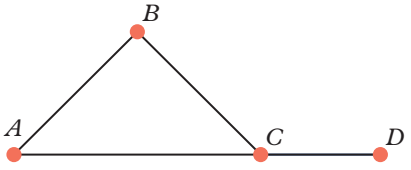
Annabel, Benjamin, Cecilia and Derek are students. Their friendships are given in the table below. Display this information in a network.

Student	Annabel	Benjamin	Cecilia	Derek
Friends	Benjamin, Cecilia	Annabel, Cecilia	Annabel, Benjamin, Derek	Cecilia

THINK

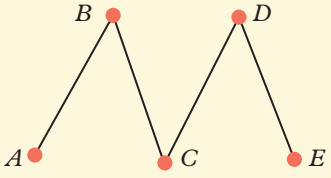
1. Draw a vertex for each student.
2. Label the vertices with the students' initials.
3. Draw an edge between students who are friends with each other.

WRITE



Worked example 9.1C Finding a walk in a graph

Find a walk from A to E in this graph.



THINK

1. Start at vertex A.
2. Move along the edges and through the vertices of the graph until you reach vertex E.
3. Write all the vertices that were visited, in order, during your walk.

WRITE

A-B-C-D-E

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Helpful hints

- ✔ Two graphs may look quite different even if they represent the same information.
- ✔ When drawing a graph, make sure that the vertices and edges are legible and easy to interpret.

Learning pathways

1–4, 6, 7, 9, 11, 13, 15–17	4–16	4–12, 15, 18, 19

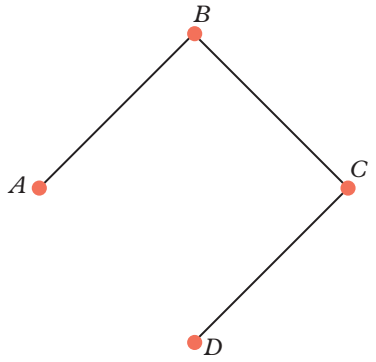
ANS p736

Exercise 9.1A: Understanding and fluency

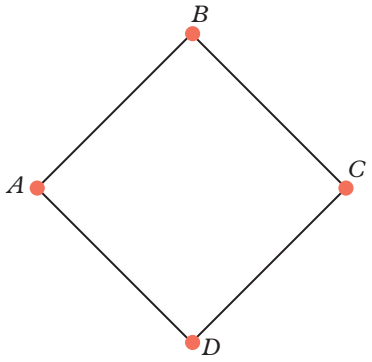
1 For each of the graphs below, state the number of:

i vertices **ii** edges.

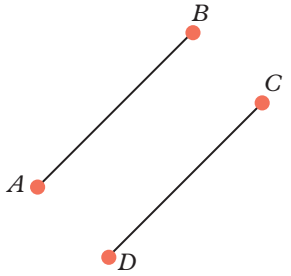
a



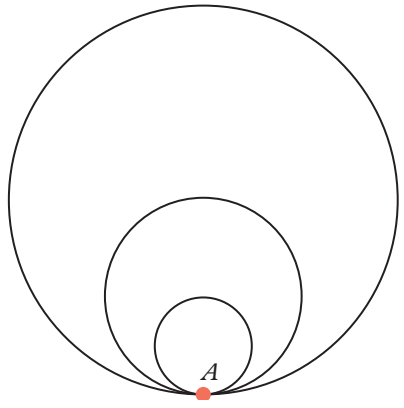
b



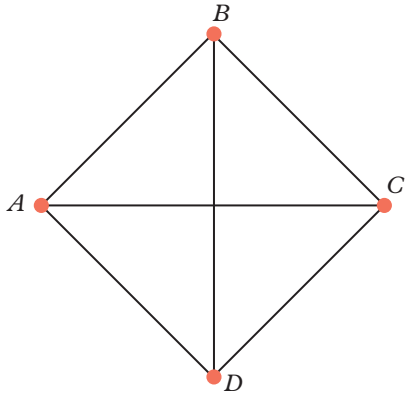
c



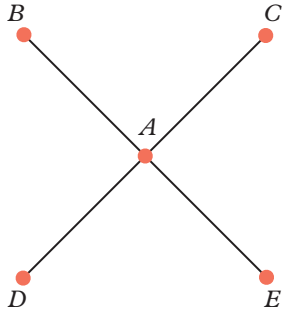
d



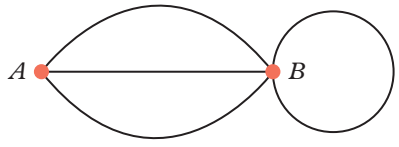
e



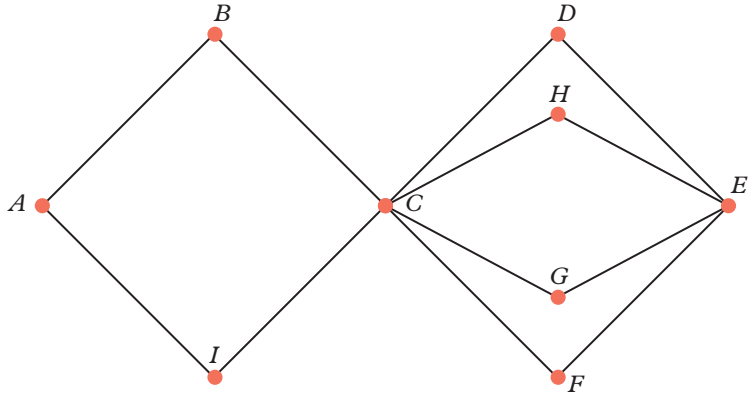
f



g



h



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WE 9.1A **2** Make a table of vertex degrees for each graph in question **1**.

3 Which graph in question **1** is disconnected?

WE 9.1B **4** A coach company is offering cheap trips between regional Victorian towns. Display the following information in a network.

Airport	Ararat (A)	Bendigo (B)	Castlemaine (C)	Daylesford (D)	Euroa (E)
Destinations	Bendigo, Daylesford	Ararat, Castlemaine, Daylesford	Bendigo, Euroa	Ararat	Castlemaine

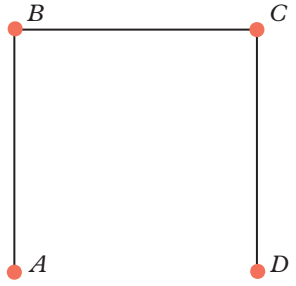
5 A house contains a printer, computer, tv and gaming console. All these devices are connected to a single router via ethernet cables.

a Draw a network to represent this information. Use vertex names *P, C, T, G* and *R* for the printer, computer, tv, gaming console and router, respectively, and use edges to represent ethernet connections between the devices.

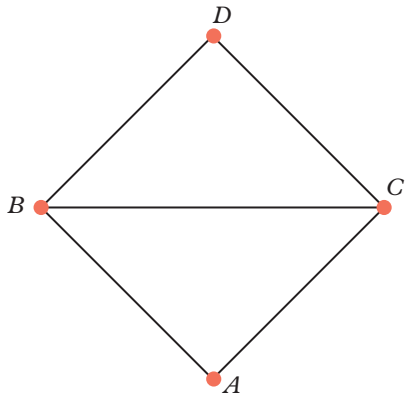
b How many ethernet connections are required at *R*?

WE 9.1C **6** Which of the following graphs contain a walk from *A* to *D*? If a walk exists, write it as a sequence of vertices. If more than one walk exists, only one sample walk needs to be provided.

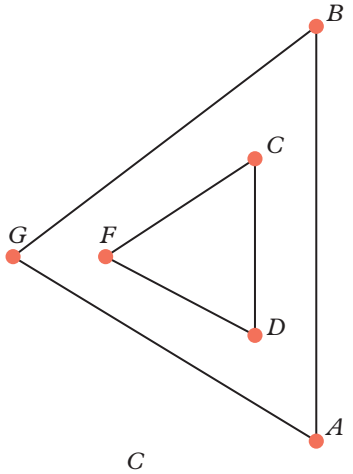
a



b



c

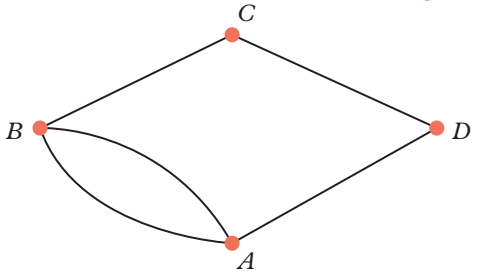


7 Consider the network on the right, where the vertices represent four people at a party and the edges represent handshakes.

a How many handshakes did each person make?

b What is the total number of handshakes?

8 What is the sum of the vertex degrees in the network from question **7**?

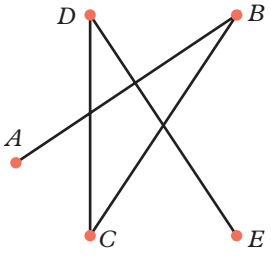


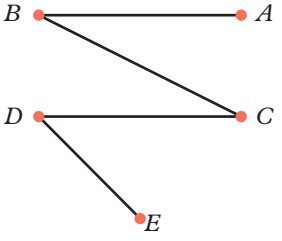
Exercise 9.1B: Problem solving and reasoning

9 A graph has five vertices and each vertex has degree 4. How many edges does it have?

10 Why is it impossible to have a graph with three vertices, where the vertex degrees are 3, 2 and 2?
Hint: If there were such a graph, how many edges must it have?

11 Explain why these two diagrams represent the same graph.





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12 Networks can be used to represent friendships within a group of people.

a Draw the graph which represents the following list of people and their friends.

Person	Uma	Vanessa	Wasim	Xavier	Yvonne	Zhang
Friends	Vanessa, Zhang	Wasim, Uma	Vanessa, Zhang, Yvonne, Xavier	Wasim, Yvonne	Wasim, Xavier	Uma, Wasim

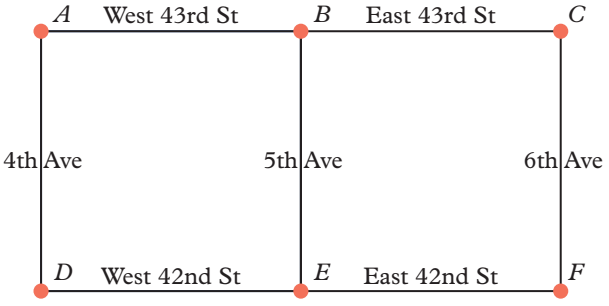
b Fill out the table of vertex degrees.

Vertex	Uma	Vanessa	Wasim	Xavier	Yvonne	Zhang
Degree						

c Uma is telling Vanessa a story about what happened to a friend of a friend of hers. Use your answer from part **a** to find out who the story is about.

d Yvonne is telling Wasim a story about what happened to a friend of a friend of hers. Use your answer from part **a** to find out who the story is *not* about.

13 The streets of Manhattan are divided into East and West by 5th Avenue. The following network is a map of Manhattan between 4th and 6th Avenues and 43rd and 42nd streets.



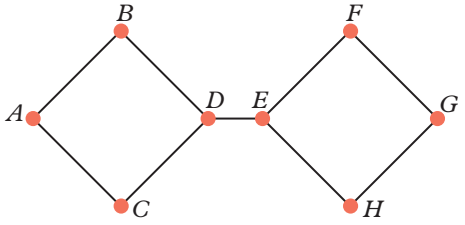
You are given some directions: Start at corner A. Follow West 43rd St to corner B. Follow 5th Avenue to corner E. Follow East 42nd St to corner F. Then follow 6th Avenue to corner C.

a Is this a valid walk? If so, where does this walk lead to?

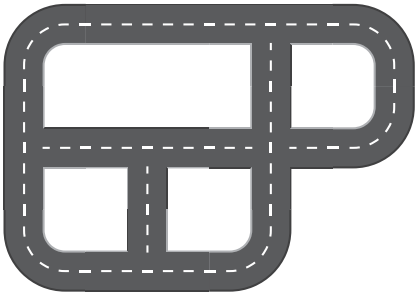
b Write another walk with the same starting and ending vertex.

14 You now have a different set of directions for the network in question **13**: Start at corner A. Follow East 42nd St to corner F. Explain why this walk is not valid.

15 Disconnect this graph by removing exactly one edge. Which edge you have removed?



16 Redraw the roads below as a network using vertices (A to E) to represent intersections and edges to represent a span of road connecting two intersections.



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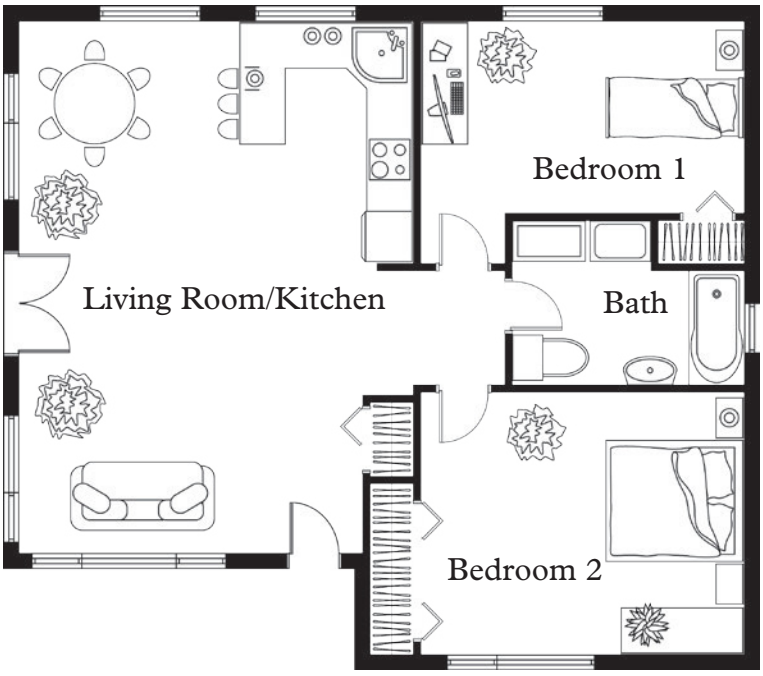
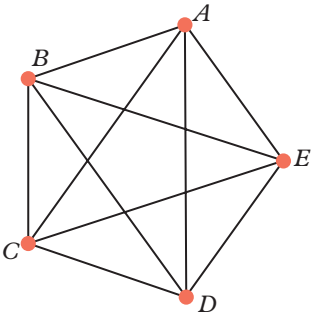
17 Draw the following house plan as a network. Use vertices to represent the Living Room/Kitchen, Bathroom, Bedroom 1 and Bedroom 2, and use edges to represent the doorways that connect the rooms.

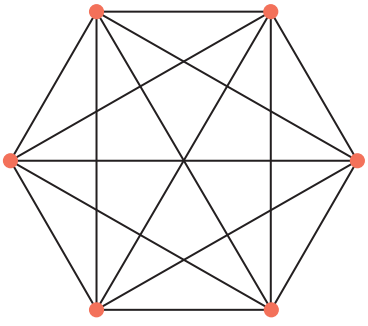


18 Copy the graph on the right and use red and blue pens to colour all the edges. Try to do this without drawing a red triangle or a blue triangle (i.e. three vertices where the edges between them are all the same colour). Hint: First, colour as many edges in red as you can without drawing a triangle, then do the same for the blue edges.



Exercise 9.1C: Challenge

19 Consider the graph below.



a Copy the graph and colour all the edges with a blue or a red pen.

b Find a red or blue triangle in your coloured graph.

c Suppose you use a graph with coloured edges to represent whether dinner guests have met before or not: blue edges connect people who have met before and red edges connect people who have not met before. If you are hosting a dinner party and want to ensure that either three people will all have met before, or three people will have never met before, what is the smallest number of people you should invite? Hint: Compare your answers to questions **18** and **19** part **b**.

Online resources:

Interactive skillsheet
Networks

Quick quiz
9.1

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Lesson 9.2

Planar graphs

Learning intentions

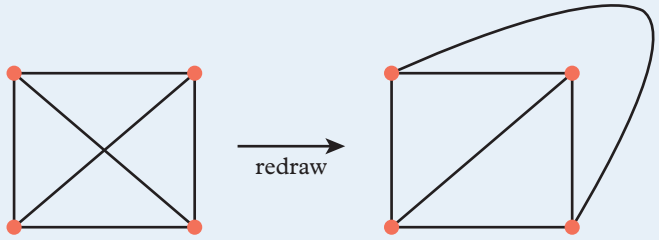
By the end of this lesson you will be able to ...

✔ determine whether a graph is planar, and use Euler's formula for connected planar graphs.

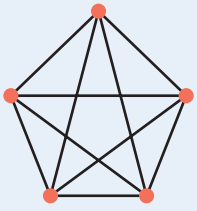
Planar graphs

Key content video
Planar graphs

- All graphs can be redrawn by moving the edges and vertices, provided that all edges still connect the same two vertices as before.
- A **planar graph** is a graph that can be drawn in two dimensions so that no two edges of the graph cross.
  - → If a graph is drawn with edges crossing but can be redrawn without edges crossing then it is a planar graph.



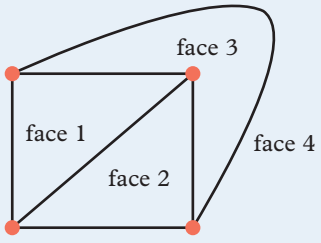
  - → A **non-planar graph** is a graph that cannot be drawn in two dimensions without some edges crossing.



- A drawing of a planar graph with no edges crossing divides the graph into regions, called **faces**, bordered by edges.
  - → There is always one infinite face, surrounding the graph.
  - → A loop forms a face surrounded by one edge.
- The number of vertices (V), the number of edges (E) and the number of faces (F) in a connected planar graph are related by **Euler's formula**:

  $F + V = E + 2$.

  For example, consider the planar graph above. After it has been redrawn, we can see that $F = 4$, $V = 4$, and $E = 6$. These values satisfy Euler's formula, since $4 + 4 = 6 + 2$.



- A **simple graph** is a graph without loops or multiple edges.
- For a simple connected graph with $V \geq 3$ vertices and E edges; if $E > 3V - 6$, then the graph is non-planar.
  - → If $E \leq 3V - 6$, then the graph *may* or *may not* be planar.

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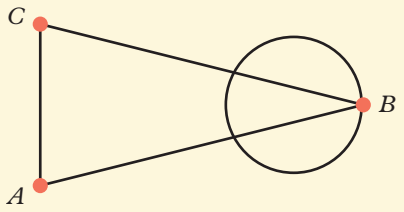
Worked example 9.2A Determining whether a graph is planar

Try to redraw the following graph without crossing edges, and hence determine whether it is planar or non-planar.



THINK

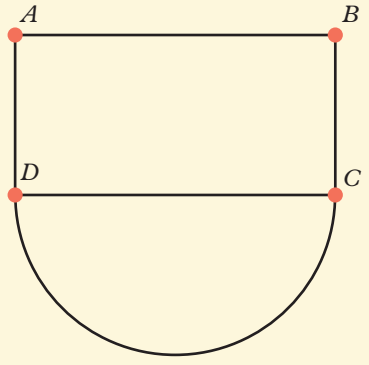
If the graph can be redrawn without crossing edges, it is a planar graph.

WRITE

The graph can be redrawn without crossing edges, so it is a planar graph.

Worked example 9.2B Using Euler's formula to find the number of edges

Consider the following connected planar graph.



a Count the number of vertices in the graph.

b Count the number of faces in the graph.

c Use Euler's formula (instead of counting) to find the number of edges in the graph.

THINK

a There are four vertices: A, B, C and D.

b There are three faces: inside the rectangle, inside the semi circle, and the infinite face surrounding the graph.

c Write Euler's formula. Substitute in the values for V and F and solve for E.

WRITE

a $V = 4$

b $F = 3$

c $F + V = E + 2$

$3 + 4 = E + 2$

$E = 5$

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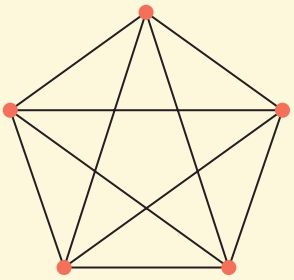
Worked example 9.2C Using $E > 3V - 6$ to prove a graph is non-planar

Consider this simple and connected graph. Use the number of edges and vertices to prove that this graph is non-planar.



THINK	WRITE
1 Count the number of vertices in the graph.	$V = 5$
2 Count the number of edges in the graph.	$E = 10$
3 If the number of edges in the graph exceeds $3V - 6$, then the graph has too many edges and cannot be planar.	$3V - 6 = 9$ As $E > 3V - 6$ $(10 > 9)$, the graph is non-planar.

Helpful hints

- ✔ If a graph can be redrawn without edges crossing, it is a planar graph.
- ✔ $3V - 6$ is the maximum number of edges a simple connected planar graph can have. If the number of edges exceeds this limit, then any drawing of the graph must have some edges crossing, and the graph cannot be planar.

Learning pathways

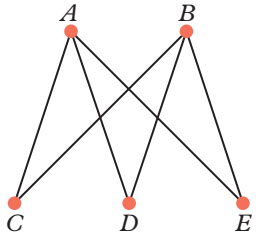
1–5, 7, 11, 14, 17	1–11, 13, 15, 17	1–13, 15–18

ANS p737

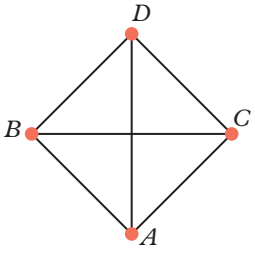
Exercise 9.2A: Understanding and fluency

WE 9.2A **1** Try to redraw the following graphs without edges crossing, and hence determine whether they are planar or non-planar.

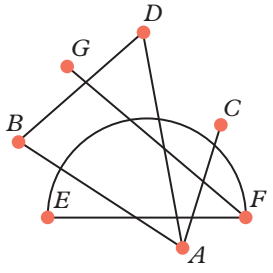
a



b



c



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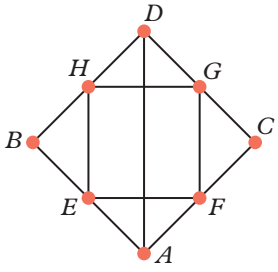

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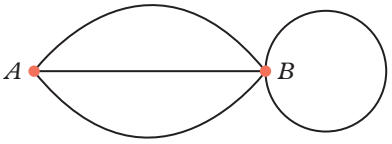
WE 9.2B **2** Consider the connected graph on the right.

a Count the number of vertices in the graph.

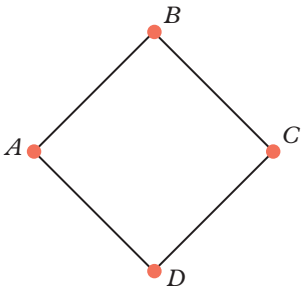
b Count the number of faces in the graph.

c Use Euler's formula (instead of counting) to find the number of edges in the graph.

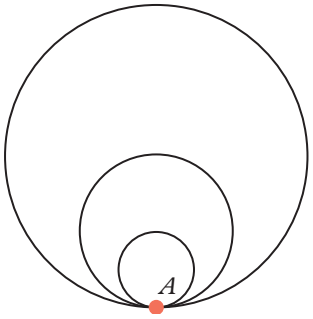


3 Count the number of edges, faces and vertices in the graphs below and show Euler's formula is satisfied.

a



b



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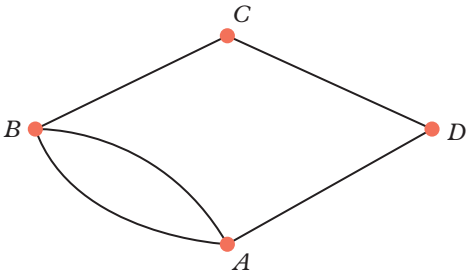


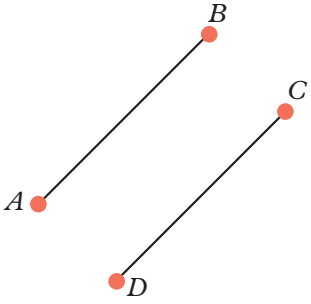
4 Which graphs in question **3** are simple? For the graphs that are not simple, state why.

5 Sketch a planar graph with 5 vertices and 4 edges. How many faces will this graph have?

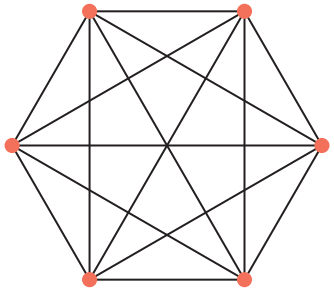
6 Use Euler's formula to complete this table for connected planar graphs.

Faces	Vertices	Edges
4	2	
6	6	
	4	5
	3	6
2		4
3		8

7 Why does the graph from **Lesson 9.1** question **1c** (reproduced on the right) not satisfy Euler's formula?



WE 9.2C **8** Consider this simple and connected graph.



Use the number of edges and vertices to show that this graph is non-planar.

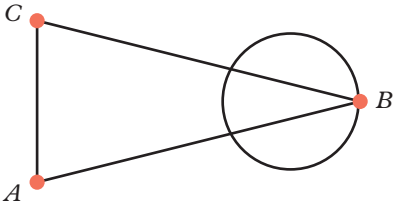
Exercise 9.2B: Problem solving and reasoning

9 Consider the planar graph from **Worked example 9.2A** (reproduced below).

a State the number of edges and vertices.

b Show that $E > 3V - 6$.

c Explain why this planar graph has more than $3V - 6$ edges.



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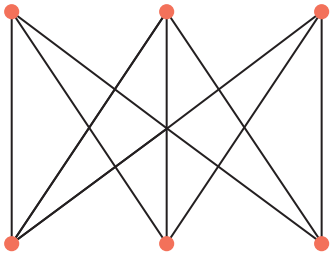
10 If there are *fewer* than $3V - 6$ edges, a graph might be planar or non-planar. For example, consider the following non-planar graph.



a Count the number of edges and vertices.

b Is this graph simple?

c Is this graph connected?

d Show that this non-planar graph has fewer than $3V - 6$ edges.

11 Two houses need to be connected to three underground utilities: power, water and internet. Is it possible to make the connections without utility lines crossing?

12 Repeat question **11** with three houses instead of two.

13 A simple connected planar graph has 5 vertices.

a What is the largest number of possible edges?

b Draw such a graph.

14 The vertex degrees of a connected planar graph are 5, 3 and 2. How many faces does the graph have?

15 Can a simple connected planar graph have 5 faces and 4 vertices? Explain your answer.

16 Faces can also have degrees; the degree of a face is the number of edges along its border.

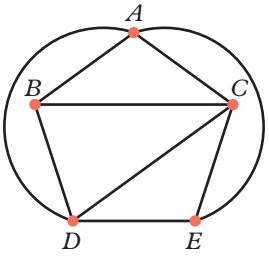
a What does a face of degree 1 look like?

b What does a face of degree 2 look like?

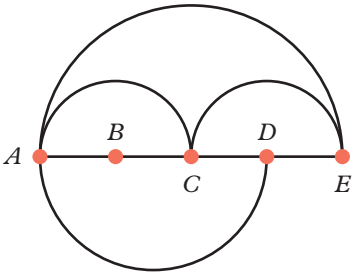
c What does a face of degree 3 look like?

17 Fáry's Theorem states that every simple planar graph can be drawn with all its edges drawn as straight line segments with no crossing edges. Redraw these simple planar graphs with all edges drawn as straight line segments with no crossing edges.

a

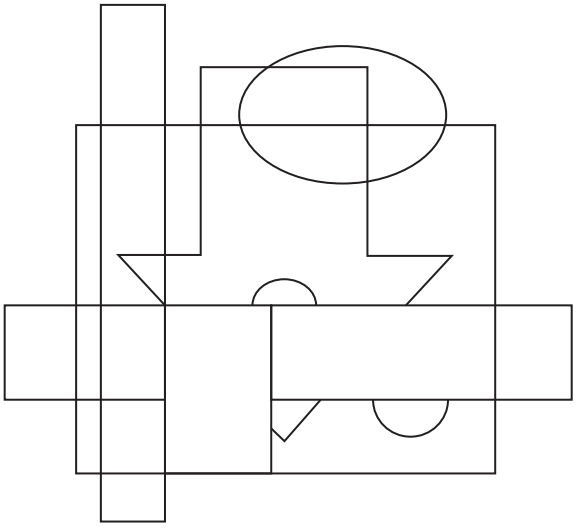


b



Exercise 9.2C: Challenge

18 The four-colour theorem states that only four colours are needed to colour a planar graph without loops so that neighbouring regions have different colours. Copy the planar graph and colour it in using only four colours. Remember to colour the outside face.



Online resources:

- **Interactive skillsheet** Planar graphs
- **BLM** Four-colour theorem
- **Quick quiz** 9.2

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Lesson 9.3

Polyhedrons

Learning intentions

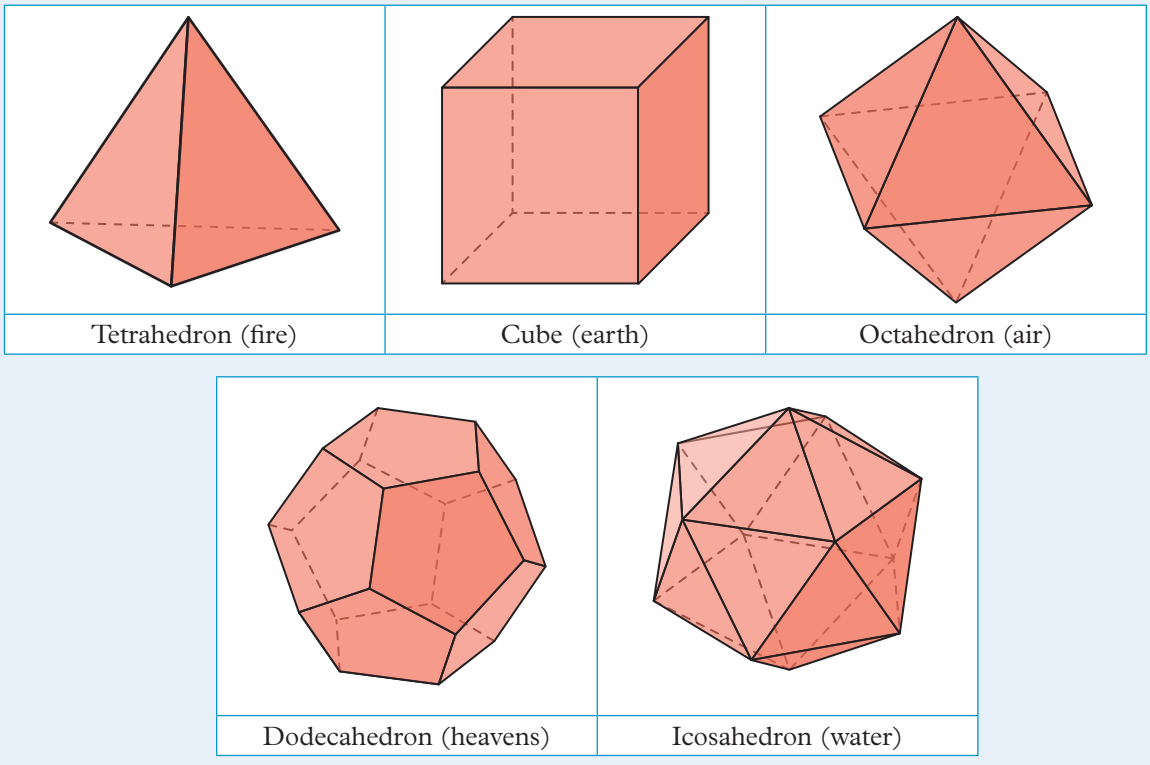
By the end of this lesson you will be able to ...

✔ draw graphs to represent convex polyhedrons, and verify Euler's formula.

Polyhedrons

Key content video
Polyhedrons and planar graphs

- A **polyhedron** is a three-dimensional object with flat polygonal faces, straight edges and vertices.
- A regular polyhedron has faces which are all identical regular polygons, with the same number of faces meeting at every vertex.
- A polyhedron is **convex** if any line segment connecting two vertices on the surface lies completely within the polyhedron.
  → Every convex polyhedron can be viewed as a connected planar graph; therefore, Euler's formula applies to convex polyhedrons.
- A **Platonic solid** is a convex regular polyhedron.
  → The ancient Greek mathematician Euclid, known as the father of geometry, proved that there are only five Platonic solids.
- The Platonic solids get their name from the famous philosopher Plato, who associated them with the four elements: fire, earth, air and water. A fifth Platonic solid was associated with the heavens and constellations.
  The Platonic solids are:



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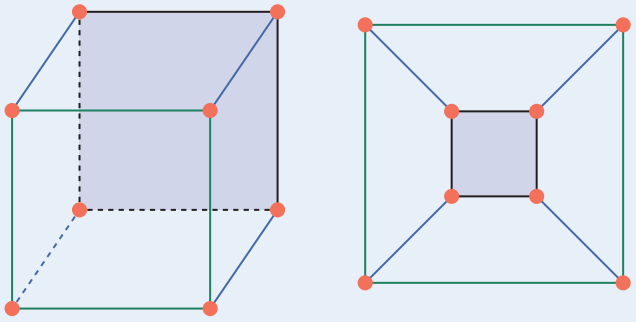
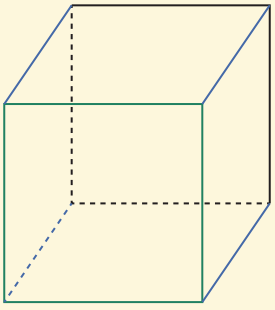
Representing convex polyhedrons as planar graphs

- Convex polyhedrons (such as the Platonic solids) can be represented by planar graphs. To represent a convex polyhedron as a planar graph, imagine punching a hole in one of the faces and peering through to the inside of the solid. You will see a simple connected planar graph with the same faces, edges and vertices as the polyhedron. For example, this is a cube and its corresponding planar graph. The edges have been coloured to help you visualise how the planar graph has been drawn.



Worked example 9.3A Representing Platonic solids as planar graphs



The cube is one of the five Platonic solids.

a Redraw the cube to show it is a planar graph.

b Count the number of vertices, edges and faces in the cube.

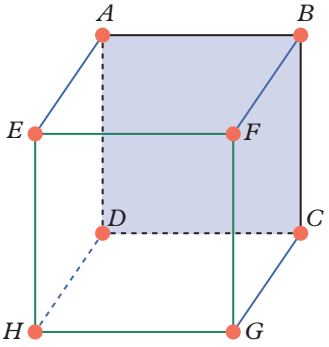
c Show that the cube satisfies Euler's formula.

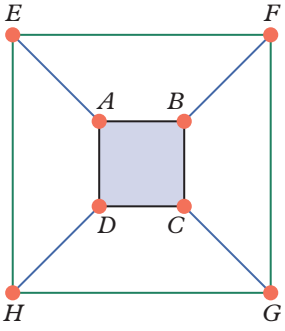
THINK

a 1 Redraw the cube, plotting points on vertices and using labels on each vertex.

2 Peer inside the cube through the 'front face' *EFGH*. You will see a planar graph corresponding to the cube. If helpful, one of the faces (like the 'back face' *ABCD*) can be shaded. The graph drawn does not have any crossed edges.

WRITE





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b Both diagrams from part **a** can be used to answer this question; they are two representations of the same graph.

c The graph is connected and planar, so Euler's formula tells us the number of edges, vertices and faces are related.

b Both the cube and the graph have $V = 8$ vertices, $E = 12$ edges, and $F = 6$ faces. (Remember to count the outside face.)

c $V + F = E + 2$

$8 + 6 = 12 + 2$

$14 = 14$

Therefore, the cube satisfies Euler's formula.

Helpful hints

✔ An alternative approach to show that each convex polyhedron is a planar graph is to flatten the polyhedron, and then rearrange your squashed polyhedron into a planar graph.

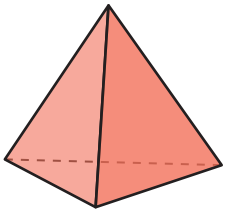
Learning pathways

1–9, 13 | 1, 2, 5–12, 14–16 | 1, 3, 5, 9–12, 14–18

ANS p738

Exercise 9.3A: Understanding and fluency

WE 9.3A **1** The tetrahedron is one of the five Platonic solids.



a Redraw the tetrahedron to show it is a planar graph.

b Count the number of vertices, edges and faces in the tetrahedron.

c Show that the tetrahedron satisfies Euler's formula.

2 Label the following statements as true or false.

a Every Platonic solid corresponds to a planar graph.

b Every planar graph corresponds to a Platonic solid.

3 Consider the types of faces a Platonic solid may have. Which Platonic solids have:

a triangular (3-sided) faces

b square (4-sided) faces

c pentagonal (5-sided) faces?

4 Consider the corners of Platonic solids at which faces meet.

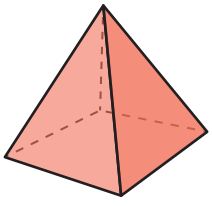
a Which Platonic solids have 3 faces meeting at each corner?

b Which have 4 faces meeting at each corner?

c Which have 5 faces meeting at each corner?

5 The game Dungeons & Dragons has five types of dice, known as d4, d6, d8, d12 and d20. The name of each die refers to the number of faces of the die. Match these dice with the five Platonic solids.

6 Draw the square pyramid, shown on the right, as a planar graph.



7 Count the number of edges, vertices and faces in a square pyramid. Hence, show that Euler's formula applies.

8 Repeat question **7** for a pentagonal pyramid.

9 Consider a convex polyhedron that has 12 faces.

a If it has 30 edges, how many vertices does it have?

b If it has 20 vertices, how many edges does it have?

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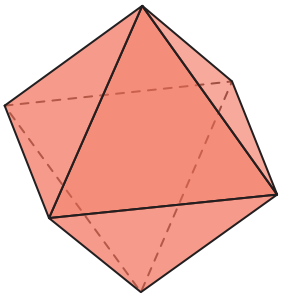
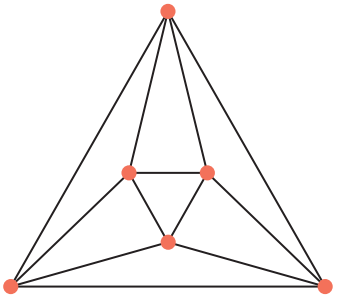
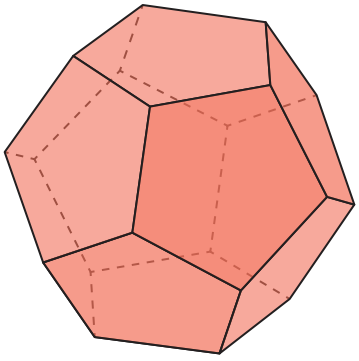
Exercise 9.3B: Problem solving and reasoning

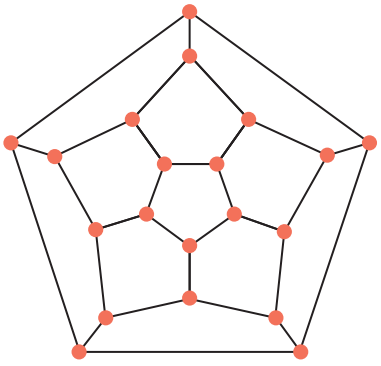
10 Peer inside the octahedron and you will see a simple connected planar graph with eight faces. State the number of vertices, and find the degree of each vertex. Hence, use the handshaking lemma to find the number of edges.



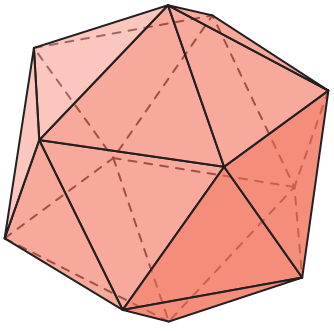


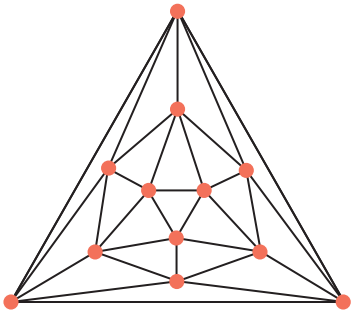
11 Peer inside the dodecahedron and you will see a simple connected planar graph with 12 faces. State the number of vertices, and find the degree of each vertex. Hence, use the handshaking lemma to find the number of edges.





12 Peer inside the icosahedron and you will see a simple connected planar graph with 20 faces. State the number of vertices, and find the degree of each vertex. Hence, use the handshaking lemma to find the number of edges.





13 Use your answers from questions **1, 10, 11** and **12** to complete the following table. Then verify that Euler's formula is satisfied for each platonic solid.

Platonic solid	**Tetrahedron**	**Cube**	**Octahedron**	**Dodecahedron**	**Icosahedron**
Vertices		8			
Faces	4	6	8	12	20
Edges		12			

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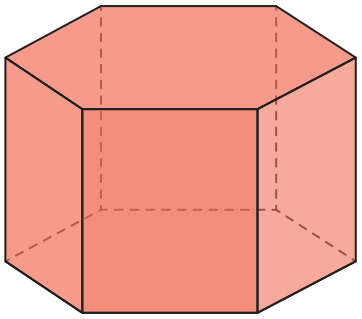
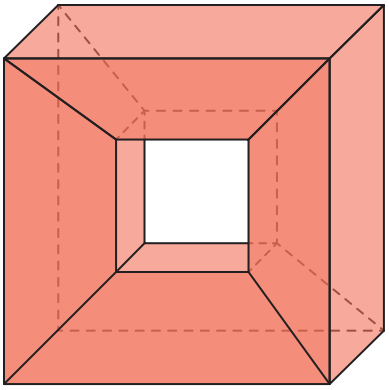
14 Consider the hexagonal prism.

a State the number of faces, vertices and edges.

b Verify that the hexagonal prism satisfies Euler's formula.



15 Consider the non-convex polyhedron shown below.
Count the number of edges, vertices and faces.
Hence, show that $V + F = E$ instead of $V + F = E + 2$.



16 A regular graph is a graph where every vertex has the same degree. Explain why graphs which represent the Platonic solids are regular graphs.

Exercise 9.3C: Challenge

17 You may have noticed that each vertex in a Platonic solid has degree $m \geq 3$, and so the handshaking lemma becomes:

$2 \times E = \text{sum of vertex degrees} = m \times V$.

If we take all the faces and add up the number of edges surrounding that face, then this also equals twice the number of edges. So, if each face is a regular polygon with $n \geq 3$ sides, then:

$2 \times E = n \times F$.

Putting this into Euler's formula and rearranging gives:

$\frac{1}{E} = \frac{1}{n} + \frac{1}{m} - \frac{1}{2}$.

a Fill out the missing entries in the following table with the value of $\frac{1}{n} + \frac{1}{m} - \frac{1}{2}$.

	$n = 3$	$n = 4$	$n = 5$
$m = 3$	$\frac{1}{3} + \frac{1}{3} - \frac{1}{2} = \frac{1}{6}$		
$m = 4$			
$m = 5$			$\frac{1}{5} + \frac{1}{5} - \frac{1}{2} = -\frac{1}{10}$

b Which entries in this table have a positive value for $\frac{1}{E}$? How do these entries relate to the Platonic solids?

c Is it possible to have a graph for which $\frac{1}{E}$ is negative?

d Does this mean Euclid was right, and there are only five Platonic solids?

18 Why don't we need to consider $m \geq 6$ in the previous question?
Hint: This would force $n < 3$; compare with **Lesson 9.2** question **16**.

Online resources:

Interactive skillsheet Platonic solids

Investigation Constructing solid shapes

Quick quiz 9.3

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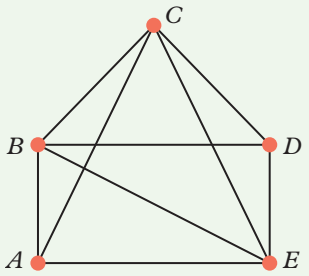
Checkpoint

Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this module.

9.1 **1** Fill out the table of vertex degrees for the network below.

Vertex	*A*	*B*	*C*	*D*	*E*
Degree					



9.1 **2** Is the network in question **1** connected?

9.1 **3** Explain why it is impossible to have a graph with three vertices, where the vertex degrees are 4, 3 and 2.

9.2 **4** Is the network in question **1** simple?

9.2 **5** Is the network in question **1** planar?

9.2 **6** How many faces does the graph in question **1** have?

9.2 **7** Consider a simple connected network of 6 vertices where each vertex has degree 5.

a How many edges does this graph have?

b Is this graph planar?

9.2 **8** Draw a network with 6 vertices, where each vertex has degree 3 and the graph is:

a planar

b non-planar.

9.3 **9** In Dungeons & Dragons, d4 represents a die with 4 sides and d20 represents a die with 20 sides. A chaotic goblin wizard taunts you and you take two d4 emotional damage.

a What is the maximum possible total of the numbers shown on two d4 dice?

b What is the smallest amount of emotional damage you can take?

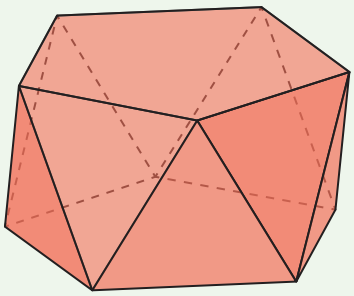
c You attempt a witty retort to the chaotic goblin wizard; however, goblins are thick-skinned creatures. You roll a d20 and if the die shows a score of 13 or more your retort is successful. Are you more likely to succeed or to fail?

9.3 **10** A pentagonal antiprism has two pentagonal faces and ten triangular faces.

a What is the sum of the vertex degrees?

b How many edges does it have?

c Verify Euler's formula for the pentagonal antiprism.



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Lesson 9.4

Special types of walks

Learning intentions

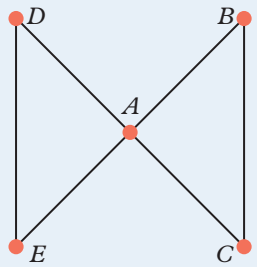
By the end of this lesson, you will be able to ...

✔ classify walks, and determine when Eulerian trails and circuits exist.

Trails, circuits, paths and cycles

Key content video
Special types of walks

- Recall from **Lesson 9.1** that a walk describes a sequence of vertices visited and the edges between them.
  - → A **trail** is a walk where every edge is different.
    For example, in the diagram shown on the right, *A-E-D-A-C* is a trail.
  - → A **path** is a trail where every vertex is different.
    For example, *A-E-D* is a path.



- There are also special names for walks that return to where they started.
  - → A **closed walk** is a walk with at least one edge, that starts and ends at the same vertex.
  - → A **circuit** is a closed walk where every edge is different.
    For example, *A-E-D-A-C-B-A* is a circuit.
  - → A **cycle** is a circuit where every vertex is different, except for the start/end vertex which must be the same. For example, *A-E-D-A* is a cycle since only the start/end vertex is repeated.

Type of walk	Repeated edges allowed	Repeated vertices allowed	Must start and finish at same vertex
Walk	✔	✔	✗
Closed walk	✔	✔	✔
Trail	✗	✔	✗
Circuit	✗	✔	✔
Path	✗	✗	✗
Cycle	✗	✗*	✔

* excluding the start/end vertex

Eulerian trails and circuits

- Trails and circuits that contain every edge in a graph exactly once are called **Eulerian trails** and **Eulerian circuits**.
- There is an easy way to know when an Eulerian trail or circuit exists: by counting the number of vertices of odd degree. Specifically, suppose your graph is connected and has n vertices of odd degree.
  - → If $n = 0$, then there must be an *Eulerian circuit* and the graph is **Eulerian**.
    For example, in the graph above right, $n = 0$ since there are 0 vertices of odd degree.
    Therefore, the graph is Eulerian and an Eulerian circuit exists, e.g. *D-A-B-C-A-E-D*.
  - → If $n = 2$, then there must be an *Eulerian trail* and the graph is **semi-Eulerian**.
    The starting and ending vertices are the vertices of odd degree.
  - → Otherwise, there is no Eulerian circuit or Eulerian trail, and the graph is not Eulerian.

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Worked example 9.4A Classifying walks

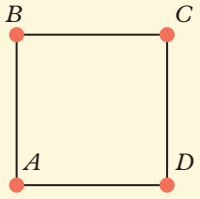
For the graph below, classify the following sequences as accurately as possible.

a *A-D-A*

b *A-D-C*

c *A-D-A-B*

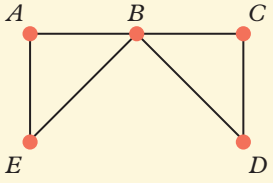
d *A-C*



THINK	WRITE
a This sequence re-uses edge *AD*, so it must be a walk. As it starts and finishes with the same vertex (*A*), it is a closed walk.	**a** This is a closed walk.
b This sequence has no repeated edges or vertices. As it starts and finishes at different vertices, it is a path.	**b** This is a path.
c This sequence re-uses edge *AD*, so it must be a walk. It starts and finishes at different vertices, so it is not closed.	**c** This is a walk.
d There is no edge to directly connect vertices *A* and *C*, so this walk is not possible.	**d** This is an invalid walk.

Worked example 9.4B Finding an Eulerian circuit in a graph

Find an Eulerian circuit, if one exists, in the following graph.



THINK	WRITE
1 Check the number of vertices with odd degree to see if an Eulerian circuit exists. There are zero vertices with an odd degree, so there must be an Eulerian circuit.	
2 Start at A, then move clockwise around the network until you return to A. This is a circuit because it starts and ends at A without re-using edges. This circuit is Eulerian because it contains every edge exactly once.	An Eulerian circuit for this network is *A-B-C-D-B-E-A*.

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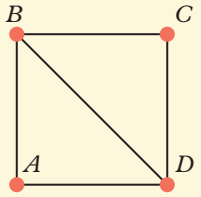
Worked example 9.4C Finding an Eulerian trail in a graph

Find an Eulerian trail, if one exists, in the graph below.



THINK

1 Check the number of vertices with odd degree to see if an Eulerian trail exists. There are two vertices with an odd degree, so there must be an Eulerian trail.

2 Start at B, then move clockwise around the network until you return to B. Then use the diagonal edge to finish at D. This is a trail because edges are not re-used. This trail is Eulerian because it contains every edge exactly once.

WRITE

An Eulerian trail for this graph is *B-C-D-A-B-D*.

Helpful hints

- ✔ A visual way to understand walks is to place your finger on a vertex, then move your finger along the graph and finish at any vertex. Everything that your finger touches (i.e. both vertices and edges) is a walk. Trails, circuits, paths and cycles are just special kinds of walks.
- ✔ If a trail or circuit contains every edge exactly once, then it is said to be Eulerian.
- ✔ If a graph has $n = 2$ vertices of odd degree, then the Eulerian trail will start at one of these vertices and end at the other vertex.
- ✔ When there are multiple edges in a graph, it can be helpful to label them (using lower-case letters) and to include the edges in the walk to indicate which edge labels are being used.

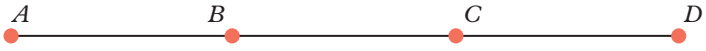
Learning pathways

1, 2, 4–11, 13, 15, 16	2, 4–12, 14, 15	2–4, 6, 7, 9–12, 14, 17

ANS p739

Exercise 9.4A: Understanding and fluency

1 The graph below consists of four vertices (labelled *A, B, C, D*).



Remember from **Lesson 9.1** that vertices are adjacent if they are connected by an edge.

a List the adjacent vertices of *B*.

b Is *A* adjacent to *C*?

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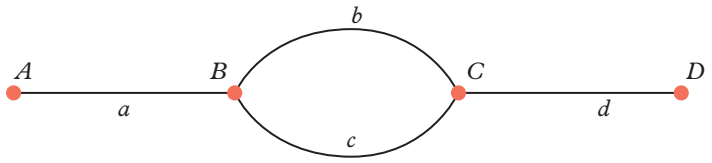
Use the following graph to answer questions **2** to **5**.



2 Consider two different walks from A to D.

One walk is: A-a-B-b-C-d-D.

The other walk is: A-a-B-c-C-d-D.

a Write the walks as sequences of vertices only.

b In this example, is it a good idea to abbreviate walks to just a sequence of vertices?

WE 9.4A **3** Classify the following sequences as accurately as possible.

a A-a-B-c-C

b A-a-B-c-C-b-B

c A-b-B-a

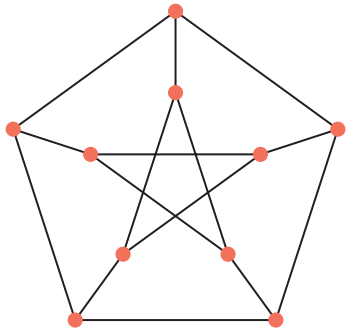
d B-c-C-b-B-c-C

4 Consider the walk A-a-B-b-C-c-B-a-A. What kind of walk is this?

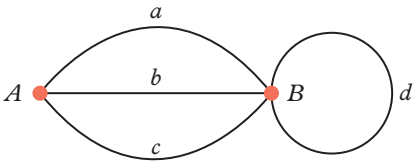
5 Find a circuit in the graph.

WE 9.4B **6** Find a different Eulerian circuit for the graph in **Worked example 9.4B**.

7 Explain why the following graph is not Eulerian.

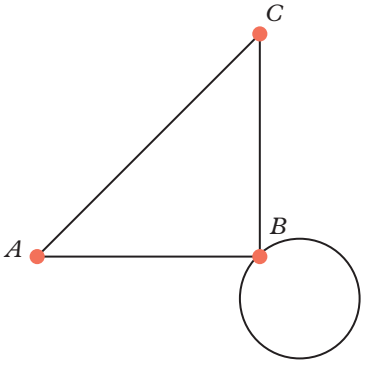


8 Explain why the following graph is semi-Eulerian.



WE 9.4C **9** Find an Eulerian trail in the graph from question **8**.

10 Explain why the following graph is Eulerian by counting the number of vertices of odd degree.



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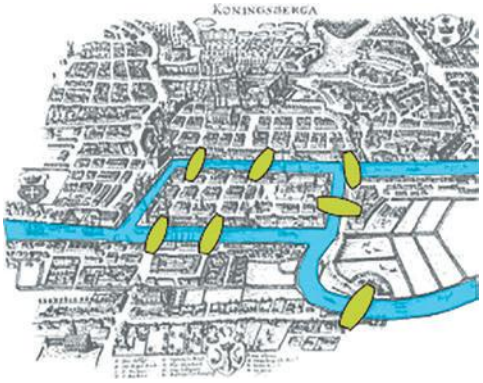
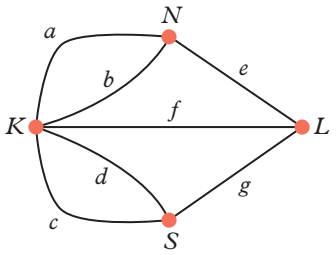
Exercise 9.4B: Problem solving and reasoning

11 The old city of Königsberg lies on the river Pregel. The city has seven bridges, as shown below. For many years, the residents of Königsberg were intrigued by a question: is it possible to find a walk through the city which crossed each of the seven bridges exactly once?





The mathematician Leonard Euler found the answer to this question, and in doing so he created the branch of mathematics now known as graph theory. Euler realised that the problem was fundamentally about how many bridges were connected to each region of the city.

The city regions are represented by vertices: N for the North side of the river, S for the south side, K for Kneiphof Island and L for Lomse Island. The bridges are represented by edges. The network above shows the city regions as vertices and the bridges between them as edges. Now we need to solve the problem, 'Does this graph have an Eulerian trail?'

a Complete the table of vertex degrees.

Vertex	N	K	S	L
Degree				

b How many vertices have odd degree?

c Is the network Eulerian, semi-Eulerian or not Eulerian?

d Is there a walk through Königsberg which crosses every bridge exactly once? Explain your answer.

12 The mayor of Königsberg wants to encourage tourism with a grand tour of the bridges. Tour groups must start and end outside the mayor's offices in the northern part of the city and traverse every bridge exactly once.

a The mayor will allow you to build one additional bridge. How can you explain to the mayor that the task is still impossible?

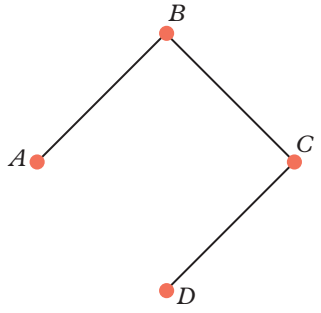
b The mayor really wants this to happen and has secured funding for two additional bridges on the condition that both of them start from the northern side of the river. Is the task now possible? Explain your answer.

c The mayor is willing to budge a little and as well as the two additional bridges will allow the tour to finish in any location, provided it starts from the mayor's offices. Is the task now possible? Explain your answer.

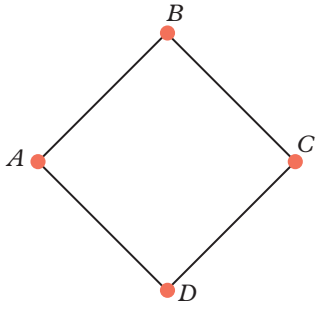
13 Counting the number of vertices of odd degrees helps identify the existence of Eulerian trails and Eulerian circuits.

a Determine whether the following graphs are Eulerian, semi-Eulerian or not Eulerian.

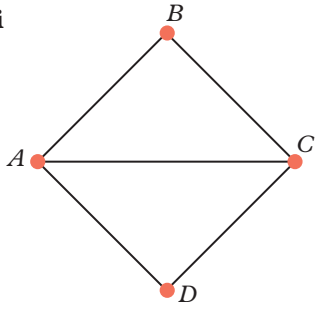
i



ii



iii



b For the graphs in part **a**, find an Eulerian circuit and/or Eulerian trail.

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14 Which Platonic solid can be represented by a graph which has an Eulerian circuit along its edges?

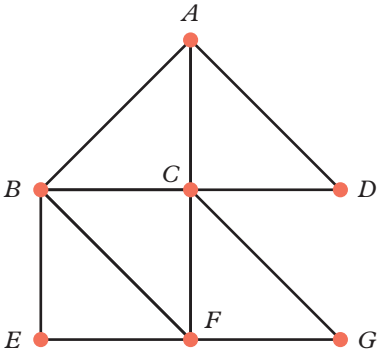
15 Design a garbage truck route that visits every street in these maps exactly once, or explain why this task is impossible.

a

b

16 After a graph has been shown to be semi-Eulerian, Fleury's algorithm can be used to identify an Eulerian trail in the graph. Fleury's algorithm runs through the following steps.

1. Start at one of the two vertices of an odd degree.
2. Choose any edge connected to this vertex, provided that deleting this edge will not disconnect the graph.
3. Add this edge to your trail, moving to the vertex at the other end of this edge.
4. Delete this edge from the graph.
5. Return to step 2 until all edges have been added to your trail.



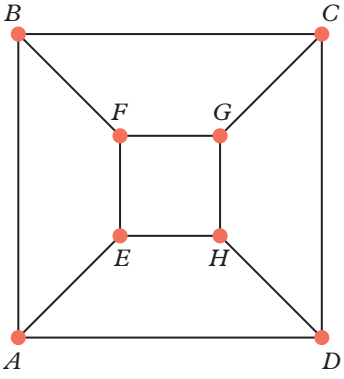
a Explain why the graph on the right is semi-Eulerian.

b Use Fleury's algorithm to identify two Eulerian trails in this graph with different starting vertices.

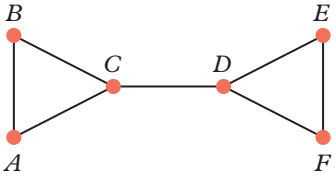
Exercise 9.4C: Challenge

17 A Hamiltonian cycle is a cycle that visits every vertex exactly once.
Unlike Eulerian circuits, a simple way to determine if a graph has a Hamiltonian cycle has never been found.

a Find a Hamiltonian cycle for the cube.



b Is there a Hamiltonian cycle in the graph below? Explain your answer.



Online resources:

Interactive skillsheet
Eulerian trails and circuits

Quick quiz
9.4

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Lesson 9.5

Applications of networks

Learning intentions

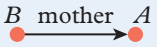
By the end of this lesson, you will be able to ...

✔ use directed edges in networks.

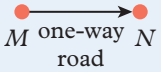
Directed networks

Key content video
Directed networks

- A **directed edge** is an edge that has a direction indicated by an arrow.
  - → For example, suppose person B is the mother of person A.
    We can represent this directional relationship with a directed edge.
    The arrow represents the direction of the 'mother of' relationship.



  - → Another example is a one-way road connecting places M and N.
    The one-way road can be represented with a directed edge.
    The diagram shows that you can travel from M to N, but not from N to M.



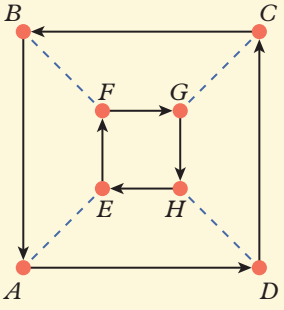
- A **directed network** is a network with directed edges.

Worked example 9.5A Using networks to represent kin systems

The Warlpiri are a people whose country is found in Central Australia around the Tanami Desert. The following kin system is inspired by the Warlpiri people and has been simplified for the purposes of this example. Suppose everyone belongs to one of eight kin groups, A, B, C, D, E, F, G or H, and that:

- Marriage is recommended between kin groups A and E, B and F, C and G, or D and H. Recommended marriages are represented by the dashed blue edges in the network below. Here and throughout we will simplify things by assuming all children are born from recommended marriages.
- The kin group of the mother determines the group of all her children.

	Kin group							
Mother	A	B	C	D	E	F	G	H
Children	D	A	B	C	F	G	H	E



The directed edges in the network on the right represent the 'mother of' relation.

What is the kin group of the parents for children in kin group A?

THINK	WRITE
1 Follow the 'mother of' arrow backwards from A to find the kin group of their mother.	Their mother is in kin group B.
2 Once you know the kin group of their mother, follow the dashed line to find the recommended kin group of her partner.	Their father is in kin group F.

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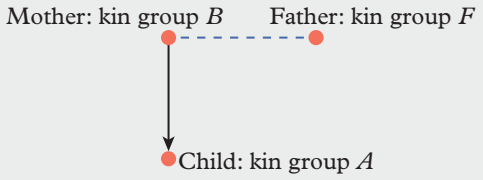
Helpful hints

✔ Draw a family tree, and then add kin groups.



Learning pathways

1–3, 6–10 | 1, 3, 5, 8, 9, 11–14 | 1, 3–5, 9, 11–13, 15

ANS p740

Exercise 9.5A: Understanding and fluency

Use the kin diagram in **Worked example 9.5A** to answer questions **1** to **5**.

WE 9.5A **1** Suppose you are in kin group D.

a What is the kin group of your siblings?

b What is the recommended kin group of your spouse?

c What is the kin group of your sister's children?

d What is the kin group of your brother's children?

2 Repeat question **1**, supposing that you are in kin group H.

3 Complete this paternity table. Assume that children are only born from recommended marriages.

	Kin group							
Father	A	B	C	D	E	F	G	H
Children	F					A		

4 Suppose you belong to kin group A. What is the kin group of:

a your mother's sister's children

b your mother's brother's children

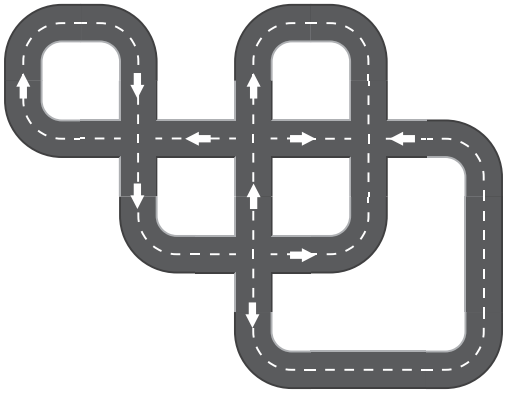
c your father's sister's children

d your father's brother's children?

5 Complete the following table.

	Kin group							
You	A	B	C	D	E	F	G	H
Cousins	A or G						G or A	

6 Draw this system of one-way streets as a directed network using labelled vertices (A to D) to represent intersections and directed edges to represent one-way streets.



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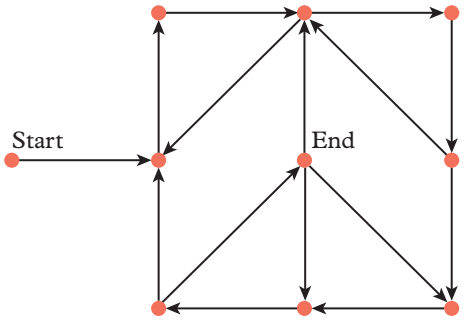
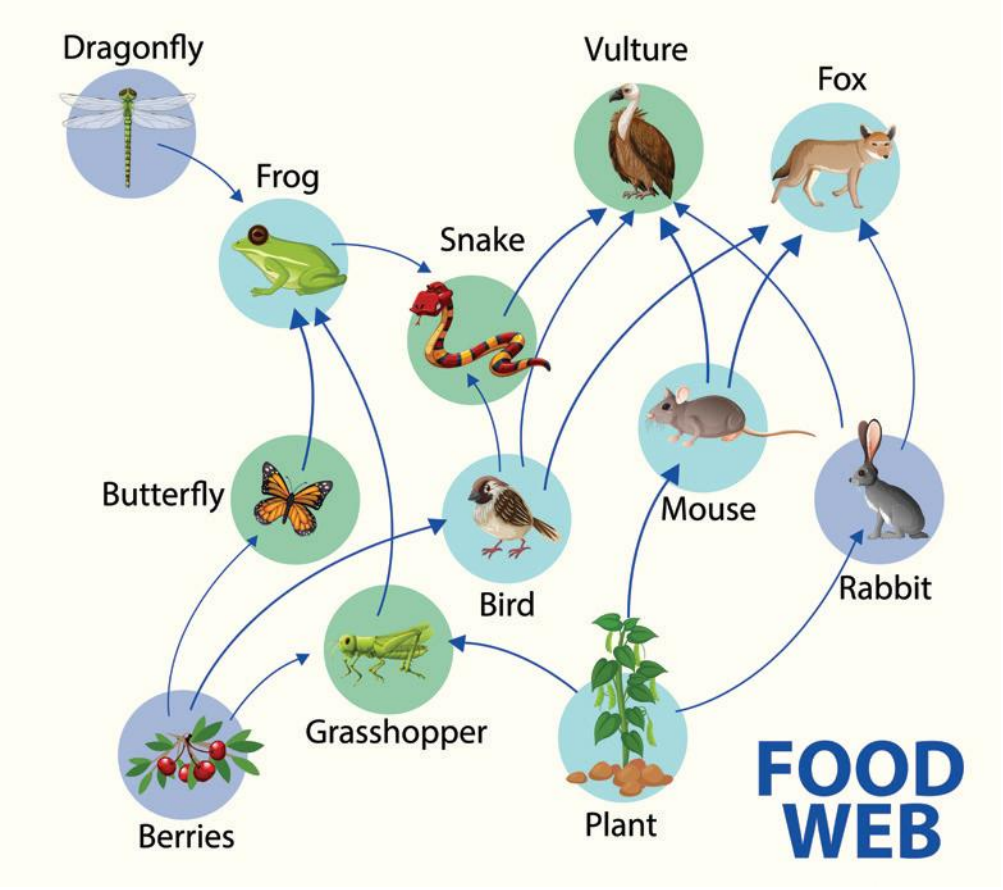
7 Find your way to the end of the labyrinth. You can only travel along edges in the direction of the arrows.



a How many vertices did you pass on your way?

b How many edges did you travel through?

8 This food web represents a simple ecosystem, where the arrow at the end of an edge represents the 'is eaten by' relationship. For example, dragonflies are eaten by frogs.



Answer these questions according to the relationships shown in the food web.

a Which creatures eat birds?

b What do frogs eat?

c Do birds eat mice?

d How many creatures do vultures eat?

e Do foxes have any predators?

9 Draw a directed network to represent the following relationships.

- Rabbits eat carrots and grass
- Grain is eaten by mice, grasshoppers and birds
- Foxes eat rabbits and birds
- Owls eat mice and grasshoppers
- Grasshoppers eat grass
- Birds eat grasshoppers

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Exercise 9.5B: Problem solving and reasoning

Use the kin diagram in **Worked example 9.5A** to answer questions **10** to **13**.

10 A child is born into kin group *A*. What is the kin group of the:

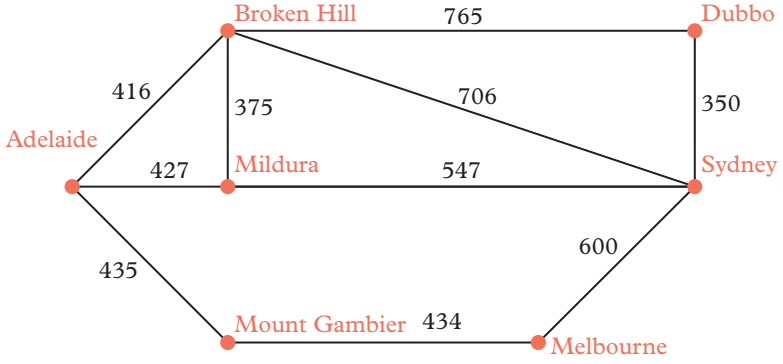
- **a** maternal grandmother
- **b** maternal grandfather
- **c** paternal grandmother
- **d** paternal grandfather?

11 Are you always in the same kin group as your paternal grandfather?
Hint: Use the table from question **3**.

12 Are you ever recommended to marry your cousin?
Hint: Use the table from question **5**.

13 You are in kin group *A* and your mother re-marries. What is the kin group of your step-sisters and step-brothers?

14 Consider the network of regional flights. Each edge has a number that represents the cost (in $) of each route. What is the cheapest route from Dubbo to Mount Gambier?



Exercise 9.5C: Challenge

15 The Martuthunira language group from the Pilbara region in Western Australia has four kinship groups (which are labelled as *A*, *B*, *C* and *D*).

Child	Mother	Father
A	*D*	*C*
B	*C*	*D*
C	*B*	*A*
D	*A*	*B*

- **a** Draw a diagram similar to **Worked Example 9.5A** for this kin relation.
- **b** If you are in kin group *A*, what is the kin group of your father's father?
- **c** Are you always in the same kin group as your mother's mother?

Online resources:

Interactive skillsheet
Directed networks

Quick quiz
9.5

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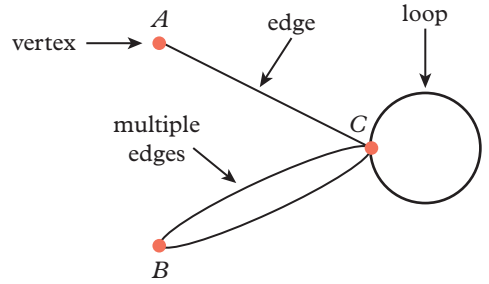
Lesson 9.6

Review: Networks

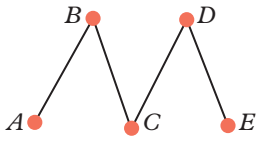
Module summary

Networks

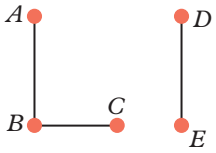


Walks

- A walk is an alternating sequence of vertices.
- A graph is connected if a walk exists between every pair of vertices in the graph.



Connected graph



Disconnected graph

Planar graphs

- Planar graphs can be drawn without edges crossing.
- There is always one infinite 'outside' face.
- Euler's formula is satisfied for all connected planar graphs:

$$F + V = E + 2$$

Handshaking lemma

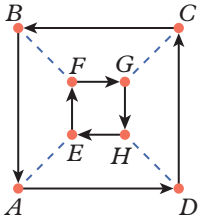
sum of vertex degrees = 2 × number of edges

Simple graphs

- Simple graphs have no loops or multiple edges.
- For simple connected graphs with $V \geq 3$ vertices and E edges:
  → If $E > 3V - 6$ graph is non-planar.
  → If $E \leq 3V - 6$ graph may or may not be planar.

Directed networks

- A directed edge has a direction indicated by an arrow.
- A directed network is a network with directed edges.

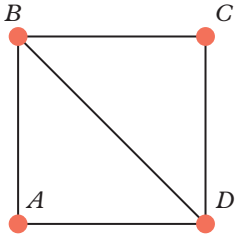


Trails and circuits

Type of walk	Repeated edges allowed	Repeated vertices allowed	Must start and finish at same vertex
Walk	✔	✔	✗
Closed walk	✔	✔	✔
Trail	✗	✔	✗
Circuit	✗	✔	✔
Path	✗	✗	✗
Cycle	✗	✗*	✔

*excluding the start/end vertex

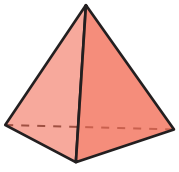
- Eulerian trails and Eulerian circuits contain every edge in the graph exactly once.
- If a connected graph has n vertices of odd degree:
  → if $n = 0$, there is an Eulerian circuit and graph is Eulerian
  → if $n = 2$, there is an Eulerian trail and graph is semi-Eulerian
  → otherwise there is no Eulerian circuit or trail, and the graph is not Eulerian.



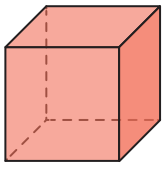
The degrees of vertices B and D are odd; there are $n = 2$ vertices of odd degree, so an Eulerian trail exists.

Polyhedrons

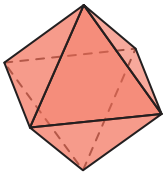
- A polyhedron is a 3D object with flat polygonal faces, straight edges and vertices.
- Euler's formula applies to convex polyhedrons because they can be viewed as connected planar graphs.
- There are five Platonic solids, all of which can be represented as planar graphs.



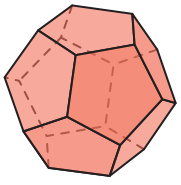
Tetrahedron



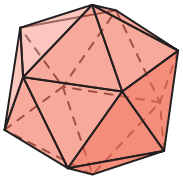
Cube



Octahedron



Dodecahedron



Icosahedron

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Review questions 9.6A: Mathematical literacy review

Review quiz
Take the review quiz to assess your knowledge of this module.

Quizlet
Test your knowledge of this module by working individually or in teams.

The following key terms are used in this module:

- adjacent
- circuit
- closed walk
- connected
- convex
- cycle
- degree
- directed edge
- directed network
- disconnected
- edge
- Eulerian
- Eulerian circuit
- Eulerian trail
- Euler's formula
- faces
- graph
- handshaking lemma
- loop
- multiple edges
- network
- path
- planar graph
- Platonic solids
- polyhedron
- semi-Eulerian
- simple graph
- trail
- vertices
- walk

1 Explain what makes a graph connected.

2 Name the five Platonic solids.

3 How can you determine the degree of a vertex?

4 What makes a circuit an Eulerian circuit?

5 Identify the key terms being referenced in each of these definitions.

a a walk without repeated edges or repeated vertices

b a graph that can be drawn without crossing edges

6 Using an example, provide a definition in your own words for the following key terms.

a cycle **b** simple graph

7 Complete the following sentences using words from the key terms list.

a A graph is said to be _____ if there is a _____ between every pair of vertices.

b A polyhedron is said to be _____ if any line segment connecting two _____ on the surface lies completely within the polyhedron.

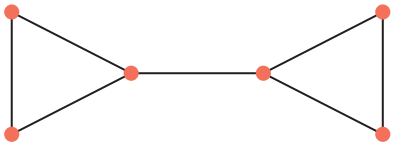
Review questions 9.6B: Multiple choice

9.1 **1** A graph has a vertex degree sequence of 3, 2, 1. How many edges does it have?

A 2 **B** 3 **C** 4 **D** 5

9.1 **2** What is the largest vertex degree in this graph?

A 1
B 2
C 3
D 4

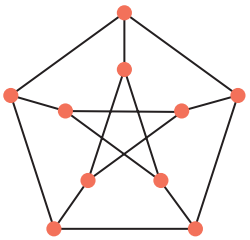


9.2 **3** How many faces does the graph from question **2** have?

A 1 **B** 2 **C** 3 **D** 4

9.2 **4** What is the largest number of edges that you can remove from this network without making the graph disconnected

A 3
B 4
C 5
D 6



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9.2 **5** What is the smallest number of edges that must be removed from the graph in question **4** to make the graph disconnected

A 1 **B** 2 **C** 3 **D** 4

9.3 **6** Which platonic solid has 12 faces?

A cube **B** octahedron **C** dodecahedron **D** icosahedron

9.3 **7** Which platonic solid has 12 vertices?

A cube **B** octahedron **C** dodecahedron **D** icosahedron

9.4 **8** Every vertex in a connected graph has an even degree. This means the graph is:

A not Eulerian **B** semi-Eulerian **C** Eulerian **D** directed

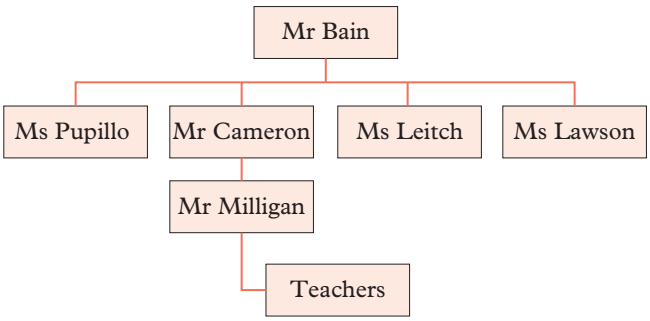
9.4 **9** Suppose three vertices in a graph have an odd degree and the rest have an even degree. This graph is:

A not Eulerian **B** semi-Eulerian **C** Eulerian **D** invalid

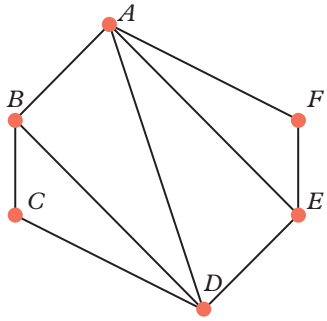
9.5 **10** The following organisational chart shows the hierarchy within Oban High School. A teacher reports a serious incident to Mr Milligan. How many steps will it take for this report to be handed up from Mr Milligan to Mr Bain?

A 1 **B** 2 **C** 3 **D** 4

Oban High School Organisation Chart



Review questions 9.6C: Short answer



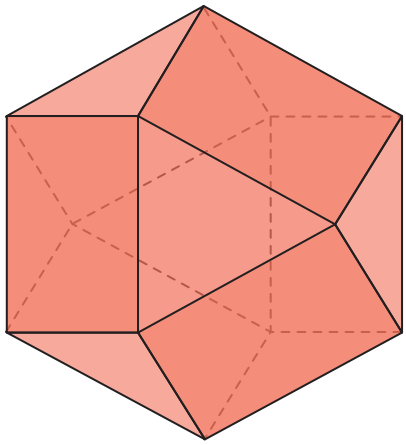
9.1 **1** Fill out the table of vertex degrees for the graph on the right.

Vertex	*A*	*B*	*C*	*D*	*E*	*F*
Degree						

9.2 **2** The graph from question **1** is planar. How many faces does it have?

9.2 **3** Use Euler's formula to find the number of edges in the graph from question **1**.

9.3 **4** The cuboctahedron is related to the cube and octahedron. According to Archimedes, it may have been known to Plato. It has 6 square faces and 8 triangular faces. What is the sum of the vertex degrees?



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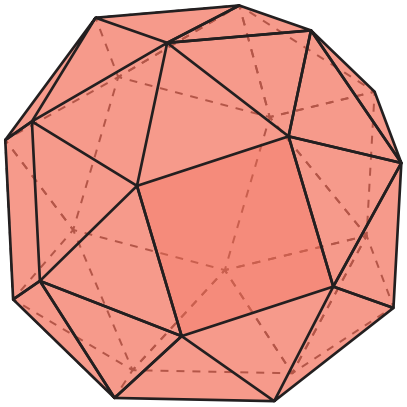
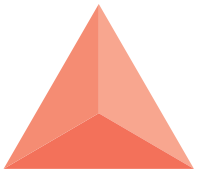
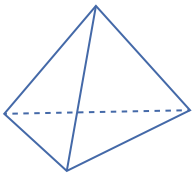
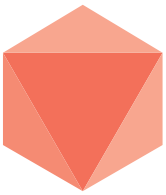
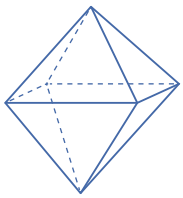
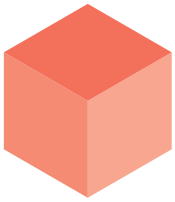
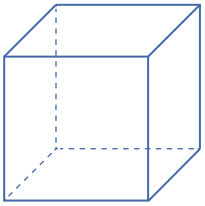
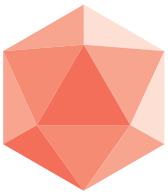
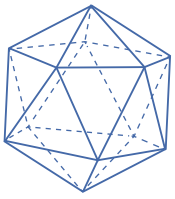
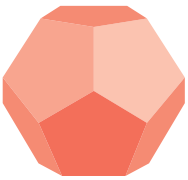
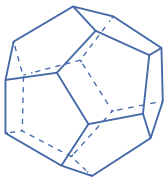
9.3 **5** The snub cube is a convex three-dimensional shape with 6 square faces and 32 triangular faces. Why is it not considered a Platonic solid?



9.3 **6** Why is it impossible for a convex polyhedron to have five faces, five vertices and ten edges?

9.3 **7** Fill out the table for the number of faces, vertices and edges of the Platonic solids.

Solid			Faces	Vertices	Edges
Tetrahedron					
Octahedron					
Cube					
Icosahedron					
Dodecahedron					

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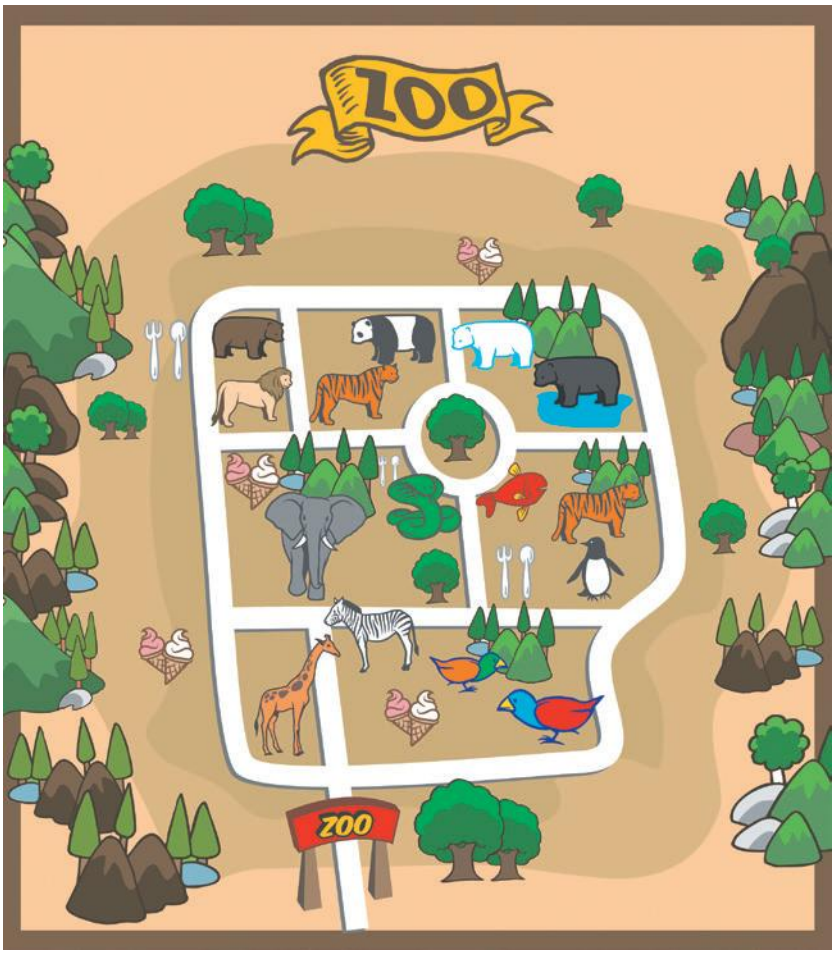
9.4 **8** Consider the following town map as a network of intersections (vertices) connected by roads (edges).

a How many vertices are in the network?

b What is the highest vertex degree?

c Can you design a bus route that visits every street exactly once?

9.4 **9** Why is it impossible to pass through every footpath in the zoo below exactly once?



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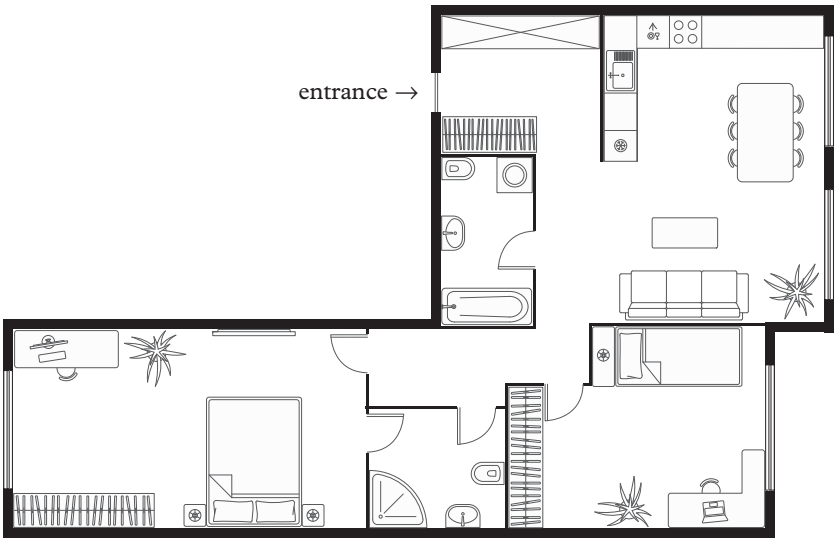
9.4 **10** You have an eccentric uncle who can only go to sleep once he has passed through and closed every internal door in his apartment, shown below. If he starts in the bathroom by the entrance, in which bedroom does he sleep at night?



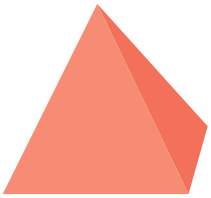
9.5 **11** By referring to the kin system in **Worked example 9.5A**, complete the table to show that you are always in the same kin group as your maternal great-great-grandmother (i.e. your mother's mother's mother's mother).

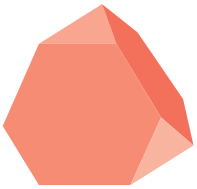
	Kin group							
You	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*
Mother	*B*							
Grandmother	*C*							
Great-grandmother	*D*							
Great-great-grandmother	*A*							

Review questions 9.6D: Mathematical modelling

1 Platonic solids can be truncated into different shapes by flattening the vertices into new faces. For example, the tetrahedron has four faces, while the truncated tetrahedron has 8 faces.

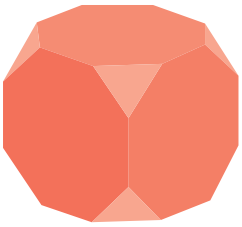
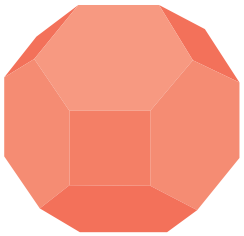
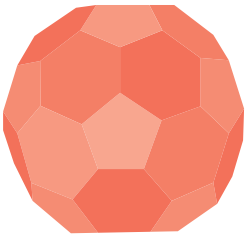
Tetrahedron





Truncated tetrahedron

a Complete the table below.

Shape	 **Truncated cube**	 **Truncated octahedron**	 **Truncated icosahedron**
Number of faces			

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b For the truncated tetrahedron:

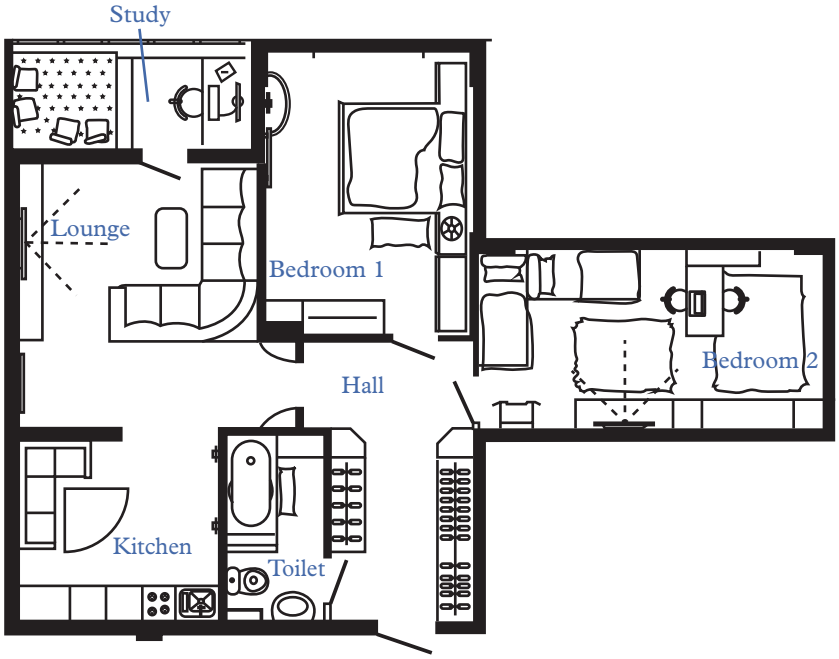
i state the number of vertices

ii state the number of edges.

c Repeat part **b** for the truncated cube.

2 Networks can be used to represent floor plans, showing the connections between adjacent rooms.

a Draw a network for the following floor plan. Use vertices to represent rooms, and represent rooms connected by a door with an edge.



b Can you walk through the house passing through every doorway exactly once? Why or why not?

Checklist

Now that you have completed this module, reflect on your ability to do the following.

I can do this	I need to review this
☐ Use terminology relating to networks/graphs ☐ Draw networks to represent real-world situations ☐ Reason with a network/graph	☐ Go back to Lesson 9.1 Introduction to networks
☐ Identify planar graphs ☐ Identify non-planar graphs ☐ Use Euler's formula for connected planar graphs	☐ Go back to Lesson 9.2 Planar graphs
☐ Think of convex polyhedrons as simple connected planar graphs ☐ Identify the five Platonic solids	☐ Go back to Lesson 9.3 Polyhedrons
☐ Classify walks in a graph/network ☐ Know when a trail or circuit is Eulerian ☐ Determine when Eulerian trails or circuits exist	☐ Go back to Lesson 9.4 Special types of walks
☐ Use directed edges in networks ☐ Model authentic real-world situations using networks	☐ Go back to Lesson 9.5 Applications of networks

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Computational Thinking

Curriculum links

Implement algorithms that use data structures using pseudocode or a general purpose programming language (VC2M10A06)

Design, test and refine solutions to spatial problems using algorithms and digital tools; communicate and justify solutions (VC2M10ASP06) **10A**

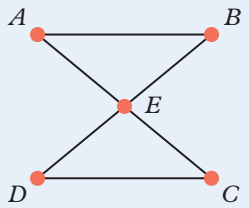
Inter-year links

Year 7	Classifying polygons
Year 8	Binary numbers
Year 8	Geometric proofs
Year 9	Triangles

Eulerian trails and circuits

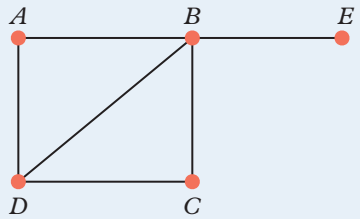
Eulerian trails and Eulerian circuits contain every edge in a graph exactly once.

In a connected graph, an Eulerian circuit exists if all vertices have even degrees. In the graph on the right, all vertices have even degrees, so an Eulerian circuit exists for this graph, which may start at any vertex.



In a connected graph, an Eulerian trail exists if there are two vertices with odd degree and all other vertices have even degree. The Eulerian trail starts at one of these vertices and finishes at the other.

In the second graph on the right, vertex D has a degree of 3, vertex E has a degree of 1 and all other vertices have even degrees. Therefore, an Eulerian trail exists for this graph, which starts at either vertex D or E and ends at the other.

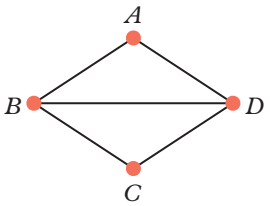


Check your understanding

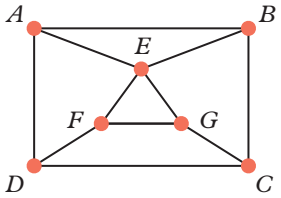
1 Answer the following questions.

a Determine if the following graphs have an Eulerian trail or circuit, or neither. Explain your reasoning.

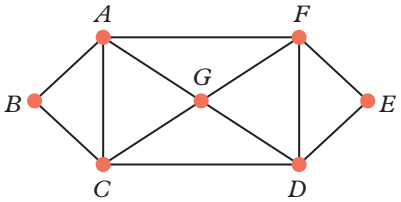
i



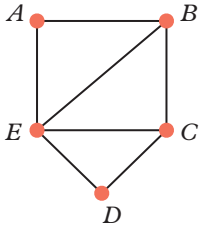
ii



iii



iv



b Give an example of an Eulerian trail or circuit for the graphs where they exist in part **a**.

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Pseudocode

Algorithms can be represented using pseudocode. The steps of the algorithm are written in plain English, with key words in bold and formatting using indentations indicating how the algorithm should be applied. The lines in a pseudocode algorithm are often numbered so we can easily refer to them. Learning to read (and potentially write) pseudocode can be difficult, so start by looking at a pseudocode representation of an algorithm that you are already familiar with.

The pseudocode algorithm below defines a function that can be used to identify whether a connected graph has an Eulerian trail, an Eulerian circuit, or neither.

```
1   define checkEulerian(graph)
2       odd_degree_vertices ← 0
3       for each vertex in graph
4           if degree(vertex) is odd then
5               odd_degree_vertices ← odd_degree_vertices + 1
6           end if
7       end for
8       if odd_degree_vertices = 0 then
9           print Eulerian circuit
10      else if odd_degree_vertices = 2 then
11          print Eulerian trail
12      else
13          print Neither
14      end if
```

The reserved (bold) terms and specific symbols tell us how to follow the algorithm. The indentations are also important as they indicate how the steps below relate to the ones above.

- **define**: this algorithm defines a function (checkEulerian) that takes the variable (graph) in brackets as input.
- **for**: this creates a loop which means any steps indented below should be repeated (here the repetition is for each vertex in the input graph). The loop stops at **end for**.
- **if** … **then**: this is a decision point where the result is either true or false. The decision point stops at **end if**. If the result is true, the indented steps below should be followed. If the result is false, go to the line after **end if**.
- **else if** and **else**: these are additional decision points that within an **if** decision point. These are written in the order that any decisions should be made. Once a result is given as true, go to the line after **end if**.
- ← (arrow pointing left): the value of the variable on the left of the arrow should be updated to the value on the right.
- **print**: the algorithm should print this text.

An additional reserved term not included in the algorithm above is **break**. This is used to exit a loop and move to the step on the line after the end of the loop (for example, after **end for**).

Check your understanding

2 Answer the following questions.

a Copy the pseudocode algorithm into your workbook, including the line numbers. Pay attention to the indentation of each line.

b Briefly explain the role of each line in the algorithm.

c Check your answers to question **1a** using the pseudocode algorithm.

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Task 1

An algorithm is said to be more efficient if it can reach the conclusion with fewer steps of computation or comparison. The pseudocode algorithm given previously is not very efficient, as it must always check every vertex in a graph, even once it has counted that there are at least three vertices of odd degree (so it can't contain an Eulerian trail or circuit).

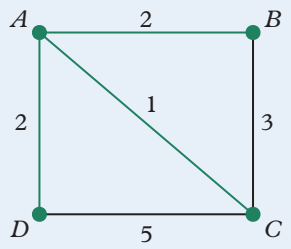
1 Modify the pseudocode algorithm above to define a new function called 'checkEulerian2' which takes a graph as input and prints 'neither' as soon as three vertices of odd degree have been counted. Think about what you should add and where to put it, what you should remove, and which lines need to be updated.

2 Suppose you input a graph with an Eulerian circuit into the original and modified algorithms. Which algorithm is more efficient? Explain, including an example in your response.

3 Suppose you input a graph where every vertex has odd degree. Which algorithm is more efficient? Explain, including an example in your response.

Minimum spanning trees

Graphs can be used to represent many real-life situations. For example, consider a representation of a road network, where vertices represent houses and edges represent a route between those houses. We can label each edge with the distance along the route between the two houses. In this instance, the distance is called the 'weight' of the edge. A graph with weighted edges is called a weighted graph.



A spanning tree of a connected graph is a graph with the same vertices, but only some of the edges, that connects all the vertices without including any cycles.

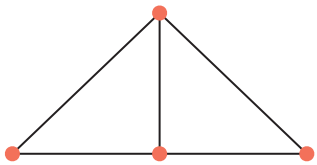
A minimum spanning tree of a weighted graph is a spanning tree of the graph where the sum of the edge weights in the tree is minimal. There may be more than one minimum spanning tree for a weighted graph, but they will all have the same minimal total edge weight. The green edges and vertices in the graph on the right show a minimum spanning tree for the graph.

Check your understanding

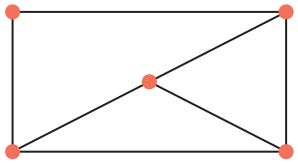
3 Answer the following questions

a Draw a spanning tree for each of the following graphs.

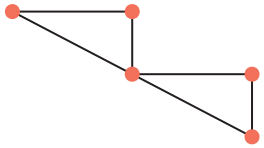
i



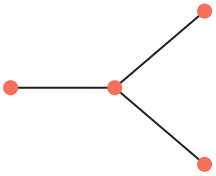
ii



iii



iv



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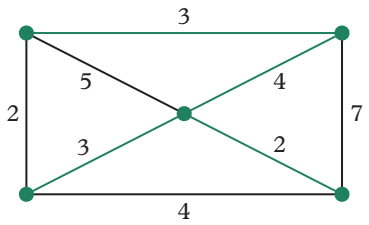
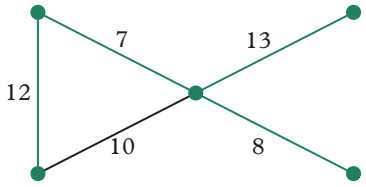
b For each of the spanning trees shown on the graphs below determine the total edge weight and whether it is a minimum spanning tree. If it is not, identify a minimum spanning tree for the graph and determine its total edge weight.

i



ii



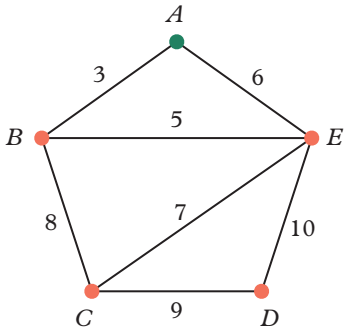
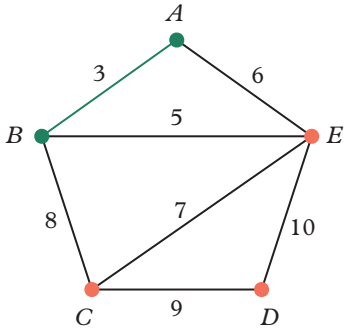
Prim's algorithm

Prim's algorithm provides a systematic approach to finding the minimum spanning tree in a connected graph.

Here is Prim's algorithm represented as pseudocode. This algorithm includes a **while** loop on line **4**. This means that you should continue applying the loop while the condition is true and move on to the line after **end while** when it is false.

```
1  input connected graph G
2        choose a vertex V
3        T ← tree consisting of vertex V
4        while number of vertices in T < number of vertices in G
5              find edge e with endpoints X in T and Y not in T with minimal weight
6              add vertex Y and edge e to T
7        end while
8        print T
```

Here is an example of applying Prim's algorithm to a connected, weighted graph.

Algorithm steps	Graph
Let the graph be G Choose a vertex. Here we choose vertex A. Add vertex A to tree T.	
Enter the **while** loop. number of vertices in T < number of vertices in G Edges with an endpoint in T: • AB, weight = 3 • AE, weight = 6 AB is the minimally weighted edge with exactly one endpoint in T. AB does not form a cycle with edges in T. Add vertex B and edge AB to T.	

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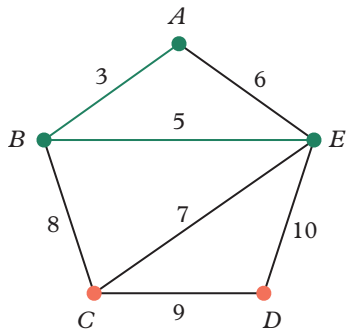
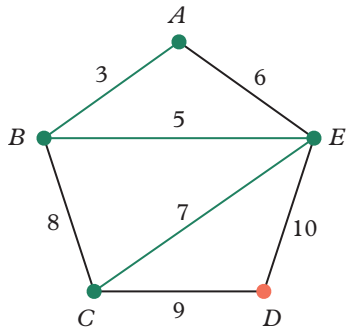
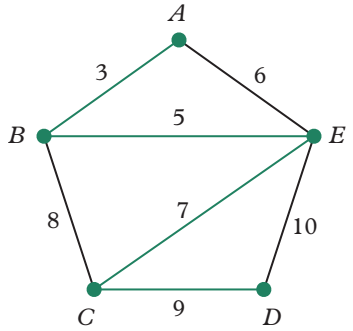
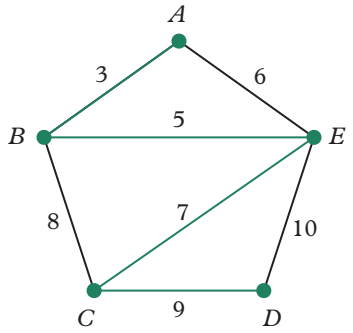
Algorithm steps	Graph
number of vertices in T < number of vertices in G Edges with an endpoint in T: • BE, weight = 5 • AE, weight = 6 (exclude, both endpoints are in T) • BC, weight = 8 BE is the minimally weighted edge with exactly one endpoint in T. BE does not form a cycle with edges in T. Add vertex E and edge BE to T.	
number of vertices in T < number of vertices in G Edges with an endpoint in T: • AE, weight = 6 (exclude, both endpoints are in T) • EC, weight = 7 • BC, weight = 8 • ED, weight = 10 EC is the minimally weighted edge with exactly one endpoint in T. EC does not form a cycle with edges in T. Add vertex C and edge EC to T.	
number of vertices in T < number of vertices in G Edges with an endpoint in T: • AE, weight = 6 (exclude, both endpoints are in T) • BC, weight = 8 (exclude, both endpoints are in T) • CD, weight = 9 • ED, weight = 10 CD is the minimally weighted edge with exactly one endpoint in T. CD does not form a cycle with edges in T. Add vertex D and edge CD to T.	
number of vertices in T = number of vertices in G Exit the **while** loop. Print the minimum spanning tree T.	

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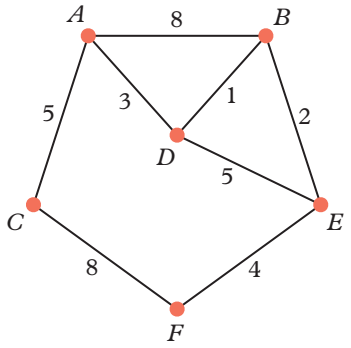
Check your understanding

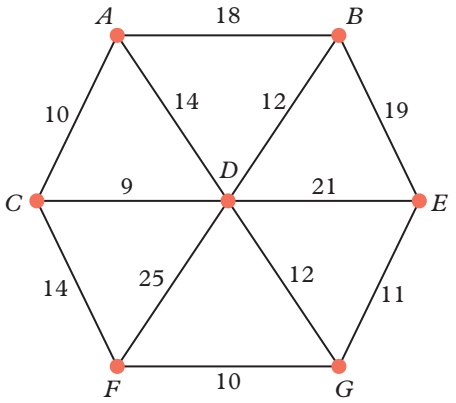
4 Answer the following questions.

a Using Prim's algorithm to find a minimum spanning tree for each of the graphs below starting with vertex *A*. Determine the total edge weight of each minimum spanning tree.

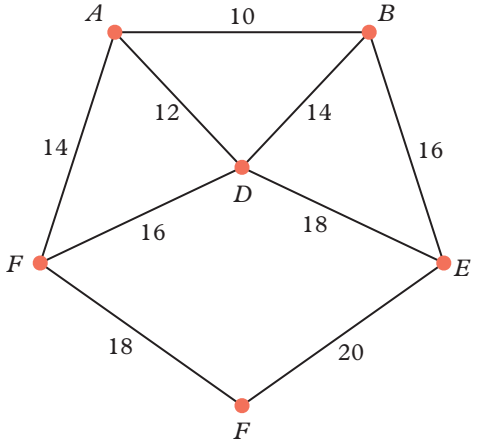
i



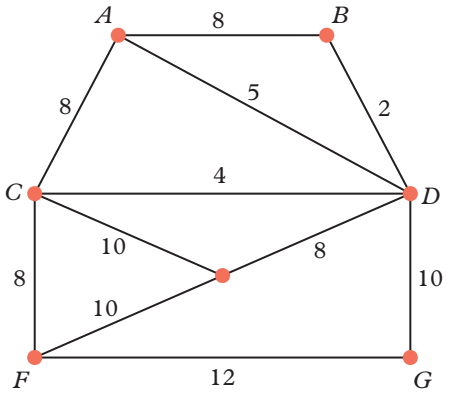
ii



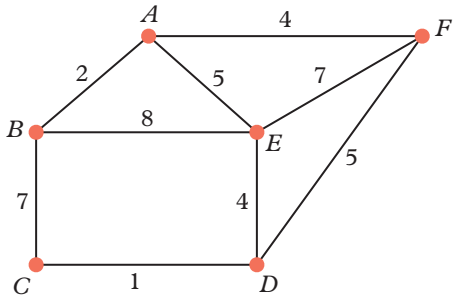
iii



iv



b Consider the following graph.

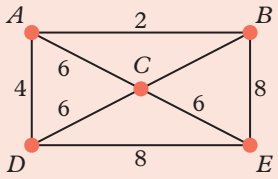


i Determine two different minimum spanning trees for this graph using Prim's algorithm. Verify that the total edge weights of the two trees are equal.

ii Change the weight of one of the edges on the graph, so that the resulting graph has a unique minimum spanning tree.

Task 2

Consider the graph on the right.



1 Find the minimum spanning tree using Prim's algorithm, starting from vertex *A*.

2 Find the minimum spanning tree using Prim's algorithm, starting from vertex *E*.

3 Compare the two minimum spanning trees you have found and discuss the impact of the choice of starting vertex when applying Prim's algorithm.

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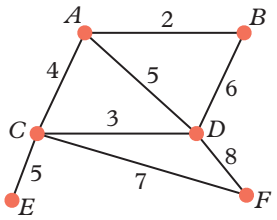
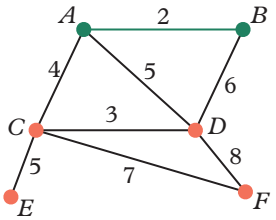
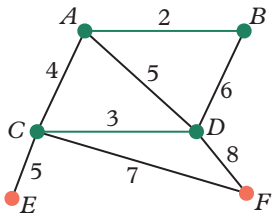
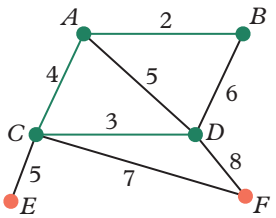
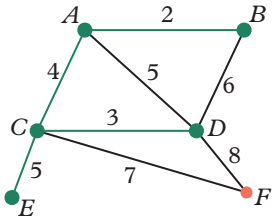
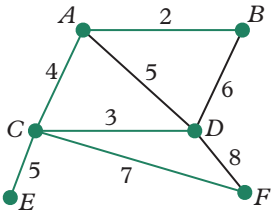
All remaining content in this Computational thinking project is 10A.

Kruskal's algorithm

Kruskal's algorithm is another algorithm which finds a minimum spanning tree of a connected graph. Here is Kruskal's algorithm presented as pseudocode. Where two edges have the same weight, the order they are chosen in the algorithm does not matter. Comments can be added are # (hash) symbols to explain the purpose of specific lines of code.

```
input connected graph G
    L ← list of all edges in G
    sort L into ascending order according to edge weights
    T ← ∅     # creates an empty list T
    while number of vertices in T < number of vertices in G
        choose edge e in L with minimal weight
        if e does not form a cycle with edges in T then
            add endpoints of e (if not already present) and e to T
        end if
        remove e from L
    end while
    print T
```

Here is a summary of an example of applying Kruskal's algorithm to a weighted graph which shows the order in which edges are added to the graph.

Graph	Step 1
	
Step 2	**Step 3**
	
Step 4	**Step 5**
	

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Check your understanding

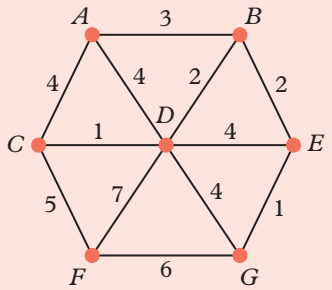
5 Answer the following questions.

a Determine the minimum spanning tree for each of the graphs in Check your understanding question **4** part **a** using Kruskal's algorithm. Write down the order in which the edges are added to each minimum spanning tree.

b For the minimum spanning trees found in part **a**, compare the order the edges are added to the tree in Prim's algorithm and Kruskal's algorithm. How does this affect the minimum spanning tree identified?

Task 3

Consider the following weighted graph.



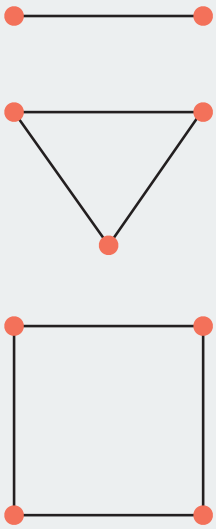
1 Find the minimum spanning tree using Prim's algorithm, starting from vertex *F*. Show all steps as you work through the algorithm.

2 Find the minimum spanning tree using Kruskal's algorithm. Show all steps as you work through the algorithm.

3 Compare your answers to questions **1** and **2**. What do you notice? What would happen if you applied Prim's algorithm starting from a different vertex?

Extension

Graphs which are not connected are said to be made up of 'connected components'. There is a path between any two vertices in a connected exponent. For example, the graph on the right has three connected components. We can find 'minimum spanning forests' for weighted graphs which are not connected. A minimum spanning forest is the graph formed by the union of the minimum spanning trees of the connected components of a weighted graph.



1 Adjust Prim's algorithm so that it can be used to find the minimum spanning forest of any weighted graph.

2 Can you use Kruskal's algorithm to find a minimum spanning forest for any weighted graph? Explain.

3 Research the reverse-delete algorithm and compare it to both Prim's algorithm and Kruskal's algorithm.

4 The three algorithms mentioned above are all 'greedy' algorithms. Research what this means and discuss how it applies to each algorithm.

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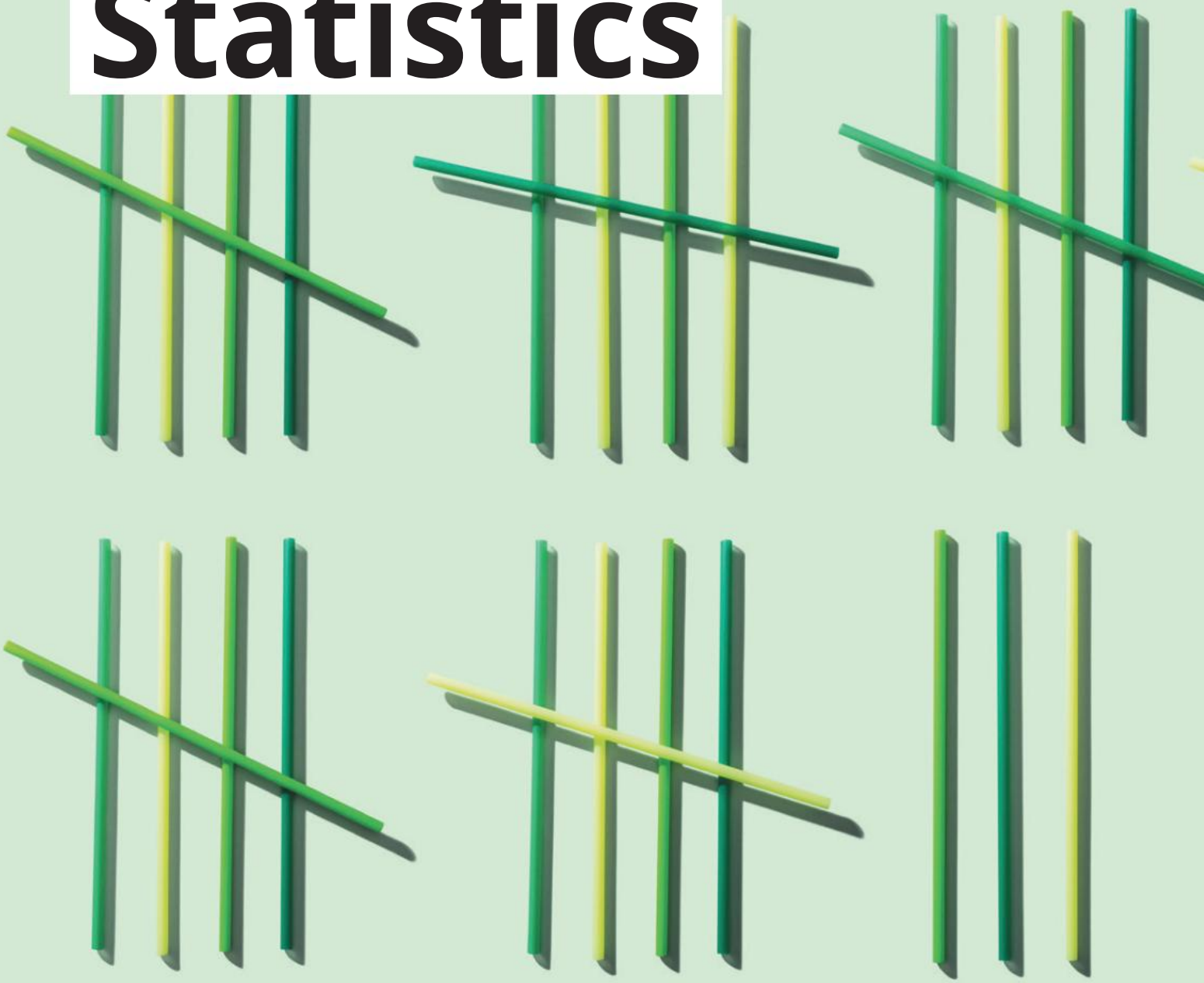
10

Statistics



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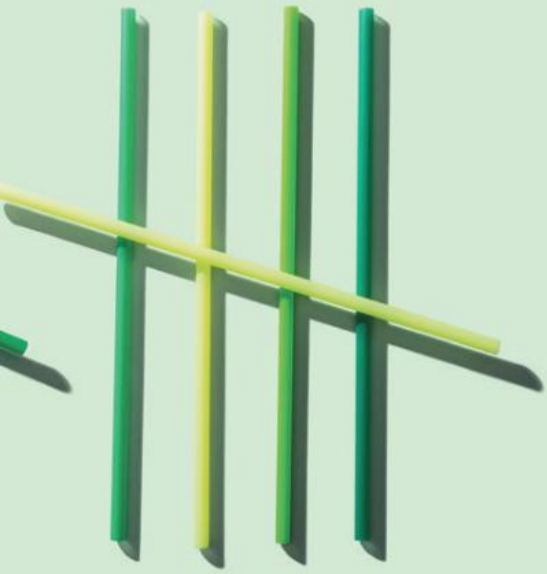

Lessons

10.1 Five-number summary and interquartile range
10.2 Boxplots
10.3 Distributions of data
10A **10.4** The mean and standard deviation
10.5 Bivariate categorical data
10.6 Scatterplots
10A **10.7** Lines of best fit
10.8 Time series
10.9 Analysing statistical reports

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ Stem-and-leaf plots
- ✔ Line graphs
- ✔ Histograms
- ✔ The Cartesian plane

Curriculum links

- Compare data distributions for continuous numerical variables ... (VC2M10ST01)
- Construct scatterplots and consider a line of good fit ... (VC2M10ST02)
- Construct two-way tables and discuss possible relationships ... (VC2M10ST03)
- Analyse claims, inferences and conclusions of statistical reports ... (VC2M10ST04)
- Plan and conduct statistical investigations ... (VC2M10ST05)
- 10A Calculate and interpret the mean and standard deviation of data ... (VC2M10AST01)
- 10A Identify measures of spread, and understand their interpretation and usefulness ... (VC2M10AST02)
- 10A Use digital tools to investigate bivariate numerical data sets ... (VC2M10AST03)
- 10A Investigate reports of studies in digital media ... (VC2M10AP02)

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Lesson 10.1

Five-number summary and interquartile range

Learning intentions

By the end of this lesson you will be able to ...

✔ calculate the five-number summary, range and interquartile range of a data set.

Inter-year links

Support	Understanding data
Year 7	10.1 Summary statistics
Year 8	10.2 Summary statistics
Year 9	7.4 Summary statistics from tables and displays

Measures of data distribution

Key content video
The five-number summary and interquartile range

- **Quartiles** divide a numerical data set into four quarters.
- The **median** is the middle value of an ordered numerical data set. The median divides a data set into two halves.
  → The median is also known as the second quartile, Q_2.
- The **lower quartile** (Q_1) is the median of the *lower half* of an ordered numerical data set.
- The **upper quartile** (Q_3) is the median of the *upper half* of an ordered numerical data set.
- When calculating the quartiles for a data set with an *odd number of data points*, the median is not included in either half of the data set.

$$\underbrace{1\ \ 2\ \ \overset{Q_1}{\boxed{3}}\ \ 4\ \ 5}_{\text{lower half}}\ \ \overset{Q_2}{\boxed{6}}\ \ \underbrace{7\ \ 8\ \ \overset{Q_3}{\boxed{9}}\ \ 10\ \ 11}_{\text{upper half}}$$

- If there is an *even number of data points*, both middle values are included in the quartile calculations.

$$\underbrace{1\ \ 2\ \ 3\ \ \overset{Q_1}{4}\ \ 5\ \ 6}_{\text{lower half}}\ \ \underbrace{7\ \ 8\ \ \overset{Q_2}{9}\ \ 10\ \ 11\ \ 12}_{\text{upper half}}$$

$$Q_1 = \frac{(3+4)}{2} = 3.5$$

$$Q_2 = \frac{(6+7)}{2} = 6.5$$

$$Q_3 = \frac{(9+10)}{2} = 9.5$$

- The **five-number summary** for a numerical data set consists of the minimum and maximum values, the lower and upper quartiles, and the median.
  → The five values that make up the five-number summary are displayed in ascending order:
  minimum, Q_1, Q_2 (median), Q_3, maximum
- The **range** of a data set can be found by subtracting the smallest value (minimum) from the largest value (maximum).
- The **interquartile range** (IQR) represents the range of the middle 50% of a data set. It can be found by subtracting the lower quartile from the upper quartile.

$$\text{IQR} = Q_3 - Q_1$$

- An **outlier** is an extreme value that is much higher or lower than the other values in a data set.
  → Outliers do not significantly impact the median or interquartile range.

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Worked example 10.1A Finding the five-number summary

Find the five-number summary for this ordered data set: 2, 2, 3, 4, 5, 6, 6, 6, 8.

THINK	WRITE
1 The data set is already ordered, so find the median by either circling the middle number or finding the average of the two middle numbers.	2 2 3 4 (5) 6 6 6 8 $Q_2 = 5$
2 Divide the data set into two equal halves, remembering not to include the median in either half.	
3 Find the lower quartile (Q_1) and the upper quartile (Q_3) by finding the median of each half of the set.	2 2 3 4 (5) 6 6 6 8 $Q_1 = 2.5$ $Q_3 = 6$
4 Identify the minimum and maximum values from the data set.	minimum = 2, maximum = 8
5 Write your answer.	2, 2.5, 5, 6, 8

Worked example 10.1B Calculating the interquartile range

Find the interquartile range for this data set: 6, 32, 7, 2, 8, 9, 9, 4, 6, 6, 21, 19, 13, 5.

THINK	WRITE
1 Arrange the values in order.	
2 Divide the data set into two halves. Because there are 14 data values, each half will contain 7 values.	2 4 5 (6) 6 6 7 8 9 9 (13) 19 21 32 Q_1 Q_3
3 Find the value of the lower and upper quartiles by finding the median of each half of the data set.	$Q_1 = 6$, $Q_3 = 13$
4 Subtract Q_1 from Q_3 to obtain the IQR.	$\text{IQR} = Q_3 - Q_1$ $= 13 - 6$ $= 7$

Helpful hints

- ✔ When placing a set of data in order, count to make sure the ordered set has the same amount of data values as the unordered set.
- ✔ If there are n values in an ordered data set, the median will be in the $\frac{n+1}{2}$th position. For example, if there are 11 values, the median will be in the $\frac{11+1}{2} = 6$th position.

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Learning pathways

1–9, 13, 14 | 2, 3, 5, 7, 9–11, 13, 15, 17(a, b) | 3(c, d), 5(c, d), 7, 9, 11, 13, 15–17

ANS p742

Exercise 10.1A: Understanding and fluency

1 Find the range for each of these data sets.

a 5, 9, 6, 12, 16, 3, 8, 19, 11, 8, 12

b 48, 3, 7, 3, 8, 4, 12, 19, 6, 6, 8, 19, 13

c 110, 167, 189, 102, 144, 117, 166

d 54, 46, 78, 11, 19, 22, 26, 57, 67

WE 10.1A **2** Find the five-number summary for each of these ordered data sets.

a 3, 3, 3, 3, 4, 5, 6, 6, 7, 8, 9

b 7, 9, 11, 12, 13, 13, 18, 19, 35, 42, 45, 46, 61, 62, 78

c 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 4, 5, 6, 6, 6, 6, 7, 7, 7, 8, 8, 9, 9

d 2, 4, 4, 6, 7, 7, 8, 8, 9, 11, 16, 16, 18, 19, 19, 19, 21, 27, 32

3 Find the five-number summary for each of these data sets.

a 4, 8, 9, 12, 7, 6, 6, 6, 7, 4, 3, 2, 8, 9, 11

b 46, 28, 16, 23, 8, 4, 46, 49, 55, 13

c 72, 81, 16, 22, 42, 55, 67, 73, 19, 24, 16, 77, 37

d 724, 169, 843, 120, 564, 786, 341, 186

4 Given the following five-number summaries, find the interquartile range each time.

a 2, 6, 10, 16, 19 **b** 17, 22, 25, 31, 33 **c** 7, 9, 13, 19, 24 **d** 34, 40, 52, 59, 66

WE 10.1B **5** Find the interquartile range for each of these data sets.

a 46, 48, 32, 12, 62, 27, 19, 34, 86, 16, 16

b 8, 1, 6, 4, 23, 5, 7, 16, 3, 3, 14, 13, 7, 5

c 123, 189, 146, 179, 98, 165, 108, 142, 152

d 19, 62, 53, 14, 8, 43, 62, 9, 46, 13, 19, 20

Exercise 10.1B: Problem solving and reasoning

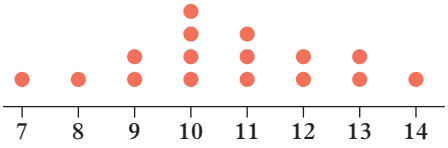
6 The stem-and-leaf plot below shows the number of cupcakes a local bakery sold per day over the course of a month. Find the range and the interquartile range and describe the spread of the set.

Stem	Leaf
4	1 2 2 6 7 8
5	1 6 6 7 9
6	1 2 3 3 6 6 8 9
7	2 4 8 9 9
8	0 1 5 5 7
9	1 2

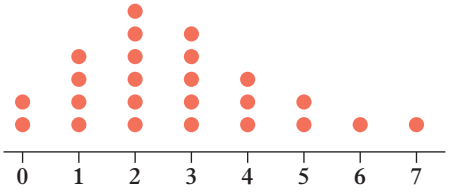
Key: 4 | 2 = 42

7 Find the five-number summary for each of these dot plots.

a



b



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8 Consider these two data sets.

Set A: 9, 31, 46, 11, 39, 24, 3, 43, 25, 10, 19, 4, 36, 14, 16

Set B: 16, 26, 29, 18, 14, 23, 14, 57, 26, 21, 16, 18, 14, 19, 19

a Find the range for each data set. What do you notice?

b Find the interquartile range for each data set.

c Explain why the interquartile range should be used to describe the difference between the spreads of these two sets.

9 Consider this data about the number of times a year a sample of Year 10 students have dyed their hair.

Number of times hair has been dyed	0	1	2	3	4	5	6
Frequency	10	2	3	7	11	9	10

a Find the range of the data.

b Find the five-number summary.

c Find the interquartile range.

10 Explain what it means if a data set has:

a a small interquartile range

b a small interquartile range and a large range

c a large interquartile range

d an interquartile range that is very similar to its range.

11 Consider this data set, which is the income (in dollars) per week of some Year 10 students.

10 120 50 60 45 85 250 80 100 65 75 20
35 45 50 80 25 10 20 50 100 45 80

a Calculate the range and the interquartile range of this data.

b Why is the range so much larger than the interquartile range?

c If the data in a set is spread evenly, the range will be roughly twice the size of the interquartile range. Explain why.

d The presence of an outlier can affect the range. Explain why, for data sets with more than 5 values, an outlier would not affect the interquartile range.

12 For each data set below:

i calculate the range and interquartile range

ii describe the spread of the data.

a 11, 8, 19, 9, 11, 15, 7, 2, 6, 17, 13, 13, 12, 16, 8, 7, 16, 5, 4, 19

b 32, 68, 43, 31, 77, 44, 67, 42, 37, 55, 59, 79, 81, 83, 11, 56, 57, 61, 82, 63

c 123, 118, 131, 141, 142, 136, 128, 127, 119, 129, 138, 127, 191, 132, 134

13 Consider this back-to-back stem-and-leaf plot showing the heights of students (in centimetres) in two classes.

a For each class, find:

i the median

ii the range

iii the interquartile range.

b Compare the heights of the students in the two classes. Which class would you say is taller? Which class would you say is more diverse?

Leaf Class A	**Stem**	**Leaf Class B**
	14	9
9	15	6 8
9 8 7 7 7 6 5 4 3 3 2 1 1	16	0 1 3 4 5 5 7 9
9 9 8 7 6 5 5 4 3 2 2 1	17	2 3 4 4 5 5 5 7 9
6 5 4 1	18	1 4 8
	19	7

Key: 1 | 6 = 16 cm

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14 Consider the set of data: 80, 50, 60, 66, 74, 71, 83, 55, 70, 79.

a Determine the five-number summary for the data.

b Determine the interquartile range and range for the data.

c It turns out the final value, 79, was instead meant to be 9.

i Determine the correct five-number summary for the data.

ii Determine the correct interquartile range and range for the data.

iii Describe the effect that changing 79 to 9 had on the median, the range, and the interquartile range.

d Explain why outliers and other extreme values have little to no effect on the interquartile range.

15 Consider this five-number summary created from a list of 10 numbers.

10, 10, 12, 14, 14

a Determine the number of possible original data sets that would have this five-number summary.

b If the maximum was increased to 15, determine the number of possible original data sets that would have this five-number summary.

Exercise 10.1C: Challenge

16 A k-quantile is a more general name for the points at which a data set is divided into smaller, equally sized parts. For example, the 4-quantiles are the quartiles Q_1, Q_2 and Q_3 because they divide the data set into four equally sized parts. A 5-quantile is called a quintile. Quintiles are often used in economics to describe how income is distributed across a population.

a Give one advantage of using quintiles instead of quartiles to describe the income distribution of a large population.

b The term quintile is also used to describe each 20% of a data set. The following data shows the income distribution in Australia between 2013 and 2020.

Quintile	2013–2014 (%)	2015–2016 (%)	2017–2018 (%)	2019–2020 (%)
Lower (0–19%)	2.7	3.2	3.1	3.1
Second (20–39%)	8.4	8.6	8.6	8.6
Third (40–59%)	16.9	16.0	16.5	16.7
Fourth (60–79%)	25.7	25.4	25.6	25.8
Highest (80–100%)	46.4			

i Copy the above table into your workbook and fill in the missing values to complete the last row.

ii Was income evenly distributed in Australia in 2019–2020? If not, what would an even distribution of income look like in the table?

c The ratio of the percentage income earned by the lowest quintile compared to the highest quintile in 2013–2014 can be calculated as follows:

$$46.4 : 2.7 = \frac{46.4}{2.7} : \frac{2.7}{2.7} \approx 17.185 : 1$$

i Calculate the income ratios between the lowest and highest quintiles for the other years.

ii Describe how the income ratios change between 2013 and 2020.

17 Determine the interquartile range for each of the following data sets.

a 21, 4, −11, −2, −24, −18, −3, −1

b $\frac{13}{6}, \frac{17}{3}, \frac{10}{3}, \frac{7}{3}, \frac{1}{4}, 5, \frac{17}{3}, \frac{2}{3}$

c $\frac{18}{3}, \frac{8}{19}, \frac{8}{3}, \frac{13}{10}, \frac{12}{13}, \frac{14}{13}, \frac{9}{11}, \frac{6}{7}$

d $-\frac{17}{18}, -\frac{1}{2}, -\frac{5}{13}, -\frac{11}{8}, -\frac{20}{7}, \frac{9}{10}, \frac{19}{14}, -\frac{13}{17}$

Online resources:

Interactive skillsheet
Five-number summary and interquartile range

Quick quiz
10.1

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Boxplots

Learning intentions

By the end of this lesson you will be able to ...

✔ understand, create and use boxplots.

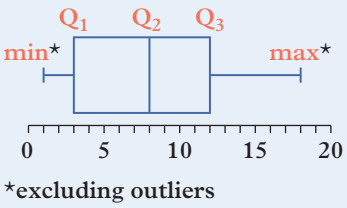
Inter-year links

Year 8	10.2 Summary statistics
Year 9	7.6 Comparing data

Boxplots

Key content video
Boxplots

- A **boxplot** (or box-and-whisker plot) can be used to represent the five-number summary visually and to display the centre (median) and spread (range and interquartile range) of a numerical data set.
- A boxplot consists of a central box with horizontal lines called 'whiskers' on either side of the box.
  - → Boxplots should always be accompanied by a clear and even scale.
  - → If the minimum or maximum value is equal to the value of the lower or upper quartile, there will be no whisker on that side of the boxplot.
  - → If the median is equal to either quartile, the median line will not be visible.

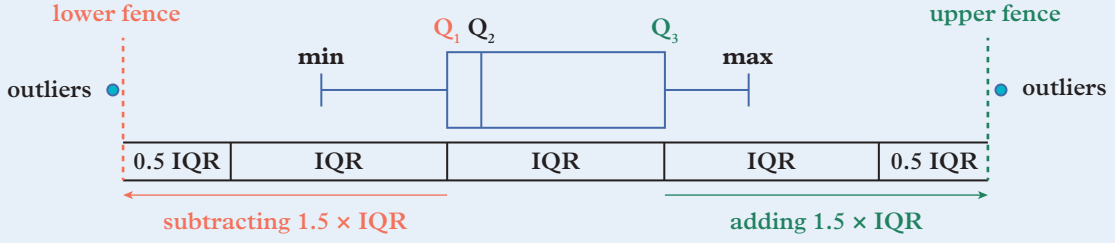


Identifying outliers

- The quartiles and the interquartile range can be used to identify outliers. If a data value is less than the **lower fence** or more than the **upper fence**, the value is considered an outlier.

  lower fence = $Q_1 - 1.5 \times IQR$ upper fence = $Q_3 + 1.5 \times IQR$

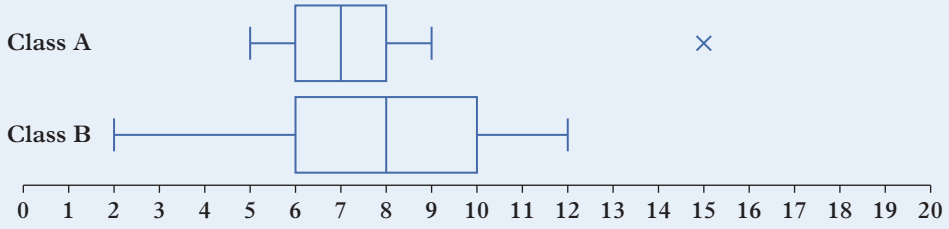
- Outliers should be identified before boxplots are drawn. On boxplots, each outlier is represented with a cross or a dot.



Using boxplots to compare data sets

- **Parallel boxplots** can be used to compare data sets on the same scale. Place one boxplot directly above the other boxplot on the same scale to directly compare key points from each set of data.

  For example, in the parallel boxplots below, the scores of Class A are more consistent than the scores of Class B, except for the outlier in Class A. However, Class B performs better on average.



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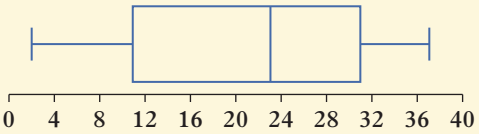
Worked example 10.2A Understanding boxplots

Use this boxplot to state the:

a five-number summary **b** interquartile range.



THINK

a The end of the left-hand whisker represents the minimum value, the left-hand side of the box represents the lower quartile, the line inside the box represents the median, the right-hand side of the box represents the upper quartile and the end of the right-hand whisker represents the maximum value.

b Subtract Q_1 from Q_3 to obtain the IQR.

WRITE

a min = 2,
$Q_1 = 11$,
$Q_2 = 23$,
$Q_3 = 31$,
max = 37

b $IQR = Q_3 - Q_1$
$= 31 - 11$
$= 20$

Worked example 10.2B Creating a boxplot

Draw a boxplot to represent this data about the number of books that people own.

45	22	19	4	14	39	152	108	39	19	66	81
122	42	38	22	18	35	144	33	27	13	31	

THINK

1 Arrange the values in order and find the value of the median, the lower quartile, the upper quartile and the interquartile range.

2 Calculate the values of the lower fence (LF) and upper fence (UF).

3 Check for outliers by determining whether any values lie below the lower fence or above the upper fence.

4 Identify the minimum and maximum values, excluding outliers.

5 Draw an even scale that covers all values, including outliers.

6 Mark in the position of the minimum and maximum values, the quartiles, the median and any outliers. Complete your boxplot using a ruler, providing a title if appropriate.

WRITE

4, 13, 14, 18, 19, 19, 22, 22, 27, 31, 33, 35, 38, 39, 39, 42, 45, 66, 81, 108, 122, 144, 152

$Q_1 = 19$, $Q_2 = 35$, $Q_3 = 66$, IQR = 47

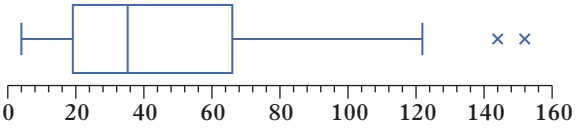
$LF = 19 - 1.5 \times 47$
$= 19 - 70.5$
$= -51.5$

$UF = 66 + 1.5 \times 47$
$= 66 + 70.5$
$= 136.5$

144 and 152 are outliers.

min = 4, max = 122

Number of books people own



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Helpful hints

- ✔ The ends of the whiskers do not represent the minimum and maximum values in a data set if there are outliers. If there are outliers, the left whisker will start at the value directly after the closest outlier to the lower fence, and the right whisker will finish at the value directly before the closest outlier to the upper fence.
- ✔ Outliers lie beyond the fences. If a value is exactly equal to the lower fence or upper fence it is not an outlier.
- ✔ The height of a boxplot has no particular significance.

Learning pathways

1, 2, 3(a–d), 4–6, 8 | 1, 3(e–h), 4, 5, 7, 8, 11, 12 | 3(e–h), 4, 5, 9, 10, 12–14

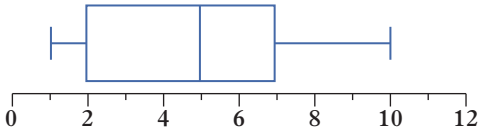
ANS p743

Exercise 10.2A: Understanding and fluency

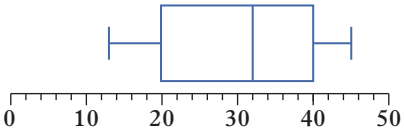
WE 10.2A **1** For each boxplot below, state the:

i five-number summary **ii** interquartile range.

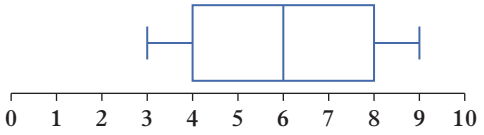
a



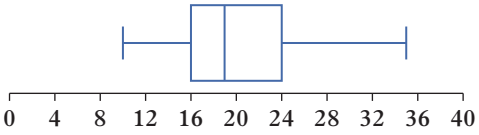
b



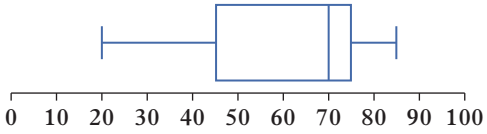
c



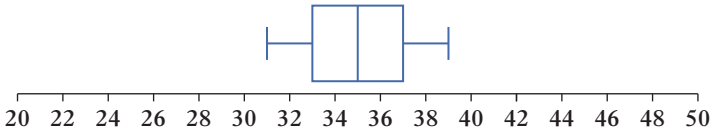
d



e



f



2 Consider this data set: 2, 6, 3, 8, 9, 10, 14, 19, 4, 8, 22, 11, 13, 17, 4, 19, 7, 7, 10.

a Arrange the data in order.

b Find the lower quartile, median and upper quartile.

c Calculate the values of the lower and upper fences.

d Determine whether there are any outliers in this data set.

e Create a boxplot for this data.

WE 10.2B **3** Draw a boxplot to represent each of these data sets.

a 3, 6, 2, 8, 4, 8, 10, 15, 6, 8, 11

b 11, 15, 14, 19, 20, 26, 17, 13, 6, 22

c 4, 7, 1, 8, 3, 7, 6, 6, 9, 3, 5, 1, 6, 9, 7, 5, 3

d 12, 36, 45, 92, 42, 85, 49, 66, 51, 19, 27, 78

e 16, 28, 9, 16, 13, 27, 18, 22, 33, 24, 9, 18, 8

f 5, 7, 9, 19, 6, 9, 8, 5, 5, 4, 3, 9, 9, 1, 6, 5, 4

g 11, 18, 49, 23, 24, 26, 37, 31

h 8, 11, 19, 21, 6, 7, 12, 18, 19, 31, 4, 6, 24, 7, 9

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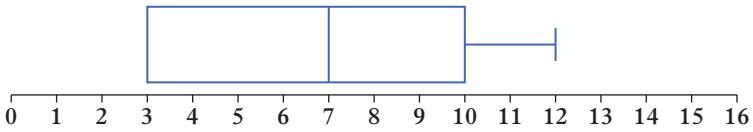
4 For each of the following boxplots, state the:

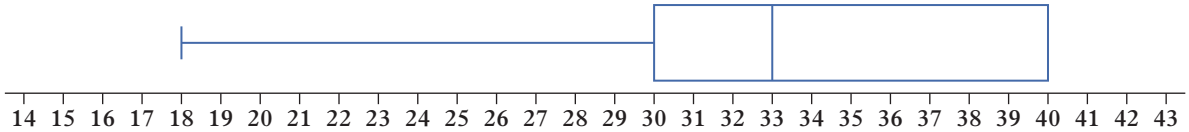
i five-number summary

ii interquartile range.

a



b



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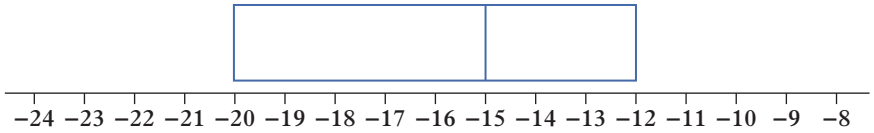

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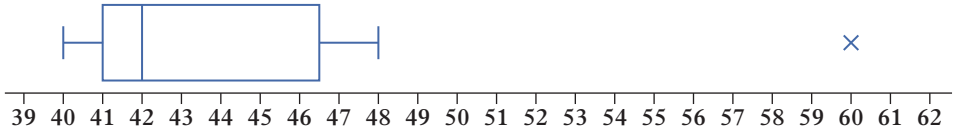

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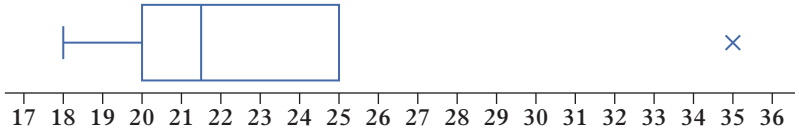

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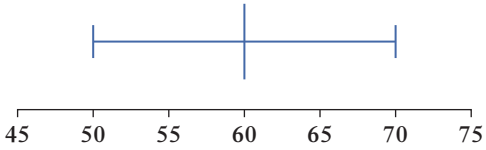
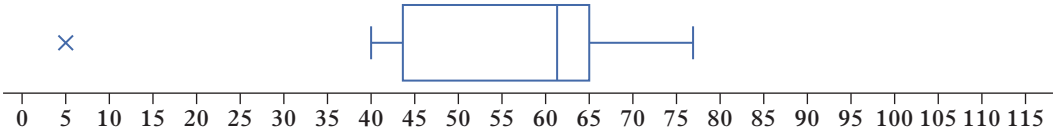
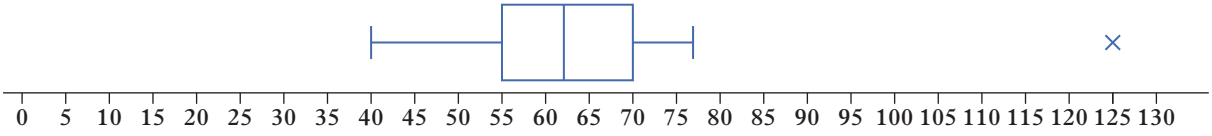


5 For each of the following boxplots, add vertical dashed lines where the upper and lower fences lie.

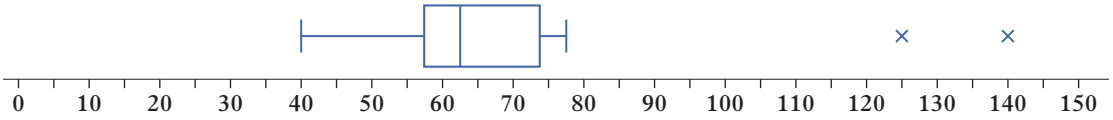
a



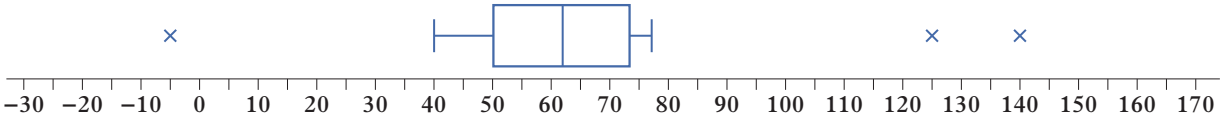
b



c



d



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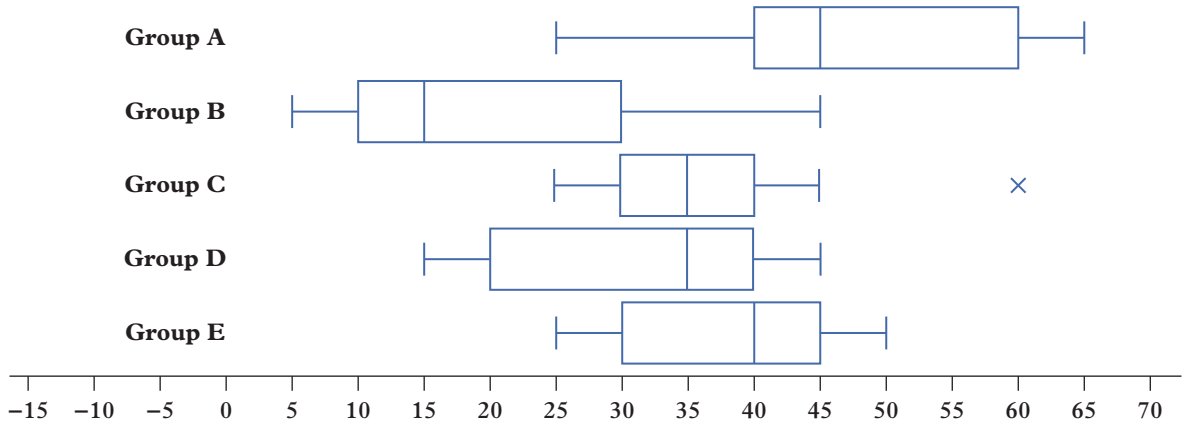
6 Consider the parallel boxplots below.



a State the group(s) with the lowest:

i minimum

ii maximum.

b At 45, state the statistic (if any) that group:

i A has

ii B has

iii C has

iv D has

v E has.

c Which group has a range equal to the interquartile range of another group?

Exercise 10.2B: Problem solving and reasoning

7 This data was collected about the number of puppies in a litter:

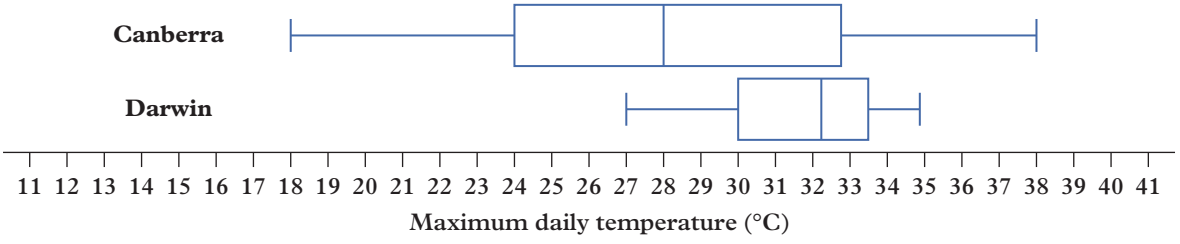
1	6	4	2	5	4	8	6	4	5	8
3	2	4	6	5	8	7	5	6	5	6
12	4	6	5	7	6	5	6	5		

a Find the lower quartile, median and upper quartile of the data set.

b Determine if there are any outliers in this data set.

c Draw a boxplot to represent the data.

8 Consider these parallel boxplots, showing the distribution of the maximum daily temperatures for a particular month in Darwin and Canberra.



a What is the centre (median) and spread (IQR) of the temperature in Darwin?

b What is the centre (median) and spread (IQR) of the temperature in Canberra?

c Write a paragraph comparing the distributions of the two sets of temperature data. Which would you say is the hotter city?

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9 Data was collected on the ages of customers at a laser tag facility on weekdays (set A) and at weekends (set B).

Set A: 19, 18, 21, 22, 23, 18, 24, 25, 22, 21, 20, 19, 18, 18, 19, 20, 19, 20, 24, 22, 21, 23, 27, 19, 21

Set B: 21, 26, 23, 19, 18, 17, 15, 6, 10, 43, 36, 26, 23, 49, 7, 11, 14, 13, 51, 38, 32, 27, 18, 12, 19, 33

a Draw a set of parallel boxplots to represent the data.

b Describe each data set in terms of its centre (median) and spread (range).

c Compare the two sets. What differences do you notice?

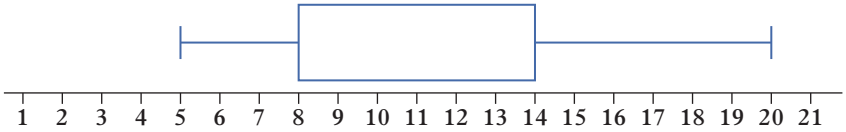
10 The five-number summary for a set of data that shows the number of customers in a shoe shop at 1 pm each day for a month is 4, 6, 7, 8, 10.

a Draw a possible boxplot for this five-number summary.

b Are other boxplots possible? Explain.

c The five-number summary for another set of data is 2, 6, 7, 8, 12. Explain why you cannot draw the boxplot for this five-number summary with only the information provided.

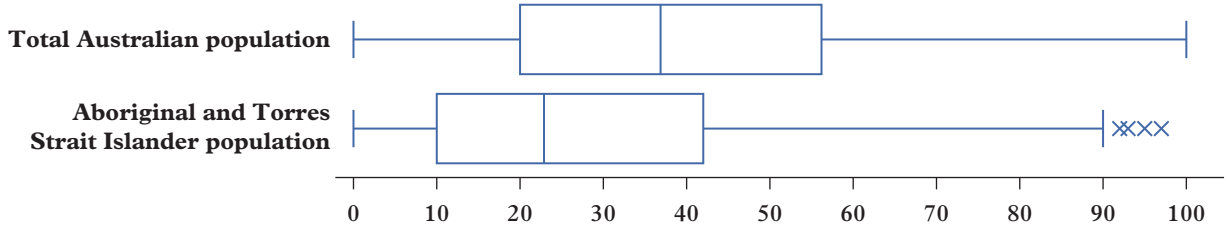
11 Consider this boxplot.



a Explain why there are two possible five-number summaries when only the boxplot (and not the original data) is provided.

b Provide the two possible five-number summaries for the data the boxplot represents.

12 The parallel boxplots below show the distribution of the ages of a sample of the total Australian population and the distribution of ages of a sample of Aboriginal and Torres Strait Islanders in 2024.



Use these parallel boxplots to compare the distributions of the ages of all Australians and the ages of Aboriginal and Torres Strait Islanders.

Exercise 10.2C: Challenge

13 Draw a set of 3 parallel boxplots, A, B and C, such that:

- the maximum of A and B are equal
- 75% of the data in B lies within the same values that 25% of the data in A lies within
- 100% of the data in C, excluding an outlier, lies within the same values that the middle 50% of the data of A lies within
- the median of B is equal to the value of the outlier of C
- the medians of A and C are equal to the minimum value of B
- A is perfectly symmetric and each quartile is equally spaced, B is negatively skewed, and C is symmetric.

14 For a particular set of data, the lower fence is 15 and the upper fence is 51. Determine the:

a interquartile range

b the upper and lower quartiles.

Online resources:

- **Interactive skillsheet** Boxplots
- **Worksheet** Boxplots
- **Investigation** How varying data values affects the interquartile range
- **CAS instructions** Boxplots
- **Quick quiz** 10.2

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Lesson 10.3

Distributions of data

Learning intentions

By the end of this lesson you will be able to ...

✔ describe distributions of data and use cumulative frequency distributions.

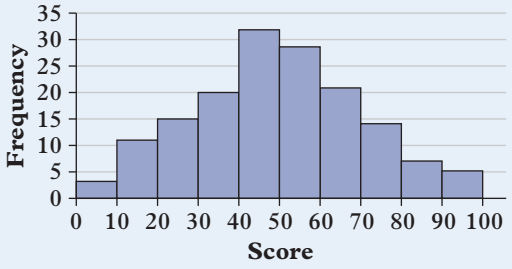
Inter-year links

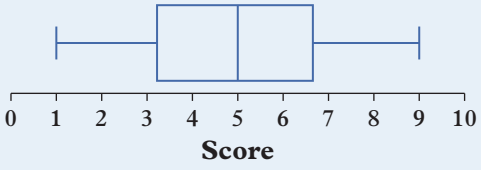
Year 9 7.5 Describing data

Symmetric and skewed distributions

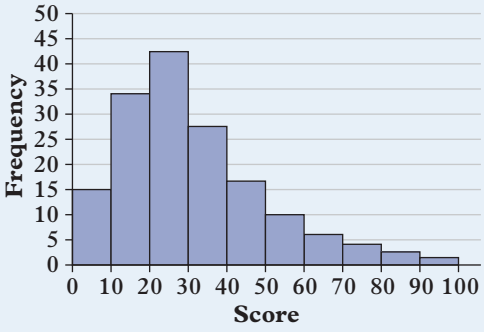
Key content video
Describing data distributions

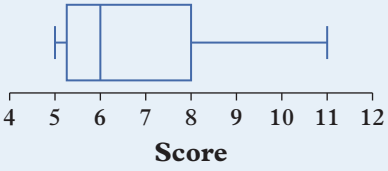
- Symmetrical or **symmetric distributions** are evenly spread on either side of a central peak. The **histogram** and boxplot below are examples of symmetric distributions.



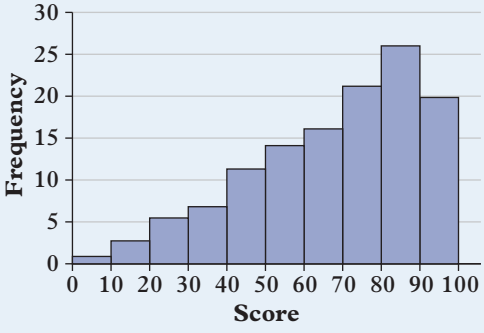


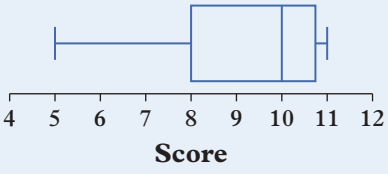
- A **skewed distribution** has a big cluster of data points on one side of the distribution. The other side of the distribution has a tail.
  - → A **positively skewed distribution** has a cluster of values around the left-hand side of the distribution and a tail tapering to the right. The histogram and boxplot below are examples of positively skewed distributions.





  - → A **negatively skewed distribution** has a cluster of values around the right-hand side of the distribution and a tail tapering to the left. The histogram and boxplot below are examples of negatively skewed distributions.





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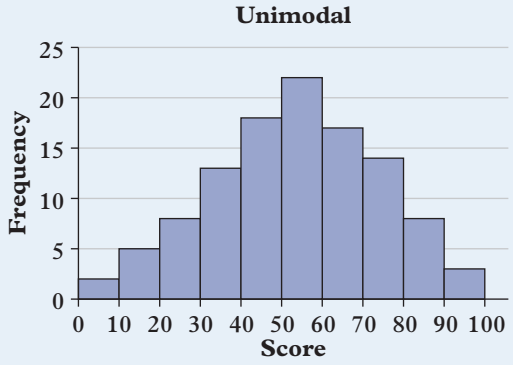
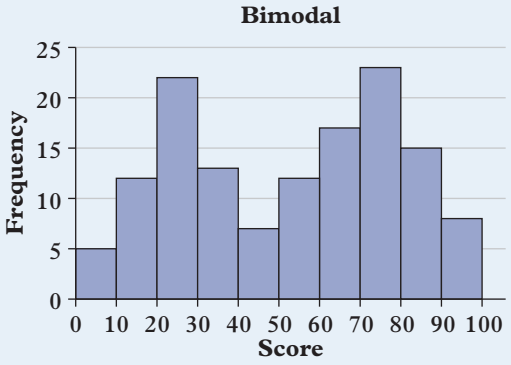
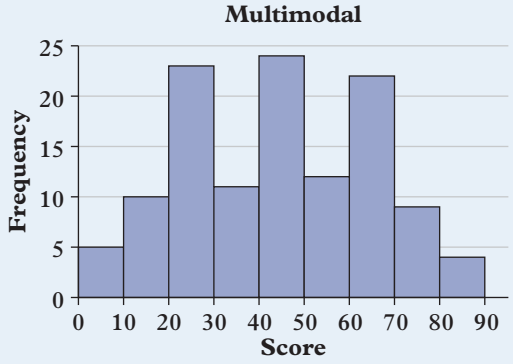
Modality

- The **modality** of a distribution describes the number of peaks in a data set.
- A distribution is **unimodal** if it has one clear peak in the data, **bimodal** if has two clear peaks in the data, and **multimodal** if it has more than two clear peaks in the data.

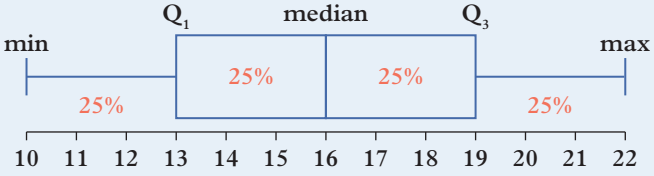






Spread

- To describe the **spread** of a data set, consider where the values in the data set lie across the distribution. Remember that 25% of a data set lies between each pair of consecutive terms in a five-number summary, so if any two values in a five-number summary are close together, 25% of values will be covered in a small range of data (i.e. packed closely together).
- Each section of a boxplot represents approximately 25% of a data set.



- To describe the centre of a data set, consider where the median score in the data set lies in the distribution.

Cumulative frequency distributions

Key content video
Cumulative frequency distributions

- The **cumulative frequency** of a numerical data set can be calculated by adding the frequency of a particular score to the sum of the frequencies of all its preceding values.
  → Cumulative frequencies can be expressed as raw numbers or as a percentage of the data set.
- A **cumulative frequency polygon**, also called an ogive (pronounced: 'oh-jive'), is a graph plotting the cumulative frequency of the values in a data set (on the vertical axis) against those values (on the horizontal axis).

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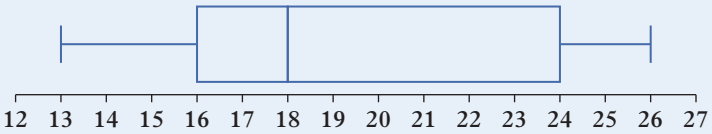
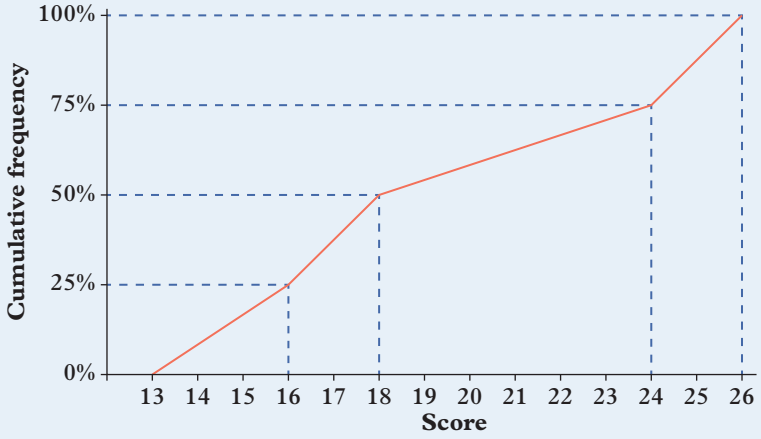
→ The data displayed in the following cumulative frequency table and boxplot are plotted on the cumulative frequency polygon below.

Score	Cumulative frequency
13	0%
16	25%
18	50%
24	75%
26	100%



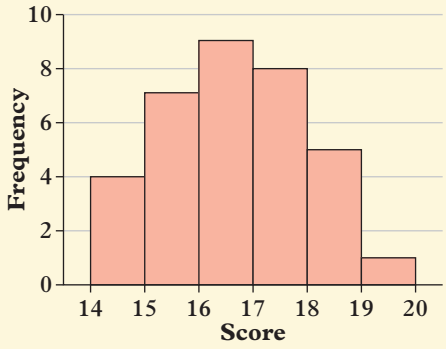


- Cumulative frequency polygons can also be used to plot grouped data. Each section of the cumulative frequency polygon should start at the lowest value of a class interval and end at the highest value of the same class interval (as shown in **Worked example 10.3C**).

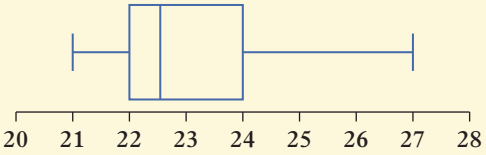
Worked example 10.3A Describing a data distribution

Describe each of the following distributions of data as symmetric, positively skewed or negatively skewed.

a



b



THINK

a Look at the shape of the histogram. Is there a skew towards one side of the distribution, or is the data approximately symmetrical?

b Each section of a boxplot represents approximately 25% of the data set. Identify whether there is a cluster of points on one side of the boxplot, or whether the distribution is symmetric.

WRITE

a Symmetric

b Positively skewed

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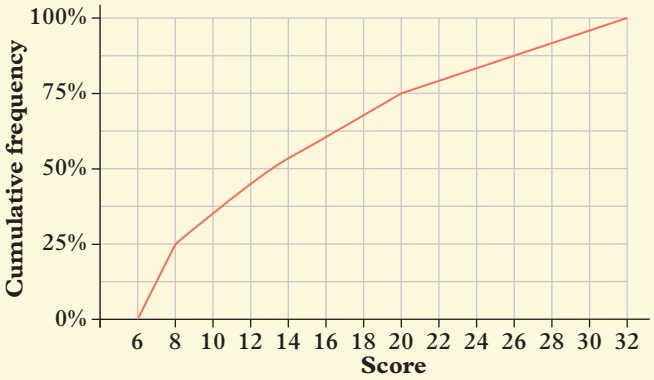
Worked example 10.3B Interpreting a cumulative frequency polygon

Use this cumulative frequency distribution to write the:

a five-number summary for the data displayed

b interquartile range for the data.



THINK

a The value at 0% represents the minimum value, the value at 25% represents the lower quartile, the value at 50% represents the median, the value at 75% represents the upper quartile and the value at 100% represents the maximum value.

b Subtract Q_1 from Q_3 to obtain the interquartile range.

WRITE

a min = 6,
$Q_1 = 8$,
$Q_2 = 13$,
$Q_3 = 20$,
max = 32

b $\text{IQR} = Q_3 - Q_1$
$= 20 - 8$
$= 12$

Worked example 10.3C Creating a cumulative frequency polygon

Using the data from this table, create a cumulative frequency polygon.

Score	Frequency	Cumulative frequency
5–<10	6	6
10–<15	4	10
15–<20	4	14
20–<25	2	16
25–<30	2	18
30–<35	1	19
35–<40	1	20

THINK

1 Draw and label axes to cover the full range of values on the horizontal axis and the cumulative frequency on the vertical axis.

2 Mark the first point with a frequency of 0 at the start of the first class interval. The next point to mark will be the point at the end of the first class interval, with a cumulative frequency equal to that of the first class interval.

3 Mark each subsequent point from the table on your graph.

4 Join the points together with straight lines.

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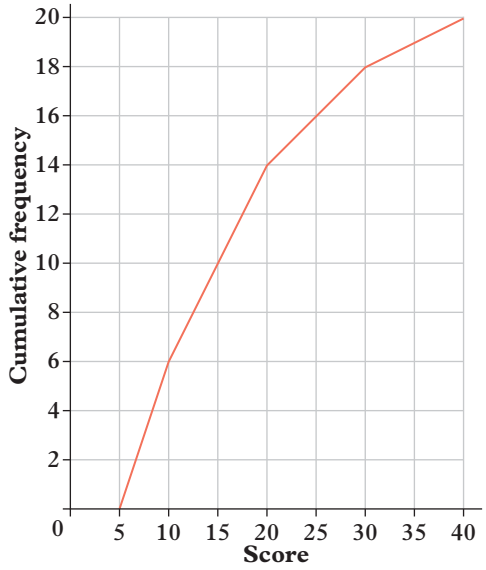
WRITE



Helpful hints

- ✔ Symmetric data does not need to be perfectly symmetric; it only needs to be approximately symmetric.
- ✔ The cumulative frequency of the final score will equal the number of pieces of data, or 100% of the data set.
- ✔ It is not possible to state the modality of a data distribution from a boxplot, since boxplots do not show the peak (or peaks) of a data distribution.

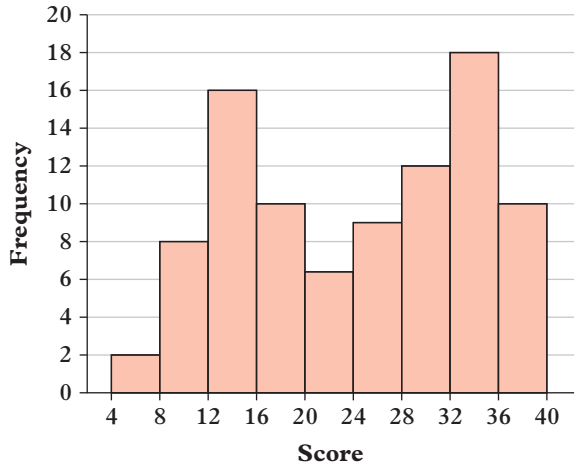
Learning pathways

1–3, 4(a–c), 5, 6, 8, 9, 11(a), 12(a)	1–3, 4(b, d), 5, 7–9, 11(b), 12(b), 14, 15	1–3, 4(b, d), 7, 10, 11(b), 12(b), 13, 14, 16

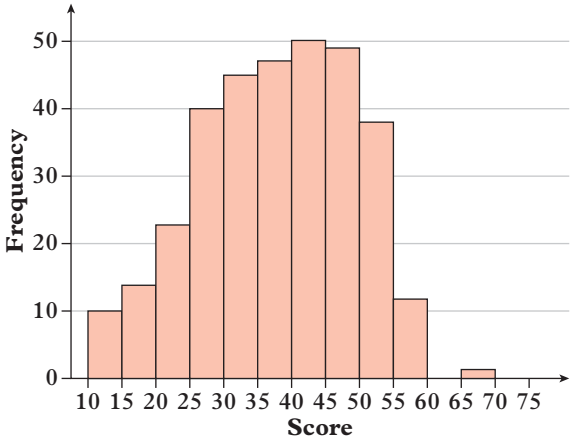
Exercise 10.3A: Understanding and fluency

1 State the modality of the following distributions of data.

a



b



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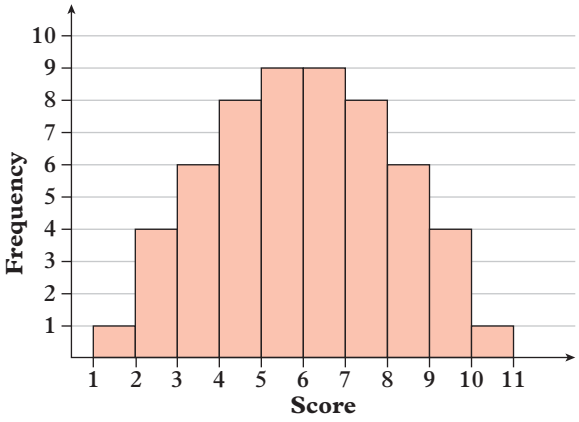
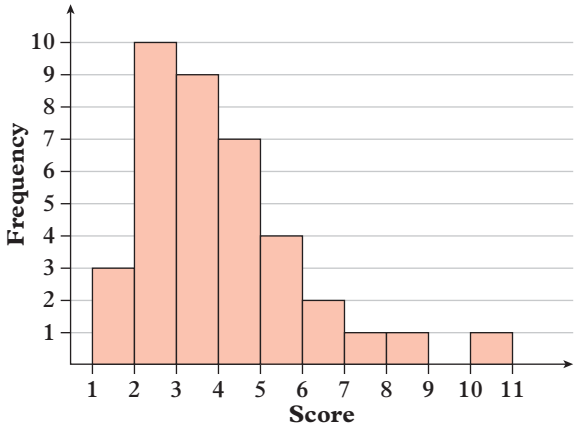
WE 10.3A **2** Describe each of the following distributions of data as symmetric, positively skewed or negatively skewed.

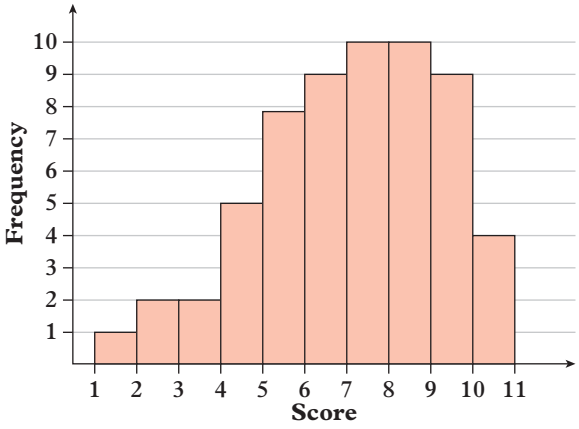
a



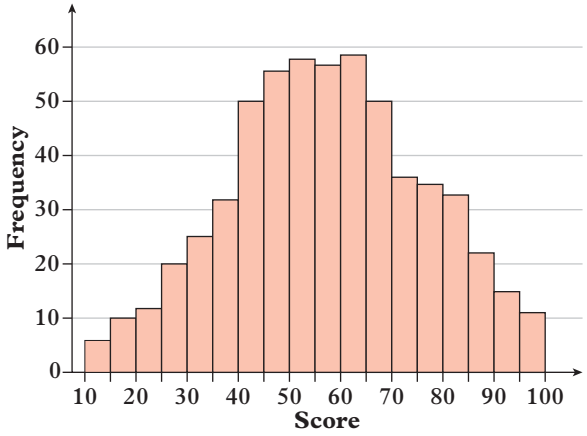
b



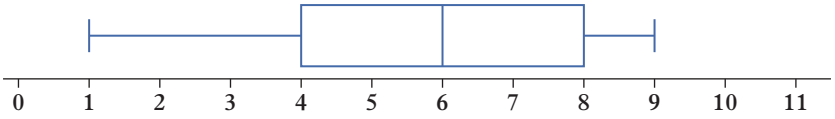
c



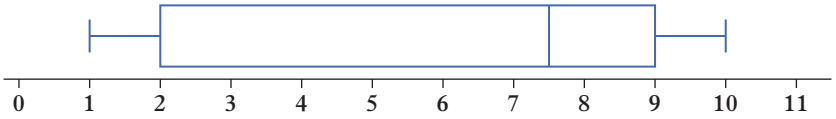
d



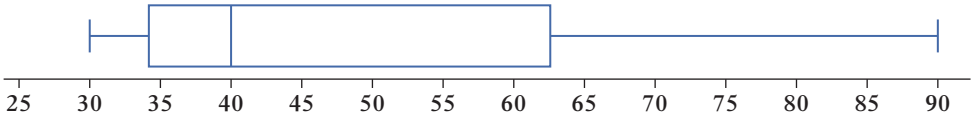
e



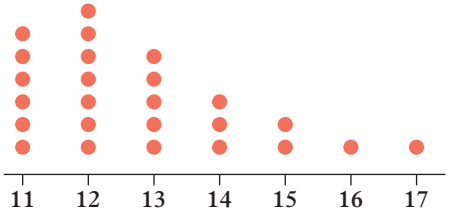
f



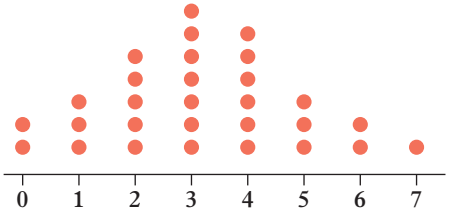
g



h



i



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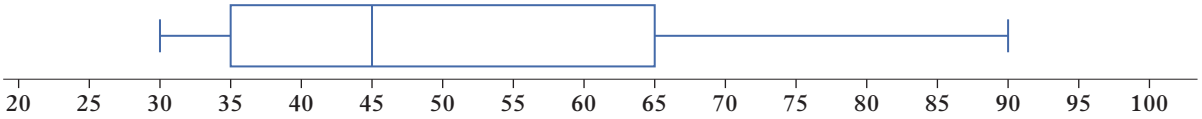
3 State the percentage of the data that lies in the following ranges of values for this boxplot.

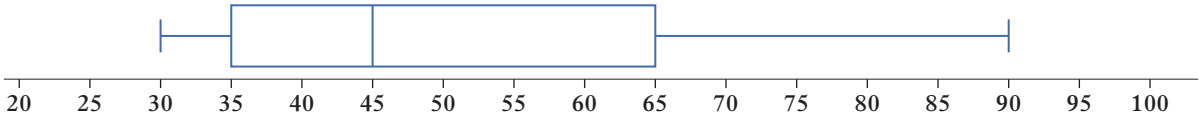


a between 35 and 45 **b** between 35 and 65 **c** between 65 and 90

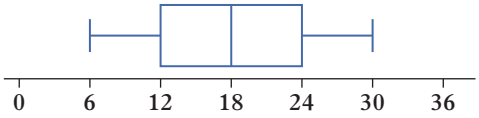
d less than 65 **e** greater than 35 **f** less than 35 or greater than 65

4 Construct a cumulative frequency polygon for the following boxplots.

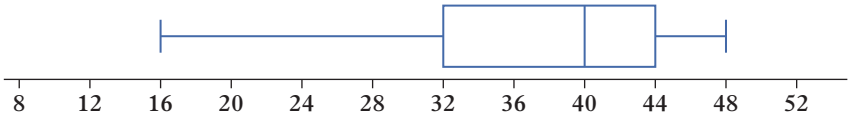
a



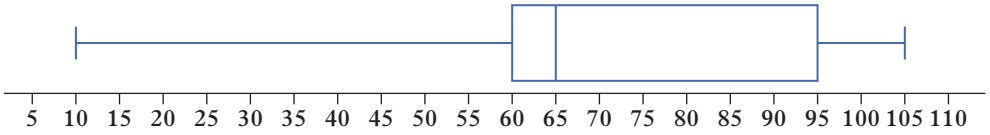
b



c



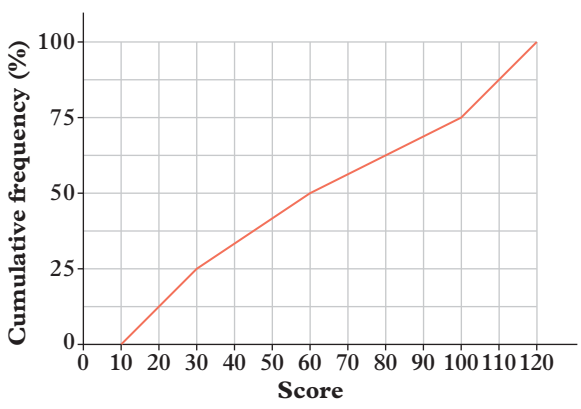
d



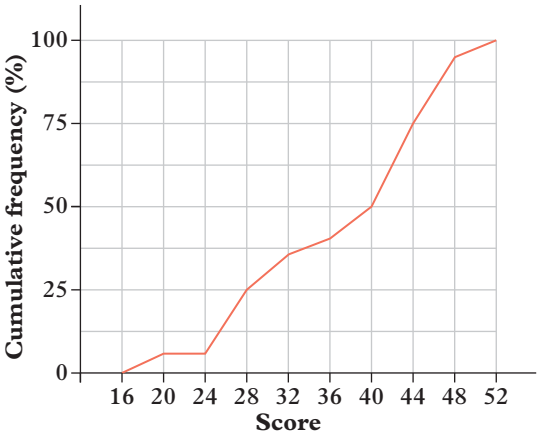
WE 10.3B **5** For each cumulative frequency polygon, state the:

i five-number summary **ii** interquartile range.

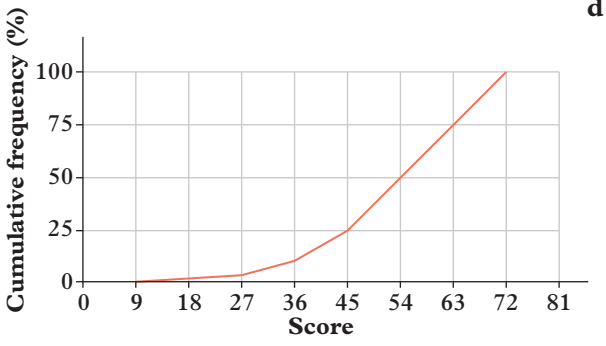
a



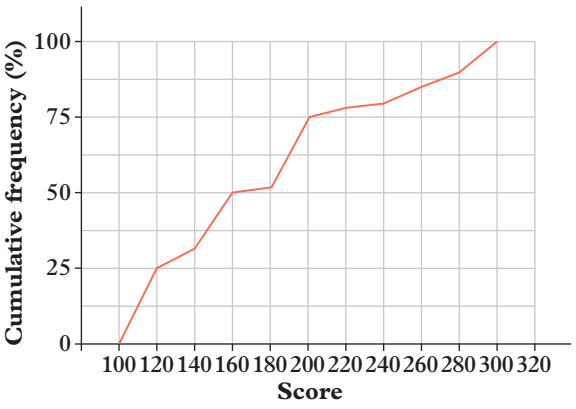
b



c



d



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WE 10.3C **6** Create a cumulative frequency polygon using the data in the table below.

Score	Frequency	Cumulative frequency	Cumulative percentage
7	5	5	25
8	3	8	40
9	2	10	50
10	5	15	75
11	3	18	90
12	2	20	100

7 For this table:

a complete the cumulative frequency and cumulative percentage columns

b construct a cumulative frequency polygon.

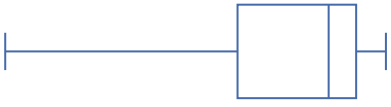
Score	Frequency	Cumulative frequency	Cumulative percentage
20–<30	10		
30–<40	40		
40–<50	50		
50–<60	20		
60–<70	10		
70–<80	20		
80–<90	10		
90–<100	40		

Exercise 10.3B: Problem solving and reasoning

8 Match each of these boxplots (**a–d**) with the correct graph (**I–IV**).

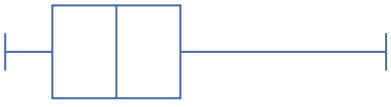
a

b

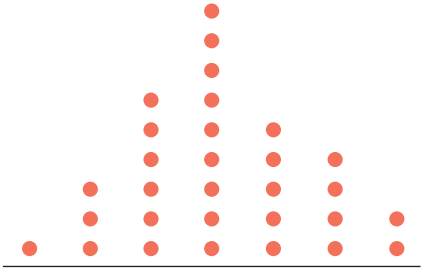


c

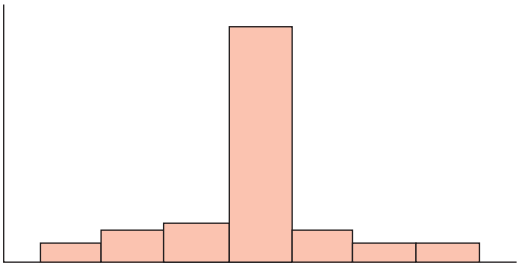
d



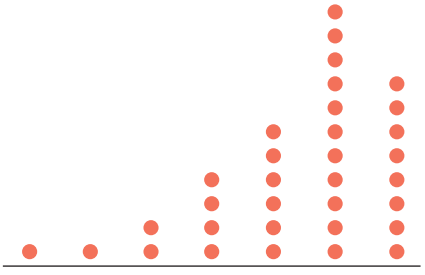
I



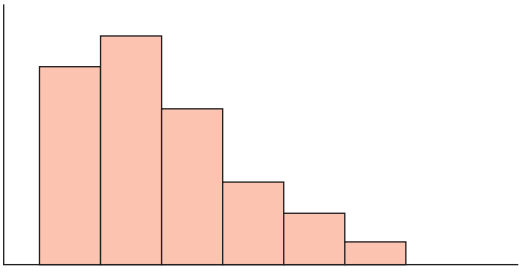
II



III



IV



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9 Consider the boxplot shown on the right.

a Interpret the boxplot by describing its:

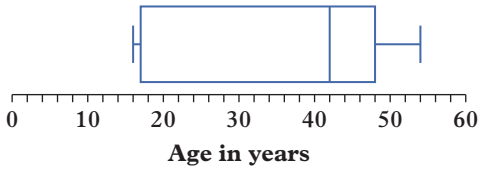
i shape

ii spread

iii centre.

b Can you suggest a reason for the unusual shape of the distribution?

Ages of people at a Year 10 orientation night



10 A vet collected the following data about the weights (in kilograms) of dogs.

5.1	11.2	35.5	17.9	21.1	84.4	42.2	51.0	9.2	8.6	7.5	18.5	19.2	21.8	22.6
24.8	13.7	13.1	16.9	29.8	58.3	42.7	2.1	3.8	9.6	10.8	46.0	1.9	35.0	39.4

a Draw a boxplot to represent the data.

b Draw a histogram with intervals of width 10 to represent the data.

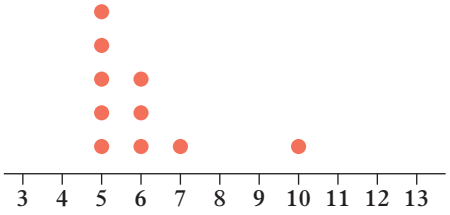
c Describe the distribution of the data.

d Which of the displays you drew in parts **a** and **b** do you think represents the data best? Explain.

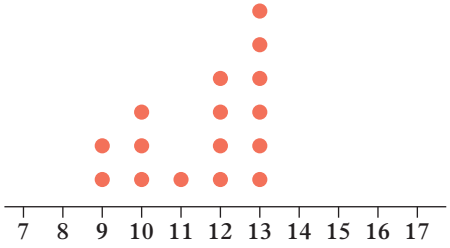
e Why would a dot plot not be as useful for looking at the distribution of the weights of these dogs?

11 Construct a cumulative frequency polygon for each of these dot plots.

a

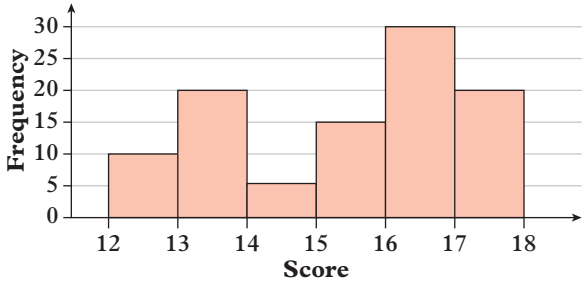


b

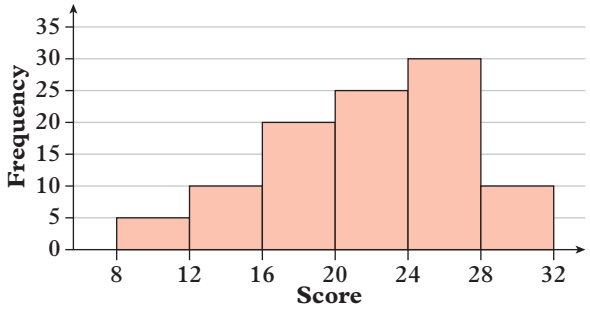


12 Construct a cumulative percentage frequency polygon for each of these histograms.

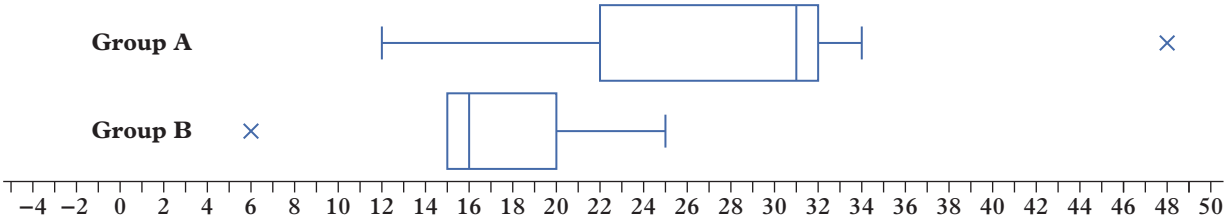
a



b



13 Consider the parallel boxplots below.



a Describe the distribution of the boxplots.

b Give two reasons why the median would be a better measure of centre than the mean when comparing these boxplots.

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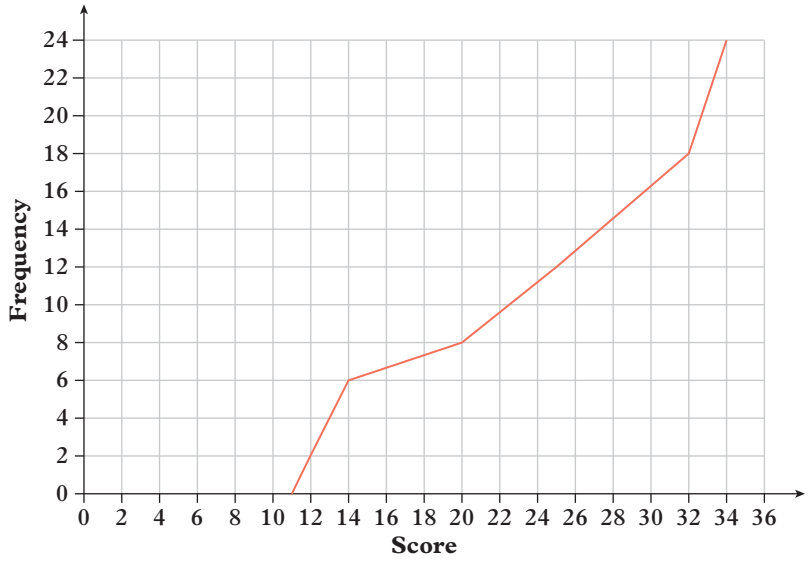
14 Consider the cumulative frequency polygon shown with a frequency scale.

a Determine the five-number summary for the data.

b Determine the interquartile range for the data.



15 Explain why no section of a cumulative frequency polygon can have a negative gradient.

Exercise 10.3C: Challenge

16 Another way to describe a data set is to use percentiles. One way to define the kth percentile in a data set is as the data value below which k% of data values fall. For example, the median is the 50th percentile as 50% of data values are below the median.

a Write the equivalent percentile for each of the following values in a data set.

i Q_1

ii Q_2

iii Q_3

b Using this definition, why doesn't it make sense to talk about the 100th percentile in a data set?

c For each of the following data sets, calculate the 30th and 70th percentiles.

Set 1: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

Set 2: 10, 20, 20, 30, 30, 30, 40, 40, 40, 40, 50, 50, 50, 50, 50, 60, 60, 60, 60, 60

Set 3: The set of all integers from 1 to 1000.

d Another way to define the kth percentile is as the data value at or below which k% of the data values fall.

i Recalculate the 30th and 70th percentiles for each of the data sets in part **c** using this definition.

ii How do the percentile values you found in **d** part **i** differ from those you found in part **c**?

iii 100 000 people chose an integer from 1 to 100 and the data is collected. The 30th and 70th percentiles are calculated using both definitions given above. How different or similar would you expect the values to be and why?

Online resources:

- **Interactive skillsheet** Describing data
- **Interactive skillsheet** Cumulative frequency polygons
- **CAS instructions** Histograms
- **Quick quiz** 10.3

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Lesson 10.4

The mean and standard deviation

Learning intentions

By the end of this lesson you will be able to ...

✔ calculate and use means and standard deviations.

Inter-year links

Year 8	10.2 Summary statistics
Year 9	7.4 Summary statistics from tables and displays

Key content video
Standard deviation

Populations and samples

- When working with data, the **population** refers to all potential pieces of data under consideration.
- A **sample** is a selection of data that is part of the population.

Mean and standard deviation

- The **mean** is a numerical average of all values in a data set.
  - → The Greek letter μ (mu) is used to represent the population mean.
  - → $\bar{x}$ (x-bar) is used to represent a sample mean.
- The **standard deviation** is a measure of how much the values in a set differ (or deviate) from the mean.
  - → A small standard deviation means that the values in a data set are close to the mean.
  - → A large standard deviation means that the values in a data set are spread out from the mean.
  - → The Greek letter σ (sigma) is used to represent the population standard deviation.
  - → The letter s is used to represent a sample standard deviation.
- A population's standard deviation (σ) can be calculated using the formula:

  $\sigma = \sqrt{\dfrac{(x_1 - \mu)^2 + (x_2 - \mu)^2 + \ldots + (x_n - \mu)^2}{n}}$, where n is the number of values in the population.
- A sample's standard deviation (s) can be calculated using the formula:

  $s = \sqrt{\dfrac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2}{n - 1}}$, where n is the number of values in the sample.
  - → When calculating the standard deviation of a sample, the sum is divided by $n - 1$ rather than n, because this tends to give a better estimate of the sample standard deviation.

Using the mean and standard deviation to describe and compare data sets

- The mean describes a central point in a data set. If a data set is positively skewed, the mean will be greater than the median, and if the data is negatively skewed, the mean will be less than the median.
- When the data is skewed, the median represents the centre of the data better than the mean.
- The standard deviation describes the spread of a data set. When comparing data sets, a larger standard deviation will have values that are more spread out than a smaller standard deviation.

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Worked example 10.4A Finding the standard deviation for a list of data

Find the standard deviation for this sample of data, correct to one decimal place:

4, 8, 12, 19, 5

THINK

1 Calculate the mean of the sample.

2 Substitute the given values into the formula for the standard deviation of a sample, and round your answer to one decimal place.

WRITE

$$\bar{x} = \frac{4 + 8 + 12 + 19 + 5}{5}$$
$$= 9.6$$

$$s = \sqrt{\frac{(4 - 9.6)^2 + (8 - 9.6)^2 + (12 - 9.6)^2 + (19 - 9.6)^2 + (5 - 9.6)^2}{5 - 1}}$$
$$= \sqrt{\frac{(-5.6)^2 + (-1.6)^2 + (2.4)^2 + (9.4)^2 + (-4.6)^2}{4}}$$
$$\approx 6.1$$

Worked example 10.4B Using a spreadsheet to find the standard deviation

Use Excel (or alternative spreadsheet software) to help you find the standard deviation for the data sample below. Give your answer correct to one decimal place.

10, 15, 11, 23, 18, 9, 22, 17, 14, 10

THINK

1 Enter each value from the data sample into a separate cell in column A, in a vertical list.

2 Use the formula =STDEV.S(A1:A10) to calculate the standard deviation of cells A1 to A10.

3 Round the standard deviation to one decimal place.

WRITE

	A
1	10
2	15
3	11
4	23
5	18
6	9
7	22
8	17
9	14
10	10
11	=STDEV.S(A1:A10)

	A
1	10
2	15
3	11
4	23
5	18
6	9
7	22
8	17
9	14
10	10
11	5.043147166

$s \approx 5.0$

Note: The formula for finding the standard deviation might vary if you are using a different spreadsheet program.

Helpful hints

- ✔ The formula for the population standard deviation using Excel is =STDEV.P(:), where the range of cells containing the data is placed within the brackets. The cell names of the first and last cells from the listed data are separated by the colon.
- ✔ Most calculators have a function for calculating the standard deviation of a list of data.

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Learning pathways

1–3, 4(a, b, c, e), 5, 7, 8, 10, 16, 17	1, 2, 3–4(d–f), 5–7, 9, 11, 13, 14, 16, 17	1, 2, 5, 9, 11, 12, 14, 15, 17–19

ANS p744

Exercise 10.4A: Understanding and fluency

WE 10.4A **1** Using a calculator, find the standard deviation for each of these sample data sets. Give your answers to one decimal place.

a 6, 3, 8, 65, 4 **b** 8, 11, 15, 3, 9, 5 **c** 7, 7, 10, 12, 17, 15

d 13, 29, 21, 22, 22, 11, 19 **e** 140, 156, 120, 99, 187, 147 **f** 159, 166, 122, 171, 136, 129

2 Using a calculator, find the standard deviation for each of these population data sets. Give your answers to one decimal place.

a 2, 2, 2, 2, 2 **b** 2, 2, 2, 8, 8, 8 **c** 20, 60, 70, 80, 100

d −11, −9, −17, −23, −15 **e** 2.2, 5.0, −0.2, −0.8, 5.0, 3.2 **f** 1, 2, 4, 16, 32, 62

WE 10.4B **3** Calculate the standard deviation for these samples of data using technology. Give your answers to one decimal place.

a 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 **b** 39, 24, 34, 34 30, 32, 38, 36, 36, 32

c 10, 100, 1000, 10 000, 100 000, 1 000 000 **d** 20, 200, 2000, 20 000, 200 000, 2 000 000

e −1 000 000, −100 000, −10 000, −1000, −100, −10 **f** $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{9}{10}, \frac{10}{11}$

4 Calculate the standard deviation for these populations of data using technology. Give your answers to one decimal place.

a 89, 104, 97, 114, 97, 90, 107, 82, 109, 103

b 1776.6, 2521.4, 2110.1, 606.5, 926.6, 2395.4, 1572.5, 12.03, 2496.3, 123.7

c 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

d 81, 82, 83, 84, 85, 86, 87, 88, 89, 90

e 5, 6, 7, 8, 9, 10, 11, 12, 13, 14

f 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256

5 For each data set below:

i decide whether the data set is from a sample or a population.

ii calculate the appropriate standard deviation.

a This data was collected about the ages of students attending a school formal by surveying everybody who attended.

16	17	18	16	17	16	15	14	16	17	18	16	17	17
16	17	16	17	15	16	17	16	17	16	16	16	15	17
18	18	17	16	17	16	17	16	17	16	16	16	17	17

b This data was collected about the number of hours 20 students in a Year 10 class (of 30 students) spent per week using social networks.

5	10	7	1	19	17	16
7	10	11	8	9	15	13
14	6	21	18	14	13	

c This data was collected about the weights of newborn babies in South Australia:

3.2	4.1	2.9	3.4	3.5	3.4	3.1
3.8	4.0	3.3	3.6	2.9	3.1	
3.5	4.2	2.6	3.0	3.6	3.4	
2.1	3.5	3.3	3.6	3.4	4.8	

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Exercise 10.4B: Problem solving and reasoning

6 Consider this data set, which shows the ages of every player in two cricket teams.

16	16	15	17	15	16	14	15	16
16	17	16	15	17	16	15	16	16
17	16	17	15	16	16	17	16	

a Find the standard deviation for the data set. Give your answers to two decimal places.

b Write a sentence about the spread of ages of players in the cricket teams.

7 Consider this data about the number of text messages sent by a sample of Year 10 students per day.

1	5	11	17	21	9	8	25	26	19	5	2	8
9	13	34	16	29	31	26	19	18	15	20	10	

a Find the mean and standard deviation for the data. Give your answers to two decimal places.

b Write a sentence describing the centre and spread of the data set.

8 The following is a list of the number of people at a shoe sale each hour over a 24-hour period:

46	26	35	49	58	13
27	48	46	39	42	48
46	51	35	29	47	41
42	42	49	45	46	51

a Find the mean for this data. Give your answers to one decimal place.

b Find the standard deviation. Give your answers to two decimal places.

c Comment on the centre and spread of the data set.

9 Data was collected about the number of minutes a sample of passengers were kept waiting for their trains on two different train lines over the course of a day.

Train line A: 4, 2, 3, 5, 6, 7, 4, 4, 6, 5, 6, 7, 9, 4, 5, 6, 6, 7, 6, 7, 9, 4, 6, 6, 1

Train line B: 2, 9, 15, 3, 4, 3, 11, 1, 1, 6, 2, 3, 4, 2, 3, 19, 2, 3, 2, 2, 2, 3, 1, 4

a For each train line, find the mean waiting time. Give your answers to one decimal place.

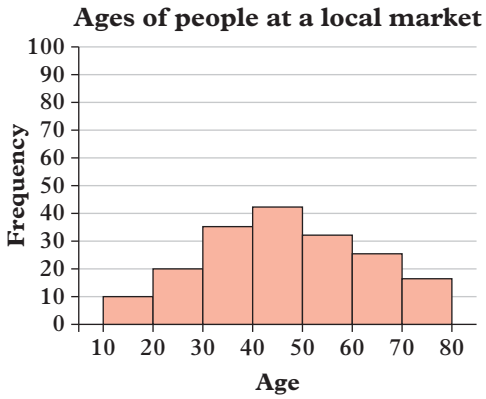
b Find the standard deviation for the data from each train line. Give your answers to two decimal places.

c Write a sentence comparing the centre and spread of the two data sets.

d On which train line would you say that the trains run late more significantly?

10 The two histograms below show data collected about the ages of people attending a local market on two different days. The data sets have the same mean and the same number of scores and have been shown on axes with the same scale for ease of comparison.

Which data set has a small standard deviation and which has a large standard deviation? Explain how you know.



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11 Consider this data collected about the price (in dollars) of a particular brand of electronic tablet bought from all stores where it is available in a city.

499	519	499	475	469	479	509	549	529	525
475	499	499	199	489	479	499	475	515	499

a Find the standard deviation for this data set.

b You may notice that one of the prices is distinctly different from the rest (an outlier). Provide a possible reason for this.

c Recalculate the standard deviation for the data set excluding the outlier.

d What do you notice about your answers for parts **a** and **c**?

e What measure of spread might you use instead of standard deviation when there is an outlier in the data set?

12 The magnitude or absolute value of x, written $|x|$, is its distance from 0 on a number line, ignoring direction. For example, $|5| = |-5| = 5$.

In a data set with n data points, the mean absolute deviation is found using the formula:

$$\text{mean absolute deviation} = \frac{|x_1 - \bar{x}| + \ldots + |x_n - \bar{x}|}{n}.$$

a Calculate the mean absolute deviation and standard deviation for each of the following population data sets.

i 1, 2, 2, 3, 4, 4, 4, 5, 6, 10 **ii** 5, 5, 5, 7, 8, 10, 15, 21

b What do you notice when comparing the mean absolute deviations and standard deviations you calculated in part **a**?

13 Another measure of spread is the median absolute deviation. In a data set with n data points and median $\tilde{x}$, the *median absolute deviation* is the median of the data set

$$|x_1 - \tilde{x}|, \ldots, |x_n - \tilde{x}|.$$

For each of the data sets in question **12** part **a**:

a find the median $\tilde{x}$ **b** calculate the median absolute deviation.

14 The range, interquartile range and standard deviation all provide different information about the spread of the distribution of data in a data set. Consider the following three sets of sample data.

Set A: 13, 15, 15, 16, 18, 18, 20, 22, 24, 25, 30

Set B: 25, 25, 26, 26, 26, 27, 27, 27, 27, 28, 28, 28, 29, 29, 30

Set C: 4, 6, 7, 10, 12, 13, 15, 15, 16, 20, 22, 50

a For each of the data sets, calculate the following.

i range **ii** interquartile range

iii standard deviation (correct to two decimal places) for each data set.

b Do any of the data sets contain outliers?

c Considering your answers to parts **a** and **b**, what are the strengths and weaknesses of each of these three measures of spread?

15 Quartiles, quantiles and percentiles divide a data set into four, five and one-hundred equally-sized parts respectively. For example, the first quartile is the value under which 25% of data points lie, and the 70th percentile is the value under which 70% of data points lie.

What are some of the advantages and disadvantages of dividing a data set up into more or less equally-sized parts?

16 Some measures of statistics are more useful than others in particular situations.

a When is the median a more useful measure of the centre of a distribution of data than the mean? Explain.

b When is the interquartile range a more useful measure of the spread of a distribution of data than the standard deviation? Explain.

17 Compare the following pairs of symmetric sample data in terms of their mean and standard deviation. Use Excel (or an alternative spreadsheet software) to help you calculate the statistics.

a **A:** 20, 24, 28, 28, 30, 30, 31, 32, 35, 40

B: 24, 25, 25, 25, 26, 27, 29, 31, 34, 36, 36, 36, 37

b **A:** 75, 81, 83, 84, 88, 92, 95, 101, 105, 109, 113

B: 64, 70, 71, 71, 72, 74, 80, 81, 83, 84, 86, 90

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18 We can compare individual data values by determining the number of standard deviations, s, the value x is above or below the mean, $\bar{x}$. That is, $z = \frac{x - \bar{x}}{s}$. These values are called standard scores (or z-scores).

For example, the table below shows that Student A scored 85 on a test where the mean score was 70 and the standard deviation was 10. Therefore, Student A's standard score is $z = \frac{85 - 70}{10} = +1.5$, meaning Student A's score is 1.5 standard deviations above the mean score.

	Class A			Class B		
	Mean	**SD**	**Student A**	**Mean**	**SD**	**Student B**
a	70	10	85	55	16	85
b	70	20	85	70	15	85
c	50	25	30	75	5	50
d	50	8	26	50	16	2

For each pair of students' results in the table above:

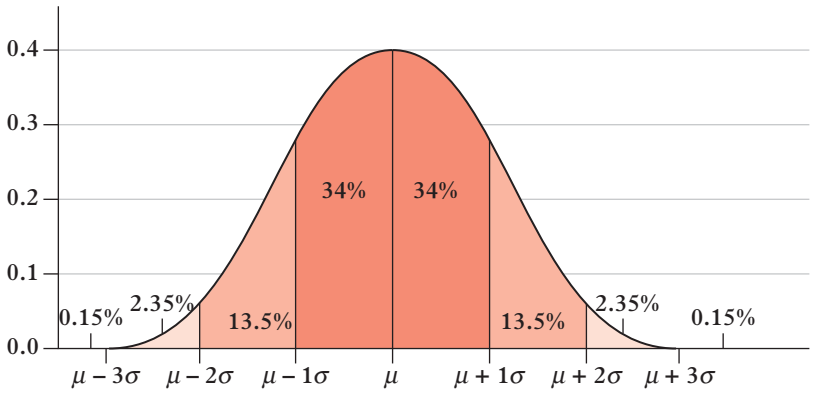
i calculate the standard scores, correct to two significant figures

ii compare their scores, using your answers from part **i**.

Exercise 10.4C: Challenge

19 A population that has a symmetric distribution where the majority of the data is centred around the mean in a bell shape is said to be normally distributed (also known as forming a bell curve). In a normal distribution, nearly all values (99.7%) will lie within three standard deviations (or 'three sigma') of the mean.

The graph below shows approximate percentages of the amount of data between different standard deviations using the 68–95–99.7% rule.



a Show that 99.7% of values lie within three sigma (three standard deviations) of the mean (μ).

b What percentage of values lie within:

i one sigma of the mean

ii two sigma of the mean?

c An example of the normal distribution can be seen in the data for IQ (intelligence quotient) scores. The mean IQ score is set to 100, and the standard deviation is set to 15.

i Redraw the graph above so that the scale on the x-axis uses these values (that is, $\mu = 100$, $\mu + 1\sigma = 115$, etc.).

ii What is the IQ range of 99.7% of the population?

iii What IQ would you need to have in order to be in the top 0.15% of the population?

iv What IQ would you need to have in order to be in the bottom 2.5% of the population?

v What percentage of the population has an IQ of 130 or greater?

vi What percentage of the population has an IQ of 85 or less?

vii A group of highly intelligent people had their IQs tested and the distribution was found to be normal, with a mean in the middle of these scores. If 95% of this group had an IQ between 120 and 152, find the mean and standard deviation of the group.

Online resources:

Interactive skillsheet — Standard deviation

CAS instructions — Standard deviation

Quick quiz — 10.4

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ANS p746

Checkpoint

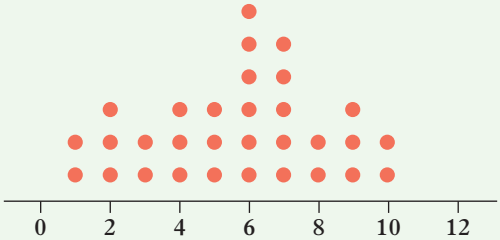
Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this module.
✔ 10
✔ 10A

10.1 **1** Determine the five-number summary for each of the following.

a 14, 16, 50, 23, 17, 24, 32, 19, 22, 32

b 40, 35, 38, 54, 51, 37, 48, 54, 55, 39, 42, 60

c



d

Stem	Leaf
5	4 7 7
6	1 3 5 8 9
7	2 3 5 7 9 9
8	0 1 2

Key: 5 | 4 = 5.4

10.1 **2** Determine the interquartile range and range for the following five-number summaries:

a 14, 18, 23, 29, 32

b 58, 64, 85, 91, 93

10.2 **3** Construct a boxplot for each of the following five-number summaries. These boxplots will have no outliers.

a 120, 130, 145, 150, 175

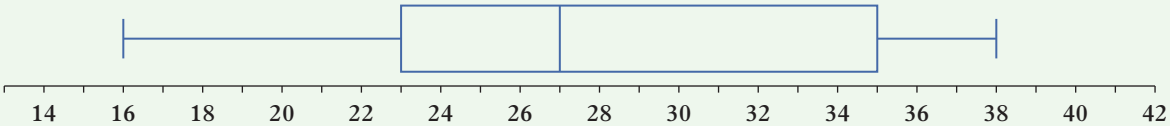
b 7, 12, 16, 22, 30

10.2 **4** For each of the following, state the:

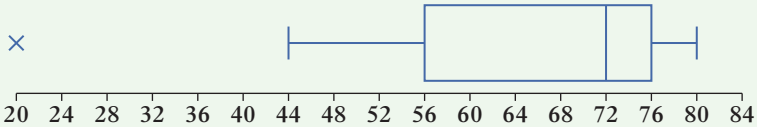
i five-number summary

ii interquartile range.

a



b

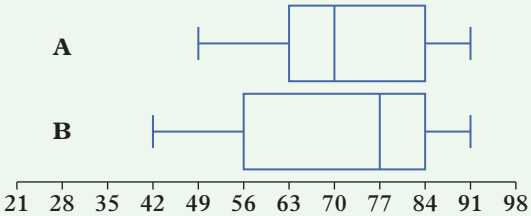


10.2 **5** Construct a boxplot for each set of data. Check each set for potential outliers.

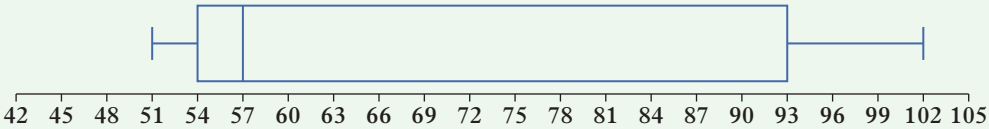
a 41, 85, 74, 67, 79, 75, 22, 84, 80, 66, 41, 79, 45, 54, 47, 84

b 56, 68, 68, 71, 56, 60, 79, 68, 76, 74, 64, 75, 62, 77, 80, 72, 76, 55, 62, 71, 72

10.2 **6** Consider the parallel boxplots shown below. Compare the distributions of the data in terms of the centre and spread.



10.3 **7** State the percentage of the data that lies in the following ranges for the boxplot shown below.



a less than 54

b between 54 and 93

c less than 93

d greater than 57

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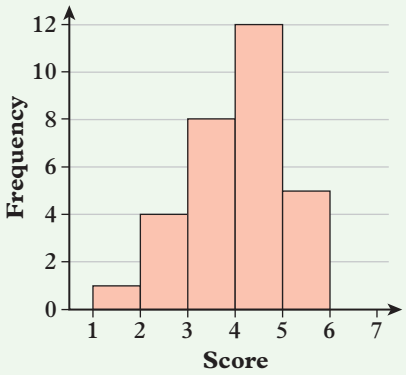
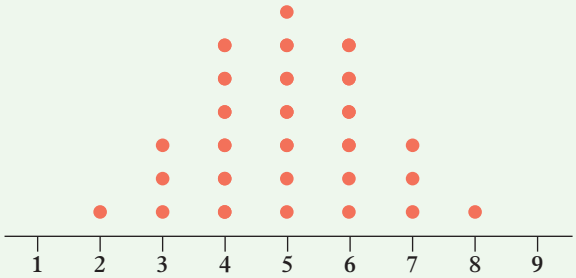
10.3 **8** State the shape of the following distributions.

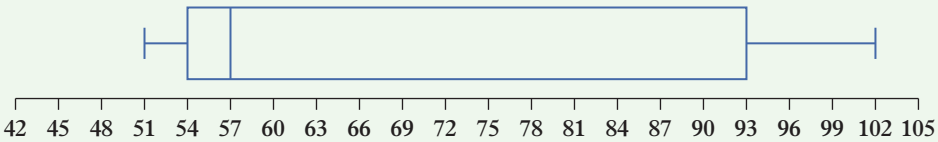
a



b



c

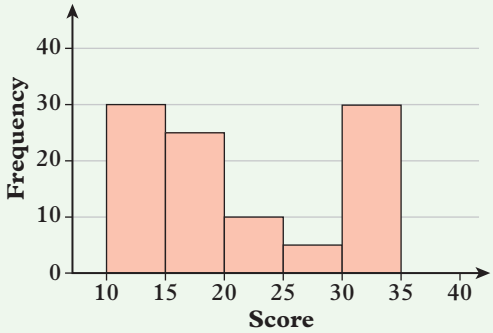


10.3 **9** Construct a cumulative frequency polygon for each of the following.

a

Score	Frequency
8	5
9	10
10	13
11	0
12	4
Total	**32**

b



10A 10.4 **10** Consider the following *sample* data sets. Use a calculator to find the following for each set, giving your answers to one decimal place where appropriate.

i mean **ii** standard deviation

a 9, 11, 13, 16, 18, 22, 24

b 35, 31, 39, 33, 40, 36, 32, 44, 36, 33, 32

c 111, 96, 84, 102, 146, 99, 118, 122, 63

d 277, 215, 278, 293, 229, 210, 234, 283, 285, 206

10A 10.4 **11** Consider the following *population* data sets. Use a calculator to find the following for each set, giving your answers to one decimal place where appropriate.

i mean **ii** standard deviation

a 11, 7, 9, 7, 14, 10, 8, 11, 5, 14, 15, 15

b 46, 48, 48, 37, 49, 38, 56, 48, 43, 46

c 97, 61, 65, 94, 77, 89, 45, 99, 48, 78, 95, 43

d 282, 334, 344, 333, 345, 297, 330, 361, 309

10A 10.4 **12** For each data set below:

i decide whether the data set is from a sample or a population.

ii calculate the appropriate standard deviation, correct to one decimal place.

a This data was collected about the ages of all people at a 40th birthday dinner.

44	43	39	40	40	12	36	67	40	41
14	41	40	6	40	69	73	40	39	37

b This data was collected about the weights, in grams, of bananas from a banana plantation.

174	225	152	178	200	201	198	194
219	173	152	162	154	179	228	212

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Lesson 10.5

Bivariate categorical data

Learning intentions

By the end of this lesson you will be able to ...

✔ decide whether there is a possible relationship between two categorical variables.

Inter-year links

Year 8 10.5 Two-way tables

Year 9 8.5 Two-way tables

Bivariate categorical data and Likert scales

- **Categorical data** is data that can be put into distinct groups or categories.
- **Bivariate data** contains two variables, where each data point in one variable is paired with a data point in the other variable.
- A **Likert scale** is a rating scale which is commonly used in surveys.
  - → When using a Likert scale, respondents are typically asked to measure their level of agreement or disagreement to a particular statement.

Key content video
Bivariate categorical data

Strongly Disagree	Disagree	Neutral	Agree	Strongly Agree
○	○	○	○	✔

Two-way tables

- **Two-way tables** can be used to display the relationship between different categories or attributes of data (such as survey responses).

  For example, the following two-way table shows the responses of 50 children and 60 adults to a survey question asking: 'How did you feel about the new Star Wars movie?'

	Very unhappy	Unhappy	Neutral	Happy	Very happy	Total
Children	4	7	12	15	12	50
Adults	11	10	15	14	10	60

- The percentages for each category can be calculated using the total number in that category.
  For example, $\frac{4}{50} \times 100\% = 8\%$ of children were very unhappy with the new movie.

	Very unhappy	Unhappy	Neutral	Happy	Very happy	Total
Children	8%	14%	24%	30%	24%	100%
Adults	18%	17%	25%	23%	17%	100%

  - → This allows us to compare the responses for different categories of respondents, even when there is not an equal number of respondents in each category. For example, 30% + 24% = 54% of children were either happy or very happy with the movie, compared to 23% + 17% = 40% of adults.

Relationships between variables

- When there is a significant difference between the percentages in each category, then we can say that there is a possible relationship between the variables.
  - → Small differences in the percentages in different categories may be due to natural variation in the data or the samples chosen.
- Patterns may also be identified when analysing data in two-way tables.
  For example, you may find that one category of respondent has responses spread out across a Likert scale, whereas another category of respondent has more responses at either end of the scale.

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Worked example 10.5A Calculating percentages for each category in a two-way table

The following two-way table shows whether a sample of Year 7 and Year 10 students agree or disagree with the statement, 'I prefer to do my homework on the same day I receive it, rather than waiting until it is due'.

	Strongly disagree	Disagree	Neutral	Agree	Strongly agree	Total
Year 7	32	9	15	9	15	80
Year 10	24	10	16	12	23	85

Convert the responses into percentages for each category of respondent, rounding values to the nearest whole number where necessary.

THINK

Divide the number for each particular response by the total number for that category, rounding your answers to the nearest whole number.

WRITE

	Strongly disagree	Disagree	Neutral	Agree	Strongly agree	Total
Year 7	$\frac{32}{80} = 40\%$	$\frac{9}{80} \approx 11\%$	$\frac{15}{80} \approx 19\%$	$\frac{9}{80} \approx 11\%$	$\frac{15}{80} \approx 19\%$	100%
Year 10	$\frac{24}{85} \approx 28\%$	$\frac{10}{85} \approx 12\%$	$\frac{16}{85} \approx 19\%$	$\frac{12}{85} \approx 14\%$	$\frac{23}{85} \approx 27\%$	100%

Worked example 10.5B Identifying possible relationships between categorical variables

The following percentage two-way table shows how different proportions of right- and left-handed people responded to the statement, 'I think the world is a fair place'.

	Strongly disagree	Disagree	Neutral	Agree	Strongly agree	Total
Right-handed	19%	27%	23%	20%	11%	100%
Left-handed	25%	31%	24%	14%	6%	100%

Comment on if there appears to be a relationship between the categorical variables.

THINK

1. Look at how the percentages change for each category across the five different levels of agreement. As well as looking at the individual values, see if there are any patterns in the data.
2. Do the differences in the proportions for each category appear to be due to natural variation or due to a relationship between the variables?

WRITE

There does appear to be a relationship between the categorical variables, with a significantly higher number of right-handed people (31%) strongly agreeing or agreeing with the statement compared to left-handed people (20%).

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Helpful hints

✔ Remember that there is a natural variation in data, and you should not expect the proportions within two categories of data to be exactly identical. Instead, look to see if there is a significant difference in the proportions within the categories of data.

Learning pathways

1–7, 9 | 2, 4, 6, 7–10 | 3, 5, 7–11

ANS p747

Exercise 10.5A: Understanding and fluency

1 Complete this two-way table.

	Primary school	High school	Total
Catch public transport	22		40
Are driven	37	11	
Walk/ride			
Total		38	102

WE 10.5A **2** The following two-way table shows whether a sample of Year 10 and Year 12 students agree or disagree with the statement, 'I prefer to spend my free time outside rather than inside'.

	Strongly disagree	Disagree	Neutral	Agree	Strongly agree	Total
Year 10	13	17	20	33	27	110
Year 12	9	13	18	24	26	90

Convert the responses into percentages for each category of respondent, rounding values to the nearest whole number where necessary.

3 The following two-way table shows whether a sample of swimmers and runners agree or disagree with the statement, 'I enjoy waking up early to start my training for the day'.

	Strongly disagree	Disagree	Neutral	Agree	Strongly agree	Total
Swimmers	13	10	14	16	11	64
Runners	20	15	13	14	10	72

Convert the responses into percentages for each category of respondent, rounding values to the nearest whole number where necessary.

WE 10.5B **4** The following percentage two-way table shows how different proportions of Australians and New Zealanders responded to the statement, 'My ideal holiday is relaxing on a beach with nothing to do'.

	Strongly disagree	Disagree	Neutral	Agree	Strongly agree	Total
Aus	13%	14%	15%	21%	37%	100%
NZ	22%	14%	20%	14%	30%	100%

Comment on if there appears to be a relationship between the categorical variables.

5 The following percentage two-way table shows how players from two different local footy clubs responded to the rule changes for the upcoming season.

	Very unhappy	Unhappy	Neutral	Happy	Very happy	Total
Team A	31%	14%	25%	18%	12%	100%
Team B	28%	16%	23%	19%	14%	100%

Comment on if there appears to be a relationship between the categorical variables.

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Exercise 10.5B: Problem solving and reasoning

6 Nikolas was conducting research into people's feelings about the nominations for the student council.

Of the 60 Year 8 students he asked, 21 were very happy with the options, 9 were happy with the options, 13 were unhappy with the options, and 17 were very unhappy with the options.

By contrast, from the 75 Year 10 students he asked, 43 were very happy with the options, 14 were happy with the options, 11 were unhappy with the options, and 7 were very unhappy with the options.

a Create a percentage two-way table to represent this information, rounding to the nearest per cent.

b Comment on if there appears to be a relationship between the categorical variables.

7 The CEO of a company has constructed this table showing the composition of her company.

	Total employees		Full-time		Part-time	
	25 and over	Under 25	25 and over	Under 25	25 and over	Under 25
Managers	18	4	14	4	4	0
Non-managers	42	61	40	31	2	30

Below are four claims made by the CEO. For each statement, state whether it is correct. Provide mathematical evidence to support your reasoning. For any incorrect claims, supply the correct figure.

a 82% of 25-and-over employees are managers.

b 6% of under-25s in the company are managers.

c 29% of employees work part-time.

d 49% of under-25 non-managers work full-time

8 Pedro is a manager of a flower store which sells bunches of flowers in either paper wrapping or with a vase. To help with his inventory planning, Pedro records the proportion of sales of the bunches he sells.

	Summer sunrise	Orchid bloom	Fond memories	Rosy outlook	Autumn days	Total
In paper	31%	10%	22%	28%	9%	100%
With vase	18%	24%	20%	26%	12%	100%

a If Pedro sells 400 bunches in paper and 150 bunches with vases, convert the percentage two-way table into a two-way frequency table.

b Answer the following questions.

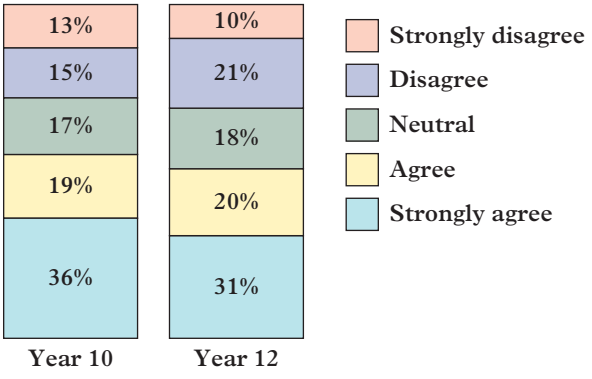
i Did any of the bunches sell more with a vase than in paper? If so, which bunches?

ii Which type of bunch had the biggest difference in percentage sales between selling in paper compared to selling in vases?

iii Which type of bunch had the biggest difference in raw sales between selling in paper compared to selling in vases?

9 Segmented bar charts can be used to show proportions within individual categories. In a segmented bar chart, the height of the total bar represents 100% of a category, and the height of each segment represents a proportion within that category.

Consider the segmented bar charts on the right, which represent the proportions of Year 10 and Year 12 students who agree with the statement, 'Summer is my favourite time of the year'.



a Comment on if there appears to be a relationship between the categorical variables.

b Convert the segmented bar chart into a percentage two-way table.

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10 A study into social media usage asked participants to rate their level of agreement with the statement, 'TikTok is my favourite social media platform'. The results for children and adults have been split up and are displayed in the following table.

	Strongly disagree	Disagree	Neutral	Agree	Strongly agree	Total
Children	27%	11%	10%	15%	37%	100%
Adults	34%	26%	15%	11%	14%	100%

a Comment on if there appears to be a relationship between the categorical variables.

b Convert the percentage two-way table into a segmented bar chart.

Exercise 10.5C: Challenge

11 Up to this point we have used our own judgement on whether there appears to be a relationship between categorical variables, however, a chi-squared test can also be used for this purpose.

A chi-squared test of independence can be used when you have two categorical variables, and you want to test whether there is a relationship between the variables. In a chi-squared test we have two hypotheses.

- Null hypothesis (H_0): There is no relationship between the categorical variables
- Alternative hypothesis (H_A): There is a relationship between the categorical variables

To conduct a chi-squared test, we need to calculate a chi-squared value and then compare this value to a critical value. We then:

- accept the null hypothesis if the chi-squared value is less than the critical value, or
- reject the null hypothesis if the chi-squared value is greater than the critical value.

Consider this two-way table, which shows two groups of attendees' responses to the annual Christmas play at a local theatre.

	Didn't enjoy	Neutral	Loved it
First-time attendees	10	24	40
Previous attendees	16	20	90

a Calculate the total number of attendees to the Christmas play.

b Calculate the row totals, that is, the total number of first-time attendees and previous attendees.

c Calculate the column totals, that is, the total number of attendees who didn't enjoy, were neutral towards, or who loved the play.

d For each cell in the table, multiply the row total by the column total and divide by the overall total (calculated in part **a**). Input these values into a new table, with the same row and column headings.

e We call the values in the table from part **d** the expected values. For each cell in the table, calculate the square of the difference between the expected value and the actual value, and divide this total by the expected value. Round these values to two decimal places and input the results into another new table, with the same row and column headings.

f Our chi-squared value is the sum of the values from part **e**. Calculate this value.

g Compare our chi-squared value to the critical value of 5.991*. Is the null hypothesis accepted or rejected?

*Note: This is not a fixed value and varies depending on the parameters of the test.

h Does this mean that there is an association between the categorical variables or not?

Online resources:

Interactive skillsheet
Bivariate categorical data

Quick quiz
10.5

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Lesson 10.6

Scatterplots

Learning intentions

By the end of this lesson you will be able to ...

✔ interpret, draw and use scatterplots and draw lines of good fit by eye.

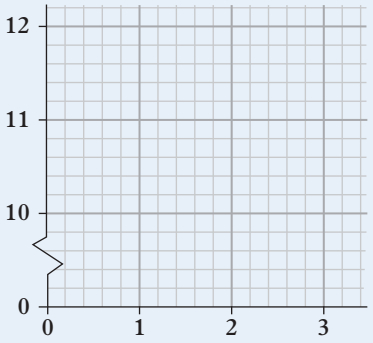
Inter-year links

Year 9 7.2 Classifying and displaying data

Scatterplots

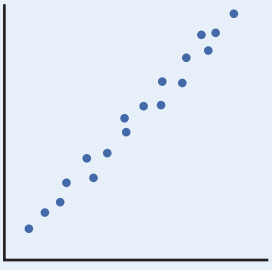
Key content video
Scatterplots

- A **scatterplot** can be used to display numerical bivariate data.
  → When plotting numerical bivariate data on a scatterplot, one variable is plotted on the horizontal axis and the other variable is plotted on the vertical axis.
- Scatterplots are usually drawn focusing on the area of the axes between which the data points lie. A break in the scale on the axes, indicated by a zigzag, can be used to indicate that the focus is on the axes a distance away from the origin.

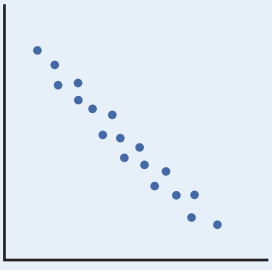


Direction

- If there is a **positive relationship** between two numerical variables, then as one variable increases in value the other variable also increases in value.
- If there is a **negative relationship** between two numerical variables, then as one variable increases in value the other variable decreases in value.



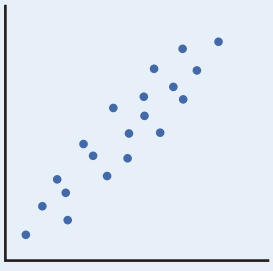
Positive



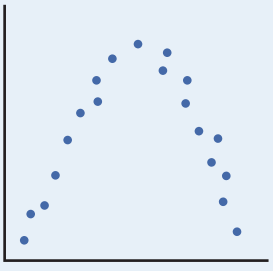
Negative

Linearity

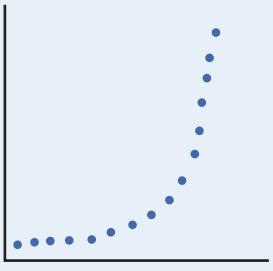
- When considering bivariate data, we need to determine whether the pattern of the data is either linear or non-linear. For a linear relationship, the points can be approximated by a straight line.



Linear



Non-linear



Non-linear

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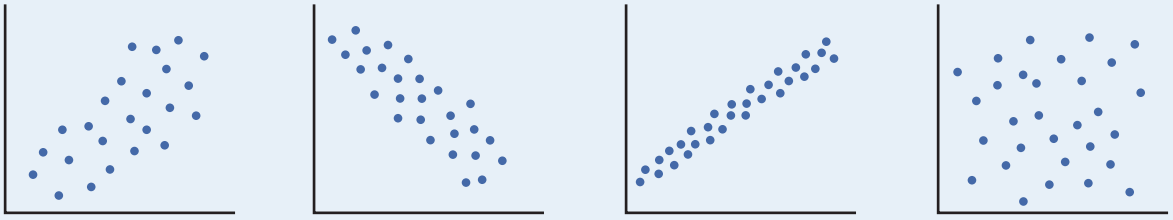
Strength

- The **correlation** between two numerical variables is a measure of the strength of the linear relationship between those variables. If the relationship is strong, the points will lie close to a straight line.
- When there is no obvious pattern in the data, then there is no correlation between the variables.



Weak **Moderate** **Strong** **No correlation**

Association and causation

- When investigating numerical bivariate data, we usually look at how one variable (the **independent variable**) affects the other variable (the **dependent variable**).
- To determine which variable is the independent variable, consider which variable may reasonably be impacted by a change in the other variable. For example, if you are investigating how temperature affects ice cream sales, then 'temperature' would be the independent variable and 'ice cream sales' would be the dependent variable.
- When plotting a scatterplot:
  - → the independent variable should be placed on the horizontal axis
  - → the dependent variable should be placed on the vertical axis.
- When two variables are correlated, then we can say that there is an association between the two variables. For example, if there is a strong linear positive correlation between two variables, we can say that, as one variable increases in value, the other variable also increases in value.
- If there is a **cause-and-effect** relationship (or 'causal' relationship) between two variables, then a change in one variable directly causes a change in the other variable. For example, a change in temperature may directly affect the amount of ice cream sales.
- An association between two variables may but does not necessarily mean that there is a cause-and-effect relationship between the two variables.
- To determine whether a relationship between two variables is potentially causal, consider whether it is reasonable to say that a change in the independent variable may cause a change in the dependent variable.
- If there is no obvious relationship between two variables, then there may be a **confounding variable** that affects both the independent and dependent variables.
  - → It is difficult to say for certain if there is a causal relationship between variables, so be careful with the language you use and look for a potential causal relationship.

Lines of good fit by eye

Key content video
Lines of good fit

- A **line of good fit** is a line that represents the relationship between the two variables on a scatterplot.
- To draw a line of good fit by eye, draw a straight line on your scatterplot so that it lies as close to as many points as possible.
  - → There should be approximately the same number of points on each side (below and above) of the line of good fit.

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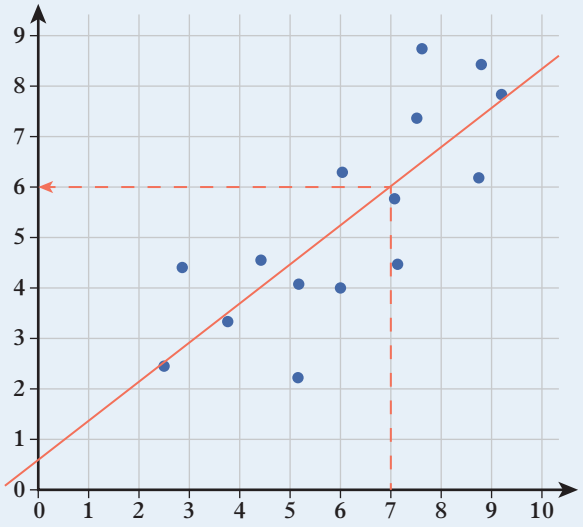
Using a line of good fit to make predictions

- A line of good fit can help predict the value of one variable when the other variable is known.
  - → As the strength of the correlation between two plotted variables increases, the predictions made using a line of good fit become more reliable.
  - → Predictions made *within* the original range of the data plotted are more reliable than predictions made *outside* the original range of data.
- To make a prediction using a line of good fit, locate the value on the axis you are predicting for, and draw a straight line (horizontal or vertical) to the line of good fit. From this point, draw another straight line (horizontal or vertical) to the other axis to get your predicted value.

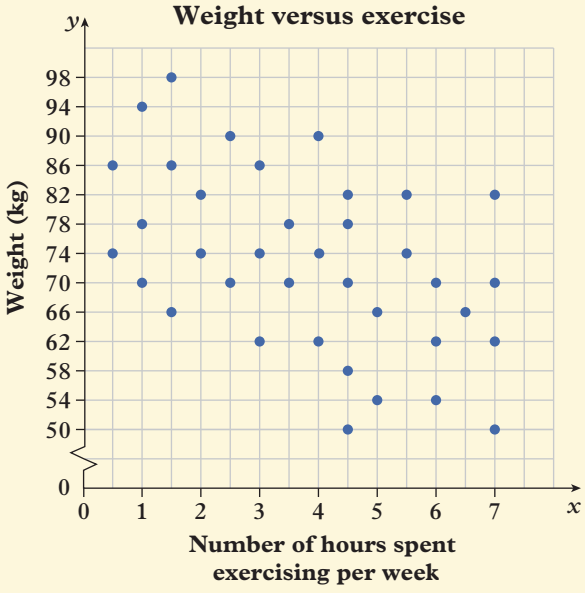


Worked example 10.6A Describing the relationship in a scatterplot

Describe the relationship shown in this scatterplot:

a in terms of its strength and direction

b in the context of the data it represents.



THINK

a There is only a weak linear correlation in the data, but as the independent variable increases, the dependent variable does decrease.

b Describe the relationship shown in the context of what it tells you about the data.

WRITE

a The scatterplot shows a weak, negative linear relationship between the two variables.

b As the number of hours spent exercising increases, weight generally decreases.

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Worked example 10.6B Drawing a scatterplot

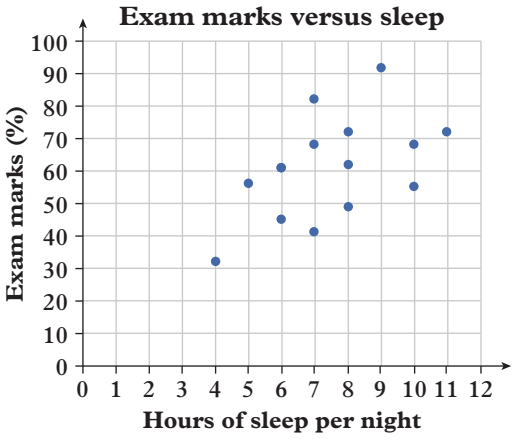
Draw a scatterplot for this data set.

Hours of sleep per night	8	6	7	9	10	11	8	7	6	5	4	8	7	6	10
Exam marks	72	61	82	92	68	72	49	68	61	56	32	62	41	45	55

THINK

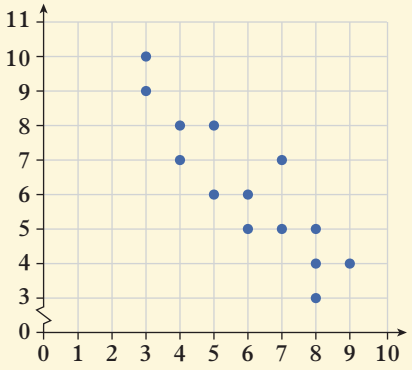
1. Determine which of the two variables is the independent variable and which variable is the dependent variable. A change in the number of hours of sleep per night could affect the exam marks, so the number of hours of sleep per night is the independent variable.
2. Use the horizontal axis for the independent variable (hours of sleep per night) and the vertical axis for the dependent variable (exam mark).
3. Identify the smallest and largest value for each variable and mark scales on the axes to cover these values. (Show breaks in the axes if the values on either axis do not begin at zero.)
4. Create a scatterplot by plotting each pair of variables from the table on the axes.

WRITE



Worked example 10.6C Drawing a line of good fit by eye

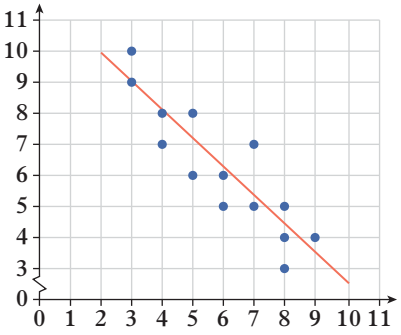
Draw a line of good fit by eye for the data shown on this scatterplot.



THINK

1. Look at the direction of the plotting of the points. There is a clear downward trend, so the line of good fit will need to be drawn in this direction.
2. Draw a straight line on the scatterplot in the direction of the data, with an approximately equal number of points below and above the line.

WRITE



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Helpful hints

- ✔ When creating a scatterplot by hand, use graph paper to ensure the scatterplots you create are appropriately scaled and the points are accurately plotted.
- ✔ A line of good fit may, but does not need to, pass through any of the points on the scatterplot.
- ✔ The strength of an association is based on how well a straight line can fit the data, not the steepness of a line of good fit.

Learning pathways

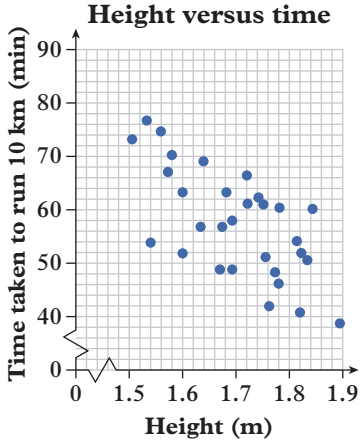
1–3, 6–10	1–7, 11, 15, 16	1–3, 9–18

Exercise 10.6A: Understanding and fluency

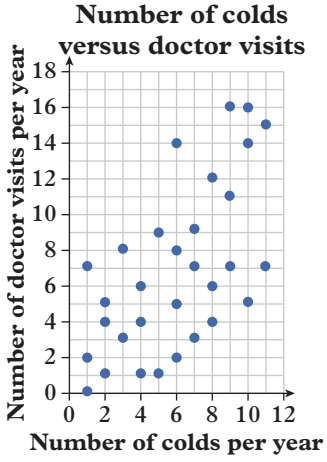
WE 10.6A **1** Describe the relationship shown in each of the scatterplots below:

i in terms of strength and direction **ii** in the context of what it shows about the data it represents.

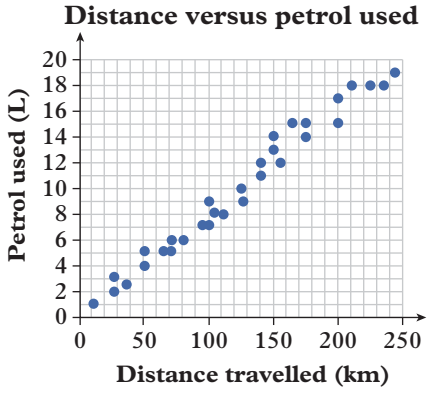
a



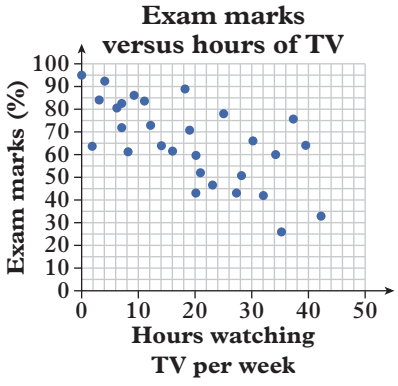
b



c



d



WE 10.6B **2** Draw a scatterplot for each of these data sets.

a

Height (cm)	150	181	166	168	175	178	172	155	161	168	179	184	188	159	167
Shoe size	4	9	9	7	9	8	7	4	5	7	7	10	12	7	6

b

Temperature (°C)	33	15	16	25	27	36	41	11	8	32	20	25	31	18	28
Number of pies sold	5	25	33	16	19	2	3	42	35	12	21	11	10	51	15

c

Number of customers	45	56	16	32	78	98	26	66	45	76	86	42	59	31	67
Daily profit ($100s)	12	16	8	19	21	48	7	32	22	29	36	13	18	16	12

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3 Describe the relationship shown in each scatterplot you drew for question **2** in the context of what it tells you about the bivariate data it represents.

WE 10.6C **4** For each of these data sets, construct a scatterplot and draw a line of good fit by eye.

a

x	3	2	7	5	6	3	1	4	10	9	7	4	8	10	1	5	7	3	8	6
y	5	4	7	6	8	2	2	3	10	9	9	8	6	8	5	4	4	6	10	5

b

x	4	7	1	2	10	5	8	6	2	9	4	3	10	7	1	9	8	5
y	5	4	9	9	2	6	4	5	7	4	8	8	1	6	8	1	5	7

c

x	6	8	3	9	11	14	5	1	15	13	8	5	11	12	7	4
y	5	10	5	6	11	9	3	2	11	13	8	7	8	10	3	1

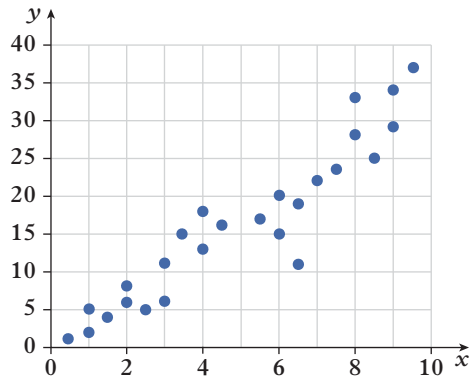
d

x	4	9	13	15	7	6	18	11	13	7	9	10	18	12	16	17
y	42	29	15	6	38	36	1	13	18	40	20	12	4	16	9	7

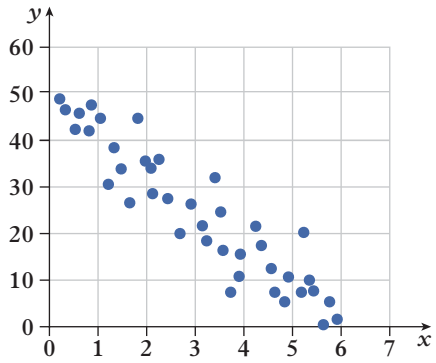
5 For each scatterplot below:

i draw a line of good fit by eye **ii** use the line of good fit you drew to predict the value of y when $x = 5$.

a



b



Note: Copies of these scatterplots can be found in the Exercise 10.6 BLM.

Exercise 10.6B: Problem solving and reasoning

6 Consider the scatterplots shown in question **1**. In which of these situations do you think there is a potential causal relationship between the independent variable and the dependent variable? Explain your thinking in each instance.

7 Marie wants to investigate the relationship between the number of different types of animals in zoos and the number of visitors that the zoos receive per day. For this investigation, she collects data from 16 different zoos.

a Predict the correlation you expect to see between the two variables in terms of linearity, strength and direction.

b Marie summarised the data in the table below. Draw a scatterplot to represent this data.

Number of different animal types	9	26	11	37	19	42	15	14	25	32	47	34	31	37	22	18
Number of visitors per day	42	261	150	541	247	387	226	291	317	401	761	396	309	311	177	151

c Describe the trend shown by the scatterplot you drew for part **b**. Compare the trend to the prediction you made in your answer to part **a**.

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8 When examining the relationship between two variables in real life, we think about how one variable changes depending on how the other changes. Height and weight are a good example of this.

a Would you say that the height of a person depends on how much they weigh? Or does the weight of a person depend on how tall the person is?

b Would you say that, as a person gets taller, this generally makes them heavier? Or, as a person gets heavier, this generally makes them taller?

c If we consider height and weight, which is:

i the dependent variable **ii** the independent variable?

9 For each of the pairs of variables **a–d** below, determine:

i if there might be a linear relationship between them

ii which is the independent variable and which is the dependent variable if a linear correlation does exist between them

iii the strength and direction of any linear correlation that exists.

iv Decide if there is a potential causal relationship between the variables, justifying your answer.

a time (years) and height of a tree (metres)

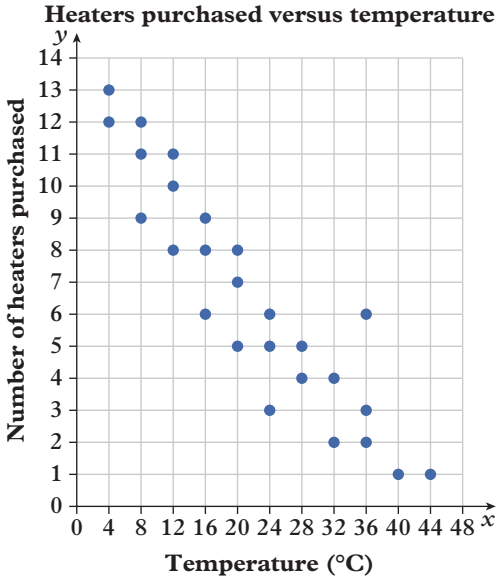
b value of a car (dollars) and age (years)

c weight of a person (kilograms) and number of languages he or she speaks

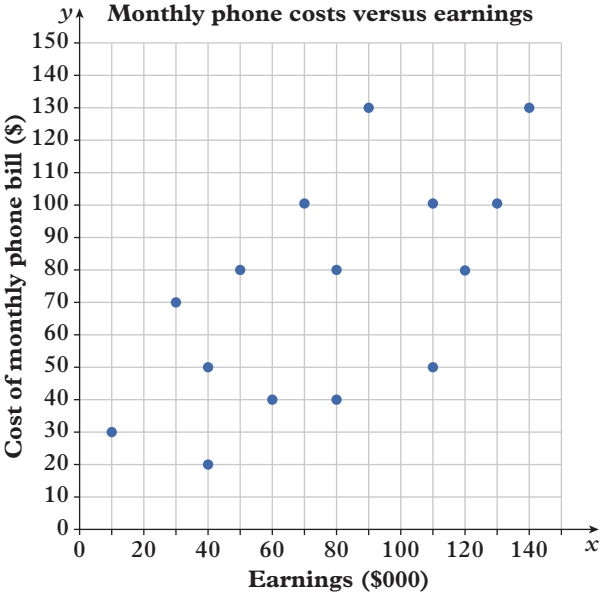
d shoe size and height of a person (metres).

10 Consider the two scatterplots below.

Scatterplot A



Scatterplot B



a For each of the scatterplots, describe the strength and direction of the linear relationship shown in the context of the data it shows.

b If the temperature on a day was 26°C, how many heaters do you predict would be bought?

c If somebody had earnings of $100 000, what would you predict their monthly phone bill to be?

d How confident are you of your predictions in parts **b** and **c**?
Explain why you would not be as confident predicting a value from scatterplot A as you would be predicting a value from scatterplot B.

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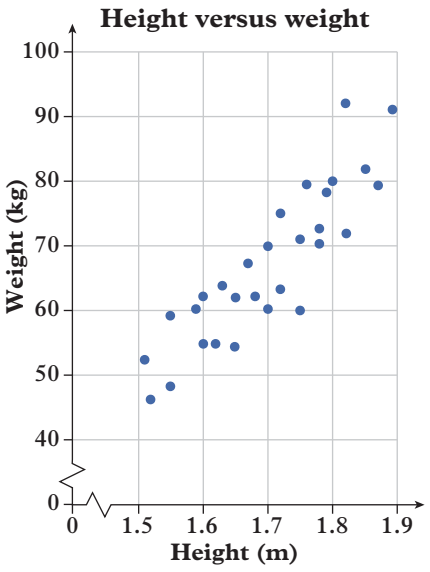
11 A strong positive linear correlation was found between the number of hours studied and the result achieved on a test.

a Does this mean that if you study for many hours you will get high marks? Explain.

b To help prevent assumptions like the one made in part **a**, there is a saying: 'correlation does not mean causation'. What does this mean?

c Explain why the saying quoted in part **b** is useful to remember, particularly when dealing with strong correlations.

12 Consider the scatterplot on the right, showing the relationship between the heights and weights of the students in a Year 10 class.



a Describe the trend shown.

b Draw a line of good fit by eye for this scatterplot.

c Use your line of good fit to predict the weight of a person who is 185 cm tall.

d How confident are you of your prediction in part **c**? Explain your answer.

13 Consider the data in this table.

Number of videos downloaded	21	6	17	21	8	49	2	31	22	13	4	6	42	36	25	9	30	17	21	14
Space available on device (GB)	13	10	8	9	18	2	14	1	5	21	22	18	9	2	10	11	13	15	3	10

a Draw a scatterplot to represent the data.

b From your scatterplot, describe the relationship between the two variables.

c On your scatterplot, draw a line of good fit by eye.

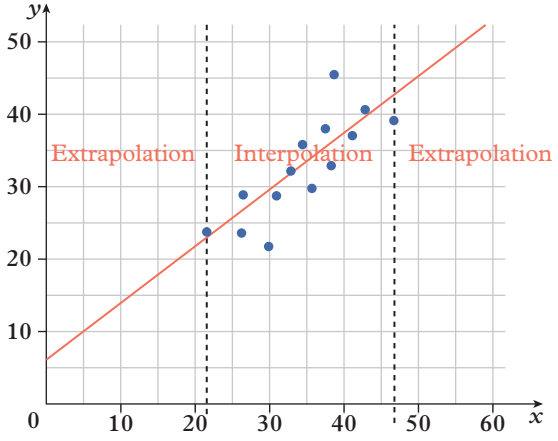
d If somebody had downloaded 30 videos, how much space would you predict there would still be available on the device?

e Explain why you can't have much confidence in your prediction from part **d**.

f Can you provide a reason why the relationship between the variables in this data set is moderate?

g Predicting a value within a data set, as you did in part **d**, is called interpolation. Sometimes you may want to predict a value outside the data set. This is called extrapolation.

Identify whether each of these predictions uses interpolation or extrapolation.



i There will be 5 GB available on a device when 40 videos have been downloaded.

ii There will be 2 GB available on a device when 60 videos have been downloaded.

iii There will be 15 GB available on a device when 20 videos have been downloaded.

iv There will be 1 GB available on a device when 50 videos have been downloaded.

h Imagine that somebody had downloaded 200 videos. If the graph continues its current downward trend, what prediction would you make about available space on the device?

i Comment on your answer to part **h**. How confident can you be about making a prediction outside the given data set?

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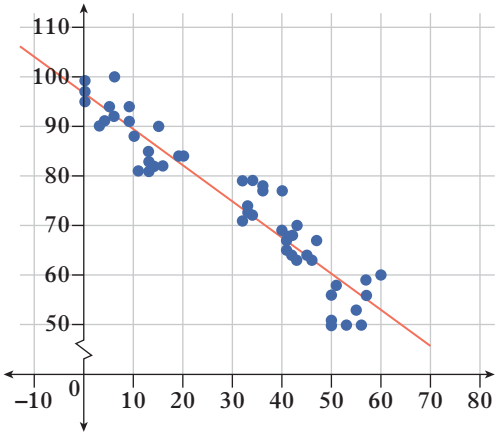
14 Consider the scatterplot and line of good fit shown.



a Predict the dependent value for each of the following, correct to the nearest 5. State whether the prediction is interpolation or extrapolation.

i $x = 10$ **ii** $x = -10$

iii $x = 25$ **iv** $x = 65$

b Predict the independent value for each of the following. State whether the prediction is interpolation or extrapolation.

i $y = 75$ **ii** $y = 55$

iii $y = 40$ **iv** $y = 100$

c For which values of x will predictions of the dependent variable be an:

i interpolation **ii** extrapolation?

d For which values of y will predictions of the independent variable be:

i interpolation **ii** extrapolation?

15 Can a scatterplot be drawn with a negative section of the x- and/or y-axis? Explain.

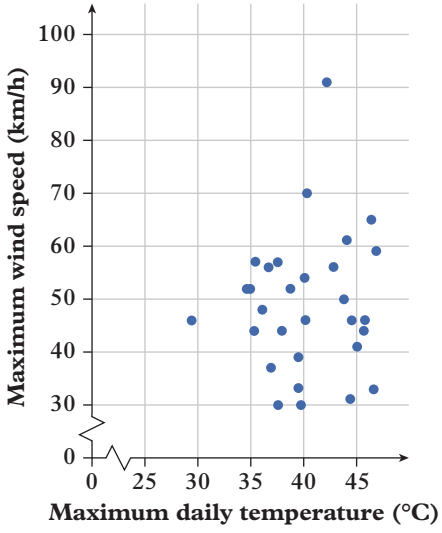
16 Explain why it is important to plot a scatterplot with the original data rather than with grouped data.

17 For each of the following:

i State whether there is a correlation between the two variables, and if there is, describe it in terms of linearity, strength and direction

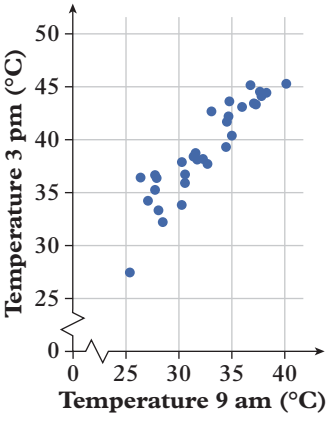
ii Decide whether there is a potential causal relationship between the variables, justifying your answers.

a The scatterplot below plots the maximum wind speed, in kilometres per hour, against the maximum temperature, in °C, for each day in January 2022 in Birdsville, Queensland.



(Source: BOM)

b The scatterplot below plots the temperature at 3 pm, in °C, against the temperature at 9 am, in °C, for each day in January 2022 in Birdsville, Queensland.



(Source: BOM)

Exercise 10.6C: Challenge

18 Look up the weather data for the previous month in your local area. Determine any correlations between the numerical variables given by constructing a variety of scatterplots with the data.

Online resources:

- **Interactive skillsheet** Scatterplots
- **BLM** Exercise 10.6
- **Desmos activity** Dapper cat
- **Quick quiz** 10.6

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Lines of best fit

Learning intentions

By the end of this lesson you will be able to...

✔ calculate equations of lines of best fit and use them to make predictions.

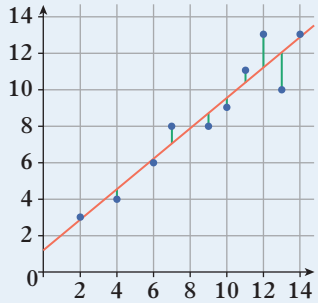
Inter-year links

Year 9 7.2 Classifying and displaying data

Least-squares regression lines

Key content video
Lines of best fit

- A **least-squares regression line** is a mathematically calculated **line of best fit** for numerical bivariate data. It is found by minimising the sum of the squares of the errors between a line of best fit and the points on a scatterplot.
- The equation for a least-squares regression line can be found easily using spreadsheet software (like Excel), a calculator or other technology.
- When the equation for a regression line is known, it can be used to make predictions about the data by substituting in values for one of the variables.
  - → As the strength of the correlation between the variables increases, the predictions made using a regression line become more reliable.
  - → Predictions made within the original range of the data (known as **interpolation**) are more reliable than predictions made outside the original range of the data (known as **extrapolation**).



Worked example 10.7A Creating a scatterplot with a line of best fit

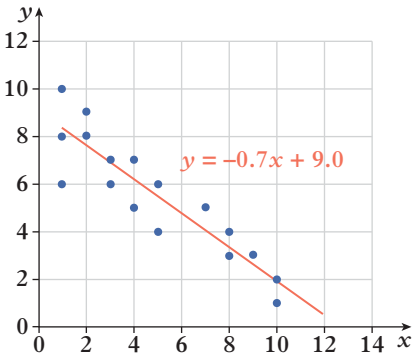
Use Excel (or alternative spreadsheet software) to graph the data from this table and find the least-squares regression line, correct to one decimal place.

x	1	5	8	3	10	2	7	4	1	9	10	3	5	2	1	8	4
y	10	4	3	6	1	8	5	7	8	3	2	7	6	9	6	4	5

THINK

1. Enter the data into an Excel spreadsheet. The values can be put in rows or columns, with each cell containing a single value.
2. To create a scatterplot, highlight the cells containing the data and select Insert → Scatter.
3. Select the graph, then use the toolbar to select Chart Design → Add Chart Element → Trendline → Linear.
4. Click on the trendline and select Trendline Options → Display Equation on chart.
5. Round the values in the equation to one decimal place.

WRITE



$y = -0.7x + 9.0$ (correct to one decimal place)

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Worked example 10.7B Using a line of best fit to make predictions

Use the equation for the line of best fit given in **Worked example 10.7A** to:

a predict the value of y when $x = 3$

b predict the value of x, to one decimal place, when $y = 5$.

THINK

a 1 Write the equation for the line of best fit.

2 Substitute the value $x = 3$ into the equation manually, or use your calculator, to find the predicted value of y.

b 1 Write the equation for the line of best fit.

2 Substitute the value $y = 5$ into the equation manually, or use your calculator, to find the predicted value of x.

3 Round the value of x to one decimal place.

WRITE

a $y = -0.7x + 9.0$

$= -0.7 \times 3 + 9.0$

$= 6.9$

The predicted value of y is 6.9.

b $y = -0.7x + 9.0$

$5 = -0.7x + 9.0$

$0.7x = 9.0 - 5$

$0.7x = 4$

$x = \frac{4}{0.7}$

$x \approx 5.7$

The predicted value of x is 5.7.

Helpful hints

✔ The least-squares regression line can be expressed in two forms: $y = mx + c$ or $y = a + bx$.

Learning pathways

1(a–c), 2–4, 6–8 | 1(b, d), 2, 3, 5, 7–9 | 1(c, d), 5, 7–11

ANS p747

Exercise 10.7A: Understanding and fluency

WE 10.7A **1** For each of these data sets, use technology to graph the data and find the equation of the line of best fit (least-squares regression line). Round the values in the equation to one decimal place.

a

x	11	6	14	8	13	7	16	3	19	8	18	3	6	4	7	10	12	1
y	32	16	49	27	40	29	58	24	62	36	64	12	23	17	24	37	42	6

b

x	3	6	4	1	2	8	9	7	4	8	7	19	9	4	3	6	2	1
y	5	11	7	2	4	10	12	12	10	15	10	16	15	5	9	9	6	5

c

x	2	9	7	8	6	4	4	6	7	3	8	2	1	6	4	9	8	3
y	19	2	5	7	9	11	12	22	2	15	4	17	23	7	15	1	3	18

d

x	2	3	4	8	6	4	7	9	2	9	6	5	3	7	8	5	1	2
y	10	13	16	22	19	14	25	33	13	26	8	20	15	20	26	22	9	8

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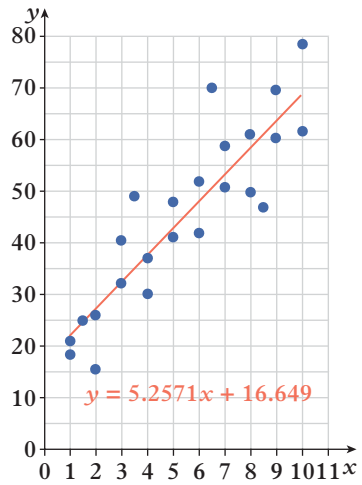
WE 10.7B **2** Use the line of best fit shown on this scatterplot to predict the value of the following (correct to one decimal place where necessary):



a x when $y = 36$
b x when $y = 73$
c y when $x = 4$
d y when $x = 9$
e y when $x = 0$
f x when $y = 30$
g y when $x = 10$.

3 Use each line of best fit you found in question **1** to predict the value of y when $x = 10$.

Exercise 10.7B: Problem solving and reasoning

4 Andrew wanted to investigate the relationship between the number of books students read in a year and their scores in an English exam.

a What relationship do you predict will exist between the number of books read and the English mark obtained?

b State which variable (number of books read or English mark obtained) is the independent variable and which is the dependent variable.

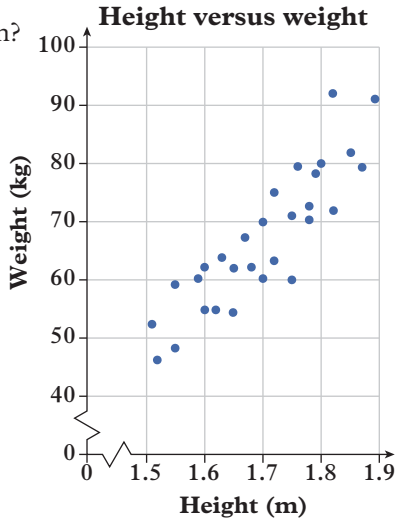
c Andrew collected the data in the table below. Create a scatterplot to represent the data.

Number of books read	11	6	21	8	2	7	9	15	16	1	0	4	1	0	12	29
English exam mark (%)	70	60	89	70	47	67	65	86	75	45	55	52	62	30	77	95

d Use technology to find a line of best fit (least-squares regression line) for this data.

e Andrew wants to know how many books he should read in a year if he is aiming for a mark of 70% in the English exam. What would you tell him?

5 Consider the scatterplot on the right, showing the relationship between the heights and weights of the students in a Year 10 class.



a Describe the trend shown.

b The line of best fit for this scatterplot passes through the points (1.65, 62) and (1.85, 82). Use these two given points and the formula $y - y_1 = m(x - x_1)$ to find the equation of the line of best fit.

c Use the equation found in part **b** to predict the weight of a person who is 185 cm tall.

6 Ms Archer is looking into the science and maths results of her class and compiles the following data.

Science result	65	59	73	42	55	81	70	68	36	49	84	68	71	51	38	77	62
Maths result	71	54	81	38	73	69	66	91	50	44	92	63	75	60	52	73	57

a Plot this data on a scatterplot.

b Use technology to find the equation of the line of best fit.

c Use your result from part **b** to predict the following, to one decimal place.

i the maths result when the science result is 48
ii the science result when the maths result is 60
iii the maths result when the science result is 78
iv the science result when the maths result is 93

d How confident are you with your predictions in part **c**? Explain your answer.

e Which of the predictions from part **c** are you least confident about? Explain your answer.

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7 Lou is a basketball fanatic and wanted to research how a basketball player's height affects their scoring ability. He collected the following data on players in his local league.

Height (cm)	183	175	186	198	169	190	177	168	172	188	177	194	181	196
Points per game	14	17	16	19	9	22	20	14	15	24	11	21	13	11

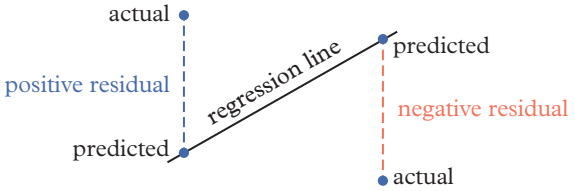
a Plot this data on a scatterplot.

b Use technology to find the equation of the line of best fit.

c How confident would you be with making predictions using your line of best fit? Explain your answer.

d Do you think there is a potential causal relationship between height and points per game? Explain your answer.

8 A residual is the signed difference between the dependent value from the original data and the predicted value from the regression line of the same independent value. That is, residual value = actual value – predicted value.



a Determine the least-squares regression line for the data below using technology. Write the coefficients correct to three significant figures.

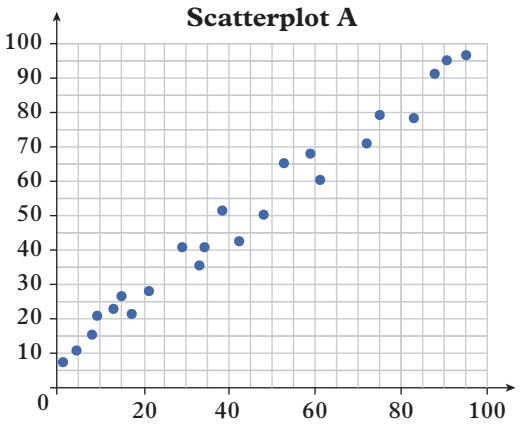
x	1	1.2	2	1.4	2.4	3	1.6	3	2.3
y	10	16.9	20	24.2	15.6	20	14.5	30	25.8

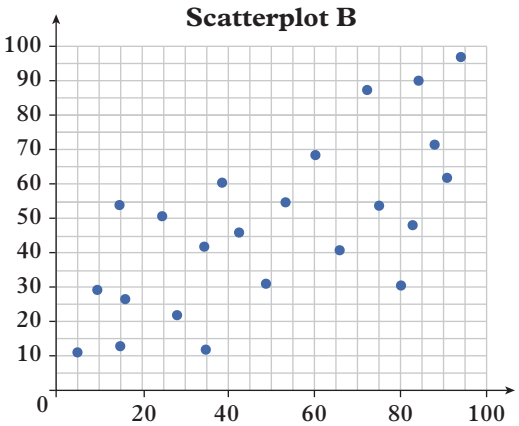
b Determine the residual values of the following data values, correct to two significant figures.

i $x = 1.2$ **ii** $x = 1.6$ **iii** $x = 2$ **iv** $x = 2.3$

c Write each residual from part **b** as a percentage of its respective actual data value, correct to one decimal place.

9 Consider these two scatterplots.





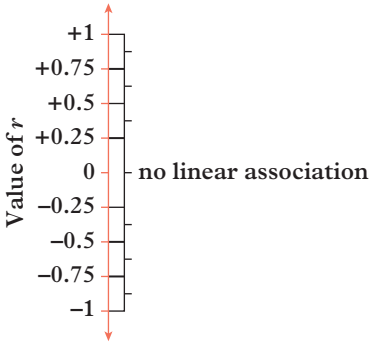
a Describe the difference between the two scatterplots.

b Draw a line of best fit by eye for each scatterplot and use it to predict the value of y when $x = 8$.

c In which prediction from part **b** do you have more confidence? Explain.

d The strength of a bivariate relationship (its correlation) can be described using the Pearson's product–moment correlation coefficient (represented by the pronumeral r). For scatterplot A, $r = +0.9$, and for scatterplot B, $r = +0.6$.

Copy and complete this figure on the right using the words strong, moderate or weak and either 'positive linear association' or 'negative linear association'.



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10 The equation of the least-squares regression line for the following set of data is $y = \frac{27}{11} + \frac{7}{11}x$.

x	1	2	3	2	5	3	6	6	4	8
y	2	6	3	4	4	5	5	6	6	9

a Determine the mean of the x-coordinates, $\bar{x}$, and the mean of the y-coordinates, $\bar{y}$.

b Hence, show that the point $(\bar{x}, \bar{y})$ lies on the least-squares regression line.

c For the equation of the least-squares regression line, $y = a + bx$, show that the y-intercept, a, can be given by $a = \bar{y} - b\bar{x}$.

d Determine the exact value of the standard deviation of the x-coordinates, s_x, and the mean of the y-coordinates, s_y. Assume the data is a sample.

e The slope of the least-squares regression line, b, is given by $b = r \times \frac{s_y}{s_x}$, where r is the correlation coefficient.

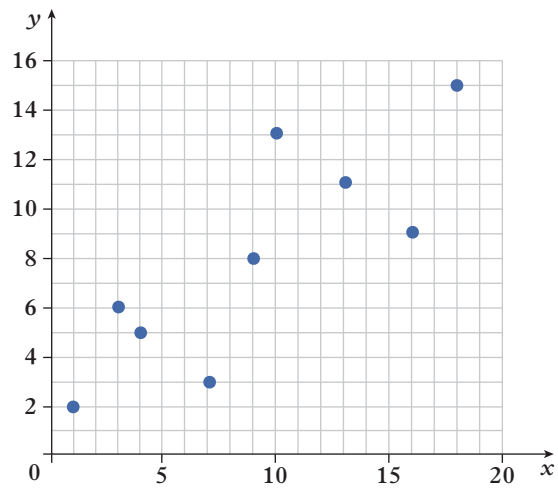
i Determine the value of the correlation coefficient, correct to four decimal places.

ii Hence, state the strength of the relationship between the two variables.

Exercise 10.7C: Challenge

11 Another common method for finding a line of best fit is called median–median regression or three-median regression.

a Copy this scatterplot and divide it vertically into three sections so that each section has the same number of points.



b For the median–median regression method, you need to find the median of each section. This is obtained by finding the median x-coordinate and the median y-coordinate and marking those points on the graph. Show that the points (3, 4), (9, 8) and (16, 11) are the medians of each section.

c Draw in the line of best fit by placing your ruler on the lower median and the upper median, then moving it $\frac{1}{3}$ of the way towards the middle median before you draw the line.

d Find the equation of the line of best fit you drew in part **c**.

e Use your calculator to find the equation of the line of best fit using linear regression. How does this equation compare to what you calculated in part **d**?

f Explain why the three-median regression line is best used when a scatterplot has one or more outliers.

Online resources:

- **Interactive skillsheet** Lines of best fit
- **Worksheet** Scatterplots and lines of best fit
- **Investigation** Our expanding universe
- **CAS instructions** Scatterplots and lines of best fit
- **Quick quiz** 10.7

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Lesson 10.8

Time series

Learning intentions

By the end of this lesson you will be able to ...

✔ understand and use time series.

Inter-year links

Year 7 5.5 Line graphs

Year 9 7.2 Classifying and displaying data

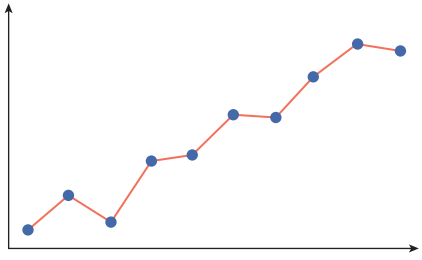
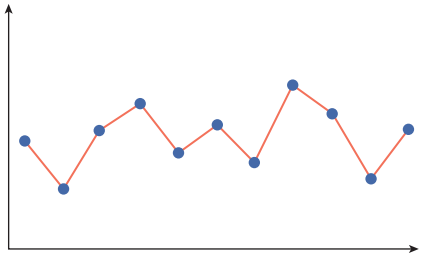
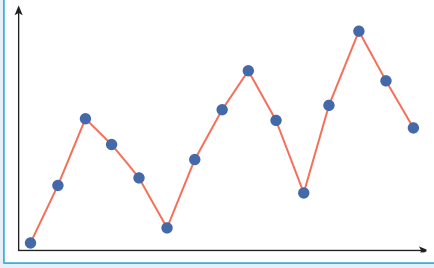
Time series

Key content video
Time series

- A **time series** is a sequence of numerical data taken at regular intervals.
- Time series data is bivariate data for which the independent variable is time.
- Time series data is usually shown as a **line graph**.

Identifying features in time series

- Time series plots can be described in terms of their trend (long term direction: upward, downward or stationary) as well as whether they have seasonality (repeated patterns).
  - → When describing the trend of a time series, you are looking for a general direction, taking into account any seasonality.
  - → Seasonality can be identified by spotting a regular repeating pattern in the points, for example, every 4th point being higher than average.
- Time series plots will almost always have some natural variation (some up-and-down bouncing) and may also have irregular fluctuations.
- Some time series follow a non-linear trend.

Sample time series graph	Trend	Seasonality
	upwards	no
	stationary	no
	upwards	yes

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Worked example 10.8A Plotting time series data

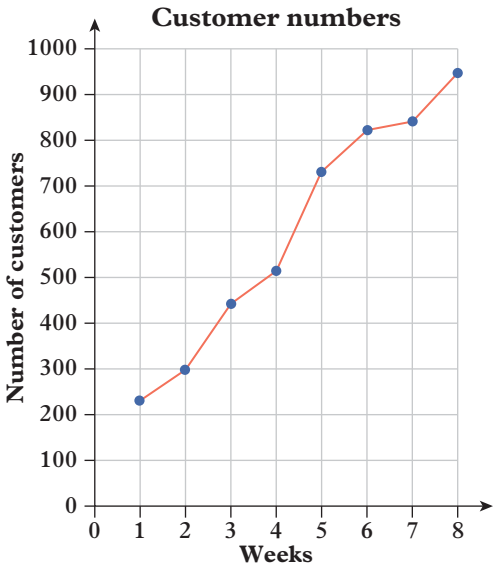
Plot the time series data from this table, showing the number of customers in a store over an 8-week period.

Week	1	2	3	4	5	6	7	8
Number of customers	229	298	442	512	731	822	841	945

THINK

Plot the data on an appropriate set of axes, joining successive points with a straight line. Ensure your graph has a title and an even scale on the axes.

WRITE

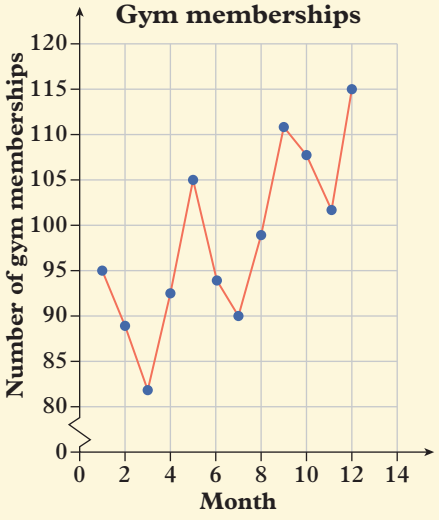


Worked example 10.8B Identifying the features of a time series

For the time series graph on the right:

a State whether the time series has seasonality.

b Describe the trend.



THINK

a Look at the pattern of the data. Is there seasonality (showing regular peaks and troughs)?

b Look at the graph. Is the overall trend upward, downward, or staying roughly the same?

WRITE

a The time series has seasonality.

b There is an upward trend.

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Worked example 10.8C Describing time series data

Describe the time series data from **Worked example 10.8A** in the context of what it shows about customer numbers during the 8 weeks.

THINK	WRITE
1 Look at the graph. Determine the trend shown in the data and whether there is seasonality.	
2 Interpret the trend the graph shows in the context of the data it represents.	As time increases, the number of customers is steadily increasing with no seasonality.

Helpful hints

- ✔ The time intervals for a time series can be in any unit (for example, hours, days, weeks or years), but the dependent variable must be measured after the same amount of time has passed for every interval.
- ✔ The natural variation in time series data should be considered when identifying features of a time series plot.

Learning pathways

1–7, 9	1(d–f), 2, 4, 5, 8–10, 12	1(d–f), 4, 8–13

Exercise 10.8A: Understanding and fluency

WE 10.8A **1** Plot each of these time series data sets.

a

Hours after midday	1	2	3	4	5	6	7	8	9	10
Temperature (°C)	34	36	37	35	34	31	27	22	19	17

b

Years of experience	1	2	3	4	5	6	7	8	9	10	11	12
Salary ($1000s)	42	44	46	49	51	54	56	59	62	65	68	72

c

Days after rain	0	5	10	15	20	25	30	35
Water storage level (L)	456	421	362	321	298	254	221	182

d

Weeks of experience	0	1	2	3	4	5	6	7
Time to complete task (min)	98	71	62	55	50	46	43	42

e

Months	1	2	3	4	5	6	7	8	9	10	11	12
Sales ($100s)	26	32	29	31	35	32	31	34	37	39	36	40

f

Weeks	1	2	3	4	5	6	7	8	9
Hours worked	36	42	38	35	38	41	48	31	38

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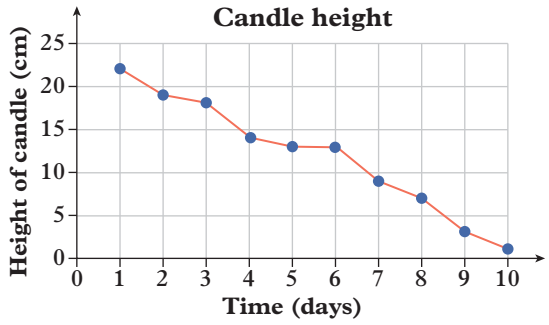
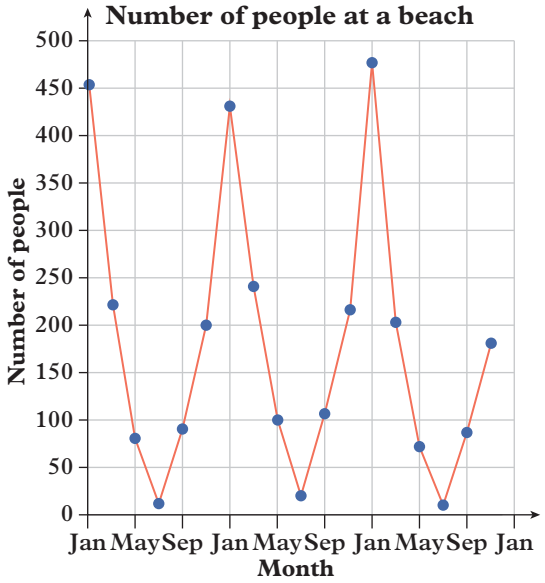
WE 10.8B **2** For each of the following time series graphs:

i State whether the time series has seasonality **ii** Describe the trend.

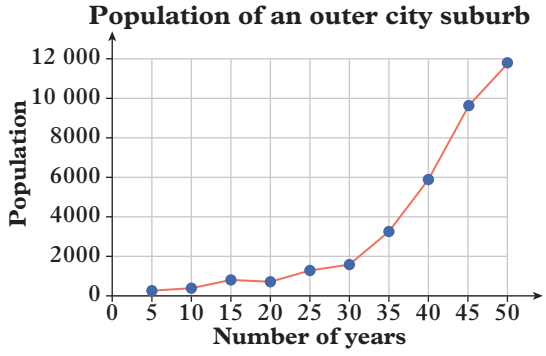
a



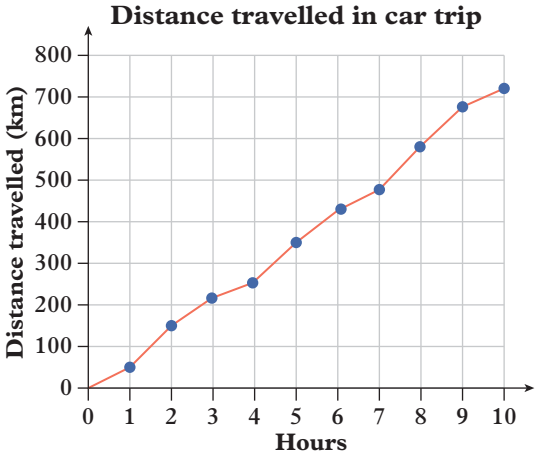
b



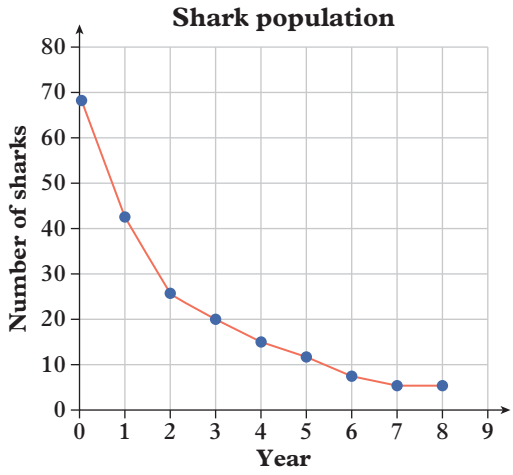
c



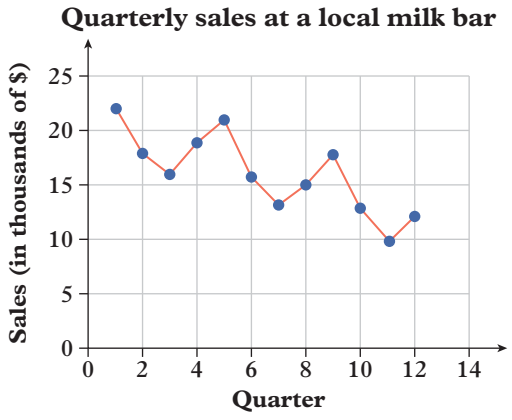
d



e



f



3 Describe the trends shown in the time series plots from question **1**.

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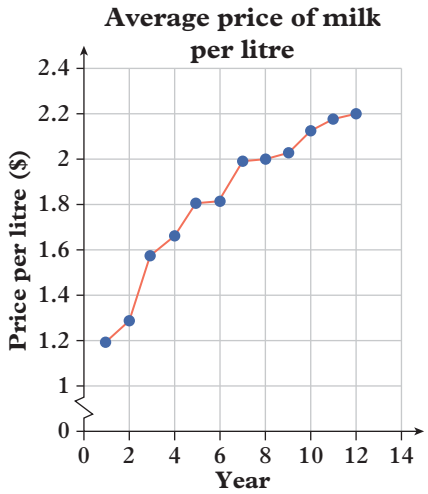
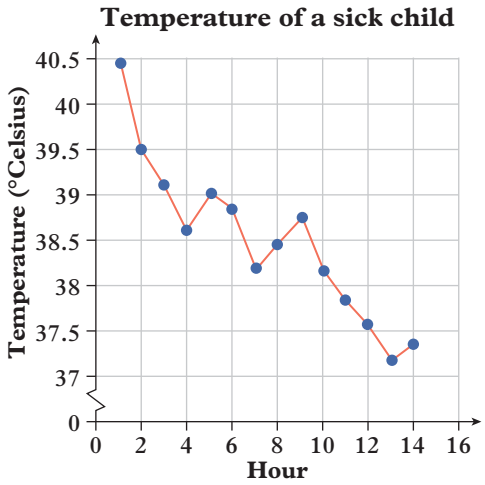
WE 10.8C **4** Describe the time series data from each of these graphs in context.

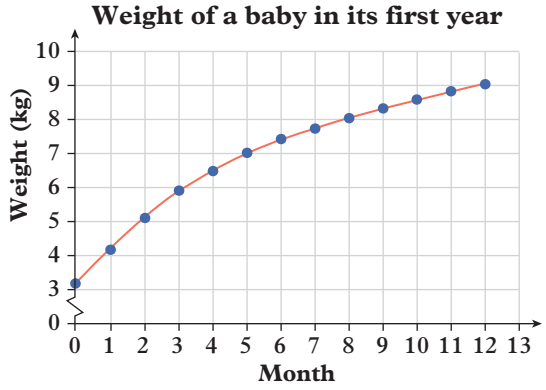
a



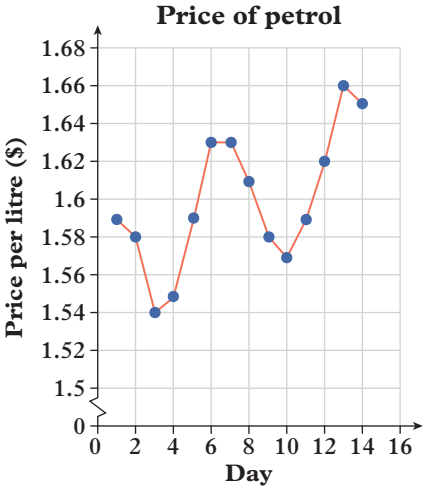
b



c



d



5 Describe each of the time series plots from question **2** in context.

6 Describe each of the time series plots from question **1** in context.

Exercise 10.8B: Problem solving and reasoning

7 Consider this data set, showing the balance (to the nearest dollar) of a bank account over a 10-week period.

Week	1	2	3	4	5	6	7	8	9	10
Balance ($)	787	542	312	116	816	678	457	298	998	749

a Plot the time series data and describe the features shown.

b Over the 10-week time period, what was:

i the minimum balance **ii** the maximum balance?

c Suggest a reason for the seasonality shown in your graph.

8 Consider the data set shown in this table. It records the growth of a tree over a number of years.

Time (years)	1	2	3	4	5	6	7	8
Tree height (m)	0.4	1.2	2.3	3.5	4.5	5.6	6.8	7.9

a Plot the time series data from the table.

b Describe the features shown in your graph.

c How tall would you predict the tree to be when it is:

i 10 years old **ii** 20 years old?

d Explain why you would be more confident about your first prediction than your second prediction.

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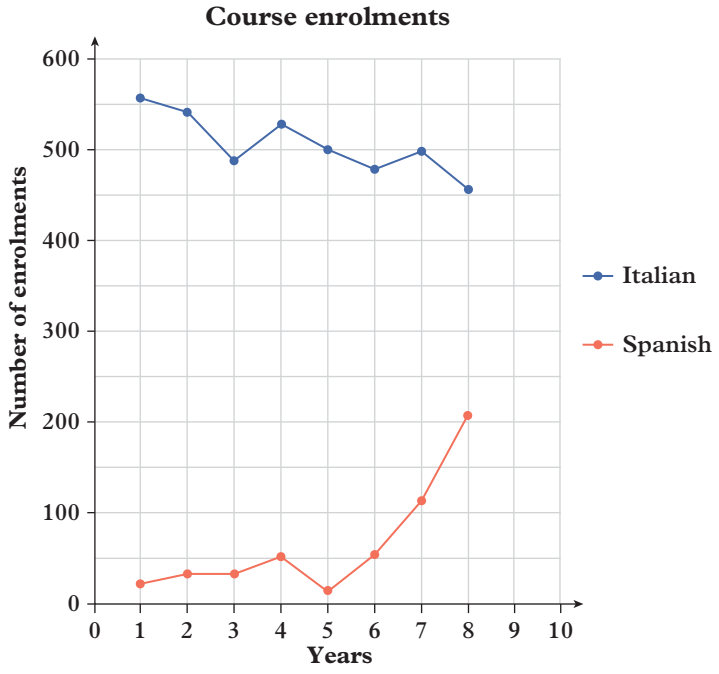
9 Consider the graph below.



a What is the graph showing?

b Describe the trend shown by each of the graph lines.

c At what point would you predict that the Spanish course would have more enrolments than the Italian course?

10 Consider the data set in this table, showing the number of hours students spent studying for a Maths exam and the marks they obtained for the exam.

Time spent studying (hours)	0	1	2	3	4	5	6	7	8	2	6
Maths exam mark (%)	35	42	56	83	71	76	88	92	70	67	72

a Explain why a scatterplot must be used to plot this data.

b Explain why this is not a time series plot, even though time is the independent variable.

c Plot the data and describe the trend shown.

d Explain why, even if the last two data points were not included in the set, a scatterplot must still be used to plot this data.

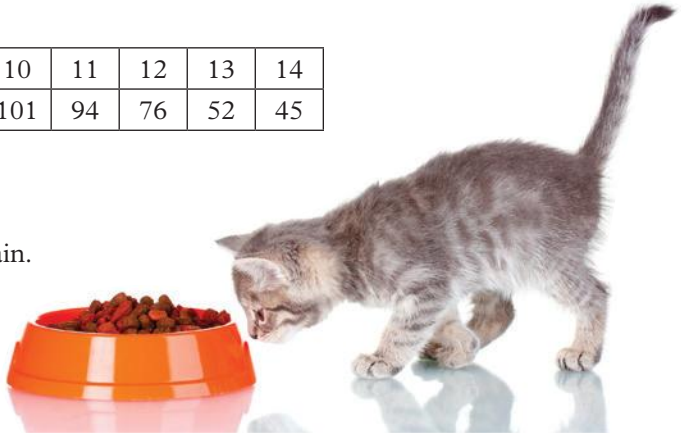
11 Consider the data set in this table, showing the number of pieces of dry cat food in a bowl after a number of hours.

Time (hours)	5	6	7	8	9	10	11	12	13	14
Cat food pieces	189	167	167	132	128	101	94	76	52	45

a Plot the time series data from the table.

b Describe the trend shown in your graph.

c Would you expect this trend to continue? Explain.



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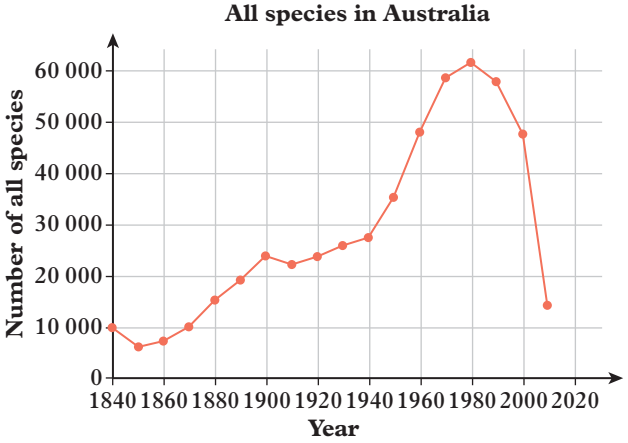
Exercise 10.8C: Challenge

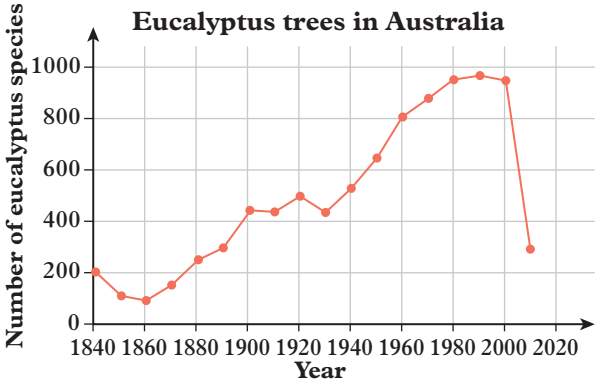
12 Below are two time series. The first shows the change in the number of all species in Australia, and the second shows the change in the number of eucalyptus species in Australia.

a Describe the changing trends of the number of all species over time.

b Is the number of species of eucalyptus associated with the number of all species? Explain.

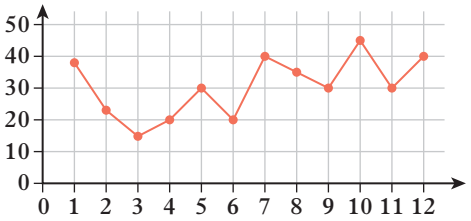


(Source: BOM)



(Source: BOM)

13 Consider the time series with seasonality shown:



To better understand the data, we can compare each seasonal value to the mean value of the seasons by finding the percentage each value is above or below the mean of all the values. For example, if the seasonal value is 250 and the mean value is 200, then the season is $\frac{250}{200} = 1.25$ times greater than the mean and so is 25% greater than the mean. This multiplier, 1.25, is called the seasonal index.

a Determine the mean value of the seasons.

b Hence, correct to two decimal places, determine the seasonal index for each season.

Season	1	2	3	4	5	6	7	8	9	10	11	12
Value	38	23	15	20	30	20	40	35	30	45	30	40
Seasonal index												

c State the percentage and direction, above or below the mean, of the following seasons compared to the mean.

i 2 **ii** 7 **iii** 10 **iv** 11

d To correct for the variation caused by the seasonality, we can increase or decrease the seasonal value by a per cent to make the value average. For example, for a value that is 25% above the mean, with a seasonal index of 1.25, multiplying by the reciprocal, $\frac{1}{1.25} = 0.8$, makes the seasonal value average. Hence, decreasing by 20% will correct for the seasonality.

Determine, correct to two decimal places, the percentage and the direction (increase or decrease) the following seasons must be increased or decreased by to correct for seasonality.

i 2 **ii** 7 **iii** 10 **iv** 11

Online resources:

Interactive skillsheet — Time series

CAS instructions — Line graphs

Quick quiz — 10.8

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Lesson 10.9

Analysing statistical reports

Learning intentions

By the end of this lesson you will be able to ...

✔ analyse statistical reports.

Inter-year links

Year 8	10.1 Collecting data and sampling methods
Year 9	7.1 Surveys and sampling methods

Key content video
Analysing statistical reports

Bias in statistical reports

- When analysing a statistical report, you need to consider the source and size of the sample and identify potential sources of **bias**.
- If a sample is not of sufficient size (at least 50 data points), then it is unlikely to represent the population. Larger sample sizes are more likely to accurately reflect the population.
- Samples should also be collected using a **random sampling method** to avoid bias. When using a random sampling method, all potential data points in the population will have an equal chance of being selected in the sample.
  - → Random sampling methods include: simple random sampling, stratified sampling and systematic sampling.
  - → Non-random sampling methods include: convenience sampling and quota sampling.

Misleading claims and graphs

- Be prepared to question the claims made in reports.
- Consider why the data has been presented as it has, and what may have been omitted.
- Look at the range of data being presented and consider whether any data has been purposefully omitted. For example, if you are looking at temperature data over a year, all months should be considered, as temperature varies due to the seasons in a year.
- Look at the scales used on the axes of graphs to be certain they are evenly spaced.
  - → Look at the value the vertical axis starts at. If it does not start at 0, vertical differences will be enlarged and the differences in heights will be exaggerated.
- If graphs or axes are improperly labelled, incorrect assumptions can be made about the data being presented.

Ethical considerations

- **Ethics** is the system of moral principles that distinguish between right and wrong behaviour.
- Ethics need to be considered throughout the statistical investigation process, from data collection through to data interpretation and reporting.
- The following ethical considerations should be made when analysing statistical reports:
  - → Was the data collected in a fair manner with the aim to find the truth as opposed to validating pre-existing beliefs?
  - → Has any of the data been manipulated or hidden?
  - → Has the data been presented in a balanced way, showing both positive and negative results?
  - → Could reporting the data cause harm to any individuals or groups of people?

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Worked example 10.9A Reviewing a published report

A new toll tunnel has just been opened to allow cars to travel from one side of the river to the other. The alternative route is via a no-toll bridge. Shortly after the tunnel opened, a newspaper reported:

'The tunnel is now so popular that the proportion of cars using the tunnel rather than the bridge is 85%'.

Public perception was that this is not the case. Some residents decided to do a car count over a long time and found 967 cars from a total of 2472 went over the bridge. Comment on the newspaper report.

THINK

1 From the collected data, calculate the probability of a car using the tunnel.

2 Compare the collected data with the reported data, and comment on the comparison.

WRITE

Using the collected data:

$$\Pr(\text{car uses bridge}) = \frac{967}{2472} \times 100\%$$
$$\approx 39\%$$

So: $\Pr(\text{car uses tunnel}) \approx 61\%$

The collected data shows that 61% of cars use the tunnel, while the newspaper reports this figure to be 85%. It seems likely that the newspaper report is exaggerated as this gap is unlikely to occur from natural variation.

Worked example 10.9B Identifying misleading features in graphs

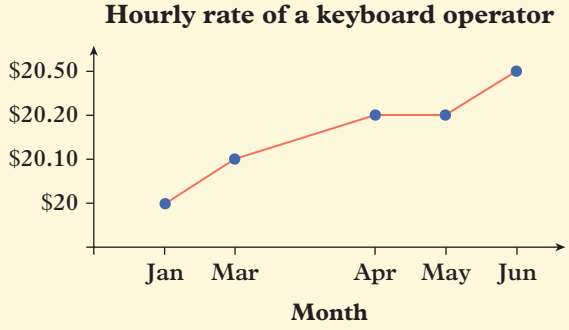
This graph displays the hourly pay rate for a keyboard operator employed by a large company. The time period covered is the first 6 months of the year.

a This graph is misleading. Give reasons why.

b The company manager claims that the hourly rate of the keyboard operator has increased dramatically over the last 6 months. The graph appears to indicate this.

i Is this true?

ii How does the graph seem to support the manager's interpretation of the data?



THINK

a Look for anything that might make the graph misleading.

b i Look at the values on the y-axis.

ii Consider the scale on the y-axis.

WRITE

a This graph is misleading for the following reasons:

- The scale on the x-axis is not uniform.
- The month of February is missing.
- The scale on the y-axis does not start at zero, is not evenly spaced, and no break is indicated.

b i The keyboard operator's hourly pay rate has only increased from \$20 to \$20.50 over the 6-month period, an increase of 50c per hour. This is not a 'dramatic' increase.

ii Because the scale on the y-axis does not start at zero, the range of \$20 to \$20.50 has been stretched out to appear larger than it would on a complete scale. This gives the impression that the increase is greater than it really is.

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Helpful hints

- ✔ When reviewing graphs, look at where the numbering on the axes starts. Heights of columns in a column graph may not be proportional to their frequencies unless the scale is accurate.
- ✔ Reports should indicate the source of statistics so that readers can investigate the data themselves.
- ✔ Be sceptical of claims made without evidence, such as, 'it has been reported that ...' or, 'according to some surveys ...'.

Learning pathways

1, 2, 4, 6, 7, 10, 11 | 1, 3, 5, 6, 8, 9, 11, 12, 15 | 2, 3, 5, 6, 8, 11–15

ANS p749

Exercise 10.9A: Understanding and fluency

WE 10.9A **1** Critically analyse each of these reports.

a A survey conducted at a gym revealed that 90% of the gym's clients felt they were getting enough exercise each week. It was later quoted that: 'Australians are getting fitter, as 90% of them say they exercise regularly each week'.

b Four teenage boys were still left on a bus when it reached the end of the line. A reporter asked them if they had part-time jobs. Two of them said they did. It was later reported that, '50% of teenagers today have part-time jobs'.

c A survey showed that four out of five children believe they do their fair share of chores at home. A report later appeared in a newspaper with the heading, 'Families lucky: 80% of children do fair share of chores'.

2 Critically analyse each of the media reports below. Suggest what research could have been done to make each claim valid.

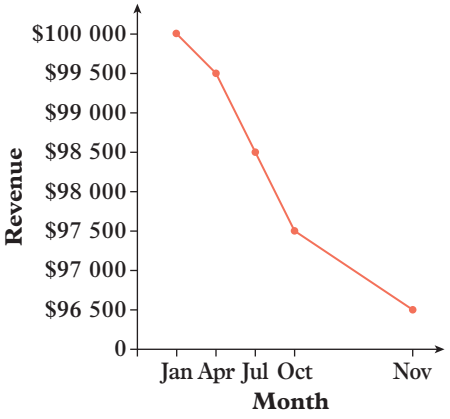
a In a study of women, having coffee with lunch seemed to reduce the risk of diabetes more than having coffee at other times of day.

b Studies suggest that physically active people have a reduced risk of Alzheimer's disease in later life.

c Studies suggest that happiness wards off heart disease.

3 A television channel invites its viewers to vote on a topic that has just been discussed on air. Later in the program, the results of the votes are revealed, such as, '98% of viewers agree that …' Discuss whether the reported figures in this type of survey are valid.

Exercise 10.9B: Problem solving and reasoning



WE 10.9B **4** The graph on the right displays the revenue a company has made during a year.

a This graph is misleading. Give reasons why.

b The chief finance officer claims that the company's revenue has declined significantly this year, but declined at a slower rate at the end of the year. The graph appears to indicate this.

i Is this true?

ii How was this effect achieved?

iii By what percentage has the revenue declined from January to November?

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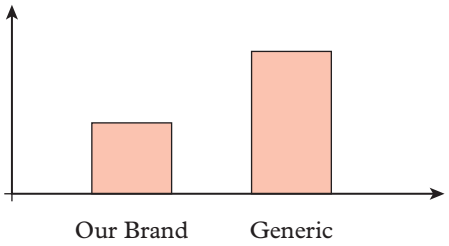
5 The following graph displays the average time a brand of paracetamol takes before relief, compared to a generic competitor.



a This graph is misleading. Give reasons why.

b The company claims their brand of paracetamol is twice as effective as their generic competitors. The graph appears to indicate this.

i Is this true?

ii How was this effect achieved?

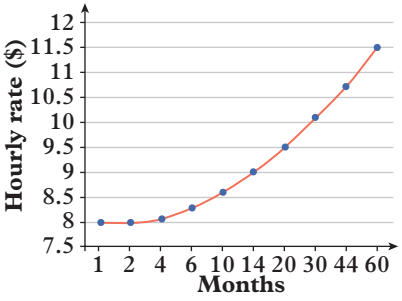
6 Fatima's employer constructed the graph on the right.

a What is the graph supposed to represent?

b Explain how the graph misrepresents the data.

c Redraw the graph as best you can so that it is fair.

Hourly rate paid vs time employed



7 Some advertisements claim that eating a good breakfast leads to success at school. Consider an alternative argument:

'Students who do not eat breakfast are more likely to be absent from school or experience lethargy.'

Does this statement mean the same thing as eating a good breakfast leading to success at school?

Why do the advertisements use one statement rather than the other?

8 We are often unaware of companies sponsoring studies.

Consider this headline: 'A hotdog a day increases risk of cancer by 21%.'

Whether this statement is true or not is open to debate. However, if the study was later revealed to have been conducted by an animal-rights group whose purpose is to convince people to be vegan, would this affect the credibility of the study? Suggest how this type of reporting of information from a biased source could be monitored.

9 A car manufacturer gets their new model tested by an independent company prior to release. The testing reveals that while the car is comfortable and good value for money, its fuel efficiency is significantly less than competitor models. How do you think the car company should use these test results when marketing their new model? Give reasons with your answer.

10 Jake wrote a report on people's attitudes towards a proposed skyscraper in a bayside suburb. His report claimed that, '71% of locals are against the building of the new skyscraper.'

Jake's report was based on a survey that he asked shoppers at the local mall to complete, with 158 of 222 respondents stating that they were either 'very against' or 'against' the building of a skyscraper.

a Does the data Jake collected appear to back up his claim?

b Explain why the data Jake collected may not be a sample from the population.

c Explain why the data Jake collected is not a random sample.

d Considering your answers to parts **a** to **c**, does the claim in Jake's report have any validity? Justify your answer.

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11 Marissa was researching attitudes of Year 10 students at her school towards the new gym facilities. For her research, Marissa inputted the names of all 185 Year 10 students into a spreadsheet and used a random-number generator to select 35 students at random to interview.

a If 5 of the students that Marissa randomly selected refused to be interviewed, is her sample still random? Justify your answer.

b If instead, 3 of the students that Marissa randomly selected were ill on the day that the interviews were conducted, is her sample still random? Justify your answer.

c Are there any other issues with the sample that Marissa selected that may bias the research results?

12 A report on vaccination rates claimed that, 'In all states and territories, vaccination rates are at their lowest in the 5–14 age group and are at their highest in the 65+ age group.' The report used the following information taken from the National Centre for Immunisation Research and Surveillance (NCIRS).

Influenza vaccination % coverage

	ACT	NSW	VIC	QLD	SA	WA	TAS	NT	AUS
0–4 years	50.0	26.8	32.0	23.9	28.0	27.3	32.9	32.2	**28.3**
5–14 years	24.4	14.9	17.1	14.7	15.6	18.6	16.2	14.1	**16.1**
15–49 years	32.7	20.6	24.8	20.4	25.0	21.7	24.7	21.9	**22.4**
50–64 years	46.1	34.1	38.5	36.1	40.7	37.4	44.5	27.2	**36.8**
65+ years	67.3	60.3	64.0	63.5	68.1	64.3	70.1	36.3	**63.1**

(Source: NCIRS)

a Are the vaccination rates at their lowest in the 5–14 age group and at their highest in the 65+ age group?

b Explain why the data in the table does not back up the claim made in the report.

13 The Oxford Press released a report with the headline, 'The poor were the real Australian victims of COVID-19'. The report used the following data from the Australian Bureau of Statistics (ABS) on the proportion of deaths from COVID-19 in Australia by socio-economic grouping.

Socio-economic group

	1 (lowest)	2	3	4	5 (highest)
Wave 2	34.5	23.3	17.7	15.3	9.2
Delta Wave	40.4	22.0	16.0	13.6	7.9

(Source: ABS)

a Explain why this is population data, not sample data.

b Is any data excluded from the report? Explain your answer.

c Is this data sufficient to back up the claim in the headline? Consider both the data and the claim in your answer.

Exercise 10.9C: Challenge

14 Confirmation bias is the tendency to favour information that confirms already existing beliefs.

Ellie is running to be the class president at her school. She conducts a survey of students in her school, asking for their opinions on a range of topics, and reports the results as favourable to her campaign.

a Discuss where confirmation bias may have occurred in either the collection or reporting of Ellie's data.

b Give another example of where confirmation bias may affect a statistical report. Explain how the confirmation bias may affect either the data being collected or how the results are reported.

15 Choose a topic and conduct a survey among the members of your class. Report on the results of your survey, justifying any claims with mathematical evidence.

Online resources:

Investigation Conducting your own statistical report

Investigation Media reports (10A)

Quick quiz 10.9

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Statistical Investigations

The statistical investigation process

Statistics can be used to answer a wide range of real-life questions. By collecting and analysing data, we can draw evidence-based conclusions, allowing us to make informed decisions.

A statistical investigation involves the following steps:

1. Identifying a question to investigate
2. Collecting data
3. Analysing the data
4. Reporting on the findings.

Data used in a statistical investigation can either be collected directly by the researcher (primary data) or taken from a reputable source (secondary data).

Curriculum link

Plan and conduct statistical investigations of situations that involve bivariate data, including where the independent variable is time; evaluate and report findings with consideration of limitations of any inferences. (VC2M10ST05)

Measuring education levels and life expectancy

The Education Index (EI) is a measure of the level of education that children and young adults receive in a country. Higher scores indicate that the population spends more years in school and university.

Life Expectancy at Birth (LEB) measures the number of years that a child born in a country is expected to live. Higher LEB scores indicate that people are expected to live longer in that country.

Investigation

Is there an association between education and life expectancy?

Find the Education Index (EI) and Life Expectancy at Birth (LEB) for a sample of ten different countries. Remember that random samples help avoid bias and give the best chance that the sample is representative of the population. You can take a random sample by finding and using a random country generator. For each country in your sample, search the internet to find the EI and LEB from a reliable source.

It is fine if some of your EI and LEB scores are a few years old or are from a different year to the scores you have found for other countries. The EI and LEB scores for an individual country do not change much over the course of one or two years, so this should have a negligible effect on the association between education and life expectancy.

Copy the table shown into your workbook and use it to record your sample data.

Enter your bivariate data into a CAS calculator or Microsoft Excel (or a similar spreadsheet program) and create a scatterplot of the data, using EI scores for the horizontal axis and LEB for the vertical axis.

Country	EI	LEB

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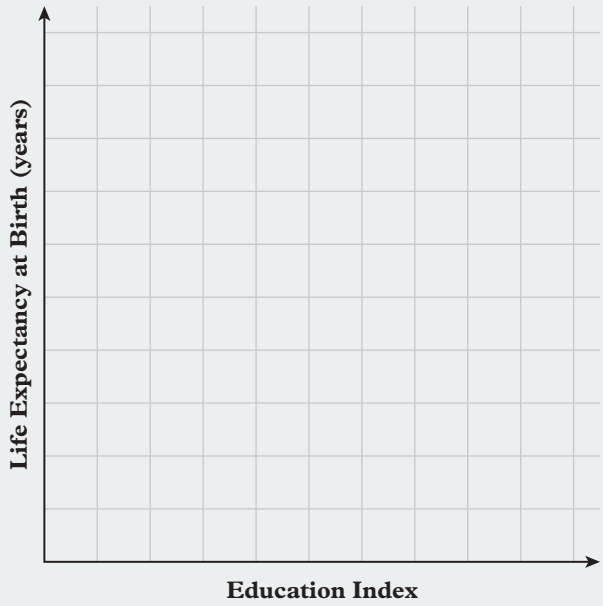
Investigation questions

1 How did you choose the ten countries in your sample? Is your sample random?
2 What sources did you use to find the Education Index (EI) and Life Expectancy at Birth (LEB) scores? Are these reliable sources? How do you know?
3 What is the highest EI in your sample? What is the lowest EI in your sample?
4 What is the highest LEB in your sample? What is the lowest LEB in your sample?
5 Discuss the limitations that come with using a small sample size.

Analysis

1 Draw the scatterplot of your data, using an appropriate scale. Education Index must be on the horizontal axis and Life Expectancy at Birth on the vertical axis. Draw a line of best fit by eye.



2 Based on your scatterplot, describe the strength and direction of the relationship between EI and LEB. Does the relationship appear to be linear?
3 Does a relationship between education level and life expectancy mean that one value causes the other? Explain why it is inappropriate to assume that a relationship between two variables implies a causal effect between the variables.
4 Use your results to write a short report on the relationship between education and life expectancy, acknowledging any uncertainties in your findings and limitations of the investigation.
5 Compare your findings with a classmate. Their scatterplot and line of good fit will be different to yours due to your classmate having a different sample of countries. Did your classmate reach the same conclusion about the relationship between education and life expectancy?

Investigate further

Repeat the investigation with a census, rather than a sample. That is, repeat the investigation by finding the EI and LEB for all countries, not just ten randomly selected countries. This will involve finding data on both measures from reliable sources and copying and pasting large tables into Microsoft Excel (or a similar spreadsheet program). Make sure the data is sorted correctly so that the measures for each country are matched. This will ensure that each individual dot on the scatterplot represents the EI and LEB for one country. Did the results of your random sample investigation match the results of your census investigation?

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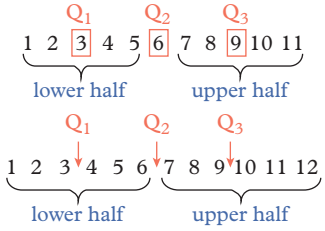
Lesson 10.10

Review: Statistics

Module summary

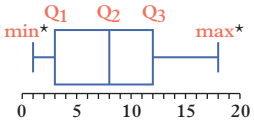
Five number summary

min, Q_1, Q_2 (median), Q_3, max



Boxplots

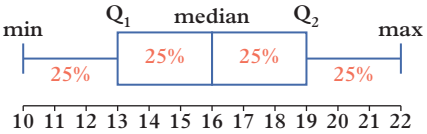
- Lower fence = $Q_1 - 1.5 \times \text{IQR}$
- Upper fence = $Q_3 + 1.5 \times \text{IQR}$
- Outliers: values that lie beyond the fences



*excluding outliers

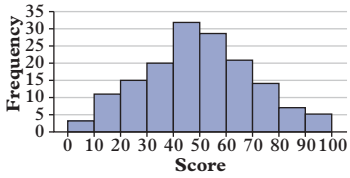
Cumulative frequency

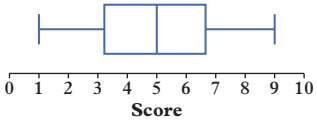
The sum of all frequencies up to and including the given value



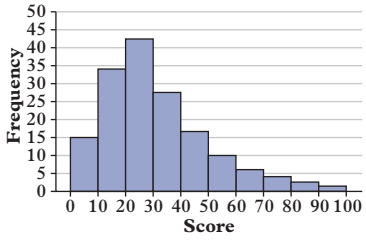
Distributions

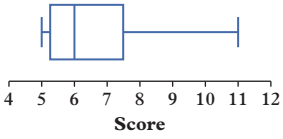
- Symmetric



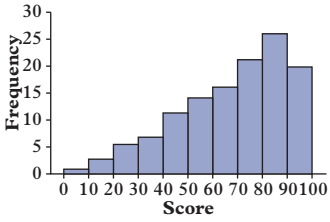


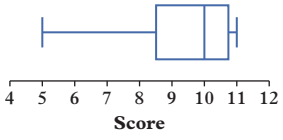
- Positively skewed





- Negatively skewed





Standard deviation

Population standard deviation:

$$\sigma = \sqrt{\frac{(x_1 - \mu)^2 + (x_2 - \mu)^2 + \ldots + (x_n - \mu)^2}{n}}$$

Sample standard deviation:

$$s = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2}{n - 1}}$$

Likert scales

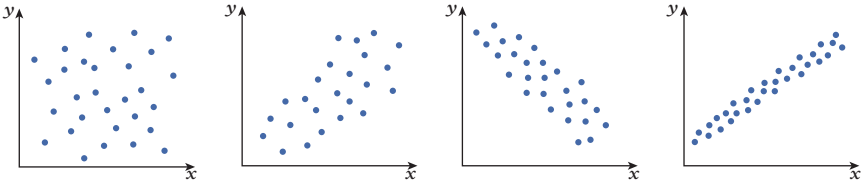
Strongly Disagree | Disagree | Neutral | Agree | Strongly Agree

Two-way tables

- Horizontal axis: independent variable
- Vertical axis: dependent variable

	Very unhappy	Unhappy	Neutral	Happy	Very happy	Total
Children	8%	14%	24%	30%	24%	100%
Adults	18%	16%	26%	24%	16%	100%

Scatterplots



- No association
- No correlation

- Weak correlation
- Positive direction

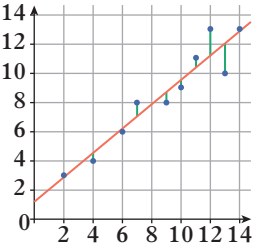
- Moderate correlation
- Negative direction

- Strong correlation
- Positive direction

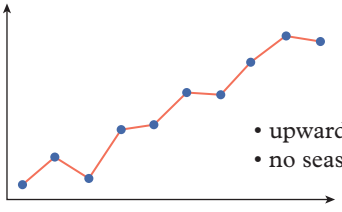
Lines of best fit

$y = mx + c$

$y = a + bx$



Time series



- upwards trend
- no seasonality

Analysing statistical reports

- Consider the size and source of the sample
- Identify potential sources of bias
- Question the claims made in reports
- Check that no data has been purposefully omitted
- Ask whether the data is presented in a fair manner

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Review questions 10.10A: Mathematical literacy review

Review quiz
Take the review quiz to assess your knowledge of this module.
✔ 10
✔ 10A

Quizlet
Test your knowledge of this module by working individually or in teams.

The following key terms are used in this module:

- bias
- bimodal
- bivariate data
- boxplot
- categorical data
- cause-and-effect
- confounding variable
- correlation
- cumulative frequency
- cumulative frequency polygon
- dependent variable
- ethics
- extrapolation
- five-number summary
- histogram
- independent variable
- interpolation
- interquartile range
- least-squares regression line
- Likert scale
- line graph
- line of best fit
- line of good fit
- lower fence
- lower quartile
- mean
- median
- modality
- multimodal
- negatively skewed distribution
- negative relationship
- numerical data
- outlier
- parallel boxplots
- population
- positive relationship
- positively skewed distribution
- quartiles
- random sampling method
- range
- sample
- scatterplot
- skewed distribution
- spread
- standard deviation
- symmetric distribution
- time series
- two-way table
- unimodal
- upper fence
- upper quartile

1 How many variables does bivariate data contain?

2 Explain the difference between a positive and a negative relationship between two numerical variables.

3 Which five values make up a five-number summary?

4 Identify the key terms being referenced in each of these definitions.

a an extreme value this is much higher or lower than the other values in a data set

b a mathematically calculated line of best fit for numerical bivariate data

5 Using an example, provide a definition in your own words for the following key terms.

a time series

b ethics

6 Complete the following sentences using words from the key terms list.

a A _____ has a cluster of values around the right-hand side of the distribution and a tail tapering to the left.

b The _____ lies 1.5 × IQR below the lower quartile, and the _____ lies 1.5 × IQR above the upper quartile.

Review questions 10.10B: Multiple choice

10.1 **1** A data set has 14 values. Which of the following is correct?

A The median and both upper and lower quartiles are values in the set.

B The median is a value in the set, but the upper and lower quartiles are not.

C The upper and lower quartiles are values in the set, but the median is not.

D None of the median, upper or lower quartiles are values in the set.

10.2 **2** In a boxplot, what does the 'box' part represent?

A mean **B** interquartile range **C** range **D** median

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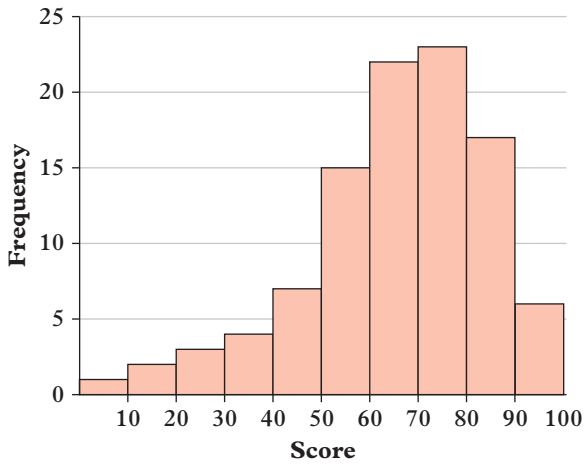
10.3 **3** Which option best describes the histogram on the right?

A positively skewed

B symmetric

C perfectly symmetric

D negatively skewed



10.3 **4** The length from the lower quartile to the upper quartile in a boxplot represents what percentage of the scores of the data set?

A 25% **B** 50% **C** 75% **D** 90%

10A 10.4 **5** What does the standard deviation in a set of data indicate?

A how large the mean is

B how the scores are spread out from the mean

C how many scores there are in the data set

D where the median lies in the set

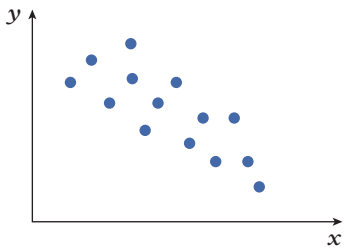
10.5 **6** Consider the following two-way table.

	Strongly disagree	Disagree	Neutral	Agree	Strongly agree	Total
Year 10	13	21	20	14	12	80
Year 12	9	17	28	31	35	120

For which response did approximately 26% of Year 10 students answer to the survey question?

A Strongly disagree **B** Disagree **C** Neutral **D** Agree

Questions **7–8** refer to the scatterplot below, which shows a linear relationship between two variables, x and y.



10.6 **7** How could you describe the linear relationship between the two variables?

A strong **B** weak **C** moderate **D** no correlation

10.6 **8** Which of these terms also describes the linear relationship between the two variables displayed on the graph?

A positive **B** negative **C** horizontal **D** vertical

10A 10.7 **9** The line of best fit for a scatterplot is $y = 1.7x + 1.3$. When x has a value of 1.5, what is the predicted value for y?

A 1.25 **B** 4.5 **C** 2.55 **D** 3.85

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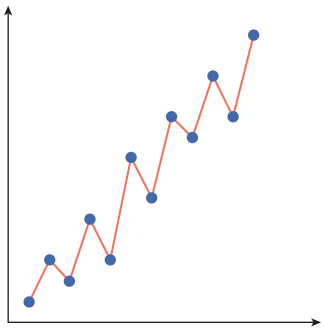
10.8 **10** Which of the following is the best description of this time series?

A increasing

B decreasing

C seasonal

D increasing and seasonal



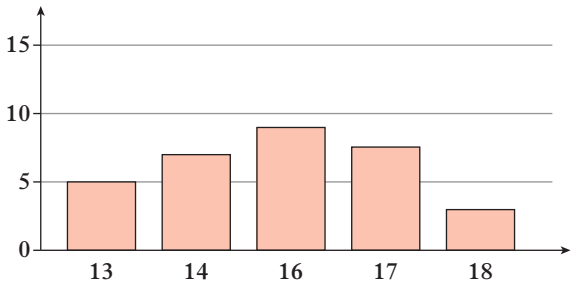
10.9 **11** The graph on the right displays the number of secondary students of different ages who joined the school band. Why is the graph misleading?

A The x-axis doesn't start at 0.

B The y-axis doesn't start at 0.

C Not all values are present on the x-axis.

D The maximum value on the y-axis is too high.



Review questions 10.10C: Short answer

10.1 **1** For the data in the stem-and-leaf plot on the right, showing the number of ice creams sold per day during the month of January, find:

a the range

b the median

c the five-number summary

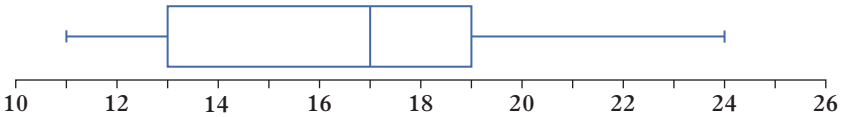
d the interquartile range.

Stem	Leaf
4	0 1 3 5 7 9
5	2 5 5 7 8
6	2 3 3 5 7 7 9 9
7	3 5 6 7 8
8	2 4 6 7 8
9	3 6

Key: 4 | 0 = 40

10.2 **2** Draw a boxplot for the data from question **1** and comment on its shape.

10.2 **3** State the five-number summary and interquartile range of the data presented in the following boxplot.



10.3 **4** Consider the data in the stem-and-left plot from question **1**.

a Create a grouped frequency table using class intervals of 10 with cumulative frequency and cumulative percentage columns. Round percentages correct to the nearest integer.

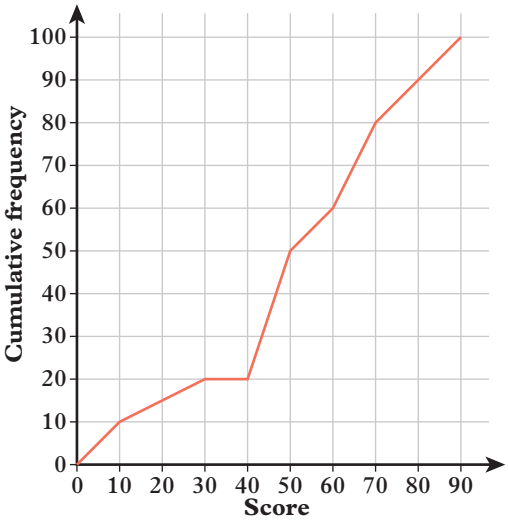
b Construct a cumulative percentage frequency polygon.

10.3 **5** State the percentage of the data that lies in the following ranges of values for the cumulative frequency polygon on the right.

a between 50 and 60

b less than 30

c greater than 70



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10A 10.4 **6** Below are the results Lily and Cassy obtained for five tests.

Lily: 72, 77, 67, 69, 75 **Cassy:** 47, 92, 87, 77, 52

a For each student, calculate:

i the mean result **ii** the standard deviation (to two decimal places).

b Whose results were more consistent?

10.5 **7** The following two-way table shows whether patients for two physiotherapists agree or disagree with the statement, 'I was happy with the level of service I received.'

	Strongly disagree	Disagree	Neutral	Agree	Strongly agree	Total
Physio 1	2	5	6	12	37	62
Physio 2	4	8	15	23	40	90

a Convert the responses into percentages for each category of respondent, rounding values to the nearest whole number where necessary.

b Comment on if there appears to be a relationship between the categorical variables.

10.6 **8** This table shows the number of children enrolled at a daycare centre over the 12 years since its opening.

Years since opening	1	2	3	4	5	6	7	8	9	10	11	12
Number of children	25	27	26	33	32	35	37	39	40	39	40	45

a Create a scatterplot for the data. **b** Describe the relationship between the two variables.

10A 10.7 **9** Consider the table of data from question **8**.

a Use technology to find the equation of the line of best fit for the scatterplot you drew in question **8** part **a**.

b Predict the number of children the daycare centre could expect to have 20 years after opening.

c Explain the problems in making predictions of this kind.

10.8 **10** This table shows a town's mean monthly temperature (°C) in the years 2004 and 2024.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
2004	28.8	29.0	27.4	26.9	23.1	21.5	19.8	20.5	24.5	24.5	27.3	27.6
2024	28.3	29.2	27.8	25.3	22.4	20.3	20.8	21.7	23.6	24.6	27.5	28.4

a Plot the two sets of data as time series on the same graph. **b** Describe the trend of each time series.

c Compare the two graphs to describe the differences between the data.

10.9 **11** A company conducts an annual staff survey for all its staff, asking for feedback in several different areas. A selection of the aggregated results of the survey are shown.

- Direction of the company: 8.0/10
- Pay and benefits: 3.9/10
- Company culture: 6.5/10
- Management performance: 8.5/10
- Workload: 4.4/10

At the all-staff meeting, the HR director highlights the strong results in the survey but doesn't mention the lower-scoring categories. Is this an ethical issue? Give reasons why or why not.

Review questions 10.10D: Mathematical modelling

Usain Bolt of Jamaica became a triple Olympic champion in the 100 m sprint at the Rio 2016 Olympic Games. A record of the winning times for this race has been kept since the early 1900s. The Olympic Games usually take place every four years but were not held during the years of World War I or World War II, so there are some gaps in the record.

Consider this table showing winning times for the 100 m sprint since 1952.

Year	1952	1956	1960	1964	1968	1972	1976	1980	1984
Time (s)	10.79	10.62	10.32	10.06	9.95	10.14	10.06	10.25	9.99

Year	1988	1992	1996	2000	2004	2008	2012	2016
Time (s)	9.92	9.96	9.84	9.87	9.85	9.69	9.63	9.81

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Answer the following questions, using the table in the previous page.

a From the raw data, describe the winning times over the period shown.

b From the data in the table, calculate (to two decimal places) the:

i mean and median **ii** range and interquartile range **iii** five-number summary.

10A **c** Find the standard deviation of the data set.

d Boxplots are useful for analysing the shape of numerical data.

i Draw a boxplot to represent the data from the table.

ii Comment on the shape of the boxplot.

iii Explain how you think the shape of the boxplot will change as future winning times are added to the data.

e Draw a scatterplot of the data.

10A **f** Calculate the equation of the line of good fit for your scatterplot.

10A **g** In the 1960s, speculation centred on the question, 'When will the 10 second barrier be broken for the 100 m sprint?' Today, the talk centres around the question, 'When will the 9.4 second barrier be broken?' Predict when this might occur.

Checklist

Now that you have completed this module, reflect on your ability to do the following.

I can do this	I need to review this
☐ Find the five-number summary for a numerical data set ☐ Calculate the range and interquartile range of a numerical data set	☐ Go back to Lesson 10.1 Five-number summary and interquartile range
☐ Interpret and create boxplots ☐ Use parallel boxplots to compare data sets	☐ Go back to Lesson 10.2 Boxplots
☐ Describe distributions of data ☐ Interpret and create cumulative frequency distributions	☐ Go back to Lesson 10.3 Distributions of data
☐ Calculate the standard deviation of a population and sample ☐ Compare two (or more) data sets using the mean and standard deviation	☐ Go back to Lesson 10.4 The mean and standard deviation [10A]
☐ Calculate percentages for categories in two-way tables ☐ Identify possible relationships between categorical variables	☐ Go back to Lesson 10.5 Bivariate categorical data
☐ Interpret and draw scatterplots ☐ Determine the strength and direction of a linear relationship ☐ Draw a line of good fit by eye	☐ Go back to Lesson 10.6 Scatterplots
☐ Calculate the least-squares regression line using technology ☐ Use a line of best fit to make predictions	☐ Go back to Lesson 10.7 Lines of best fit [10A]
☐ Plot and describe time series data ☐ Identify the trends shown in time series data	☐ Go back to Lesson 10.8 Time series
☐ Identify potential sources of bias in statistical reports ☐ Identify where claims or graphs may be misleading	☐ Go back to Lesson 10.9 Analysing statistical reports

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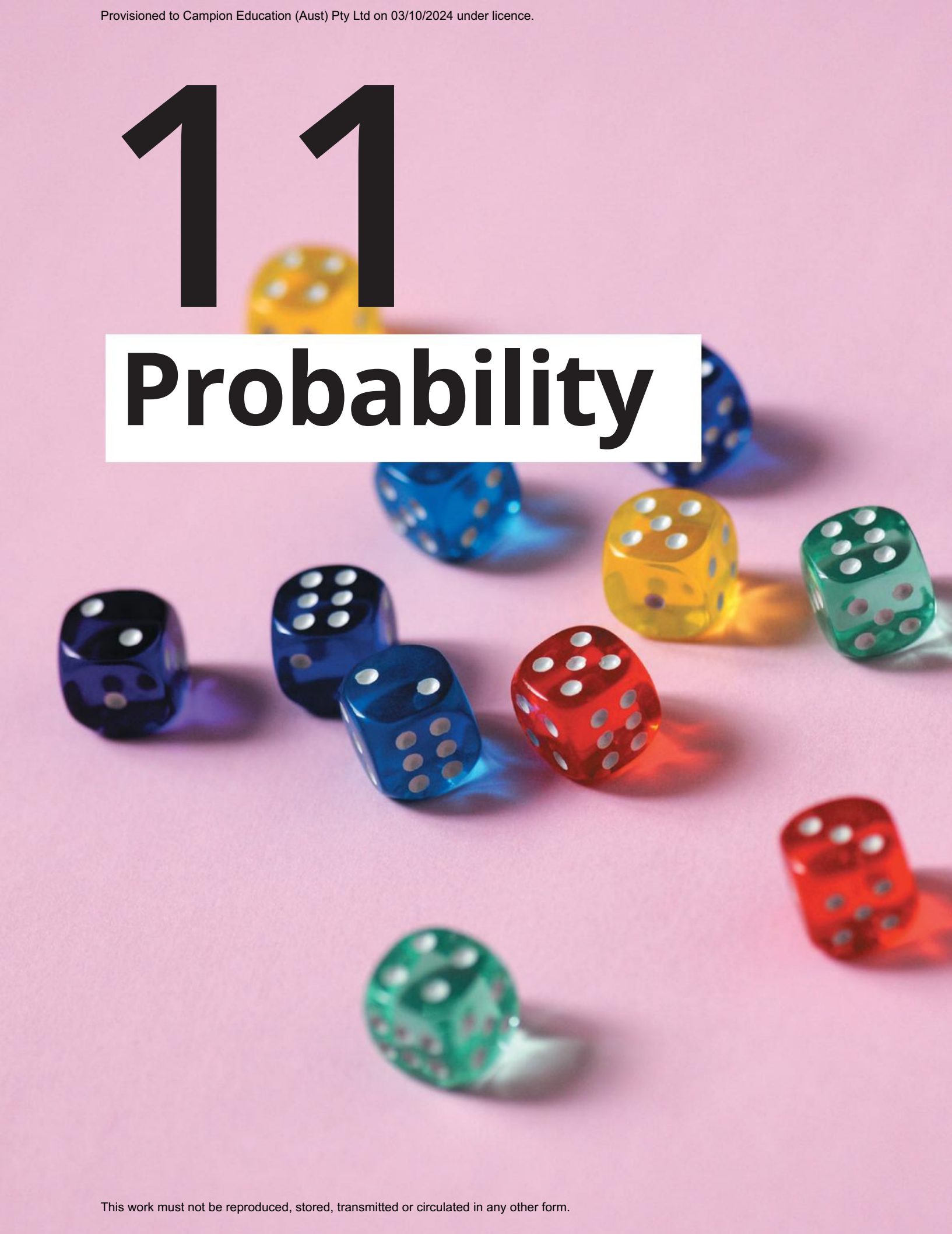
11 Probability



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Lessons

11.1 Theoretical probability
11.2 Experiments with and without replacement
11.3 Two-way tables and Venn diagrams
11.4 Conditional probability
11.5 Independence
10A **11.6** Permutations
10A **11.7** Combinations

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ Calculating theoretical probability
- ✔ Adding and subtracting fractions
- ✔ Multiplying and dividing fractions

Curriculum links

- Use the language of 'if ... then ...', 'given', 'of' and 'knowing that' to investigate conditional statements and identify common mistakes in interpreting such language, and describe and interpret situations involving conditional probability; design and conduct simulations using digital tools to model conditional probability and interpret results (VC2M10P01)
- Describe the results of two- and three-step chance experiments, both with and without replacements, assign probabilities to outcomes and determine probabilities of events; investigate the concept of independence (VC2M10P02)
- 10A Explore counting principles, and factorial notation as a representation that provides efficient counting in multiplicative contexts, including calculations of probabilities (VC2M10AP01)

© VCAA

Materials

- ✔ Calculator

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Lesson 11.1

Theoretical probability

Learning intentions

By the end of this lesson you will be able to ...

✔ calculate the probabilities of outcomes in multi-step experiments.

Inter-year links

Year 7	10.5 Theoretical probability
Year 8	10.4 Theoretical probability
Year 9	8.1 Two-step chance experiments

Theoretical probability

Key content video
Theoretical probability

- The **sample space** of an **experiment** is a list of all the different possible **outcomes** of that experiment.
- The different outcomes of an experiment may not be equally likely to occur.
- An **event** is a set of one or more outcomes from the sample space of an experiment.
- The probabilities of all outcomes within a sample space sum to 1.
- If all the outcomes of an experiment are equally likely to occur, the probability of an event occurring can be calculated using the formula:

$$\Pr(\text{event}) = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$$

- The **complement** of an event, A, is the event that A does not occur.
  - → The complement of A is written as A′, and is sometimes read as 'not A'.
  - → The sum of the values of the probabilities of complementary events is 1.

    $\Pr(A) + \Pr(A') = 1$

    For example, if $\Pr(A) = \frac{1}{4}$, then $\Pr(A') = \frac{3}{4}$.

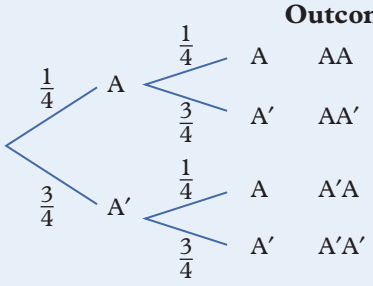
Mutually exclusive events

- If two events are **mutually exclusive** then they cannot both occur at the same time.
  For example, when rolling a standard six-sided die, the events 'rolling an odd number' and 'rolling an even number' are mutually exclusive. Their sample spaces are {1, 3, 5} and {2, 4, 6} respectively. No element exists in both sample spaces.

Tree diagrams

Key content video
Arrays and tree diagrams

- **Tree diagrams** can be used to display the outcomes of multi-step experiments.
  - → The possible outcomes for each step of the experiment are represented by the branches.
  - → The branches might not all have the same probability.
  - → The final outcomes are listed at the ends of the branches.
  - → The probability of the final outcomes can be calculated by multiplying together the individual probabilities along the branches.



Outcomes	Probability
AA	$\frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$
AA′	$\frac{1}{4} \times \frac{3}{4} = \frac{3}{16}$
A′A	$\frac{3}{4} \times \frac{1}{4} = \frac{3}{16}$
A′A′	$\frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$

- If there are two potential outcomes at any step of a multi-step experiment, then the outcomes will be complementary and their probabilities will sum to 1.

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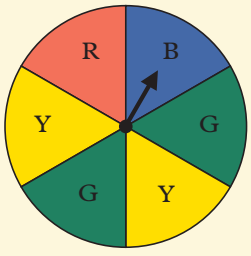
Arrays

- The sample space of a two-step experiment can be displayed in an **array**.
- Arrays can be used to calculate the probability of an experiment.
  - → To do this, all the cells in the array must be equally likely to occur.
  - → Divide the number of cells that satisfy the given event by the total number of cells in the array.
- The table below shows the outcomes for the two-step experiment of flipping a coin and then rolling a fair six-sided die.
  - → The outcomes of the first step are the headings of the rows and the outcomes of the second step are the headings of the columns.
  - → Reading across the columns and then down the rows to each cell will help you list all the possible combinations of outcomes.

		Rolling a fair six-sided die					
		1	**2**	**3**	**4**	**5**	**6**
Flipping a coin	**H**	(H, 1)	(H, 2)	(H, 3)	(H, 4)	(H, 5)	(H, 6)
	T	(T, 1)	(T, 2)	(T, 3)	(T, 4)	(T, 5)	(T, 6)

Worked example 11.1A Identifying the sample space and calculating theoretical probability

The following spinner divided into equally sized sectors (two yellow, two green, one red, one blue) is spun and the colour that the pointer lands on is recorded.



a List the sample space for this experiment.

b Calculate the probability of the spinner landing on a green sector.

THINK

a List all of the possible outcomes for this experiment inside curly brackets (also called braces).

b **1** The spinner has six equally sized sections, each of which has an equally likely chance of the pointer landing on it.

2 Divide the number of equally sized sections that are green by the total number of equally sized sections and simplify.

WRITE

a {red, blue, green, yellow}

b $\Pr(\text{green}) = \frac{2}{6}$

$= \frac{1}{3}$

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Worked example 11.1B Calculating the probability of multi-step experiments with equally likely outcomes

A coin is flipped twice. Calculate the probability that the outcome will be:

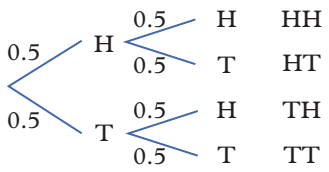
a exactly one head **b** no heads **c** at least one head.

THINK

1. Draw a tree diagram to represent all the different outcomes of this experiment.
2. Write the individual probabilities on the branches of the tree diagram. For each flip of the coin there are two different possible outcomes which are equally likely to occur, therefore, they each have a $\frac{1}{2}$ or 0.5 chance of occurring.
3. The probabilities of the final outcomes can be determined by multiplying the probabilities along the related branches together.
4. Identify the required final outcomes.
5. All final outcomes have the same probability of occurring, so the required probability can be determined by multiplying the number of final outcomes by their probability.

WRITE

Outcomes



All final outcomes have a probability of:
$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

a Favourable outcomes are HT and TH.

$$\Pr(\text{exactly one head}) = 2 \times \frac{1}{4} = \frac{2}{4} = \frac{1}{2}$$

b The favourable outcome is TT.

$$\Pr(\text{no heads}) = \frac{1}{4}$$

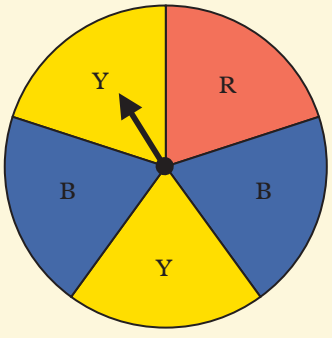
c Favourable outcomes are HH, HT and TH.

$$\Pr(\text{at least one head}) = 3 \times \frac{1}{4} = \frac{3}{4}$$

Note: The probability of this event could also be found by subtracting the probability of the complementary event ('no heads') from 1.

Worked example 11.1C Calculating the probability of multi-step experiments without equally likely outcomes

The spinner on the right divided into equally sized sectors (two blue, two yellow, one red) is spun twice, with the colour the pointer lands on being recorded both times. Calculate the probability of landing on a blue sector both times.



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THINK

1 Calculate the probabilities of landing on each colour. There are 5 equally sized sectors, so the probability of landing on a blue sector is $\frac{2}{5}$, and so on.

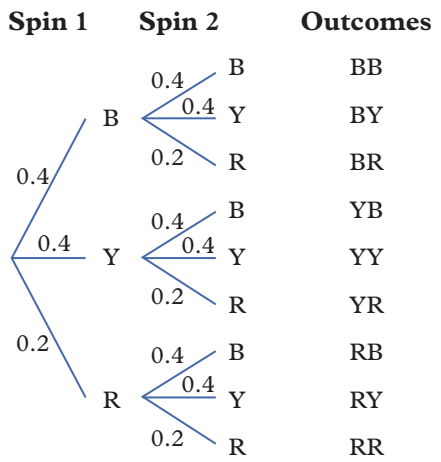
WRITE

$\Pr(\text{blue}) = \Pr(\text{B})$
$= \frac{2}{5}$
$= 0.4$

$\Pr(\text{yellow}) = \Pr(\text{Y})$
$= \frac{2}{5}$
$= 0.4$

$\Pr(\text{red}) = \Pr(\text{R})$
$= \frac{1}{5}$
$= 0.2$

2 Draw a tree diagram. Mark the individual probabilities on the branches and show all outcomes.



3 Identify the outcome with two blue sectors.

4 Calculate the probability of this outcome by multiplying the probabilities on each branch together.

$\Pr(\text{two blues}) = 0.4 \times 0.4$
$= 0.16$

Helpful hints

- ✔ The word 'event' has a specific meaning in probability that is separate to its meaning in everyday English. Read the definition carefully and make sure you do not get confused.
- ✔ Remember that you can only use the theoretical probability formula,

  $$\Pr(\text{event}) = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}},$$

  if the outcomes you are dealing with are equally likely to occur.
- ✔ Addition of decimals is often easier than addition of fractions, and multiplication of fractions is often easier than multiplication of decimals. Therefore, depending on the context, it may be useful to write your probabilities as decimals or fractions.
  For example, evaluating $0.25 \times 0.7 \times 0.8$ is more difficult than $\frac{1}{4} \times \frac{7}{10} \times \frac{4}{5}$, whereas $0.15 + 0.125$ is easier to evaluate than $\frac{3}{20} + \frac{1}{8}$.
- ✔ Be careful not to represent decimals incorrectly. For example, $\frac{1}{3}$ is not equal to 0.33.

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Learning pathways

1–8, 10, 12, 13, 17, 18	1, 2, 6, 7, 9, 10, 14, 17, 18	1, 2, 8, 10, 11, 14, 15–17, 19, 20

ANS p750

Exercise 11.1A: Understanding and fluency

WE 11.1A **1** List the sample space for each of these experiments.

- **a** rolling an eight-sided die (with sides numbered 1–8) and recording the uppermost number
- **b** randomly selecting a letter from the word AUSTRALIA
- **c** spinning a spinner with two red, three white and four blue sections (all sections being equal in size) and recording the colour

2 State whether the outcomes in the sample spaces for the experiments in question **1** are equally likely to occur or not.

3 A letter is selected at random from the word MATHEMATICS.

- **a** List the sample space for this experiment.
- **b** Calculate the probability of selecting an M.

WE 11.1B **4** A multiple-choice question consists of two parts. Each part has an equally likely chance of being true or false. What is the probability that:

- **a** the answers are both true
- **b** the answers are both false
- **c** one answer is true, and one answer is false?

5 In a room there are two lights, each with its own on/off switch.

- **a** Draw a tree diagram to show all the possible combinations of on or off the two lights could be.
- **b** If it is equally likely that each light is either on or off, what is the probability that the lights will:
  - **i** both be on
  - **ii** both be off
  - **iii** not be in the same mode?

WE 11.1C **6** Two fair six-sided dice are rolled. Each time a 6 is rolled on either of the dice, it is recorded.

- **a** If A represents the event 'rolling a 6', what does A′ represent?
- **b** Draw a tree diagram showing the probabilities of the different outcomes of the experiment. Write the probabilities for each step on the branches.
- **c** Use your tree diagram to find the probability of rolling:
  - **i** a double 6
  - **ii** at least one 6.

7 The experiment in question **6** is repeated. This time, all the numbers rolled on both dice are recorded.

- **a** Explain how this experiment differs from the experiment in question **6**.
- **b** How many outcomes are possible in this case?
- **c** Draw an array to represent the experiment, and use it to determine the probability that the two numbers rolled are:
  - **i** the same
  - **ii** both prime numbers.

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Exercise 11.1B: Problem solving and reasoning

8 Three eight-sided dice (numbered 1–8) are thrown and the number rolled on each die is noted. It is recorded whether each number is even or odd.

Use a tree diagram to find the probability of rolling:

a no odd numbers

b one or two odd numbers

c more odd numbers than even numbers.

9 'Unders and overs' is a game that is played by rolling two six-sided dice. The numbers rolled are added together to give a total. Players predict whether the total will be under 7, equal to 7 or over 7.

a Draw an array to show all the possible outcomes when rolling two dice.

b Colour your array from part **a** to show which outcomes are:

i under 7 **ii** equal to 7 **iii** greater than 7.

c Calculate the probability of the prediction being correct if you predicted:

i under 7 **ii** equal to 7 **iii** greater than 7.

d Describe what your strategy would be if you played this game.

10 A ten-sided die (numbered 1–10) is rolled and the number facing up is recorded. Let A represent the event 'rolling an even number', B represent the event 'rolling a multiple of 4', and C represent the event 'rolling a prime number'.

a List the outcomes in each event: A, B and C.

b Which two events are mutually exclusive?

c Find each of the following:

i Pr(A) **ii** Pr(B) **iii** Pr(C).

11 When rolling two unbiased dice of the same kind, the probability of rolling a double depends on the number of sides on each die.

a Consider rolling a pair of standard six-sided dice.

i How many outcomes are possible when the two dice are rolled?

ii How many of these outcomes are doubles? List them.

iii What is the probability of rolling a double?

b Consider rolling a pair of eight-sided dice. Repeat the questions from part **a**.

c Write a general statement for the probability of rolling a double using a pair of dice with n sides.

12 A marble collection is made up of clear and opaque marbles. There are 15 clear marbles, and the probability of choosing an opaque marble from the collection is $\frac{2}{5}$. How many opaque marbles are in the collection?

13 Two fair six-sided dice are thrown and the product of the two numbers rolled is recorded.

a What is the probability that this number is:

i odd **ii** prime **iii** a square number?

b From the three outcomes given in part **a**, what is:

i the most likely outcome **ii** the least likely outcome?

14 A game involves a regular six-sided die and a fair coin. The die is rolled, and the number obtained is recorded. The coin is then flipped. If the coin lands on heads, then the die score is doubled, whereas if the coin lands on tails, then the die score remains the same.

a List the sample space of possible scores for this game.

b Explain why each outcome in the sample space is not equally likely to occur.

c Find the probability that a player achieves a score of:

i 10 **ii** 4 **iii** at least 8.

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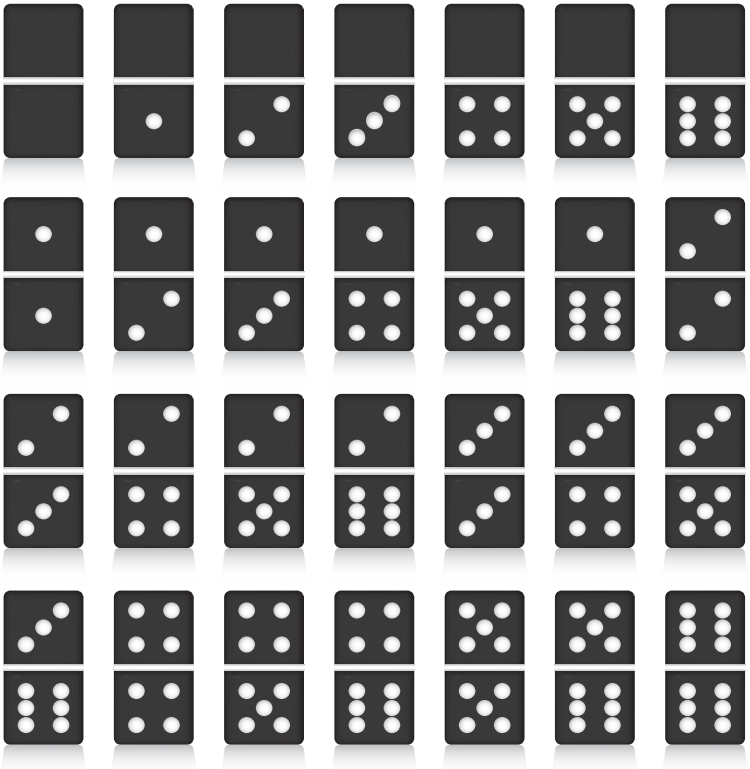
15 Two standard six-sided dice are rolled and the two numbers facing up are recorded. What is the probability that the sum of the numbers is greater than the product of the numbers?

16 A set of dominoes contains tiles which have been divided in two with numbers represented by dots (similar to dice). A blank space on a tile represents 0.



a How many tiles are in the complete set of dominoes (shown above)?

b Explain the order of the tiles in the photo.

c For each of the values (0, 1, 2, 3, 4, 5 and 6), count the number of tiles on which the value appears. If the value appears on both halves of a tile, count the tile once. You should get the same number of tiles for each of the values from 0 to 6. What is this number?

d One tile is chosen at random. What is the probability that it shows:

- **i** double 6
- **ii** the same number twice
- **iii** two different numbers
- **iv** one even number and one odd number
- **v** at least one prime number
- **vi** two numbers, one of which is a multiple of the other (not including zero as a multiple)?

17 A coin is biased so that the chance of it showing a tail when flipped is 0.7.

a Draw a tree diagram to show the outcomes and their probabilities if the coin is flipped three times.

b From your tree diagram, what is:

- **i** the most likely outcome
- **ii** the least likely outcome?

c What is the probability of flipping:

- **i** only one head
- **ii** only one tail
- **iii** more heads than tails?

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18 Snakes and ladders is played on a board with numbered grid squares. Ladders connect certain pairs of squares together, and snakes connect other pairs of squares together. The aim of the game is to progress from square 1 to square 100 by rolling a fair six-sided die and moving the number of squares rolled. You must move up a ladder or down a snake if you land on the squares that show the base of a ladder or the head of a snake.

Imagine a grid on which the first ladder base is on square 7.

a What is the minimum number of die rolls which would take you to square 7?

b What is the maximum number of die rolls which could take you to square 7?

c Consider the minimum number of rolls required.

i List the die rolls which would take you to square 7.

ii How many different options are there?

Exercise 11.1C: Challenge

19 A bag contains Christmas decorations of four different colours: red, green, gold and silver. There are four silver decorations and eight gold decorations. One decoration is drawn at random from the bag. The probability it is silver or gold is 0.6. If there are three green decorations, how many red decorations are in the bag?

20 Consider the following target.

a Discuss whether the chance of landing on a red section is the same as that of landing on a white section.

b The diameter of the innermost red circle (the bullseye) is 2 cm, and the widths of each of the subsequent rings is 2 cm. What is the radius of each of the rings?

c Calculate the area of each of the rings in terms of π.

d Assuming the arrow is equally likely to hit any point within the target, calculate (to three decimal places) the probability of the arrow landing:

i in the bullseye

ii in the outermost red ring.

Online resources:

- **Interactive skillsheet** Complementary events
- **Interactive skillsheet** Arrays
- **Investigation** Sicherman dice
- **Quick quiz** 11.1
- **Interactive skillsheet** Tree diagrams
- **Worksheet** Probability and tree diagrams

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Lesson 11.2

Experiments with and without replacement

Learning intentions

By the end of this lesson you will be able to ...

✔ calculate the probabilities of outcomes in experiments with and without replacement.

Inter-year links

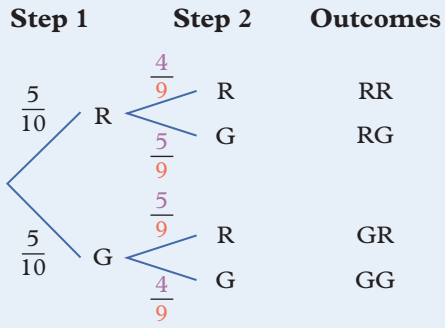
Year 8	10.4 Theoretical probability
Year 9	8.2 Experiments with replacement
Year 9	8.3 Experiments without replacement

Experiments with and without replacement

Key content video
Experiments with replacement

Key content video
Experiments without replacement

- When multi-step experiments involve items being selected from a group, the item can either be *replaced* or *not replaced* after each selection.
- If the items in a multi-step experiment involving selection are *replaced*, the probabilities for each step of the experiment remain the same.
- If the items in a multi-step experiment involving selection are *not replaced*, the probabilities for each step of the experiment change.
- The outcomes for each step of a multi-step experiment *with replacement* are independent, as the probabilities for the second step are not dependent on the outcome of the first step.
- The outcomes for each step of a multi-step experiment *without replacement* are dependent (not independent), as the probabilities for the second step depend on the outcome of the first step.
- For experiments *without replacement*, the number of items to select from decreases by 1 after each selection. For example, the following tree diagram represents two selections, without replacement, from a bag containing 5 red balls and 5 green balls.



The number of outcomes in the sample space is reduced for the second selection.

Depending on the first selection, the number of successful outcomes may also be reduced.

- Arrays can be used to show the reduced sample space for experiments without replacement. For example, if two letters are picked at random from the word 'SUN' without replacement, the sample space would be:

		Second letter		
		S	**U**	**N**
First letter	**S**	–	(S, U)	(S, N)
	U	(U, S)	–	(U, N)
	N	(N, S)	(N, U)	–

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Worked example 11.2A Calculating probability for experiments with replacement

A barrel contains coloured balls: two red, one blue and three green. One ball is drawn from the barrel, its colour is noted, then it is replaced. A second ball is then chosen and its colour is also noted.

a Draw a tree diagram to represent the situation.

b Determine the probability that the two balls selected are the same colour.

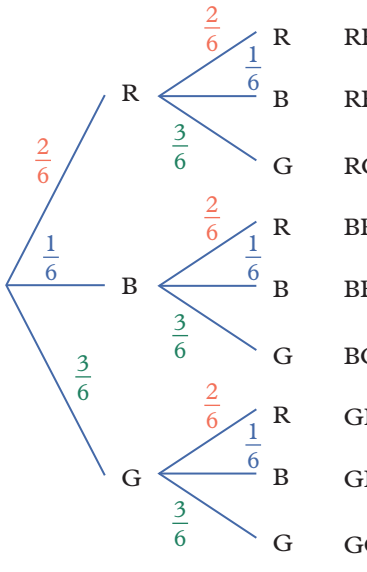
c Determine the probability that a red and green ball are selected in any order.

THINK

a Draw a tree diagram, writing all possible outcomes along with their probabilities. Because the selected ball is replaced after the first selection, the probabilities for the second selection remain the same.

WRITE

a



Outcomes	Probability
RR	$\frac{2}{6} \times \frac{2}{6} = \frac{4}{36}$
RB	$\frac{2}{6} \times \frac{1}{6} = \frac{2}{36}$
RG	$\frac{2}{6} \times \frac{3}{6} = \frac{6}{36}$
BR	$\frac{1}{6} \times \frac{2}{6} = \frac{2}{36}$
BB	$\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$
BG	$\frac{1}{6} \times \frac{3}{6} = \frac{3}{36}$
GR	$\frac{3}{6} \times \frac{2}{6} = \frac{6}{36}$
GB	$\frac{3}{6} \times \frac{1}{6} = \frac{3}{36}$
GG	$\frac{3}{6} \times \frac{3}{6} = \frac{9}{36}$

b On your tree diagram, identify the final outcomes in which the two balls are the same colour. Add the probabilities for those outcomes to find the required probability, and then simplify.

b Favourable outcomes are RR, BB and GG.

$\Pr(\text{RR}) = \frac{4}{36}$

$\Pr(\text{BB}) = \frac{1}{36}$

$\Pr(\text{GG}) = \frac{9}{36}$

$$\begin{aligned}\Pr(\text{same colour}) &= \frac{4}{36} + \frac{1}{36} + \frac{9}{36}\\ &= \frac{14}{36}\\ &= \frac{7}{18}\end{aligned}$$

c Identify the final outcomes in which a red and a green ball are selected. Add the probabilities for these outcomes to find the required probability, and then simplify.

c Favourable outcomes are RG and GR.

$\Pr(\text{RG}) = \frac{6}{36}$

$\Pr(\text{GR}) = \frac{6}{36}$

$$\begin{aligned}\Pr(\text{red and green}) &= \frac{6}{36} + \frac{6}{36}\\ &= \frac{12}{36}\\ &= \frac{1}{3}\end{aligned}$$

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Worked example 11.2B Calculating probability for experiments without replacement

Repeat the experiment from **Worked example 11.2A**, but this time assume the first ball is not replaced before the second ball is chosen.

THINK

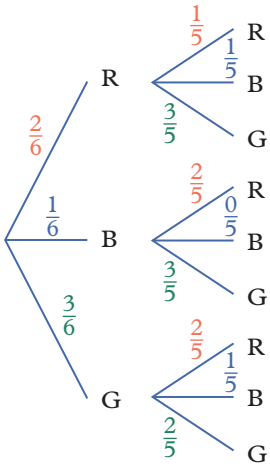
a Draw a tree diagram. The number of balls from which the second ball is chosen decreases by 1, so the possible outcomes at the second step are different from the possible outcomes for the first step.

b On your tree diagram, identify the final outcomes in which the two colours are the same. Add the probabilities for those outcomes to find the required probability and simplify. (Note the outcome BB is not possible, because there is only one blue ball.)

c Identify the final outcomes in which a red and a green ball are selected. Add the probabilities for these outcomes to find the required probability, and then simplify.

WRITE

a



Outcomes	Probability
RR	$\frac{2}{6} \times \frac{1}{5} = \frac{2}{30}$
RB	$\frac{2}{6} \times \frac{1}{5} = \frac{2}{30}$
RG	$\frac{2}{6} \times \frac{3}{5} = \frac{6}{30}$
BR	$\frac{1}{6} \times \frac{2}{5} = \frac{2}{30}$
BB	$\frac{1}{6} \times \frac{0}{5} = 0$
BG	$\frac{1}{6} \times \frac{3}{5} = \frac{3}{30}$
GR	$\frac{3}{6} \times \frac{2}{5} = \frac{6}{30}$
GB	$\frac{3}{6} \times \frac{1}{5} = \frac{3}{30}$
GG	$\frac{3}{6} \times \frac{2}{5} = \frac{6}{30}$

b Favourable outcomes are RR and GG.

$\Pr(\text{RR}) = \frac{2}{30}$

$\Pr(\text{GG}) = \frac{6}{30}$

$$\begin{aligned}\Pr(\text{same colour}) &= \frac{2}{30} + \frac{6}{30}\\ &= \frac{8}{30}\\ &= \frac{4}{15}\end{aligned}$$

c Favourable outcomes are RG and GR.

$\Pr(\text{RG}) = \frac{6}{30}$

$\Pr(\text{GR}) = \frac{6}{30}$

$$\begin{aligned}\Pr(\text{red and green}) &= \frac{6}{30} + \frac{6}{30}\\ &= \frac{12}{30}\\ &= \frac{2}{5}\end{aligned}$$

Helpful hints

- ✔ The sum of the probabilities at each step of a multi-step experiment is always equal to 1.
- ✔ For multi-step experiments, it is often helpful to leave fractions unsimplified until the final answer. In **Worked example 11.2A**, the probability that the balls are the same colour was calculated as $\frac{4}{36} + \frac{1}{36} + \frac{9}{36}$, which is easy to add. However, if each fraction were simplified then the denominators would all be different, which would make addition more difficult.

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Learning pathways

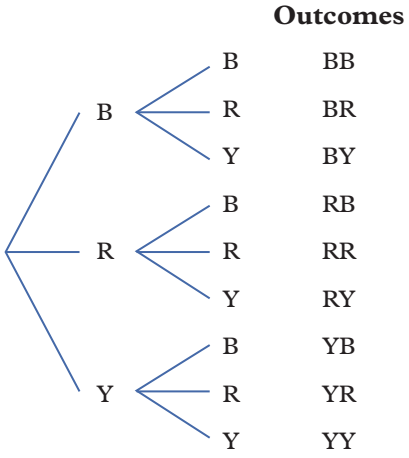
1–3, 5–7, 10, 12 | 3, 4, 8, 9, 11, 13–15, 17, 18 | 4, 7, 9, 11, 13–19

ANS p752

Exercise 11.2A: Understanding and fluency

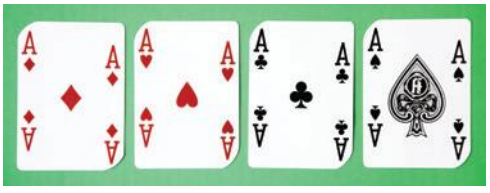
1 A bag contains 22 coloured marbles: 6 blue, 8 red and 8 yellow. One marble is drawn from the bag, its colour is noted, then it is replaced. A second marble is then chosen and its colour is also noted.

a Copy and complete the tree diagram representing this situation by adding probabilities to the branches.



b Determine the probability that the two chosen marbles are the same colour.

WE 11.2A **2** A standard deck of playing cards consists of four suits (hearts, diamonds, clubs and spades) with 13 cards in each suit (52 cards in total).



One card is chosen from a deck and its suit is noted. It is replaced, then a second card is chosen and the suit noted.

a Draw a tree diagram to represent this experiment.

b Determine the probability that two hearts are selected.

c Determine the probability that a club and a diamond are selected in any order.

d Determine the probability that two cards of the same suit are drawn.

e Determine the probability that two cards of different suits are drawn.

3 John removes the loose coins from his pockets each night and throws them into a tin. There are 75 silver coins and 25 gold coins in the tin.

John picks out a coin from the tin and notes its colour. After replacing it in the tin, he repeats this process another two times.

a Determine the probability of John drawing out:

i a silver coin

ii a gold coin.

b Draw a tree diagram to display this experiment. Write the individual probabilities on the branches of the tree diagram.

c Use your tree diagram to calculate the probability of John drawing out:

i all silver coins

ii no silver coins

iii at least one coin of each colour.

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4 Consider the digits 1, 2, 3, 4 and 5. From these five digits, two are randomly selected to create a two-digit number. The same digit may be selected both times.

a Draw a tree diagram to represent this situation and use it to help you write all possible outcomes.

b How many outcomes are there?

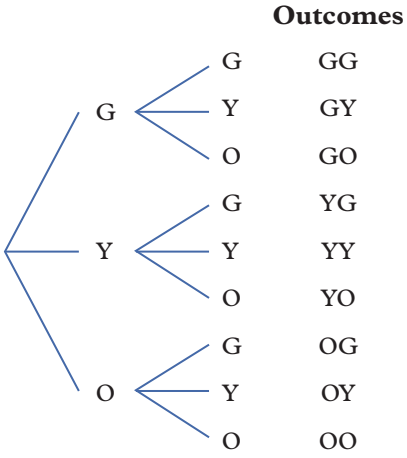
c How many of these outcomes have two digits which are the same?

d Calculate the probability that the number has:

i two different digits **ii** two digits the same **iii** the first digit less than the second.

5 A bag contains 16 coloured marbles: 6 green, 5 yellow and 5 orange. One marble is drawn from the bag, its colour is noted and it is not replaced. A second marble is then chosen and its colour is also noted.

a Copy and complete the tree diagram representing this situation by adding probabilities to the branches.



b Determine the probability that the two chosen marbles are the same colour.

WE 11.2B **6** A bag contains 8 cubes: two black, three white and three grey. One cube is drawn and its colour is noted. A second cube is drawn without the first cube being replaced and its colour is noted.

a Draw a tree diagram to represent the situation.

b Determine the probability that the two cubes selected are the same colour.

c Determine the probability that a black and a white cube are selected in any order.

7 The letters of the word PROBABILITY are written on separate pieces of paper and placed in a box. One letter is drawn from the box and it is recorded whether the letter is a consonant or a vowel. A second letter is then drawn without replacing the first letter. Determine the probability that:

a the letters are both vowels

b one letter is a vowel and the other is a consonant.

8 From the set of all 26 letters in the English alphabet, two letters are drawn. Find the probability that the letters are both vowels or both consonants, if the letters were drawn:

a with replacement **b** without replacement.

9 Consider again the situation described in question **4**, but this time, the same digit cannot be selected both times.

a Draw a tree diagram to represent this situation and use it to help you write all possible outcomes.

b How many outcomes are there?

c Calculate the probability that the number has:

i two digits the same **ii** two different digits **iii** the first digit smaller than the second.

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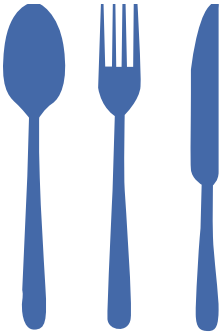
10 Sam has a small bag of lollies. The bag contains 10 red, 8 blue, 6 orange, 4 yellow and 12 green lollies.

a Sam prefers the red lollies. What is the probability that, if he chooses one lolly at random, it will be red?

b Sam eats all the red lollies. He then chooses another lolly at random, eats it, then chooses another one at random and eats it. What is the probability that the last two lollies Sam ate were the same colour?

11 You are setting the table for you and your friend to have dinner. You each need a knife, fork and spoon. The cutlery drawer in the kitchen contains 15 knives, 10 forks and 15 spoons.



Without looking in the drawer, you choose three pieces of cutlery at random.

a What is the chance that your first choice was a knife?

b If your first choice is a knife, what is the probability that your next choice will be a fork and the next choice a spoon?

c What is the probability of choosing a knife, a fork and a spoon in any order?

d Assume that you have already chosen a knife, fork and spoon for yourself. What is the probability that you will now choose these three items for your friend in your next three selections?

12 Consider a pencil case containing 30 pencils: 10 black, 2 red, 4 blue, 3 green and 11 yellow.

Two pencils are taken out one after the other without replacement. The chance that the first pencil chosen is black, then the next one is red is calculated as:

$$\begin{aligned}\Pr(\text{black then red}) &= \Pr(\text{black}) \times \Pr(\text{red})\\ &= \frac{10}{30} \times \frac{2}{29}\\ &= \frac{2}{87}\end{aligned}$$

a Use this information to help you determine the probability that:

i the first pencil chosen is black, then the next one is not red

ii the first pencil chosen is black, then the next one is red or blue

iii the first pencil chosen is black, then the next one is red, blue or green

iv both pencils are black

v both pencils are red

vi the first pencil chosen is red, then the next one is blue.

b Investigate to see whether undertaking the two events in the reverse order affects the probability by determining the probability that:

i the first pencil chosen is red, then the next one is black

ii the first pencil chosen is blue, then the next one is red.

c Comment on your answers to part **b**.

13 A bag contains 5 balls: three white and two black. Three balls are drawn, and each ball is placed back in the bag immediately after it has been drawn.

a Find the probability that all three balls drawn are black.

b Find the probability that all three balls drawn are white.

14 As with question **13**, a bag contains 5 balls: three white and two black. This time, three balls are drawn without replacement.

a Explain why at least one white ball will be drawn.

b Find the probability that all three balls drawn are white.

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Exercise 11.2B: Problem solving and reasoning

15 Explain why the events in multi-step experiments with replacement are independent, and the events in multi-step experiments without replacement are dependent.

16 Five balls labelled with numbers 1–5 are placed in a bag. Two balls are selected, at random, one at a time.

a Write the sample space for the sum of the two numbers if the first ball is:

i replaced after being selected **ii** not replaced after being selected.

b Find the probability that the sum of the two balls chosen is 4 if the first ball is:

i replaced after being selected **ii** not replaced after being selected.

c Find the probability that the sum of the two balls chosen is 6 if the first ball is:

i replaced after being selected **ii** not replaced after being selected.

17 It is not always necessary to draw tree diagrams to determine the number of outcomes of an experiment. First, consider forming a two-digit number from the digits 1 to 5 *with replacement* (as described in question **4**). There are five digits to choose from when writing the first digit. You can use all the digits again for the second digit, so there are five from which to choose again. This means that there are 5×5 possible outcomes.

Next, consider forming a two-digit number from the digits 1 to 5 *without replacement*. There are five digits to choose from when writing the first digit. Since you are not able to repeat any of the digits, there are only four digits from which to choose the second digit. This means that there are 5×4 possible outcomes.

a Use this method to determine the number of possible outcomes when forming a three-digit number from the digits 1 to 5:

i with replacement **ii** without replacement.

b Repeat part **a** using the digits 1 to 9 to form a three-digit number.

Exercise 11.2C: Challenge

18 A restaurant offers a menu of six entrées, six main meals and six desserts.

a How many different combinations of three-course meals (consisting of an entrée, main and dessert) does the restaurant offer?

b During the evening, the kitchen runs out of one of the entrées and two of the desserts. What is the decrease in the number of combinations available?

19 Suppose that car number plates contain six digits, and the digits can be repeated.

a Number plates can start with the digit 0.

i How many different digits can be used in each place of the six-digit number?

ii How many different number plates could be produced using this system?

b A new system was introduced whereby each number plate contains three digits followed by three letters.

i How many number plates would be possible using this system?

ii Explain why the system with a mixture of digits and letters for number plates is better than the system with six digits.

Online resources:

Interactive skillsheet Experiments with replacement

Interactive skillsheet Experiments without replacement

Investigation Replacement and sample sizes

Investigation Birthday surprises

Quick quiz 11.2

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Lesson 11.3

Two-way tables and Venn diagrams

Learning intentions

By the end of this lesson you will be able to ...

✔ understand and use two-way tables and Venn diagrams.

Inter-year links

Year 8	10.5 Two-way tables
Year 8	10.6 Venn diagrams
Year 9	8.5 Two-way tables
Year 9	8.6 Venn diagrams

Set notation

Key content video
Set notation

- A set is a collection of distinct objects.
- An element is a member of a set. The number of elements in set A is written as $n(A)$.
- The **universal set** contains all the elements under consideration and is represented by ξ (the Greek letter xi, pronounced 'zai').
- The **intersection** of two sets, A and B, is the set of elements in both A and B, and is denoted by $A \cap B$.
- The **union** of two sets, A and B, includes all elements from both sets, and is denoted by $A \cup B$.
- The complement of set A is written as A'. This includes everything from the universal set that is not in set A (not A).
- A **subset** is a smaller set within a set.
  - → If set A is a subset of set B, this can be written as $A \subset B$.
  - → If set A is not a subset of set B, this can be written as $A \not\subset B$.
- The null set, or the empty set, is the set with no elements; that is, $\{\}$. The symbol for the null set is $\varnothing$.
- If two sets, A and B, are mutually exclusive, their intersection has no elements: $A \cap B = \varnothing$.

Venn diagrams

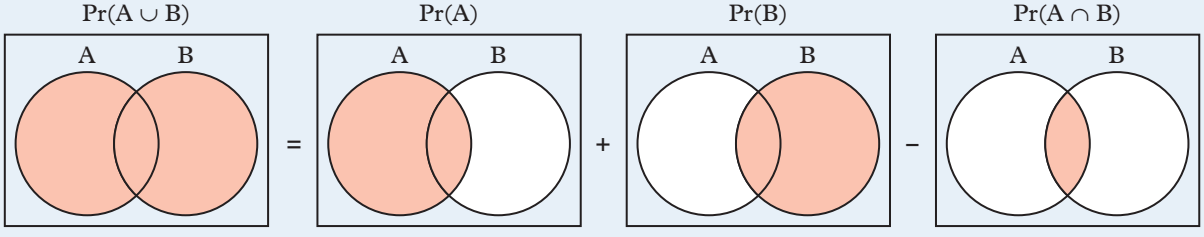
- Each circle in a **Venn diagram** represents a different set of data, and the rectangle represents the entire data set (the universal set, ξ).
- Sometimes Venn diagrams may be drawn with the circles being completely separated, which should be interpreted as meaning that the events are mutually exclusive.

The addition rule for probability

Key content video
The addition rule for probability

- When two sets (A and B) overlap, the following formula can be used:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$



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Two-way tables

- A two-way table is a means of displaying the relationship between two sets of data. The sets are represented by the rows or columns of the two-way table.

	School A	School B	Total
Musical instrument	10	4	14
No musical instrument	3	13	16
Total	13	17	30

Key content video
Understanding two-way tables

Key content video
Creating and completing two-way tables

Key content video
Converting between two-way tables and Venn diagrams

Worked example 11.3A Converting a two-way table to a Venn diagram

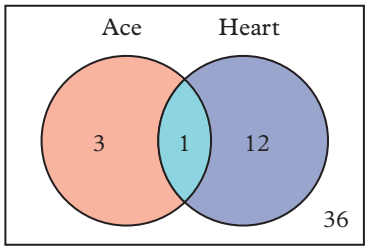
This two-way table details information about the cards found in a standard deck of playing cards. Convert the two-way table into a Venn diagram.

	Ace (A)	Not ace (A′)	Total
Heart (H)	1	12	13
Not heart (H′)	3	36	39
Total	4	48	52

THINK

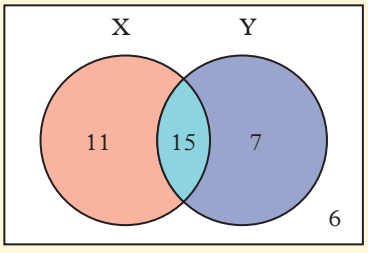
1. Draw two circles on your Venn diagram to represent the two sets. Label each circle with one of the possible options from each set.
2. Fill in the sections of the Venn diagram with the information from the two-way table.
3. Check that the total of the numbers in the Venn diagram equals the total in the bottom right-hand corner of the two-way table.

WRITE



Worked example 11.3B Converting a Venn diagram to a two-way table

Convert this Venn diagram into a two-way table.



THINK

1. Create a table with columns and rows for the sets in the Venn diagram and their complements.
2. Add a row and a column for the totals.
3. Fill in the inner cells of the two-way table with the information from the Venn diagram.
4. Calculate the values in the total row and column, and check the bottom right-hand corner equals the total of the values in the Venn diagram.

WRITE

	X	X′	Total
Y	15	7	22
Y′	11	6	17
Total	26	13	39

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Worked example 11.3C Using the addition rule for probability

Given that $\Pr(A) = 0.35$, $\Pr(B) = 0.75$ and $\Pr(A \cap B) = 0.2$, find $\Pr(A \cup B)$.

THINK

1 Substitute the given values into the addition rule for probability.

2 Simplify and write your answer.

WRITE

$$\begin{aligned}\Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B)\\ &= 0.35 + 0.75 - 0.2\\ &= 0.9\end{aligned}$$

Helpful hints

- ✔ Converting between two-way tables and Venn diagrams can help you to visualise problems in a different way.
- ✔ The addition rule for probability can be rearranged to find the values of $\Pr(A)$, $\Pr(B)$ or $\Pr(A \cap B)$ as necessary.
- ✔ Two-way tables provide information about the totals of each category, which isn't displayed in a Venn diagram.

Learning pathways

1–10, 13	2–4, 6, 8–11, 15, 16	2, 6, 8–12, 14, 16–18

ANS p753

Exercise 11.3A: Understanding and fluency

1 Consider the whole numbers from 1 to 12 inclusive. Then the universal set, ξ, is $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. Let A be the set of odd numbers and B be the set of prime numbers.

a Write the list of elements in the set:

i A **ii** B

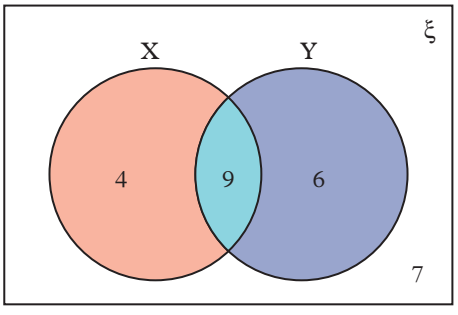
iii $A \cap B$ **iv** $A \cup B$

b One number from the universal set is selected at random. Find:

i $\Pr(A)$ **ii** $\Pr(B')$

iii $\Pr(A \cap B)$ **iv** $\Pr(A \cup B)$

2 Consider the Venn diagram on the right.



a Find:

i $n(X)$ **ii** $n(X \cup Y)$

iii $n(X')$ **iv** $n(X \cap Y)$

v $n(\xi)$ **vi** $n(X' \cap Y)$

b Calculate each of these probabilities.

i $\Pr(X)$ **ii** $\Pr(X')$

iii $\Pr(X \cap Y)$ **iv** $\Pr((X \cup Y)')$

v $\Pr(X' \cap Y')$ **vi** $\Pr(X \cap Y')$

vii $\Pr(X \cup Y')$ **viii** $\Pr(Y \cup Y')$

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3 A survey of 50 people was conducted to collect data relating to the types of movies those people like. The movie categories were comedy, drama and animation. The results were as follows.

- 26 like comedy
- 24 like drama
- 33 like animation
- 13 like comedy and drama
- 17 like drama and animation
- 14 like comedy and animation
- 8 like all three types of movies.

Draw a Venn diagram to display this data.

4 Use your Venn diagram from question **3** to help you find the probability that a person randomly chosen:

a likes comedy

b likes comedy, but doesn't like drama

c likes drama, but doesn't like comedy

d likes comedy or drama, but doesn't like animation

e likes any of these movie forms.

WE 11.3A **5** The following two-way table details information about the sports that 140 students play. Convert the two-way table into a Venn diagram.

	Netball (N)	Not netball (N′)	Total
Field hockey (F)	22	10	32
Not field hockey (F′)	45	63	108
Total	67	73	140

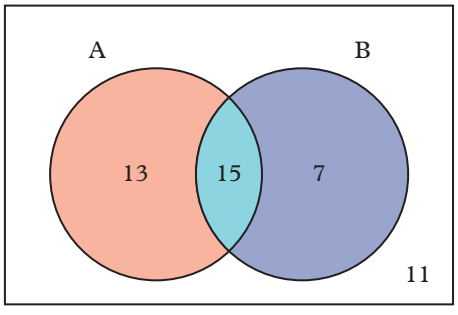
6 200 Year 10 students were asked if they have subscriptions to Netflix and Spotify. The results are partially displayed in the two-way table below.

	Netflix (N)	Not Netflix (N′)	Total
Spotify (S)	85	62	147
Not Spotify (S′)			53
Total	126		200

a Fill in the missing numbers in the two-way table.

b Convert the completed two-way table into a Venn diagram.

WE 11.3B **7** Convert the following Venn diagram into a two-way table.



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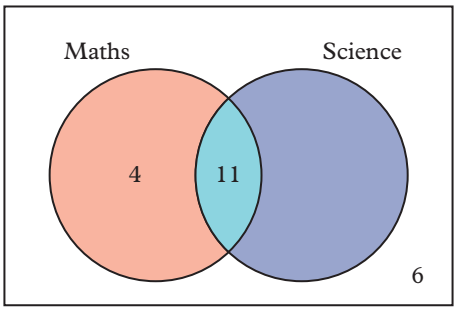
8 A class of 28 students were each asked if they enjoyed studying mathematics and science. The results are partially displayed in the Venn diagram on the right.



a The number of students who enjoy studying science, but not maths, is missing. What is this number?

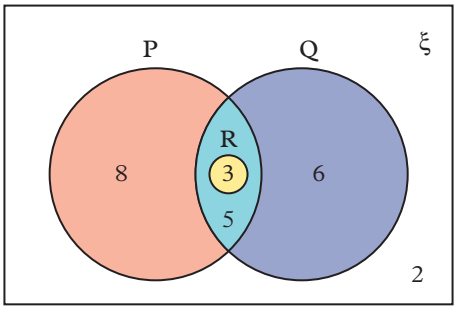
b Convert the completed Venn diagram into a two-way table.

WE 11.3C **9** Given that $\Pr(A) = 0.6$, $\Pr(B) = 0.3$ and $\Pr(A \cap B) = 0.2$, find $\Pr(A \cup B)$.

10 Find the probability of A, given that $\Pr(B) = \frac{1}{3}$, $\Pr(A \cup B) = \frac{4}{9}$ and $\Pr(A \cap B) = \frac{1}{6}$.

Exercise 11.3B: Problem solving and reasoning

11 Consider the Venn diagram on the right.



a Decide whether each of the following statements is true for the Venn diagram and explain your reasoning.

i $R \subset P$ **ii** $R \subset Q$

iii $R \not\subset (P \cap Q)$ **iv** $R \subset (P \cup Q)$

v $R \not\subset \xi$ **vi** $R \subset Q \subset \xi$

b Use the Venn diagram to help you calculate each of these probabilities.

i $\Pr(P \cap R)$ **ii** $\Pr(P \cap Q')$

iii $\Pr(P' \cap Q')$ **iv** $\Pr(P \cap Q \cap R)$

12 One hundred people took part in a survey to determine the readership of three monthly magazines (A, B and C). The results were as follows:

- 59 read A
- 68 read B
- 59 read C
- 42 read A and B
- 45 read B and C
- 39 read A and C
- 27 read all three magazines.

a Draw a Venn diagram to show the results of the survey.

b Find the number of people who:

i read magazines A or B **ii** read magazine C but not magazine B

iii read exactly two of the magazines **iv** read two magazines at most

v do not read magazine A **vi** read only one magazine

vii do not read B or C **viii** do not read three magazines.

13 As with question **1**, let the universal set, ξ, be $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. Let A be the set of odd numbers, and let B be the set of prime numbers.

a Write the list of elements in the set:

i A' **ii** $A' \cap B$

iii $(A \cap B)'$, the complement of the set $A \cap B$.

b One number from the universal set is selected at random. Find:

i $\Pr(A' \cap B)$ **ii** $\Pr(A' \cap B')$ **iii** $\Pr(A' \cup B')$.

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14 In the theory for this lesson, Venn diagrams were used in the proof of the addition rule. Draw Venn diagrams to help you prove that $\Pr((A \cap B)') = \Pr(A' \cup B')$.

15 For two events, A and B, it is known that $\Pr(A) = 0.75$, $\Pr(B) = 0.3$ and $\Pr(A' \cap B) = 0.1$.

a Find $\Pr(A')$.

b Use the addition rule to find $\Pr(A' \cup B)$.

16 For two events, A and B, it is known that $\Pr(A) = 0.3$, $\Pr(B) = 0.6$ and $\Pr(A \cap B) = 0.2$. These values have been entered into the following two-way probability table.

	A	**A′**	**Total**
B			0.6
B′	0.2		
Total	0.3		1

a Complete the missing values in the two-way probability table.

b Use the table from part **a** to help you find:

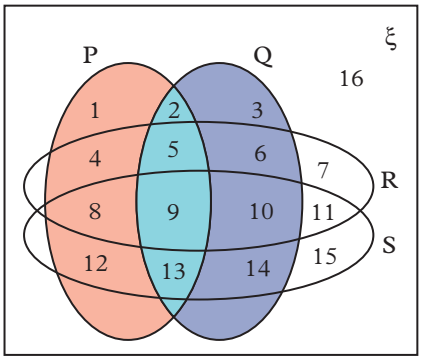
i $\Pr(A' \cap B')$

ii $\Pr(A' \cap B)$

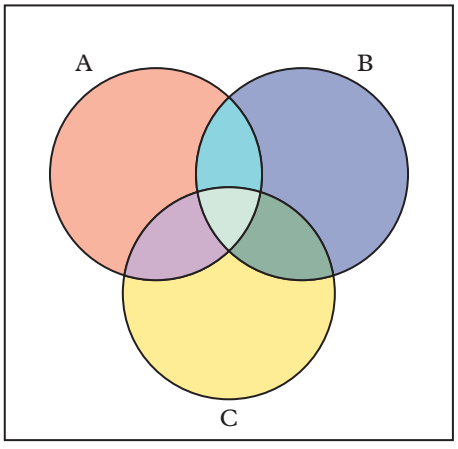
iii $\Pr(A' \cup B')$.

Exercise 11.3C: Challenge

17 This Venn diagram shows the relationship between four sets, P, Q, R and S. Determine the sum of the elements within the region $(P \cap Q) \cup (R \cap S')$.



18 Use the following Venn diagram to find an additional rule formula for $\Pr(A \cup B \cup C)$. That is, express $\Pr(A \cup B \cup C)$ in terms of the probabilities of A, B, C and their intersections.



Online resources:

- **Interactive skillsheet** Two-way tables
- **Interactive skillsheet** Set notation
- **Interactive skillsheet** The addition rule for probability
- **Quick quiz** 11.3
- **Interactive skillsheet** Venn diagrams

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ANS p755

Checkpoint

Checkpoint quiz
Take the checkpoint quiz to check your knowledge of the first part of this module.

11.1 **1** List the sample space for each of the following experiments.

a A fair coin is tossed.

b A standard six-sided die is rolled.

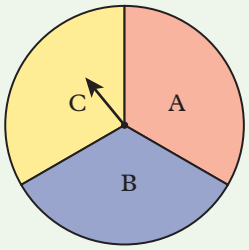
c A card is picked from a standard 52-card deck and its suit noted.

d A letter is selected at random from the word LOLLIPOP.

11.1 **2** A letter from the word CHANCE is selected at random. Find the probability that the selected letter is:

a 'H' **b** 'C' **c** a vowel **d** not a vowel.

11.1 **3** The spinner on the right is spun twice.



a Draw a tree diagram to represent the possible outcomes of the two spins.

b Find the probability that the spinner lands on 'A' both times.

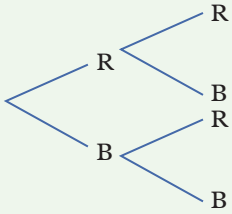
11.1 **4** A biased coin, which has a 60% chance of landing on heads, is tossed twice.

a Draw a tree diagram to represent this situation.

b Find the probability the coin lands on:

i heads both times **ii** heads once and tails once.

11.2 **5** A bag contains 3 red balls and 4 black balls. A ball is randomly selected from this bag and its colour is noted. The ball is then replaced before another ball is randomly selected from the bag.
The tree diagram on the right shows the possible outcomes of the experiment.



a Copy and complete the tree diagram by writing the probabilities on the branches.

b Find the probability that both balls selected are red.

11.2 **6** Two balls are selected at random from a bag containing 2 black balls and 3 white balls.
Find the probability that both balls selected are black if:

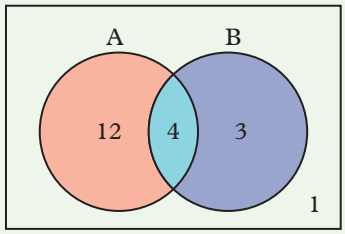
a the first ball is replaced before the second ball is selected

b the first ball is not replaced before the second ball is selected.

11.2 **7** A card is randomly selected from a standard 52-card deck, and its suit is noted. The card is then replaced, and another card is drawn at random. Find the probability that the two cards are:

a both hearts **b** the same suit **c** different suits.

11.3 **8** Consider the Venn diagram on the right.



Find:

a Pr(A)

b Pr(B)

c $\Pr(A \cap B)$

d $\Pr(A' \cap B)$.

11.3 **9** Consider the incomplete two-way table on the right.

	M	M′	Total
N	6	11	
N′		15	
Total			40

a Fill in the missing numbers in the two-way table.

b Convert the two-way table to a Venn diagram.

11.3 **10** The two-way table, below right, shows data gathered from a class of 24 students about which brand of phone they have and which operating system they use on their home computer.
One of the 24 students is selected at random.
Find the probability that this student:

	Apple	Samsung	Other	Total
Macintosh	8	1	0	9
Windows	6	4	5	15
Total	14	5	5	24

a has an Apple phone

b has Windows on their home computer

c has an Apple phone and Macintosh on their home computer

d has Windows on their home computer but not a Samsung phone.

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Lesson 11.4

Conditional probability

Learning intentions

By the end of this lesson you will be able to ...

✔ calculate conditional probabilities.

Inter-year links

Year 8	10.5 Two-way tables
Year 8	10.6 Venn diagrams
Year 9	8.3 Experiments without replacement

Conditional probability

Key content video
Conditional probability

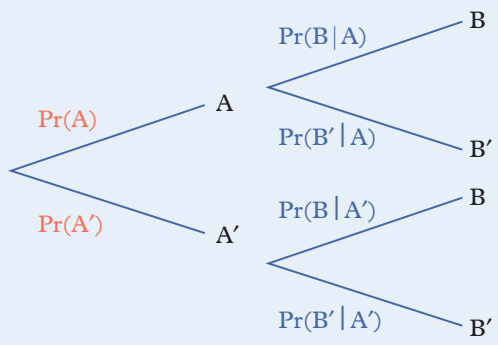
- **Conditional probability** is the probability of an event occurring, given the knowledge that another event has occurred.
  - → Conditional probability is often about a single trial of an experiment. For example, if a standard six-sided die is rolled, what is the probability the outcome is odd, given that the outcome is prime?
  - → Conditional probability restricts the sample space to a subset of the universal set. In the above example, the information 'given that the outcome is prime' restricts the sample space to $\{2, 3, 5\}$ from $\{1, 2, 3, 4, 5, 6\}$. This gives Pr(odd given prime) $= \frac{2}{3}$, since two of the three primes are odd.
- $\Pr(A|B)$ is the probability that event A occurs given that event B has occurred. This can be read as 'the probability of A given B'.
- $\Pr(B|A)$, which represents the probability that event B occurs given that event A has occurred, is different from $\Pr(A|B)$.
- Conditional probability can be calculated by using the following formulas:

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} \quad \text{or} \quad \Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)}$$

- The above formulas can be rearranged to give:

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B|A) \quad \text{or} \quad \Pr(A \cap B) = \Pr(B) \times \Pr(A|B)$$

  - → This explains why we multiply the probabilities on branches in tree diagrams:



$\Pr(A \cap B) = \Pr(A) \times \Pr(B|A)$

$\Pr(A \cap B') = \Pr(A) \times \Pr(B'|A)$

$\Pr(A' \cap B) = \Pr(A') \times \Pr(B|A')$

$\Pr(A' \cap B') = \Pr(A') \times \Pr(B'|A')$

- Conditional probability can be identified when any of the following terms are used in a probability question: 'if ... then ...', 'given ...' or 'knowing that ...'
- When calculating conditional probability using two-way tables or Venn diagrams, the words 'of' and 'from' indicate the set from which the choice is being made. For example, 'from the group who …'

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Worked example 11.4A Calculating conditional probability using a tree diagram

A sandwich tray contains 30 sandwiches, 10 of which are ham (H) and 20 of which are salad (S). Jack takes two sandwiches at random, one after the other.

a Draw a tree diagram to display the possible outcomes of Jack's two choices.

b If Jack's first sandwich is ham, what is the probability that his second sandwich will also be ham?

c Knowing that Jack's first choice is a salad sandwich, what is the probability that his second one will be ham?

d What is the probability of Jack's second sandwich being salad, given that his first sandwich is ham?

THINK

a Draw a tree diagram to display all the outcomes. The probabilities for the second choice depend on the outcome of the first choice. Write the probabilities on the branches of your tree diagram.

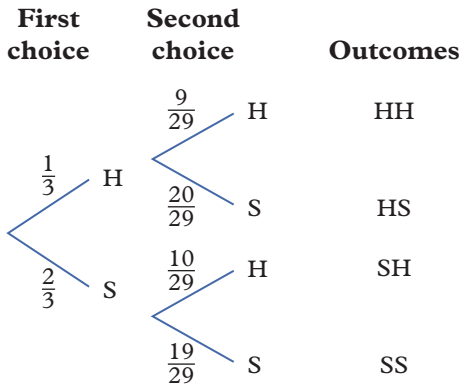
b In this case, you are looking for the probability that the second choice is H, given that the first choice is H. Follow the branches H then H on the tree diagram. The probability on the second branch is the probability of the first and second sandwiches both being ham.

c In this case, you are looking for the probability that the second choice is H, given that the first choice is S. Follow the branches S then H on the tree diagram. The probability on the second branch is the probability of the second sandwich being ham if the first sandwich was salad.

d In this case, you are looking for the probability that the second choice is S, given that the first choice is H. Follow the branches H then S on the tree diagram. The probability on the second branch is the probability of the second sandwich being salad if the first sandwich was ham.

WRITE

a H represents ham and S represents salad.



b $\Pr(\text{Ham second} \mid \text{Ham first}) = \frac{9}{29}$

c $\Pr(\text{Ham second} \mid \text{Salad first}) = \frac{10}{29}$

d $\Pr(\text{Salad second} \mid \text{Ham first}) = \frac{20}{29}$

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Worked example 11.4B Calculating conditional probability using a two-way table

The table below shows the results of a survey taken at a library to find out whether people under 18 and people 18 or over prefer using Facebook or Twitter (or neither).

	Under 18	**18 or over**	**Total**
Facebook	20	30	50
Twitter	5	10	15
Neither	10	5	15
Total	35	45	80

Use the two-way table to find the relative frequency that a person:

a prefers Facebook, given they are 18 or over

b is 18 and over, given they prefer Facebook

c Explain why your answers to parts **a** and **b** are not equal.

THINK

a **1** Identify the total number of people surveyed who are 18 or over by looking at the total number in the '18 or over' column.

2 Identify the number of people who prefer Facebook in the '18 or over' column.

3 Divide the number of people who are 18 or over who prefer Facebook by the total number of people who are 18 or over, and give the answer as a fraction, simplifying if possible.

b **1** Identify the total number of people who prefer Facebook by looking at the Facebook row.

2 Identify the number of people who are 18 or over in the 'Facebook' row.

3 Divide the number of people who prefer Facebook who are 18 or over by the total number of people who prefer Facebook, and give the answer as a fraction, simplifying if possible.

c Identify that parts **a** and **b** are asking for information about different groups of people who were surveyed. Recall that $\Pr(A|B)$ is usually not equal to $\Pr(B|A)$.

WRITE

a total number of people 18 or over surveyed = 45

number of people 18 or over who prefer Facebook = 30

$$\Pr(\text{prefer Facebook}|\text{18 or over}) = \frac{30}{45} = \frac{2}{3}$$

b total number of people who prefer Facebook = 50

number of the people who prefer Facebook who are 18 or over = 30

$$\Pr(\text{18 and over}|\text{prefer Facebook}) = \frac{30}{50} = \frac{3}{5}$$

c The selections are made from different groups of people. In **a**, the selection is made from those aged 18 or over, while in **b**, the selection is made from those who prefer Facebook.

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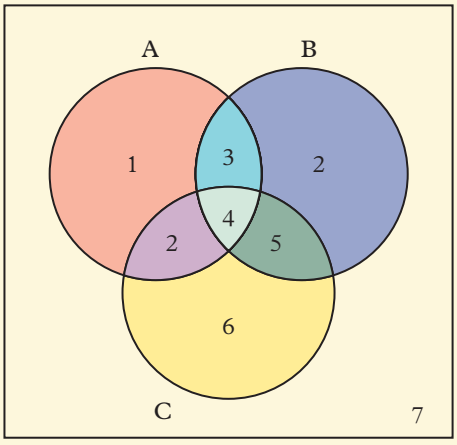
Worked example 11.4C Calculating conditional probability using a Venn diagram

This Venn diagram shows the distribution of a group of students who study Art (A), Business Studies (B) and Computing (C). Use the diagram to determine the relative frequency that a student studies:

a all three of these subjects, given that they study at least two of the subjects

b Art and Computing, given that they do not study Business Studies.



THINK

a **1** Identify the number of students in the 'given' set. This is the total in all the overlapping sections of the diagram.

2 Identify the number of students within this group who study all three subjects.

3 Express the two numbers you identified in parts **1** and **2** as a fraction, and simplify if possible.

b **1** Identify the number who do not study Business Studies. This is all those outside the circle B.

2 Identify the number of students within this group who study Art and Computing.

3 Express the two numbers you identified in parts **1** and **2** as a fraction, and simplify if possible.

WRITE

a number of students studying two or more subjects = 3 + 4 + 5 + 2

= 14

number of students studying all three subjects = 4

$\Pr(\text{all three subjects} \mid \text{two subjects}) = \frac{4}{14}$

$= \frac{2}{7}$

b number of students not studying Business Studies = 1 + 2 + 6 + 7

= 16

number of students in this group studying Art and Computing = 2

$\Pr(\text{Art and Computing} \mid \text{not Business Studies})$

$= \frac{2}{16}$

$= \frac{1}{8}$

Helpful hints

- ✔ When working with conditional probability, remember to determine the number of elements in the conditional (or given) set first.
- ✔ When calculating conditional probability, the probability of the condition always goes on the denominator.
- ✔ $\Pr(A \mid B)$ does not equal $\Pr(B \mid A)$. The formulas to calculate the two conditional probabilities have the same numerators, but different denominators.

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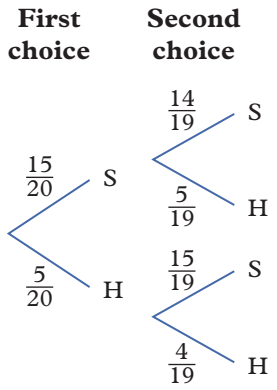
Learning pathways

1, 3–8, 10, 12, 14, 15 | 2, 3, 5–7, 9–11, 13, 14, 16, 18 | 3, 5, 6, 9, 11, 14, 16, 17, 19, 20

ANS p755

Exercise 11.4A: Understanding and fluency



1 A box of 20 chocolates contains 15 chocolates with soft centres and 5 chocolates with hard centres. Two chocolates are taken at random, one after the other. The tree diagram on the right represents the outcomes of the two choices.

a Explain the probabilities of $\frac{15}{20}$ and $\frac{5}{20}$ on the first two branches of the tree diagram.

b Explain the probabilities on the four branches for the four possible second choices.

c If the first chocolate has a hard centre, what is the probability that the second chocolate will also have a hard centre?

d Knowing that the first chocolate has a soft centre, what is the probability the second chocolate will have a hard centre?

e What is the probability that the second chocolate has a soft centre, given the first chocolate will did not?

WE 11.4A **2** A bag contains 6 balls: two black and four white. A ball is selected at random, its colour noted and then another ball is selected without the first ball being replaced.

a Draw a tree diagram to display this situation.

b If the first ball is white, what is the probability the second ball will also be white?

c Knowing that the first ball is black, what is the probability the second ball is white?

d What is the probability that the second ball is black, given that the first ball is white?

3 A box contains 7 balls: four green and three red. Three balls are selected without replacement.

a Draw a tree diagram to display this situation.

b If the first ball is green, what is the probability the second ball will also be green?

c Knowing that the first two balls are both green, what is the probability that the third ball will also be green?

d What is the probability that the third ball is green, given that a green ball and then a red ball were drawn in the first two selections?

WE 11.4B **4** Consider the two-way table from **Worked example 11.3A**. Use this two-way table to determine the relative frequency of a card being:

a not an Ace, given it is a heart

b not an Ace, given it is not a heart.

	Ace (A)	Not Ace (A′)	Total
Heart (H)	1	12	13
Not heart (H′)	3	36	39
Total	4	48	52

5 The two-way table on the right shows the modes of transport a group of students use to get to school. State the number of:

a primary school students who catch public transport to school

b high school students who are driven to school

c students who walk or ride to school

d students who are in high school.

	Primary school	High school	Total
Catch public transport	15	27	42
Are driven	22	18	40
Walk/ride	4	14	18
Total	41	59	100

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6 Using the table from question **5**, calculate the relative frequency of a student:

a catching public transport to school, given they are at primary school

b being at high school, given they are driven to school

c walking or riding to school

d being at high school

e being driven to school, given they are at high school.

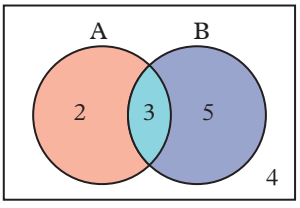
7 Consider the Venn diagram on the right.

a How many elements are in set B?

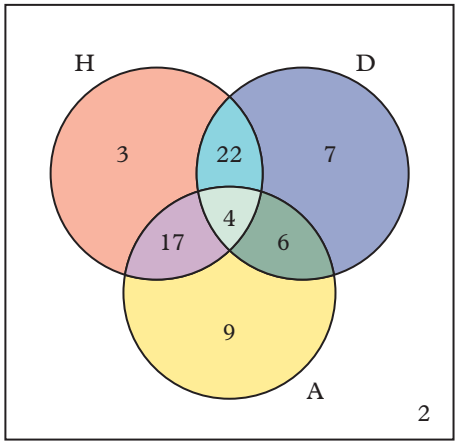
b How many of the elements in set B are also in set A?

c Determine Pr(A | B).

d Using similar reasoning, determine Pr(B | A).



WE 11.4C **8** The following Venn diagram shows the distribution of a group of 70 students who were asked if they enjoy watching horror films (H), drama films (D) and action films (A).



Use the Venn diagram to determine the relative frequency that a student enjoys:

a all three genres of film, given that they enjoy at least two of the film genres

b horror films, given that they enjoy action films.

9 Twenty members of the Swiss national soccer team were asked which languages they speak fluently out of German (G), French (F) and Italian (I). Their responses are shown in the Venn diagram on the right.

a One of the twenty players is selected at random. Find the probability that they speak:

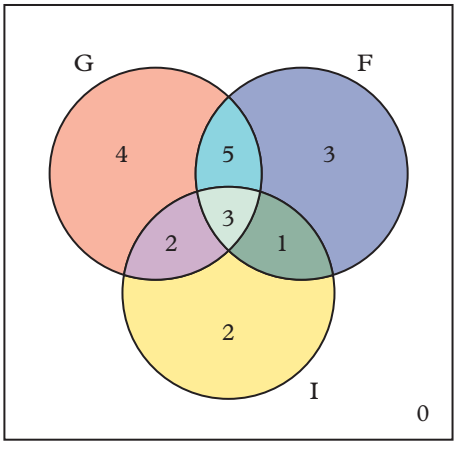
i German

ii German, but not French

iii German, given that they speak French

iv French, given that they speak German

v all three languages, given that they speak at least two of the languages.



b An Italian-speaking player is selected at random. Find the probability that this player can also speak French.

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10 For two events, A and B, it is known that $\Pr(A|B) = 0.4$, $\Pr(A \cap B) = 0.1$ and $\Pr(A) = 0.3$.

a Find $\Pr(B|A)$. Give your answer as a fraction.

b Find $\Pr(B)$. Give your answer as a decimal.

Exercise 11.4B: Problem solving and reasoning

11 A computer generates random questions for students to use for practice. From collected data, the chance that a student gets the first question correct is 0.8. If the student gets this question correct, the computer generates a harder question, with the probability of the student getting this one correct being reduced by 0.1. If the student gets the first question wrong, the computer gives the student a question of the same level of difficulty as the first question.

a Draw a tree diagram to display the possible outcomes of the two practice questions.

b Find the probability, expressed as a decimal, that the student gets the second question:

i correct, given they got the first question correct

ii correct, given they got the first question wrong

iii wrong, given they got the first question correct.

12 At a small firm with 50 employees, some work full-time and others work part-time. There are 15 employees who work from home (WFH) in total and 30 of the 34 full-time employees are hybrid workers who split their time between the office and at home.

a Draw a two-way table to record this data. Complete all the entries in the table.

b How many employees are:

i full-time employees who work from home?

ii part-time?

iii hybrid workers?

c Find the relative frequency of an employee:

i working part-time, given they are a hybrid worker

ii being a hybrid worker, given they work part-time

iii being a WFH employee, given they work full-time

iv working full-time, given they are a WFH employee.

d Explain the differences in your answers for the following parts of part **c**:

i parts **i** and **ii**

ii parts **iii** and **iv**.

13 The two-way table on the right shows the number and types of paper clips in a pile on a desk.

	Small	**Large**	**Total**
Silver	27	35	
Coloured	18	10	
Total			

a Copy and complete the totals for the rows and columns.

b Determine whether each of these statements is true, showing the calculations you used. (Percentages are rounded to the nearest whole number.) For those statements which are not true, show working to provide a correction.

i There is an equal number of small and large paper clips.

ii There are fewer coloured paper clips than silver ones.

iii The percentage of small paper clips which are silver is 44%.

iv The percentage of coloured paper clips which are large is 22%.

v The whole pile contains 69% silver paper clips.

14 The most common mistakes when working with conditional probability occur when the language used in a question is not correctly interpreted.

a Let A = 'able to dunk a basketball' and let B = 'over 6 foot tall' to explain why there is a difference between $\Pr(A)$ and $\Pr(A|B)$, using the word 'given' in your answer.

b Let A = 'exercises daily' and let B = 'is an Olympic athlete' to explain why there is a difference between $\Pr(A|B)$ and $\Pr(B|A)$, using the words 'if' and 'then' in your answer.

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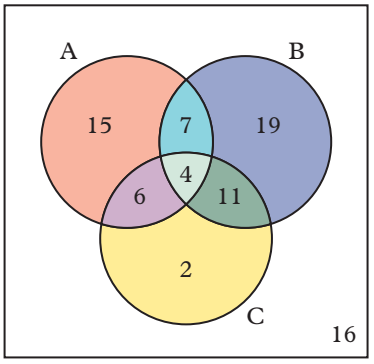
15 A jar contains black and white balls. Two balls are drawn without replacement. The probability of selecting a black ball on the first draw is 0.4, and the probability of selecting a second black ball, given that the first ball is black, is 0.375. What is the probability of selecting a white ball on the second draw, given that the first ball is a black one? Give your answer as a decimal.

16 Consider the Venn diagram on the right which shows survey data gathered from 80 people.



a Find each of the following.

i $n(\mathrm{A})$ **ii** $n(\mathrm{B})$ **iii** $n(\mathrm{C}')$

b One person from the 80 is selected at random. Find:

i $\Pr(\mathrm{B}|\mathrm{A})$ **ii** $\Pr(\mathrm{A}|\mathrm{B})$ **iii** $\Pr(\mathrm{A}'|\mathrm{C}')$.

c Explain how the answers to part **a** are helpful in answering part **b**.

17 For two events, X and Y, it is known that $\Pr(\mathrm{X}|\mathrm{Y}) = 0.2$ and $\Pr(\mathrm{Y}|\mathrm{X}) = 0.5$.

a Find:

i $\Pr(\mathrm{X}'|\mathrm{Y})$ **ii** $\Pr(\mathrm{Y}'|\mathrm{X})$.

b If $\Pr(\mathrm{X} \cap \mathrm{Y}) = 0.05$, then find $\Pr(\mathrm{X})$ and $\Pr(\mathrm{Y})$.

18 There are b blue marbles and r red marbles in a bag, and there are at least two marbles of each colour. Two marbles are drawn without replacement.

a Draw a tree diagram to represent this situation. Include all probabilities in terms of b and r on the branches.

b Find the probability that the second ball drawn is blue. Comment on this value.

Exercise 11.4C: Challenge

19 If an event, A, relates to the second step of a multi-step experiment, then it may be necessary to split A into cases in order to determine its probability. That is, $\Pr(\mathrm{A}) = \Pr(\mathrm{A} \cap \mathrm{B}) + \Pr(\mathrm{A} \cap \mathrm{B}')$, meaning that A can either occur with B or without B, where B relates to the first step of the experiment. Then, use the conditional probability formula, $\Pr(\mathrm{A}) = \Pr(\mathrm{B}) \times \Pr(\mathrm{A}|\mathrm{B}) + \Pr(\mathrm{B}') \times \Pr(\mathrm{A}|\mathrm{B}')$.

Imagine that a soccer team has a 40% chance of winning their upcoming game if it rains and a 70% chance of winning their upcoming game if it does not rain.

a Find the probability that the team wins their upcoming game if there is a 20% chance of rain. Give your answer as a decimal.

b Find the probability that it rains on the day of their upcoming game if their chance of winning the game is 0.5. Give your answer as a fraction.

20 Baye's rule is a useful conditional probability formula which states that $\Pr(\mathrm{A}|\mathrm{B}) = \dfrac{\Pr(\mathrm{B}|\mathrm{A}) \times \Pr(\mathrm{A})}{\Pr(\mathrm{B})}$.

Baye's rule can be proved using the conditional probability formulas for $\Pr(\mathrm{A}|\mathrm{B})$ and $\Pr(\mathrm{B}|\mathrm{A})$.

a Prove Baye's rule using $\Pr(\mathrm{A}|\mathrm{B}) = \dfrac{\Pr(\mathrm{A} \cap \mathrm{B})}{\Pr(\mathrm{B})}$ and $\Pr(\mathrm{B}|\mathrm{A}) = \dfrac{\Pr(\mathrm{A} \cap \mathrm{B})}{\Pr(\mathrm{A})}$.

b Two boxes each contain white balls and black balls. Box X contains 7 white and 3 black, whereas Box Y contains 2 white and 3 black. One box is chosen at random and then a ball is selected from it at random. Let X be the event 'selecting box X' and W be the event 'selecting a white ball'.

i Find $\Pr(\mathrm{W})$. Give your answer as a decimal.

ii Use Baye's rule to find $\Pr(\mathrm{X}|\mathrm{W})$, which is the probability that Box X was chosen, given that the selected ball is white. Give your answer as a fraction.

Online resources:

- **Interactive skillsheet** Conditional probability
- **Worksheet** Conditional probability
- **Investigation** Bertrand's box
- **Quick quiz** 11.4

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Lesson 11.5

Independence

Learning intentions

By the end of this lesson you will be able to ...

✔ determine whether two events are independent or dependent.

Inter-year links

Year 9	8.5 Two-way tables
Year 9	8.6 Venn diagrams

Independent events

Key content video
Independent events

- Two events are **independent** if one event occurring has no effect on the probability of the other event occurring.
- If A and B are independent events, then $\Pr(A|B) = \Pr(A)$, since the fact that B occurred does not change the probability of A.
  - → If A and B are independent events, it is also true that:
    $\Pr(B|A) = \Pr(B)$ and $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$
- The concept of independence can be used to determine if the different steps of a multi-step chance experiment are independent of each other.
  - → The outcomes for each step of a chance experiment *with replacement* are *independent*, as the probabilities for the second step are not dependent on the outcome of the first step.
  - → The outcomes for each step of a chance experiment *without replacement* are *not independent*, as the probabilities for the second step depend on the outcome of the first step.
- The concept of independence can also be used to determine if real-life events are independent or not; for example, if the weather outside has an effect on whether people take public transport on their commute to work.
- When A and B are independent events, then $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$.

Worked example 11.5A Testing for independence

For two events, A and B, it is known that $\Pr(A) = 0.6$, $\Pr(B) = 0.75$ and $\Pr(A \cap B) = 0.5$.

a Evaluate:

i $\Pr(A|B)$ **ii** $\Pr(B|A)$.

b Are A and B independent or dependent events?

THINK

a Use the conditional probability formula to evaluate each probability.

b Use your answers to part **a** to check if $\Pr(A|B)$ is equal to $\Pr(A)$, or if $\Pr(B|A)$ is equal to $\Pr(B)$. If yes, then the events are independent. If not, the events are dependent.

WRITE

a **i** $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)} = \dfrac{0.5}{0.75} = \dfrac{2}{3}$

ii $\Pr(B|A) = \dfrac{\Pr(A \cap B)}{\Pr(A)} = \dfrac{0.5}{0.6} = \dfrac{5}{6}$

b A and B are dependent events, since $\Pr(A|B)$ is not equal to $\Pr(A)$. Equivalently, $\Pr(B|A)$ is not equal to $\Pr(B)$.

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Worked example 11.5B Using a two-way table to determine whether two events are independent

The two-way table below shows whether people who prefer cats or dogs have a preference for reading or exercise.

	Cats	Dogs	Total
Reading	31	18	49
Exercise	16	35	51
Total	47	53	100

Are the events 'prefers cats' and 'prefers exercise' independent?

THINK

1. Calculate the probability of a person chosen at random preferring exercise.
2. Calculate the probability of a person chosen at random preferring exercise, given that they prefer cats.
3. Compare the two probabilities. If they are similar, the two events are independent. If there is a significant difference, the two events are dependent.

WRITE

$$\Pr(\text{exercise}) = \frac{51}{100} = 0.51$$

$$\Pr(\text{exercise} \mid \text{cats}) = \frac{16}{47} \approx 0.34$$

The events 'prefers cats' and 'prefers exercise' are dependent.

Helpful hints

- ✔ The equation $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$ is not a formula, but rather a test. This means that you can use the equation to test if A and B are independent. If you are told that A and B are independent, then you can use the equation as a formula. However, if you are not told that A and B are independent, then you can find $\Pr(A \cap B)$ using the addition rule formula or the conditional probability formulas, which are always true. These are:
  - → $\Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B)$
  - → $\Pr(A \cap B) = \Pr(B) \times \Pr(A \mid B)$
  - → $\Pr(A \cap B) = \Pr(A) \times \Pr(B \mid A)$
- ✔ Natural variation in real-life events means it can be difficult to prove independence, but you can state whether real-life events appear to be independent or not.

Learning pathways

1–7, 10, 12	1, 2, 4, 7–13	2, 4, 9–15

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ANS p757

Exercise 11.5A: Understanding and fluency

WE 11.5A **1** For two events, A and B, it is known that $\Pr(A) = 0.4$, $\Pr(B) = 0.5$ and $\Pr(A \cap B) = 0.2$.

a Evaluate:

i $\Pr(A|B)$ **ii** $\Pr(B|A)$.

b Are A and B independent or dependent events?

2 For two independent events, M and N, it is known that $\Pr(M) = \frac{2}{3}$ and $\Pr(N) = \frac{1}{4}$. Find each of the following.

a $\Pr(M|N)$ **b** $\Pr(N|M)$ **c** $\Pr(M \cap N)$

3 The two-way table on the right shows data gathered from 21 students about their preferred sport (out of cricket and football) and their preferred subject at school (out of Maths and English). Are the events 'prefers cricket' and 'prefers Maths' independent?

	Cricket	Football	Total
Maths	6	8	14
English	3	4	7
Total	9	12	21

WE 11.5B **4** The two-way table below shows data gathered from 40 people about their preferred ice cream flavour (out of chocolate, strawberry, vanilla) and their preferred chocolate variety (out of milk, dark, white).

	Chocolate	Strawberry	Vanilla	Total
Milk	8	7	10	25
Dark	4	2	1	7
White	3	0	5	8
Total	15	9	16	40

Decide whether the following pairs of events are independent or dependent.

a 'prefers vanilla' and 'prefers milk' **b** 'prefers chocolate' and 'prefers milk'

c 'prefers strawberry' and 'prefers dark' **d** 'prefers white' and 'prefers chocolate'

5 Consider the two-way table on the right.

	A	A′	Total
B	9	3	12
B′	6	2	8
Total	15	5	20

a Evaluate:

i $\Pr(A)$

ii $\Pr(A|B)$.

b Based on the results from part **a**, are A and B independent or dependent events?

6 Consider the two-way table on the right.

	X	X′	Total
Y	14	6	20
Y′	4	6	10
Total	18	12	30

a Evaluate:

i $\Pr(Y)$

ii $\Pr(Y|X)$.

b Based on the results from part **a**, are X and Y independent or dependent events?

7 For two events, A and B, it is known that $\Pr(A) = 0.6$ and $\Pr(B) = 0.8$. If A and B are independent events, then find:

a $\Pr(A \cap B)$ **b** $\Pr(A \cup B)$

8 Alexei is told that $\Pr(A) = 0.6$, $\Pr(B) = 0.7$ and $\Pr(A \cup B) = 0.8$. He is asked to find $\Pr(A \cap B)$ and calculates the answer to be 0.42.

a What was Alexei's mistake?

b What is the actual value of $\Pr(A \cap B)$, and how must it be calculated?

9 Explain the difference between independent events and mutually exclusive events.

10 In each case, decide if A and B are independent events or dependent events.

a $\Pr(A) = 0.4$, $\Pr(B) = 0.3$, $\Pr(A \cap B) = 0.2$ **b** $\Pr(A) = \frac{3}{5}$, $\Pr(B) = \frac{2}{3}$, $\Pr(A \cap B) = \frac{2}{5}$

c $\Pr(A) = 0.5$, $\Pr(B) = 0.8$, $\Pr(A \cap B') = 0.1$ **d** $\Pr(A) = 0.6$, $\Pr(B) = 0.5$, $\Pr(A|B) = 0.3$

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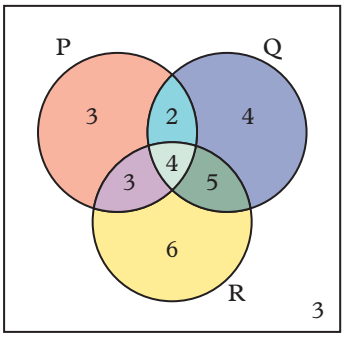
Exercise 11.5B: Problem solving and reasoning

11 Consider the Venn diagram on the right.
Which pairs of events are independent? Justify your answer with appropriate calculations.



12 Independent events in statistics are rare, due to random variations.
For example, outside of the Mathematics classroom we would say that someone's mode of transport to school and their favourite school subject are 'independent,' since it does not follow that knowing that a student takes a train to school would tell us anything about their favourite subject.
However, it does not follow that the events 'takes train' and 'prefers Maths' would be independent events in a survey of Year 10 students.
Consider the following data which shows the results of a survey from 30 students who were asked their preferred movie genre (out of action, comedy, drama) and whether their birthday occurs on an even calendar date or an odd calendar date.

	Action	**Comedy**	**Drama**	**Total**
Odd	7	5	2	14
Even	5	3	2	10
Total	12	8	4	24

a One student is selected at random. Find:
i Pr(Action) **ii** Pr(Action | Odd) **iii** Pr(Comedy) **iv** Pr(Comedy | Even).
b Explain why 'prefers action movies' and 'born on an odd date' are independent events in this survey.
c Explain why 'born on an even date' and 'prefers comedy movies' are dependent events in this survey.
d Explain why it does not make sense to claim that being born on an even date makes someone more or less likely to prefer comedy movies.

13 'Independent events' and 'independent trials' are often confused in probability. Independent trials are separate steps of a multi-step experiment where each step does not affect the probabilities at any other step. If the trials in a multi-step experiment are independent, then this will inevitably lead to an event relating to one trial being independent of an event relating to another trial.
Consider a bag containing six balls: 4 white and 2 black. Two balls are randomly selected from the bag.
Let W1 be the event 'the first ball is white' and W2 be the event 'the second ball is white'.
a Explain why W1 and W2 are independent events if the first ball is replaced before choosing the second ball.
b Find Pr(W2 | W1) if the first ball is:
i replaced after being selected **ii** not replaced after being selected.
c Find the probability that the two balls selected are white if they are:
i replaced after being selected **ii** not replaced after being selected.

Exercise 11.5C: Challenge

14 A and B are independent events with Pr(A) = Pr(B). Find Pr(A) if $\Pr(A \cup B) = 0.84$.

15 Venn diagrams are a useful visual tool to represent probabilities.
a Use Venn diagrams to show to that $\Pr(A) = \Pr(A \cap B) + \Pr(A \cap B')$.
b Prove that if A and B are independent events, then A and B′ are also independent.

Online resources:

Interactive skillsheet
Independence

Quick quiz
11.5

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Simulations

Counterintuitive results

Some results in probability can be counterintuitive. Simulations can be used to help convince us that these results are accurate.

A famous example of a counterintuitive result in probability is the birthday problem. The birthday problem asks us to find the probability that, in a group of n people, two or more of them will share a birthday (month and day, regardless of year). The counterintuitive result of the birthday problem is that a surprisingly small group size is required for a 50% chance that two or more people in the group share a birthday.

Curriculum links

- ... design and conduct simulations using digital tools to model conditional probability and interpret results. (VC2M10P01)
- Devise and use algorithms and simulations to solve mathematical problems. (VC2M10AA02) **10A**

Simulation

In simulating the birthday problem, we will make some assumptions for the sake of simplicity. First, let's assume that no one in the group will be born on February 29 (which is the extra day added to a leap year). Let's also assume that a person is equally likely to be born on any day of the year. More complex simulations can be done without making these assumptions, but the difference in results is negligible.

Instructions

Open a new Microsoft Excel document or a similar spreadsheet program.

Code	Explanation
In cell A1, type in '=RANDBETWEEN(1,365)' and press Enter.	This will randomly generate a whole number from 1 to 365 with equal chance. The numbers will represent birthdays, meaning 1 represents January 1st, 2 represents January 2 and so on until we have 365 representing December 31st.
Drag the bottom right corner of cell A1 down to cell A50.	This will simulate 50 birthdays.
In cell B1, type in '=COUNTIF(A:A,A1)' and press Enter.	This will count the number of times that the value in cell A1 appears in column A. If it appears more than once, it is a duplicate birthday.
Drag the bottom right corner of cell B1 down to B50.	This will check all entries in column A for duplicates.

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Code	Explanation
In cell D1, type in '=COUNTIF(B:B,">=2")' and press Enter.	This will check if there are any entries in column B which have a value of 2 or more. The value in D1 will therefore show the total number of people in the group who share a birthday with another member of the group. Your spreadsheet should look similar to the image below. A B C D 1 91 1 2 2 360 1 3 182 1 4 354 1 5 83 1 6 92 1 7 66 1 8 65 1 9 233 1 10 335 1 11 111 2 12 196 1

Questions

1 Explain why a group size of 366 ensures that the probability of two or more people in the group sharing a birthday is 1.

2 The values in column B help us determine if there are duplicate birthdays in the simulation. Explain why it is not possible to have exactly one number in column B which is greater than 1.

3 In your simulation of 50 birthdays, how many members in the group share a birthday with another member in the group? How can you tell?

You may notice that every time you execute a formula, the whole spreadsheet refreshes. This can also be done manually by pressing F9. Refreshing the spreadsheet means all the formulas are executed again and so all the randomly generated numbers are re-generated.

4 Press F9 many times, focusing on the value in cell D1. Is this number usually zero or greater than zero? What does this tell us about the likelihood of at least two people sharing a birthday in a group of 50 people?

5 Guess the group size which results in approximately a 50-50 chance of having at least two people in the group with the same birthday.

6 Reduce the number of entries in columns A and B until it seems like there is approximately a 50-50 chance of there being duplicate values in column A. This can be done by gradually deleting cells from columns A and B and then pressing F9 repeatedly, keeping note of the proportion of times the value in D1 is zero.

Write down your estimate and then search on the internet for the solution to see how close you were.

Extension

The birthday problem can be extended by investigating the probability that three or more people in a group will share the same birthday. This can be done by changing the formula in cell D1 to '=COUNTIF(B:B,">=3")'.

Using a similar method to question **6**, estimate the number of people required to be in the group before the probability of three or more sharing a birthday exceeds 50%. To add extra birthdays, highlight the last entries in columns A and B together and drag down the bottom right of this highlighted pair.

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Lesson 11.6

Permutations

Learning intentions

By the end of this lesson you will be able to ...

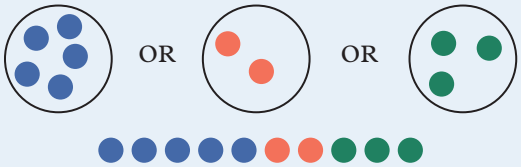
✔ apply the addition and multiplication principles to counting problems, and calculate probabilities involving permutations.

Inter-year links

Year 8	10.4 Theoretical probability
Year 9	8.1 Two-step chance experiments

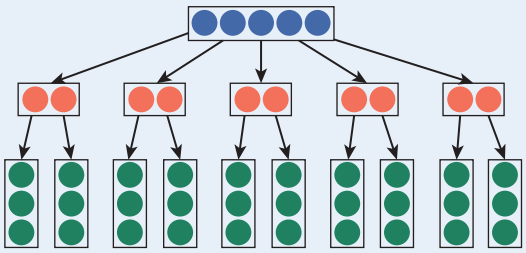
Counting principles

- If there are m elements in set A and n elements in set B, then there are $m + n$ elements to choose from when selecting one element from A or B. This is called the **addition principle**.



5 + 2 + 3 = 10 possibilities

- When choosing one element from set A (with m elements) and one element from set B (with n elements), there are $m \times n$ ways to select the two elements. This is called the **multiplication principle**.
- The addition and multiplication principles can be used together. For example, when choosing between 2 types of bread (white or multigrain) and between 3 spreads (jam, honey or butter), there are $(1 + 1) \times (1 + 1 + 1) = 2 \times 3 = 6$ choices in total.



5 × 2 × 3 = 30 possibilities

Permutations

Key content video
Permutations

- A **permutation** of a set is an ordering of the elements in the set. The number of permutations of a set depends on the number of elements in the set
- The number of permutations of a set with n elements is:

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1.$$

This value, $n!$, is read as 'n factorial'. It is found using the multiplication principle and removing one choice after each selection.

ABC BCA CAB
ACB BAC CBA

$3! = 3 \times 2 \times 1 = 6$

- When ordering r elements from a set with n elements, where $r \leq n$, the number of permutations is:

$$^{n}P_{r} = \frac{n!}{(n-r)!}.$$

This number, $^{n}P_{r}$, is read as 'n permute r', and is found by cancelling the last $(n - r)$ factors of $n!$, as only the first r factors are used when ordering r objects.

AB BC CA
AC BA CB

$^{3}P_{2} = \frac{3!}{(3-2)!} = \frac{6}{1} = 6$

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Worked example 11.6A Using the addition and multiplication principles

a How many choices of dish do you have if a menu has 5 chicken dishes, 4 beef dishes, 3 pork dishes, 2 seafood dishes and 3 vegetarian dishes?

b At a restaurant, a banquet consists of a soup, an entrée, a main and a dessert. If the menu lists 2 soups, 3 entrées, 5 mains and 2 desserts, how many different possible banquets are there?

THINK	WRITE
a Determine if each group of options is sequential (multiplication principle) or simultaneously available (addition principle). The options are simultaneously available. Apply the appropriate principle to determine the number of choices.	**a** $5 + 4 + 3 + 2 + 3 = 17$ choices
b Determine if each group of options is sequential (multiplication principle) or simultaneously available (addition principle). The options are sequential. Apply the appropriate principle to determine the number of choices.	**b** $2 \times 3 \times 5 \times 2 = 60$ choices

Worked example 11.6B Finding the number of permutations

a In how many ways can you arrange five toys in a row?

b In how many ways can you arrange seven playing cards in a row?

THINK	WRITE
a Evaluate the appropriate factorial.	**a** $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ ways
b Evaluate the appropriate factorial.	**b** $7! = 7 \times 6 \times 5! = 7 \times 6 \times 120 = 5040$ ways

Worked example 11.6C Permutations of a selection

In how many ways can three out of a total of eight toys be arranged in a row?

THINK	WRITE
1 Write down the factorial quotient to evaluate.	$^8P_3 = \frac{8!}{(8-3)!}$
2 Write the factorials in the numerator and denominators as their products. Cancel the common factors.	$= \frac{8!}{5!}$ $= \frac{8 \times 7 \times 6 \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}$
3 Evaluate the remaining product.	$= 8 \times 7 \times 6$ $= 336$ ways

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Worked example 11.6D Finding probabilities using permutations

What is the probability that an arrangement of four of the letters of the word COUNT begins with two vowels?

THINK	WRITE
1 Use the appropriate factorial quotient to find the number of ways to arrange four of the letters of the word COUNT.	${}^5P_4 = \frac{5!}{(5-4)!} = \frac{5!}{1!} = 120$
2 Draw four boxes to represent the positions of letters to arrange. Inside each box, write down the number of available choices for the corresponding position. If the first two letters are vowels, there are 2 options (O or U) for the first letter and 1 for the second letter. The third and fourth letters can be any of the 3 remaining letters. There are 3 options for the third letter and 2 for the fourth.	$\boxed{2}\ \boxed{1}\ \boxed{3}\ \boxed{2}$
3 Multiply the numbers in the boxes together to find the number of ways to arrange four letters of the word COUNT where the first two letters are vowels.	$2 \times 1 \times 3 \times 2 = 12$
4 Divide the number of arrangements beginning with two vowels by the total number of arrangements of four letters to find the desired probability.	$\Pr(\text{begins with two vowels}) = \frac{12}{120}$ $= \frac{1}{10}$

Helpful hints

- ✔ Contexts that involve the addition principle usually can be described in terms of 'or' as in 'this or that', while contexts that involve the multiplication principle usually can be described in terms of 'and' as in 'this and that'.
- ✔ Some calculators have a function for factorials and permutations that can be used to quickly evaluate both.
- ✔ The technique shown in **Worked example 11.6D** of drawing boxes to represent the number of positions and then filling in the number of possibilities at each position is a useful way to solve many problems involving permutations.

Learning pathways

1–8,10, 12, 13(a, b)	1(b, c), 3(c–f), 4(a,b), 5(c–f), 6–8, 9(a,b), 10–14	1(c, d), 3(e,f), 5(e,f), 6,7,9, 10–17

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Exercise 11.6A: Understanding and fluency

WE 11.6A **1** Answer the following using counting principles.

a How many choices for a dish do you have if a menu has 7 chicken dishes, 9 beef dishes, 5 pork dishes, 6 seafood dishes and 8 vegetarian dishes?

b How many choices of yoghurt brand do you have if a grocery store has 5 brands of vanilla yoghurt, 4 brands of strawberry yoghurt and 8 other brands of yoghurts?

c How many choices of game to buy do you have if there are 8 RPGs, 9 Metroidvania games, 5 party games, 4 online multiplayer games and 3 racing games?

d At a restaurant, a banquet consists of a soup, an entrée, a main and a dessert. If the menu lists 4 soups, 6 entrées, 3 mains and 3 desserts, how many choices of banquet are there?

e At the corner store, there are 8 different yoghurts, 7 different drinks and 10 different snacks on offer. In how many ways could you buy one of each item?

f There is a videogames sale online and you have decided to buy one of 8 RPGs, one of 9 Metroidvania games, one of 5 party games, one of 4 online multiplayer games and one of 3 racing games. In how many ways could you buy one of each type of game?

2 Evaluate the following factorials.

a $3!$ **b** $4!$ **c** $6!$ **d** $10!$

WE 11.6B **3** Answer the following permutation problems.

a In how many ways could a playlist of seven songs be played on shuffle?

b In how many ways could you arrange nine toys in a row?

c In how many ways could you arrange a hand of five playing cards in a row?

d Ten people compete in a 100-metre race. In how many ways can they place from first to tenth?

e In an octave there are twelve notes, each a semitone apart. In how many ways can they be played in a row?

f In a monster capture game, you have a party of six different monsters. In how many ways can they be ordered in your line-up?

4 Evaluate the following.

a ${}^{6}P_{3}$ **b** ${}^{10}P_{2}$ **c** ${}^{10}P_{5}$ **d** ${}^{7}P_{5}$ **e** ${}^{6}P_{5}$ **f** ${}^{30}P_{2}$

WE 11.6C **5** Answer the following permutation problems.

a In how many ways could six out of seven songs in a playlist be played on shuffle?

b In how many ways could you arrange four out of a total of nine toys in a row?

c In how many ways could you arrange three out of a total of five playing cards in a row?

d Ten people compete in a 100-metre race. In how many ways can the gold, silver and bronze medals be given to the first three people to complete the race?

e In an octave there are twelve notes, each a semitone apart. In how many ways can eight of them be played in a row?

f In a monster capture game, you have a party of six different monsters. In how many ways can four of them be ordered in your line-up?

6 A collection of symbols is made from the 26 uppercase and 26 lowercase English letters, the 24 uppercase and 24 lowercase Greek letters and the 10 Hindu–Arabic numerals. Determine the probability that:

a a randomly selected symbol is an English letter

b a randomly selected symbol is an uppercase letter

c a randomly selected symbol is a numeral or an English letter

d a randomly selected symbol is not a numeral

e a randomly selected pair of symbols is an English letter or a Greek letter in either order if each symbol is replaced before the next one is selected

f a randomly selected pair of symbols consists of two English letters if the first symbol is not replaced before the second symbol is selected.

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WE 11.6D **7** Find the probabilities of the following.

a an arrangement of four of the letters of the word SAMPLE beginning with two vowels

b an arrangement of four of the letters of the word PIECE beginning with two vowels

c an arrangement of five of the letters of the word PLEASANT beginning with two vowels

Exercise 11.6B: Problem solving and reasoning

8 Consider the following working out: $\frac{10!}{7!} = \frac{10 \times 9 \times 8 \times \cancel{7!}}{\cancel{7!}} = 10 \times 9 \times 8 = 720$.

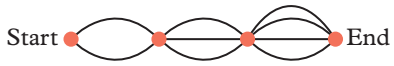
a Explain why we can write 10! as $10 \times 9 \times 8 \times 7!$ and how this helps to simplify the fraction.

b Use this method to evaluate the following:

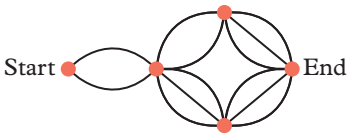
i $\frac{12!}{10!}$ **ii** $\frac{20!}{17!}$ **iii** $\frac{100!}{99!}$ **iv** ${}^{34}P_3$ **v** ${}^{51}P_2$ **vi** ${}^{1000}P_3$

9 Determine the number of ways to get from the start to the end of each network by only moving from left to right along the edges.

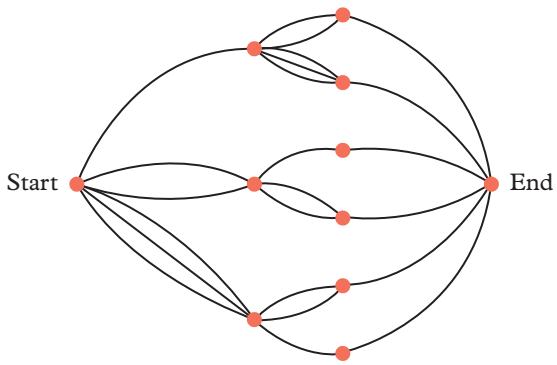
a



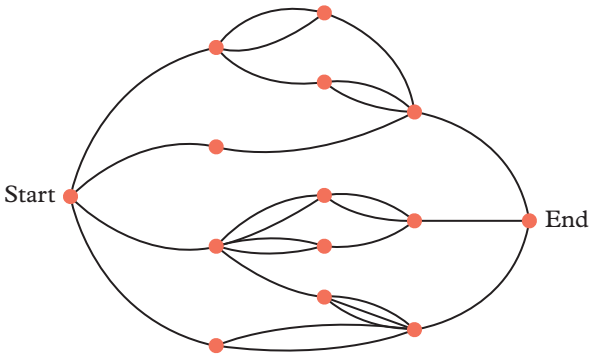
b



c

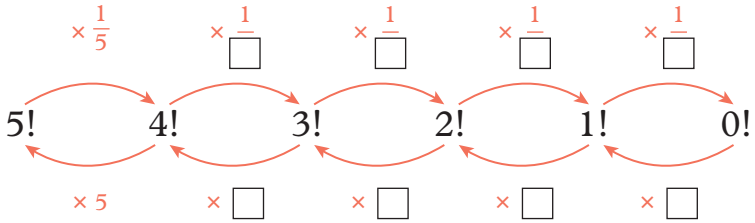


d



10 Consider the factorial of zero: 0!.

a Copy and complete the following.



b Use the pattern to evaluate 0!.

c Explain why it is important that 0! is this value in the case of ${}^{n}P_{n}$.

11 The number of permutations is reduced if some of the objects being arranged are indistinguishable. For example, when considering the permutations of the letters A, A and B, we cannot distinguish between AAB and AAB as being different without some external marker that tells you which A is which. So, we need to remove the repeated arrangements from the list of all possible arrangements. State the number of unique ways the following letters can be written in a row.

a AAB **b** AABC **c** AABB **d** AABBBCCCC

12 Five people (A, B, C, D, E) stand in a line for a group photograph in a random order.

a In how many different ways can the five people line up?

b What is the probability that A and B are standing in the first two positions (in either order)?

c What is the probability that C and D are standing in the third and fourth positions (in either order), given that A and B are standing in the first two positions?

d What is the probability that B and C are standing in the second and third positions (in either order), given that A is standing the first position?

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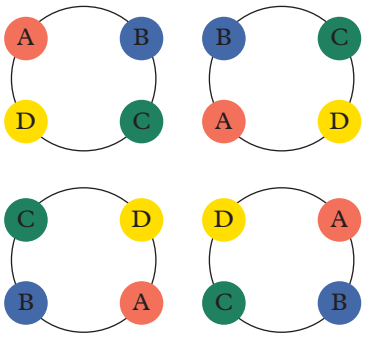
13 The number of permutations when arranging n objects in a loop is less than the number of permutations when arranging n objects in a row, as the loop does not have a distinct start and end. This means that the diagrams on the right all represent the same arrangement.



a In how many unique ways can the letters ABCD be placed in a loop?

b In how many unique ways can the letters ABCDE be placed in a loop?

c In how many unique ways can the letters ABCDEFGHI be placed in a loop?

d In how many unique ways can n different letters be placed in a loop?

14 Six people (A, B, C, D, E, F) are standing in a circle in a random order.

a In how many different ways can the people be standing in the circle?

b What is the probability that A and B are standing next to each other?

c What is the probability that C and D are standing next to each other, given that A and B are also standing next to each other?

d What is the probability that B and C are standing next to each other, given that neither of them are standing next to A?

15 A group of seven friends go out for dinner and sit around a circular table.

a In how many ways could they sit around the circular table?

b If there are three couples that want to sit next to their partners, in how many ways could they sit around the circular table?

c If everyone sits down at the table randomly, what is the probability that all the couples are seated next to their partner?

d The group want to take a photo together and decide to have four of them stand behind the other three.

i How many ways could they take the photo?

ii If the couples want to stand next to their partners, in how many ways could they take the photo?

iii If everyone takes a position randomly, what is the probability that all the couples are standing next to their partner?

Exercise 11.6C: Challenge

16 Simplify the following, where n is a natural number.

a $\dfrac{n!}{(n+1)!}$ **b** $\dfrac{(n+3)!}{(n-3)!}$ **c** $\dfrac{(n+3)!}{n!(n^2+3n+2)}$

d $\dfrac{1}{(n-1)!} - \dfrac{1}{n!}$ **e** $\dfrac{{}^{n+1}\mathrm{P}_n}{{}^{n}\mathrm{P}_{n-1}}, n \geq 1$ **f** $\dfrac{{}^{n+5}\mathrm{P}_n}{{}^{n}\mathrm{P}_{n-3}}, n \geq 3$

17 There exist variations of factorials that are products of descending sequences. An online calculator may be required to evaluate the following numbers with all of their digits.

a The double factorial, $n!!$, is the product of either all of the positive even or odd numbers less than or equal to n. For example, $9!! = 9 \times 7 \times 5 \times 3 \times 1$ and $8!! = 8 \times 6 \times 4 \times 2$. Evaluate the following.

i 9!! **ii** 8!! **iii** 15!!

b The primorial, $n\#$, is the product of all the prime numbers less than or equal to n. For example, $12\# = 11 \times 7 \times 5 \times 3 \times 2$. Evaluate the following.

i 12# **ii** 15# **iii** 20#

c The superfactorial, $n\$$, is the product of the first n factorials. For example, $5\$ = 5! \times 4! \times 3! \times 2! \times 1!$. Evaluate the following.

i 2$ **ii** 3$ **iii** 5$

Online resources:

Interactive skillsheet Counting principles

Interactive skillsheet Permutations

Quick quiz 11.6

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Combinations

Learning intentions

By the end of this lesson you will be able to ...

✔ use Pascal's triangle to evaluate combinations, and calculate probabilities involving combinations.

Inter-year links

Year 8	10.4 Theoretical probability
Year 9	8.3 Experiments without replacement

Subsets

- A subset is a set contained within another set. Subsets of a set include the null set or empty set, Ø, and the set itself.
  For example, the subsets of the set $\{a, b, c\}$ are Ø, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$ and $\{a, b, c\}$.
- There are 2^n subsets of a set containing n elements.
  For example, the set $\{a, b, c\}$ has 3 elements and therefore has $2^3 = 8$ different subsets.

Combinations

Key content video
Combinations

- A **combination** is a selection of the elements of a set where the order of the selection is not considered. Combinations are always selections made *without replacement*.
  For example, a hand of five playing cards from a pack of 52 cards is a combination because the order of the cards doesn't matter.
- All combinations are subsets of the set they are selected from.
- The number of combinations of r elements from a set of n elements is:

$$^{n}C_{r} = \frac{n!}{r!(n-r)!} = \frac{^{n}P_{r}}{r!}.$$

  This value, $^{n}C_{r}$, is read as 'n choose r'. It is found by dividing $^{n}P_{r}$ by the number of ways to permute the r selected objects.
- The addition and multiplication principles apply when considering combinations from more than one set.
- Pascal's triangle is a triangular diagram where the entry in row n position r is equal to $^{n}C_{r}$.
  - → The first row is labelled as row 0 and consists of only a 1.
  - → Numbers in subsequent rows are found by adding the two numbers diagonally above to the left and right.
  - → The first and last entry of every row is a 1.
  - → Entries in each row are numbered from 0 to n.
  - → The first seven rows, rows 0 to 6, are shown below.

Row 0							1						
Row 1						1		1					
Row 2					1		2		1				
Row 3				1		3		3		1			
Row 4			1		4		6		4		1		
Row 5		1		5		10		10		5		1	
Row 6	1		6		15		20		15		6		1
	$^{6}C_{0}$		$^{6}C_{1}$		$^{6}C_{2}$		$^{6}C_{3}$		$^{6}C_{4}$		$^{6}C_{5}$		$^{6}C_{6}$

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Worked example 11.7A Counting subsets

Determine the number of different subsets there are in a set with:

a 6 elements

b 10 elements.

THINK

a Evaluate two to the power of the number of elements in the set.

b Evaluate two to the power of the number of elements in the set.

WRITE

a $2^6 = 64$ subsets

b $2^{10} = 1024$ subsets

Worked example 11.7B Finding the number of combinations

In how many ways can you choose five out of a total of eight toys?

THINK

1. Write down the factorial quotient to evaluate.
2. Write the factorials in the numerator and denominators as their products. Cancel out the common factors.
3. Evaluate the remaining product.

WRITE

$${}^8C_5 = \frac{8!}{5! \times 3!}$$

$$= \frac{8 \times 7 \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1} \times \cancel{3} \times \cancel{2} \times \cancel{1}}$$

$$= 8 \times 7$$

$$= 56 \text{ ways}$$

Worked example 11.7C Using combinations and counting principles

A grocery store has five varieties of cola and five varieties of lemonade.

a If you need to buy three different types of cola and three different types of lemonade, how many different combinations of drinks could you buy?

b If you need to buy four different types of cola and two different types of lemonade, how many different combinations of drinks could you buy?

THINK

a 1. Use the multiplication rule to multiply the combinations for each type of drink.
2. Evaluate the product.

b 1. Use the multiplication rule to multiply the number of combinations for each type of drink.
2. Evaluate the product.

WRITE

a ${}^5C_3 \times {}^5C_3$

$= 10 \times 10$

$= 100$ combinations

b ${}^5C_4 \times {}^5C_2$

$= 5 \times 10$

$= 50$ combinations

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Helpful hints

- ✔ Ask yourself if order is important (permutations) or if order is not important (combinations) in the scenario. This may be implied rather than explicitly stated. For example, there are ${}^{25}C_2$ ways to choose two student leaders from a group of 25 students, but there are ${}^{25}P_2$ ways to choose a captain and then a vice-captain, since the order of selection matters in this case.
- ✔ Pascal's triangle can be used to quickly evaluate combinations without factorials.
- ✔ Some scientific calculators have a function which can be used to quickly evaluate ${}^{n}C_r$.

Learning pathways

1–4, 6–8, 10	1(a–c), 2(c–e), 3(a,b), 4(a–c), 5, 6, 7(a), 8–13	2(c–e), 4(d–f), 5(a,f), 6(a), 7(a), 8, 9, 10(a–c), 11–15

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Exercise 11.7A: Understanding and fluency

WE 11.7A **1** Determine the number of different subsets there are in a set with:

a 5 elements **b** 8 elements **c** 12 elements
d 15 elements **e** 20 elements **f** 30 elements.

WE 11.7B **2** Answer the following combination problems.

a In how many ways could you choose four out of a total of nine toys?
b How many sets of three cards out of a hand of five playing cards are there?
c Ten people compete in a 100-metre race. How many different groups of three people can take a podium place (gold, silver and bronze)?
d In an octave there are twelve notes, each a semitone apart. How many ways are there to choose eight different notes?
e In a monster capture game, you have a party of six different monsters. How many groups of four different monsters can be chosen?
f You have a playlist of twenty songs. How many ways are there to choose eight songs from the playlist?

3 Evaluate the following combinations.

a ${}^{6}C_3$ **b** ${}^{10}C_2$ **c** ${}^{10}C_5$ **d** ${}^{7}C_5$ **e** ${}^{6}C_5$ **f** ${}^{30}C_{28}$

WE 11.7C **4** Answer the following problems using combinations and counting principles.

a A grocery store has eight varieties of chips and seven varieties of dips. If you need to buy four different bags of chips and five different dips, how many different combinations could you buy?
b Savi is creating a playlist from two albums. She wants five tracks from the first album that has eight tracks and six tracks from the second album that has ten tracks. How many different playlists can she make if she only plays on shuffle (that is, she does not care about the order they are played in)?
c Brayden is packing clothes to take on holidays. He wants to pack three of his eight pairs of shorts, two of his five pairs of pants, and seven of his fifteen t-shirts. How many different sets of clothes could he bring on his holiday?
d Eleven players per team are needed on the field for a game of soccer. If a team wants eight of their twenty senior members on the team and three of their 10 junior members, how many different groups of members can be on the field together?
e Martine is planning to travel around Victoria and New South Wales. She has found a 'top ten places to visit' list for both states. She decides to visit six in Victoria and four in New South Wales. How many different combinations of places to visit are there?
f Xuehong is deciding which games to buy this year and has decided to buy three of eight RPGs, two of nine Metroidvania games and one of five party games. How many sets of games could Xuehong buy this year?

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5 Answer the following problems using counting techniques.

a In a game of five-card draw poker, you have the option to discard any number of the five cards you are dealt, including discarding none. How many different selections of cards can you discard?

b Shipra has ten items on her wish list and recently received her pay. How many different selections of the items on her wish list can she choose to buy, including opting to not buy any right now?

c In a deck-building card game, you build your deck as you play, purchasing cards using in-game resources. There are eight cards available for purchase this turn and any number of them can be purchased, or you can decline to buy any. Provided you have enough resources, how many different selections of the available cards could you choose to purchase?

d A game requires players to work as a team to remove as many of the nine jewels from a glass case as possible, while not alerting the opposing players who have their eyes shut and are counting to 30. How many different selections of jewels could be stolen in a round, including failing to steal any?

e There are six bags of groceries to bring in from the car. How many possible combinations of those bags could be brought in together first if at least one bag must be carried?

f Five pieces of chocolate are placed on a plate. How many ways are there to eat at least one chocolate on the plate?

6 Answer the following problems using counting techniques.

a An album with ten tracks is played on shuffle. Determine the probability that tracks 4, 5, 7 and 8 are played in any order, if four tracks are played in total.

b In an octave there are twelve notes, each a semitone apart. A random series of four notes in an octave are played. Determine the probability that the notes played are numbers 1, 5, 8 and 12 in any order.

7 Answer the following problems using counting techniques.

a A team of four players is needed to be selected from seven seniors and four juniors. Determine the probability that two seniors and two juniors are selected.

b A bag contains five red counters and eight blue counters. Determine the probability that when six counters are selected at random, four red and two blue counters are selected.

c A puzzle has eight '3-in-1-out' pieces, nine '2-in-2-out' pieces, and four '1-in-3-out' pieces left to be placed in the puzzle. Determine the probability that the next seven pieces to be placed in the puzzle are four '3-in-1-out', two '2-in-2-out' and one '1-in-3-out' in any order.

Exercise 11.7B: Problem solving and reasoning

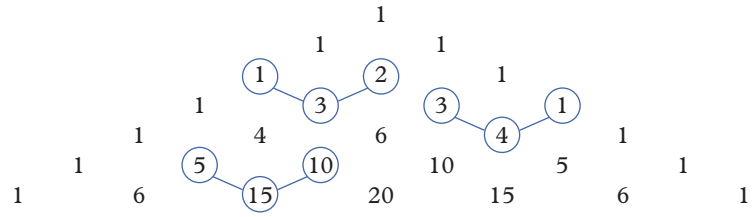
8 A netball team consists of six seniors and eight juniors. Seven players are randomly selected to start the game.

a Determine the probability that all the seniors are selected.

b Determine the probability that the order the players were selected alternated between junior and senior.

c Suppose the size of the team can vary. Determine the probability that a team of seven is selected out of all possible selections, given that a team must have at least 2 players.

9 Below are the first few rows of Pascal's triangle. The circled values are connected by addition.



a Write each of the circled values' sum in terms of the combinations they represent. Use the form ${}^{n}C_{r}$.

b Let the bottom circled value be represented by ${}^{n}C_{r}$. Write the equation of the similar type of sum to part **a** in terms of n and r. (This relationship is known as Pascal's identity, which is named after Blaise Pascal.)

c Use Pascal's identity to state the value ${}^{n}C_{r}$ which can be found given the following information.

i ${}^{10}C_{4}$ and ${}^{10}C_{5}$ **ii** ${}^{20}C_{15}$ and ${}^{20}C_{14}$ **iii** ${}^{16}C_{14}$ and ${}^{17}C_{14}$

iv ${}^{22}C_{3}$ and ${}^{21}C_{2}$ **v** ${}^{52}C_{29}$ and ${}^{53}C_{30}$ **vi** ${}^{99}C_{49}$ and ${}^{99}C_{48}$

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10 Use the given values to evaluate the following.

a ${}^{5}C_{3}$ given ${}^{5}P_{3} = 60$ and $3! = 6$

b ${}^{8}C_{4}$ given ${}^{8}P_{4} = 1680$ and $4! = 24$

c ${}^{10}C_{8}$ given ${}^{10}P_{8} = 1\,814\,400$ and $80! = 40\,320$

d ${}^{5}P_{4}$ given ${}^{5}C_{4} = 5$ and $4! = 24$

e ${}^{7}P_{5}$ given ${}^{7}C_{5} = 21$ and $5! = 120$

f ${}^{11}P_{3}$ given ${}^{11}C_{3} = 165$ and $3! = 6$

11 Consider sums of combinations in the form ${}^{n}C_{r}$.

a Evaluate the following sums.

i ${}^{1}C_{0} + {}^{1}C_{1}$

ii ${}^{2}C_{0} + {}^{2}C_{1} + {}^{2}C_{2}$

iii ${}^{3}C_{0} + {}^{3}C_{1} + {}^{3}C_{2} + {}^{3}C_{3}$

iv ${}^{4}C_{0} + {}^{4}C_{1} + {}^{4}C_{2} + {}^{4}C_{3} + {}^{4}C_{4}$

v ${}^{5}C_{0} + {}^{5}C_{1} + {}^{5}C_{2} + {}^{5}C_{3} + {}^{5}C_{4} + {}^{5}C_{5}$

b Describe the pattern you see in your answers to part **a**.

c Explain why this pattern holds by viewing combinations as subsets.

12 Complete the following steps to find the relationship between ${}^{n}C_{r}$ and ${}^{n}C_{n-r}$.

a Write down the first and last number of every row of Pascal's triangle.

b Hence, state the values of ${}^{n}C_{0}$ and ${}^{n}C_{n}$.

c Describe the connection between the row number (starting from row 0) and the second number from both ends of every row of Pascal's triangle.

d Hence, state the values of ${}^{n}C_{1}$ and ${}^{n}C_{n-1}$.

e Describe the relationship between ${}^{n}C_{r}$ and ${}^{n}C_{n-r}$ where $r \le n$.

13 The general form of Pascal's identity is ${}^{n}C_{r} = {}^{n-1}C_{r-1} + {}^{n-1}C_{r}$.

a Prove Pascal's identity using the definition ${}^{n}C_{r} = \dfrac{n!}{r!(n-r)!}$.

Hint: Write an expression for each term and factorise the expression on the right.

b Another way to prove Pascal's identity is to consider an element, x, from a set of size n.

i How many elements are there in the set, not including x?

ii How many ways are there to choose r elements from the set if x is one of the r elements? Explain.

iii How many ways are there to choose r elements from the set if x is not one of the r elements? Explain.

iv Use your previous answers to prove Pascal's identity.

Exercise 11.7C: Challenge

14 The nth triangular number is the sum of the first n natural numbers. For example, the fifth triangular number is $1 + 2 + 3 + 4 + 5 = 15$. Use Pascal's identity to explain why ${}^{n+1}C_{2}$ is a triangular number for $n \ge 1$.

15 Complete the following steps to be able to expand the expression $(x + 1)^{n}$, where n is a natural number.

a Expand $(x + 1)^{2}$.

b Expand $(x + 1)^{3}$ by expanding $(x + 1)(x + 1)^{2}$.

c Expand $(x + 1)^{4}$ by expanding $(x + 1)(x + 1)^{3}$.

d Describe what the values of the coefficients are related to.

e Use the idea in part **d** to:

i expand $(x + 1)^{7}$

ii determine the coefficient of the x^{4} term in the expansion of $(x + 1)^{20}$.

Online resources:

- **Interactive skillsheet** Subsets
- **Interactive skillsheet** Combinations
- **Investigation** Combinations and coefficients
- **Quick quiz** 11.7

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Lesson 11.8

Review: Probability

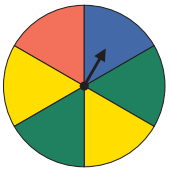
Module summary

Theoretical probability

- When all outcomes are equally likely:

$$\Pr(\text{event}) = \frac{\text{number of outcomes in event}}{\text{number of outcomes in the sample space}}$$

$$\Pr(\text{green}) = \frac{2}{6} = \frac{1}{3}$$



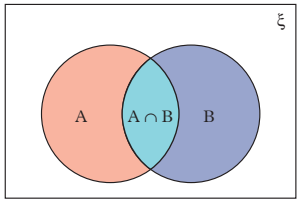
- The sum of probabilities of complementary events is 1:
  $\Pr(A) + \Pr(A') = 1$
- If two events are mutually exclusive then they cannot both occur at the same time.

Conditional probability

$\Pr(A|B)$ = the probability that event A occurs given that event B has occurred.

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} \qquad \Pr(A|B) \neq \Pr(B|A)$$

Venn diagrams



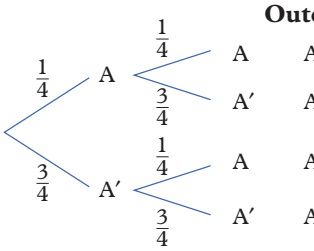
Independent events

- Two events are independent if the outcome of one event has no effect on the outcome of the other event.
- If A and B are independent events, then $\Pr(A|B) = \Pr(A)$.
- If A and B are independent events, then $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$.
- The outcomes for each step of a chance experiment *with replacement* are *independent*.
- The outcomes for each step of a chance experiment *without replacement* are *not independent*.

Set notation

- A set is a collection of distinct objects.
- ξ = the universal set containing all the elements under consideration
- $A \cap B$ = the intersection of two sets, A and B
- $A \cup B$ = the union of two sets, A and B
- A' = the complement of set A, everything from ξ that is not in A
- $n(A)$ = the numbers of elements in set A
- $A \subset B$ = 'set A is a subset of set B'

Tree diagrams



Outcomes	Probabilities
AA	$\frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$
AA′	$\frac{1}{4} \times \frac{3}{4} = \frac{3}{16}$
A′A	$\frac{3}{4} \times \frac{1}{4} = \frac{3}{16}$
A′A′	$\frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$

Arrays

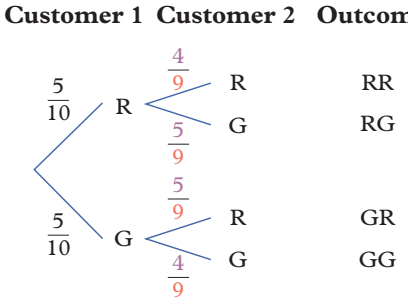
		Rolling a standard six-sided die					
		1	2	3	4	5	6
Flipping a coin	H	(H, 1)	(H, 2)	(H, 3)	(H, 4)	(H, 5)	(H, 6)
	T	(T, 1)	(T, 2)	(T, 3)	(T, 4)	(T, 5)	(T, 6)

Two-way tables

	School A	School B	Total
Musical instrument	10	4	14
No musical instrument	3	13	16
Total	13	17	30

Experiments without replacement

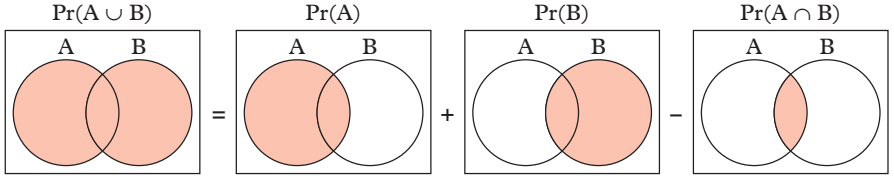
When multi-step experiments involve items being selected from a group, if the items are not replaced after each selection, then the probabilities will change.



Permutations and combinations

- The number of ways to arrange n distinct objects is:
  $n! = n \times (n-1) \times \ldots \times 2 \times 1.$
- The number of ways of arranging r objects from n distinct objects is:
  $${}^{n}P_{r} = \frac{n!}{(n-r)!}$$
- The number of ways of choosing r objects from n distinct objects is:
  $${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

Additional rule of probability



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Review questions 11.8A: Mathematical literacy review

Review quiz
Take the review quiz to check your knowledge of this module.
✔ 10
✔ 10A

Quizlet
Test your knowledge of this module by working individually or in teams.

The following key terms are used in this module:

- addition principle
- addition rule for probability
- array
- combination
- complement
- conditional probability
- element
- event
- experiment
- experiments with replacement
- experiments without replacement
- independent events
- intersection
- multiplication principle
- mutually exclusive events
- outcome
- Pascal's triangle
- permutation
- sample space
- set
- subset
- tree diagram
- two-way table
- union
- universal set
- Venn diagram

1 How are the different steps in a multi-step experiment displayed in a tree diagram?

2 Explain why the outcomes for each step of a multi-step experiment with replacement are independent.

10A **3** Explain the multiplication principle.

4 Identify the key terms being referenced in each of these definitions.

a one of the possible outcomes or a set of possible outcomes for an experiment

b the likelihood of an event occurring based on the occurrence of a previous event

5 Using an example, provide a definition in your own words for the following key terms.

a mutually exclusive events

b intersection

6 Complete the following sentences using words from the key terms list.

a The _____ of an event A is the event that A does not occur.

10A **b** A _____ of a set is an ordering of the elements in the set.

Review questions 11.8B: Multiple choice

11.1 **1** A coin is flipped three times. How many outcomes would appear on a tree diagram representing this experiment?

A 2 **B** 4 **C** 6 **D** 8

11.1 **2** A letter is randomly selected from a word. The sample space for the experiment is {A, E, L, P}. Which of the following could be the word?

A HELP **B** PLEASE **C** PLEAD **D** APPLE

11.2 **3** A standard deck of cards contains 52 cards, with 13 cards of each of four suits (clubs, diamonds, hearts and spades). One card is drawn from a deck of playing cards and its suit is noted. It is replaced, then a second card is drawn. What is the probability two diamonds are drawn?

A $\frac{1}{4}$ **B** $\frac{1}{13}$ **C** $\frac{1}{16}$ **D** $\frac{1}{17}$

11.2 **4** Repeat the experiment from question **3**, but this time, assume the first card is not replaced before the second card is drawn. What is the probability that two diamonds are drawn?

A $\frac{1}{4}$ **B** $\frac{1}{13}$ **C** $\frac{1}{16}$ **D** $\frac{1}{17}$

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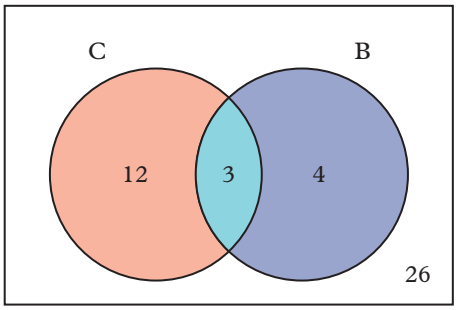
11.3 **5** The following Venn diagram shows the number of students who play chess (C) and badminton (B).



One of these students is selected at random. What is the probability that they play chess?

A $\frac{1}{15}$ **B** $\frac{12}{19}$ **C** $\frac{1}{3}$ **D** $\frac{4}{15}$

11.3 **6** Based on the following incomplete two-way table, what is Pr(A)?

	A	**A′**	**Total**
B	12		24
B′		16	
Total			50

A 0.44 **B** 0.5 **C** $\frac{6}{11}$ **D** $\frac{14}{25}$

11.4 **7** Based on the Venn diagram in question **5**, what is the probability that a student plays badminton, given that they also play chess?

A $\frac{1}{3}$ **B** $\frac{7}{45}$ **C** $\frac{1}{5}$ **D** $\frac{3}{7}$

11.4 **8** For two events, A and B, it is known that $\Pr(A) = 0.45$, $\Pr(B) = 0.6$ and $\Pr(A \cup B) = 0.85$. What is $\Pr(A|B)$?

A $\frac{1}{3}$ **B** $\frac{1}{4}$ **C** $\frac{1}{5}$ **D** $\frac{2}{3}$

11.5 **9** If A and B are independent events with $\Pr(A) = 0.7$ and $\Pr(B) = 0.4$, then what is $\Pr(A \cup B)$?

A 0.28 **B** 0.3 **C** 0.55 **D** 0.82

11.5 **10** The following two-way table shows the data gathered from 50 repetitions of a scientific experiment. Which pair of events is independent?

	Cold	**Warm**	**Hot**	**Total**
Success	12	8	0	20
Inconclusive	4	5	1	10
Failure	3	12	5	20
Total	19	25	6	50

A 'Cold' and 'Success'
B 'Warm' and 'Inconclusive'
C 'Hot' and 'Failure'
D 'Hot' and 'Success'

10A 11.6 **11** In how many ways could three out of a total of seven playing cards be arranged in a row?

A 7! **B** 3! **C** $\frac{7!}{3!}$ **D** $\frac{7!}{4!}$

10A 11.7 **12** How many ways are there to choose 10 student representatives if there are 120 students in the year level?

A $\frac{120!}{110!}$ **B** $\frac{120!}{110!10!}$ **C** 120! **D** $\frac{120!}{10!}$

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Review questions 11.8C: Short answer

11.1 **1** A ten-sided die (with sides numbered 1–10) is rolled and the number facing up is recorded.

a List the sample space.

b If A is the event that the number facing up is odd and B is the event that the number facing up is a multiple of 3, then list the elements in the set:

i A **ii** B **iii** $A \cap B$ **iv** $A \cup B$ **v** $A \cap B'$.

11.1 **2** A bag contains the 26 letters of the alphabet. One letter is randomly chosen, whether it's a vowel or a consonant is recorded, and then the letter is replaced. This is repeated another two times. Draw a tree diagram to find the probability of choosing:

a no vowels **b** at least one vowel **c** three consonants **d** no more than two consonants.

11.2 **3** A three-digit number is made using the digits 0 to 9. The digits can be repeated.

a Describe any restrictions on the formation of such a three-digit number.

b How many three-digit numbers are possible?

c What is the probability the number will be 111?

d What is the probability that all the digits in the number will be the same?

11.2 **4** The experiment in question **3** is repeated, this time not allowing any repetition of the digits.

a Describe any restrictions on the formation of the number

b How many numbers are possible in this case?

c What is the probability the number will be 107?

d How many numbers with a value greater than 500 are possible?

11.3 **5** Sets A, B and C overlap, and can be described as follows:

- $n(A) = 14$
- $n(B) = 13$
- $n(C) = 15$
- $n(A \cap B) = 6$
- $n(B \cap C) = 5$
- $n(A \cap C) = 7$
- $n(A \cap B \cap C) = 4$

Draw a Venn diagram to display this information.

11.3 **6** The following two-way table shows the number of iPhone users in a group of students.

	Year 8	Year 10	Total
iPhone	20		
Other type of phone		12	
Total	35		82

State the number of:

a Year 8 students who do not use an iPhone

b students who do not use an iPhone

c Year 10 students

d students who use an iPhone.

11.4 **7** Two pens are taken from a box containing four blue pens, five black pens and two red pens. Calculate the probability that:

a the second pen is red, given the first pen is blue **b** the second pen is red, given the first pen is black

c the two pens are the same colour **d** the two pens are different colours.

11.4 **8** A letter is chosen at random from the word CONDITIONAL. Find the probability that the chosen letter is:

a C **b** N

c O, given that it is a vowel **d** A vowel, given that it is O

e A vowel, given that it is adjacent to an N.

11.5 **9** For two events, A and B, it is known that $\Pr(A) = \frac{1}{6}$ and $\Pr(A \cap B) = \frac{1}{12}$. If A and B are independent events, then find:

a $\Pr(A|B)$ **b** $\Pr(B)$ **c** $\Pr(A \cup B)$

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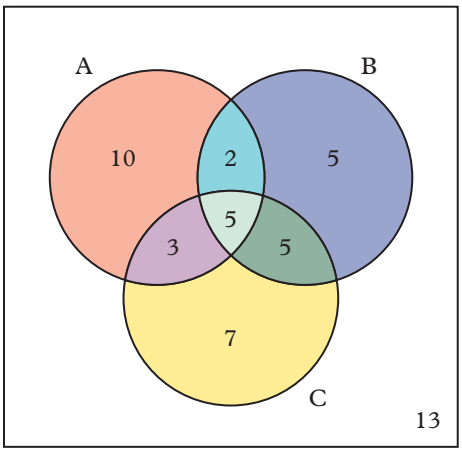
11.5 **10** Consider the following Venn diagram.



a Calculate the probability of:

i A **ii** $A \cap B$ **iii** $A \cup B$ **iv** $B' \cap C$.

b Which pair of events is independent? Justify your answer.

10A 11.6 **11** What is the probability that:

a an arrangement of six of the letters of the word PROBABILITY begins with a vowel

b an arrangement of six of the letters of the word PROBABILITY begins with four vowels in any order

c an arrangement of six of the letters of the word PROBABILITY begins with four vowels in alphabetical order (AIIO)?

10A 11.7 **12** In a cricket match, there are eleven players on each team. The team consists of the captain plus a random combination of ten of the eight senior players and seven junior players from the cricket club.

a Determine the probability that four junior players and six senior players are selected.

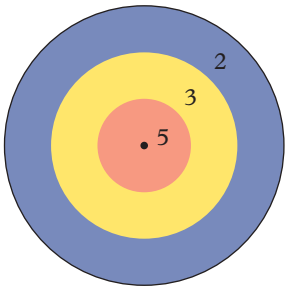
b Determine the probability that all seven junior players are selected.

c What are the restrictions on the numbers of junior and senior players that could make up the team?

d What is the most likely number of senior players in the team out of all the possible team selections?

Review questions 11.8D: Mathematical modelling

1 An archery target consists of an inner circle of radius 10 cm, encircled by two rings, each with width 10 cm. An arrow which lands in the outermost ring is worth 2 points, an arrow which lands in the middle ring is worth 3 points, and an arrow which lands in the inner circle is worth 5 points, as shown in the diagram.



An archer shoots two arrows, both of which hit this target.

a Find the fraction of the target area that is taken up by the region worth:

i 5 points **ii** 3 points **iii** 2 points.

b Write the sample space for the sum of the two scores from the archer's two shots.

c Assume that each arrow has an equally likely chance of hitting anywhere on the target. Find the probability that, from their two shots, the archer scores:

i 10 points **ii** exactly 5 points **iii** at least 8 points.

d It is not realistic to assume that the archer has an equally likely chance to hit any point on the target. Instead, assume that, for each shot, the archer has a 20% chance of hitting the outermost ring, a 50% chance of hitting the middle ring, and a 30% chance of hitting the inner circle.

Find the probability that, from their two shots, the archer scores:

i 10 points **ii** exactly 6 points **iii** no more than 5 points.

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e It is reasonable to think that the number of points the archer scores on the second shot is dependent on their score from the first shot. Assume that the archer learns from their first shot and will therefore score at least as many points on the second shot as they did on their first shot.

If the archer scores 2 points on their first shot, then the probabilities for their second shot remain 20%, 50%, and 30% for scoring 2, 3, or 5 points, respectively. However, if the archer scores 3 points on their first shot, then they have an even chance of scoring 3 or 5 points on their next shot. If the archer scores 5 points on their first shot, then they will score 5 points again on their second shot.

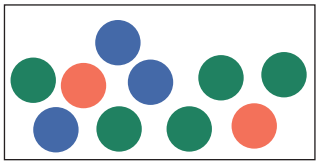
Under these conditions, find the probability that, from their two shots, the archer scores:

i 10 points **ii** exactly 8 points **iii** at least 6 points.

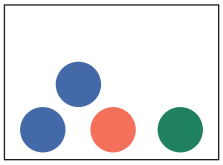
2 Two boxes each contain balls.

- Box A contains 5 green, 2 red and 3 blue balls.
- Box B contains 1 green, 1 red, and 2 blue balls.

Box A



Box B



a If one box is chosen at random and then one ball is chosen at random from it, find the probability that:

i Box B was chosen

ii the ball is green

iii the ball is green and was chosen from Box B

iv the ball is green or Box B was chosen.

b If two balls are randomly selected, one from each box, then find the probability that:

i both balls are red

ii at least one red ball is chosen

iii two red balls are chosen, given that at least one red ball was chosen.

c If one box is chosen at random and then two balls are chosen from that box (without replacement), find the probability that:

i both balls are blue

ii both balls are blue, given that box B was chosen

iii box B was chosen, given that both balls are blue.

10A **d** Both boxes are emptied onto a table and the balls are jumbled. The same number of each colour ball as there were originally are randomly placed back into Boxes A and B.

How many possible combinations of each colour ball are there that could have been placed into Box A?

10A **e** Find the probability that:

i the green balls in Box A are the same green balls which were originally in the box

ii the red balls in Box A are the same red balls which were originally in the box

iii the blue balls in Box A are the same blue balls which were originally in the box

iv all of the balls in Box A are the same as they were originally.

10A **f** Repeat parts **d** and **e** with Box B.

10A **g** Comment on your answer to part **f**.

10A **h** Find the probability that all three sets of coloured balls in Boxes A and B are different to what they were originally. Why is this the most likely outcome?

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Checklist

Now that you have completed this module, reflect on your ability to do the following.

I can do this	I need to review this
☐ Calculate probabilities of experiments with or without equally likely outcomes ☐ Calculate theoretical probabilities using tree diagrams ☐ Calculate theoretical probabilities using arrays	☐ Go back to Lesson 11.1 Theoretical probability
☐ Calculate probabilities of multi-step experiments with and without replacement	☐ Go back to Lesson 11.2 Experiments with and without replacement
☐ Understand and use set notation ☐ Convert two-way tables to Venn diagrams ☐ Convert Venn diagrams to two-way tables ☐ Use the addition rule for probability	☐ Go back to Lesson 11.3 Two-way tables and Venn diagrams
☐ Understand conditional statements ☐ Calculate conditional probability using tree diagrams ☐ Calculate conditional probability using two-way tables ☐ Calculate conditional probability using Venn diagrams	☐ Go back to Lesson 11.4 Conditional probability
☐ Understand the difference between independent and dependent events ☐ Calculate the probability of independent and dependent events	☐ Go back to Lesson 11.5 Independence
☐ Understand and apply the addition and multiplication principles to counting problems ☐ Evaluate factorials ☐ Apply factorial notation to counting problems involving permutations ☐ Apply counting principles and permutations to simplify probability calculations	☐ Go back to Lesson 11.6 Permutations [10A]
☐ Apply factorial notation to counting problems involving combinations ☐ Understand Pascal's triangle and how to construct it ☐ Apply Pascal's triangle to evaluate combinations ☐ Apply counting principles and combinations to simplify probability calculations	☐ Go back to Lesson 11.7 Combinations [10A]

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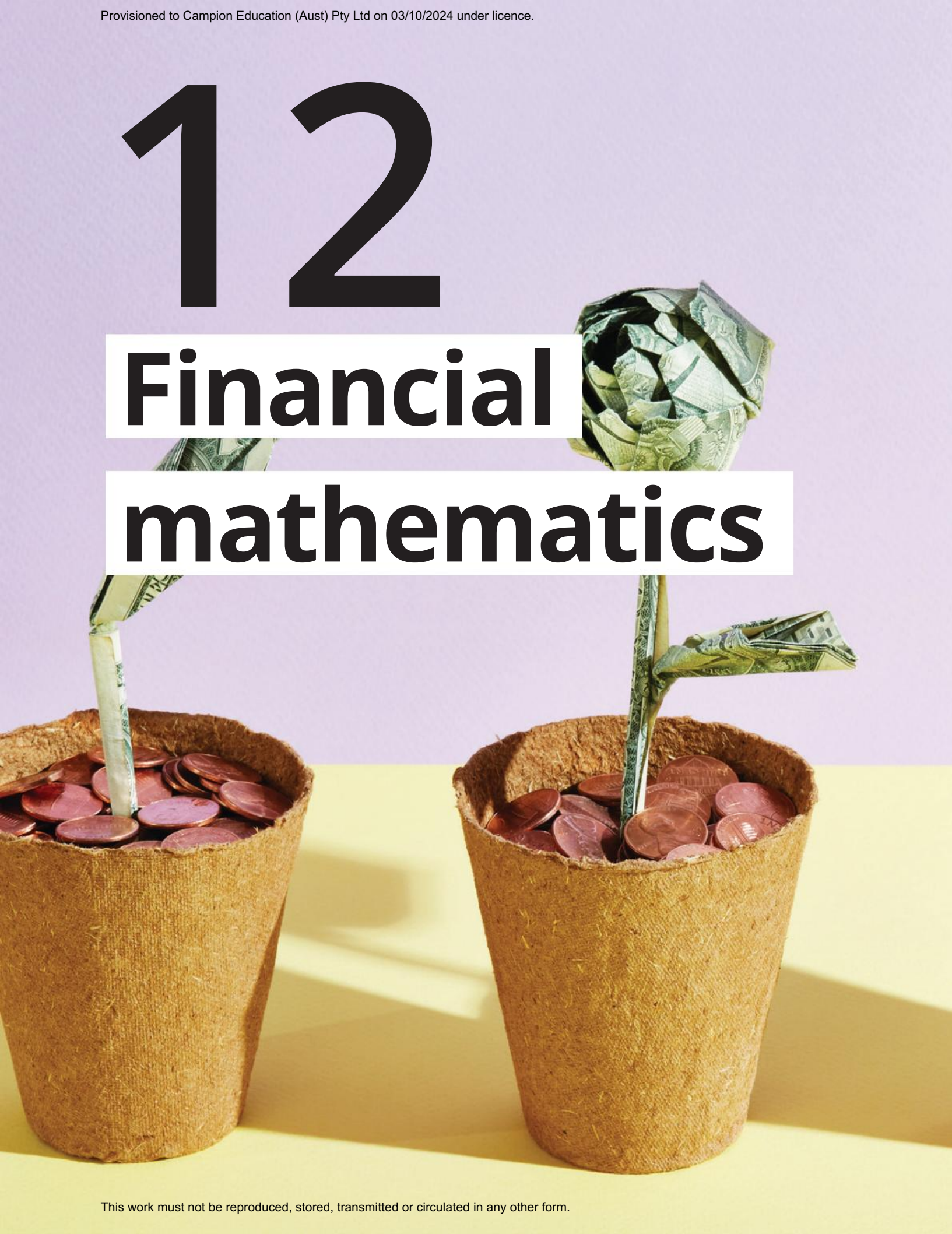
12 Financial mathematics



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oxforddigital **This module is available on Oxford Digital.**

Lessons

12.1 Earning money
12.2 Simple interest
12.3 Compound interest
12.4 Compound interest calculations
12.5 Depreciation

Prerequisite skills

Diagnostic pre-test
Take the diagnostic pre-test to assess your knowledge of the prerequisite skills listed below.

Interactive skillsheets
After completing the diagnostic pre-test, brush up on your knowledge of the prerequisite skills by using the interactive skillsheets.

- ✔ Substitution
- ✔ Multiplying and dividing decimals
- ✔ Rounding decimals
- ✔ Calculating percentages
- ✔ Rates

Curriculum links

- Use mathematical modelling to solve applied problems involving inverse proportion, growth and decay, including in financial contexts to establish the compound interest formula as repeated applications of simple interest; formulate problems, choosing to apply linear, quadratic or exponential models; interpret solutions in terms of the situation; evaluate and modify models as necessary and report assumptions, methods and findings (VC2M10A15)

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Materials

- ✔ Calculator

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Semester 2 review

Short answer

oxforddigital These questions are available on Oxford Digital.

ANS p759

Mathematical modelling

1 Surjan recently received a $1000 bonus and is considering two investments. They are listed below.

- **Investment A:** A simple interest rate of 7% per annum
- **Investment B:** A compound interest rate of 5% per annum, compounding annually

a If Surjan plans to invest his $1000 for 10 years, which investment should he choose?

b Write the rule that gives the value, V_A and V_B, of each investment after n years.

c Complete the following table of values to show the performance of each investment. Write each value correct to the nearest cent.

n	0	5	10	15	20
V_A					
V_B					

d Use the values from your completed table to help you sketch graphs on the same set of axes showing the value of each investment against the number of years, for $0 \le n \le 20$.

e Use the graphs you drew for part **d** to help you determine after which year Investment B would be worth more than Investment A.

f After how many years will Investment A be worth more than Investment B is at 5 years?

g Surjan is offered an alternative investment, described below.

Investment C: Simple interest rate of 5% per annum, with $150 added to the account when it is first opened.

i After how many years would Investment A and Investment C be worth the same amount?

ii What would be the value of each investment at that time?

h After how many years would Investment B be worth the same amount as Investment A was worth after 5 years? Write your answer:

i as an exact value

ii correct to two decimal places.

2 Erica is planning a hike through a local national park. She plans to:

- drive to the carpark, at point P
- then walk on a bearing of 036.9°T for 10 km towards the lower bridge, at point L
- then walk on a bearing of 108.4°T for 6.3 km towards the upper bridge, at point U
- then walk on a bearing of 233.1°T for 10 km towards the new bridge, at point N.

From the new bridge there is then a 4 km track due west back to the carpark.

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a Draw a diagram to show Erica's hike plan.

b Determine the length of the planned hike.

c What is the average minimum speed Erica must walk to make it back to her car in 6 hours?

d Erica reaches the upper bridge and sees a warning sign that says it is a 50 m drop to the ground below. From the upper bridge, Erica can see a kangaroo on the ground at an angle of depression of 12°. If Erica's eye height is 172 cm, determine the horizontal distance to the kangaroo, correct to the nearest metre.

e The park has multiple tracks that can be taken from the carpark. The tracks vary in difficulty. Hikers often mark which way they went on a signpost, and this has allowed the park rangers to determine the following:

- From point P, 20% of hikers walk to point R and 80% walk to point L.
- From point L, 5% of hikers walk to point U and 95% walk to point A.
- From point R, 10% of hikers walk to point U and 90% walk to point A.
- They all then follow tracks that lead to point N.

Determine the probability that a randomly selected hiker took the path:

i through points P, R, U and N

ii through points P, L, A and N

iii through points P, R, A and N

iv through points P, L, U and N, given they walked to point U.

f Draw a network that represents the paths between the points P, R, L, U, A and N.

g Consider the network drawn in part **f**.

i Is the network planar?

ii Why is there no Eulerian circuit in the network? Is there an Eulerian trail?

iii Suppose there is a path from all points to N. Is the corresponding network planar?

3 A teacher decided to retest the class two weeks after a recent test to see how much the students were able to remember. The scores the 24 students scored on the two tests are given in the table below.

The teacher hoped to be able to predict the retest scores from the test scores.

Test score (%)	95	90	90	85	85	80	80	80	75	75	75	70	70	70	65	65	65	60	60	60	55	50	40
Retest score (%)	80	85	70	50	65	65	50	90	65	60	40	80	60	75	60	65	35	35	70	50	50	30	30

a State the independent variable.

b Construct a scatterplot of the scores from the retest (vertical axis) against the scores from the original test (horizontal axis).

c Describe the correlation between the scores for the retest and the scores for the original test.

10A **d** Use technology to find a line of good fit (least-squares regression line) for this data. Round the values in the equation to two decimal places.

10A **e** If a student scored 70% for the original test, use the line of best fit to predict the student's score for the retest. State whether or not each prediction is reliable.

10A **f** The teacher claims that students who scored 75% or more originally did better on the retest than students who scored less than 75%. Evaluate the teacher's claim.

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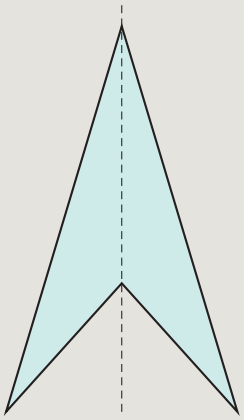
ANS p759

EXPLORATIONS 2

1 Thorn polygons

A *thorn polygon* is a shape that has at least one axis of symmetry and the maximum possible number of interior reflex angles. The symmetrical dart shown is an example of a thorn quadrilateral.



a How many reflex angles can a triangle have? What is another name for a thorn triangle?

b There are two different types of thorn pentagon: one with the reflex angles adjacent and one with them separated. Draw an example of each type.

c Explain why every thorn heptagon has at least one pair of adjacent reflex angles.

d Explain why no thorn octagon can have two obtuse angles.

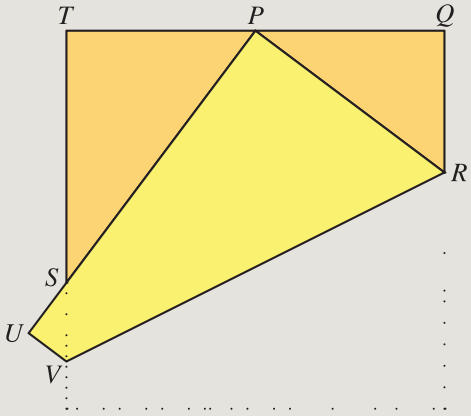
e Show that an n-sided thorn polygon has exactly $n - 3$ reflex angles.
Hint: This requires two steps: 1) show that a symmetrical n-sided polygon with $n - 3$ reflex angles always exists, and 2) show that more than $n - 3$ reflex angles is impossible.

f Find the maximum number of sides in a thorn polygon in which no three consecutive angles are all reflex angles.

g Show that, for every $n \geq 3$, there exists an n-sided thorn polygon that can be bisected along its line of symmetry to form two congruent thorn polygons.

2 Know when to fold 'em

A square piece of origami paper has side a length of 1 unit. The bottom-right corner is folded to a point P on the top edge, as shown, creating triangles PQR, PTS and SUV.



a Suppose P is at the midpoint of the top edge of the square.

i Set up an equation to find the length of QR.

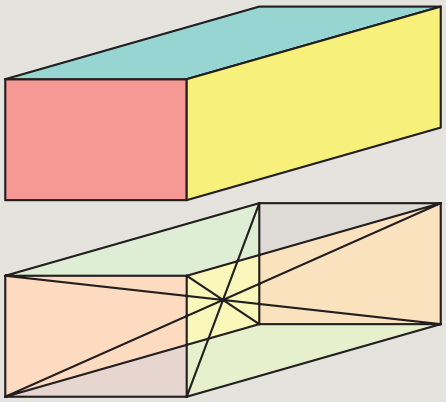
ii Use similarity to find the side lengths of triangle SUV.

iii Calculate the area of the folded yellow region of the paper.

b Find all possibilities for the length of PQ, given that triangle SUV has side lengths in the ratio 7 : 24 : 25.

c Show that, for any Pythagorean triple (a, b, c) with $a^2 + b^2 = c^2$, there are two different positions for P such that triangle SUV has sides in the ratio $a : b : c$. What is the second position for the ratio in part **a**?

3 Six pyramids



a A 4 cm × 6 cm × 12 cm rectangular prism is divided into six right rectangular-based pyramids, as shown. Each face of the prism is the base of one pyramid and the centre of the prism is the apex of each pyramid.

i Find the volume of each pyramid.

ii For each pyramid, find the sum of the lengths of its eight edges.

iii Find the surface area of each pyramid, to 2 decimal places.

b Given an 85 cm × 132 cm × 720 cm rectangular prism, show that its six pyramids each have integer surface area, in square centimetres.

c Let (a, b, c) be any Pythagorean triple satisfying $a^2 + b^2 = c^2$ and define $u = 2a(4b^2 - c^2)$, $v = 2b(4a^2 - c^2)$ and $w = 8abc$.
Show that the six pyramids of a u cm × v cm × w cm rectangular prism each have integer surface areas, in square centimetres.

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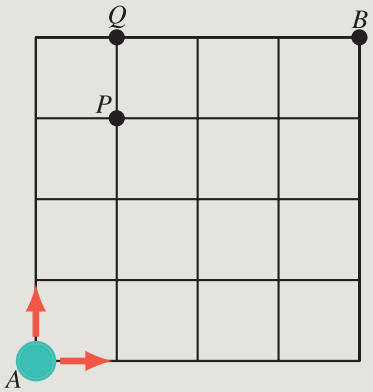
4 Chance encounters

In the 4 × 4 grid shown, a counter is placed at A. Each second, the counter is moved along a grid line to an adjacent grid point, always increasing its distance from A. Where there are two possible moves, they are equally likely. After 8 seconds, the counter reaches point B.



a Explain why there is a one-in-four chance that the counter starting at A will be at P after 4 seconds.

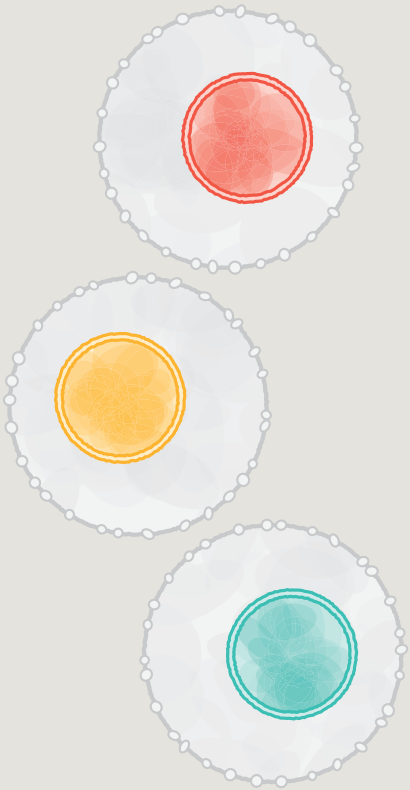
b What is the probability that the counter starting at A will be at Q after 5 seconds?

c A second counter is placed at B. Each second, it is moved along a grid line, always increasing its distance from B and with each move equally likely. The two counters start moving from A and B at the same time.

- **i** Show on a diagram all the grid points at which the counters can meet. Explain why there are no others.
- **ii** What is the probability that the counters meet?
- **iii** The game is changed so that each counter has probability $\frac{1}{3}$ of moving vertically and probability $\frac{2}{3}$ of moving horizontally. Is it more or less likely that the counters will meet?

5 Algal bloom

Dr Al Goldblum discovered that a certain species of poisonous singled-celled algae has three different forms, called kirons, lirons and mirons. A kiron lives for one week and then changes into a liron. A liron also lives for one week and then changes into a miron. One week later, a miron divides into one kiron and one liron.



A spreadsheet is a useful way to track the numbers of kirons, lirons, mirons and total algae over time. Dr Al always takes measurements at the same time of the week: 12 pm on Wednesday.

a In one experiment, Dr Al started in week 1 with a single kiron in a flask of water.

- **i** What will be the total number of algae in the flask in week 10?
- **ii** In which week will the total number of algae first exceed 100?

b Show that, from week 6 onwards, the number of kirons is equal to the sum of the numbers of kirons one week earlier and five weeks earlier. Is this also true for lirons and mirons?

c In another experiment, Dr Al started in week 1 with an equal number of kirons and mirons, but no lirons. Some weeks later, there were exactly 100 algae in total. What is the smallest possible number of algae at the beginning of the experiment?

d Dr Al's goal is to develop a herbicide to kill the poisonous algae. Batch X31 destroys all lirons within one day, but it does not affect the other two forms. It remains active for several weeks.

- **i** On Wednesday in week 11, the X31 herbicide is added to the flask that started with a single kiron in week 1. During which week will all forms of algae be completely eradicated?
- **ii** If X31 is added to a lake infested with all three forms, what is the maximum number of days it will take to eradicate the algae?

Explorations inspired by the Australian Maths Trust's competitions and programs: www.amt.edu.au

AUSTRALIAN MATHS TRUST

Problemo
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[STEAM project 1]

How can we use technology so that we improve the lives of people in the world's poorest nations?

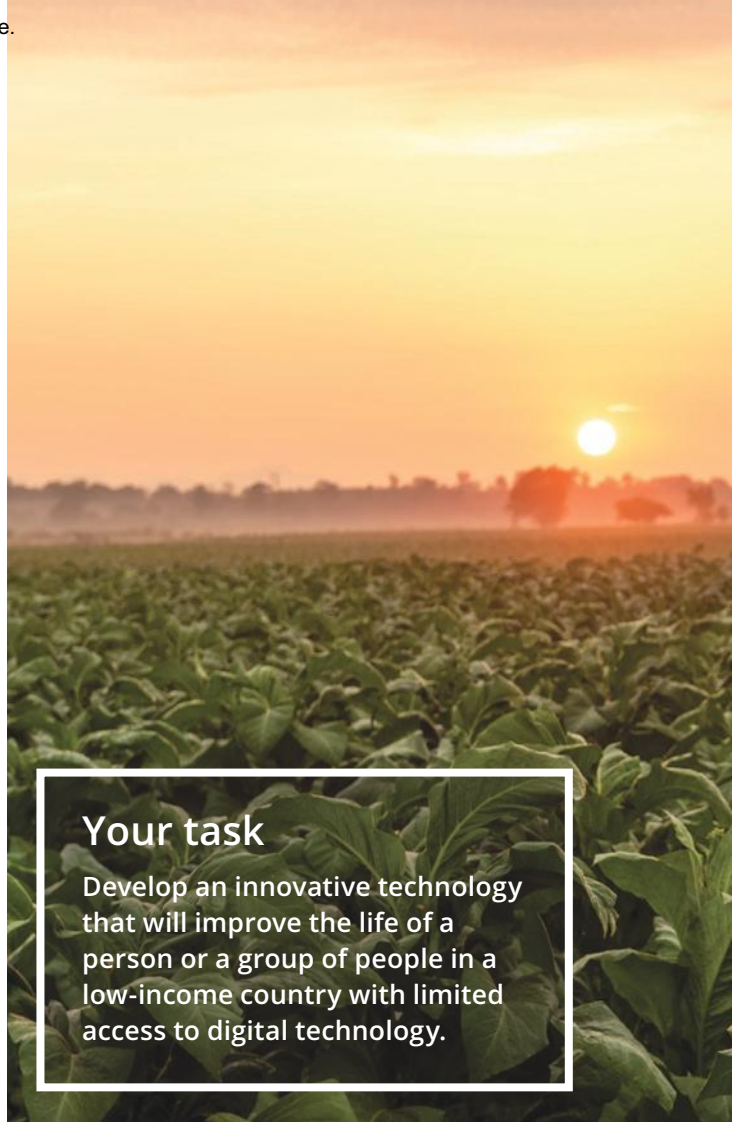
In Australia we are surrounded by technology every day. It is in the phones we use, the televisions we watch and the cars we drive. The term 'technology' is used for any machinery or equipment that applies the scientific knowledge we have discovered. Wheels and computers are both examples of technology.

In high-income countries, emergency response teams often rely on technological data supplied by electronic sensors to respond to natural disasters such as storms, fires and plagues. Drones might be used to conduct search and rescue operations. Doctors can use technology to remotely diagnose people who are sick and to perform operations that save lives.

At the end of 2017, the number of high-speed mobile subscriptions in member countries of the Organisation for Economic Co-operation and Development (OECD) reached a milestone: more subscriptions than the number of people. These mobile phones have been used to alert people to natural disasters or to call for help in the event of floods and fires.

Technology is not just used for communication during natural disasters. It is also used to create medicine, improve farming practices and for education.

However, not everyone has access to technology.



Your task

Develop an innovative technology that will improve the life of a person or a group of people in a low-income country with limited access to digital technology.

Figure 1 Technology can be used to enhance and improve agricultural practices.

The digital divide

The term 'digital divide' is used to describe the gap between those who have access to digital technology – such as mobiles, computers and the internet – and those who do not. The Australian Bureau of Statistics has identified that almost 2.6 million Australians do not use the internet and cannot access technology in an emergency. Access is even lower in lower-income countries throughout Africa and Asia.

The OECD has identified that targeted innovation that uses technology can boost productivity, increase economic growth and help solve problems in society.

Figure 2 Technology has made attending a doctor's appointment easier and more accessible for people who may have difficulty attending in person.

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HUMANITIES

In Geography this year, you will be learning about the spatial variations in human wellbeing globally. You will need to explore a range of factors that lead to inequalities, such as social, political, economic and technological differences. In Commerce, you will explore variations in living standards between countries and how living standards can be improved.

To complete this task successfully, you will need to research the initiatives of international governments and non-government organisations (NGOs) aimed at improving human wellbeing, particularly regarding health. You should consider how technology could be effectively accessed, resourced and used by a group of people to address their health concerns.

MATHS

In Maths this year, you will extend your skills in representing, comparing and interpreting data. You will use digital technology to work with data but also perform calculations by hand.

To complete this task successfully, you will need to find data to quantify the problem, to cost your interventions and to calculate a quantitative, evidence-based estimate of the likely benefits of your interventions. You will need to use skills in performing proportionality and other calculations with very large numbers, using scientific notation.

You will find relevant mathematical and statistical concepts in Module 10 'Statistics' of *Oxford Maths 10/10A Victorian Curriculum (Second edition)*.

SCIENCE

In Science this year, you will learn how an understanding of evolution can contribute to the selection of desired traits (such as drought resistance) in plants and animals. You will also learn how genetic engineering can be used to develop medicines that will cure cancers and prevent disease.

To complete this task successfully, you will need to consider how the values and needs of different societies can influence the focus of scientific research. You will also need to consider the ethics of the technology that you will be offering to your selected individual or group of people.

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[STEAM project 1]

The design cycle

To successfully complete this task, you will need to complete each of the phases of the design cycle.

Discover

When designing solutions to a problem, you need to know who you are helping and what they need. The people you are helping, who will use your design, are called your end-users.

Consider the following questions to help you empathise with your end-users:

- Who am I designing for?
- What problems are they facing? Why are they facing them?
- What do they need? What do they not need?
- What does it feel like to face these problems? What words would you use to describe these feelings?

To answer these questions, you may need to investigate using different resources or conduct interviews or surveys.

Define

Before you start to design your innovative technology, you need to define the parameters you are working towards.

Define your version of the problem

Rewrite the problem so that you describe the group you are helping, the problem they are experiencing and the reason it is important to solve it. Use the following phrase as a guide.

'How can we help (the group) to solve (the problem) so that (the reason)?'

Determine the criteria

1. Describe the limitations in energy, communications, transport and support personnel that will need to be considered as part of the solution.
2. Describe how many copies of the solution prototype will need to be made to make a difference in the country you have chosen.
3. Identify who could pay for the construction of the solution prototypes.
4. Describe the social culture that is experienced by the individuals and groups who are affected by the problem. Why might some technologies be viewed as unwanted or even dangerous?

Ideate

Once you know who you're designing for and you know what the criteria are, it's time to get creative!

Outline the criteria or requirements your technological design must fulfil (e.g. cost, size and weight for transportation, and cultural appropriateness).

Brainstorm at least one idea per person that fulfils the criteria.

Remember that there are no bad ideas at this stage. One silly thought could lead to a genius innovation!

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Build

Each group member should select one design to draw. Label each part of the design. Include the materials that will be used for its construction.

Include in the individual designs:

a a detailed diagram of the design

b a description of how it will change the life of your selected individual or group

c an outline of any similar designs that are already available to buy

d an outline of why your idea or design is better than others that are already available.

Present your design to your group.

Build the prototype

Choose one solution and build two or three prototypes. The prototype may be full size, or it may be a scale model (10 cm = 1 m).

Use the following questions as a guideline for your prototype solution.

- What materials or technology will you need to build or represent your prototype solution?
- What skills will you need to construct your prototype design? Does your group have these skills, or who can teach you these skills?
- How will you make sure your prototype design is able to be used by your selected individual or group? Will they need training?
- How will you display or describe the way the prototype design will work?

Test

Prototype 1

Use the scientific method to design an experiment that will test the effectiveness and strength of your first prototype solution.

You will test the prototype more than once to compare results, so you will need to control your variables between tests.

What criteria will you use to determine the success of your prototype? Conduct your tests and record your results in an appropriate table.

Prototype 2

If your prototype will be used to help an individual, then you will need to generate a survey to test whether the prototype is appropriate for their use. (How would they use it? Would it make their work easier or harder? Would they consider buying it?)

Prototype 3

Use the information you have obtained from testing the first two versions to adapt your last prototype to be more effective and usable for the group you are helping. You may want to use the first two prototypes to demonstrate how the design has been improved over time.

Communicate

Present your design to the class as though you are trying to get your peers to invest in your designed solution.

In your presentation, you will need to:

- outline the situation of the country in which your selected individual or group lives
- outline the challenges faced by your selected individual or group
- include a working model or a detailed series of diagrams with a description of how the prototype of the solution will be used
- include a description of how you changed your design prototype as a result of testing or feedback
- include a description of how the design prototype will improve the lives of your selected individual or group.

Online resources:

Student booklet
This helpful booklet will guide you step-by-step through the project.

What is the design cycle?
This video will help you to better understand each phase in the design cycle.

How to manage your project
This 'how-to' video will help you to manage your time throughout the design cycle.

How to pitch your idea
This 'how-to' video will help you with the 'Communicate' phase of your project.

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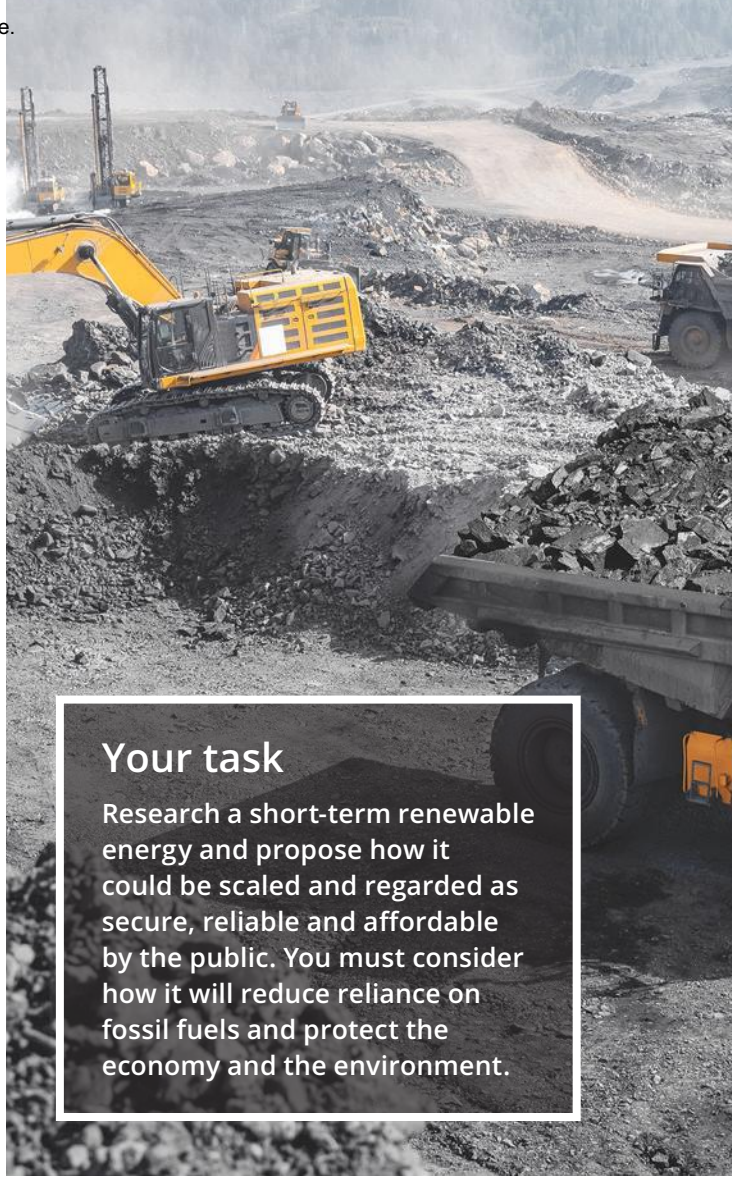
[STEAM project 2]

How can Australia reduce its reliance on fossil fuels so that we protect the environment and the economy?

Fossil fuels such as coal, oil and gas are made from fossilised, decomposed organisms that aged over millions of years in the Earth's crust. They contain carbon, which can be burned for energy. Due to the length of time it takes for these fuels to form as part of the carbon cycle, they are classified as long-term renewable sources of energy, sometimes called non-renewable.

Australia is a major user, producer and exporter of fossil fuels. Nearly 80 per cent of Australia's electricity is generated from coal and gas. Seventy-five per cent of coal mined in Australia is exported, making Australia the largest net exporter of this fuel in the world. In 2019, it was reported that Australia was the world's third-largest exporter of fossil fuels. Economically, the production and export of fossil fuels contributes hugely to Australia's GDP, and the mining industry is an important employer.

Coal emits higher amounts of CO_2 than oil or gas when used to produce energy. Measuring fossil fuel exports according to their potential to emit CO_2 makes Australia's carbon footprint per capita one of the largest in the world. This is contentious globally, particularly for nations most affected by a changing climate.



Your task

Research a short-term renewable energy and propose how it could be scaled and regarded as secure, reliable and affordable by the public. You must consider how it will reduce reliance on fossil fuels and protect the economy and the environment.

Figure 1 Most of Australia's energy is generated from coal. Almost 80 per cent of the coal produced in Australia is from open-cut mines.

Renewable alternatives

To protect both the environment and the economy, Australia needs to focus more on short-term renewable energy. When deciding on energy alternatives, it is important that energy supply be secure, reliable and affordable.

Short-term renewable energy sources include hydropower, solar power, wind power, bioenergy and ocean energy. Australia's landscape is suitable for many of these alternatives, but large investments in technology are required. As we invest more in the technology that makes short-term renewable energy possible, the better we get at making it, the more affordable it becomes and the more demand for renewable energy grows (a cyclical process).

Figure 2 A solar farm in Canberra. Solar energy is a source of renewable energy.

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HUMANITIES

In Geography this year, you will learn about environmental change and management. You will explore the environmental, technological and economic factors that have influenced the change and the consequences of human actions on the sustainability of the environment.

In Commerce, you will investigate how the performance of Australia's economy is measured and how Australia's economic growth has depended on natural resources. You will explore the impact that environmental policies can have on Australia's economy and living standards.

To complete this task successfully, you will need to understand how stakeholders such as governments, communities and businesses can work together to initiate environmental change and management plans that protect both the environment and the economy.

MATHS

In Maths this year, you will extend your skills in representing and interpreting data, including univariate, bivariate and multivariate data sets. This will include critical consideration of media reports that use statistics and present graphs. You will use digital technology to work with data but also perform calculations by hand.

To complete this task successfully, you will need to find data to quantify the problem, to cost your interventions and to calculate a quantitative, evidence-based estimate of the likely benefits of your interventions. You will need to have skills in performing proportionality and other calculations with very large numbers, using scientific notation.

You will find relevant mathematical and statistical concepts in Module 10 'Statistics' of *Oxford Maths 10/10A Victorian Curriculum (Second edition)*.

SCIENCE

In Science this year, you will learn about the impacts of fossil fuel combustion reactions in the production of carbon dioxide and carbon monoxide. You will also examine how increased reliance on this form of energy has affected the way carbon cycles through Earth's spheres, and how the resulting increase in greenhouse gases (including carbon dioxide) has led to enhanced global warming, which is contributing to melting sea ice and permafrost, rising sea levels and an increased number of extreme weather events.

To complete this task successfully, you will need to consider how energy that is generated can be used efficiently.

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[STEAM project 2]

The design cycle

To successfully complete this task, you will need to complete each of the phases of the design cycle.

Discover

When designing solutions to a problem, you need to know who you are helping (your end-users) and what they need.

Consider the following questions to help you empathise with your end-users:

- Who am I designing for? Will I be helping the government or members of the public?
- What problems are they facing? Why are they facing them?
- What do they need? What do they not need?

To answer these questions, you may need to investigate using different resources or conduct interviews or surveys.

Define

Before you start to design your solution for the potential replacement of fossil fuels, you need to define the parameters you are working towards.

Define your version of the problem

Rewrite the problem so that you describe the group you are helping, the problem they are experiencing and the reason it is important to solve it. Use the following phrase as a guide.

'How can we help (the group) to solve (the problem) so that (the reason)?'

Determine the criteria

1. What type of energy source are you trying to replace? How much of it is currently used and how is it used?
2. How will the renewable energy be used? Will it be easy for the user to access?
3. Will the renewable energy require many changes in the vehicles or equipment being used? Who will pay for this change in infrastructure? How much will it cost?
4. How long will it take to generate the resources needed to make this renewable energy resource accessible and affordable for most people?

Ideate

Once you know who you're designing for and you know what the criteria are, it's time to get creative!

Outline the criteria or requirements your design must fulfil (e.g. type of equipment, number and amount of materials, area that needs to be covered).

Brainstorm at least one idea per person that fulfils the criteria.

Remember that there are no bad ideas at this stage. One silly thought could lead to a genius innovation!

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Build

Each team member should draw one individual design. Label each part of the design. Include the materials that will be used to construct a model of the design. Also include descriptions of:

a what you see as the biggest problem with the energy source that you are replacing

b the renewable energy that you propose could be used instead

c how this renewable energy could be scaled up so that it can be used more effectively.

Present your design to your group.

Build the prototype

As a group, choose one design and plan how to model or build it. You may need to produce two or three to-scale prototypes for your group's design. Keep each iteration so that you can show the progress of your ideas.

Use the following questions as a guideline for your prototype.

- How will you replicate or model the renewable energy source?
- How will you model how the renewable energy will be used?
- What are the limitations of the renewable energy source? Will it produce enough energy for the equipment that currently uses fossil fuels?
- Calculate the number, density or requirements of energy sources in your area. How will your model provide for these demands?

Test

Use the scientific method to design an experiment that will test the limitations of your renewable energy prototype idea. You will need to model and test more than one prototype to compare results, so you will need to consider all variables between tests.

What criteria will you use to determine the success of your renewable energy prototype?

If your prototype will be used by a particular group of individuals, then you will need to generate a survey to test whether the prototype is appropriate for their use. (How would they use the alternative energy source? Would it make their life easier or harder? Would they consider buying it? How much would they be prepared to pay to access this form of energy?)

Conduct your tests and record your results in an appropriate table.

Communicate

Present your design to the class as though you are trying to get your peers to invest in your alternative energy design.

In your presentation, you will need to:

- outline the energy needs of the selected individual or group you are supporting
- outline the energy challenges faced by your selected individual or group
- create a working model or a detailed series of diagrams, with a description of how the design prototype will be used to replace the current energy demands
- describe how you changed your design prototype as a result of testing or feedback
- describe how the renewable energy prototype will improve the life of your selected individual or group
- estimate the cost of production for each element of your energy design
- estimate the number of each element of your energy design required in your local government area
- estimate the total implementation cost to individuals or to local, state or national government bodies
- compare how this energy system could be implemented in developed and developing countries.

Online resources:

Student booklet
This helpful booklet will guide you step-by-step through the project.

What is the design cycle?
This video will help you to better understand each phase in the design cycle.

How to manage your project
This 'how-to' video will help you to manage your time throughout the design cycle.

How to define a problem
This 'how-to' video will help you to narrow your ideas down and define a specific problem.

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Answers

MODULE 1 Algebra

EX p6 1.1 Exponents

1 **a** a^4 **b** b^{20} **c** c^{15} **d** d
e y^{10} **f** m^{14} **g** n^{14} **h** h^7
i x^6 **j** p^{18} **k** w^3 **l** a^{10}

2 **a** 1
b **i** 1 **ii** 1 **iii** 1 **iv** 1

3 **a** 3 **b** 1 **c** 8 **d** 1
e 1 **f** 2 **g** 0 **h** 2
i 6 **j** 1 **k** 1 **l** 1

4 **a** a^5b^5 **b** k^9p^9 **c** $64c^3$ **d** $7^{13}y^{13}$ **e** x^4y^4
f c^7d^7 **g** $9w^2$ **h** 6^6g^6 **i** 2^8p^8

5 **a** $\frac{a^4}{b^4}$ **b** $\frac{f^7}{g^7}$ **c** $\frac{k^6}{64}$
d $\frac{5^6}{x^6}$ **e** $\frac{d^{-2}}{c^{-2}}$ **f** $\frac{m^{-5}}{3^{-5}}$

6 **a** $5x^5$ **b** $6a^7$ **c** $9x^2$ **d** $8x^{12}$ **e** $4a^3$
f $2x^5$ **g** 2 **h** $30x^{12}$ **i** $3a^3$

7 **a** $8y^4$ **b** $24x^9$ **c** $72k^5$ **d** $45b^{12}$ **e** $60a^{20}$
f $10c^5$ **g** $2n^2$ **h** v^6 **i** $25m^{13}$

8 **a** p^9 **b** $18t^{20}$ **c** $25m^{39}n^{48}$ **d** $\frac{1}{j^6}$

9 **a** false; $x^7 \times x \times x^7 = x^{7+1+7} = x^{15}$
b false; $(5a)^3 = 5^3 \times a^3 = 125 \times a^3$
c true; $-k^0 = -(1) = -1$
d false; $a^2b^3 \times a^3b^4 = a^{2+3}b^{3+4} = a^5b^7$
e true; $x^7 \times y^4 \div y^6 = \frac{x^7y^4}{y^6} = \frac{x^7\cancel{y^4}^{1}}{\cancel{y^6}_{y^2}} = \frac{x^7}{y^2}$
f false; $\left(\frac{m}{n}\right)^5 = \frac{m^5}{n^5}$
g false; $\frac{w^6 \times w^6}{w^{12}} = \frac{w^{6+6}}{w^{12}} = \frac{w^{12}}{w^{12}} = 1$
h true; $\frac{(b^5)^4 \times b^2}{(b^3)^7} = \frac{b^{20} \times b^2}{b^{21}} = \frac{b^{22}}{b^{21}} = b$
i false; $\frac{a^8b^5}{a^4b^3} \times \frac{a^2b^2}{a^9b^4} = \frac{a^{8+2}b^{5+2}}{a^{4+9}b^{3+4}} = \frac{a^{10}b^7}{a^{13}b^7} = \frac{1}{a^3}$

10 **a** 9 **b** 0 **c** 10 **d** 4 **e** 7
f 9 **g** 6 **h** 5 **i** 8

11 2^5

12 3^{2x}

13 $\frac{5^{4x}}{3^{3x}}$

14 **a** $x = 4$ and $y = 8$ **b** $x = -2, y = 12$

15 **a** $\frac{2^{13} \times 3^{11} \times 5^9}{30^7} = \frac{2^{13} \times 3^{11} \times 5^9}{(2 \times 3 \times 5)^7}$
$= \frac{2^{13} \times 3^{11} \times 5^9}{2^7 \times 3^7 \times 5^7}$
$= 2^6 \times 3^4 \times 5^2$
$= (2^3 \times 3^2 \times 5)^2$

b $a = 5 \times 7$ or $a = 5 \times 7^3$ or $a = 5 \times 7^5$ or $a = 5 \times 7^7$ or $a = 5^3 \times 7$ or $a = 5^3 \times 7^3$ or $a = 5^3 \times 7^5$ or $a = 5^3 \times 7^7$ or $a = 5^5 \times 7$ or $a = 5^5 \times 7^3$ or $a = 5^5 \times 7^5$ or $a = 5^5 \times 7^7$

c The prime factorisation of any square number can be written as a product of squares of prime numbers.

16 50

17 **a** $(x + y)^2 = x^2 + y^2 + 2xy$. Since $x, y \neq 0$, $2xy \neq 0$.
b **i** $(2 + 3)^3 = 125 \neq 35 = 2^3 + 3^3$
ii $(4 + 5)^4 = 6561 \neq 881 = 4^4 + 5^4$
iii $(1 + 5)^5 = 7776 \neq 3126 = 1^5 + 5^5$
c **i** $125 = 35 + 3 \times m$, $m = 30$
ii $(x + y)^3 = x^3 + y^3 + 3 \times (x^2y + xy^2)$
iii (For example) $x = 1, y = 3, n = 4$
$(1 + 3)^4 = 1^4 + 3^4 + 4 \times \frac{87}{2}$, but $m = \frac{87}{2}$ is not an integer

EX p12 1.2 Negative exponents

1 **a** 1 **b** $-\frac{1}{2}$ **c** 3 **d** $-\frac{1}{5d}$
e $\frac{1}{2w}$ **f** $2n$ **g** $\frac{xy}{z}$ **h** $\frac{a^2}{b}$

2 **a** x^3 **b** a^{-5} **c** 1 **d** y^4
e a^5 **f** m^7 **g** n^{-1} **h** n^5
i k^{12} **j** d^{-15} **k** y **l** n^2

3 **a** $\frac{1}{x^3}$ **b** a^5 **c** 1 **d** $\frac{1}{y^4}$
e $\frac{1}{a^5}$ **f** $\frac{1}{m^7}$ **g** n **h** $\frac{1}{n^5}$
i $\frac{1}{k^{12}}$ **j** d^{15} **k** $\frac{1}{y}$ **l** $\frac{1}{n^2}$

4 **a** b^{-1} **b** n^2m^{-1} **c** 1 **d** $-x^5y^{-5}$
e $b^{-1}c^2$ **f** $ab^{-3}c^3$ **g** $-y^{-1}$ **h** yz^{-1}

5 **a** $\frac{1}{x^5}$ **b** $\frac{d^4}{c^9}$ **c** $\frac{j^5}{k^3}$ **d** $\frac{y^8}{x}$
e $\frac{2a^3}{b^6}$ **f** $\frac{3}{p^4q^5}$ **g** $\frac{w^4y^7}{x^5}$ **h** $\frac{m^5}{k^3n^8}$
i $\frac{6d^2f}{e^7}$ **j** $\frac{11}{a^3b^5c^9}$ **k** x^3 **l** b^4c^6
m $4m^3p^7$ **n** $\frac{b^2}{a^4}$ **o** $\frac{4y^5}{x^2}$ **p** $\frac{d^8e^4}{c}$

6 **a** $\frac{1}{x^4y^4}$ **b** $\frac{1}{4^6a^6}$ **c** $\frac{1}{9n}$
d $\frac{1}{5^3p^3} = \frac{1}{125p^3}$ **e** $\frac{1}{9p^2}$ **f** $\frac{1}{k^5m^5}$
g $\frac{64}{x^3}$ **h** $\frac{d}{7}$

7 **a** $\frac{1}{x^6}$ **b** $\frac{4}{y^6}$ **c** w^{12} **d** $4x^{14}$
e $\frac{9}{a^8}$ **f** $2m$ **g** $8h^{12}$ **h** $\frac{30}{c^9}$
i $2d^3$ **j** $\frac{2}{t^{16}}$ **k** x **l** $\frac{12}{a^8}$

8 **a** $\frac{1}{x^2}$ **b** $\frac{2}{x^5}$ **c** $42x$ **d** $\frac{20}{x^4}$
e $\frac{1}{x^2}$ **f** $2x^9$ **g** $\frac{1}{4x^9}$ **h** $\frac{3}{5x^4}$
i $\frac{1}{x^{11}}$ **j** $\frac{10}{x^6}$ **k** $\frac{3}{x^7}$ **l** $\frac{1}{x^{21}}$

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9 **a** x^{-2} **b** y^2 **c** $a^{-4}b^2$ **d** m^2n^{-6}
e $\frac{1}{64}x^{-6}$ **f** p^2q^2 **g** $123g^{234}$ **h** $4t^3$
i $10^{10}u^{-10}$ **j** $\frac{1}{3}pq^{-2}$ **k** $\frac{1}{9}x^2$ **l** $\frac{81}{8}n^3m^{-3}$

10 **a** $a^{11}b^7$ **b** $3m^{12}n^6$ **c** $\frac{d^{12}}{h^{11}}$ **d** $a^{30}b^{12}$
e u^3wx^{17} **f** $125c^4n$ **g** $m^{34}n$ **h** $\frac{1}{y^7}$

11 **a** $\frac{1}{25}$ **b** $\frac{1}{27}$ **c** 40
d 144 **e** $\frac{1}{8}$ **f** $\frac{13}{9}$

12 **a** false, $\frac{a^5}{a^9} = \frac{1}{a^4}$ **b** false, $\frac{x^3}{x^6} = \frac{1}{x^3}$
c true **d** false, $\left(\frac{3a^2b^{-1}}{c^3}\right)^2 = \frac{9a^4}{b^2c^6}$
e true **f** true
g false, $\frac{3(x^{-2}y^1)^3}{x^3y^{-2}z} = \frac{3y^5}{x^9z}$ **h** true

13 **a** 3 **b** −4 **c** −2 **d** 3 **e** 1 **f** 5

14 **a** $x^{2a+5}y^{4b+1}$ **b** $m^{2x+4}n^3$

15 **a** x^{3a-4} **b** y^{3m+5} **c** a^{x+3} **d** b^{x+4}

16 **a** $\frac{3a^2b^2}{2c}$ **b** $\frac{3}{2b^8c^2}$

17 **a** $\frac{8c^{12}}{27a^6b^9}$ **b** Answers may vary

18 B

19 $\frac{1}{y}$

20 **a** 1 and −1
b There is no number n such that $0 \times n = 1$.

EX 1.3 Simplifying
p17

1 **a** **i** 3 **ii** Sample answer: $-4abcd$
b **i** −4 **ii** Sample answer: $2mn$
c **i** 1 **ii** Sample answer: $5xy^2$
d **i** 9 **ii** Sample answer: $-3k^2m^4p$

2 **a** −6 **b** −3 **c** 10 **d** 9
e −52 **f** −24 **g** −15 **h** −9
i $\frac{21}{4}$ **j** $-\frac{1}{6}$ **k** $-\frac{41}{4}$ **l** $-\frac{1}{2}$

3 **a** $9x$ **b** $-10ab$ **c** $10y^2$
d $8m^2n$ **e** $12a + 9d$ **f** $8k + 2m$
g $\frac{2}{a} - 3y$ **h** $\frac{7a}{b} - \frac{4a^2}{3}$ **i** $\frac{3x^2}{y} + \frac{2xy}{7}$
j $\frac{19a}{24} - \frac{ab}{4}$ **k** $\frac{3m^2n}{4} - \frac{8mn^2}{15}$ **l** $\frac{7xy}{10} - \frac{x}{4} + \frac{y}{3}$

4 **a** $11a$ **b** $-4n + 2n^2 + m - mn$
c $3x^2 - 2x + 1$ **d** $-3x^2y + 3xy - 5$
e $\frac{21p^2}{2a} - \frac{9p}{4a^2} - \frac{9}{b} + \frac{2p^3}{b^2}$
f $\frac{11a^2b}{3c} + \frac{7a^2}{6c} + \frac{3ab^2}{5c} - \frac{b^2}{4c}$
g $\frac{5ab}{c} + \frac{2ac}{b} + \frac{3bc}{a}$ **h** $\frac{25ab}{12c}$
i $\frac{17x^2y}{3z} + \frac{13xy}{3z}$ **j** $\frac{491amp}{90b}$

5 **a** $10abcd$ **b** $-21mnxy$ **c** $4k^2p$
d ac **e** $6n$ **f** $\frac{2x}{y}$

6 **a** $4x^5y^3$ **b** $6a^6b^3$ **c** $\frac{24m^7}{n^2}$
d $\frac{25c}{2}$ **e** $\frac{5x^4y^2}{4}$ **f** $\frac{m^7n^5}{3}$
g $6x^4y^3$ **h** $\frac{2x^2y^2}{5}$ **i** $\frac{27a^3}{160}$

7 **a** $\frac{18x^3y^5}{z^5}$ **b** $\frac{2z^3}{9xy^3}$ **c** $\frac{4}{ab^2c^3}$
d $\frac{16}{a^5b^4c^3}$ **e** $\frac{a^3b^3c^3}{x^3y^5z^7}$ **f** $\frac{xy^3z^5}{abc}$

8 **a** 1550 cm² **b** 0.62 m²

9 **a** $7x + 6y$ **b** $6x + 4y - 6$ **c** $20y - 12$

10 **a** 61 cm **b** 44 cm **c** 28 cm

11 The answers are square units:
a $42ab$ **b** $4a^2$ **c** $10ab$
d $9\pi a^2$ **e** $24ab + \frac{9}{8}\pi b^2$ **f** $37ab$

12 **a** 504 m² **b** 64 m² **c** 120 m²
d 144π m² **e** $288 + \frac{81\pi}{8}$ **f** 444 m²

13 **a** $3\pi x^2$
b Any value larger than about 6.5147 mm and smaller than 7.2836 mm. Some possible values are: 6.7 mm, 7 mm and 7.1 mm.
c $6\pi x$ **d** 126.3 mm, 131.9 mm, 133.8 mm

14 **a** $-3a + 15ab$ **b** $24xy^2 - x$
c $-6a^2 + 25a^2b^6$ **d** $16x^2y^2 + 13y^2$
e $23a^2b^2$ **f** $\frac{5y^2}{2} + 7xz$

15 **a** $abc - 6c$ m³ **b** $2ab + 2ac + 2bc - 12$ m²

EX 1.4 Expanding
p21

1 **a** $3a + 21$ **b** $5b - 20$ **c** $12 - 2c$
d $4 + 4d$ **e** $-6e - 18$ **f** $-7f + 14$
g $-3g + 3$ **h** $-45 + 9h$ **i** $x^2 + 2x$
j $2y^2 - 8y$ **k** $-k^2 - 5k$ **l** $-3p^2 + 3p$
m $3mn + km$ **n** $12ab + 24ac$ **o** $-w^2 + 8wx$
p $-4p - 10p^2$

2 **a** $3a - 12$ **b** $-5b + 15$ **c** $8x + 12$
d $-14y - 28$ **e** $10a - 5a^2$ **f** $2m^3 + 6m$
g $24a^2b + 8ab$ **h** $8wx - 4w^2$ **i** $15k^2 - 10k^4$
j $12x^3y - 6x^2y^4$ **k** $15c^2d - 10cd^2$ **l** $-8a^3c + 6a^2bc$

3 **a** $9x + 10$ **b** $4y - 13$ **c** $mn + 3m$
d $8x^2 - 5x$ **e** $2p^3 - 3p$ **f** $x^7 - 3x^5$

4 **a** $6x + 2$ **b** $m^2 + 14m - 18$
c $6a^2 + 14a - 6$ **d** $k^5 + 7k^2 + 5$
e $3p^6 - 4p^4 + 12p$ **f** $10y^8 + y^3 + 8y$

5 **a** $ab + 2a + 4b + 8$ **b** $cd + 6c + 3d + 18$
c $mn + 5m + 7n + 35$ **d** $xy + 8x + 6y + 48$
e $kp - k + 5p - 5$ **f** $fg + 3f - 2g - 6$
g $xy - 4x - 9y + 36$ **h** $2mn + 2m + 3n + 3$
i $12xy - 28x + 15y - 35$
j $15ac - 12ad - 10bc + 8bd$ **k** $y^2 + 8y - 9$
l $30 - 11k + k^2$ **m** $12p^2 + 52p + 35$
n $8x^2 + 10x - 3$ **o** $15y^2 - 26y + 8$

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6 **a** $2a^2 + 6a - ab - 3b$ **b** $4t^2 - 3tu - u^2$
c $2x^2 + 11xy + 12y^2$ **d** $-x^2 + x + 20$
e $6x^3 + 18x^2y + xy + 3y^2$ **f** $ab^4 - 2a^2b^2 + b^2 - 2a$
g $ab + ac + b^2 + bc$ **h** $b^2 + bd + 3bc + 3cd$
i $3a - ac + 6b - 2bc$ **j** $3a^4b + 2a^3 + 3ab^3 + 2b^2$
k $5x^5 - 2x^4y - 15xz + 6yz$ **l** $c^4 - 16a^6b^2$

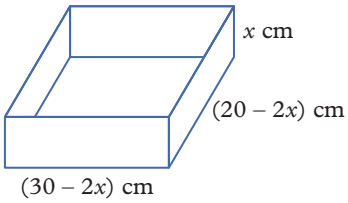
7 **a** **i** $x^2 - 4$ **ii** $k^2 - 49$ **iii** $m^2 - 64$
iv $w^2 - 36$ **v** $y^2 - 1$ **vi** $a^2 - b^2$
b The product of the two binomial factors produces an expression that is the difference of two squared numbers or pronumerals. The first term in each binomial factor is the same, and the second term in one factor is the negative of the second term in the other factor.
c In the expression $a^2 - b^2$, the terms a^2 and b^2 are both squares, and there is a minus sign between them indicating 'find the difference'.
d No; multiplication is commutative.
e **i** $a^2 - 9$ **ii** $x^2 - 100$ **iii** $m^2 - n^2$
iv $9 - x^2$ **v** $1 - d^2$ **vi** $4x^2 - 25$
vii $4 - 9k^2$ **viii** $16g^2 - h^2$ **ix** $25y^2 - 4w^2$

8 **a** **i** $x^2 + 6x + 9$ **ii** $y^2 + 16y + 64$
iii $a^2 + 2ab + b^2$ **iv** $k^2 - 10k + 25$
v $p^2 - 12p + 36$ **vi** $x^2 - 2xy + y^2$
b The two binomial factors are the same. The product of factors like these produces an expression made up of the squares of the two terms in the factor, and twice the product of the two terms in the factor.
c The expression $(a + b)^2$ is the square of $(a + b)$.
d **i** $a^2 + 4a + 4$ **ii** $x^2 + 10x + 25$
iii $p^2 + 8p + 16$ **iv** $y^2 + 20y + 100$
v $k^2 + 18k + 81$ **vi** $m^2 + 2mn + n^2$
vii $25 + 10x + x^2$ **viii** $1 + 4d + 4d^2$
ix $9w^2 + 42w + 49$
e The expression $(a - b)^2$ is the square of $(a - b)$.
f **i** $a^2 - 6a + 9$ **ii** $b^2 - 8b + 16$
iii $x^2 - 18x + 81$ **iv** $n^2 - 22n + 121$
v $4 - 4x + x^2$ **vi** $g^2 - 2gh + h^2$
vii $16w^2 - 8w + 1$ **viii** $25 - 20y + 4y^2$
ix $9m^2 - 12mp + 4p^2$

9 **a** $x^4 - 4$ **b** $x^4 - 25$ **c** $x^4 - y^4$
d $x^4 + 8x^2 + 16$ **e** $x^4 - 6x^2 + 9$ **f** $x^6 - 2x^3 + 1$

10 **a** $2x^3 + 9x^2 + 7x - 3$
b $2x^3 + 5x^2 + x - 2$
c $x^3 - 4x^2 + 3x + 2$
d $2x^4 + x^3 + 3x^2 + 2x - 2$
e $2x^4 - 6x^3 - x^2 + 9x - 3$
f $3x^4 + 5x^3 - 5x^2 - 6x$

11 **a** **i** $(30 - 2x)$ cm
ii $(20 - 2x)$ cm
iii x cm
b

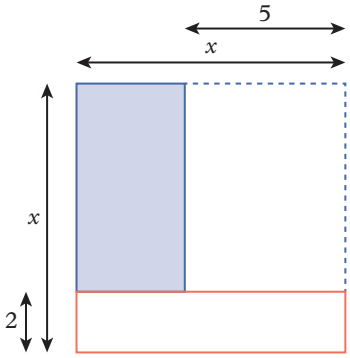


c **i** $x(30 - 2x)(20 - 2x) = 4x^3 - 100x^2 + 600x$
ii $30 \times 20 - 4x^2 = 600 - 4x^2$ (using the net)
d **i** 1008 cm^3 **ii** 564 cm^2

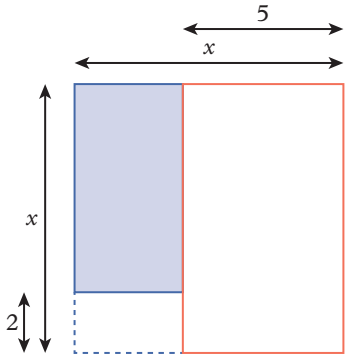
12 **a** $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$
b $x^2 + 4xy + 6x + 4y^2 + 12y + 9$
c $4a^2 - 9b^2 - c^2 + 6bc$

13 **a** $x^3 + 3x^2 + 3x + 1$
b $x^3 - 9x^2 + 27x - 27$
c $x^4 + 8x^3 + 24x^2 + 32x + 16$
d $x^4 - 2x^2y^2 + y^4$

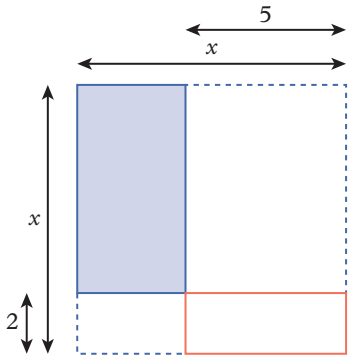
14 **a** **i**



ii



iii



b The rectangle with area 10 is subtracted as part of the $2x$ rectangle and as part of the $5x$ rectangle. Since the rectangle with area 10 is subtracted twice when it should only be subtracted once, it must be added back once.

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EX p27 1.5 Algebraic fractions

1 **a** $\frac{x}{2}$ **b** $\frac{1}{y}$ **c** 4 **d** $\frac{2x}{3}$
e $\frac{x+3}{5}$ **f** $2(x-5)$ **g** $x-4$ **h** $x+1$

2 **a** $\frac{x+5}{x+1}$ **b** $\frac{x-6}{x-2}$ **c** $\frac{x-1}{x+1}$
d $\frac{x+2}{x-8}$ **e** 1 **f** 1
g $\frac{2(x-1)}{x+3}$ **h** $\frac{x+8}{4(x+9)}$ **i** $\frac{x+4}{3(x-12)}$

3 **a** $\frac{2}{y}$ **b** $2b$ **c** $\frac{5}{a}$ **d** $\frac{x}{2}$
e $\frac{1}{3m}$ **f** $\frac{n+7}{2}$ **g** $3y$ **h** $\frac{x}{x+1}$
i $\frac{5(y-3)}{y-4}$ **j** $\frac{t+2}{r+2}$ **k** $\frac{x-1}{x-4}$ **l** $\frac{p-2}{3(p+2)}$

4 **a** $\frac{1}{2}$ **b** 5 **c** 1
d 3 **e** 5 **f** x

5 **a** $\frac{5c+3}{12}$ **b** $\frac{7m-5}{12}$ **c** $\frac{3x-10}{10}$ **d** $\frac{12k+5}{20}$
e $\frac{29x-7}{40}$ **f** $\frac{-x+2}{6}$ **g** $\frac{b+10}{24}$ **h** $\frac{-5}{12}$

6 **a** $\frac{5x+13}{6}$ **b** $\frac{11x-8}{30}$ **c** $\frac{3x-7}{8}$
d $\frac{x+10}{12}$ **e** $\frac{x-12}{24}$ **f** $\frac{4x+1}{10}$
g $\frac{13x-6}{15}$ **h** $\frac{7x+17}{18}$ **i** $\frac{23x+5}{6}$
j $\frac{7x+15}{20}$ **k** $\frac{40-10x}{21}$ **l** $\frac{3x^2+1}{40}$

7 **a** $\frac{2a+1}{a-4}$ **b** $\frac{x+1}{x-3}$
c $\frac{2}{t}$ **d** $\frac{m+2}{(m-3)(m+7)}$
e $\frac{(x+3)(x+5)}{(x-3)}$ **f** $\frac{1}{x+5}$
g $\frac{3(2y+3)}{y+3}$ **h** $\frac{m(m+1)}{(m-3)(m-1)}$
i $\frac{c^2+6}{c+2}$ **j** $\frac{n-p}{n}$
k $\frac{y^2}{x^2(x+1)}$ **l** $\frac{10(a^2+b^2)}{a-b}$

8 **a** $\frac{2(x-5)}{x-2}$ **b** $\frac{(x+8)(x-6)}{(x-3)(x+6)}$
c $\frac{(3x+2)(5x+1)}{(x+1)(x+2)}$ **d** $\frac{x}{2(x+8)}$

9 **a** $\frac{5m+3}{m(m+1)}$ **b** $\frac{4k^2+3}{k(k^2+1)}$
c $\frac{4y-7}{(y+2)(y-3)}$ **d** $\frac{x^2+2x+6}{x(x+3)}$
e $\frac{t^2+t+10}{(t+5)(t-1)}$ **f** $\frac{8x+5}{x(2x+1)}$
g $\frac{2x-1}{(x+2)(x+1)}$ **h** $\frac{-2h+11}{(2h-1)(h+2)}$
i $\frac{2m^2+3}{(m-3)(2m+1)}$ **j** $\frac{a^3+12a+4}{a(3a+1)}$
k $\frac{x^2-3x-6}{2(x+1)}$ **l** $\frac{2x+1}{x(x+1)}$

10 $a = x-3$

11 $b = 3x+1$

12 **a** $\frac{(1-x^4)}{x^2}$ **b** $\frac{(x^4+2x^2+1)}{x^2}$
c $\frac{1}{x-1}$ **d** $\frac{1}{2x^2}$

13 **a** $\frac{1}{x(x+1)}$
b **i** $\frac{1}{42}$ **ii** $\frac{1}{90}$
iii $\frac{1}{9900}$ **iv** $\frac{1}{1\,001\,000}$

14 **a** $\frac{x}{y(y+1)}$
b **i** $\frac{5}{42}$ **ii** $\frac{7}{110}$
iii $\frac{67}{10\,100}$ **iv** $\frac{11}{999\,000}$

15 **a** $\frac{x+1}{(x+2)(x-2)}$ **b** $\frac{2x^2-4x+4}{(x+1)(x-1)}$
c $\frac{x+4}{(x+2)(x+1)}$ **d** $\frac{4x^3-8x^2+3}{x(x-1)(x-2)}$
e $\frac{x^2+10x+4}{x(x+2)(x-2)}$ **f** $\frac{x^3+7x^2+9x}{(x+1)(x+2)(x+3)}$

16 **a** -1 **b** -4 **c** $\frac{x-3}{x+4}$
d $-x-1$ **e** $\frac{(5-x)}{(x+5)}$ **f** $\frac{2}{(2-x)}$

17 ± 1

EX p30 Checkpoint

1 **a** x^9 **b** y^4 **c** $2m$
d a^{10} **e** $9x^2y^4$ **f** $\frac{8a^3}{b^6}$

2 **a** 1 **b** 2 **c** 1 **d** 3

3 **a** $\frac{1}{y^3}$ **b** $\frac{4}{x^2}$ **c** $\frac{1}{6a^7}$
d $2b^3$ **e** $\frac{1}{2a^2b^5}$ **f** $\frac{5d^4}{c^3}$

4 $\frac{12}{a^7b}$

5 **a** $6x$ **b** $8m^2$
c $7a+5b$ **d** $-3c+5d$
e $5a^2-2a$ **f** $-2x^2y+3xy+xy^2$

6 **a** $-\frac{x}{6}$ **b** $-\frac{6x}{a}+\frac{3y^2}{b}$ **c** $\frac{3x}{2y}$
d $-\frac{x}{6y}$ **e** $\frac{5x^2}{9y^2}$ **f** $\frac{4}{5}$

7 **a** $3a+21$ **b** $6y+7$
c $3x-8$ **d** $-7x+10$

8 **a** a^2+3a+2 **b** $x^2+2x-24$
c k^2-25 **d** $2x^2+x-6$

9 $x^3+2x^2y-xy-2y^2$

10 **a** $\frac{5x}{6}$ **b** $\frac{17y+2}{6}$
c $\frac{5m^2-4m+5}{20}$ **d** $\frac{11x+1}{21}$

11 **a** $\frac{a}{3}$ **b** $\frac{x+4}{5}$
c $\frac{1}{x+7}$ **d** $\frac{m-4}{m}$

12 $\frac{x^2-2x+6}{(x-2)(x+1)}$

EX p35 1.6 Factorising

1 **a** $5(a+7)$ **b** $6(5b+1)$ **c** $3(8+11c)$
d $4(3d+2)$ **e** $2(4e-9)$ **f** $7(5f-3)$

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g $8(3 + 2g)$ **h** $10(1 - 4h)$ **i** $9(j + k)$
j $4(4w - 7x)$ **k** $m(m + 2)$ **l** $n(n - 6)$
m $a(5 + a)$ **n** $p(9 - p)$ **o** $3q(q + 1)$
p $2r(r - 4)$

2 a $3(x - 4)$ **b** $3(2y - 5)$
c $8x(x - 2)$ **d** $2k(2k + 3)$
e $c(5 + 11c)$ **f** $4u(2 - u)$
g $3m(2m - 1)$ **h** $y^2(1 + 5y)$
i $2t(7 - 4t)$ **j** $3p^2(3p + 1)$
k $a(4b + 3)$ **l** $tu(5t - 3)$
m $2mn(2m + n)$ **n** $3xy(2x + 1)$
o $3c(5c + 7d)$ **p** $6a^2bc(3b^2 - 2a^2c)$

3 a b, b
b $-5x, -5x, -5x$

4 a $-6(mn + 3)$ **b** $-3b(a + c)$
c $-y(y + 10)$ **d** $-4k(1 - 2k)$
e $-10a(2a^2 + 3)$ **f** $-2xy(6x - 7)$
g $-w^5(1 - w^2)$ **h** $-2b^2(2b - 1 + 3b^2)$

5 a $y(x - 5) + 2(x - 5) = (x - 5) \times y + (x - 5) \times 2$
$= (x - 5)(y + 2)$
b $4a(3 + k^2) - 9(3 + k^2) = (3 + k^2) \times 4a + (3 + k^2) \times (-9)$
$= (3 + k^2)(4a - 9)$

6 a $(x + 3)(a + 5)$ **b** $(k - 2)(m + 4)$
c $(y + 5)(y + 2)$ **d** $(x - 1)(x + 9)$
e $(k + 6)(k - 3)$ **f** $(p - 9)(p - 6)$
g $(4 - a)(7 + a)$ **h** $(2n - 5)(3n + 4)$
i $(y^2 + 2)(x + 8)$ **j** $(a - 4)(a^2 + 3)$
k $(y^2 + 1)(2x - 5)$ **l** $(7 - x^3)(5x + 2)$

7 a $(a + 4)(b + 3)$ **b** $(x - 7)(x + 2y)$
c $(n + 5)(m - 6)$ **d** $(a + 2)(a + 5)$
e $(x - 1)(x + 4)$ **f** $(c + 8)(c - 3)$
g $(2 - y)(3y + 4)$ **h** $(p + 2)(p^2 + 5)$
i $(x^2 + 1)(x^2 + 3)$ **j** $(m^2 + 2)(2m - 3)$
k $(1 + k^2)(4 + k^3)$ **l** $(y - 3)(x + 5)$

8 a $x^2 - 25$
b Write each term as a square; that is, $x^2 - 25 = x^2 - 5^2$. Write the product of the sum and difference of the two values that are squared, x and 5. The factorised form is $(x + 5)(x - 5)$.

9 a $(x + 2)(x - 2)$ **b** $(a + 6)(a - 6)$
c $(10 + y)(10 - y)$ **d** $(8b + 3)(8b - 3)$
e $(5 + 7p)(5 - 7p)$ **f** $(2a + 9b)(2a - 9b)$
g $3(m + 1)(m - 1)$ **h** $2(2k + 3)(2k - 3)$
i $(xy + 4w)(xy - 4w)$ **j** $(m^3 + n)(m^3 - n)$
k $4(a^5 + 3b^3)(a^5 - 3b^3)$ **l** $(h + 8)(h - 2)$
m $4(1 - x)$ **n** $(1 + a + b)(1 - a - b)$
o $(y - 3)(7 - y)$ **p** $9(2x - 3)$

10 a $(x + \sqrt{7})(x - \sqrt{7})$ **b** $(a + \sqrt{13})(a - \sqrt{13})$
c $(\sqrt{19} + y)(\sqrt{19} - y)$ **d** $(4k + \sqrt{5})(4k - \sqrt{5})$
e $(p + 8 + \sqrt{2})(p + 8 - \sqrt{2})$
f $(m - 4 + \sqrt{6})(m - 4 - \sqrt{6})$
g $(x + 7 + \sqrt{3})(x + 7 - \sqrt{3})$
h $(y - 1 + \sqrt{19})(y - 1 - \sqrt{19})$
i $(d - 17 + \sqrt{17})(d - 17 - \sqrt{17})$

11 a $x^2 + 10x + 25$
b Find the square root of the first and last terms in the expansion ($\sqrt{x^2} = x$ and $\sqrt{25} = 5$). Note that the middle term is twice the product of these two values, and is positive because it is preceded by a '+' sign ($+10x$). Write the factorised form as the square of the sum of x and 5; that is, $(x + 5)^2$.

12 a $y^2 - 16y + 64$
b Find the square root of the first and last terms in the expansion ($\sqrt{y^2} = y$ and $\sqrt{64} = 8$). Note that the middle term is twice the product of these two values, and is negative because it is preceded by a '–' sign ($-16y$). Write the factorised form as the square of the difference between y and 8; that is, $(y - 8)^2$.

13 a $(x + 6)^2$ **b** $(m + 4)^2$ **c** $(p + 10)^2$
d $(y - 3)^2$ **e** $(x - 7)^2$ **f** $(g - 2)^2$
g $(1 + a)^2$ **h** $(5 - x)^2$ **i** $(9 - b)^2$
j $(w + x)^2$ **k** $(k - m)^2$ **l** $(3x + 2)^2$

14 a **i** $(x + 10)$ cm **ii** $(2x + 3)$ cm
b **i** $4x$ cm **ii** $6(x + 1)$ cm

15 a $\frac{a}{b}$ **b** $\frac{x}{3y}$ **c** $c^2 - 4b$
d mn **e** $\frac{x - y}{2}$ **f** $\frac{3}{x}$
g $\frac{3x}{4}$ **h** $\frac{m + 7}{5m}$ **i** $\frac{a + b}{a - b}$
j $\frac{x + 3}{x - 3}$ **k** $\frac{4a + 5}{2a}$ **l** $\frac{2x + 7}{7 - 2x}$

16 a $\frac{6}{x - 7}$ **b** 1

17 a $(n + 2)$ and $(n + 4)$ **b** $3n + 6$
c $3(n + 2)$; this expression is 3 times the middle odd number.
d $425 \times 3 = 1275$
e $n + (n + 2) + (n + 4) + (n + 6) + (n + 8) = 5n + 20$
$= 5(n + 4)$
The expression describes a value that is 5 times the middle number.
f $2675 \times 5 = 13\,375$

18 a The sum of k odd numbers (where k is an even number)
$= k \times$ (the average of the two middle odd numbers)
or $k \times (n + k - 1)$
or $k \times$ (the average of the first and last of the odd numbers)
b 10 (the average of 81 and 83) $= 10 \times 82 = 820$
or $10(73 + 10 - 1) = 820$ or $10[(73 + 91) \div 2] = 820$

19 15

20 a 399 **b** 896 **c** 6391 **d** 2484

21 a 101 **b** 50
c $50 \times 101 = 5050$ **d** $\frac{n}{2}(n + 1)$

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EX p40 1.7 Factorising quadratic expressions

1 **a** 7 and 1 **b** 8 and 2 **c** 9 and 5
d 10 and 6 **e** 9 and 4 **f** 8 and 3

2 **a** $(x + 7)(x + 1)$ **b** $(x + 8)(x + 2)$
c $(x + 9)(x + 5)$ **d** $(x + 10)(x + 6)$
e $(x + 9)(x + 4)$ **f** $(x + 8)(x + 3)$

3 **a** $(x + 2)(x + 1)$ **b** $(x + 3)(x + 2)$
c $(x + 10)(x + 1)$ **d** $(x + 9)(x + 3)$
e $(x + 5)(x + 2)$ **f** $(x + 7)(x + 2)$
g $(x + 3)(x + 3) = (x + 3)^2$ **h** $(x + 11)(x + 1)$
i $(x + 9)(x + 2)$ **j** $(x + 8)(x + 4)$
k $(x + 9)(x + 7)$ **l** $(x + 10)(x + 4)$
m $(x + 6)(x + 5)$ **n** $(x + 11)(x + 2)$
o $(x + 12)(x + 3)$

4 **a** 8 and –1 **b** –6 and –1 **c** –7 and 5
d 5 and –3 **e** –4 and 2 **f** –6 and –5

5 **a** $(x + 8)(x - 1)$ **b** $(x - 6)(x - 1)$
c $(x - 7)(x + 5)$ **d** $(x + 5)(x - 3)$
e $(x - 4)(x + 2)$ **f** $(x - 6)(x - 5)$

6 **a** $(x + 6)(x - 2)$ **b** $(x - 10)(x + 2)$
c $(x - 3)(x - 2)$ **d** $(x + 9)(x - 3)$
e $(x - 8)(x - 5)$ **f** $(x - 5)(x + 4)$
g $(x + 7)(x - 3)$ **h** $(x - 11)(x - 3)$
i $(x - 7)(x + 4)$ **j** $(x + 4)(x - 3)$
k $(x - 9)(x + 2)$ **l** $(x - 8)(x - 7)$

7 **a** $2(x + 8)(x + 1)$ **b** $3(x + 3)(x + 2)$
c $7(x + 4)(x - 2)$ **d** $-5(x + 3)(x + 4)$
e $-4(x - 5)(x - 2)$ **f** $-(x + 6)(x - 1)$
g $2(x - 2)(x + 9)$ **h** $19(x - 3)(x + 1)$
i $-3(x - 13)(x + 2)$ **j** $-2(x - 1)(x - 2)$
k $\frac{1}{2}(x - 1)(x + 4)$ **l** $\frac{1}{3}(x + 6)(x - 3)$

8 **a** $x + 4$ **b** $\frac{1}{x - 2}$ **c** $\frac{x}{x - 2}$
d $\frac{x - 5}{x + 3}$ **e** $\frac{2(x - 5)}{x + 2}$ **f** $\frac{x}{x + 2}$
g $\frac{x + 1}{1 - x}$ **h** $\frac{2(x - 3)}{x + 4}$ **i** $\frac{x - 5}{2(x + 2)}$

9 **a** 2 **b** $\frac{x - 6}{3}$ **c** $\frac{5x - 2}{6}$ **d** $\frac{29x - 19}{24}$

10 **a** $(x + 11)(x + 4)$ **b** $(x - 5)(x + 3)$
c $(x + 2)^2$ **d** $3(x - 4)(x - 2)$
e $5(x + 3)(x - 3)$ **f** $2x(x - 4)$

11 **a** $x^2 + 3^2$ is the sum of two squares.
b $(x - 3)^2 + 4^2$ is the sum of two squares.
c $(x + 1)^2 + 2^2$ is the sum of two squares.
d There are no numbers that add to –4 and multiply to 5.
e The expression simplifies to $x^2 + 2^2$, which is the sum of two squares.
f There are no numbers that add to 12 and multiply to 40.

12 **a** $2x^2$ m^2 and $(12x + 18)$ m^2
b $(2x^2 + 12x + 18)$ m^2
c $(x + 3)$ m

13 $(x + 6)$ m

14 **a** **i** $x^2 + 2xy + y^2$ **ii** $x^2 + 5xy + 6y^2$
b **i** $(x + 2)(x + 8)$ **ii** $(x + 2y)(x + 8y)$
c **i** $(x + 5)(x + 2)$ **ii** $(x + 5y)(x + 2y)$
d **i** $(x - 4y)(x - 2y)$
ii $(y - 7x)(y - 2x)$ or $(7x - y)(2x - y)$
iii $(4x + y)(x + y)$

15 **a** $x(x + 6)$ **b** not possible
c not possible **d** not possible
e not possible **f** $(x + 1)(x + 5)$
g not possible **h** not possible
i $(x + 2)(x + 4)$ **j** $(x + 3)^2$

16 **a** $k = 5, 8, 9$ (positive) or $k = 0$ or $k = -7, -16, -27, -40, \ldots$ (negative)
The negative case can be expressed as any number of the form $n(-n - 6) = -n(n + 6)$, where n is any positive integer.
b $k = 19, 36, 51, 64, 75, 84, 91, 96, 99, 100$ (positive) or $k = 0$ or $k = -21, -44, -69, -96, \ldots$ (negative)
The negative case can be expressed as any number of the form $-n(n + 20)$, where n is any positive integer.
The positive case can be expressed as any number of the form $n(20 - n)$, where n is positive and less than 20.

17 **a** $(x^2 + 4)(x + 2)(x - 2)$ **b** $4(4a + b)(4a - b)$
c $(a - x + y)(a - x - y)$

EX p46 1.8 Completing the square

1 **a** **i** $x^2 + 6x + 9$ **ii** $(x + 3)(x + 3) = (x + 3)^2$
b **i** $x^2 - 18x + 81$ **ii** $(x - 9)(x - 9) = (x - 9)^2$
c **i** $(x + 4)^2$ **ii** $(x + 2)^2$ **iii** $(x - 6)^2$
iv $(x + 8)^2$ **v** $(x - 5)^2$ **vi** $(x - 10)^2$

2 **a** 9 **b** 25 **c** 16
d 4 **e** 1 **f** 64

3 **a** $x^2 + 14x + 49$ **b** $x^2 - 6x + 9$
c $x^2 + 18x + 81$ **d** $x^2 + 20x + 100$
e $x^2 - 12x + 36$ **f** $x^2 + 22x + 121$

4 **a** $(x + 7)^2$ **b** $(x - 3)^2$ **c** $(x + 9)^2$
d $(x + 10)^2$ **e** $(x - 6)^2$ **f** $(x + 11)^2$

5 $(x + 3)^2 - (\sqrt{2})^2 = (x + 3 + \sqrt{2})(x + 3 - \sqrt{2})$

6 **a** $(x + 4 + \sqrt{7})(x + 4 - \sqrt{7})$
b $(x - 5 + \sqrt{11})(x - 5 - \sqrt{11})$
c $(x + 2 + \sqrt{10})(x + 2 - \sqrt{10})$
d $(x + 1 + \sqrt{6})(x + 1 - \sqrt{6})$
e $(x - 3 + \sqrt{13})(x - 3 - \sqrt{13})$
f $(x - 6 + \sqrt{23})(x - 6 - \sqrt{23})$
g $(x + 5 + \sqrt{29})(x + 5 - \sqrt{29})$
h $(x - 7 + \sqrt{3})(x - 7 - \sqrt{3})$

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i $(x+7)(x+1)$

j $\left(x+\frac{3}{2}-\frac{\sqrt{5}}{2}\right)\left(x+\frac{3}{2}+\frac{\sqrt{5}}{2}\right)$

k $\left(x-\frac{5}{2}-\frac{\sqrt{13}}{2}\right)\left(x-\frac{5}{2}+\frac{\sqrt{13}}{2}\right)$

l $\left(x+\frac{11}{2}-\frac{\sqrt{85}}{2}\right)\left(x+\frac{11}{2}+\frac{\sqrt{85}}{2}\right)$

7 a $2(x+2-\sqrt{3})(x+2+\sqrt{3})$

b $-(x+3-\sqrt{11})(x+3+\sqrt{11})$

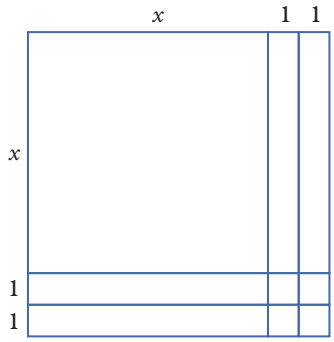
c $-5(x-1-\sqrt{2})(x-1+\sqrt{2})$

d $3\left(x-\frac{5}{2}-\frac{\sqrt{17}}{2}\right)\left(x-\frac{5}{2}+\frac{\sqrt{17}}{2}\right)$

8 a x by $y-x$ **b** x^2

c y by $y-x$ **d** y^2

9 a



b x^2+4x+4 **c** $(x+2)^2$

10 a $x(x+6)$

b $(x+3+\sqrt{8})(x+3-\sqrt{8})$

c $(x+3+\sqrt{7})(x+3-\sqrt{7})$

d $(x+3+\sqrt{6})(x+3-\sqrt{6})$

e $(x+3+\sqrt{5})(x+3-\sqrt{5})$

f $(x+1)(x+5)$

g $(x+3+\sqrt{3})(x+3-\sqrt{3})$

h $(x+3+\sqrt{2})(x+3-\sqrt{2})$

i $(x+2)(x+4)$

j $(x+3)^2$

11 a $x^2+7x+6=\left(x+\frac{7}{2}\right)^2-\left(\frac{5}{2}\right)^2$

$=\left(x+\frac{7}{2}+\frac{5}{2}\right)\left(x+\frac{7}{2}-\frac{5}{2}\right)$

$=(x+6)(x+1)$

b $6\times 1=6$ and $6+1=7$, so the factors are $(x+6)$ and $(x+1)$.

12 a $c>16$ **b** $c>25$ **c** $-2<b<2$

13 a Can be written as $(x+3)^2+1$, which is the sum of two squares.

b Can be written as $(x-4)^2+5$, which is the sum of two squares.

c The leading coefficient is not a highest common factor.

d The leading coefficient is not a highest common factor.

e It can be written as $2(x+1)^2+2$, which is the sum of two squares.

14 a $2(x+7)(x-4)$

b not possible as it is the sum of two squares: $\left(x-\frac{3}{2}\right)^2+\left(\frac{\sqrt{11}}{2}\right)^2$

c $x(x+6)(x-3)$

d $-(x-4)(x+3)$

e $-2(x+7)(x-7)$

f not possible as it is the sum of two squares: $(2x)^2+5^2$

15 a $2\left(x+3-\sqrt{\frac{13}{2}}\right)\left(x+3+\sqrt{\frac{13}{2}}\right)$

b $2\left(x+\frac{7}{4}-\frac{\sqrt{41}}{4}\right)\left(x+\frac{7}{4}+\frac{\sqrt{41}}{4}\right)$

c $3\left(x+\frac{11}{6}-\frac{\sqrt{133}}{6}\right)\left(x+\frac{11}{6}+\frac{\sqrt{133}}{6}\right)$

16 a $\left(x+\frac{b}{2}+\sqrt{\frac{b^2}{4}-c}\right)\left(x+\frac{b}{2}-\sqrt{\frac{b^2}{4}-c}\right)$

b $\frac{b^2-4c}{4}$

EX 1.9 Factorising non-monic quadratic expressions

p51

1 a i $ac=4, b=5$ **ii** 1, 4 **iii** $x+4x$

b i $ac=18, b=11$ **ii** 2, 9 **iii** $2x+9x$

c i $ac=16, b=17$ **ii** 1, 16 **iii** $x+16x$

d i $ac=-24, b=10$ **ii** $-2, 12$ **iii** $-2x+12x$

e i $ac=-18, b=-7$ **ii** $-9, 2$ **iii** $-9x+2x$

f i $ac=30, b=-13$ **ii** $-3, -10$ **iii** $-3x-10x$

2 a $(3x+2)(x+3)$ **b** $(4x+1)(x+4)$

c $(2x+1)(x+2)$ **d** $(2x-3)(3x+1)$

e $(3x-2)(x+4)$ **f** $(5x-3)(x-2)$

3 a $(2x+3)(x+2)$ **b** $(3x+2)(x+1)$

c $(2x+1)(2x+5)$ **d** $(7x+2)(x+1)$

e $(5x+4)(x+2)$ **f** $(2x+3)(x+3)$

g $(3x+2)(2x+1)$ **h** $(5x+1)(2x+3)$

i $(4x+5)(2x+1)$

4 a $(3x+1)(x-3)$ **b** $(4x-3)(x+2)$

c $(2x-5)(x-2)$ **d** $(5x+4)(x-1)$

e $(4x-1)(2x+3)$ **f** $(2x-3)(2x+5)$

g $(7x-4)(x-2)$ **h** $(4x-1)(3x+1)$

i $(3x+4)(x-5)$

5 a $2(2x+3)(x+1)$ **b** $3(4x+1)(x+2)$

c $2(5x-1)(x+4)$ **d** $(4x+5)(2x+3)$

e $3(2x+5)(x-2)$ **f** $4(3x+1)(x-1)$

g $-2(2x+1)(x+1)$ **h** $-3(4x-1)(x+2)$

i $-(3x+4)(2x+3)$

6 a $2x+3$ **b** $\frac{1}{x+3}$ **c** $2x+7$

d $\frac{2x-1}{x+5}$ **e** $\frac{5x+2}{2x-3}$ **f** -4

7 a $\frac{5x-3}{x-3}$ **b** $\frac{2x+7}{2x+1}$

c 1 **d** $\frac{(2x+7)(x+4)}{(2x-7)(x-4)}$

e $\frac{5x+2}{2}$ **f** $\frac{x+1}{x-3}$

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8 **a** $3(x+3+\sqrt{2})(x+3-\sqrt{2})$
b $2(x-5+\sqrt{5})(x-5-\sqrt{5})$
c $5(x+1+\sqrt{7})(x+1-\sqrt{7})$
d $3(x+2+\sqrt{3})(x+2-\sqrt{3})$
e $2\left(x-3+\sqrt{\frac{21}{2}}\right)\left(x-3-\sqrt{\frac{21}{2}}\right)$
f $4\left(x+1+\frac{\sqrt{5}}{2}\right)\left(x+1-\frac{\sqrt{5}}{2}\right)$
g $6\left(x-2+\sqrt{\frac{29}{6}}\right)\left(x-2-\sqrt{\frac{29}{6}}\right)$
h $3\left(x+4+\sqrt{\frac{47}{3}}\right)\left(x+4-\sqrt{\frac{47}{3}}\right)$
i $9\left(x-1+\frac{\sqrt{7}}{3}\right)\left(x-1-\frac{\sqrt{7}}{3}\right)$

9 $\frac{-14}{3x+4}$

10 **a** $(4x-1)(x+2)$
b $-4x^2-7x+2=-(4x^2+7x-2)=-(4x-1)(x+2)$
$=(-4x+1)(x+2)$
$=(1-4x)(x+2)$
c **i** $(3-2x)(x+1)$
ii $(1-3x)(x-5)=(3x-1)(5-x)$
iii $(7-5x)(x+2)$ **iv** $(2-x)(4x+3)$
v $(1-2x)(3x+2)$ **vi** $(1-5x)(5x-1)$

11 **a** $\sqrt{9x^2-4x-5}$ m **b** $(9x+5)(x-1)$

12 **a** $(x+4-\sqrt{11})(x+4+\sqrt{11})$
b $2\left(x+2-\sqrt{\frac{3}{2}}\right)\left(x+2+\sqrt{\frac{3}{2}}\right)$
c $3(x+1)\left(x+\frac{5}{3}\right)$ or $(x+1)(3x+5)$

13 Parts **a**, **d** and **e** cannot be factorised.
b $(5x-4)(3x+2)$
c $3(x-2+\sqrt{3})(x-2-\sqrt{3})$
f $-3(2x-9)(x-7)$

14 **a** Room A: $(2x+3)$ m by $(4x-5)$ m
Room B: $(4x-5)$ m by $(4x-5)$ m
b $(24x^2-38x+10)$ m^2
c $(6x-2)$ m by $(6x-2)$ m
d $[(6x-2)^2-(4x-5)^2]$ m^2
e $(10x-7)(2x+3)$ m^2
f $36-\frac{24}{x}+\frac{4}{x^2}$
g 28

15 **a** **i** $(x^2+5)(x^2+2)$
ii $(x^2+12)(x^2+2)$
iii $(x^2+3)(x^2+1)$
b **i** $(x^2+1)(x-1)(x+1)$
ii $(x^2+4)(x-2)(x+2)$
iii $(x^2+9)(x-3)(x+3)$
c **i** $(x-2)(x+2)(x-3)(x+3)$
ii $(x-1)(x+1)(x-5)(x+5)$
iii $(x-1)(x+1)(x-\sqrt{10})(x+\sqrt{10})$

16 **a** $a\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a}+c$
b $a\left(x+\frac{b+\sqrt{b^2-4ac}}{2a}\right)\left(x+\frac{b-\sqrt{b^2-4ac}}{2a}\right)$
c b^2-4ac

EX p55

Review: Algebra

oxforddigital **Answers for this lesson can be found on Oxford Digital.**

MODULE 2 Real numbers

EX p64

2.1 Simplifying surds

1 **a** 4 **b** 9 **c** 7 **d** 2
e 10 **f** 5 **g** 18 **h** 44
i 1 **j** 2

2 **a** $9\sqrt{7}$ **b** $4\sqrt{3}$ **c** $2\sqrt{6}$ **d** $5\sqrt{10}$
e $3\sqrt{2}$ **f** $6\sqrt{5}$ **g** $8\sqrt{11}$ **h** $30\sqrt{5}$
i $6\sqrt{10}$

3 **a** 11 **b** 30 **c** −32 **d** −1
e 24 **f** 14 **g** 10 **h** 5
i 13 **j** −6 **k** 11 **l** 9

4 **a** $2\sqrt{7}$ **b** $5\sqrt{2}$ **c** $2\sqrt{6}$ **d** $3\sqrt{3}$
e $4\sqrt{3}$ **f** $5\sqrt{3}$ **g** $3\sqrt{7}$ **h** $2\sqrt{15}$
i $2\sqrt{14}$ **j** $3\sqrt{5}$ **k** $8\sqrt{2}$ **l** $6\sqrt{3}$
m $6\sqrt{5}$ **n** $7\sqrt{3}$ **o** $9\sqrt{2}$ **p** $5\sqrt{5}$
q $15\sqrt{2}$ **r** $12\sqrt{2}$ **s** $3\sqrt{30}$ **t** $9\sqrt{7}$

5 **a** $10\sqrt{5}$ **b** $6\sqrt{2}$ **c** $12\sqrt{3}$ **d** $27\sqrt{2}$
e $16\sqrt{5}$ **f** $9\sqrt{10}$ **g** $60\sqrt{6}$ **h** $-12\sqrt{3}$
i $28\sqrt{6}$ **j** $-14\sqrt{5}$ **k** $64\sqrt{3}$ **l** $60\sqrt{7}$
m $-20\sqrt{7}$ **n** $21\sqrt{11}$ **o** $-24\sqrt{17}$ **p** $1000\sqrt{3}$
q $4\sqrt{5}$ **r** $3\sqrt{6}$ **s** $16\sqrt{62}$ **t** $3\sqrt{11}$

6 $5\sqrt{27}=5\times\sqrt{27}$
$=5\times\sqrt{9\times3}$
$=5\times\sqrt{9}\times\sqrt{3}$
$=5\times3\times\sqrt{3}$
$=15\sqrt{3}$

7 **a** $2\sqrt{11}$ **b** $5\sqrt{5}$ **c** $2\sqrt{30}$ **d** $6\sqrt{2}$
e $5\sqrt{6}$ **f** $-6\sqrt{6}$ **g** $3\sqrt{14}$ **h** $4\sqrt{15}$
i $-9\sqrt{2}$ **j** $7\sqrt{5}$ **k** $4\sqrt{3}$ **l** $2\sqrt{3}$
m $-8\sqrt{3}$ **n** $60\sqrt{5}$ **o** $6\sqrt{21}$ **p** $9\sqrt{2}$
q $-160\sqrt{2}$ **r** −20 **s** $-10\sqrt{10}$ **t** $3\sqrt{2}$

8 **c**, **d**, **e** are surds

9 **a** $4\sqrt{5}$ cm **b** $6\sqrt{5}$ cm **c** $2\sqrt{13}$ cm **d** $12\sqrt{11}$ m

10 **a** $3\sqrt{6}=\sqrt{3^2}\times\sqrt{6}$
$=\sqrt{9}\times\sqrt{6}$
$=\sqrt{9\times6}$
$=\sqrt{54}$
b To be simplified to a whole number coefficient it must have first been expressed as the square root of a perfect square.
c They are equal.

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d Put the entire surd and the original surd into your calculator and press equal. You should get the same answer.

e $\sqrt{48}$

11 a $\sqrt{45}$ **b** $\sqrt{12}$ **c** $\sqrt{32}$ **d** $\sqrt{252}$
e $\sqrt{160}$ **f** $\sqrt{275}$ **g** $\sqrt{384}$ **h** $\sqrt{125}$
i $\sqrt{240}$ **j** $-\sqrt{128}$ **k** $\sqrt{1008}$ **l** $-\sqrt{75}$

12 a $\sqrt{a^2 b}$

c Square the whole part then multiply it by the surd part to find the number that will be under the root sign in the entire surd. If the whole part is a negative, the entire surd will also be a negative.

13 a $\sqrt{m^2 n}$ **b** $\sqrt{49a}$ **c** $\sqrt{5x^2}$ **d** $\sqrt{100xy}$
e $\sqrt{2cd^2}$ **f** $\sqrt{36abc}$ **g** $\sqrt{a^3 b^2}$ **h** $\sqrt{x^4 y^3 z^2}$

14 a $x\sqrt{y}$ **b** xy **c** $b\sqrt{a}$ **d** $xz\sqrt{y}$
e x^2 **f** $x^2\sqrt{y}$ **g** $x^2 y$ **h** $a^2 b\sqrt{c}$

15 a $2^2 \times 3$ **b** $\sqrt{2^2 \times 3} = \sqrt{2^2} \times \sqrt{3} = 2 \times \sqrt{3} = 2\sqrt{3}$

16 a 2×3^3 **b** $\sqrt{3^3 \times 2} = \sqrt{3^2 \times 3 \times 2}$ **c** $3\sqrt{6}$

17 a $3\sqrt{2}$ **b** $2\sqrt{7}$ **c** $4\sqrt{2}$ **d** $3\sqrt{3}$
e $2\sqrt{6}$ **f** $5\sqrt{2}$ **g** $6\sqrt{5}$ **h** $16\sqrt{2}$
i $15\sqrt{7}$ **j** $16\sqrt{10}$ **k** $28\sqrt{3}$ **l** $18\sqrt{5}$
m $6\sqrt{5}$ **n** $35\sqrt{3}$ **o** $54\sqrt{2}$ **p** $5\sqrt{5}$
q $30\sqrt{2}$ **r** $12\sqrt{2}$

18 a $\sqrt{18}$ m $= 3\sqrt{2}$ m **b** $\sqrt{2}$ m

c The side length of the 18 m² square is three times the side length of the 2 m² square. The area of a square divided into nine smaller squares has a total area of $3 \times 3 = 9$ smaller squares, which is the specified amount.

d i $\sqrt{27}$ m $= 3\sqrt{3}$ m **ii** $\sqrt{48}$ m $= 4\sqrt{3}$ m
iii $\sqrt{50}$ m $= 5\sqrt{2}$ m **iv** $\sqrt{75}$ m $= 5\sqrt{3}$ m

19 a i $2\sqrt{2}$ **ii** $3\sqrt{3}$ **iii** $5\sqrt{5}$ **iv** $a\sqrt{a}$
b i $4\sqrt{2}$ **ii** $9\sqrt{3}$ **iii** $25\sqrt{5}$ **iv** $a^2\sqrt{a}$
c i $2\sqrt[3]{2}$ **ii** $3\sqrt[3]{3}$ **iii** $5\sqrt[3]{5}$ **iv** $a\sqrt[3]{a}$
d i $2\sqrt[4]{2}$ **ii** $3\sqrt[4]{3}$ **iii** $5\sqrt[4]{5}$ **iv** $a\sqrt[4]{a}$

20 a $\sqrt{5}, \sqrt{10}, \sqrt{21}, \sqrt{22}, \sqrt{27}, \sqrt{30}, \sqrt{42}, \sqrt{46}$

b $\sqrt{6}, 3, \sqrt{11}, 2\sqrt{3}, 2\sqrt{6}, 6, \sqrt{39}, 4\sqrt{3}$

c $4, 2\sqrt{5}, \sqrt{22}, 3\sqrt{3}, \sqrt{43}, 2\sqrt{11}, 3\sqrt{6}, \sqrt{83}$

d $2\sqrt{6}, 4\sqrt{5}, 3\sqrt{14}, 2\sqrt{57}, 12\sqrt{2}, 3\sqrt{43}, 20, 7\sqrt{10}$

2.2 Multiplying and dividing surds

EX p69

1 a $\sqrt{35}$ **b** $\sqrt{33}$ **c** $-\sqrt{14}$ **d** $-\sqrt{105}$
e $\sqrt{65}$ **f** $-\sqrt{22}$ **g** $\sqrt{21}$ **h** $\sqrt{95}$
i $3\sqrt{77}$ **j** $20\sqrt{6}$ **k** $-28\sqrt{22}$ **l** $-36\sqrt{30}$
m $66\sqrt{34}$ **n** $64\sqrt{42}$ **o** $-24\sqrt{33}$ **p** $132\sqrt{15}$

2 a 7 **b** 3 **c** 2 **d** 6
e 5 **f** 11 **g** 10 **h** 9

3 a The answer is rational. **b** a

4 a 9 **b** −77 **c** 40 **d** 60
e $3\sqrt{10}$ **f** $30\sqrt{30}$ **g** $-32\sqrt{26}$ **h** $3\sqrt{33}$
i 84 **j** 15 **k** $-6\sqrt{26}$ **l** $\sqrt{105}$
m $72\sqrt{42}$ **n** $\frac{3\sqrt{195}}{8}$ **o** $-5\sqrt{70}$ **p** $-11\sqrt{85}$

5 a $\sqrt{17}$ **b** $\sqrt{3}$ **c** 1 **d** $-\sqrt{5}$
e $\sqrt{6}$ **f** $\sqrt{7}$ **g** $4\sqrt{6}$ **h** $2\sqrt{5}$
i $-3\sqrt{5}$ **j** $2\sqrt{5}$ **k** $-4\sqrt{10}$ **l** $\frac{\sqrt{5}}{2}$

6 a 4 **b** 2 **c** 3 **d** 2

7 a $-12\sqrt{5}$ **b** $5\sqrt{3}$ **c** $-3\sqrt{2}$ **d** $2\sqrt{5}$
e $4\sqrt{7}$ **f** $-\frac{7\sqrt{5}}{3}$ **g** $\frac{3}{\sqrt{3}}$ **h** $\frac{\sqrt{5}}{5}$

8 $6\sqrt{3} \times 3\sqrt{2} = 6 \times \sqrt{3} \times 3 \times \sqrt{2}$
$= 6 \times 3 \times \sqrt{3} \times \sqrt{2}$
$= 18 \times \sqrt{3 \times 2}$
$= 18\sqrt{6}$

9 $\frac{24\sqrt{40}}{8\sqrt{20}} = \frac{24}{8} \times \frac{\sqrt{40}}{\sqrt{20}}$
$= 3 \times \sqrt{\frac{40}{20}}$
$= 3\sqrt{2}$

10 a 6 **b** $\frac{3\sqrt{5}}{4}$ **c** $\frac{11\sqrt{17}}{7\sqrt{21}}$
d $-\frac{4\sqrt{3}}{3\sqrt{5}}$ or $-\frac{4\sqrt{5}}{5\sqrt{3}}$ **e** $\frac{9\sqrt{5}}{4}$
f $-\frac{2}{3}$ **g** −48 **h** $\frac{\sqrt{15}}{16\sqrt{2}}$ **i** 1

11 a 3 **b** $\frac{9}{10}$ **c** 24 **d** $\frac{1}{3}$
e −60 **f** 2 **g** 84 **h** 576
i $\frac{24}{5}$ **j** $\frac{1}{8}$ **k** $-\frac{4}{3}$ **l** $-\frac{7}{3}$

12 a $2\sqrt{10}$ **b** $\frac{\sqrt{3}}{5}$ **c** $\frac{8}{5}$
d $-\frac{11\sqrt{2}}{8}$ **e** $\frac{8\sqrt{2}}{3}$ **f** $-\frac{2\sqrt{6}}{5}$
g $\frac{\sqrt{5}}{9}$ **h** $-\frac{9\sqrt{7}}{5}$ **i** $-\frac{10\sqrt{11}}{9}$

13 a 5 **b** 8 **c** 2 **d** 7
e 9 **f** 80 **g** 54 **h** 192
i 891

14 a rational

b $\sqrt{a} \times \sqrt{a} = (\sqrt{a})^2 = \sqrt{a^2} = a$

c $\sqrt{80}, \sqrt{54}, \sqrt{192}, \sqrt{891}$

d The number beneath the square root of an entire surd is equal to the square of the surd.

15 a x **b** ab **c** $6y$ **d** $x + y$

16 a $\frac{25}{7}$ **b** $\frac{1}{5}$ **c** $\frac{3}{2}$ **d** $\frac{10}{13}$

17 a Method 1: $1680\sqrt{6}$ Method 2: $42\sqrt{2} \times 40\sqrt{3}$

b yes

c Use Method 2 only when the surd part can be simplified. Method 2 has the added advantage that the numbers are smaller.

18 a $6\sqrt{2}$ **b** $\sqrt{2}$ **c** $\frac{3\sqrt{13}}{5}$ **d** $\frac{6\sqrt{5}}{7}$
e $-7\sqrt{3}$ **f** $\frac{8\sqrt{3}}{3}$ **g** $-16\sqrt{5}$ **h** $\frac{\sqrt{6}}{5}$
i $\frac{-2\sqrt{21}}{3}$ or $\frac{-2\sqrt{7}}{\sqrt{3}}$

19 a $\frac{11\sqrt{2}}{8}$ **b** $8\sqrt{6}$ **c** $\frac{-7\sqrt{2}}{16}$ **d** $8\sqrt{3}$
e $-\frac{15\sqrt{3}}{2}$ **f** $\frac{6\sqrt{2}}{5}$ **g** −8 **h** $-\frac{32\sqrt{6}}{11}$

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20 $ab\sqrt{bc}$

21 **a** **i** 50.00 cm^2 **ii** 50 cm^2
b **i** 137.48 cm^2 **ii** $30\sqrt{21}\text{ cm}^2$
c **i** 19.44 cm^2 **ii** $3\sqrt{42}\text{ cm}^2$
d **i** 169.71 mm^2 **ii** $120\sqrt{2}\text{ mm}^2$
e **i** 6.00 cm^2 **ii** 6 cm^2
f **i** 23.24 m^2 **ii** $6\sqrt{15}\text{ m}^2$
g **i** 2.51 cm^2 **ii** $\frac{4\pi}{5}\text{ cm}^2$
h **i** 37.70 cm^2 **ii** $12\pi\text{ cm}^2$
i **i** 0.39 m^2 **ii** $\frac{\pi}{8}\text{ m}^2$

22 The perimeter of the square is $4\sqrt{a}$ and the area is a. Equating the perimeter and area to form the equation $4\sqrt{a} = a$, which has the solutions $a = 0$, $a = 16$. Since the square roots of 0 and 16 are not surds, there is no value of a such that the perimeter and area are equal if the side length is a surd.

23 **a** $4\sqrt{5}\text{ m}$ **b** $4\sqrt{17}\text{ m}$ **c** $12\sqrt{2}\text{ m}$
d $8\sqrt{5}\text{ m}$ **e** $2\sqrt{3}\text{ m}$ **f** $\frac{10\sqrt{7}}{3}\text{ m}$

24 **a** $\sqrt{5}$ **b** $\sqrt{6}$ **c** $\sqrt{35}$
d $\sqrt{14}$ **e** $\sqrt{39}$ **f** $\sqrt{66}$

EX p74 2.3 Adding and subtracting surds

1 **a** yes **b** no **c** no **d** yes
e no **f** no **g** yes **h** yes

2 **a** $7\sqrt{3} + \sqrt{2}$ **b** $4\sqrt{3} - \sqrt{2}$
c $-3\sqrt{10} - 4\sqrt{7} - \sqrt{6} + 2\sqrt{3}$ **d** $8\sqrt{13} + 4\sqrt{39} - \sqrt{3}$
e $13\sqrt{2} - 6\sqrt{5}$ **f** $4\sqrt{6} + 4\sqrt{5} + 1$
g $3\sqrt{11} + 4\sqrt{55} - \sqrt{5}$ **h** $\sqrt{21} - \sqrt{3}$
i $7\sqrt{6} + 3\sqrt{5} + 5$ **j** $-7\sqrt{15} - 3\sqrt{7}$

3 **a** $\sqrt{2}, 4\sqrt{2}, 3\sqrt{2}$ **b** none
c $3\sqrt{12}, \sqrt{3}, 2\sqrt{27}$ **d** $9\sqrt{32}, 4\sqrt{8}, \sqrt{128}, 5\sqrt{200}$

4 **a** $13\sqrt{2}$ **b** $24 - 4\sqrt{3}$ **c** $6\sqrt{6} + 3\sqrt{5}$
d $28\sqrt{7}$ **e** $59\sqrt{2}$ **f** $28 - 4\sqrt{2}$

5 **a** $-15\sqrt{3}$ **b** $-6\sqrt{5}$ **c** $-4\sqrt{7}$
d $5\sqrt{3}$ **e** $23\sqrt{2}$ **f** 0

6 **a** $4\sqrt{5}$ **b** 0
c $4\sqrt{6}$ **d** $\sqrt{2}$
e $-19\sqrt{6} + 8\sqrt{5}$ **f** $-25\sqrt{3} + 2\sqrt{2}$
g $16\sqrt{2} - 30\sqrt{3}$ **h** 0
i $6\sqrt{3} - 2\sqrt{7} + 10\sqrt{2} - 4\sqrt{5}$ **j** $3\sqrt{5} - 2\sqrt{6} - 6\sqrt{3}$

7 **a** $5\sqrt{5}$ **b** $42\sqrt{3}$
c $29\sqrt{2} - 80\sqrt{3}$ **d** $18\sqrt{6} - 5\sqrt{5} - 14\sqrt{3}$
e $66\sqrt{7} - 8\sqrt{6} + 112\sqrt{3}$
f $10\sqrt{2} - 24\sqrt{5} + 60\sqrt{6} - 12\sqrt{10}$
g $3\sqrt{3} - 25\sqrt{5} - 61$
h $88\sqrt{2} - 113\sqrt{3}$

8 **a** 0 **b** $-22\sqrt{x}$
c $7\sqrt{x} - 6\sqrt{3x}$ **d** $-5\sqrt{2xy}$
e $20\sqrt{6xy}$ **f** $-x\sqrt{6y}$
g $12y\sqrt{3x}$ **h** $34x\sqrt{y} - 5x\sqrt{3y}$
i $8x\sqrt{5xy} + 10\sqrt{3xy} - 5x\sqrt{5y} - 24x\sqrt{3y}$

9 **a** $12 + 6\sqrt{3}$ **b** $8\sqrt{5} - 32$
c $20 - 30\sqrt{2}$ **d** $12\sqrt{2} + 8\sqrt{3}$
e $10\sqrt{6} - 6\sqrt{5}$ **f** $-10\sqrt{42} + 15\sqrt{70} - 10\sqrt{7}$
g $45\sqrt{2} + 12\sqrt{15}$ **h** $70 - 60\sqrt{7}$
i $12\sqrt{33} - 30\sqrt{77} + 66$ **j** $3\sqrt{5} - 10 + 25\sqrt{3}$
k $-36\sqrt{2} + 6\sqrt{30} - 81\sqrt{5}$

10 **a** $10 + 20\sqrt{6} + 6\sqrt{3} + 36\sqrt{2}$
b $15\sqrt{10} - 24\sqrt{2} + 30\sqrt{5} - 48$
c $28\sqrt{10} + 56\sqrt{30} - 24 - 48\sqrt{3}$
d $12\sqrt{3} + 15\sqrt{2} + 12\sqrt{6} + 30$
e $24\sqrt{6} - 288 - 15\sqrt{15} + 90\sqrt{10}$
f $56\sqrt{5} - 48\sqrt{30} + 21\sqrt{10} - 36\sqrt{15}$

11 **a** $31 + 10\sqrt{6}$ **b** $16 - 6\sqrt{7}$
c $57 + 12\sqrt{15}$ **d** $159 - 24\sqrt{42}$

12 **a** $(a + b)^2 = (a + b)(a + b)$
$= a^2 + ab + ba + b^2$
$= a^2 + 2ab + b^2$
b $(a - b)^2 = (a - b)(a - b)$
$= a^2 - ab - ba + b^2$
$= a^2 - 2ab + b^2$

13 **a** 10 **b** 116 **c** 10 **d** -67

14 **a** The terms in the second bracket are exactly the same as the terms in the first bracket, but the operation sign between the terms in the second bracket is the inverse of the sign between the terms in the first bracket.
b Rational, because they can be written as the difference of two squares, and the surds become rational numbers after being squared each time.
c $(a + b)(a - b) = a^2 - ab + ba - b^2$
$= a^2 - b^2$
$(a - b)(a + b) = a^2 + ab - ba - b^2$
$= a^2 - b^2$

15 Sample answer: LHS $= \sqrt{1} + \sqrt{4}$
$= 1 + 2$
$= 3$
RHS $= \sqrt{1 + 4}$
$= \sqrt{5}$
LHS $\neq$ RHS

16 **a** $(32\sqrt{10} + 15\sqrt{30} - 12\sqrt{6})\text{ m}^2$
b $(72\sqrt{5} - 40 + 10\pi)\text{ cm}^2$
c $(15 + 3\sqrt{3} + 2.5\sqrt{2} - 0.5\sqrt{6})\text{ m}^2$

17 **a** $180\sqrt{2} - 18\sqrt{5}\text{ cm}^2$ **b** $409 + 80\sqrt{6}\text{ m}^2$
c $(151 - 24\sqrt{7})\pi\text{ cm}^2$ **d** $(7 + 4\sqrt{2})\pi\text{ cm}^2$

18 **a** $(10\sqrt{10} + 16\sqrt{5})\text{ m}$ **b** $(36 + 2\pi\sqrt{5})\text{ cm}$

19 **a** $8\sqrt{6}$ **b** $3\sqrt{15}$ **c** $\frac{73\sqrt{3}}{18}$
d $2\sqrt{10} + 6\sqrt{5}$ **e** $\sqrt{11} - 3\sqrt{7}$ **f** $6\sqrt{6} - 6$

20 $25 + 3\sqrt{2} + 3\sqrt{3} + \sqrt{5} + \sqrt{6} + \sqrt{7} + \sqrt{10} + \sqrt{11} + \sqrt{13} + \sqrt{14} + \sqrt{15}$

21 **a** **i** add $5\sqrt{5}$ **ii** add $\sqrt{3} + 7$
iii multiply $\sqrt{2}$ **iv** multiply $2\sqrt{5}$

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b **i** $23\sqrt{5}, 28\sqrt{5}, 33\sqrt{5}, 38\sqrt{5}, 43\sqrt{5}, 48\sqrt{5}$

ii $6\sqrt{3}+28, 7\sqrt{3}+35, 8\sqrt{3}+42, 9\sqrt{3}+49, 10\sqrt{3}+56, 11\sqrt{3}+63$

iii $12\sqrt{2}, 24, 24\sqrt{2}, 48, 48\sqrt{2}, 96$

iv $400\sqrt{5}, 4000, 8000\sqrt{5}, 80\,000, 160\,000\sqrt{5}, 1\,600\,000$

c **i** $255\sqrt{5}$ **ii** $65\sqrt{3}+315$

iii $93\sqrt{2}+186$ **iv** $168\,421\sqrt{5}+1\,684\,210$

22 **a** **i** $\frac{15\sqrt{2}+30}{8}=\frac{15\sqrt{2}}{8}+\frac{15}{4}$

ii $2+2\sqrt{2}$

b **i** $\frac{3\sqrt{5}+4\sqrt{3}+6\sqrt{2}}{6}=\frac{\sqrt{5}}{2}+\frac{2\sqrt{3}}{3}+\sqrt{2}$

ii $\frac{3\sqrt{5}}{2}$

EX 2.4 Rationalising the denominator

p79

1 **a** **i** $\frac{1}{2}$ **ii** $\frac{3}{4}$ **iii** $\frac{3}{4}$ **iv** $\frac{2}{5}$

b The fraction appearing first in the multiplication remained unchanged.

c It is the same as 1, so each fraction has been multiplied by 1.

2 **a** **i** $\frac{2}{\sqrt{3}}$ **ii** $\frac{6}{3\sqrt{3}}$ **iii** $\frac{2\sqrt{3}}{3}$ **iv** $\frac{4}{2\sqrt{2}}$

b part **iii**

3 **a** $\frac{\sqrt{6}}{2}$ **b** $\frac{2\sqrt{7}}{7}$ **c** $\sqrt{3}$ **d** $\frac{4\sqrt{35}}{7}$

e $\frac{5\sqrt{2}}{6}$ **f** $\frac{7\sqrt{5}}{20}$ **g** $\frac{2\sqrt{15}}{5}$ **h** $-\frac{\sqrt{5}}{2}$

i $\frac{\sqrt{7}}{49}$ **j** $\frac{2\sqrt{110}}{5}$ **k** $\frac{4\sqrt{10}}{3}$ **l** $\frac{9\sqrt{30}}{20}$

4 **a** $-\sqrt{3}$ **b** $-\frac{\sqrt{10}}{2}$ **c** $\frac{4\sqrt{14}}{21}$ **d** $\sqrt{3}$

e $\frac{6\sqrt{5}}{7}$ **f** -6 **g** $\frac{5\sqrt{x}}{x}$ **h** $\frac{\sqrt{y}}{y}$

i $\frac{2x\sqrt{2y}}{y}$ **j** $\frac{\sqrt{xy}}{xy}$ **k** $\frac{\sqrt{2xy}}{2xy}$ **l** $-\frac{\sqrt{xyz}}{2xyz}$

5 **a** $3-\sqrt{2}$ **b** $2+\sqrt{3}$ **c** $\sqrt{6}+5$

d $-4-3\sqrt{5}$ **e** $-3\sqrt{5}-2$ **f** $4\sqrt{7}-1$

g $1-2\sqrt{3}$ **h** $\sqrt{2}-\sqrt{3}$ **i** $5+\sqrt{3}$

j $5\sqrt{6}+2$ **k** $-3\sqrt{7}-2$ **l** $-5+2\sqrt{5}$

6 **a** $2-\sqrt{3}$ **b** $\frac{9+3\sqrt{2}}{7}$ **c** $\frac{30-5\sqrt{5}}{31}$

d $\frac{-9-12\sqrt{5}}{71}$ **e** $\frac{5\sqrt{2}-6}{7}$ **f** $\frac{12-2\sqrt{3}}{11}$

g $\frac{3\sqrt{5}+3}{4}$ **h** $-6\sqrt{42}+15\sqrt{7}$ **i** $4\sqrt{6}+8\sqrt{2}$

j $\sqrt{15}-2\sqrt{3}$ **k** $\frac{10\sqrt{5}-3\sqrt{30}}{23}$ **l** $\frac{6\sqrt{10}+12\sqrt{5}}{5}$

7 **a** $\frac{2\sqrt{6}-3\sqrt{2}+2\sqrt{3}-3}{3}$

b $\frac{20-5\sqrt{2}+8\sqrt{3}-2\sqrt{6}}{14}$

c $\frac{3+\sqrt{5}+3\sqrt{3}+\sqrt{15}}{4}$

d $\frac{3\sqrt{6}-3+\sqrt{30}-\sqrt{5}}{5}$

e $\frac{6\sqrt{15}-4\sqrt{3}-6\sqrt{5}+4}{41}$

f $\frac{10\sqrt{10}-15\sqrt{2}-2\sqrt{5}+3}{11}$

8 **a** $\frac{x^2+x\sqrt{y}}{x^2-y}$ **b** $\frac{x+y\sqrt{x}}{x-y^2}$

c $\frac{x^2\sqrt{x}+xy\sqrt{x}-xy-y^2}{x^3-y^2}$ **d** $\frac{x^2-2x\sqrt{y}+y}{x^2-y}$

9 **a** $\frac{3\sqrt{2}}{2}$ **b** $\frac{3\sqrt{2}+1}{2}$

10 **a** $\frac{3\sqrt{5}+1}{5}$ **b** $\frac{4\sqrt{3}-1}{3}$ **c** $\frac{9\sqrt{2}+4\sqrt{3}}{6}$

d $\frac{\sqrt{3}+9\sqrt{2}}{12}$ **e** $\frac{9\sqrt{5}-\sqrt{6}}{9}$ **f** $\frac{12\sqrt{6}+5\sqrt{15}}{60}$

11 **a** $\frac{\sqrt{5}+1}{4}+\frac{3\sqrt{5}-3}{4}$ **b** $\frac{2\sqrt{5}-1}{2}$

12 **a** $2\sqrt{6}$ **b** -8

c $\frac{2\sqrt{6}+5\sqrt{3}-3}{5}$ **d** $5\sqrt{2}-4\sqrt{5}-13$

e $\frac{106\sqrt{3}-5\sqrt{6}}{238}$ **f** $\frac{16\sqrt{3}-25}{13}$

13 **a** $\frac{\sqrt{2}}{2}$ **b** $\frac{\sqrt{3}}{15}$ **c** $\frac{\sqrt{7}}{2}$

d $\frac{4\sqrt{3}}{9}$ **e** $\frac{\sqrt{6}-2}{2}$ **f** $\frac{7-3\sqrt{5}}{2}$

14 **a** 2 cm^2 **b** 1 cm^2 **c** $\frac{1}{\sqrt{2}}\text{ cm}$

d $\frac{1}{2}$ **e** $\frac{1}{2}\sqrt{2}\text{ cm}$

f The area of the shaded rectangle is given by both $\frac{1}{\sqrt{2}}$ and $\frac{1}{2}\sqrt{2}$. Therefore, they are equal because rationalising $\frac{1}{\sqrt{2}}$ gives $\frac{1}{2}\sqrt{2}$.

g $\frac{7}{10}\sqrt{10}\text{ cm}$

15 **a** **i** $\sqrt{2}-\sqrt{3}, -\sqrt{2}+\sqrt{3}$

ii $\sqrt{5}+\sqrt{3}, -\sqrt{5}-\sqrt{3}$

iii $2\sqrt{5}-5\sqrt{6}, -2\sqrt{5}+5\sqrt{6}$

iv $2\sqrt{5}+3\sqrt{7}, -2\sqrt{5}-3\sqrt{7}$

b **i** -1 **ii** 1 **iii** $-2\sqrt{6}-5$

iv $2\sqrt{6}-5$ **v** 1

c **i** $\sqrt{5}+\sqrt{3}$ or $-\sqrt{5}-\sqrt{3}$

ii $2\sqrt{2}-5\sqrt{5}$ or $-2\sqrt{2}+5\sqrt{5}$

iii $\sqrt{11}+3\sqrt{2}$ or $-\sqrt{11}-3\sqrt{2}$

iv $5\sqrt{13}-4\sqrt{3}$ or $-5\sqrt{13}+4\sqrt{3}$

d **i** $\frac{\sqrt{5}+\sqrt{3}}{2}$ **ii** $\frac{5\sqrt{5}-2\sqrt{2}}{117}$

iii $\frac{\sqrt{11}+3\sqrt{2}}{7}$ **iv** $\frac{4\sqrt{3}-5\sqrt{13}}{277}$

e **i** Lucy multiplied by the conjugate that is the negative of the denominator, so it did not rationalise. The product is not the difference of two squares, it is the negative of the square of the denominator.

ii Lucy could use the conjugate of her new denominator $-2\sqrt{6}-5$ and multiply her final line by $\frac{-2\sqrt{6}-5}{-2\sqrt{6}-5}$.

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f **i** $\sqrt{5}-\sqrt{3}$ **ii** $-4\sqrt{6}-8$

iii $\dfrac{\sqrt{15}-\sqrt{6}}{3}$ **iv** $\dfrac{3\sqrt{30}-10\sqrt{2}}{7}$

v $2\sqrt{5}+2\sqrt{2}$

vi $\dfrac{15\sqrt{6}-10\sqrt{10}-3\sqrt{3}+2\sqrt{5}}{7}$

vii -1

viii $\dfrac{x^2\sqrt{x}-x\sqrt{y}+xy\sqrt{x}-y\sqrt{y}}{x^3-y}$

g $2\sqrt{6}$

h $\dfrac{\sqrt{6}(\sqrt{2}+\sqrt{3}-\sqrt{5})}{3}=\dfrac{2\sqrt{3}+3\sqrt{2}-\sqrt{30}}{3}$

EX p82 Checkpoint

1 **a** surd **b** not surd **c** not surd
d not surd **e** surd **f** not surd
g not surd **h** surd

2 **a** $2\sqrt{2}$ **b** $3\sqrt{5}$ **c** $5\sqrt{7}$ **d** $10\sqrt{6}$

3 **a** $10\sqrt{3}$ **b** $21\sqrt{6}$ **c** $150\sqrt{2}$ **d** $80\sqrt{5}$

4 **a** $\sqrt{35}$ **b** $-2\sqrt{77}$ **c** $-90\sqrt{2}$ **d** $12\sqrt{14}$

5 **a** $\sqrt{6}$ **b** $-\dfrac{2\sqrt{17}}{3}$ **c** $-\dfrac{3}{2}$ **d** $\dfrac{5\sqrt{3}}{2}$
e $-\dfrac{\sqrt{30}}{3}$ **f** $\dfrac{6\sqrt{74}}{37}$

6 **a** 6 **b** $\dfrac{2\sqrt{6}}{9}$

7 **a** $12\sqrt{2}$ **b** $-6\sqrt{2}$
c $12\sqrt{3}+18\sqrt{5}$ **d** $17\sqrt{11}-2\sqrt{7}$

8 **a** $13\sqrt{2}$ **b** $2\sqrt{3}$
c $8\sqrt{6}+14\sqrt{3}$ **d** $17\sqrt{10}-17\sqrt{2}$

9 **a** $6\sqrt{3}$ **b** $18\sqrt{6}-18\sqrt{2}$
c $2\sqrt{7}+3\sqrt{5}-\sqrt{35}-6$
d $15\sqrt{15}+10\sqrt{10}+8\sqrt{6}+36$
e $30\sqrt{2}-54\sqrt{30}$
f $10\sqrt{154}-12\sqrt{14}+160\sqrt{11}-192$

10 **a** $\dfrac{5\sqrt{2}}{2}$ **b** $4\sqrt{3}$ **c** $\dfrac{2\sqrt{35}}{7}$ **d** $\dfrac{\sqrt{66}}{6}$

11 **a** $5\sqrt{2}-5$ **b** $\dfrac{\sqrt{3}+1}{2}$
c $\dfrac{4\sqrt{5}+30}{41}$ **d** $\dfrac{-13\sqrt{7}-35}{14}$
e $\dfrac{9\sqrt{5}-47}{164}$ **f** $\dfrac{-\sqrt{10}-8\sqrt{5}-2\sqrt{3}-8\sqrt{6}}{62}$

EX p85 2.5 Fractional exponents

1 **a** $\sqrt{5}$ **b** $\sqrt[3]{12^2}$ **c** $\sqrt[4]{x^3}$
d $\sqrt[3]{32^2}$ **e** $\sqrt[4]{100}$ **f** $\sqrt[5]{x^4}$

2 **a** $6^{\frac{1}{2}}$ **b** $x^{\frac{2}{3}}$ **c** $5^{\frac{3}{2}}$
d $x^{\frac{3}{2}}$ **e** $x^{\frac{5}{2}}$ **f** $x^{\frac{3}{4}}$

3 **a** $x^{\frac{3}{5}}$ **b** $x^{\frac{1}{2}}$ **c** $x^{\frac{7}{8}}$ **d** $x^{\frac{17}{30}}$
e x **f** $x^{\frac{11}{12}}$ **g** $x^{\frac{1}{2}}$ **h** $x^{\frac{1}{2}}$

4 **a** $x^{\frac{4}{11}}$ **b** $x^{\frac{1}{4}}$ **c** $x^{\frac{1}{14}}$ **d** $x^{\frac{1}{20}}$
e $x^{\frac{1}{6}}$ **f** $x^{\frac{1}{10}}$ **g** $x^{\frac{7}{4}}$ **h** 1

5 **a** $x^{\frac{15}{28}}$ **b** $x^{\frac{1}{5}}$ **c** 1 **d** $x^{\frac{3}{10}}$
e $x^{\frac{1}{18}}$ **f** $x^{\frac{1}{2}}$ **g** x^{12} **h** $x^{\frac{1}{35}}$

6 **a** $10x^{\frac{7}{12}}$ **b** $30x^{\frac{31}{40}}$ **c** $-36x^{\frac{41}{63}}$ **d** $4x^{\frac{1}{12}}$
e $2x^{\frac{9}{40}}$ **f** $4x^{\frac{1}{30}}$ **g** $6x^{\frac{5}{6}}$ **h** $\dfrac{1}{4}x^{\frac{1}{10}}$

7 **a** $5^{\frac{3}{5}}x^{\frac{1}{5}}$ **b** $16^{\frac{1}{4}}x^{\frac{1}{10}}$ **c** $27^{\frac{2}{3}}x^{\frac{1}{4}}$ **d** $2x^{\frac{1}{5}}y^{\frac{3}{8}}$
e $4x^{\frac{2}{15}}y^{\frac{2}{7}}$ **f** $5x^{\frac{1}{4}}y^{\frac{3}{10}}$ **g** $5xy^{\frac{3}{10}}$ **h** $7x^{\frac{1}{3}}y^{\frac{20}{33}}$

8 **a** $28x^{\frac{7}{12}}y^{\frac{19}{20}}$ **b** $-72x^{\frac{3}{2}}y$ **c** $45x^{\frac{19}{24}}y^{\frac{7}{5}}$
d $5x^{\frac{2}{5}}y^{\frac{1}{6}}$ **e** $2x^{\frac{7}{40}}y^{\frac{4}{33}}$ **f** $x^{\frac{1}{14}}y^{\frac{1}{35}}$

9 **a** $a^m \times a^n = a^{m+n}$ **b** 3
c 3 **d** The answers are the same.
e **i** 2 **ii** 5 **iii** 10 **iv** x
f The fractional power of $\dfrac{1}{2}$ performs the same operation as finding the square root ($\sqrt{}$).

10 **a** 3 **b** 3
c The answers are the same.
d **i** 2 **ii** 5 **iii** 10 **iv** x
e $\sqrt[3]{}$ is a cube root and $\sqrt{}$ is a square root.
f The fractional power of $\dfrac{1}{3}$ performs the same operation as finding the cube root ($\sqrt[3]{}$).

11 **a** $(x^2)^{\frac{1}{3}} = x^{2\times\frac{1}{3}} = x^{\frac{2}{3}}$ and $\left(x^{\frac{1}{3}}\right)^2 = x^{\frac{1}{3}\times 2} = x^{\frac{2}{3}}$
b Since $(x^2)^{\frac{1}{3}}$ and $\left(x^{\frac{1}{3}}\right)^2$ are both equal to $x^{\frac{2}{3}}$, we only need to show that $x^{\frac{2}{3}}$ is equal to $\sqrt[3]{x^2}$ and $\left(\sqrt[3]{x}\right)^2$. So $x^{\frac{2}{3}} = x^{\frac{1}{3}\times 2} = \left(x^{\frac{1}{3}}\right)^2 = \left(\sqrt[3]{x}\right)^2$ and $x^{\frac{2}{3}} = x^{2\times\frac{1}{3}} = (x^2)^{\frac{1}{3}} = \sqrt[3]{x^2}$.

12 $x^{\frac{m}{n}} = x^{m\times\frac{1}{n}} = (x^m)^{\frac{1}{n}} = \sqrt[n]{x^m}$ and $x^{\frac{m}{n}} = x^{\frac{1}{n}\times m} = \left(x^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{x}\right)^m$

13 **a** 5 **b** 6 **c** 1 **d** 12
e 10 **f** 14 **g** 2 **h** 3
i 5 **j** 1 **k** 1 **l** 2

14 **i** The answers would be the reciprocal of the original answers.
ii **a** $\dfrac{1}{5}$ **b** $\dfrac{1}{6}$ **c** 1 **d** $\dfrac{1}{12}$
e $\dfrac{1}{10}$ **f** $\dfrac{1}{14}$ **g** $\dfrac{1}{2}$ **h** $\dfrac{1}{3}$
i $\dfrac{1}{5}$ **j** 1 **k** 1 **l** $\dfrac{1}{2}$
iii Parts **g**, **h**, **i**, **j**, **k** and **l** can have a negative base as the root is odd (3 or 5). The cube root and 5th root of a negative number are real numbers as $(-a)^3 = -a$ so $\sqrt[3]{-a} = -a$, but the square root of a negative number is not a real number.
iv **g** $-\dfrac{1}{2}$ **h** $-\dfrac{1}{3}$ **i** $-\dfrac{1}{5}$
j -1 **k** -1 **l** $-\dfrac{1}{2}$

15 **a** $3\sqrt{2}$ **b** $2\sqrt{2}$ **c** $2\sqrt{6}$ **d** $6\sqrt{3}$
e $2\sqrt{38}$ **f** $3\sqrt[3]{2}$ **g** $4\sqrt[3]{2}$ **h** $8\sqrt{5}$
i $9\sqrt{3}$ **j** $12\sqrt{3}$ **k** $5\sqrt{5}$ **l** $6\sqrt{6}$

16 **a** $x^{\frac{3}{2}}y^{\frac{1}{2}}$ **b** $x^{\frac{2}{3}}y^{\frac{1}{3}}$ **c** $x^{\frac{3}{2}}y^{\frac{5}{2}}$
d $3^{\frac{1}{2}}x^{\frac{1}{2}}y^{\frac{3}{2}}$ **e** $4x^{-\frac{1}{2}}y^{\frac{1}{2}}$ **f** $x^{\frac{4}{3}}y^{\frac{2}{3}}$

17 **a** $108x^{\frac{9}{20}}y^{\frac{13}{15}}$ **b** $\dfrac{1}{5}x^{\frac{2}{9}}y^{\frac{7}{20}}$ **c** $2x^{\frac{1}{5}}y^{\frac{1}{4}}$
d $6x^4y^2$ **e** $x^{\frac{3}{2}}y^{\frac{1}{2}}$ **f** $x^{\frac{1}{2}}y^{\frac{5}{6}}$
g $96^{\frac{1}{2}}x^2y^{\frac{1}{2}}z^4$ **h** $x^4y^{\frac{2}{9}}z^{\frac{2}{3}}$

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18 **a** $4x\sqrt{y}$ **b** $\dfrac{2y^3\sqrt{3y}}{x\sqrt{x}}$ **c** $xy\sqrt{3x}$

d $3\sqrt[3]{xy^2}$ **e** $2\sqrt[4]{x^3y^2z}$ **f** $\dfrac{7y^2\sqrt{7y}}{\sqrt{x}}$

19 **a** $18x^{\frac{5}{12}}y^{\frac{7}{15}}$ **b** $\dfrac{72y^{\frac{1}{12}}}{x^{\frac{5}{12}}}$ **c** $\dfrac{x^{\frac{2}{3}}}{y^{\frac{1}{2}}}$

d $\dfrac{1}{4x^{\frac{1}{15}}y^{\frac{7}{20}}}$ **e** $\dfrac{1}{x^{\frac{1}{12}}y^{\frac{3}{4}}}$ **f** $\dfrac{16}{x^{\frac{8}{5}}y^5}$

g $10x^{\frac{7}{30}}$ **h** $\dfrac{y^{\frac{13}{30}}}{3^{\frac{1}{2}}x^{\frac{1}{24}}}$

20 **a** $\dfrac{3^{\frac{2}{3}}}{9}$ or $3^{-\frac{4}{3}}$ **b** $x = -\dfrac{2}{3}$

21 **a** $x = -\dfrac{5}{2}$ **b** $x = 11$ **c** $x = -\dfrac{1}{2}$

22 **a** Both denominators specify how many repeated parts the base is broken into. When multiplying by a fraction, the parts are added; when raising to the power, the parts are multiplied. Both numerators describe the number of those repeated parts to operate together.

b **i** 3 **ii** 27 **iii** 243

iv 823 543 **v** 59 049 **vi** 8

c **i** $81^{\frac{1}{4}} = \sqrt[4]{81}$ **ii** $81^{\frac{3}{4}} = \left(\sqrt[4]{81}\right)^3$

iii $81^{\frac{5}{4}} = \left(\sqrt[4]{81}\right)^5$ **iv** $343^{\frac{7}{3}} = \left(\sqrt[3]{343}\right)^7$

v $81^{\frac{5}{2}} = \left(\sqrt[2]{81}\right)^5 = \left(\sqrt{81}\right)^5$ **vi** $256^{\frac{3}{8}} = \left(\sqrt[8]{256}\right)^3$

d The denominator of the exponent and the root of a surd both specify how many repeated parts the base is broken into. The numerator of the exponent and the exponent of a surd describe the number of those repeated factors of the base to multiply together.

23 **a** $\sqrt{ab} = (ab)^{\frac{1}{2}}$
$= a^{\frac{1}{2}} \times b^{\frac{1}{2}}$
$= \sqrt{a} \times \sqrt{b}$

b $\sqrt{\dfrac{a}{b}} = \left(\dfrac{a}{b}\right)^{\frac{1}{2}}$
$= \dfrac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}$
$= \dfrac{\sqrt{a}}{\sqrt{b}}$

24 **a** $\sqrt{10^5}$ **b** $\sqrt[4]{2^{17}}$ **c** $\sqrt[25]{3^4}$ **d** $\sqrt[200]{5^{1111}}$

EX p90 2.6 Logarithms

1 **a** $\log_3(27) = 3$ **b** $\log_4(4) = 1$ **c** $\log_5(625) = 4$

d $\log_{10}(1000) = 3$ **e** $\log_6(36) = 2$ **f** $\log_9(729) = 3$

g $\log_3(81) = 4$ **h** $\log_2(128) = 7$

2 **a** $2^5 = 32$ **b** $3^2 = 9$ **c** $5^2 = 25$

d $7^2 = 49$ **e** $4^2 = 16$ **f** $2^{10} = 1024$

g $6^3 = 216$ **h** $10^2 = 100$

3 **a** 4 **b** 4 **c** 3 **d** 3 **e** 2 **f** 3

g 2 **h** 5 **i** 4 **j** 4 **k** 2 **l** 4

m 3 **n** 3 **o** 2 **p** 5

4 **a** **i** $\dfrac{1}{6}$ **ii** $\log_6\left(\dfrac{1}{6}\right) = -1$

b **i** $\dfrac{1}{4}$ **ii** $\log_4\left(\dfrac{1}{4}\right) = -1$

c **i** $\dfrac{1}{4}$ **ii** $\log_2\left(\dfrac{1}{4}\right) = -2$

d **i** $\dfrac{1}{32}$ **ii** $\log_2\left(\dfrac{1}{32}\right) = -5$

e **i** $\dfrac{1}{25}$ **ii** $\log_5\left(\dfrac{1}{25}\right) = -2$

f **i** $\dfrac{1}{10}$ **ii** $\log_{10}\left(\dfrac{1}{10}\right) = -1$

g **i** $\dfrac{1}{1000}$ **ii** $\log_{10}\left(\dfrac{1}{1000}\right) = -3$

h **i** $\dfrac{1}{27}$ **ii** $\log_3\left(\dfrac{1}{27}\right) = -3$

i **i** 2 **ii** $\log_{\frac{1}{2}}(2) = -1$

j **i** 9 **ii** $\log_{\frac{1}{3}}(9) = -2$

k **i** 125 **ii** $\log_{\frac{1}{5}}(125) = -3$

l **i** 100 **ii** $\log_{0.01}(100) = -1$

5 **a** −3 **b** −3 **c** −2 **d** −1

e −4 **f** −6 **g** −3 **h** −2

6 **a** **i** 16 **ii** $\dfrac{1}{16} = 0.0625$

b no real number

c **i** no real number **ii** no real number

iii no real number **iv** no real number

v no real number **vi** no real number

d There is no real logarithm of a negative number to any positive integer base.

7 **a** no

b **i** no **ii** no **iii** no

iv no **v** no

c There is no power of base 1 that gives an answer other than 1, so there is no logarithm of any other number to base 1.

8 **a** Raise 2 to the power of 7. Then, take the logarithm with base 2 of 2^7. The result is 7.

b **i** 7 **ii** 5 **iii** 9

iv 140 **v** −6 **vi** 123

c The logarithm with base a is the exponent to raise a to get a^x. x is the exponent to raise a to get a^x. Therefore, $\log_a(a^x) = x$.

9 **a** Take the logarithm to base 2 of 7. Then, raise 2 to the power of that. The result is 7.

b **i** 7 **ii** 5 **iii** 11

iv 531 **v** 8 **vi** 147

c The logarithm to base a is the exponent, y, to raise a to get x. We raise a to that exponent, y. Therefore, $a^{\log_a(x)} = x$.

10 **a** $\log_x(64) = 2$
$x^{\log_x(64)} = x^2$
$64 = x^2$
$x = 8$

b 8, as $x > 0$ so $x = -8$, while a solution to $x^2 = 64$ is not a solution for $\log_x(64) = 2$.

c $x^2 = 8^2$ **d** It is the same.

11 **a** 2 **b** 5 **c** 4 **d** 2

e 2 **f** 3 **g** 4 **h** 2

12 **a** $\log_4(x) = 3$
$4^{\log_4(x)} = 4^3$

b 64 $x = 4^3$

c It can be easily solved by multiplying the base by itself the number of times indicated by the exponent.

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13 **a** 27 **b** 25 **c** 6 **d** 10 000
e 32 **f** 64 **g** $\frac{1}{27}$ **h** $\frac{1}{4}$

14 **a** $(a^2)^{\frac{1}{2}} = a^{2\times\frac{1}{2}} = a^1 = a$
$(a^{\frac{1}{2}})^2 = a^{\frac{1}{2}\times 2} = a^1 = a$

b $12^2 = 144$
$12 = 144^{\frac{1}{2}}$
$\log_{144}(12) = \log_{144}(144^{\frac{1}{2}})$
$\log_{144}(12) = \frac{1}{2}$

c $\frac{1}{x}$

15 **a** $\frac{1}{3}$ **b** $\frac{1}{6}$ **c** $\frac{1}{4}$ **d** $\frac{1}{9}$

16 **a** 1 **b** 1 **c** 1 **d** 1

17 **a** $\frac{1}{2}$ **b** 2 **c** $-\frac{3}{2}$ **d** $\frac{7}{3}$
e $\frac{3}{2}$ **f** $\frac{3}{8}$ **g** 2 **h** $-\frac{1}{3}$

EX p95 2.7 Logarithm laws

1 **a** 1 **b** 1 **c** 1 **d** 4
e 3 **f** 0 **g** 0 **h** 0

2 **a** $\log_{10}(3) + \log_{10}(5)$ **b** $\log_2(6) + \log_2(3)$
c $\log_5(2.5) + \log_5(10)$ **d** $\log_2(x) + \log_2(y)$

3 **a** $\log_3(12)$ **b** $\log_{10}(15)$ **c** $\log_2(50)$
d $\log_4(72)$ **e** $\log_a(15)$ **f** $\log_2(xy)$

4 **a** $\log_{10}(5) - \log_{10}(7)$ **b** $\log_2(5) - \log_2(3)$
c $\log_5(11) - \log_5(20)$ **d** $\log_2(a) - \log_2(b)$

5 **a** $\log_3\left(\frac{12}{5}\right)$ **b** $\log_{10}\left(\frac{50}{15}\right)$ **c** $\log_2\left(\frac{25}{10}\right)$
d $\log_4\left(\frac{36}{9}\right)$ **e** $\log_a\left(\frac{3}{15}\right)$ **f** $\log_2\left(\frac{x}{y}\right)$

6 **a** 4 **b** 2 **c** 5 **d** 4 **e** 2 **f** 3
g 2 **h** 2 **i** 9 **j** 2 **k** 2 **l** 4

7 **a** 2 **b** 2 **c** 3

8 **a** $2 \times \log_3(7)$ **b** $9 \times \log_5(2)$
c $3 \times \log_4(8)$ **d** $4 \times \log_9(11)$

9 **a** 5 **b** 2 **c** 5 **d** 8
e 9 **f** 12 **g** 10 **h** 2

10 **a** −2 **b** −1 **c** −2 **d** −14
e −30 **f** −21 **g** $-\frac{1}{3}$ **h** −5

11 **a** 1 **b** 4 **c** 4 **d** $\frac{1}{2}$ **e** 1 **f** 1

12 **a** 3 **b** 2 **c** 2 **d** −1 **e** −2 **f** 3

13 **a** 3 **b** 2 **c** 6 **d** 4 **e** 1
f 7 **g** 5 **h** 3 **i** 3 **j** 3

14 **a** 3 **b** 3 **c** 9 **d** 4 **e** 216 **f** 64
g −3 **h** 36 **i** 1 **j** $\frac{1}{2}$ **k** $\frac{1}{2}$ **l** 4

15 **a** 0.5
b $\log_{10}(9) = \log_{10}(3^2) = 2\log_{10}(3)$
c $\frac{\log_{10}(3)}{\log_{10}(9)} = \frac{\log_{10}(3)}{2\log_{10}(3)} = \frac{1}{2} = 0.5$

16 Hailey, because $\frac{\log_3(8)}{\log_3(2)} = \frac{3\log_3(2)}{\log_3(2)} = 3$

17 **a** 4 **b** $\frac{3}{2}$ **c** 3 **d** $\frac{3}{4}$ **e** $\frac{2}{3}$ **f** 3

18 **a** **i** $\log_5(2x) + \log_5(3) - \log_5(12) = 0$
$\log_5(6x) - \log_5(12) = 0$
$\log_5\left(\frac{6x}{12}\right) = 0$
$\log_5\left(\frac{x}{2}\right) = 0$
ii $5^0 = \frac{x}{2}, \frac{x}{2} = 1, x = 2$
b **i** $\log_5(2x) + \log_5(3) = \log_5(6x) = \log_5(12)$
ii $6x = 12, x = 2$

19 **a** 3 **b** 4 **c** 20 **d** $\frac{2}{3}$
e $\frac{1}{14}$ **f** $\frac{25\,000}{3}$ or $8333\frac{1}{3}$

20 **a** 1.545 **b** 0.431 **c** 1.114

21 $x = 4$ or $x = 5$

22
$$\log_{10}(2^{2x+1}) = \log_{10}(4)$$
$$(2x + 1)\log_{10}(2) = \log_{10}(2^2)$$
$$(2x + 1)\log_{10}(2) = 2\log_{10}(2)$$
$$2x + 1 = 2$$
$$2x = 1$$
$$x = \frac{1}{2}$$

23 $y = 77$

EX p99 2.8 Exponential equations

1 **a** $x = 9$ **b** $x = 6$ **c** $x = -3$ **d** $x = 1$
e $x = 8$ **f** $x = 4$ **g** $x = 7$ **h** $x = -10$
i $x = 3$ **j** $x = 5$ **k** $x = 6$ **l** $x = 0$

2 **a** $x = 3$ **b** $x = 2$ **c** $x = 5$ **d** $x = 4$
e $x = 7$ **f** $x = 3$ **g** $x = 2$ **h** $x = -1$

3 **a** $x = \log_2(11)$ **b** $x = \log_5(92)$
c $x = \log_{92}(5)$ **d** $x = -\log_{14}(4)$
e $x = -\log_3(2)$ **f** $x = \log_7\left(\frac{53}{2}\right)$
g $x = \frac{1}{2}\log_5(17)$ **h** $x = \frac{1}{3}\log_{10}(75)$
i $x = \log_8(33)$ **j** $x = \frac{1}{6}\log_{11}(55)$
k $x = \log_{17}(12)$ **l** $x = -\frac{1}{4}\log_{12}\left(\frac{46}{9}\right)$

4 **a** $x = \log_8(10) + 1$ **b** $x = \log_3(12) - 1$
c $x = \log_5(2) - 3$ **d** $x = \log_2(6) + 5$
e $x = -\log_2(48) + 3$ **f** $x = -\log_9(20) + 2$
g $x = -\log_6(4) + 3$ **h** $x = -\log_2(30) + 6$
i $x = \frac{1}{2}\log_4(24) - \frac{3}{2}$ **j** $x = \frac{1}{4}\log_3(87) - \frac{1}{4}$
k $x = \frac{1}{5}\log_7(3) - \frac{2}{5}$ **l** $x = \frac{1}{4}\log_{10}\left(\frac{3}{2}\right) - \frac{3}{4}$

5 **a** $x = 2$ **b** $x = 3, x = -3$
c $x = 2, x = -2$

6 **a** $x = 2, x = 3$ **b** $x = 3, x = 4$
c $x = 6, x = 4$ **d** $x = 0, x = 1, x = 2$
e $x = -5, x = -3$ **f** $x = 1, x = \frac{5}{3}$

7 **a** 2 and 3 **b** 5 and 6 **c** 1 and 2
d 3 and 4 **e** 1 and 2 **f** 2 and 3
g 1 and 2 **h** 4 and 5

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8 **a** $\frac{9}{32} \times 64^x = 27$ **b** $x = \log_{64}(96)$

9 **a** The solution to the equation would be $x = \log_3(-1)$. However, the logarithm of a negative number is not a real number. Therefore, the equation has no real solution.

b The solution to the equation would be $x = \log_3(0)$. However, the logarithm of 0 is not a defined value. Therefore, the equation has no real solution.

10 **a** $x = 3$ **b** $x = 1$

c $x = \log_{10}(15)$ **d** $x = 1, x = 2$

e $x = \log_3(7)$

11 **a** 100°C **b** 61°C **c** 4.4 **d** 20°C

e decay; temperature decreases as the amount of time increases

12 **a** 2 **b** 82 **c** 3.14

d population growth; population increases as the amount of time increases

13 **a** **i** 8.5 billion **ii** 11.9 billion

b 2053. It will reach 15 billion after 103.96 years, or very close to the end of 2053.

c The model predicted 7.1 billion, so the 2010 figure was slightly less than the prediction; there have been changes to birth and death rates so, despite the population continuing to grow, the overall rate of population growth has decreased slightly since 1950.

14 **a** 45 bacteria

b **i** $3\log_2(10)$ hours **ii** 10 hours

c 750 000 000 bacteria

d 5025 bacteria

e

t	66	67	68	69	70
N	204	202	200	198	196

f 68 hours

g $\log_{0.99}\left(\frac{1}{2}\right) - 1 = \log_{0.99}\left(\frac{50}{99}\right)$ hours

15 **a** $k = 0.13$ **b** 11 minutes

c **i** 1.5 min or 1 min 30 sec or 90 sec

ii $m = 0.28$

16 $x = -4, x = -2, x = 3, x = 7$

17 **a** **i** $\frac{\log_{10}(7)}{\log_{10}(3)}$ **ii** $\frac{\log_{10}(3)}{\log_{10}(7)}$ **iii** $\frac{\log_{10}(45)}{\log_{10}(11)}$

iv $\frac{\log_{10}(12)}{\log_{10}(101)}$ **v** $\frac{\log_{10}(24)}{\log_{10}(19)}$ **vi** $\frac{\log_{10}(b)}{\log_{10}(a)}$

b **i** $\log_9(8)$ **ii** $\log_8(9)$ **iii** $\log_5(18)$

iv $\log_{15}(6)$ **v** $\log_2(75)$ **vi** $\log_t(v)$

c $2^x = 54$

d
$$x = \log_2(54)$$
$$2^x = 54$$
$$\log_{10}(2^x) = \log_{10}(54)$$
$$x\log_{10}(2) = \log_{10}(54)$$
$$x = \frac{\log_{10}(54)}{\log_{10}(2)}$$

e
$$x = \log_b(a)$$
$$b^x = b^{\log_b(a)}$$
$$b^x = a$$
$$\log_{10}(b^x) = \log_{10}(a)$$
$$x\log_{10}(b) = \log_{10}(a)$$
$$x = \frac{\log_{10}(a)}{\log_{10}(b)}$$

18 **a** 1.05 **b** 2.11 **c** 13.0 **d** 0.726

e −3.18 **f** 1.03 **g** 0.377 **h** 4.87

i −0.215

EX p104

Review: Real numbers

oxforddigital **Answers for this lesson can be found on Oxford Digital.**

MODULE 3 Linear relationships

EX p113

3.1 Solving linear equations

1 **a** $x = \frac{19}{3}$ **b** $x = -40$ **c** $x = \frac{2}{7}$

d $x = \frac{24}{5}$ **e** $x = -53$ **f** $x = -3$

g $x = -\frac{5}{3}$ **h** $x = -8$ **i** $x = \frac{29}{12}$

j $x = -\frac{19}{2}$ **k** $x = -\frac{43}{6}$ **l** $x = \frac{62}{5}$

m $x = -\frac{32}{9}$ **n** $x = \frac{15}{2}$ **o** $x = -\frac{7}{3}$

2 **a** $x = -8\frac{1}{2}$ **b** $x = 1.7$ **c** $x = 4$

d $x = -2.464$ **e** $x = 2.4$ **f** $x = \frac{15}{77}$

g $x = \frac{1}{2}$ **h** $x = -\frac{1}{3}$ **i** $x = -\frac{20}{21}$

3 **a** $x = 2$ **b** $x = 5$ **c** $x = -3$

d $x = 1$ **e** $x = -7$ **f** $x = -2$

g $x = 3$ **h** $x = 4$ **i** $x = \frac{3}{7}$

j $x = -4$ **k** $x = 2$ **l** $x = -5$

4 **a** $x = -5$ **b** $x = 3$ **c** $x = -2$

d $x = 6$ **e** $x = -\frac{1}{3}$ **f** $x = 7$

g $x = 3$ **h** $x = \frac{1}{9}$ **i** $x = 6$

5 **a** $x = 5$ **b** $x = -2$ **c** $x = 3$

d $x = -1$ **e** $x = 6$ **f** $x = 9$

g $x = -4$ **h** $x = 8$ **i** $x = 4$

6 **a** $x = -1$ **b** $x = 2$ **c** $x = 5$

d $x = \frac{1}{2}$ **e** $x = \frac{4}{15}$ **f** $x = -2$

7 **a** $x = 4$ **b** $x = \frac{1}{4}$ **c** $x = 5$

d $x = -5$ **e** $x = -3$ **f** $x = 1$

8 **a** $x = b - a$ **b** $x = k + p$ **c** $x = \frac{d}{c}$

d $x = \frac{h-g}{3}$ **e** $x = \frac{y-5}{4}$ **f** $x = mn - a$

g $x = \frac{5+2w}{7}$ **h** $x = a(b-c)$ **i** $x = \frac{5c-3d}{8}$

j $x = \frac{e+f}{2}$ **k** $x = \frac{v-2y}{2}$ **l** $x = p(w-q)$

9 **a** Decide that n = the number of weeks of saving. Any letter will do.

b $24n + 105 = 219$ **c** $n = 4.75$

d It will take Liam 5 weeks to save for his new tennis racquet.

10 **a** Let m be the cost of a makeup kit.

$4m + 29 = 387$

$m = 89.5$

Cost: \$89.50

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b Let the length be l m and the breadth b m.

$l = b + 12$

$2l + 2b = 88$

$2(b + 12) + 2b = 88$

$4b + 24 = 88$

$b = 16, l = 28$

length: 28 m, breadth: 16 m

c Let their scores be l, s and b, respectively.

$l = s + 5, b = s - 3$

$l + s + b = (s + 5) + s + (s - 3)$

$3s + 2 = 29$

$s = 9, l = 14, b = 6$

Lisa: 14 goals, Sara: 9 goals, Bindi: 6 goals

d Let n = the number of people

$27n + 400 = 1800; n = 51.851\ldots$

51 people

11 a $6d + 3.8 = 4d + 9.2$, where d is the cost of a dumpling

b \$2.70

12 a $A = 50$ cm^2 **b** $h = 7$ mm

c $a = 7$ m **d** $a = \frac{2A}{h} - b$

e i $a = 52$ cm **ii** $a = 7.6$ m

13 a $x = \frac{y}{k}$ **b** $k = \frac{y}{x}$

c any four pairs where $\frac{y}{x} = 4$

d any four pairs where $kx = 36$

14 a $K = C + 273.15$

b i 273.15 K **ii** 373.15 K

iii 300 K **iv** 453.15 K

c $C = K - 273.15$

d i 226.85°C **ii** 726.85°C

iii −253.15°C **iv** −93.15°C

15 a i −17.78°C **ii** 37.78°C

iii 10°C **iv** 30°C

b $F = \frac{9}{5}C + 32$ or $F = 1.8C + 32$

c i 86°F **ii** 14°F

iii 68°F **iv** 356°F

16 a $h = 3.45$ **b** $h = \frac{2E - mv^2}{2mg}$ or $h = \frac{E}{mg} - \frac{v^2}{2g}$

c $m = 60.38$ **d** $m = \frac{2E}{2gh + v^2}$

17 $5x + 20 = 3x + 50$; $x = 15$; 15, 17, 19, 21, 23

EX p118

3.2 Solving linear equations with algebraic fractions

1 a $x = -8$ **b** $x = 4$ **c** $x = -2$

d $x = 2$ **e** $x = -1$ **f** $x = 4$

g $x = -7$ **h** $x = 3$ **i** $x = 5$

2 a $x = -8$ **b** $x = 31$ **c** $x = 8$

d $x = 1$ **e** $x = 17$ **f** $x = 13$

g $x = -4\frac{2}{3}$ **h** $x = 6\frac{1}{7}$ **i** $x = -2\frac{1}{8}$

j $x = 4$ **k** $x = -5$ **l** $x = 4\frac{3}{10}$

3 a $x = 2$ **b** $x = -5$ **c** $x = -6$

d $x = -4$ **e** $x = 4$ **f** $x = \frac{15}{2}$

g $x = \frac{10}{3}$ **h** $x = \frac{2}{35}$ **i** $x = 2$

j $x = -1$ **k** $x = \frac{1}{4}$ **l** $x = -\frac{12}{5}$

m $x = -\frac{11}{8}$ **n** $x = \frac{35}{2}$ **o** $x = \frac{5}{6}$

4 a $x = 3$ **b** $x = -3$ **c** $x = 12$

d $x = -\frac{3}{7}$ **e** $x = \frac{14}{15}$ **f** $x = \frac{1}{11}$

g $x = \frac{3}{5}$ **h** $x = -\frac{18}{25}$ **i** $x = 1$

5 a $x = \frac{8}{17}$ **b** $x = -33$ **c** $x = -1$

d $x = 16$ **e** $x = \frac{4}{25}$ **f** $x = 1$

6 a $x = -16$ **b** $x = -\frac{1}{2}$ **c** $x = -\frac{37}{40}$

d $x = \frac{11}{5}$ **e** $x = \frac{31}{37}$ **f** $x = -\frac{17}{5}$

7 a $x = 10$ **b** $x = \frac{1}{10}$

8 a $x = ky$ **b** $x = \frac{k}{y}$ **c** $x = ac - b$

9 a $k = \frac{x}{y}$ **b** $k = xy$

c $b = ac - x, a = \frac{x + b}{c}$

10 a $x(2t + 1) = t(g - x)$

$2tx + x = tg - tx$

$x(3t + 1) = tg$

$x = \frac{tg}{3t + 1}$

b $x = \frac{25}{8} = 3.125$

c $x(2(5) + 1) = 5(10 - x)$

$11x = 50 - 5x$

$16x = 50$

$x = \frac{50}{16} = \frac{25}{8} = 3.125$

11 a i $x = \frac{y}{mk}$ **ii** $k = \frac{y}{mx}$ **iii** $m = \frac{y}{kx}$

b i $x = \frac{ym}{k}$ **ii** $k = \frac{ym}{x}$ **iii** $m = \frac{kx}{y}$

c i $x = \frac{y - m}{k}$ **ii** $k = \frac{y - m}{x}$

iii $m = y - kx$

12 a $y = -\frac{A}{B}x + \frac{D}{B}$

b $m = -\frac{A}{B}$ and $c = \frac{D}{B}$

c $A = -mB$ and $D = cB$

d $-px + qy = qc$ or $px - qy = -qc$

e $y = mx - mh + k$

13 a $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$

$\frac{1}{c} = \frac{1}{a} + \frac{1}{b}$

$\frac{1}{c} = \frac{b}{ab} + \frac{a}{ab}$

$\frac{1}{c} = \frac{a + b}{ab}$

$c = \frac{ab}{a + b}$

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b **i** 3 cm **ii** 400 ohms

c $\frac{1}{a}+\frac{1}{b}=\frac{1}{c}$

$\frac{1}{a}=\frac{1}{c}-\frac{1}{b}$

$\frac{1}{a}=\frac{b}{bc}-\frac{c}{bc}$

$\frac{1}{a}=\frac{b-c}{bc}$

$a=\frac{bc}{b-c}$

d **i** 40 cm **ii** 16 ohms

14 **a** $mt+a=nt+b$ **b** 5 minutes

c $t=\frac{b-a}{m-n}$ or $t=\frac{a-b}{n-m}$

d 5 minutes

15 **a** $c=\frac{ab}{a+b}$ **b** $a=\frac{bc}{b-c}$

16 **a** $x=\frac{33}{5}$ **b** $x=-\frac{41}{34}$ **c** $x=\frac{33}{52}$

17 **a** $x=\frac{p-qr}{nr-m}=\frac{qr-p}{m-nr}$

b $x=\frac{b^2-2aby-a^2}{a^2y-2ab-b^2y}=\frac{a^2+2aby-b^2}{b^2y+2ab-a^2y}$

c $x=-\frac{1}{y}$

EX p124 3.3 Sketching linear graphs

oxforddigital **Answers for this lesson can be found on Oxford Digital.**

EX p132 3.4 Determining linear equations

1 **a** 4 **b** −2 **c** $\frac{1}{2}$

d −1 **e** 5 **f** −3

2 **a** **i** 4 **ii** $y=4x+2$

b **i** 1 **ii** $y=x-7$

c **i** $\frac{1}{2}$ **ii** $y=\frac{1}{2}x+5$

d **i** 10 **ii** $y=10x+6$

e **i** −3 **ii** $y=-3x-5$

f **i** $-\frac{1}{4}$ **ii** $y=-\frac{1}{4}x+4$

3 **a** $y=-2x+4$ **b** $y=x+3$

c $y=2x-6$ **d** $y=-5x+5$

e $x=5$ **f** $y=-2$

4 **a** $y=3x+1$ **b** $y=2x+3$

c $y=-4x-2$ **d** $y=-x+9$

e $y=5x+9$ **f** $y=-3x-1$

g $y=\frac{6}{5}x-14$ **h** $y=-\frac{3}{4}x+1$

i $y=\frac{5}{2}x+\frac{3}{2}$ **j** $y=-\frac{7}{9}x-\frac{2}{3}$

5 **a** $y=2x+1$ **b** $y=6x-28$

c $y=-3x+10$ **d** $y=-2x+4$

e $y=x-1$ **f** $y=-x-6$

g $y=\frac{7}{3}x-\frac{16}{3}$ **h** $y=-\frac{7}{5}x-\frac{24}{5}$

i $y=\frac{1}{6}x+\frac{23}{6}$

6 **a** $y=5x-3$ **b** $y=-2x+4$ **c** $y=\frac{4}{5}x+2$

7 **a** $y=3x-11$ **b** $y=-2x+6$ **c** $y=x+2$

8 **a** $y=2x-5$ **b** $y=\frac{1}{2}x-5$

c $y=-5x+3$ **d** $y=-\frac{1}{5}x+3$

e $y=4x-2$ **f** $y=\frac{3}{2}x+\frac{1}{2}$

g $y=4$ **h** $y=3x$

i $x=-3$ **j** $y=6x-12$

k $y=\frac{5}{12}x+10$ **l** $y=-\frac{4}{7}x+\frac{31}{7}$

9 **a** $m=\frac{5-2}{4-(-2)}$ and $m=\frac{2-5}{-2-4}$

$m=\frac{3}{6}$ $m=\frac{-3}{-6}$

$m=\frac{1}{2}$ $m=\frac{1}{2}$

b Juan has not vertically aligned the corresponding x- and y-coordinates of the same point, so he will end up with the negative of the actual gradient, $-m=-\frac{1}{2}$.

c $m=\frac{y_1-y_2}{x_1-x_2}$

$=\frac{y_1-y_2}{x_1-x_2}\times\frac{-1}{-1}$

$=\frac{-(y_1-y_2)}{-(x_1-x_2)}$

$=\frac{-y_1+y_2}{-x_1+x_2}$

$=\frac{y_2-y_1}{x_2-x_1}$

10 **a** $2=\frac{1}{2}(-2)+c$ and $5=\frac{1}{2}(4)+c$

$2=-1+c$ $5=2+c$

$c=3$ $c=3$

b Juan has substituted the x-coordinate of one of the points for the gradient and the corresponding y-coordinate for the y-intercept.

11 **a** $c=k-mh$

b $y=mx+k-mh$

c $y=mx+k-mh$

$y=mx-mh+k$

$y=m(x-h)+k$

12 **a** $y=\frac{3}{4}(x-8)+2$

b $y=-\frac{7}{3}(x-15)-6$

c $y=\frac{1}{4}(x+3)+5$ or $y=\frac{1}{4}(x-1)+6$

d $y=-\frac{8}{5}(x+6)+5$ or $y=-\frac{8}{5}(x+1)-3$

e $y=-\frac{1}{2}(x+4)+2$ or $y=-\frac{1}{2}(x-2)-1$

f $y=0(x-4)+3$ or $y=0(x+7)+3$

g $y=-\frac{3}{4}(x-4)+0$ or $y=-\frac{3}{4}(x-0)+3$

h $y=\frac{2}{5}(x+5)+0$ or $y=\frac{2}{5}(x-0)+2$

13 **a** $y=\frac{3}{4}x-4$ **b** $y=-\frac{7}{3}x+29$ **c** $y=\frac{1}{4}x+\frac{23}{4}$

d $y=-\frac{8}{5}x-\frac{23}{5}$ **e** $y=-\frac{1}{2}x$ **f** $y=3$

g $y=-\frac{3}{4}x+3$ **h** $y=\frac{2}{5}x+2$

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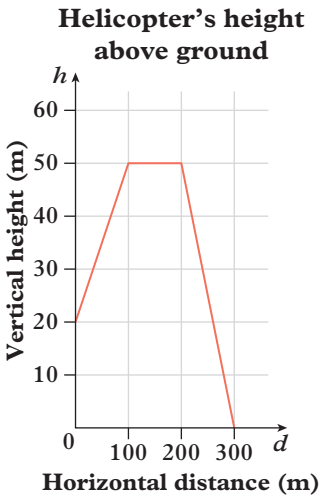
14 a **i** translation of 6 units up
ii translation of 3 units left
iii translation of 2 units left and 2 units up
b **i** $y = 2x + 6$
ii $y = 2(x + 3)$
iii $y = 2(x + 2) + 2$
c **i** translation of 1 unit left and 2 units down
ii translation of 1 unit right and 2 units up
iii translation of 1 unit right and 2 units up
iv translation of 4 units left and 6 units up
v translation of $\frac{3}{2}$ units right and $\frac{4}{3}$ units up
vi translation of 4 units right and 2 units up

15 a $y = \frac{1}{200}x + 50$
b $y = -\frac{1}{15}x + 6$
c $y = \frac{1}{5}x$

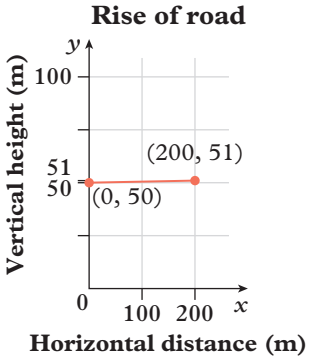
16 a



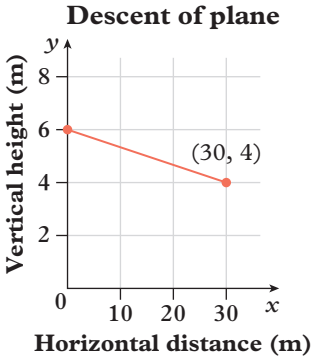
b In the order of the flight: $h = 0.3d + 20$, $h = 50$, $h = -0.5d + 150$
c **i** 29 m **ii** 44 m **iii** 50 m **iv** 30 m
d $66\frac{2}{3}$ m and 220 m

17 a $x - 2y = 4$ **b** $y = -23$
c $2x + 3y + 20 = 0$ **d** $12x - 17y = 0$ or $17y = 12x$

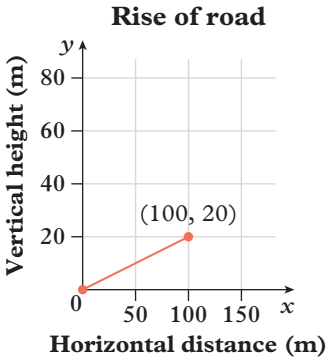
18 a $y = \frac{1}{200}x + 50$, for $0 \le x \le 200$



b $y = -\frac{1}{15}x + 6$, for $0 \le x \le 30$



c $y = \frac{1}{5}x$, for $0 \le x \le 100$



19 a **i** $\frac{3}{2}x + \frac{7}{2}y = 1$ **ii** $3x + 7y = 2$
b **i** $-\frac{5}{3}x + \frac{2}{3}y = 1$ **ii** $-5x + 2y = 3$
c **i** $\frac{5}{3}x - \frac{5}{2}y = 1$ **ii** $10x - 15y = 6$
d **i** $-2x - 3y = 1$ **ii** $2x + 3y = -1$
e **i** $\frac{1}{7}x + \frac{1}{11}y = 1$ **ii** $11x + 7y = 77$
f **i** $x - y = 1$ **ii** $x - y = 1$

EX p137 Checkpoint

1 a $x = 21$ **b** $x = -9$
c $x = -\frac{31}{2}$ **d** $x = -\frac{3}{7}$

2 a $x = 19$ **b** $x = \frac{5}{7}$
c $x = \frac{1}{30}$ **d** $x = \frac{57}{10}$

3 a $x = 3$ **b** $x = -\frac{1}{2}$

4 a $x = \frac{c - by}{a}$ **b** $x = \frac{f}{dg} + \frac{e}{d} = \frac{f + eg}{dg}$
c $x = \frac{jk + mn}{m - j}$ **d** $x = \frac{rs + q}{rt + p}$

5 a $x = \frac{5}{2}$ **b** $x = \frac{1}{3}$
c $x = -7$
d $x = \frac{41}{21}$

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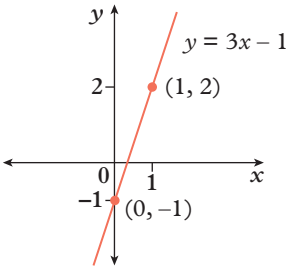
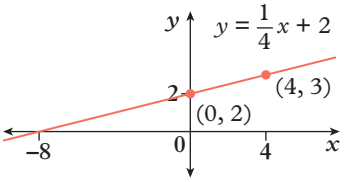
6 a



b



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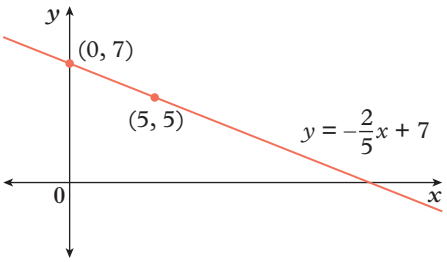

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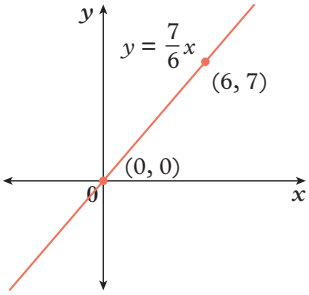
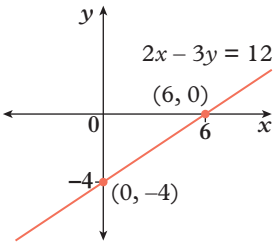
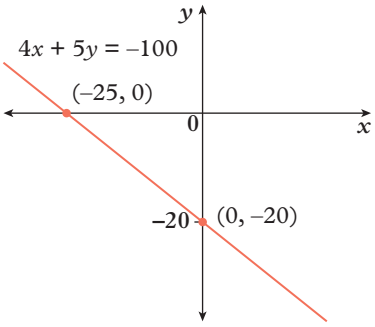


7 a



b



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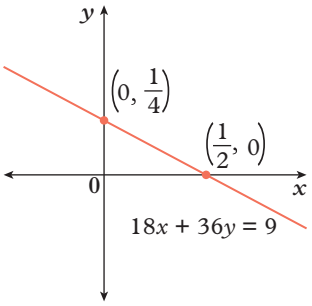
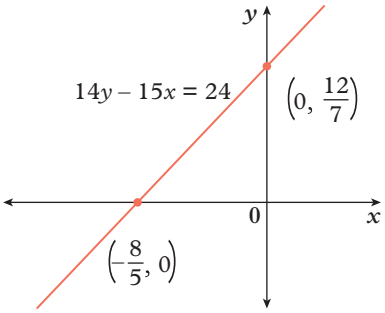
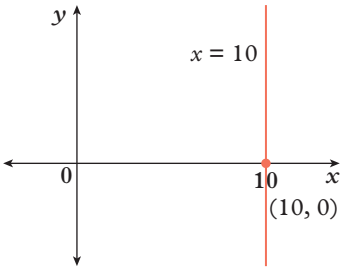


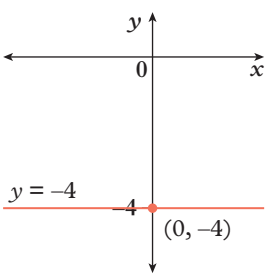
d



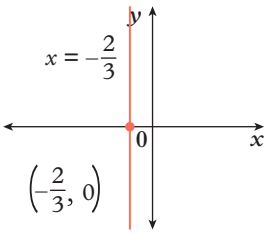
8 a



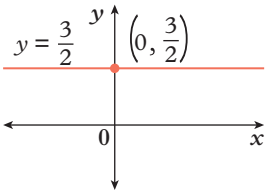
b



c



d



9 a $\frac{1}{2}$ **b** $\frac{1}{5}$ **c** $-\frac{9}{4}$ **d** $-\frac{7}{4}$

10 a $y = \frac{x}{2} + \frac{13}{2}$ **b** $y = \frac{x}{5} - \frac{3}{5}$

c $y = -\frac{9x}{4} + \frac{13}{4}$ **d** $y = -\frac{7}{4}x + \frac{13}{4}$

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11 **a** $y = 2x + 11$ **b** $y = \frac{x}{2} + 8$

c $y = x + 9$ **d** $y = -2x + 3$

e $y = -\frac{x}{2} + 6$

12 **a** $y = 2x - 3$ **b** $y = -\frac{x}{2} - \frac{1}{2}$

EX p141 3.5 Parallel and perpendicular lines

1 **a** **i** gradients: 2 and 2

ii gradients: $-\frac{1}{3}$ and $-\frac{1}{3}$

b Both pairs of lines are parallel.

c The gradients of parallel lines are always equal.

d **i** parallel

ii parallel

iii not parallel

2 **a** **i** gradients: 3 and $-\frac{1}{3}$

ii gradients: $\frac{3}{4}$ and $-\frac{4}{3}$

b Both products are –1.

c Both pairs of lines are perpendicular.

d The product of the gradients of two perpendicular lines is always –1.

e $m_2 = -\frac{1}{m_1}$

f **i** perpendicular

ii not perpendicular

iii perpendicular

3 **a** $-\frac{1}{5}$ **b** $\frac{1}{8}$ **c** 1

d –6 **e** $-\frac{5}{4}$ **f** $\frac{2}{3}$

4 **a** **i** $y = -3x + 11$ **ii** -3 **iii** $\frac{1}{3}$

b **i** $y = 4x + 7$ **ii** 4 **iii** $-\frac{1}{4}$

c **i** $y = \frac{5}{3}x + \frac{2}{3}$ **ii** $\frac{5}{3}$ **iii** $-\frac{3}{5}$

d **i** $y = -\frac{3}{2}x + 4$ **ii** $-\frac{3}{2}$ **iii** $\frac{2}{3}$

e **i** $y = \frac{5}{2}x - 2$ **ii** $\frac{5}{2}$ **iii** $-\frac{2}{5}$

f **i** $y = -\frac{8}{5}x - \frac{11}{5}$ **ii** $-\frac{8}{5}$ **iii** $\frac{5}{8}$

g **i** $y = \frac{1}{3}x + \frac{5}{3}$ **ii** $\frac{1}{3}$ **iii** -3

h **i** $y = -\frac{1}{5}x + \frac{3}{5}$ **ii** $-\frac{1}{5}$ **iii** 5

i **i** $y = x + 7$ **ii** 1 **iii** -1

5 **a** parallel; same gradient of 4

b perpendicular; negative reciprocal gradients of $\frac{2}{3}$ and $-\frac{3}{2}$

c perpendicular; negative reciprocal gradients of $\frac{3}{8}$ and $-\frac{8}{3}$

d neither; gradients of –2 and 2

e neither; gradients of –5 and $-\frac{1}{5}$

f parallel; same gradient of $\frac{2}{7}$

g neither; gradients of $\frac{2}{3}$ and $-\frac{2}{3}$

h perpendicular; negative reciprocal gradients of –2 and $-\frac{1}{2}$

6 **a** $y = 4x - 13$ **b** $y = -3x - 6$

c $y = -x - 2$ **d** $y = \frac{1}{2}x - 3$

e $y = -\frac{1}{3}x + 5$ **f** $y = \frac{2}{7}x + 11$

g $y = -\frac{5}{4}x - 5$ **h** $y = \frac{5}{6}x + \frac{39}{2}$

7 **a** $y = -4x + 5$; $y = -4x + 18$

b $y = -\frac{1}{2}x + 3$; $y = -\frac{1}{2}x - 3$

c $y = \frac{7}{3}x - \frac{5}{3}$; $y = \frac{7}{3}x + 6$

d $y = \frac{3}{5}x + \frac{3}{20}$; $y = \frac{3}{5}x - 3$

8 **a** $y = -\frac{1}{5}x - 7$ **b** $y = \frac{1}{7}x + 3$

c $y = -x - 3$ **d** $y = \frac{3}{2}x + 10$

e $y = -\frac{9}{4}x + 8$ **f** $y = 3x + 30$

g $y = \frac{6}{5}x - \frac{54}{5}$ **h** $y = -\frac{7}{4}x - 46$

9 **a** $y = -5x + 4$; $y = \frac{1}{5}x + 9$

b $y = \frac{1}{3}x - 2$; $y = -3x - 33$

c $y = -\frac{9}{7}x + \frac{13}{7}$; $y = \frac{7}{9}x + 47$

d $y = \frac{2}{5}x + \frac{7}{5}$; $y = -\frac{5}{2}x - 100$

10 **a** $y = -\frac{1}{2}x + 3$

b $y = x + 3$

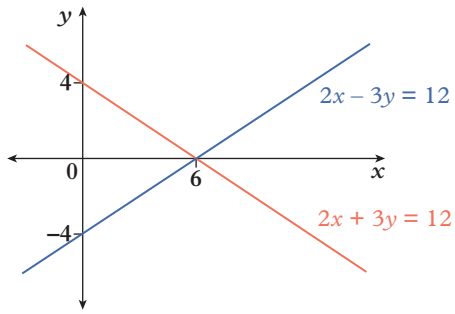
c $y = -\frac{2}{3}x + 3$

11 **a** **i** $y = 2x - 4$ **ii** $y = -\frac{1}{2}x - 4$

b **i** $y = -x - 4$ **ii** $y = x - 4$

c **i** $y = \frac{3}{2}x - 4$ **ii** $y = -\frac{2}{3}x - 4$

12 **a** **i**



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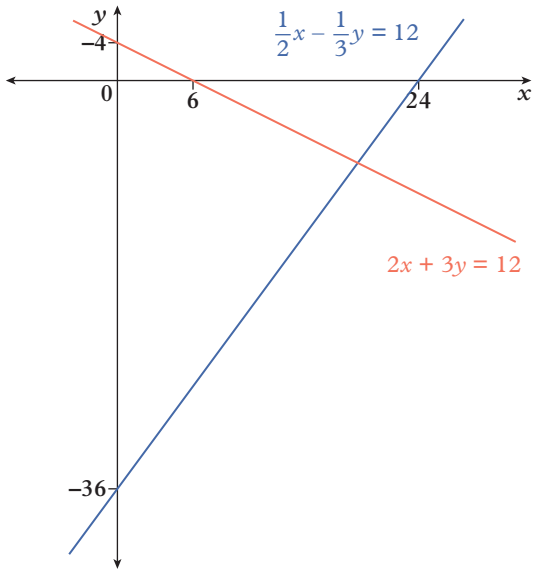
ii



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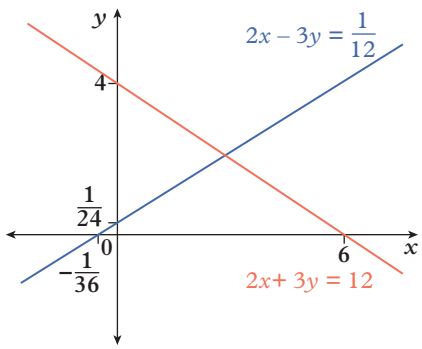

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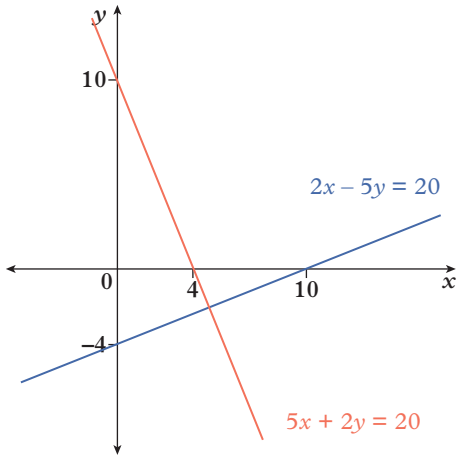
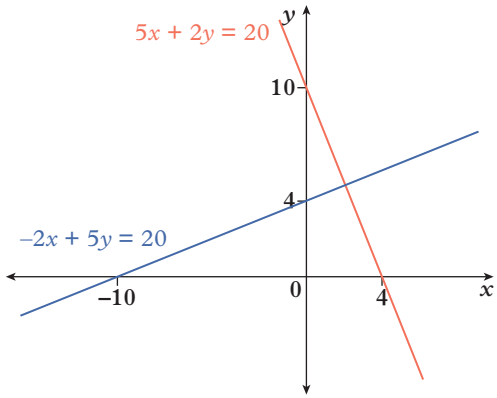
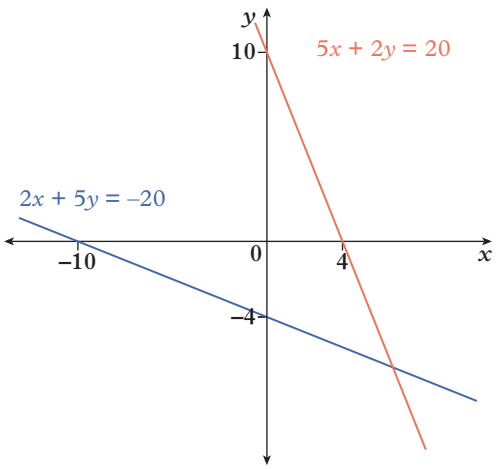


v



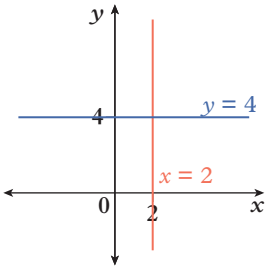
vi



b **ii, iv, v**

c $\frac{1}{a}x - \frac{1}{b}y = 1$ or $-\frac{1}{a}x + \frac{1}{b}y = 1$ or $bx - ay = 1$ or $-bx + ay = 1$

13 a



b **i** Aamira is correct because lines that meet at right angles are, by definition, perpendicular.

ii Jack is incorrect as you cannot divide by zero, even if you divide out that factor.

c **i** Vertical lines are perpendicular to horizontal lines because they meet at a 90° angle.

ii Vertical lines are perpendicular to horizontal lines but we cannot use the formula $m_1 \times m_2 = -1$ because the gradient of a vertical line is undefined.

iii Vertical lines are perpendicular to horizontal lines, so the equations $x = a$ and $y = c$ are perpendicular.

14 The gradient of the line joining (2, –3) and (4, 5) is $\frac{5-(-3)}{4-2} = \frac{8}{2} = 4$.
The gradient of $y = 4x - 7$ is 4.
So the lines are parallel.

15 The gradient of the line joining (–11, –7) and (–1, –2) is $\frac{-2-(-7)}{-1-(-11)} = \frac{5}{10} = \frac{1}{2}$.
The gradient of $y = -2x + 5$ is –2.
The product of the gradients is –1, so the lines are perpendicular.

16 a $\frac{2}{3}$

b **i** $y = \frac{2}{3}x - 5$ **ii** $y = -\frac{3}{2}x + 2$

17 a $y = -3x$

b $y = \frac{1}{3}x + 9$

c $y = \frac{1}{3}x - 1$

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18 $y = -x + 1$, $y = -x + 2$, $y = x + 1$, $y = x + 2$

19 a Yes, both have a gradient of $\frac{3}{7}$.

b No; AC has a gradient of $\frac{5}{4}$, while BD has a gradient of $-\frac{2}{3}$.

20 a The gradient of AB is 3; the gradient of BC is $-\frac{1}{3}$, so line segments AB and BC are perpendicular. This makes triangle ABC right-angled.

b The length of AB is $\sqrt{90}$ units; the length of BC is $\sqrt{40}$ units. So the triangle is not isosceles right-angled.

c 27.2 units

d 30 square units

21 a $AD \parallel BC$, each line segment has a gradient of $\frac{1}{2}$.

$AB \parallel DC$, each line segment has a gradient of 2.

length of AD = length of BC = length of AB = length of $DC = \sqrt{5}$ units

Note: It is only necessary to prove all sides equal in length, but it can also be shown by proving both pairs of opposite sides parallel and two adjacent sides equal in length.

b $EF \parallel HG$; both line segments have a gradient of $\frac{3}{7}$.

$HE \parallel GF$; both line segments have a gradient of 2.

22 gradient of KN = gradient of $LM = -\frac{1}{2}$

gradient of MN = gradient of $LK = 2$

So opposite sides are parallel, and adjacent sides are at right angles.

$KL = LM = MN = NK = 2\sqrt{5} \approx 4.5$ units,

perimeter $= 8\sqrt{5} \approx 17.9$ units,

area $= (2\sqrt{5})^2 = 20$ square units

23 gradient of PS = gradient of $QR = 1$

gradient of PQ = gradient of $SR = -1$

So opposite sides are parallel, and adjacent sides are at right angles.

$PS = QR = 3\sqrt{2}$ units, $PQ = SR = 4\sqrt{2}$ units,

perimeter $= 14\sqrt{2} \approx 19.8$ units

area $= 3\sqrt{2} \times 4\sqrt{2} = 24$ square units

24 $y = x - 5$

25 length of YX = length of $YZ = 7.1$ units

$WX = WZ = \sqrt{425} = 5\sqrt{17} \approx 20.6$ units

This is all that is required. Extra information is as follows: The gradient of $XZ = -\frac{3}{4}$ and the gradient of $WY = \frac{4}{3}$, so the diagonals are at right angles. The midpoint of XZ is $(3, 5)$. This point also lies on the diagonal WY, which has the equation $y = \frac{4}{3}x + 1$.

26 Gradients of the line segments are: $AB = \frac{1}{2}$, $BC = -3$, $CD = \frac{1}{2}$, $DA = \frac{6}{5}$, $AC = 0$, $BD = \frac{9}{11}$

This indicates that the quadrilateral is a trapezium (one pair of parallel sides).

3.6 Simultaneous linear equations

EX p150

1 a is a solution **b** is a solution

c is not a solution **d** is a solution

e is not a solution **f** is not a solution

2 a i $(4, 3)$ **ii** $x = 4, y = 3$

b i $(2, -4)$ **ii** $x = 2, y = -4$

c i $(-5, -2)$ **ii** $x = -5, y = -2$

d i $(-2, 3)$ **ii** $x = -2, y = 3$

3 a $x = 5, y = 4$ **b** $x = 2, y = 6$

c $x = 6, y = 4$ **d** $x = -2, y = 5$

e $x = -1, y = 3$ **f** $x = -3, y = 1$

g $x = 3, y = 2$ **h** $x = 4, y = -2$

i $x = -2, y = -3$ **j** $x = -5, y = 6$

k $x = 2, y = -2$ **l** $x = 3, y = 5$

m $x = -4, y = -1$ **n** $x = 1, y = -4$

o $x = -6, y = -5$ **p** $x = -1, y = 2$

4 a $x = 2, y = 7$ **b** $x = 4, y = 1$

c $x = -1, y = 3$ **d** $x = 6, y = 5$

e $x = -4, y = -2$ **f** $x = 2, y = 12$

g $x = 1, y = 3$ **h** $x = -3, y = 11$

i $x = 5, y = -5$ **j** $x = 5, y = 3$

k $x = -4, y = -1$ **l** $x = 5, y = 6$

5 a $x = 1, y = 8$ **b** $x = 6, y = 4$

c $x = -3, y = -5$ **d** $x = 5, y = -1$

e $x = -1, y = 4$ **f** $x = 4, y = -7$

g $x = 2, y = 1$ **h** $x = -2, y = 6$

6 a $x = 2, y = 5$ **b** $x = -3, y = 1$

c $x = 1, y = -5$

7 a $x = 5, y = 3$ **b** $x = -2, y = 9$

c $x = 3, y = -7$

8 a $x = -4, y = -2$ **b** $x = 6, y = 2$

c $x = -1, y = 2$ **d** $x = -1, y = 5$

e $x = 5, y = 3$ **f** $x = -3, y = -4$

9 a $x = 3, y = 2$ **b** $x = 2, y = -1$

c $x = -4, y = 7$ **d** $x = 5, y = 3$

e $x = -3, y = -4$ **f** $x = -2, y = 6$

g $x = 1, y = 4$ **h** $x = -3, y = 1$

i $x = 2, y = -1$ **j** $x = -4, y = 2$

k $x = 5, y = 5$ **l** $x = -3, y = -2$

10 a $x = 2, y = 0$ **b** $x = \frac{1}{3}, y = -3$

c $x = \frac{3}{2}, y = \frac{2}{5}$ or $x = 1.5, y = 0.4$

d $x = -9, y = -7$ **e** $x = 4, y = 2$

f $x = -2, y = -3$ **g** $x = 3, y = 10$

h $x = 5, y = -2$ **i** $x = 2, y = -1$

j $x = -0.8, y = 2.3$

k $x = -\frac{1}{4}, y = 5$ or $x = -0.25, y = 5$

l $x = 28, y = 41$

11 Both graphs have a gradient of 2. This means the lines are parallel, so there is no point of intersection.

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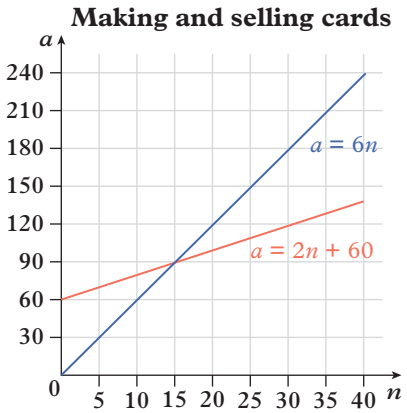
12 a To make n cards, the cost of plain white cards is $2 \times n$ or $2n$. So, the amount of money spent (in \$) is the cost of the plain white cards plus the start-up cost of \$60.

b Each card sells for \$6, so for selling n cards, the amount of money received (in \$) is $6 \times n$ or $6n$.

c



d (15, 90); this shows the number of cards that need to be sold (15) so that the amount of money received is the same as the amount of money spent (\$90).

e 15 cards **f** 16 to 40 cards

13 a $2l + 2w = 810$ or $2(l + w) = 810$

b $l = 2w$

c 270 cm long and 135 cm wide

14 a **i** yes **ii** yes **iii** yes

b They are equivalent equations. When graphed, the position of the lines is identical; that is, one line lies on top of the other. There are an infinite number of points where the lines intersect so there are infinite solution.

c No. For example, (1, 2) is not a solution to the given simultaneous equations.

15 a Strategy A: Equation ③ is $10x - 6y = -2$.

① – ③: $-3x = -12$, $x = 4$

Substituting $x = 4$ into ②: $20 - 3y = -1$, $y = 7$

The solution is $x = 4$, $y = 7$.

Strategy B: Equation ④ is $-10x + 6y = 2$.

① + ④: $-3x = -12$, so $x = 4$

Substituting $x = 4$ into ②: $20 - 3y = -1$, $y = 7$

The solution is $x = 4$, $y = 7$.

Strategy C: Equation ⑤ is $35x - 30y = -70$ and equation ⑥ is $-35x - 21y = -7$.

⑤ – ⑥: $-9y = -63$, so $y = 7$

Substituting $y = 7$ into ②: $5x - 21 = -1$, so $x = 4$

The solution is $x = 4$, $y = 7$.

The strategies all provide the same solution of $x = 4$ and $y = 7$.

b Some possible answers are:

Multiply equation ① by 5 to form equation ⑦ and multiply equation ② by –7 to form equation ⑧. Add equations ⑦ and ⑧.

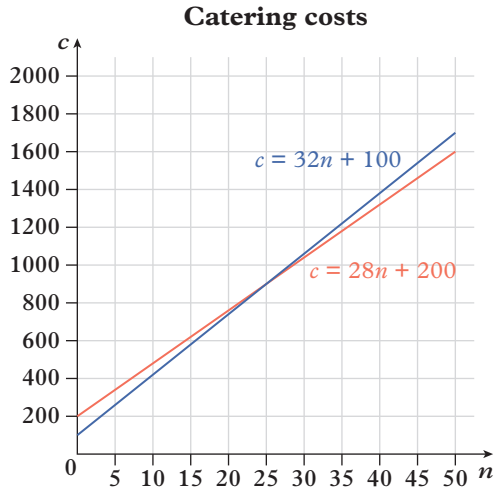
Graph equations ① and ② and identify the coordinates of the point of intersection.

Rearrange equation ② to obtain $3y = 5x - 1$ and substitute that into equation ①. In other words, substitute $5x - 1$ for $3y$ in $7x - 2(3y) = -14$.

16 n = number of people and c = catering cost

a $c = 28n + 200$ **b** $c = 32n + 100$

c



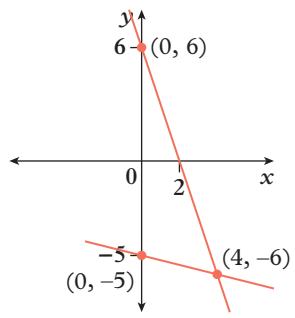
d Cool Food Club is cheaper for 18 people; at $n = 18$, the line for Cool Food Club is lower than the line for Angie's Catering. This represents a lower cost for 18 people.

e Angie's Catering

f 25 people at a cost of \$900

g When catering for fewer than 25 people, Cool Food Club is cheaper. When catering for more than 25 people, Angie's Catering is cheaper. The cost is the same when catering for 25 people.

17 a



b $y = -\frac{1}{4}x - 5$, $y = -3x + 6$

c $x = 4$, $y = -6$

d If $x = 4$ and $y = -6$:

For $y = -\frac{1}{4}x - 5$, RHS $= -\frac{1}{4} \times 4 - 5 = -6 =$ LHS.

For $y = -3x + 6$, RHS $= -3 \times 4 + 6 = -6 =$ LHS.

18 a $15 + (-5) = 10$ and $15 + 5 = 20$

b $22 + (-12) = 10$ and $22 + (-2) = 20$

c Since there are three variables but only two equations, if we add, subtract or substitute, the resulting equation will still have two variables in it when we need it to have only one.

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19 a

$$\begin{aligned} ax + by &= c \\ ax + by - (ax + qy) &= c - (r) \\ by - qy &= c - r \\ (b - q)y &= c - r \\ y &= \frac{c - r}{b - q} \\ ax + b\left(\frac{c - r}{b - q}\right) &= c \\ ax + \frac{bc - br}{b - q} &= \frac{c(b - q)}{b - q} \\ ax &= \frac{cb - cq}{b - q} - \frac{bc + br}{b - q} \\ ax &= \frac{cb - cq - bc - br}{b - q} \\ ax &= \frac{br - cq}{b - q} \\ x &= \frac{br - cq}{ab - aq} \end{aligned}$$

b $x = \frac{4 \times 7 - 9 \times 6}{3 \times 4 - 3 \times 6} = \frac{13}{3}$ and $y = \frac{9 - 7}{4 - 6} = -1$

c

$$\begin{aligned} ax + by &= c \ (\times p) \\ apx + bpy &= cp \\ px + qy &= r \ (\times a) \\ apx + aqy &= ar \\ apx + bpy - (apx + aqy) &= cp - (ar) \\ bpy - aqy &= cp - ar \\ (bp - aq)y &= cp - ar \\ y &= \frac{cp - ar}{bp - aq} \\ ax + b\left(\frac{cp - ar}{bp - aq}\right) &= c \\ ax + \frac{bcp - abr}{bp - aq} &= \frac{c(bp - aq)}{bp - aq} \\ ax &= \frac{bcp - acq}{bp - aq} - \frac{bcp - abr}{bp - aq} \\ ax &= \frac{bcp - acq - bcp + abr}{bp - aq} \\ ax &= \frac{abr - acq}{bp - aq} \\ x &= \frac{br - cq}{bp - aq} \end{aligned}$$

d $x = -13$ and $y = 12$

EX 3.7 Solving linear inequalities
p159

oxforddigital **Answers for this lesson can be found on Oxford Digital.**

EX Review: Linear relationships
p164

oxforddigital **Answers for this lesson can be found on Oxford Digital.**

MODULE 4 Non-linear relationships

EX 4.1 Solving quadratic equations
p173

1 a $x = -2$ or $x = 4$ **b** $x = -7$ or $x = -1$
c $x = 3$ or $x = 6$ **d** $x = \pm 9$
e $x = 0$ or $x = 5$ **f** $x = 0$ or $x = \frac{1}{2}$
g $x = 0$ or $x = -1$ **h** $x = 0$ or $x = -2$
i $x = 0$ or $x = \frac{3}{4}$ **j** $x = \pm\sqrt{2}$
k $x = \pm\sqrt{5}$ **l** $x = \pm 4\sqrt{3}$
m $x = -1\frac{1}{2}$ or $x = 4$ **n** $x = \pm\frac{1}{3}$
o $x = -\frac{2}{7}$ or $x = 1\frac{1}{4}$

2 a $x = -3$ or $x = -4$ **b** $x = -5$ or $x = 2$
c $x = 1$ or $x = 8$ **d** $x = \pm 5$
e $x = \pm 9$ **f** $x = \pm 2$
g $x = 0$ or $x = 3$ **h** $x = 0$ or $x = -1$
i $x = 0$ or $x = -4$ **j** $x = 0$ or $x = \frac{2}{5}$
k $x = 0$ or $x = 6$ **l** $x = 0$ or $x = -10$

3 a $x = -6$ or $x = 2$ **b** $x = 1$ or $x = 4$
c $x = \pm 3$ **d** $x = -5$ or $x = 8$
e $x = 0$ or $x = 9$ **f** $x = 2 \pm \sqrt{3}$
g $x = 0$ or $x = 4$ **h** $x = 2$ or $x = 9$
i $x = -3$ or $x = 6$ **j** $x = -7$ or $x = 3$
k $x = 4$ or $x = 6$ **l** $x = -3$

4 a Both sides of the original equation have been divided by 3, so the equation remained in balance and the values of the variables are unchanged.
b, c $x = -6$ or $x = 8$

5 a $x = -5$ or $x = 6$ **b** $x = 2$ or $x = 4$
c $x = -8$ or $x = -1$ **d** $x = 3$ or $x = -9$
e $x = \pm 4$ **f** $x = 0$ or $x = -7$
g $x = 0$ or $x = 2$ **h** $x = 0$ or $x = -1\frac{2}{3}$
i $x = 1\frac{1}{2}$ or $x = -10$

6 a $x = -2$ or $x = -1$ **b** $x = -5$ or $x = 3$
c $x = 1$ or $x = 4$ **d** $x = -3$ or $x = 2$
e $x = -2$ or $x = 6$ **f** $x = -8$ or $x = 7$
g $x = \pm 2$ **h** $x = \pm 3$
i $x = 0$ or $x = \frac{1}{2}$

7 a $x = 2$ **b** $x = 3 \pm \sqrt{3}$ **c** $x = -1, x = 3$
d $x = 6 \pm 2\sqrt{10}$ **e** $x = 1 \pm \sqrt{2}$ **f** $x = 6 \pm \sqrt{33}$

8 a $x = -1 \pm \sqrt{7}$ **b** $x = 2 \pm \sqrt{2}$
c $x = -4 \pm \sqrt{5}$ **d** $x = 6 \pm 2\sqrt{2}$
e $x = -10 \pm 5\sqrt{3}$ **f** $x = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$
g $x = -3 \pm \sqrt{2}$ **h** $x = -1 \pm \sqrt{5}$
i $x = 2 \pm \sqrt{3}$ **j** $x = -4 \pm \sqrt{10}$
k $x = 3 \pm \sqrt{\frac{21}{2}}$ or $x = 3 \pm \frac{\sqrt{42}}{2}$
l $x = -\frac{1}{2} \pm \sqrt{\frac{13}{20}}$ or $x = -\frac{1}{2} \pm \frac{\sqrt{65}}{10}$

9 a $x = -5$ or $x = -\frac{3}{2}$ **b** $x = \frac{2}{5}$ or $x = 4$
c $x = -\frac{2}{3}$ or $x = \frac{1}{2}$ **d** $x = -7$ or $x = \frac{3}{4}$
e -1 and $\frac{1}{3}$ **f** $x = -\frac{3}{7}$ or $x = 2$

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10 Dividing both sides of the first equation by 3 gives the equivalent equation $x(x-2)=0$ and this does not affect the solutions (with $x=2$ and $x=0$ being a solution for both equations). Dividing both sides of the equation by $3x$ loses the solution $x=0$. In addition, since the value of x is not fixed, dividing by $3x$ includes dividing by 0, which is undefined.

11 **a** two **b** one **c** two
d one **e** two **f** none

12 **a** $x=\pm 3$ **b** $x=-11$ or $x=7$
c $x=4$ **d** no solutions
e $x=0$ or $x=-4$ **f** $x=3\pm\sqrt{10}$

13 **a** **i** 8 m **ii** 6 m
b 6 m **c** 3 s
d It is not possible to have negative time values.

14 **a** 25 kg **b** 9 kg **c** 5 weeks
d No. If the amounts were the same each week, the relationship would be linear.

15 **a** $x(x+3)=70$ or $x^2+3x-70=0$
b $x=-10$ or $x=7$. The feasible solution is $x=7$, while $x=-10$ is not feasible because it is not possible to have a negative length.
c Width 7 m, length 10 m

16 $x(x-5)=66$, where x is the length in centimetres. The dimensions are length 11 cm, width 6 cm.

17 **a** $x=\pm\sqrt{6}$ **b** $x=-\frac{1}{2}$

18 11

EX 4.2 The quadratic formula

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1 **a** none **b** two **c** two
d none **e** one **f** two

2 **a** **i** 64 **ii** two **iii** $x=-3$ or $x=5$
b **i** 0 **ii** one **iii** $x=-2$
c **i** 196 **ii** two **iii** $x=-7$ or $x=7$
d **i** 324 **ii** two **iii** $x=0$ or $x=6$
e **i** 8 **ii** two **iii** $x=3-\sqrt{2}$ or $x=3+\sqrt{2}$
f **i** -8 **ii** none **iii** no solutions

3 **a** **i** $x=\frac{7\pm\sqrt{29}}{2}$ **ii** $x\approx 0.81$ or $x\approx 6.19$
b **i** $x=\frac{1\pm\sqrt{13}}{6}$ **ii** $x\approx -0.43$ or $x\approx 0.77$
c **i** $x=\frac{-1\pm\sqrt{33}}{4}$ **ii** $x\approx -1.69$ or $x\approx 1.19$
d **i** $x=\frac{-1\pm\sqrt{109}}{6}$ **ii** $x\approx -1.91$ or $x\approx 1.57$
e **i** $x=\frac{-5\pm\sqrt{97}}{12}$ **ii** $x\approx -1.24$ or $x\approx 0.40$
f **i** $x=1\pm\sqrt{2}$ **ii** $x\approx -0.41$ or $x\approx 2.41$
g **i** $x=-4\pm 2\sqrt{3}$ **ii** $x\approx -7.46$ or $x\approx -0.54$
h **i** $x=\frac{3\pm 2\sqrt{2}}{2}$ **ii** $x\approx 0.09$ or $x\approx 2.91$
i **i** $x=2\pm\sqrt{6}$ **ii** $x\approx -0.45$ or $x\approx 4.45$
j **i** $x=\frac{-5\pm 2\sqrt{10}}{5}$ **ii** $x\approx -2.26$ or $x\approx 0.26$
k **i** $x=\frac{-3\pm\sqrt{97}}{4}$ **ii** $x\approx -3.21$ or $x\approx 1.71$
l **i** $x=-1\pm 2\sqrt{2}$ **ii** $x\approx -3.83$ or $x\approx 1.83$

4 **a** $x=\frac{-3\pm\sqrt{13}}{2}$ **b** no solutions **c** $x=-3\pm\sqrt{5}$
d no solutions **e** $x=\frac{-5\pm\sqrt{17}}{4}$ **f** $x=4$
g $x=-1\pm\sqrt{5}$ **h** $x=\frac{-7\pm\sqrt{61}}{2}$ **i** $x=-4\pm\sqrt{26}$
j $x=\frac{1\pm\sqrt{21}}{2}$ **k** no solutions **l** $x=\frac{-\sqrt{5}\pm 3}{2}$

5 **a** $m=\frac{-3\pm\sqrt{37}}{2}$ **b** $u=-2\pm\sqrt{5}$
c $r=\frac{-9\pm\sqrt{77}}{2}$ **d** $y=-1$ or $y=3$
e $h=\frac{-7\pm\sqrt{61}}{6}$ **f** $w=3\pm\sqrt{14}$

6 $x=\frac{3\pm\sqrt{13}}{2}$

7 **a, b** $x=2$ or $x=3$
c In cases such as this, the method described in part **a** is often found to be quicker and easier.
d If integer solutions exist, the method described in part **a** is usually quicker.

8 **a** There is no real number that is the square root of a negative number, so we cannot obtain a result using the quadratic formula. There are no solutions.
b The square root of zero is zero, so there will be only one distinct value obtained using the quadratic formula: $x=\frac{-b\pm 0}{2a}=-\frac{b}{2a}$
c The square root of a positive number gives a positive result so there will be two distinct solutions:
$x=\frac{-b-\text{positive}}{2a}$ or $x=\frac{-b+\text{positive}}{2a}$.

9 Step 3: $x^2+\frac{b}{a}x+\frac{c}{a}=0$

Step 4: $x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}=0$

Step 5: $\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}=0$

Step 7: $\left(x+\frac{b}{2a}\right)^2-\left(\frac{b^2-4ac}{4a^2}\right)^2=0$

Step 10: $\left(x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right)\left(x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right)=0$

Step 11: $x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}=0$ or $x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}=0$

Step 12: $x=-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}$ or $x=-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}$

10 **a**
$$\begin{aligned} a(x-h)^2+k&=0\\ a(x-h)^2&=-k\\ (x-h)^2&=-\frac{k}{a}\\ x-h&=\pm\sqrt{-\frac{k}{a}}\\ x&=h\pm\sqrt{-\frac{k}{a}}\end{aligned}$$
b The quadratic equation would have:
two real solutions if k and a have different signs,
one real solution if $k=0$ and
no real solutions if k and a have the same signs.

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c **i** two **ii** one **iii** zero

d **i** $x = 5 \pm \frac{1}{2}$ **ii** $x = -4$

11 **a** \$620 **b** 59

12 length 81.4 m, width 61.4 m

13 length 369 mm, width 131 mm

14 **a** **i** $\Delta = 33$. Solutions are irrational.

ii $\Delta = 25$. Solutions are rational.

iii $\Delta = 9$. Solutions are rational.

iv $\Delta = 41$. Solutions are irrational.

v $\Delta = 49$. Solutions are rational.

b The quadratic formula is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
If $\sqrt{b^2 - 4ac}$ is an integer and a, b and c are integers then the solutions are rational. $\sqrt{b^2 - 4ac}$ is an integer if $b^2 - 4ac$, the discriminant, is a square number, since the square root of a square number is an integer.

15 **a** **i** $(-2, 0)$ and $(-5, 0)$ **ii** $(2, 0)$ and $(4, 0)$

iii no x-intercepts **iv** $(4, 0)$

v no x-intercepts

b The discriminant relates to the number of solutions to a quadratic when it equals zero, and therefore the number of x-intercepts of its graph.

16 **a** $k = 3$

b **i** $k < 3$ **ii** $k > 3$

17 **a** $k = 1$ **b** $k < 2$

c $k < -9$ **d** $k = \frac{9}{4}$

e $k < -6$ and $k > 6$ **f** $-10 < k < 10$

18 **a** $b^2 - 20a$

b $b^2 = 20a$

c $a = 5, b = 10$

EX 4.3 Sketching parabolas

p186

oxforddigital **Answers for this lesson can be found on Oxford Digital.**

EX 4.4 Sketching circles

p192

oxforddigital **Answers for this lesson can be found on Oxford Digital.**

EX 4.5 Sketching exponential relationships

p200

oxforddigital **Answers for this lesson can be found on Oxford Digital.**

EX Checkpoint

p203

1 **a** $x = 0$ or $x = 2$ **b** $x = -3$ or $x = 4$

c $x = -1$ or $x = 1$ **d** $x = -\frac{3}{2}$ or $x = 5$

2 **a** $x = 0$ or $x = -5$ **b** $x = 0$ or $x = 6$

c $x = -2$ or $x = -4$ **d** $x = 2$ or $x = 20$

e $x = -17$ or $x = 1$ **f** $x = -3$ or $x = 2$

3 **a** $x = \pm 2$ **b** $x = \pm\sqrt{7}$ **c** $x = \pm\sqrt{22}$

d $x = \pm 3$ **e** $x = \pm\sqrt{5}$ **f** $x = \pm\sqrt{6}$

4 **a** $\Delta = 13$. Therefore, there are two solutions.

b $\Delta = 0$. Therefore, there is one solution.

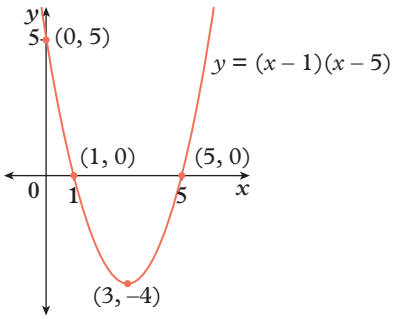
c $\Delta = 129$. Therefore, there are two solutions.

d $\Delta = -7$. Therefore, there are no solutions.

5 **a** $x = \frac{-3 \pm \sqrt{5}}{2}$ **b** $x = 1$ **c** $x = \frac{3 \pm \sqrt{13}}{2}$

d $x = \frac{-1 \pm \sqrt{41}}{10}$ **e** $x = -\frac{3}{2}$ **f** $x = -5 \pm \sqrt{31}$

6



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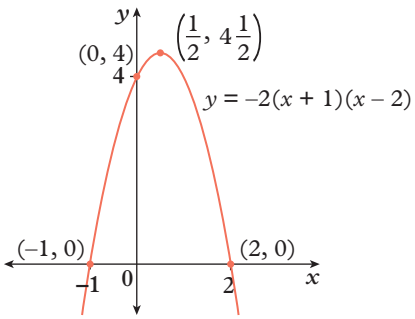
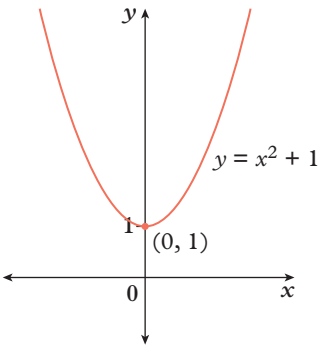


8 **a**



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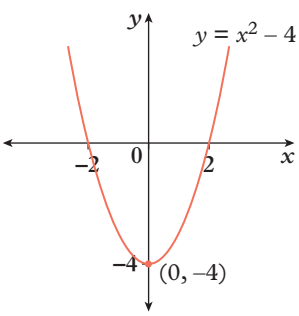

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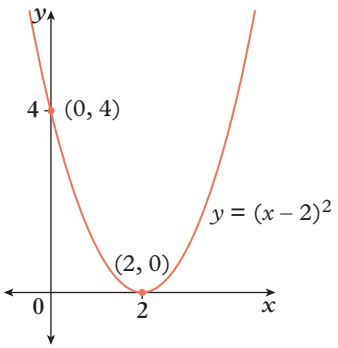
c



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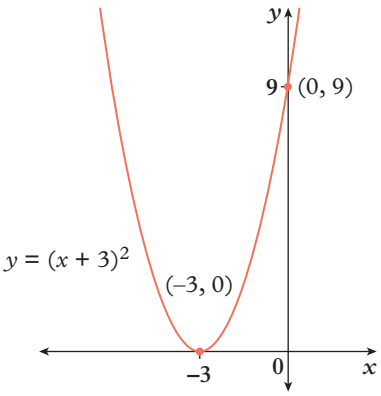

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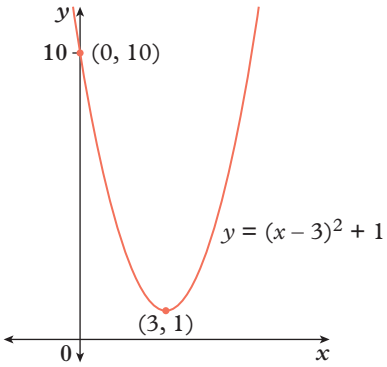

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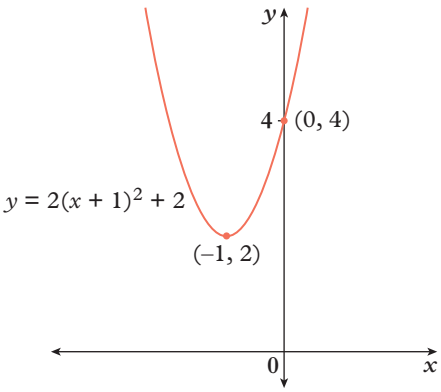

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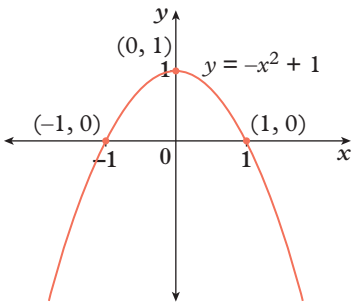

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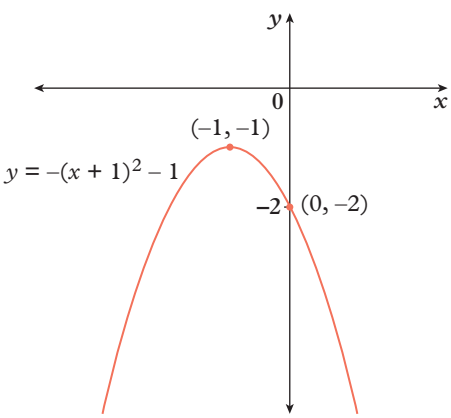

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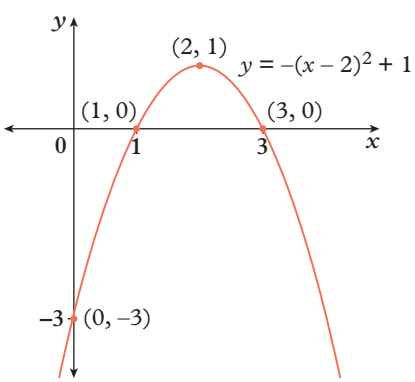

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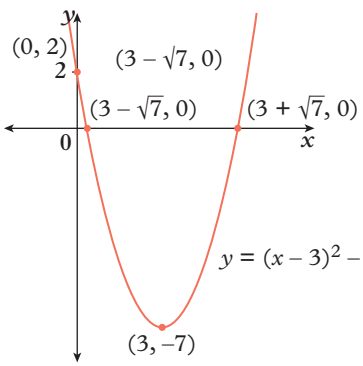

10 a $y = -(x + 3)^2 + 1$ **b** $y = -3(x - 4)^2 - 2$

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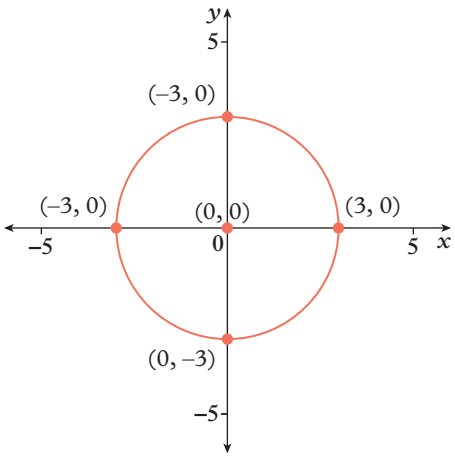

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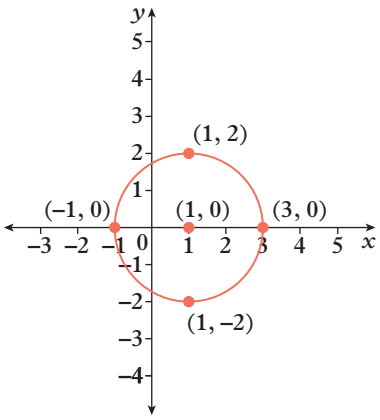

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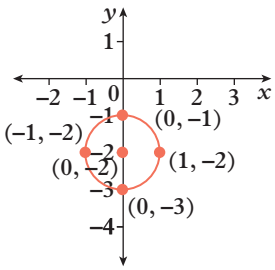

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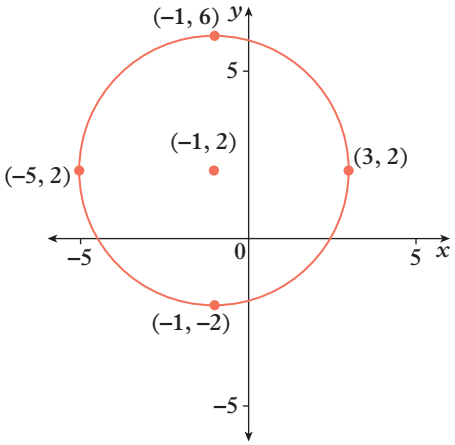

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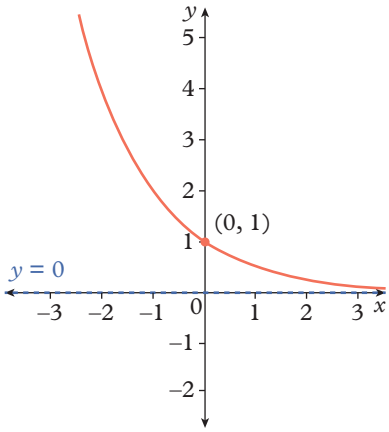

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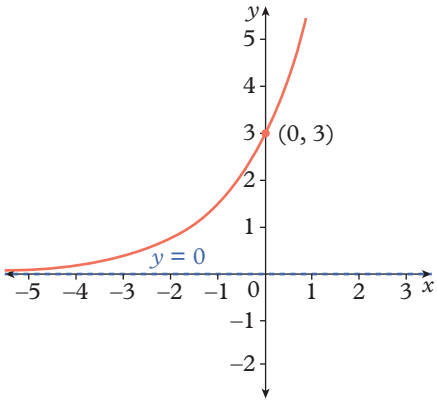

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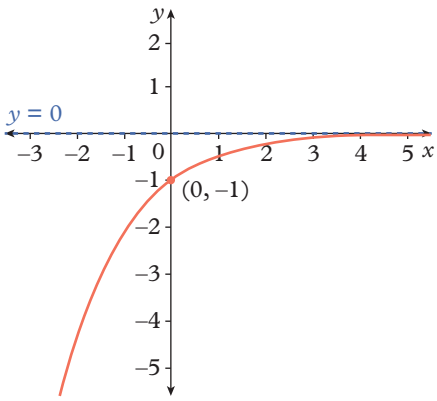

EX 4.6 Direct and inverse proportion

p206

1	**a**	4	**b**	1.5	**c**	0.5	
	d	5	**e**	9	**f**	0.4	
2	**a**	2	**b**	1.2	**c**	1.5	**d** 0.5
3	**a**	6	**b**	4	**c**	15	
	d	12	**e**	6	**f**	1	
4	**a**	2	**b**	5	**c**	0.5	**d** 8

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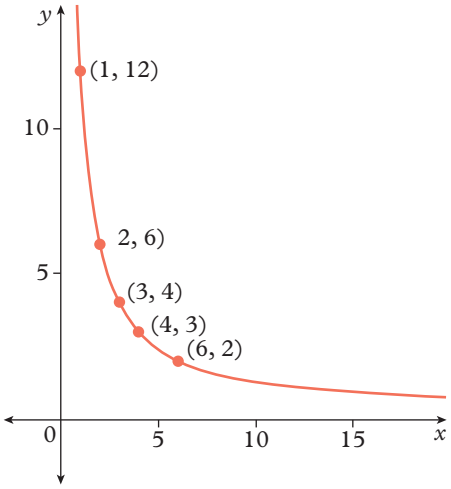
5 a



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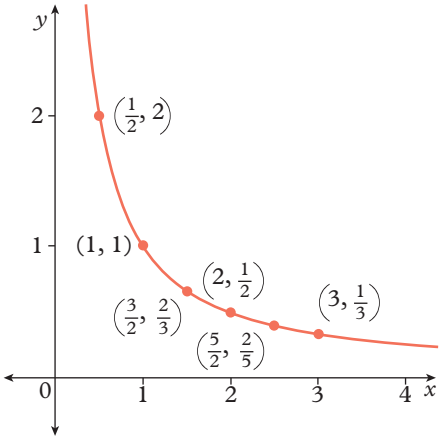

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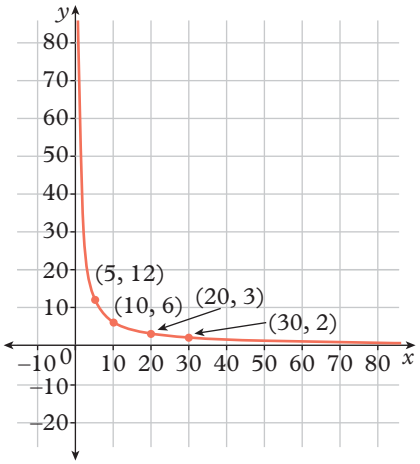

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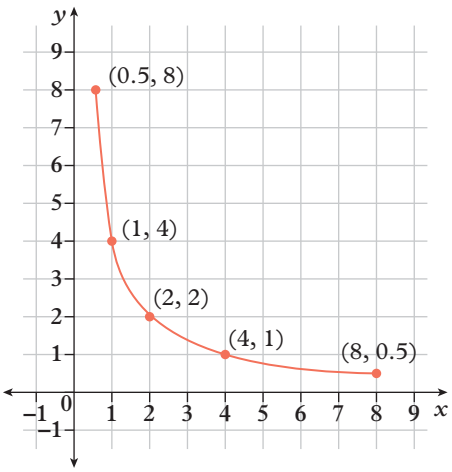

6 **a** 12 **b** 1 **c** 60

7 **a** directly **b** neither **c** inversely
d neither **e** directly **f** inversely

8 **a** inversely proportional
b 4 **c** 1, 0.5

9 **a** The graph does not pass through (0, 0) therefore not direct variation. The graph is a straight line, therefore not inverse variation.

b The graph is not a straight line, therefore not direct variation. The graph intersects the vertical and horizontal axes therefore not inverse variation.

c The graph does not pass through (0, 0) or y decreases as x increases, therefore not direct variation. The graph is a straight line, therefore not inverse variation.

d The graph is not a straight line, therefore not direct variation. The x-and y-coordinates of the points on the graph do not multiply to give the same number, therefore not inverse variation.

10 7.5 days

11 **a** 6 **b** 6

c It will always be the same. If y is inversely proportional to x then $y = \frac{k}{x}$ for some real number, k. Rearranging to make x the subject gives $x = \frac{k}{y}$, which is the same constant of variation.

12 **a** \$171 **b** 171

c The total cost, T, is fixed. Therefore, the cost per person, C, will be $\frac{T}{n}$, where n is the number of people. This gives the equation $C = \frac{T}{n}$, which shows that C is inversely proportional to n and that T is the constant of variation.

13 **a**, **b**, **c** Yes. $xy = k$.

14 If the distance is fixed, it is a constant value, d, so speed is inversely proportional to time.

15 **a** 3.2, 3.1, 3.1, 3.25, 3.24, 3.15, 3.1

b 3.16

c He is trying to approximate π, which is roughly 3.14. Therefore, he is relatively close.

16 **a** 4.8

b **i** 0.192 W/m^2 **ii** 22 cm

c 75% (i.e. the light intensity is one quarter of its previous value)

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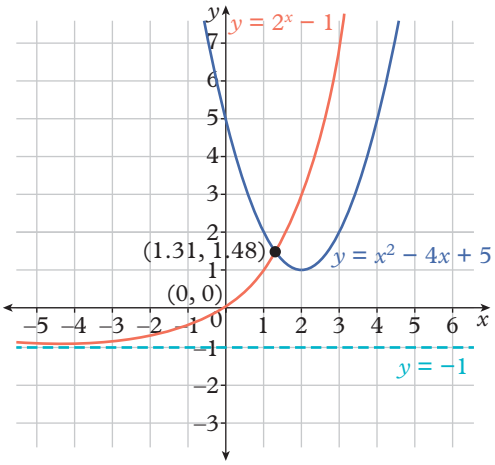
17 a If $x \propto y$ and $y \propto z$, then $x = k_1 y$ and $y = k_2 z$. Therefore, $x = k_1 k_2 z$, which shows that $x \propto z$.

b If $a \propto \frac{1}{b}$ and $b \propto \frac{1}{c}$, then $a = \frac{k_1}{b}$ and $b = \frac{k_2}{c}$. Therefore, $a = \frac{k_1}{k_2/c} = \frac{k_1 c}{k_2} = \frac{k_1}{k_2}c$, which shows that $a \propto c$.

Computational thinking: Solving non-linear equation using digital tools

1 a It can't be solved directly using methods discussed in this book.

b i–ii



iii The x-coordinate of the point of intersection is the solution to the equation in part **a**.

c $x = -4.38$ and $x = 2$

d One solution

2 a The answer should be the same.

b Spreadsheet programs like Excel have the ability to copy down cell entries and automatically update variables used in formulas in a predictable way. We can apply this feature to the bisection method as it iterates the same steps over and over again without having to update the value of variables at each iteration.

EX p216 4.7 Sketching hyperbolas

oxforddigital **Answers for this lesson can be found on Oxford Digital.**

EX p221 4.8 Graphs and transformations

oxforddigital **Answers for this lesson can be found on Oxford Digital.**

EX p230 4.9 Solving non-linear equations

1 a $x = -1$ or $x = 3$ **b** $x = -2$ or $x = 1$

c $x = -5$ or $x = 0$ **d** $x = -1$ or $x = 2$

e $x = 2$ **f** $x = -3$ or $x = 0$

2 a i E **ii** $x = -2$

b i B **ii** $x = 1$ or $x = 4$

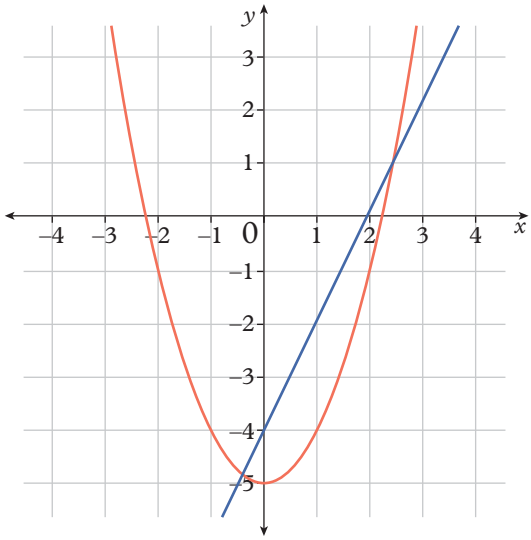
c i D **ii** $x = -5$ or $x = 0$

d i C **ii** $x = -5$ or $x = 0$

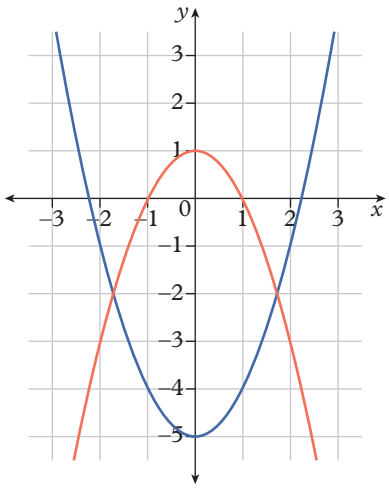
e i F **ii** $x = -3$ or $x = 0$

f i A **ii** $x = 0$ or $x = 4$

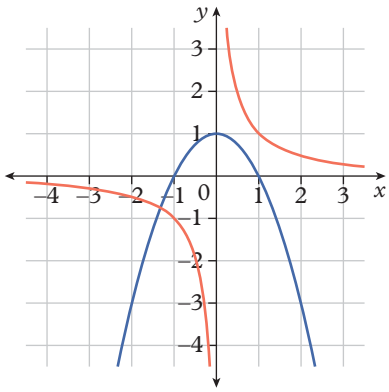
3 a 2 solutions



b 2 solutions



c 1 solution



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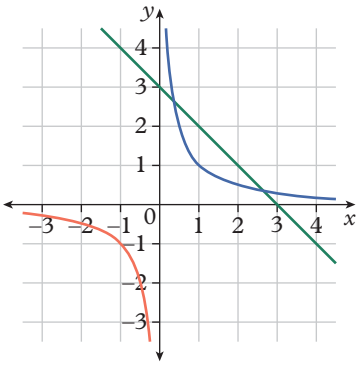
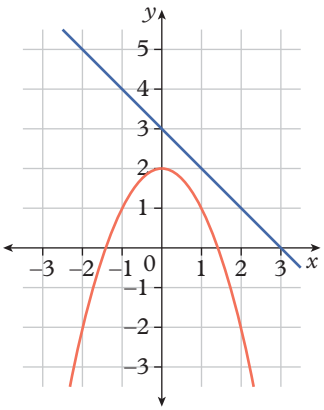
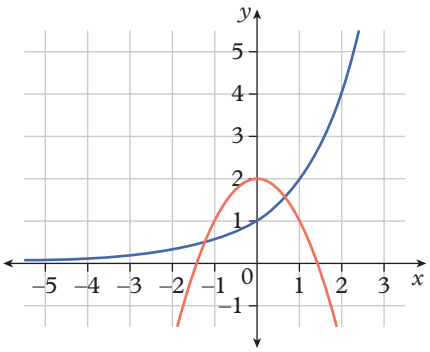
d 2 solutions



e 0 solutions



f 2 solutions



4 **a** $x = -6$ or $x = 5$ **b** $x = \frac{-1 \pm \sqrt{33}}{4}$

c $x = 0$ or $x = -5$ **d** $x = \frac{4 \pm \sqrt{22}}{3}$

e $x = 2$ **f** $x = \frac{-1 \pm \sqrt{127}}{6}$

5 **a** $(-4, 3)$ or $(-3, 4)$ **b** $(0, -3)$ or $(3, 0)$

c $(-6, 8)$ **d** $(-3, 6)$ or $(-1, 8)$

e $(14, -2)$ or $(17, 1)$ **f** $(-2, 4)$

6 **a** **i** $-7, 7$ **ii** $x = -1.14$

b **i** $-3, 1$ **ii** $x = -0.79$

c **i** $-1, 6$ **ii** $x = 2.16$

d **i** $-3, 2$ **ii** $x = 0.69$

e **i** $10, -1$ **ii** $x = 3.93$

f **i** $-19, 39$ **ii** $x = 5.37$

7 **a** $x = 6.14$ **b** $x = 3.79$

c $x = -4.16$ **d** $x = -2.19$

e $x = -0.59$ **f** $x = -0.17$

8 **a** $(2, 1), (6, 9)$ **b** $(1, 6), (3, 2)$

c $(2, 5)$ **d** $(2, 0), (3, 0)$

e $(-2, 0), \left(-5, \frac{3}{2}\right)$ **f** $(3, 12), (7, 8)$

9 **a** $a = 2$ **b** $a = 3$

c $a = 2$ **d** $a = 7, b = -21$

e $a = 4, b = 2$ **f** $a = 5, b = 4$

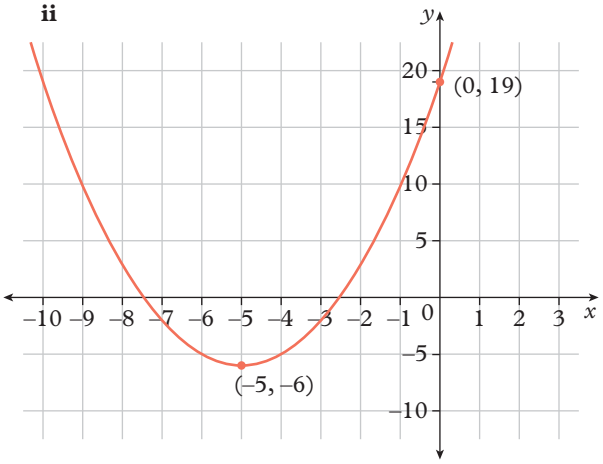
10 **a** $x = \frac{-6 \pm \sqrt{21}}{5}$

b $x = -2.12$ or $x = -0.28$

c 0.171% and 0.0229%

11 **a** **i** $x^2 + 10x + 19 = 0$

ii

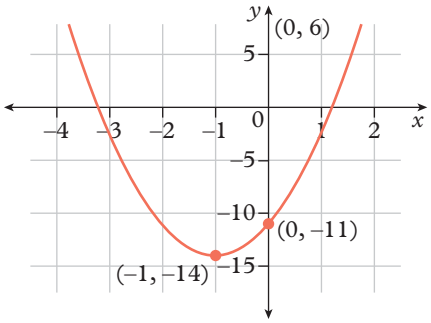


iii $-8, -7$ and $-3, -2$

iv $x = -7.45$ or $x = -2.55$

b **i** $3x^2 + 6x - 11 = 0$

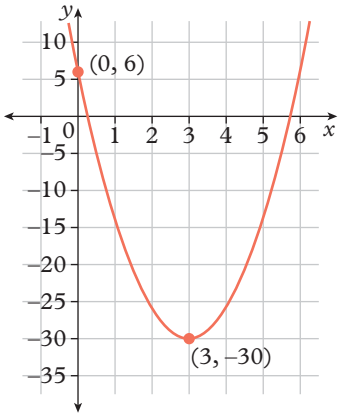
ii



iii $-4, -3$ and $1, 2$ **iv** $x = -3.15$ or $x = 1.16$

c **i** $4x^2 - 24x + 6 = 0$

ii



iii $0, 1$ and $5, 6$

iv $x = 0.26$ or $x = 5.74$

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12 **a** 1.41 **b** 1.73
c 2.24 **d** 2.65

13 **a** $\left(\frac{-\sqrt{-2+2\sqrt{5}}}{2}, \frac{-1+\sqrt{5}}{2}\right), \left(\frac{\sqrt{-2+2\sqrt{5}}}{2}, \frac{-1+\sqrt{5}}{2}\right)$

b $\left(\frac{-\sqrt{10+2\sqrt{5}}}{2}, \frac{-1+\sqrt{5}}{2}\right), \left(\frac{-\sqrt{10+2\sqrt{5}}}{2}, \frac{-1+\sqrt{5}}{2}\right),$
$\left(\frac{-\sqrt{10-2\sqrt{5}}}{2}, \frac{-1-\sqrt{5}}{2}\right), \left(\frac{-\sqrt{10-2\sqrt{5}}}{2}, \frac{-1-\sqrt{5}}{2}\right)$

c No points of intersection

d $(-\sqrt{5}, -1), (0, 4), (\sqrt{5}, -1)$

e $\left(\frac{-\sqrt{31+2\sqrt{193}}}{4}, \frac{7+\sqrt{193}}{4}\right),$
$\left(\frac{\sqrt{31+2\sqrt{193}}}{4}, \frac{7+\sqrt{193}}{4}\right),$
$\left(\frac{-\sqrt{31-2\sqrt{193}}}{4}, \frac{7-\sqrt{193}}{4}\right),$
$\left(\frac{\sqrt{31-2\sqrt{193}}}{4}, \frac{7-\sqrt{193}}{4}\right)$

f $\left(\frac{-2-\sqrt{-2+2\sqrt{101}}}{2}, \frac{3+\sqrt{101}}{2}\right),$
$\left(\frac{-2+\sqrt{-2+2\sqrt{101}}}{2}, \frac{3+\sqrt{101}}{2}\right)$

14 **a** $x = 1.5$

b The equation $x^2 - 3x + 2.25 = 0$ can be written as $(x - 1.5)^2 = 0$ and so it does not change sign (stays non-negative) at the solution $x = 1.5$. Since the bisection method requires two x-values where one produces a positive value and the other a negative, the bisection method will not be able to find this solution in the manner discussed in this lesson.

15 **a** $(-5, -12), (5, 12), (-12, -5), (12, 5)$

b $(-4, -3), (-3, 4), (3, 4), (4, -3)$

16 **a** $(2.322, 5)$

b $(-2.863, 0.137), (2.445, 5.445)$

c $(0.543, 1.457)$

d no intersection

e $(-1.392, 0.217), (1.686, 6.371)$

f $(-0.567, 1.865)$

17 **a**

x	−2	−1	0	1	2	3	4	5
x^2	4	1	0	1	4	9	16	25
2^x	0.25	0.5	1	2	4	8	16	32

Therefore, $x = 2$ and $x = 4$.

b As at $x = -1$, $x^2 > 2^x$ but at $x = 0$, $x^2 < 2^x$. This means the expressions must be equal at some value in between −1 and 0.

c $x = -0.77$

EX 4.10 Rates of change

p236

1 **b**, **d**, **f**

2 **a** 3 **b** 7 **c** 1.5
d $\frac{2}{3}$ **e** −2 **f** $\frac{1}{90}$

3 **a** −5 **b** 46.125 **c** 4.95 **d** −2

4 **a** −5.9, −5.99, −5.999, hence $m_T \approx -6$

b 34.481, 32.2408, 32.0240, hence $m_T \approx 32$

c 4.95, 2.84605, 2.322986, hence $m_T \approx 2.3$

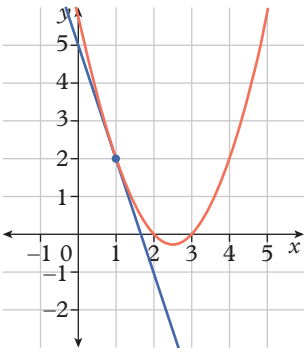
d −2, −2.99, −2.9999, hence $m_T \approx -3$

5 **a** **i** y-intercept = 2 **ii** −1
b **i** y-intercept = 1 **ii** −3
c **i** y-intercept = 1 **ii** 0
d **i** y-intercept does not exist **ii** undefined

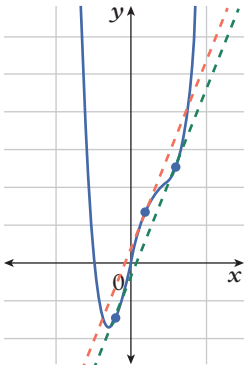
6 **a** 2 km/min **b** 1 km/min **c** 0.5 km/min
d 1 km/min **e** 1 km/min

7 **a** −2.5, −2.9, −2.99, hence $m_T \approx -3$

b



8 Not necessarily, the graph of a function could have a gradient of 5 at three different x values without being linear such as the one below.



9 **a** The rate starts at zero and steadily increases (becomes more positive).

b The rate starts at zero and steadily decreases (becomes more negative).

c The rate decreases (becomes less positive) until it is zero then continues decreasing (becomes more negative).

d The rate starts at nearly zero and increases at an increasing rate.

e The rate decreases (becomes less positive) until it is zero then continues decreasing (becomes more negative), then begins increasing (becomes less negative) until it is zero again then continues increasing (becomes more positive).

f The rate alternates between decreasing and increasing repeatedly.

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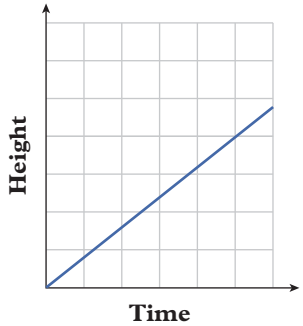
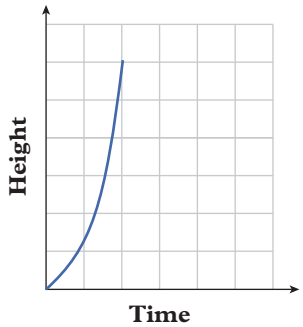
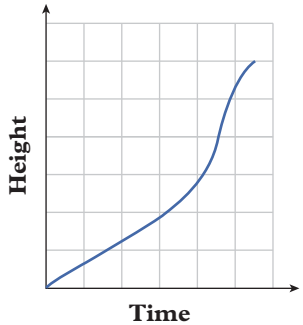
10 a



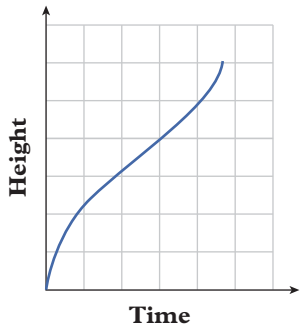
b



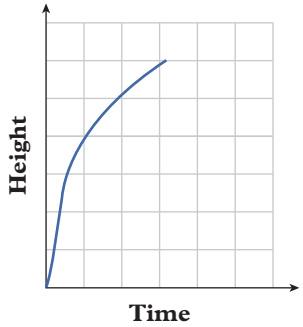
c



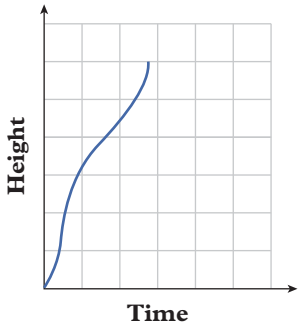
d



e



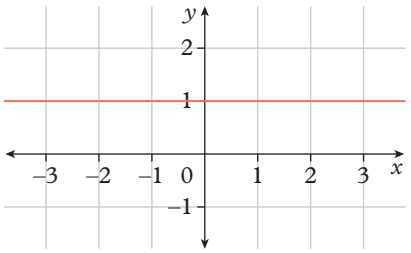
f



11 a $h + 6$ **b** $h^2 + 3h + 3$ **c** $3h + 5$

12 a 6 **b** $h^2 + 3$ **c** 5

13 a



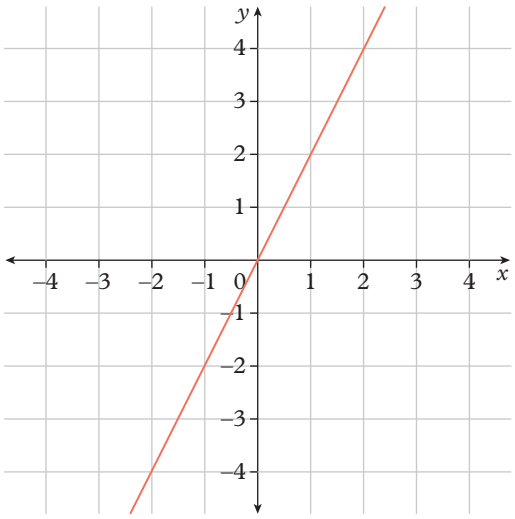
b No, the gradient of a linear relationship in the form $y = x + b$ is 1, the graph of the gradient is always a horizonal line passing through 1 on the vertical axis. The graph of the gradient is the same for different values of b. Similarly, the gradient of linear relationship in the form $y = ax + b$ is a, the graph of the gradient is always a horizonal line passing through a on the vertical axis regardless of the value of b.

c Gradient between (–2, 5) and (–1, 2) is –3
Gradient between (–1, 2) and (0, 0) is –1
Gradient between (0, 0) and (1, 2) is 1
Gradient between (1, 2) and (2, 5) is 3

d

x	–1.5	–0.5	0	0.5	1.5
Gradient	–3	–1	0	1	3

e



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f C

g No, the graph of the gradient of $y = x^2 + 1$ and $y = x^2 - 1$ are the same. If a quadratic relationship $y = a(x - h)^2$ is translated k units up or down, the rates of change are still the same for a given x value, the graph of the gradient remains the same regardless the value of k.

EX p242

Review: Non-linear relationships

oxforddigital **Answers for this lesson can be found on Oxford Digital.**

MODULE 5 Polynomials

EX p252

5.1 Functions and relations

oxforddigital **Answers for this lesson can be found on Oxford Digital.**

EX p260

5.2 Polynomials

1 a i 4 **ii** 3 **iii** 5
iv $2x^3$ **v** 2 **vi** 3
vii non-monic

b i 6 **ii** 5 **iii** −3
iv $4x^5$ **v** 4 **vi** −2
vii non-monic

c i 4 **ii** 4 **iii** 1
iv $-5x^4$ **v** −5 **vi** 5
vii non-monic

d i 7 **ii** 7 **iii** 0
iv x^7 **v** 1 **vi** −1
vii monic

e i 5 **ii** 6 **iii** 9
iv $-7x^6$ **v** −7 **vi** 2
vii non-monic

f i 3 **ii** 10 **iii** 3
iv $-11x^{10}$ **v** −11 **vi** 0
vii non-monic

2 a $5x^3 - 3x^2 - 4x + 2$ **b** $x^5 - 2x^4 + 4x^2$
c $2x^4 + 6x^3 + 2x^2 - 3$ **d** $5x^2 - 11x + 9$
e $3x^3 + 5x - 3$ **f** $-4x^4 - 4x^3 + 6x^2 + x$

3 a 2 **b** −8 **c** 16 **d** −68

4 a 127 **b** −5 **c** 15 **d** 97

5 a $-\frac{1}{2}$ **b** $\frac{137}{27} = 5\frac{2}{27}$ **c** $7 - \sqrt{2}$
d $\frac{13\sqrt{3} + 18}{9} = \frac{13\sqrt{3}}{9} + 2$

6 a $2x^3 + 2x^2 - 4x - 4$ **b** $2x^3 - 4x^2 + 10x - 8$
c $-2x^3 + 4x^2 - 10x + 8$ **d** $6x^3 - 3x^2 + 9x - 18$
e $-6x^2 + 14x - 4$ **f** $4x^3 + x^2 - x - 10$
g $2x^3 - 13x^2 + 31x - 14$ **h** $6x^3 - 9x^2 + 23x - 22$
i $6x^5 - 17x^4 + 20x^3 - 41x^2 + 48x - 12$
j 0 **k** 56 **l** −38

7 a $x^5 + 2x^4 + 7x^3$ **b** $4x^4 - 20x^3 + 8x^2$
c $-6x^7 + 24x^5 - 6x^3$ **d** $x^3 + x^2 - 2x + 12$
e $x^4 + 3x^3 - 8x^2 - 3x + 7$
f $x^7 + 4x^6 - 4x^4 - 10x^3 + 8x^2 + 3x - 2$

8 a $3x^3 + 15x^2 + 18x$ **b** $-7x^4 - 21x^3 + 70x^2$
c $x^3 + 10x^2 + 27x + 18$ **d** $x^3 - 6x^2 + 5x + 12$
e $3x^3 - 23x^2 + 44x - 20$ **f** $8x^3 - 2x^2 - 113x - 28$

9 a $x^6 + 8x^3 + 16$ **b** $x^4 + 6x^3 + 9x^2$
c $2x^5 - 8x^3 + 8x$ **d** $x^6 - 10x^5 + 25x^4$
e $x^3 + 6x^2 + 12x + 8$ **f** $8x^3 - 36x^2 + 54x - 27$
g $x^4 + 12x^3 + 54x^2 + 108x + 81$
h $x^4 - 4x^3 + 6x^2 - 4x + 1$

10 a polynomial **b** not a polynomial
c not a polynomial **d** not a polynomial
e polynomial **f** not a polynomial
g polynomial **h** not a polynomial

11 Polynomials of degree 3 can have a maximum of 4 terms; polynomials of degree 5 can have a maximum of 6 terms.

12 a i $n + 1$ **ii** 1 **iii** n **iv** $2n$
b i n **ii** n **iii** $n + m$

13 $k = -4$

14 $k = 11$

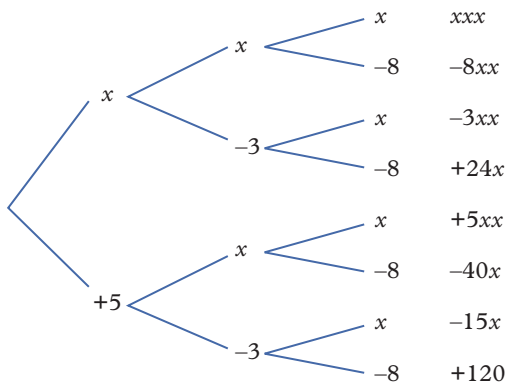
15 $a = -1, b = 4$

16 a $a^3 - 3a^2 + 2a + 1$
b $-27a^3 - 27a^2 - 6a + 1$
c $a^6 - 3a^4 + 2a^2 + 1$
d $-a^6 - 3a^4 - 2a^2 + 1$
e $a^3 + 3a^2 + 2a + 1$
f $8a^3 - 24a^2 + 22a - 5$

17
$$\begin{aligned}(2x-1)^4 &= (2x-1)(2x-1)(2x-1)(2x-1)\\ &= (2x-1)(2x-1)(4x^2-4x+1)\\ &= (2x-1)(8x^3-8x^2+2x-4x^2+4x-1)\\ &= (2x-1)(8x^3-12x^2+6x-1)\\ &= 16x^4-24x^3+12x^2-2x-8x^3+12x^2-6x+1\\ &= 16x^4-32x^3+24x^2-8x+1\end{aligned}$$

18 a $x^6 - 6x^4 + 12x^2 - 8$
b $x^6 + 2x^4 - 2x^3 + x^2 - 2x + 1$
c $16x^{12} + 32x^9 + 24x^6 + 8x^3 + 1$

19 a $x^3 - 6x^2 - 31x + 120$



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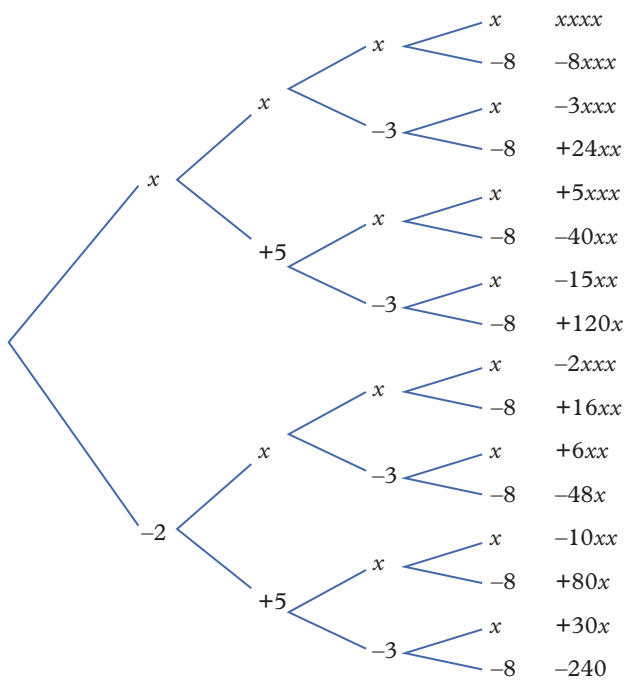
b $x^4 - 8x^3 - 19x^2 + 182x - 240$



EX 5.3 Dividing polynomials

p264

1 **a** quotient $x + 3$, remainder 5
b quotient $x - 8$, remainder -6
c quotient $5x + 6$, remainder 10
d quotient $9x - 1$, remainder 18
e quotient $7x + 2$, remainder -20
f quotient $6x - 7$, remainder 100
g quotient $-x - 53$ remainder 343
h quotient $-9x + 4$ remainder 2

2 **a** quotient $x^2 + x - 2$, remainder 1
b quotient $x^2 + 2x + 5$, remainder -6
c quotient $x^2 + 8x + 12$, remainder 9
d quotient $x^2 - 7x + 8$, remainder -7
e quotient $2x^2 - x - 3$, remainder 2
f quotient $3x^2 + x - 6$, remainder -4
g quotient $5x^2 + 3x + 3$, remainder -6
h quotient $4x^2 + 5x - 13$, remainder 12
i quotient $x^3 + x^2 - 5x + 3$, remainder 5
j quotient $x^3 - 3x^2 - 4x - 9$, remainder -21

3 **a** quotient $x^2 - 3x + 10$, remainder -9
b quotient $2x^2 + x + 2$, remainder -2
c quotient $4x^3 + 4x^2 + x + 1$, remainder -4
d quotient $3x^3 - 6x^2 + 12x - 24$, remainder 7

4 **a** **i** $x + 3 + \dfrac{5}{x+7}$
ii $(x + 7)(x + 3) + 5$
b **i** $x - 8 - \dfrac{6}{x-4}$
ii $(x - 4)(x - 8) - 6$
c **i** $5x + 6 + \dfrac{10}{x-2}$
ii $(x - 2)(5x + 6) + 10$
d **i** $9x - 1 + \dfrac{18}{x-4}$
ii $(x - 4)(9x - 1) + 18$
e **i** $7x + 2 - \dfrac{20}{2x+7}$
ii $(2x + 7)(7x + 2) - 20$
f **i** $6x - 7 + \dfrac{100}{10x+3}$
ii $(10x + 3)(6x - 7) + 100$

5 **a** **i** $x^2 + x - 2 + \dfrac{1}{x-3}$
ii $(x - 3)(x^2 + x - 2) + 1$
b **i** $x^2 + 2x + 5 - \dfrac{6}{x+1}$
ii $(x + 1)(x^2 + 2x + 5) - 6$
c **i** $x^2 + 8x + 12 + \dfrac{9}{x-2}$
ii $(x - 2)(x^2 + 8x + 12) + 9$
d **i** $x^2 - 7x + 8 - \dfrac{7}{x+4}$
ii $(x + 4)(x^2 - 7x + 8) - 7$
e **i** $2x^2 - x - 3 + \dfrac{2}{x+2}$
ii $(x + 2)(2x^2 - x - 3) + 2$
f **i** $3x^2 + x - 6 - \dfrac{4}{x-1}$
ii $(x - 1)(3x^2 + x - 6) - 4$

6 **a** $3x^3 + 2x^2 + 3x - 4$ remainder -2
b $12x^3 + 10x^2 + 8x + 2$ remainder 6
c $x^3 - x + 2$ remainder 4
d $2x^3 + x^2 - 14x - 30$ remainder 156
e $50x^3 - 146x^2 + 455x - 1370$ remainder 8226
f $24x^3 - 50x^2 + 101x - 269$ remainder 1620

7 **a** $x^2 - x - \frac{1}{2}$ remainder -4
b $3x^2 + 3x + 1$ remainder 1
c $4x^3 + 2x^2 + 4x + 2$ remainder -10
d $4x^3 + 2x^2 - 2x - 1$ remainder -11
e $-3x^3 - 3x^2 - 5x - 5$ remainder 3
f $3x^3 + 6x^2 + 14x + 22$ remainder 88

8 **a** $3x^2 + 7 + \dfrac{6}{x+2}$ **b** $2x^2 - 5x + \dfrac{9}{x-3}$
c $x^3 - 5x^2 - 8 + \dfrac{50}{x+6}$ **d** $x^3 + 12 + \dfrac{50}{x-7}$

9 **a** $(x - 3)(x + 5) + 6$
b $(x - 3)(x^2 - 4x + 9) + 6$
c $(x - 3)(x^2 - 6x + 2) - 11$
d $(x - 3)(x^3 + 8x^2 + 9x + 4) + 3$

10 **a** $4 - \dfrac{5}{x+2}$ **b** $9 + \dfrac{23}{x-3}$
c $1 - \dfrac{6}{x-2}$ **d** $5 + \dfrac{29}{2x-5}$

11 **a** $n - 1$
b $n - m$
c The degree of the remainder is less than the degree of the divisor $x^2 - 3$, so it has at most a linear term and a constant term and is of the form $ax + b$.

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12 a $\dfrac{(x+4)(x^2+5x+3)+5}{x+4}$

b $\dfrac{(x+4)(x^2+5x+3)}{x+4}+\dfrac{5}{x+4}=x^2+5x+3+\dfrac{5}{x+4}$

The fraction $\dfrac{(x+4)(x^2+5x+3)}{x+4}$ can be simplified by dividing the numerator and denominator by their common factor $(x+4)$.

13 a $k=24$

b linear factor $(x-2)$; quadratic factor (x^2-x-12)

$P(x)=(x-2)(x^2-x-12)$

$=(x-2)(x-4)(x+3)$

14 a $x^3+7x^2-26x-72$

b $(x+2)(x^2+5x-36)=(x+2)(x-4)(x+9)$

15 a quotient $x-3$, remainder $x-1$

b quotient x^2+x-4, remainder $6x-7$

c quotient $2x^2+2$, remainder -5

16 a $f(a)=0$

b $f(a)=\dfrac{P(a)}{D(a)}=\dfrac{b}{0}$, which is undefined as the denominator is zero.

c $f(x)=Q(x)$

EX 5.4 Remainder and factor theorems

p269

1 a 14 **b** 14; same answer

c 24 **d** 24; same answer

e $P(4)=64+16-40+8=48$; the remainder is 48.

f

$$\begin{array}{r|l} & x^2+5x+10 \\ \hline x-4 & x^3+x^2-10x+8 \\ & \underline{x^3-4x^2} \\ & 5x^2-10x+8 \\ & \underline{5x^2-20x} \\ & 10x+8 \\ & \underline{10x-40} \\ & 48 \end{array}$$

Long division produces a quotient of $x^2+5x+10$ and remainder of 48.

2	**a** 0	**b** 14	**c** −16	**d** 0
3	**a** 12	**b** 15	**c** 15	**d** 0
4	**a** 12	**b** −48	**c** 0	**d** 630
5	**a** no	**b** yes	**c** no	**d** no
	e no	**f** yes	**g** no	**h** yes
6	**a** no	**b** no	**c** yes	**d** no
	e no	**f** yes	**g** yes	**h** no

7 a $(x-1)$ or $(x+3)$ or $(x+6)$

b $(x-2)$ or $(x-3)$ or $(x+4)$

c $(x-3)$ or $(x+3)$ or $(x-4)$

d $(x+2)$ or $(x+3)$ or $(x-5)$

e $(x-1)$ or $(x-3)$

f $(x+2)$ or $(x-4)$

g $(x-2)$

h $(x+3)$

8 a $(x-1)(x+2)(x+5)$

b $(x-1)(x-2)(x+7)$

c $(x-2)(x+2)(x+5)$

d $(x+3)(x-4)(x-4)$ or $(x+3)(x-4)^2$

e $(x-2)(x-3)(x+3)$

f $(x-2)(2x+1)(x+2)$

g $(x+2)(4x-5)(x-3)$

h $(x-4)(3x+2)(2x-1)$

i $(x-1)(4x-3)(3x+1)$

9 a $(x-1)(x+3)(x+6)$

b $(x-2)(x-3)(x+4)$

c $(x-3)(x+3)(x-4)$

d $(x+2)(x+3)(x-5)$

e $(x-1)(3x+2)(x-3)$

f $(x+2)(2x-1)(x-4)$

g $(x-2)(4x+1)(2x-3)$

h $(x+3)(2x-1)(2x-3)$

10 a $k=1$, 3 or 5

b $P(x)=(x+1)(x+3)(x+5)$

11 a 2 **b** 3 **c** 4 **d** 7 **e** 12 **f** n

12 a 3

b −24; the constant term of $P(x)$ will be the product of the constant terms of the three linear factors, i.e. $2\times-3\times4=-24$

c −2, 3, −4; they are factors of −24

13 a $P(-1)=0$, so $(x+1)$ is a factor

b $P(2)=0$ so $(x-2)$ is a factor

c $P(-3)=0$ so $(x+3)$ is a factor

d $P(x)=(x+1)(x-2)(x+3)$

e $(x+1)(x-2)(x+3)$

$=(x+1)(x^2+x-6)$

$=x^3+x^2-6x+x^2+x-6$

$=x^3+2x^2-5x-6$

14 a $(x+1)$, $(x+4)$ and $(x-2)$

b $P(x)=(x+1)(x+4)(x-2)$

c $(x+1)(x+4)(x-2)$

$=(x+1)(x^2+2x-8)$

$=x^3+2x^2-8x+x^2+2x-8$

$=x^3+3x^2-6x-8$

15 a $(x-1)$, $(x+2)$ and $(x-5)$

b $P(x)=(x-1)(x+2)(x-5)$

16 $(x+1)(x-3)(x+4)$

17 a $x=a$ **b** $P(a)=r$

c The product $(a-a)Q(a)$ in the expression for $P(a)$ is equal to zero, so the value of $P(a)$ is equal to the remaining term, r, which is the remainder of the division $P(x)\div(x-a)=\dfrac{(x-a)Q(x)+r}{x-a}$

$=Q(x)+\dfrac{r}{x-a}$

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d **i** $x = \frac{a}{b}$

ii $F\left(\frac{a}{b}\right) = h$

iii The product $\left(b \times \frac{a}{b} - a\right)G\left(\frac{a}{b}\right)$ in the expression for $F\left(\frac{a}{b}\right)$ is equal to zero, so the value of $F\left(\frac{a}{b}\right)$ is equal to the remaining term, h, which is the remainder of the division

$$F(x) \div (bx - a) = \frac{(bx - a)G(x) + h}{bx - a}$$
$$= G(x) + \frac{h}{bx - a}$$

18 **a** $a = -15$ **b** $b = 5$

c $c = 31$ **d** $m = 6$ and $n = -3$

e $a = -33, b = -15$

19 **a** You need to substitute x values that are fractions $\left(\frac{1}{3}\text{ and } -\frac{3}{2}\right)$; the polynomial factorises to $(3x - 1)(2x + 3)(x + 1)$.

b There is only one linear factor; the polynomial factorises to $(x + 2)(x^2 + 3x + 4)$.

20 Sample answer: $x^3 + x^2 + 2x - 11$

21 **a** $(x - 1)(x + 1)(x + 2)(x - 3)$

b $(x - 1)(x + 2)(x + 3)(x + 4)$

c $(x + 1)(x + 2)(2x - 1)(x + 4)$

22 **a**

$$\begin{array}{r|l} & x^2 + 9 \\ x^2 + 4 & x^4 + 13x^2 + 36 \\ x^2(x^2 + 4) & -(x^4 + 4x^2) \\ & +9x^2 + 36 \\ 9(x^2 + 4) & -(9x^2 + 36) \\ & 0 \end{array}$$

b $(x^2 + 4)(x^2 + 9)$

EX Checkpoint

p271

1 **a**, **c** and **f** are functions

2 **a** polynomial

i 7 **ii** $4x^7$ **iii** 4

b not a polynomial; $-9x^{-100}$

c polynomial

i 81 **ii** $\frac{5y^{81}}{9}$ **iii** $\frac{5}{9}$

d not a polynomial; $-42\sqrt{y^3}$, $27y^{\frac{2}{3}}$

3 **a** $5x^3 - x^2 - 6x + 6$

b $5x^3 - 5x^2 + 10x - 8$

c $30x^3 - 18x^2 + 12x - 6$

d $10x^5 - 46x^4 + 63x^3 - 39x^2 + 22x - 7$

4 **a** quotient: $x + 6$, remainder: -5

b quotient: $x^2 - 5x + 12$, remainder: 9

c quotient: $2x - 6$, remainder: 10

d quotient: $2x^2 + 5x - 2$, remainder: -4

5 **a** $x + 12 - \frac{57}{x + 4}$

b $5x^2 + 21x + 128 + \frac{769}{x - 6}$

c $5x - 6 + \frac{-20x + 31}{x^2 + 4}$

6 **a** 20 **b** 0 **c** 84 **d** 0

7 **a** is not a factor **b** is a factor

c is not a factor **d** is a factor

8 **a** $(x - 1)(x + 4)(x - 7)$ **b** $(x + 1)(x + 2)(x - 3)$

9 **a** $5(x - 4)(x - 5)(x + 6)$

b $(x - 2)(x + 3)(x + 5)(x - 7)$

EX 5.5 Solving polynomial equations

p274

1 **a** $x = -5, -2$ or 4 **b** $x = -4, -1$ or 3

c $x = -3$, 2 or 6 **d** $x = -2$, 0 or 9

e $x = -1, \frac{1}{2}$ or 1 **f** $x = -5, -3\frac{1}{2}$ or $\frac{2}{3}$

2 **a** $x = -7, -3$, 1 or 4 **b** $x = -6$, 2, 3 or 5

c $x = -4, -2$ or -1 **d** $x = -5$, 0 or 6

e $x = -4, -\frac{1}{3}$, 0 or 4 **f** $x = -2\frac{1}{2}$ or $\frac{3}{4}$

3 **a** $(x + 1)(x + 2)(x + 6) = 0$; $x = -6, -2$ or -1

b $(x - 3)(x - 5)(x + 4) = 0$; $x = -4$, 3 or 5

c $(x - 2)(x - 4)(x - 2) = 0$ or $(x - 2)^2(x - 4) = 0$; $x = 2$ or 4

d $(x + 4)(x + 7)(x - 1) = 0$; $x = -7, -4$ or 1

e $(x + 3)(x + 4)(x - 4) = 0$; $x = -4, -3$ or 4

f $x(x - 1)(x + 1)(x - 1) = 0$ or $x(x - 1)^2(x + 1) = 0$; $x = -1$, 0 or 1

g $(x - 5)(x - 5)(x - 5) = 0$ or $(x - 5)^3 = 0$; $x = 5$

h $(x + 2)(x + 3)(x + 3) = 0$ or $(x + 2)(x + 3)^2 = 0$; $x = -3$ or -2

i $(x - 4)(x + 2)(x - 3) = 0$; $x = -2$, 3 or 4

j $(x + 1)(x + 1)(x + 1) = 0$ or $(x + 1)^3$; $x = -1$

k $15x(x + 3)(x + 6) = 0$; $x = -6, -3$ or 0

l $2(x - 2)(x - 1)(x + 3) = 0$; $x = -3$, 1 or 2

4 **a** $x = -4, -3$ or 1 **b** $x = -2$, 2 or 5

c $x = -6, -1$ or 6 **d** $x = -7, -3$ or 0

e $x = -2, -1$ or 3 **f** $x = -2$

g $x = -3$ or 3 **h** $x = -4$ or -1

i $x = 4$

5 **a** $x = -2, -1$ or 4 **b** $x = -1$, 1 or 5

c $x = -2$ or 2 **d** $x = -3, -2$ or -1

6 **a** $x = -3, -2$ or 3 **b** $x = -4$, 3 or 5

c $x = -1$, 1 or 4 **d** $x = -2$ or 3

7 **a** $x = 1, -1 - \sqrt{2}$ or $-1 + \sqrt{2}$

b $x = -2, -2 - \sqrt{3}$ or $-2 + \sqrt{3}$

c $x = -3, 1 - \sqrt{5}$ or $1 + \sqrt{5}$

d $x = 4, -4 - \sqrt{11}$ or $-4 + \sqrt{11}$

8 **a** $(x - 1)$ or $(x + 1)$ or $(x + 3)$ or $(x - 2)$.

b Sample answer: $P(x) = (x - 1)(x^3 + 2x^2 - 5x - 6)$

c Sample answer: The linear factor is $(x + 1)$, and the quadratic factor is $(x^2 + x - 6)$.

d Sample answer: $(x + 3)(x - 2)$

e $P(x) = (x - 1)(x + 1)(x + 3)(x - 2)$; $x = -3, -1$, 1 or 2

9 **a** $x = -1$, 1, 2 or 3 **b** $x = -2, -1$, 3 or 4

c $x = -4, -3, -2$ or 2 **d** $x = -3, -2$, 1 or 5

e $x = -2$, 1, 2 or 4 **f** $x = -3, -2$, 2 or 3

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10 **a** 1 **b** 2 **c** 3
d 4 **e** 6 **f** n

11 **a** $P(x) = (x-1)(x+2)(x-3)$
b $x = -2$, 1 or 3

12 $x^3 + x^2 - 2x - 8 = 0$ has factorised form $(x-2)(x^2+3x+4) = 0$, because the quadratic factor $x^2 + 3x + 4$ cannot be factorised.
So, $(x-2)(x^2+3x+4) = 0$ has one solution ($x = 2$) because there is only one linear factor.

13 $x^3 - 5x^2 + 3x + 9 = 0$ has factorised form $(x+1)(x-3)(x-3) = 0$, which has two solutions ($x = -1, 3$) because two of the linear factors are the same.

14 **a** $x(2x)(x+2) = 2x^3 + 4x^2$
b $2x^3 + 4x^2 = 192$; $x^3 + 2x^2 - 96 = 0$;
$(x-4)(x^2+6x+24) = 0$; $x = 4$;
height 4 cm, length 8 cm, width 6 cm

15 $\frac{1}{2}x(5x)(x-1) = 450$; $x = 6$;
base 6 cm, height 5 cm, length 30 cm

16 **a** **i** $(x-1)(x+1)(x-2)(x+3) = 0$
ii $(x-1)^2(x+1)(x-2)(x+3) = 0$ or
$(x-1)(x+1)^2(x-2)(x+3) = 0$ or
$(x-1)(x+1)(x-2)^2(x+3) = 0$ or
$(x-1)(x+1)(x-2)(x+3)^2 = 0$
iii $2(x-1)^2(x+1)(x-2)(x+3) = 0$ or
$2(x-1)(x+1)^2(x-2)(x+3) = 0$ or
$2(x-1)(x+1)(x-2)^2(x+3) = 0$ or
$2(x-1)(x+1)(x-2)(x+3)^2 = 0$
iv $5(x-1)^3(x+1)(x-2)(x+3) = 0$ or
$5(x-1)(x+1)^3(x-2)(x+3) = 0$ or
$5(x-1)(x+1)(x-2)^3(x+3) = 0$ or
$5(x-1)(x+1)(x-2)(x+3)^3 = 0$ or
$5(x-1)^2(x+1)^2(x-2)(x+3) = 0$ or
$5(x-1)^2(x+1)(x-2)^2(x+3) = 0$ or
$5(x-1)^2(x+1)(x-2)(x+3)^2 = 0$ or
$5(x-1)(x+1)^2(x-2)^2(x+3) = 0$ or
$5(x-1)(x+1)^2(x-2)(x+3)^2 = 0$ or
$5(x-1)(x+1)(x-2)^2(x+3)^2 = 0$
b $a(x-1)^p(x+1)^q(x-2)^r(x+3)^{n-p-q-r} = 0$, where p, q, r are non-zero, positive integers such that $p + q + r \le n$.

17 $P(x) = (x+1)(x-2)(x-4)(x+3)$; $x = -3, -1, 2$ or 4

18 **a** $x = -2, -1, 1$ or 5 **b** $x = -3, -2, 1, 2$ or 4

19 $x = \sqrt{2}, -\sqrt{2}$, or -2

EX p280 5.6 Sketching graphs of polynomials using intercepts

oxforddigital **Answers for this lesson can be found on Oxford Digital.**

EX p286 5.7 Sketching graphs of polynomials using transformations

oxforddigital **Answers for this lesson can be found on Oxford Digital.**

EX p291 Review: Polynomials

oxforddigital **Answers for this lesson can be found on Oxford Digital.**

MODULE 6 Geometry

EX p301 6.1 Geometric properties

1 **a** $a = 146°$ **b** $b = 24°$
c $c = 57°$, $d = 57°$, $e = 123°$ **d** $d = 75°$
e $e = 109°$, $f = 71°$, $g = 74°$
f $g = 71°$, $h = 19°$, $i = 71°$, $j = 90°$

2 **a** $a = 117°$ **b** $b = 109°$
c $c = 49°$ **d** $d = 129°$
e $e = 63°$, $f = 117°$, $g = 117°$, $h = 63°$ **f** $f = 126°$

3 **a** $a = 97°$
b $b = 88°$, $c = 88°$, $d = 92°$
c $c = 58°$, $d = 302°$
d $d = 52°$, $e = 71°$, $f = 57°$
e $d = 180°$, $e = 30°$, $f = 115°$, $g = 51°$
f $f = 67°$, $g = 64°$, $h = 116°$, $i = 91°$

4 **a** $a = 76°$ **b** $b = 60°$
c $c = 32°$, $d = 116°$ **d** $d = 160°$
e $e = 115°$ **f** $f = 37°$, $g = 75°$, $h = 68°$

5 **a** $a = 55°$ **b** $b = 117°$, $c = 63°$, $d = 117°$
c $c = 30°$, $d = 118°$ **d** $d = 45°$
e $e = 119°$, $f = 25°$
f $f = 58°$, $g = 98°$, $h = 98°$, $i = 122°$

6 **a** 3240° **b** 8640° **c** 17 640°

7 **a** $n - 2$
b The sum of the interior angles of the triangles that a polygon can be dissected into is equal to the sum of the interior angles of the polygon. As the interior angle sum of a triangle is 180°, the interior angle sum of an n-sided polygon is $(n-2) \times 180°$.

8 **a** 18 **b** 10 **c** 36 **d** 1000

9 **a** 18° **b** 27° **c** 22.5°
d 9.69° **e** 21° **f** 17.63°

10 If a side length in a triangle was greater than the sum of the side lengths of the other two sides, then the other two side lengths would not be able to meet at a point if each one was attached to a different end of the longer side length. Therefore, a triangle could not be formed with these side lengths.

11 **a** 6 cm $< x <$ 18 cm **b** 2 cm $< y <$ 26 cm
c 11 mm $< z <$ 29 mm

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12 **a** 8.9 cm < p < 15.5 cm **b** 7.5 cm < q < 13.2 cm
c 23.5 mm < r < 35 mm

13 **a** 120° **b** 360° **c** 90° **d** 360°
e The exterior angles of a convex polygon can be joined together to form a full revolution, which is 360°.

14 **a** false; angles at a point add to 360°.
b false; corresponding angles on parallel lines are equal.
c false; complementary angles add to 90°.
d true
e false; the sum of an interior angle and the adjacent exterior angle is 180°
f true

15 **a** $b = 30°$, $c = 44°$, $d = 74°$, $e = 29°$
b $c = 72°$ **c** $d = 109°$, $e = 8°$

16 **a** $a = 60°$, $b = 30°$, $c = 90°$ **b** $d = 35°$
c $e = 79°$ **d** $f = 163°$
e $g = 220°$ **f** $h = 154.5°$

17 **a** 330° **b** 185° **c** 225° **d** 351°
e 54° **f** 87° **g** 70° **h** 296°
i 325° **j** 300°

EX 6.2 Geometric proofs

p307

1 **a**

Statements	Reasons
$\angle EOF + \angle FOB = 180°$	Angles on a straight line sum to 180°
$\angle BOC + \angle FOB = 180°$	Angles on a straight line sum to 180°
$\angle EOF + \angle FOB = \angle BOC + \angle FOB$	Transitive law
$\therefore \angle EOF = \angle BOC$ ■	Subtraction property of equality

b

Statements	Reasons
$\angle EOF + \angle AOF + \angle AOB = 180°$	Angles on a straight line sum to 180°
$\angle BOC + \angle AOB + \angle AOF = 180°$	Angles on a straight line sum to 180°
$\angle EOF + \angle AOF + \angle AOB = \angle BOC + \angle AOF + \angle AOB$	Transitive law
$\therefore \angle EOF = \angle BOC$ ■	Subtraction property of equality

2

Statements	Reasons
$\angle GKJ + \angle JKL = 180°$	Angles on a straight line sum to 180°
$\angle NKL + \angle JKL = 180°$	Angles on a straight line sum to 180°
$\angle GKJ + \angle JKL = \angle NKL + \angle JKL$	Transitive law
$\therefore \angle GKJ = \angle NKL$ ■	Subtraction property of equality

3

Statements	Reasons
$\angle ABC = \angle EFB$	Corresponding angles on parallel lines are equal
$\angle ABC + \angle CBD = 180°$	Angles on a straight line sum to 180°
$\angle DBF + \angle CBD = 180°$	Angles on a straight line sum to 180°
$\angle ABC + \angle CBD = \angle DBF + \angle CBD$	Transitive law
$\angle ABC = \angle DBF$	Subtraction property of equality
$\angle EFB + \angle BFG = 180°$ $\angle EFB = \angle ABC = \angle DBF$	Angles on a straight line sum to 180°
$\therefore \angle DBF + \angle BFG = 180°$ ■	Transitive law

4

Statements	Reasons
$\angle DBA + \angle ABC + \angle CBE = 180°$	Angles on a straight line sum to 180°
$\angle DBA = \angle BAC$	Alternate angles on parallel lines are equal
$\angle CBE = \angle BCA$	Alternate angles on parallel lines are equal
$\therefore \angle BAC + \angle ABC + \angle BCA = 180°$ ■	Transitive law

5

Statements	Reasons
$\angle EFH + \angle DFE = 180°$	Angles on a straight line sum to 180°
$\angle DFE + \angle DEF + \angle FDE = 180°$	Interior angle sum of a triangle is 180°
$\angle EFH + \angle DFE = \angle DFE + \angle DEF + \angle FDE$	Transitive law
$\therefore \angle EFH = \angle DEF + \angle FDE$ ■	Subtraction property of equality

6

Statements	Reasons
X lies on $\overleftrightarrow{AB}$	Given
Y lies on $\overleftrightarrow{CD}$	Given
$\angle AXY = 90°$	Given
$\angle CYZ = 90°$	Given
$\angle CYX + \angle CYZ = 180°$	Angles on a straight line are supplementary
$\angle CYX = 180° - \angle CYZ$ $= 180° - 90°$ $= 90°$	Subtraction law of equality
$\angle AXY + \angle CYX = 180°$	Addition law of equality
$\therefore \overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ ■	If co-interior angles are supplementary, they lie on parallel lines

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7

Statements	Reasons
$\angle MPQ + \angle QPN = 180°$	Supplementary angles sum to 180°
$\angle MPQ + \angle QPN = \angle MPN$	Angle addition postulate
$\angle MPN = 180°$	Transitive law
$\therefore MN$ is a straight line ■	Angles on a straight line sum to 180°

8 **a** Parallel lines will never meet, no matter how far they extend. Lines which are not parallel will meet at some point.

b $\angle EAB$ and $\angle ACB$

c If AC is equal to 0, the points A and C would be the same point. Any value less than zero is meaningless.

d $\angle EAB + \angle BAC = 180°$

But: $\angle EAB = x$

So: $x + \angle BAC = 180°$

That is: $\angle BAC = 180° - x$

e $(180° - x) + x + y = 180°$

f $180° + y = 180°$ simplifies to $y = 0°$

g If $\angle ABC$ is 0°, line BC would lie on top of line BA, so points A and C would lie on top of each other.

h This contradicts the assumption that $AC > 0$. In other words, the lines must be parallel.

i Assume that $\angle EAB$ and $\angle ACB$ are equal corresponding angles, and that DG is not parallel to HF.

Also assume that the length of AC is greater than zero.

In $\triangle ABC$: $\angle ABC + \angle ACB + \angle CAB = 180°$

But: $\angle CAB = 180° - \angle EAB$

So: $\angle ABC + \angle ACB + (180° - \angle EAB) = 180°$

It was assumed that $\angle EAB = \angle ACB$.

This relationship then reduces to $\angle ABC = 0°$.

This implies that $AC = 0$.This contradicts the statement that $AC > 0$. So the lines DG and HF must be parallel.

9 **a** If two lines cut by a transversal are not parallel, then the corresponding angles are not equal.

b

Statements	Reasons
$AB \nparallel CB$	Given
$\angle EAB$ and $\angle ACB$ are corresponding angles	Given
ABC is a triangle	Three points that do not form a line form a triangle
$\angle EAB = \angle ABC + \angle ACB$	The exterior angle of a triangle is equal to the sum of the two non-adjacent interior angles
$\angle ABC \neq 0°$	Angles cannot be equal to 0°
$\angle ABC + \angle ACB \neq 0° + \angle ACB$	Angle addition postulate
$\angle EAB \neq 0° + \angle ACB$	Transitive law
$\therefore \angle EAB \neq \angle ACB$ ■	Subtraction property of equality

10 If $a = b$, then $(a - b) = 0$. Dividing by $(a - b)$ is actually dividing by 0, which gives an answer which is undefined.

EX p314

6.3 Congruence and similarity

oxforddigital **Answers for this lesson can be found on Oxford Digital.**

EX p323

6.4 Proofs and quadrilaterals

oxforddigital **Answers for this lesson can be found on Oxford Digital.**

EX p326

Checkpoint

1 **a** $w = 49°$, $x = 104°$ **b** $y = 144°$

2 **a** $m = 68°$ **b** $j = 65°$, $k = 65°$

3 **a** $p = 135°$ **b** $q = 72°$

4

Statements	Reasons
AB is a line segment	Given
CD is a line segment	Given
$\angle AEC + \angle CEB = 180°$	Angles on a straight line sum to 180°
$\angle CEB + \angle DEB = 180°$	Angles on a straight line sum to 180°
$\angle AEC + \angle CEB = \angle CEB + \angle DEB$	Transitive law
$\therefore \angle AEC = \angle DEB$ ■	Subtraction law of equality

5

Statements	Reasons
$\angle ABE = \angle DCF$	Given
$\angle ABE = \angle CBE$	Vertically opposite angles are equal
$\angle DCF = \angle CBE$	Transitive law
$\therefore AB \parallel CD$ ■	If two lines are cut by a transversal such that corresponding angles are equal, then the lines are parallel

6 **a** $a = 8$ cm, $b = 7$ cm

b $x = 105°$, $y = 20°$, $z = 9$ cm

7 **a** $g = 27°$, $h = 1.3$ cm

b $p = 1.4$ cm, $q = 6.25$ cm

8 **a**

Statements	Reasons
$AD = DE$	Given
$AC = CE$	Given
CD is common to both $\triangle ADC$ and $\triangle EDC$	Common side
$\triangle ADC \cong \triangle EDC$	SSS (congruence)
$\therefore \angle ACD = \angle ECD$ ■	Corresponding angles in congruent triangles are equal

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b

Statements	Reasons
$AB \parallel EF$	Given
$\angle DAB = \angle DEF$	Corresponding angles on parallel lines are equal
$\angle DBA = \angle DFE$	Corresponding angles on parallel lines are equal
$\angle ADB = \angle EDF$	Common angle
$\therefore \triangle ADB \sim \triangle EDF$ ■	AAA (similarity)

9

Statements	Reasons
$\angle ADC = 90°$	Given $AD \perp CD$
$\angle BCD = 90°$	Given $BC \perp CD$
$\angle ADC + \angle BCD = 180°$	Addition property of equality
$AD \parallel BC$	If co-interior angles are supplementary, they lie on parallel lines
$ABCD$ is a parallelogram	A parallelogram is a quadrilateral with two pairs of parallel sides
$ABCD$ is a rectangle	A rectangle is a parallelogram with a right angle
$\therefore AC = BD$ ■	Diagonals of a rectangle are equal in length

10

Statements	Reasons
$\angle SPQ = \angle RST$	Given
$PQ \parallel SR$	If two lines are cut by a transversal such that corresponding angles are equal, then the lines are parallel
$PS \parallel QR$, $\angle RST = \angle SRQ$	If alternate angles are congruent, they lie on parallel lines
$\therefore PQRS$ is a parallelogram ■	If a quadrilateral has two pairs of opposite sides that are parallel, it is a parallelogram

EX p331

6.5 Circle geometry: angles

1 **a** 45° **b** 120° **c** 74°

2 **a** 90° **b** 18° **c** 45°

3 **a** $x = 32°$ **b** $x = 27°$ **c** $a = 9°$, $c = 6°$

4 In the following answers, CT stands for circle theorem.

a $x = 34.5°$, CT2 and angle sum of a triangle

b $x = 109°$, CT1 **c** $x = 21°$, CT3

d $a = 14°$, $b = 7°$, CT3

e $x = 22.5°$, CT2 and triangle angle sum

f $a = 26\frac{1}{3}°$, $e = 9\frac{7}{8}°$, CT3

5 **a** 12.2° **b** 76° **c** 32° **d** 68° **e** 55° **f** 29°

6 **a** Each one is a radius of the circle.

b $\triangle AOB$ is isosceles.

c $\angle AOD = \angle ABO + \angle OAB = 2\angle ABO$

d $\triangle AOB$ is isosceles, so $\angle OBC = \angle OCB$.

$\angle DOC = \angle OBC + \angle OCB = 2\angle OBC$

e $\angle AOC = \angle AOD + \angle DOC = 2\angle ABO + 2\angle OBC = 2(\angle ABO + \angle OBC) = 2\angle ABC$

7 **a** 180°

b Since the angle at the centre is twice the angle at the circumference subtended by the same arc, the angle at the circumference is equal to $\frac{1}{2}$ of 180°, or 90°.

8 **a** $\angle WOZ$ is an angle at the centre of the circle, while $\angle WXZ$ is an angle at the circumference. Both are subtended by the same arc, and so circle theorem 1 applies so that $\angle WOZ = 2\angle WXZ$.

b Similarly, $\angle WOZ$ and $\angle WYZ$ are both subtended by the same arc, so circle theorem 1 applies and $\angle WOZ = 2\angle WYZ$.

c It follows that $\angle WYZ = \angle WXZ$.

9 **a** Obtuse $\angle AOC$ is an angle at the centre of the circle, subtended by the same arc as angle x at the circumference.

Similarly, the reflex angle $\angle AOC$ is an angle at the centre of the circle, subtended by the same arc as angle y at the circumference.

In both cases, circle theorem 1 applies, so that $\angle AOC = 2x$ and $\angle AOC = 2y$.

b obtuse $\angle AOC$ + reflex $\angle AOC = 360°$

That is: $2x$ + reflex $\angle AOC = 360°$

or: reflex $\angle AOC = 360° - 2x$

c Since: reflex $\angle AOC = 2y$

$2y = 360° - 2x$

Simplifying: $y = 180° - x$

d obtuse $\angle BOD = 2\angle BCD$

reflex $\angle BOD = 2\angle BAD$

obtuse $\angle BOD$ + reflex $\angle BOD = 360°$ (angles at a point)

So: $2\angle BAD + 2\angle BCD = 360°$

Rearranging and simplifying: $\angle BAD = 180° - \angle BCD$

e **i** $x = 142°$, $y = 99°$ **ii** $a = 45°$, $y = 38°$

iii $x = 30°$, $y = 22.5°$

10 **a** $\angle ADC + \angle ABC = 180°$ (opposite angles of a cyclic quadrilateral)

So: $\angle ADC = 180° - \angle ABC$

b supplementary angles on a straight line

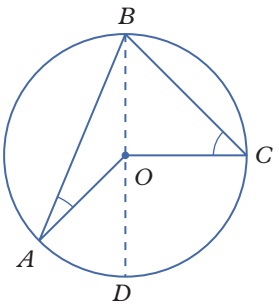
c both are equal to the same expression ($180° - \angle ABC$)

d **i** $a = 61°$

ii $a = 81°$, $b = 92°$, $c = 99°$, $d = 88°$

iii $x = 15°$, $z = 29.75°$

11 Circle theorem 1



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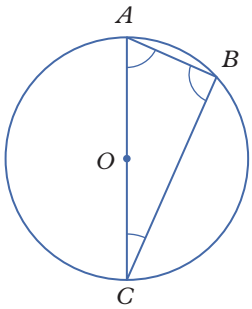
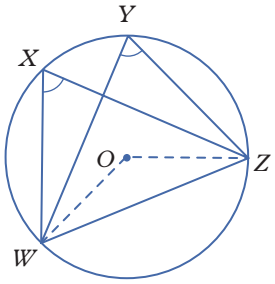
Statements	Reasons
$OA = OB = OC$	Radii of the circle
$\triangle AOB$ and $\triangle COB$ are isosceles	If two sides in a triangle are equal, the triangle is isosceles
$\angle ABO = \angle OAB$ $\angle CBO = \angle OCB$	Angles opposite equal sides in an isosceles triangle are equal
$\angle AOD = \angle ABO + \angle OAB$ $\angle DOC = \angle CBO + \angle OCB$	The exterior angle of a triangle is equal to the sum of the two opposite interior angles
$\angle AOD = 2\angle ABO$ $\angle DOC = 2\angle CBO$	Transitive law, substitution and simplifying
$\angle AOC = \angle AOD + \angle DOC$	Angle addition postulate
$\angle AOC = 2\angle ABO + 2\angle CBO$ $= 2(\angle ABO + \angle CBO)$	Transitive law and factorising
$\angle ABC = \angle ABO + \angle CBO$	Angle addition postulate
$\therefore \angle AOC = 2\angle ABC$ ■	Transitive law

Circle theorem 2



Statements	Reasons
$\angle AOC = 180°$	A straight angle equals 180°
$\angle AOC = 2\angle ABC$	Circle theorem 1
$2\angle ABC = 180°$	Transitive law
$\therefore \angle ABC = 90°$ ■	Solving equation

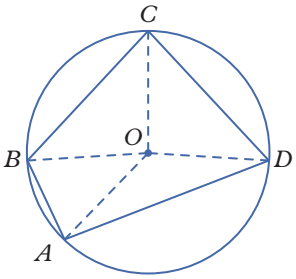
Circle theorem 3



Statements	Reasons
$\angle WOZ = 2\angle WXZ$	Circle theorem 1
$\angle WOZ = 2\angle WYZ$	Circle theorem 1
$2\angle WXZ = 2\angle WYZ$	Transitive law
$\therefore \angle WXZ = \angle WYZ$ ■	Simplify equation

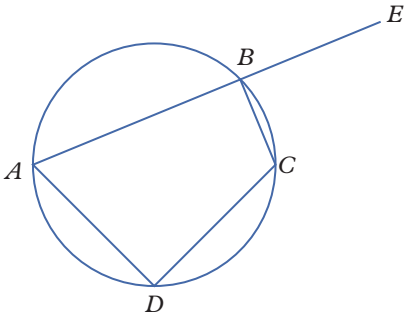
Circle theorem 4

The labels $\angle_O$ and $\angle_R$ are used to label obtuse and reflex angles, respectively.



Statements	Reasons
$\angle_R COA = 2\angle CBA$	Circle theorem 1
$\angle_O COA = 2\angle CDA$	Circle theorem 1
$\angle_R COA + \angle_O COA = 360°$	Angles at a point add to 360°
$2\angle CBA + 2\angle CDA = 360°$	Transitive law
$\angle CBA + \angle CDA = 180°$	Simplify equation
$\angle_R BOD = 2\angle BAD$	Circle theorem 1
$\angle_O BOD = 2\angle BCD$	Circle theorem 1
$\angle_R BOD + \angle_O BOD = 360°$	Angles at a point add to 360°
$2\angle BAD + 2\angle BCD = 360°$	Transitive law
$\therefore \angle BAD + \angle BCD = 180°$ ■	Simplify equation

Circle theorem 5



Statements	Reasons
$\angle ABC + \angle ADC = 180°$	Circle theorem 4
$\angle ABC + \angle CBE = 180°$	Angles on a straight line add to 180°
$\angle ABC + \angle ADC =$ $\angle ABC + \angle CBE$	Transitive law
$\therefore \angle ADC = \angle CBE$ ■	Subtraction property of equality

12 **a** $\angle DBC = 66°$ **b** $\angle DEG = 19°$
c $\angle ODC = 38°$ **d** $\angle OAB = 62°$

EX p336

6.6 Circle geometry: chords

oxforddigital **Answers for this lesson can be found on Oxford Digital.**

EX p343

6.7 Circle geometry: tangents and secants

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EX Review: Geometry
p348

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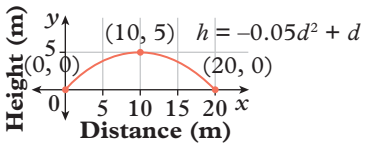
EX Semester 1 review: Mathematical modelling
p352

1 **a** 20 m wide

b maximum height = 5 m, 10 m away from the entrance

c



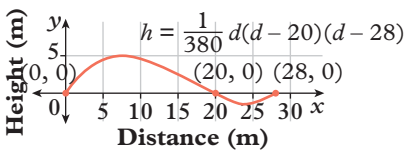
d **i** $10 - 2\sqrt{5}$ m or $10 + 2\sqrt{5}$ m

ii 5.53 m or 14.47 m

e $h = \frac{1}{380}d(d - 20)(d - 28)$

f $h = \frac{1}{380}(d^3 - 48d^2 + 560d)$

g



h **i** maximum height = 5.06 m, 7.67 m away from the entrance

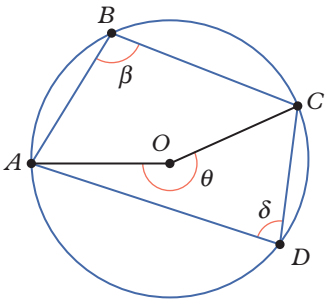
ii maximum height = 1.02 m, 24.33 m away from the entrance

2 **a** Square, rhombus, kite **b** Kite

c Let K be a kite $ABCD$ such that $AB = AD$ and $BC = CD$. Consider the two triangles $\triangle ABC$ and $\triangle ADC$. Since $AB = AD$, $BC = CD$ and AC is a shared edge, $\triangle ABC \cong \triangle ADC$ by the test SSS. This means that the angles $\angle ADC$ and $\angle ABC$ are equal since they are in corresponding positions in the two triangles.

d **i** Cyclic quadrilateral. Opposite pairs of internal angles are supplementary.

ii Let $ABCD$ be a cyclic quadrilateral with centre O. Let the interior angles $\angle ABC = \beta$ and $\angle ADC = \delta$ and the angle at O subtended by the arc containing B be θ, as shown. Circle theorem 1 implies that $\theta = 2\beta$. Since angles at a point sum to 360°, circle theorem 1 also implies that $360° - \theta = 2\delta$. Substituting $\theta = 2\beta$ and rearranging implies $360° = 2\delta + 2\beta$. Dividing by 2 gives $\delta + \beta = 180°$. The same logic implies that $\angle BAD + \angle BCD = 180°$. Hence, opposite pairs of angles in a cyclic quadrilateral are supplementary.



iii From part **c**, $\triangle ABC \cong \triangle ADC$ and angles $\angle ADC$ and $\angle ABC$ are equal. Since $ABCD$ is a cyclic quadrilateral, opposite pairs of angles are supplementary. Since $\angle ADC$ and $\angle ABC$ are opposite angles, $\angle ADC + \angle ABC = 2\angle ABC = 2\angle ADC = 180°$. Dividing by two gives $\angle ABC = \angle ADC = 90°$. Since both $\triangle ABC$ and $\triangle ADC$ contain a right angle, they are congruent right-angled triangles that make up $ABCD$.

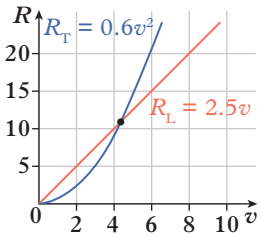
3 **a** **i** $k = 2.5$; $R_L = 2.5v$

ii $R_L = 50$ N

b **i** $k = 0.6$; $R_T = 0.6v^2$

ii $R_T = 240$ N

c

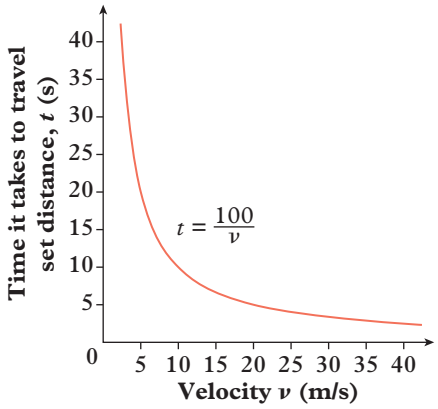


d $v = 0$ m/s; $v = \frac{25}{6} = 4\frac{1}{6}$ m/s

e $k = 100$; $v = \frac{100}{t}$

f 100 m

g



h As the velocity increases the time it takes to travel 100 m approaches zero, but it never reaches zero as $v = \frac{100}{t}$ and $\frac{100}{0}$ is undefined.

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EX p354

EXPLORATIONS 1

1 a 4 red bricks

b 2 purple bricks

2 a

×	2	7	10	5	3
9	18	63	90	45	27
4	8	28	40	20	12
6	12	42	60	30	18
8	16	56	80	40	24
1	2	7	10	5	3

b

×	6	4	2	8
10	60	40	20	80
12	72	48	24	96
14	84	56	28	112
16	96	64	32	128

c Four tables fit the given criteria:

×	10	7	5	3	2
6	60	42	30	18	12
4	40	28	20	12	8
9	90	63	45	27	18
8	80	56	40	24	16

×	10	7	5	3	8
6	60	42	30	18	48
4	40	28	20	12	32
9	90	63	45	27	72
2	20	14	10	6	16

×	10	7	5	9	2
6	60	42	30	54	12
4	40	28	20	36	8
3	30	21	15	27	6
8	80	56	40	72	16

×	10	7	5	3	2
6	60	42	30	54	48
4	40	28	20	36	32
9	30	21	15	27	24
8	20	14	10	18	16

3 a $\sqrt{3} - \sqrt{2}$

b i $\dfrac{\sqrt{a} - \sqrt{b}}{a - b}$

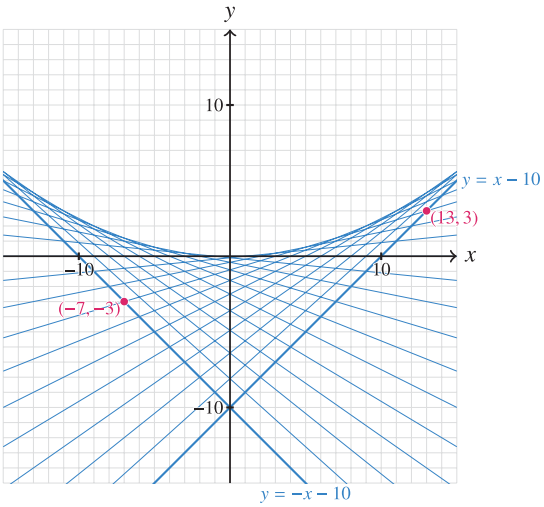
ii It fails when $a = b$. Then $\dfrac{\sqrt{a}}{2a}$ is the correct simplified fraction.

c $\dfrac{2\sqrt{3} + 3\sqrt{2} - \sqrt{30}}{12}$

d i $\dfrac{a\sqrt{b} + b\sqrt{a} - \sqrt{abc}}{2ab}$

ii $\dfrac{1}{\sqrt{3} + \sqrt{5} + \sqrt{8}} = \dfrac{3\sqrt{5} + 5\sqrt{3} - 2\sqrt{30}}{30}$

4 a



b i $y = \dfrac{3}{10}x - \dfrac{9}{10}$

ii Show that $\dfrac{1}{40}x^2 = \dfrac{3}{10}x - \dfrac{9}{10}$ simplifies to $(x - 6)^2 = 0$, which has exactly one solution.

c i $Q(k - 10, -k)$

ii $y = \dfrac{k}{10}x - \dfrac{k^2}{10}$

iii Simplify to $(x - 2k)^2 = 0$ or show that the discriminant is zero.

d Show that $ax^2 = \dfrac{k}{d}x - \dfrac{k^2}{d}$ has discriminant equal to zero when $a = \dfrac{1}{4d}$.

5 a 1, 32, 243, 1024, 3125. The last digit of the power is equal to the base. Yes, this also works for other fifth powers.

b $n(n-1)(n+1)(n^2+1)$. Show that 2, 3 and 5 each divide at least one of the four factors. If n is of the form $5k \pm 2$, show that 5 divides $n^2 + 1$.

c Since $n^5 - n$ ends in 0, n^5 and n must have the same last digit.

d Explain why the second and third factors are divisible by 2 and 4, or vice versa.

e Any n that is equal to, one more than, or one less than a multiple of 9.

f $n^9 - n = (n^5 - n)(n^4 + 1)$ is a multiple of 30. It is true for all 1st, 9th, 13th, …, $(4m + 1)$th, … powers.

MODULE 7 Pythagoras' theorem and trigonometry

EX p360

7.1 Pythagoras' theorem

1 a right-angled

b acute-angled

c acute-angled

d right-angled

e right-angled

f obtuse-angled

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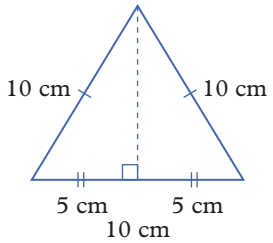
2 **a** $5\sqrt{2}$ mm **b** $\sqrt{13}$ cm **c** 75 mm
d $2\sqrt{286}$ cm **e** $\sqrt{52.65}$ cm **f** $\sqrt{3.95}$ cm
g $\sqrt{75.33}$ cm **h** $\sqrt{17.49}$ cm **i** $\sqrt{66.28}$ cm

3 **a** $2\sqrt{2}$ cm **b** $30\sqrt{2}$ mm **c** $13\sqrt{2}$ cm
d $35\sqrt{2}$ cm

4 $x\sqrt{2}$ cm

5 **a**



b $5\sqrt{3}$ cm
c $25\sqrt{3}$ cm^2

6 $\frac{\sqrt{3}}{4}x^2$ cm^2

7 **a** 51.45 cm **b** 9.52 cm **c** 39.38 cm
d 131.23 cm **e** 92.66 cm **f** 73.03 cm

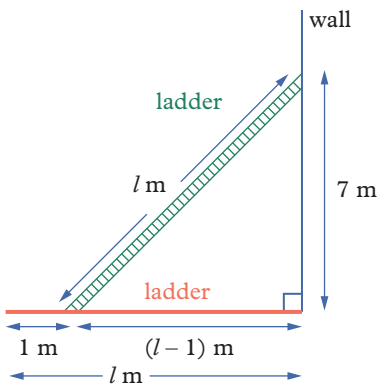
8 **a** obtuse **b** acute **c** obtuse
d acute **e** acute **f** acute

9 **a** 11.2 cm, 22.4 cm
b 7.9 cm, 23.7 cm
c 4.9 cm, 24.5 cm

10 38 m

11 **a** 6.4 m **b** 38.4 m **c** 74.6 m
d **i** 75.7 m **ii** 82 m

12 **a**



b 25 m

13 **a** $y = -\frac{1}{2}x$, $y = \frac{1}{3}x + \frac{5}{3}$, $y = 7x + 15$
b (–2, 1)
c $2\sqrt{5}$ units, $\sqrt{10}$, units, $5\sqrt{2}$ units

EX p365

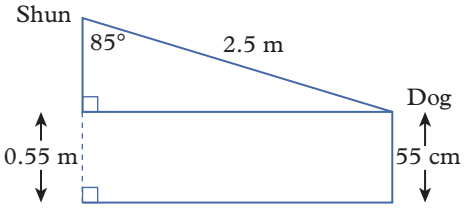
7.2 Trigonometry

1 **a** $p = 5.2$ cm, $q = 4.3$ cm
b $t = 6.3$ mm, $s = 13.6$ mm
c $x = 18.1$ cm, $a = 16.0$ cm
d $k = 41.2$ mm, $m = 26.0$ mm
e $r = 8.6$ mm, $k = 9.2$ mm
f $s = 4.6$ cm, $t = 6.8$ cm

2 **a** 22° **b** 32° **c** 67°
d 50° **e** 64° **f** 1°

3 **a** $B = 66°$, $C = 24°$ **b** $T = 29°$, $P = 61°$
c $G = 32°$, $H = 58°$ **d** $K = 29°$, $M = 61°$
e $D = 27°$, $R = 63°$ **f** $D = 40°$, $F = 50°$

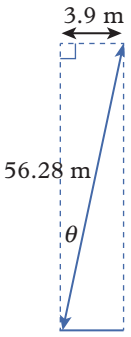
4 **a**



b 249 cm **c** 77 cm

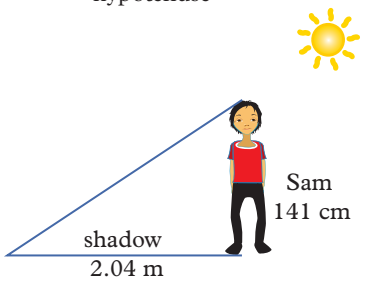
5 $l = 230$ mm, $w = 76$ mm

6 **a**



b $\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$ **c** 4°

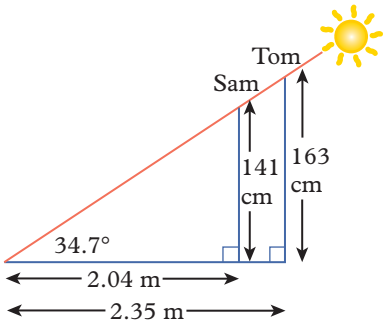
7 **a**



b Tom 34.7°, Sam 34.7°. The base angle represents the angle of elevation of the sun from the ground at the given time.

c **i** If the two triangles have the same base angle and a right angle then all three corresponding angles are equal, meaning the triangles are similar.

ii



8 **a** In the first and third questions Ji-Ho has written the trigonometric ratios in the wrong order, with the numerator and denominator needing to be reversed. In the second question Ji-Ho has selected the wrong trigonometric ratio to use.

b When writing trigonometric ratios, make sure to follow the SOH CAH TOA rule

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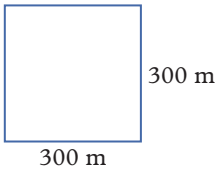
9 a



b 424.3 m **c** 424.3 m

d The answers are identical.

10 a 1.7 m **b** $l = 100$ cm, $w = 80$ cm

11 a For Route 1 **i** 1 : 2 **ii** 50%

For Route 2 **i** 5 : 12 **ii** 41.7%

For Route 3 **i** 13 : 30 **ii** 43.3%

b Route 1, 26.6°; Route 2, 22.6°; Route 3, 23.4°

c Route 2, Route 3, Route 1

12 a $x = 13.16$ cm **b** $x = 46.24°$

c $x = 19.66$ cm **d** $x = 145.32°$

13 20.78 cm

14 a i $x = 40$ cm, $y = 34.64$ cm

ii $x = 80.52$ cm, $y = 59.69$ cm

b i 692.8 cm^2 **ii** 2403.3 cm^2

15 a 67.5° **b** octagon **c** 80 cm

16 a i almost 0 **ii** The ratio is almost 0.

b i a is almost equal in length to c.

ii The ratio is almost 1.

c The sine increases from 0 at 0° to 1 at 90°.

d The cosine ratio decreases from 1 at 0° to 0 at 90°.

e The tangent is 0 at 0°, and becomes increasingly large (approaching infinity) as the angle approaches 90°.

17, 18 The sine ratio has a minimum value of 0 and a maximum value of 1.

The cosine ratio has a minimum value of 0 and a maximum value of 1.

The tangent ratio has a minimum value of 0 with no maximum value.

7.3 Applications of trigonometry

EX p373

1 a $x = 117°$, $y = 63°$ **b** $x = 58°$, $y = 302°$

c $x = 45°$, $y = 45°$

d $x = 19°$, $y = 60°$, $z = 281°$

e $v = 153°$, $w = 57°$, $x = 150°$, $y = 30°$, $z = 282°$

f $v = 118°$, $w = 99°$, $x = 40.5°$, $y = 37°$, $z = 282.5°$

2 a $w = 95.7$ m **b** $x = 27.0$ m

c $z = 148.2$ m **d** $p = 121.7$ m

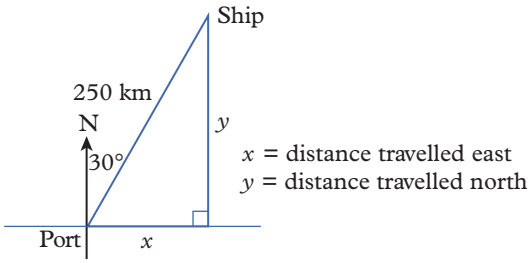
e $s = 254.3$ m **f** $y = 235.2$ m

3 a 25 m **b** 144 m

4 a i 150°T **ii** 308°T **iii** 005°T

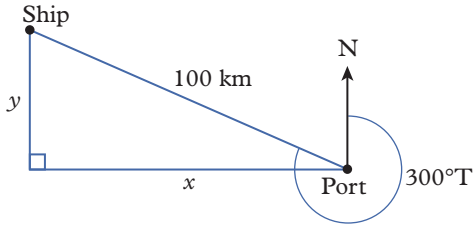
b i S20°W **ii** S80°E **iii** N60°W

5 a i



ii 125 km east, 217 km north

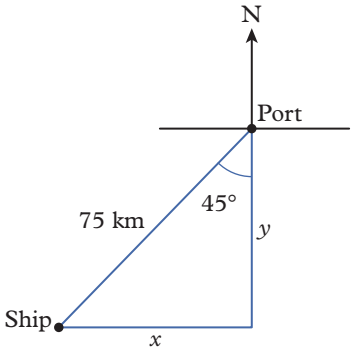
b i



x = distance travelled west
y = distance travelled north

ii 87 km west, 50 km north

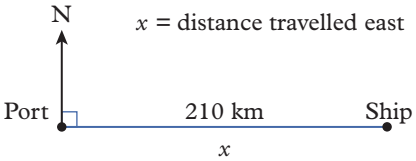
c i



x = distance travelled west
y = distance travelled south

ii 53 km west, 53 km south

d i



ii 210 km east, 0 km north

6 a i 250 km on a bearing of S30°W

ii 250 km on a bearing of 210°T

b i 100 km on a bearing of S60°E

ii 100 km on a bearing of 120°T

c i 75 km on a bearing of N45°E

ii 75 km on a bearing of 045°T

d i 210 km on a bearing west

ii 210 km on a bearing of 270°T

7 a 10.39 m **b** 0.4 m/s

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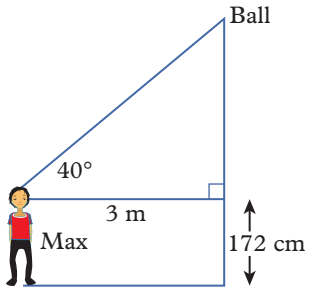
8 a



b 2.52 m (252 cm)

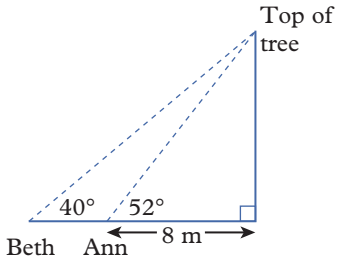
c 4.24 m (424 cm)

d The height of the ball is not significantly larger than the height of the eye level. If Max was measuring the height of a cliff, the height of the eye level may not be as important. If we do not consider the height of a person's eye in this situation, the error would be too big.

9 The angle of elevation and the angle of depression cannot be obtuse, because they are always measured from the line of sight pointing towards the other point. An obtuse angle would point backwards and therefore be measured facing the opposite direction.

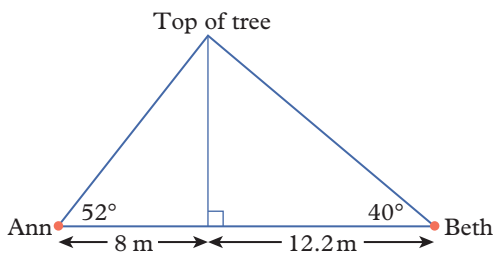
10 a 25.87 m **b** 0.55 m

11 a



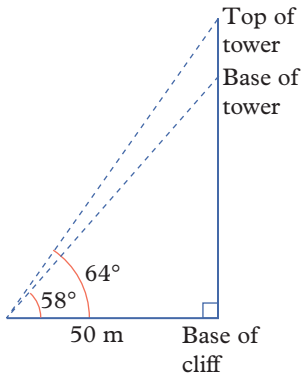
b 10.2 m **c** 12.2 m **d** 4.2 m

e i



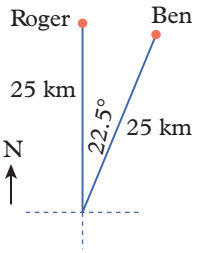
ii 20.2 m

12 a



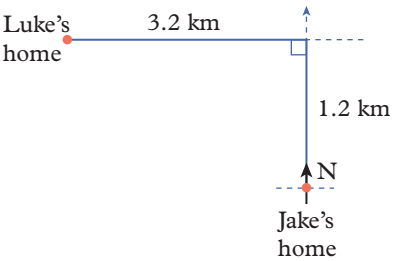
b 22.5 m

13 a



b 23.1 km **c** 9.6 km **d** 9.8 km

14 a



b 3.4 km **c** 291°T **d** 111°T

e The bearing and its back-bearing differ by 180°.

f i 3.4 km

ii The bearing from Jake's house to Luke's house is 291°T.

The bearing from Luke's house to Jake's house is 111°T.

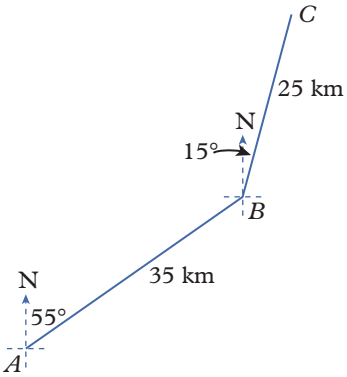
g The answers should be the same, but may vary slightly depending on the accuracy of the scale drawing.

15 a 60° **b** 45° **c** 40° **d** 10°

e i The angles in a triangle add to 180°; subtracting the acute measured angle and the right angle from 180° gives the angle of elevation. Alternatively, the acute measured angle and the angle of elevation are complementary as they are the non-right angles in a right-angled triangle.

ii The acute measured angle is vertically opposite the other non-right angle in the right-angled triangle. Alternatively, the obtuse measured angle forms a straight angle with the other non-right angle, so they are supplementary and must add to 180°. Then find the angle of depression by subtracting the acute measured angle and the right angle from 180° (or by finding the complement of the acute measured angle).

16 a



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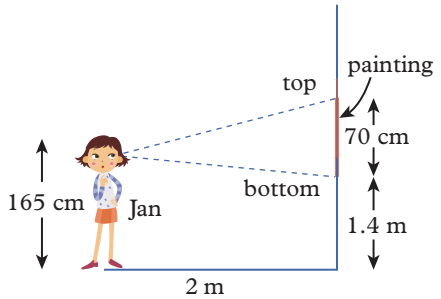
b **i** 20.1 km **ii** 24.1 km **iii** 44.2 km
c **i** 28.7 km **ii** 6.5 km **iii** 35.2 km
d **i** 56.5 km **ii** 039°T **iii** 219°T
e **i** 56.5 km, bearing of C from A is 039°T, bearing of A from C is 219°T
f The answers should be the same, but may vary slightly depending on the accuracy of the scale drawing.

17 **a** 26.6 km **b** 062°T

18 **a**



b 12.7° **c** 7.1° **d** 13.3°

19 139 m

20 **a** 610 km
b **i** 26 km **ii** 12 km
c Yes, total distance would be 3552 km, total fuel was enough to fly 3560 km.

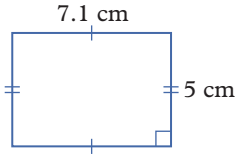
EX p379

7.4 Three-dimensional problems

1 **a** 21.21 cm **b** 25.98 cm **c** 35.26°

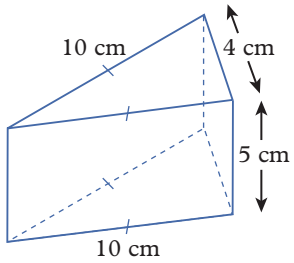
2 **a** diagonal of base (d):
$d = \sqrt{20^2 + 15^2}$
$d = \sqrt{625}$
$d = 25$ cm
diagonal of the box (e):
$e = \sqrt{10^2 + 25^2}$
$e = \sqrt{725}$
$e = 26.93$ cm
The length of the rod is greater than the length of the diagonal of the box, so a 30 cm rod will not fit into the box.
b 21.80°

3 **a** rectangle:

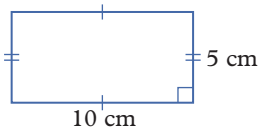


b Area of exposed face = 35.5 cm²; area of cube face = 25 cm². So the area of the exposed face is 1.4 times greater than the area of one of the cube's faces.

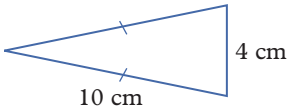
4 **a**



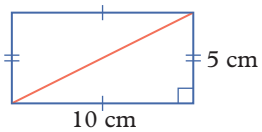
b **i**



ii



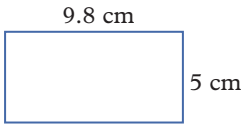
c



i 11.2 cm **ii** 26.57°
iii The angles are complementary.
$90° - 26.57° = 63.43°$

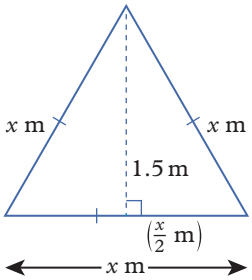
d **i** 9.8 cm **ii** 19.6 cm²

e **i**



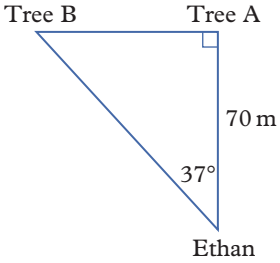
ii Cut side: $9.8 \times 5 = 49$ cm²
Other side: $10 \times 5 = 50$ cm²
iii 11 cm

5 **a**

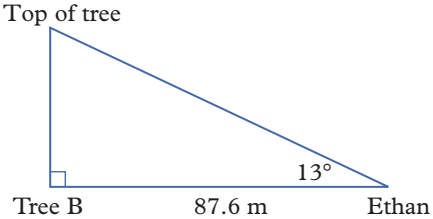


b 1.73 m
c Jack requires $2 \times 1.73 = 3.46$ m of material to cover the tent. $3.46 < 3.5$, there is sufficient material.

6 **a**



b 87.6 m
c Top of tree



d 20.2 m

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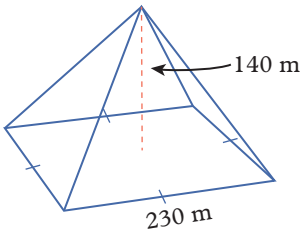
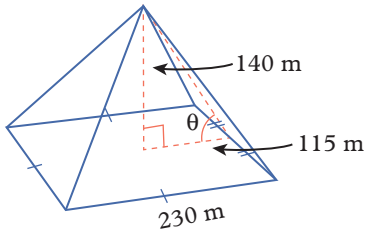
7 **a**



b 325.27 m **c** 40.72°

d **i**



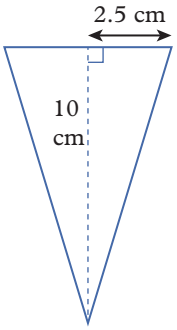
ii 50.60°

8 **a** 19.1 cm

b Internal diagonal = 22.6 cm. The entire spoon could fall below the rim, because the internal diagonal of the saucepan is longer than the spoon.

9 **a** 2.5 cm

b



c 10.3 cm

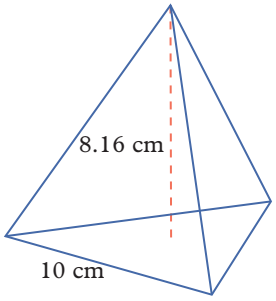
10 9 streamers equalling the full slant length of the tree

11 **a** (2, 2, 2), (2, 5, 5), (5, 2, 2), (5, 2, 5) and (5, 5, 5)

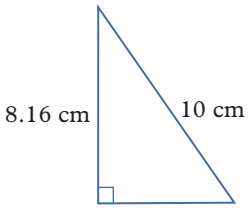
b $\sqrt{27}$

12 distance = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

13 **a, b**

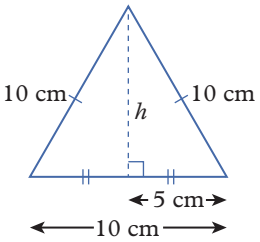


c **i**



ii 54.7°

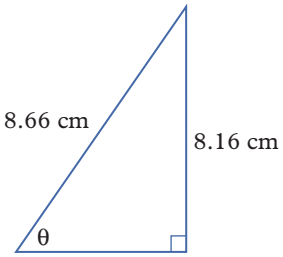
d



h = height of face

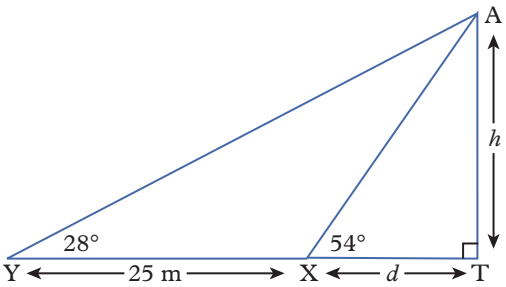
i 8.66 cm

ii



The angle (θ) between the base and a face of the tetrahedron is 70.4°.

14 **a**



b $h = d \times \tan(54°)$

c $h = (25 + d) \times \tan(28°)$

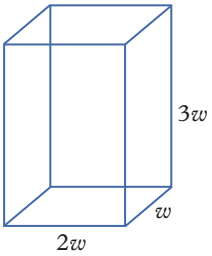
d 15.7 m **e** 21.7 m

15 **a** $\sqrt{w^2 + l^2}$, $\sqrt{l^2 + h^2}$, $\sqrt{w^2 + h^2}$

b $\sqrt{w^2 + l^2 + h^2}$

16 **a** $l = 2w$; $h = 3w$

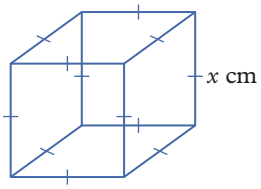
b



c $\sqrt{5}w$ **d** $\sqrt{14}w$ **e** 53°

17 3.6 cm

18 **a**



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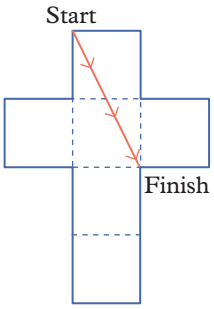
b **i**



ii $\sqrt{5}x$ cm

19 **a** 29.2 m **b** 36.8°

EX p388 7.5 The sine and area rules

oxforddigital **Answers for this lesson can be found on Oxford Digital.**

EX p393 Checkpoint

1 **a** yes **b** yes **c** no **d** yes

2 **a** **i** $\sqrt{41}$ cm **ii** 6.40 cm
b **i** $6\sqrt{14}$ mm **ii** 22.45 mm
c **i** $10\sqrt{5}$ mm **ii** 22.36 mm
d **i** $20\sqrt{2}$ cm **ii** 28.28 cm

3 **a** $x = 12.0$ mm, $y = 6.0$ mm, $\theta = 26.6°$
b $x = 9.3$ mm, $y = 6.3$ mm, $\theta = 42.7°$
c $x = 17.4$ mm, $y = 14.1$ mm, $\theta = 35.9°$

4 **a** $x = 63.8°$, $y = 26.2°$, $z = 20.1$ cm
b $x = 50.9°$, $y = 39.1°$, $z = 24.6$ mm

5 **a** 095°T **b** 145°T **c** 293.55°T

6 **a** 28.2 m **b** 26.6° **c** 15.8 m **d** 8.6 m

7 **a** 78.7 m east and 216.1 m south **b** 056°T

8 **a** $x = 5$ cm, $y = 7.8$ cm
b $r = 5.3$ cm, $y = 9.8$ cm

9 **a** $x = 9.0$ mm, $z = 12.0$ mm
b $x = 63.4°$, $y = 22.4$ cm, $z = 12.6°$

10 **a** $x = 43°$, $y = 4.7$ cm, $z = 5.6$ cm
b $y = 23.5°$, $x = 37.5°$, $z = 11.0$ mm
c $x = 128°$, $y = 16.4$ cm, $z = 13.2$ cm
d $y = 30.2°$, $x = 84.8°$, $z = 8.9$ mm

11 **a** $y = 68.0°$, $x = 60.0°$, $z = 11.2$ mm or $y = 112.0°$, $x = 16.0°$, $z = 3.6$ mm
b $y = 85.5°$, $x = 65.5°$, $z = 16.7$ cm or $y = 94.5°$, $x = 56.5°$, $z = 15.3$ cm

12 **a** 126.61 mm^2 **b** 662.82 cm^2 **c** 253.03 cm^2
d 117.42 mm^2

EX p398 7.6 The cosine rule

1 **a** 0.5 **b** −0.5 **c** 0.7660
d −0.7660 **e** 0.1736 **f** −0.1736
g 0.9272 **h** −0.9272

2 **a** 3.6 cm **b** 8.8 cm **c** 22.5 cm
d 13.7 cm **e** 56.2 mm **f** 44.9 cm
g 55.4 mm **h** 14.7 m **i** 15.5 cm

3 **a** $A = 47.2°$, $B = 78.1°$, $C = 54.6°$
b $A = 81.8°$, $B = 66.0°$, $C = 32.2°$
c $A = 40.7°$, $B = 87.9°$, $C = 51.5°$
d $A = 56.9°$, $B = 84.2°$, $C = 38.9°$

4 **a** 109.2° **b** 48.6°
c 38.7° **d** 144.5°

5 **a** $a = 18.5$ cm, $B = 94°$, $C = 38°$
b $e = 36.1$ mm, $D = 60°$, $F = 48°$
c $h = 48.2$ cm, $G = 53°$, $I = 68°$
d $j = 40.7$ m, $K = 50°$, $L = 67°$
e $n = 5.3$ m, $M = 41°$, $P = 60°$
f $s = 19.3$ mm, $Q = 51°$, $R = 60°$

6 **a** We would need to know the magnitude of a second angle, either B or C.
b 55.8 cm
c **i** Sample answer: use $\frac{\sin(A)}{a} = \frac{\sin(B)}{b}$ to find B. Subtract the sum of A and B from 180° to find C.
ii $B = 35°$, $C = 59°$
d $B = 35°$, $C = 59°$
e The answers are the same.

7 **a** $c^2 = a^2 + b^2 - 2ab\cos(C)$
b 0
c $c^2 = a^2 + b^2$; this is Pythagoras' theorem.
d **i** If C is an acute angle, then $0 < \cos(C) < 1$, so the adjustment term, $-2ab\cos(C)$, is negative. Therefore, the adjustment term decreases the sum of the squares of a and b, so $a^2 + b^2 - 2ab\cos(C) < a^2 + b^2$.
ii If C is an obtuse angle, then $-1 < \cos(C) < 0$, so the adjustment term, $-2ab\cos(C)$, is positive as the product of two negative numbers is positive. Therefore, the adjustment term increases the sum of the squares of a and b, so: $a^2 + b^2 - 2ab\cos(C) > a^2 + b^2$.

8 **a** $c^2 = h^2 + (b - x^2)$
b $h^2 = a^2 - x^2$
c $c^2 = a^2 - x^2 + (b - x)^2$
$= a^2 - x^2 + b^2 - 2bx + x^2$
$= a^2 + b^2 - 2bx$
d $x = a\cos(C)$
f $a^2 = b^2 + c^2 - 2bc\cos(A)$
$b^2 = a^2 + c^2 - 2ac\cos(B)$
g **i** $\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$
ii $\cos(B) = \frac{a^2 + c^2 - b^2}{2ac}$
iii $\cos(C) = \frac{a^2 + b^2 - c^2}{2ab}$
h **i** two side lengths and the angle between them
ii three side lengths

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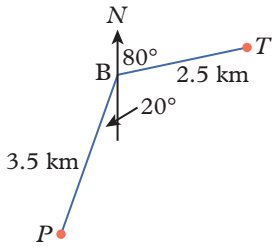
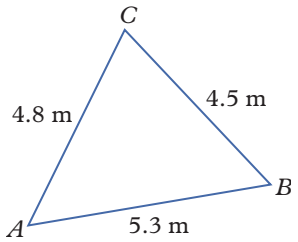
9 a



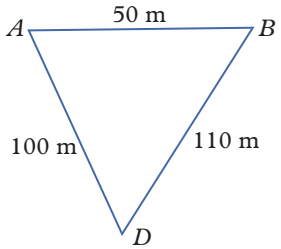
b 120° **c** 5.2 km **d** 25°
e N45°E **f** 35° **g** S45°W

10 a



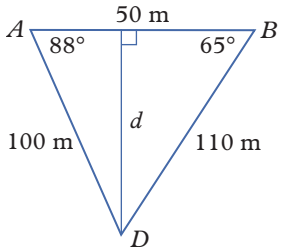
b C **c** 69° **d** 10.1 m²

11 a



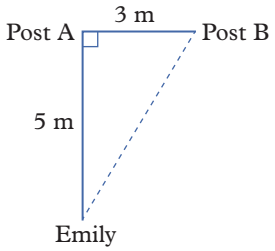
b 65° and 88°

c



d 99.9 m

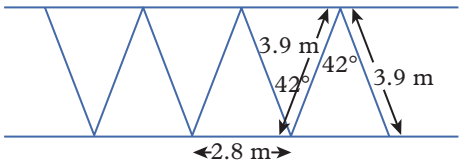
12 a



b 5.8 m
c 31°
d Emily has a greater chance of scoring a goal. She now has a larger angle of 35° within which she can score a goal.

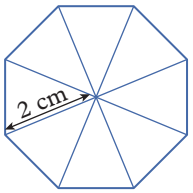
13 a 3.9 m **b** 3.7 m

14 a 42°
b



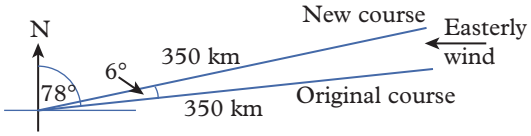
c 3.6 m

15 a



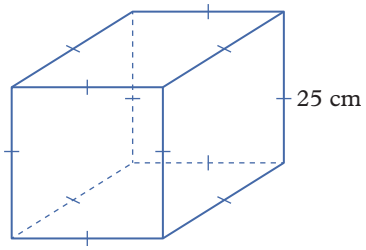
b i 45° **ii** 67.5°
c 12.2 cm

16 a



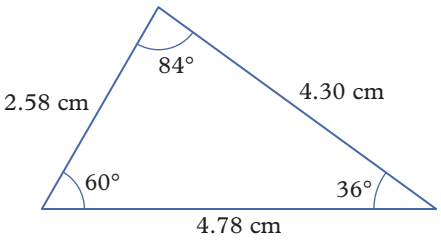
b 36.6 km

17 a



b $25\sqrt{2}$ cm **c** $25\sqrt{3}$ cm

18 a 36°, 60°, 84°
b i 4.78 cm
ii



19 a $(38\sin(63°))^2 + (24 - 38\cos(63°)) = x^2$
$x = 34.5$ cm
b $(41\sin(116°))^2 + (17 - 41\cos(116°))^2 = x^2$
$x = 50.8$ cm
c $(b\sin(C))^2 + (a - b\cos(C))^2 = c^2$
$(b\sin(C))^2 + a^2 - 2a(b\cos(C)) + (b\cos(C))^2 = c^2$
$(b\sin(C))^2 + (b\cos(C))^2 + a^2 - 2ab\cos(C) = c^2$
$a^2 + b^2 - 2ab\cos(C)^2 = c^2$,
since $(b\sin(C))^2 + (b\cos(C))^2 = b^2$

20 a i, ii $r\sqrt{2}$ units
b Both methods give the same answer.

21 a $x^2 - 6\cos(64°)x - 16 = 0$ **b** $x = -2.9$, $x = 5.5$
c No, as one of the solutions, $x = -2.9$, is negative. Therefore, the unknown side length is 5.5 cm.
d i $x = 2.3$, $x = 5.2$

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ii Yes, as both are positive values. This triangle has two possible lengths for the third side, 2.3 cm and 5.2 cm, as it is an ambiguous triangle as the shorter known side length is opposite the known angle.

iii The sine rule would first need to determine the possible angles opposite the other known side length, then determine the corresponding third angle(s) in the triangle using the interior angle sum of a triangle, and then determine the possible lengths of the unknown side length separately. The cosine rule can directly find both possible lengths of the unknown side by solving a trinomial quadratic equation, which is a harder process than using the sine rule multiple times. The other angles in the triangle can then be determined if needed.

e $11^2 = x^2 + x^2 - 2xx\cos(103°)$
$11^2 = 2x^2 - 2x^2\cos(103°)$
$x = 7.0$ cm since x must be positive

22 Area $= 5\frac{\sqrt{3}}{4} - \sqrt{3}\cos(\theta)$

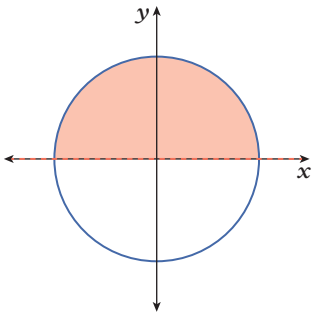
23 a It is when the triangle inequality is not true.

b We proceed by contradiction. Suppose $a \geq b + c$. If a, b and c are the side lengths of a triangle, then $0 < a, b, c$ because distances are positive. This means that the square of the first inequality holds:
$a^2 \geq (b + c)^2 = b^2 + c^2 + 2bc$.
The cosine rule states that $a^2 = b^2 + c^2 - 2bc\cos(A)$.
Combining with the previous inequality, we have:
$a^2 = b^2 + c^2 - 2bc\cos(A) \geq b^2 + c^2 + 2bc$.
Simplifying (since all values are positive), we have:
$-\cos(A) \geq 1$.
This implies that $\cos(A) = -1$, as $-1 \leq \cos(A) \leq 1$. But $\cos(A) = -1$ implies $A = 180°$. This is a contradiction because there can't be a 180° angle in a triangle as the interior angle sum of a triangle is 180° and there can't be 0° angles in a triangle.
Hence, the original assumption that $a \geq b + c$ is false, which proves the triangle inequality.

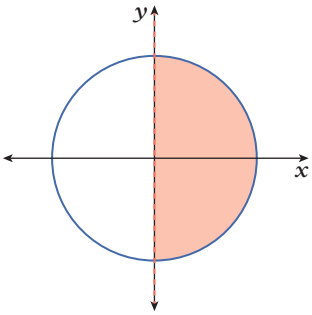
EX p408

7.7 The unit circle

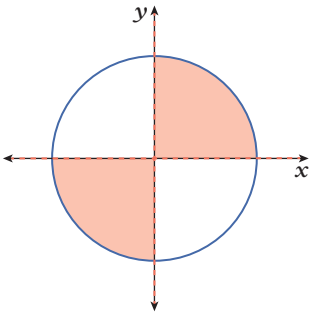
1 a



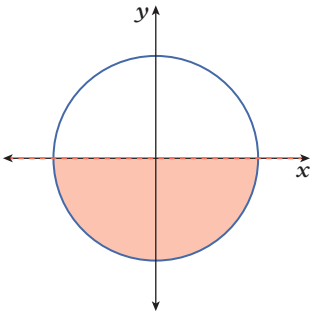
b



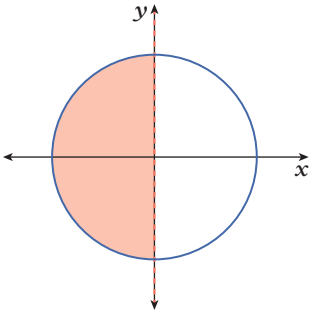
c



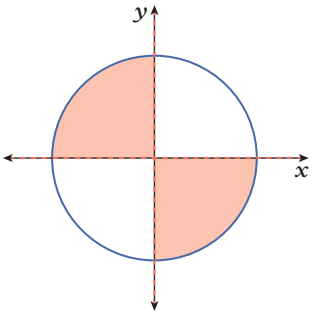
d



e



f



2 **a** quadrant 2 **b** quadrant 3 **c** quadrant 2
d quadrant 4 **e** quadrant 3 **f** quadrant 4
g quadrant 2 **h** quadrant 2 **i** quadrant 3
j quadrant 4

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3 a positive b negative c negative
d positive e positive f negative
g negative h positive i negative
j negative

4 a 40° b 88° c 20° d 10°
e 58° f 45° g 80° h 80°
i 80° j 30° k 15° l 10°
m 40° n 75° o 61°

5 a $-\sin(61°)$ b $\cos(56°)$ c $-\tan(85°)$
d $\sin(16°)$ e $-\cos(70°)$ f $-\cos(20°)$
g $\tan(20°)$ h $-\tan(2°)$ i $-\sin(75°)$
j $\sin(5°)$

6 a i $1 = \cos(0°)$ ii $0 = \sin(0°)$
b i $0 = \cos(90°)$ ii $1 = \sin(90°)$
c i $-1 = \cos(180°)$ ii $0 = \sin(180°)$
d i $0 = \cos(270°)$ ii $-1 = \sin(270°)$
e i $1 = \cos(360°)$ ii $0 = \sin(360°)$

7 a $\tan(\theta) = \frac{y}{x}$
b The gradient of the radius is its rise over its run, $\frac{y}{x}$. The tangent of θ is also equal to this ratio; therefore, the tangent of the angle is equal to the gradient of the radius.
c $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$
d i $\tan(20°)$ ii $\tan(152°)$ iii $\frac{1}{\tan(218°)}$
iv $\tan(347°)$
e i 0 ii undefined iii 0
iv undefined v 0

8 a i $\sin(40°)$ ii $\sin(40°)$
b i $\cos(170°)$ ii $-\cos(10°)$
c i $\tan(80°)$ ii $\tan(80°)$
d i $\sin(270°)$ ii -1
e i $\cos(57°)$ ii $\cos(57°)$
f i $\cos(304°)$ ii $\cos(56°)$
g i $\tan(200°)$ ii $\tan(20°)$
h i $\cos(180°)$ ii -1

9 a i $\sin(315°)$ ii $-\sin(45°)$
b i $\cos(135°)$ ii $-\cos(45°)$
c i $\tan(225°)$ ii $\tan(45°)$
d i $\sin(45°)$ ii $\sin(45°)$
e i $\cos(212°)$ ii $-\cos(32°)$
f i $\tan(284°)$ ii $-\tan(76°)$
g i $\cos(90°)$ ii 0
h i $\sin(80°)$ ii $\sin(80°)$

10 a $\sin(158°) = \sin(22°) = -\sin(202°) = -\sin(338°)$
b $\cos(213°) = \cos(147°) = -\cos(33°) = -\cos(327°)$
c $\tan(285°) = \tan(105°) = -\tan(75°) = -\tan(255°)$

11 a $-0.8090... = -0.8090...$,
$-0.587\,785... = -0.587\,785...$,
$1.376\,38... = 1.376\,38...$
b 360°
c The reference angle, when the angle is in the fourth quadrant, is calculated using $360° - \theta°$. The same has been applied to 234° as $360° - 234° = 126°$. The RHS of each equation uses the sign of the trigonometric function in the fourth quadrant to ensure the expressions are equal.

12 a $(2\sin(46°), -2\cos(46°))$
b $(-2\sin(50°), -2\cos(50°))$
c $(-6\sin(40°), 6\cos(40°))$
d $(20\sin(80°), -20\cos(80°))$
e $(-15\sin(17°), 15\cos(17°))$
f $(-75\sin(10°), -75\cos(10°))$

13 a $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$
b i $\sin(\theta) = \pm\sqrt{1-(\cos(\theta))^2}$
ii $\cos(\theta) = \pm\sqrt{1-(\sin(\theta))^2}$
c i $\cos(50°) = 0.6428$
ii $\sin(116°) = 0.8988$
iii $\cos(177°) = -0.9986$
iv $\sin(240°) = -0.8660$

14 a $\cos(60°)$ b $\cos(38°)$
c $\sin(66°)$ d $\sin(45°)$
e i $\cos(-68°)$ ii $\cos(292°)$ iii $\cos(68°)$
f i $\sin(-261°)$ ii $\sin(99°)$ iii $\sin(81°)$
g i $\cos(-121°)$ ii $\cos(239°)$ iii $-\cos(59°)$
h i $\cos(105°)$ ii $\cos(105°)$ iii $-\cos(75°)$
i i $\sin(-392°)$ ii $\sin(328°)$ iii $-\sin(32°)$
j i $\sin(645°)$ ii $\sin(285°)$ iii $-\sin(75°)$

7.8 Exact values

EX p416

oxforddigital **Answers for this lesson can be found on Oxford Digital.**

7.9 Trigonometric graphs

EX p424

oxforddigital **Answers for this lesson can be found on Oxford Digital.**

7.10 Solving trigonometric equations

EX p429

1 a 90° b $\theta = 90°, 450°$
2 a $\theta = 180°$ b $\theta = 180°, 540°$
3 a $\theta = 0°, 180°, 360°$
b $\theta = 0°, 180°, 360°, 540°, 720°$
4 a $\theta = 90°, 270°$ b $\theta = 0°, 180°, 360°$
c $\theta = 270°$ d $\theta = 0°, 360°$
5 a $\theta = 30°, 330°, 390°, 690°$
b $\theta = 225°, 315°, 585°, 675°$
c $\theta = 150°, 330°, 510°, 690°$
d $\theta = 240°, 300°, 600°, 660°$
e $\theta = 45°, 315°, 405°, 675°$
f $\theta = 45°, 225°, 405°, 585°$
6 a $\theta = 114°, 246°$ b $\theta = 197°, 343°$
c $\theta = 42°, 222°$ d $\theta = 37°, 143°$

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e $\theta = 53°, 307°$ **f** $\theta = 98°, 278°$

7 **a** $90° \le \theta \le 270°$ **b** $60° \le \theta \le 300°$
c $60° \le \theta \le 120°$ **d** $45° \le \theta \le 135°$
e $150° \le \theta \le 210°$ **f** $270° < \theta < 630°$

8 **a** $210° \le \theta \le 690°$ **b** $120° \le \theta \le 600°$
c $225° \le \theta \le 675°$ **d** $135° \le \theta \le 585°$

9 **a–c** When each of these equations is simplified the sine or cosine value is greater than 1. Sine and cosine can only take values in the range –1 to 1.

EX p442

10 **a** yes **b** yes **c** yes **d** yes
e No, when the equation is simplified the cosine value is less than –1.
f yes
g No, when the equation is simplified the cosine value is greater than 1.
h yes
i No, when the expression is simplified the sine value is less than –1.

11 **a** 1 **b** 0 **c** 1 **d** 1 **e** 1 **f** 1

12 **a** $\theta = 90°$ **b** $\theta = 0°$ **c** $\theta = 135°, 225°$
d $\theta = 240°$ **e** $\theta = 225°$ **f** $\theta = 150°$

13 **a** $\theta = 22.5°, 67.5°$ **b** $\theta = 45°, 135°$
c $\theta = 45°, 105°, 165°$

14 **a** $\theta = 420°, 660°$
b $\theta = 135°, 945°$
c $\theta = 300°, 750°$

15 **a** 180° **b** 360° **c** 240°
d 540° **e** 360° **f** 480°

16 **a** $\theta = -300°, -240°$ **b** $\theta = -225°, -135°$
c $\theta = -300°, -120°$ **d** $\theta = -90°$
e $\theta = -270°, -90°$ **f** $\theta = 0°, -180°, -360°$

17 **a** $x = 0, \pi, 2\pi$ **b** $x = \frac{\pi}{6}, \frac{11\pi}{6}$ **c** $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$
d $x = \frac{\pi}{4}, \frac{3\pi}{4}$ **e** $x = \pi$ **f** $x = \frac{\pi}{6}, \frac{7\pi}{6}$

18 **a** $\frac{\sin(\theta)}{\cos(\theta)} = \frac{\cos(\theta)}{\cos(\theta)}$
$\tan(\theta) = 1$
b **i** $\theta = 45°, 225°$ **ii** $\theta = 60°, 240°$
iii $\theta = 150°, 330°$ **iv** $\theta = 30°, 210°$

19 **a** $\theta = 15°, 75°, 195°, 255°$ **b** $\theta = 360°$
c $\theta = 180°, 300°, 360°, 480°, 540°$
d $\theta = -5\pi, -3\pi, 3\pi, 5\pi, 11\pi$

20 **a** $x = -\frac{1}{2}, 2$ **b** $\theta = 210°, 330°$

21 **a** $x = \frac{4}{3}, \frac{3}{2}$
b There are no solutions because $-1 \le \sin(\theta) \le 1$ and both solutions to part **a** are greater than 1.
c There are no solutions because $-1 \le \cos(\theta) \le 1$ and both solutions to part **a** are greater than 1.
d There are infinitely many solutions.

EX p433

Review: Pythagoras' theorem and trigonometry

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MODULE 8 Measurement

8.1 Area review

1 **a** 700 cm **b** 4.5 km **c** 8 400 000 m²
d 0.29 cm² **e** 2.4 ha **f** 880 mm
g 0.003 42 km² **h** 1 022 000 cm **i** 300 ha

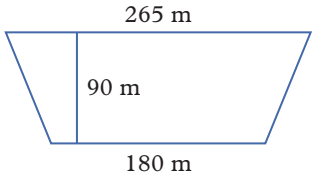
2 **a** 42 m² **b** 348.5 cm²
c 843.6 mm² **d** 164.4 cm²

3 **a** 59.4 cm² **b** 515.3 mm² **c** 2875.1 cm²

4 **a, b** chimney: trapezium, 1.8 m²
roof: triangle, 6.5 m²
front wall: rectangle, 15.54 m²
front door: rectangle, 3.9 m²
windows: square, 0.49 m² each window
c multiply by 4
d **i** 21.88 m² **ii** 1.5 L

5 **a** 168 cm² **b** 24 m²
c 6.4 cm² **d** 348 cm²
e 933.9 cm² **f** 219.5 cm²

6 **a** 20 025 m²



b 2 ha

7 **a** Sample answer: In the left-hand half on the yin-yang symbol the blue section is a semicircle with a diameter equal to the radius of the whole circle, and in the right-hand half of the yin-yang symbol the white section is a semicircle with a diameter equal to the radius of the whole circle, therefore the blue and white sections have an equal area.
b 904.8 cm²

8 **a** Subtract the area of the inner circle from the area of the outer circle.
b 100.5 m²
c **i** 115.5 m² **ii** 58.9 m² **iii** 125.7 m²

9 **a** 75.4 cm² **b** 60°
c 12.6 cm² **d** 44.0 cm²

10 **a** 60π cm² **b** 40π cm² **c** 3 : 2

11 92 cm²

12 **a** $5\sqrt{3}$ cm **b** $25\sqrt{3}$ cm²
c $150\sqrt{3}$ cm² **d** $\frac{3\sqrt{3}}{2}a^2$

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EX p448

8.2 Surface area review

1 a 262 cm^2 b 1350 cm^2 c 6144 mm^2
d 34.56 m^2 e 6156 mm^2 f 23.88 m^2
g 876 mm^2 h 332 cm^2 i 53.66 m^2

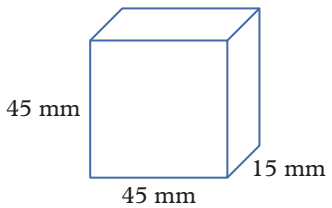
2 a 662.8 cm^2 b 2310.1 mm^2 c 117.4 cm^2

3 383.04 cm^2

4 a 1809.6 cm^2 b 883.4 cm^2 c 11 309.7 mm^2

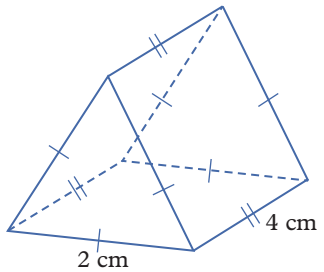
5 a 8224 cm^2 b 334.2 cm^2 c 470.4 cm^2

6 a 1350 mm^2
b 12 150 mm^2
c

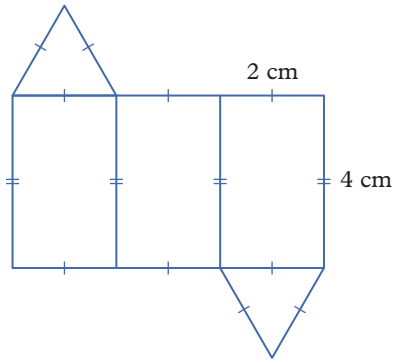


d 6750 mm^2
e 5400 mm^2; 24 surfaces of the dice are hidden within the structure ($24 \times 15^2 = 5400$).

7 a



b Sample answer:



8 a 570.2 cm^2
b No. The pipe is open, so there is no top or bottom. This means $2\pi r^2$ is removed from formula.

9 279.9 cm^2

10 a 7191.3 cm^2
b No. The curved surface area will be double, but the one circular end, which needs painting, has the same area for both poles.
c 14 259.9 cm^2
$2 \times 7191.3 = 14\,382.6$ cm^2
This is 122.7 cm^2 larger than 14 259.9 cm^2.
This difference is the area of one end, which is not doubled.
This supports the answer to part **b**.

11 a 730.1 cm^2 b 650.8 cm^2

12 a 407.8 m^2 b 4815.8 m^2

13 1311 cm^2

14 208.9 cm^2

15 a 9 m^2 b 4 cans

EX p453

8.3 Volume review

1 a 12 500 000 cm^3 b 240 cm^3
c 34.2 m^3 d 550 000 000 mm^3
e 0.000 067 2 m^3 f 9000 mm^3
g 7.52 cm^3 h 8 740 000 cm^3
i 0.1429 m^3 j 0.073 m^3

2 a 2508 cm^3 b 16 128 mm^3
c 5.8 m^3 d 43 264 cm^3
e 689.1 cm^3 f 6.2 m^3
g 96 140 mm^3 or 96.1 cm^3 h 29 440 cm^3
i 1539 m^3 j 53.5 cm^3
k 74 088 m^3 l 175 cm^3

3 a 1256.6 cm^3 b 172 240.2 mm^3
c 16 155.6 m^3 d 290.8 cm^3
e 178 623.1 cm^3 or 0.2 m^3 f 3698.5 cm^3

4 a 0.064 m^3 b 64 000 cm^3

5 a 1102.5 cm^3 b 35 224 mm^3 c 15.58 m^3

6 a 12.48 m^3 b 10 560 mm^3 c 676.2 cm^3
d 9000 cm^3 e 17 090.3 mm^3 f 25 249.0 cm^3

7 5193.4 cm^3

8 a rectangular prism, triangular prism, cylinder
b 97 cm^3

9 a 206.2 cm^3 b 360 cm^3 c 1687.5 cm^3

10 a 176 cm^3 b 0.176 kg c \$3.70

11 a 45 000 cm^3 b 39.6 L c 0.4 cm

12 14 cm

13 a 21 cm b 34 mm

14 a second tank b 1178 L

15 a $\pi r^2 h$
b $\frac{1}{4}$ the volume c 4 times the volume
d $\frac{1}{2}$ the volume e 2 times the volume

16 41 471 litres

17 a annulus, area = 84.8 cm^2
b 2121 cm^3
c Subtract the volume of the inner cylinder from the volume of the outer cylinder.

18 The roof of a house is in the shape of a triangular prism, so that all the rainfall that falls on the roof is collected in the gutters that surround the roof and filter into a water tank.
a 117.1728 m^2 b 1758 L c 99.5 cm

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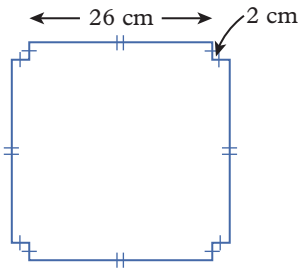
19 a



b length = 26 cm, width = 26 cm, height = 2 cm

c 1352 cm^3

d The box would be 24 cm × 24 cm × 3 cm, giving a volume of 1728 cm^3. The volume of the box would increase.

20 a circular nut: 107.71 mm^3
hexagonal nut: 144.68 mm^3

b hexagonal nut

EX p459

8.4 Compounding errors

1 a 1.92 cm **b** 0.01575

2 a 0.3946% **b** 0.3954%

3 a $\frac{20}{\pi}$ cm **b** 6.4 cm **c** $\frac{400}{\pi}$ cm^2
d 128.68 cm^2 **e** 1.06%

4 a 37°
b $\tan^{-1}\left(\frac{3}{4}\right)$
c 0.35%

5 a 12.6 cm^2 **b** 0.2676% **c** 50.3 cm^2
d 0.06867% **e** 75.5 cm^2 **f** 0.1350%

6 a 55.42 cm^2 **b** 0.5747%
c 1.1569%. The error in the measurement of the diameter is compounded when calculating the area of the coaster.

7 a Sample answer: A margin of error is stated so there is an understanding that the poll is unlikely to be 100% accurate.
b Sample error: The margin of error could be decreased by increasing the sample size or by ensuring that the sample accurately reflects the voting population.

8 a James: 20 km/h; Karen: 12 km/h
b James: 5.6 m/s; Karen: 3.3 m/s
c i 9.375 minutes
ii 562 seconds = 9.363 minutes
d 0.128%

9 a Jemima: 0.6; Sister: 0.8 **b** 17.9 minutes
c 18 minutes **d** 0.1 minutes

10 a \$0.002 242 **b** \$0.81

11 a \$14.95; under payment
b

Number of packs	Exact price ($)	Payment amount ($)	Error ($)
1	14.9625	14.95	–0.0125
2	29.925	29.95	–0.025
3	44.8875	44.90	–0.0125
4	59.85	59.85	0

c –\$0.6125

EX p461

Checkpoint

1 a 4.5 cm **b** 4 m
c 4530 mm **d** 40 cm^2
e 5 200 000 m^2 **f** 6.8 m^2

2 a 13.125 m^2
b 306 cm^2

3 a 219 cm^2 **b** 25 cm^2

4 41.7 m^2

5 697.8 cm^2

6 2850 cm^2

7 7634.1 cm^3

8 a 4000 mm^3 **b** 4 cm^3

9 1292.655 cm^3

10 a 0.038 46 **b** 3.846%

11 a 8137 m
b 0.1276%
c The error seems large in magnitude, but the percentage error is not large for high school study purpose and the radius of the Earth is not a constant value.

EX p465

8.5 Logarithmic scales

1 a 12 **b** –5 **c** 3
d 8 **e** –7 **f** 3

2 a 3 units to the right of 1
b 6 units to the left of 1
c 1 unit to the right of 1
d 2.30 units to the right of 1
e 1.70 units to the left of 1
f 3.10 units to the left of 1.

3 a 100 000 **b** 0.1 **c** 2510
d 0.002 00 **e** 1.26 **f** 0.000 025 1

4 a 4.215×10^4 **b** 4.215×10^5
c 4.215×10^7 **d** 0.729
e 7.29 **f** 729

5 a 4.215×10^2 **b** 4.215×10^1
c 4.215×10^{-2} **d** 0.007 29
e 0.000 729 **f** 0.000 000 729

6 a 6 orders higher **b** 3 orders lower
c 15 orders lower **d** 2 orders higher
e 3 orders lower **f** 12 orders higher

7 a divide by 1000
b multiply by 10 000
c multiply by 100 000 000 000 (or 10^{11})
d multiply by 1 995 262
e divide by 126
f divide by 125 892 541 179 (or $10^{11.1}$)

8 a i $7.293\,85 \times 10^3$ **ii** $1.052\,357\,9 \times 10^5$
iii 2.135×10^{-3} **iv** 5.0123×10^4
v 6.1290×10^1 **vi** 3.2793×10^{-6}
b i 3 **ii** 5 **iii** –3
iv 4 **v** 1 **vi** –6

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c **i** 3.863 **ii** 5.022 **iii** −2.671
iv 4.700 **v** 1.787 **vi** −5.484

9 **a** Increasing a value by 0.5 orders of magnitude corresponds to the value becoming 3.16 times greater.
b Increasing a value by 1.5 orders of magnitude corresponds to the value becoming 31.6 times greater.
c Decreasing a value by 2.5 orders of magnitude corresponds to the value becoming 316 times smaller.
d Decreasing a value by 3.5 orders of magnitude corresponds to the value becoming 3160 times smaller.
e Multiplying a value by 2 increases its order of magnitude by 0.301.
f Multiplying a value by 4 increases its order of magnitude by 0.602.
g Dividing a value by 8 decreases its order of magnitude by 0.903.
h Dividing a value by 16 decreases its order of magnitude by 1.20.

10 **a** 4.724 units to the right of 1
b 6.276 units to the left of 1
c 6.676 unit to the left of 1
d 8.990 units to the right of 1
e 5.154 units to the left of 1
f 0.979 units to the left of 1

11 **a** **i** 5.5 **ii** 3.1 **iii** 4.9
iv 2.8 **v** 7.3 **vi** 8.2
b **i** 3 162 278 μm **ii** 1 000 000 μm
iii 3 162 000 μm **iv** 32 μm
v 63 095 734 μm **vi** 19 953 μm
c **i** 10 **ii** 3.162 **iii** 31.62
iv 63.10 **v** 1 259 000 **vi** 1.259

12 **a** The difference in values between the three countries, ranging from 100 to 10 000 000, is so great that a linear graph would not be able to show much detail and it would be difficult to see the difference between countries B and C.
b The number of cases becomes 10 times greater every 100 days.
c **i** 50 days **ii** 200 days **iii** 500 days
iv 31 623 cases **v** 1000 cases
d The growth in cases has slowed down below the point where the cumulative number of cases in country B is the same as if the number of cases had been increasing at a rate of 10 times the amount of cases every 100 days.

13 **a** linear **b** logarithmic
c logarithmic **d** linear
e logarithmic **f** linear

14 **a** The logarithm of a negative number is not a real number.
b The logarithm of zero is undefined.

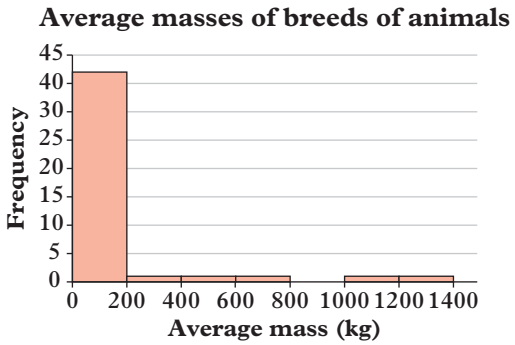
15 **a** **i**

Class interval	Frequency
0–<200	42
200–<400	1
400–<600	1
600–<800	1
800–<1000	0
1000–<1200	1
1200–<1400	1

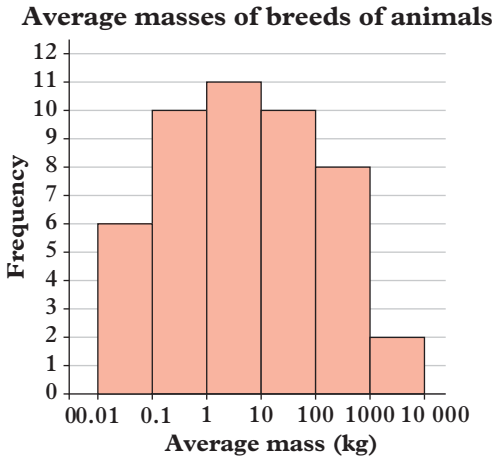
ii

Class interval	Frequency
0.01–<0.1	6
0.1–<1	10
1–<10	11
10–<100	10
100–<1000	8
1000–<10 000	2

b **i**



ii



c Linear scale: positively skewed
Logarithmic scale: approximately symmetric
d The logarithmic scale spreads out the average masses in the 0–<200 column (42 out of 47 masses) into five different columns and groups the data in the remaining columns (with only 5 masses) into two columns.

16 **a** 1 day **b** 1 day **c** constant
d decreasing **e** increasing

17 **a** 2 hours

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b With the base-two logarithmic scale you can more easily identify the amount of the active substance remaining after a high number of hours, however it is more difficult to identify the amount of active substance remaining for periods between each half-life interval (every two hours).

18 a **i** 10 **ii** 10 **iii** 10 000
iv 100 000 **v** 1 000 000 000 **vi** 1.259
b **i** −10 dB **ii** 10 dB **iii** −20 dB
iv −6 dB **v** 0 dB **vi** 20 dB
c **i** 5000 W **ii** 10.0 W **iii** 99.8 W
iv 500 000 W **v** 240 000 000 W
vi 24 000 W

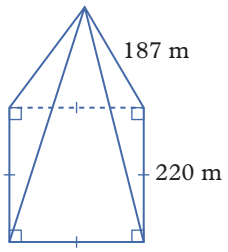
EX p472 8.6 Surface area of pyramids and cones

1 a 2744 mm² **b** 118 cm² **c** 4856 mm²
d 408 cm² **e** 3234 m² **f** 7605 cm²

2 a 809.4 cm² **b** 1578.8 mm² **c** 9259.9 cm²

3 1890 m²

4 a



b 66 539.0 m²

5 a 195.6 cm² **b** 914.3 mm² **c** 4875.0 cm²
d 460.1 cm² **e** 619.2 mm² **f** 129.2 cm²

6 a 9123.2 mm² **b** 2801.5 cm² **c** 624.4 cm²
d 282.7 cm² **e** 628.3 cm² **f** 1582.2 mm²

7 20 cm

8 slant height = 18.2 cm, SA = 1641.0 cm²

9 a 27.7 cm² **b** surface area = $x^2\sqrt{3}$

10 339.3 cm²

11 a 8
b 679.0 mm²
c Each pyramid, including the base, has SA of 535.5 mm². So, SA of two pyramids is 1071.0 mm².
d The answers to parts **b** and **c** are not equal in this form. If you subtract the areas of the bases of the two pyramids (which are covered) the answers are the same.
1071.0 mm² − 2 × 196 mm² = 679 mm²

12 a 162.7 cm² **b** 6579.8 mm² **c** 1847.3 cm²

13 580.8 cm²

14 a **i** 471.2 cm² **ii** 7.5 cm
b **i** $\theta = 180°$ **ii** $\theta = 120°$ **iii** $\frac{\theta}{360}r$

15 a 300 cm² **b** $\frac{2}{3}$
c **i** 16.2 cm **ii** 424.3 cm² **iii** 52.9%

EX p477 8.7 Volume of pyramids and cones

1 a 1615 cm³ **b** 37 762.7 mm³
c 11.4 cm³ **d** 6700.7 cm³
e 28.9 cm³ or 28 918.7 mm³
f 38.2 m³ **g** 3033.3 cm³
h 93.6 cm³ **i** 5118.8 mm³

2 a 65 m³ **b** 1239.3 cm³ **c** 1760 cm³

3 2 592 100 m³

4 a 823 825.1 mm³
b 1747.7 cm³
c 119.7 cm³ or 119 703.1 mm³
d 5026.5 cm³
e 5277.9 mm³
f 53.1 m³ or 53 136 105.4 cm³

5 16 222.4 cm³

6 15 519.2 mm³

7 a 8.5 cm; 25.5 cm²
b **i** 93.5 cm³
ii 93 500 mm³
c $V = \frac{1}{3} \times \frac{1}{2}bh_1h_2$, where h_1 and h_2 represent the height

8 a 7.0 cm **b** 2.9 cm

9 5.2 L

10 a 37 447.8 mm² **b** height of cone
c 566 755.9 mm³

11 a 47.1 cm³ **b** 188.5 cm³
c When the radius is doubled and the height remains the same, the volume increases by a factor of 4.
d 94.2 cm³
e When the height is doubled and the radius remains the same, the volume is doubled.
f The formula for the volume of a cone involves r^2 so, when r is doubled, r^2 and the volume increase by a factor of 2^2 (or 4). The formula also involves h so, when h is doubled, this increases the volume by a factor of 2.

12 1.2%

13 a 6031.9 cm³ **b** 7680 cm³
c 1648.1 cm³ **d** 4 − π : 4

14 a 261.8 cm³ **b** 7.9 cm

15 a Four faces are trapeziums; two faces are squares.
b 7291.7 cm³ **c** 466.7 cm³ **d** 6825 cm³

16 3.0 m³

17 a 225 723.4 mm³ **b** 226 mL

18 One possible answer is shown here.
For a truncated square-based pyramid:
$V = \frac{1}{3}L^2(h_1 + h_2) - \frac{1}{3}l^2h_1$
For a truncated cone:
$V = \frac{1}{3}\pi R^2(h + H) - \frac{1}{3}\pi r^2 H$
or: $V = \frac{\pi}{3}([h + H]R^2 - Hr^2)$

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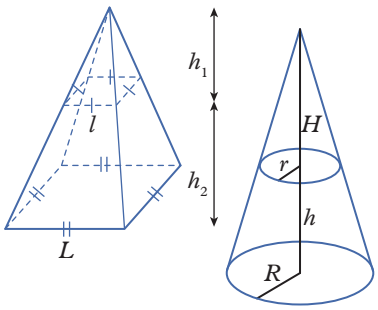

EX p483

8.8 Surface area and volume of spheres

1 a 1256.6 mm² b 8494.9 cm²
c 2290.2 cm² d 33 979.5 mm²

2 a 3631.7 cm² b 7543.0 mm²
c 10.2 m² d 8992.0 cm²

3 a 4188.8 mm³ b 73 622.2 cm³
c 10 306.0 cm³ d 588 977.4 mm²

4 a 20 579.5 cm³ b 61 600.9 mm³
c 3.1 m³ d 80 178.9 cm³

5 11 cm

6 a 6 cm b 9 cm
c 13.0 cm d 16.0 cm

7 a 100π cm²
b i 25π cm² ii 50π cm² iii 75π cm²

8 a i 508.9 cm² ii 1526.8 cm³
b i 3041.1 mm² ii 22 301.1 mm³
c i 4561.6 mm² ii 22 301.1 mm³
d i 1490.1 cm² ii 7649.3 cm³
e i 82.0 cm² ii 53.8 cm³
f i 9954.9 mm² ii 71 896.7 mm³

EX p488

9 a 18 145.8 cm² b 229 847.3 cm³

EX p498

10 a 33 cm b 6.7 cm

11 a

Planet	Diameter (km)	Total surface area (km²)	Volume (km³)
Mercury	4878	7.5×10^7	6.1×10^{10}
Venus	12 104	4.6×10^8	9.3×10^{11}
Earth	12 756	5.1×10^8	1.1×10^{12}
Mars	6787	1.4×10^8	1.6×10^{11}
Jupiter	142 800	6.4×10^{10}	1.5×10^{15}
Saturn	120 000	4.5×10^{10}	9.0×10^{14}
Uranus	51 118	8.2×10^9	7.0×10^{13}
Neptune	49 528	7.7×10^9	6.4×10^{13}

b The volume of Earth is about 18 times the volume of Mercury.
c The surface area of Jupiter is about 125 times the surface area of Earth.
d 362 942 012 km²

12 a 8.6 cm b 116 cm³

13 a 10 744.2 cm³ b 10.7 L
c approximately 4.3 s
d It needs to be assumed that the rate of inflation is a constant rate; and that there is no time lost between filling one balloon and the next.

14 a 658.5 cm³
b dimensions of canister: diameter = 6.8 cm, height = 27.2 cm, volume = 987.8 cm³
c 329.3 cm³
d There would be more unused space because the cylinder would fit inside the rectangular prism.

15 132 mL

16 a 16 278.9 cm³ b 4308.2 cm³ c 738.2 cm³

17 3 : 1 : 4 : 2

18 a 33 510.3 mm³ or 33.5 cm³ b 0.08 g/cm³

19 a $V = \frac{1}{3}nAr$
b $nA = 4\pi r^2$
c The bases of all n pyramids cover the surface area of the sphere. Since nA is equal to n times the surface area of the base of one pyramid, it must be equal to the surface area of the sphere.
d It takes a large number of pyramids to approximate a sphere. For example, $n = 6$ only gives you a cube.

20 a i 20 cm ii 3811.2 cm³
b i 8.7 cm ii 1720.7 cm³

Review: Measurement

oxforddigital **Answers for this lesson can be found on Oxford Digital.**

MODULE 9 Networks

9.1 Introduction to networks

1 a i 4 ii 3
b i 4 ii 4
c i 4 ii 2
d i 1 ii 3
e i 4 ii 6
f i 5 ii 4
g i 2 ii 4
h i 9 ii 12

2 a

Vertex	A	B	C	D
Degree	1	2	2	1

b

Vertex	A	B	C	D
Degree	2	2	2	2

c

Vertex	A	B	C	D
Degree	1	1	1	1

d

Vertex	A
Degree	6

e

Vertex	A	B	C	D
Degree	3	3	3	3

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f

Vertex	*A*	*B*	*C*	*D*	*E*
Degree	4	1	1	1	1

g

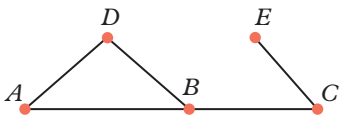
Vertex	*A*	*B*
Degree	3	5

h

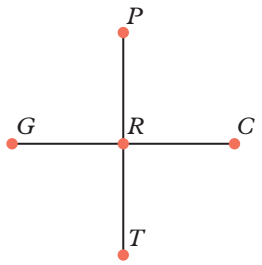
Vertex	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*	*I*
Degree	2	2	6	2	4	2	2	2	2

3 Graph **c**

4



5 a



b 4

6 a Sample answer: *A*-*B*-*C*-*D* (other walks from *A* to *D* exist)

b Sample answer: *A*-*B*-*D* (other walks from *A* to *D* exist)

c There is no walk from *A* to *D*.

7 a

Person	*A*	*B*	*C*	*D*
Number of hands shook	3	3	2	2

b 5 (the number of edges)

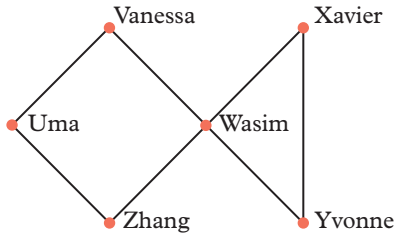
8 10 (twice the number of edges)

9 $4 \times 5 \div 2 = 10$

10 Such a graph would have $\frac{3 + 2 + 2}{2} = 3.5$ edges, which is impossible.

11 A graph is defined by its vertices and edges, and both these diagrams contain the same vertices and edges, therefore they represent the same graph.

12 a



b

Vertex	Uma	Vanessa	Wasim	Xavier	Yvonne	Zhang
Degree	2	2	4	2	2	2

c Wasim (or Uma) **d** Uma

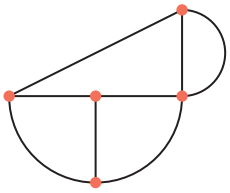
13 a Yes; the walk leads to corner *C*.

b Sample answer: *A*-*B*-*C*.

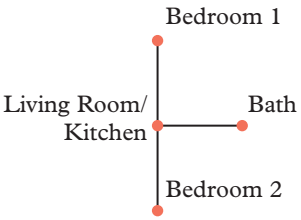
14 There are missing directions. East 42nd St does not connect to corner *A*.

15 Remove the edge between *D* and *E*.

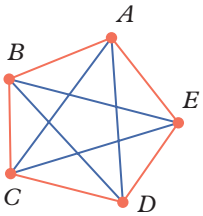
16



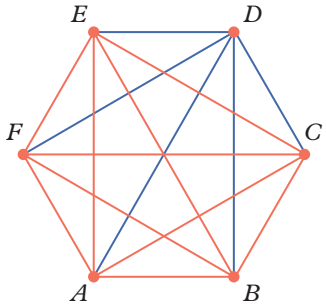
17



18 Sample answer:



19 a Sample answer:



b Sample answer: *ACE*.

c Question **18** shows you need to invite more than 5 people, since it contains no red or blue triangle. No matter how you answer question **19a** there is always a red or blue triangle, so you need 6 dinner party guests.

9.2 Planar graphs

EX p504

1 a planar **b** planar

c planar **d** planar

2 a 2 **b** 4

c 4

3 a $V = 4, F = 2, E = 4; 4 + 2 = 4 + 2$

b $V = 1, F = 4, E = 3; 1 + 4 = 3 + 2$

c $V = 4, F = 3, E = 5; 4 + 3 = 5 + 2$

4 Only graph **a** is simple. Simple graphs have no loops or multiple edges. Graph **b** is not simple as it has loops. Graph **c** is not simple as it has multiple edges between *A* and *B*.

5 Sample answer:

The graph will only have one face.

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6

Faces	Vertices	Edges
4	2	4
6	6	10
3	4	5
5	3	6
2	4	4
3	7	8

7 Because it is not connected.

8 A planar graph with 6 vertices has at most $3V - 6 = 12$ edges. This graph has 15 edges, so it cannot be planar.

EX p509

9 **a** The graph has $E = 4$ edges and $V = 3$ vertices.

b $E = 4 > 3 = 3V - 6$

c The restriction $E > 3V - 6$ only applies to simple graphs; this graph is not simple.

10 **a** $E = 9$, $V = 6$

b yes (no loops or multiple edges)

c yes

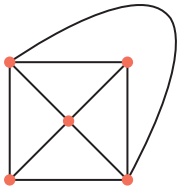
d $E = 9$ is less than $3V - 6 = 12$

11 Yes (see **9.2** question **1a**)

12 No (see **9.2** question **10**)

13 **a** If the graph is planar, then it has at most $3V - 6 = 9$ edges.

b Sample answer:



14 4

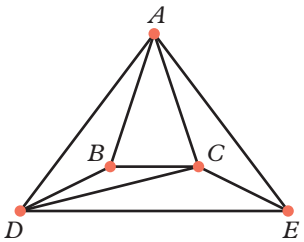
15 No. If such a graph did exist, then $E = F + V - 2 = 5 + 4 - 2 = 7$. This exceeds the limit of $3V - 6 = 6$ for planar graphs.

16 **a** a loop

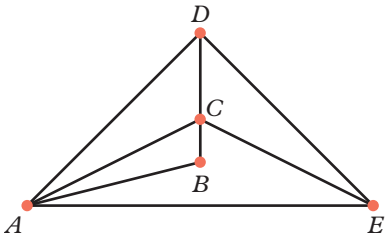
b a pair of multiple edges

c a triangle

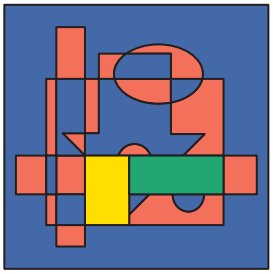
17 **a**



b

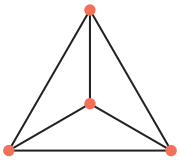


18 Sample answer:



9.3 Polyhedrons

1 **a**



b Both the tetrahedron and the graph have $V = 4$ vertices, $E = 6$ edges, and $F = 4$ faces.

c $V + F = E + 2$

$4 + 4 = 6 + 2$

$8 = 8$

Therefore, the tetrahedron satisfies Euler's formula.

2 **a** true

b false

3 **a** tetrahedron, octahedron, icosahedron

b cube

c dodecahedron

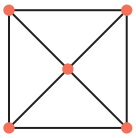
4 **a** tetrahedron, cube, dodecahedron

b octahedron

c icosahedron

5 d4: tetrahedron, d6: cube, d8: octahedron, d12: dodecahedron, d20: icosahedron

6



7 There are 5 vertices, 5 faces, and 8 edges.

$V + F = E + 2$

$5 + 5 = 8 + 2$

$10 = 10$

Therefore, the square pyramid satisfies Euler's formula.

8 A pentagonal pyramid has 6 vertices, 6 faces, and 10 edges.

$V + F = E + 2$

$6 + 6 = 10 + 2$

$12 = 12$

Therefore, the pentagonal pyramid satisfies Euler's formula.

9 **a** 20 **b** 30

10 6 vertices, each of degree 4. Hence, ther are $\frac{6 \times 4}{2} = 12$ edges

11 20 vertices, each of degree 3. Hence, ther are $\frac{20 \times 3}{2} = 30$ edges

12 12 vertices, each of degree 5. Hence, ther are $\frac{12 \times 5}{2} = 30$ edges

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13

Platonic solid	Tetra-hedron	Cube	Octa-hedron	Dodeca-hedron	Icosahe-dron
Vertices	4	8	6	20	12
Faces	4	6	8	12	20
Edges	6	12	12	30	30

14 a 8 faces, 12 vertices, and 18 edges

b $V + F = E + 2$

$12 + 8 = 18 + 2$

$20 = 20$

Therefore, the hexagonal prism satisfies Euler's formula.

15 The non-convex polyhedron has 16 vertices, 16 faces, and 32 edges. Hence, $V + F = E$.

16 A Platonic solid is a regular polyhedron with the same number of faces meeting at every vertex, therefore every vertex has the same degree and the graphs which represent the Platonic solids are regular graphs.

17 a

	$n = 3$	$n = 4$	$n = 5$
$m = 3$	$\frac{1}{3}+\frac{1}{3}-\frac{1}{2}=\frac{1}{6}$	$\frac{1}{4}+\frac{1}{3}-\frac{1}{2}=\frac{1}{12}$	$\frac{1}{5}+\frac{1}{3}-\frac{1}{2}=\frac{1}{30}$
$m = 4$	$\frac{1}{3}+\frac{1}{4}-\frac{1}{2}=\frac{1}{12}$	$\frac{1}{4}+\frac{1}{4}-\frac{1}{2}=0$	$\frac{1}{5}+\frac{1}{4}-\frac{1}{2}=-\frac{1}{20}$
$m = 5$	$\frac{1}{3}+\frac{1}{5}-\frac{1}{2}=\frac{1}{30}$	$\frac{1}{4}+\frac{1}{5}-\frac{1}{2}=-\frac{1}{20}$	$\frac{1}{5}+\frac{1}{5}-\frac{1}{2}=-\frac{1}{10}$

b The five entries with positive values of $\frac{1}{E}$ correspond to the five platonic solids.

c No, the number of edges must be zero or greater.

d Yes, provided this table contains all the positive entries (see question **17**).

18 Suppose we want $\frac{1}{n}+\frac{1}{m}-\frac{1}{2}>0$. If m is large, then n must be small. Concretely, if $m \geq 6$ then $\frac{1}{n}>\frac{1}{2}-\frac{1}{m}\geq\frac{1}{2}-\frac{1}{6}=\frac{1}{3}$, so $3 > n$. This means $n = 2$ or $n = 1$. By **9.2** question **16**, we have multiple edges when $n = 2$ and loops when $n = 1$. Both cases are excluded because the graph is simple.

EX p512

Checkpoint

1

Vertex	A	B	C	D	E
Degree	3	4	4	3	4

2 yes

3 A network with vertex degrees 4, 3 and 2 has $\frac{(4+3+2)}{2} = 4.5$ edges, and we must have a whole number of edges.

4 yes (no multiple edges or loops)

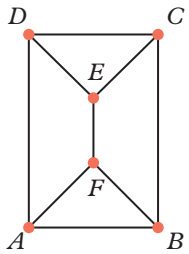
5 yes (can be redrawn without edges crossing)

6 6

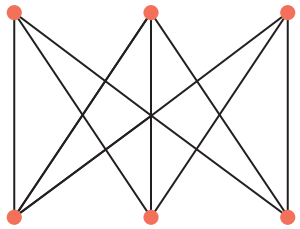
7 a 15

b No, a planar network with 6 vertices can have at most $3v - 6 = 12$ edges, so this network cannot be planar.

8 a Sample answer:



b Sample answer:



9 a 8 **b** 2 **c** More likely to fail

10 a $4 \times 10 = 40$

b 20

c $V + F = E + 2$

$10 + 12 = 20 + 2$

$22 = 22$

Therefore, Euler's formula is satisfied.

EX p515

9.4 Special types of walks

1 a A and C **b** No

2 a A-B-C-D and A-B-C-D

b No, because two different walks can have the same vertex sequence.

3 a path **b** trail

c invalid walk **d** walk

4 Closed walk

5 Sample answer: B-C-B

6 Sample answer: B-E-A-B-C-D-B

7 This network is not Eulerian because it has 10 vertices of odd degree.

8 This network is semi-Eulerian because it has 2 vertices of odd degree.

9 Sample answer: A-B-A-B-B

10 This network is Eulerian because no vertices have odd degree.

11 a

Vertex	N	K	S	L
Degree	3	5	3	3

b 4 **c** not Eulerian

d No, because the graph has too many vertices of odd degree.

12 a After adding one bridge, there will still be two vertices of odd degree.

b No, since vertex N (and at least one other vertex) will still have an odd degree.

c Yes, adding new bridges from N to S and N to L will make the graph semi-Eulerian, so there is a trail that crosses every bridge exactly once.

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13 **a** **i** semi-Eulerian **ii** Eulerian
iii semi-Eulerian
b Sample answers:
i *A-B-C-D* **ii** *A-B-C-D-A* **iii** *A-B-C-D-A-C*

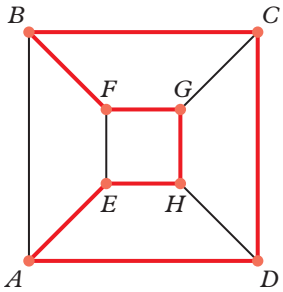
14 octahedron

15 **a** Sample answer:

b Not possible, since there are 4 intersections with odd degrees.

16 **a** There are exactly two vertices (*A* and *C*) with an odd degree.
b Sample answers:
A-D-C-A-B-E-F-B-C-F-G-C and *C-G-F-E-B-F-C-B-A-D-C-A*

17 **a** Sample answer:



b There is no Hamiltonian cycle because to return to the starting vertex the central edge must be crossed twice.

EX p520

9.5 Applications of networks

1 **a** *D* **b** *H* **c** *C* **d** *E*

2 **a** *H* **b** *D* **c** *E* **d** *C*

3

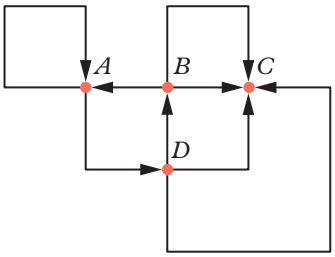
	Kin Group							
Father	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*
Children	*F*	*G*	*H*	*E*	*D*	*A*	*B*	*C*

4 **a** *A* **b** *G* **c** *G* **d** *A*

5

	Kin group							
You	*A*	*B*	*C*	*D*	*E*	*F*	*G*	*H*
Cousins	*A* or *G*	*B* or *H*	*C* or *E*	*D* or *F*	*E* or *C*	*F* or *D*	*G* or *A*	*H* or *B*

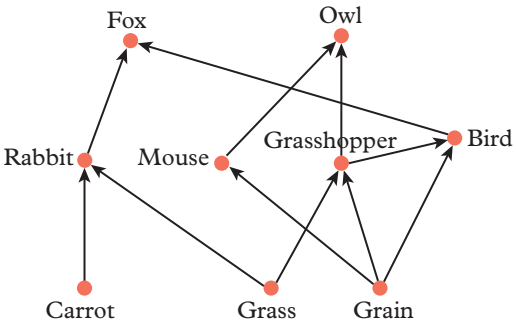
6 Sample answer:



7 **a** 10 **b** 9

8 **a** snakes, vultures and foxes
b dragonflies, butterflies and grasshoppers
c no **d** 4 **e** no

9

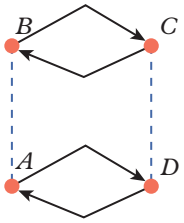


10 **a** *C* **b** *G* **c** *E* **d** *A*

11 yes **12** no **13** also *A*

14 via Sydney and Melbourne ($1384 total)

15 **a**



b *A*
c yes

EX p524

Review: Networks

oxforddigital **Answers for this lesson can be found on Oxford Digital.**

Computational thinking: Graph algorithms

1 **a** **i** Eulerian trail. Exactly two vertices have odd degree.
ii Neither. More than two vertices have odd degree.
iii Eulerian circuit. All vertices have even degree.
iv Eulerian trail. Exactly two vertices have odd degree.
b Answers may vary.
i *B–A–D–B–C–D*
iii *A–B–C–A–F–E–D–F–G–D–C–G–A*
iv *B–A–E–B–C–D–E–C*

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2 **a** Copy pseudocode from text.

b **1** The algorithm defines a function which takes a connected graph as input.

2 Set the variable odd_degree_vertices to 0.

3 Start a for loop which considers all vertices in the graph.

4 Check if the degree of a vertex is odd.

5 If the degree is odd, add 1 to the value of odd_degree_vertices.

6 End the decision point started on line **5**.

7 End the for loop started on line **3**.

8 Check if odd_degree_vertices = 0.

9 If odd_degree_vertices = 0, print 'Eulerian circuit'. If true, go to line **14**.

10 Check if odd_degree_vertices = 2.

11 If odd_degree_vertices = 2, print 'Eulerian trail'. If true, go to line **14**.

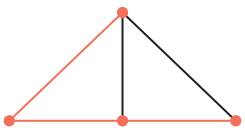
12 If neither lines **8** nor **10** are evaluated as true, go to line **13**.

13 Print 'Neither'.

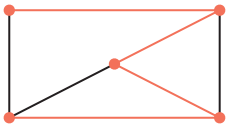
14 End the decision point started on line **9**.

c Same as question **1** part **a**

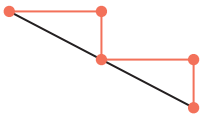
3 **a** **i**



ii

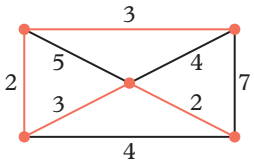


iii

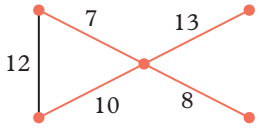


iv

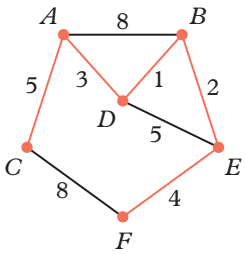
b **i** Total edge weight = 12. No, this is not a minimum spanning tree. Here is a minimum spanning tree with a total edge weight of 10.



ii Total edge weight = 12 + 7 + 8 + 13 = 40. No, this is not a minimum spanning tree. Here is a minimum spanning tree with a total edge weight of 38.

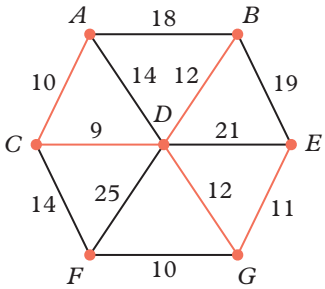


4 **a** **i**



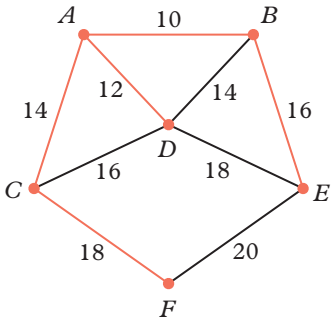
Total edge weight = 15

ii



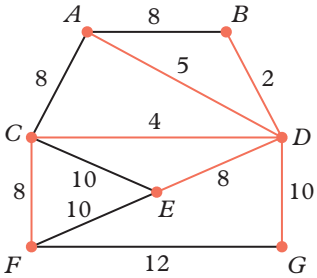
Total edge weight = 64

iii



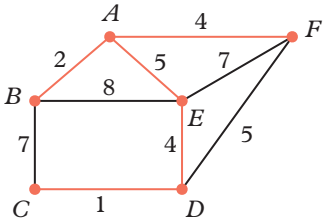
Total edge weight = 70

iv



Total edge weight = 37

b **i**



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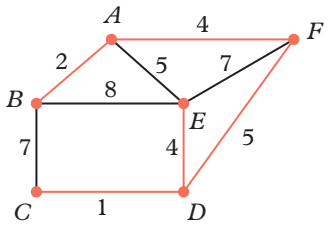

The total edge weights of the two trees are both equal to 16.

ii Change the weight of either edge *AE* or edge *DF* so that their weights are different and include the one with the smallest weight in the minimum spanning tree.

5 a The minimum spanning trees are the same as in question **4a**.

i *BD*, *BE*, *AD*, *EF*, *AC*

ii *CD*, *AC*, *FG*, *EG*, *BD*, *DG*. *AC* and *FG*, and *BD* and *DG* can occur in any order.

iii *AB*, *AD*, *AC*, *BD*, *CF*

iv *BD*, *CD*, *AD*, *CF*, *DE*, *DG*. *CF* and *DE* can occur in any order.

b The orders are different in parts **i**, **ii** and **iv** because in Prim's algorithm, only edges that have at least one endpoint already in the graph can be added, while in Kruskal's algorithm, a minimally weighted edge which does not form a cycle is added. For example, in part **i**, the edge *AD* is added first in Prim's algorithm but the edge *BD* is added first in Kruskal's algorithm. For part **iii**, the orders are the same but would differ if any of vertices *C*, *D*, *E*, or *F* were chosen as the starting vertex. Both algorithms result in the same minimum spanning trees as each of these graphs have a unique minimum spanning tree.

MODULE 10 Statistics

EX p542

10.1 Five-number summary and interquartile range

1 a 16 **b** 45 **c** 87 **d** 67

2 a 3, 3, 5, 7, 9 **b** 7, 12, 19, 46, 78

c 1, 3, 5, 7, 9 **d** 2, 7, 11, 19, 32

3 a 2, 4, 7, 9, 12

b 4, 13, 25.5, 46, 55

c 16, 20.5, 42, 72.5, 81

d 120, 177.5, 452.5, 755, 843

4 a 10 **b** 9 **c** 10 **d** 19

5 a 32 **b** 9 **c** 56.5 **d** 36

6 Range = 51; IQR = 23. The difference between the least number of cupcakes sold in the month and the most number of cupcakes sold in the month is 51. On half the days, the number of cupcakes sold falls within a range of 23 (from 56 to 79 cupcakes sold).

7 a 7, 9.5, 10.5, 12, 14

b 0, 1.5, 2.5, 4, 7

8 a Both sets have a range of 43.

b IQR of set A = 26, IQR of set B = 10

c because they have the same range

9 a 6

b minimum = 0, Q_1 = 2, Q_2 (median) = 4, Q_3 = 5, maximum = 6

c 3

10 a If the IQR is small, 50% of scores are grouped closely around the median.

b If the IQR is small and the range is large, 50% of the scores are grouped closely around the median, while the 25% of scores lying below the lower quartile and the 25% of scores above the upper quartile are spread out.

c If the IQR is large, 50% of the scores are spread widely around the median.

d If the IQR is similar to the range, the range of the 25% of scores lying below the lower quartile plus the range of the 25% of scores above the upper quartile is the same as the range of the middle 50% of scores.

11 a range = \$240, IQR = \$45

b because there is an outlier

c The IQR represents the middle 50% of scores; those scores between Q_1 and Q_3. If the spread is even, the remaining 50% of scores will have a spread of the same size as the IQR.

d The IQR indicates a region within which the middle 50% of scores lie and the outlier will fall outside this range.

12 a i range = 17, IQR = 8.5

ii The data are spread fairly uniformly throughout the range.

b i range = 72, IQR = 30

ii The scores are packed more closely in the IQR. There appears to be an outlier of 11 in the lower quarter of the scores.

c i range = 73, IQR = 11

ii Scores are packed more closely in the IQR. There appears to be an outlier of 191 in the upper quarter of the scores.

13 a **Class A** **i** 171.5 **ii** 27 **iii** 12

Class B **i** 172.5 **ii** 48 **iii** 12.5

b The median and IQR of heights in the two classes is quite similar, with Class B having a slightly larger median; however, the range of heights in Class B is greater. Class B has the tallest and shortest members of the group, indicating it is a more diverse group.

14 a 50, 60, 70.5, 79, 83 **b** IQR = 19, range = 33

c i 9, 55, 68, 74, 83 **ii** IQR = 19, range = 74

iii The median decreased slightly from 70.5 to 68. The range increased significantly from 33 to 74. The interquartile range remained at 19.

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d Since outliers and extreme values can only be at the ends of a list of data, they are not close enough to the position of the quartiles to impact the interquartile range.

15 a 14 **b** 28

16 a Using quintiles instead of quartiles provides a more detailed breakdown of the income distribution and gives a clearer view of the disparities between low and high income earners.

b i

Quintile	2015–2016 (%)	2017–2018 (%)	2019–2020 (%)
Lower (0–19%)	3.2	3.1	3.1
Second (20–39%)	8.6	8.6	8.6
Third (40–59%)	16.0	16.5	16.7
Fourth (60–79%)	25.4	25.6	25.8
Highest (80–100%)	46.8	46.2	45.8

ii No, an even distribution of income would have 20 as every entry in the table, as each 20% of the population would earn 20% of income.

c i 14.625 : 1, 14.903 : 1, 14.774 : 1

ii There is a decrease after 2013–2014 and then the ratio remains relatively steady at approximately 14.75 : 1.

17 a 16 **b** $\frac{47}{12}$ **c** $\frac{5293}{4620}$ **d** $\frac{13\,267}{9360}$

EX p547

10.2 Boxplots

1 a i 1, 2, 5, 7, 10 **ii** IQR = 5

b i 13, 20, 32, 40, 45 **ii** IQR = 20

c i 3, 4, 6, 8, 9 **ii** IQR = 4

d i 10, 16, 19, 24, 35 **ii** IQR = 8

e i 20, 45, 70, 75, 85 **ii** IQR = 30

f i 31, 33, 35, 37, 39 **ii** IQR = 4

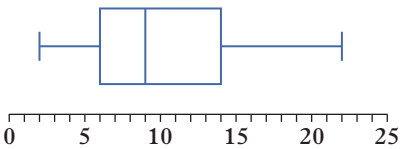
2 a 2, 3, 4, 4, 6, 7, 7, 8, 8, 9, 10, 10, 11, 13, 14, 17, 19, 19, 22

b $Q_1 = 6$, median = 9, $Q_3 = 14$

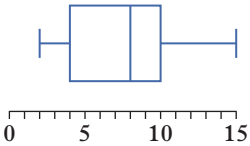
c lower fence = –6, upper fence = 26

d no outliers

e



3 a



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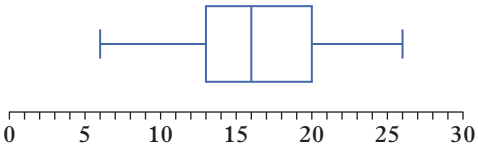

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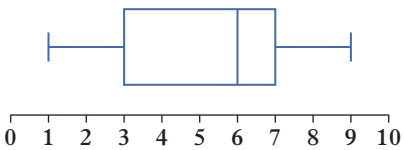

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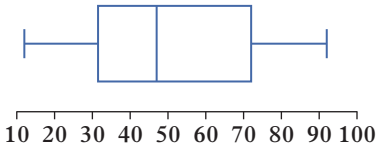

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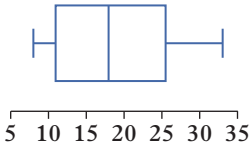

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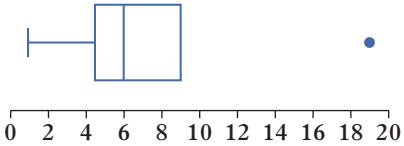

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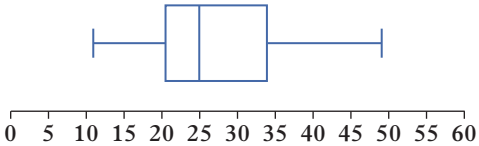

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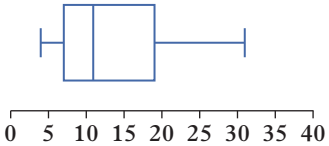


4 a i 3, 3, 7, 10, 12 **ii** 7

b i 18, 30, 33, 40, 40 **ii** 10

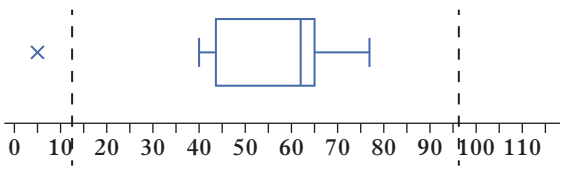
c i –20, –20, –15, –12, –12 **ii** 8

d i 40, 41, 42, 46.5, 60 **ii** 5.5

e i 18, 20, 21.5, 25, 35 **ii** 5

f i 50, 60, 60, 60, 70 **ii** 0

5 a



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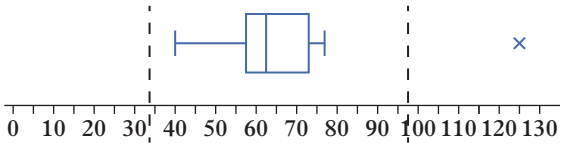

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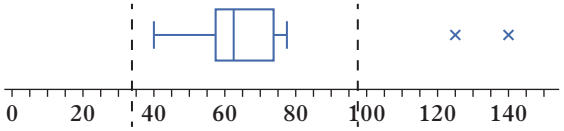

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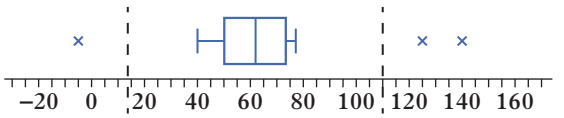

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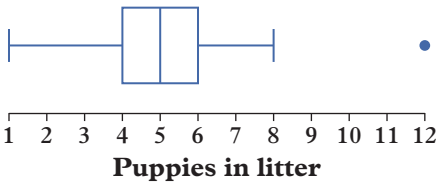
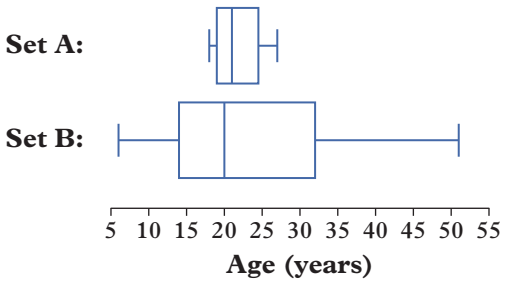
6 **a** **i** B **ii** B, D
b **i** median **ii** maximum **iii** none
iv maximum **v** upper quartile
c Group C

7 **a** $Q_1 = 4$, median = 5, $Q_3 = 6$ **b** 12 is an outlier.
c



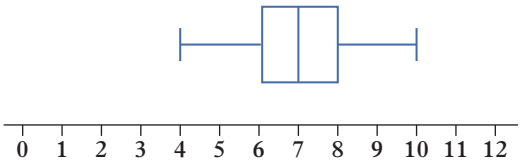
8 **a** median ≈ 32.2°C, IQR ≈ 3.5°C
b median ≈ 28°C, IQR ≈ 8.7°C
c Darwin's median maximum daily temperature during this month was approximately 32.2°C, which was hotter than Canberra's median maximum daily temperature, 28°C. Darwin's maximum daily temperature was less variable with an interquartile range of approximately 3.5°C compared to Canberra's larger interquartile range of approximately 8.7°C. While Canberra had the largest maximum daily temperature for the month, Darwin is, on average, hotter than Canberra.

9 **a**



b **Set A:** The centre (median) = 21, the spread is 18–27 years, positively skewed. The range of ages is small, indicating that, during the week, the centre is used mainly by young people.
Set B: The centre (median) = 20, the spread is 6–51 years old, positively skewed. The range of ages is much greater during the weekend, indicating that families most probably visit then.
c The range of ages is much greater in set B than in set A, although the median ages are quite similar. More people in a younger and older age range visit during the weekend than attend on weekdays.

10 **a**

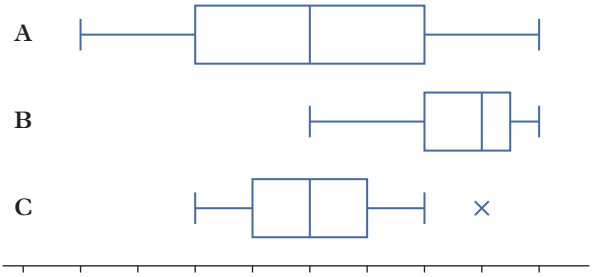


b No, as the minimum value (4) is greater than the lower fence (3) and the maximum value (10) is less than the upper fence (11), so the data contains no outliers.
c The minimum value (2) is less than the lower fence (3) and the maximum value (12) is greater than the upper fence (11) so they are both outliers. Therefore, we would need to know at least the next value above the minimum and less than the maximum value (provided they are not also outliers) to know the location of the ends of the whiskers of the boxplot.

11 **a** Since the median line is not visible, it is equal to either the upper quartile or lower quartile so that it overlaps the line for either quartile.
b 5, 8, 8, 14, 20 or 5, 8, 14, 14, 20

12 The median age of all Australians is approximately 37 years, which is greater than the median age of Aboriginal and Torres Strait Islanders (approximately 23 years). The interquartile range of the ages of all Australians is approximately 37 years, which is greater than the interquartile range of the ages of Aboriginal and Torres Strait Islander peoples (approximately 32 years).

13



14 **a** IQR = 9 **b** $Q_1 = 28.5$, $Q_3 = 37.5$

10.3 Distributions of data

EX p555

oxforddigital **Answers for this lesson can be found on Oxford Digital.**

10.4 The mean and standard deviation

EX p563

1 **a** 26.8 **b** 4.3 **c** 4.1 **d** 6.1 **e** 30.3 **f** 20.7
2 **a** 0.0 **b** 3.0 **c** 26.5 **d** 4.9 **e** 2.3 **f** 21.8
3 **a** 3.0 **b** 4.4 **c** 401 083.6
d 802 167.1 **e** 401 083.6 **f** 0.1
4 **a** 9.5 **b** 922.3 **c** 2.9 **d** 2.9 **e** 2.9 **f** 73.3
5 **a** **i** population **ii** 0.85
b **i** sample **ii** 5.20
c **i** sample **ii** 0.54
6 **a** 0.78
b The spread of ages of players in the cricket teams are tightly packed around the mean value, with all bar one player being between 15 and 17 years of age.
7 **a** mean = 15.88, standard deviation = 9.18
b The data are grouped pretty evenly with a large standard deviation and the mean being very close to halfway between the minimum and maximum.
8 **a** 41.7 **b** 9.75
c The spread of data is 13 to 58, with the mean at about 42. The mean is towards the higher end of the data, indicating that the data has a negative skew. The standard deviation of scores from the mean is about 10.
9 **a** the mean for train line A = 5.4
the mean for train line B = 4.5

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b the standard deviation for train line A = 1.89
the standard deviation for train line B = 4.57

c The mean for train line A is higher than that for train line B, indicating trains are generally later on line A. Standard deviation for train line A is lower than that for train line B, indicating scores are bunched closely around the mean, so that line A is consistently late. Line B has four outliers of 9, 11, 15 and 19 minutes, which have increased the mean and the standard deviation. Without these outliers, the standard deviation decreases to 1.23 and mean for train line B decreases to 2.65.

d The outliers for train line B might have been caused by unusual circumstances. If these outliers are disregarded, it could be said that the train on line A runs late more often. If the outliers are not disregarded, the train on Line B, on average, runs earlier but is very unpredictable.

10 The first histogram has a larger standard deviation than the second, because the scores are more spread out.

11 a 68.51

b The shop may be a discount shop or might be having a sale; the tablet might be a discontinued old model.

c 20.98

d Standard deviation is much lower with the outlier excluded.

e interquartile range

12 a i $\sigma \approx 2.43$, mean absolute deviation = 1.74

ii $\sigma \approx 5.38$, mean absolute deviation = 4.375

b mean absolute deviation $\leq \sigma$

13 a i $\tilde{x} = 4$ **ii** $\tilde{x} = 7.5$

b i 1.5 **ii** 2.5

14 a i Set A: 17, Set B: 5, Set C: 46

ii Set A: 9, Set B: 2, Set C: 9.5

iii Set A: 5.16, Set B: 1.47, Set C: 12.04

b Set C contains an outlier

c Sample answer: The range gives an easy-to-interpret value for the spread of a data set but is sensitive to outliers and doesn't provide any insight into how the data points are clustered between the highest and lowest values.

The interquartile range isn't impacted by outliers, but only provides an insight into where half of the data set lies and doesn't provide any insight into how the data points are clustered.

The standard deviation provides a measure of how tightly clustered the values in a data set are, but doesn't provide information about the full range of the data and is sensitive to outliers, particularly for smaller samples of data.

15 Sample answer: Dividing a data set up into more equally-sized pieces requires more time and effort, but provides a closer insight into the spread of the data and the nuances throughout a data set.

16 a The median is a better measure of the centre than the mean when the data has outliers or is skewed as both of these affect the value of the mean.

b The interquartile range is a better measure of the spread than the standard deviation when the data has outliers or is skewed as both of these affect the value of the mean and hence the standard deviation.

17 a Group A has a mean of 29.8, which is less than group B's mean of 30.1. Group A has a standard deviation of 5.5, which is greater than group B's standard deviation of 5.1. Therefore, on average, group A is less than and more variable than group B.

b Group A has a mean of 93.3, which is greater than group B's mean of 77.2. Group A has a standard deviation of 12.4, which is greater than group B's standard deviation of 7.9. Therefore, on average, group A is greater than and more variable than group B.

18 a i $z_A = 1.5$, $z_B = 1.875$

ii Student B scored 1.875 standard deviations above their class mean, which is greater than the 1.5 standard deviations Student A scored above their class mean. This means both students achieved better results than most of their class, with Student B performing better than Student A.

b i $z_A = 0.75$, $z_B = 1$

ii Student B scored 1 standard deviation above their class mean, which is greater than the 0.75 standard deviations Student A scored above their class mean. This means both students achieved better results than their class average, with Student B performing better than Student A.

c i $z_A = -0.8$, $z_B = -5$

ii Student B scored 5 standard deviations below their class mean, which is less than the 0.8 standard deviations Student A scored below their class mean. This means both students achieved worse results than their class average, with Student B performing worse than Student A.

d i $z_A = -3$, $z_B = -3$

ii Student B scored 3 standard deviations below their class mean, which is the same as the 3 standard deviations Student A scored below their class mean. This means both students achieved significantly worse results than their class average.

19 a Adding the percentages shown in the given graph:

2.35% + 13.5% + 34% + 34% + 13.5% + 2.35% = 99.7%

This covers the region of three sigma either side of the mean.

b i 68% **ii** 95%

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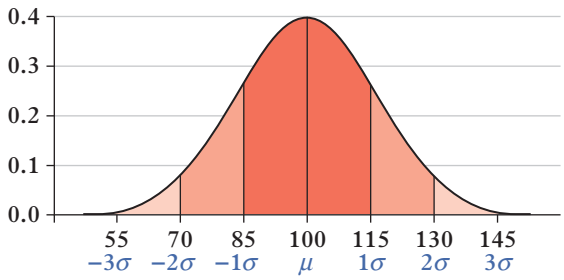
c **i**



ii 55 to 145 **iii** higher than 145

iv less than 70 **v** 2.5% **vi** 16%

vii mean = 136, standard deviation = 8

EX p567

Checkpoint

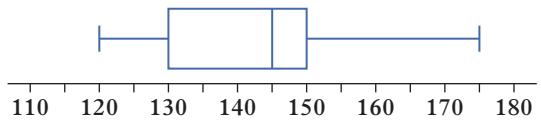
1 **a** 14, 17, 22.5, 32, 50 **b** 35, 38.5, 45, 54, 60

c 1, 4, 6, 7, 10 **d** 5.4, 6.2, 7.2, 7.9, 8.2

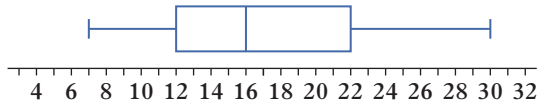
2 **a** IQR = 11, range = 18

b IQR = 27, range = 35

3 **a**



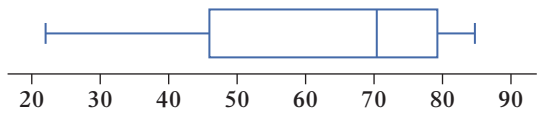
b



4 **a** **i** 16, 23, 27, 35, 38 **ii** 12

b **i** 20, 56, 72, 76, 80 **ii** 20

5 **a**



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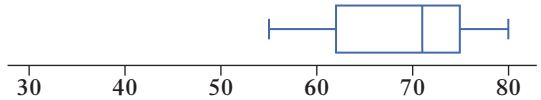


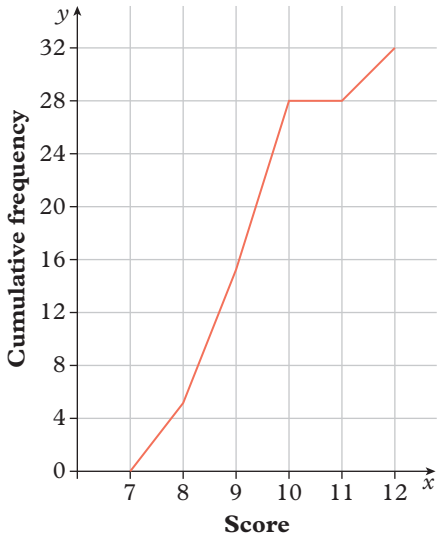
6 The median of Group A is 70, which is less than the median of Group B, 77. The interquartile range of Group A is 21, which is less than the interquartile range of Group B, 28. Therefore, Group A is less than Group B on average, and Group A's scores are less variable (or more consistent) than Group B.

7 **a** 25% **b** 50% **c** 75% **d** 50%

8 **a** negatively skewed **b** symmetric

c positively skewed

9 **a**



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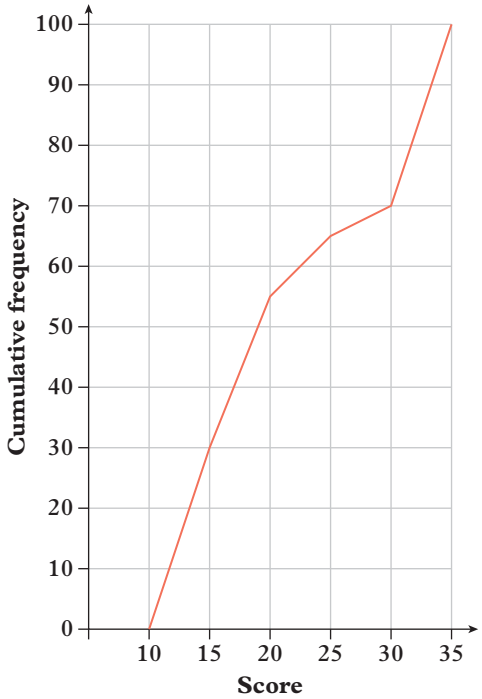

10 **a** **i** 16.1 **ii** 5.6

b **i** 35.5 **ii** 4.0

c **i** 104.6 **ii** 23.7

d **i** 251 **ii** 35.2

11 **a** **i** 10.5 **ii** 3.3

b **i** 45.9 **ii** 5.2

c **i** 74.3 **ii** 20.4

d **i** 326.1 **ii** 23.8

12 **a** **i** population **ii** 16.3

b **i** sample **ii** 25.8

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EX 10.5 Bivariate categorical data

p571

1

	Primary school	High school	Total
Catch public transport	22	18	40
Are driven	37	11	48
Walk/ride	5	9	14
Total	64	38	102

2

	Strongly disagree	Disagree	Neutral	Agree	Strongly agree	Total
Year 10	12%	15%	18%	30%	25%	100%
Year 12	10%	14%	20%	27%	29%	100%

3

	Strongly disagree	Disagree	Neutral	Agree	Strongly agree	Total
Swimmers	20%	16%	22%	25%	17%	100%
Runners	28%	21%	18%	19%	14%	100%

4 There does appear to be a relationship between the categorical variables, with a significantly higher number of Australians (58%) agreeing or strongly agreeing with the statement compared to New Zealanders (44%).

5 There does not appear to be a relationship between the categorical variables, with similar numbers from Team A and Team B in all responses.

6 a

	Very unhappy	Unhappy	Happy	Very happy	Total
Year 8	28%	22%	15%	35%	100%
Year 10	9%	15%	19%	57%	100%

b There does appear to be a relationship between the categorical variables, with a significantly lower number of Year 8 students (50%) happy or very happy about the options compared to Year 10 students (76%).

7 a incorrect, $\frac{18}{60} \times 100\% = 30\%$

b correct, $\frac{4}{65} \times 100\% \approx 6\%$

c correct, $\frac{36}{125} \times 100\% \approx 29\%$

d incorrect, $\frac{31}{61} \times 100\% \approx 51\%$

8 a

	Summer sunrise	Orchid bloom	Fond memories	Rosy outlook	Autumn days	Total
In paper	124	40	88	112	36	400
With vase	27	36	30	39	18	150

b i No, all bunches sold more in paper wrapping than with a vase.

ii Orchid bloom (14%)

iii Summer sunrise (97)

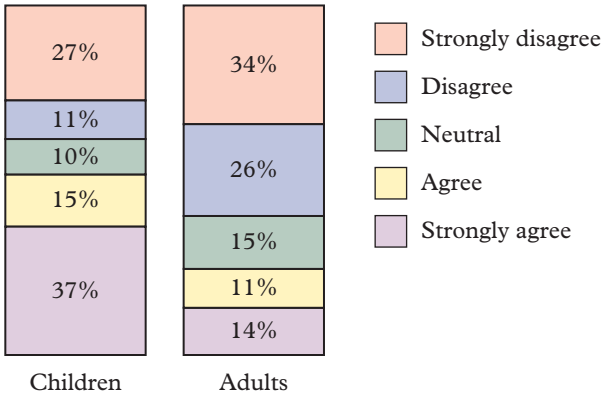
9 a There does not appear to be a relationship between the categorical variables, with only small differences between Year 10 and Year 12 for all responses.

b

	Strongly disagree	Disagree	Neutral	Agree	Strongly agree	Total
Year 10	13%	15%	17%	19%	36%	100%
Year 12	10%	21%	18%	20%	31%	100%

10 a There does appear to be a relationship between the categorical variables, with a significantly higher number of children (52%) agreeing or strongly agreeing with the statement compared to adults (25%).

b



11 a 200

b first time attendees: 74, previous attendees: 126

c didn't enjoy: 26, neutral: 44, loved it: 130

d

	Didn't enjoy	Neutral	Loved it
First-time attendees	9.62	16.28	48.1
Previous attendees	16.38	27.72	81.9

e

	Didn't enjoy	Neutral	Loved it
First-time attendees	0.02	3.66	1.36
Previous attendees	0.01	2.15	0.80

f 8.00

g The null hypothesis is rejected

h Yes, there is an association between the categorical variables

EX 10.6 Scatterplots

p578

oxforddigital **Answers for this lesson can be found on Oxford Digital.**

EX 10.7 Lines of best fit

p584

1 a $y = 3.1x + 4.6$ b $y = 0.8x + 4.5$

c $y = -2.4x + 23.8$ d $y = 2.3x + 5.9$

2 a 3.7 b 10.7 c 37.7 d 63.9

e 16.6 f 2.5 g 69.2

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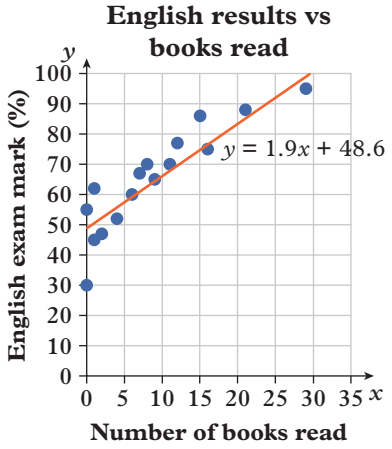
3 **a** 35.6 **b** 12.5 **c** –0.2 **d** 28.9

4 **a** a strong, positive relationship

b independent variable = number of books read
dependent variable = English exam mark

c, d

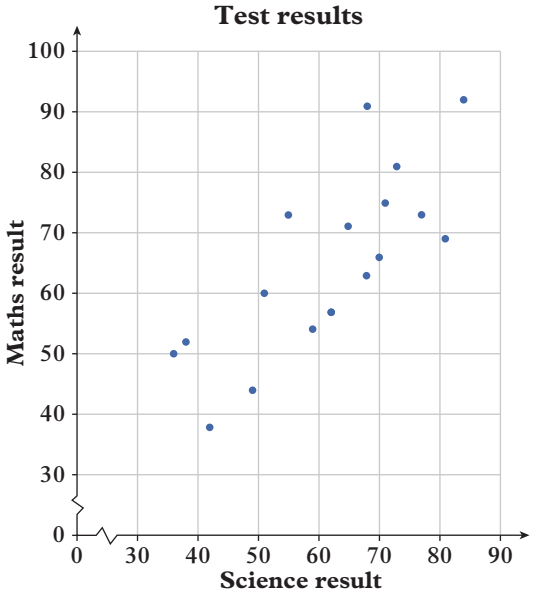


e The prediction based on the data would be that he should read about 11 books, but this does not guarantee an English exam mark of 70%.

5 **a** The trend is moderately strong and positive. As height increases, weight also increases.

b $y = 100x - 103$ **c** 82 kg

6 **a**



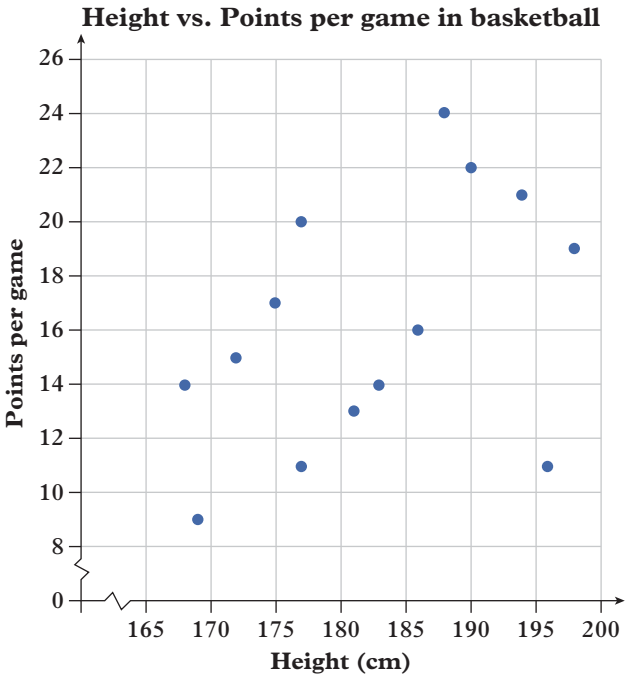
b maths result = 0.8112 × science result + 15.116

c **i** 54 **ii** 55 **iii** 78 **iv** 96

d Somewhat confident, there is a moderate correlation between the variables.

e Part iv, this prediction is outside the original range of the data.

7 **a**



b points per game = 0.2047 × height – 21.209

c Not very confident; there is only a weak correlation between the variables

d Potential causal relationship; taller basketball player will be able to reach closer to the hoop and score more points

8 **a** $y = 9.69 + 5.01x$

b **i** 1.2 **ii** –3.2 **iii** 0.29 **iv** 4.6

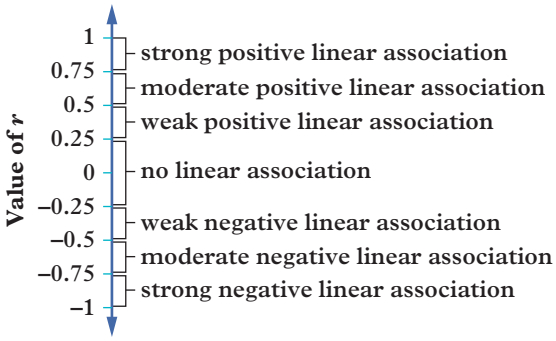
c **i** 7.1% **ii** –22.1% **iii** 1.45% **iv** 17.8%

9 **a** Relationship between the two variables is strong and positive in scatterplot A, but weak and positive in scatterplot B.

b scatterplot A = 18, scatterplot B = 15
(answers may vary)

c You should have more confidence in your prediction from scatterplot A because it's a stronger correlation and $x = 8$ is close to several other points that lie close to the line of best fit.

d



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10 **a** $\bar{x} = 4, \bar{y} = 5$

b $y = \frac{27}{11} + \frac{7}{11}x$

$= \frac{27}{11} + \frac{7}{11} \times 4$

$= \frac{27}{11} + \frac{28}{11}$

$= \frac{55}{11}$

$= 5$

c $y = a + bx$

substitute $x = \bar{x}$ and $y = \bar{y}$

$\bar{y} = a + b\bar{x}$

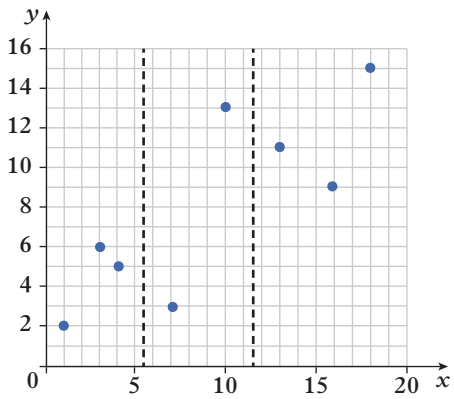
$a = \bar{y} - b\bar{x}$

d $s_x = \frac{\sqrt{44}}{3} = \frac{4\sqrt{11}}{3}, s_y = \frac{\sqrt{34}}{3}$

e **i** $r = 0.7239$

ii moderate

11 **a**

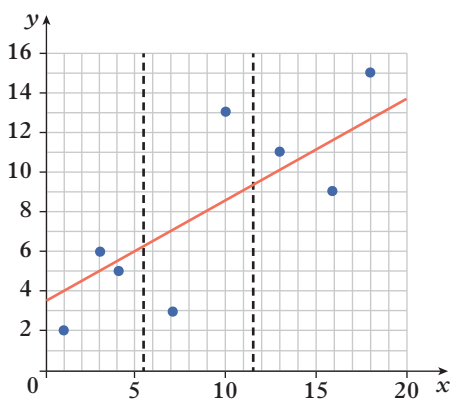


b For first section: median x-coordinate = 3, median y-coordinate = 5

For second section: median x-coordinate = 9, median y-coordinate = 8

For third section: median x-coordinate = 16, median y-coordinate = 11

c



d $y = 0.5x + 3.5$

e $y = 0.5x + 3.7$; this equation is very close to the equation you found in part **d**.

f The effect of extreme values (outliers) is minimised by finding and using the median value of that particular third.

EX p590 10.8 Time series

oxforddigital **Answers for this lesson can be found on Oxford Digital.**

EX p597 10.9 Analysing statistical reports

1 **a** This survey has used results from one gym to represent the entire population of Australia, meaning it is extremely biased.

b Two boys out of four does not represent a sufficient sample on which to base the claim. Information such as socio-economic backgrounds, ages, suburbs are not considered.

c If only five children were surveyed, the sample is not large enough to represent the larger population claimed. If the reference is to four-fifths of a larger population, more details are needed about what is considered a 'fair share' of chores and of whom that population was made up. Also, the children stated they 'believe they do' their fair share, but the report simply says they 'do' their fair share.

2 **a** Sample answer: They have provided no evidence to support this claim and no sample size is given. They should be able to answer the following questions: Did any of the women surveyed have diabetes, and if so, what type and to what degree? Were women who had coffee at other times of day surveyed too? Is there a biological link between coffee and diabetes?

b Sample answer: No evidence has been provided to support this claim and no sample size is given. They should be able to answer the following questions: How was the survey conducted? If the information was collected from a sample containing some people with Alzheimer's disease, is the information reliable? What timeline was followed? Was it conducted over many years and were the results monitored? Is there a biological link between being physically active and Alzheimer's disease?

c Sample answer: In order to determine the accuracy of data, information is required about the source of the survey results, who conducted the survey, how it was conducted, the sample size surveyed and the population from which the sample was chosen.

3 Sample answer: Not necessarily. The sample only includes people who watch the show and from those people it only includes those viewers who decide to call.

4 **a** The scale on the x-axis is not uniform, and the scale on the y-axis does not start at zero, with no break indicated.

b **i** The revenue declined from \$100 000 to \$96 500, a loss of \$3500 which is not a significant decline. The decrease from October to November is a decrease of \$1000 over the course of a month. This is about 30% of the total loss over eleven months, which is not a slower rate.

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ii Since the scale on the y-axis does not start at zero, the range of \$20 to \$20.50 can be stretched out. This gives the impression that the increase is greater than it really is.

iii 3.5%

5 **a** This graph is misleading as the scale on the y-axis is not labelled so we do not know if it starts at zero or not or if the scale on the y-axis is uniform or not.

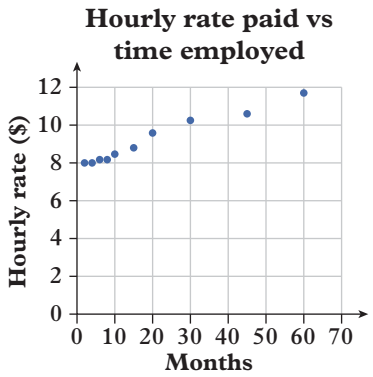
b **i** The graph is said to display the time taken before relief, however, the claim that the brand is twice as effective is not necessarily related to the time before relief.

ii By not labelling the y-axis, the company can make the graph look like a bigger bar represents a longer time for the generic brand to start acting. The bar appears twice as large but with the lack of scaling we cannot verify if the value it actually represents is twice as large.

6 **a** Fatima's hourly rate of pay over 60 months of employment

b The scale on the x-axis is not uniform and the scale on the y-axis doesn't start at the origin.

c



7 No. Success in school depends on more variables than just whether or not a student eats breakfast. Those who use the first statement may be trying to sell breakfast foods.

EX p614

8 The source of information, the sample used to obtain the results and the source of funding for such surveys should be stated with any reported results.

9 Sample answer: The company should report both the negative and positive outcomes of the independent testing, particularly because fuel efficiency is something potential buyers would be interested in. However, they do not need to compare the fuel efficiency to their competitors' models.

10 Sample answers:

a Yes, the data does appear to back up Jake's claim.

b The population is all people from the local area, and some people at the mall may not be from the local area.

c There is not an equal chance of everyone in the population being selected. Some people are more likely to be at the mall than other people.

d No, this is not a random sample of the population, therefore Jake's claim is not valid.

11 Sample answers:

a No, people with particular views may be more or less likely to refuse to be interviewed, so this is not a random sample.

b No, people with particular views may be more or less likely to be ill, so this is not a random sample.

c Yes, the sample size is too small to accurately represent the population. A sample size of at least 50 should be used.

12 **a** Yes

b This data is only for influenza vaccination rates, not for all vaccination rates.

13 Sample answers:

a The ABS collects data from the whole Australian population.

b Yes, this only contains data from two waves of COVID-19.

c No, the full set of data about COVID-19 should be used, and this data only reports on the proportion of deaths from COVID-19 and doesn't contain information about other impacts.

14 **a** Sample answer: Ellie may have asked leading questions in her survey, which would make it more likely for people to answer in the way that she wanted.

b Teacher to check.

15 Teacher to check.

EX p603

Review: Statistics

oxforddigital **Answers for this lesson can be found on Oxford Digital.**

MODULE 11 Probability

11.1 Theoretical probability

1 **a** {1, 2, 3, 4, 5, 6, 7, 8} **b** {A, U, S, T, R, L, I}

c {red, white, blue}

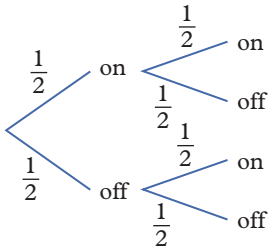
2 **a** equally likely **b** not equally likely

c not equally likely

3 **a** {M, A, T, H, E, I, C, S} **b** $\frac{2}{11}$

4 **a** 0.25 **b** 0.25 **c** 0.5

5 **a**



b **i** 0.25 **ii** 0.25 **iii** 0.5

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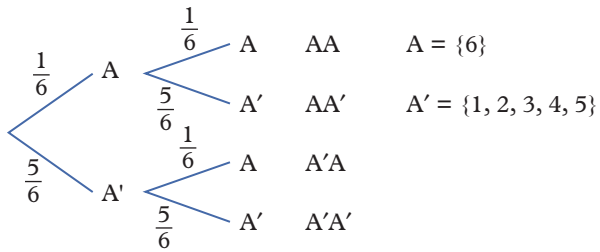
6 **a** not rolling a 6

b



c **i** $\frac{1}{36}$ **ii** $\frac{11}{36}$

7 **a** Each outcome now has an equal chance of occurring.

b 36

c **i** $\frac{1}{6}$ **ii** $\frac{1}{4}$

		2nd roll					
		1	**2**	**3**	**4**	**5**	**6**
1st roll	**1**	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
	6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

8 **a** 0.125 **b** 0.75 **c** 0.5

9 **a, b** **i** Cells coloured red **ii** Cells coloured orange

iii Cells coloured green

		2nd roll					
		1	**2**	**3**	**4**	**5**	**6**
1st roll	**1**	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
	5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
	6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

c **i** There are 15 favourable outcomes out of 36 possible outcomes; $\frac{15}{36} = \frac{5}{12}$.

ii There are 6 favourable outcomes out of 36 possible outcomes; $\frac{6}{36} = \frac{1}{6}$.

iii There are 15 favourable outcomes out of 36 possible outcomes; $\frac{15}{36} = \frac{5}{12}$.

d The most successful strategy would be to predict a number greater than (or less than) 7.

10 **a** A = {2, 4, 6, 8, 10}, B = {4, 8}, C = {2, 3, 5, 7}

b B and C

c **i** $\frac{1}{2}$ **ii** $\frac{1}{5}$ **iii** $\frac{2}{5}$

11 **a** **i** 36

ii 6 outcomes are doubles; (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)

iii $\frac{1}{6}$

b **i** 64

ii 8 outcomes are doubles, (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7), (8, 8)

iii $\frac{1}{8}$

c Pr(doubles) = $\frac{1}{n}$

12 10 marbles

13 **a** **i** $\frac{1}{4}$ **ii** $\frac{1}{6}$ **iii** $\frac{2}{9}$

b **i** For the outcomes from part **a**, an odd number is most likely.

ii For the outcomes from part **a**, a prime number is least likely.

14 **a** {1, 2, 3, 4, 5, 6, 8, 10, 12}

b Some scores can only be achieved one way, whereas others can be achieved two ways.

c **i** $\frac{1}{12}$ **ii** $\frac{1}{6}$ **iii** $\frac{1}{4}$

15 $\frac{11}{36}$

16 **a** 28

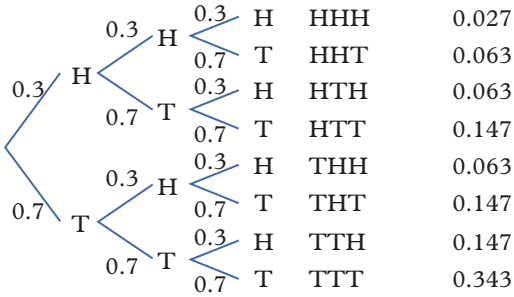
b The rows of tiles are grouped, in order from zero to six, according to the value shown on the top half of the tiles. Within each of these ordered groups, the values of the bottom half of the tiles have also been placed in order.

c 7

d **i** $\frac{1}{28}$ **ii** $\frac{1}{4}$ **iii** $\frac{3}{4}$

iv $\frac{3}{7}$ **v** $\frac{9}{14}$ **vi** $\frac{13}{28}$

17 **a**



Outcomes	**Probability**
HHH	0.027
HHT	0.063
HTH	0.063
HTT	0.147
THH	0.063
THT	0.147
TTH	0.147
TTT	0.343

b **i** TTT **ii** HHH

c **i** 0.441 **ii** 0.189 **iii** 0.216

18 **a** 2 **b** 7

c **i** (6, 1), (5, 2), (4, 3), (3, 4), (2, 5), (1, 6)

ii 6

19 5 red decorations

20 **a** The chance of landing on a red section is not the same as the chance of landing on a white section. There are more red bands than white and the combined area of the red bands appears larger than the combined area of the white bands.

b radii of circles from inner circle out: 1 cm, 3 cm, 5 cm, 7 cm, 9 cm

c bullseye = π cm^2, inner white band = 8π cm^2, middle red band = 16π cm^2, outer white band = 24π cm^2, outermost red band = 32π cm^2

d **i** 0.012 **ii** 0.395

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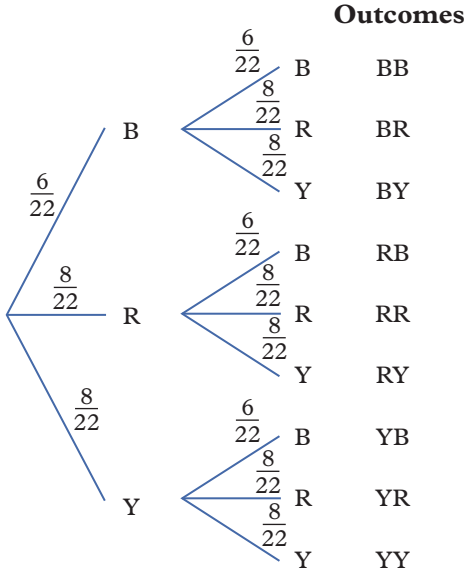
EX 11.2 Experiments with and without replacement

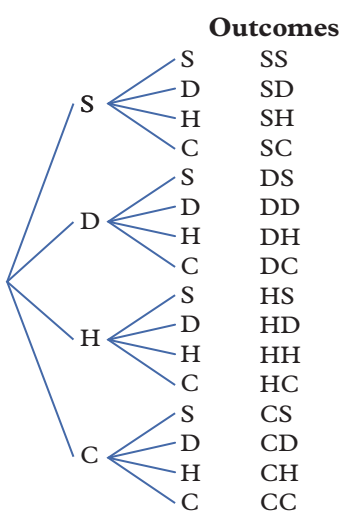
p621

1 a



b $\frac{41}{121}$

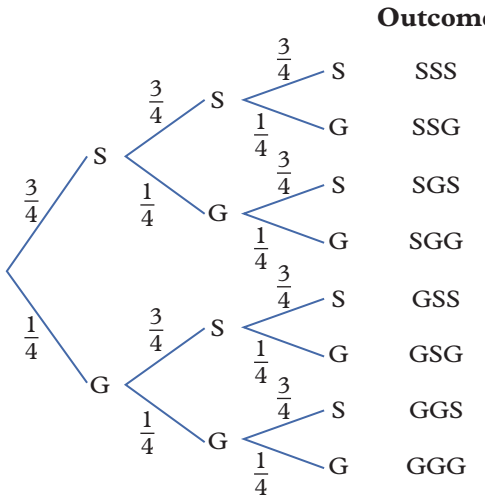
2 a



b $\frac{1}{16}$ **c** $\frac{1}{8}$ **d** $\frac{1}{4}$ **e** $\frac{3}{4}$

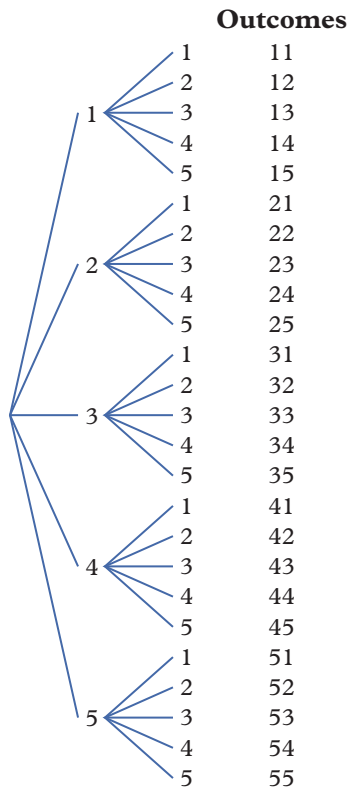
3 a i $\frac{3}{4}$ **ii** $\frac{1}{4}$

b



c i $\frac{27}{64}$ **ii** $\frac{1}{64}$ **iii** $\frac{9}{16}$

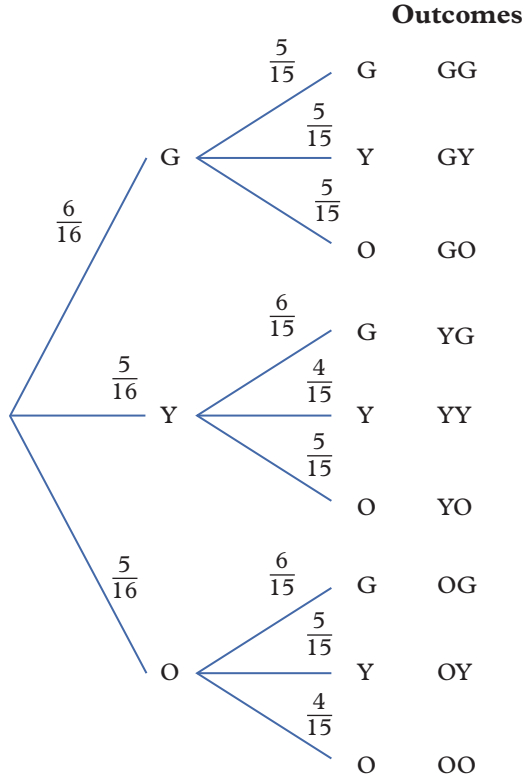
4 a



b 25 **c** 5

d i $\frac{4}{5}$ **ii** $\frac{1}{5}$ **iii** $\frac{2}{5}$

5 a



b $\frac{7}{24}$

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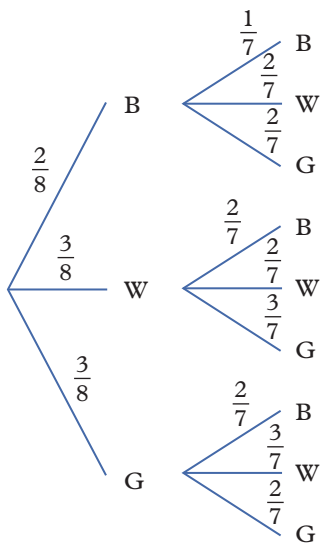
6 a

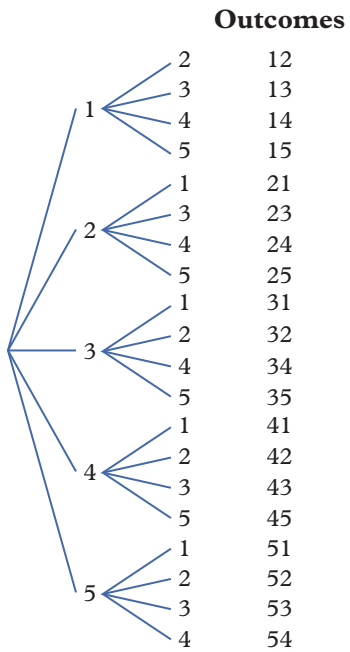


b $\frac{1}{4}$ **c** $\frac{3}{14}$

7 a $\frac{6}{55}$ **b** $\frac{28}{55}$

8 a $\frac{233}{338}$ **b** $\frac{44}{65}$

9 a



b 20

c i 0 **ii** 1 **iii** $\frac{1}{2}$

10 a $\frac{1}{4}$ **b** $\frac{23}{87}$

11 a $\frac{3}{8}$ **b** $\frac{25}{247}$ **c** $\frac{225}{988}$ **d** $\frac{42}{185}$

12 a i $\frac{9}{29}$ **ii** $\frac{2}{29}$ **iii** $\frac{3}{29}$ **iv** $\frac{3}{29}$ **v** $\frac{1}{435}$ **vi** $\frac{4}{435}$

b i $\frac{2}{87}$ **ii** $\frac{4}{435}$

c Undertaking the events in reverse order does not affect the overall probability of the events occurring.

13 a 0 **b** $\frac{27}{125}$

14 a There are only two non-white balls in total.

b $\frac{1}{10}$

15 The probabilities of the events in a multi-step experiment with replacement are not affected by the results of the previous steps, whereas the probabilities of the events in a multi-step experiment without replacement are affected by the results of the previous steps.

16 a i {2, 3, 4, 5, 6, 7, 8, 9, 10}

ii {3, 4, 5, 6, 7, 8, 9}

b i $\frac{3}{25}$ **ii** $\frac{1}{10}$

c i $\frac{1}{5}$ **ii** $\frac{1}{5}$

17 a i 125 **ii** 60

b i 729 **ii** 504

18 a 216 **b** 96

19 a i 10 **ii** 1 000 000

b i 17 576 000

ii You can form more possibilities with this system, because the alphabet has 26 letters, enabling more combinations than the 10 digits.

EX p627

11.3 Two-way tables and Venn diagrams

1 a i {1, 3, 5, 7, 9, 11} **ii** {2, 3, 5, 7, 11}

iii {3, 5, 7, 11} **iv** {1, 2, 3, 5, 7, 9, 11}

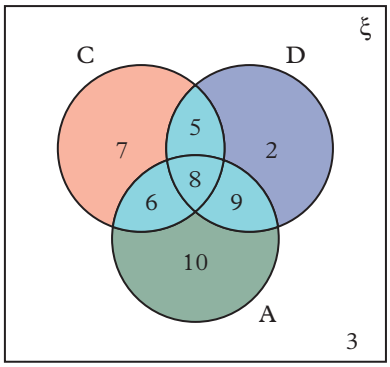
b i $\frac{1}{2}$ **ii** $\frac{7}{12}$ **iii** $\frac{1}{3}$ **iv** $\frac{7}{12}$

2 a i 13 **ii** 19 **iii** 13 **iv** 9 **v** 26 **vi** 6

b i $\frac{1}{2}$ **ii** $\frac{1}{2}$ **iii** $\frac{9}{26}$ **iv** $\frac{7}{26}$

v $\frac{7}{26}$ **vi** $\frac{2}{13}$ **vii** $\frac{10}{13}$ **viii** 1

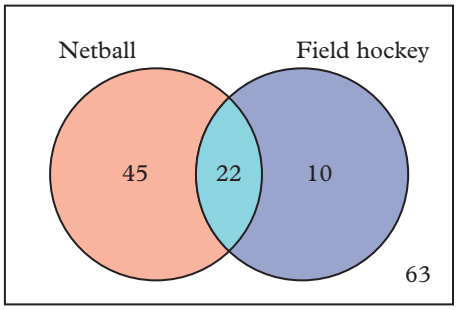
3



4 a $\frac{13}{25}$ **b** $\frac{13}{50}$ **c** $\frac{11}{50}$

d $\frac{7}{25}$ **e** $\frac{47}{50}$

5



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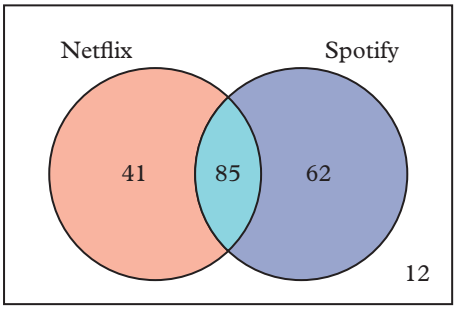
6 a

	Netflix (N)	Not Netflix (N′)	Total
Spotify (S)	85	62	147
Not Spotify (S′)	41	12	53
Total	126	74	200

b



7

	A	A′	Total
B	15	7	22
B′	13	11	24
Total	28	18	46

8 a 7

b

	Maths (M)	Not maths (M′)	Total
Science (S)	11	7	18
Not science (S′)	4	6	10
Total	15	13	28

9 0.7

10 $\frac{5}{18}$

11 a i True; set R is completely within set P so it is a subset of set P.

ii True; set R is completely within set Q so it is a subset of set Q.

iii False; set R is completely within the intersection of sets P and Q so it is a subset of the intersection of sets P and Q.

iv False; set R is completely within the union of sets P and Q so it is a subset of the union of sets P and Q.

v False; set R is within the universal set, ξ, so it is a subset of the universal set.

vi True; set R is within set Q, which is within the universal set so it is a subset of both set Q and the universal set, ξ.

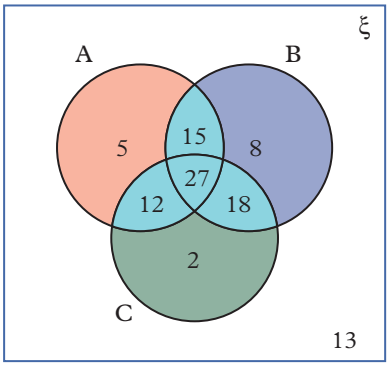
b i $\frac{1}{8}$

ii $\frac{1}{3}$

iii $\frac{1}{12}$

iv $\frac{1}{8}$

12 a



b i 85 **ii** 14 **iii** 45 **iv** 73

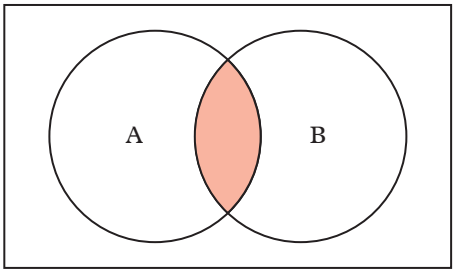
v 41 **vi** 15 **vii** 18 **viii** 73

13 a i {2, 4, 6, 8, 10, 12} **ii** {2}

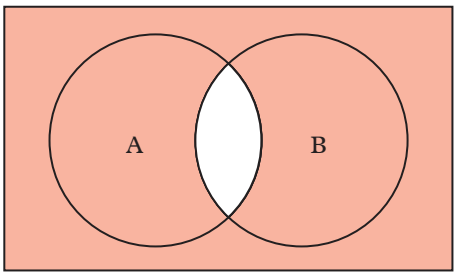
iii {1, 2, 4, 6, 8, 9, 10, 12}

b i $\frac{1}{12}$ **ii** $\frac{5}{12}$ **iii** $\frac{2}{3}$

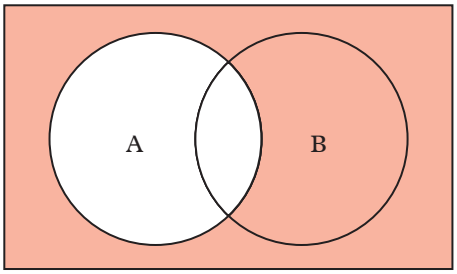
14 Pr(A ∩ B) is:

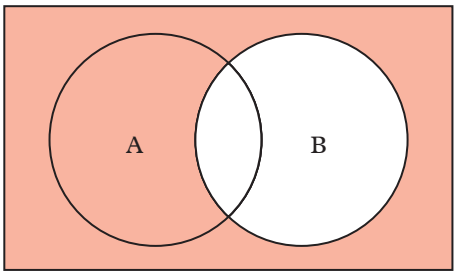


So Pr((A ∩ B)′) is:



Pr(A′) and Pr(B′) are:





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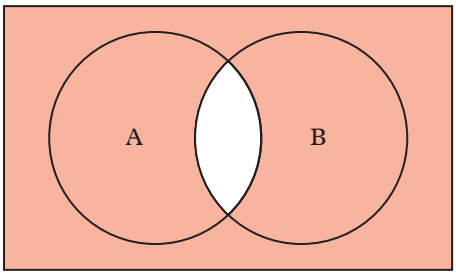
Hence, $\Pr(A' \cup B')$ is:



15 **a** 0.25 **b** 0.45

16 **a**

	A	A′	Total
B	0.1	0.5	0.6
B′	0.2	0.2	0.4
Total	0.3	0.7	1

b **i** 0.2 **ii** 0.5 **iii** 0.9

17 $P \cap Q = \{2, 5, 9, 13\}$

$R \cap S' = \{4, 5, 6, 7\}$

$\{2, 5, 9, 13\} \cup \{4, 5, 6, 7\} = \{2, 4, 5, 6, 7, 9, 13\}$

$2 + 4 + 5 + 6 + 7 + 9 + 13 = 46$

18 $\Pr(A \cup B \cup C) = \Pr(A) + \Pr(B) + \Pr(C) - \Pr(A \cap B) - \Pr(A \cap C) - \Pr(B \cap C) + \Pr(A \cap B \cap C)$

EX p631

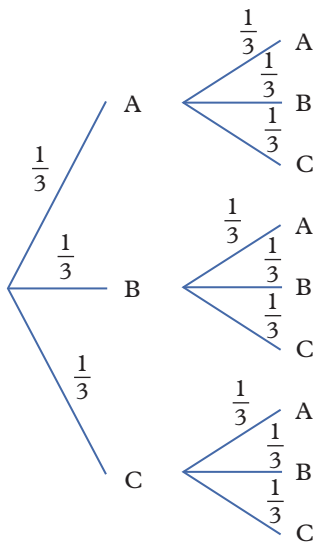
Checkpoint

1 **a** {H, T} **b** {1, 2, 3, 4, 5, 6}

c {club, diamond, heart, spade} **d** {L, O, I, P}

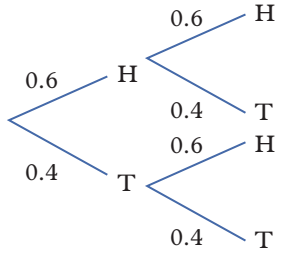
2 **a** $\frac{1}{6}$ **b** $\frac{1}{3}$ **c** $\frac{1}{3}$ **d** $\frac{2}{3}$

3 **a**



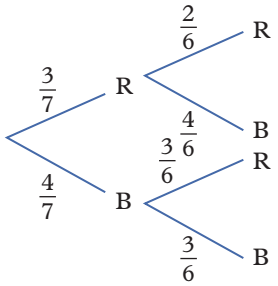
b $\frac{1}{9}$

4 **a**



b **i** 0.36 **ii** 0.48

5 **a**



b $\frac{1}{7}$

6 **a** $\frac{4}{25}$ **b** $\frac{1}{10}$

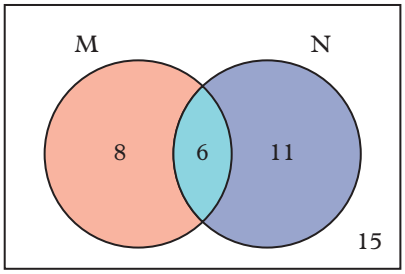
7 **a** $\frac{1}{16}$ **b** $\frac{1}{4}$ **c** $\frac{3}{4}$

8 **a** $\frac{4}{5}$ **b** $\frac{7}{20}$ **c** $\frac{1}{5}$ **d** $\frac{3}{20}$

9 **a**

	M	M′	Total
N	6	11	17
N′	8	15	23
Total	14	26	40

b



10 **a** $\frac{7}{12}$ **b** $\frac{5}{8}$ **c** $\frac{1}{3}$ **d** $\frac{11}{24}$

EX p636

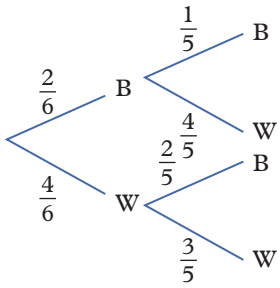
11.4 Conditional probability

1 **a** The probability $\frac{15}{20}$ means there are 15 soft centres out of a total of 20 chocolates. The probability $\frac{5}{20}$ means there are 5 hard centres out of the 20 chocolates.

b The branches on the tree indicate that if 1 soft centre is eaten, there are 14 soft centres and 5 hard centres left out of the remaining 19 chocolates. They also show that, if it had been 1 hard centre that had been eaten, there would have been 15 soft centres and 4 hard centres left out of the remaining 19 chocolates.

c $\frac{4}{19}$ **d** $\frac{5}{19}$ **e** $\frac{15}{19}$

2 **a**



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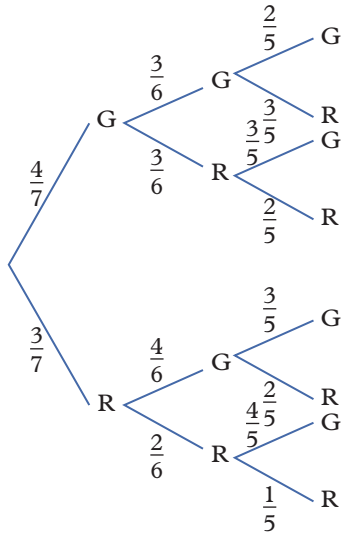
b $\frac{3}{5}$ **c** $\frac{4}{5}$ **d** $\frac{2}{5}$

3 a



b $\frac{1}{2}$ **c** $\frac{2}{5}$ **d** $\frac{3}{5}$

4 a $\frac{12}{13}$ **b** $\frac{12}{13}$

5 a 15 **b** 18 **c** 18 **d** 59

6 a $\frac{15}{41}$ **b** $\frac{9}{20}$ **c** $\frac{9}{50}$ **d** $\frac{59}{100}$ **e** $\frac{18}{59}$

7 a 8 **b** 3 **c** $\frac{3}{8}$ **d** $\frac{3}{5}$

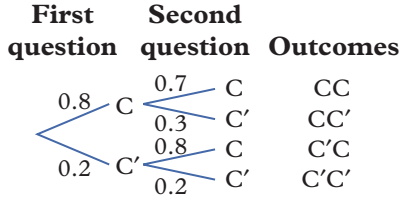
8 a $\frac{4}{49}$ **b** $\frac{7}{12}$

9 a i $\frac{7}{10}$ **ii** $\frac{3}{10}$ **iii** $\frac{2}{3}$ **iv** $\frac{4}{7}$ **v** $\frac{3}{11}$

b $\frac{1}{2}$

10 a $\frac{1}{3}$ **b** 0.25

11 a



b i 0.7 **ii** 0.8 **iii** 0.3

12 a

	Hybrid	WFH	Total
Full-time employees	30	4	34
Part-time employees	5	11	16
Total	35	15	50

b i 4 **ii** 16 **iii** 35

c i $\frac{5}{35}$ **ii** $\frac{5}{16}$ **iii** $\frac{2}{17}$ **iv** $\frac{4}{15}$

d i In part **i**, the selection is made from hybrid workers (total of 35), whereas in part **ii** the selection is made from part-time employees (total of 16).

ii In part **iii**, the selection is made from full time employees (total of 34), whereas in part **iv** the selection is made from WFH employees (total of 15).

13 a

	Small	Large	Total
Silver	27	35	62
Coloured	18	10	28
Total	45	45	90

b i True; there are 45 of each.

ii True; there are 28 coloured paper clips but there are 62 silver paper clips.

iii False; there are 27 small silver paper clips out of a total of 45 small paper clips. $\frac{27}{45} = 60\%$

iv False; there are 10 large coloured paper clips out of a total of 28 coloured paper clips. $\frac{10}{28} \approx 36\%$

v True; there are 62 silver paper clips out of a total of 90 paper clips. $\frac{62}{90} \approx 69\%$

14 a Pr(A) means the probability of A occurring, whereas Pr(A|B) means the probability of A occurring given that B has occurred. Therefore, when comparing Pr(A) and Pr(A|B) we are looking at two different sets. In the first case, Pr(A), we are looking at the whole population, whereas in the second case, Pr(A|B), we are only looking at the proportion of the population which meets the criteria Pr(B).

Given our example, this is the difference between the proportion of the whole population who can dunk a basketball (which will be very small), compared to the proportion of people over 6 foot tall who can dunk a basketball (which will be slightly larger).

b Pr(A|B) means the probability of A occurring if B has occurred, where Pr(B|A) means the probability of B occurring if A has occurred. Therefore, when comparing (Pr(A|B) to Pr(B|A) we are looking at two different sets. In the first case, Pr(A|B), we are looking at the proportion of the population which meets the criteria Pr(B), whereas in the second case, Pr(B|A), we are looking at looking at the proportion of the population which meets the criteria Pr(A).

Given our example, this is the difference between the proportion of Olympic athletes who exercise daily (which will be very high), compared to the proportion of people who exercise daily who are Olympic athletes (which will be very low).

15 0.625

16 a i 32 **ii** 41 **iii** 57

b i $\frac{11}{32}$ **ii** $\frac{11}{41}$ **iii** $\frac{35}{57}$

c They help to restrict the sample space, giving the appropriate denominator for the conditional probability.

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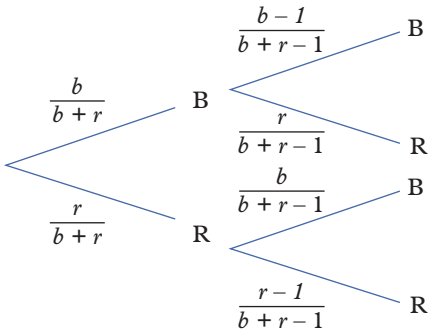
17 a **i** 0.8 **ii** 0.5

b $\Pr(X) = 0.1$ and $\Pr(Y) = 0.25$

18 a



b $\frac{b}{b+r}$. This is equal to the probability that the first ball is blue. The denominator from the conditional probabilities in the second step cancels leaving this simplified fraction.

19 a 0.64 b $\frac{2}{3}$

20 a Start with $\Pr(B|A) = \frac{\Pr(A \cap B)}{\Pr(A)}$

Multiplying through by $\Pr(A)$ gives $\Pr(A \cap B) = \Pr(B|A) \times \Pr(A)$.

Substituting this result into

$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}$ gives

$\Pr(A|B) = \frac{\Pr(B|A) \times \Pr(A)}{\Pr(B)}$, which is Baye's rule.

b **i** 0.55 **ii** $\frac{7}{11}$

EX p642

11.5 Independence

1 a **i** 0.4 **ii** 0.5

b independent

2 a $\frac{2}{3}$ **b** $\frac{1}{4}$ **c** $\frac{1}{6}$

3 Yes. $\Pr(\text{cricket}) = \frac{9}{21} = \frac{3}{7}$ and $\Pr(\text{cricket}|\text{Maths}) = \frac{6}{14} = \frac{3}{7}$

4 a independent **b** dependent

c dependent **d** independent

5 a **i** $\frac{3}{4}$ **ii** $\frac{3}{4}$

b independent

6 a **i** $\frac{2}{3}$ **ii** $\frac{7}{9}$

b dependent

7 a 0.48 **b** 0.92

8 a He assumed A and B were independent events and evaluated $\Pr(A) \times \Pr(B)$.

b 0.38. It must be calculated using the addition rule: $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

9 Mutually exclusive events are events that cannot occur at the same time while independent events have no effect on each other occurring.

10 a dependent **b** independent

c independent

d dependent

11 Q and R, P and Q

$\Pr(Q) \times \Pr(R) = \frac{9}{30} = \Pr(Q \cap R)$

$\Pr(P) \times \Pr(Q) = \frac{1}{5} = \Pr(P \cap Q)$

12 a **i** $\frac{1}{2}$ **ii** $\frac{1}{2}$ **iii** $\frac{1}{3}$ **iv** $\frac{3}{10}$

b because $\Pr(\text{Action}|\text{Odd}) = \Pr(\text{Action})$

c because $\Pr(\text{Comedy}|\text{even}) \neq \Pr(\text{Comedy})$

d There is no logical reason that these two things should be connected. They appear dependent due to random variation.

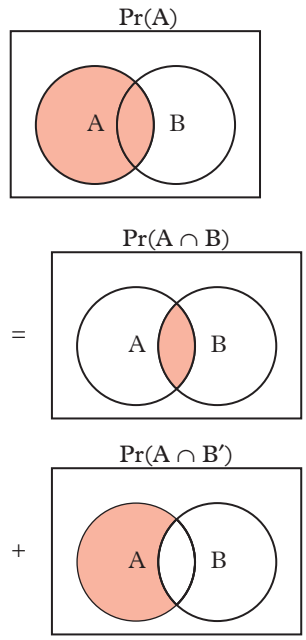
13 a The colour of the first ball selected will not affect the probability that the second ball is white.

b **i** $\frac{2}{3}$ **ii** $\frac{3}{5}$

c **i** $\frac{4}{9}$ **ii** $\frac{2}{5}$

14 $\Pr(A) = 0.6$

15 a



b $\Pr(A) \times \Pr(B') = \Pr(A) \times (1 - \Pr(B))$

$= \Pr(A) - \Pr(A) \times \Pr(B)$

$= \Pr(A) - \Pr(A \cap B)$

$= \Pr(A \cap B')$

EX p649

11.6 Permutations

1 a 35 **b** 17 **c** 29 **d** 216 **e** 560 **f** 4320

2 a 6 **b** 24 **c** 720

d 3 628 800

3 a 5040 **b** 362 880 **c** 120

d 3 628 800 **e** 479 001 600 **f** 720

4 a 120 **b** 90 **c** 30 240

d 2520 **e** 720 **f** 870

5 a 7 **b** 3024 **c** 60

d 720 **e** 19 958 400 **f** 360

6 a $\frac{26}{55}$ **b** $\frac{5}{11}$ **c** $\frac{31}{55}$

d $\frac{10}{11}$ **e** $\frac{1248}{3025}$ **f** $\frac{1326}{5995}$

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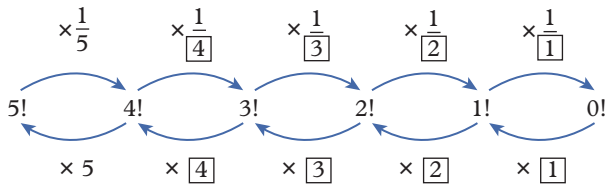
7 **a** $\frac{1}{15}$ **b** $\frac{3}{10}$ **c** $\frac{3}{28}$

8 **a** $10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$
$= 10 \times 9 \times 8 \times (7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1)$
$= 10 \times 9 \times 8 \times 7!$
This shows that 7! is a common factor of the numerator and denominator that can be divided out to simplify the fraction.

b **i** 132 **ii** 6840 **iii** 100
iv 35 904 **v** 2550 **vi** 997 002 000

9 **a** 24 **b** 30 **c** 20 **d** 16

10 **a**



b $0! = 1$

c ${}^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!}$ but we know that ${}^nP_n$ means from n objects permute n of them, which is the same as $n!$ and so $n! = \frac{n!}{0!}$ which is only true if $0! = 1$.

11 **a** 3 **b** 12 **c** 6 **d** 1260

12 **a** 120 **b** $\frac{1}{10}$ **c** $\frac{1}{3}$ **d** $\frac{1}{6}$

13 **a** 6 **b** 24 **c** 40 320 **d** $(n-1)!$

14 Six people (A, B, C, D, E, F) are standing in a circle in a random order.

a 120 **b** $\frac{7}{15}$ **c** $\frac{1}{2}$ **d** $\frac{1}{2}$

15 **a** 720 **b** 48 **c** $\frac{1}{15}$

d **i** 5040 **ii** 96 **iii** $\frac{2}{105}$

16 **a** $\frac{1}{n+1}$ **b** $(n+3)(n+2)n(n+1)(n-1)(n-2)$

c $n+3$ **d** $\frac{n-1}{n!} = \frac{1}{n(n-2)!}$

e $n+1$ **f** $\frac{(n+5)(n+4)(n+3)(n+2)(n+1)}{20}$

17 **a** **i** 945 **ii** 384 **iii** 2 027 025
b **i** 2310 **ii** 30 030 **iii** 9 699 690
c **i** 2 **ii** 12 **iii** 34 560

EX p654

11.7 Combinations

1 **a** 32 **b** 256 **c** 4096
d 32 768 **e** 1 048 576 **f** 1 073 741 824

2 **a** 126 **b** 10 **c** 120
d 495 **e** 15 **f** 125 970

3 **a** 20 **b** 45 **c** 252
d 21 **e** 6 **f** 435

4 **a** 1470 **b** 11 760 **c** 3 603 600
d 15 116 400 **e** 44 100 **f** 10 080

5 **a** 32 **b** 1024 **c** 256
d 512 **e** 63 **f** 31

6 **a** $\frac{1}{210}$ **b** $\frac{1}{495}$

7 **a** $\frac{21}{55}$ **b** $\frac{35}{429}$ **c** $\frac{28}{323}$

8 **a** $\frac{1}{429}$ **b** $\frac{8}{429}$ **c** $\frac{3432}{16\,369}$

9 **a** ${}^2C_0 + {}^2C_1 = {}^3C_1$, ${}^3C_2 + {}^3C_3 = {}^4C_3$, ${}^5C_1 + {}^5C_2 = {}^6C_2$

b ${}^nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1}$

c **i** ${}^{11}C_5$ **ii** ${}^{21}C_{15}$ **iii** ${}^{16}C_{13}$
iv ${}^{21}C_3$ **v** ${}^{52}C_{30}$ **vi** ${}^{100}C_{49}$

10 **a** 10 **b** 70 **c** 45 **d** 120 **e** 2520 **f** 990

11 **a** **i** 2 **ii** 4 **iii** 8 **iv** 16 **v** 32

b They are all powers of 2.

c Since a combination is really a subset, the sum of all values ${}^nC_r$ for each possible r must be equal to the number of subsets of a set of n elements, which is equal to 2^n.

12 **a** 1 and 1 **b** ${}^nC_0 = 1$ and ${}^nC_n = 1$

c The row number and the second number in from both ends are the same.

d ${}^nC_1 = n$ and ${}^nC_{n-1} = n$

e They are equal, ${}^nC_r = {}^nC_{n-r}$

13 **a**
$$\begin{aligned}
{}^{n-1}C_{r-1} + {}^{n-1}C_r &= \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-r-1)!} \\
&= \frac{(n-1)!}{(r-1)!\,(n-r-1)!}\left(\frac{1}{n-r} + \frac{1}{r}\right) \\
&= \frac{(n-1)!}{(r-1)!(n-r-1)!}\left(\frac{n}{(r(n-r)}\right) \\
&= \frac{n(n-1)!}{r(r-1)!(n-r)(n-r-1)!} \\
&= \frac{n!}{r!(n-r!)} \\
&= {}^nC_r
\end{aligned}$$

b **i** $n-1$

ii ${}^{n-1}C_{r-1}$. Since x is one of the r chosen elements, there must be $r-1$ elements chosen from the remaining $n-1$ elements in the set.

iii ${}^{n-1}C_r$. There are $n-1$ elements in the set not including x from which all r elements must be chosen.

iv The element x is either one of the r chosen elements or not. The number of ways of choosing r elements from a set of size n must equal the sum of the ways to include or exclude x from the r elements, which are ${}^{n-1}C_{r-1}$ and ${}^{n-1}C_r$, respectively. Hence, ${}^{n-1}C_{r-1} + {}^{n-1}C_r = {}^nC_r$.

14 Starting with ${}^2C_2 = 1$ and using Pascal's identity,
${}^3C_2 = {}^2C_1 + {}^2C_2 = 2 + 1 = 3$,
${}^4C_2 = {}^3C_1 + {}^3C_2 = 3 + 3 = 3 + (2 + 1) = 6$,
${}^5C_2 = {}^4C_1 + {}^4C_2 = 4 + 6 = 4 + (3 + 2 + 1) = 10$,
we can generalise to ${}^{n+1}C_2 = {}^nC_1 + {}^nC_2$
$= n + ((n-1) + (n-2) + \cdots$
$3 + 2 + 1)$.

15 **a** $x^2 + 2x + 1$ **b** $x^3 + 3x^2 + 3x + 1$

c $x^4 + 4x^3 + 6x^2 + 4x + 1$

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d The coefficients of $(x + 1)^n$ are the values in the nth row of Pascal's triangle.

e **i** $x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 7x + 1$

ii 4845

EX Review: Probability

p656

oxforddigital **Answers for this lesson can be found on Oxford Digital.**

Semester 2 review: Mathematical modelling

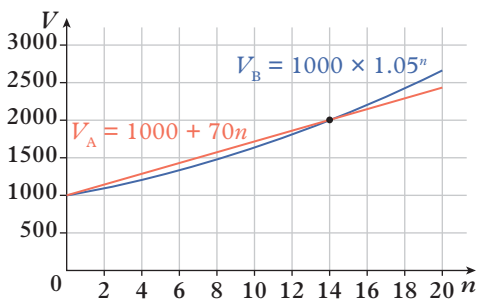
1 **a** Investment A

b $V_A = 1000 + 70n$ and $V_B = 1000 \times 1.05^n$

c

n	0	5	10	15	20
V_A	1000	1350	1700	2050	2400
V_B	1000	1276.28	1628.89	2078.93	2653.30

d



e After the 14th year

f After 4 years

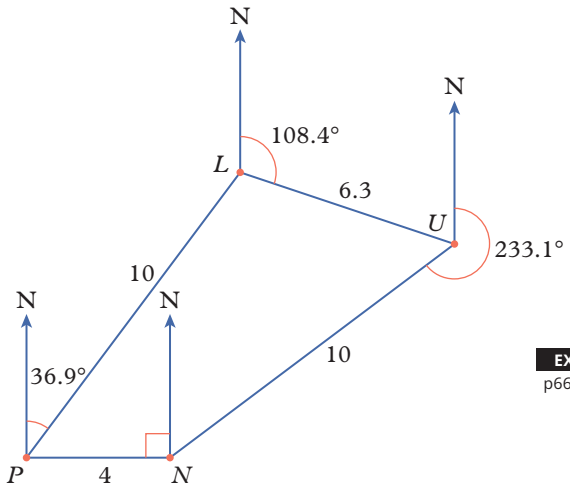
g **i** After 12 years

ii \$1840

h **i** $\log_{1.05}\left(\frac{27}{20}\right)$ years or $\frac{\log(1.05)}{\log(1.35)}$

ii 6.15 years

2 **a**



b 30.3 km

c 5.05 km/h

d 243 m

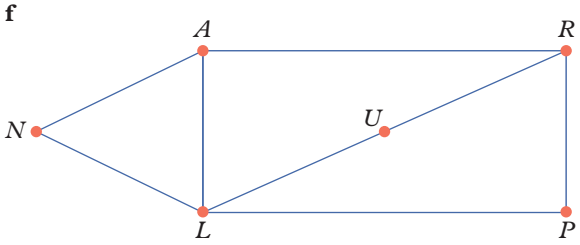
e **i** 2%

ii 76%

iii 18%

iv $\frac{2}{3}$

f



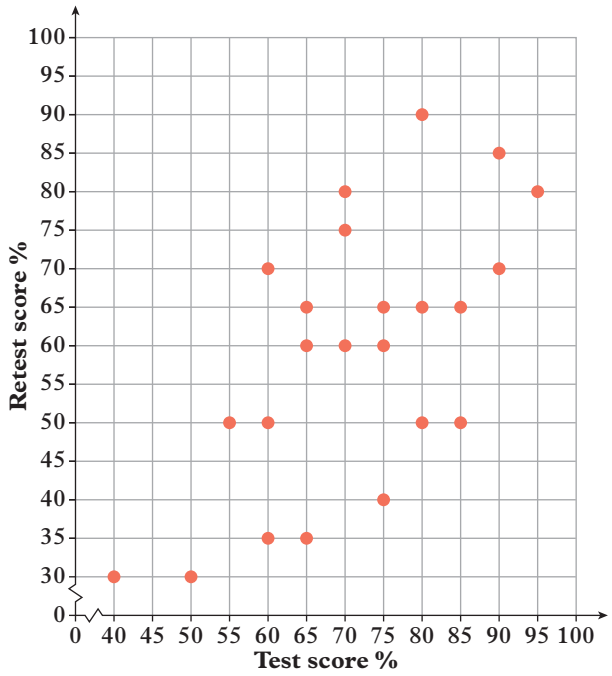
g **i** yes

ii There are two vertices of odd degree in the graph. Yes, there is an Eulerian trail.

iii no

3 **a** test scores

b



c moderate, positive, linear

d $y = 0.81x + 1.44$

e **i** 58%

ii reliable

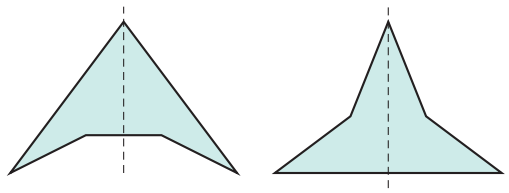
f There are 11 students who scored at or above 75% on the original test. Of these, 8 scored better on the retest than the median of those who scored less than 75% on the original test. The remaining 3 students of the 11 who scored at or above 75% on the original test received scores on the retest that were worse than a majority of those who scored less than 75% on the original test. Therefore, while the teacher's claim is somewhat accurate, the moderate correlation means that there are students whose results do not fit the teacher's claim.

EX EXPLORATIONS 2

p668

1 **a** None, so a thorn triangle is just an isosceles triangle.

b



c Thorn heptagons have 4 reflex angles. Trying to keep them all separated would require at least 4 non-reflex angles.

d Thorn octagons have 5 reflex angles, leaving less than 180° to share among the remaining 3 angles.

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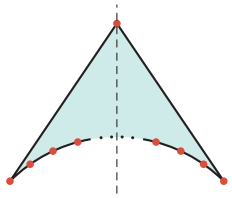
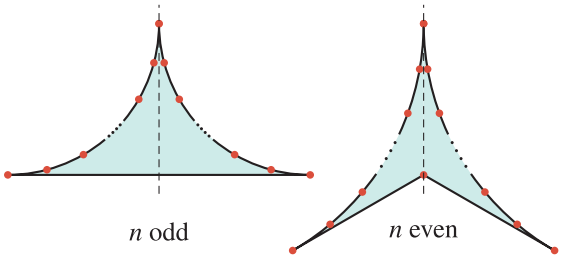
e 1) Place $n - 1$ equally spaced points along a circular arc, as shown. There are other strategies.



2) The angle sum of an n-gon is $180(n - 2)$, which is strictly less than the sum of $n - 2$ reflex angles.

f 9

g Consider equally spaced points along two identical circular arcs. Treat the odd and even cases separately.



2 **a** **i** $\frac{3}{8}$ **ii** $SU = \frac{1}{6}$, $SV = \frac{5}{24}$, $UV = \frac{1}{8}$ **iii** $\frac{3}{8}$

b $PQ = \frac{1}{7}$ or $\frac{3}{4}$

c $PQ = \frac{a}{b + c}$, so for 4 : 3 : 5, $PQ = \frac{1}{3}$

3 **a** **i** 48 cm^3 **ii** 48 cm, 60 cm, 64 cm

iii 88.78 cm^2, 118.10 cm^2, 153.21 cm^2

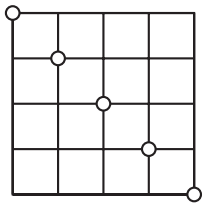
b It is enough to show the triangular faces have areas 15 555 cm^2, 23 925 cm^2 and 28 260 cm^2.

c Show that the perpendicular heights of the triangular faces are the integers c^3, $a(4b^2 + c^2)$ and $b(4a^2 + c^2)$.

4 **a** There are $2^4 = 16$ combinations of horizontal and vertical moves after 4 seconds. Four of these finish at P.

b $\frac{3}{16}$

c **i** The counters can only meet when they are both halfway; that is, after 4 moves:



ii $\frac{35}{128} \approx 0.273$

iii Less likely, with probability $\frac{1120}{6561} \approx 0.171$

5 **a** **i** 9 **ii** week 19

b If there were k kirons, ℓ lirons and m mirons 5 weeks ago, then 1 week ago there were $\ell + m$ kirons and this week there are $k + \ell + m$ kirons. Yes, it is also true for lirons and mirons.

c 8 algae (4 kirons and 4 mirons)

d **i** week 13 **ii** 15 days

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Glossary

AAA
the abbreviation for angle–angle–angle. Triangles are similar if all three angles are equal. The condition AAA does not necessarily mean the triangles are congruent

AAS
the abbreviation for angle–angle–side. Triangles are congruent if there are two pairs of corresponding equal angles and two corresponding sides are the same length

absolute error
the magnitude of the difference between the actual value and an estimated value

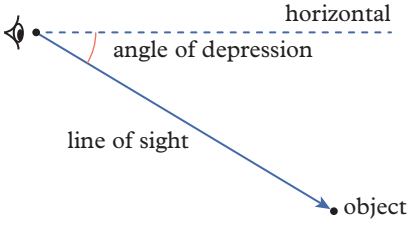
addition principle
the number of ways to choose one element from either a set containing m elements or a set containing n elements is $m + n$

adjacent vertices
the vertices on either side of an edge in a graph/network

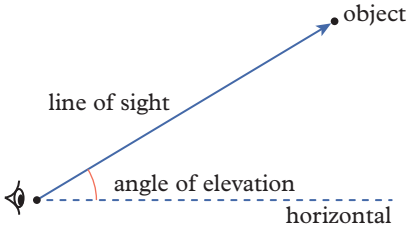
algebraic fractions
fractions that contain at least one pronumeral

amplitude
the vertical distance from the mean value of a sine or cosine graph to its maximum or minimum turning points; the graph of $y = \sin(\theta)$ and the graph of $y = \cos(\theta)$ both have an amplitude of 1

angle of depression
the angle between a horizontal line and an observer's line of sight to an object that is below the horizontal



angle of elevation
the angle between a horizontal line and an observer's line of sight to an object that is above the horizontal



angle–side relationship
this relationship states that in a triangle, the longest side is always opposite the largest angle

arc
part of a circle's circumference, which is also used to mark angles on diagrams

area
the amount of space enclosed by a two-dimensional (2D) shape

array
a collection of objects or numbers arranged in ordered rows and columns

asymptote
an imaginary boundary that a graph approaches but never reaches

average rate of change
the gradient of the line connecting the two endpoints of an interval

axis of symmetry
an imaginary line that divides a symmetrical shape or graph so that one side is a reflection of the other

base
a number that is repeatedly multiplied by itself; for example, in 2^4 the base is 2

base-ten
the traditional number system we use which uses the digits 0 to 9

bias
an inclination towards one person (group of people, things or ideas) over another

bimodal
a distribution is bimodal if it has two clear peaks in the data

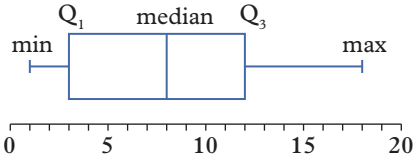
binomial
an algebraic expression for the sum or the difference of two terms

bisection method
a numerical method to solve equations which works by reducing the size of an interval which must contain a solution

bivariate data
data consisting of two variables

bonus
an incentive or reward for good performance, which is given on top of a salary or wage

boxplot
a visual representation of the five-number summary of a numerical data set along a number line. The 'box' represents the middle 50% of ordered data, with vertical lines indicating the first quartile, median and third quartile. The 'whiskers' extend from the left of the box to the minimum value and from the right of the box to the maximum value
see also five-number summary



capacity
the maximum amount (usually of fluid or gas) that a three-dimensional object (container) can hold; usually measured in litres (L) or millilitres (mL)

categorical data
data that can be put into distinct groups or categories

cause-and-effect
a relationship between two variables is a cause-and-effect relationship if the change in one variable causes a change in the other variable

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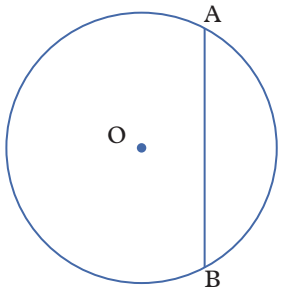
chord
a line segment joining two points on the circumference of a circle. In the diagram below, AB is a chord



circuit
a trail in a graph/network with the same first and last vertices

circumference
the perimeter of a circle

closed walk
a walk in a graph/network with the same first and last vertices

co-domain
the set onto which the elements of the domain are mapped

combination
a selection of the elements in a set where the order of selection does not matter. The number of combinations of size $r \le n$ from a set containing n elements is:

$${}^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

commission
a payment which is a proportion of the total value of sales

compass bearing
direction describing one position from another by using the number of degrees from north or south towards east or west that the second position is from the first; for example, S43°E represents a direction that is 43° east of due south

complement
the complement of an event A is the event that A does not occur. The sum of the probabilities of an event and its complement is 1

complementary angles
angles that add to make a right angle (90°)

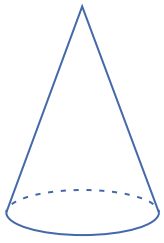
completing the square
the process of changing a quadratic expression from the general form, $ax^2 + bx + c$, to the form $a(x-h)^2 + k$, by applying the 'perfect square' rule

composite shape
a shape made up of more than one simple shape

compound interest
a type of interest whereby the amount of interest earned or charged is based on the present value of the loan or investment

conditional probability
the likelihood of an event occurring based on the outcome of a previous event

cone
a three-dimensional (3D) object with a curved surface tapering from its circular base to an apex



confounding variable
an unmeasured variable that affects both the independent and dependent variables, explaining the relationship between these two variables

congruent
two figures are congruent if they are identical in shape and size, but not necessarily in position or orientation

conjugate
the conjugate of a binomial surd expression is formed by changing the operation sign between the two terms to the opposite operation sign (+ to – and – to +); for example, the conjugate of $(3 + \sqrt{2})$ is $(3 - \sqrt{2})$. When a binomial surd expression is multiplied by its conjugate, the product is a rational number

connected (graph)
a graph where a walk exists between every pair of vertices

constant of proportionality
if the relationship between two quantities x and y is $y = kx$ or $y = \frac{k}{x}$, k is the constant of proportionality

constant term
a term without any pronumeral component that has a fixed value

conversion factor
a multiplier for converting a quantity expressed in one set of units into another quantity expressed in a different set of units; for example, multiply by 10 to convert from cm to mm

convex polyhedron
a polyhedron in which any line segment connecting two vertices on the surface lies completely within the polyhedron

correlation
the description of the strength (weak, moderate, strong) and direction (positive, negative) of the relationship between bivariate data displayed in a scatterplot

cosine (cos)
in trigonometry, the cosine of an angle in a right-angled triangle equals the ratio of the length of the adjacent side to the length of the hypotenuse

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

cosine rule
the relationship between three side lengths (for example, a, b and c) of a triangle ($\triangle ABC$) and one of the triangle's angles (A):

$a^2 = b^2 + c^2 - 2bc\cos(A)$.

Variations of this rule apply to other angles of the triangle:

$b^2 = a^2 + c^2 - 2ac\cos(B)$ and

$c^2 = a^2 + b^2 - 2ab\cos(C)$

coterminal angles
angles that differ by a multiple of 360°

cross-section
the shape of the surface revealed when we cut straight through an object. When the 'slice' is cut parallel to the base of a prism or cylinder, the cross-section has the same shape and size as that base

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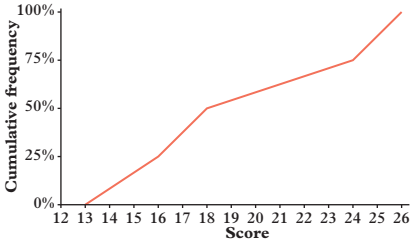
cumulative frequency
for any frequency distribution, the sum of all the frequencies up to the current point. Cumulative frequency can be displayed in a table or on a graph

see also cumulative frequency polygon

cumulative frequency polygon
a graph of a cumulative distribution, displaying data values or class intervals on the horizontal axis and the cumulative frequencies on the vertical axis



cycle
a walk in a graph/network without repeated edges or repeated vertices and which starts and finishes at the same vertex

cylinder
a three-dimensional (3D) object with a uniform circular cross-section parallel to its circular base

degree
1 the highest power of any variable in a polynomial; for example, the polynomial $4x^3 - 2x^2 + 5x + 6$ has degree 3
2 the number of edges connected to a vertex in a graph/network

dependent variable
a variable whose value depends on the value of another variable in a relationship between two variables. The dependent variable is listed second in a table of values and in pairs of coordinates, and is shown on the vertical axis of the Cartesian plane

depreciating asset
an asset with a limited life expectancy that is expected to decline in value over the time it is used and which can be depreciated for a tax deduction

depreciation
the decline in the value of a depreciating asset over time

difference of two squares
when an expression is seen to be the difference of two perfect squares, $a^2 - b^2$, it can be factorised as: $a^2 - b^2 = (a + b)(a - b)$

dilation
a transformation that can enlarge or compress the graph of a function in the x- or y-direction

directed edge
an edge in a graph/network that has a direction indicated by an arrow

directed network
a network with directed edges

directly proportional
two variables are said to be directly proportional if $y = kx$, where k is the constant of variation. An increase in one quantity causes a corresponding increase in the other quantity, or a decrease in one quantity results in a corresponding decrease in the other quantity. Proportionality is denoted using the symbol $\propto$

disconnected (graph)
a graph where there exists a pair of vertices with no valid walk between them

discriminant
a value which is used in the quadratic formula to find the solution of quadratic equations; $\Delta = b^2 - 4ac$

distributive law
the result of first adding several numbers and then multiplying the sum by another number is the same as first multiplying each of the several numbers separately by that number and then adding the products: $a(b + c) = ab + ac$

dividend
the number or expression that is divided or distributed in a division problem

divisor
a number or expression that divides into another number or expression (dividend); for example, in $(4x^3 - 2x^2 + 5x + 6) \div 2x$, the divisor is $2x$

domain
the set from which the mapping occurs in a relation

edge (network)
a relation between two vertices in a graph/network

element
a member of a set; for example, 5 is an element of $\{1, 3, 5, 7, 9\}$

elimination method
the algebraic method used to solve simultaneous equations containing more than one variable by forming a new equation with all but one of the variables removed

equidistant
at equal istances from a point

error
the difference between the exact value and the estimated value of a quantity

ethics
the system of moral principles that distinguishes between right and wrong behaviour

Eulerian
a graph/network is Eulerian if it contains an Eulerian circuit

Eulerian circuit
a circuit containing every edge in a graph/network exactly once

Eulerian trail
a trail containing every edge in a graph/network exactly once

Euler's formula
a formula that states that in a connected planar graph, the sum of the number of faces and vertices is equal to two more than the number of edges; $F + V = E + 2$

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event
a subset of the sample space for a random experiment. It is one of the possible outcomes or sets of outcomes for the experiment; for example, for the experiment of rolling a die, a single-outcome event might be 'rolling a 2', and an event that includes a set of outcomes might be 'rolling a 2 or a 4'

expand
to distribute products over addition and subtraction, removing brackets in the process

see also distributive law

expansion of a perfect square
a pattern observable when expanding a factorised expression in which a factor is multiplied by itself, represented as:

$(a + b)^2 = (a + b)(a + b)$
$= a^2 + 2ab + b^2$

or: $(a - b)^2 = (a - b)(a - b)$
$= a^2 - 2ab + b^2$

experiment
a repeatable procedure with a clearly defined set of possible results

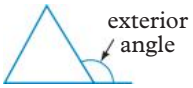
exponent
for a value expressed in exponent form, the exponent indicates the number of times the base is multiplied by itself as a repeated multiplication; for example, in 2^4 the exponent is 4 and $2^4 = 2 \times 2 \times 2 \times 2$

exponent form
a shorter form of writing a repeated multiplication, in which a number is written with a base and an exponent; for example, 2^4 is written in exponent form and means $2 \times 2 \times 2 \times 2$

exponent laws
the set of laws that can be used to simplify calculations involving numbers in exponent form

exponential relationship
a relationship between two variables, one of which is an index; for example, $y = 2^x$

exterior angle
an angle between any side of a shape and a line extended from the adjacent side



extrapolation
a prediction made outside the original range of the data

faces
the regions of a planar graph that are bordered by edges

factor theorem
the polynomial $P(x)$ has a factor of $(x - a)$ if $P(a) = 0$

factorise
the opposite process to expanding; to express an expression as a product of its factors

five-number summary
five pieces of information related to a set of numerical data: minimum value, lower quartile, median, upper quartile and maximum value

flat-rate depreciation
a type of depreciation whereby the depreciating asset is depreciated by a fixed amount each year

function
a function, f, maps each element of one set, S, to precisely one element of a second set, T

function notation
a way of expressing a relationship between two variables in the form $f(x) = [\text{rule}]$; for example, $f(x) = x^2$ is the function which squares a number represented by the variable x

general form
a quadratic relationship written in the form $y = ax^2 + bx + c$

gradient (m)
the measure of the steepness of a straight line; or the numerical measure of the slope of a graph

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

gradient–intercept form
a form of the equation of a line, $y = mx + c$, where m is the gradient and c is the y-value of the y-intercept of the corresponding linear graph

graph
see network

handshaking lemma
proposition that states that in every graph/network, the sum of vertex degrees is twice the total number of edges

histogram
a graphical display of data using bars or columns of different heights, similar to a bar chart, but showing the frequency of numerical data grouped into ranges or class intervals, with the horizontal scale boundaries located at the edges of the columns, rather than at their centres

hypotenuse
the longest side in a right-angled triangle, located opposite the right angle

included angle area formula
the area of a triangle with two known side lengths b and c and an included angle A can be found using the formula:

$\text{Area} = \frac{1}{2} bc \sin(A)$

income
money received over a set period of time in exchange for work or investment

income tax
the tax which is paid on an individual's taxable income, calculated by using a tax table or online tax calculator

independent events
two events are considered independent when the outcome of one event does not affect the probability of the other event occurring

independent variable
the quantity in a relationship between variables which does not rely on any other variable for its value and is not affected by changes in another variable. The independent variable is the value listed first in a table of values and in pairs of coordinates, and the value shown along the horizontal axis on a Cartesian plane

index (plural: indices)
see exponent

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instantaneous rate of change
the gradient at a given point

intercept form
a quadratic relationship written in the form $y = a(x - x_1)(x - x_2)$

interest
a fee paid for an amount borrowed or invested, usually calculated as a percentage of the amount that has been borrowed or invested

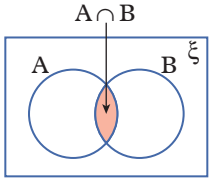
interest rate
the rate, usually expressed as a percentage, at which interest is charged for a loan or paid for an investment

interior angle
an angle inside a plane shape

interpolation
a prediction made within the original range of the data

interquartile range
a measure of spread for a set of data, calculated by subtracting the lower quartile (Q_1) from the upper quartile (Q_3); that is, IQR = $Q_3 - Q_1$. The interquartile range indicates the spread or range of the middle 50% of the data

intersection
the intersection of two sets ($A \cap B$) is the set of elements that appear in both set A and set B



interval
the set of two real numbers lying between two endpoints, which may or may not be included; for example, $0 < x \leq 2$ has lower bound 0 and upper bound 2

inversely proportional
if two variables are inversely proportional, then as one variable increases the other variable decreases at a proportional rate. The product of the two variables is constant

investment
money deposited into a bank, financial institution or other business for which interest is paid to the person depositing the money (the investor)

irrational number
a number that cannot be written in fraction form; irrational numbers are non-terminating decimals that are not recurring

leading coefficient
the coefficient of the leading term in a polynomial; for example, the leading coefficient in the polynomial $4x^3 - 2x^2 + 5x + 6$ is 4

see also leading term

leading term
the term with the highest power in a polynomial; for example, the leading term in the polynomial $4x^3 - 2x^2 + 5x + 6$ is $4x^3$

least-squares regression line
a mathematically calculated line of best fit that minimises the sum of the squares of the errors between the line of best fit and the points on a scatterplot

leave loading
an extra payment on top of annual leave that is usually calculated as 17.5% of normal pay for this period

like surds
surd terms that have the same number or pronumeral/s under the root sign when in simplest form; for example, $2\sqrt{5}$, $-\sqrt{5}$ and $7\sqrt{5}$ are like surds

like terms
terms that contain the same pronumeral(s) raised to the same power(s)

Likert scale
A rating scale which is commonly used in surveys, often consisting of five possible responses

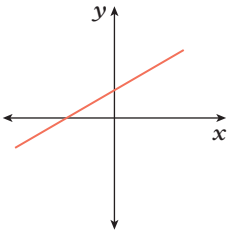
line graph
a graph consisting of line segments that join together to represent the relationship between two variables, with time usually represented on the horizontal axis

line of best fit
a line that best represents the relationship between the two variables on a scatterplot. Lines of best fit are usually drawn as straight lines

line of good fit
a line that represents the relationship between the two variables on a scatterplot

linear equation
an equation that contains only a pronumeral term (or terms) for which the highest power is one; for example, $2x + 3 = 6$ is a linear equation

linear graph
the graph of a linear relationship. The graph is a straight line



linear inequality
an inequality containing pronumerals raised only to the power of 1 with no two pronumerals multiplied together; for example, $2x + 3 > 1$

linear relationship
a relationship between two variables that produces a linear graph

loan
an amount of money lent by a bank or other institution to a person for a limited time (the term of the loan) for which the recipient (the borrower) must pay interest to the lender

logarithm (log)
the power (exponent) to which a given number (the base) must be raised in order to produce the required number; for example, in $2^3 = 8$, the logarithm is 3 and can be written as $\log_2(8) = 3$

logarithm of a power
the logarithm of a term that is expressed as a power. The related logarithm law is: $\log_a(x^n) = n \times \log_a(x)$

logarithm of a product
the logarithm of the product of two terms. The related logarithm law is: $\log_a(xy) = \log_a(x) + \log_a(y)$

logarithm of a quotient
the logarithm of the quotient of two terms. The related logarithm law is: $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$

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logarithmic scale
a non-linear scale where the position of a value is marked using the logarithm of the value instead of the actual value

loop
an edge in a graph/network with the same vertex at both ends

lower fence
the lower end point in an interval beyond which any lower data points are considered outliers. The lower fence lies 1.5 × IQR below the lower quartile
see also interquartile range (IQR)

lower quartile
the median of the first half (or lower half) of ordered numerical data, denoted by Q_1

maximum turning point
a point at which a parabola changes direction and at which it has its maximum y-value

mean
a measure of the centre of a data set, calculated by adding all of the data values together and dividing by the number of values

median
a measure of the centre of a data set (the 'middle value'), determined by placing the data in ascending order (smallest to largest) and finding the middle number of the ordered set if there is an odd number of values, and the average of the two middle numbers if there is an even number of values

Medicare levy
a 2% tax on an individual's taxable income that helps to fund the health care system

minimum turning point
a point at which a parabola changes direction and at which it has its minimum y-value

modality
the number of peaks in a numerical data set

monic
when the leading coefficient of an expression or equation is equal to one
see also leading coefficient

multimodal
a distribution is multimodal if it has more than two clear peaks in the data

multiple edges
two (or more) edges in a graph/network with the same endpoints

multiplication principle
the number of ways to choose one element from a set containing m elements and one element from a set containing n elements is $m \times n$

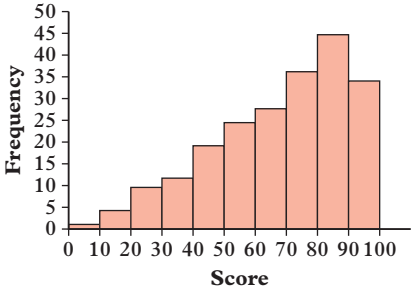
multiplicity
the number of times a factor is repeated in the factorisation of a polynomial into linear factors

mutually exclusive events
two or more events that cannot occur at the same time

negative exponent
an exponent with a negative value, indicating repeated division

negative relationship
(between bivariate data) a relationship between two sets of numerical variables such that, as one variable increases in value, the other variable decreases in value

negatively-skewed distribution
a distribution that is skewed away from what would be a symmetrical distribution, having a tail on the lower end of the distribution (which is the left of the distribution when graphed). The mean of this type of distribution is less than the median



net
a 2D plan that can be folded to form a 3D object

network
a collection of objects and the relations between them

non-linear relationship
a relationship between two variables where a change in one variable does not produce a uniform change in the other variable

non-monic
when the leading coefficient of an expression or equation is not equal to one
see also leading coefficient

non-planar graph
a graph/network that cannot be drawn in two dimensions without some edges crossing

Null Factor Law
the law stating that, if the product of two factors is 0, one or both of the factors must be equal to 0

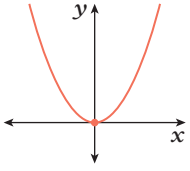
oblique pyramid
a pyramid with its apex (highest point) not directly above the centre of its base
see also pyramid

order of magnitude
the power of ten of a number in base-ten

outcome
a possible result of a probability experiment

outlier
extreme piece of data that is much higher or lower than the rest of the data in a numerical data set

parabola
the approximately U-shaped graph of a quadratic relationship. A parabola can open up, down, left, right, or in any other direction, but is always symmetrical



parallel boxplots
two or more boxplots placed directly above each other on the same number line for comparison

parallel lines
lines that are always the same distance apart and that never meet. Parallel lines are labelled with matching arrowheads

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path
a walk in a graph/network without repeated edges or repeated vertices

penalty rates
higher rates of pay that are given to employees for working particular hours or days, such as weekends, public holidays or overtime

percentage error
the absolute error as a percentage of the exact value

period
the horizontal distance covered by one full cycle of a periodic graph

periodic function
a function or relationship that repeats itself continuously in cycles;
for example, the sine, cosine and tangent functions are periodic functions

permutation
an ordering of some or all of the elements in a set. There are $n!$ permutations of a set containing n elements. The number of permutations of $r \leq n$ elements in a set containing n elements is:
$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

perpendicular lines
line that meet to form a right angle (90°). Perpendicular lines are labelled using a small box at the right angle

piece work
income which is paid for each item (or group of items) produced or action performed

planar graph
a graph/network that can be drawn without crossing edges

Platonic solid
a convex, three-dimensional (3D) shape where each face is an identical regular polygon

point of inflection
point where a curve changes between being 'concave up' and 'concave down'

point of intersection
the point on a Cartesian plane where two graphs meet

polygon
a two-dimensional (2D) shape with straight edges

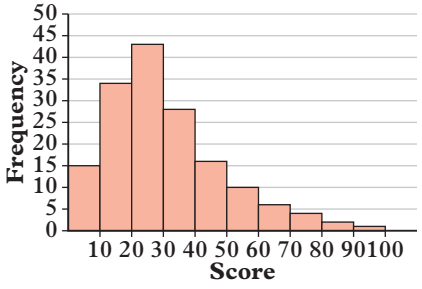
polyhedron
a three-dimensional (3D) object with flat polygonal faces, straight edges and vertices

polynomial
an expression with terms containing one variable only. Each term in a polynomial contains the variable raised to a power that is a positive integer or zero; for example, $4x^3 - 2x^2 + 5x + 6$ is a polynomial ($4x^3 - 2x^2 + 5x + 6x^0$)

population
the complete set of individuals or things about which statistical information is being collected

positive relationship
(between bivariate data) a relationship between two sets of numerical variables such that, as one variable increases in value, the other variable also increases in value

positively-skewed distribution
a distribution that is skewed away from what would be a symmetrical distribution, having a tail on the upper end of the distribution (which is on the right of the distribution when graphed). The mean of this type of distribution will be greater than the median

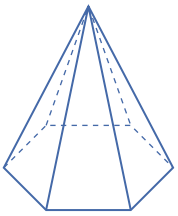


principal
a sum of money invested, or an amount of money borrowed as a loan

prism
object with two ends that are identical polygons and joined by straight edges
see also polygon

proof
a series of logical statements showing that a statement is true in all cases

pyramid
a three-dimensional (3D) object whose base is a polygon that is joined to triangular faces, which all meet at a single point (the apex)



Pythagoras' theorem
the theorem stating that the square of the hypotenuse of a right-angled triangle is equal to the sum of the squares of the two other sides:
$c^2 = a^2 + b^2$, where c is the length of the hypotenuse and a and b are lengths of the other two sides

quadrant
one of the four equal regions of a Cartesian plane indicated by the x- and y-axes and numbered anti-clockwise from the positive direction of the x-axis

quadratic
an equation (or expression) that contains a pronumeral term or terms with a highest power of 2; the general form of a quadratic equation is
$ax^2 + bx + c = 0$, for constants a, b and c with $a \neq 0$

quadratic formula
the formula used to solve equations of the form $ax^2 + bx + c = 0$, where a, b and c are constants and $a \neq 0$. Solutions are found using $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

quadrilateral
a polygon with four sides

quartiles
the values that divide a numerical data set into four quarters

quotient
the result of dividing one number or algebraic expression by another

radian
a unit for measurement of angles, where there are 2π radians in a revolution

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random sampling method
a sampling method whereby every data point has an equal chance of being selected and where bias in the selection of data is avoided

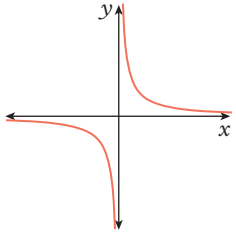
range
1 the subset of the co-domain to which elements of the domain are actually matched to
2 a measure of spread determined by finding the difference between the highest value and lowest value in a numerical data set

rational number
a number that can be written as a fraction with both the numerator and denominator being integers (whole numbers). The set of rational numbers includes whole numbers, fractions, terminating decimals and recurring decimals

real number
a number that can be represented by a point on a number line. Real numbers consist of all rational and irrational numbers

reciprocal
a value which, when multiplied by the given value, equals 1; for example, the reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$

rectangular hyperbola
an open curved graph line consisting of two branches whose ends continue to move apart from each other and begin to approximate, but never reach, two lines known as asymptotes; for example, the graph of $y = \frac{1}{x}$ is a hyperbola (see below)



reducing-balance depreciation
a type of depreciation whereby the depreciating asset is depreciated by a constant percentage each year

reference angle
the acute angle between the x-axis and the line connecting the origin to a point on the unit circle

reflection
a transformation in which a shape or object is reflected in a mirror line to produce its exact mirror image (sometimes referred to as a 'flip')

regular polygon
a polygon with all sides equal in length and all interior angles equal in size

regular tetrahedron
a tetrahedron with four faces that are congruent equilateral triangles
see also tetrahedron

relation
a mapping between the elements of two sets

relative error
the ratio between the absolute error and the exact value of a quantity

remainder
the part 'left over' when dividing a number or expression into equal groups

remainder theorem
if the polynomial $P(x)$ is divided by the linear expression $(x - a)$, the remainder will be $P(a)$

RHS
1 abbreviation for right angle–hypotenuse–side. Right-angled triangles are congruent if their hypotenuses are the same length and another pair of side lengths are equal
2 abbreviation for right angle–hypotenuse–side. Right-angled triangles are similar if their hypotenuses are in the same ratio as another pair of sides

right cone
a cone with its apex (highest point) directly above the centre of the base
see also cone

right pyramid
a pyramid with its apex (highest point) directly above the centre of the base
see also pyramid

root
a solution to a polynomial equation

royalty
a payment made to a person or company for the use of their assets

salary
a fixed income which is paid to an employee on a regular basis

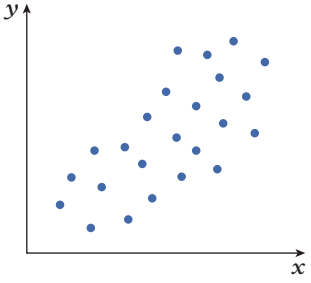
salvage value
the value of a depreciating asset after it has been depreciated

sample
a subset of a population. In statistical investigations a selection from a population is sometimes used to represent the whole population

sample space
the list of all the different outcomes possible for a probability experiment; for example, the sample space when rolling a die is $\{1, 2, 3, 4, 5, 6\}$

SAS
1 abbreviation for side–angle–side. Triangles are congruent if two corresponding side lengths are equal and the angle between them is equal
2 abbreviation for side–angle–side. Triangles are similar if two corresponding side lengths are in the same ratio and angle between them is equal

scatterplot
a graph that displays a relationship between two variables in which the points are plotted but not joined



scientific notation
a value written in the form $a \times 10^m$, where $1 \le a < 10$ or $-10 < a \le -1$ and m is an integer

secant
1 a line that passes through at least two points of a non-linear graph
2 a straight line that cuts a circle twice

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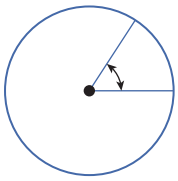
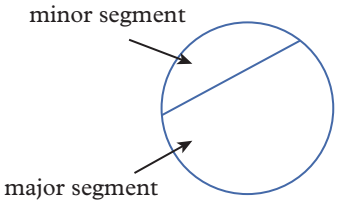
sector
a portion of a circle formed by two radii and part of the circumference



segment
the region enclosed in a circle by a chord and an arc
see also chord
see also arc



semi-Eulerian
a graph is semi-Eulerian if it contains an Eulerian trail

set
a collection of distinct objects or elements
see also element

similar
two figures or objects are similar if they have exactly the same shape. Similar shapes do not need to be the same size

simple graph
a graph/network without loops or parallel edges

simple interest
a type of interest whereby the amount of interest earned or charged is based on the principal of the investment or loan

simultaneous linear equations
equations involving two or more unknowns that have the same values in those equations

sine (sin)
in trigonometry, the sine of an angle in a right-angled triangle equals the ratio of the length of the opposite side to the length of the hypotenuse

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

sine rule
the relationship between the side lengths (for example, a, b and c) and angles (A, B and C) of a triangle ($\triangle ABC$):

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

skewed distribution
a data distribution with a cluster of data points on one side of the distribution and a tail on the other side of the distribution

sphere
a three-dimensional (3D) object shaped like a ball where every point on its surface is equidistant from its centre

spread
a statistic describing how widely the data values in a set are distributed. Measures of spread include the range and interquartile range

SSS
1 abbreviation for side–side–side. Triangles are congruent if all three side lengths are equal
2 abbreviation for side–side–side. Triangles are similar if all three side lengths are in the same ratio

standard deviation
a measure of spread representing the difference between individual data values and the mean of a numerical data set

subject
the value that appears by itself on one side of an equation

subset
a set contained within another set

substitution
replacement of a pronumeral with a number

substitution method
the algebraic method used to solve simultaneous equations containing more than one variable, by rearranging to make one of the variables the subject of an equation equivalent to the first equation and substituting the value of that variable from that equation into the second equation

supplementary angles
angles that add to make a straight angle (180°)

surd
an irrational root of an integer that cannot be simplified to remove the root; for example, $\sqrt{2}$

surface area (SA)
the area of the surface of an object. The surface area of a prism is the sum of the areas of all the faces of that prism

symmetric distribution
a distribution with similar frequencies on either side of a central peak

tangent
1 the tangent (tan) of an angle in a right-angled triangle equals the ratio of the length of the opposite side to the length of the adjacent side

$$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

2 a straight line that touches but does not cross a curve
3 a line that touches the circumference of a circle at only one point

tax deductions
expenses that can be subtracted from a person's total income such as work-related expenses or gifts and donations

taxable income
an individual's total income minus any tax deductions, which is used to calculate the amount of income tax payable

tetrahedron
a pyramid with a triangular base
see also pyramid

time period
the regular interval of time at which interest is calculated and added to the principal

time series
a sequence of numerical data taken at regular time intervals

trail
a walk in a graph/network without repeated edges

transformations
the general name for translations, reflections, rotations and dilations

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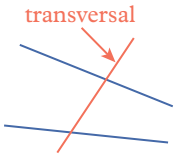
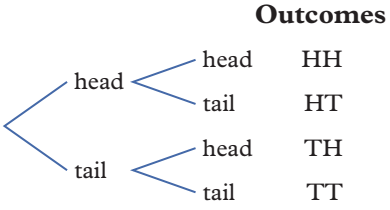
translation
a transformation in which a shape, graph or object is moved in a straight line without turning or changing size. The movement is often described as a number of units up, down, left or right

transversal
a line that crosses (intersects) two or more lines



tree diagram
a display of outcomes for a multi-step experiment; for example, a tree diagram can be used to display the possible outcomes of flipping two coins (as shown below)



triangle
a polygon with three sides

triangle inequality
the sum of the lengths of two sides of a triangle is greater than the length of the third side

trigonometry
the study of the relationships between the angles and side lengths of a triangle

trinomial
an algebraic expression of the form $ax^2 + bx + c$, where a, b and c are constants and $a \neq 0$.

true bearing
the direction from one position to another, described by a number of degrees from north in a clockwise direction. The angle of a true bearing is written as three digits followed by T (or true); for example, 090°T is due east, 180°T is due south, and 270°T is due west

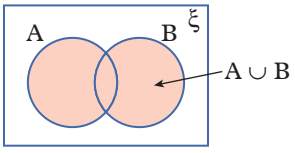
turning point
the point on a graph where a parabola changes direction

turning point form
the general form of a quadratic equation, $y = a(x - h)^2 + k$, where a is the dilation factor, and the coordinates of the turning point are (h, k)

two-way table
a display of the outcomes of a survey or experiment that shows the relationship between two attributes

unimodal
a distribution is unimodal if it has one clear peak in the data

union
the union of two sets ($A \cup B$) is the set of elements that appear in A or B or both



unit circle
a circle drawn on the Cartesian plane with its centre at the origin and a radius of 1 unit. The unit circle can be used to define trigonometric functions

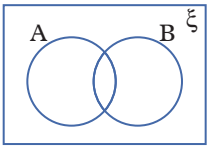
universal set
the set containing all elements. Everything inside the rectangle of a Venn diagram belongs to the universal set. The symbol for the universal set is ξ

upper fence
the upper end point in an interval beyond which any greater data points are considered outliers. The upper fence lies 1.5 × IQR above the upper quartile

see also interquartile range (IQR)

upper quartile
the median of the upper half of an ordered set of numerical data, denoted by Q_3

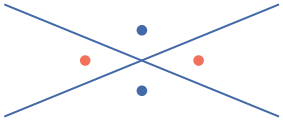
Venn diagram
a display of the relationship between different sets of data. A rectangle represents the universal set, ξ, and each set of data within the universal set is represented by a circle



vertex
1 an alternative name for a turning point; *see also* turning point
2 in a graph or a network, an object represented by a point and labelled with a capital letter

vertical line test
a test that determines whether the graph of a relation is also a function. If a vertical line intersects or touches a graph at more than one point, then the graph is not a function

vertically opposite angles
a pair of equal angles that are not adjacent to (next to) each other, formed where two lines intersect. They are sometimes called 'X angles'. There are two pairs of vertically opposite angles in the diagram below



volume
the quantity of space that a three-dimensional (3D) object occupies

wage
a fixed rate of income an employee receives for working a given time period; for example, $25 per hour

walk
a sequence of vertices and the edges between them in a graph/network

zero
a value, a, such that, when substituted into an expression, the result evaluates to zero

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Graphs can take many different forms. For example, the graphs of some trigonometric functions form repeating, wave-like patterns like those shown here.

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