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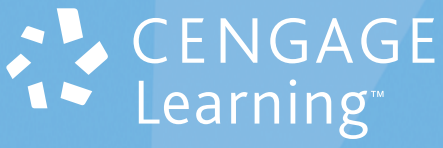
VICmaths

VCE UNITS 1 + 2



specialist mathematics 11

Greg Neal
Sue Garner
George Dimitriadis
Stephen Swift



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— **Sarah,** Secondary School Teacher

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VICmaths

VCE UNITS 1 + 2



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Sue Garner
George Dimitriadis
Stephen Swift
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specialist mathematics 11

Nelson

Nelson VICmaths Specialist Mathematics 11
1st Edition
Greg Neal
Sue Garner
George Dimitriadis
Stephen Swift
ISBN 9780170448338

Publishers: Robert Yen, Dirk Strasser
Project editor: Tanya Smith
Editors: Karen Chin, Geoffrey Brent
Series cover design: Watershed Art & Design (Leigh Ashforth)
Series text design: Alba Design (Rina Gargano)
Series designer: Nikita Bansal
Permissions researcher: Liz McShane
Production controller: Karen Young
Typeset by: Nikki M Group Pty Ltd

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National Library of Australia Cataloguing-in-Publication Data
A catalogue record for this book is available from the National Library of Australia.

Cengage Learning Australia
Level 5, 80 Dorcas Street
Southbank VIC 3006 Australia

Cengage Learning New Zealand
Unit 4B Rosedale Office Park
331 Rosedale Road, Albany, North Shore 0632, NZ

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Printed in China by 1010 Printing International Limited.
1 2 3 4 5 6 7 26 25 24 23 22

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To the student

Nelson VICmaths is your best friend when it comes to studying Specialist Mathematics in Year 11. It has been written to help you maximise your learning and success this year. Every explanation, every exam hack and every worked example has been written with the exams in mind.



To the teacher

Now there's a better way to VCE maths mastery.

Nelson VICmaths 11–12 is a new VCE mathematics series that is backed by research into the science of learning. The design and structure of the series have been informed by teacher advice and evidence-based pedagogy, with the focus on preparing VCE students for their exams and maximising their learning achievement.

- Using **backwards learning design**, this series has been built by analysing past VCE exam questions and ensuring that all theory and examples are precisely mapped to the VCE Study Design.
- To reduce the **cognitive load** for learners, explanations are clear and concise, using the technique of **chunking** text and including diagrams and infographics.
- The student book and workbook combination has been designed for **mastery** of the learning content.
- The Exercise structure of **Recap, Mastery** and **Exam Practice** leads students from procedural fluency to **higher-order thinking** using the learning technique of **interleaving**.
- **Exam practice** includes exam-style questions and graded past VCE exam questions with success percentages based on **VCAA performance data**.
- The cumulative structure of Exercise **Recaps** and chapter-based **Cumulative Examinations** is built on the learning and memory techniques of **spacing** and **retrieval**.

About the authors

Greg Neal has taught for over 40 years and was Head of Mathematics at Ballarat High School. He has been a VCAA examination assessor for Mathematical Methods for over 20 years and has expertise with CAS technology. Greg has written a range of student resources for Further Mathematics, Mathematical Methods and Specialist Mathematics.

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Neale Woods coordinated the *Using CAS* content. He is a CAS specialist who often presents at conferences and conducts technology workshops for teachers and students. Neale has been teaching mathematics for over 45 years, including for Virtual Schools Victoria.

Darren Smyth (maffsguru.com) created the VCE question analysis videos.

Study Design grid

Area of study	*Nelson VICmaths Specialist Mathematics 11* chapter	
UNIT 1		
1 Algebra, number and structure		
Proof and number	2	Proof and number
Graph theory	3	Graph theory
Logic and algorithms	4	Logic and algorithms
2 Discrete mathematics		
Sequences and series	1	Sequences and series
Combinatorics	6	Counting techniques
Matrices	5	Matrices
UNIT 2		
1 Data analysis, probability and statistics		
Simulation, sampling and sampling distributions	11	Sampling distributions
2 Space and measurement		
Trigonometry	7	Trigonometric identities
Transformations	12	Transformations and matrices
Vectors in the plane	10	Vectors
3 Algebra, number and structure		
Complex numbers	9	Complex numbers
4 Functions, relations and graphs		
	8	Graphing functions and relations

About this book

In each chapter

Study Design coverage and extracts are shown at the start of the chapter, along with a listing of **Nelson MindTap chapter resources**.

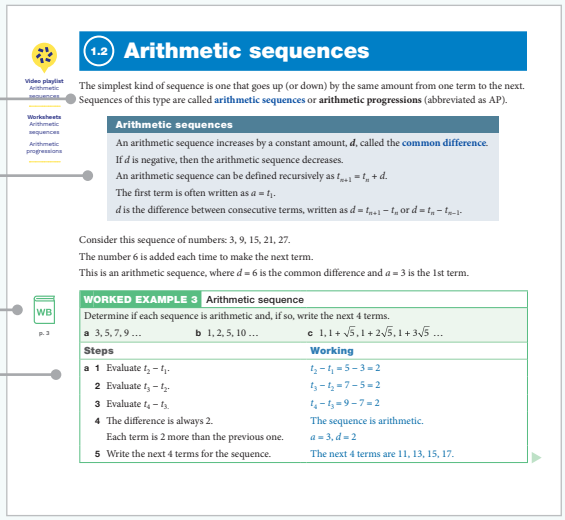


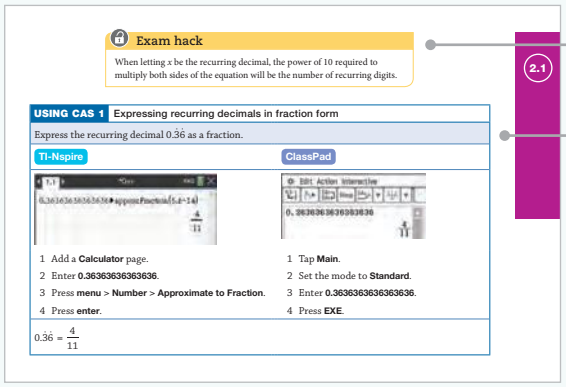
Important words and phrases are printed in blue and listed in the **Glossary and index** at the back of the book.

Important facts and formulas are highlighted in a shaded box.

Links to scaffolded Matched examples in the Mastery Workbook (WB).

Worked examples are explained clearly step-by-step, with the mathematical working shown on the right-hand side.



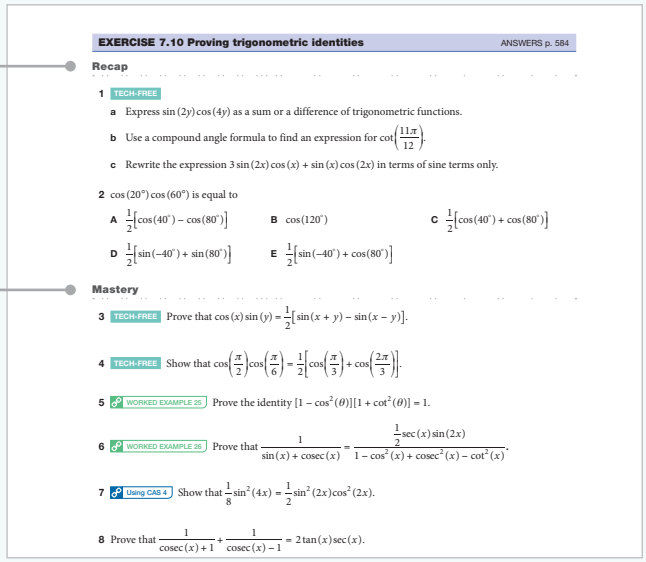


Exam hacks highlight valuable exam hints and common student errors.

Using CAS provides clear instructions for TI-Nspire and Casio ClassPad calculators.

Graded exercises, including **Recap**, **Mastery** and **Exam practice** questions, are linked to worked examples and Using CAS, and include past VCE exam and exam-style questions.

Recap questions revise skills from the previous exercise, **Mastery** questions provide skill practice, while **Exam practice** applies learned skills to VCE exam and exam-style problems.



EXERCISE 7.10 Proving trigonometric identities ANSWERS p. 584

Recap

1 TECH-FREE

a Express $\sin(2y)\cos(4y)$ as a sum or a difference of trigonometric functions.

b Use a compound angle formula to find an expression for $\cot\left(\frac{11\pi}{12}\right)$.

c Rewrite the expression $3\sin(2x)\cos(x) + \sin(x)\cos(2x)$ in terms of sine terms only.

2 $\cos(20°)\cos(60°)$ is equal to

A $\frac{1}{2}\left[\cos(40°) - \cos(80°)\right]$ B $\cos(120°)$ C $\frac{1}{2}\left[\cos(40°) + \cos(80°)\right]$

D $\frac{1}{2}\left[\sin(-40°) + \sin(80°)\right]$ E $\frac{1}{2}\left[\sin(-40°) + \cos(80°)\right]$

Mastery

3 TECH-FREE Prove that $\cos(x)\sin(y) = \frac{1}{2}[\sin(x+y) - \sin(x-y)]$.

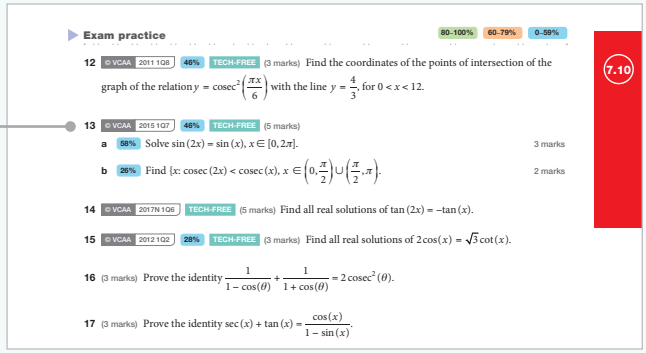
4 TECH-FREE Show that $\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{6}\right) = \frac{1}{2}\left[\cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{2\pi}{3}\right)\right]$.

5 WORKED EXAMPLE 25 Prove the identity $[1 - \cos^2(\theta)][1 + \cot^2(\theta)] = 1$.

6 WORKED EXAMPLE 26 Prove that $\frac{1}{\sin(x) + \operatorname{cosec}(x)} = \frac{\frac{1}{2}\sec(x)\sin(2x)}{1 - \cos^2(x) + \operatorname{cosec}^2(x) - \cot^2(x)}$.

7 Using CAS 4 Show that $\frac{1}{8}\sin^2(4x) = \frac{1}{2}\sin^2(2x)\cos^2(2x)$.

8 Prove that $\frac{1}{\operatorname{cosec}(x) + 1} + \frac{1}{\operatorname{cosec}(x) - 1} = 2\tan(x)\sec(x)$.



Exam practice 80–100% 60–79% 0–59%

7.10

12 © VCAA 2011 1Q8 46% TECH-FREE (3 marks) Find the coordinates of the points of intersection of the graph of the relation $y = \operatorname{cosec}^2\left(\frac{\pi x}{6}\right)$ with the line $y = \frac{4}{3}$, for $0 < x < 12$.

13 © VCAA 2015 1Q7 46% TECH-FREE (5 marks)

a 58% Solve $\sin(2x) = \sin(x)$, $x \in [0, 2\pi]$. 3 marks

b 26% Find $\{x: \operatorname{cosec}(2x) < \operatorname{cosec}(x), x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)\}$. 2 marks

14 © VCAA 2017N 1Q6 TECH-FREE (5 marks) Find all real solutions of $\tan(2x) = -\tan(x)$.

15 © VCAA 2012 1Q2 28% TECH-FREE (3 marks) Find all real solutions of $2\cos(x) = \sqrt{3}\cot(x)$.

16 (3 marks) Prove the identity $\frac{1}{1 - \cos(\theta)} + \frac{1}{1 + \cos(\theta)} = 2\operatorname{cosec}^2(\theta)$.

17 (3 marks) Prove the identity $\sec(x) + \tan(x) = \frac{\cos(x)}{1 - \sin(x)}$.

© VCAA 2015 1Q7 **46%**

Past VCE exam questions are clearly tagged and graded by colour-coded success percentages based on VCAA data presented in the order: Exam 1, Exam 2A, Exam 2B.

KEY

2021	2021 exam year
<year>N	Northern hemisphere exam
<year>S	VCAA sample exam
TECH-FREE	Technology not permitted
1	Exam 1 (tech-free)
2A	Exam 2 Section A (multiple-choice)
2B	Exam 2 Section B

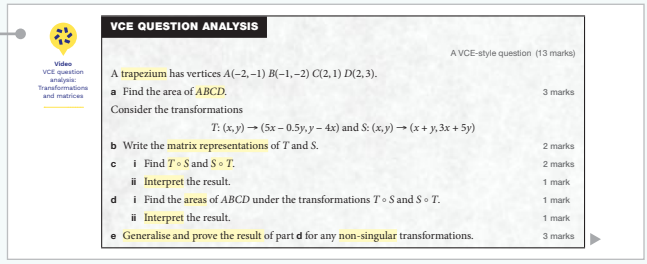
For multiple-choice questions and other 1-mark questions, the success percentage is the percentage of students who answered correctly. For questions worth more than 1 mark, the success percentage is the mean percentage result scored by students.

Success percentage range	Example
80–100% (straightforward)	83%
60–79% (standard)	69%
0–59% (complex)	45%

Note: Questions from Northern hemisphere and VCAA sample exams do not have success percentages.

At the end of each chapter

VCE question analysis leads students through a VCE-style exam question that exemplifies the chapter, discussing how to approach the question, providing advice on interpreting the question and common student errors.



Video VCE question analysis: Transformations and matrices

VCE QUESTION ANALYSIS

A VCE-style question (13 marks)

A trapezium has vertices $A(-2, -1)$ $B(-1, -2)$ $C(2, 1)$ $D(2, 3)$.

a Find the area of $ABCD$. 3 marks

Consider the transformations

$T: (x, y) \to (5x - 0.5y, y - 4x)$ and $S: (x, y) \to (x + y, 3x + 5y)$

b Write the matrix representations of T and S. 2 marks

c i Find $T \circ S$ and $S \circ T$. 2 marks

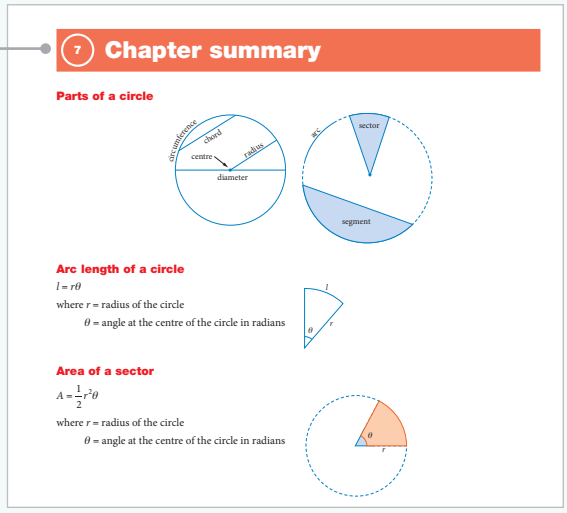
ii Interpret the result. 1 mark

d i Find the areas of $ABCD$ under the transformations $T \circ S$ and $S \circ T$. 1 mark

ii Interpret the result. 1 mark

e Generalise and prove the result of part d for any non-singular transformations. 3 marks

Chapter summary for easy reference.



7 Chapter summary

Parts of a circle

Arc length of a circle

$l = r\theta$

where r = radius of the circle

θ = angle at the centre of the circle in radians

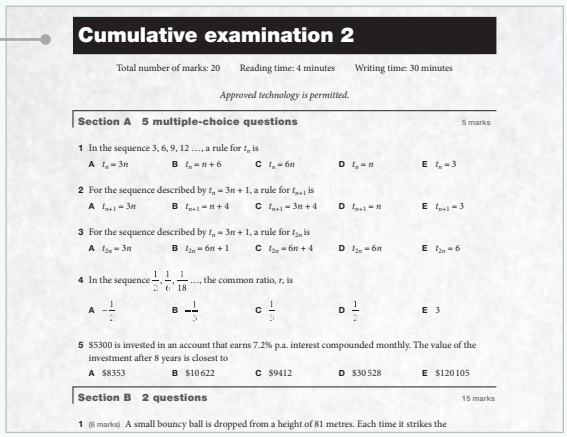
Area of a sector

$A = \frac{1}{2}r^2\theta$

where r = radius of the circle

θ = angle at the centre of the circle in radians

Cumulative examinations 1 and **2** are mini-exams based on the format of the VCE examinations 1 and 2, with around 50% of questions focusing on the chapter in which they appear.



Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions 5 marks

1 In the sequence 3, 6, 9, 12 …, a rule for t_n is

A $t_n = 3n$ B $t_n = n + 6$ C $t_n = 6n$ D $t_n = n$ E $t_n = 3$

2 For the sequence described by $t_n = 3n + 1$, a rule for t_{n+1} is

A $t_{n+1} = 3n$ B $t_{n+1} = n + 4$ C $t_{n+1} = 3n + 4$ D $t_{n+1} = n$ E $t_{n+1} = 3$

3 For the sequence described by $t_n = 3n + 1$, a rule for t_{2n} is

A $t_{2n} = 3n$ B $t_{2n} = 6n + 1$ C $t_{2n} = 6n + 4$ D $t_{2n} = 6n$ E $t_{2n} = 6$

4 In the sequence $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}$ …, the common ratio, r, is

A $-\frac{1}{2}$ B $\frac{1}{3}$ C $\frac{1}{3}$ D $\frac{1}{2}$ E 3

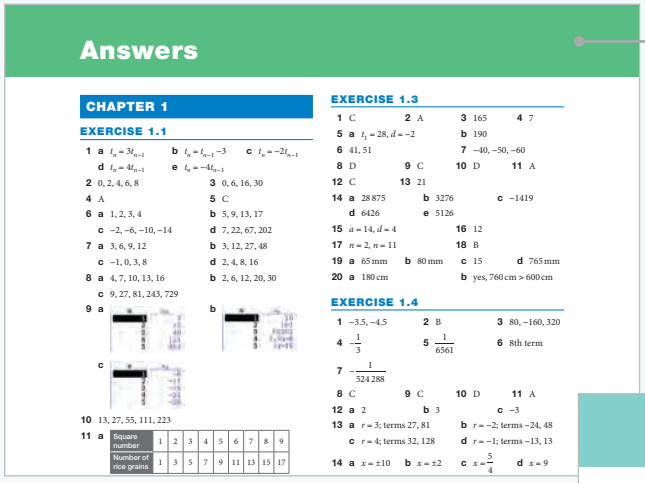
5 $5300 is invested in an account that earns 7.2% p.a. interest compounded monthly. The value of the investment after 8 years is closest to

A $8353 B $10 622 C $9412 D $30 528 E $120 105

Section B 2 questions 15 marks

1 (6 marks) A small bouncy ball is dropped from a height of 81 metres. Each time it strikes the

At the end of the book



Answers

CHAPTER 1

EXERCISE 1.1

1 a $t_n = 3t_{n-1}$ b $t_n = t_{n-1} - 3$ c $t_n = -2t_{n-1}$
d $t_n = 4t_{n-1}$ e $t_n = -4t_{n-1}$

2 0, 2, 4, 6, 8 **3** 0, 6, 16, 30

4 A **5** C

6 a 1, 2, 3, 4 b 5, 9, 13, 17
c −2, −6, −10, −14 d 7, 22, 67, 202

7 a 3, 6, 9, 12 b 3, 12, 27, 48
c −1, 0, 3, 8 d 2, 4, 8, 16

8 a 4, 7, 10, 13, 16 b 2, 6, 12, 20, 30
c 9, 27, 81, 243, 729

9 a b c

10 13, 27, 55, 111, 223

11 a

Square number	1	2	3	4	5	6	7	8	9
Number of rice grains	1	3	5	7	9	11	13	15	17

EXERCISE 1.3

1 C **2** A **3** 165 **4** 7

5 a $t_1 = 28, d = -2$ b 190

6 41, 51 **7** −40, −50, −60

8 D **9** C **10** D **11** A

12 C **13** 21

14 a 28 875 b 3276 c −1419
d 6426 e 5126

15 $a = 14, d = 4$ **16** 12

17 $n = 2, n = 11$ **18** B

19 a 65 mm b 80 mm c 15 d 765 mm

20 a 180 cm b yes, 760 cm > 600 cm

EXERCISE 1.4

1 −3.5, −4.5 **2** B **3** 80, −160, 320

4 $-\frac{1}{3}$ **5** $\frac{1}{6561}$ **6** 8th term

7 $-\frac{1}{524\,288}$

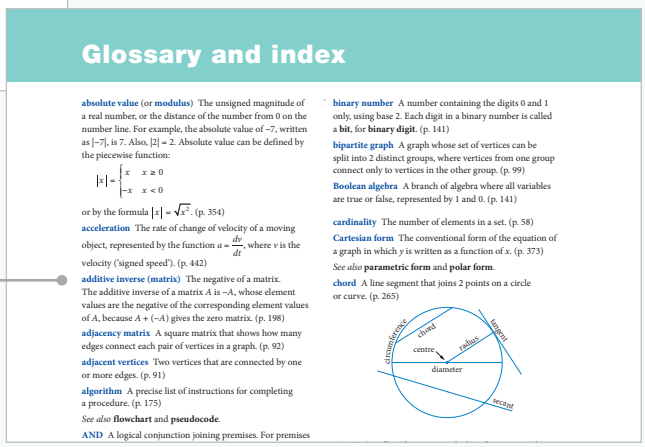
8 C **9** C **10** D **11** A

12 a 2 b 3 c −3

13 a $r = 3$; terms 27, 81 b $r = -2$; terms −24, 48
c $r = 4$; terms 32, 128 d $r = -1$; terms −13, 13

14 a $x = \pm 10$ b $x = \pm 2$ c $x = \frac{5}{4}$ d $x = 9$

Answers (with worked solutions provided on Nelson MindTap for teachers to allocate to students).



Glossary and index

absolute value (or **modulus**) The unsigned magnitude of a real number, or the distance of the number from 0 on the number line. For example, the absolute value of −7, written as $|-7|$, is 7. Also, $|2| = 2$. Absolute value can be defined by the piecewise function:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

or by the formula $|x| = \sqrt{x^2}$. (p. 354)

acceleration The rate of change of velocity of a moving object, represented by the function $a = \frac{dv}{dt}$, where v is the velocity ('signed speed'). (p. 442)

additive inverse (matrix) The negative of a matrix. The additive inverse of a matrix A is $-A$, whose element values are the negative of the corresponding element values of A, because $A + (-A)$ gives the zero matrix. (p. 198)

adjacency matrix A square matrix that shows how many edges connect each pair of vertices in a graph. (p. 92)

adjacent vertices Two vertices that are connected by one or more edges. (p. 91)

algorithm A precise list of instructions for completing a procedure. (p. 175)

See also **flowchart** and **pseudocode**.

AND A logical conjunction joining premises. For premises

binary number A number containing the digits 0 and 1 only, using base 2. Each digit in a binary number is called a **bit**, for **binary digit**. (p. 141)

bipartite graph A graph whose set of vertices can be split into 2 distinct groups, where vertices from one group connect only to vertices in the other group. (p. 99)

Boolean algebra A branch of algebra where all variables are true or false, represented by 1 and 0. (p. 141)

cardinality The number of elements in a set. (p. 58)

Cartesian form The conventional form of the equation of a graph in which y is written as a function of x. (p. 373)

See also **parametric form** and **polar form**.

chord A line segment that joins 2 points on a circle or curve. (p. 265)

A combined **Glossary and index**.

Nelson MindTap

Nelson MindTap is an online learning space that provides students with tailored learning experiences. Access tools and content that make learning simpler yet smarter to help you achieve VCE maths mastery.

Nelson MindTap includes an eText with integrated interactives and online assessment.

Margin links in the student book signpost multimedia student resources found on MindTap.

Video playlists

Worksheets

Skillsheets

Nelson MindTap for students:

- **Watch** video tutorials featuring expert teacher advice to unpack new concepts and develop your understanding.
- **Revise** using quizzes, worksheets and skillsheets to practise your skills and build your confidence.
- **Navigate** your own path, accessing the content, analytics and support as you need it.

Nelson MindTap for teachers*:

- Tailor content to different learning needs – assign directly to the student, or the whole class.
- Monitor progress using assessment tools like Gradebook and Reports.
- Integrate content and assessment directly within your school's LMS for ease of access.
- Access topic tests, teaching plans and worked solutions to each exercise set.

*Complimentary access to these resources is only available to teachers who use this book as part of a class set, book hire or booklist. Contact your Cengage Education Consultant for information about access and conditions.

Nelson VICmaths 11–12 series



Companion resources

WB

Mastery Workbook

- Step-by-step scaffolding to guide students towards mastery of the course content.
- Write-in support that encourages students to show working.
- Matched examples for *every* worked example in the student book.
- Full integration between student book and workbook.
- Answers (worked solutions for teachers on Nelson MindTap).

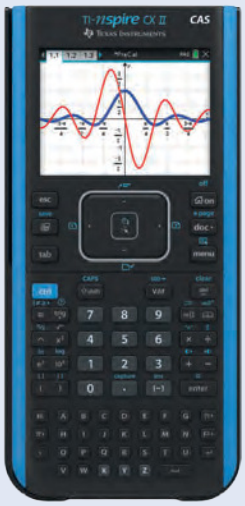
Exam*plus*

- Create and simulate exam-like conditions in minutes.
- Save time with an extensive bank of filterable and difficulty-graded questions for Year 12.
- Over 1000 past VCAA and unseen exam-style questions and solutions all in the one place.
- Extensively researched and user-tested.

TI-Nspire CAS introduction

The latest TI-Nspire model is TI-Nspire CX II CAS. When purchasing a new handheld, you also gain access to the student software. If you purchase a used handheld then you can pay an additional fee for the student software. Alternatively, you can connect your handheld to your computer using the TI-Nspire™ CX II Connect web-based app, which enables you to perform a variety of functions such as screen captures, file transfers and operating system updates.

Note that there is also TI-Nspire non-CAS technology available. It is **vital** that you use the CAS technology.



TI-Nspire CX II CAS

Student Book instructions

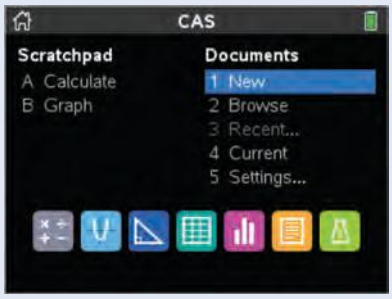
The instructions in this student book use words instead of symbols. Most keys on the keypad are clearly labelled with a word or an abbreviation. Four words that are used to represent less obvious keys are:

Word	Key
home	on
catalog	
template	
Scratchpad	

Several keys have a second function highlighted in blue above the key. For example, press **ctrl + x^2** to access the square root function $\sqrt{\ }$.

Applications

The applications available are outlined below.



Press **home** to view the home page. The **Scratchpad** options on the left are available for quick calculations and graphing. The **Document** options on the right are used for navigation. The seven icons on the bottom are the main applications.

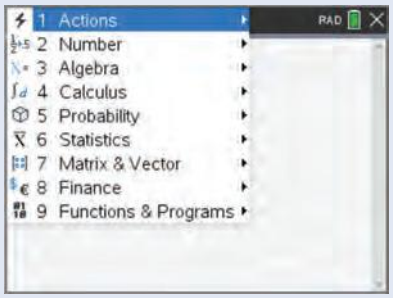


When you select **Documents > New** from the home page, a list of the seven applications plus two additional menu options will be displayed. From any application, press **ctrl + I** (for insert) to display this list and add a new page to the document.

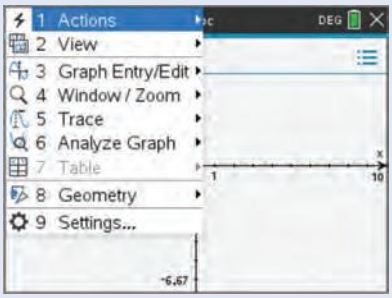
Menus

The instructions in this student book primarily use the **Calculator, Graphs, Lists & Spreadsheet** and **Data & Statistics** applications (see **Hints** on page xvii). The following figures show the initial menu options for these four applications. These menu options link to submenu options, which are not shown below. On the handheld and software, the applications are referred to as **Documents**. In the student book instructions, the applications are referred to as **pages** of the document (e.g. Add a **Graphs** page).

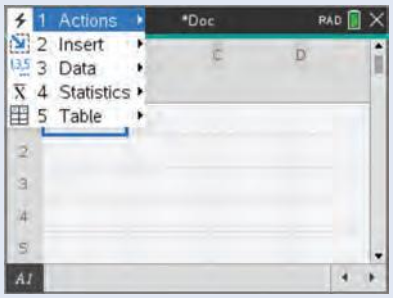
Calculator



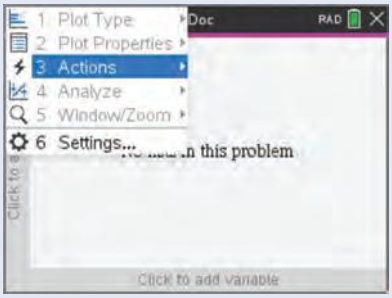
Graphs



Lists & Spreadsheet



Data & Statistics



If your document contains more than one page, move among them by clicking on the numbered tabs at the top of the screen. Alternatively, press **ctrl + left arrow** or **ctrl + right arrow**. To view all the pages of a document, press **ctrl + up arrow**.

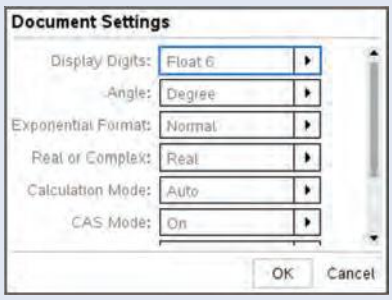
All menu and submenu options include numbers. The student book instructions do not include numbers. For example, the instruction in a **Calculator** page for clearing all calculations is 'press **menu > Actions > Clear History'**. The shortcut is 'press **menu > 1 > 5'**. For efficiency, you are encouraged to learn the sequence of numbers for frequently used commands.

Document Settings

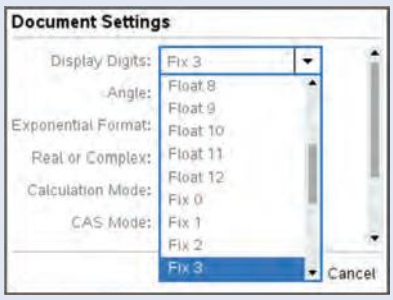
The document settings can be accessed in the following ways:

1. From the **home** page, press **Settings > Document Settings**.
2. From a document page, press the **doc** key or click on **Doc** at the top of the page, then select **Settings & Status > Document Settings**.
3. Click on the **battery icon** in the top right-hand corner of the page, then select **Document Settings**.

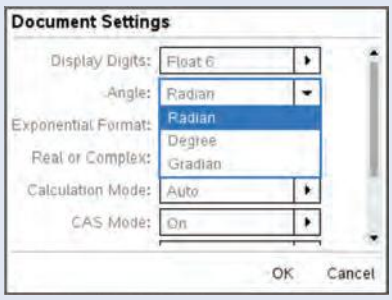
The document settings shown below are the primary ones you will be using.



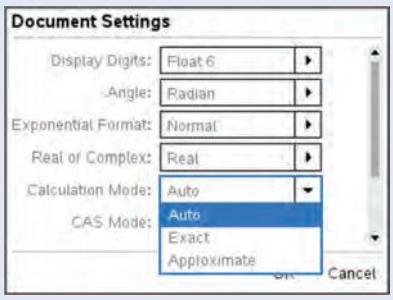
The screen above shows the default document settings. The **Display Digits** field is set to **Float 6**, which means up to 6 significant figures will be displayed.



Click in any field to display the options. The screen above shows the **Display Digits** options. Scroll down to select a specific number of decimal places.



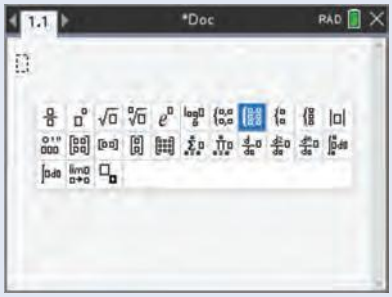
There are three **Angle** options shown above. Select either **Radian** or **Degree**. These two angle settings can be toggled at any time by clicking on **DEG** or **RAD** in the top right-hand corner of the screen.



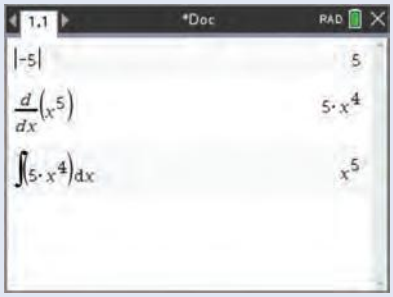
There are 3 **Calculation Mode** options shown above. It is recommended that you keep the default setting of **Auto**. If you require an approximate or decimal answer, press **ctrl + enter** or include a decimal point in your calculation.

Templates

The **template** key is located to the right of the **9** key.



Press **template** and to view the template options. Most templates will be directly inserted but for some, you will be prompted with a dialogue box. For example, after selecting the **3 piecewise function** template, you will be prompted for the number of function pieces.

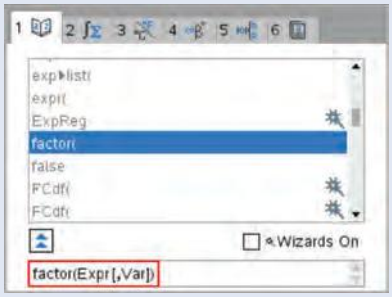


The screen above shows three examples using templates: an **absolute value**, a **derivative** and an **integral**. Many of the templates can be accessed using keys on the keyboard, e.g. fraction, square root, etc.

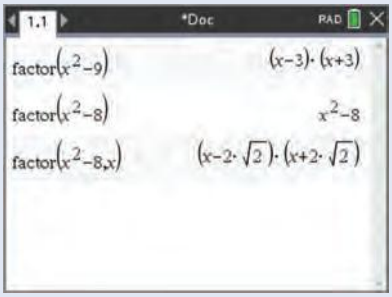
Catalog

The **catalog** key is located to the right of the **template** key. In addition to using the menus and submenus, you can access all the commands using the catalog. The advantage of using the catalog is that it shows the parameters required for each command at the bottom of the screen. Optional parameters appear in square brackets.

For example:



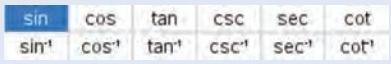
Press **catalog** and ensure tab **1** is selected. Press **F** to jump to the commands starting with F. Scroll down to **factor(**. The parameters for the factor command appear at the bottom of the screen (see **red** rectangle). **Expr** is a required parameter. The square brackets around **Var** mean it is an optional parameter.



When you **factor** the expression $x^2 - 9$, the solution is $(x - 3)(x + 3)$. When you factor $x^2 - 8$, the expression remains unchanged. However, if you include the optional parameter, **x**, the expression will factorise to $(x - 2\sqrt{2})(x + 2\sqrt{2})$,

Symbols

All symbols are listed at the end of the **catalog**. To access the symbol palette, press **ctrl + catalog.** Frequently used symbols are available in mini-palettes by pressing the keys shown below.



Press **trig** to access the trigonometry functions.



Press **ctrl** + = to access the inequalities and the **constraint** symbol.

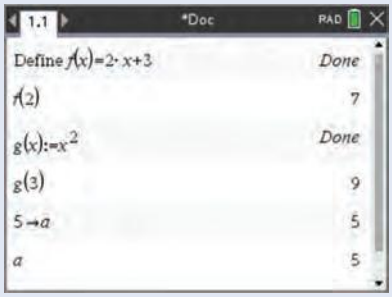


Press **π** to access commonly used symbols.

Hints

The instructions below provide a few hints to assist with using the 5 main applications.

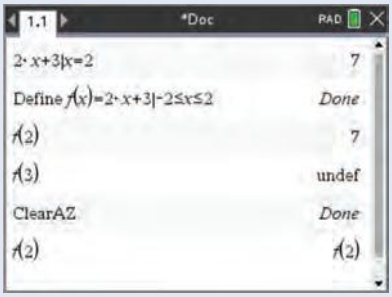
Calculator



There are a few options available to define or store functions and variables.

1. Press **menu > Actions > Define**.
2. Press **ctrl + template** for the **:=** symbol.
3. Press **ctrl + var** for the **store** symbol.

Press **var** to view the list of defined and stored variables.

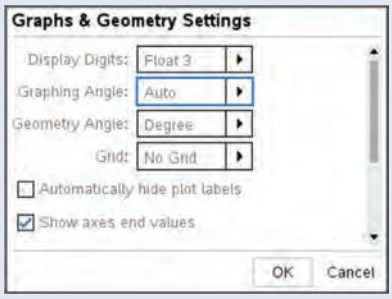


Press **ctrl + =** to access the **constraint** symbol (|).

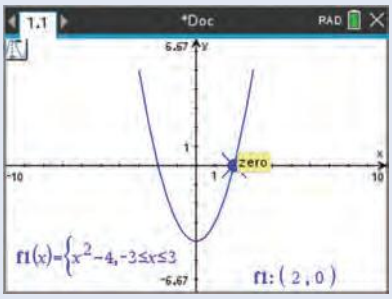
1. Use **constraint** for substitution.
2. Use **constraint** to limit the domain of a function.

Press **menu > Actions > Clear a-z** to delete defined variables.

Graphs

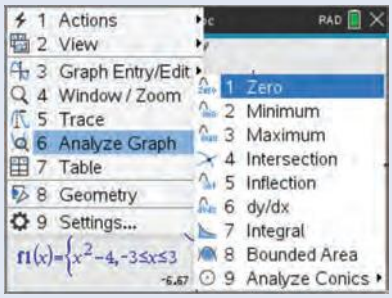


Press **menu > Settings** to view and/or change the settings for the **Graphs** and **Geometry** applications. The default setting for **Display Digits** is **Float 3** so change this if you need greater accuracy. The **Graphing Angle** is set to **Auto** but can be changed to **Degree** or **Radian**.

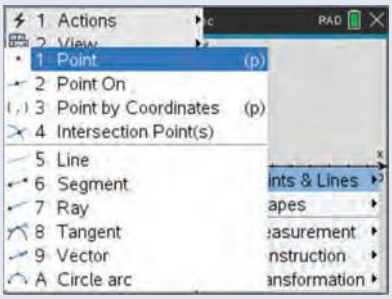


The **constraint** command can also be used in a **Graphs** page to specify a domain.

Press **menu > Trace > Graph Trace** to identify points on a graph.

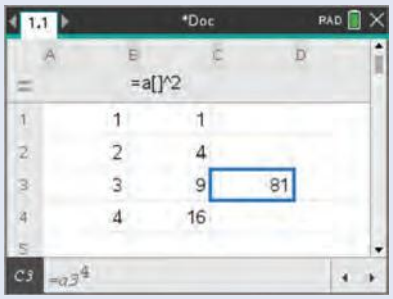


A second option for identifying key points on a graph is to press **menu > Analyze Graph**. These options prompt you for a **lower bound** and **upper bound** to locate the point.

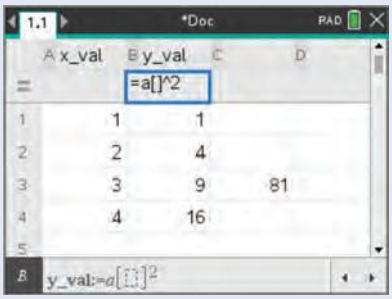


A third option is to press **menu > Geometry > Points & Lines**. Locate points of intersection or place points on the graphs and move them along the graph. Click on a coordinate to manually change a value.

Lists & Spreadsheet

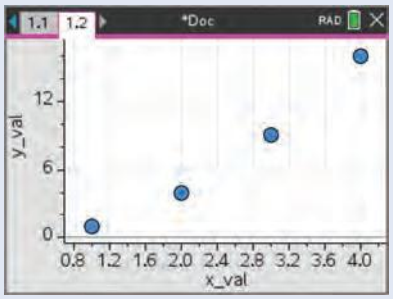


The columns in **Lists & Spreadsheet** application can be used for lists. Above, the values in the list in column **B** are the squares of the values of the list in column **A**. It can also be used as a spreadsheet. The value in cell **C3** is the value in cell **A3** raised to the power of 4.

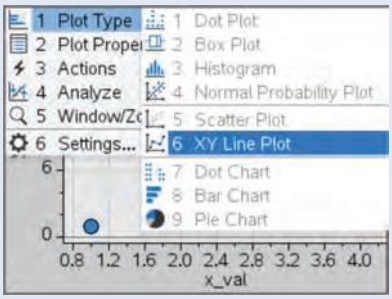


Calculations using the lists can be completed using the column headings **A, B, C,** etc.
Lists need to be labelled to be used in other applications. The list **A** above has been labelled **x_val** and list **B** labelled **y_val**. List names cannot have spaces so spaces are replaced by asterisks. Alternatively, press **ctrl + space bar** to insert the **underscore** character instead of a space.

Data & Statistics

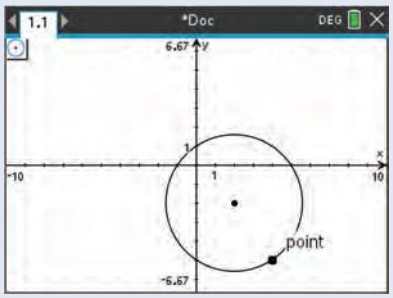


The **Data & Statistics** application is reliant on lists generated in other applications. It is designed for ungrouped data. The plot above displays the **x_val** and **y_val** lists from the **Lists & Spreadsheet** application.



Press **menu > Plot Type** to view the various graphing options. Options 1 to 4 are for univariate data (one list). Options 5 and 6 are for bivariate data (two lists). Options 7 to 9 are for categorical data (one list).

Geometry

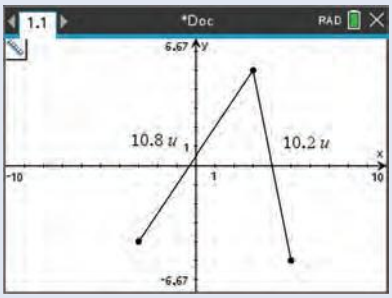


Add a **Graphs** page and press **esc** to hide the **Graph Entry Line**.

Press **menu > View > Grid > Dot Grid**.

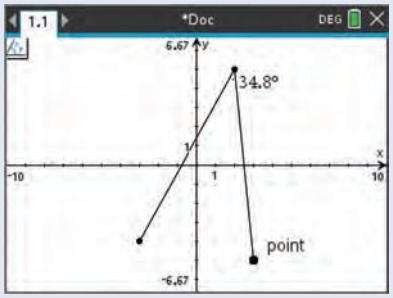
Press **menu > Geometry > Shapes** to view the submenu options. Select **Circle**.

Click on a centre and another point to draw a circle.



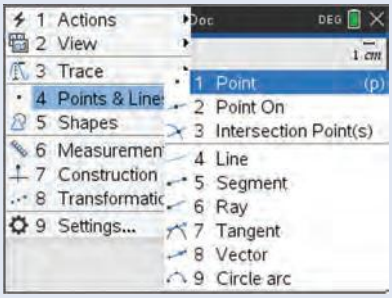
Press **menu > Geometry > Points & Lines** to view the submenu options. Select **Segment** then draw the segments.

Press **menu > Geometry > Measurement > Length** then click on the segments to measure them.



Press **menu > Geometry > Measurement > Angle** then click on the three points to measure the angle.

The tool being used is shown in the top left-hand corner of the screen. Press **esc** to remove a tool, which will then allow you to move objects around the screen.



Another option available is the **Geometry** page, which is restricted to plane geometry. Some of the menu options are shown above.

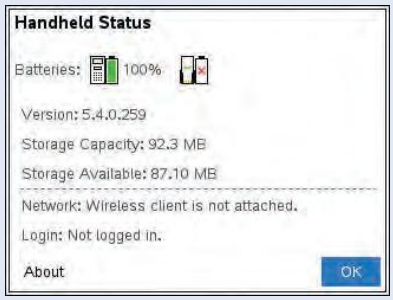
There is no grid or axes available on a Geometry page but an analytical Geometry page can be inserted.

Operating systems

Ensure the latest operating system is installed on your handheld and software.

Installing the latest operating system is relatively straightforward. Using the USB cable provided, connect the handheld to a computer with the student or computer link software installed. Select **Help > Check for OS Updates**. If you see a message that a new OS is available, follow the links to install it. Alternatively, go to the TI website at https://education.ti.com/ to download the latest operating system. Select **Tools > Install OS** then select the downloaded file.

To determine the version of your operating system, press **home > Settings > Status**. At the time of publication, the operating system for the CX II is version **5.4.0.259**.



Casio ClassPad introduction

The latest model of the Casio ClassPad is the fx-CP400. The connectivity software Screen Receiver, Share Assistant and Program Link Software can be downloaded for free. The ClassPad Manager software emulator is a separate program available at an additional cost.



Casio ClassPad

Student Book instructions

The instructions in this student book use words instead of symbols. ClassPad tools are located at the top of the screen. These tools vary with each application. Initially, these instructions will show a tool enclosed in a red rectangle with the corresponding word highlighted in red. Examples are shown below:

Word	Tool
Graph	
View Window	
Table	
Table Input	

Applications

The applications available are outlined below.



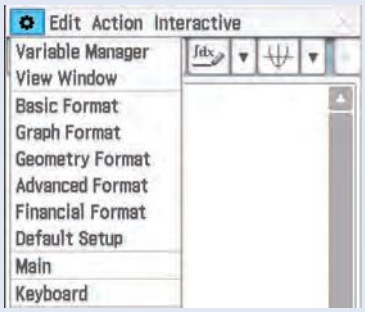
Tap **Menu** to view the applications.



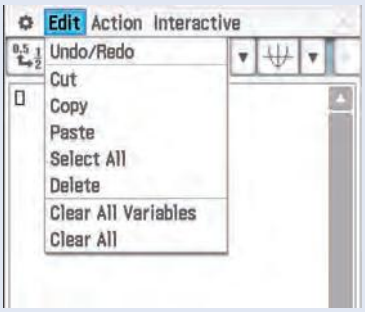
Slide the scroll bar at the bottom of the screen to access the full list.

Menus

The instructions in this student book will primarily use the **Main, Statistics, Geometry, Graph&Table and Conics** and applications (see **Applications** on page xxiv). All these applications have the ⚙ and **Edit** menus available at the top left of the screen.

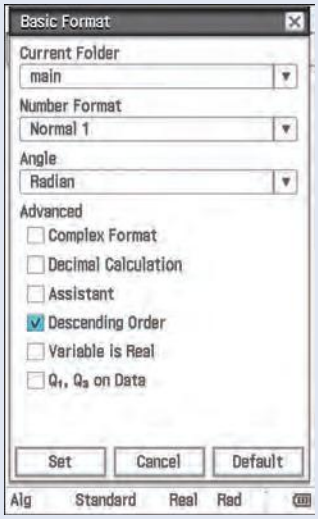


Tap ⚙ to view the system menu. The menu options allow you to manage variables and format applications.

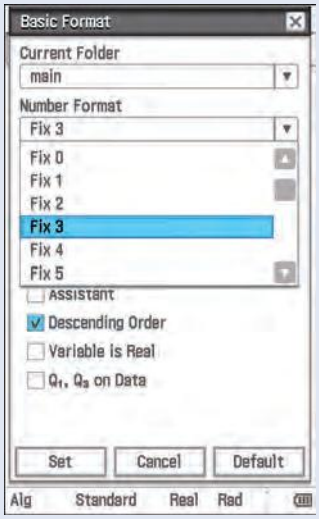


Tap **Edit** to view the edit menu. The menu options allow you to cut, copy, paste and delete screen content and clear variables. The **Edit** menu varies with each application.

Document Settings



Tap ⚙ > **Basic Format**. The screen above shows the default document settings. The **Number Format** field is set to **Normal 1**.

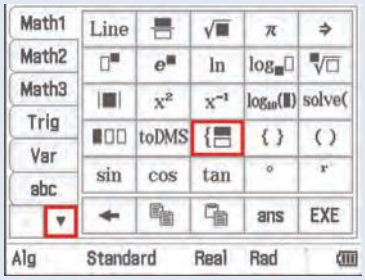


Tap the **Number Format** field to display the options. The screen above shows the **Display Digits** options. Scroll down to fix a specific number of decimal places.

The settings **Standard/Decimal, Real/Cplx** and **Rad/Deg/Gra** can be toggled at the bottom of the screen. The recommended settings are **Standard, Real** and **Rad**.

Keyboard

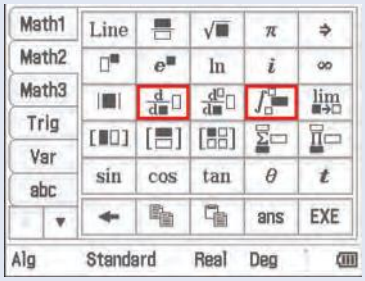
There are 9 soft keyboards available.



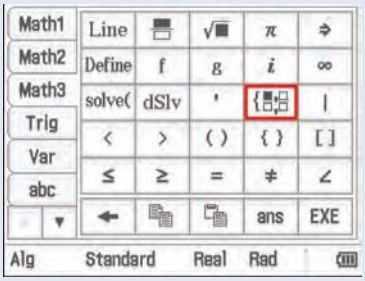
Press **Keyboard** to view the soft keyboards. The **Math1** soft keyboard is shown above with the **simultaneous equation** template highlighted in red. Tap the left tabs to access the other keyboards. Press the **down arrow** to view the second screen.



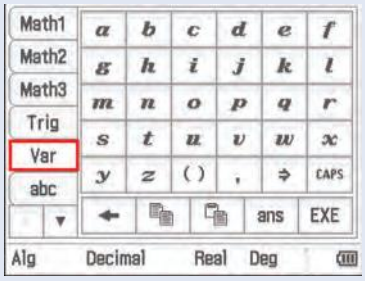
All functions can be accessed from **Catalog**. Tap on the letters at the top to jump through the list. Press the **up arrow** to return to the first screen.



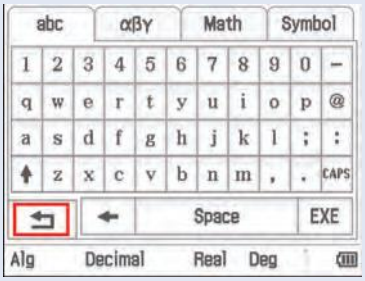
Tap **Math2** to display the soft keyboard. The **derivative** and **integral** templates highlighted in red.



The **Math3** template is shown above with the **piecewise** template highlighted in red. Use this keyboard for inequalities.



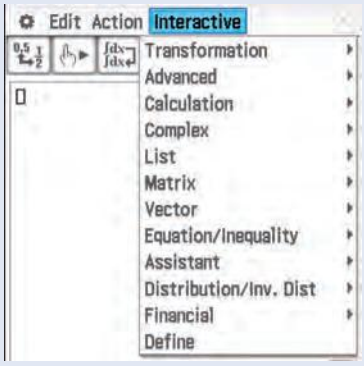
Tap **Var** to access the variables. Use variables, not letters in your algebraic calculations.



Tap **abc** to access letters and symbols. Use letters to name functions and matrices, etc. Tap the tabs at the top of the screen to access the range of symbols. Press **back** to return to the main screen.

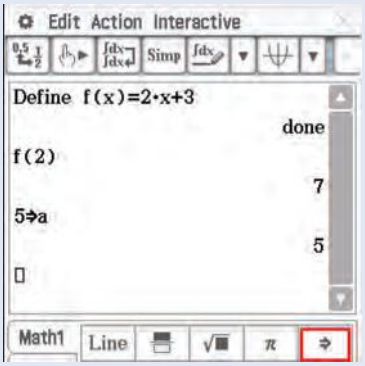
Applications

Main



In **Main**, there are two menus available to enter functions, **Active** and **Interactive**. With the **Interactive** menu, users are prompted with a dialogue box to enter the parameters. With few exceptions, the instructions in this student book are written using the **Interactive** menu.

For **derivative** and **integral** it is easier, and matches other working, to use **Interactive > Calculation > diff or summa** (∫) rather than the icons in Math2.



To define a function, enter and highlight the expression then tap **Interactive > Define**. The **store** (⇒) arrow is available from the **Math** and **Trig** soft keyboards. **Store** can be used to store values.

Use the **abc** soft keyboard to store values.

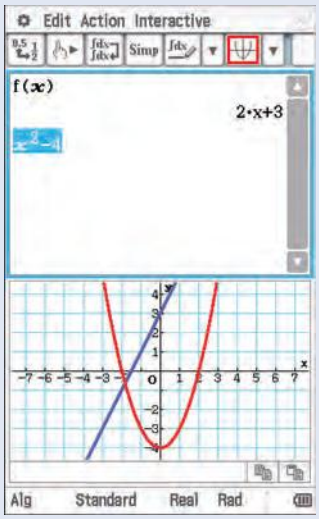
All algebraic working uses **Var** and not **abc**

Define a function will be over-saved with a new defined function.

In contrast, stored variables needs to be cleared using **Variable Manager**.

Graphing

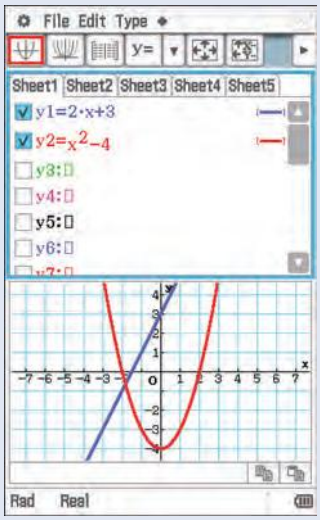
There are 2 main options for graphing functions and relations.



In **Main**, tap **Graph** to open the graph window. Enter and highlight the function or expression and drag it into the **Graph** window from either the right or left side.

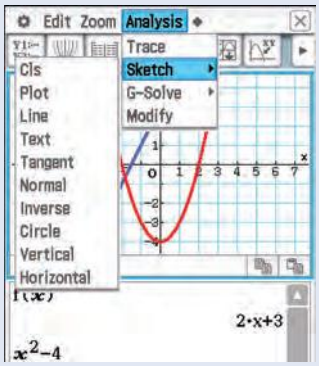
If the function has a restricted domain, it must be dragged from the left side of the screen.

The significance of **Main** is that the algebra is on the top and matching graphs are on the bottom of the twin screens.

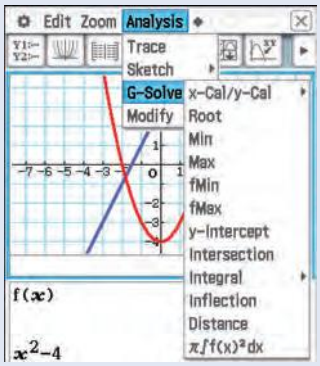


Graph&Table is used much less frequently because it requires rearrangement in terms of **y = ...**

Tap **Menu > Graph&Table.** Enter the function then tap **Graph**. The instructions in this student book use the **Main** option in preference to the **Graph&Table** option.

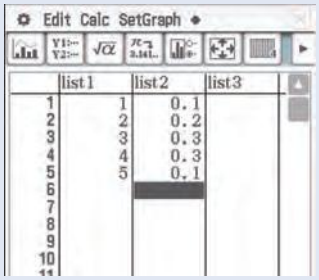


There are several menu options available to analyse graphs. Tap **Analysis > Trace** and press the **arrow** keys to move along the graph. Tap **Analysis > Sketch** for the options shown above.

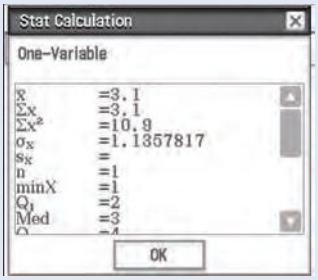


Tap **Analysis > G-Solve** to identify key features of a graph. For example, select **Root** to find the *x*-intercepts and select **Intersection** to locate points of intersection of 2 graphs. **G-Solve** gives decimal values. Use **Main** to calculate exact values.

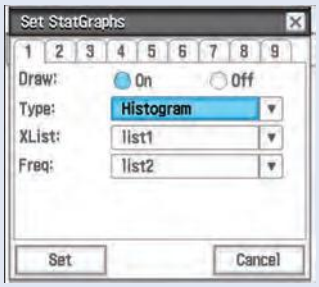
Statistics



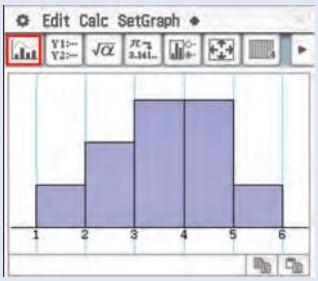
Tap **Menu > Statistics** to open the Statistics application. The default settings show **list1**, **list2**, **list3**, etc. If required, tap on the list name to enter a new heading.



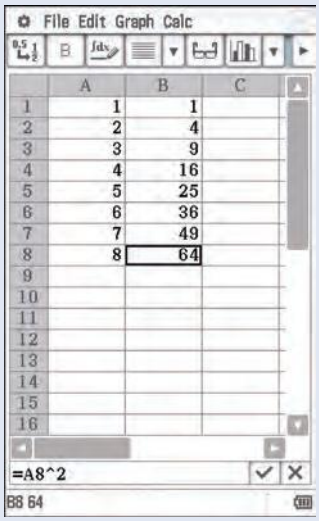
Tap **Calc** to view the calculation menu options. Tap **One-Variable** for the dialogue box for statistical analysis. The screen above displays the one variable statistics for **list1** for the frequencies in **list2**.



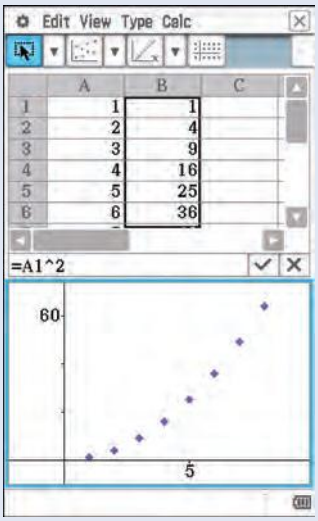
Tap **SetGraph** to set up the statistics graphs. Tap **Type:** to view the graphing options for statistical graphs. When finished, tap **Set**.



Tap **Graph** to display the statistical graph. The graph above displays a histogram of the **list1** values for the frequencies in **list2**.



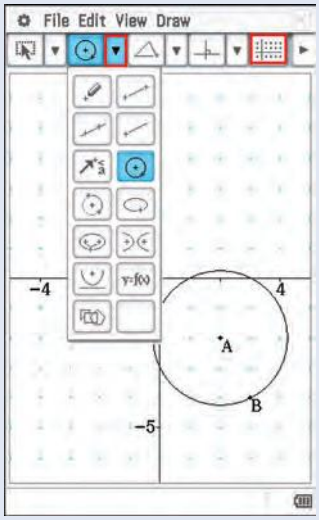
Tap **Menu > Spreadsheet** to open the Spreadsheet application. Values and formulas are entered using standard spreadsheet techniques. Column **A** shows the numbers 1 to 8 and Column **B** shows the square of the numbers.



Highlight the required data and tap **Calc** to access the range of statistical functions.

Highlight the data and tap **Graph** to access the range of graphs available. The graph above shows a **Scatterplot** of the squared values.

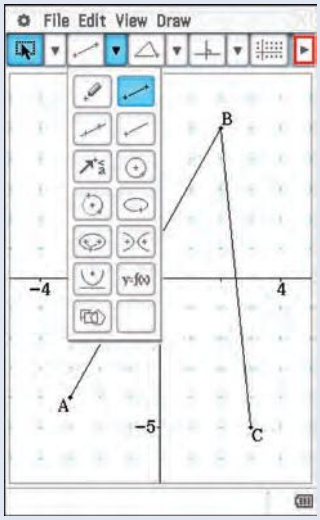
Geometry



Tap the **Grid** tool to toggle through the various grid options. The other tools at the top of the screen can also be accessed by tapping on the **Draw** menu.

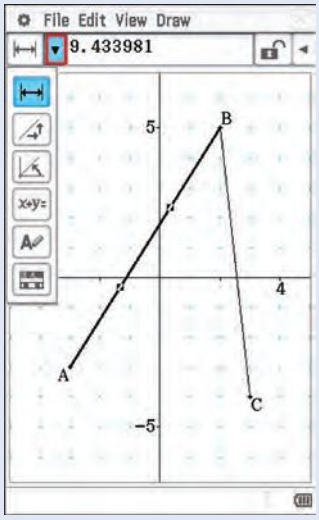
Tap the basic object **down arrow** to view the objects.

Select the circle object. Tap a centre and another point to draw a circle.



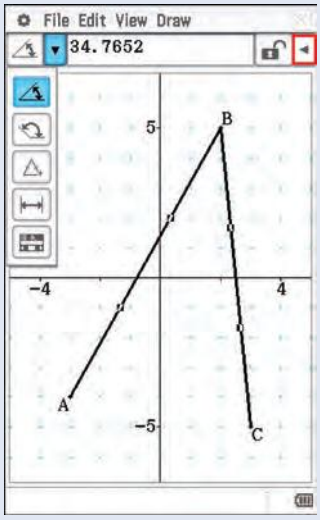
Select the line segment object then draw segments.

To measure or find equations of objects, tap the **right arrow** to view additional tools.



Tap to highlight an object then tap the **down arrow** to view the measurement box options.

The length of the line is shown at the top of the screen.



Tapping two objects gives different measurement box objects.

The angle between the line segments is shown at the top of the screen.

Tap the **left arrow** to return to the first screen.

Operating systems

Ensure you have the latest operating system is installed on your handheld.

Tap **Settings** (located in the bottom left corner of the screen) then select **Version**. At the time of publication, the latest version is **02.01.7001**.

To download the latest operating system, go to Australian Shriro website at http://www.casio.edu.shriro.com.au/classpad.php. Using the USB cable provided, connect the handheld to a computer. Start the installation program and follow the prompts. Some of these prompts will be on the computer and others will be on the handheld.



CHAPTER

1 SEQUENCES AND SERIES

Study Design coverage

Nelson MindTap chapter resources

1.1 Sequences
- **Using CAS 1:** Generating sequences
- **Using CAS 2:** Recurrence relations

1.2 Arithmetic sequences

1.3 Arithmetic series

1.4 Geometric sequences
- Convergence of a geometric sequence

1.5 Geometric series

1.6 The infinite geometric series
- Convergence of an infinite geometric series

1.7 Linear recurrence relations
- Defining a sequence using the previous term
- Defining a sequence using an explicit rule
- **Using CAS 3:** Linear recurrence relations
- **Using CAS 4:** Generating linear recurrence relations

1.8 Applications of recurrence relations
- The Tower of Hanoi
- Fibonacci numbers
- Compound interest
- Annuities
- Reducing balance loans
- Population growth

VCE question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2

Study Design coverage

UNIT 1, AREA OF STUDY 2: DISCRETE MATHEMATICS

Sequences and series

- definitions of sequences and series, arithmetic and geometric sequences and their partial sums
- the limiting behaviour as $n \to \infty$ of the terms t_n in a geometric sequence and dependence on the value of the common ratio
- sequences generated by recursion
- solution of first order linear recurrence relations of the form $t_{n+1} = at_n + b$, $a \neq 0$, with constant coefficients and their application to financial problems and population modelling.

VCE Mathematics Study Design 2023–2027 p. 60, © VCAA 2022

Video playlists (9):

1.1 Sequences
1.2 Arithmetic sequences
1.3 Arithmetic series
1.4 Geometric sequences
1.5 Geometric series
1.6 The infinite geometric series
1.7 Linear recurrence relations
1.8 Applications of recurrence relations
VCE question analysis Sequences and series

Worksheets (17):

1.2 Arithmetic sequences • Arithmetic progressions
1.3 Arithmetic series • Sequences and series • Arithmetic series assignment • Arithmetic series extension
1.4 Geometric sequences • Classifying sequences • Geometric progressions 1 • Geometric progressions 2 • Modelling arithmetic and geometric sequences
1.5 Geometric series 1 • Geometric series 2 • Mixed sequences and series • Geometric series assignment • Geometric series problems
1.6 Limiting sum of a geometric series

Nelson MindTap

To access resources above, visit
cengage.com.au/nelsonmindtap

Video playlist
Sequences

1.1 Sequences

A **sequence** is an ordered list of numbers such as 5, 2, −1, −4 … The 1st term, 2nd term, and nth term are usually written as t_1, t_2 and t_n respectively, so in the sequence above, $t_1 = 5$, $t_2 = 2$, $t_3 = -1$ and so on.

A sequence may be defined by listing the terms in order, by giving the rule as a function (whose domain is the natural numbers), or by a recurrence relation.

A **recurrence relation**, also called a **recursive formula**, gives each term as a rule involving the previous term (or terms).

For example, the sequence 2, 4, 6, 8 … has $t_1 = 2$, $t_2 = t_1 + 2$, $t_3 = t_2 + 2$ and so on.

So the recurrence relation is $t_n = t_{n-1} + 2$, where $t_1 = 2$.

This can also be written as $t_{n+1} = t_n + 2$.

The sequence 1, 3, 9, 27 … has $t_1 = 1$, $t_2 = t_1 \times 3$, $t_3 = t_2 \times 3$ and so on, so the recurrence relation is $t_n = 3t_{n-1}$, where $t_1 = 1$.

WB p. 1

WORKED EXAMPLE 1 Recurrence relations

Find the recurrence relation for the sequence 5, 2, −1, −4 …

Steps	Working
1 State t_1.	$t_1 = 5$
2 Find a pattern for the terms.	$t_2 = t_1 - 3$
	$t_3 = t_2 - 3$
	$t_4 = t_3 - 3$
3 Write the recurrence relation.	$t_n = t_{n-1} - 3$

WB p. 2

WORKED EXAMPLE 2 Finding terms in a sequence

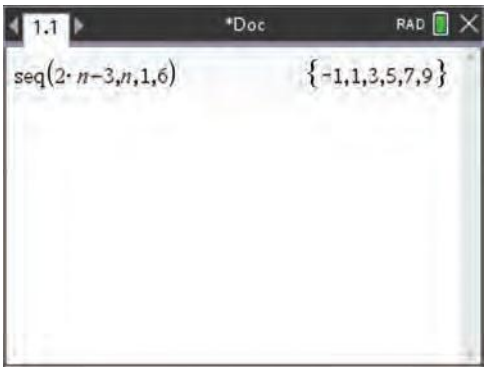
Find the first 5 terms for the sequence defined by the rule $t_n = 2n^2 + 2n$, where n is a natural number.

Steps	Working
1 State t_1.	$t_1 = 2(1)^2 + 2(1) = 4$
2 Substitute values of n to write the terms.	$t_2 = 2(2)^2 + 2(2) = 12$
	$t_3 = 2(3)^2 + 2(3) = 24$
	$t_4 = 2(4)^2 + 2(4) = 40$
	$t_5 = 2(5)^2 + 2(5) = 60$
3 Write the first 5 terms for the sequence.	4, 12, 24, 40, 60

USING CAS 1 Generating sequences

Generate the first 6 terms of the sequence defined by the rule $t_n = 2n - 3$.

TI-Nspire

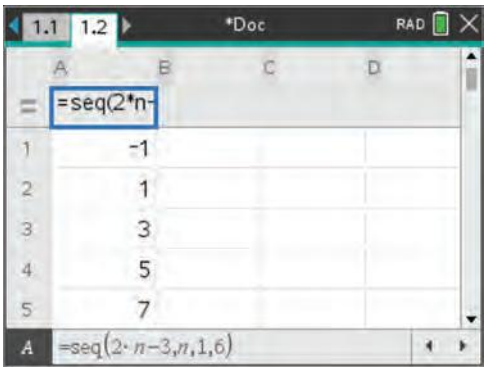


1 Add a **Calculator** page.

2 Press **menu > Statistics > List Operations > Sequence**.

3 Enter **2n–3,n,1,6**.

4 Press **enter** to display the sequence.



5 Sequences can also be generated in a **List & Spreadsheet** page.

6 Place the cursor in the cell immediately under **A** then press **=**.

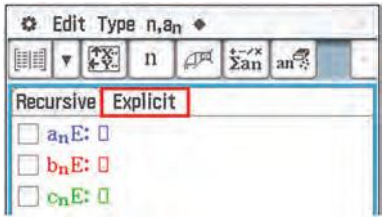
7 Press **catalog** then press the **S** key to jump to the functions starting with S.

8 Scroll down to select **seq(**.

9 Enter **2n–3,n,1,6**.

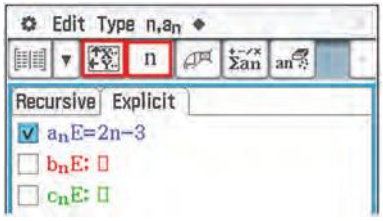
10 Press **enter** to display the sequence.

ClassPad



1 Tap **Menu > Sequence**.

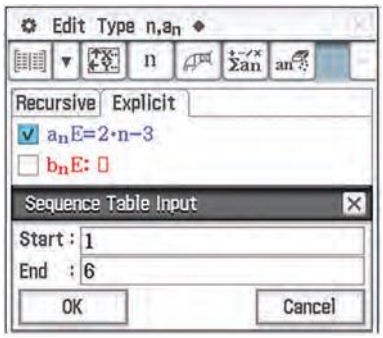
2 Tap the **Explicit** tab.



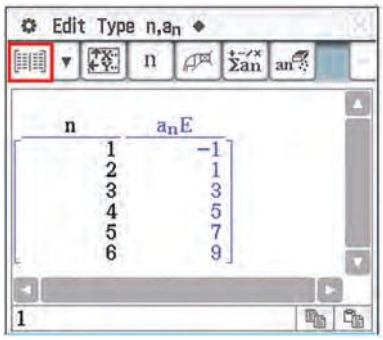
3 Use the **n** tool to enter the expression **2n–3**.

4 Press **EXE** to select the expression.

5 Tap the **Sequence Table Input** tool.



6 In the dialogue box, enter a **Start:** value of **1** and an **End:** value of **6**, then tap **OK**.



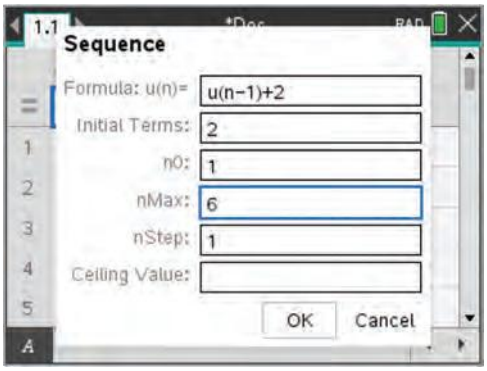
7 Tap **Table** to display the sequence in the lower window.

USING CAS 2 Recurrence relations

Generate the sequence defined by the recurrence relation $t_n = t_{n-1} + 2$, where $t_1 = 2$.

Plot the graph of the sequence.

TI-Nspire



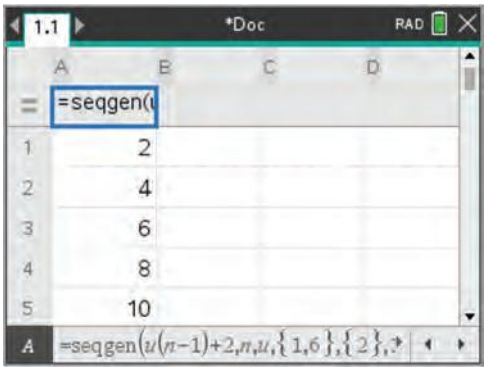
1 Add a **Lists & Spreadsheet** page.

2 Place the cursor in the cell immediately under **A**.

3 Press **menu > Data > Generate Sequence**.

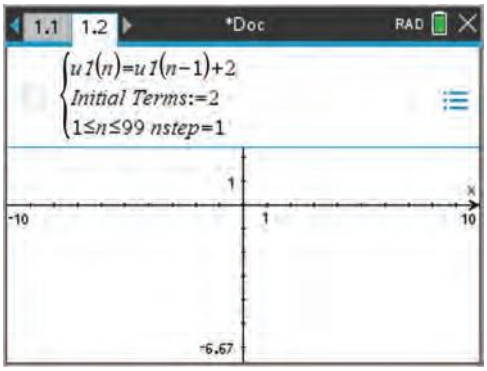
4 Enter the expression and values shown above.

5 Click **OK**.



6 The first 6 terms of the recurrence relation will be displayed in column **A**.

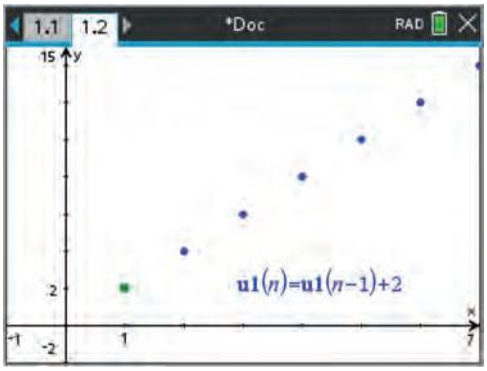
Alternatively, use the **seq(** function shown in Using CAS 1 to generate the list.



7 Add a **Graphs** page.

8 Press **menu > Graph Entry/Edit > Sequence**.

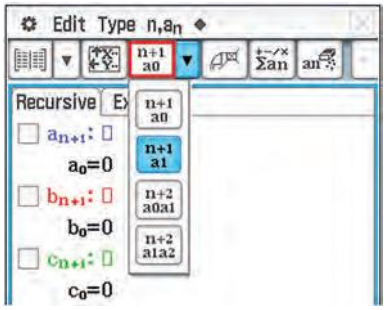
9 Enter the expression and the initial term as shown above, then press **enter**.



10 The first 6 points of the recurrence relation will be displayed.

11 Adjust the window settings to suit.

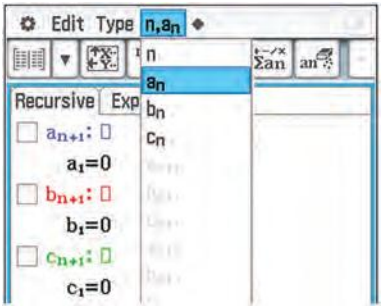
ClassPad



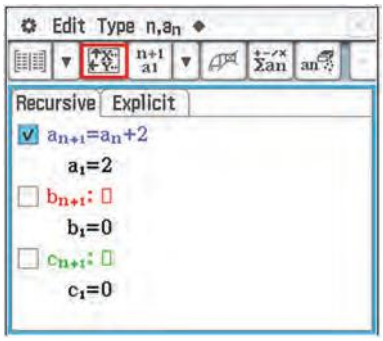
1 Tap **Menu > Sequence**.

2 Tap the **Recursive** tab.

3 Tap the **Sequence Type** tool and select the second option from the dropdown menu.



4 Tap the **n,a_n** menu heading at the top of the screen and select **a_n**.

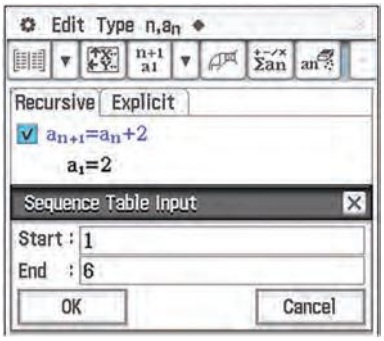


5 Enter **+2** to complete the recurrence relation **a_n + 2**.

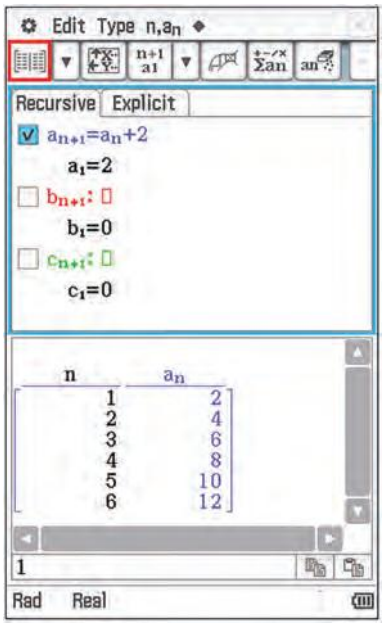
6 Set **a_1 = 2**.

7 Press **EXE** to select the sequence.

8 Tap the **Sequence Table Input** tool.

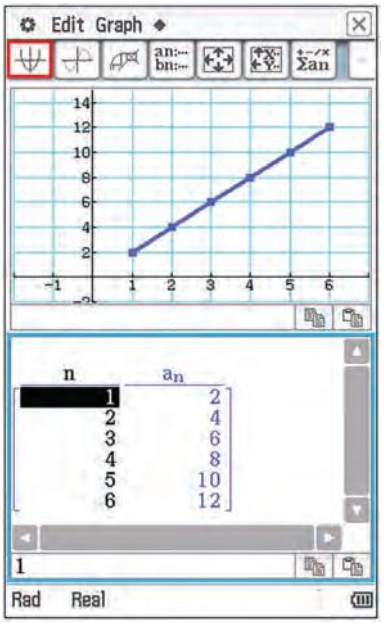


9 In the dialogue box, enter a **Start:** value of **1** and an **End:** value of **6**, then tap **OK**.



10 Tap the **Table** tool.

11 The first 6 terms of the recurrence relation will appear in the lower window.



12 Tap **Graph**.

13 The connected graph of the first 6 points will be displayed in the upper window.

14 Adjust the window settings to suit.

15 Tap **Analysis > Trace** to locate the individual points.

EXERCISE 1.1 Sequences

ANSWERS p. 564

Mastery

1 WORKED EXAMPLE 1 Find the recurrence relation for each sequence.

a 1, 3, 9, 27 …

b 10, 7, 4, 1, −2 …

c 10, −20, 40, −80 …

d 1, 4, 16, 64 …

e −1, 4, −16, 64 …

2 WORKED EXAMPLE 2 Find the first 5 terms for the sequence defined by the rule $t_n = 2n - 2$, where n is a natural number.

3 Find the first 4 terms for the sequence defined by the rule $t_n = 2n^2 - 2$, where n is a natural number.

4 Using CAS 1 Using CAS, the first 4 terms of the sequence of numbers defined by the rule $t_n = 2n - 3$ are

A −1, 1, 3, 5

B −1, 1, 3

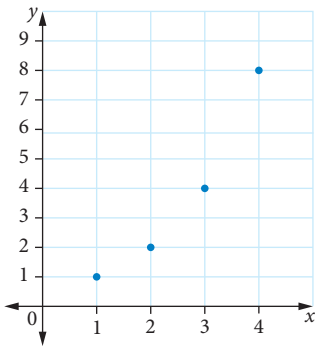
C 5, 7, 9, 11, 13

D 11, 13, 15, 17, 19

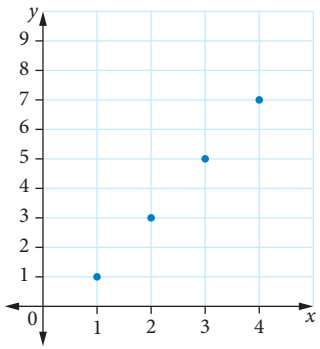
E 5, 7, 9, 11

5 Using CAS 2 Using CAS, the graph of the sequence defined by the recurrence relation $t_n = t_{n-1} - 2$, where $t_1 = 1$, is

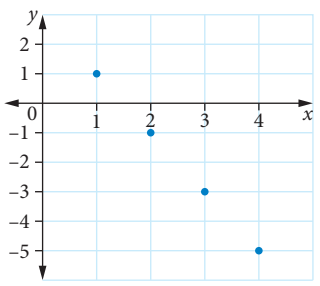
A



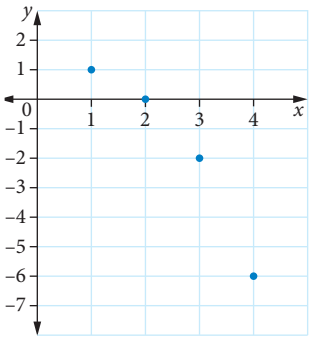
B



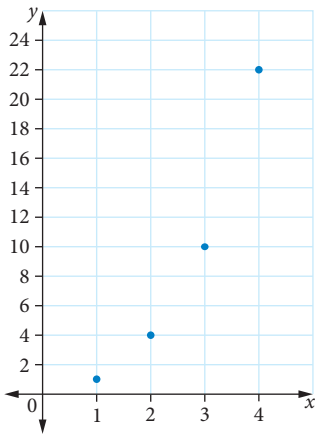
C



D



E



6 TECH-FREE List the first 4 terms of the sequence for each recurrence relation.

a $t_1 = 1, t_n = t_{n-1} + 1$ **b** $t_1 = 5, t_n = t_{n-1} + 4$

c $t_1 = -2, t_n = t_{n-1} - 4$ **d** $t_1 = 7, t_n = 3t_{n-1} + 1$

7 TECH-FREE For each sequence defined by the given rule, list the first 4 terms.

a $t_n = 3n$ **b** $t_n = 3n^2$ **c** $t_n = n^2 - 2n$ **d** $t_n = 2^n$

8 TECH-FREE For each rule, find the first 5 terms of the given sequence.

a $t_n = 3n + 1$ **b** $t_n = n^2 + n$ **c** $t_n = 3^{n+1}$

9 For each rule, use CAS to find the first 5 terms of the given sequence.

a $t_1 = 1, t_n = 3t_{n-1} + 1$ **b** $t_1 = 3, t_n = (t_{n-1})^2 + 1$ **c** $t_1 = -1, t_n = t_{n-1} - 5$

Exam practice

10 TECH-FREE (2 marks) Given $a_n = 2a_{n-1} + 1$ and that $a_1 = 6$, write the first 5 terms of the sequence.

11 TECH-FREE (5 marks) A grain of rice is placed on the 1st square of a chessboard, 3 grains are placed on the 2nd square, 5 grains are placed on the 3rd and so on.

a Copy and complete this table. 2 marks

Square number	1	2	3	4	5	6	7	8	9
Number of rice grains	1	3	5						

b How many rice grains will be placed on the 12th square if the same pattern is followed? 1 mark

c If the squares of the chessboard were filled by placing one grain of rice in the 1st square, 2 in the 2nd, 4 in the 3rd and so on, how many grains of rice would be placed on the 12th square? Would they fit? 2 marks

12 The first 5 terms for the sequence defined by the rule $t_n = n - 2$, where n is a natural number, are

A 1, 2, 3, 4, 5 **B** −1, 0, 1, 2, 3 **C** 0, 1, 2, 3, 4

D 0, 1, 2, 3 **E** 2, 4, 6, 8, 10

13 Which table shows the first 5 terms of the sequence defined by the rule $t_n = 5n$?

A

n	t_n
1	5
2	10
3	15
4	20
5	25

B

n	t_n
1	6
2	11
3	16
4	21
5	26

C

n	t_n
1	6
2	7
3	8
4	9
5	10

D

n	t_n
1	6
2	7
3	8
4	9
5	10
6	11

E

n	t_n
1	5
2	10
3	15
4	20
5	25
6	30

14 The recurrence relation that can be defined for the sequence 5, 0, –5, –10 … is

A $t_n = t_{n-1} + 5$, where $t_1 = 5$ **B** $t_n = t_{n-1} - 3$, where $t_1 = 1$

C $t_n = t_{n-1} - 5$, where $t_1 = 1$ **D** $t_n = t_{n-1} - 5$, where $t_1 = 5$

E $t_n = 5t_{n-1}$, where $t_1 = 5$

15 The recurrence relation for the sequence 5, 25, 125, 625 … is

A $t_n = t_{n-1} + 5$, where $t_1 = 5$ **B** $t_n = 5t_{n-1}$, where $t_1 = 1$

C $t_n = t_{n-1} - 5$, where $t_1 = 1$ **D** $t_n = t_{n-1} - 5$, where $t_1 = 5$

E $t_n = 5t_{n-1}$, where $t_1 = 5$

16 The first 4 terms for the sequence defined by the rule $t_n = n^2$ are

A 1, 2, 3, 4 **B** 0, 1, 2, 3 **C** 1, 4, 9, 16 **D** 1, 4, 9, 16, 25 **E** 2, 4, 6, 8

17 The first 4 terms for the sequence defined by the rule $t_n = 2n + 1$, where n is a natural number, are

A 1, 2, 3, 4 **B** 3, 5, 7, 9 **C** 0, 1, 2, 3 **D** 1, 3, 5, 7 **E** 2, 3, 4, 5

Video playlist
Arithmetic sequences

Worksheets
Arithmetic sequences

Arithmetic progressions

1.2 Arithmetic sequences

The simplest kind of sequence is one that goes up (or down) by the same amount from one term to the next. Sequences of this type are called **arithmetic sequences** or **arithmetic progressions** (abbreviated as AP).

Arithmetic sequences

An arithmetic sequence increases by a constant amount, ***d***, called the **common difference**.

If d is negative, then the arithmetic sequence decreases.

An arithmetic sequence can be defined recursively as $t_{n+1} = t_n + d$.

The first term is often written as $a = t_1$.

d is the difference between consecutive terms, written as $d = t_{n+1} - t_n$ or $d = t_n - t_{n-1}$.

Consider this sequence of numbers: 3, 9, 15, 21, 27.

The number 6 is added each time to make the next term.

This is an arithmetic sequence, where $d = 6$ is the common difference and $a = 3$ is the 1st term.

WB p. 3

WORKED EXAMPLE 3 Arithmetic sequence

Determine if each sequence is arithmetic and, if so, write the next 4 terms.

a 3, 5, 7, 9 … **b** 1, 2, 5, 10 … **c** $1, 1+\sqrt{5}, 1+2\sqrt{5}, 1+3\sqrt{5}$ …

Steps	Working
a 1 Evaluate $t_2 - t_1$.	$t_2 - t_1 = 5 - 3 = 2$
2 Evaluate $t_3 - t_2$.	$t_3 - t_2 = 7 - 5 = 2$
3 Evaluate $t_4 - t_3$.	$t_4 - t_3 = 9 - 7 = 2$
4 The difference is always 2.	The sequence is arithmetic.
Each term is 2 more than the previous one.	$a = 3, d = 2$
5 Write the next 4 terms for the sequence.	The next 4 terms are 11, 13, 15, 17.

b 1 Evaluate $t_2 - t_1$ and $t_3 - t_2$ and $t_4 - t_3$.	$t_2 - t_1 = 2 - 1 = 1$ $t_3 - t_2 = 5 - 2 = 3$ $t_4 - t_3 = 10 - 5 = 5$
2 The increases from one term to the next are different.	The sequence is NOT arithmetic.
c 1 Evaluate $t_2 - t_1$ and $t_3 - t_2$ and $t_4 - t_3$.	$t_2 - t_1 = 1 + \sqrt{5} - 1 = \sqrt{5}$ $t_3 - t_2 = 1 + 2\sqrt{5} - (1 + \sqrt{5}) = \sqrt{5}$ $t_4 - t_3 = 1 + 3\sqrt{5} - (1 + 2\sqrt{5}) = \sqrt{5}$
2 Each term is $\sqrt{5}$ more than the previous one.	The sequence is arithmetic. $a = 1, d = \sqrt{5}$
3 Write the next 4 terms for the sequence.	The next 4 terms are $1 + 4\sqrt{5}, 1 + 5\sqrt{5}, 1 + 6\sqrt{5}, 1 + 7\sqrt{5}$.

For any arithmetic sequence with common difference d,

$t_1 = a$

$t_2 = a + d$

$t_3 = t_2 + d = (a + d) + d = a + 2d$

$t_4 = t_3 + d = (a + 2d) + d = a + 3d$

This gives a formula for the **general term** or ***n*th term, t_n, of an arithmetic sequence**.

General term of an arithmetic sequence

$$t_n = a + (n - 1)d$$

where a is the first term and d is the common difference.

WORKED EXAMPLE 4 Terms in an arithmetic sequence

Find the 20th term of the arithmetic sequence 4, 11, 18 …

Steps	Working
1 Write the first term, a.	$a = 4$
2 Find the common difference, d.	$d = 11 - 4 = 7$
3 Write the number of terms, n.	$n = 20$
4 Substitute the values into the formula $t_n = a + (n - 1)d$.	$t_n = a + (n - 1)d$ $t_{20} = 4 + (20 - 1) \times 7$ $= 137$
Check the answer by listing the first 20 terms of the arithmetic sequence.	The 20th term of the sequence is 137.

WB p. 4

WB p. 5

WORKED EXAMPLE 5 Finding a term of an arithmetic sequence

Which term of the arithmetic sequence 3, 7, 11 … is equal to 111?

Steps	Working
1 Write the first term, a, and the common difference, d.	$a = 3$ $d = 7 - 3 = 4$
2 Write the formula for the nth term and make it equal to 111.	$t_n = a + (n - 1)d = 111$
3 Substitute the values into the formula and solve for n.	$3 + (n - 1)4 = 111$ $3 + 4n - 4 = 111$ $4n - 1 = 111$ $4n = 112$ $n = 28$
4 Write the answer.	The 28th term in the sequence is 111.

Check the answer by listing the first 28 terms of the arithmetic sequence.

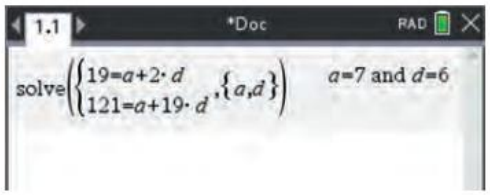
WB p. 6

WORKED EXAMPLE 6 Finding an arithmetic sequence

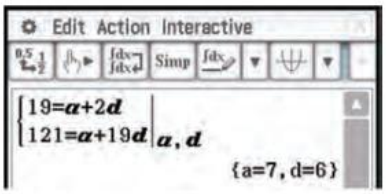
Find the first 4 terms of the arithmetic sequence with a 3rd term of 19 and a 20th term of 121.

Steps	Working
1 Write t_3 in the form $a + (n - 1)d$.	$t_3 = a + 2d = 19$
2 Write t_{20} in the form $a + (n - 1)d$.	$t_{20} = a + 19d = 121$
3 Solve simultaneously to find a and d.	$a = 7, d = 6$

TI-Nspire



ClassPad



4 Use $a = 7$ and $d = 6$ to write the AP.	7, 13, 19, 25.

EXERCISE 1.2 Arithmetic sequences

ANSWERS p. 564

Recap

1 TECH-FREE Using the recurrence relation $a_n = 2n + 3$, find the first 5 terms of the sequence, where n is a natural number.

2 Find the next 3 terms in the sequence: $-1, -10, -19$ …

Mastery

3 WORKED EXAMPLE 3 Test whether each sequence is arithmetic. If so, write the next 2 terms.

a $1 + \sqrt{2}, 1 + 2\sqrt{2}, 1 + 3\sqrt{2}$ … **b** 1, 1, 2, 3, 5, 8 … **c** $1, -5, -11$ …

4 WORKED EXAMPLE 4 Find the 10th term of the arithmetic sequence $1, -5, -11$ …

5 WORKED EXAMPLE 5 Which term of the arithmetic sequence 1, 3, 5, 7 … is equal to 11?

6 WORKED EXAMPLE 6 The first 2 terms of the arithmetic sequence with a 3rd term of 10 and a 10th term of 20 are

A 10, 12 **B** 20, 22 **C** $\frac{10}{7}, \frac{50}{7}$ **D** $\frac{10}{7}, \frac{20}{7}$ **E** $\frac{50}{7}, \frac{60}{7}$

7 Which sequence is NOT an arithmetic sequence?

A $1, 1+\sqrt{2}, 1+2\sqrt{2}, 1+3\sqrt{2}$ … **B** 1, 3, 5, 7, 9 … **C** $1, \frac{3}{2}, 2, \frac{5}{2}$ …

D 1, −2, 4, −8 … **E** 3, −1, −5, −9 …

8 Which sequence is an arithmetic sequence?

A $1, \sqrt{2}, 1+2\sqrt{2}, 1+3\sqrt{2}$ … **B** 1, 3, 6, 10, 15 … **C** $1, \frac{3}{2}, 2, \frac{5}{2}$ …

D 1, −2, 4, −8 … **E** 1, 1, 2, 3, 5, 8 …

9 TECH-FREE Write the first 3 terms for each arithmetic sequence.

a $a = 3, d = 5$ **b** $a = 6, d = -3$ **c** $a = -5, d = -2$

10 TECH-FREE Write the 10th term for each arithmetic sequence.

a 1, 5, 9, 13 … **b** 4, −1, −6, −11 … **c** 1.3, 0.9, 0.5, …

11 TECH-FREE Find the number of terms in each arithmetic sequence.

a 6, 10, 14 … 82 **b** 9, 15, 21 … 195 **c** 14, 23, 32 … 554

12 Find the formula for the general term (t_n) for each arithmetic sequence.

a $d = 2, t_5 = 15$ **b** $a = 7, t_8 = 35$ **c** $a = 3, t_{11} = -53$

Exam practice

13 TECH-FREE (2 marks) Find the 35th term of the sequence 213, 208, 203 …

14 TECH-FREE (2 marks) Find the 17th term of the sequence $a + 5d, a + 12d, a + 19d$ …

15 TECH-FREE (2 marks) For a particular arithmetic sequence, $t_1 = 8$ and $t_{14} - t_4 = 30$. Find the value of t_{20}.

16 Which term of the arithmetic sequence −2, −4, −6, −8 … is equal to −30?

A 3rd **B** 10th **C** 13th **D** 14th **E** 15th

17 The arithmetic sequence with a 4th term of 1 and a 12th term of 10 is

A $-\frac{7}{2}, -\frac{19}{8}, -\frac{5}{4}, -\frac{1}{8}$ … **B** $\frac{7}{2}, \frac{19}{8}, \frac{5}{4}, \frac{1}{8}$ … **C** $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1$ …

D $\frac{19}{8}, \frac{10}{8}, \frac{1}{8}, 1$ … **E** $-\frac{19}{8}, -\frac{10}{8}, -\frac{1}{8}, 1$ …

18 The common difference for the arithmetic sequence $1, -\frac{1}{2}, -2, -\frac{7}{2}$ … is

A $d = -\frac{5}{2}$ **B** $d = -\frac{3}{2}$ **C** $d = -\frac{1}{2}$ **D** $d = \frac{1}{2}$ **E** $d = \frac{3}{2}$

19 The 10th term of the arithmetic sequence 2, 11, 20 … is

A 27 **B** 29 **C** 81 **D** 83 **E** 92

20 (3 marks) The 7th floor of a building is 23 metres above street level and the 13th floor is 41 metres above street level. All floors have the same height. How high above street level is the top of the first floor (ground floor), and what is the height of each floor?



Video playlist
Arithmetic series

Worksheets
Arithmetic series

Sequences and series

Arithmetic series assignment

Arithmetic series extension

1.3 Arithmetic series

The **sum** of the terms in a sequence is called a **series**, represented by S_n, called the **sum to *n* terms**.

$S_n = t_1 + t_2 + t_3 + t_4 + \ldots + t_n$

For example, for the sequence 1, 2, 5, 10 …

$t_4 = 10$

but $S_4 = 1 + 2 + 5 + 10 = 18$.

For an arithmetic series:

$S_n = a + (a + d) + (a + 2d) + (a + 3d) + \ldots + [a + (n - 1)d]$

If we call the last term of the series l, then we can rewrite this as:

$S_n = a + (a + d) + (a + 2d) + (a + 3d) + \ldots + (l - 3d) + (l - 2d) + (l - d) + l$

Adding the first and last terms gives $a + l$.

Adding the 2nd and 2nd-last terms gives $(a + d) + (l - d) = a + l$.

Adding the 3rd and 3rd-last terms gives $(a + 2d) + (l - 2d) = a + l$.

Continuing this way, we get pairs of terms that each add up to $a + l$.

There are n terms in S_n, so there are $\frac{1}{2}n$ pairs of terms that each add up to $a + l$.

This proof also works for an odd number of terms. If n is odd, then the middle term is paired with itself and then halved.

So $S_n = \frac{1}{2}n(a + l)$, more commonly written as $S_n = \frac{n}{2}[a + l]$.

Because $l = t_n = [a + (n - 1)d]$,

$S_n = \frac{n}{2}[a + a + (n - 1)d] = \frac{n}{2}[2a + (n - 1)d]$

Arithmetic series

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

or $$S_n = \frac{n}{2}[a + l]$$

where a is the first term, d is the common difference and l is the last term.

Also, for any series:

$S_n = S_{n-1} + t_n$

WORKED EXAMPLE 7 Arithmetic series

Evaluate the arithmetic series 9 + 14 + 19 + 24 + … for 10 terms.

Steps	Working
1 Write down the information provided.	$a = 9$, $d = 5$ and $n = 10$
2 Evaluate the sum of the 10 terms using the formula for S_n.	$S_n = \frac{n}{2}[2a + (n-1)d]$
	$S_{10} = \frac{10}{2}[2(9) + (10-1)5]$
	$= 315$
Check the answer by adding the first 10 terms of the arithmetic sequence.	

WB p. 7

WORKED EXAMPLE 8 Terms in an arithmetic series

If 10 + 13 + 16 + … + t_n = 198, how many terms are there in the series?

Steps	Working
1 Write down the information provided.	$a = 10$, $d = 3$, $S_n = 198$
2 Substitute values into S_n to find n.	$S_n = \frac{n}{2}[2a + (n-1)d]$
	$198 = \frac{n}{2}[2(10) + (n-1)3]$
	$198 = \frac{n}{2}[20 + 3n - 3]$
	$198 = \frac{n}{2}[3n + 17]$
3 Solve the equation for n using CAS.	$n = 9$ or $-\frac{44}{3}$
4 n must be a natural number.	$n = 9$
Check the answer by adding the first 9 terms of the arithmetic sequence.	There are 9 terms in the series.

WB p. 8

The formula for a term in an AP: $t_n = a + (n-1)d$, combined with the formula for the sum of an AP: $S_n = \frac{n}{2}[2a + (n-1)d]$, can be used to find new information about the sequence.

WORKED EXAMPLE 9 Finding an arithmetic series

The 6th term of an arithmetic series is 23 and the sum of the first 10 terms is 210.

Find the sum of the first 20 terms.

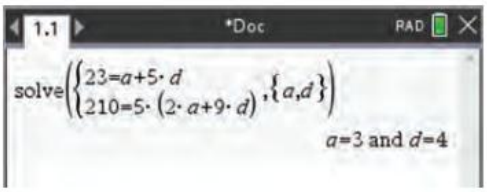
Steps	Working
1 Write down the information provided.	$t_6 = 23$, $S_{10} = 210$
2 Substitute values into the formulas for t_n and S_n.	$t_n = a + (n-1)d$
	$t_6 = a + (6-1)d = 23$
	$a + 5d = 23$ … [1]
	$S_n = \frac{n}{2}[2a + (n-1)d]$
	$S_{10} = \frac{10}{2}[2a + (10-1)d] = 210$
	$5(2a + 9d) = 210$ … [2]

WB p. 9

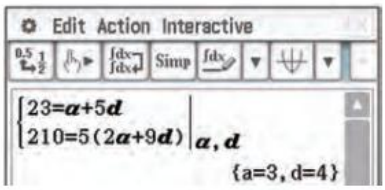
3 Use CAS to solve simultaneous equations [1] and [2] to find a and d.

$a = 3, d = 4$

TI-Nspire



ClassPad



4 Calculate the value of S_{20} using S_n.

$a = 3, d = 4$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$S_{20} = \frac{20}{2}[2(3) + (20-1)4]$

$= 820$

EXERCISE 1.3 Arithmetic series

ANSWERS p. 564

Recap

1 Which sequence is NOT an arithmetic sequence?

A $1, 1 + x, 1 + 2x, 1 + 3x \ldots$ **B** $11, 21, 31 \ldots$ **C** $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \ldots$

D $-10, -20, -30 \ldots$ **E** $-3, -1, 1, 3 \ldots$

2 Which sequence is an arithmetic sequence?

A $1 + x, 1 + 2x, 1 + 3x \ldots$ **B** $1, 3, 6, 10, 15 \ldots$ **C** $\frac{1}{4}, \frac{1}{8}, \frac{1}{16} \ldots$

D $1, -2, 4, -8 \ldots$ **E** $1, 1, 2, 3, 5, 8 \ldots$

Mastery

3 WORKED EXAMPLE 7 TECH-FREE Evaluate the arithmetic series $3 + 6 + 9 + \ldots$ for 10 terms.

4 WORKED EXAMPLE 8 TECH-FREE If $1 + 6 + 11 + \ldots + t_n = 112$, how many terms are there in the series?

5 WORKED EXAMPLE 9 The 5th term of an arithmetic series is 20 and the sum of the first 5 terms is 120.

a Find t_1 and the common difference, d. **b** Find the sum of the first 10 terms.

6 TECH-FREE Find the next 2 terms in the AP: 11, 21, 31 …

7 TECH-FREE Find the next 3 terms in the AP: −10, −20, −30 …

8 The sum of the first 15 terms of the arithmetic series $60 + 56 + 52 + \ldots$ is

A −2280 **B** 4 **C** 60 **D** 480 **E** 1320

9 For the series $1 + 11 + 21 + \ldots + t_n = 561$, the number of terms in the series is

A 2.5 **B** 10.5 **C** 11 **D** 21 **E** 561

10 The 4th term of an arithmetic series is 2 and the sum of the first 10 terms is 120. The sum of the first 20 terms is

A $-\frac{9860}{3}$ **B** −32 **C** 120 **D** $\frac{2720}{3}$ **E** 2360

11 For the table, the recurrence relation a_{n+1} and a_o respectively are given by

n	a_n
1	1
2	3
3	7
4	15
5	31

A $a_{n+1} = 2a_n + 1, a_o = 1$ **B** $a_{n+1} = 2a_n, a_o = 0$

C $a_{n+1} = 2a_n - 1, a_o = 0$ **D** $a_{n+1} = a_n + 1, a_o = 1$

E $a_{n+1} = 2a_n + 1, a_o = 0$

12 In an arithmetic progression where $a = 3$ and $d = \frac{1}{2}$, the 5th term is

A $3\frac{1}{2}$ **B** 4 **C** 5 **D** $6\frac{1}{2}$ **E** 7

13 TECH-FREE How many terms of the series $45 + 47 + 49 + \ldots + t_n$ give a sum of 1365?

14 TECH-FREE Evaluate each arithmetic series.

a $15 + 20 + 25 + \ldots + 535$ **b** $9 + 17 + 25 + \ldots + 225$ **c** $5 + 2 - 1 - \ldots - 91$

d $81 + 92 + 103 + \ldots + 378$ **e** $9 + 14 + 19 + \ldots + 224$

Exam practice

15 TECH-FREE (2 marks) The sum of the first 5 terms of an arithmetic series is 110 and the sum of the first 10 terms is 320. Find the first term and the common difference.

16 TECH-FREE (2 marks) How many terms of the series $80 + 73 + 66 + \ldots$ must be added to give a sum of 498?

17 TECH-FREE (2 marks) The sum of n terms of the series $18 + 15 + 12 + \ldots$ is 33. Find all possible values of n.

18 The sum of the first 5 terms of an arithmetic series is 35 and the sum of the next 5 terms is 160. The first term and the common difference are respectively

A $a = \frac{1}{5}, d = \frac{18}{5}$ **B** $a = -3, d = 5$ **C** $a = \frac{18}{5}, d = \frac{1}{5}$

D $a = 3, d = 5$ **E** $a = \frac{1}{5}, d = -\frac{18}{5}$

19 (6 marks) A furnishing company makes thread with the following thicknesses, in order, 2, 9, 16, 23, 30 … measured in millimetres.

a Find the thickness of the company's 10th thread. 1 mark

b If put all together side by side, find the total thickness of the company's smallest 5 threads. 2 marks

c If the largest thread in this sequence has a thickness of 100, how many individual thicknesses does the company produce? 2 marks

d Find the sum of all thicknesses the company produces. 1 mark

20 (4 marks) A 'tiny' house is built in a hillside 6 metres high above a road. Unfortunately, the house does not have proper foundations, causing it to slip down the hill. The slippage follows an arithmetic sequence. By the end of the first year, the house had slipped down the hill 4 centimetres, and by the end of the second year, the house had slipped down the hill a further 16 centimetres.

a How far does the house move down the hill over 5 years? 2 marks

b After 10 years, will the house end up on the road? 2 marks

Video playlist
Geometic sequences

Worksheets
Geometric sequences
Classifying sequences
Geometric progressions 1
Geometric progressions 2
Modelling arithmetic and geometric sequences

1.4 Geometric sequences

In the book and film, *Pay It Forward*, an 11-year-old boy starts a movement to do good for others. He decides to help 3 people in a way that will change their lives, and then asks each one of them to pay it forward to 3 others, and so on. If successful, then the number of people affected by this movement will grow exponentially.

Number of rounds	Number of people reached
1	$3 = 3^1$
2	$9 = 3^2$
3	$27 = 3^3$
4	$81 = 3^4$
5	$243 = 3^5$

The numbers 3, 9, 27, 81, 243 … represent a **geometric sequence** or **geometric progression** (abbreviated as GP). Instead of adding the same number each time, we multiply by the same number (in this case 3) in each instance.

Geometric sequences

A geometric sequence increases by multiplying a constant amount, ***r***, called the **common ratio**.

If r is a fraction between -1 and 1, then the geometric sequence decreases.

A geometric sequence can be defined recursively as $t_{n+1} = rt_n$.

The first term is often written as $a = t_1$.

r is the ratio between consecutive terms, written as $r = \frac{t_{n+1}}{t_n}$ or $r = \frac{t_n}{t_{n-1}}$.

Consider this sequence of numbers: 5, 15, 45, 135, 405.

Each term is multiplied by 3 to make the next term.

This is a geometric sequence, where $r = 3$ is the common ratio and $a = 5$ is the first term.

WB
p. 10

WORKED EXAMPLE 10 Common ratio of geometric sequence

Determine the common ratio for each geometric sequence.

a 4, 12, 36, 108 …

b $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}$ …

Steps	Working
a 1 Evaluate $\frac{t_2}{t_1}$ and $\frac{t_3}{t_2}$ and $\frac{t_4}{t_3}$.	$\frac{t_2}{t_1} = \frac{12}{4} = 3$ $\frac{t_3}{t_2} = \frac{36}{12} = 3$ $\frac{t_4}{t_3} = \frac{108}{36} = 3$
2 The ratio is always 3. Each term is 3 times the previous one.	Common ratio, $r = 3$.

b **1** Evaluate $\frac{t_2}{t_1}$ and $\frac{t_3}{t_2}$.

$$\frac{t_2}{t_1} = \frac{-\frac{1}{2}}{1} = -\frac{1}{2}$$

$$\frac{t_3}{t_2} = \frac{\frac{1}{4}}{-\frac{1}{2}} = -\frac{1}{2}$$

2 Each term is $-\frac{1}{2}$ times the previous one. Common ratio, $r = -\frac{1}{2}$.

For any geometric sequence with common ratio, r,

$t_2 = rt_1 = ra = ar$

$t_3 = rt_2 = r \times ar = ar^2$

$t_4 = rt_3 = r \times ar^2 = ar^3$

This gives the formula for the **general term** or ***n*th term, t_n, of a geometric sequence**.

General term of a geometric sequence

$$t_n = ar^{n-1}$$

where a is the first term and r is the common ratio.

WORKED EXAMPLE 11 Terms of a geometric sequence

What is the 7th term in the geometric sequence 6, –30, 150 …?

Steps	Working
1 Write the first term, a.	$a = 6$
2 Find the common ratio, r.	$r = \frac{150}{-30} = \frac{-30}{6} = -5$
3 Write the number of terms, n.	$n = 7$
4 Substitute the values into the formula.	$t_n = ar^{n-1}$ $t_7 = 6 \times (-5)^{7-1}$ $= 6 \times (-5)^6$ $= 93\,750$
5 Write the answer.	The 7th term is 93 750.

WB p. 11

WORKED EXAMPLE 12 Finding a term of a geometric sequence

Which term of the geometric sequence –2, 1, $-\frac{1}{2}$ … is equal to $-\frac{1}{128}$?

Steps	Working
1 Write the first term, a, and the common ratio, r.	$a = -2$ $r = \frac{1}{-2} = -\frac{1}{2}$
2 Substitute into the formula, make $t_n = -\frac{1}{128}$ and solve for n using CAS.	$t_n = ar^{n-1}$ $t_n = -2 \times \left(-\frac{1}{2}\right)^{n-1} = -\frac{1}{128}$ $n = 9$
3 Write the answer.	$-\frac{1}{128}$ is the 9th term of the geometric sequence.

WB p. 12

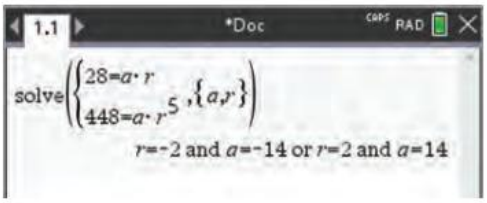
WB p. 13

WORKED EXAMPLE 13 Finding a geometric sequence

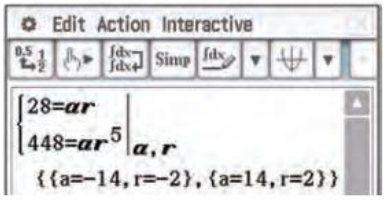
The 2nd and 6th terms of a geometric sequence are 28 and 448 respectively.
What is the 10th term of the sequence?

Steps	Working
1 Write t_2 and t_6 in the form $t_n = ar^{n-1}$ and set them equal to 28 and 448 respectively.	$t_2 = ar^{2-1} = 28$ $ar = 28$ … [1] $t_6 = ar^{6-1} = 448$ $ar^5 = 448$ … [2]
2 Use CAS to solve these simultaneous equations to find a and r.	$a = -14, r = -2$ or $a = 14, r = 2$

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3 Use $a = -14, r = -2$ or $a = 14, r = 2$ to find the 10th term of the sequence.	$t_{10} = ar^9$ $= (-14)(-2)^9$ or $(14)(2)^9$ $= 7168$ (the same in both cases)

Convergence of a geometric sequence

What happens to the terms of a geometric sequence as $n \to \infty$?

Compare the sequences

3, 6, 12, 24, 48 … and 8, 4, 2, 1, $\frac{1}{2}$ …

as $n \to \infty$.

The size of the terms in the first sequence get bigger. They are said to **diverge**.

The size of the terms in the second sequence get smaller (and closer to 0). They are said to **converge** to 0 because the common ratio is a fraction; that is, for $-1 < r < 1$. As $n \to \infty$, $t_n \to 0$.

Geometric sequences with $-1 < r < 1$ converge to 0.

Geometric sequences with $r > 1$ or $r < -1$ diverge.
Their values increase in size without bound.

What happens to the terms if $r = 1$ or -1?

WB p. 14

WORKED EXAMPLE 14 Convergence in a geometric sequence

Do the terms in the geometric sequence $\frac{5}{2}, \frac{5}{6}, \frac{5}{18}$ … converge?

Steps	Working
1 Find the common ratio using $r = \frac{t_2}{t_1} = \frac{t_3}{t_2}$.	$r = \frac{\frac{5}{6}}{\frac{5}{2}} = \frac{1}{3}$ and $r = \frac{\frac{5}{18}}{\frac{5}{6}} = \frac{1}{3}$
2 $r = \frac{1}{3}$ is between -1 and 1.	Since r falls within the range $-1 < r < 1$, the terms of the geometric sequence converge.

EXERCISE 1.4 Geometric sequences

ANSWERS p. 564

Recap

1 Find the next 2 terms in the arithmetic progression $-0.5, -1.5, -2.5 \ldots$

2 In an arithmetic sequence, where $a = -3$ and $d = -2$, the 10th term is

A -23 **B** -21 **C** -15 **D** 21 **E** 1536

Mastery

3 TECH-FREE Find the next 3 terms in the geometric sequence $-10, 20, -40 \ldots$

4 WORKED EXAMPLE 10 TECH-FREE Find the common ratio for the geometric sequence $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27} \ldots$

5 WORKED EXAMPLE 11 TECH-FREE What is the 9th term in the geometric sequence $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27} \ldots$?

6 WORKED EXAMPLE 12 TECH-FREE Which term of the geometric sequence $-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8} \ldots$ is equal to $\frac{1}{256}$?

7 WORKED EXAMPLE 13 The 2nd term of a geometric sequence is $-\frac{1}{2}$ and the 6th term is $-\frac{1}{32}$. Find the 20th term.

8 WORKED EXAMPLE 14 The terms in the sequence $\frac{3}{2}, \frac{1}{2}, \frac{1}{6} \ldots$

A are equal **B** get larger **C** converge
D diverge **E** are all divisible by 2

9 Which sequence is a geometric sequence?

A $x, 2x, 3x, 4x \ldots$ **B** $11, 21, 31 \ldots$ **C** $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \ldots$
D $-10, -20, -30 \ldots$ **E** $-3, -1, 1, 3 \ldots$

10 Which sequence is NOT an arithmetic sequence?

A $x, 2x, 3x, 4x \ldots$ **B** $a, 3a, 5a, 7a \ldots$ **C** $\frac{1}{2}, \frac{3}{4}, 1 \ldots$
D $1, -2, 4, -8 \ldots$ **E** $1, 2, 3, 4 \ldots$

11 For the table, the recurrence relation a_{n+1} and a_o respectively are given by

A $a_{n+1} = a_n - 21, a_o = 2$ **B** $a_{n+1} = 2a_n, a_o = 2$
C $a_{n+1} = a_n - 21, a_o = 1$ **D** $a_{n+1} = 2a_n - 21, a_o = 1$
E $a_{n+1} = a_n + 21, a_o = 2$

n	a_n
1	−19
2	−40
3	−61
4	−82
5	−103

12 Find the common ratio for each sequence.

a $1, 2, 4, 8 \ldots$ **b** $-4, -12, -36, -108 \ldots$ **c** $-4, 12, -36, 108 \ldots$

13 Find the common ratio and the next 2 terms in each of these geometric sequences.

a 1, 3, 9 … **b** 3, −6, 12 … **c** $\frac{1}{2}$, 2, 8 … **d** 13, −13, 13 …

14 Find all possible values of x in these geometric sequences.

a 2, x, 50 … **b** $\frac{1}{2}$, x, 8 … **c** 5, $2\frac{1}{2}$, x … **d** 81, 27, x …

Exam practice

15 TECH-FREE (2 marks) Write the 10th term of the geometric sequence −5, 10, −20 …

16 TECH-FREE (2 marks) Write the 15th term of the arithmetic sequence 1.041, 1.042, 1.043 …

17 TECH-FREE (2 marks) Which term of the geometric sequence 4, 12, 36 … is equal to 78 732?

18 TECH-FREE (2 marks) Which term of the geometric sequence 4, 20, 100 … is equal to 12 500?

19 TECH-FREE (2 marks) Which term of the geometric sequence starting with 6, 36, 216 … is equal to 7776?

20 TECH-FREE (2 marks) Which term of the sequence 8, 4, 2 … is $\frac{1}{128}$?

21 TECH-FREE (2 marks) The 3rd term in a geometric sequence is 18 and the 7th term is 1458. Find the first term and the common ratio.

22 TECH-FREE (2 marks) Find x if 5, x, 45 form a geometric sequence.

23 TECH-FREE (2 marks) The 4th term of a geometric sequence is −648 and the 5th term is 3888. Find the 7th term.

24 TECH-FREE (2 marks) The 1st and 5th terms of a geometric sequence are 9 and 2304. Find the 2 possible values of the common ratio.

25 TECH-FREE (4 marks) State whether each geometric sequence converges or diverges as $n \to \infty$.

a 1000, 500, 250, 125 … **b** 5, 10, 20, 40 …

c −8, 4, −2, 1 … **d** 0.004, 0.04, 0.4, 4 …

26 The 10th term in the geometric sequence 6, −3, 1.5 … is

A $-\frac{3}{512}$ **B** $-\frac{3}{256}$ **C** $\frac{1}{1024}$ **D** $\frac{3}{512}$ **E** $\frac{3}{256}$

27 The position of the term of the geometric sequence 18, 6, 2 … that is equal to $\frac{2}{243}$ is

A 1st **B** 4th **C** 7th **D** 8th **E** no solution

28 The 3rd and 7th terms of a geometric sequence are 3 and 12 respectively. If the common ratio is positive, then the 10th term of the sequence is

A $-24\sqrt{2}$ **B** −24 **C** 24 **D** $24\sqrt{2}$ **E** 48

1.5 Geometric series

S_n is the **sum to *n* terms of a series**.

$$S_n = t_1 + t_2 + t_3 + t_4 + \ldots + t_n$$

For a geometric series:

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} \quad \ldots [1]$$

If we multiply both sides of this equation by r, we get:

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \ldots + ar^n \quad \ldots [2]$$

[2] – [1] gives:

$$\begin{aligned} rS_n - S_n &= ar + ar^2 + ar^3 + ar^4 + \ldots + ar^n - [a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}] \\ &= ar^n - a \end{aligned}$$

$$S_n(r-1) = a(r^n - 1)$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

Geometric series

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

where a is the first term and r is the common ratio.

If $r < 1$, it is more convenient to use this equivalent formula:

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

to ensure positive values in the numerator and denominator (but either formula will work because they are equivalent).

WORKED EXAMPLE 15 Geometric series

Evaluate the geometric series $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \ldots$ for 7 terms.

Steps	Working
1 Write down the information provided.	$a = \frac{1}{3}, r = \frac{1}{9} \div \frac{1}{3} = \frac{1}{3}, n = 7$
2 Evaluate the sum of the 7 terms using the formula for S_n.	$S_n = \frac{a(1 - r^n)}{1 - r}$ $S_7 = \frac{\frac{1}{3}\left[1 - \left(\frac{1}{3}\right)^7\right]}{1 - \frac{1}{3}}$ $= \frac{1093}{2187}$

1.5

Video playlist
Geometic series

Worksheets
Geometric series 1
Geometric series 2
Mixed sequences and series
Geometric series assignment
Geometric series problems

WB
p. 15

WB p. 16

WORKED EXAMPLE 16 Number of terms in a geometric series

If $2 + 4 + 8 + \ldots + t_n = 2046$, how many terms are there in this geometric series?

Steps	Working
1 Write down the information provided.	$a = 2, r = \frac{4}{2} = 2, S_n = 2046$
2 Substitute values into formula for S_n.	$S_n = \frac{a(r^n - 1)}{r - 1}$ $2046 = \frac{2(2^n - 1)}{2 - 1}$
3 Solve for n.	$n = 10$
4 Write the answer.	There are 10 terms in the series.

WB p. 17

WORKED EXAMPLE 17 Finding a term in a geometric series

The common ratio of a geometric series is 4 and the sum of the first 5 terms is 3069.
Find the first term.

Steps	Working
1 Write down the information provided.	$r = 4, S_5 = 3069$
2 Substitute values into $S_n = \frac{a(r^n - 1)}{r - 1}$.	$S_n = \frac{a(r^n - 1)}{r - 1}$ $S_5 = \frac{a(4^5 - 1)}{4 - 1} = 3069$
3 Solve for a.	$a = 9$
4 Write the answer.	The first term of the geometric series is 9.

EXERCISE 1.5 Geometric series

ANSWERS p. 565

Recap

1 TECH-FREE Find the next 3 terms in the GP $\frac{1}{2}, \frac{1}{4}, \frac{1}{8} \ldots$

2 TECH-FREE Find the next 2 terms in the GP $\ldots -1, 5, -25 \ldots$

Mastery

3 WORKED EXAMPLE 15 TECH-FREE Evaluate the geometric series $2 + 6 + 18 + \ldots$ for 7 terms.

4 WORKED EXAMPLE 16 If $1 + 4 + 16 + \ldots + t_n = 1365$, how many terms are there in this geometric series?

5 WORKED EXAMPLE 17 The common ratio of a geometric series is $\frac{1}{2}$ and the sum of the first 5 terms is $\frac{93}{16}$. Find the first term.

Exam practice

6 TECH-FREE (2 marks) Find the sum of the first 7 terms of the geometric sequence 5, 15, 45 …

7 TECH-FREE (2 marks) Find the 4th term of the geometric series $5 + 5.5 + 6.05 + \ldots$

8 TECH-FREE (2 marks) Find t_5 for the geometric series $16 - 12 + 9 - 6\frac{3}{4} + \ldots$

9 TECH-FREE (2 marks) Find S_7 for the geometric sequence $2\frac{2}{3}$, 4, 6 …

10 TECH-FREE (2 marks) Find the 9th term of the geometric series $7 + 14 + 28 + \ldots$

11 TECH-FREE (2 marks) How many terms are there in the geometric series if $-3 + 6 - 12 + \ldots = 255$?

12 TECH-FREE (2 marks) Calculate 2 possible sums of the first 4 terms of the geometric sequence where $t_3 = 20$ and $t_5 = 80$.

13 TECH-FREE (3 marks) In a geometric series, the 4th and 6th terms are 56 and 224 respectively. Find 2 possible values for the sum of the first 5 terms.

14 The geometric series $1 + 3 + 9 + 27 + \ldots$ when evaluated for 10 terms is

A 40 **B** 121 **C** 9841 **D** 29 524 **E** 88 572

15 If $1 + 3 + 9 + 27 + \ldots + t_n = 1093$, how many terms are there in this geometric series?

A 4 **B** 5 **C** 6 **D** 7 **E** 8

16 The sum of a geometric series for the first 2 terms is $\frac{3}{2}$ and the sum of the first 5 terms is $\frac{31}{16}$. If the common ratio is positive, then the first term is

A $\frac{1}{2}$ **B** 1 **C** $\frac{3}{2}$ **D** 2 **E** 5

17 (2 marks) Find the number of terms needed for the sum to exceed 1 000 000 in the series $4 + 16 + 64 + \ldots$

18 (3 marks) A ball is dropped from a height of 24 metres and rebounds to a height of 16 metres. If it continues to rebound to two-thirds of the previous height each time, find the total distance it travels after bouncing exactly 5 times.

19 (4 marks) Lucia earned $2 000 for her first month's work in a casual job. Her wage increases by 5% each month. Find

a her wage after 6 months. 2 marks

b her total earnings over the 6 months. 2 marks

20 (3 marks) The sum of the first 5 terms in a geometric sequence is 77 and the sum of the next 5 terms is −2464. Find the 4th term of the sequence.

21 (3 marks) How many terms of the geometric sequence 3, 4, $5\frac{1}{3}$, $7\frac{1}{9}$ … must be taken for the sum to exceed 500?

Video playlist
The infinite geometric series

Worksheet
Limiting sum of a geometric series

1.6 The infinite geometric series

A student walks halfway across the classroom, then half of that, then half of that again. Do they reach the other side of the classroom? This scenario is an example of an infinite geometric series with the terms $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\ldots$

If we add these first 6 terms, we get $\frac{63}{64}$. The sum of this series will never actually equal 1, but very close to it. Taking this scenario literally, the student will never actually reach the other side of the classroom. This is an example of a **limiting sum**, a value towards which an infinite series approaches.

Convergence of an infinite geometric series

On page 20, we learned that the terms of a geometric sequence **converge** to 0 when the common ratio is a fraction; that is, for $-1 < r < 1$. Because the terms are getting smaller, a *geometric series* will also **converge** to a limiting sum if r is a fraction.

$$S_n = \frac{a(1-r^n)}{1-r}$$

However, if r is a fraction and the power of a fraction is a smaller fraction, as $n \to \infty$, r^n will become smaller and approach 0, and the limiting sum will be $S_\infty = \frac{a(1-0)}{1-r} = \frac{a}{1-r}$.

Infinite geometric series

$$S_\infty = \frac{a}{1-r}$$

where $-1 < r < 1$.

For $-1 < r < 1$, the sum S_n approaches a limiting value as $n \to \infty$.

Geometric series with $-1 < r < 1$ converge to a limiting sum.

Geometric series with $r > 1$ or $r < -1$ diverge. Their sums increase in size without bound.

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p. 18

WORKED EXAMPLE 18 Infinite geometric series

Find the limiting sum of the infinite geometric series $\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\frac{1}{128}+\ldots$

Steps	Working
1 Find a and r.	$a = \frac{1}{2}$ $r = \frac{1}{8} \div \frac{1}{2} = \frac{1}{4}$
2 Substitute values into the formula for S_∞ to find the limiting sum.	$S_\infty = \frac{a}{1-r}$ $S_\infty = \frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{2}{3}$
3 Write the answer.	The limiting sum is $\frac{2}{3}$.

We can use a calculator or spreadsheet to check the answer by evaluating the sum as a decimal for many terms:

$$\frac{1}{2}+\frac{1}{8}+\frac{1}{32}+\frac{1}{128}+\frac{1}{512}\ldots = 0.666\,01\ldots$$

and $\frac{2}{3} = 0.666\,66\ldots$

The sum is heading towards the limit, but since the sequence is infinite, it will never actually get there.

WORKED EXAMPLE 19 Recurring decimals

Express the recurring decimal $0.\dot{3}\dot{5}$ as a fraction.

Steps	Working
1 Expand $0.\dot{3}\dot{5}$ as an infinite geometric series and find a and r.	$0.\dot{3}\dot{5} = 0.35 + 0.0035 + 0.000\,035 + \ldots$ $a = 0.35,\ r = 0.01$
2 Substitute values into $S_\infty = \frac{a}{1-r}$ to find the limiting sum.	$S_\infty = \frac{a}{1-r}$ $S_\infty = \frac{0.35}{1-0.01}$ $= \frac{0.35}{0.99}$ $= \frac{35}{99}$
3 Write the answer.	$0.\dot{3}\dot{5} = \frac{35}{99}$

Check your answer by dividing 35 by 99 on your calculator.

EXERCISE 1.6 The infinite geometric series

ANSWERS p. 565

Recap

1 TECH-FREE Find the sum of the first 5 terms in the geometric progression $-1, 5, -25 \ldots$

2 In a geometric sequence where the 4th term is 1 and the 6th term is 20, the 7th term is

A $\frac{\sqrt{5}}{200}, 2\sqrt{5}$ **B** $-\frac{\sqrt{5}}{200}, -2\sqrt{5}$ **C** 400 **D** $\pm 40\sqrt{5}$ **E** $40\sqrt{5}$

Mastery

3 WORKED EXAMPLE 18 TECH-FREE Find the limiting sum of the infinite geometric series

$3 + 1\frac{1}{2} + \frac{3}{4} + \frac{3}{8} + \ldots$

4 WORKED EXAMPLE 19 TECH-FREE Express the recurring decimal $0.\dot{0}\dot{3}$ as a fraction.

5 The limiting sum of the infinite geometric series, where the first term is 1 and the common ratio is $\frac{2}{5}$, is

A $\frac{2}{5}$ **B** $\frac{3}{5}$ **C** 1 **D** $\frac{5}{3}$ **E** $\frac{5}{2}$

WB p. 19

6 The recurring decimal $0.\dot{7}$ expressed as a fraction is

A $\frac{7}{10}$ **B** $\frac{7}{9}$ **C** 1 **D** $\frac{9}{7}$ **E** $\frac{10}{7}$

7 TECH-FREE Which of the following series has a limiting sum?

a $9 + 3 + 1 + \ldots$ **b** $\frac{1}{4} + \frac{1}{2} + 1 + \ldots$ **c** $16 - 4 + 1 + \ldots$

d $\frac{2}{3} + \frac{7}{9} + \frac{49}{54} + \ldots$ **e** $1 + \frac{2}{3} + \frac{4}{9} + \ldots$ **f** $\frac{5}{8} + \frac{1}{8} + \frac{1}{40} + \ldots$

8 TECH-FREE Write each recurring decimal as a fraction.

a $0.\dot{4}$ **b** $0.\dot{6}$ **c** $0.\dot{2}\dot{5}$

Exam practice

9 TECH-FREE (2 marks) In an infinite geometric series, the 1st and 2nd terms are 3 and 1 respectively. Find the limiting sum of the series.

10 TECH-FREE (4 marks) In an infinite geometric series, the 4th and 6th terms are 27 and 3 respectively.

a Find possible values of a and r. 2 marks

b Find the limiting sum of the series. 2 marks

11 TECH-FREE (4 marks) Find the limiting sum of each series.

a $40 + 20 + 10 + \ldots$ **b** $320 + 80 + 20 + \ldots$

c $100 - 50 + 25 + \ldots$ **d** $6 + 3 + 1\frac{1}{2} + \ldots$

12 (6 marks) For the geometric series $k + k^2 + k^3 + \ldots$

a for what values of k does the limiting sum exist? 2 marks

b find the limiting sum of the series when $k = -\frac{2}{3}$. 2 marks

c evaluate k if the limiting sum of the series is 3. 2 marks

13 (2 marks) The 1st term of a geometric sequence is 54 and the 4th term is 16. Find the common ratio and the limiting sum.

14 (2 marks) A geometric series has a limiting sum of 5 and a first term of 3. Find the common ratio.

15 (2 marks) The limiting sum of a geometric series is $9\frac{2}{3}$ and the common ratio is $\frac{2}{5}$. Find the first term of the series.

16 (3 marks) A geometric series has a 3rd term of 12 and a 4th term of –3. Find t_1, r and S_∞.

17 (3 marks) The sum $x + \frac{x}{4} + \frac{x}{16} + \ldots$ has a limit of $\frac{7}{8}$. Find the value of x.

1.7 Linear recurrence relations

A **linear recurrence relation** is an equation that relates a term in a sequence to previous terms using recursion. The use of the word 'linear' refers to the fact that the equation is linear and does not have powers of t (that are greater than 1).

For example:

- the sequence 1, 3, 5, 7, 9 … is defined by $t_n = t_{n-1} + 2$ and $t_1 = 1$.
- the sequence $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81} \ldots$ is defined by $t_n = \frac{1}{3}t_{n-1}$ and $t_1 = \frac{1}{3}$.
- the sequence 2, 1, −2, −11, −38 … is defined by $t_{n+1} = 3t_n - 5$ with $t_1 = 2$.

Defining a sequence using the previous term

The next term in a sequence can be determined using a formula involving the previous term for a recurrence relationship.

In general, $t_{n+1} = at_n + b$.

For example, to write down the first 5 terms of the recurrence relationship defined by $x_{n+1} = 2x_n + 3$ where $x_1 = 6$, we start with 6, multiply by 2 and then add 3, to get 6, 15, 33, 69, 141.

In contrast, going from the sequence to the recurrence relationship, we can use the terms $16, -4, 1, -\frac{1}{4}, \frac{1}{16}$ to get the recurrence relationship $t_{n+1} = -\frac{1}{4}t_n$ where $t_1 = 16$.

Video playlist
Linear recurrence relations

Defining a sequence using an explicit rule

The terms in a sequence can be defined by an explicit rule stated in terms of n.

Explicit rule	Sequence terms	Sequence
$t_n = 3n$	$t_1 = 3, t_2 = 6, t_3 = 9, t_4 = 12 \ldots$	3, 6, 9, 12 …
$t_n = 3^n$	$t_1 = 3, t_2 = 9, t_3 = 27, t_4 = 81 \ldots$	3, 9, 27, 81 …
$t_n = 3n - 1$	$t_1 = 2, t_2 = 5, t_3 = 8, t_4 = 11 \ldots$	2, 5, 8, 11 …
$t_n = 9 \times 2^{n-1} - 3$	$t_1 = 6, t_2 = 15, t_3 = 33, t_4 = 69 \ldots$	6, 15, 33, 69 …

WORKED EXAMPLE 20 General term of a sequence

Find the rule for the nth term of the sequence 1, 4, 9, 16, 25 … in terms of n.

Steps

1 Consider $n \in \{1, 2, 3, 4, 5\}$ and compare with the terms of the sequence.

2 The terms are the squares of n.

Working

n	1	2	3	4	5
t_n	1	4	9	16	25

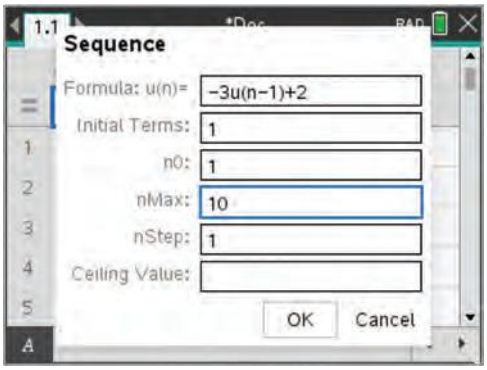
$t_n = n^2$

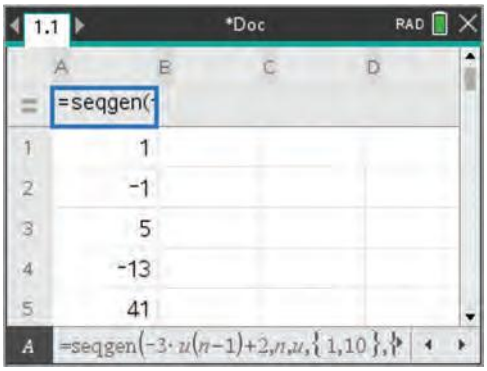
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USING CAS 3 Linear recurrence relations

Generate the first 10 terms of the sequence of numbers defined by the recurrence relation $t_{n+1} = -3n + 2$ starting at $n = 1$.

TI-Nspire





1 Add a **Lists & Spreadsheet** page.

2 Place the cursor in the cell immediately under **A**.

3 Press **menu > Data > Generate Sequence**.

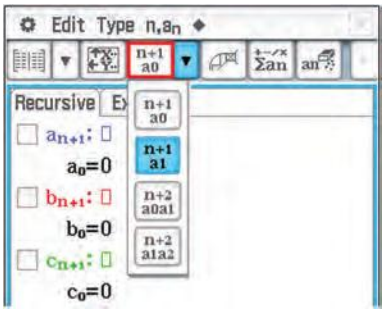
4 Enter the expression and values shown above.

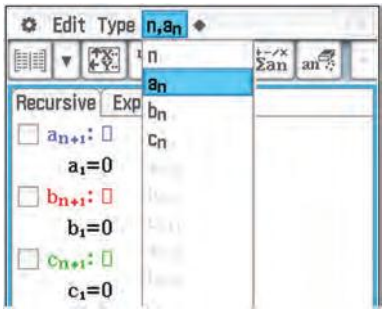
5 Select **OK**.

6 The first 10 terms of the recurrence relation will be displayed in column **A**.

7 Scroll down to see the rest of the sequence.

ClassPad



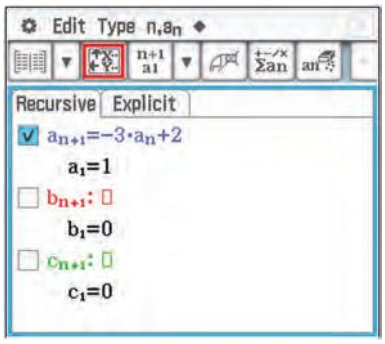


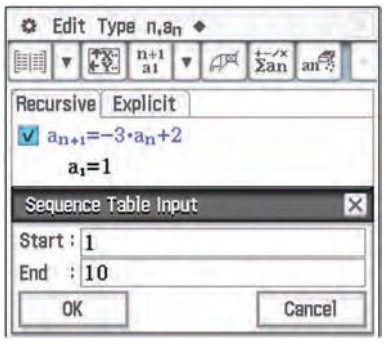
1 Tap **Menu > Sequence**.

2 Tap the **Recursive** tab.

3 Tap the **Sequence Type** tool and select the second option from the dropdown menu.

4 Tap the **n,a_n** menu heading at the top of the screen to select **a_n**.





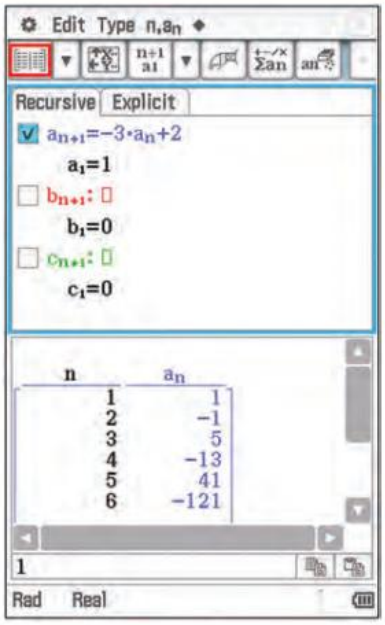
5 Enter $t_{n+1} = -3a_n + 2$.

6 Set $a_1 = 1$.

7 Tick to select the sequence.

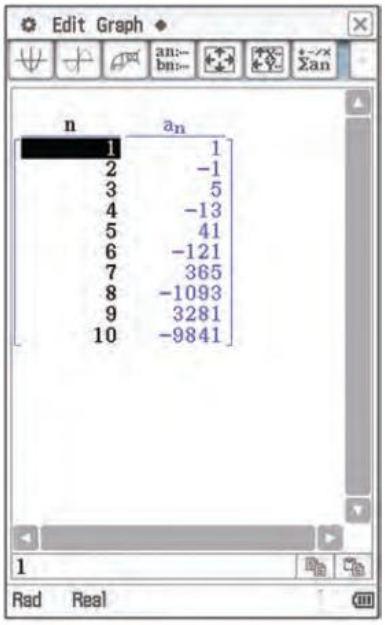
8 Tap the **Sequence Table Input** tool.

9 In the dialogue box, enter a **Start:** value of **1** and an **End:** value of **10**, then tap **OK**.



10 Tap the **Table** tool.

11 The first 10 terms of the recurrence relation will appear in the lower window.



12 Highlight the lower windows and tap **Resize** to view all 10 terms.

WORKED EXAMPLE 21 Linear recurrence relation

Use the recurrence relation to find the first 5 terms of the sequence $t_1 = 3$ and $t_{n+1} = \frac{1}{2}t_n + 5$.

Steps	Working
1 The first term is given.	$t_1 = 3$
2 Find t_n for $n = 2, 3, 4$ using $t_n = \frac{1}{2}t_{n-1} + 5$.	$t_2 = \frac{1}{2}t_1 + 5 = \frac{13}{2}$
	$t_3 = \frac{1}{2}t_2 + 5 = \frac{33}{4}$
	$t_4 = \frac{1}{2}t_3 + 5 = \frac{73}{8}$
	$t_5 = \frac{1}{2}t_4 + 5 = \frac{153}{16}$
3 List the first 5 terms.	$3, \frac{13}{2}, \frac{33}{4}, \frac{73}{8}, \frac{153}{16}$

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WB

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WORKED EXAMPLE 22 The explicit rule of a linear recurrence relation

A sequence of numbers is defined by the linear recurrence relation $t_{n+1} = 4t_n - 3$, with $t_1 = 2$.
Find the explicit rule for t_n in terms of n only.

Steps	Working
1 Write t_1, t_2, t_3, t_4, t_5 using the rule to find a pattern, but do not calculate.	$t_1 = 2$ $t_2 = 4 \times 2 - 3$ $t_3 = 4(4 \times 2 - 3) - 3$ $t_4 = 4[4(4 \times 2 - 3) - 3] - 3$ $t_5 = 4\{4[4(4 \times 2 - 3) - 3] - 3\} - 3$
2 Separate the parts of t_5.	$t_5 = 4^4 \times 2 - 4^3 \times 3 - 4^2 \times 3 - 4^1 \times 3 - 1 \times 3$
3 Factorise appropriately.	$= 4^4 \times 2 - 3(4^3 + 4^2 + 4^1 + 1)$ $= 4^4 \times 2 - 3(1 + 4^1 + 4^2 + 4^3)$
4 Use the geometric series formula $S_n = \frac{a(r^n - 1)}{r - 1}$.	$= 4^4 \times 2 - 3 \times \frac{1(4^4 - 1)}{4 - 1}$ $= 4^4 \times 2 - 3 \times \frac{4^4 - 1}{3}$
5 Simplify.	$= 4^4 \times 2 - 4^4 + 1$ $= 4^4(2 - 1) + 1$ $t_5 = 4^4 + 1$
6 Generalise.	The general term is $t_n = 4^{n-1} + 1$.

We can see that the general term of any linear recurrence relation of the form $t_{n+1} = at_n + b$ can be worked out in the same way. Repeatedly multiplying the first term by a will always give a geometric sequence. The rest will always be a geometric series so the general term will be of the form $t_n = a^{n-1}t_1 + \frac{b(a^{n-1} - 1)}{a - 1}$, which will simplify to the form $ka^{n-1} + c$, where k and c are constants.

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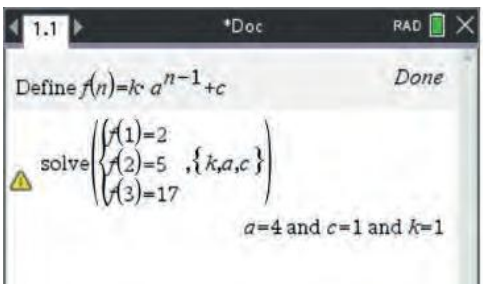
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WORKED EXAMPLE 23 The explicit rule of a linear recurrence relation using CAS

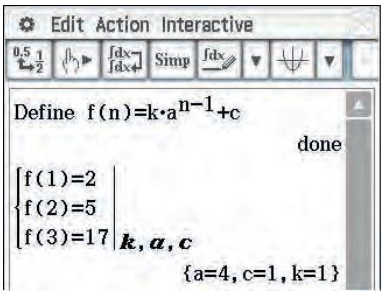
A sequence of numbers is defined by the linear recurrence relation $t_{n+1} = 4t_n - 3$, with $t_1 = 2$.
Find the explicit rule for t_n in terms of n only.

Steps	Working
1 Calculate t_1, t_2, t_3 using the rule.	$t_1 = 2$ $t_2 = 4 \times 2 - 3 = 5$ $t_3 = 4 \times 5 - 3 = 17$
2 The explicit rule is of the form $t_n = ka^{n-1} + c$ Define this function and solve for k, a and c.	

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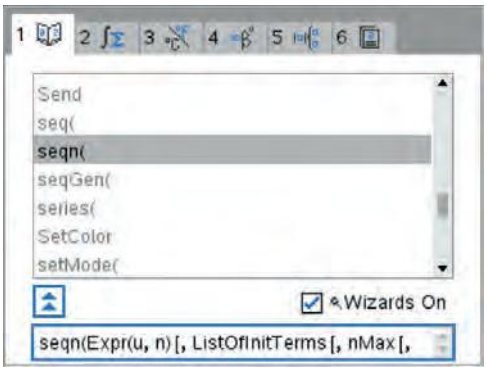
3 Write the answer.	The general term is $t_n = 4^{n-1} + 1$.

USING CAS 4 Generating linear recurrence relations

Generate the first 10 terms of each recurrence relation:

$t_{n+1} = 3t_n - 2$ with $t_1 = 4$, $t_{n+1} = 0.6t_n + 3$ with $t_1 = 2$, $t_{n+1} = -2t_n + 3$ with $t_1 = 4$ and $t_{n+1} = -0.4t_n + 1$ with $t_1 = 5$ and comment on whether they converge or diverge.

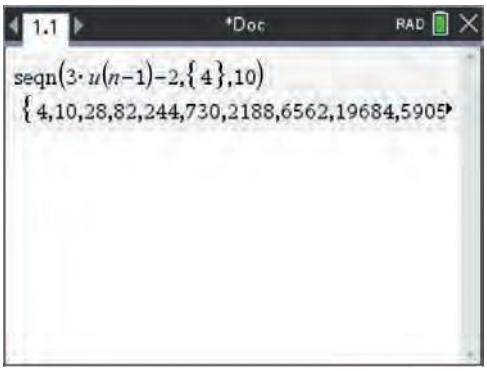
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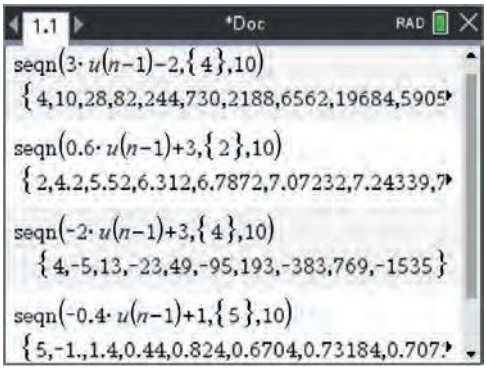
1 Add a **Calculator** application.

2 Press **catalog > S** to jump to the functions starting with S.

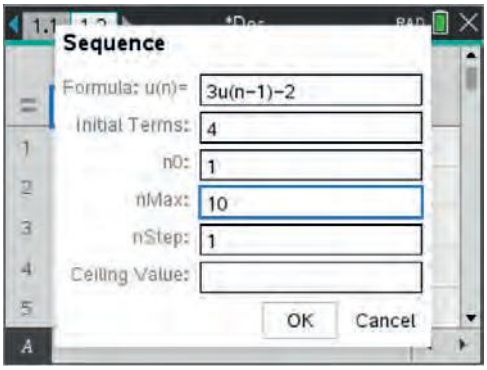
3 Scroll down and select **seqn(**.



4 Enter the expression and parameters for the recursive sequence, as shown above.



5 Repeat steps 1 to 4 to determine the other sequences.

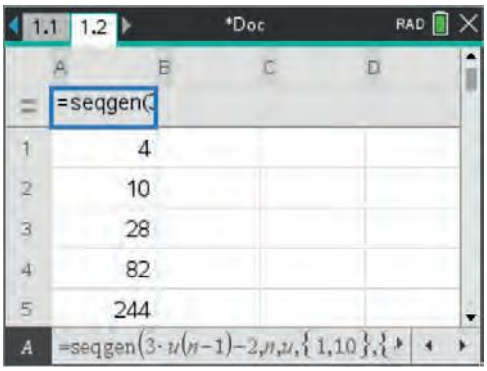


6 Alternatively, add a **Lists & Spreadsheet** application.

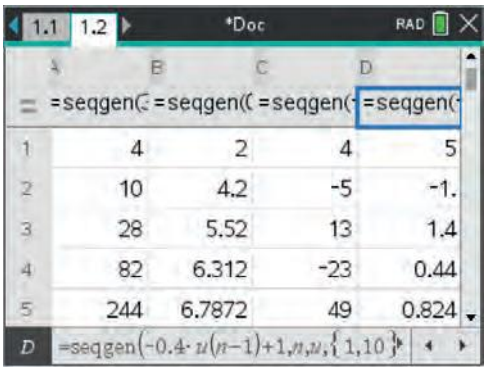
7 Place the cursor in the cell immediately under the column **A** heading.

8 Press **menu > Data > Generate Sequence**.

9 In the dialogue box, enter the expression and the parameters as shown above and select **OK**.



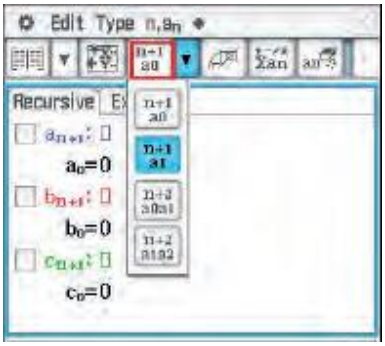
10 The sequence will be displayed in column A.



11 Repeat steps 6 to 9 to determine the other sequences.

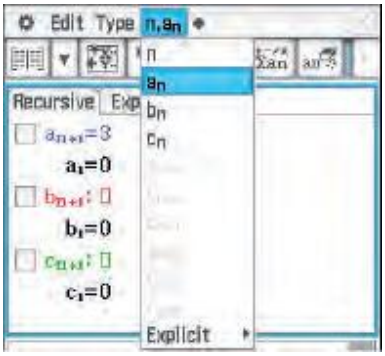
The 2nd and 4th recurrence relations with $a = 0.6$ and $a = -0.4$ appear to converge, but the 1st and 3rd with $a = 3$ and -2 appear to diverge.

ClassPad



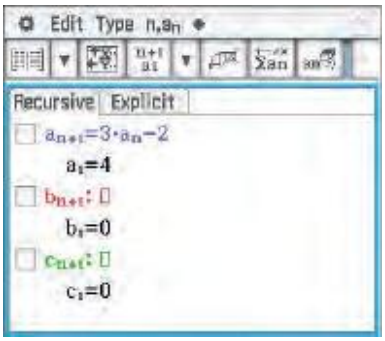
1 Open the **Sequence** application.

2 Tap the **n+1** tool and select the second menu item to change **a0** to **a1**.

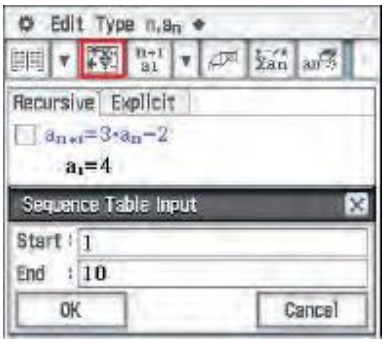


3 In the a_{n+1} line, enter **3**.

4 Tap the **n,a_n** menu and select a_n.



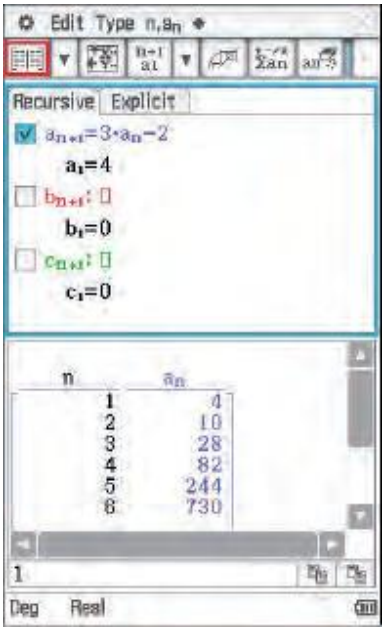
5 Complete the rest of the recursive relation, as shown above.



6 Tap the **Sequence Table Input** tool.

7 In the dialogue box, set **Start:** to 1 and **End:** to 10.

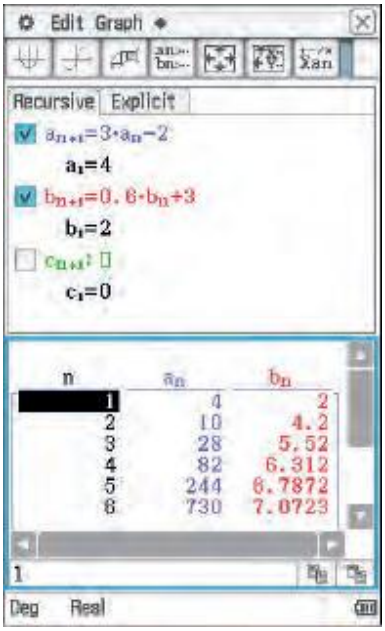
8 Tap **OK**.



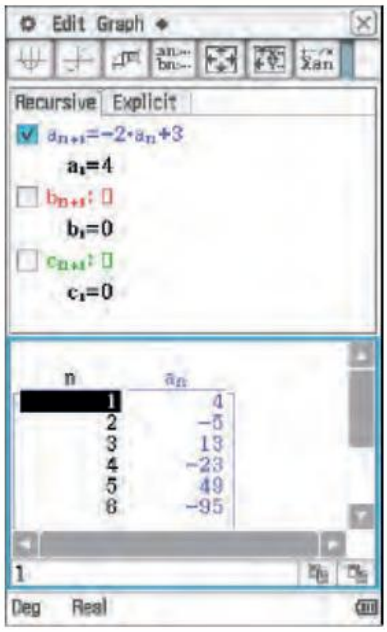
9 Tap to select the sequence.

10 Tap the **Table** tool.

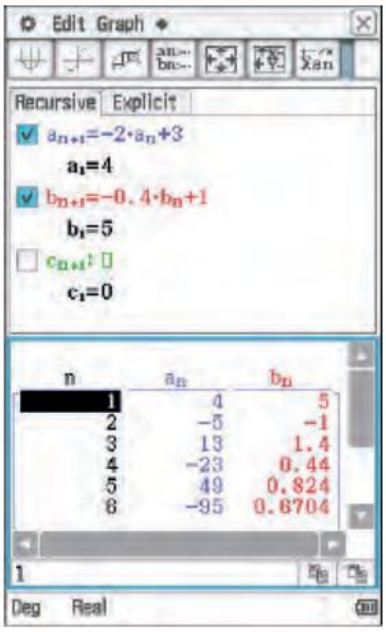
11 The sequence will appear in the lower window.



12 Follow the steps above to determine the second sequence using b_{n+1} and b_1.



13 The maximum number of sequences available is 3 so enter the third sequence using $\mathbf{a_{n+1}}$ and $\mathbf{a_n}$.



14 Enter the 4th sequence using $\mathbf{b_{n+1}}$ and $\mathbf{b_1}$.

The 2nd and 4th recurrence relations with $a = 0.6$ and $a = -0.4$ appear to converge, but the 1st and 3rd with $a = 3$ and -2 appear to diverge.

Geometric sequences with $-1 < r < 1$ converge to 0, and their limiting sum is $\frac{a}{1-r}$ as $n \to \infty$.

A linear recurrence relation of the form $t_{n+1} = at_n + b$ behaves in a similar way.

As $n \to \infty$, $a^{n-1} \to 0$, so $a^{n-1}t_1 + \frac{b(1-a^{n-1})}{1-a} \to \frac{b}{1-a}$.

EXERCISE 1.7 Linear recurrence relations

ANSWERS p. 565

Recap

1 The limiting sum of the infinite geometric series $3 + 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$ is

A $\frac{2}{9}$ **B** $\frac{3}{2}$ **C** 3 **D** $\frac{9}{2}$ **E** 9

2 The recurring decimal $0.\dot{5}$ expressed as a fraction is

A $\frac{3}{9}$ **B** $\frac{5}{9}$ **C** $\frac{6}{10}$ **D** $\frac{3}{2}$ **E** 6

Mastery

3 WORKED EXAMPLE 20 TECH-FREE Find the rule for the nth term of the sequence $-1, 3, 7, 11 \ldots$ in terms of n.

4 WORKED EXAMPLE 21 TECH-FREE Use the recurrence relation to find the first 5 terms of the sequence $t_1 = 10$ and $t_{n+1} = 2t_n - 5$.

5 WORKED EXAMPLE 22 TECH-FREE For each linear recurrence relation, find the explicit rule for t_n in terms of n only.

a $t_{n+1} = 3t_n + 7$, with $t_1 = 4$

b $t_{n+1} = -4t_n + 3$, with $t_1 = 2$

c $t_{n+1} = -2t_n - 1$, with $t_1 = -4$

d $t_{n+1} = 5t_n - 3$, with $t_1 = -7$

e $t_{n+1} = \frac{1}{2}t_n + 4$, with $t_1 = 10$

f $t_{n+1} = -\frac{2}{3}t_n - 2$, with $t_1 = 6$

g $t_{n+1} = -\frac{3}{5}t_n - 3$, with $t_1 = -4$

h $t_{n+1} = \frac{3}{4}t_n - 1$, with $t_1 = 5$

6 Using CAS 3 Does the sequence with the linear recurrence relation $t_{n+1} = 0.8t_n - 3$ with $t_1 = 10$ converge or diverge? If it converges, what is the limit?

7 Does the sequence for $t_{n+1} = 1.2t_n + 4$ with $t_1 = 6$ converge or diverge? If it converges, what is the limit?

8 WORKED EXAMPLE 23 Express the general term of each linear recurrence relation in terms of n only.

a $t_{n+1} = -2t_n + 3$, with $t_1 = 4$

b $t_{n+1} = 0.6t_n - 2$, with $t_1 = 4$

c $t_{n+1} = -0.3t_n + 1$, with $t_1 = -5$

d $t_{n+1} = 5t_n - 8$, with $t_1 = 6$

e $t_{n+1} = 0.15t_n + 0.8$, with $t_1 = 1$

f $t_{n+1} = -3.2t_n + 1.6$ with $t_1 = 4$

g $t_{n+1} = 0.9t_n + 1.2$, with $t_1 = -2.5$

h $t_{n+1} = 1.8t_n + 1.8$, with $t_1 = 0$

9 Using CAS 4 Describe whether a sequence generated by $t_{n+1} = at_n + b$ is arithmetic or geometric, and if geometric, whether it converges or diverges, if

a $a = 1$ and $b \neq 0$ **b** $a > 1$ or $a < -1$ and $b = 0$ **c** $-1 < a < 1$, $a \neq 0$ and $b = 0$.

10 For each relation, state whether the sequence converges or diverges. If it converges, state the limit.

a $t_{n+1} = 0.2t_n + 1$, with $t_1 = 4$

b $t_{n+1} = -2t_n + 1$, with $t_1 = -5$

c $t_{n+1} = 0.8t_n - 0.5$, with $t_1 = -3$

d $t_{n+1} = -1.5t_n - 3$, with $t_1 = 4$

e $t_{n+1} = -2.4t_n - 5$, with $t_1 = -1$

f $t_{n+1} = 0.6t_n - 1.2$, with $t_1 = 3$

g $t_{n+1} = -0.75t_n + 9$, with $t_1 = 1$

h $t_{n+1} = 1.8t_n + 0.7$, with $t_1 = -1.9$

Exam practice

11 TECH-FREE (2 marks) Find the general rule for the sequence 5, –10, 20, –40, 80 …

12 TECH-FREE (2 marks) What is the recurrence relation for the sequence 1, 5, 21, 85, 341 …?

13 TECH-FREE (2 marks) What is the general rule for the linear recurrence relation $t_{n+1} = -3t_n + 5$, $t_1 = 4$?

14 TECH-FREE (2 marks) Find the recurrence rule and the limit for the sequence 5, 6, 6.8, 7.44, 7.952 …

15 The 4th term of the linear recurrence relation given by $t_{n+1} = 4t_n - 3$, $t_1 = 5$ is

A 37 **B** 65 **C** 257 **D** 325 **E** 1025

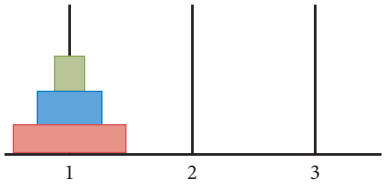
16 The limit of the linear recurrence relation $t_{n+1} = 0.6t_n - 2$, $t_1 = -4$ is

A –10 **B** $-6\frac{2}{3}$ **C** –5 **D** $-3\frac{1}{3}$ E –2

1.8 Applications of recurrence relations

The Tower of Hanoi

The Tower of Hanoi is a disc puzzle that is also known as the Tower of Brahma or the Lucas Tower. It is a mathematical game that consists of 3 rods and a number of discs of different diameters.



Video playlist
Applications of recurrence relations

The objective of the game is to shift the entire stack of discs from rod 1 to rod 3 following these rules:

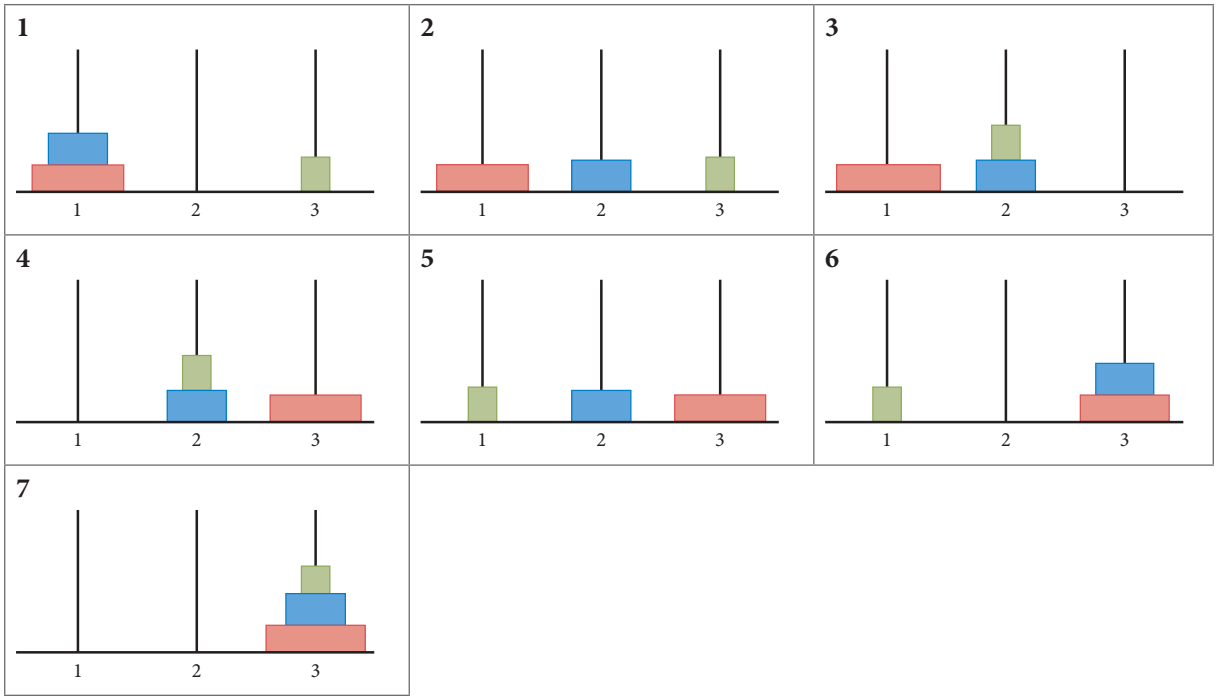
1 Only one disc can be moved at a time.

2 A disc cannot be placed on the top of smaller discs.

The number of moves required to solve the Tower of Hanoi puzzle with n discs is $2^n - 1$.

Number of discs	Minimum number of moves
1	$2^1 - 1 = 1$
2	$2^2 - 1 = 3$
3	$2^3 - 1 = 7$
4	$2^4 - 1 = 15$
5	$2^5 - 1 = 31$

For example, solving the Tower of Hanoi puzzle with 3 discs requires 7 moves.



The terms of the Tower of Hanoi sequence are 1, 3, 7, 15, 31 …

Can you see the pattern between the terms, the rule for each new term?

$T_n = 2T_{n-1} + 1$, where $T_1 = 1$.

Some more examples of linear recurrence equations are:

Recurrence relations	Initial values	Solutions
$F_n = F_{n-1} + F_{n-2}$	$F_1 = F_2 = 1$	Fibonacci number
$P_n = P_{n-2} + P_{n-3}$	$P_1 = P_2 = P_3 = 1$	Padovan sequence
$P_n = 2P_{n-1} + P_{n-2}$	$P_1 = 0, P_2 = 1$	Pell number

Fibonacci numbers

Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21 …, where each term is the sum of the 2 terms before it, or $F_n = F_{n-1} + F_{n-2}$. For the rule to work, we need to define the initial values $F_1 = 1$ and $F_2 = 1$.

When we take any 2 consecutive Fibonacci numbers, their ratio becomes closer and closer to a number called the golden ratio $\frac{1+\sqrt{5}}{2} = 1.6180\ldots$, which has many applications in nature, design and architecture.

Fibonacci number	Ratio
2	$\frac{2}{1} = 1$
3	$\frac{3}{2} = 1.5$
5	$\frac{5}{3} = 1.666\,666\,666\ldots$
8	$\frac{8}{5} = 1.6$
13	$\frac{13}{8} = 1.625$
21	$\frac{21}{13} = 1.615\,384\,615\ldots$
…	…
233	$\frac{233}{144} = 1.618\,055\,556\ldots$
377	$\frac{377}{233} = 1.618\,025\,751\ldots$

Compound interest

Recurrence relations can be used to solve problems involving finance and population.

Compound interest and population growth are closely related as examples of exponential growth. In both cases, the increase in one year depends on the current amount. If the increase in a year is 15%, then the amount in any year is 1.15 times the amount in the previous year. The number 1.15 is called the **growth factor**.

WORKED EXAMPLE 24 Compound interest calculation 1

An investment of $3500 earns 10.5% p.a. compound interest over 7 years.

a Write a recurrence relation for the amount at the end of each year.

b Use the relation to write an expression for the amount at the end of 7 years.

Steps	Working
a 1 Write the interest as a decimal.	interest rate = 0.105
2 Write the growth factor.	growth factor = 1.105
3 Write the recurrence relationship for A_n, the amount after n years.	$A_{n+1} = 1.105A_n$, where $A_0 = 3500$
b 1 Write the amounts after the first 3 years.	$A_1 = 1.105 \times 3500$ $A_2 = 1.105 \times (1.105 \times 3500)$ $A_3 = 1.105 \times [1.105 \times (1.105 \times 3500)]$
2 Generalise to A_n, the amount after n years.	$A_n = 1.105^n \times 3500$
3 Write the expression for the amount after 7 years.	The amount after 7 years is given by $\$3500 \times 1.105^7$.

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After 7 years, $\$3500 \times 1.105^7$ is about $7041, which is just over double the initial amount. By contrast, simple interest would give a little over $6000.

Compound interest

The **growth factor** for compound interest at interest rate r is $(1 + r)$, where r is a decimal.

The linear recurrence relationship for the amount A_n accumulated is $A_{n+1} = (1 + r)A_n$.

The amount invested, A_0, is called the **principal**.

The final amount after n time periods is given by $A_n = A_0(1 + r)^n$.

If compound interest is calculated monthly, then the annual interest rate must be divided by 12, and the time period, n, is measured in months. For example, an interest rate of 9% p.a. is calculated at $\frac{9}{12} = 0.75\%$ per month, and each month, the amount would be multiplied by 1.0075 to calculate the new amount.

WORKED EXAMPLE 25 Compound interest calculation 2

Find an expression for the value of an investment of $5000 under compound interest at 6% p.a. calculated monthly after 11 years 6 months.

Steps	Working
1 Write the monthly interest as a decimal.	interest rate $= \frac{0.06}{12} = 0.005$
2 Write the growth factor.	growth factor = 1 + 0.005 = 1.005
3 Write the recurrence relationship for A_n, the amount after n months.	$A_{n+1} = 1.005A_n$, where $A_0 = 5000$
4 Write the amount after n months.	$A_n = 5000 \times 1.005^n$
5 Find n for the time 11 years 6 months.	$n = 12 \times 11 + 6 = 138$ (months)
6 Write an expression for the value after 11 years and 12 months.	The value after 11 years and 6 months is $\$5000 \times 1.005^{138}$.

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After 11 years 6 months, $\$5000 \times 1.005^{138}$ is about $9951, nearly double the original.

Annuities

Linear recurrence relationships are more important for regular investments of the same amount over a period of time, called an **annuity** (for example, superannuation).

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WORKED EXAMPLE 26 Return on a regular investment

Zac starts work at age 18 and puts $2400 per year into a superannuation account earning 6% p.a. interest at the end of each year.

a Write a recurrence relation for his investment each year.

b Find an expression for the value of his investment after n years.

c Write an expression for the value of his investment when he turns 25.

Steps	Working
a Use A_n for the value at the beginning of the nth year.	$A_{n+1} = 1.06A_n + 2400$, where $A_1 = 2400$
b 1 Write expressions for the first few values.	$A_2 = 1.06 \times 2400 + 2400$ $A_3 = 1.06(1.06 \times 2400 + 2400) + 2400$ $A_4 = 1.06[1.06(1.06 \times 2400 + 2400) + 2400] + 2400$
2 Simplify the expression for A_4.	$= 2400(1.06^3 + 1.06^2 + 1.06 + 1)$ $= 2400(1 + 1.06 + 1.06^2 + 1.06^3)$
3 Use the geometric series formula $S_n = \frac{a(r^n - 1)}{r - 1}$.	$= 2400 \times \frac{1(1.06^4 - 1)}{1.06 - 1}$ $= 2400\left(\frac{1.06^4 - 1}{0.06}\right)$
4 Generalise the expression.	$A_n = 2400\left(\frac{1.06^n - 1}{0.06}\right)$
5 Write the answer.	The value after n years is $2400\left(\frac{1.06^n - 1}{0.06}\right)$.
c When he turns 25, he will have been working for 7 years.	The value of Zac's investment when he turns 25 would be given by $2400\left(\frac{1.06^7 - 1}{0.06}\right)$.

After 7 years, $\$2400\left(\frac{1.06^7 - 1}{0.06}\right) \approx \$20\,145$, so he has accumulated $\$20\,145 - 7 \times \$2400 = \$3345$ interest on the 7 payments made.

More frequent deposits and interest calculations give more interest than annual payments.

Annuities

An amount, D, invested at regular intervals at compound interest rate, r, over n periods accumulates to the amount $D\left[\frac{(1 + r)^n - 1}{r}\right]$.

This kind of investment is called an **annuity** and the value $\frac{(1 + r)^n - 1}{r}$ is called the **annuity factor**.

Reducing balance loans

Financial institutions like banks lend money where the interest is calculated on the amount still owing, while the loan is being paid off in regular instalments weekly, fortnightly or monthly.

The interest is calculated *before* the payments are subtracted from the loan. Reducing balance loans are modelled by recurrence relations.

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WORKED EXAMPLE 27 Reducing balance loan

Ali and Fatima borrow $500 000 at 9% p.a. interest to buy a house, making payments of $60 000 a year.

a Write a recurrence relation for the amount left to pay after each year.

b Find an expression for the amount left to pay after n years.

c Write an expression for the amount left to pay after 10 years.

Steps	Working
a Use A_n for the amount left to pay at the end of the nth year.	$A_{n+1} = 1.09A_n + 60\,000$, where $A_1 = 500\,000$
b 1 Write expressions for the first few values.	$A_2 = 1.09 \times 500\,000 - 60\,000$ $A_3 = 1.09(1.09 \times 500\,000 - 60\,000) - 60\,000$ $A_4 = 1.09[1.09(1.09 \times 500\,000 - 60\,000) - 60\,000] - 60\,000$
2 Simplify the expression for A_4.	$= 500\,000 \times 1.09^3 \times 60\,000(1.09^2 + 1.09 + 1)$ $= 2400(1 + 1.09 + 1.09^2)$
3 Use the geometric series formula $S_n = \frac{a(r^n - 1)}{r - 1}$	$= 500\,000 \times 1.09^3 - 60\,000 \times \frac{1.09^3 - 1}{1.09 - 1}$ $= 500\,000 \times 1.09^3 - 60\,000\left(\frac{1.09^3 - 1}{0.09}\right)$
4 Generalise the expression.	$A_n = 500\,000 \times 1.09^n - 60\,000\left(\frac{1.09^n - 1}{0.09}\right)$
5 Write the answer.	The amount to pay after n years is $500\,000(1.09^n) - 60\,000\left(\frac{1.09^n - 1}{0.09}\right)$
c Find the amount after 10 years.	The amount to pay after 10 years is $500\,000(1.09^{10}) - 60\,000\left(\frac{1.09^{10} - 1}{0.09}\right)$.

After 10 years, $\$500\,000(1.09^{10}) - 60\,000\left(\frac{1.09^{10} - 1}{0.09}\right) \approx \$272\,106$. Ali and Fatima will have paid $10 \times \$60\,000 = \$600\,000$. They will pay off the loan after another 7 years. Altogether they will have paid nearly $1 million dollars by then, for a $500 000 loan. For 25-year and 30-year mortgages, people normally end up paying much more than double the loan.

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WORKED EXAMPLE 28 Reducing balance loan repayments

Find an expression for the annual payment L for a loan of \$400 000 over 30 years at 8% p.a. compound interest.

Steps	Working
1 Write a recurrence expression for the amount A_n.	$A_{n+1} = 1.08A_n - L$
2 Write expressions for the first few values.	$A_2 = 1.08 \times 400\,000 - L$ $A_3 = 1.08(1.08 \times 400\,000 - L) - L$ $A_4 = 1.08[1.08(1.08 \times 400\,000 - L) - L] - L$
3 Simplify the expression for A_4.	$= 1.08^3 \times 400\,000 - L(1.08^2 + 1.08 + 1)$ $= 1.08^3 \times 400\,000 - L(1 + 1.08 + 1.08^2)$
4 Use the geometric series formula $S_n = \frac{a(r^n - 1)}{r - 1}$.	$= 1.08^3 \times 400\,000 - L \times \frac{1.08^3 - 1}{1.08 - 1}$ $= 400\,000(1.08^3) - L\left(\frac{1.08^3 - 1}{0.08}\right)$
5 Generalise the expression.	$A_n = 400\,000 \times 1.08^n - L\left(\frac{1.08^n - 1}{0.08}\right)$
6 The loan is paid off when the amount owing is \$0.	The loan is paid when $A_n = 0$, at $n = 30$.
7 Write the equation.	$0 = 400\,000(1.08^{30}) - L\left(\frac{1.08^{30} - 1}{0.08}\right)$
8 Solve for L.	$L\left(\frac{1.08^{30} - 1}{0.08}\right) = 400\,000(1.08^{30})$ $L = 400\,000(1.08^{30})\left(\frac{0.08}{1.08^{30} - 1}\right)$
9 Write the answer.	The payment required is $400\,000(1.08^{30})\left(\frac{0.08}{1.08^{30} - 1}\right)$.

The annual payment is $\$400\,000(1.08^{30})\left(\frac{0.08}{1.08^{30} - 1}\right)$, which is about \$35 531. Over 30 years, the total paid is $30 \times \$35\,531 = \$1\,065\,930$, so the interest paid is \$665 930.

Population growth

WORKED EXAMPLE 29 Population modelling

The population of elephants in Kruger National Park in South Africa in 2015 was 17 000. Assuming an increase of 5% each year, write an expression for the likely population in 2025.

Steps	Working
1 Write a recurrence expression for the amount A_n.	$P_{n+1} = 1.05P_n$
2 Write expressions for the first few years after 2015, where n is the number of years after 2015.	$P_1 = 1.05 \times 17\,000$ $P_2 = 1.05 \times (1.05 \times 17\,000)$ $P_3 = 1.05 \times [1.05 \times (1.05 \times 17\,000)]$
3 Simplify the expression for P_3.	$= 1.05^3 \times 17\,000$
4 Generalise the expression.	$P_n = 1.05^n \times 17\,000$
5 Write an expression for 2025 ($n = 10$).	$P_{10} = 1.05^{10} \times 17\,000$
6 Write the answer.	The population in 2025 will be $1.05^{10} \times 17\,000$.

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The population will be $1.05^{10} \times 17\,000 \approx 27\,700$. However, unlimited population growth such as this is unsustainable, and limitations of the environment are likely to impact on the rate of growth. A very large number of elephants are also likely to inflict considerable damage on the park, so culling may need to be used.

A simple population model fails to predict the population over longer periods. It doesn't take into account limitations such as the available food or predation.

VCE QUESTION ANALYSIS

Video VCE question analysis: Sequences and series

A VCE-style question (10 marks)

A deep-sea diver is trapped in an underground cave, facing a sudden deluge of water.

The water comes up to a height of 1 metre, which is half of the height of the roof of the cave.

After an hour, another rush of water occurs, and the water level in the cave rises by $\frac{1}{3}$ metre.

After a second hour, another deluge raises the water level by $\frac{1}{9}$ metre.

a Find the amount that the water level rises by at the end of the third hour. 1 mark

b Find the water level at the end of the second hour. 2 marks

c Find the amount the water level rises by at the end of the fifth hour. 2 marks

d Find the total height of water at the end of the fifth hour. 2 marks

e If this process continues indefinitely, will the cave be filled with water and be a danger to the diver? Give reasons. 3 marks

Reading the question

- The question is about a sequence of numbers.
- The question asks, step-by-step, what is happening to the level of water in the cave.
- The limiting level is considered.

Thinking about the question

- List the sequence of numbers to try to observe a pattern.
- Test whether the sequence is arithmetic or geometric.
- If desired, use the table function or sequence function on CAS.
- The questions can use step-by-step calculations or use the sequence formulas.
- The questions ask either about the term of a sequence or the sum of a sequence.
- Part **e** requires clear and adequate mathematical reasoning.

Worked solution (✓ = 1 mark)

a Initial height of water is 1 metre.

After first hour, the water level rises by $\frac{1}{3}$.

After second hour, the water level rises by $\frac{1}{9}$.

The sequence of numbers is $1, \frac{1}{3}, \frac{1}{9} \ldots$, so a geometric sequence where $r = \frac{1}{3}$.

Following the pattern, after the third hour the water rises by $\frac{1}{27}$ metre. ✓

or

$a = 1, r = \frac{1}{3}, n = 4$ (for the third hour as $t_1 = 1$ is 0th hour)

Geometric sequence, $t_n = ar^{n-1}$:

$t_4 = ar^{4-1}$

$= 1 \times \left(\frac{1}{3}\right)^3$

$= \mathbf{\frac{1}{27}}$ ✓

b Water level $= 1 + \frac{1}{3} + \frac{1}{9}$ ✓

The water level at the end of the second hour is $\frac{13}{9}$ or $1\frac{4}{9}$ m. ✓

or

$a = \frac{1}{3}, r = \frac{1}{3}, n = 3$ ✓

Geometric series, $S_n = \frac{a(1 - r^n)}{1 - r}$:

$S_3 = \frac{1\left(1 - \left(\frac{1}{3}\right)^3\right)}{1 - \frac{1}{3}}$

$= \frac{\frac{26}{27}}{\frac{2}{3}}$

$= \mathbf{\frac{13}{9}}$ **or** $\mathbf{1\frac{4}{9}}$ **m** ✓

c The sequence of numbers is $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}, \ldots$ ✓

Following the pattern, after the fifth hour the water rises by $\frac{1}{243}$ m. ✓

or

$a = \frac{1}{3}, r = \frac{1}{3}, n = 6$ ✓

Geometric sequence, $t_n = ar^{n-1}$:

$t_6 = 1 \times r^{6-1}$

$\therefore t_6 = 1 \times \left(\frac{1}{3}\right)^5 = \mathbf{\frac{1}{243}\ m}$ ✓

d Water level $= 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} \ldots$ ✓

The water level at the end of the fifth hour is $\frac{364}{243}$ or $1\frac{121}{243}$ m. ✓

or

Geometric sequence, $S_n = \frac{a(1-r^n)}{1-r}$:

$S_6 = \frac{1\left(1-\left(\frac{1}{3}\right)^6\right)}{1-\frac{1}{3}}$ ✓

$= \mathbf{\frac{364}{243}\ or\ 1\frac{121}{243}\ m}$ ✓

e Limiting sum, $S_\infty = \frac{a}{1-r}$ ✓

$S_\infty = \frac{1}{1-\frac{1}{3}}$

$= \frac{3}{2}$

$= 1.5$ m ✓

Since the height of the cave is 2 m and the limiting height of the water is 1.5 m, the water will NOT fill the cave, and the diver will be safe. There will be 0.5 m left for air for the diver. ✓

EXERCISE 1.8 Applications of recurrence relations

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Recap

1 The 6th term of the sequence generated by the linear recurrence $t_{n+1} = 5t_n + 2$ with $t_1 = 3$ is

A 485 **B** 1457 **C** 2187 **D** 10 937 **E** 54 687

2 The limit of the sequence generated by the linear recurrence $t_{n+1} = 0.7t_n + 6$ with $t_1 = 4$ is

A $5\frac{5}{7}$ **B** $8\frac{4}{7}$ **C** $13\frac{1}{3}$ **D** $17\frac{1}{2}$ **E** 20

Mastery

3 WORKED EXAMPLE 24 An investment of $6000 earns 9% p.a. compound interest over 8 years.

- **a** Write a recurrence relation for the amount at the end of each year.
- **b** Use the relation to write an expression for the final amount.
- **c** What is the final amount, correct to the nearest dollar?

4 An investment of $2500 earns 11.5% p.a. compound interest over 6 years.

- **a** Write a recurrence relation for the amount at the end of each year.
- **b** Use the relation to write an expression for the final amount.
- **c** What is the final amount, to the nearest dollar?

5 An investment of $4000 earns 12% p.a. interest compounded monthly over 20 years.

- **a** Write a recurrence relation for the amount at the end of each month.
- **b** Use the relation to write an expression for the final amount.
- **c** What is the final amount, to the nearest dollar?

6 WORKED EXAMPLE 25

- **a** Find an expression for the value of an investment of $8000 at 9% p.a. interest compounded monthly, after 7 years 8 months.
- **b** Compare the final amount with the initial value of the investment.

7 WORKED EXAMPLE 26 Each year $1400 is put into an investment, earning an average of 10% p.a. compound interest over 15 years.

- **a** Write a recurrence relation for the investment at the end of each year.
- **b** Find an expression for the value of this investment after n years.
- **c** Find the final amount to the nearest dollar.

8 Each month, $300 is put into an investment earning an average of 9.6% interest compounding monthly over a period of 10 years.

- **a** Write a recurrence relation for the investment at the end of each month.
- **b** Find an expression for the value of this investment after n years.
- **c** Find the final amount to the nearest dollar.

9 WORKED EXAMPLE 27 Sienna has a $350 000 loan at 11% p.a. compound interest, making payments of $42 000 per year.

- **a** Write a recurrence relation for the amount left to pay after each year.
- **b** Find an expression for the amount left to pay after n years.
- **c** Find the amount left to pay after 10 years, to the nearest dollar.
- **d** Find the amount left to pay after 20 years, to the nearest dollar.

10 Toby has a $600 000 loan at 10.8% p.a. interest compounded monthly, with payments of $6000 per month.

- **a** Write a recurrence relation for the amount left to pay after each month.
- **b** Find an expression for the amount left to pay after n months.
- **c** Find the amount left to pay after 10 years to the nearest dollar.
- **d** Find the amount left to pay after 20 years to the nearest dollar.

11 WORKED EXAMPLE 28

a Find an expression for the annual payment, L, for a loan of $340 000 over 30 years at 9% p.a. compound interest.

b Calculate the total amount paid, to the nearest dollar.

c Calculate the total interest paid, to the nearest dollar.

12 **a** Find an expression for the monthly payment, L, for a loan of $460 000 over 20 years at 8.4% p.a. interest compounding monthly.

b Calculate the total amount paid, to the nearest dollar.

c Calculate the total interest paid, to the nearest dollar.

13 WORKED EXAMPLE 29 The population of feral cats on an island off the Australian coast after an eradication campaign was thought to be 20. Without predators, the population is estimated to grow by 25% per year. A population of more than 120 will cause damage to the native wildlife.

a Write an expression for the population in n years.

b Find the population 5, 10 and 15 years later.

14 The population of mice in a 20-hectare wheat crop was 1000 at the beginning of spring (September). The population of mice grew by 30% per week. A population of 200 per hectare or more would cause significant damage to the crop.

a Write an expression for the population of mice after n weeks.

b This crop is due to be harvested at the beginning of November. Would the crop be significantly damaged by mice if no action is taken?

Exam practice

15 TECH-FREE (2 marks) An investment of $9000 earns 7% p.a. compound interest over 10 years.

a Write the linear recurrence relation for the amount after each year.

b Use the relation to find an expression for the amount after 10 years.

16 The value of an investment of $8000 at 7.2% p.a. compound interest after 8 years is closest to

A $8500 **B** $12 600 **C** $14 000 **D** $18 400 **E** $61 000

17 Julio puts $2000 each year into an industry superannuation fund earning an average of 12.5% p.a. compound interest (after fees). After 30 years it would be worth about

A $15 500 **B** $67 500 **C** $68 500 **D** $532 000 **E** $548 000

18 A university student borrowed $35 000 to pay for their final year's fees. The student agrees to pay back $3500 per year at 8% compound interest. About how much would be left to pay after 20 years?

A $0 **B** $3000 **C** $6000 **D** $25 800 **E** $26 700

19 The annual payment for a 25-year loan of $800 000 at 9.6% p.a. compound interest is closest to

A $76 800 **B** $81 000 **C** $85 500 **D** $86 400 **E** $108 800

20 What would a population of 30 million people become after 10 years at a growth rate of 2% per year?

A 30.6 million **B** 36 million **C** 36.6 million **D** 44.6 million **E** 185.7 million

1 Chapter summary

Sequences and series

- A **sequence** is a list of numbers in a certain order. The numbers in a sequence are called its **terms**.
- The ***n*th term** of a sequence is t_n.

Arithmetic sequences

- An **arithmetic sequence** or arithmetic progression (AP) has a common difference, d.
- d is the difference between successive terms, written as $d = t_n - t_{n-1}$.
- An arithmetic sequence can be defined recursively as $t_n = t_{n-1} + d$.
- The general term is $t_n = t_1 + (n-1)d$.

Arithmetic series

- The sum of the terms in a sequence is called a **series**.
- For an **arithmetic series**: $S_n = \frac{n}{2}[2a + (n-1)d]$ or $S_n = \frac{n}{2}[a + l]$,

  where a is the first term, d is the common difference and l is the last term.

Geometric sequences

- A **geometric sequence** or geometric progression (GP) has a common ratio, r.
- r is the ratio between successive terms, written as $r = \frac{t_n}{t_{n-1}}$.
- A geometric sequence can be defined recursively as $t_n = rt_{n-1}$.
- The general term is $t_n = ar^{n-1}$.

Geometric series

For a **geometric series**: $S_n = \frac{a(r^n - 1)}{r - 1}$ for $r > 1$ or $S_n = \frac{a(1 - r^n)}{1 - r}$ for $r < 1$

where a is the first term and r is the common ratio.

The infinite geometric series

$S_\infty = \frac{a}{1 - r}$, where $-1 < r < 1$.

Linear recurrence relations

A **linear recurrence relation** is a linear equation that relates a term in a sequence to previous terms using recursion. It is of the form $t_{n+1} = at_n + b$.

- The general term of a linear recurrence relation is $t_n = a^{n-1}t_1 + \frac{b(a^{n-1} - 1)}{a - 1}$, which simplifies to the form $ka^{n-1} + c$ where k and c are constants.
- The linear recurrence relation for compound interest and population growth is $A_{n+1} = (1 + r)A_n$. The principal amount A_0 grows to $A_n = A_0(1 + r)^n$ after n periods.
- An **annuity** of amount D invested at compound interest rate r after n periods accumulates to $D\left[\frac{(1 + r)^n - 1}{r}\right]$.
- With a **reducing balance loan**, the interest is calculated on the amount still owing *before* each payment is subtracted from the loan.

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 (2 marks)

a Is the sequence $-11, -\frac{23}{2}, -12 \ldots$ arithmetic? 1 mark

b Is the sequence $-11, -\frac{11}{2}, -12 \ldots$ geometric? 1 mark

2 (1 mark) Find the limiting sum of the geometric series

$3 + \frac{1}{2} + \frac{1}{12} + \frac{1}{72} + \ldots$

3 (2 marks) For the sequence of numbers 1, 3, 9, 27 …, find

a the 5th term 1 mark

b the sum of the first 5 terms. 1 mark

4 (2 marks) For the sequence of numbers 1, −1, −3 …, find

a the 5th term 1 mark

b the sum of the first 5 terms. 1 mark

5 (1 mark) Find the limiting sum of the geometric series

$-3 - 1\frac{1}{2} - \frac{3}{4} - \frac{3}{8} - \ldots$

6 (2 marks) $2000 is invested every year into an annuity that earns 8% p.a. interest at the end of the year. Write an expression for the value of the annuity after 25 years.

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 In the sequence 3, 6, 9, 12 …, a rule for t_n is

A $t_n = 3n$ **B** $t_n = n + 6$ **C** $t_n = 6n$ **D** $t_n = n$ **E** $t_n = 3$

2 For the sequence described by $t_n = 3n + 1$, a rule for t_{n+1} is

A $t_{n+1} = 3n$ **B** $t_{n+1} = n + 4$ **C** $t_{n+1} = 3n + 4$ **D** $t_{n+1} = n$ **E** $t_{n+1} = 3$

3 For the sequence described by $t_n = 3n + 1$, a rule for t_{2n} is

A $t_{2n} = 3n$ **B** $t_{2n} = 6n + 1$ **C** $t_{2n} = 6n + 4$ **D** $t_{2n} = 6n$ **E** $t_{2n} = 6$

4 In the sequence $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}$ …, the common ratio, r, is

A $-\frac{1}{2}$ **B** $-\frac{1}{3}$ **C** $\frac{1}{3}$ **D** $\frac{1}{2}$ **E** 3

5 \$5300 is invested in an account that earns 7.2% p.a. interest compounded monthly. The value of the investment after 8 years is closest to

A \$8353 **B** \$10 622 **C** \$9412 **D** \$30 528 **E** \$120 105

Section B 2 questions

15 marks

1 (6 marks) A small bouncy ball is dropped from a height of 81 metres. Each time it strikes the ground, it rebounds to two-thirds of the height before it.

a Find the height that the ball reaches after the 1st bounce. 1 mark

b Find the height that the ball reaches after the 5th bounce. 1 mark

c Find the total distance that the ball travels just before the 6th bounce. 2 marks

d Assuming that the ball continues to bounce indefinitely, find the total distance travelled by the ball. 2 marks

2 (9 marks) Bill and Ben are twins who are planning on running a marathon together. One week before the marathon, they undertake 7 daily training sessions. Bill conducts his training runs according to an arithmetic sequence, while Ben conducts his training runs according to a geometric sequence. They both run 10 km on the first run of the week. In total, Bill runs 175 km across the 7 runs. Ben has run a total of 21 km after 2 sessions.

a Find the common difference for Bill's daily runs. 1 mark

b Find the common ratio for Ben's daily runs. 1 mark

c Find the length of the run on the 3rd day for each runner. 2 marks

d How far does Ben run in total, to the nearest kilometre? 2 marks

e How far would Bill run on the 10th day if he had continued his training plan? 1 mark

f How far would Ben run on the 10th day if he had continued his training plan, correct to one decimal place? 2 marks

CHAPTER

2

PROOF AND NUMBER

Study Design coverage

Nelson MindTap chapter resources

2.1 Real and complex numbers

Complex numbers

Using CAS 1: Expressing recurring decimals in fraction form

Closure laws

2.2 Set notation

The symbols of set notation

Venn diagrams

2.3 Prime numbers and rational numbers

Prime factorisation

Rational numbers, integer properties and proofs

2.4 Methods of proof

Proof by counterexample

Direct proof

Proof by contradiction

Proof by contrapositive

2.5 Proof by mathematical induction

VCE question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2

9780170448338

Study Design coverage

UNIT 1, AREA OF STUDY 1: ALGEBRA, NUMBER AND STRUCTURE

Proof and number

- number systems for the natural numbers, N, integers, Z, rational numbers, Q, real numbers, R, and complex numbers, C, and their fundamental properties and structure
- set notation and operations including element, intersection, union, complement sub-set and power set
- prime numbers, the fundamental theorem of arithmetic, and proof that there are infinitely many prime numbers
- conversion between fraction and decimal forms of rational numbers
- introduction to principles of proof including propositions and quantifiers, examples and counter-examples, direct proof, proof by contradiction, and proof using the contrapositive and mathematical induction
- simple proofs involving, for example, divisibility, sequences and series, inequalities and irrationality.

VCE Mathematics Study Design 2023–2027 p. 58, © VCAA 2022

Video playlists (6):

2.1 Real and complex numbers
2.2 Set notation
2.3 Prime numbers and rational numbers
2.4 Methods of proof
2.5 Proof by mathematical induction
VCE question analysis Proof and number

Worksheets (6):

2.1 Rational and irrational numbers • Decimal representation
2.4 Counterexamples • Contrapositive
2.5 Proof by induction • Mathematical induction

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

Video playlist Real and complex numbers

Worksheet Rational and irrational numbers

2.1 Real and complex numbers

Our number system evolved to keep pace with new mathematical ideas and developments. Over time, the exclusive use of whole numbers was extended to include zero, negative numbers, fractions, decimals, rational and irrational numbers, and complex numbers.

- The **set** of **natural numbers**, N, consists of 'counting numbers' or the positive whole numbers: $\{1, 2, 3, 4 ...\}$. An **element**, n, of this set N can be defined as $\{n: n \in N\}$ or $\{n: n \in 1, 2, 3, 4 \ldots\}$.
- The set of **integers**, Z, consists of positive and negative whole numbers and zero: $\{\ldots -3, -2, -1, 0, 1, 2, 3 \ldots\}$. Each element is $n \in Z$.
- The set of **rational numbers**, Q, consists of numbers that can be expressed as a fraction $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

  Each element, n, of Q is given as $\{n: n \in Q\}$ or $\{n: n = \frac{p}{q}, p \in Z, q \in Z\backslash\{0\}\}$.

  Every integer, n, is a rational number since it can be written in fraction form, $\frac{n}{1}$.

The symbol $\in$ means 'is an element of'.

- **Terminating decimals** and **recurring decimals** are also rational numbers because each can be written in fraction form; for example:

$$2.5 = \frac{5}{2} \qquad 0.14 = \frac{7}{50} \qquad 3.\dot{1} = \frac{28}{9} \qquad 0.\dot{3}84\,615\dot{} = \frac{5}{13}$$

- Each element of the set of **irrational numbers**, Q', cannot be expressed as a fraction. Irrational numbers include **surds** such as $\sqrt{2}$ and $\sqrt[3]{5}$, and **transcendental numbers** such as π and e.

The set of rational numbers, Q, is a **subset** (part) of the set of real numbers, R, and the set of natural numbers, N, is a subset of the set of integers, Z. The symbol $\subset$ means 'is a subset of', so we write $Q \subset R$ and $N \subset Z$.

The rational and irrational numbers comprise the set of **real numbers**, R.

Complex numbers

What is the solution to the equation $x^2 = -25$?

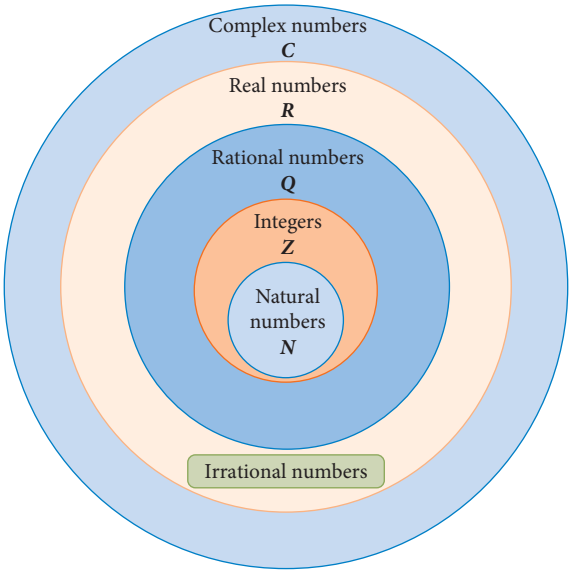
This equation has no solution for x in R because there is no real number whose square is -25.

However, let's write $x^2 = -25$ as $x^2 = 25 \times (-1)$. This becomes $x = \pm\sqrt{25} \times \sqrt{-1} = \pm 5\sqrt{-1}$.

Now suppose there is a number, i, with the property $i = \sqrt{-1}$ or $i^2 = -1$.

The solution to the equation $x^2 = -25$ will therefore be $x = \pm 5i$.

The **imaginary number**, i, is the basis of the set of **complex numbers**, C.



By using $i = \sqrt{-1}$, we can deal with square roots of negative numbers.

A complex number is of the form $z = a + bi$, where a and b are real numbers.

Some examples of complex numbers are $4 + 7i$, $1 - i$, $3i$ and 10.

All real numbers are also complex numbers, so the set of real numbers is a subset of the set of complex numbers ($R \subset C$).

This can be shown to be true by letting $b = 0$, so $z = a + bi$ becomes $z = a$, which is a real number.
The diagram above shows that each number system is a subset of another number system.

Thus, $N \subset Z \subset Q \subset R \subset C$ and $Q' \subset R$.

WB

p. 29

Worksheet
Decimal representation

WORKED EXAMPLE 1 Show that a recurring decimal is a rational number

Express each recurring decimal in fraction form.

a $2.\dot{3}$ **b** $0.\dot{4}\dot{7}$ **c** $1.2\dot{3}4\dot{5}$

Steps	Working
a 1 Let x be the recurring decimal.	Let $x = 2.\dot{3}$ … [1]
2 Multiply both sides by 10 since there is only one recurring digit.	$10x = 23.\dot{3}$ … [2]
3 Subtract equation [1] from equation [2].	$10x - x = 23.\dot{3} - 2.\dot{3}$ $9x = 21$
4 Solve the equation and write the answer.	$x = \frac{21}{9} = 2\frac{1}{3}$ $2.\dot{3} = 2\frac{1}{3}$
b 1 Let x be the recurring decimal.	Let $x = 0.\dot{4}\dot{7}$ … [1]
2 Multiply both sides by 100 since there are 2 recurring digits.	$100x = 47.\dot{4}\dot{7}$ … [2]
3 Subtract equation [1] from equation [2].	$100x - x = 47.\dot{4}\dot{7} - 0.\dot{4}\dot{7}$ $99x = 47$
4 Solve the equation and write the answer.	$x = \frac{47}{99}$ $0.\dot{4}\dot{7} = \frac{47}{99}$
c 1 Let x be the recurring decimal.	Let $x = 1.2\dot{3}4\dot{5}$
2 Multiply both sides by 10 so that the first digit after the decimal point is a recurring digit.	$10x = 12.\dot{3}4\dot{5}$ … [1]
3 Multiply both sides of equation [1] by 1000 since there are 3 recurring digits to eliminate.	$10\,000x = 12\,345.\dot{3}4\dot{5}$ … [2]
4 Subtract equation [1] from equation [2].	$10\,000x - 10x = 12\,345.\dot{3}4\dot{5} - 12.\dot{3}4\dot{5}$ $9990x = 12\,333$
5 Solve the equation and write the answer.	$x = \frac{12\,333}{9990}$ $= 1\frac{781}{3330}$ $1.2\dot{3}4\dot{5} = 1\frac{781}{3330}$

You can check your answer using CAS by converting $2\frac{1}{3}$ to a decimal.

Notice how the recurring digits are eliminated.

Exam hack

When letting x be the recurring decimal, the power of 10 required to multiply both sides of the equation will be the number of recurring digits.

USING CAS 1 Expressing recurring decimals in fraction form

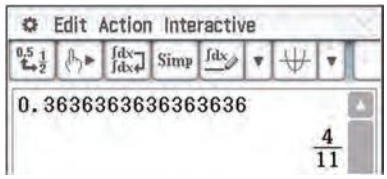
Express the recurring decimal $0.\dot{3}\dot{6}$ as a fraction.

TI-Nspire



1 Add a **Calculator** page.
2 Enter **0.36363636363636**.
3 Press **menu** > **Number** > **Approximate to Fraction**.
4 Press **enter**.

ClassPad



1 Tap **Main**.
2 Set the mode to **Standard**.
3 Enter **0.3636363636363636**.
4 Press **EXE**.

$0.\dot{3}\dot{6} = \frac{4}{11}$

WORKED EXAMPLE 2 Describing number sets

State the smallest subset of C (N, Z, Q, R or C) that contains each number or set of numbers.

a $\sqrt{121}$

b $\{x : x^2 > 1\}$ ← The set of all numbers x such that $x^2 > 1$.

c $16^{\frac{1}{2}} + 0.\dot{4}$

d $\{x: -\infty < x < \infty\}$ ← The set of all numbers x that lie between $-\infty$ and ∞.

e $3i$

Steps	Working
a 1 Simplify the expression if possible.	$\sqrt{121} = 11$
2 Determine the smallest subset of C that contains 11.	11 is a natural number. Therefore, $\sqrt{121} \in N$.
b 1 Simplify the inequality.	$x^2 > 1$ means all values of x satisfying $x < -1$ and $x > 1$.
2 Determine the smallest subset of C that contains these values.	$\{x: x^2 > 1\}$ contains most of the real numbers, so $\{x: x^2 > 1\} \subset R$. $\in$ means 'is an element of'. $\subset$ means 'is a subset of'.
c 1 Simplify if possible.	$16^{\frac{1}{2}} + 0.\dot{4} = 4 + 0.\dot{4} = 4.\dot{4}$
2 Determine the smallest subset of C that contains $4.\dot{4}$.	$4.\dot{4}$ is a recurring decimal, so it is a rational number. $16^{\frac{1}{2}} + 0.\dot{4} \in Q$
d 1 Simplify if possible.	$\{x: -\infty < x < \infty\}$ is all the real numbers.
2 Determine the smallest subset of C.	The smallest subset of C is the set of real numbers, R.
e Determine the smallest subset of C.	$3i$ is a complex number. Hence, the smallest subset is the set of complex numbers, C. $3i \in C$

WB p. 31

Closure laws

The sum of any 2 natural numbers, such as 2 and 3, is another natural number, in this case, 5. We say that the set of natural numbers, N, is **closed** under addition because when the operation of addition is performed on any 2 elements of the set, it produces an element that is also in that set.

However, the set of natural numbers is not closed under subtraction. For example, using 2 and 3, subtraction produces $2 - 3 = -1$, which is not a natural number.

WB p. 32

WORKED EXAMPLE 3 Deciding if a set is closed

Determine if the set of integers, Z, is closed under each of the following operations.

a Subtraction **b** Multiplication **c** Division

Steps	Working
a 1 Show the operation in terms of elements of the set of integers.	Let $a, b \in Z$. For subtraction, we have $c = a - b$.
2 State the conclusion.	c is another integer, so the set of integers is closed under subtraction.
b 1 Show the operation in terms of elements of the set.	Let $a, b \in Z$. Let $c = a \times b$.
2 State the conclusion.	$c \in Z$ (c is another integer), so the set of integers is closed under multiplication.
c 1 Show the operation in terms of elements of the set.	Let $a, b \in Z$. Let $c = \frac{a}{b}$.
2 State the conclusion.	c is not necessarily an integer (c could be a fraction, or undefined if $b = 0$), so the set of integers is not closed under division.

WB p. 33

WORKED EXAMPLE 4 Use closure properties of a set to prove closure of another set

Given that the set of integers, Z, is closed under addition and multiplication, prove that the set of rational numbers, Q, is closed under addition.

Steps	Working
1 Represent x and y as a fraction.	Let $x = \frac{a}{b}$ and $y = \frac{c}{d}$, where a, b, c and d are integers and b and d are not equal to 0.
2 Sum x and y.	$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$
3 $\frac{ad + bc}{bd}$ will be rational if $ad + bc$ and bd are integers and $bd \neq 0$. Explain why $ad + bc$ and bd are integers and $bd \neq 0$.	Because addition and multiplication are closed under the set of integers, ad, bc and bd are all integers, so $ad + bc$ is an integer. Because b and d are not equal to 0, $bd \neq 0$.
4 State the conclusion.	$ad + bc$ and bd are integers and $bd \neq 0$, so $\frac{ad + bc}{bd}$ is rational and can be written in the form $\frac{e}{f}$ where $e, f \in Z, f \neq 0$. Therefore, the set of rational numbers, Q, is closed under addition.

EXERCISE 2.1 Real and complex numbers

ANSWERS p. 566

Mastery

1 WORKED EXAMPLE 1 Express each recurring decimal in fraction form.

a $4.\dot{6}\dot{5}$ **b** $0.2\dot{5}6\dot{7}$

2 Using CAS 1 Use CAS to convert each recurring decimal to a fraction.

a $0.\dot{1}$ **b** $0.58\dot{3}$ **c** $3.1\dot{2}\dot{8}$ **d** $1.\dot{7}14285\dot{7}$ **e** $1.4\dot{9}$

3 WORKED EXAMPLE 2 State the smallest subset of C (N, Z, Q, R or C) that contains each number or set of numbers.

a $\{x: (x-4)(x+2)=0\}$ **b** $\sqrt{\frac{4}{9}}$

c $\{x: x=2n, n \in N\}$ **d** $\frac{1}{2}(1+i)^2 - i + \frac{1}{2}$

4 WORKED EXAMPLE 3 Show that the set of rational numbers is closed under addition, subtraction and multiplication.

5 WORKED EXAMPLE 4 Show that if a rational number is divided by a non-zero rational number, then the result is also a rational number.

Exam practice

6 TECH-FREE (2 marks) Write $2.0\dot{3}$ as a proper fraction.

7 TECH-FREE (2 marks) Explain, with an example, why the sum or difference of 2 recurring decimals is a rational number.

8 Which expression is equivalent to a terminating decimal?

A $\frac{(\sqrt{\pi})^2}{\pi(\sqrt{5})^4}$ **B** $\frac{\sqrt[3]{64}}{\sqrt{49}}$ **C** $\frac{0.\dot{2}}{0.\dot{3}}$ **D** $4\frac{1}{2} \div \frac{21}{22}$ **E** $1 + 0.\dot{6} + 0.0\dot{5}$

9 Select from the following sets the smallest subset of R that contains the set $\left\{x: x = 1 + \frac{1}{k}, k \in N\right\}$.

A N **B** R **C** Z **D** Q **E** C

10 Select from the following sets the smallest subset of C that contains the set $\{x: x^2 = a^2, a \in N\}$.

A C **B** Z **C** R **D** Q **E** N

11 (4 marks) Decide whether each statement is always true. If the statement is not always true, provide a counterexample.

a For $n \in Z$, $n^2 \in Z$. 1 mark

b If $n = 2m - 1$ and $n \in N$, then $m \in N$. 1 mark

c If $n^2 \in N$, then $n \in N$. 1 mark

d Terminating decimals are closed under addition and subtraction. 1 mark

12 TECH-FREE (4 marks) State all values of $n \in N$ so that the value of i^n will be

a 1 **b** -1

13 (4 marks)

a Provide an example where the product of 2 irrational numbers is a rational number. 2 marks

b Given that the sum of surds of the form $\sqrt{x}$, where $x \in N$, is an irrational number, explain why the product of a surd of the form $\sqrt{x}$ by a natural number is an irrational number. 2 marks

14 (3 marks) Explain under what conditions the sum of 2 fractions of the form $\frac{a}{b}$ and $\frac{c}{d}$ will be a natural number.

15 TECH-FREE (3 marks) Determine the positive values of k so that the solutions for x in the equation $\sqrt{a^2 - b^2 x} = k$ are complex.

16 (2 marks) Provide an example where the sum of 2 irrational numbers is a rational number.

Video playlist
Set notation

2.2 Set notation

The symbols of set notation

The notation '$N \subset Z$' is read as the set of natural numbers, N, is a subset of the set of integers, Z'.

A common list of symbols used in set notation is presented below.

Notation	Meaning	Example
{ }	the set	$\{A\}$ the set A
U	universal set	U = {real numbers}
$\varnothing$, { }	null or empty set	{vowels in 'sky'} = $\varnothing$
$\in$	is an element of	$2 \in$ {natural numbers}
$\notin$	is not an element of	$1.5 \notin$ {integers}
$\cap$	intersection	$A = \{1, 2\}$, $B = \{2, 3, 4\}$, $A \cap B = \{2\}$
$\cup$	union	$A = \{1, 2\}$, $B = \{3\}$, $A \cup B = \{1, 2, 3\}$
$\subset$	is a subset of	$A = \{1, 2\}$, $B = \{1, 2, 3, 4\}$, $A \subset B$
A'	complement of set A	A = {vowels in the word 'out'} A' = {consonants in the word 'out'}
$n(A)$	number of elements of a set	$A = \{4, 2, 8\}$, $n(A) = 3$

A set is usually denoted by a capital letter and contains a collection of objects whose elements are listed or described using { }, which are called braces, curly brackets or squiggly brackets. Thus, D = {Tuesday, Thursday} and D = {days of the week whose first letter is T} represent the same set comprising 2 elements. We say Tuesday $\in D$ (Tuesday is an element of set D) and Thursday $\in D$ (Thursday is an element of set D). The number of elements in a set is its **cardinality**, so $n(D) = 2$.

The null or empty set, $\varnothing$, contains no elements, as is the case for the set M = {months of the year that have more than 31 days}.

The **intersection**, $\cap$, of 2 or more sets consists of elements that are common to all those sets.

So if $A = \{1, 2, 5\}$, $B = \{2, 4\}$ and $C = \{2, 5\}$, then $A \cap B \cap C = \{2\}$.

The **union**, $\cup$, of 2 or more sets is a set that contains the elements of all the sets, with duplicate elements listed once only. Hence, $A \cup B = \{1, 2, 4, 5\}$ and $A \cup B \cup C = \{1, 2, 4, 5\}$. A set from which all subsets are defined is the **universal set**, U. If $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8\}$, a possible universal set is $U = \{\text{natural numbers from 1 to 9 inclusive}\}$.

WORKED EXAMPLE 5 Working with sets

$U = \{\text{natural numbers from 1 to 10 inclusive}\}$

$A = \{\text{even numbers from 2 to 8 inclusive}\}$

$B = \{\text{odd numbers from 3 to 9 inclusive}\}$

$C = \{\text{multiples of 4 from 1 to 10 inclusive}\}$

Find

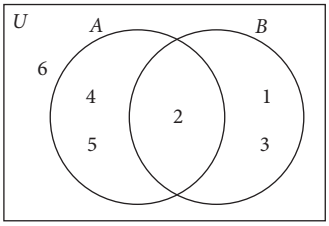
a $A \cup B$ **b** $B \cap C$ **c** $(A \cap (B \cup C))'$ **d** $n(A \cup C)'$

Steps	Working
a 1 List the elements of sets A and B.	$A = \{2, 4, 6, 8\}$ $B = \{3, 5, 7, 9\}$
2 To find the union, list the elements in A and B, omitting duplicate elements.	$A \cup B = \{2, 3, 4, 5, 6, 7, 8, 9\}$
b 1 List the elements of set C.	$C = \{4, 8\}$
2 Find the elements common to sets B and C.	There are no elements in common. $B \cap C = \varnothing$
c 1 List the elements of $A \cap (B \cup C)$.	$B \cup C = \{3, 4, 5, 7, 8, 9\}$ $A \cap (B \cup C) = \{4, 8\}$
2 Find the **complement**, which is all elements in U that are not in $A \cap (B \cup C)$.	$(A \cap (B \cup C))' = \{1, 2, 3, 5, 6, 7, 9, 10\}$
d 1 Find the elements of $A \cup C$.	$A \cup C = \{2, 4, 6, 8\}$
2 List the elements of $(A \cup C)'$.	$(A \cup C)' = \{1, 3, 5, 7, 9, 10\}$
3 State the number of elements in $(A \cup C)'$.	$n(A \cup C)' = 6$

WB p. 34

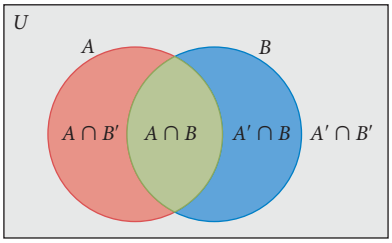
Venn diagrams

In the 1880s, the English mathematician John Venn proposed a way of illustrating set relationships diagrammatically. Known as a **Venn diagram**, sets are displayed using circles, with a rectangle representing the universal set and overlapping circles show intersections.



As an example, the relationship between the 3 sets $U = \{1, 2, 3, 4, 5, 6\}$, $A = \{2, 4, 5\}$ and $B = \{1, 2, 3\}$ is shown on the right.

The general case for 2 intersecting sets, A and B, is illustrated below.



WB

p. 35

WORKED EXAMPLE 6 Venn diagrams

Show each set as a Venn diagram.

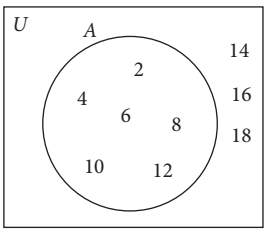
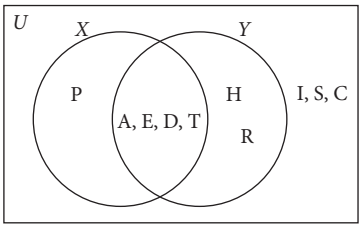
a U = {even numbers between 1 and 20}

A = {even numbers from 2 to 12 inclusive}

b U = {letters in the word DISPATCHER}

X = {letters in the word TAPED}

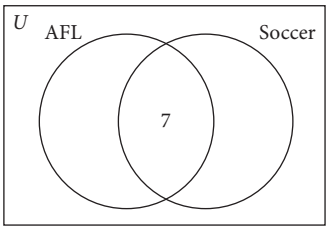
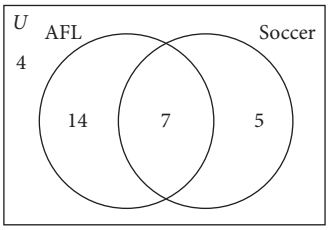
Y = {letters in the word THREAD}

Steps	Working
a 1 Only one circle is needed and it contains all the elements given. Determine the elements of this set.	Set A is the even numbers from 2 to 12. A = {2, 4, 6, 8, 10, 12}
2 Identify the elements in the universal set that are not in the given set. That is, find the complement of the set.	U = {2, 4, 6, 8, 10, 12, 14, 16, 18} The elements remaining in the universal set is the complement of A. A' = {14, 16, 18}.
3 Draw the Venn diagram.	
b 1 Decide if there is an intersection between the 2 subsets of the universal set.	Sets X and Y intersect. $X \cap Y$ = {A, E, D, T}.
2 Find the elements remaining in each set that are not part of the intersection of the universal set.	Letters remaining in set X are {P}. Letters remaining in set Y are {H, R}.
3 Find the elements remaining in the universal set.	U = {D, I, S, P, A, T, C, H, E, R} Identify the elements that are not in $X \cup Y$. $(X \cup Y)'$ = {I, S, C}
4 Draw the Venn diagram.	

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WORKED EXAMPLE 7 Cardinal number with 2 sets

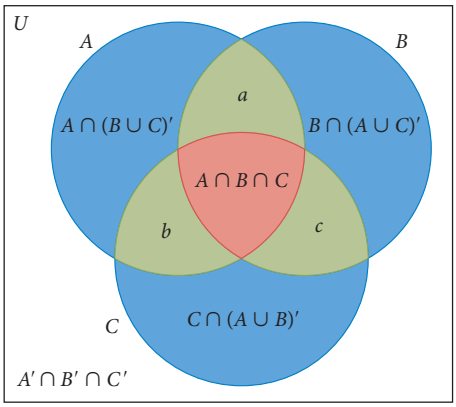
A survey of 30 people found that 21 like AFL, 12 like soccer and 7 like both sports. Display this information as a Venn diagram.

Steps	Working
1 Display the given information on a Venn diagram.	Label the sets 'AFL' and 'Soccer'. The intersection has 7 people that like both sports. 
2 Calculate all other required information.	12 people in total like soccer. Hence, $7 + x = 12$, where $x = 5$ is the number of people who only like soccer. 21 people in total like AFL. Hence, $7 + y = 21$, where $y = 14$ is the number of people who prefer only AFL. 30 people were surveyed, so $5 + 7 + 14 + z = 30$, where $z = 4$ is the number of people who do not like either sport.
3 Display the remaining information on the Venn diagram.	

WB
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The Venn diagram involving 3 intersecting sets, A, B and C, is illustrated below.

$a \cup (A \cap B \cap C) = A \cap B$
$b \cup (A \cap B \cap C) = A \cap C$
$c \cup (A \cap B \cap C) = B \cap C$



The diagram above can be useful as a reference, but it can also be efficient to find values by thinking in terms of regions rather than symbols. For example, think of $a \cup (A \cap B \cap C)$ in the diagram as the intersection of A and B.

WB

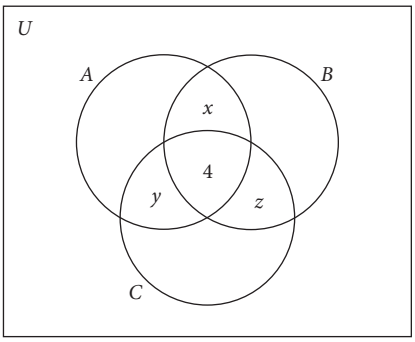
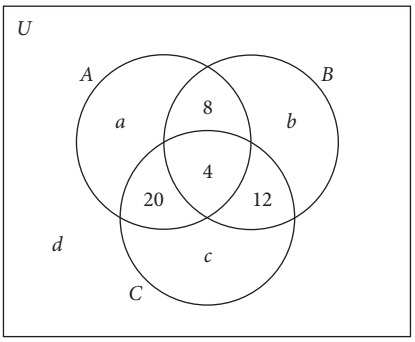
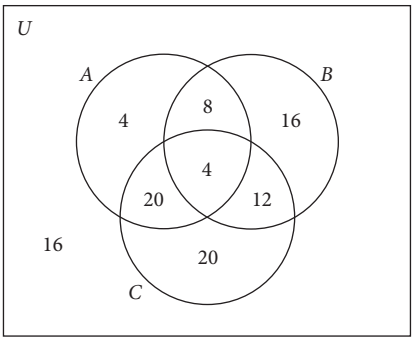
p. 37

WORKED EXAMPLE 8 Cardinal number with 3 sets

Use the following information to show a completed Venn diagram.

$n(U) = 100$ $n(A) = 36$ $n(B) = 40$ $n(C) = 56$

$n(A \cap B) = 12$ $n(A \cap C) = 24$ $n(B \cap C) = 16$ $n(A \cap B \cap C) = 4$

Steps	Working
1 Display the known information as a Venn diagram. Determine the values of some intersections that can be found immediately using the given information.	Let A, B and C be the 3 sets. The intersection of all 3 sets is 4.  $n(A \cap B) = 12$, so $x + 4 = 12$, $x = 8$. $n(A \cap C) = 24$, so $y + 4 = 24$, $y = 20$. $n(B \cap C) = 16$, so $z + 4 = 16$, $z = 12$.
2 Find the remaining values of the intersections and the number of elements outside the 3 sets.	 $n(A) = 36$, $a + 20 + 4 + 8 = 36$, so $a = 4$. $n(B) = 40$, $b + 8 + 4 + 12 = 40$, so $b = 16$. $n(C) = 56$, $c + 20 + 4 + 12 = 56$, so $c = 20$. $n(U) = 100$, $4 + 8 + 4 + 16 + 20 + 12 + 20 + d = 100$ $d = 16$
3 Draw the Venn diagram.	

9780170448338

EXERCISE 2.2 Set notation

ANSWERS p. 566

Recap

1 $99(1.\dot{2}\dot{3} + 0.\dot{1}i)$ is equivalent to

A $122 + 11i$ **B** $99.99 + 9.9i$ **C** $123 + 99i$ **D** $124 + 10i$ **E** $122 + 99i$

2 Choose the correct statement.

A $\dfrac{\sqrt{9}+\sqrt{7}}{\sqrt{25}}$ is rational.

B $2.\dot{3}2\dot{5}$ is equivalent to $\dfrac{2.325}{1000}$.

C The sum of 2 rational numbers cannot be an integer.

D The set of recurring decimals is a subset of the set of rational numbers.

E The sum of 2 complex numbers cannot be a real number.

Mastery

3 WORKED EXAMPLE 5

U = {natural numbers from 5 to 20 inclusive}

A = {even numbers from 8 to 20 inclusive}

B = {odd numbers from 7 to 17 inclusive}

C = {multiples of 5 from 10 to 20 inclusive}

Find

a $A \cap B \cap C$ **b** $(A \cup B)'$ **c** $A \cap (C \cup B)$ **d** $n(A' \cap B)'$

4 WORKED EXAMPLE 6 Show the following sets as a Venn diagram.

U = {letters in 'DO PICK MATHS'}

B = {letters in 'TOM'S PICK'}

C = {letters in 'MICK'S HAT'}

5 Display the information given below as a Venn diagram.

U = {●, ○, ▲, △, ⬬, ⬭, ■, □, ➞, ⇨, ✚, ✜}

A = {straight-sided shapes}

B = {shaded shapes}

6 Show the 3 sets given as a Venn diagram.

$U = R$, $A = Q$, $B = N$

where R is the set of real numbers, Q is the rational numbers, and N is the natural numbers.

7 Show the 3 sets described as a Venn diagram.

U = {letters of the alphabet from A to J}

X = {vowels from A to H inclusive}

Y = {consonants from A to H inclusive}

8 WORKED EXAMPLE 7 A survey involved asking 35 people their drinking preferences regarding tea and coffee. Of the people surveyed, 16 people stated that they like to drink tea, 20 drink coffee and 6 people said they like to drink both beverages. Display this information as a Venn diagram.

9 WORKED EXAMPLE 8 Use the following information to show a completed Venn diagram.

$n(U) = 32$ $n(A) = 20$ $n(B) = 15$ $n(C) = 13$

$n(A \cap B) = 9$ $n(A \cap C) = 8$ $n(B \cap C) = 7$ $n(A \cap B \cap C) = 6$

Exam practice

10 TECH-FREE (3 marks) Most residents in a housing complex of 36 families read newspapers.

A survey of the 36 families revealed that 20 families read *The Times*, 25 families read the *Daily Grind* and 2 families do not read either paper.

Determine the number of families that read both newspapers.

11 TECH-FREE (8 marks)

U = {the first 20 positive integers}

P = {the first 8 prime numbers}

$Q = \{1 \leq n \leq 10, n \in N\}$

$R = \{n = 2m + 1, 0 < m \leq 9, m \in N\}$

Find

a $P \cap Q$ **b** $P \cap Q \cap R$ **c** $(P \cup Q)' \cap R$ **d** $P' \cap Q' \cap R'$

12 (3 marks) For sets X, Y and universal set U, if X is a subset of Y, show on a Venn diagram the region given by $(X \cap Y)' \cap Y$.

13 TECH-FREE Let U = {natural numbers from 1 to 10 inclusive}, A = {even numbers from 2 to 10 inclusive} and B = {multiples of 5 from 1 to 10 inclusive}.

$(A' \cup B)' \cap B$ is

A $A \cap B$ **B** $A' \cap B$ **C** $\varnothing$ **D** $A' \cup B$ **E** U

14 TECH-FREE A survey of 50 adults revealed their ownership of cars and motorbikes. Of the 50 adults, 40 adults own a car and 18 adults own a motorbike. However, 7 adults said they do not own a car or a motorbike.

The number of people who own both a car and a motor bike is

A 9 **B** 14 **C** 15 **D** 25 **E** 32

15 (3 marks) A survey of 30 people returned the following information in relation to their preference for drinking milk, tea and coffee.

- Each person drinks at least 1 of the beverages.
- 2 people said they drink all 3 beverages.
- 5 people drink both tea and coffee.
- 6 people drink tea and coffee.
- 10 people stated that they drink tea.
- 5 people indicated that they only drink milk.
- 8 people drink milk and coffee.

Represent this information as a Venn diagram using 3 overlapping circles.

16 (7 marks) From a poll of 30 diners, it was found that

- 20 had an entrée.
- 15 ordered a main meal.
- 17 had dessert.
- 12 had an entrée and a main meal.
- 6 diners ordered a main meal and a dessert.
- 9 diners had an entrée and dessert.
- 4 customers had all 3 items.

a Draw a Venn diagram showing all the information. 3 marks

b Find the number of diners who only had an entrée. 1 mark

c How many diners had ordered mains and dessert but no entrée? 1 mark

d How many people did not order dessert or had all 3 items? 1 mark

e How many diners did not order anything? 1 mark

17 (7 marks) An electronic toy is designed to hop, skip and jump. As part of quality control, a sample of 100 faulty toys is inspected. It is found that

- 60% can hop.
- 40% can skip.
- 55% can jump.
- 15% can hop and skip.
- 35% can hop and jump.
- 15% can skip and jump.
- 5% can hop, skip and jump.

a Draw the Venn diagram that includes all the information given. 3 marks

b Find the percentage of toys that can only perform 2 of the 3 possible movements. 2 marks

c Determine the percentage of toys that can't do any of the intended movements. 2 marks

2.3 Prime numbers and rational numbers

Video playlist
Prime numbers and rational numbers

Prime factorisation

A **prime number** is a natural number greater than 1 whose only factors are 1 and itself. For example, 31 is prime because its factors are 1 and 31 only. The first 10 prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23 and 29. Later in this chapter it will be shown that there exists an infinite number of prime numbers.

A **composite number** is a natural number greater than 1 that has more than 2 factors. For example, 8 is composite because it has 4 factors: 1, 2, 4 and 8.

The number 1 is neither prime nor composite.

The fundamental theorem of arithmetic

The **fundamental theorem of arithmetic** states that every **integer** greater than 1 is either prime or can be written as a unique product of **prime numbers**. For example:

$20 = 2 \times 2 \times 5 = 2^2 \times 5$

$64\,478\,886\,875 = 5 \times 5 \times 5 \times 5 \times 13 \times 13 \times 19 \times 19 \times 19 \times 89 = 5^4 \times 13^2 \times 19^3 \times 89$

'Unique product' means that there is only one way of writing the number as a product of primes.

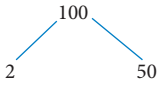
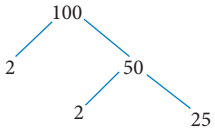
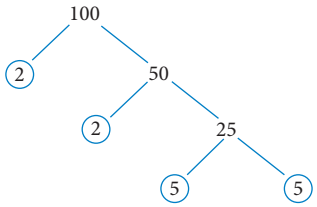
One way to find all the prime factors of a number is to construct a **factor tree**.

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WORKED EXAMPLE 9 Using a factor tree

a Construct a factor tree to express 100 as the product of prime numbers.

b State all the factors of 100 that are greater than 1.

Steps	Working
a 1 Find 2 factors whose product is 100.	Suppose we use 2×50. 
2 Repeat step 1 for any factor that is not a prime number. 2 is prime, but 50 is not and can be written as 2×25.	
3 Keep repeating step 1 until all factors are prime numbers. 2 is prime, but 25 can be written as 5×5. Stop, because all factors are prime.	
4 Write the number as the product of prime numbers (circled in the factor tree).	$100 = 2^2 \times 5^2$
b 1 The factors of 100 include all the prime factors and combinations of the products of the factors, including powers.	2 | 5 $2 \times 5 = 10$ | $2^2 = 4$ $5^2 = 25$ | $2^2 \times 5 = 20$ $2 \times 5^2 = 50$ | $2^2 \times 5^2 = 100$
2 State the factors.	The factors of 100 that are greater than 1 are 2, 4, 5, 10, 20, 25, 50, 100.

A factor tree can also be used to find a square root and to simplify surds.

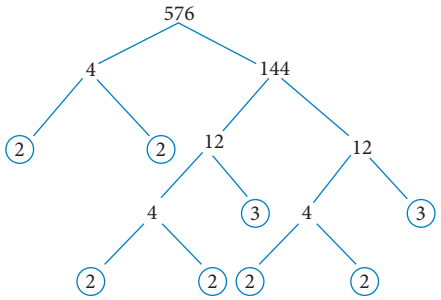
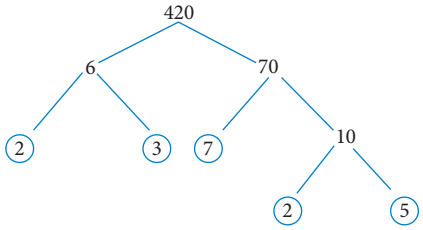
Exam hack

For questions not permitting CAS, choose 2 factors so that one of them is as small as possible. This may take more steps to complete the factor tree but calculation errors will be minimised.

WORKED EXAMPLE 10 Using a factor tree to find roots and simplify surds

a Use a factor tree to find $\sqrt{576}$.

b Simplify $\sqrt{420}$.

Steps	Working
a 1 Construct a factor tree whose product of prime factors is 576.	 $576 = 2^6 \times 3^2$
2 Evaluate the square root using the powers of the prime numbers.	$\sqrt{576} = \sqrt{2^6 \times 3^2}$ $= 2^3 \times 3^1$ $= 24$
b 1 Write 420 as the product of prime numbers.	 $420 = 2^2 \times 3 \times 5 \times 7$
2 Simplify the surd using the powers of the prime numbers.	$\sqrt{420} = \sqrt{2^2 \times 3 \times 5 \times 7}$ $= 2^1 \times \sqrt{3 \times 5 \times 7}$ $= 2\sqrt{105}$

WB
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Rational numbers, integer properties and proofs

Recall that a rational number is a ratio $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

An **even number** is exactly divisible by 2. The set of even numbers is {0, ±2, ±4, ±6 …}, with each element, n, defined by $n = 2k$, $k \in Z$.

An **odd number** leaves a remainder of 1 when it is divided by 2. The set of odd numbers is {±1, ±3, ±5 …}, with each element, n, defined by $n = 2k + 1$, $k \in Z$.

Even and odd numbers alternate, so 1 added to, or subtracted from, an even number produces an odd number. Also, 1 added to, or subtracted from, an odd number gives an even number.

Properties of sums, differences and products of odd and even numbers

Some useful properties of the sum, difference and product of odd and even numbers:

- even number ± even number = even number
- odd number ± odd number = even number
- even number ± odd number = odd number
- odd (or even number) × even number = even number
- odd number × odd number = odd number

WB

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WORKED EXAMPLE 11 Proofs involving integers

a Prove that if m and n are both even numbers, then $m + n$ is even.

b Prove that if m and n are both odd numbers, then $m \times n$ is odd.

Steps	Working
a 1 Write each number as an element of its set.	m and n are even numbers, so let $m = 2k_1$, $n = 2k_2$, where k_1 and k_2 are integers.
2 Write an expression for $m + n$ and show that it is even.	$m + n = 2k_1 + 2k_2$ $= 2(k_1 + k_2)$ Since k_1 and k_2 are integers, $k_1 + k_2$ is also an integer. Then $2(k_1 + k_2)$ is even because it is divisible by 2.
3 State the conclusion.	Therefore, if m and n are both even numbers, then $m + n$ is also even.
b 1 Write each number as an element of its set.	m and n are odd numbers. Let $m = 2k_1 + 1$, $n = 2k_2 + 1$, $k_1, k_2 \in Z$.
2 Write an expression for $m \times n$ and show that it is odd.	$m \times n = (2k_1 + 1)(2k_2 + 1)$ $= 4k_1k_2 + 2k_1 + 2k_2 + 1$ $= 4k_1k_2 + 2(k_1 + k_2) + 1$ $4k_1k_2$ and $2(k_1 + k_2)$ are even because they are divisible by 2. So $4k_1k_2 + 2(k_1 + k_2)$ is even because it is the sum of 2 even numbers (from part **a**). So $4k_1k_2 + 2(k_1 + k_2) + 1$ is odd because it is an even number + 1.
3 State the conclusion.	Therefore, if m and n are both odd numbers, then $m \times n$ is also odd.

WB

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WORKED EXAMPLE 12 Using basic properties of integers to prove other properties

a Given that the product of an odd and an even number is even, prove that $n^2 + n$ is even if $n \in Z$.

b Hence, prove that if n^2 is even, then n is even.

Steps	Working
a 1 Write $n^2 + n$ as a product by factorising.	$n^2 + n = n(n + 1)$
2 Use the given property.	$n(n + 1)$ is the product of 2 consecutive integers, so one of them must be odd, and the other even. So the product must be even.
3 State the conclusion.	Therefore, $n^2 + n$ is even.
b Use the result from part **a**.	The sum $n^2 + n$ is even. The sum of 2 even numbers is another even number. So given that n^2 is even, n must be even also.

EXERCISE 2.3 Prime numbers and rational numbers

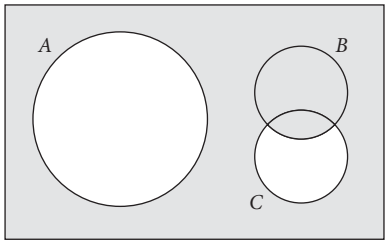
ANSWERS p. 567

Recap

1 $U = \{\text{integers between } -4 \text{ and } 5\}$, $A = \{n = 2m, 1 \leq m \leq 2, m \in N\}$ and $B = \{-3, -1, 0, 4\}$. So $(A' \cap B')'$ is

A $\{-3, -1, 1, 3\}$ **B** $\{2, 1, 0, 1, 4\}$ **C** $\{-3, -1, 0, 2, 4\}$ **D** $\{-3, -1, 4\}$ **E** $\{-2, -1, 0, 3\}$

2



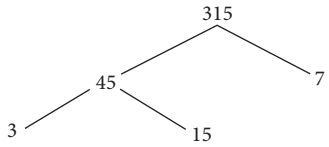
The unshaded region represents

A $A' \cup B' \cup C'$ **B** $(A' \cap B')' \cap C$ **C** $(A \cap B')' \cup C$ **D** $(A' \cup B)' \cup C$ **E** $A \cup (B' \cap C)$

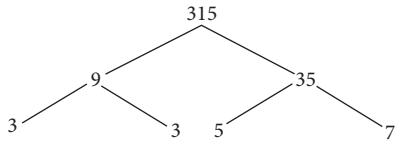
Mastery

3 Which factor tree displays the correct sequence of the prime factors of 315?

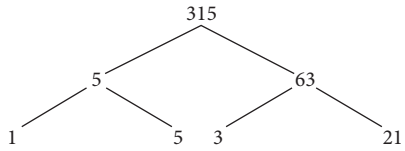
A



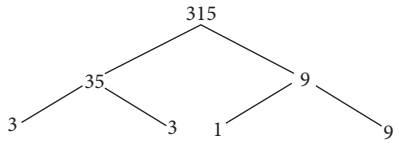
B



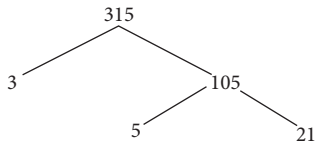
C



D



E



4 If $432 = 2^4 \times 3^3$, state all factors of 432 in ascending order.

5 WORKED EXAMPLE 9 Use a factor tree to find all prime factors of 525. Hence, write down all factors of 525 that are greater than 1.

6 WORKED EXAMPLE 10

a Use a factor tree to find $\sqrt{1764}$.

b Use a factor tree to simplify $\sqrt{1350}$.

7 Simplify $\sqrt{4\frac{7}{32}}$ using a factor tree.

Exam practice

8 (3 marks) Prove that n^n is odd if n is odd.

9 (4 marks)

a Explain why every prime number, $p > 3$, can be written in the form $p = 6k \pm 1$, $k \in N$. 2 marks

b Hence, show that for $p > 3$, p^2 is odd. 2 marks

10 (2 marks) Explain using the properties of odd numbers why for any prime number $p > 2$, p^n is odd.

11 (3 marks) Prove that the product of any 2 prime numbers greater than 2 is odd.

12 WORKED EXAMPLE 11 (3 marks) Prove that if m is an even number and n is an odd number, then $m \times n$ is an even number.

13 WORKED EXAMPLE 12 (3 marks) Prove that for $n \in N$, $n^2 + 3n$ is even.

14 (5 marks)

a Prove that the product of 2 even numbers is an even number. 2 marks

b Hence, prove that 2^n, $n \in N$, is even. 2 marks

c Use your results in parts **a** and **b** to explain why $(2^n + 1)^2$, $n \in N$ is always odd. 1 mark

15 For $a, b, c \in N$, the value of $(a^b)^c$ is always odd when

A a is even and c is odd **B** b is even and c is odd **C** a, b and c are even

D a is odd and b is odd **E** b is odd

16 The number of distinct prime factors for the number 384 is

A 2 **B** 3 **C** 4 **D** 5 **E** 6

Video playlist
Methods of proof

2.4 Methods of proof

In mathematics, a **theorem** is a statement that has been proved to be always true, such as Pythagoras' theorem for right-angled triangles, $a^2 + b^2 = c^2$. On the other hand, a **conjecture** is a statement that seems to be true but has not been proved conclusively. For example, in 1742 the German mathematician Christian Goldbach proposed that every even number greater than 2 can be expressed as the sum of 2 prime numbers, such as $8 = 3 + 5$ and $24 = 11 + 13$. Goldbach's conjecture is still to be proved or disproved.

Worksheet
Counterexamples

Proof by counterexample

Some conjectures can be disproved by a **counterexample**. To disprove a conjecture only requires one counterexample. For example, consider the conjecture that $n^2 - n + 41$, for $n \in N$, produces only prime numbers. The expression is true for values of n up to 40, but using $n = 41$ produces the number 1681, which is not prime ($1681 = 41 \times 41$). This is sufficient proof that $n^2 - n + 41$ does not generate only primes.

WORKED EXAMPLE 13 Proof by counterexample

Prove that each statement is false by providing a counterexample.

a All prime numbers are odd.

b If n is an integer greater than 4 and n^2 is divisible by 4, then n is divisible by 4.

c For $a, b \in R$, if $a^2 = b^2$, then $a = b$.

Steps		Working
a	**1** Find a prime number that is not odd.	2 is prime but is not odd.
	2 State the conclusion.	Therefore, not all prime numbers are odd.
b	**1** Try values of n such that n^2 is divisible by 4, until n is not divisible by 4.	For $n = 6$, $n^2 = 6^2$ is divisible by 4, but 6 is not divisible by 4.
	2 State the conclusion.	If n is an integer and n^2 is divisible by 4, n is not always divisible by 4.
c	**1** Find values for a and b for which $a^2 = b^2$ but $a \neq b$.	If $a = 1$ and $b = -1$, then $a^2 = b^2 = 1$, but $a \neq b$.
	2 State the conclusion.	For $a, b \in R$, if $a^2 = b^2$, then it is not always true that $a = b$.

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Direct proof

In **direct proof**, one or more true statements are used to arrive at one or more true conclusions. The method takes the form, **If** (or **given that**) $(S_1, S_2 \ldots S_m)$ **then** $(C_1, C_2, C_3 \ldots C_n)$, where each of the S_m statements is accepted as true and used to reach C_n logically valid conclusions. In many cases, applying this method of proof usually only involves one or 2 statements from which a single conclusion is drawn.

Worked examples 11 and 12 in the previous section provided proofs involving integers using the direct proof method.

WORKED EXAMPLE 14 Applying direct proof

a Given that the sum of 2 odd numbers is even, show that for odd m, $m^2 + 3m$ is even.

b Show that if $n \in N$ is even, then $3n + 5$ is odd.

Steps		Working
a	**1** Factorise the expression to obtain the product of 2 quantities.	$m^2 + 3m = m(m + 3)$
	2 Apply the given information.	m is odd and 3 is odd, so $m + 3$ is even. Write $m + 3 = 2k$, $k \in Z$
	3 Write the complete expression to include the change.	$m^2 + 3m = 2mk$
	4 Draw the conclusion.	$2mk$ is divisible by 2, so it is even. Hence, $m^2 + 3m$ is even.
b	**1** Write the expression using the information provided.	k is even, so let $n = 2k$, $k \in N$ $3n + 5 = 3(2k) + 5$ $= 6k + 5$
	2 Write the result in a form from which the required conclusion can be made.	$6k + 5 = (6k + 4) + 1$ $= 2(3k + 2) + 1$ $= 2p + 1$, $p \in N$
	3 State the conclusion.	$2p + 1$ is in the form of an odd number, and hence $3n + 5$ is odd.

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Proof by contradiction

Some conjectures can be proved using **proof by contradiction**.

The conjecture is assumed to be **false**, which leads to a conclusion that contradicts the original assumption.

A contradiction is a statement that says the opposite.

WB p. 44

WORKED EXAMPLE 15 Applying proof by contradiction to surds

Prove by contradiction that $\sqrt{2}$ is irrational.

Steps	Working
1 Assume that the statement is false.	Assume that $\sqrt{2}$ is rational and can be written as $\sqrt{2} = \frac{a}{b}$, where $a, b \in Z$ have no common factors other than 1.
2 Rearrange the equation and state a conclusion.	$a^2 = 2b^2$ $2b^2$ is divisible by 2, so it is even. This means that a^2 is also even and hence a is even (proved in Worked example 12).
3 Obtain another equation.	Since a is even, let $a = 2c$, $c \in Z$. Then $a^2 = 2b^2$ becomes $4c^2 = 2b^2$ or $2c^2 = b^2$.
4 State a conclusion.	$2c^2$ is divisible by 2, so it is even. This means that b^2 is also even and hence b is even.
5 State the contradiction.	Both a and b are even and therefore have 2 as a common factor, but this contradicts the initial assumption that a and b have no common factors other than 1. Hence, the original assumption that $\sqrt{2}$ is rational is false.
6 State the conclusion.	Therefore, $\sqrt{2}$ is irrational.

WB p. 45

WORKED EXAMPLE 16 Proof by contradiction applied to prime numbers

Prove by contradiction that there is an infinite number of prime numbers.

Steps	Working
1 Assume that the statement is false.	Assume that there is a finite number of prime numbers, say n of them.
2 Write the given information.	Let the n prime numbers be $p_1, p_2, p_3, p_4 \ldots p_n$, $n \in N$, where $p_1 = 2$, $p_2 = 3$, $p_3 = 5 \ldots$ Therefore, all numbers greater than p_n are composite (not prime).
3 Create a number greater than p_n and assume that it is composite.	Let P be the product of the primes plus 1. $P = p_1 \times p_2 \times p_3 \times \ldots \times p_n + 1$ If P is composite, then it is divisible by at least one of $p_1, p_2, p_3, p_4 \ldots p_n$. But when P is divided by any of these primes, there is a remainder of 1.
4 State the contradiction.	P cannot be both divisible by a number and have remainder 1.
5 State the conclusion.	This contradicts the original assumption that there is a finite number of primes. The assumption is false. Therefore, there is an infinite number of prime numbers.

WORKED EXAMPLE 17 Proof by contradiction involving the set of integers

Prove by contradiction that $a^2 - 4b \neq 2$, $a, b \in Z$.

Steps	Working
1 Assume the statement is false.	Assume that $a^2 - 4b = 2$ for some $a, b \in Z$.
2 Rearrange and factorise the equation.	$a^2 = 2 + 4b$ $= 2(1 + 2b)$
3 Draw a conclusion.	a^2 is even because $2(1 + 2b)$ is divisible by 2. Therefore, a is even.
4 Express a as an even number and substitute.	Let $a = 2c$, $c \in Z$. Substitute into $a^2 - 4b = 2$: $(2c)^2 - 4b = 2$ $4c^2 - 4b = 2$ $c^2 - b = \frac{1}{2}$
5 State the contradiction.	This is impossible because $b, c \in Z$, so $c^2 - b$ must be an integer, not $\frac{1}{2}$. Hence, our original assumption that $a^2 - 4b = 2$ is false.
6 State the conclusion.	Therefore, $a^2 - 4b \neq 2$, $a, b \in Z$.

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Exam hack

When using proof by contradiction, think:
'If I assume this is true, I must conclude something that is false.'

Proof by contrapositive

Proof by contrapositive begins with considering the negation of each statement. Hence, with statements P and Q, to prove '**if** P **then** Q' becomes showing '**if not** Q **then not** P'.

Worksheet Contrapositive

WORKED EXAMPLE 18 Proof by contrapositive

Prove that for $n \in N$, if $3n + 5$ is odd, then n is even.

Steps	Working
1 Express the statement 'if P then Q' as 'if not Q then not P'.	P: $3n + 5$ is odd. Q: n is even. Not Q is 'n is odd'. Not P is '$3n + 5$ is even'. 'If n is odd, then $3n + 5$' is even'.
2 Prove 'if not Q then not P'.	If n is odd, then $3n$ is odd because odd × odd = odd. $3n$ is odd and 5 is odd, so $3n + 5$ is even because odd + odd = even.
3 State the conclusion.	If n is odd, then $3n + 5$ is even. Therefore, if $3n + 5$ is odd, then n is even.

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EXERCISE 2.4 Methods of proof

ANSWERS p. 568

Recap

1 Select the set below that is the minimum subset of R that contains $\{x : x^2 - a = 0, a \in N\}$.

A R **B** C **C** N **D** Q **E** Z

2 The fraction $\frac{2050}{999}$ as a recurring decimal is

A $2.05\dot{2}\dot{2}$ **B** $2.\dot{0}5\dot{2}$ **C** $\dot{2}.\dot{0}5\dot{2}$ **D** $2.\dot{0}5\dot{2}\dot{2}$ **E** $2.\dot{0}5\dot{2}\dot{2}$

Mastery

3 WORKED EXAMPLE 13 Prove that each statement is false by providing a counterexample.

a All shapes with 4 sides and 4 right angles are squares.

b If the sum of 2 integers is even, then one of the integers must be even.

c For $n \in R$, if n^2 is rational, then n is rational.

4 WORKED EXAMPLE 14 Prove that for all odd numbers, n, $n^2 + 3n + 5$ is odd.

5 WORKED EXAMPLE 15 Prove by contradiction that $\sqrt{3}$ is not a rational number.
(You may assume the property that if n^2 is divisible by 3, then n is also divisible by 3.)

6 WORKED EXAMPLE 16 The value $P = p_1 \times p_2 \times p_3 \times \ldots \times p_n + 1$ in Worked example 16 may not necessarily be a prime number.

a What is the value of P for $n = 6$?

b Show that your answer to part **a** is not a prime number.
Hint: Use the CAS **factor( )** function.

c If P is composite, explain why it is still true that there is an infinite number of primes.

Exam practice

7 WORKED EXAMPLE 17 (3 marks) Generalise the proof in Worked example 17 by proving by contradiction that $a^2 - 2^n b \neq 2$, where $a, b \in Z$, $n \in N$.

8 WORKED EXAMPLE 18 (3 marks) Use the contrapositive to prove that for $n \in Z$, if $n^2 - 6n + 5$ is even, then n is odd.

9 A counterexample to show that the statement '*All snakes bite*' is false would be

A Snakes don't bite.

B I've never seen a snake bite.

C If it bites, then it is a snake.

D My pet snake doesn't bite.

E Snakes bite when they fear attack.

10 Proof by contradiction is to be used for the statement:

'The sum of the interior angles of a triangle is 180°.'

The first part of the proof is to assume that

A The sum of the exterior angles of a triangle is 180°.

B The sum of the interior angles of a triangle is more than 180°.

C The sum of the interior angles of a triangle is less than 180°.

D The sum of the interior angles of a triangle is not 180°.

E The product of the interior angles of a triangle is 180°.

11 A proof by contradiction is shown here with some missing words in the first line:

Assume that _____.

Squares have 4 equal sides and 4 right angles.

Therefore, rhombuses have 4 equal sides and 4 right angles.

But rhombuses do not necessarily have 4 right angles, so the assumption cannot be true.

Therefore, rhombuses are not squares.

The first line of the above proof is

A Assume that squares are rhombuses.

B Assume that squares are not squares.

C Assume that rhombuses are squares.

D Assume that rhombuses are not squares.

E Assume that squares and rhombuses have 4 equal sides and 4 right angles.

12 (5 marks) Consider $A = \left\{\frac{1}{n}\right\}$, where n is a positive integer.

a Use a counterexample to show that A is not closed under

i addition **ii** subtraction **iii** division. 3 marks

b Use direct proof to show that A is closed under multiplication. 2 marks

13 (4 marks) Find a counterexample for each statement below to demonstrate that the statement is false.

a If $x^2 = 9$, then $x = 3$. 1 mark

b The statement $x^2 - 3x + 2 = 0$ is true for $x = 1, 2, 3 \ldots$ 1 mark

c All lines that never meet are parallel. 1 mark

d If an animal has 8 legs, then it is a spider. 1 mark

14 (2 marks) A Fermat number is a positive integer of the form $F_n = 2^n + 1$, where n is a non-negative integer.

Goldbach conjectured that all Fermat numbers are prime. Find a counterexample by trying values of n less than 10 that will prove the conjecture to be false.

'Non-negative' means positive or zero.

15 (4 marks) Decide whether each statement below is true or false. If false, give a counterexample.

a If the square of an integer is even, then the integer must be even. 1 mark

b If $x^2 \geq 4$, then $x \geq 2$. 1 mark

c If $a > b$ and $c < b$, then $a > c$. 1 mark

d If a number is prime, then it is odd. 1 mark

16 (3 marks) Follow the steps in Worked example 15 to prove by contradiction that $\sqrt[3]{2}$ is irrational.

17 (3 marks) Prove by contradiction that for $n \in N$, if $n^2 - 2n + 7$ is even, then n must be odd.
Hint: Start by assuming that $n^2 - 2n + 7$ is even and n is not odd.

18 (3 marks) Prove by contradiction that there do not exist integers a and b for which $18a + 6b = 1$.

19 (3 marks) Use proof by contradiction to show that there are no rational solutions to the equation $x^3 + x + 1 = 0$.

2.5 Proof by mathematical induction

Video playlist
Proof by mathematical induction

Worksheets
Proof by induction

Mathematical induction

Deduction is logical reasoning that begins with known conditions and assumptions, from which you prove a statement or formula to be true.

Induction is logical reasoning that begins with one *specific* fact or statement, from which you prove a *general* statement or formula to be true. **Mathematical induction** is used to prove that a statement or formula is true for an infinite set of consecutive values of n, such as all natural numbers $\boldsymbol{N}$ (positive integers), after first proving that it is true for an initial value, such as $n = 1$.

Proof by mathematical induction requires 4 steps:

1 Test the statement is true for the first value of n, such as $n = 1$ for natural numbers.

2 Assume the statement is true for some value of n, say k.

3 Use the assumption above to prove that the statement is true for $n = k + 1$ (the next value of n): this is the **induction step**.

4 State your conclusion in terms of the initial assumption:

'Hence, if the result is true for $n = k$, then it is true for $n = k + 1$.

The statement is true for $n = 1$, so by mathematical induction the statement is true for all positive integers n.'

Mathematical induction works because of a 'domino effect' after we have proved that if a statement is true for $n = k$, then it is also true for the next integer $n = k + 1$.

Because we have shown the statement to be true for $n = 1$,

then it is also true for $n = 1 + 1 = 2$,

therefore it is also true for $n = 2 + 1 = 3$,

$n = 3 + 1 = 4$, and so on for all positive integers, n.

Proof by mathematical induction

Step 1 (base step): Show true for the first value of n, such as $n = 1$.

Step 2 (assumption): Assume true for some integer $n = k$, where k is in the required domain for n.

Step 3 (induction): Prove true for $n = k + 1$.

Step 4 (conclusion): Conclude that the statement is true for all required integers n.

WORKED EXAMPLE 19 Proof by induction of a series formula

Prove by induction that $1 + 2 + 3 + \ldots + n = \frac{n}{2}(n + 1)$, $n \in N$.

Steps	Working
1 Prove the base step.	Let $P(n) = 1 + 2 + 3 + \ldots + n = \frac{n}{2}(n + 1)$. Prove that the formula is true for $n = 1$. LHS is 1. RHS is $\frac{1}{2} \times (1 + 1) = 1$. LHS = RHS, so the statement is true for $n = 1$.
2 Assume the statement is true for some integer $n = k$, where k is positive.	Assume the formula is true for $n = k$ $(k \geq 1)$. Assume $P(k) = 1 + 2 + 3 + \ldots + k = \frac{k(k + 1)}{2}$. Need to prove $P(k + 1) = \frac{(k + 1)(k + 2)}{2} = \frac{k^2 + 3k + 2}{2}$
3 Prove that the statement is true for $n = k + 1$, substituting the above assumption.	$(1 + 2 + 3 + \ldots + k) + (k + 1) = P(k) + (k + 1)$ $= \frac{k(k + 1)}{2} + (k + 1)$ $= \frac{k(k + 1) + 2k + 2}{2}$ $= \frac{k^2 + k + 2k + 2}{2}$ $= \frac{k^2 + 3k + 2}{2}$ $= P(k + 1)$
4 State the conclusion.	If the statement is true for $n = k$, then it is true for $n = k + 1$. The statement is true for $n = 1$, so by mathematical induction the statement is true for all $n \in N$.

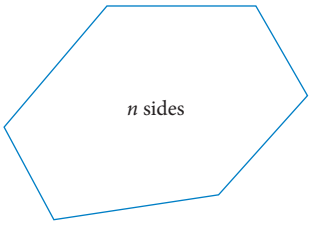
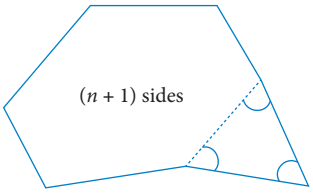
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WORKED EXAMPLE 20 The sum of the interior angles of a polygon

Prove by induction that for $n \geq 3$, the sum of the interior angles of an n-sided polygon is $180(n - 2)$ degrees.

Steps	Working
1 Prove the base step.	Let $P(n) = 180(n - 2)$. Prove that the formula is true for $n = 3$ (triangle). $P(3) = 180(3 - 2) = 180°$, so true for $n = 3$.
2 Assume the statement is true for some integer $n = k$, where $k \geq 3$.	Assume the formula is true for $n = k$ $(k \geq 3)$. $P(k) = 180(k - 2)$ Need to prove $P(k + 1) = 180(k + 1 - 2)$ $= 180(k - 1)$. The diagrams below show what happens to the angle sum of a polygon when an extra side is added to it. The second diagram shows that an extra side can be added by removing 1 of the sides and replacing it with 2 sides, adding a triangle shape to the polygon.   The diagram shows that the angle sum of the polygon increases by the angle sum of a triangle, which is 180°.
3 Prove that the statement is true for $n = k + 1$, substituting above assumption.	So $P(k + 1) = P(k) + 180$ $= 180(k - 2) + 180$ $= 180k - 360 + 180$ $= 180k - 180$ $= 180(k - 1)$
4 State the conclusion.	The statement is true for $n = k + 1$ if it is true for $n = k$. The statement is true for $n = 3$, so by mathematical induction the statement is true for all integers $n \geq 3$.

WORKED EXAMPLE 21 Divisibility proof

Use proof by induction to show that for all natural numbers, $6^n + 4$ is divisible by 5.

Steps	Working
1 Verify the base step.	Let $P(n) = 6^n + 4$. True for $n = 1$: $6^1 + 4 = 10$ is divisible by 5.
2 Assume the statement is true for some positive integer $n = k$.	Assume $6^k + 4$ is divisible by 5 ($k \geq 1$). So let $6^k + 4 = 5M$, where M is a natural number. $6^k = 5M - 4$ Need to prove that $6^{k+1} + 4$ is divisible by 5.
3 Prove that the statement is true for $n = k + 1$.	$P(k + 1) = 6^{k+1} + 4$ $= 6^k(6) + 4$ $= (5M - 4)(6) + 4$ $= 30M - 24 + 4$ $= 30M - 20$ $= 5(6M - 4)$ which is divisible by 5.
4 State the conclusion.	The statement is true for $n = k + 1$ if it is true for $n = k$, and as it is true for $n = 1$, then by mathematical induction it is true for all natural numbers, n.

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WORKED EXAMPLE 22 Inequality proof

Use proof by induction to show that $2^n > n^2$ for integers $n > 4$.

Steps	Working
1 Show the base step.	True for $n = 5$: $2^5 = 32$ and $5^2 = 25$ $32 > 25$ so statement is true for $n = 5$.
2 Assume the statement is true for some integer $n = k$.	Assume for $k > 4$, $2^k > k^2$. Required to show $2^{k+1} > (k + 1)^2$
3 Prove that the statement is true for $n = k + 1$.	$2^{k+1} = 2^k(2)$ $= 2^k + 2^k$ $> k^2 + k^2$, since $2^k > k^2$ $> k^2 + 4k$, since $k > 4$ $= k^2 + 2k + 2k$ $> k^2 + 2k + 8$, since $k > 4$ $> k^2 + 2k + 1$ $= (k + 1)^2$ Hence $2^{k+1} > (k + 1)^2$.
4 State the conclusion.	The statement is true for $n = k + 1$ if it is true for $n = k$. Since it is true for $n = 5$, then by mathematical induction it is true for all integers $n > 4$.

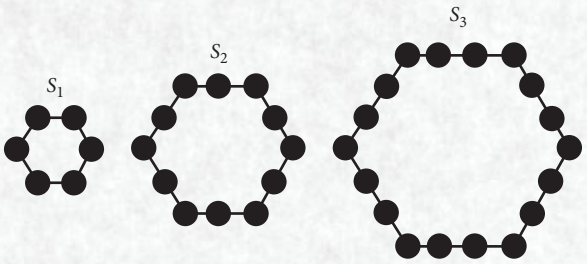
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Video
VCE question analysis: Proof and number

VCE QUESTION ANALYSIS

A VCE-style question (10 marks)

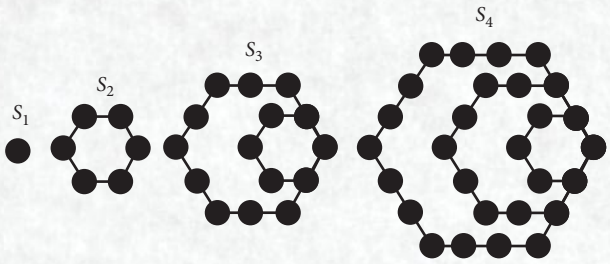
Below are hexagon shapes created using circles.



a **i** Let $C(n)$ be the number of circles in shape S_n. State the relationship between $C(n)$ and n. 1 mark

ii Use mathematical induction to prove the equation in part **i**. 2 marks

b Suppose the hexagons are constructed as shown below.



i Let $S(n) = an^2 + bn + c, a, b \in R$ be the number of circles in shape S_n.

Use technology or another method to obtain the values $a = 2$, $b = -1$ and $c = 0$. 2 marks

ii The sequence of differences, $S(n) - S(n-1)$, is arithmetic.

List the first five terms of the sequence and state the rule for the nth term in the form $t(n) = dn + e$, $(d, e \in R)$. 2 marks

iii Hence, use proof by induction to prove the equation given in part **i**. 3 marks

Reading the question

- In part **a**, you are required to 'state the relationship'. This will mean writing an equation.
- In part **b iii**, 'hence' requires that the answer to part **b ii** is to be used as part of the induction proof.
- Full marks for part **a** and part **b** rely on correctly establishing the pattern in creating the shapes. Be clear about what the patterns are.

Thinking about the question

- The question has a common theme in providing proofs by using the method of induction, so be clear about the steps involved. Show each step clearly, providing explanations as required.
- In part **b i**, you have the choice of working out the answer algebraically or by using a calculator.

  If technology will be used, provide sufficient explanation and results to justify the 2 marks given for the question.
- Follow instructions precisely. For example, in part **b ii**, it asks for 5 terms. Writing 4 or 6 terms will be marked as incorrect and is also time-wasting.

Worked solution (✓ = 1 mark)

a **i** $n = 1, C(1) = 6$

$n = 2, C(2) = 12$

$n = 3, C(3) = 18$

$\boldsymbol{C(n) = 6n}$ ✓

ii $C(n) = 6n$

True for $n = 1$.

LHS is 1.

RHS is $6 \times 1 = 6$.

Assume the formula is true for $n = k$, $(k \geq 1)$.

Assume $C(k) = 6k$:

Need to prove that $C(k + 1) = 6(k + 1)$

$= 6k + 6$

As the hexagon shape increases and n increases by 1, 6 dots are added to the shape (one for each side),

$C(k + 1) = C(k) + 6$

$= 6k + 6$ ✓

If the formula is true for $n = k$, then it is also true for $n = k + 1$.

The formula is true for $n = 1$, so by mathematical induction it is true for all positive integers n. ✓

b **i** $S(n) = an^2 + bn + c$

$n = 1, S(1) = 1, \quad a + b + c = 1$

$n = 2, S(2) = 6, \quad 4a + 2b + c = 6$

$n = 3, S(3) = 15, \quad 9a + 3b + c = 15$ ✓

Subtract the first equation from the second equation and subtract the second equation from the third equation.

$3a + b = 5$

$5a + b = 9$

Subtracting these equations gives:

$2a = 4$, so $a = 2$.

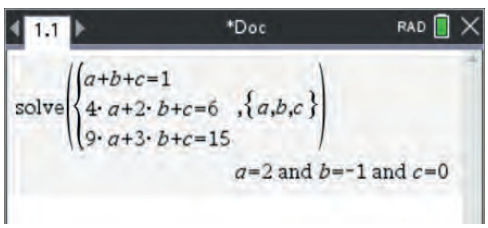
$3a + b = 5$, so $b = -1$.

$a + b + c = 1$, so $c = 0$.

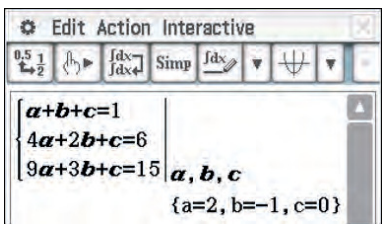
Hence, $a = 2, b = -1, c = 0$. ✓

Or using CAS:

TI-Nspire



ClassPad



ii $S(n) = 2n^2 - n$

$S(1) = 1, S(2) = 6, S(3) = 15, S(4) = 28, S(5) = 45, S(6) = 66$

Calculate the difference between successive term:

$S(2) - S(1) = 6 - 1 = 5$

$S(3) - S(2) = 15 - 6 = 9$

$S(4) - S(3) = 28 - 15 = 13$

$S(5) - S(4) = 45 - 28 = 17$

$S(6) - S(5) = 66 - 45 = 21$

$t(1) = 5, t(2) = 9, t(3) = 13, t(4) = 17, t(5) = 21$ ✓

Find the values for d and e in the equation $t(n) = dn + e$.

The sequence is arithmetic, so use $t(n) = a + (n - 1)d$, where the first term is $a = 5$ and the common difference is $d = 4$.

This gives $t(n) = 4n + 1$. ✓

iii $S(n) = 2n^2 - n$

True for $n = 1$:

$S(1) = 1$ ✓

Assume the formula is true for $n = k$, $(k \geq 1)$.

Assume $S(k) = 2k^2 - k$

Need to prove that $S(k + 1) = 2(k + 1)^2 - (k + 1)$

$= 2k^2 + 4k + 2 - k - 1$

$= 2k^2 + 3k + 1$

$S(k + 1) = S(k) + (4k + 1)$ from part **ii**.

The number of circles in each shape is the sum of the number of circles in the previous shape and additional circles given by $4n + 1$. ✓

If the formula is true for $n = k$, then it is true for $n = k + 1$.

The formula is true for $n = 1$, so by mathematical induction it is true for all positive integers n. ✓

EXERCISE 2.5 Proof by mathematical induction

ANSWERS p. 568

Recap

1 The contrapositive to the statement 'If it's my birthday, then I get cake' is

A 'When it's my birthday, I get cake.'

B 'When I get cake, it's my birthday.'

C 'If it's not my birthday, I don't get cake.'

D 'If it's not my birthday, I still get cake.'

E 'If I don't get cake, then it's not my birthday.'

2 To prove by contradiction the statement, 'For all positive integers, n, $3^n + 1$ is not divisible by 2', the assumption to make can be

A 'For all positive integers, n, if $3^n + 1$ is even then n is not divisible by 2'.

B 'For some positive integer, n, $3^n + 1$ is divisible by 2'.

C 'For some negative integers, n, $3^n + 1$ is not divisible by 2'.

D 'For all positive integers, n, if n is not divisible by 2 then $3^n + 1$ is odd'.

E 'For some positive integers, n, if $3^n + 1$ is odd then it is not divisible by 2'.

Mastery

3 WORKED EXAMPLE 19 Prove by induction that the number of non-overlapping interior triangles formed by connecting the vertices of an n-sided polygon is $n - 2$, where $n > 2$.

4 WORKED EXAMPLE 20 Prove by induction that $4 + 9 + 14 + \ldots + (5n - 1) = \frac{n}{2}(5n + 3)$, $n \in N$.

5 Use induction to show that $2 + 4 + 6 + \ldots + 2n = n(n + 1)$, $n \in N$.

6 Show by the method of induction that $1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$, $n \in N$.

7 WORKED EXAMPLE 21 Prove by induction that for all natural numbers, n, $9^n - 1$ is divisible by 8.

8 Prove by induction that for all natural numbers, n, $11^n - 6$ is divisible by 5.

9 Use proof by induction to show that $1 + 4 + 7 + 10 + \ldots + (3n - 2) = \frac{n(3n - 1)}{2}$, where $n \in N$.

Exam practice

10 (3 marks) Prove by induction that for all natural numbers, $3^n - 1$ is divisible by 2.

11 (3 marks) Prove by induction that for all natural numbers greater than 1, $n^3 - n$ is divisible by 3.

12 (3 marks) Prove by induction that $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n}{6}(n + 1)(2n + 1)$.

13 (3 marks) Prove by induction that $2^0 + 2^1 + 2^2 + \ldots + 2^{n-1} = 2^n - 1$.

14 (4 marks) Prove by induction that $7^n - 4^n$ is a multiple of 3.

15 (3 marks) Prove by induction that $3^{2n} + 11$ is divisible by 4 for all $n \geq 1$.

16 (4 marks) Prove by induction that $n^3 - 7n + 9$ is divisible by 3 for all positive integers, n.

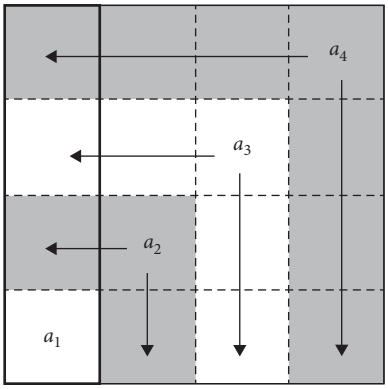
17 (4 marks) Prove by induction that $11^{n+1} + 12^{2n-1}$ is divisible by 133 for all positive integers, n.

18 WORKED EXAMPLE 22 (4 marks) Prove by induction that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$ for all positive integers, n.

19 (4 marks) The diagram shows groups of squares labelled a_1, a_2, a_3, a_4.

Let the sum $s_n = a_1 + a_2 + a_3 + \ldots + a_n$, where a_n is the number of squares in the nth group.

Use proof by induction to show that $s_n = n^2$.



2 Chapter summary

Real and complex numbers

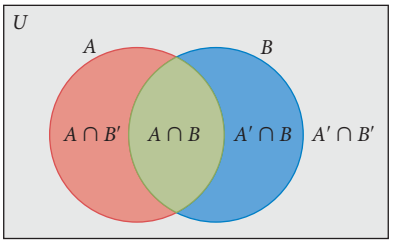
- The set of **natural numbers**, $N = \{1, 2, 3, 4 \ldots\}$, contains the positive whole numbers.
- The set of **integers**, $Z = \{\ldots -3, -2, -1, 0, 1, 2, 3 \ldots\}$, contains the positive and negative whole numbers and 0.
- The **even numbers** are $\{0, \pm 2, \pm 4, \pm 6 \ldots\}$, $n = 2k$, $k \in Z$, which are divisible by 2.
- The **odd numbers** are $\{\pm 1, \pm 3, \pm 5 \ldots\}$, $n = 2k + 1$, $k \in Z$, which leave a remainder of 1 when divided by 2.
- even number ± even number = even number
- even number ± odd number = odd number
- odd number ± odd number = even number
- odd (or even number) × even number = even number
- odd number × odd number = odd number
- The set of **rational numbers** contains numbers that can be written as a fraction $\frac{p}{q}$, where p and q are integers and $q \neq 0$.
- **Terminating decimals** such as 8.87 and **recurring decimals** such as $4.565\,656\ldots$ (or $4.5\dot{6}\dot{6}$) are rational numbers because they can be written in fraction form.
- **Irrational numbers** such as π and surds cannot be written as fractions.
- The rational numbers and irrational numbers together make up the set of **real numbers**, R.
- Sets of numbers are **subsets** of other sets. $\subset$ means 'is a subset of'. For example, $Q \subset R$ and $R \subset C$.
- A set is **closed** under a given operation if the operation always produces an element that is also in the same set. For example, the set of natural numbers is closed under the operations of addition and multiplication, but not under subtraction and division.
- A **complex number**, $z \in C$, is of the form $z = a + bi$, $a, b \in R$, where $i^2 = -1$ (or $i = \sqrt{-1}$).

Set notation

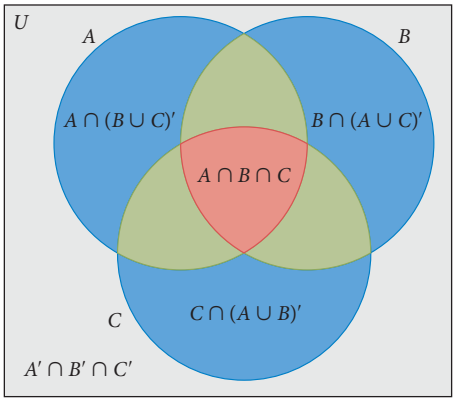
- A **Venn diagram** can be used to show sets.

Notation	Meaning
$\{\,\}$	the set
U	universal set
$\varnothing, \{\,\}$	null or empty set
$\in$	is an element of
$\notin$	is not an element of
$\cap$	intersection
$\cup$	union
$\subset$	is a subset of
A'	complement of set A
$n(A)$	cardinality of a set

- For 2 sets, A and B that have common elements, the Venn diagram is as follows.



- For 3 sets, A, B and C that have common elements, the Venn diagram is as follows.



Prime numbers and rational numbers

- A **prime number** is a natural number greater than 1 whose only factors are 1 and itself.
- A **composite number** is a natural number greater than 1 that has more than 2 factors.
- The **fundamental theorem of arithmetic** states that every integer greater than 1 is either prime or can be written as a unique product of prime numbers.
- A **factor tree** shows all prime factors.
- Even numbers are divisible by 2 and odd numbers leave a remainder of 1 after division by 2.

Methods of proof

- A **theorem** is a true statement, and a **conjecture** is a statement that seems to be true but has not been proved. Some conjectures can be disproved by providing a **counterexample**, one example that does not agree with the conjecture.
- When using **direct proof**, given information (true statements) are used to sequentially reach a logical conclusion.
- **Proof by contradiction** assumes the opposite is true, which then leads to an impossible conclusion.
- **Proof by contrapositive** takes the negation of each statement. To prove '**if** P **then** Q', show '**if not** Q **then not** P'.

Proof by mathematical induction

Step 1 (base step): Show true for the first value of n, such as $n = 1$.

Step 2 (assumption): Assume true for some integer $n = k$, where k is in the required domain for n.

Step 3 (induction): Prove true for $n = k + 1$.

Step 4 (conclusion): Conclude that the statement is true for all required integers n.

For example, to prove that $2 + 4 + 6 + 8 + \ldots + 2n = n(n + 1)$, $n \in N$,

Step 1: The equation is true for $n = 1$, $2 = 1(1 + 1)$.

Step 2: Assume the equation is true for some positive integer $n = k$.

Let $P(k) = 2 + 4 + 6 + 8 + \ldots + 2k = k(k + 1)$.

Step 3: Need to prove that $P(k + 1) = (k + 1)(k + 2)$:

$$\begin{aligned} P(k + 1) &= (2 + 4 + 6 + 8 + \ldots + 2k) + 2(k + 1) \\ &= k(k + 1) + 2(k + 1) \\ &= (k + 1)(k + 2) \end{aligned}$$

Step 4: We have proved the equation is true for $n = k + 1$ if it is true for $n = k$. It is true for $n = 1$, so by mathematical induction it is true for all $n \in N$.

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 (2 marks) Let $z = a + bi$, where a and b are real numbers and $b \neq 0$.
Show that z^3 is real if $b^2 = 3a^2$.

2 (2 marks)

a Write 490 as the product of prime numbers. 1 mark

b Hence, list the factors of 490 that are between 1 and 490. 1 mark

3 (2 marks) 85 people were surveyed regarding their exercise habits. Fifty-nine people stated that walking was part of their schedule and 56 of them said running formed part of their exercise routine. In addition, 5 people answered that they neither walked nor ran when exercising.
Draw the Venn diagram that shows this information.

4 (2 marks) A ball is dropped from a height of 100 cm, with each bounce producing a height that is 80% of the previous height. If the ball is allowed to bounce forever, what total distance travelled will the ball approach?

5 (2 marks) The sum of the first 4 terms of an arithmetic series is 26 and the sum of the first 6 terms of the same series is 57. Determine the value of the first term of the series and the common difference.

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 If a is a surd of the form $\sqrt{x}$, where $x \in N$, and b is an integer, then select the set below that is the minimum subset of the real numbers that contains $a^2 + b^2$.

A irrational numbers **B** real numbers **C** positive integers

D positive real numbers **E** natural numbers

2 To show that the sum of 2 positive odd numbers is an even number, we can start with

A $m = 2k_1$, $n = 2k_2 - 1$, $(k_1 \in N, k_2 \in N)$, $m + n = 2(k_1 + k_2) - 1$

B $m = 2k_1 - 1$, $n = 2k_2 - 2$, $(k_1 \in N, k_2 \in N)$, $m + n = 2(k_1 + k_2) - 1$

C $m = 2k_1 - 1$, $n = 2k_2 - 1$, $(k_1 \in N, k_2 \in N)$, $m + n = 2(k_1 + k_2 - 1)$

D $m = 2k_1$, $n = 2k_2$, $(k_1 \in N, k_2 \in N)$, $m + n = 2(k_1 + k_2)$

E $m = k_1 - 1$, $n = k_2 - 1$, $(k_1 \in N, k_2 \in N)$, $m + n = k_1 + k_2 - 2$

3 A survey of 45 people showed that 20 of them drink tea, 30 drink coffee and 3 do not drink either. The number of people who drink both tea and coffee is

A 3 **B** 5 **C** 7 **D** 8 **E** 9

4 The nth term of a sequence is given by $t_n = t_{n-1} + 3\frac{1}{4}$. If $t_3 = 5$, then t_5 is

A $8\frac{1}{4}$ **B** $9\frac{1}{2}$ **C** $10\frac{3}{4}$ **D** $11\frac{1}{2}$ **E** $12\frac{3}{4}$

5 The 2nd term of a geometric sequence is 6 and the 4th term of a geometric sequence is $\frac{3}{8}$.

Possible values for the first term and the common ratio are

A $a = 4$, $r = \frac{1}{4}$ **B** $a = -12$, $r = \frac{3}{4}$ **C** $a = -24$, $r = -\frac{1}{4}$

D $a = 24$, $r = -\frac{3}{4}$ **E** $a = -8$, $r = \frac{1}{2}$

Section B 6 questions

15 marks

1 (3 marks) Provide a proof by contradiction that for all $a > 0$, $b > 0$ with $a \neq b$, $\frac{a}{b} + \frac{b}{a} \geq 2$.

Hint: Assume $\frac{a}{b} + \frac{b}{a} < 2$.

2 (2 marks) Use a factor tree to simplify $\sqrt{2016}$.

3 (2 marks) Use direct proof to show that all prime numbers greater than 2 are odd.

4 (2 marks) Given that the product of 2 natural numbers is a natural number, prove that the product of 2 recurring decimals, r_1 and r_2 is a rational number.

5 (3 marks) The sum of n terms of the arithmetic sequence a, $a + 19$, $a + 38$ … is 315.

This is the same as the sum of n terms of the geometric sequence a, $2a$, $4a$, $8a$ …

Calculate the value of a and the value of n.

6 (3 marks)

a The number of residents in Mathsville at the start of 2020, 2021 and 2022 was 5000, 5500 and 6050 respectively. Assuming the sequence is geometric, express the population in the form $P(t) = P_0 \times B^t$, where P_0 and B are constants and $P(t)$ is the population after t years, where $t = 0$ represents January 1 2020. 2 marks

b In which year and month will the population reach twice the January 1 2020 population? 1 mark

CHAPTER

3

GRAPH THEORY

Study Design coverage

Nelson MindTap chapter resources

3.1 Graphs
- Lists and adjacency matrices
- Road maps and graphs
- The handshaking lemma

3.2 Types of graphs

3.3 Planar graphs and Euler's formula
- Euler's formula for planar graphs

3.4 Bipartite graphs and trees
- Bipartite graphs

3.5 Walks, paths, trails, cycles and circuits
- The Königsberg bridge problem

VCE question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2



Study Design coverage

UNIT 1, AREA OF STUDY 1: ALGEBRA, NUMBER AND STRUCTURE

Graph theory

- vertices and edges for undirected graphs, including multiple edges and loops, and their representation using lists, diagrams and adjacency matrices
- examples of graphs from a range of contexts such as molecular structure, electrical circuits, social networks, utility connections and their use to discuss types of problems in graph theory including existence problems, construction problems, counting problems and optimisation problems
- the degree of a vertex and the result that the sum of all the vertex degrees is equal to twice the number of edges (the handshaking lemma)
- simple graphs, isomorphism, subgraphs, connectedness, complete graphs and the complement of a graph
- bipartite graphs, trees, and regular graphs (including the Platonic graphs)
- planar graphs and related proofs and applications such as:
  - Euler's formula $v + f - e = 2$ for simple connected planar graphs
  - the complete graph K_n on n vertices has $\frac{n(n-1)}{2}$ edges
  - a regular graph with n vertices each of degree r has $\frac{nr}{2}$ edges
  - the planarity of various types of graphs, including all trees, complete graph on n vertices, if $n \leq 4$, and $K_{m,n}$, the complete bipartite graph with m, n vertices, if $m \leq 2$ or $n \leq 2$
- equivalent conditions for a simple graph with n vertices to be a tree
- trails and circuits, Euler circuits and Euler trails, Hamiltonian cycles and paths, and the Königsberg bridge problem.

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Video playlists (6):

3.1 Graphs
3.2 Types of graphs
3.3 Planar graphs and Euler's formula
3.4 Bipartite graphs and trees
3.5 Walks, paths, trails, cycles and circuits

VCE question analysis Graph theory

Worksheets (8):

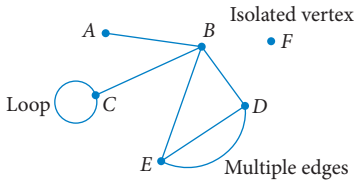
3.1 Adjacency matrices 1 • Adjacency matrices 2
3.2 Types of graphs
3.3 Planar graphs
3.5 Eulerian graphs • Eulerian circuits
• Eulerian trails and circuits
• Hamiltonian paths and cycles

Nelson MindTap

To access resources above, visit
cengage.com.au/nelsonmindtap

3.1 Graphs

In graph theory, a **graph** is a diagram consisting of **vertices** (points) connected by **edges** (lines).
The graph can be used to represent things like electrical circuits, utility connections or social networks.
A graph and a table of basic terms in graph theory are shown here.



Basic terms in graph theory

Term	Definition	Reference to the diagram
Vertex	A point or node on a graph.	The graph has 6 vertices: *A, B, C, D, E, F*
Edge	An interval connecting 2 vertices. The edge *AB* joins vertex *A* to vertex *B*.	The graph has 7 edges: *AB, BC, CC, BE, BD, DE, DE*
Adjacent vertices	Two vertices connected by an edge.	Vertices *A* and *B* are adjacent, as there is an edge that connects *A* to *B*.
Degree of a vertex	The number of edges connected to the vertex.	Four edges are connected to vertex *B* so the degree of vertex *B* is 4.
Loop	An edge that connects a vertex to itself.	There is a loop at vertex *C*, which is written as the edge *CC*.
Multiple edges	Two or more edges connecting the same vertices.	Vertices *D* and *E* have multiple edges.
Isolated vertex	A vertex that is not connected to any other vertices, so it has degree 0.	*F* is an isolated vertex.

Video playlist
Graphs

WORKED EXAMPLE 1 Describing vertices and edges

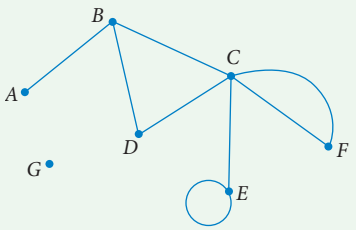
Consider the graph shown.

a Which vertices are adjacent to vertex *D*?

b Which vertex has a loop?

c Which vertex is isolated?

d Which pair of vertices are connected by multiple edges?



Steps	Working
a The vertices adjacent to vertex *D* are connected by an edge.	Vertices *C* and *B* are adjacent to vertex *D*.
b A loop connects a vertex to itself.	Vertex *E* has a loop.
c The isolated vertex is not connected to any other vertex.	Vertex *G* is isolated.
d Vertices connected by more than 1 edge have multiple edges.	Vertices *C* and *F* have multiple edges.

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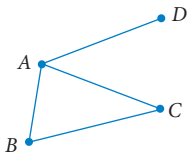
Worksheets
Adjacency matrices 1

Adjacency matrices 2

Lists and adjacency matrices

A graph can be represented by **lists** that show the set of vertices and the set of edges.

The set of vertices = {*A*, *B*, *C*, *D*} and the set of edges = {*AD*, *AB*, *AC*, *BC*} describe the graph shown.



A graph can also be represented by an **adjacency matrix**, a square matrix of order $n \times n$ where n is the number of vertices.

The names of the vertices are written on the left or right of the rows and above the columns. The cells in the matrix have the value 0 if there is no edge connecting the vertices, and the value 1 for every edge connecting the vertices.

The adjacency matrix for the graph above is

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Adjacency matrices

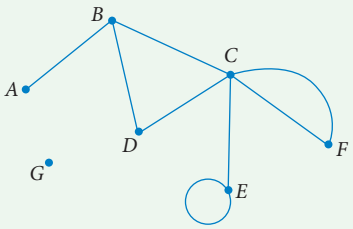
When representing a graph using an adjacency matrix:

- a '0' represents no edge connecting the vertices.
- a '1' represents 1 edge connecting the vertices.
- a number greater than 1 is used when there are multiple edges connecting the vertices.
- a '1' at row x, column x indicates a loop at vertex x.

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WORKED EXAMPLE 2 Using lists to represent a graph

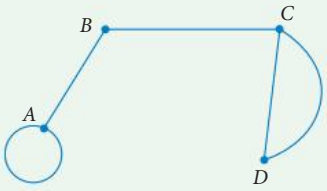
Represent the graph using lists.



Steps	Working
1 Write the list of vertices as a set.	vertices = {*A*, *B*, *C*, *D*, *E*, *F*, *G*}
2 There are 8 edges. Each edge is written using the letters of the vertices it connects. The loop at *E* is written as *EE* and the multiple edges between *C* and *F* are shown by writing *CF* twice.	edges = {*AB*, *BD*, *BC*, *CD*, *CE*, *CF*, *CF*, *EE*}

WORKED EXAMPLE 3 Represent the graph using an adjacency matrix

Use an adjacency matrix to represent the graph shown.



Steps	Working
1 Draw a 4 × 4 matrix and label the rows and columns *A*, *B*, *C* and *D*.	$\begin{matrix} & A & B & C & D \\ A & \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \\ B & \\ C & \\ D & \end{matrix}$
2 Row *A* Vertex *A* is connected to itself with a loop and to vertex *B*. Write '1' at cell *AA* and cell *AB*. Write '0' in the other 2 cells.	
3 Complete the other rows in the same way.	$\begin{matrix} & A & B & C & D \\ A & \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \\ B & \\ C & \\ D & \end{matrix}$ Cell *AB* is in row *A* and column *B*. In an adjacency matrix, the value in cell *AB* is equal to the value in cell *BA*.

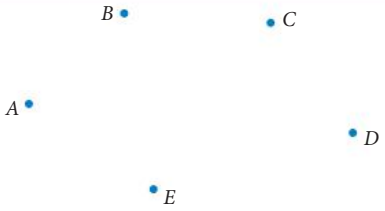
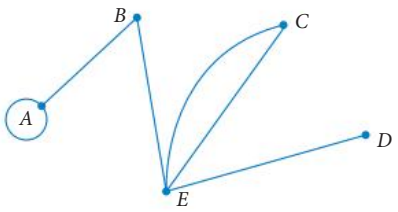
WB p. 54

WORKED EXAMPLE 4 Drawing a graph from lists

Draw the directed graph from the lists:

vertices = {*A*, *B*, *C*, *D*, *E*}

edges = {*AA*, *AB*, *BE*, *CE*, *CE*, *DE*}

Steps	Working
1 Vertices = {*A*, *B*, *C*, *D*, *E*} Draw the 5 vertices *A*, *B*, *C*, *D* and *E*.	
2 Edges = {*AA*, *AB*, *BE*, *CE*, *CE*, *DE*} Draw a loop at *A* for the edge *AA*. Draw a single edge between vertex *A* and vertex *B* for edge *AB* and repeat for edges *BE* and *DE*. Draw 2 edges between vertices *C* and *E* for the edges *CE* and *CE*.	 Any graph with the same vertices connected by the same edges is correct. Other correct graphs will appear different if the vertices are in a different position.

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Road maps and graphs

A map shows the roads that connect towns or landmarks. These routes are a graph and can often be redrawn in a simpler form and summarised as lists or an adjacency matrix.

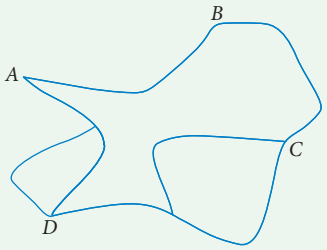
Drawing a road map as a graph

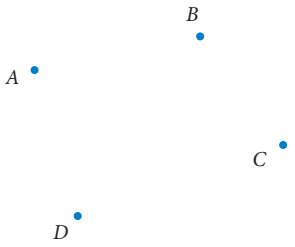
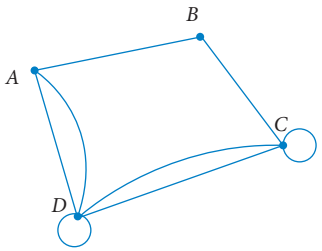
- List the vertices. These may be towns or landmarks.
- List the edges. These are all the possible paths between the vertices. A path from vertex A to vertex A is drawn as a loop.
- Use the 2 lists to draw the graph.

WB
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WORKED EXAMPLE 5 Drawing a map as a graph

The map shows the roads that connect towns A, B, C and D. Draw this map as a graph.



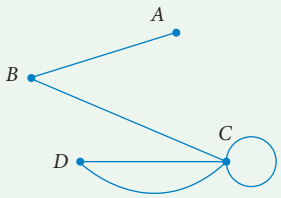
Steps	Working
1 Each of the 4 towns is a vertex. List the 4 vertices.	vertices = $\{A, B, C, D\}$
2 Each possible path between the towns is an edge.	
From town A, there is one path to town B and 2 paths to town D.	edges from vertex A = $\{AB, AD, AD\}$
From town B, there is 1 path to town C.	edges from vertex B = $\{BC\}$
From town C, there is a path back to vertex C and 2 paths to town D.	edges from vertex C = $\{CC, CD, CD\}$
From town D, there is 1 path back to town D.	edges from vertex D = $\{DD\}$
3 Draw the 4 vertices.	
4 Draw all the edges. The graph contains loops at vertices C and D.	

The degree of a vertex

- The **degree** (or **order**) **of a vertex** is the number of edges connected to the vertex.
- A loop adds 2 to the degree of a vertex.
- The degree of vertex A is written as $\deg(A)$.

WORKED EXAMPLE 6 Finding the degree of the vertices of graph

Find the degree of each of the vertices in the graph shown.



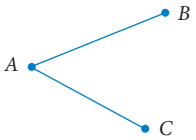
Steps	Working
1 Only 1 edge is connected to vertex A, so the degree is 1.	$\deg(A) = 1$
2 Two edges are connected to vertex B, so the degree is 2.	$\deg(B) = 2$
3 Three edges and a loop are connected to vertex C, so the degree is $3 + 2 = 5$.	$\deg(C) = 5$
4 Two edges are connected to vertex D, so the degree is 2.	$\deg(D) = 2$

WB
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The handshaking lemma

An important property of graphs involves the degrees of a graph and is known as the **handshaking lemma**. When the sum of degrees of a graph is found, each edge is counted twice, once for each end of the edge. The sum of the degrees of the graph is equal to twice the number of edges.

The graph on the right represents 3 people A, B and C who shake hands. The vertices represent the people and the edges represent the handshakes. Person A shakes hands with person B and person C.



The degree of a vertex indicates the number of hands a person has shaken.

$\deg(A) = 2$, $\deg(B) = 1$ and $\deg(C) = 1$

degree sum $= 2 + 1 + 1$
$= 4$

$2 \times$ the number of edges $= 2 \times 2$
$= 4$

The handshaking lemma

In any graph, the sum of the degrees of the vertices is equal to twice the number of edges.

degree sum $= 2e$, where e = the number of edges

A consequence of this is the number of vertices with odd degree in any graph must be even.

WB

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WORKED EXAMPLE 7 Applying the handshaking lemma

A graph with 9 edges has 6 vertices *A, B, C, D, E* and *F*. The table below shows the degree of each vertex. Find the degree of vertex *F*.

Vertex	Degree
A	4
B	3
C	2
D	1
E	5
F	*x*

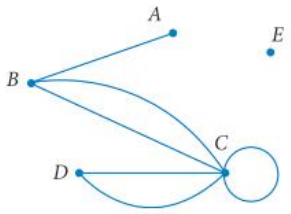
Steps	Working
1 Substitute $e = 9$ into the formula.	degree sum $= 2e = 2 \times 9 = 18$
2 Add the degrees of the vertices in the table.	$4 + 3 + 2 + 1 + 5 + x = 18$ $15 + x = 18$ $x = 3$
	The degree of vertex *F* is 3.

EXERCISE 3.1 Graphs

ANSWERS p. 570

Mastery

1 WORKED EXAMPLE 1 In the graph shown



a which vertex is isolated?

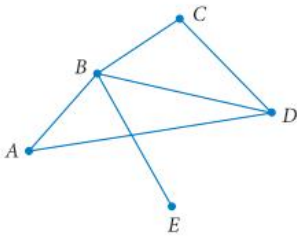
b which vertex has a loop?

c which vertices have multiple edges?

d which vertices are adjacent to vertex *C*?

2 WORKED EXAMPLE 2 Write the lists that represent the graph in question **1**.

3 WORKED EXAMPLE 3 Write the adjacency matrix that defines this graph.

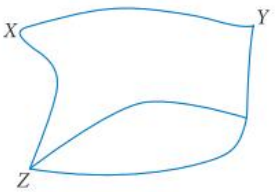


4 WORKED EXAMPLE 4 Draw the undirected graph from the lists:

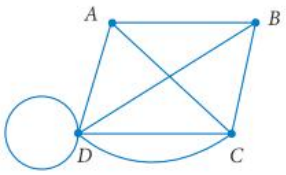
vertices = {*A, B, C, D*}

edges = {*AB, AC, AD, BD, BC, CD, CD*}

5 WORKED EXAMPLE 5 The map shows roads connecting towns *X*, *Y* and *Z*. Draw an undirected graph to represent this map.



6 WORKED EXAMPLE 6 Find the degree of each vertex in this graph.



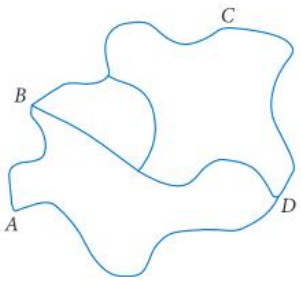
7 WORKED EXAMPLE 7 A graph with 8 edges has 6 vertices A, B, C, D, E and F. The table shows the degree of each vertex, except for vertex A.

Find the degree of vertex A.

Vertex	Degree
A	
B	2
C	2
D	1
E	3
F	4

Exam practice

8 TECH-FREE (6 marks) The diagram shows a map of roads between towns A, B, C and D.



a List the vertices and edges in the graph. 2 marks

b Represent the graph with an adjacency matrix. 2 marks

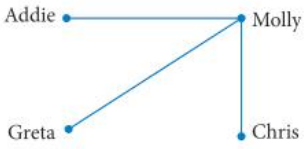
c Draw the graph. 2 marks

9 TECH-FREE (3 marks) In a competition, members of a team work together to complete a series of challenges. The members of one team are Addie (A), Molly (M), Greta (G) and Chris (C).

In one of the challenges, these 4 team members are only allowed to communicate directly with each other as indicated by the edges of the following network.

The adjacency matrix also shows the allowed lines of communication.

$$\begin{matrix} A & M & G & C \end{matrix}$$
$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & x & 0 & 0 \\ 0 & y & 0 & 0 \end{bmatrix} \begin{matrix} A \\ M \\ G \\ C \end{matrix}$$



a Explain the meaning of a zero in the adjacency matrix. 1 mark

b Find the values of x and y. 2 marks

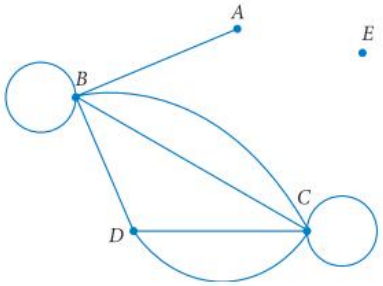
10 TECH-FREE (3 marks) In a physical education class, 4 students, Trent (T), Amelia (A), Nick (N) and Maddie (M) play games of table tennis.

- Trent plays against Amelia and Maddie.
- Amelia plays against Trent, Maddie and Nick.
- Nick plays against Amelia and Maddie.

a Draw a graph to represent the games played. 2 marks

b Find deg (N) and explain the meaning of the degree. 1 mark

11 In this graph, the number of vertices and the number of edges respectively is



A 5, 10

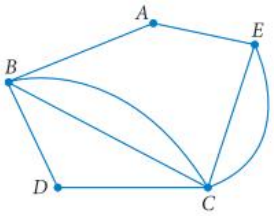
B 5, 8

C 5, 6

D 4, 8

E 4, 6

12 The lists that describe the following graph are



A vertices = $\{A, B, C, D, E\}$, edges = $\{AB, AE, BC, BC, BD, CD, CE, CE\}$

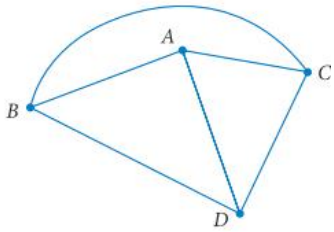
B vertices = $\{A, B, C, D, E\}$, edges = $\{AB, AE, BC, BD, CD, CE\}$

C vertices = $\{A, B, C, D\}$, edges = $\{AB, AE, BC, BC, BD, CD, CE, CE\}$

D vertices = $\{A, B, C, D, E\}$, edges = $\{AB, AE, BA, BC, BC, BD, CB, CB, CD, CE, CE, EA, EC, EC\}$

E vertices = $\{A, B, C, D, E\}$, edges = $\{AB, BC, BC, BD, CD, CE, CE\}$

13 Consider the graph on the right.



An adjacency matrix that can be used to represent the graph is

A
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

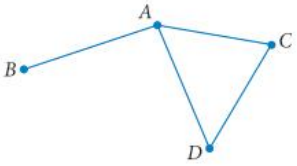
B
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

C
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

D
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

E
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

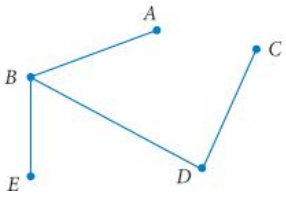
14 The sum of the degrees of the vertices in the graph is



A 4 **B** 6 **C** 7

D 8 **E** 10

15 The number of vertices with odd degrees in this graph is



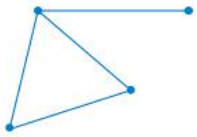
A 1 **B** 2 **C** 3

D 4 **E** 5

3.2 Types of graphs

Types of graphs

A **simple graph** has no loops or multiple edges.

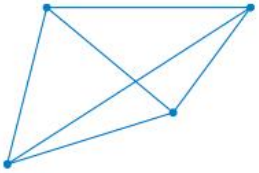


A **connected graph** contains no isolated vertices. Every vertex is reachable from any other vertex.

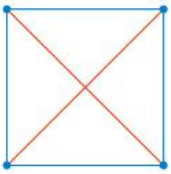


A **complete graph** is a simple graph where every vertex is connected to every other vertex.

A complete graph K_n with n vertices has $\frac{n(n-1)}{2}$ edges.

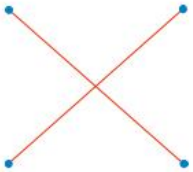


The complete graph K_4 has 4 vertices and $\frac{4 \times 3}{2} = 6$ edges.

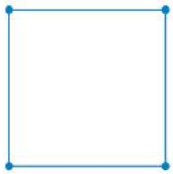


Complete graph K_4

The **complement** of a graph G, written as $\overline{G}$, contains the same vertices as G and all of the edges for the complete graph that are not present in G.



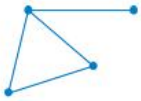
Graph G



Complement of G

A **subgraph** is part of another graph.

It contains some or all of the vertices and edges in the original graph.

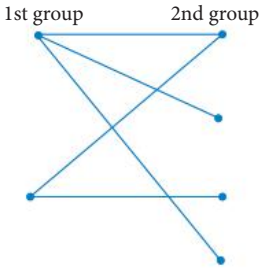


Subgraph



Original graph

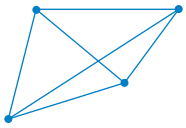
A **bipartite graph** has vertices in 2 groups and its edges connect vertices in the 1st group to vertices in the 2nd group.



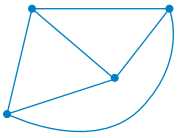
Video playlist
Types of graphs

Worksheet
Types of graphs

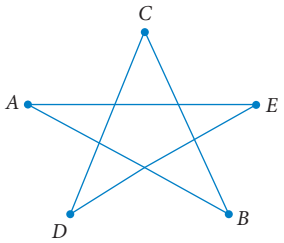
A **planar graph** is a simple, connected graph that has no edges that cross or can be redrawn so that no edges cross.



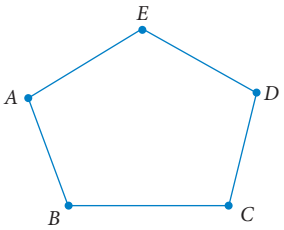
This graph is planar since it can be redrawn so no edges cross.



Two graphs are **isomorphic** if they contain exactly the same vertices with edges connected exactly the same way. Isomorphic graphs will have the same lists and the same adjacency matrix.



These 2 graphs are isomorphic because they have the same vertices with edges connected the same way.

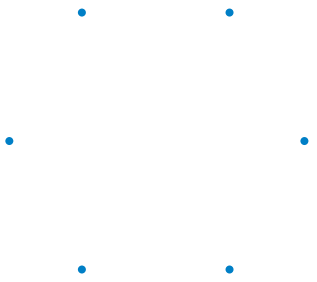


A **regular graph** has vertices that all have the same degree. If every vertex has degree r, the graph is regular of degree r.

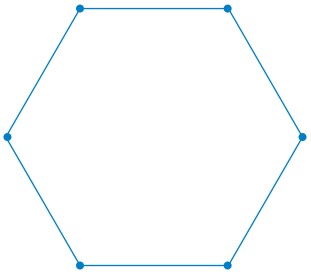
All complete graphs with v vertices are regular of degree $v - 1$.

A regular graph with n vertices each of degree r has $\frac{nr}{2}$ edges.

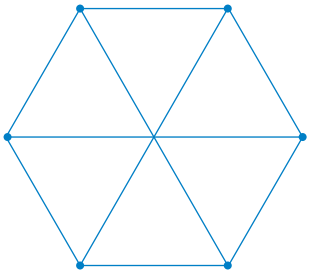
Regular of degree 0



Regular of degree 2



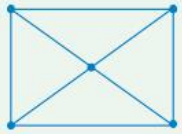
Regular of degree 3



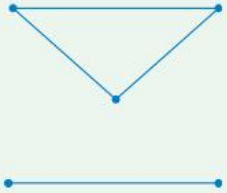
WORKED EXAMPLE 8 Classifying graphs

Classify the following graphs as simple, connected, complete, planar, bipartite or a combination of these.

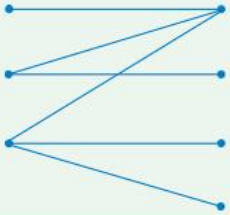
a



b



c



Steps	Working
a Simple: no loops or multiple edges. Connected: every vertex can be reached from every other vertex. Planar: it can be drawn so that no edges cross.	The graph is simple, connected and planar.
b Simple: no loops or multiple edges. Not connected: it is not possible to reach the lower 2 vertices from the top 3 vertices. Not planar: it is not connected.	The graph is simple.
c Simple: no loops or multiple edges. Connected: every vertex can be reached from every other vertex. Bipartite: there are 2 groups of vertices with edges connecting the vertices in 1 group to the vertices in the other group. Planar: it can be redrawn so that no edges cross.	The graph is simple, connected, planar and bipartite.

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WORKED EXAMPLE 9 Finding the number of edges in a complete graph

How many edges would there be in a complete graph with 10 vertices?

Steps	Working
A complete graph with n vertices has $\frac{n(n-1)}{2}$ edges. Substitute $n = 10$ into the formula.	The number of edges $= \frac{10(10-1)}{2} = \frac{90}{2} = 45$

WB p. 60

WORKED EXAMPLE 10 Vertices and edges in a complete graph

Determine the number of vertices and edges for the complete graph K_{12}.

Steps	Working
The complete graph K_n has n vertices and $\frac{n(n-1)}{2}$ edges. Substitute $n = 12$ into the formulas.	The complete graph K_{12} has 12 vertices. The number of edges $= \frac{12(12-1)}{2} = \frac{132}{2} = 66$ K_{12} has 12 vertices and 66 edges.

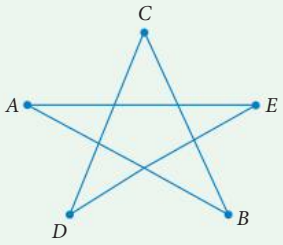
WB p. 61

WB

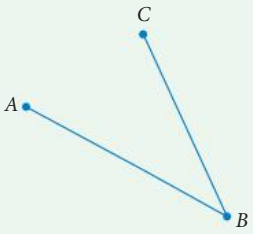
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WORKED EXAMPLE 11 Subgraphs

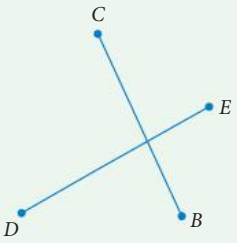
Which of the following graphs is not a subgraph of the graph shown?



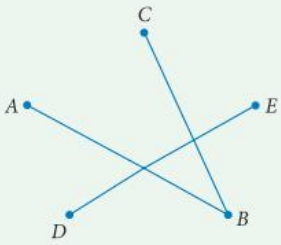
A



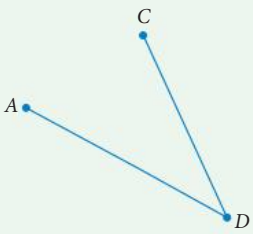
B



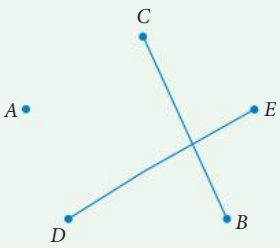
C



D



E



Steps	Working
The original graph has	Option **D** has
vertices = {*A*, *B*, *C*, *D*, *E*}	vertices = {*A*, *C*, *D*}
edges = {*AB*, *AE*, *BC*, *CD*, *DE*}	edges = {*AD*, *CD*}
A subgraph must also contain vertices and edges chosen from these sets.	This is not a subgraph since *AD* was not an edge in the original graph.
	The correct answer is option **D**.

WB

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WORKED EXAMPLE 12 Finding the number of edges in a regular graph

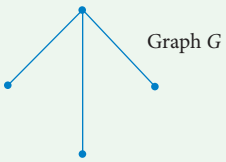
Determine the number of edges for a regular graph with 7 vertices of degree 4.

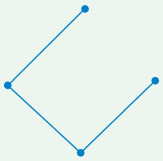
Steps	Working
A regular graph with n vertices each of degree r has $\frac{nr}{2}$ edges.	number of edges $= \frac{nr}{2}$
Substitute $n = 7$ and $r = 4$ into the formula.	$= \frac{7 \times 4}{2}$
	$= 14$
	There are 14 edges.

9780170448338

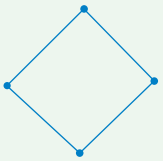
WORKED EXAMPLE 13 Identifying the complement of a graph

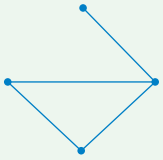
The complement of graph G is

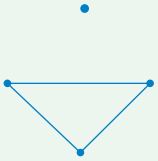


A 

B 

C 

D 

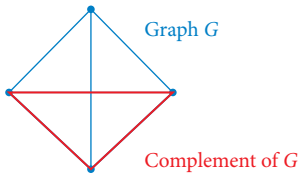
E 

Steps

The complete graph K_4 has 6 edges.

The complement of G contains the other 3 edges that are in the complete graph K_4 but are not present in G.

Working



The answer is option **E**.

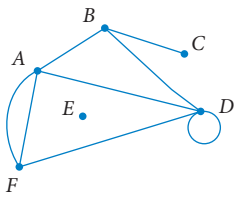
WB
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EXERCISE 3.2 Types of graphs

ANSWERS p. 570

Recap

1 For the graph shown, which statement is NOT true?



A Vertex E is isolated.

B Vertices A and F have multiple edges.

C The graph has 6 vertices.

D The graph has 7 edges.

E There is a loop at vertex D.

2 The degree of vertex C defined by the lists

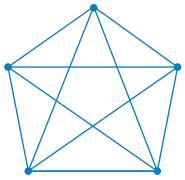
vertices = $\{A, B, C, D\}$

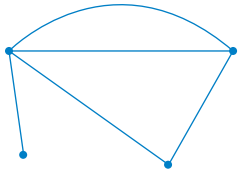
edges = $\{AB, BB, BC, BC, BD, CC, CD, CD\}$ is

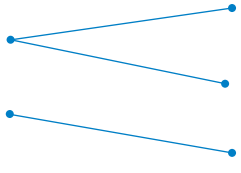
A 2 **B** 3 **C** 4 **D** 5 **E** 6

Mastery

3 WORKED EXAMPLE 8 Classify each graph as simple, connected, complete, planar, bipartite or a combination of these.

a 

b 

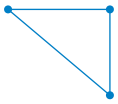
c 

4 WORKED EXAMPLE 9 How many edges would there be in a complete graph with

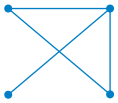
a 6 vertices? **b** 12 vertices? **c** 7 vertices?

5 WORKED EXAMPLE 10 Determine the number of vertices and edges for the complete graph K_{15}.

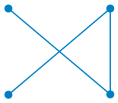
6 WORKED EXAMPLE 11 This graph is a subgraph of which of the following graphs?



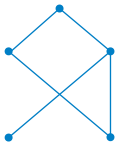
A



B

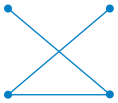


C



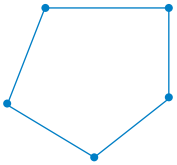
D

E



7 WORKED EXAMPLE 12 Determine the number of edges for a regular graph with 8 vertices of degree 4.

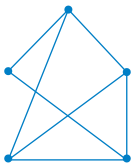
8 WORKED EXAMPLE 13 Draw the complement of this graph.



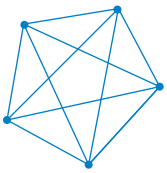
Exam practice

9 TECH-FREE (3 marks) Classify the following graphs as simple, connected, complete, planar, bipartite or a combination of these.

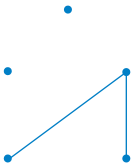
a



b



c



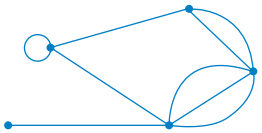
10 TECH-FREE (2 marks)

a Determine the number of edges in a regular graph with 8 vertices of degree 3. 1 mark

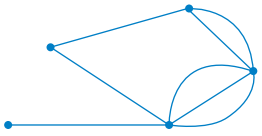
b A regular graph with n vertices of degree 4 has 14 edges. Find the value of n. 1 mark

11 (2 marks) The smallest number of edges are removed from the graph to transform it into a simple graph.

Find the sum of the degrees of all the vertices for this simple graph.



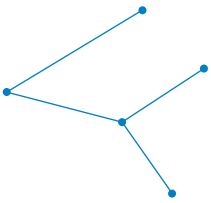
12 The smallest number of edges that must be removed to make this graph simple is



A 1 **B** 2 **C** 3

D 4 **E** 8

13 The smallest number of edges that must be added to make this graph complete is

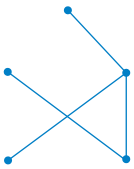


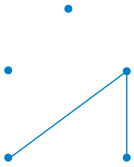
A 2 **B** 4 **C** 6
D 8 **E** 10

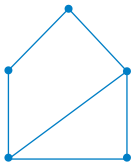
14 The number of edges needed to make a complete graph with 9 vertices is

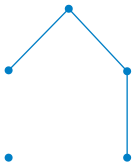
A 8 **B** 18 **C** 36
D 45 **E** 72

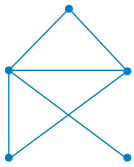
15 This graph is a subgraph of which of the following graphs?

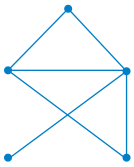


A


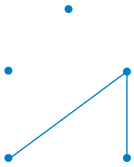
B


C


D


E


16 The smallest number of edges that must be added to make this graph connected is



A 1 **B** 2 **C** 4
D 8 **E** 10

17 The number of vertices and edges in the complete graph K_{20} is

A 20 vertices, 171 edges **B** 20 vertices, 190 edges **C** 20 vertices, 380 edges
D 190 vertices, 20 edges **E** 19 vertices, 171 edges

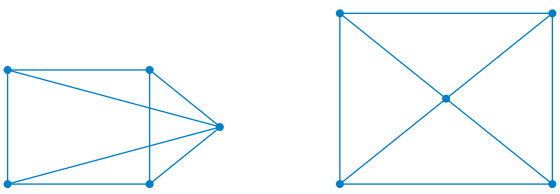
3.3 Planar graphs and Euler's formula

Video playlist
Planar graphs and Euler's formula

Worksheet
Planar graphs

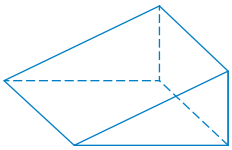
Planar graphs

A **planar graph** is a connected graph that can be drawn in a plane so that no edges cross.

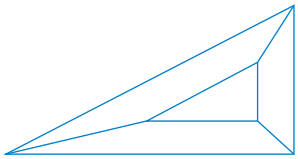


The graph above on the right is planar because an isomorphism exists where none of the edges cross. When the graph is redrawn so the edges do not cross, it is called a **plane drawing**.

Many 3-dimensional shapes and all prisms can be drawn as planar graphs.



Triangular prism



Planar graph

Complete planar graphs

The only complete graphs that are planar have 4 vertices or less.

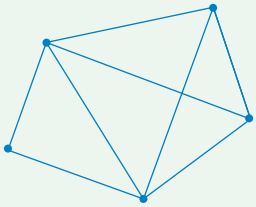
K_n is planar for $n \leq 4$, $n \in N$.

WB
p. 65

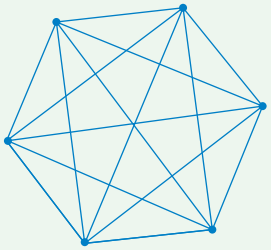
WORKED EXAMPLE 14 Identifying planar graphs

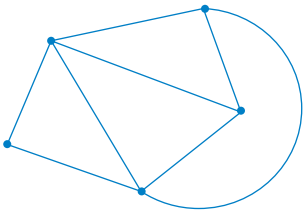
Which of the connected graphs below are planar?

a



b

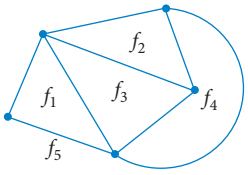


Steps	Working
a The graph can be redrawn so that no edges cross.	 This graph is planar.
b The graph cannot be drawn so that no edges cross. This is the complete graph K_6 so it cannot be planar.	This graph is not planar.

Euler's formula for planar graphs

Any plane drawing of a planar graph divides the plane into **faces**. A **face** is an enclosed region of the plane. The region outside the graph, however, is considered to be a face. It is unbounded and is called the **infinite face**.

In this planar graph, the faces labelled f_1, f_2, f_3 and f_4 are all bounded faces, while f_5 is the unbounded or infinite face.



9780170448338

The Swiss mathematician Leonard Euler discovered a formula that can be used to determine whether a graph is planar, a relationship between the vertices, edges and faces of a planar graph.

Euler's rule for planar graphs

Euler's formula states that for a graph to be planar,

$$v + f - e = 2$$

where v = the number of vertices

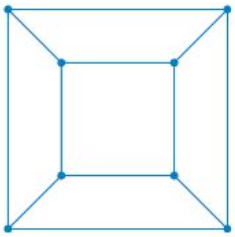
f = the number of faces and

e = the number of edges.

We can verify the formula using the planar graph shown.

number of vertices + number of faces − number of edges = 8 + 6 − 12

= 2



WORKED EXAMPLE 15 Using Euler's rule with an adjacency matrix

WB p. 66

The adjacency matrix represents a planar graph.

Determine the number of faces.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ & 0 & 1 & 1 \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} \end{array}$$

Steps	Working
1 The number of edges is 6, found by adding the numbers above or on the leading diagonal of the matrix.	$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$
The graph has 4 vertices $\{A, B, C, D\}$ and 6 edges $\{AB, AC, AD, BC, BD, CD\}$.	The graph has 4 vertices and 6 edges.
2 Substitute into Euler's rule to determine the number of faces.	$v + f - e = 2$ $4 + f - 6 = 2$ $-2 + f = 2$ $f = 4$ There are 4 faces.

WORKED EXAMPLE 16 Applying Euler's rule

WB p. 67

A connected graph has 5 vertices and 7 edges.

Determine the number of faces.

Steps	Working
Use Euler's rule to determine the number of faces. $v + f - e = 2$	$v + f - e = 2$ $5 + f - 7 = 2$ $f = 4$

EXERCISE 3.3 Planar graphs and Euler's formula

ANSWERS p. 571

Recap

1 The number of edges in a complete graph with 15 vertices is

A 14 **B** 15 **C** 30 **D** 105 **E** 225

2 The number of faces in the planar graph defined by the adjacency matrix is

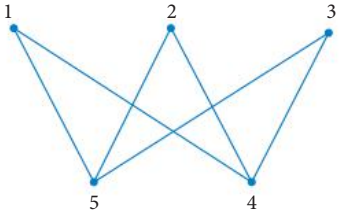
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

A 1 **B** 2 **C** 3 **D** 4 **E** 5

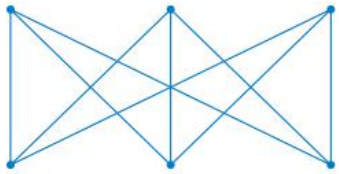
Mastery

3 WORKED EXAMPLE 14 State whether each graph is planar.

a



b



4 WORKED EXAMPLE 15 Determine the number of faces for the adjacency matrices below.

a

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

b

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

5 WORKED EXAMPLE 16

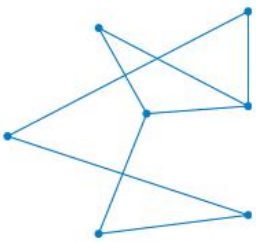
a A connected planar graph has 6 vertices and 9 edges. Determine the number of faces.

b A connected planar graph has 7 vertices and 5 faces. Determine the number of edges.

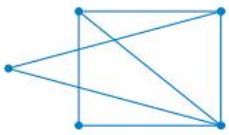
Exam practice

6 TECH-FREE (4 marks) Redraw the following planar graphs as plane graphs where the edges do not cross.

a



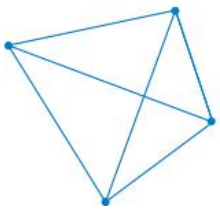
b



c



d



7 A connected planar graph has 15 edges that divide the plane into 8 distinct regions.
Its number of vertices is

A 7 **B** 9 **C** 11 **D** 23 **E** 25

8 A connected planar graph has 14 edges. The graph could have

A 7 vertices and 9 faces **B** 8 vertices and 10 faces **C** 6 vertices and 11 faces
D 5 vertices and 9 faces **E** 5 vertices and 13 faces

9 A connected planar graph has 11 edges and 11 faces. The number of vertices is equal to

A 2 **B** 3 **C** 9 **D** 12 **E** 22

10 The adjacency matrix represents a connected planar graph with 4 vertices.
Its number of faces is

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix} \end{array}$$

A 3 **B** 5 **C** 6
D 12 **E** 14

11 The adjacency matrix for a connected graph is

A
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

B
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

C
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

D
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

E
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix} \end{array}$$

12 A connected planar graph has an equal number of vertices and faces.
If its number of edges is 10, then the number of faces must be

A 5 **B** 6 **C** 8 **D** 10 **E** 12

13 A connected planar graph has 8 vertices and 13 edges. The number of edges that are removed to leave a connected graph with 8 vertices and 4 faces is

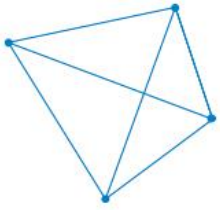
A 2 **B** 3 **C** 4 **D** 5 **E** 10

14 For which graph does Euler's formula NOT apply?

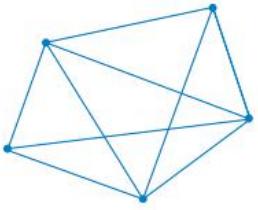
A



B



C



D



E



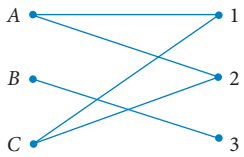
Video playlist
Bipartite graphs and trees

3.4 Bipartite graphs and trees

Bipartite graphs

A **bipartite graph** has its vertices in 2 distinct sets. The edges join vertices in the first set to vertices in the second set. Bipartite graphs can be used to show the link between workers and the tasks they are allocated, students in a class and the football teams they support, or cipher where the letters in the alphabet correspond to a particular code.

In the bipartite graph below, there are 3 tasks $\{1, 2, 3\}$ that can be performed by 3 people $\{A, B, C\}$. The edges show which people are qualified to perform which tasks.



The bipartite graph shows:

A can perform tasks 1 and 2.

B can perform task 3.

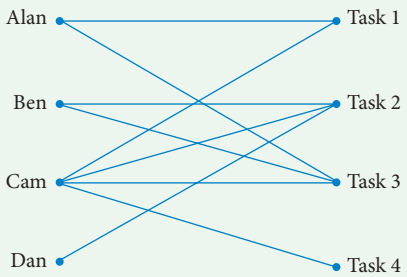
C can perform tasks 1 and 2.

WB
p. 68

WORKED EXAMPLE 17 Interpreting a bipartite graph

The bipartite graph shows the tasks 4 workers are qualified to perform.

Determine which of the following is a valid allocation.



A

Alan	Task 4
Ben	Task 3
Cam	Task 2
Dan	Task 1

B

Alan	Task 3
Ben	Task 4
Cam	Task 2
Dan	Task 1

C

Alan	Task 3
Ben	Task 2
Cam	Task 1
Dan	Task 4

D

Alan	Task 1
Ben	Task 3
Cam	Task 4
Dan	Task 2

E

Alan	Task 1
Ben	Task 3
Cam	Task 2
Dan	Task 4

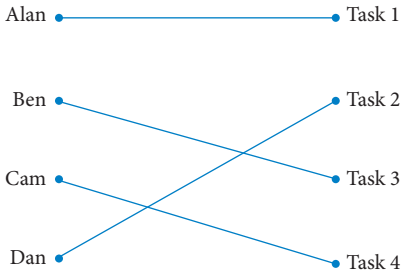
Steps

There must be an edge between the worker and the task for the allocation to be valid.

Each option must be checked on the bipartite graph to see if there is an edge between the worker and the task number.

Working

Option **D** shows the only valid allocation.



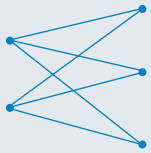
9780170448338

The complete bipartite graph $K_{m,n}$

The complete bipartite graph $K_{m,n}$ has 2 sets of vertices with

m vertices in the first set

and n vertices in the second set.



The complete bipartite graph $K_{m,n}$ has $m \times n$ edges.

The complete bipartite graph $K_{2,3}$ is shown.

In a complete bipartite graph, every vertex in the first set is connected to every vertex in the second set. There are no edges between vertices in the first set or between vertices in the second set.

WORKED EXAMPLE 18 Maximum edges for a complete bipartite graph

Find the maximum number of edges for a complete bipartite graph with 6 vertices.

Steps

1 Determine the different possible values of m and n in the complete bipartite graph $K_{m,n}$.

Working

Since $m + n = 6$, the possible values of m and n are

$m = 1, n = 5$

$m = 2, n = 4$

$m = 3, n = 3$

$m = 4, n = 2$

$m = 5, n = 1$

2 The complete bipartite graph $K_{m,n}$ has $m \times n$ edges.

m	n	$m \times n$ edges
1	5	5
2	4	8
3	3	9
4	2	8
5	1	5

The maximum number of edges is 9.

Trees

A **cycle** is a path in a graph that starts and finishes at the same vertex.

A **tree** is a simple connected graph that does not contain any cycles.

In a tree, there is only 1 path between any 2 vertices.

A tree with n vertices contains $n - 1$ edges.



This tree with 8 vertices has 7 edges.

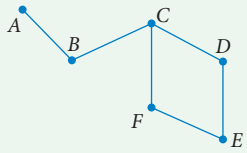
WB p. 69

WB p. 70

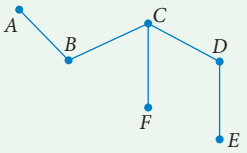
WORKED EXAMPLE 19 Identifying trees

Which of the following graphs are trees?

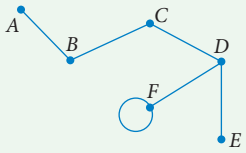
a



b



c



Steps	Working
a This graph is not a tree as it contains a cycle *CDEFC*.	This graph is not a tree.
b This graph is a tree as it is a simple connected graph with no cycles.	This graph is a tree.
c This graph is not a tree. It is not a simple graph since it contains a loop at vertex *F*.	This graph is not a tree.

WB p. 71

WORKED EXAMPLE 20 Finding the number of vertices in a tree

Find the number of vertices in a tree with 9 edges.

Steps	Working
A tree with n vertices contains $n - 1$ edges. Substitute $e = 9$ into the formula $e = v - 1$.	$e = v - 1$ $9 = v - 1$ $v = 9 + 1 = 10$ There are 10 vertices.

EXERCISE 3.4 Bipartite graphs and trees

ANSWERS p. 571

Recap

80–100% 60–79% 0–59%

1 A connected planar graph has an equal number of vertices and faces. If the number of edges is 12, the number of faces must be equal to

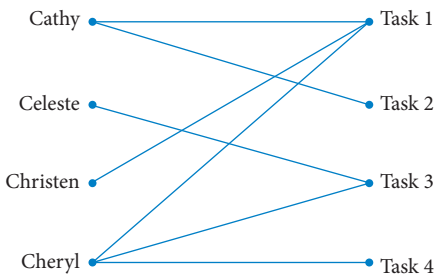
A 5 **B** 6 **C** 7 **D** 8 **E** 14

2 © VCAA 2018 Further Mathematics 1B2Q3 76% A planar graph has five faces. This graph could have

A eight vertices and eight edges. **B** six vertices and eight edges. **C** eight vertices and five edges.
D eight vertices and six edges. **E** five vertices and eight edges.

Mastery

3 WORKED EXAMPLE 17 The bipartite graph shows the tasks that 4 workers are qualified to perform.



Determine which of the following is a valid allocation.

A

Cathy	Task 4
Celeste	Task 3
Christen	Task 2
Cheryl	Task 1

B

Cathy	Task 2
Celeste	Task 1
Christen	Task 3
Cheryl	Task 4

C

Cathy	Task 2
Celeste	Task 3
Christen	Task 1
Cheryl	Task 4

D

Cathy	Task 3
Celeste	Task 1
Christen	Task 4
Cheryl	Task 2

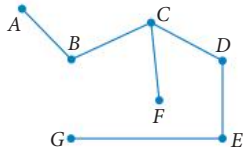
E

Cathy	Task 1
Celeste	Task 2
Christen	Task 3
Cheryl	Task 4

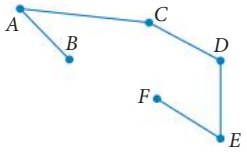
4 WORKED EXAMPLE 18 Find the maximum number of edges for a complete bipartite graph with 10 vertices.

5 WORKED EXAMPLE 19 Which of the following graphs are trees?

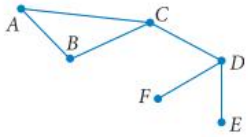
a



b



c



6 WORKED EXAMPLE 20

a Determine the number of edges for a tree with 12 vertices.

b Determine the number of vertices for a tree with 20 edges.

Exam practice

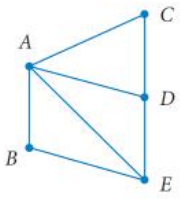
7 TECH-FREE (2 marks) Two basketball teams compete and at the end of the game each player on the first team shakes hands with each player on the second team. If the first team had 6 players and the second team had 7 players, how many handshakes were there?

8 TECH-FREE (2 marks) Workers A, B, C and D are each to be assigned one task from tasks 1, 2, 3 and 4. The matrix representing the workers that are qualified to perform the tasks is shown.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Draw a bipartite graph and hence determine a valid allocation.

9 TECH-FREE (4 marks) The graph below shows the roads that connect 5 towns.



a Which town is adjacent to every other town? 1 mark

b What is the smallest number of roads required so that each town can be reached from every other town? 1 mark

c How many edges must be removed from the graph to produce a tree? 1 mark

d How many edges must be added to change this graph into the complete graph K_5? 1 mark

10 TECH-FREE (3 marks) Car parts are delivered from a factory to 4 warehouses. The warehouses supply these parts to 12 different retailers. Each retailer is only supplied by one warehouse.

a What type of graph can be used to represent this situation? 1 mark

b How many vertices does this graph have? 1 mark

c How many edges does this graph have? 1 mark

11 Two badminton teams compete againsteach other. Each team has 6 players and each player from the first team must play each player in the second team. The number of games played is

A 6 **B** 12 **C** 15 **D** 30 **E** 36

12 A manager is directly in charge of 5 department heads and each department head is directly in charge of 4 staff. This structure could be represented graphically using

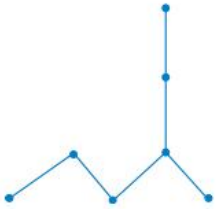
A a complete graph **B** a tree

C a cycle **D** a regular graph

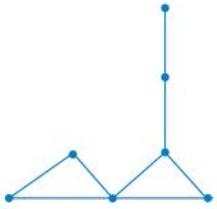
E a complete bipartite graph

13 Which of the following is a tree?

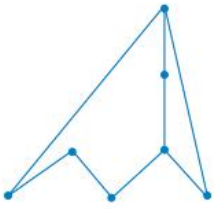
A



B



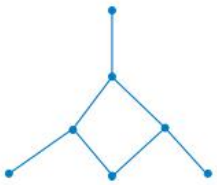
C



D



E



14 Which complete bipartite graph is a tree?

A $K_{3,5}$ **B** $K_{3,3}$ **C** $K_{3,1}$ **D** $K_{2,5}$ **E** $K_{5,5}$

15 One vertex and n edges are removed from the complete bipartite graph $K_{3,3}$ to produce the graph shown.



The value of n is

A 1 **B** 2 **C** 3 **D** 6 **E** 9

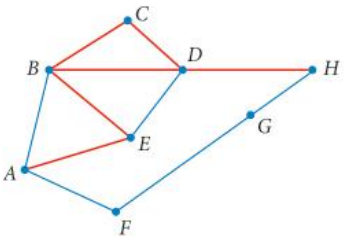
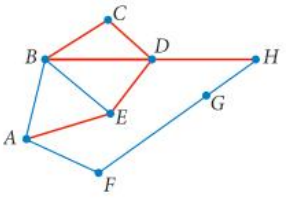
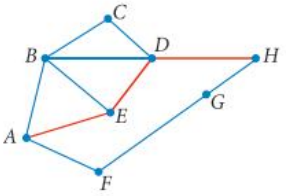
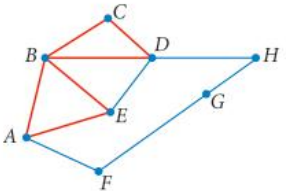
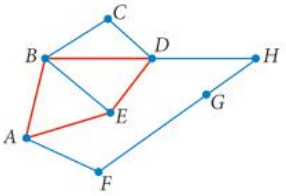
3.5 Walks, paths, trails, cycles and circuits

Travelling and exploring a graph involves moving from one vertex to another through an edge and is described by listing the vertices in order. The terms that are used when discussing travelling and exploring are listed below with examples.



Video playlist
Walks, paths, trails, cycles and circuits

Walks, paths trails, cycles and circuits

Definition	Example
A **walk** is any sequence of vertices in a graph along edges, which may include repeated vertices and repeated edges. An **open walk** starts and finishes at different vertices. A **closed walk** starts and finishes at the same vertex.	 *A–E–B–D–C–B–D–H* is a walk.
A **trail** is a walk with no repeated edges, such as a trail of a gardener sweeping the paths in a park.	 *A–E–D–B–D–H* is a trail.
A **path** is a trail with no repeated vertices (and edges), such as a postal van collecting mail from post boxes.	 *A–E–D–H* is a path.
A **circuit** is a trail that starts and finishes at the same vertex. It has no repeated edges.	 *B–A–E–B–D–C–B* is a circuit.
A **cycle** is a path that starts and finishes at the same vertex. It has no repeated edges or vertices except for the start/finish vertex.	 *B–A–E–D–B* is a cycle.

	Repeated vertices and edges	No repeated edges	No repeated edges or vertices
Start and end anywhere	Walk	Trail	Path
Start and end at the same place	Closed walk	Circuit	Cycle

Worksheets
Eulerian graphs
Eulerian circuits
Eulerian trails and circuits

Eulerian trails and circuits

- An **Eulerian trail** is a walk that includes every edge once and starts and finishes at different vertices.
- For an Eulerian trail to exist, the connected graph must have exactly 2 vertices of odd degree.
- To find the Eulerian trail, we must start at one of the vertices of odd degree and finish at the other vertex of odd degree.
- An **Eulerian circuit** is a walk that includes every edge once and starts and finishes at the *same* vertex.
- An Eulerian circuit exists if *all* vertices are of even degree.

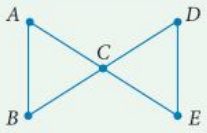
Exam hack

To determine if an Eulerian trail or circuit exists, always calculate the degree of every vertex.

WB p. 72

WORKED EXAMPLE 21 Classifying travelling

For the graph shown, classify the following walks as either an open walk, a closed walk, a trail, a path, a circuit, a cycle or a combination of these.



a $A–C–E–D–C–B$

b $A–B–C–A$

c $B–A–C–E$

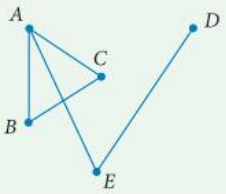
d $A–B–C–D–E–C–A$

Steps	Working
a A trail has no repeated edges and starts and finishes at different vertices.	$A–C–E–D–C–B$ is an open walk and a trail.
b A cycle is a path that starts and finishes at the same vertex.	$A–B–C–A$ is a closed walk, a circuit and a cycle.
c A trail has no repeated edges and starts and finishes at different vertices. A path is a trail where there are no repeated vertices.	$B–A–C–E$ is an open walk, a trail and a path.
d A circuit starts and finishes at the same vertex, has no repeated edges but can have repeated vertices.	$A–B–C–D–E–C–A$ is a closed walk and a circuit.

WB p. 73

WORKED EXAMPLE 22 Finding Eulerian trails

An Eulerian trail starting at vertex A exists for the graph shown.



a Find the finishing vertex.

b Find the possible Eulerian trails.

Steps	Working
a **1** Find the degree of each vertex.	$A:3, B:2, C:2, D:1, E:2$
2 The Eulerian trail must start and finish at the vertices with odd degree.	The Euler path finishes at D.
b Trace all the possible paths that start at A, finish at D and include every edge in the graph.	Eulerian trails: $A–B–C–A–E–D$ $A–C–B–A–E–D$

The Königsberg bridge problem

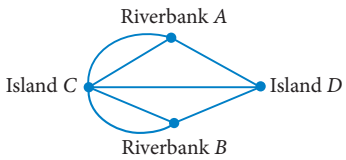


© Historic Cities Research Project.
Courtesy of Ozgur Tufekci.

Königsberg, a town on the Preger River, was the name for the historic Prussian city that is now Kaliningrad, Russia. Within the town, there are 2 river islands that are connected to the banks with 7 bridges.

The citizens of Königsberg came up with a game. The goal was to walk across all 7 bridges, crossing the islands only once, without ever repeating a single bridge. This proved to be a very frustrating problem, so they asked Leonard Euler to solve this problem.

Euler realised that in the Königsberg problem, the exact layout of the city or the choice of route taken is irrelevant. The only thing that is important is how things are connected. This allowed him to turn the complicated map of the city into a network or graph.



In this graph, the land masses are the vertices, the bridges are the edges and the number of links to the land mass is the degree of the vertex.

Euler observed that if every link (bridge) is to be crossed over only once, then each vertex (land mass) except for the start and end vertices must have an even number of links attached to it.

That's because whenever we enter one by a link, we need to leave it by another. So the vertex needs 2 links if we visit it once, 4 if we visit it twice and so on. The only vertices that can have an odd number of edges attached to them are the vertices where the walk starts and ends.

In this graph, $\deg(A) = 3$, $\deg(B) = 3$, $\deg(C) = 5$ and $\deg(D) = 3$. As all the degrees are odd, it is impossible to create a trail that travels along each edge exactly once.

Leonard Euler essentially proved the very first theorem in graph theory.

Hamiltonian paths and cycles

- A **Hamiltonian path** is a walk that includes *every* vertex in a graph exactly once and starts and finishes at *different* vertices.
- A **Hamiltonian cycle** is a walk that includes *every* vertex in a graph exactly once and starts and finishes at the *same* vertex.
- To determine whether a Hamiltonian path or cycle exists, we must use trial and error.

Worksheet
Hamiltonian paths and cycles

WB

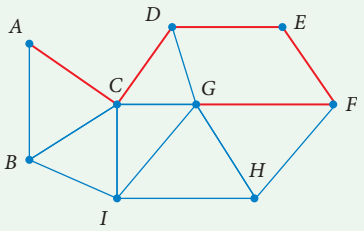
p. 74

WORKED EXAMPLE 23 Finding Hamiltonian paths and cycles

The start of a Hamiltonian path is shown in red.

a Complete the Hamiltonian path.

b Complete the Hamiltonian cycle.



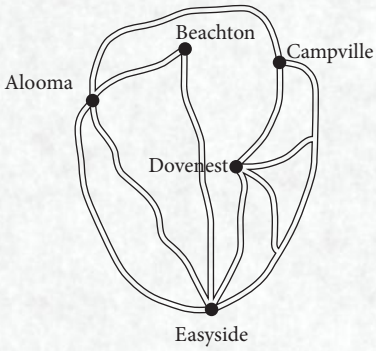
Steps	Working
a The Hamiltonian path must visit each vertex in the graph exactly once and finish at a different vertex to A.	The Hamiltonian path is $A–C–D–E–F–G–H–I–B$.
b The Hamiltonian cycle must visit each vertex in the graph exactly once and finish at the vertex A.	The Hamiltonian cycle is $A–D–E–F–G–H–I–B–A$.

Video
VCE question analysis: Graph theory

VCE QUESTION ANALYSIS

© VCAA 2016 Further Mathematics 2B2Q1, 2 2016 Further Mathematics Examination 2
Networks and decision mathematics, Questions 1, 2 (6 marks)

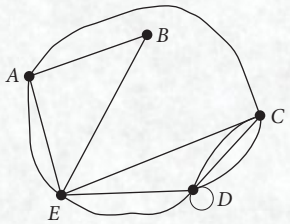
1 (3 marks) A map of the roads connecting five suburbs of a city, Alooma (A), Beachton (B), Campville (C), Dovenest (D) and Easyside (E), is shown.



a Starting at Beachton, which two suburbs can be driven to using only one road? 1 mark

A graph that represents the map of the roads is shown.

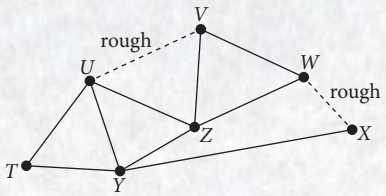
One of the edges that connects to vertex E is missing from the graph.



b **i** Name the missing edge. 1 mark

ii Explain what the loop at D represents in terms of a driver who is departing from Dovenest. 1 mark

2 (3 marks) The suburb of Alooma has a skateboard park with seven ramps. The ramps are shown as vertices *T, U, V, W, X, Y* and *Z* on the graph below.



The tracks between ramps *U* and *V* and between ramps *W* and *X* are rough, as shown on the graph.

a Nathan begins skating at ramp *W* and follows an Eulerian trail.

At which ramp does Nathan finish? 1 mark

b Zoe begins skating at ramp *X* and follows a Hamiltonian path. The path she chooses does not include the two rough tracks.

Write down a path that Zoe could take from start to finish. 1 mark

c Birra can skate over any of the tracks, including the rough tracks.

He begins skating at ramp *X* and will complete a Hamiltonian cycle.

In how many ways could he do this? 1 mark

Reading the question

- Highlight the text that indicates the first diagram is a map which is represented by an incomplete graph below.
- Highlight the word indicating an explanation is required in question **1b ii**.
- The second question involves Eulerian trails and Hamiltonian paths and cycles. Highlight which type of walk is required in each part of question **2**.

Thinking about the question

- You will need to know how to interpret a map and be able to convert it into a graph.
- You will need to know the conditions in a graph when Eulerian trails, and Hamiltonian paths and cycles are possible.
- You are also required to list the paths for these walks.

Worked solution (✓ = 1 mark)

1 a Starting at Beachton, the towns **Alooma and Easyside** can be reached using only 1 road. ✓

b i There are 3 paths on the map that go from Easyside to Dovenest; however, only 2 are shown on the graph.

The missing edge is ***ED***. ✓

ii **The loop is required at *D*** on the graph since it is possible to drive on a road from Dovenest to Dovenest without passing through another town. ✓

2 a The degrees of the vertices are $\deg(T) = 2$, $\deg(U) = 4$, $\deg(V) = 3$, $\deg(W) = 3$, $\deg(X) = 2$, $\deg(Y) = 4$ and $\deg(Z) = 4$.

Eulerian trail starts and finishes at odd degree vertices.

Start at *W* and finish at *V*. ✓

b Hamiltonian paths are ***X–Y–T–U–Z–W–V* or *X–Y–T–U–Z–V–W***. ✓

c There are 4 possible Hamiltonian cycles.

X–Y–T–U–Z–V–W–X ***X–W–V–Z–U–T–Y–X***

X–Y–T–U–V–Z–W–X ***X–W–Z–V–U–T–Y–X*** ✓

Student performance 80–100% 60–79% 0–59%

1 a 83%

b i 53% Many students did not answer this question.

ii 30% Questions requiring an explanation are often poorly done.

2 a 72%

b 77%

c 31% Each Hamiltonian cycle can be reversed.

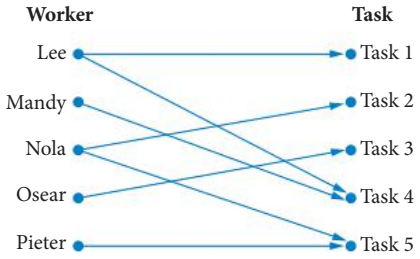
EXERCISE 3.5 Walks, paths trails, cycles and circuits

ANSWERS p. 571

Recap

80–100% 60–79% 0–59%

1 © VCAA 2016 Further mathematics 1B2Q1 97% Lee, Mandy, Nola, Oscar and Pieter are each to be allocated one particular task at work. The bipartite graph below shows which task(s), 1–5, each person is able to complete.



Each person completes a different task. Task 4 must be completed by

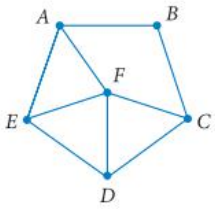
A Lee. **B** Mandy. **C** Nola. **D** Oscar. **E** Pieter.

2 The maximum number of edges for a complete bipartite graph with 13 vertices is

A 7 **B** 30 **C** 36 **D** 42 **E** 169

Mastery

3 WORKED EXAMPLE 21 For the graph shown, classify the following walks as either an open walk, a closed walk, a trail, a path, a circuit, a cycle, or a combination of these.

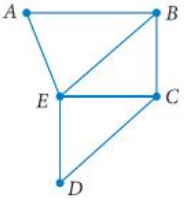


a $A–F–C–B–A$

b $A–F–E–D–F–C$

c $C–F–D–E–A–B$

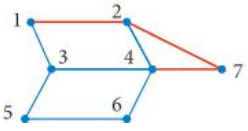
4 WORKED EXAMPLE 22 An Eulerian trail, starting at vertex B, exists for the following graph.



a Find the finish vertex.

b Find a possible Eulerian trail.

5 WORKED EXAMPLE 23 The start of a Hamiltonian path is shown in red.

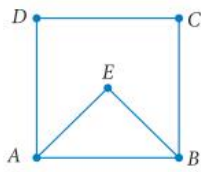


a Complete a possible Hamiltonian path.

b Complete a possible Hamiltonian cycle.

Exam practice

6 TECH-FREE (3 marks) Using the graph shown, determine

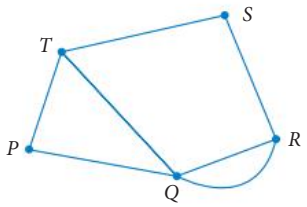


a an Eulerian trail.

b a Hamiltonian path.

c a Hamiltonian cycle.

7 TECH-FREE (5 marks) A highway patrol must check all the roads between towns P, Q, R, S and T as shown on the diagram. The patrol must also check each road no more than once.

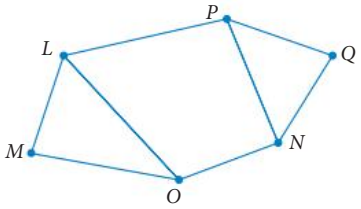


a Determine the trail the highway patrol must take so that they do not travel on any road more than once. 2 marks

b What is the name of this trail? 1 mark

c A back road also exists between 2 of the towns and is not shown on the diagram. If the highway patrol starts and finishes at town T, find the towns connected by the back road so that each road is checked at most once. 2 marks

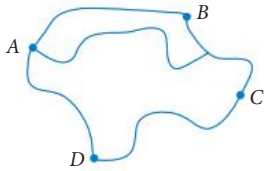
8 TECH-FREE (2 marks) A council employee must collect money from the parking ticket machines in the city each morning. Each ticket machine is represented by a vertex on the graph.



a If the employee must start and finish at different ticket machines, visiting each ticket machine once, what kind of walk is this? 1 mark

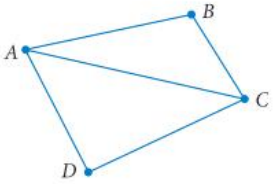
b If the employee must start and finish at ticket machine L, visiting each ticket machine once, what kind of walk is this? 1 mark

9 TECH-FREE (3 marks) Four different towns A, B, C and D are shown on the map.



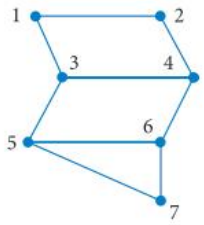
a In how many ways can a bus travel from town A to town C without travelling on a road more than once? 1 mark

b The map is redrawn as an undirected graph below but is missing an edge. What edge is missing? 1 mark



c If the bus drives along each edge of the graph exactly once, at what towns must the bus start and finish? 1 mark

10 Which of the following is a Hamiltonian cycle?



A 1–2–4–3–6–5–7

B 1–2–4–6–7–3–5–1

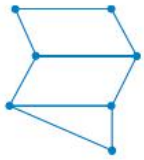
C 1–2–4–6–7–5–3–1

D 1–2–4–6–7–5–3

E 1–3–5–6–7–4–2–1

11 Which of the following graphs has an Eulerian circuit?

A



B



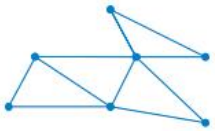
C



D



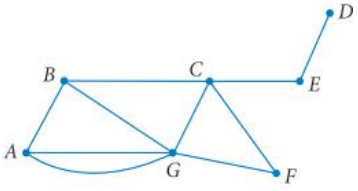
E



12 How many of the complete graphs K_4, K_5, K_6, K_7 and K_8 have an Eulerian circuit?

A 1 **B** 2 **C** 3 **D** 4 **E** 5

13 Which edge can be added to the graph to create an Eulerian trail?



A *AC* **B** *EF* **C** *ED*

D *DF* **E** *BD*

14 An Eulerian trail through a network starts at vertex *M* and finishes at vertex *S*.

Which of the following statements is true?

A In the network, there are 3 vertices of degree 3.

B The degree of vertex *M* is 4.

C The sum of the degrees of the graph is 11.

D The trail could have included vertex *S* more than once.

E The sum of the degrees of vertices *M* and *S* is 5.

15 A connected planar graph has 6 vertices *A, B, C, D, E* and *F*.

The degree of each vertex is shown in the table.

Vertex	Degree
A	2
B	4
C	2
D	2
E	4
F	2

Which of the following statements is true?

A The graph has an Eulerian trail.

B If one edge is added, the graph will have an Eulerian trail.

C If one edge is added, the graph will have an Eulerian circuit.

D Euler's formula $v + f - e = 2$ could not be used.

E The graph has 16 edges.

3 Chapter summary

Term	Definition
Vertex	A point or node on a graph.
Edge	An interval connecting 2 vertices.
Adjacent vertices	Two vertices connected by an edge.
Degree of a vertex	The number of edges connected to the vertex.
Loop	An edge that connects a vertex to itself.
Multiple edges	Two or more edges connecting the same vertices.
Isolated vertex	A vertex that is not connected to any other vertices, so it has degree 0.

Lists and adjacency matrices

A graph can be defined by lists that show the set of vertices and the set of edges.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

An **adjacency matrix** is a **square matrix** of order $n \times n$, where n is the number of vertices.

The number at row P and column Q shows the number of edges connecting P and Q.

Road maps and graphs

To draw a road map as a graph:

- List the vertices: these may be towns or landmarks.
- List the edges: these are all the possible paths between the vertices. A path from vertex A to vertex A is drawn as a loop.
- Use the 2 lists to draw a graph.

The degree of a vertex

- The **degree** (or order) of a vertex is the number of edges connected to the vertex.
- A loop adds 2 to the degree of a vertex.
- The degree of vertex A is written as $\deg(A)$.

The handshaking lemma

- In any graph, the sum of the degrees of the vertices is equal to twice the number of edges.
- Degree sum = $2e$, where e is the number of edges.
- The number of vertices with odd degree in any graph must be even.

Types of graphs

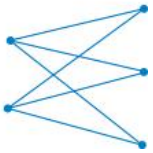
- A **simple graph** has no loops or multiple edges.
- A **connected graph** contains no isolated vertices. Every vertex is reachable from any other vertex.
- A **complete graph** is simple and has every vertex connected to every other vertex. A complete graph K_n with n vertices has $\frac{n(n-1)}{2}$ edges.
- The **complement** (or inverse) of a graph G, written as $\overline{G}$, has the same vertices as G, and all of the edges for the complete graph that are *not* present in G.
- A **subgraph** is part of another graph, and contains some or all of the vertices and edges from the original graph.
- A **bipartite graph** has its vertices in 2 distinct sets and its edges connect vertices in the first set to vertices in the second set.
- A **planar graph** is a simple, connected graph that has no edges that cross, or which can be redrawn so that no edges cross.
- Two graphs are **isomorphic** if they contain exactly the same vertices with edges connected in exactly the same way.
- A **regular graph** has vertices that all have the same degree. If every vertex has degree r, the graph is regular of degree r. All complete graphs with v vertices are regular of degree $v - 1$. A regular graph with n vertices, each of degree r, has $\frac{nr}{2}$ edges.

Planar graphs

- A **planar graph** is a connected graph that has no edges that cross.
- The only complete graphs that are planar have 4 vertices or less.
  K_n is planar for $n \leq 4$, $n \in \boldsymbol{N}$.
- **Euler's formula** states that for a graph to be planar: $v + f - e = 2$
  where v = the number of vertices, f = the number of faces, e = the number of edges.

The complete bipartite graph $K_{m,n}$

- The complete bipartite graph $K_{m,n}$ has 2 sets of vertices with m vertices in the first set and n vertices in the second set.
- The complete bipartite graph $K_{m,n}$ has $m \times n$ edges.
  The complete bipartite graph $K_{2,3}$ is shown.



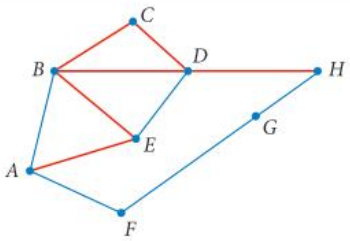
- In a complete bipartite graph, every vertex in the first set is connected to every vertex in the second set. There are no edges between the vertices in the first set or between the vertices in the second set.

Trees

- A **cycle** is a path in a graph that starts and finishes at the same vertex.
- A **tree** is a simple connected graph that does not contain any cycles.
- In a tree, there is only one path between any 2 vertices.
- A tree with n vertices contains $n - 1$ edges.

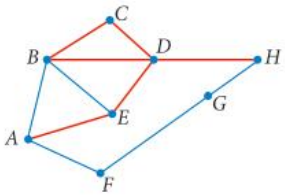
Walks, trails, paths, circuits and cycles

A **walk** is any sequence of vertices of a graph along edges.
An **open walk** starts and finishes at different vertices.
A **closed walk** starts and finishes at the same vertex.



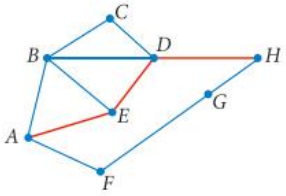
A–E–B–D–C–B–D–H is a walk.

A **trail** is a walk with no repeated edges, such as a trail of a gardener sweeping the paths in a park.



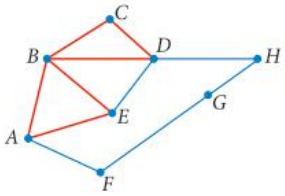
A–E–D–B–C–D–H is a trail.

A **path** is a trail with no repeated vertices (and edges), such as the path of a postal van collecting mail from post-boxes.



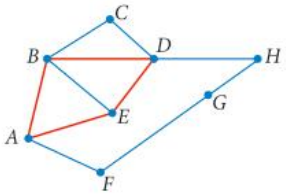
A–E–D–H is a path.

A **circuit** is a trail that starts and finishes at the same vertex. It has no repeated edges.



B–A–E–B–D–C–B is a circuit.

A **cycle** is a path that starts and finishes at the same vertex. It has no repeated edges or vertices.



B–A–E–D–B is a cycle.

	Repeated vertices and edges	No repeated edges	No repeated edges or vertices
Start and end anywhere	Walk	Trail	Path
Start and end at the same place	Closed walk	Circuit	Cycle

Eulerian trails and circuits

- An **Eulerian trail** is a walk that includes *every* edge once and starts and finishes at *different* vertices. For an Eulerian trail to exist, the connected graph must have *exactly 2* vertices of odd degree. To find the Eulerian trail, we must start at one vertex of odd degree and finish at the other vertex of odd degree.
- An **Eulerian circuit** is a walk that includes *every* edge once and starts and finishes at the *same* vertex. An Eulerian circuit exists if *all* vertices are of even degree.

Hamiltonian paths and cycles

- A **Hamiltonian path** is a walk that includes *every* vertex in a graph exactly once, and starts and finishes at *different* vertices.
- A **Hamiltonian cycle** is a walk that includes *every* vertex in a graph exactly once, and starts and finishes at the *same* vertex.
- To determine whether a Hamiltonian path or cycle exists, we must use trial and error.

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

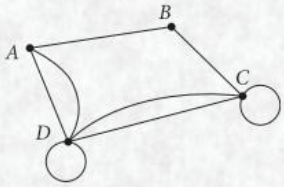
TECH-FREE *Technology is NOT permitted.*

1 (2 marks) For the complete graph K_{10}, find

a the number of vertices. 1 mark

b the number of edges. 1 mark

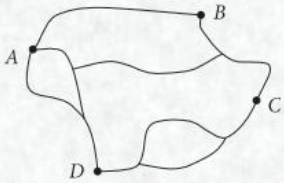
2 (2 marks) For the connected graph shown, write



a the lists 1 mark

b the adjacency matrix. 1 mark

3 (2 marks) Write the lists for the map below.



4 (2 marks) Express the recurring decimal $0.\dot{1}\dot{8}$ in fraction form.

5 (2 marks) Find the tenth term of a geometric sequence whose fourth term is 1024 and seventh term is 128.

Cumulative examination 2

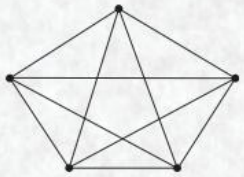
Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 © VCAA 2016 Further Mathematics 1B2Q3 MODIFIED The following graph with five vertices is the complete graph K_5.



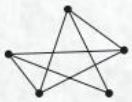
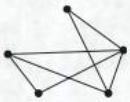
Edges are removed so that the graph becomes a tree. The minimum number of edges removed is

A 4 **B** 5 **C** 6 **D** 9 **E** 10

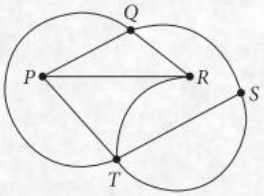
2 © VCAA 2016 Further Mathematics 1B2Q5 Consider the planar graph below.



Which one of the following graphs can be redrawn as the planar graph above?

A  **B**  **C**  **D**  **E** 

3 © VCAA 2018 Further Mathematics 1B2Q4 Consider the graph below.



Which one of the following is **not** a path for this graph?

A $P–R–Q–T–S$ **B** $P–Q–R–T–S$ **C** $P–R–T–S–Q$

D $P–T–Q–S–R$ **E** $P–T–R–Q–S$

4 The solution to the equation $|2x - 1| = 7$ is

A $x = 4$ **B** $x = 0.5$ **C** $x = -3, 4$

D $x = -3$ **E** $x = 0$

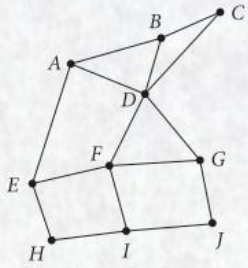
5 The number of distinct prime factors of 2184 is

A 3 **B** 4 **C** 5 **D** 7 **E** 12

Section B 2 questions

15 marks

1 © VCAA 2018 Further Mathematics 2B2Q2 (3 marks) In one area of the town of Zenith, a postal worker delivers mail to 10 houses labelled as vertices A to J on the graph below.



a Which one of the vertices on the graph has degree 4? 1 mark

For this graph, an Eulerian trail does not currently exist.

b For an Eulerian trail to exist, what is the minimum number of extra edges that the graph would require? 1 mark

c The postal worker has delivered the mail at F and will continue her deliveries by following a Hamiltonian path from F. List a possible Hamiltonian path for the postal worker. 1 mark

2 (12 marks) Angie and Jen are both preparing for the Great Victorian Bike Ride. They record the distances they ride each week for 8 weeks prior to the event.

Angie uses the recurrence formula $t_{n+1} = 2t_n + 1$, $t_1 = 2$ where t_n is the total distance, in kilometres, that Angie rides in week n.

Jen uses a geometric sequence to determine the distance she rides for training.

She rides 400 km in week 1 and 200 km in week 2.

Find

a the distance Angie rides in each of weeks 1, 2 and 3. 2 marks

b the distance Angie rides in week 8. 1 mark

c the distance Jen rides in week 8. 1 mark

d the total distance Angie rides training for the Great Victorian Bike Ride. 2 marks

e the total distance Jen rides training for the Great Victorian Bike Ride. 2 marks

f the week in which Angie covers more distance than Jen. 1 mark

g whether Angie or Jen covers more distance in training. 1 mark

h the recurrence formula for the distance Jen rides in training each week. 2 marks

CHAPTER

4

LOGIC AND ALGORITHMS

Study Design coverage

Nelson MindTap chapter resources

4.1 The building blocks of logic
- Logical arguments
- Logical connectives

4.2 Truth tables, tautologies and contradictions
- Truth tables
- Tautologies and contradictions

4.3 Binary numbers
- Converting between base 2 and base 10
- **Using CAS 1:** Converting between base 2 and base 10
- Binary arithmetic

4.4 Boolean algebra and truth tables
- Converting truth tables to Boolean expressions
- Boolean operators in search engines and databases

4.5 Karnaugh maps and truth tables
- Construction of a Karnaugh map from a truth table
- Minimising the product sum using a Karnaugh map

4.6 Electronic gates and circuits
- Notation for logic gates
- Multiple logic gates
- Simplifying logic gates

4.7 Algorithms and pseudocode
- Algorithms
- Flowcharts
- Pseudocode

VCE question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2

Study Design coverage

UNIT 1, AREA OF STUDY 1: ALGEBRA, NUMBER AND STRUCTURE

Logic and algorithms

- propositions, connectives, truth values, truth tables and Karnaugh maps
- Boolean algebras
- binary number systems
- tautologies, validity and proof patterns and the application of these to proofs in natural language and laws and properties of Boolean algebra, the algebra of sets and propositions
- logic gates and circuits, and simplification of circuits
- definition of an algorithm and the fundamental constructs needed to describe algorithms: sequence, decision (selection, choice, if … then … blocks) and repetition (iteration and loops)
- construction and implementation of basic algorithms incorporating the fundamental constructs using pseudocode.

VCE Mathematics Study Design 2023–2027 p. 59, © VCAA 2022

Video playlists (8):

4.1 The building blocks of logic
4.2 Truth tables, tautologies and contradictions
4.3 Binary numbers
4.4 Boolean algebra and truth tables
4.5 Karnaugh maps and truth tables
4.6 Electronic gates and circuits
4.7 Algorithms and pseudocode
VCE question analysis Logic and algorithms

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

Video playlist
The building blocks of logic

4.1 The building blocks of logic

Aristotle (384–322 BC) is recognised as the architect of formal **logic**. Logic is concerned with deciding what is true based on **valid reasoning**. To do this, definitions and rules of logic are used.

Logical arguments

Consider the following conversation between Goran and Amy.

Goran:	All horses have 4 legs.	1
Amy:	My cat has 4 legs.	2
Goran:	Therefore, your cat is a horse.	3
Amy:	Is it?	4
Goran:	I could be lying.	5
Amy:	Please don't lie.	6

A sentence that is either true or false is a **statement**.

For Goran and Amy's conversation, sentences 1, 2 and 5 are true and 3 is false. Sentences 4 and 6 are not statements because they cannot be said to be true or false in any context.

A **premise** is a statement from which a **conclusion** can be drawn.

Sentences 1 and 2 are premises and 3 is the conclusion (though an incorrect conclusion).

The conversation between Goran and Amy consists of premises and a conclusion. It is a **logical argument** (or **argument**).

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WORKED EXAMPLE 1 Understanding statements, premises and conclusions

The following is a logical argument between Kris and Joe.

Kris:	How are you?	1
Joe:	I feel good.	2
Kris:	I failed the exam.	3
Joe:	That's because you weren't prepared.	4
Kris:	I failed because the examiner marks hard.	5
Joe:	You don't spend enough time revising.	6
Kris:	Yes. That will help.	7
Joe:	See you later.	8

a Name the statements from which no conclusion is drawn.

b Name the premises and the conclusion(s).

c Identify the sentences that are not statements.

Steps	Working
a Statements are either true or false and do not lead to a conclusion being made.	Sentences 2, 3 and 6 are statements that are not used to form conclusions.
b A premise is a statement from which a conclusion can be drawn.	Sentence 3 is a premise because it is used to draw conclusions 4 and 5. Sentence 6 is a premise since it leads to conclusion 7.
c Identify the sentences that cannot be considered to be true or false.	Sentences 1 and 8 cannot be said to be true or false.

Logical connectives

An **atomic proposition** consists of one statement. A **compound proposition** consists of 2 or more atomic propositions connected by **logical connectives**, also called **logical operators**. The main logical connectives are listed in the table.

Logical connective	Meaning	Symbol
Conjunction	**AND**	$\wedge$
Negation	**NOT**	$\neg$
Disjunction	**OR**	$\vee$
Conditional	if then	$\rightarrow$
Biconditional	if and only if	$\leftrightarrow$
Equivalence	is identical to	$\equiv$

Equivalence ($\equiv$) and equality ($=$) are sometimes used interchangeably. Equivalence refers to 'same value', whilst equality refers to 'same in every way'. For example, $1 + 2 = 3$ and $5 - 2 = 3$ are equivalent statements, whereas 'one plus two is three' and '$1 + 2 = 3$' are equal statements.

A statement is normally referenced by a letter and its value is represented by a **propositional variable**, which can be either T (true) or F (false). T and F are **propositional constants**.

The statement '3 is a number' is always true, but 'Today is Monday' is only true if today is actually Monday.

The **biconditional statement** $X \leftrightarrow Y$ means $(X \rightarrow Y) \wedge (Y \rightarrow X)$.

Suppose X is 'a triangle is isosceles' and Y is 'the triangle has 2 equal sides'. Then $X \rightarrow Y$ reads 'If a triangle is isosceles, then it has 2 equal sides' and is true.

Also, $Y \rightarrow X$ is 'If the triangle has 2 equal sides, then it is isosceles' and is also true.

Here are some atomic propositions:

- Let A be 'the sky is blue'.
- Let B be 'the sun is shining'.
- Let C be 'it is raining'.
- Let D be 'I wear a raincoat'.
- Let E be 'I am using an umbrella'.

Some of the possible compound propositions using these atomic propositions are shown in the following table.

Symbolic form	Statement form	Correct grammar form
$A \wedge B$	**The sky is blue** AND **the sun is shining**	The sky is blue and the sun is shining.
$A \wedge \neg B$	**The sky is blue** AND NOT **it is raining**	The sky is blue and it is not raining.
$C \rightarrow D$	If **it is raining** then **I wear a raincoat**	If it is raining, I am wearing a raincoat.
$E \rightarrow (C \wedge D)$	If **I am using an umbrella** then **it is raining** AND **I wear a raincoat**	If I am using an umbrella, then it is raining and I am wearing a raincoat.
$D \leftrightarrow E$	If **I wear a raincoat** then **I am using an umbrella** AND If **I am using an umbrella** then **I wear a raincoat**	If I wear a raincoat, then I am using an umbrella, and if I am using an umbrella, then I am wearing a raincoat.
$\neg C \rightarrow A \wedge B$	If NOT **it is raining** then **the sky is blue** AND **the sun is shining**	If it is not raining, then the sky is blue and the sun is shining.
$\neg A \wedge \neg(\neg B)$	NOT **it is raining** AND NOT NOT **the sky is blue**	It is not raining and the sky is blue.

WB
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WORKED EXAMPLE 2 Identifying logical connectives

Let A be 'it is school holidays' B be 'it is a weekend'
C be 'we go to school' D be 'it is a public holiday'

Write each compound proposition using correct grammar.

a $A \wedge D$ **b** $D \rightarrow \neg C$ **c** $\neg B \rightarrow C$ **d** $\neg C \rightarrow (A \vee B)$

Steps	Working
a Write the conjunction in statement form.	$A \wedge D$ means A AND D . **It is school holidays** and **it is a public holiday**.
b 1 Write in statement form.	$\neg C$ means 'NOT **we go to school**'. $D \rightarrow \neg C$ is 'if **it is a public holiday**, then NOT **we go to school**'
2 Write using correct grammar.	If it is a public holiday, we don't go to school.
c 1 Explain each part of the compound proposition.	$\neg B$ means 'NOT B' and $\rightarrow C$ means 'then C'.
2 Write in statement form.	If NOT **it is a weekend**, then **we go to school**.
3 Write using correct grammar.	If it is not a weekend, we go to school.
d 1 Write each part of the conditional statement using proper grammar.	$\neg C$ means **we do** not **go to school**. $A \vee B$ is **it is school holidays** or **it is a weekend**.
2 Write the entire proposition using correct grammar.	If we do not go to school, then it is either school holidays or a weekend.

EXERCISE 4.1 The building blocks of logic

ANSWERS p. 573

Mastery

1 WORKED EXAMPLE 1 Use the logical argument below to answer the questions.

Edith:	Old men are grumpy.	1
Riley:	I'm an old man.	2
Edith:	Then you must be grumpy.	3
Riley:	Ouch!	4
Edith:	Have you been bitten by a mosquito?	5
Riley:	Mosquitoes bite on a hot day.	6
Edith:	I think you were bitten because you have fair skin.	7

a Which statements are conclusions?

b Which sentences are not statements?

c Which statements are premises?

d How many statements does Edith make?

2 WORKED EXAMPLE 2

Let A be 'odd number' B be 'number divisible by 2'

C be 'sum of 2 even numbers' D be 'even number'

E be 'sum of 2 odd numbers'

'Correct grammar' means to write the statement in a naturally flowing, easy-to-follow way.

Write each compound proposition using correct grammar.

a $D \rightarrow B$

b $B \rightarrow (C \vee E)$

c $\neg(A \vee D) \rightarrow \neg B$

Exam practice

3 (3 marks) Let X be 'I study for tests', let Y be 'the test will be easy', and let Z be 'I will pass the test'. Use the statements X, Y and Z to write statements for each of the following.

a 'The test will be easy, so I will pass.' 1 mark

b 'If the test is hard, I won't pass.' 1 mark

c 'If I do not pass the test or I do not study, it means the test will be hard.' 1 mark

4 If A is 'Today is Monday', and B is 'it will rain', which statement corresponds to $\neg A \wedge B$?

A Today is Monday or it will not rain.

B Today is not Monday and it will rain.

C Today is not Monday or it will rain.

D When it is Monday, it will rain.

E If it is not Monday, it will rain.

5 Let X be the statement 'a student who walks to school', let Y be the statement 'a student who does not ride their bike to school' and let Z be 'a student who gets enough exercise'. The statement 'If a student does not walk or ride to school, then they do not get enough exercise' can be written as

A $\neg X \vee Y \rightarrow Z$

B $\neg X \wedge Y \rightarrow \neg Z$

C $\neg X \wedge Y \rightarrow \neg Z$

D $\neg X \vee \neg X \rightarrow Z$

E $\neg Y \wedge \neg X \rightarrow Z$

6 If $\neg P \vee Q \rightarrow R$ represents the sentence, 'If it doesn't rain or if we have umbrellas to use, we can go on a picnic', then statements for P, Q and R can be

A	P: It will not rain.	Q: We have umbrellas.	R: We can go on a picnic.
B	P: It will not rain.	Q: We have umbrellas.	R: We can't go on a picnic.
C	P: It will rain.	Q: We don't have umbrellas.	R: We can't go on a picnic.
D	P: It won't rain.	Q: We don't have umbrellas.	R: We can't go on a picnic.
E	P: It will rain.	Q: We have umbrellas.	R: We can go on a picnic.

7 From the 2 atomic propositions 'it is raining' and 'I am wet', which compound proposition cannot be made using logical connectives?

A It is raining and I am wet.
B It is raining or I am wet.
C It is raining and I am not wet.
D If it is not raining, then I am wet.
E Sometimes it is raining when I am wet.

8 Choose the most appropriate conclusion for the following argument.

All men are mortal.

Socrates is a man.

A All men are Socrates.
B Socrates is not mortal.
C A man is mortal.
D Socrates is mortal.
E If the man is Socrates, he is mortal.

9 (2 marks)

a Give an example to show that $\neg(\neg P) \equiv P$. 1 mark

b Provide an example to show that $P \vee \neg P$ is always true. 1 mark

10 (4 marks)

a A and B are statements. List all combinations of $\neg$ and $\rightarrow$ that can be used with A and B. 2 marks

b Express the combinations containing 2 negations as meaningful sentences using the statements

A: do exercise

B: live longer 2 marks

11 (6 marks) For each of the following, state if the biconditional property applies. For parts **b** and **c**, also give the statements X and Y that can be used.

a X: numbers divisible by 6 Y: numbers divisible by 2 1 mark

b If I am sleeping, then I am breathing. 2 marks

c A car with a flat tyre shouldn't be driven. 2 marks

d X: $1 + 1 = 2$ Y: The sun sets in the west. 1 mark



Exam hack

To decide if a statement is biconditional, think of showing that 'left to right is the same as right to left'.

12 (8 marks) Use the following to write each statement using connectives.

P: Pigs fly. B: Bill will pass the test. H: Hand in homework.
C: Carmelo is a good dancer. E: Eloise will pass the test. F: Fang can fly a plane.
W: The roads are wet. R: It is raining.

a Pigs don't fly. 1 mark

b Neither Bill nor Eloise will pass the test. 1 mark

c Bill will not pass the test if he does not hand in homework. 1 mark

d Carmelo is a good dancer and Fang can fly a plane. 1 mark

e If it is raining and the roads are wet, Fang can't fly and pigs can. 2 marks

f Bill will pass the test and Eloise will fail the test if and only if pigs fly or it is not raining. 2 marks

13 (2 marks) Give 2 statements, A and B, that show $\neg(A \vee B) \equiv \neg A \wedge \neg B$.

4.2 Truth tables, tautologies and contradictions

Truth tables

Let P be a statement.

Using the conditional constants T (true) and F (false), we can show P as a **truth table**, which displays all possible truth values of the variables involved.

P	$\neg P$
T	F
F	T

The table is read as 'If P is true, then $\neg P$ is false' and 'If P is false, then $\neg P$ is true'.

The statement $P \wedge Q$ is true if P is true and if Q is true. For example, with P: '1 + 1 = 2' and Q: '2 + 3 = 4', $P \wedge Q$ is false even though P is true.

The statement $P \vee Q$ is true if either P is true or if Q is true. For P: '1 + 1 = 2' and Q: '2 + 3 = 4', $P \vee Q$ is true even though Q is false.



Video playlist
Truth tables, tautologies and contradictions



p. 77

WORKED EXAMPLE 3 Truth table involving conjunction

Construct a truth table for $A \wedge B$.

Steps

1 Set up the main structure of the truth table.

Working

There are 2 variables, A and B, and each can take the values T and F.

A	B
T	T
T	F
F	T
F	F

2 Add a column for the value of each combination of T and F.

$A \wedge B$ means both A and B must be true at the same time. So when A is T and B is T, $A \wedge B$ is T.

All other combinations give F.

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

WB

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WORKED EXAMPLE 4 Truth table involving NOT and disjunction

Construct a truth table to show that $\neg(A \vee \neg B) \equiv \neg A \wedge B$.

Steps	Working
1 Set up the main structure of the truth table.	There are 2 column headings for variables A and B. We also require columns for $\neg A$, $\neg B$, $A \vee \neg B$, $\neg(A \vee \neg B)$ and $\neg A \wedge B$.

A	B	$\neg A$	$\neg B$	$A \vee \neg B$	$\neg(A \vee \neg B)$	$\neg A \wedge B$

2 Complete columns 1 and 2 with all possible combinations of T and F for A and B.

Then fill in columns 3 and 4 by negating (taking the opposite of) each truth value in columns 1 and 2.

A	B	$\neg A$	$\neg B$	$A \vee \neg B$	$\neg(A \vee \neg B)$	$\neg A \wedge B$
T	T	F	F			
T	F	F	T			
F	T	T	F			
F	F	T	T			

3 Complete the columns for $A \vee \neg B$ and $\neg(A \vee \neg B)$.

For $A \vee \neg B$ to be true, either A or $\neg B$ must be true. Then negate each value to obtain $\neg(A \vee \neg B)$.

A	B	$\neg A$	$\neg B$	$A \vee \neg B$	$\neg(A \vee \neg B)$	$\neg A \wedge B$
T	T	F	F	T	F	
T	F	F	T	T	F	
F	T	T	F	F	T	
F	F	T	T	T	F	

4 Complete the last column $\neg A \wedge B$.

$\neg A \wedge B$ is true when both are true, otherwise $\neg A$ and B is false.

A	B	$\neg A$	$\neg B$	$A \vee \neg B$	$\neg(A \vee \neg B)$	$\neg A \wedge B$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	F	T	T
F	F	T	T	T	F	F

5 State the conclusion.

$\neg(A \vee \neg B)$ and $\neg A \wedge B$ have the same truth values, so $\neg(A \vee \neg B) \equiv \neg A \wedge B$.

9780170448338

WORKED EXAMPLE 5 The conditional statement

Construct a truth table for the conditional statement:

'If I am elected class president, then there will be no more homework.'

Steps	Working
1 Write the statements required for the truth table.	The conditional statement is of the form $A \rightarrow B$, which is read as 'If A, then B'. Let A be 'I am elected class president'. Let B be 'there will be no more homework'.
2 Decide what each column of the truth table represents.	The first 2 columns will be for A and B, and a column is needed for $A \rightarrow B$.

A	B	$A \rightarrow B$
T	T	
T	F	
F	T	
F	F	

3 Determine the result of each pair of truth values.	Suppose A is true and B is true. This means that you are elected and therefore there will be no homework. So $A \rightarrow B$ is true. Suppose A is true and B is false. This means that you are elected, but you lied about no more homework. So $A \rightarrow B$ is false. If A is false and B is true, you are not elected but there is no homework, which was the aim. So $A \rightarrow B$ is true. Now suppose A is false and B is false. You are not elected and there is homework. But you have not lied because you were not elected to be able to stop homework. So $A \rightarrow B$ is true.
4 Summarise the results as a truth table.	

A	B	$A \rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

WB
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Tautologies and contradictions

The statement 'I am a boy or I am not a boy' is always true. If a proposition is true for all values of its variables, it is a **tautology**.

The statement 'Tomorrow will be yesterday' is always false. When a proposition is always false, it is a **contradiction**.

The tautology $A \vee \neg A$ is the first truth table shown on the right.

The contradiction $A \wedge \neg A$ is the second truth table shown on the right.

A	$\neg A$	$A \vee \neg A$
T	F	T
F	T	T

A	$\neg A$	$A \wedge \neg A$
T	F	F
F	T	F

WB

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WORKED EXAMPLE 6 Proving tautologies

Show with a truth table that $A \rightarrow (A \vee B)$ is a tautology.

Steps

1 Write the statements required for the truth table.

Complete columns 1 and 2 with the 4 combinations of T and F.

A	B	$A \vee B$	$A \rightarrow (A \vee B)$
T	T		
T	F		
F	T		
F	F		

2 Complete the table by working out the truth value of each pair of truth values.

$A \vee B$ is true if either A or B is true.

$A \rightarrow (A \vee B)$ is false only when A is true and $A \vee B$ is false (see Worked example 5).

A	B	$A \vee B$	$A \rightarrow (A \vee B)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

3 State the conclusion.

$A \rightarrow (A \vee B)$ is true for all values of A and B, so it is a tautology.

WB

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WORKED EXAMPLE 7 Proving a contradiction

Show with a truth table that the biconditional statement $(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$ is a contradiction.

Steps

1 Write the statements required for the truth table.

Complete columns 1 to 5, remembering that $A \wedge B$ is true when both A and B are true.

Complete the values for $\neg A \vee \neg B$.

$\neg A \vee \neg B$ is true when either $\neg A$ or $\neg B$ is true.

A	B	$A \wedge B$	$\neg A$	$\neg B$	$\neg A \vee \neg B$	$(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$
T	T	T	F	F	F	
T	F	F	F	T	T	
F	T	F	T	F	T	
F	F	F	T	T	T	

2 $\leftrightarrow$ means $A \wedge B$ and $\neg A \vee \neg B$ are both true or both false. This is never the case, so $(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$ is false in all cases.

A	B	$A \wedge B$	$\neg A$	$\neg B$	$\neg A \vee \neg B$	$(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$
T	T	T	F	F	F	F
T	F	F	F	T	T	F
F	T	F	T	F	T	F
F	F	F	T	T	T	F

3 State the conclusion.

$(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$ is always false, so it is a contradiction.

9780170448338

EXERCISE 4.2 Truth tables, tautologies and contradictions

ANSWERS p. 573

Recap

1 Let A be 'it will rain today' and let B be 'I am happy'.
In correct grammar, $\neg A \wedge B$ is

A It will rain today but I am not happy.
B It won't rain today or I will be happy.
C It won't rain today and I am happy.
D It will rain today when I am happy.
E It will rain today or I am happy.

2 Choose the conclusion that can be made from the 2 given statements below.
All circles are round.
This shape is not a triangle.

A Triangles are not round.
B This shape is not round.
C Shapes that are not circles are triangles.
D This shape might be a circle.
E This shape is round.

Mastery

3 For $(P \wedge \neg P) \wedge (Q \wedge \neg Q)$, it is always true that

A $Q \wedge \neg Q$ is a tautology.
B $P \wedge \neg P$ is a tautology.
C $(P \wedge \neg P) \wedge (Q \wedge \neg Q)$ is a tautology.
D $(P \wedge \neg P) \wedge (Q \wedge \neg Q)$ is a biconditional statement.
E $(P \wedge \neg P) \wedge (Q \wedge \neg Q)$ is false.

4 WORKED EXAMPLE 3 Construct a truth table for $A \wedge \neg B$.

5 WORKED EXAMPLE 4 Use a truth table to show that $\neg(\neg A \wedge \neg B) \equiv A \vee B$.

6 WORKED EXAMPLE 5 Use the statements P: 'I work hard' and Q: 'I will pass the test' to construct a truth table for the conditional statement 'If I don't work hard, I won't pass the test.'

Exam hack

Interpret words such as won't, doesn't, isn't, can't, mustn't, to mean $\neg$.

7 WORKED EXAMPLE 6 Show with a truth table that $((A \rightarrow B) \wedge \neg B) \rightarrow \neg A$ is a tautology.

8 WORKED EXAMPLE 7 Show with a truth table that $(P \vee Q) \wedge (\neg P \wedge \neg Q)$ is a contradiction.

Exam practice

9 Let P be 'I will go to the beach' and let Q be 'it is not raining'. The sentence 'I will not go to the beach if it is raining' can be written as

A $P \rightarrow Q$ **B** $\neg P \rightarrow Q$ **C** $P \rightarrow \neg Q$ D $\neg P \rightarrow \neg Q$ **E** $\neg Q \rightarrow \neg P$

10 $\neg(A \vee B) \wedge (A \vee B)$ is

A a contradiction **B** a tautology **C** a biconditional statement
D a conditional statement **E** a disjunction

11 Choose the statement that would best describe $(A \wedge B) \vee A$.

A I go to school and do my work.
B I go to school or I do my work.
C When I go to school, I do my work.
D Either I go to school or I do my work.
E Either I go to school and do my work or I just go to school.

12 The truth table for $(P \vee Q) \wedge P$ is

A

P	Q	$P \vee Q$	$(P \vee Q) \wedge P$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

B

P	Q	$P \vee Q$	$(P \vee Q) \wedge P$
T	T	F	T
T	F	T	F
F	T	T	F
F	F	T	F

C

P	Q	$P \vee Q$	$(P \vee Q) \wedge P$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

D

P	Q	$P \vee Q$	$(P \vee Q) \wedge P$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

E

P	Q	$P \vee Q$	$(P \vee Q) \wedge P$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

13 (3 marks) Construct a truth table to show that $A \rightarrow B \equiv \neg A \vee B$.

14 (3 marks) Use a truth table to show that $\overline{ABC} = \overline{A} + \overline{B} + \overline{C}$, where $\overline{A}$ means $\neg A$ and ABC means A and B and C.

15 (3 marks) Use truth tables to prove that $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$ and $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$.

16 (3 marks) Construct a truth table to show that $((A \rightarrow B) \wedge \neg B) \rightarrow \neg A$ is a tautology.

17 (3 marks) Prove that $P \wedge (\neg P \wedge Q)$ is a contradiction.

18 (3 marks) Prove that $P \rightarrow (P \vee Q)$ is a tautology.

19 (4 marks) Copy and complete this truth table and state whether $(P \rightarrow Q) \wedge (Q \rightarrow R) \rightarrow (P \rightarrow R)$ is a tautology or a contradiction.

$P \rightarrow Q$	$P \rightarrow R$	$Q \rightarrow R$	$(P \rightarrow Q) \wedge (Q \rightarrow R)$	$(P \rightarrow Q) \wedge (Q \rightarrow R) \rightarrow (P \rightarrow R)$

20 (4 marks) Prove the logical equivalence $(P \rightarrow R) \vee (Q \rightarrow R) \equiv (P \wedge Q) \rightarrow R$.

4.3 Binary numbers

A **binary digit**, called a **bit**, is one of the digits 0 or 1, and a **binary number** consists of a combination of binary digits. Computers are typically 32-bit or 64-bit machines, which is a measure of the number of CPU (central processing unit) register memory addresses available, where each memory address is referenced by a bit. In a 32-bit machine, the maximum storage is $2^{32} = 4\,294\,967\,296$ bits, and in a 64-bit machine, there is potential for $2^{64} = 18\,446\,744\,073\,709\,551\,616$ bits of storage. Later we will see that binary numbers play a significant role in **Boolean algebra**.

Converting between base 2 and base 10

In the decimal (base 10) system, the digits of a number represent coefficients of powers of 10. For example, $2485 = 2 \times 10^3 + 4 \times 10^2 + 8 \times 10^1 + 5 \times 10^0$. In the base 2 system, the digits represent coefficients of powers of 2. Thus, $1010 = 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$, which is 10 in base 10. This can be written as $1010_2 = 10_{10}$. Base 2 numbers can be expressed as base 10 numbers by repeated division by 2.

Video playlist
Binary numbers

WB
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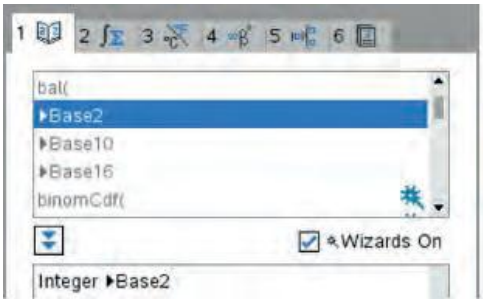
WORKED EXAMPLE 8 Change between base 2 and base 10

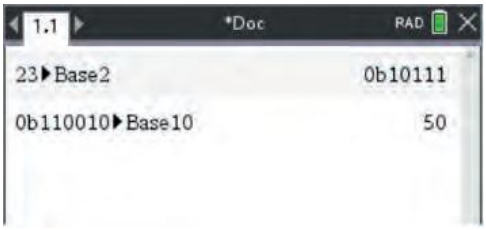
a Change 23 to base 2. **b** Express 110010 in base 10 notation.

Steps		Working
a **1**	Divide the dividend by 2 and write the remainder.	$23 \div 2 = 11$ remainder 1
2	Divide the new dividend by 2 and write the remainder.	$11 \div 2 = 5$ remainder 1
3	Repeat the method of division by 2 until the dividend is 0.	$5 \div 2 = 2$ remainder 1 $2 \div 2 = 1$ remainder 0 $1 \div 2 = 0$ remainder 1
4	Write the digits in reverse order.	10111
5	State the answer.	$23_{10} = 10111_2$
b **1**	Use the digits as the coefficients of powers of 2.	$110010 = 1 \times 2^5 + 1 \times 2^4 + 0 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$ $= 50$
2	State the answer.	$110010_2 = 50_{10}$

USING CAS 1 Converting between base 2 and base 10

TI-Nspire





1 Enter a base 10 number.

2 Press **catalog** and select tab **1**.

3 Press **B** to jump to the start of the functions starting with B.

4 Scroll down and select **Base2**.

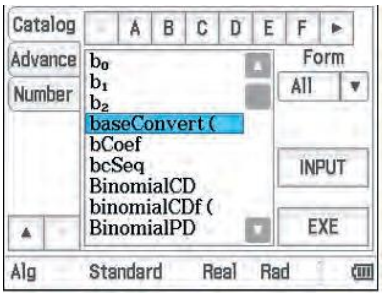
5 Press **enter**.

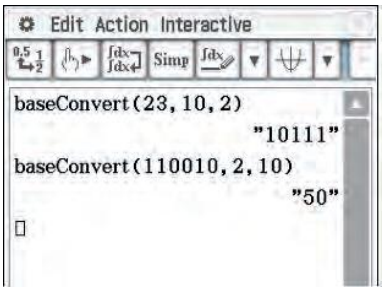
6 The number will convert to base 2. The **0b** in front of the number indicates it is a binary number.

7 To convert a base 2 number to base 10, enter **0b** followed by the binary number.

8 From the catalog, select **Base10**.

ClassPad





1 Open the **Keyboard** and select **Catalog**.

2 Tap **B** to jump to the start of the functions starting with B.

3 Scroll down and select **baseConvert(.**

4 After the function, enter **23, 10, 2**. This converts 23 from base 10 to base 2.

5 Repeat to convert **110010** from base 2 to base 10.

Binary arithmetic

Addition and subtraction

Addition and subtraction is performed in a similar way as the familiar methods involving base 10. However, we ensure that each digit is expressed as either 0 and 1.

WB p. 83

WORKED EXAMPLE 9 Addition and subtraction using base 2

Express the answer to each of the following in base 2.

a $111_2 + 101_2$ **b** $1101_2 - 1011_2$

Steps	Working
a 1 Set out the addition in the required layout.	$+\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 0 & 1 \\ \hline & & \end{array}$
2 Perform the addition, ensuring that each sum digit is binary (0 or 1). If the sum is greater than 1, then divide by 2 and carry the answer.	1 + 1 = 2, 2 ÷ 2 = 1 remainder 0. The carry is 1. $+\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 0_1 & 1 \\ \hline & & \not{2}0 \end{array}$ 1 + 1 = 2, 2 ÷ 2 = 1 remainder 0. The carry is 1. $+\begin{array}{ccc} 1 & 1 & 1 \\ 1_1 & 0_1 & 1 \\ \hline & \not{2}0 & \not{2}0 \end{array}$ 1 + 1 + 1 = 3, 3 ÷ 2 = 1 remainder 1. The carry is 1. $+\begin{array}{cccc} & 1 & 1 & 1 \\ & 1_1 & 0_1 & 1 \\ \hline 1 & 1 & \not{2}0 & \not{2}0 \end{array}$
3 State the answer.	$111_2 + 101_2 = 1100_2$
b 1 Set out the subtraction in the required layout.	$-\begin{array}{cccc} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ \hline & & & 0 \end{array}$
2 Perform the subtraction, converting digits to binary form if necessary.	'Borrow 1' from the column containing the next available non-zero digit. Each 1 that is borrowed is worth 2. $-\begin{array}{cccc} 1 & \not{1}0 & \not{0}2 & 1 \\ 1 & 0 & 1 & 1 \\ \hline & & & 0 \end{array}$ Complete the subtractions. $-\begin{array}{cccc} 1 & \not{1}0 & \not{0}2 & 1 \\ 1 & 0 & 1 & 1 \\ \hline 0 & 0 & 1 & 0 \end{array}$
3 State the answer.	$1101_2 - 1011_2 = 10_2$ (consecutive zeroes at the start can be omitted).

Multiplication

The steps to be performed in base 2 multiplication are the same as for multiplication using base 10. Remember that the sum of the digits added in each column after the multiplication are base 2.

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WORKED EXAMPLE 10 Multiplication using base 2

a Perform the multiplication $111_2 \times 11_2$.

b Write $111_2 \times 11_2$ and the answer using base 10.

Steps	Working
a 1 Set out the multiplication in the required form.	$\begin{array}{cccc} & & 1 & 1 & 1 \\ \times & & & 1 & 1 \\ \hline \\ \hline \\ \hline \end{array}$
2 Perform the multiplication.	$\begin{array}{cccc} & & 1 & 1 & 1 \\ \times & & & 1 & 1 \\ \hline & & 1 & 1 & 1 \\ & 1 & 1 & 1 & 0 \\ \hline \end{array}$
3 Perform the addition.	$\begin{array}{cccc} & & 1 & 1 & 1 \\ \times & & & 1 & 1 \\ \hline & & 1 & 1 & 1 \\ + & {\scriptstyle 1}\ 1^{1} & 1^{1} & 1 & 0 \\ \hline & 1 & \cancel{2}0 & \cancel{3}1 & \cancel{2}0 & 1 \end{array}$
4 State the answer.	$111_2 \times 11_2 = 10101_2$
b 1 Write each binary number in base 10.	$111_2 = 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 7_{10}$ $11_2 = 1 \times 2^1 + 1 \times 2^0 = 3_{10}$
2 Write the binary answer in base 10 form.	$10101_2 = 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 21_{10}$
3 State the answer.	$7 \times 3 = 21$

Division

The method of long division can be used to divide one binary number by another binary number. It is a little more involved than normal long division because comparisons of numbers are made using base 2.

WORKED EXAMPLE 11 Division using base 2

Carry out the division $10101_2 \div 11_2$.

Steps	Working
1 Display the divisor and the dividend in the form of long division.	$11\overline{)10101}$
2 Starting from the left, see if the divisor divides the first digit of the dividend.	11 is greater than 1, so it does not divide into 1. $\begin{array}{r} 0 \\ 11\overline{)10101} \end{array}$
3 Multiply by the dividend, subtract and bring down the next digit.	$\begin{array}{r} 0 \\ 11\overline{)10101} \\ 0 \\ \hline 10 \end{array}$
4 Check to see if the divisor divides the result. $11_2 = 3_{10}$ and $10_2 = 2_{10}$.	11 > 10, so it does not divide into 10. $\begin{array}{r} 00 \\ 11\overline{)10101} \\ 0 \\ \hline 10 \end{array}$
5 Multiply by the dividend, subtract and bring down the next digit.	$\begin{array}{r} 00 \\ 11\overline{)10101} \\ 0 \\ \hline 10 \\ 0 \\ \hline 101 \end{array}$
6 Check to see if the divisor divides the result. $101_2 = 5_{10}$, so 3 goes into 5 once.	11 does divide into 101, once. $\begin{array}{r} 001 \\ 11\overline{)10101} \\ 0 \\ \hline 10 \\ 0 \\ \hline 101 \end{array}$
7 Multiply by the dividend, subtract and bring down the next digit.	$\begin{array}{r} 001 \\ 11\overline{)10101} \\ 0 \\ \hline 10 \\ 0 \\ \hline 101 \\ 11 \\ \hline 0100 \end{array}$

The answers to the divisions will always be 0 or 1, depending on whether 11 is greater than or smaller than the dividend respectively.

WB p. 85

8 Check to see if the divisor divides the result.

11 is less than 0100, so 3 goes into 4 once.

$$\begin{array}{r} 0011 \\ 11\overline{)10101} \\ \underline{0} \\ 10 \\ \underline{0} \\ 101 \\ \underline{11} \\ 0100 \end{array}$$

9 Multiply by the dividend, subtract and bring down the next digit.

$$\begin{array}{r} 0011 \\ 11\overline{)10101} \\ \underline{0} \\ 10 \\ \underline{0} \\ 101 \\ \underline{11} \\ 0100 \\ \underline{11} \\ 00011 \end{array}$$

10 Check to see if the divisor divides the result.

11 divides into 00011 (once).

$$\begin{array}{r} 00111 \\ 11\overline{)10101} \\ \underline{0} \\ 10 \\ \underline{0} \\ 101 \\ \underline{11} \\ 0100 \\ \underline{11} \\ 00011 \end{array}$$

11 Multiply by the dividend and subtract.

Stop when the subtraction answer is 0 (no remainder).

$$\begin{array}{r} 00111 \\ 11\overline{)10101} \\ \underline{0} \\ 10 \\ \underline{0} \\ 101 \\ \underline{11} \\ 0100 \\ \underline{11} \\ 00011 \\ \underline{11} \\ 00000 \end{array}$$

12 State the result.

$10101_2 \div 11_2 = 00111_2$ (or 111_2)

Binary fractions

Similar to decimal fractions, binary numbers have digits to the right of the binary point that represent negative powers of 2. For 0.1101_2, start from the first digit after the binary point and write $1 \times 2^{-1} + 1 \times 2^{-2} + 0 \times 2^{-3} + 1 \times 2^{-4}$. This becomes $1 \times 0.5 + 1 \times 0.25 + 0 \times 0.125 + 1 \times 0.0625 = 0.8125$. Hence, $0.1101_2 = 0.8125_{10}$.

To express a binary fraction as a decimal, the whole number part (the digits to the left of the binary point) and the fraction part (the digits to the right of the binary point) are converted separately.

To convert a decimal to binary form, the whole number part is converted in the same way as we have done previously. However, the fractional part is converted by repeatedly multiplying by 2, omitting the whole number part and noting the 'carry' each time. This is repeated until the value becomes 0 or it is apparent that there is a sequence of repeating digits. This will mean a binary number with an infinite number of digits after the binary point.

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p. 87

WORKED EXAMPLE 12 Converting between decimal and binary fractions

a Write 1101.101_2 as a base 10 number.

b Express 75.125 in binary form.

c State 0.2 in binary form.

Steps	Working
a 1 Convert the whole part to a decimal.	$1101_2 = 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$ $= 13_{10}$
2 Convert the fraction part to a decimal.	$0.101_2 = 1 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3}$ $= 0.625_{10}$
3 State the answer.	Add the whole part to the fraction part. $13_{10} + 0.625_{10} = 13.625_{10}$
b 1 Convert the whole number part to a decimal.	Start with 75, successively divide by 2 and write the remainder each time. $75_{10} = 1001011_2$
2 Convert the fraction part to a decimal.	Start with 0.125, successively multiply by 2 and keep the integer part. $0.125 \times 2 = 0.25$ 0.25 carry 0 $0.25 \times 2 = 0.5$ 0.5 carry 0 $0.5 \times 2 = 1.0$ 0.0 carry 1 $0.125_{10} = 001_2$
3 State the answer.	Add the whole part to the fraction part. $75.125_{10} = 1001011.001_2$
c 1 Convert the fraction part to a decimal.	Start with 0.2, multiply by 2 and keep the integer part. $0.2 \times 2 = 0.4$ 0.4 carry 0 $0.4 \times 2 = 0.8$ 0.8 carry 0 $0.8 \times 2 = 1.6$ 0.6 carry 1 $0.6 \times 2 = 1.2$ 0.2 carry 1 $0.2 \times 2 = 0.4$ 0.4 carry 0 $0.4 \times 2 = 0.8$ 0.8 carry 0 $0.8 \times 2 = 1.6$ 0.6 carry 1 $0.6 \times 2 = 1.2$ 0.2 carry 1 The digits 0011 repeat.
2 State the answer.	$0.2_{10} = 0.001100110011\ldots_2$

EXERCISE 4.3 Binary numbers

ANSWERS p. 574

Recap

1 The truth table for $A \wedge \neg B$ is

A

A	B	$\neg B$	$A \wedge \neg B$
T	T	T	F
T	F	F	T
F	T	T	F
F	F	F	F

B

A	B	$\neg B$	$A \wedge \neg B$
T	T	F	T
T	F	T	F
F	T	F	T
F	F	T	T

C

A	B	$\neg B$	$A \wedge \neg B$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

D

A	B	$\neg B$	$A \wedge \neg B$
T	T	F	T
T	F	T	T
F	T	T	T
F	F	T	F

E

A	B	$\neg B$	$A \wedge \neg B$
T	T	T	F
T	F	T	F
F	T	T	T
F	F	F	F

2 $(\neg A \wedge \neg B) \vee (A \vee B)$ is

A a contradiction **B** an equivalence **C** a negation

D a conjunction **E** a tautology

Mastery

3 Using CAS 1

a Convert 2119 to base 2.

b Write 11001_2 as a base 10 number.

4 WORKED EXAMPLE 8 Perform each conversion.

a 35 to base 2

b 11101_2 to base 10

5 WORKED EXAMPLE 9 Perform each calculation, expressing the answer in base 2.

a $1001_2 + 111_2$

b $1110_2 - 1100_2$

6 WORKED EXAMPLE 10 Write the answer to $101_2 \times 10_2$

a as a base 2 number.

b as a base 10 number.

7 WORKED EXAMPLE 11 Evaluate $11011_2 \div 11_2$.

8 WORKED EXAMPLE 12

a Write 10111.011_2 as a base 10 number.

b Express 5.25 in binary form.

c Write 0.3 in binary form.

9 Write the answer to $6_{10} \times 1011_2$ as a base 2 number.

10 Find the sum of all binary numbers consisting of 3 bits. State your answer in binary form.

11 A clock using 24-hour time has hours and minutes represented by 8-bit registers, as shown below.

Hours									Minutes							
0	0	0	1	0	1	0	0	:	0	0	1	0	1	1	0	0

a Using a 12-hour clock format, what is the time currently shown on the clock?

b The clock is to be set for 10.47 am. What will be displayed in each register?

12 **a** Show that for $abc.de_2$, $abc_2 = 4a + 2b + c$ and $de_2 = \frac{1}{4}(2d + e)$.

b Given that $a + b = 2$, $a > c$ and $d + e = 2$, find the base 10 equivalent of $abc.de_2$. Express the answer as a fraction.

Exam practice

13 (3 marks) Suppose 0 and 1 represent the outcome that a mother has given birth to a son and to a daughter respectively. Hence, the sequence 011 means 1 son and 2 daughters. Suppose a mother has 4 children.

a Assuming no twins, triplets or quadruplets, how many combinations are there if both birth order and gender is taken into account? 1 mark

b Consider the combinations containing 2 boys and 2 girls. If these binary sequences are converted into base 10, what is the difference between the minimum and maximum value? 2 marks

14 (2 marks) Show that the maximum value of a binary number consisting of n bits is $2^n - 1$.

15 (2 marks) There are 4 light switches in a room, with each switch controlling 1 light. The current status of the switches is 0000, where 0 means the switch is off and 1 means the switch is on. Switches can only be turned on/off in groups of 3. Show, using 0 and 1, the minimum set of sequences to apply so that all lights will come on.

16 In base 10, the result of $1010_2 + 1011_2$ is

A 18 **B** 19 **C** 20 **D** 21 **E** 22

17 In base 10, the result of $10101010_2 + 1250_{10}$ is

A 1024 **B** 1118 **C** 1210 **D** 1376 **E** 1420

18 (3 marks) Let $n_{10} = ab11_2$ and $m_{10} = 11ab_2$. If $m_{10} = n_{10} + 2$, show that $6a + 3b = -7$.

19 (3 marks) A 'fuzzy bit' is a bit whose value constantly changes according to some condition. The value of a particular fuzzy bit is determined from the internal clock of the computer, which is read as a positive integer. The time is divided by 3. If the remainder is 0 or 1, the fuzzy bit is set to this value. Otherwise, the remainder is divided by 2 and the fuzzy bit is set to the remainder of this second calculation. Show that clock times that satisfy $6t - 4$, where t is a whole number, produce fuzzy bits of value 0.

Video playlist
Boolean algebra and truth tables

4.4 Boolean algebra and truth tables

In truth tables, each variable is assigned one of 2 values, true (T) or false (F). Around 1850, the English mathematician George Boole devised a system of representing logic using numbers and symbols. In Boolean algebra, true and false are represented by 1 and 0 respectively and the logical connectives **AND**, **OR** and **NOT** are represented by (•), (+) and (−) respectively.

Boolean algebra is used extensively in statistics, set theory and electronics circuit theory.

Truth tables for AND, OR and NOT are shown below with their equivalent Boolean form.

Truth table

AND			OR			NOT	
P	Q	$P \wedge Q$	P	Q	$P \vee Q$	P	$\neg P$
F	F	F	F	F	F	F	T
F	T	F	F	T	T	T	F
T	F	F	T	F	T		
T	T	T	T	T	T		

Boolean

AND			OR			NOT	
P	Q	$P \cdot Q$	P	Q	$P + Q$	P	$\overline{P}$
0	0	0	0	0	0	0	1
0	1	0	0	1	1	1	0
1	0	0	1	0	1		
1	1	1	1	1	1		

Note that other ways of denoting $\neg P$ and $\overline{P}$ are P' and $\sim P$. It is also common practice to omit the centre dot in $P \cdot Q$, as long as it is understood that PQ means 'P and Q' and not normal multiplication.

The Boolean laws given below are commonly used to simplify expressions.

The Boolean laws

1 $A \cdot A = A$ **2** $A + A = A$ **3** $A \cdot 1 = A$

4 $A + 1 = 1$ **5** $A \cdot 0 = 0$ **6** $A + 0 = A$

7 $A \cdot \overline{A} = 0$ **8** $A + \overline{A} = 1$ **9** $\overline{(\overline{A})} = A$

10 $A + \overline{A} \cdot B = A + B$ **11** $\overline{A} + A \cdot B = \overline{A} + B$

Associative laws:	$(A + B) + C = (A + B + C)$	$(A \cdot B) \cdot C = A \cdot (B \cdot C)$
Commutative laws:	$A + B = B + A$	$A \cdot B = B \cdot A$
Distributive laws:	$A \cdot (B + C) = A \cdot B + A \cdot C$	$A + (B \cdot C) = (A + B)(A + C)$
De Morgan's laws:	$\overline{A \cdot B} = \overline{A} + \overline{B}$	$\overline{A + B} = \overline{A} \cdot \overline{B}$

Boolean laws can be proved using truth tables. The first 5 are shown below.

$A \cdot A = A$	
A	$A \cdot A$
0	0
1	1

$A + A = A$	
A	$A + A$
0	0
1	1

$A \cdot 1 = A$	
A	$A \cdot 1$
0	0
1	1

$A + 1 = 1$	
A	$A + 1$
0	1
1	1

$A \cdot 0 = 0$	
A	$A \cdot 0$
0	0
1	0

WORKED EXAMPLE 13 Proof using Boolean algebra

Use Boolean algebra to show that $(X + Y)(X + \overline{Y}) = X$.

Steps	Working	
1 Apply the distributive law.	$(X + Y)(X + \overline{Y})$ $= X \bullet X + X \bullet \overline{Y} + Y \bullet X + Y \bullet \overline{Y}$	
2 Simplify by applying the appropriate Boolean laws.	$= X + X \bullet \overline{Y} + Y \bullet X + 0$ $= X + X \bullet \overline{Y} + Y \bullet X$ $= X(1 + \overline{Y} + Y)$ $= X(\overline{Y} + 1 + Y)$ $= X(1 + Y)$ $= X \bullet 1$ $= X$	$(A \bullet A = A, A \bullet \overline{A} = 0)$ $(A + 0 = A)$ $(A \bullet 1 = A)$ $(A + B = B + A)$ $(A + 1 = 1)$ $(A + 1 = 1)$ $(A \bullet 1 = A)$

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WORKED EXAMPLE 14 Applying Boolean laws to the distributive law

Simplify $(X\overline{Y} + \overline{Y}Z) \bullet (X\overline{Y} + \overline{Z})$.

Steps	Working	
1 Apply the distributive law.	$(X\overline{Y} + \overline{Y}Z) \bullet (X\overline{Y} + \overline{Z})$ $= X\overline{Y} \bullet X\overline{Y} + X\overline{Y} \bullet \overline{Z} + \overline{Y}Z \bullet X\overline{Y} + \overline{Y}Z \bullet \overline{Z}$	
2 Simplify by applying the appropriate Boolean laws.	$= X\overline{Y} + X\overline{Y} \bullet \overline{Z} + \overline{Y}Z \bullet X + 0$ $= X\overline{Y} + X\overline{Y} \bullet \overline{Z} + \overline{Y}Z \bullet X$ $= X\overline{Y}(1 + \overline{Z} + Z)$ $= X\overline{Y}(1)$ $= X\overline{Y}$	$(A \bullet A = A, A \bullet \overline{A} = 0)$ $(A + 0 = A)$ (factorise) $(A + 1 = 1)$ $A \bullet 1 = A$
3 State the answer.	$(X\overline{Y} + \overline{Y}Z) \bullet (X\overline{Y} + \overline{Z}) = X\overline{Y}$	

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Converting truth tables to Boolean expressions

In Boolean notation, false and true are replaced by 0 and 1 respectively. There are 8 possible combinations of 0 and 1 that involve *A*, *B* and *C*, and the truth table shown here can be used to determine the **sum of products (SOP)** that provides the Boolean expression.

Suppose that some combination of logical connectives produced the result given in the Output column.

A	*B*	*C*	Output
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

First, write AND expressions using the variables *A*, *B* and *C* for each row whose output is 1.

These expressions are known as products. The sum of products involves finding the sum (OR) of each output, which is 1.

A	*B*	*C*	Output	
0	0	0	0	
0	0	1	0	
0	1	0	1	$\overline{A}B\overline{C}$
0	1	1	1	$\overline{A}BC$
1	0	0	0	
1	0	1	0	
1	1	0	1	$AB\overline{C}$
1	1	1	1	ABC

Now write an OR expression to find $\bar{A}B\bar{C} + \bar{A}BC + AB\bar{C} + ABC$.

This is minimised as follows:

$$\begin{aligned}\bar{A}B\bar{C} + \bar{A}BC + AB\bar{C} + ABC &= \bar{A}B(\bar{C} + C) + AB(\bar{C} + C)\\ &= (\bar{C} + C)(\bar{A}B + AB)\\ &= 1(\bar{A}B + AB)\\ &= \bar{A}B + AB\\ &= B(\bar{A} + A)\\ &= B \bullet 1\\ &= B\end{aligned}$$

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WORKED EXAMPLE 15 Sum of products for a 3-variable truth table

The truth table is shown for 3 variables.

Show that the SOP is $CA + CB$.

A	B	C	Output
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Steps **Working**

1 Write the AND expression (product) for the output in each row that is 1.

A	B	C	Output	
0	0	0	0	
0	0	1	0	
0	1	0	0	
0	1	1	1	$\bar{A}BC$
1	0	0	0	
1	0	1	1	$A\bar{B}C$
1	1	0	0	
1	1	1	1	ABC

2 State the sum of products (SOP).

The sum of products is $\bar{A}BC + A\bar{B}C + ABC$.

3 Minimise the product sum.

$$\begin{aligned}\bar{A}BC + A\bar{B}C + ABC &= C(\bar{A}B + A\bar{B} + AB)\\ &= C(\bar{A}B + A(\bar{B} + B)\\ &= C(\bar{A}B + A)\\ &= C(A + B)\\ &= CA + CB\end{aligned}$$

WORKED EXAMPLE 16 Finding the sum of products

Use a truth table to find the sum of products (SOP) for $(A + B + C)(\overline{A} + B)$.

Steps | **Working**

1 Complete the truth table.

A	B	C	$\overline{A}$	$A + B + C$	$\overline{A} + B$	$(A + B + C) \cdot (\overline{A} + B)$
0	0	0	1	0	1	0
0	0	1	1	1	1	1
0	1	0	1	1	1	1
0	1	1	1	1	1	1
1	0	0	0	1	0	0
1	0	1	0	1	0	0
1	1	0	0	1	1	1
1	1	1	0	1	1	1

2 Write the product for each row that is not 0.

State the expression for each row whose product is 1.

A	B	C	$\overline{A}$	$A + B + C$	$\overline{A} + B$	$(A + B + C) \cdot (\overline{A} + B)$	
0	0	0	1	0	1	0	
0	0	1	1	1	1	1	$\overline{A}\overline{B}C$
0	1	0	1	1	1	1	$\overline{A}B\overline{C}$
0	1	1	1	1	1	1	$\overline{A}BC$
1	0	0	0	1	0	0	
1	0	1	0	1	0	0	
1	1	0	0	1	1	1	$AB\overline{C}$
1	1	1	0	1	1	1	ABC

3 State the sum of products.

The SOP is $\overline{A}\overline{B}C + \overline{A}B\overline{C} + \overline{A}BC + AB\overline{C} + ABC$.

4 Simplify the product sum.

$$\begin{aligned}
&\overline{A}\overline{B}C + \overline{A}B\overline{C} + \overline{A}BC + AB\overline{C} + ABC \\
&= \overline{A}C(\overline{B} + B) + \overline{A}B\overline{C} + AB(\overline{C} + C) \\
&= \overline{A}C + \overline{A}B\overline{C} + AB \\
&= \overline{A}(C + B\overline{C}) + AB \\
&= \overline{A}(C + B) + AB \\
&= \overline{A}C + \overline{A}B + AB \\
&= \overline{A}C + (\overline{A} + A)B \\
&= \overline{A}C + B
\end{aligned}$$

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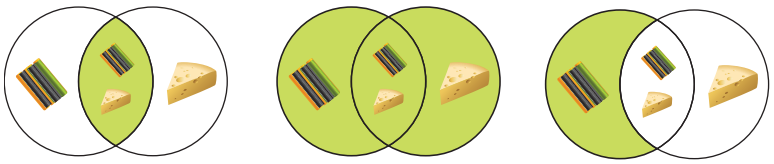
Boolean operators in search engines and databases

Internet search engines, library catalogues and spreadsheet programs perform Boolean searches to retrieve information. They use algorithms based on the Boolean operators AND, OR and NOT, with associated functions such as **XOR** and parentheses.

Boolean operators define the relationships between search terms by restricting or broadening the available records.

AND finds results containing all items to which it is applied. It reduces the number of searches.
OR finds either of the 2 defined items or both. It is useful for expanding a search by including synonyms.
NOT finds all items other than the defined item(s) and reduces the number of searches.
XOR (exclusive OR) is an instruction to find results that contain exactly one of the items to which it is applied (either but not both).

The functions of AND, OR and NOT can be illustrated with **Venn diagrams**.



AND 'chalk **and** cheese'

OR 'chalk **or** cheese'

NOT 'chalk but **not** cheese'

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WORKED EXAMPLE 17 Application of Boolean operators

Write Boolean instructions that will perform each of the following searches.

a Vitamins or minerals in apples

b Citrus fruit, excluding oranges

c Flowers and trees, but excluding roses and eucalypts

Steps	Working
a 1 There are 3 keywords.	The keywords are 'vitamins', 'minerals', 'apples'.
Decide which Boolean operators to use.	'vitamins' and 'minerals' is ORed within parantheses and ANDed with 'apples'.
2 Write the Boolean search query.	(vitamins OR minerals) AND apples
b 1 There are 2 keywords/phrases.	The key phrase is 'citrus fruit' and the keyword is 'oranges'.
Decide which Boolean operators to use.	Enclose citrus fruit in quotes to perform an exact search for the phrase.
	'Excluding' means NOT. We exclude oranges from the search of citrus fruit.
2 Write the Boolean search query.	citrus fruit NOT oranges

c 1 There are 4 keywords/phrases.
Decide which Boolean operators to use.

Keywords/phrases are 'flowers', 'trees', 'roses', 'eucalypts'.

'flowers' and 'trees' are ANDed in brackets. 'roses' and 'eucalypts' are ORed.

NOT is used with brackets to incorporate the exclusion.

2 Write the Boolean search query.

(flowers AND trees) NOT (roses OR eucalypts).

Note: Other answers are possible here, because the English expression is ambiguous.

With Boolean instructions, we interpret 'flowers and trees' to mean that each result must contain both flowers and trees. But if somebody asks for 'websites about flowers and trees', this could mean that they want to see 'websites about flowers and websites about trees', which is equivalent to 'websites about (flowers OR trees)'.

Similarly, 'excluding roses and eucalypts' might mean 'excluding sites with roses AND excluding sites with eucalypts', or it might mean 'excluding sites that have both roses and eucalypts'.

Sometimes it is not possible to tell exactly what is meant without further information. This kind of ambiguity is one of the reasons for using Boolean instructions that clearly define what is meant.

Boolean operators are used in spreadsheets to perform calculations. The main ones are AND, OR, NOT, IF, XOR, TRUE and FALSE.

Consider part of a spreadsheet shown below. The entries in column C are obtained by ORing corresponding cells in columns A and B. For example, C1 is obtained using =OR(A1,B1). Each entry in column D is obtained by ANDing corresponding cells in columns A and B. D1 is obtained using =AND(A1,B1).

	A	B	C	D
1	0	0	FALSE	FALSE
2	1	1	TRUE	TRUE
3	1	0	TRUE	FALSE
4	1	1	TRUE	TRUE
5	1	0	TRUE	FALSE

WB

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WORKED EXAMPLE 18 Applying the conditional statement

A conditional statement takes the form IF (logical test, value if true, value if false).

The negation, NOT, is of the form NOT(logical test).

Complete the table according to the following:

a In column C, each cell has '1' if corresponding entries in columns A and B are the same, and '0' if cells in columns A and B are different.

b In column D, each cell is the negation of its corresponding cell in column C.

	A	B	C	D
1	Adam	Adams		
2	Peter	Paul		
3	Elsa	Elsa		
4	Stavros	Stavros		
5	Yen	Kho		

Steps

a **1** Determine the 'logical test', 'value if true' and 'value if false' in IF (logical test, value if true, value if false).

2 Write the statement for IF (logical test, value if true, value if false).

Working

'logical test' is to compare cell entries in columns A and B. This is A1 = B1, A2 = B2 …

'value if true' will be 1 and 'value if false' is set to 0.

In cell C1, type IF(A1 = B1,1,0) and use the fill down function to fill cells C2 to C5.

	A	B	C	D
1	Adam	Adams	0	
2	Peter	Paul	0	
3	Elsa	Elsa	1	
4	Stavros	Stavros	1	
5	Yen	Kho	0	

b Use NOT(logical test) applied to column C.

'logical test' is the cells in column C.

In D1, type =NOT(C1) and use the fill down function to fill cells D2 to D5.

	A	B	C	D
1	Adam	Adams	0	1
2	Peter	Paul	0	1
3	Elsa	Elsa	1	0
4	Stavros	Stavros	1	0
5	Yen	Kho	0	1

EXERCISE 4.4 Boolean algebra and truth tables

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Recap

1 3.125 in binary form is

A 11.001 **B** 110.01 **C** 1.1001 **D** 11.010 **E** 011.101

2 0.0101_2 as a decimal is

A 0.301 25 **B** 0.310 25 **C** 0.3125 **D** 3.0125 **E** 3.125

9780170448338

Mastery

3 WORKED EXAMPLE 13 Use Boolean algebra to show that $A(\overline{A} + B) + \overline{A}B = B$.

4 WORKED EXAMPLE 14 Simplify $(\overline{A}\overline{B} + A)(\overline{A} + \overline{B}) + \overline{A}$.

5 WORKED EXAMPLE 15 The truth table is shown for 3 variables.

Show that the sum of products (SOP) is $B + \overline{A}C$.

A	B	C	Output
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

6 WORKED EXAMPLE 16 Use a truth table to find the SOP for $(\overline{A} + B)(A + B)$.

7 WORKED EXAMPLE 17 Write a Boolean instruction that will perform each search.

a Cars that are either blue or white in colour.

b Trees that are not deciduous.

c Apple trees with red or green apples or grafted pear trees with green and yellow pears.

8 WORKED EXAMPLE 18 The EXCLUSIVE OR Boolean function XOR looks for results containing one keyword, but not both.

Complete the table according to the following conditions:

1 In column D, each cell has 1 if corresponding entries in columns A and B are the same, and 0 if they are different.

2 In column E, each cell has 1 if corresponding entries in columns A and C are the same, and 0 if they are different.

3 In column F, each cell is XORed with the entries in columns D and E.

	A	B	C	D	E	F
1	Adelaide	Hobart	Sydney			
2	Perth	Perth	Perth			
3	Melbourne	Darwin	Darwin			
4	Sydney	Sydney	Sydney			
5	Brisbane	Canberra	Brisbane			

Exam practice

9 (3 marks) Show, using a Boolean algebra truth table, that the associative law $A(BC) = (AB)C$ holds.

10 (3 marks) Show that $(A + B + \overline{D})(A + B + D) = A + B$.

11 $\overline{A}\overline{B}C + \overline{A}BC + A\overline{B}C + ABC$ minimises to

A $\overline{B}C$ **B** AC **C** C **D** $\overline{A}$ **E** B

12 The next 2 steps in the simplification of $X\overline{Y} + \overline{X}Z + \overline{Y}$ would probably be to use, in order, which Boolean laws?

A $\overline{A} + A \bullet B = \overline{A} + B$ and $A + \overline{A} = 1$

B $A + 0 = A$ and $A \bullet (B + C) = A \bullet B + A \bullet C$

C $A \bullet (B + C) = A \bullet B + A \bullet C$ and $A + 1 = 1$

D $\overline{A \bullet B} = \overline{A} + \overline{B}$ and $A + \overline{A} = 1$

E $\overline{A} + A \bullet B = \overline{A} + B$ and $A + 0 = A$

13 (3 marks) Use De Morgan's laws to show that $\overline{\overline{A}\overline{B} + \overline{C} + \overline{D} + \overline{E}}$ is equivalent to $(A + B)CDE$.

14 The SOP for the truth table is

A $\overline{A}B$ **B** AB **C** $\overline{A}\overline{B}$

D $\overline{A}$ **E** $\overline{B}$

A	B	Output
0	0	1
0	1	1
1	0	0
1	1	0

15 To perform a search in a web browser for wild cats or domesticated dogs, which of the following search queries will be the most effective?

A (wild and cats) or (dogs not wild)

B 'wild cats' not 'wild dogs'

C 'wild and cats' and dogs not wild

D wild cats and domesticated dogs

E not tame cats and dogs not wild

16 (3 marks) Show that $\overline{ABC} + \overline{AB}C + A\overline{BC} + A\overline{B}C = \overline{A} + \overline{B} + \overline{C} + A\overline{B}$.

17 (3 marks) Minimise $\overline{A}\overline{B}\overline{C} + A\overline{B}\overline{C} + \overline{A}\overline{B}C + A\overline{B}C$.

18 (4 marks) By defining Y^2 to mean $Y \bullet Y$, Y^3 to mean $Y \bullet Y \bullet Y$ and so on, deduce that $(A + B)^N = A^N + B^N + AB$, where N is a positive whole number.

Hint: $(x + y)^n = x^n + c_1x^{n-1}y^1 + c_2x^{n-2}y^2 + c_2x^{n-3}y^3 + \ldots + y^n$, where c_1, c_2 … are constants.

19 (4 marks) A truth table is constructed using N variables, A_1, A_2, A_3, A_4 … A_N.

Explain why it is the case that after each product is obtained using the conjunction of the N variables, the sum of products is $A_1A_2A_3A_4 \ldots A_N$.

20 (4 marks) An internet search is conducted to find information about wind, rain and fire.

Show all combinations of these keywords as a Venn diagram consisting of 3 overlapping circles.

4.5 Karnaugh maps and truth tables

Construction of a Karnaugh map from a truth table

Applying Boolean laws to minimise expressions is not always straightforward. An alternative method, introduced in 1953 by the American physicist Maurice Karnaugh, is to use a **Karnaugh map**, or **K-map**. It is another way of representing and minimising a Boolean function having many variables. You may have used Karnaugh maps in statistics or in probability investigations involving Venn diagrams (two-way tables).

Consider the Boolean truth table below comprising variables A and B with $A + B$ as the output values. Its K-map structure is shown next to it. The value in each cell of a Karnaugh map refers to the variables of the row and column.

Truth table		
A	B	$A + B$
0	0	0
0	1	1
1	0	1
1	1	1

K-map		
A \ B	0	1
0	0	1
1	1	1



Video playlist
Karnaugh maps and truth tables

The possible values for A and B (0 and 1) are entered in the first column and first row respectively. The outputs $A + B$ are shown in the cells. For example, $A = 1$, $B = 0$ has output 1, which is placed in the cell circled. There are $2 \times 2 = 4$ output values.

A truth table consisting of 3 variables, A, B and C, with outputs ABC, is shown below. In the K-map, the values of A are in the first column and the 4 combinations for BC are in the first row. There are $2 \times 4 = 8$ values for the output. The order of the values of AB is called the Gray code sequence. Only one value changes from one cell to the next.

Truth table			
A	B	C	ABC
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1

K-map				
A \ BC	00	01	11	10
0	0	0	0	0
1	0	0	1	0

The structure of a 4-variable K-map is shown below. The truth values for AB are in column 1 and those for CD are in row 1. There are $4 \times 4 = 16$ values for the product. Notice that the values of both AB and CD are in Gray code sequence.

K-map				
AB \ CD	00	01	11	10
00				
01				
11				
10				

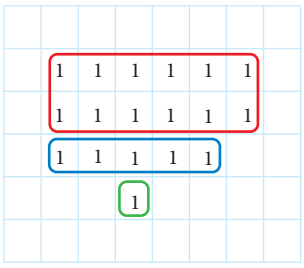
In general, a K-map comprising N variables contains 2^N output cells.

Minimising the product sum using a Karnaugh map

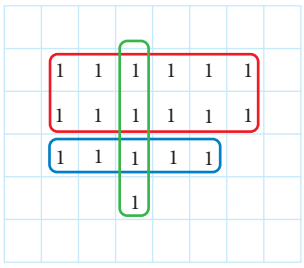
In a similar way to Boolean laws, a K-map has rules to minimise an expression. These rules may seem challenging at first, but with practice they provide a more efficient, visual way of minimising the Boolean expression. The aim is to create groups of cells with only 1s that have no change in at least 1 variable.

After the K-map is drawn, combine cells in groups according to the following rules:

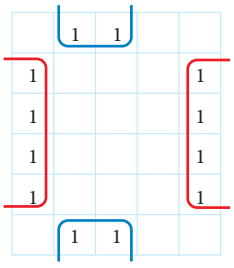
1. Put cells with 1s only in squares or rectangles so that the number of cells in each group is a power of 2: that is, 1, 2, 4, 8, 16 …
2. Make each square or rectangle of cells as large as possible, but with no 0 cells.
3. Groups can overlap.
4. Entire columns and/or rows can be swapped so that boxes with a value of 1 are next to each other.
5. The first column is considered to be next to the last column and the first row is next to the last row ('wrapping').
6. Every square containing 1 must be in a group, even if that 'group' is itself.
7. There should be as few groups as possible.



3 groups



3 groups



Wrapping

We can apply the procedure to this truth table from Worked example 15 on page 152, where the minimised expression was $CA + CB$.

A	B	C	Output
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Here is the K-map for variables A, B, C.

The minimum number of groups needed is 2.

In the red group, A changes but B and C do not ($B = 1$, $C = 1$).

This means that A is not counted. This group represents the product term BC.

In the green group, A and C do not change ($A = 1$, $C = 1$). The product term is AC.

The sum of the product terms is $BC + AC$.

A \ BC	00	01	11	10
0	0	0	1	0
1	0	1	1	0

WORKED EXAMPLE 19 Using a Karnaugh map to minimise SOP

Use the K-map to find the minimised expression for the sum of products.

AB \ CD	00	01	11	10
00	1	1	0	0
01	1	1	1	0
10	0	1	0	1
11	1	1	1	1

Steps | **Working**

1 Identify the groups to use.

AB \ CD	00	01	11	10
00	1	1	0	0
01	1	1	1	0
11	1	1	1	1
10	0	1	1	0

2 Write the expression for each group.

For the blue group, going across and down, $A = 0$ and $C = 0$ do not change.
This represents $\overline{A}\,\overline{C}$.

For the green group, B and D do not change.
This represents BD.

For the red group, A and B do not change.
This is AB.

For the black group, A and D do not change, which is represented by AD.

3 Write the expression for the SOP.

The SOP is $\overline{A}\,\overline{C} + BD + AD + AB$.

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WORKED EXAMPLE 20 Constructing a Karnaugh map

Construct a K-map from the truth table and find the minimised expression for the SOP.

A	B	C	D	Product
0	0	0	0	1
0	0	0	1	0
0	0	1	0	1
0	0	1	1	1
0	1	0	0	1
0	1	0	1	0
0	1	1	0	1
0	1	1	1	1
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	0
1	1	0	0	0
1	1	0	1	1
1	1	1	0	1
1	1	1	1	1

Steps | **Working**

1 Complete the K-map.

AB \ CD	00	01	11	10
00	1	0	1	1
01	1	0	1	1
11	0	1	1	1
10	0	0	0	0

2 Identify groups and determine the product terms.

In the orange group, $A = 0$ and $D = 0$ do not change. This is $\overline{A}\,\overline{D}$.

For the blue group, $A = 0$ and $C = 1$ do not change. This is $\overline{A}C$.

In the green group, $B = 1$ and $C = 1$ are not changing. This is BC.

Finally, in the black group, $A = 1$, $B = 1$ and $D = 1$ do not change. This is ABD.

AB \ CD	00	01	11	10
00	1	0	1	1
01	1	0	1	1
11	0	1	1	1
10	0	0	0	0

3 Write the expression for the SOP.

Add the expressions.

The SOP is $\overline{A}\,\overline{D} + \overline{A}C + BC + ABD$.

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WORKED EXAMPLE 21 Determine the sum of products from a truth table

a Find the SOP for the truth table shown.

b Deduce what the SOP will be if the negated value of the product cells in the truth table are used.

A	B	C	Product
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Steps	Working
a 1 Complete the K-map.	
2 Identify the groups and calculate the product terms. In the only group, $C = 1$ does not change. This is C.	
3 Find the SOP.	The SOP is C.
b Write the negation.	The SOP will also be C because the new combination of products will be the same as the original combination.

A \ BC	00	01	11	10
0	0	1	1	0
1	0	1	1	0

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EXERCISE 4.5 Karnaugh maps and truth tables

ANSWERS p. 574

Recap

1 In Boolean algebra, $\overline{B}A + AC + \overline{B} + C$ minimises to

A $\overline{B} + \overline{C}$ **B** $\overline{B} + C + \overline{C}$ **C** $\overline{B} + A$ **D** $\overline{B} + C$ **E** $\overline{B} + \overline{A} + C$

2 The sum of products (SOP) of the outputs in the table is

A $A\overline{B} + \overline{A}B$

B $\overline{A}\overline{B} + \overline{A}B + AB$

C $\overline{A}\overline{B} + A\overline{B} + AB$

D $\overline{A}\overline{B} + \overline{A}B$

E $AB + A\overline{B} + \overline{A}B$

A	B	Output
0	0	1
0	1	1
1	0	0
1	1	1

Mastery

3 WORKED EXAMPLE 19 Use the K-map shown to obtain the SOP.

AB \ CD	00	01	11	10
00	0	1	0	0
01	0	0	1	0
11	1	1	1	1
10	0	1	1	0

4 WORKED EXAMPLE 20 Construct a K-map from the truth table and use it to obtain the minimised expression for the SOP.

A	B	C	Product
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

5 WORKED EXAMPLE 21

a Evaluate the SOP for the truth table shown.

b Deduce the SOP when the negated values of the product cells in the truth table are used.

A	B	C	D	Product
0	0	0	0	1
0	0	0	1	0
0	0	1	0	1
0	0	1	1	0
0	1	0	0	1
0	1	0	1	0
0	1	1	0	1
0	1	1	1	0
1	0	0	0	1
1	0	0	1	0
1	0	1	0	1
1	0	1	1	0
1	1	0	0	1
1	1	0	1	0
1	1	1	0	1
1	1	1	1	0

Exam practice

6 (3 marks) Show that a 2-variable truth table with all values equal to 1 also has 1 as its SOP.

7 (3 marks) The SOP for a K-map may be obtained in different ways. Verify this statement by finding 2 different expressions for the SOP using the K-map shown.

AB \ CD	00	01	11	10
00	1	0	0	1
01	1	1	0	0
11	0	1	1	0
10	0	0	1	1

8 The number of product cells in a 5-variable K-map is

A 16 **B** 24 **C** 32 **D** 48 **E** 72

9

A	B	Output
0	1	1
1	1	0
0	0	1
1	0	0

The K-map for the truth table above could be

A

A \ B	1	0
1	0	0
0	1	1

B

A \ B	0	1
0	1	0
1	1	0

C

A \ B	0	1
0	0	1
1	0	1

D

A \ B	0	1
1	1	0
0	0	1

E

A \ B	1	0
0	1	0
1	1	0

10 Choose the correct K-map for this truth table.

A	B	C	Output
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	0

A

A \ BC	00	01	11	10
0	1	1	1	1
1	0	1	0	1

B

A \ BC	00	01	11	10
0	1	0	1	1
1	0	1	0	0

C

A \ BC	00	01	11	10
0	1	1	1	0
1	1	1	1	1

D

A \ BC	00	01	11	10
0	1	0	1	1
1	0	1	0	1

E

A \ BC	00	01	11	10
0	1	1	1	0
1	0	1	0	0

11 Which product term does the group shown in the K-map represent?

A \ BC	00	01	11	10
0	0	1	0	1
1	0	1	0	1

A $A + C$ **B** $A\overline{B}$ **C** AC **D** $\overline{A}B$ **E** $A\overline{B}C$

12 Choose the correct statement concerning the K-map given below.

X \ YZ	00	01	11	10
0	1	0	1	0
1	0	1	0	0

A Cells 000 and 101 can form a group by wrapping.

B Cells 000 and 101 are adjacent cells in a group.

C The product term for the last row is $X\overline{Y}\overline{Z}$.

D The SOP is $\overline{X}\overline{Y}\overline{Z} + \overline{X}YZ + X\overline{Y}Z$.

E Column 000 and column 011 contain a useful group.

13 (2 marks) Explain why the shaded cells in the K-map shown is not a useful choice to have as a group.

AB \ CD	00	01	11	10
00	0	1	1	1
01	1	0	1	1
11	0	1	1	1
10	0	0	0	0

14 (3 marks) For variables A, B and C, determine the SOP for $(A \vee B) \wedge C$.

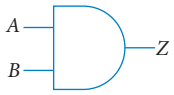
15 (2 marks) Determine the SOP for $(\neg P \vee R) \vee [P \wedge (\neg R \vee Q)]$.

4.6 Electronic gates and circuits

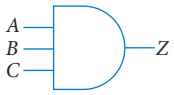
Boolean algebra, truth tables and K-maps can be applied to examine digital circuits – the heart of digital electronics and computer processing.

Notation for logic gates

A **logic gate** is a Boolean function defined by its truth table. A digital circuit comprises a set of logic gates that perform Boolean logic to achieve a specific task. The Boolean variables 0 and 1 (also known as 'lows' and 'highs' respectively) are used to represent the state of a voltage through the circuit. Logic gates are represented by symbols to indicate the type of Boolean function and the number of input variables. There is only one output variable. Below are AND logic gates showing 2 and 3 input variables A, B and C, with Z as the output variable. Also shown is the truth table controlling the value of the output. For 3 inputs, output Z will only be true (have value 1) when A, B and C are 1.



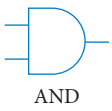
A	B	AB
0	0	0
0	1	0
1	0	0
1	1	1

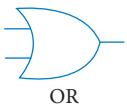


A	B	C	ABC
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1

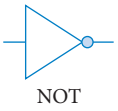
Below is a summary of the basic logic gates for 2 input variables.

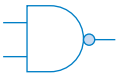
Note that a circle indicates negation (NOT).

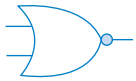

AND

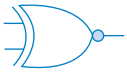

OR


XOR (exclusive OR)


NOT


NAND (NOT AND)


NOR (NOT OR)


XNOR (NOT exclusive OR)

The NOT logic gate (also called an **inverter**) can have only one input (0 or 1). We have previously established its truth table using true and false values. In Boolean notation, the truth table with variable A is shown on the right.

A	$\overline{A}$
0	1
1	0

The truth table for the XOR logic gate using 2 inputs is shown.

For A XOR B to be true, either A or B must be true, but not both.

The truth tables for NAND, NOR and XNOR can be found by taking the negation of the truth values for AND, OR and XOR respectively.

A	B	A XOR B
0	0	0
0	1	1
1	0	1
1	1	0

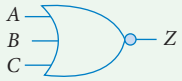
Video playlist
Electronic gates and circuits

WB

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WORKED EXAMPLE 22 Truth table for a NOR gate

Construct the truth table for the logic gate shown.



Steps	Working
1 Identify the logic gate required.	The logic gate is NOR, with 3 input variables and Z as the output variable.
2 Construct the truth table.	Complete the truth diagram for OR and take the negation of the values.

A	B	C	OR $A + B + C$	NOR $\overline{A + B + C}$
0	0	0	0	1
0	0	1	1	0
0	1	0	1	0
0	1	1	1	0
1	0	0	1	0
1	0	1	1	0
1	1	0	1	0
1	1	1	1	0

Multiple logic gates

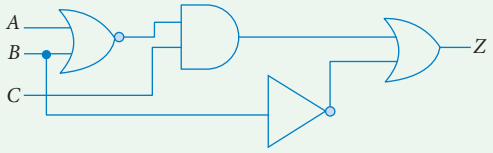
When more than one logic gate is required to represent an expression, it is better to work from left to right.

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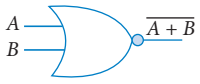
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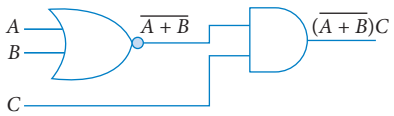
WORKED EXAMPLE 23 Find the output function from a circuit

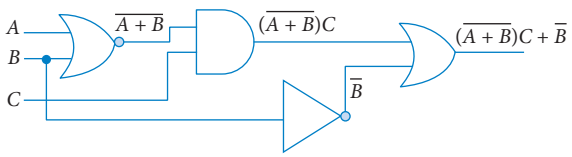
Determine the simplified output function, Z, for the circuit shown.



Steps	Working
1 Start from the left and identify the first logic gate.	The first gate is NOR for variables A and B. This is $\overline{A + B}$.
2 Describe the next gate.	The next gate ANDs C with the output from $\overline{A + B}$ to give $(\overline{A + B})C$.
3 Describe the remainder of the circuit. Note that a solid dot is used to show where 2 wires intersect. Otherwise, it is taken that the wires do not touch.	NOT B is ORed with $(\overline{A + B})C$ to give $(\overline{A + B})C + \overline{B}$ as the final output. The circle indicates that B takes 2 paths.
4 State the output for the circuit.	The circuit has output $(\overline{A + B})C + \overline{B}$. This simplifies to $\overline{B}$.
5 Use Boolean laws to simplify.	$(\overline{A + B})C + \overline{B} = (\overline{A}\,\overline{B})C + \overline{B}$ $= \overline{B}(\overline{A}C + 1)$ $= \overline{B}$



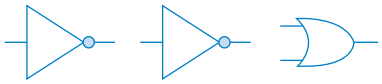


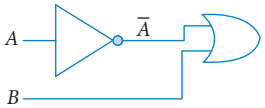


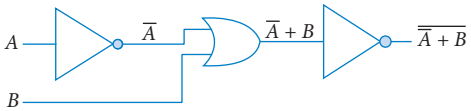
WORKED EXAMPLE 24 Constructing a logic gate using NOT, OR and AND

Show $\overline{\overline{A} + B}$ as a logic gate diagram.

Steps	Working
1 Identify the logic gates required.	$\overline{A}$ is NOT A. $\overline{A} + B$ is $\overline{A}$ OR B. $\overline{\overline{A} + B}$ is NOT $\overline{A} + B$. We need 2 NOT gates and 1 OR gate.
2 Join the gates in the correct order.	$\overline{A}$ and B is connected to an OR gate. Negate the output.







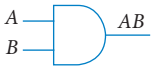
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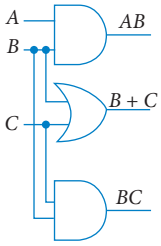
We could use fewer gates by using a NAND gate instead of AND followed by NOT.

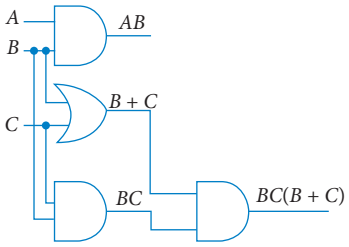
WORKED EXAMPLE 25 Logic gate using OR and AND

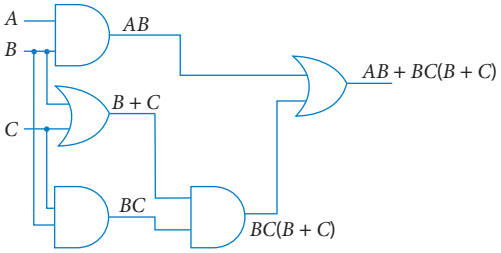
Show $AB + BC(B + C)$ as a logic gate diagram.

Steps	Working
1 Working left to right, identify the first section and gate.	AB requires a 2-variable AND gate using A and B as inputs.
2 Identify the next section and gate.	$BC(B + C)$ OR is used with inputs B and C and with $B + C$. $B + C$ requires an OR gate with B, C as inputs. BC has an AND gate.
3 Identify the next section and gate.	$BC(B + C)$ is BC AND $(B + C)$.
4 Complete the circuit diagram by combining the main sections.	$AB + BC(B + C)$ is AB OR $BC(B + C)$. Use an OR gate.





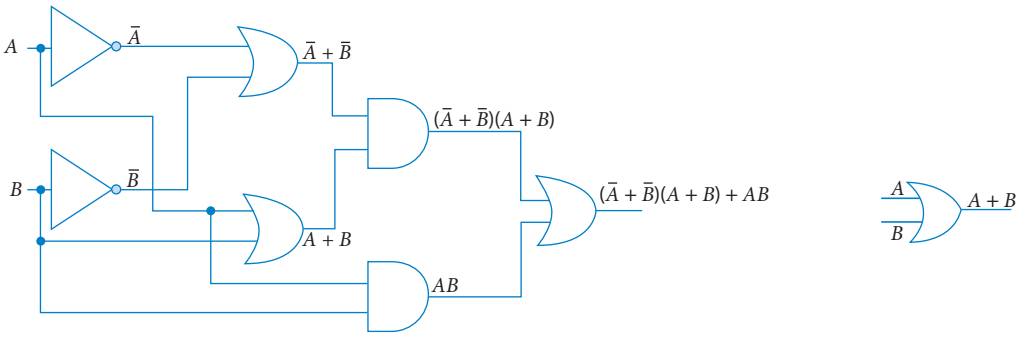




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Simplifying logic gates

The logic circuit below takes variables A and B and outputs $(\overline{A} + \overline{B})(A + B) + AB$, which can be simplified to $A + B$ using a truth table or Karnaugh map.



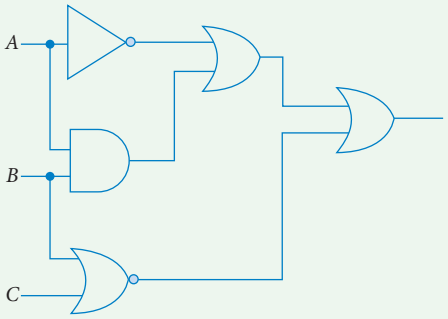
This means that the circuit on the left can be replaced by the simplified circuit on the right, which consists of one logic gate whose output is $A + B$. This is preferred, because a simplified circuit is easier to understand and in real life is more cost-efficient to make.

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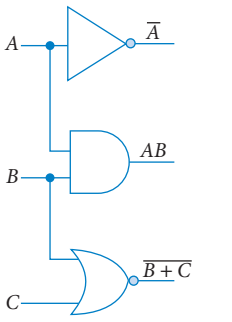
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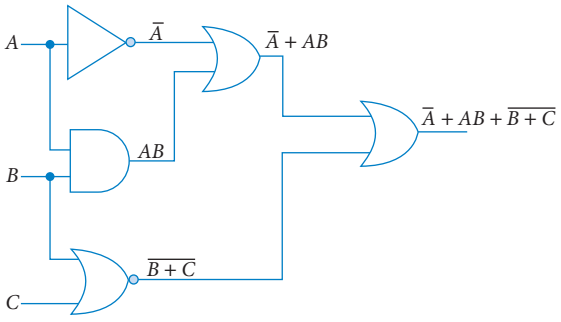
WORKED EXAMPLE 26 Simplifying a circuit

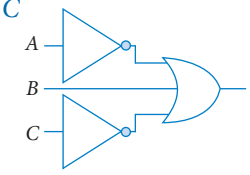
Draw a simplified version of the circuit shown below.



Steps	Working
1 Working from left to right, describe each gate.	A is negated, AND is applied to A and B, and NOR is applied to B and C.
	AB and $\overline{A}$ are ORed. An OR gate is then applied using inputs $\overline{A + B}$ and $\overline{A} + AB$.
2 Simplify the final output using any method.	$(\overline{A} + AB) + (\overline{B + C})$ $= \overline{A} + B + \overline{B} \cdot \overline{C} = \overline{A} + B + \overline{C}$
3 Draw the simplified circuit.	







9780170448338

EXERCISE 4.6 Electronic gates and circuits

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Recap

1 The SOP for the K-map is

A $A\overline{B} + BC$ **B** C **C** $\overline{B}\overline{C} + BC$

D $\overline{A}C$ **E** $\overline{A}$

A \ BC	00	01	11	10
0	0	1	1	0
1	0	1	1	0

2 Which grouping gives $\overline{A}\overline{C}D + C\overline{D} + AC$ as the sum of products?

A

AB \ CD	00	01	11	10
00	0	1	1	0
01	0	1	1	0
11	0	0	1	1
10	0	0	1	1

B

AB \ CD	00	01	11	10
00	0	1	1	0
01	0	1	1	0
11	0	0	1	1
10	0	0	1	1

C

AB \ CD	00	01	11	10
00	0	1	1	0
01	0	1	1	0
11	0	0	1	1
10	0	0	1	1

D

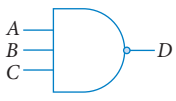
AB \ CD	00	01	11	10
00	0	1	0	1
01	0	1	0	1
11	0	0	1	1
10	0	0	1	1

E

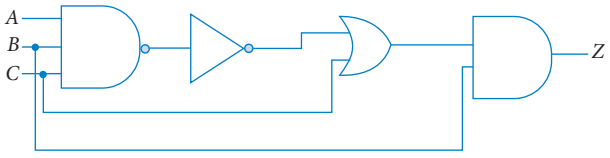
AB \ CD	00	01	11	10
00	0	1	1	0
01	0	1	1	0
11	0	0	1	1
10	0	0	1	1

Mastery

3 WORKED EXAMPLE 22 Construct the truth table for the logic gate shown.



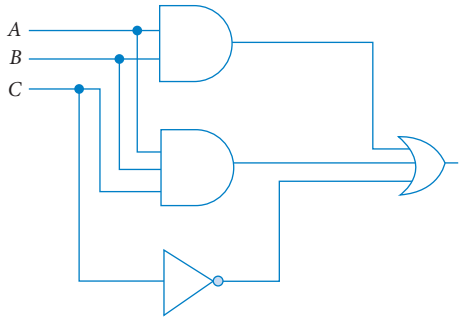
4 WORKED EXAMPLE 23 Determine the output function, Z, for the circuit shown.



5 WORKED EXAMPLE 24 Draw a logic gate diagram to show $\overline{A\overline{B}}$.

6 WORKED EXAMPLE 25 Show $A\overline{B} + \overline{C}(\overline{A}(B + C))$ as a logic gate diagram.

7 WORKED EXAMPLE 26 Draw a simplified version of the circuit shown below.



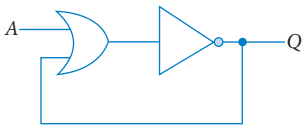
Exam practice

8 (4 marks)

a For input variables X and Y, let A be X NAND Y. Write the simplified expression for $Q = A$ OR A. 2 marks

b Draw the circuit for the output function, Q. 2 marks

9 (2 marks) Describe why the output, Q, in the circuit shown might not be constant for a constant value of A.

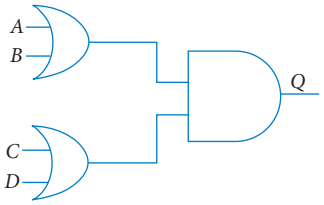


10 The output function, Q, in the circuit shown is

A $(A + B) + (C + D)$ **B** $(AB) + (CD)$

C $(AB)C + D$ **D** $A + B(C + D)$

E $(A + B)(C + D)$

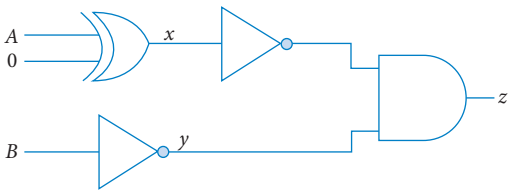


11 When $A = 1$ and $B = 0$, the Boolean values x, y and z are, respectively

A 0, 1, 1 **B** 1, 1, 1

C 1, 1, 0 **D** 1, 0, 1

E 0, 1, 0



12 The 2 circuits below produce the same output, Q.

This means

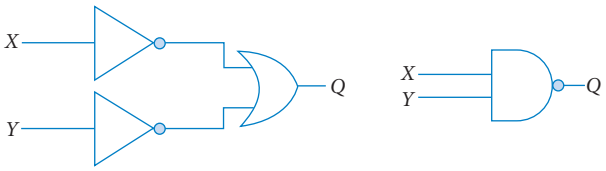
A $\overline{X} + \overline{Y} = X\overline{Y}$

B $XY = \overline{XY}$

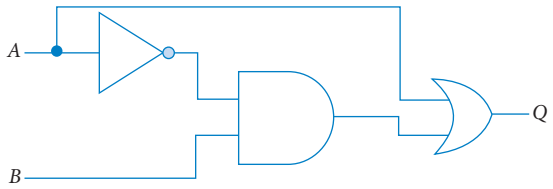
C $X(\overline{X} + \overline{Y}) = \overline{Y}(X + Y)$

D $\overline{X} + \overline{Y} = \overline{XY}$

E $\overline{X}\overline{Y} = XY$



13



The truth table for the circuit shown is

A

A	B	Q
0	0	0
0	1	1
1	0	1
1	1	1

B

A	B	Q
0	0	0
0	1	1
1	0	0
1	1	1

C

A	B	Q
0	0	0
0	1	0
1	0	0
1	1	1

D

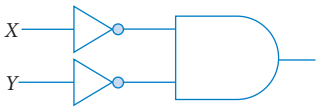
A	B	Q
0	0	1
0	1	1
1	0	1
1	1	1

E

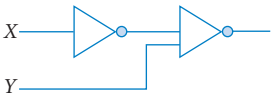
A	B	Q
0	0	0
0	1	0
1	0	0
1	1	0

14 Which circuit diagram is possible?

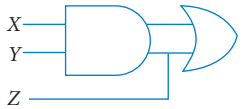
A



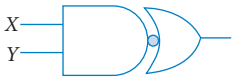
B



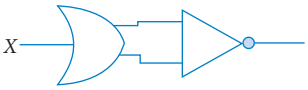
C



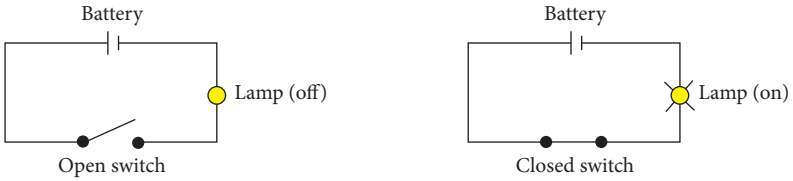
D



E



15 (4 marks) A switch in a wiring diagram can be either open or closed and represented by 0 and 1 respectively. In the open position, no current will flow through the wire and the lamp will not light.



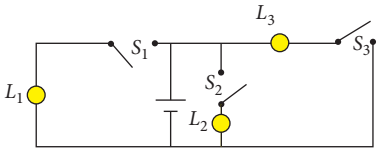
Let S_1 and S_2 be the values (0 or 1) of 2 switches.

a Draw a wiring diagram using one battery so that the globe shines only when $S_1 + S_2$ is true. 2 marks

b Draw a wiring diagram using one battery so that the globe shines only when S_1S_2 is true. 2 marks

16 (6 marks) A circuit consists of one battery, 3 globes and 3 switches, as shown.

Decide for each case, which globe(s) will be lit.



a $S_1\overline{S}_2\overline{S}_3$ 2 marks

b $\overline{S}_1\overline{S}_2S_3$ 2 marks

c $\overline{S}_1S_2\overline{S}_3$ 2 marks

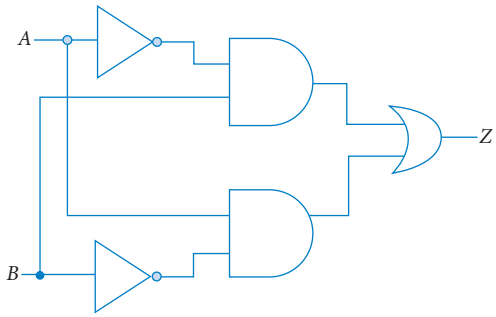
17 (5 marks)

a Use De Morgan's laws to deduce that for N input variables, V_1, V_2, V_3 … V_N, $\overline{V_1V_2V_3\ldots V_N} = \overline{V}_1 + \overline{V}_2 + \overline{V}_3 + \ldots + \overline{V}_N$. 3 marks

b Explain why an additional $(N - 1)$ logic gates are required to construct a circuit representing the calculation $\overline{V}_1 + \overline{V}_2 + \overline{V}_3 + \ldots + \overline{V}_N$ compared to a circuit representing $\overline{V_1V_2V_3\ldots V_N}$. 2 marks

18 (6 marks)

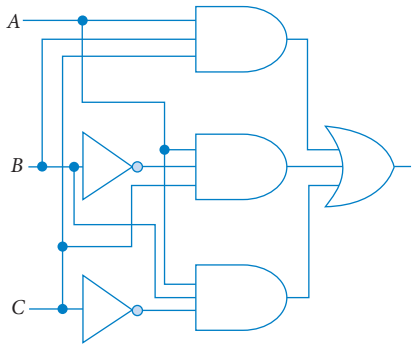
a Obtain an expression for the output, Z, for the circuit shown. 2 marks



b Construct a truth table for Z. 3 marks

c Which basic logic gate is represented by the truth table? 1 mark

19 (4 marks) Use the circuit shown to draw its equivalent simplified circuit.

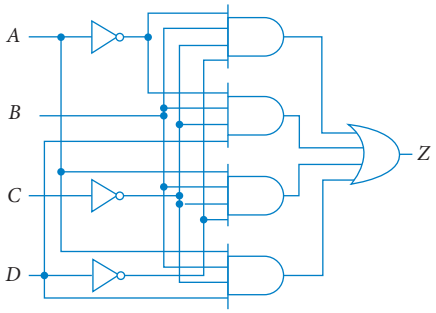


20 (4 marks) Use De Morgan's laws to help you draw the section of the circuit represented by Z in the expression $\overline{A\overline{B} + \overline{A}B} = \overline{A}\overline{B} + Z$.

21 (5 marks)

a Derive the output function, Z, for this circuit. 2 marks

b Simplify the expression obtained in part **a** and draw the circuit. 3 marks



4.7 Algorithms and pseudocode

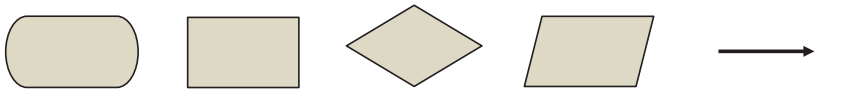
Algorithms

An **algorithm** is a set of instructions for successfully completing a task. The success of activities such as baking a cake, finding your way through a maze, and mowing the lawn relies on executing a defined sequence of steps that produce the desired outcomes. If the algorithm is efficient, the cake will be delicious, you will find your way out of the maze, and the lawn will be cut evenly.

An algorithm can be a set of instructions using everyday language, a **flowchart** or **pseudocode**.

Flowcharts

A flowchart uses a prescribed set of symbols to display the instructions of an algorithm.



Start/end | Process | Decision | Input/output | Direction connector

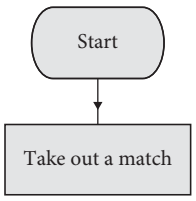
Video playlist
Algorithms and pseudocode

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WORKED EXAMPLE 27 Designing a flowchart

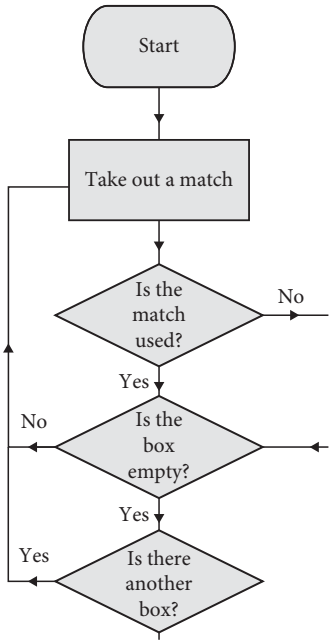
Draw a flowchart to show the following steps in the algorithm to light a match.

- Open a matchbox
- Take out a match
- If the match is used, take out another match
- Scratch the match against the striking surface
- If the match doesn't light and there is another match, repeat the method above.

Steps	Working
1 Use a start/end symbol and perform the first instruction.	Use a process symbol to 'take out a match'. 

2 Show the decisions to be made after a match is taken out.

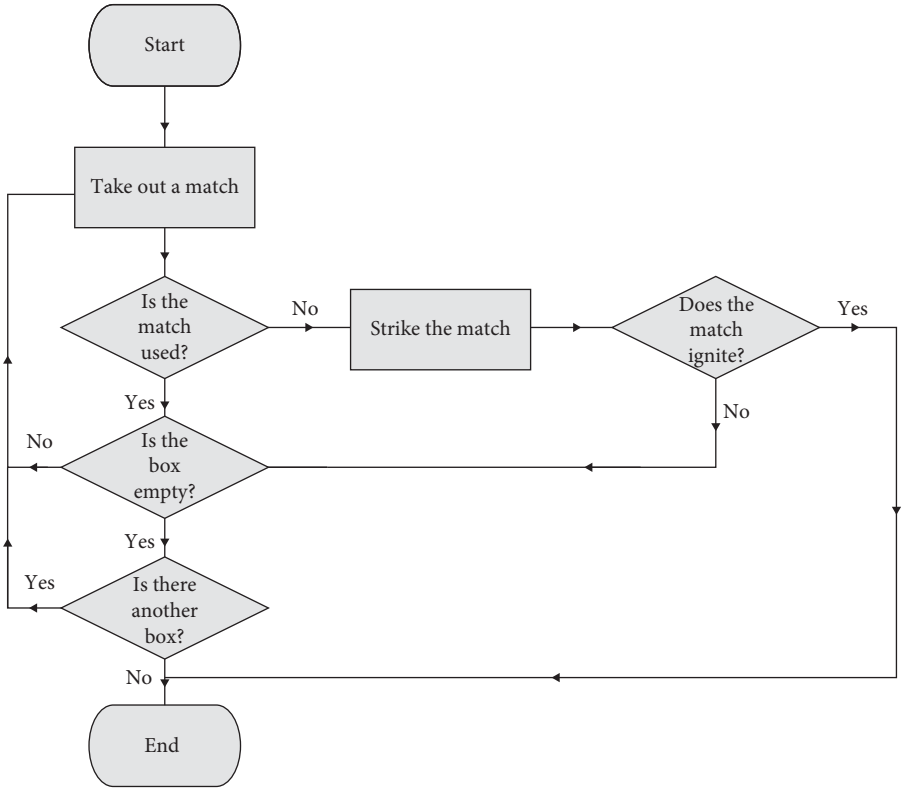
Use decision boxes to decide what to do if the match is used, the matchbox is empty or if there is no other matchbox.



3 Complete the flowchart.

Show what happens after the match is struck.

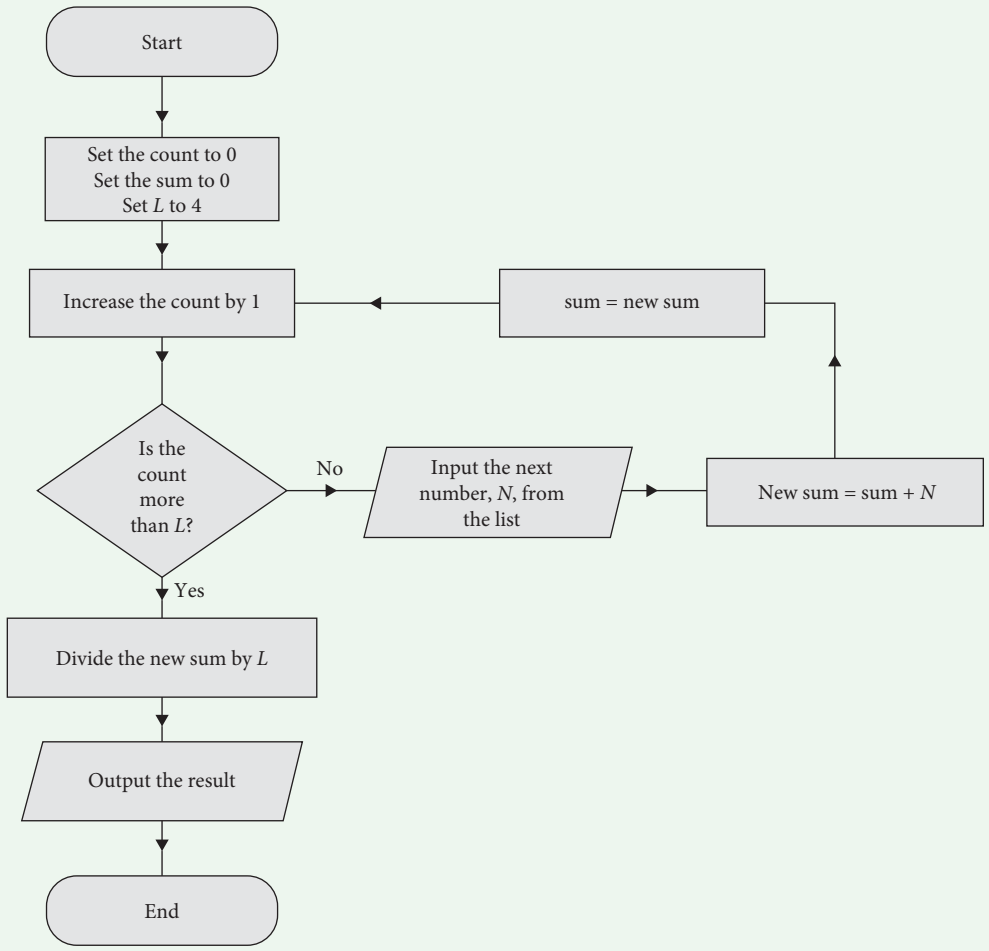
Ensure that lines with arrows point in the right direction and connect to the appropriate sections of the flowchart.



9780170448338

WORKED EXAMPLE 28 Interpret a flowchart

The flowchart below describes an algorithm.



a State the values for count, sum and new sum when $N = 5, 6, 1, 8$.

b Describe what the algorithm is calculating.

Steps	Working
a 1 State the initial values.	count = 0 sum = 0 $L = 4$
Apply the flowchart instructions for the first value of count.	**count** is increased by 1, so count = 0 + 1 = 1.
2 Follow the correct path for the decision.	The count (1) is not more than L (4), so the first number from the list (5) is used. This value is added to the value of the sum, so the new sum is 0 + 5 = 5 and sum is then updated to match the new sum. (Without this step, the sum would never get updated from zero, so the new sum would just be the last number in the list.).
3 Repeat the loop until the count value is greater than the N value.	count = 1 + 1 = 2 $N = 6$, new sum = 5 + 6 = 11 count = 2 + 1 = 3 $N = 1$, new sum = 11 + 1 = 12 count = 3 + 1 = 4 $N = 8$, new sum = 12 + 8 = 20

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	4 Follow the final set of instructions prior to terminating the program.	count = 4 + 1 = 5 count is more than L, so follow the 'yes' path at the decision box. final sum = new sum ÷ L = 20 ÷ 4 = 5 The output is 5.
b	**1** Describe the situation by looking at what happens in the process symbols.	For each value of the count from 1 to 4, a number, N, is taken from the list and added to the previous sum. In the final step, the sum is divided by how many numbers were used.
	2 State the conclusion.	The algorithm finds the arithmetic mean of a given set of 4 numbers.

Pseudocode

In a previous example on page 175, a set of easily understood instructions were used to create a flowchart describing how to light a match. In a similar way, pseudocode consists of a combination of technical terms and commonly used words that provide an informal but effective way of depicting the flow and operation of a task. Think of pseudocode as the 'planning' stage that guides how the actual computer program will be developed. Because it is not tied to any specific programming language, pseudocode allows flexibility in choosing words and phrases. However, by adopting a number of conventions, a better understanding of the basic operations required to accomplish the task can be achieved.

The following are commonly accepted practices:

- Keywords and commands are capitalised.
- One statement is given in each line.
- Indent sections dependent on a condition (for example, IF) or that will be iterated.

Below are the main commands that will be used with pseudocode.

INPUT
read data
 	
OUTPUT/PRINT
print data
	INTEGER Name[N]
	Reserves N locations to store values.
		IF (a given statement is true)
		THEN (perform action 1)
		ELSE (perform action 2)
		ENDIF

DO
<procedure>
WHILE (condition is true)
ENDWHILE
	FOR variable = first value TO last value
	<procedure>
	ENDFOR
		REPEAT
		<procedure>
		UNTIL (condition is true)

+ addition – subtraction / division * multiplication ^ power	< less than > greater than ≤ less than or equal to ≥ greater than or equal to	

WORKED EXAMPLE 29 Interpreting pseudocode

a Describe what the program below is doing.

```
INTEGER array [10]                          line 1
count ← 0                                   line 2
FOR loop = 1 to 10                          line 3
    IF loop MOD 3 = 1                       line 4
    THEN                                    line 5
        count ← count + 1                   line 6
        array[count] ← loop                 line 7
    ENDIF                                   line 8
ENDFOR                                      line 9
PRINT "The numbers divisible by 3 are"      line 10
FOR numeral = 1 TO count                    line 11
    PRINT array [numeral]                   line 12
ENDFOR                                      line 13
```

b State the output for each cycle of the PRINT command.

Steps	Working
a Describe what is happening in each section.	INTEGER array [10] is initialised to store up to 11 numbers. (line 1) A counter is set to 0. (line 2) A loop from 1 to 10 is to be used. (line 3) Each of the numbers 1 to 10 is divided by 3 and if the remainder is 1, the counter is incremented by 1 and the value stored in the array. (lines 4 – 7) When the loop count is 10, the procedure finishes. (line 9) A descriptive heading for the output is printed. (line 10) The numbers from 1 to 10 that have remainder 1 after division by 3 are printed. (lines 11, 12, 13)
b 1 Show the output from the first PRINT statement.	The output is *The numbers divisible by 3 are*
2 Write the output from the second PRINT command.	The output is 1, 4, 7, 10

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WORKED EXAMPLE 30 Pseudocode involving addition of numbers

Write pseudocode that finds the sum of 10 numbers and prints the result.

Steps	Working

1 Initialise the sum of the numbers and set up a loop to read the numbers.	There are 10 numbers to read, one at a time.
	current sum ← 0
	FOR count = 1 TO 10
	  INPUT next number from list
	  procedure
	ENDFOR

2 The procedure keeps a running total of the numbers.	current sum ← 0 FOR count = 1 TO 10 INPUT next number from list current sum ← current sum + next number from list ENDFOR
3 Print the result.	PRINT "The sum of the 10 numbers is" PRINT current sum

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WORKED EXAMPLE 31 Pseudocode using random numbers, modulus, product

The function A MOD B returns the remainder after A is divided by B, for example, 8 MOD 3 = 2.

The function RAND(A) generates a random whole number from 1 to A, for example, RAND(5) = 3.

Using these functions, write pseudocode that generates 3 random even numbers from 2 to 10 and then finds their product.

Steps	Working
1 Initialise the product of the numbers and create a loop for the random numbers.	There are 3 random numbers to generate. current product ← 1 FOR count = 1 TO 3 procedure ENDFOR

2 Check to see if the random number is even. Print the even number.	A number is even if the remainder after division by 2 is 0.
	current product ← 1
	FOR count = 1 TO 3
	  REPEAT
	  random number ← RAND(10)
	  remainder ← random number MOD 2
	  UNTIL remainder = 0
	  PRINT random number
	ENDFOR

3 Update the product and output the result.	current product ← 1
	FOR count = 1 TO 3
	  REPEAT
	  random number ← RAND(10)
	  remainder ← random number MOD 2
	  UNTIL remainder = 0
	  PRINT random number
	  current product ← current product * random number
	ENDFOR
	PRINT current product

WORKED EXAMPLE 32 Integer array

Use pseudocode to store the numbers 1 to 100 in reverse order.

Steps	Working
1 Store the numbers as an array.	Use INTEGER number[100] and a loop to store the values. INTEGER number[100] FOR count = 1 TO 100 number[count] ← count ENDFOR

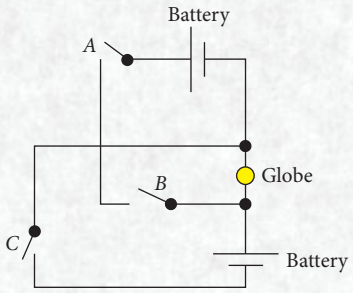
2 Create a second array and store the numbers in reverse order.	The second array will be the elements of the first array in reverse order. To do this, subtract each number from 101.
	INTEGER number[100]
	FOR count = 1 TO 100
	  number[count] ← count
	ENDFOR
	INTEGER reverse number[100]
	FOR count = 1 TO 100
	  reverse number[count] ← number[101 – count]
	ENDFOR

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VCE QUESTION ANALYSIS

A VCE-style question (8 marks)

Below is a circuit consisting of three switches, *A*, *B*, *C*, two batteries and one globe.



a Taking 0 to mean an open switch and 1 to mean a closed switch, complete a truth table to show the status of the globe (0 = unlit, 1 = lit). 3 marks

b Use your answers in part **a** to complete the K-map below. 2 marks

BC / A				

c Use the K-map in part **b** to find the SOP in simplified form. 2 marks

d Draw the simplified logic gate diagram for the circuit. 1 mark

Video
VCE question analysis: Logic and algorithms

Reading the question

- Read each question carefully, because it covers different areas of the topic and each answer is required for subsequent questions.
- Understand keywords and the methods to be used. For instance, K-map means Karnaugh map.
- Be clear about the notation of the problem. For example, the truth table will consist of 0/1 and not T/F.
- Boolean laws could be used in part **d** to ensure a simplified logic gate is obtained.

Thinking about the question

- In part **a**, construct the truth table by deciding whether the circuit is complete for the given combination of open/closed switches.
- The K-map in part **b** directly uses the results from part **a**.
- To find the sum of products (SOP), ensure that the minimum number of groups consisting of 2^n numbers are chosen. Simplify the SOP expression.
- For part **c**, find groups where the variables do not change.

Worked solution (✓ = 1 mark)

a Complete columns 1, 2 and 3 with the 8 combinations representing the status (open/closed) of each switch.

Use the circuit diagram to determine the output (globe on/off) for each combination of open and closed switches.

✓ (for row 1 column 1 headings)

✓✓ (for cell entries)

A	B	C	Output
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

b Complete the K-map using the output values from part **a**.

✓ (for row 1 column 1 headings)

✓ (for correct cell values)

A \ BC	00	01	11	10
0	0	1	1	0
1	1	1	1	1

c For the blue box, $C = 1$ does not change as we go across and down. So C.

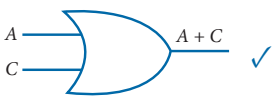
For the red, $A = 1$ is not changing, so A.

Hence the SOP is $A + C$. ✓

A \ BC	00	01	11	10
0	0	1	1	0
1	1	1	1	1

✓

d To construct the circuit, OR is required.



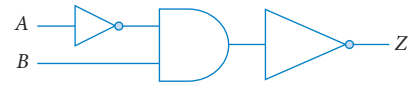
✓

EXERCISE 4.7 Algorithms and pseudocode

ANSWERS p. 576

Recap

1

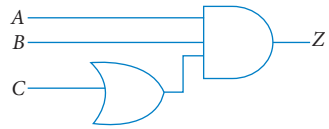


The output, Z, is

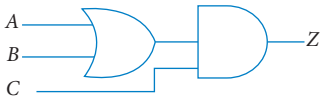
A $A\overline{B}$ **B** $\overline{A + B}$ **C** $\overline{A\overline{B}}$ **D** $\overline{A}\overline{C}$ **E** $A + \overline{B}$

2 Which logic gate represents $A(B + C)$?

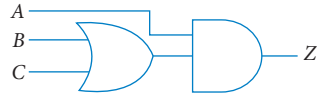
A



B



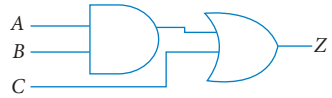
C



D



E

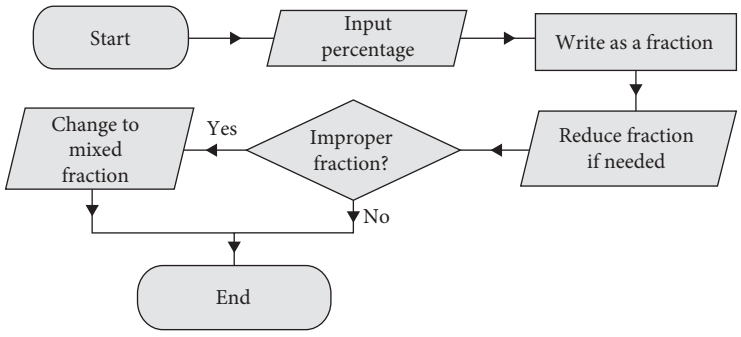


Mastery

3 WORKED EXAMPLE 27 Draw a flowchart to boil an egg for 3 minutes. Include the following points:

- Check if there are eggs.
- Place egg in saucepan, fill with water and boil.
- Stop after 3 minutes.
- Check the boiled egg. If the shell is broken, start over.

4 Choose the correct statement that applies to the following flowchart.



A Errors are likely to occur at the decision box.

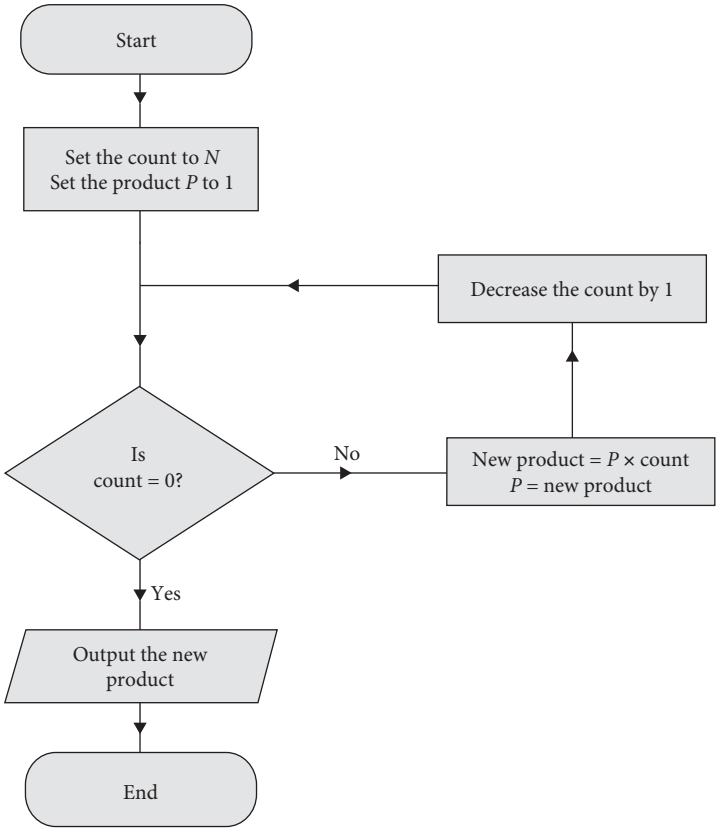
B For an improper fraction, the algorithm is always using 4 stages (excluding Start/End).

C The algorithm can handle all percentages.

D Excluding Start/End, ending with a mixed fraction can have 4 stages.

E The final stage (excluding End) is always the decision box.

5 WORKED EXAMPLE 28



a In the flowchart shown, state the progressive values of **count** and **new product** when $N = 4$.

b Explain what the flowchart is doing.

6
```
fact ← 1
FOR loop = 1 to 3
   fact ← fact + fact * loop
ENDFOR
PRINT fact
```

The output produced by the above algorithm is

A 6 **B** 18 **C** 24 **D** 36 **E** 39

7 WORKED EXAMPLE 29

a Describe what the program shown is doing.

b State the output for each cycle of the PRINT command.

INTEGER V[3]	line 1
count ← 3	line 2
FOR loop = 1 TO count	line 3
length ← loop	line 4
width ← 2*length	line 5
height ← length^2	line 6
V[loop] ← length*width*height	line 7
ENDFOR	line 8
FOR loop = 1 TO count	line 9
PRINT V[loop]	line 10
ENDFOR	line 11

8 WORKED EXAMPLE 30 Write pseudocode that finds the sum of the squares of 5 random natural numbers less than 10 and prints the result.

Exam practice

9 WORKED EXAMPLE 31 (3 marks) Write pseudocode instructions that create, store and print 5 sets of 2 random whole numbers from 1 to 20 so that the first number is even and the second number is odd.

10 Which of the following statements is correct?

A 5 MOD8 = 3.

B A ← 6/2 means A = 3.

C INTEGER[6] reserves 6 spaces to store numbers.

D RAND(2) can return any of 0, 1 or 2.

E The output from PRINT 1 + 2 is 1 + 2.

11 Given that $B > A$, the maximum difference between RAND(A) and RAND(B) is

A $B - 1$ **B** $A - B$ **C** $B - A$ **D** 1 **E** $A + B$

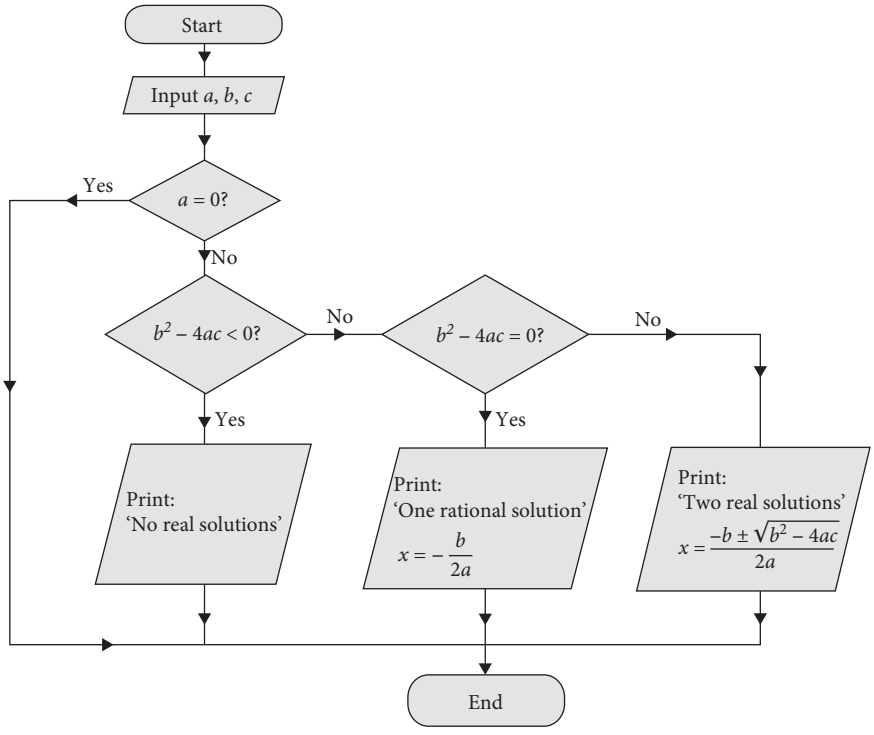
12 WORKED EXAMPLE 32 (3 marks) Use pseudocode to store, in reverse order, all whole numbers from 1 to 50 that are divisible by 3.

13 (4 marks) Write a program using pseudocode that reads the first 10 terms of the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13 … and finds the ratio of each term to the next.

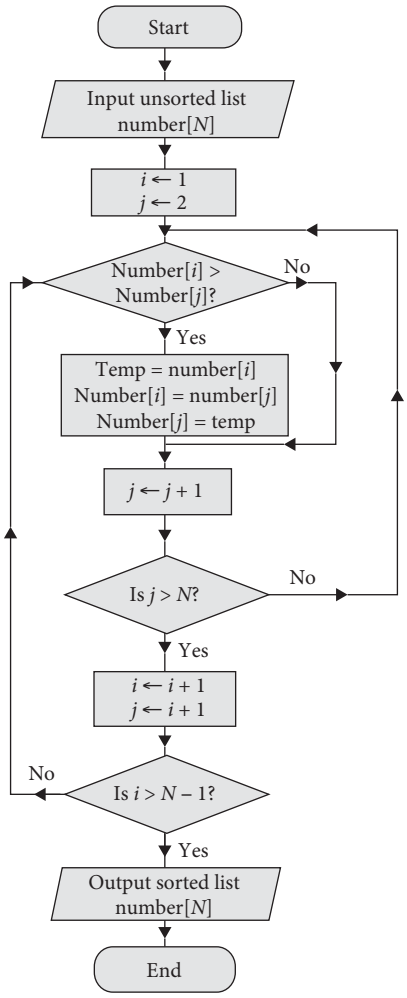
14 (4 marks) Write an algorithm that describes the following game.

A random number from 1 to 10 inclusive is generated, and if it is even, the game stops. If the number is odd, a random number from 1 to 2 inclusive is used. If the number is 1, the game stops, otherwise the game begins again.

15 (4 marks) Use the flowchart to write pseudocode that shows the solutions to a quadratic equation.



16 (4 marks) This flowchart describes an algorithm to sort N numbers in ascending order.



Suppose the unsorted list is 4, 3, 2, 1. Show the sequence of partially ordered lists created by the algorithm before the final output of 1, 2, 3, 4.

17 (8 marks) The only divisors of a prime number, P, are 1 and P. The following algorithm uses pseudocode checks to see if a number is prime.

INPUT P	line 1
count ← 0	line 2
FOR I = 2 TO P – 1	line 3
IF P MOD I = 0	line 4
THEN count ← count + 1	line 5
ENDIF	line 6
ENDFOR	line 7
IF count = 0	line 8
THEN PRINT "The number is prime."	line 9
ELSE PRINT "The number is not prime."	line 10
ENDFOR	line 11

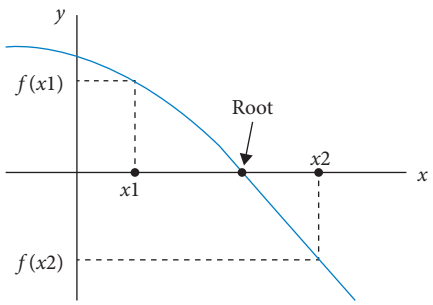
a How many cycles are carried out by the loop in line 3? 2 marks

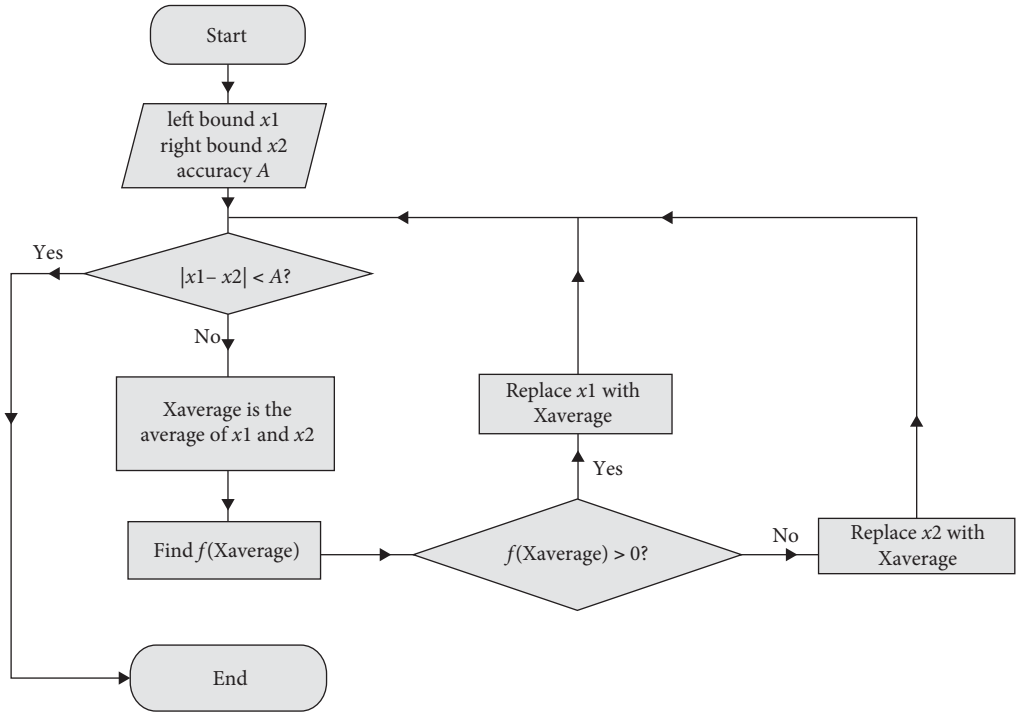
b Explain why it is not necessary for line 3 to be FOR I = 1 TO P. 2 marks

c Explain the conditional statement in line 8. 2 marks

d Discuss problems that could arise when using this algorithm for $P = 2$. 2 marks

18 (6 marks) The x-intercept (root) of the function $y = f(x)$ can be approximated by using the bisection method. This method takes two x values, $x1$ and $x2$, that are to the left and to the right of the desired x-intercept respectively and where $f(x1)$ and $f(x2)$ have opposite signs. The average (Xaverage) of $x1$ and $x2$ is used to find $f(x1)$ and $f(x2)$, which determines the new x-bounds. With each iteration, the 2 bounds converge to the exact root. The flowchart below illustrates the method. For this problem we have assumed that $f(x1) > 0$ and $f(x2) < 0$ as shown in the diagram, but the method could be extended to cover the reverse case.





a Develop an algorithm of the bisection method using pseudocode.

You might need to use the function, $|X|$, which takes the positive value of X. For example, $|-3| = 3$. 3 marks

b Apply your algorithm to find a root of the function $y = 3x - x^2$. Use start values $x1 = 1$, $x2 = 4$ and A = 0.1. (Note: the exact root is $x = 3$.) 3 marks

4 Chapter summary

Logical arguments

- When something is either true or false, it is a **statement**. ('Today is hot.')
- True and false are **propositional constants**.
- A **premise** is a statement from which a **conclusion** is drawn.
  ('All circle shapes are round. A coin is a circle shape. Therefore, a coin is round.')

Logical connectives (logical operators)

Logical connective	Meaning	Symbol
Conjunction	AND	$\wedge$
Negation	NOT	$\neg$
Disjunction	OR	$\vee$
Conditional	if then	$\rightarrow$
Biconditional	if and only if	$\leftrightarrow$
Equivalence	is identical to	$\equiv$

Compound propositions

An **atomic proposition** is a single statement. ('It rains in winter.')

A **compound proposition** consists of at least 2 atomic propositions connected by **logical connectives**.

- A = 'Yani likes tea', B = 'Kim likes apples'
  $A \wedge B$ = 'Yani likes tea **and** Kim likes apples'
- A = 'I go to the market', B = 'I watch TV'
  $A \vee B$ = 'I go to the market **or** I watch TV'
- A = 'It will rain today', $\neg A$ = 'It will **not** rain today'
- A = 'I use an umbrella', B = 'I will not get wet'
  $A \rightarrow B$ = '**If** I use an umbrella, **then** I will not get wet'
- A = '$x + 3 = 8$', B = '$x = 5$'
  $A \rightarrow B$ = '**If** $x + 3 = 8$, **then** $x = 5$'
  $B \rightarrow A$ = '**If** $x = 5$, **then** $x + 3 = 8$'
- A = '$4 + 5 = 9$', B = '$3 + 6 = 9$'
  $A \equiv B$

Truth table

- A truth table displays all possible truth values and the value of the logical operator(s) used.

A	B	$A \vee B$
False	False	False
False	True	True
True	False	True
True	True	True

- A proposition that is always true is a **tautology**. ('If $x > 1$, then $2x > 1$')
- A proposition that is always false is a **contradiction**. ('All squares are triangles.')

Boolean laws

- The Boolean system uses 1 and 0 in place of true and false. The logical connectives AND, OR and NOT are replaced by •, + and – respectively.

A	B	$A \vee B$
0	0	0
0	1	1
1	0	1
1	1	1

- **Boolean algebra** includes the following laws:

1 $A \bullet A = A$	**7** $A \bullet \overline{A} = 0$
2 $A + A = A$	**8** $A + \overline{A} = 1$
3 $A \bullet 1 = A$	**9** $\overline{(\overline{A})} = A$
4 $A + 1 = 1$	**10** $A + \overline{A} \bullet B = A + B$
5 $A \bullet 0 = 0$	**11** $\overline{A} + A \bullet B = \overline{A} + B$
6 $A + 0 = A$	

Associative laws:	$(A + B) + C = (A + B + C)$	$(A \bullet B) \bullet C = A \bullet (B \bullet C)$
Commutative laws:	$A + B = B + A$	$A \bullet B = B \bullet A$
Distributive laws:	$A \bullet (B + C) = A \bullet B + A \bullet C$	$A + (B \bullet C) = (A + B)(A + C)$
De Morgan's laws:	$\overline{A \bullet B} = \overline{A} + \overline{B}$	$\overline{A + B} = \overline{A} \bullet \overline{B}$

Sum of products

The **sum of products (SOP)** is the Boolean function obtained by taking the disjunction (OR) of the output values of each row. The Boolean laws and properties are then applied to SOP to obtain the minimised function, as is shown in the example below.

A	B	$\overline{A}(A + B) + B$	
0	0	0	$\overline{A}\overline{B}$
0	1	1	$\overline{A}B$
1	0	0	$A\overline{B}$
1	1	1	AB

$$\text{SOP} = \overline{A}B + AB = \left(\overline{A} + A\right)B = 1 \bullet B = B$$

Binary numbers

The binary digits are represented by 0 and 1, called bits, and a combination of binary digits is a **binary number**.

A base 10 number is expressed in base 2 by keeping the remainders after repeated division by 2.

For example, the sequence of remainders after 53 is divided by 2 is 1,0,1,0,1,1. The binary number for 53 is the reverse of the sequence of remainders, namely, 110101.

To convert from base 2 to base 10, express the binary number as the sum of powers of 2 whose coefficients comprise the binary number.

For example, $110001 = 1 \times 2^5 + 1 \times 2^4 + 0 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 49_{10}$.

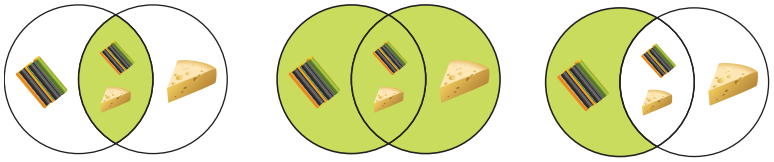
In a binary fraction, digits to the right of the binary point represent negative powers of 2.

Thus, $0.110_2 = 1 \times 2^{-1} + 1 \times 2^{-2} + 0 \times 2^{-3} = 0.75_{10}$.

Boolean operators in search engines and databases

The Boolean operators AND, OR and NOT are used to search the internet by restricting or broadening the available records in an organised way.

- **AND** finds results containing all items.
- **OR** finds either of the 2 defined items or both.
- **NOT** finds all items other than the defined item(s).



AND 'chalk **and** cheese'

OR 'chalk **or** cheese'

NOT 'chalk but **not** cheese'

Karnaugh maps

A **Karnaugh map** or **K-map** displays the Boolean values of the variables and their products. The second table below is a K-map corresponding to the truth table above it.

A	B	C	$A\bar{B}C$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

A \ BC	00	01	10	11
0	0	0	0	0
1	0	1	0	0

A K-map is simplified to minimise its sum of products by grouping them so that

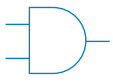
- When one dimension of the table represents more than one variable (as with BC in the example above), the order of the values follows Gray code.
- The number of cells with a value of 1 is a power of 2.
- Each square or rectangle of '1' cells is as large as possible, but with no zero cells.
- Groups can overlap, to make them larger.

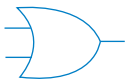
Y and Z do not change as we go from top to bottom in the vertical group and X does not change as we go from left to right in the horizontal group. SOP is $X + YZ$.

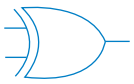
X \ YZ	00	01	11	10
0	0	0	1	0
1	1	1	1	1

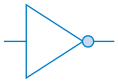
Digital circuits

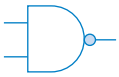
- A digital circuit comprises a set of **logic gates** that perform Boolean logic. The Boolean variables 0 and 1 are also known as 'lows' and 'highs' respectively. For 2 input variables, the logic gates are the following:

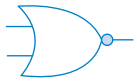

AND

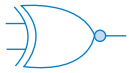

OR


XOR (exclusive OR)

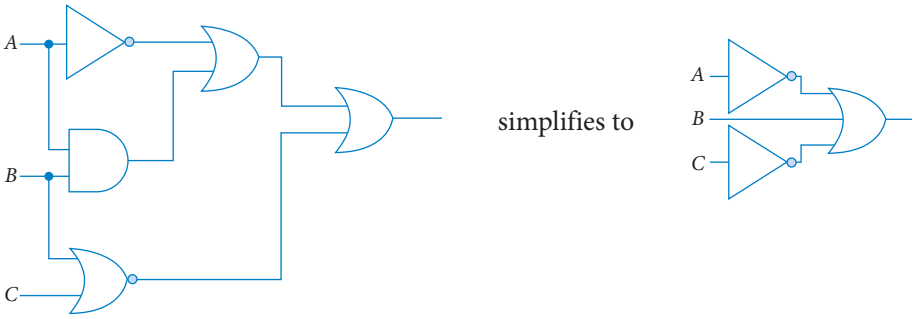

NOT


NAND (NOT AND)


NOR (NOT OR)


XNOR (NOT exclusive OR)

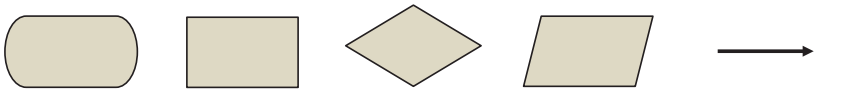
- There is only one output variable. The NOT logic gate (**inverter**) has only one input.
- A simplified circuit consists of the minimum number of logic gates possible.



Algorithms, flowcharts and pseudocode

An **algorithm** is a prescribed set of instructions that provides the solution to a problem. It can be presented as a schematic diagram (flowchart) or as a sequence of text-based instructions (pseudocode).

A **flowchart** uses a common set of symbols to display instructions.


Start/end | Process | Decision | Input/output | Direction connector

Pseudocode has its own set of commonly used commands, procedures and operators. These include

- INPUT, OUTPUT, PRINT, INTEGER Name[size]
- A ← B (A is assigned the value of B)
- IF <condition> THEN <procedure 1> ELSE <procedure 2> ENDIF
- DO WHILE ENDWHILE (A loop that is repeated while a given condition remains true)
- FOR TO ENDFOR (A loop that repeats a procedure a given number of times)
- REPEAT UNTIL (Similar to FOR TO ENDFOR. The loop finishes when the given condition is satisfied).
- + (addition), – (subtraction), / (division), * (multiplication), ^ (power)
- < (less than), > (greater than), ≤ (less than or equal to), ≥ (greater than or equal to)

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 (2 marks) Let A be the statement, 'I am thirsty' and let B be the statement, 'I drink'.
Write each of the following in sentence form.

a $A \leftrightarrow B$. 1 mark

b $\neg A \rightarrow \neg B$. 1 mark

2 (2 marks) Construct the truth table for $\neg A \rightarrow B$.

3 (2 marks) Use Boolean algebra to simplify $A(\overline{A + \overline{B}})$.

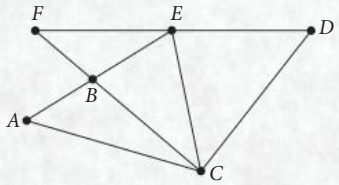
4 (2 marks) The first term of an arithmetic sequence is –100 and the 51st term is 100.

a What is the value of d, the common difference of the arithmetic sequence? 1 mark

b A geometric series whose first term is 3 has a common ratio $\frac{1}{d}$.
Determine the infinite sum of the series. 1 mark

5 (2 marks)

a Explain why an Eulerian circuit can be formed on this graph. 1 mark



b Write down a path that describes an Eulerian circuit. 1 mark

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 Consider the logical argument below.

All trees have leaves. 1
My book has leaves. 2
Therefore my book is a tree. 3

The conclusion and a premise are, respectively

A 1 and 3 **B** 2 and 3 **C** 3 and 1
D 2 and 1 **E** 1 and 2

2 Let A be 'I do not run' and let B be 'I am tired'.

In symbolic form, which of the following cannot be true?

A $\neg A \rightarrow B$ **B** $A \vee B$ **C** $B \rightarrow \neg A \wedge A$
D $\neg B \rightarrow \neg A$ **E** $B \leftrightarrow A$

3 If $a_2 + 1001_2 = 20_{10}$, then a is

A 1001 **B** 1010 **C** 1011 **D** 1101 **E** 1110

4 The sum of the degrees of the vertices in a graph consisting of straight edges and 2 loops is 16. The number of straight edges in the graph is

A 4 **B** 6 **C** 7 **D** 8 **E** 10

5 If $U = \{x: 0 < x \leq 15, x \in N\}$, A = {even numbers from 2 to 12 inclusive} and B = {multiples of 3 from 3 to 15 inclusive}, then $n((A \cup B)')$ is

A 3 **B** 4 **C** 5 **D** 6 **E** 7

Section B 5 questions

15 marks

1 (3 marks) When A and B are integers with $A > B > 0$, the fraction $\frac{A}{B}$ can be written as a mixed fraction $a\frac{b}{B}$, where a and b are integers, $0 \leq b < B$.

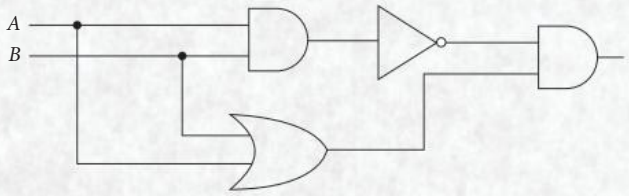
Consider the following 2 commands defined in pseudocode:

a = A DIV B and b = A MOD B

a Use pseudocode to write an algorithm that uses t minutes as input, converts it to H hours and T minutes and outputs the results. 2 marks

b Verify that your algorithm works using $t = 142$. 1 mark

2 (5 marks)



a Complete the truth table for the circuit shown above. 3 marks

b State the SOP. 1 mark

c Draw a simplified circuit to replace the circuit above. 1 mark

3 (2 marks) Prove that if $m \in N$ is divisible by 5, then $(m + 1)^2 - 1$ is also divisible by 5.

4 (2 marks) A planar graph has 7 edges and there are twice as many vertices as faces. Draw such a planar graph.

5 (3 marks) When twins Janet and Marty were born, their parents opened bank accounts for them, depositing the same amount of money into each account at the start. Each subsequent year, Janet's account increased by $750, and Marty's account increased by 5%. The deposits continued until Janet turned 16 years and her total amount of money was 2.2 times her initial amount.

Calculate, corrrect to the nearest dollar, the amount of money each twin had at this time.

MATRICES

5

Study Design coverage

Nelson MindTap chapter resources

5.1 Matrix operations and algebra

Matrix operations

Using CAS 1: Matrix operations

Matrix algebra

5.2 Determinants

Diagonal matrices

Triangular matrices

General determinants

Using CAS 2: Determinants

5.3 Matrix equations

Using CAS 3: Simple matrix equations

Inverses and matrix equations

Using CAS 4: Matrix equations

VCE question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2



Study Design coverage

UNIT 1, AREA OF STUDY 2: DISCRETE MATHEMATICS

Matrices

- matrix notation, dimension and the use of matrices to represent data
- matrix operations and algebra
- determinants and matrix equations, and simple applications.

VCE Mathematics Study Design 2023–2027 p. 60, © VCAA 2022

Video playlists (4):

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5.2 Determinants
5.3 Matrix equations
VCE question analysis Matrices

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Nelson MindTap

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5.1 Matrix operations and algebra

Video playlist Matrix operations and algebra

A **matrix** is a rectangular array of numbers or symbols enclosed by brackets, symbolised by a capital letter; for example, $A = \begin{bmatrix} 2 & -4 & 60 & 1 \\ 5 & 2.8 & -70 & 0 \end{bmatrix}$. The **dimensions of the matrix** are the number of rows m and columns n, shown as $m \times n$. The number in row i and column j of the matrix A is shown as a_{ij}. It is an **element** of the matrix. In general, $A = (a_{ij})$, $B = (b_{ij})$, etc.

The matrix A above is a 2×4 matrix with $a_{13} = 60$.

A **square matrix** with n rows and n columns has dimension n.

A **column matrix** is an $n \times 1$ matrix: it has only one column; for example, $X = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$.

A **row matrix** is a $1 \times n$ matrix: it has only one row; for example, $B = \begin{bmatrix} 0 & -0.27 & 5 \end{bmatrix}$.

Data is often shown as a matrix. Linear equations, vectors, prices or other quantities are often shown as matrices. Row and column matrices are usually shown with only one subscript number, so $x_2 = -4$.

WORKED EXAMPLE 1 Matrix dimensions and elements

Consider the matrices $P = \begin{bmatrix} 2 & 4 & 6 & 3 \\ 1 & 0 & -1 & -2 \\ -3 & 5 & 4 & -4 \end{bmatrix}$, $Q = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$ and $R = \begin{bmatrix} 6 \\ 0 \\ -6 \end{bmatrix}$.

a What are the dimensions of each matrix above?

b What are p_{12}, p_{32}, q_{21} and r_3?

Steps	Working
a Write the number of rows and columns.	P is a 3×4 matrix, Q is a square matrix of dimensions 2×2 and R is a column matrix with dimensions 3×1.
b Select the correct rows and columns.	$p_{12} = 4$, $p_{32} = 5$, $q_{21} = 1$ and $r_3 = -6$

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Worksheet
Addition and subtraction of matrices

Matrix operations

Consider the numbers of paint kits ordered from a supplier by 3 schools (A, B and C) for 2 months: January (J) and February (F). The paint kits are 'standard (S)' and 'deluxe (D)'. The number of kits for each school in the order A, B and C for standard and deluxe were as follows. Matrix J has them marked below and to the right.

$$J = \begin{bmatrix} 26 & 24 \\ 34 & 20 \\ 13 & 25 \end{bmatrix} \begin{matrix} A \\ B \\ C \end{matrix}, \quad F = \begin{bmatrix} 17 & 15 \\ 15 & 8 \\ 5 & 12 \end{bmatrix} \begin{matrix} A \\ B \\ C \end{matrix}$$
$$\begin{matrix} S & D \end{matrix} \qquad \begin{matrix} S & D \end{matrix}$$

To get T, the matrix for the total number of paint kits ordered, we add the elements of each matrix together.

If February's orders were tripled, we would multiply each element in F by 3.

Basic matrix operations

For addition and subtraction, matrices must be of the same dimensions.

- To **add matrices** of the same dimensions, add the corresponding elements.
- To **subtract matrices** of the same dimensions, subtract the corresponding elements.
- To find the **negative of a matrix**, change the sign of every element.
- To **multiply a matrix by a real number** (a scalar), multiply every element by the number.

Given $A = (a_{ij})$, $B = (b_{ij})$ and $c \in R$:

- $A + B = (a_{ij} + b_{ij})$
- $A - B = (a_{ij} - b_{ij})$
- $-A = (-a_{ij})$
- $cA = (ca_{ij})$

The negative has to have every sign changed so that $A - B = A + (-B)$.

Every element $0_{ij} = 0$ for the **zero matrix**, 0. The dimension of the zero matrix is usually taken to be the same as other matrices in an expression.

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WORKED EXAMPLE 2 Basic operations

Consider the matrices $P = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -5 & 4 \end{bmatrix}$, $Q = \begin{bmatrix} -3 & 2 & 0 \\ 1 & -4 & -1 \end{bmatrix}$ and $R = \begin{bmatrix} 6 & -2 & 5 \\ 0 & 3 & -3 \end{bmatrix}$.

Find $2P$, $-Q$, $P + Q$, $Q + (-Q)$, $R - P$ and $3Q + 2R$.

Steps	Working
1 Multiply every element by 2.	$2P = \begin{bmatrix} 2 \times 1 & 2 \times 0 & 2 \times 2 \\ 2 \times 3 & 2 \times (-5) & 2 \times 4 \end{bmatrix}$ $= \begin{bmatrix} 2 & 0 & 4 \\ 6 & -10 & 8 \end{bmatrix}$
2 Change the sign of every element.	$-Q = -\begin{bmatrix} -3 & 2 & 0 \\ 1 & -4 & -1 \end{bmatrix}$ $= \begin{bmatrix} 3 & -2 & 0 \\ -1 & 4 & 1 \end{bmatrix}$
3 Add corresponding elements.	$P + Q = \begin{bmatrix} 1 + (-3) & 0 + 2 & 2 + 0 \\ 3 + 1 & -5 + (-4) & 4 + (-1) \end{bmatrix}$ $= \begin{bmatrix} -2 & 2 & 2 \\ 4 & -9 & 3 \end{bmatrix}$
4 Add the matrices. $Q + (-Q) = 0$, no matter what the particular matrix Q happens to be.	$Q + (-Q) = \begin{bmatrix} -3 & 2 & 0 \\ 1 & -4 & -1 \end{bmatrix} + \begin{bmatrix} 3 & -2 & 0 \\ -1 & 4 & 1 \end{bmatrix}$ $= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $Q + (-Q) = 0$
5 Subtract corresponding elements.	$R - P = \begin{bmatrix} 6 - 1 & -2 - 0 & 5 - 2 \\ 0 - 3 & 3 - (-5) & -3 - 4 \end{bmatrix}$ $= \begin{bmatrix} 5 & -2 & 3 \\ -3 & 8 & -7 \end{bmatrix}$
6 Perform the multiplications first, then the addition.	$3Q + 2R = 3\begin{bmatrix} -3 & 2 & 0 \\ 1 & -4 & -1 \end{bmatrix} + 2\begin{bmatrix} 6 & -2 & 5 \\ 0 & 3 & -3 \end{bmatrix}$ $= \begin{bmatrix} -9 & 6 & 0 \\ 3 & -12 & -3 \end{bmatrix} + \begin{bmatrix} 12 & -4 & 10 \\ 0 & 6 & -6 \end{bmatrix}$ $= \begin{bmatrix} 3 & 2 & 10 \\ 3 & -6 & -9 \end{bmatrix}$

The negative of *any* matrix is the **additive inverse** of the matrix: $A + (-A) = 0$.

The paint kits mentioned earlier each contain tubes of paint, nylon and sable hair brushes. The compositions and prices the schools pay per kit are broken down as shown below.

Kit	Paint tubes	Nylon brushes	Sable brushes	Kit price
Standard	5	3	1	$11
Deluxe	9	2	3	$25

9780170448338

$J = \begin{bmatrix} 26 & 24 \\ 34 & 20 \\ 13 & 25 \end{bmatrix} \begin{matrix} A \\ B \\ C \end{matrix}$ has the orders for each school. $P = \begin{bmatrix} 11 \\ 25 \end{bmatrix} \begin{matrix} S \\ D \end{matrix}$ has the prices.

$\begin{matrix} S & D \end{matrix}$

The order of the kits in the rows of J is the same as the order of their prices in the column of P. The bill for each school in J is the sum of the products of the numbers in the rows of J and the numbers in the column of P. We can write the bill matrix as

$$B = \begin{bmatrix} 26 \times 11 + 24 \times 25 \\ 34 \times 11 + 20 \times 25 \\ 13 \times 11 + 25 \times 25 \end{bmatrix} = \begin{bmatrix} 886 \\ 874 \\ 768 \end{bmatrix}$$

so the bills are \$886, \$874 and \$768 for schools A, B and C respectively.

$C = \begin{bmatrix} 5 & 3 & 1 \\ 9 & 2 & 3 \end{bmatrix} \begin{matrix} S \\ D \end{matrix}$ shows the prices of the contents of the kits.

The rows of J are in the same order as the columns of C. The quantity of, say, sable brushes in school B's order is the sum of the products of the numbers in row 2 of J with the numbers in column 3 of C: $34 \times 1 + 20 \times 3 = 94$ sable brushes. It makes sense to put this calculation in q_{23} of the matrix Q of the quantities of paints and brushes in the kits for each school. Performing all the calculations

gives $Q = \begin{bmatrix} 346 & 126 & 98 \\ 350 & 142 & 94 \\ 290 & 89 & 88 \end{bmatrix}$.

The product of the numbers in the ith row of J and the jth column of C are added to give q_{ij} for i from 1 to 3 and j from 1 to 3.

Matrix multiplication

The product $A \times B$ is possible if the number of columns in A is the same as the number of rows in B. In that case, they are **conformable matrices**.

The $m \times n$ matrix A multiplied by the $n \times p$ matrix B gives the $m \times p$ matrix AB.

We **multiply matrices** A and B by multiplying the elements of each row of A by the elements of each column of B and then adding them together.

The symbol Σ means 'the sum of', so we can write that for the matrix multiplication $AB = C$, $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$, which reads 'the element in the ith row and jth column of the product matrix C is the sum of the products of the elements in ith row of A and corresponding elements in the jth column of B'.

WORKED EXAMPLE 3 Matrix multiplication

Consider the matrices $P = \begin{bmatrix} 5 & 2 & -1 \\ 3 & -4 & 0 \end{bmatrix}$ and $Q = \begin{bmatrix} 6 & -2 \\ 4 & -5 \\ 1 & -3 \end{bmatrix}$.

Find PQ and QP.

Steps	Working
1 Write the product PQ.	$PQ = \begin{bmatrix} 5 & 2 & -1 \\ 3 & -4 & 0 \end{bmatrix} \times \begin{bmatrix} 6 & -2 \\ 4 & -5 \\ 1 & -3 \end{bmatrix}$
2 Each element is done as *row* × *column*.	$= \begin{bmatrix} 5 \times 6 + 2 \times 4 + (-1) \times 1 & 5 \times (-2) + 2 \times (-5) + (-1) \times (-3) \\ 3 \times 6 + (-4) \times 4 + 0 \times 1 & 3 \times (-2) + (-4) \times (-5) + 0 \times (-3) \end{bmatrix}$
	$= \begin{bmatrix} 37 & -17 \\ 2 & 14 \end{bmatrix}$

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Worksheets
Multiplying matrices
Matrix multiplications

3 Write the product QP.

$$QP = \begin{bmatrix} 6 & -2 \\ 4 & -5 \\ 1 & -3 \end{bmatrix} \times \begin{bmatrix} 5 & 2 & -1 \\ 3 & -4 & 0 \end{bmatrix}$$

4 Each element is done as *row* × *column*.

$$= \begin{bmatrix} 6\times 5+(-2)\times 3 & 6\times 2+(-2)\times(-4) & 6\times(-1)+(-2)\times 0 \\ 4\times 5+(-5)\times 3 & 4\times 2+(-5)\times(-4) & 4\times(-1)+(-5)\times 0 \\ 1\times 5+(-3)\times 3 & 1\times 2+(-3)\times(-4) & 1\times(-1)+(-3)\times 0 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 20 & -6 \\ 5 & 28 & -4 \\ -4 & 14 & -1 \end{bmatrix}$$

PQ and *QP* are not the same. The order of the product of matrices is important. The results of products in different orders may be different.

Worked example 3 shows that the products of matrices in different orders may be different. They might not even be of the same dimensions!

Consideration of $A = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 \\ 0 & 4 \end{bmatrix}$ shows A and B are *not* conformable, but B and A are conformable. That is, AB does not exist, but $BA = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ does.

Matrix operations follow the same order as ordinary arithmetic, normally working from left to right.

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WORKED EXAMPLE 4 Multiple operations

$A = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ and $C = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

Find

a $A(B + C)$ **b** $AB + C$

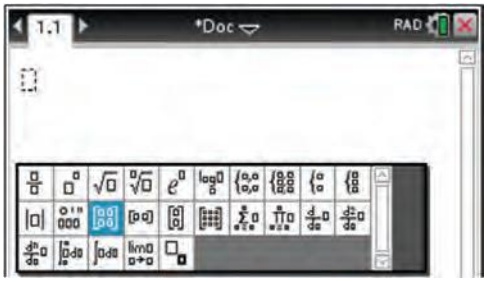
Steps	Working
a 1 Perform the operation in brackets first.	$A(B+C) = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}\left(\begin{bmatrix} 5 \\ 6 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix}\right)$ $= \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 8 \end{bmatrix}$
2 Perform the remaining operation and simplify.	$= \begin{bmatrix} 2\times 2-1\times 8 \\ 0\times 2+3\times 8 \end{bmatrix}$ $= \begin{bmatrix} -4 \\ 24 \end{bmatrix}$
b 1 Perform multiplication before addition.	$AB + C = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ $= \begin{bmatrix} 2\times 5-1\times 6 \\ 0\times 5+3\times 6 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ $= \begin{bmatrix} 4 \\ 18 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix}$
2 Perform the remaining operation and simplify.	$= \begin{bmatrix} 1 \\ 20 \end{bmatrix}$

USING CAS 1 Matrix operations

Consider the matrices $A = \begin{bmatrix} 3 & 4 \\ 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -7 \\ 2 & -5 \end{bmatrix}$, $C = \begin{bmatrix} 5 & -7 \\ 2 & -3 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Find $A + B$, $2C - D$, AB, BA, DA, BD, CD and BC.

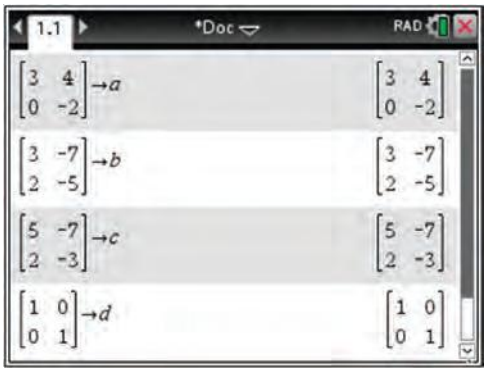
TI-Nspire



1 Press the **template** key and select the **2×2** matrix template.

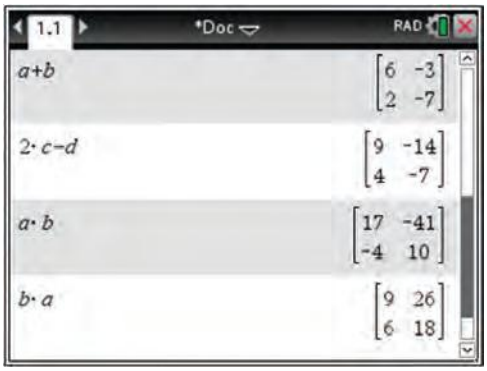
2 Alternatively, press **menu** > **Matrix & Vector** > **Create.**

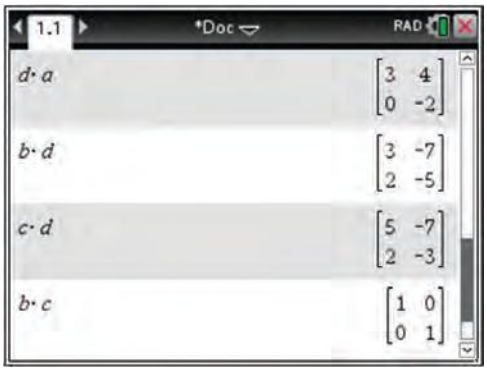
3 In the dialogue box, select **2** rows and **2** columns.



4 Enter the first matrix and press **ctrl** + **var** to store it into **a**.

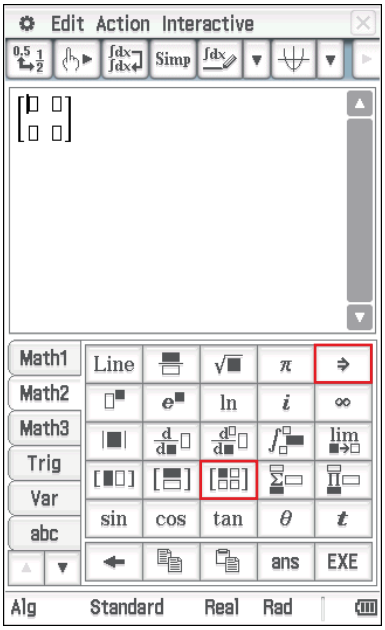
5 Repeat to store the other matrices into **b**, **c** and **d**.





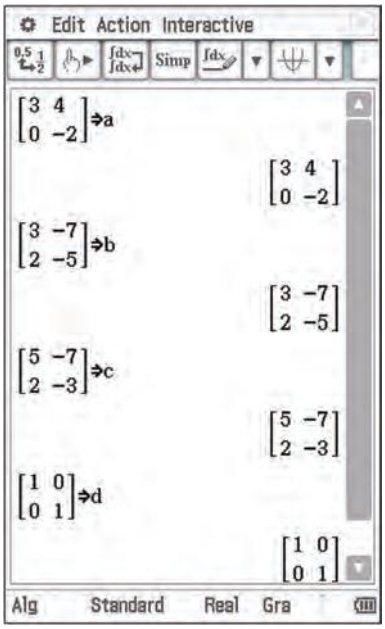
6 Complete the matrix calculations as shown. Make sure you use the **×** key between the letters to avoid entering the variables *ab*, *ba*, etc.

ClassPad



1 Open the **Keyboard** > **Math2**.

2 Tap the **2×2 matrix** template to create a 2×2 matrix. Note the location of the **store** arrow above, which will be used in the next screen.

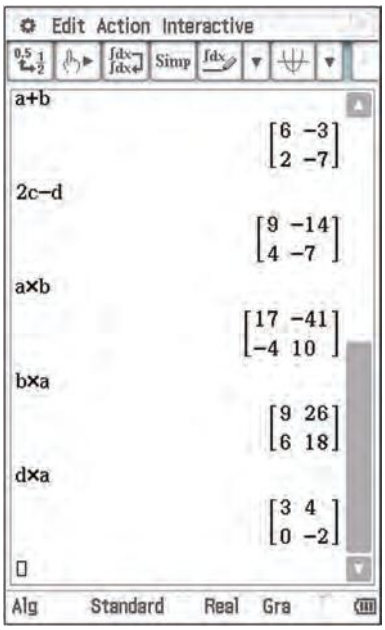


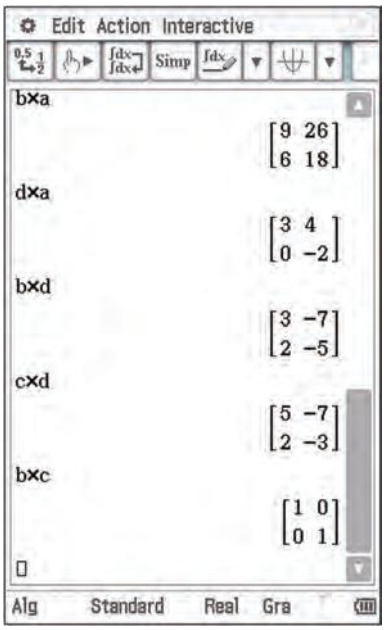
3 Enter the first matrix.

4 Open the **Keyboard** > **Math1** or **Math2**.

5 Tap **store** followed by **a**.

6 Repeat to store the other matrices into **b, c** and **d**.





7 Complete the matrix calculations as shown. Make sure you use the **×** key between the letters to avoid entering *ab*, *ba*, etc.

Matrix algebra

Since matrix addition is just ordinary addition of the elements, it is **commutative** so the order of addition of 2 matrices makes no difference: $A + B = B + A$.

It is also **associative**: the order of addition of 3 matrices makes no difference:

$(A + B) + C = A + (B + C)$.

Scalar multiplication is also commutative and associative: $xA = Ax$, $x(yA) = (xy)A$ and $(xA)B = x(AB)$.

Scalar multiplication is **distributive over addition**: $x(A + B) = xA + xB$ for all $x \in R$.

For ordinary addition, the number 0 is the **identity**: $x + 0 = 0 + x = x$ for all $x \in R$.

Under ordinary addition, every number has an **inverse**, another number that adds to give the identity: $x + (-x) = 0$ for all $x \in R$.

Matrix addition has an identity: the **zero matrix** 0, which has 0 for every entry. This matrix is also called the **additive identity**. There are also **additive inverses**: $A + (-A) = 0$.

For example, $\begin{bmatrix} 1 & 2 & -1 \\ -3 & 4 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ -3 & 4 & -2 \end{bmatrix}$

and $\begin{bmatrix} 3 & 0 & 1 \\ -4 & 6 & -3 \end{bmatrix} + \begin{bmatrix} -3 & 0 & -1 \\ 4 & -6 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

The matrix laws for addition and scalar multiplication mean we can collect like terms in the usual way. However, we cannot change the order of matrix products, so AB and BA have to be done separately.

WORKED EXAMPLE 5 Matrix simplification

Simplify each of the following.

a $3A + 2B - 4A + 5B$

b $4(C - 2G) - 2(3C - 5G)$

c $XY + 2YX - 3YX + 4XY$

d $3PQ + 4P - 3Q + 2PQ + P + QP - 2Q$

Steps	Working
a 1 Rearrange using the commutative law.	$3A + 2B - 4A + 5B$ $= 3A + (-4)A + 2B + 5B$
2 Use the distributive law and simplify.	$= (3 + (-4))A + (2 + 5)B$ $= -1A + 7B$
3 Use the commutative law.	$= 7B + (-A) = 7B - A$
b 1 Use the distributive law.	$4(C - 2G) - 2(3C - 5G)$ $= 4C - 8G - 6C + 10G$
2 Use the commutative law and the distributive law.	$= 10G - 8G + 4C - 6C$ $= (10 - 8)G + (4 - 6)C$
3 Simplify.	$= 2G - 2C$
c 1 Rearrange.	$XY + 2YX - 3YX + 4XY$ $= 1XY + 4XY + 2YX - 3YX$
2 Use the distributive law and simplify.	$= (1 + 4)XY + (2 - 3)YX$ $= 5XY - YX$
d 1 Rearrange.	$3PQ + 4P - 3Q + 2PQ + P + QP - 2Q$ $= 3PQ + 2PQ + P + 4P - 3Q - 2Q + QP$
2 Use the distributive law and simplify.	$= (3 + 2)PQ + (1 + 4)P - (3 + 2)Q + QP$ $= 5PQ + 5P - 5Q + QP$

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In ordinary algebra, the order of multiplication doesn't make any difference to the result. However, the result of multiplication of matrices can depend on the order. Matrix multiplication is **not commutative**. In general, $AB \neq BA$.

When 3 matrices are multiplied, it doesn't matter whether we do the first 2 and then multiply by the third, or do the second 2 and then multiply the first by the answer. Matrix multiplication *is* associative: $(AB)C = A(BC)$.

Matrix multiplication is distributive over addition (left and right):

$A(B + C) = AB + AC$ and $(A + B)C = AC + BC$

We can use the distributive law to expand matrix expressions and factorise where the order is the same in each product.

Note that $AB + 3B \neq (A + 3)B$ because we cannot add a matrix and a real number.

$A + 3$ makes no sense.

Ordinary multiplication has the identity 1: $1 \times x = x \times 1 = x$ for all $x \in R$.

Matrix multiplication is not commutative, so only *square* matrices have multiplicative identities.
The **multiplicative identity** for a square matrix of dimension n is I_n, the matrix of dimension n that has 1s along its leading diagonal and 0s elsewhere:

$I_n = (i_{mn})$, where $i_{mn} = 0$ for $m \neq n$ and $i_{mn} = 1$ for $m = n$.

This matrix is called the **identity matrix**.

For example, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

The n of I_n is usually left off, so $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

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WORKED EXAMPLE 6 Expansion of matrix expressions

Expand each of the following.

a $A(2B - 3C)$ **b** $3X(2Y + 5I)$ **c** $(2P + 3Q)(2P - 3Q)$

Steps	Working
a 1 Use the distributive law.	$A(2B - 3C)$ $= A(2B) + A(-3C)$
2 Use the associative law.	$= 2AB - 3AC$
b 1 Use the distributive law.	$3X(2Y + 5I)$ $= (3X)(2Y) + (3X)(5I)$
2 Use the associative law (twice) and simplify.	$= (3 \times 2)XY + (3 \times 5)XI$ $= 6XY + 15XI$
3 Use the identity law $AI = A$.	$= 6XY + 15X$
c 1 Use the distributive law.	$(2P + 3Q)(2P - 3Q)$ $= 2P(2P - 3Q) + 3Q(2P - 3Q)$
2 And again.	$= 2P \times 2P + 2P \times -3Q + 3Q \times 2P + 3Q \times -3Q$
3 Simplify.	$= 4PP - 6PQ + 6QP - 9QQ$
4 Write $PP = P^2$ and QQ as Q^2.	$= 4P^2 - 6PQ + 6QP - 9Q^2$

Generally, $PQ \neq QP$ so in the above example, we cannot cancel out $-6PQ$ and $6QP$.

This is also important when we are trying to factorise.

WORKED EXAMPLE 7 Factorisation of matrix expressions

Factorise each of the following as much as possible.

a $8ABC - 12ADC$ **b** $4PQ + 5Q$ **c** $4X^2 - Y^2$ **d** $9M^2 + 12MN - 12NM - 16N^2$

Steps		Working
a 1	Use the distributive law for scalar multiplication.	$8ABC - 12ADC$ $= 4(2ABC - 3ADC)$
2	Use the associative law.	$= 4(A(2BC) + A(-3DC))$
3	Use the left distributive law.	$= 4A(2BC - 3DC)$
4	Use the right distributive law.	$= 4A(2B - 3D)C$
b 1	Use $Q = IQ$ to insert the identity on the correct side.	$4PQ + 5Q$ $= 4PQ + 5IQ$
2	Use the right distributive law.	$= (4P + 5I)Q$
c 1	Insert $2XY - 2XY$.	$4X^2 - Y^2$ $= 4X^2 + 2XY - 2XY - Y^2$
2	Use the left and right distributive laws.	$= 2X(2X - Y) - (2X - Y)Y$
d 1	Use the left distributive law.	$9M^2 + 12MN - 12NM - 16N^2$ $= 3M(3M + 4N) - 4N(3M + 4N)$
2	Use the right distributive law.	$= (3M - 4N)(3M + 4N)$

Matrix multiplication is not commutative so in the above example:

$8ABC - 12ADC \neq 4AC(2B - 3D)$, $4X^2 - Y^2 \neq (2X - Y)(2X + Y)$

and $9M^2 + 12MN - 12NM - 16N^2 \neq (3M + 4N)(3M - 4N)$.

Of course, $4PQ + 5Q \neq (4P + 5)Q$ as $4P + 5$ makes no sense.

EXERCISE 5.1 Matrix operations and algebra

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Mastery

1 WORKED EXAMPLE 1 What are the dimensions of each matrix below?

a $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & -3 \end{bmatrix}$ **b** $B = \begin{bmatrix} 0 & 1 \\ -3 & 3 \\ -4 & 5 \end{bmatrix}$ **c** $C = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

d $D = \begin{bmatrix} 6 & 2 & -1 & 4 \\ 3 & -2 & -3 & -5 \\ -4 & 0 & 5 & 1 \end{bmatrix}$ **e** $E = \begin{bmatrix} 2 & 0 & -1 \end{bmatrix}$

For the matrices above, find the values of

f a_{12} **g** a_{21} **h** b_{21} **i** b_{12}

j d_{32} **k** d_{23} **l** d_{31}

2 WORKED EXAMPLE 2 Consider the matrices $X = \begin{bmatrix} -2 & 3 & -2 \\ 1 & 0 & -2 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & -4 & 5 \\ 6 & 1 & 4 \end{bmatrix}$ and $Z = \begin{bmatrix} 2 & -5 & 3 \\ 5 & 2 & -1 \end{bmatrix}$. Find

a $X + Y$ **b** $-3Y$ **c** $Z - X$ **d** $2X + 3Y$

e $Z - 3Y$ **f** $2Y + 4Z$ **g** $3(Y + Z)$ **h** $3Y + 3Z$

i $2Z - 3Y + X$ **j** $X - 3Y + 2Z$ **k** $(X + Y) + Z$ **l** $X + (Y + Z)$

m $2(X + Y) + 3Z$ **n** $2X + Y + 3Z$ **o** $3(Z - 2Y) + Z$ **p** $3Z - 2(Y + Z)$

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3 WORKED EXAMPLE 3 Consider the matrices $P = \begin{bmatrix} 1 & 3 & 4 \\ -2 & 0 & -1 \end{bmatrix}$, $Q = \begin{bmatrix} 2 & -3 \\ 0 & 4 \\ -2 & -1 \end{bmatrix}$, $R = \begin{bmatrix} 4 & -5 \\ -1 & 2 \\ -3 & -2 \end{bmatrix}$, $S = \begin{bmatrix} 3 & -2 \\ -3 & 4 \end{bmatrix}$, $T = \begin{bmatrix} 4 & -20 \end{bmatrix}$, $U = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$, $V = \begin{bmatrix} 1 & 6 \end{bmatrix}$ and $W = \begin{bmatrix} -4 & 1 & -2 \\ 5 & 3 & -5 \end{bmatrix}$.

Find each matrix product or state why it is not possible to calculate.

a PQ	**b** PR	**c** RQ	**d** PS
e SP	**f** TQ	**g** TR	**h** RT
i VS	**j** SW	**k** PU	**l** SQ
m WQ	**n** QW	**o** QS	**p** TU

4 Consider the matrices $E = \begin{bmatrix} 4 & 3 \\ -5 & -4 \end{bmatrix}$, $F = \begin{bmatrix} -2 & 4 \\ -4 & 8 \end{bmatrix}$, $G = \begin{bmatrix} 4 & 2 \\ -8 & -4 \end{bmatrix}$, $H = \begin{bmatrix} 3 & 2 \\ 5 & 6 \end{bmatrix}$ and $K = \begin{bmatrix} -2 & 4 \\ -3 & -5 \end{bmatrix}$.

Find each matrix product.

a EF	**b** FG	**c** GH	**d** HK
e EE	**f** F^2	**g** G^2	**h** H^2
i $K(EH)$	**j** $(KE)H$	**k** $H(KE)$	**l** $(HK)F$
m $H(E + 2G)$	**n** $2E(H + 3K)$	**o** $3KF$	**p** $3FK$

5 WORKED EXAMPLE 4 $A = \begin{bmatrix} 1 & 3 \\ 4 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 5 & -1 \\ 2 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, $D = \begin{bmatrix} -5 & 0 \\ -2 & 5 \end{bmatrix}$, $E = \begin{bmatrix} -2 & -3 \\ 3 & 5 \end{bmatrix}$ and $F = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$. Simplify each expression.

a AC	**b** $(A + B)C$	**c** $(D + 3E)F$	**d** $F(D + 3E)$
e $FD + 3E$	**f** $A(B + 3I)C$	**g** DC	**h** CD
i $2A + 3B(D + E)$	**j** $2A + 3BD + E$	**k** $(2D - A)F$	**l** BC
m $A + BC$	**n** $(D + E)F$	**o** $(2D - 3I)C$	**p** $2I + (D - 3B)E$

6 Using CAS 1 Consider the matrices $A = \begin{bmatrix} 5 & -3 \\ 6 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 5 & -7 \\ 2 & -3 \end{bmatrix}$, $D = \begin{bmatrix} -3 & 2 \\ 4 & -5 \end{bmatrix}$, $E = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & -5 \\ -3 & -4 & 2 \end{bmatrix}$, $F = \begin{bmatrix} 5 & 2 & -1 \\ 3 & 0 & -3 \\ 4 & 4 & 5 \end{bmatrix}$, $G = \begin{bmatrix} 3 & 0 & 2 \\ -3 & 4 & 6 \\ 0 & -4 & 7 \end{bmatrix}$ and $H = \begin{bmatrix} 0 & 2 & 5 \\ 6 & 1 & -3 \\ -8 & -3 & 2 \end{bmatrix}$.

Find

a CD	**b** DC	**c** AB	**d** BA
e $A(B + 3I)$	**f** $A^2 - B^2$	**g** $2A - 3(D - C)$	**h** $(A + I)(A - I)$
i EF	**j** FE	**k** $2EF + 4GH$	**l** $(E + F)(E - F)$
m $(E - F)(E + F)$	**n** $H(3I - 2G)$	**o** $3E + FG$	**p** $G^2 - H^2$

7 WORKED EXAMPLE 5 Simplify each matrix expression.

a $3A + AB + 4A + 2BA$ **b** $5XY - 5X + 3XY + 4YX$ **c** $2AB + 3B + 5A + B - BA$

d $PQ + 4P + PQ - P + QP$ **e** $2P - 3Q + 3P - 4Q + 5P$ **f** $PQ + 2QP + 5PQ - 2P$

g $8A + 2B - 10AB + 2B$ **h** $FG + 2F - FG - 3F + G$ **i** $4H + 2HK - 3K - HK$

j $5G + 4P - GP + P$ **k** $AY + 2AY + 4Y - 3Y$ **l** $3MN - 2NM + 3MN + NM$

m $PS + SP + 2PS - S$ **n** $3V + 2VW + 5V - 3WV$ **o** $TU - 3TU + 4T + 3U$

8 WORKED EXAMPLE 6 Expand each matrix expression.

a $3(4A - 5B)$ **b** $2A(A + 3B)$ **c** $2M(M - N + 3I)$

d $2I(X - 3Y + 2I + XY)$ **e** $4M(M + N)N$ **f** $(3H - 4G)(3H + 4G)$

g $(2A - 5B)^2$ **h** $2PQ(P - 2Q)$ **i** $A(A + 2B) - B(A - 2B)$

j $G(G - H) + (G + H)H$ **k** $2D(K + M) - 3K(M + D)$ **l** $(M - 3I)(M + 2I)$

m $(2X + I)(3X - 2I)$ **n** $(X + Y)(X^2 - XY + Y^2)$ **o** $X(2X + 4Y) - 3(2X + 3Y)Y$

9 WORKED EXAMPLE 7 Factorise each matrix expression.

a $3X + 2XY$ **b** $2XY - 3Y$ **c** $4AB + 3BC$

d $8PQ - 6RQ$ **e** $2MN + 5MP$ **f** $3A^2 + 4AB + B^2$

g $3S^2 - 2ST + 3TS - 2T^2$ **h** $5M^2 + 2MN + M$ **i** $A + 2BA + 3BA^2$

j $A^2 - 2A + I$ **k** $3X^2 + 7X + 2I$ **l** $3A^2 + 7AB + 2B^2$

m $2MP - 2PM + P^2 - M^2$ **n** $2M^2 + 2P^2 - PM - 4MP$ **o** $A^3 + 2A^2B + 2AB^2 + B^3$

Exam practice

10 TECH-FREE (2 marks) Find $\begin{bmatrix} 2 & 4 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -4 & 5 \end{bmatrix}$.

11 TECH-FREE (3 marks) Show that $\begin{bmatrix} 1 & 3 & -5 \\ 0 & 5 & -8 \\ 2 & -2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ -16 & 13 & 8 \\ -10 & 8 & 5 \end{bmatrix}$.

12 Given $A = \begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -1 \\ -3 & -5 \end{bmatrix}$, $3A - 2B + I =$

A $\begin{bmatrix} 0 & -4 \\ 18 & 14 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & -3 \\ 19 & 14 \end{bmatrix}$ **C** $\begin{bmatrix} -1 & -4 \\ 18 & 13 \end{bmatrix}$

D $\begin{bmatrix} -8 & -1 \\ 17 & 17 \end{bmatrix}$ **E** $\begin{bmatrix} -8 & -1 \\ 17 & 18 \end{bmatrix}$

13 The additive inverse of $\begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & 3 \end{bmatrix}$ is

A not possible because it is not square

B $\begin{bmatrix} -2 & -1 & 1 \\ 0 & 2 & -3 \end{bmatrix}$

C $\begin{bmatrix} -2 & -1 & -1 \\ 0 & -2 & -3 \end{bmatrix}$

D $\begin{bmatrix} -2 & 0 \\ -1 & 2 \\ 1 & -3 \end{bmatrix}$

E $\begin{bmatrix} -2 & 0 \\ -1 & -2 \\ -1 & -3 \end{bmatrix}$

14 Given $A = \begin{bmatrix} 3 & -2 \\ 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -2 & 3 \end{bmatrix}$, AB is

A not possible: columns are not the same

B $\begin{bmatrix} 6 & 7 \\ 8 & 2 \end{bmatrix}$

C $\begin{bmatrix} 6 & 8 \\ 7 & 2 \end{bmatrix}$

D $\begin{bmatrix} 6 & 8 \\ 7 & 2 \\ -9 & -1 \end{bmatrix}$

E $\begin{bmatrix} 6 & 7 & -9 \\ 8 & 2 & -1 \end{bmatrix}$

15 Which of the following statements about matrices is true?

A $A^2 = 0 \Rightarrow A = 0$

B $A^2 - B^2 = (A - B)(A + B)$

C $9A^2 - 4I^2 = (3A - 2I)(3A + 2I)$

D $A^2 = I \Rightarrow A = I$

E $4A + 8 = 4(A + 2)$

Video playlist
Determinants

Worksheet
Inverse and determinant of a matrix

5.2 Determinants

The **multiplicative inverse** of a (square) matrix A, if it exists, is the matrix B such that the product of A and B is the identity: $AB = BA = I$.

$\begin{bmatrix} 5 & 4 \\ 9 & 7 \end{bmatrix}\begin{bmatrix} -7 & 4 \\ 9 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 3 & 0 & -1 \\ 5 & -3 & 6 \\ 2 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0 & -1 & 3 \\ -2 & -8 & 23 \\ -1 & -3 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Both $\begin{bmatrix} 5 & 4 \\ 9 & 7 \end{bmatrix}$ and $\begin{bmatrix} 3 & 0 & -1 \\ 5 & -3 & 6 \\ 2 & -1 & 2 \end{bmatrix}$ have an inverse.

However, $\begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}$ does not have a multiplicative inverse.

If we write $\begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we get, among other things, that $4a + 2c = 1$ and $6a + 3c = 0$, which is impossible. This means that $\begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}$ has no inverse.

When it exists, the multiplicative inverse of A is written as A^{-1}. As additive inverses always exist, the multiplicative inverse is just called the **inverse**.

A matrix that has no inverse is **singular**.

A matrix that has an inverse is **invertible** or **non-singular**.

Diagonal matrices

How can we tell if a matrix is singular or non-singular?

$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is a **diagonal matrix**. It has non-zero entries only in the diagonal.

A diagonal matrix is a square matrix A such that $a_{ij} = 0$ for $i \neq j$.

WORKED EXAMPLE 8 Inverse of a diagonal matrix

Find the inverse of $P = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$, if it exists.

Steps	Working
1 Choose variables for a 2 × 2 matrix.	Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
2 Find the product of P and X.	$PX = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $= \begin{bmatrix} 3a & 3b \\ -2c & -2d \end{bmatrix}$
3 Write $PX = I$.	$\begin{bmatrix} 3a & 3b \\ -2c & -2d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $3a = 1, 3b = 0, -2c = 0, -2d = 1$
4 Solve for a, b, c, d.	$a = \frac{1}{3}, b = 0, c = 0, d = -\frac{1}{2}$
5 Write the answer.	P is invertible and $P^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$.

From the above example, it is obvious that a diagonal matrix in which all the diagonal elements are non-zero will be non-singular.

If $A = \begin{bmatrix} a_1 & 0 & 0.. \\ 0 & a_2 & 0.. \\ 0 & 0 & \ddots \\ \vdots & \vdots & \end{bmatrix}$, where $a_1, a_2, a_3 \ldots a_n$ are non-zero, then A is non-singular.

Furthermore, $A^{-1} = \begin{bmatrix} \frac{1}{a_1} & 0 & 0.. \\ 0 & \frac{1}{a_2} & 0.. \\ 0 & 0 & \ddots \\ \vdots & \vdots & \end{bmatrix} = \begin{bmatrix} a_1^{-1} & 0 & 0.. \\ 0 & a_2^{-1} & 0.. \\ 0 & 0 & \ddots \\ \vdots & \vdots & \end{bmatrix}$.

For a diagonal matrix, the product $a_1a_2a_3 \ldots a_n$ is called the **determinant** of A and is written as $|A|$ or $\det(A)$.

Since $a_1a_2a_3 \ldots a_n \neq 0 \Leftrightarrow a_1 \neq 0, a_2 \neq 0, a_3 \neq 0 \ldots a_n \neq 0$, the diagonal matrix A is non-singular if and only if $\det(A) \neq 0$.

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WORKED EXAMPLE 9 Determinant of a diagonal matrix

Use the determinants to find the nature of the following matrices.

a $A = \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}$ **b** $B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **c** $C = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$

Steps	Working
a 1 State the reason and multiply the diagonal elements to find the determinant.	A is diagonal, so $\det(A) = a_{11}a_{22}$ $= 4 \times (-3)$ $\det(A) = -12$
2 State the nature of A.	A is invertible.
b 1 State the reason and multiply the diagonal elements to find the determinant.	B is diagonal, so $\det(B) = b_{11}b_{22}b_{33}$ $= 3 \times 4 \times 0$ $\det(B) = 0$
2 State the nature of B.	B is singular.
c 1 State the reason and multiply the diagonal elements to find the determinant.	C is diagonal, so $\lvert C \rvert = c_{11}c_{22}c_{33}c_{44}$ $= 5 \times (-3) \times 1 \times (-4)$ $\lvert C \rvert = 60$
2 State the nature of C.	C is non-singular.

Triangular matrices

The matrix $\begin{bmatrix} 2 & 4 & -3 \\ 0 & -1 & 2 \\ 0 & 0 & 5 \end{bmatrix}$ is a **triangular matrix**. All the elements on one side of the diagonal are zero.

$A = (a_{ij})$ is **triangular** if one of $a_{ij} = 0$ for $i > j$ or $a_{ij} = 0$ for $a < j$ is true.

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WORKED EXAMPLE 10 Inverse of a triangular matrix

Find the inverse of $P = \begin{bmatrix} 2 & 0 \\ -3 & 5 \end{bmatrix}$ and $Q = \begin{bmatrix} 2 & 4 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, if it exists.

Steps	Working
1 Choose variables for a 2×2 matrix for the inverse of P.	Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
2 Find PX.	$PX = \begin{bmatrix} 2a & 2b \\ -3a + 5c & -3b + 5d \end{bmatrix}$
3 Write $PX = I$.	$\begin{bmatrix} 2a & 2b \\ -3a + 5c & -3b + 5d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
4 Solve for a, b, c and d.	$2a = 1 \Rightarrow a = 0.5$ $\quad 2b = 0 \Rightarrow b = 0$ $-3(0.5) + 5c = 0 \Rightarrow c = 0.3$ $\quad 0 + 5d = 1 \Rightarrow d = 0.2$
5 Write the answer.	$P^{-1} = \begin{bmatrix} 0.5 & 0 \\ 0.3 & 0.2 \end{bmatrix}$

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6 Choose variables for a 3 × 3 matrix for the inverse of Q.	Let $X = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$
	$QX = \begin{bmatrix} 2a + 4d - 3g & 2b + 4e - 3h & 2c + 4f - 3i \\ -d + 2g & -e + 2h & -f + 2i \\ 0g & 0h & 0i \end{bmatrix}$
	$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
7 Solve for a, b, c, etc.	$0i = 1 \Rightarrow$ there is no solution for i
8 Write the result.	There is no solution for X, so Q is singular.

From the above example, there is no solution if any of the diagonal entries of a triangular matrix is zero. The diagonal entries must be non-zero for the inverse of a triangular matrix to exist. The *null side* of the diagonal in the original matrix is *also null in the inverse*.

For a *triangular* matrix $A = (a_{ij})$, the determinant is defined as $\det(A) = a_{11}a_{22}a_{33} \ldots a_n$.

A triangular matrix is invertible if and only if $|A| = \det(A) \neq 0$.

WORKED EXAMPLE 11 Determinant of a triangular matrix

Use the determinants to find the nature of the following matrices.

a $A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$ **b** $B = \begin{bmatrix} 3 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & 0 & -4 \end{bmatrix}$ **c** $C = \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Steps	Working
a 1 State the reason and multiply the diagonal elements to find the determinant.	A is triangular, so $\|A\| = a_{11}a_{22}$ $= 2 \times 3$ $\|A\| = 6$
2 State the nature of A.	A is non-singular.
b 1 State the reason and multiply the diagonal elements to find the determinant.	B is triangular, so $\det(B) = b_{11}b_{22}b_{33}$ $= 3 \times 2 \times (-4)$ $\det(B) = -24$
2 State the nature of B.	B is invertible.
c 1 State the reason and multiply the diagonal elements to find the determinant.	C is triangular, so $\|C\| = c_{11}c_{22}c_{33}c_{44}$ $= 1 \times 0 \times 4 \times 1$ $\|C\| = 0$
2 State the nature of C.	C is singular.

Unfortunately, the conditions for a general matrix to have an inverse are not as simple as for diagonal or triangular matrices. This means that the determinant is not as simple either.

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General determinants

For the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is $|A| = ad - bc$.

If $|A| \neq 0$, the inverse is given by $\dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

If A is diagonal, then $b = c = 0$, so $|A| = ad$, $A^{-1} = \dfrac{1}{ad} \begin{bmatrix} d & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad} & 0 \\ 0 & \frac{a}{ad} \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{bmatrix}$.

If A is triangular, then $b = 0$ or $c = 0$, so $|A| = ad$, $\dfrac{1}{ad} \begin{bmatrix} d & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad} & 0 \\ 0 & \frac{a}{ad} \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{bmatrix}$.

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WORKED EXAMPLE 12 Determinant and inverse of a 2 × 2 matrix

Find the determinant and inverse of each matrix, if they exist.

a $A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix}$ **c** $C = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$

Steps	Working
a 1 Use $ad - bc$ for the determinant.	$\|A\| = 2 \times 3 - 4 \times 1 = 2$
2 State the outcome.	$\|A\| \neq 0$, so A is invertible.
3 Find the inverse.	$A^{-1} = \frac{1}{\|A\|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix}$
4 Write the answer.	$A^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1.5 & -2 \\ -0.5 & 1 \end{bmatrix}$
b 1 Use $ad - bc$ for the determinant.	$\begin{vmatrix} 4 & -3 \\ 8 & -6 \end{vmatrix} = -24 - (-24) = 0$
2 State the result.	The determinant of $\begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix}$ is 0, so it is singular and has no inverse.
c 1 Use $ad - bc$ for the determinant.	$\|C\| = 2 \times 0 - 3 \times 1 = -3 \neq 0$
2 $\|C\| \neq 0$ so find the inverse.	$C^{-1} = \frac{1}{-3}\begin{bmatrix} 0 & -3 \\ -1 & 2 \end{bmatrix}$
3 Write the answer.	$C^{-1} = -\frac{1}{3}\begin{bmatrix} 0 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$

Notice in Worked example 12 part **c** that a general matrix can have a zero in the diagonal, yet be non-singular. That is quite different to diagonal and triangular matrices.

Notice also that there are non-trivial matrices like $\begin{bmatrix} 3 & -2 \\ 4 & -3 \end{bmatrix}$ that are their own inverse. This is different to real numbers, where 1 and −1 are the only numbers that are their own inverses.

The rules for working out higher order inverses are quite complex.

For a 2×2 matrix, 3 separate calculations (2 products and a difference) are needed.

The number of separate calculations rises quickly with the dimension.

Number of separate calculations for the determinant of a matrix

Dimension	2	3	4	5	6
Calculations	3	18	144	1440	17 280

It is not practical to calculate the determinant of most matrices with dimension greater than 3.

Determinants

The determinant of a 1×1 matrix $A = (a_{11})$ is the element: $|A| = a_{11}$.

The **minor** M_{ij} of an $n \times n$ matrix A is the $(n-1) \times (n-1)$ matrix obtained by omitting the ith row and jth column.

$A = (a_{ij}) \Rightarrow M_{ij} = (a_{pq}), p \in \{1, 2 \ldots i-1, i+1 \ldots n\}, q \in \{1, 2 \ldots j-1, j+1 \ldots n\}$

The **cofactor** A_{ij} is the *signed* determinant of M_{ij}: $A_{ij} = (-1)^{i+j}|M_{ij}|$.

The **determinant** of A, $|A| = \det(A) = \sum_{j=1}^{n} a_{ij}A_{ij}$ (for row i) or $\sum_{i=1}^{n} a_{ij}A_{ij}$ (for column j).

The value of the determinant is independent of the row or column chosen for the expansion.

In this course, we will not be required to calculate the determinant of a higher order of matrix by hand unless there are simplifying conditions.

The definition is hard to understand, but it means we choose a row or column to expand along, and use the cofactors of that row or column.

If $(i + j)$ is even, $(-1)^{i+j} = 1$.

If $(i + j)$ is odd, $(-1)^{i+j} = -1$.

WORKED EXAMPLE 13 Determinant of a 3 × 3 matrix

Find the determinant of each matrix.

a $A = \begin{bmatrix} 2 & 3 & -2 \\ 2 & 0 & -1 \\ 4 & -4 & 3 \end{bmatrix}$ **b** $B = \begin{bmatrix} 2 & 4 & 1 \\ 4 & 5 & -2 \\ -3 & -4 & 1 \end{bmatrix}$

Steps	Working
a 1 Choose the row or column with the easiest numbers.	Row 2 has the easiest numbers, including a 0.
2 Use the formula to expand along row 2.	$\|A\| = \sum_{j=1}^{3} a_{2j}A_{2j}$ $= 2A_{21} + 0A_{22} + (-1)A_{23}$ $= 2A_{21} - A_{23}$
3 Find A_{21}: miss out row 2 and column 1.	$A_{21} = (-1)^{2+1}\begin{vmatrix} 3 & -2 \\ -4 & 3 \end{vmatrix}$ $= -1 \times (3 \times 3 - (-2) \times (-4))$ $= -1$
4 Find A_{23}: miss out row 2 and column 3.	$A_{23} = (-1)^{2+3}\begin{vmatrix} 2 & 3 \\ 4 & -4 \end{vmatrix}$ $= -1 \times (-8 - 12)$ $= 20$
5 Find $\det(A)$ using the expansion along row 2.	$\|A\| = 2A_{21} - A_{23}$ $= 2 \times (-1) - 20$ $\|A\| = -22$

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b **1** Choose the row or column with the easiest numbers.

Column 3 has the easiest numbers.

2 Use the formula to expand along column 3.

$$|B| = \sum_{j=1}^{3} b_{i3}B_{i3}$$
$$= 1 \times B_{13} + (-2) \times B_{23} + 1 \times B_{33}$$
$$= B_{13} - 2B_{23} + B_{33}$$

3 Find B_{13}: miss out row 1 and column 3.

$$B_{13} = (-1)^{1+3}\begin{vmatrix} 4 & 5 \\ -3 & -4 \end{vmatrix}$$
$$= 1 \times (-16 - (-15))$$
$$= -1$$

4 Find B_{23}: miss out row 2 and column 3.

$$B_{23} = (-1)^{2+3}\begin{vmatrix} 2 & 4 \\ -3 & -4 \end{vmatrix}$$
$$= -1 \times (-8 - (-12)$$
$$= -4$$

5 Find B_{33}: miss out row 3 and column 3.

$$B_{33} = (-1)^{3+3}\begin{vmatrix} 2 & 4 \\ 4 & 5 \end{vmatrix}$$
$$= 1 \times (10 - 16)$$
$$= -6$$

6 Find det(B) using the expansion along column 3.

$$|B| = B_{13} - 2B_{23} + B_{33}$$
$$= -1 - 2 \times (-4) + (-6)$$
$$|B| = 1$$

There is a 'shortcut' method to find the determinant of a 3 × 3 matrix.

1 For $M = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$, write the elements *twice*: $\begin{matrix} a & b & c & a & b & c \\ p & q & r & p & q & r \\ x & y & z & x & y & z \end{matrix}$.

2 Starting with the first a, b and c, multiply diagonally and add: $aqz + brx + cpy$.

3 With the other a, b and c, multiply back diagonally and add: $ary + bpz + cqx$.

4 Subtract the answers to get the determinant: $|M| = (aqz + brx + cpy) - (ary + bpz + cqx)$.

Exam hack

Choose the row or column with a zero to save calculations.

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WORKED EXAMPLE 14 Determinant of a 3 × 3 matrix using shortcut

Find the determinant of $G = \begin{bmatrix} 3 & 5 & -2 \\ -4 & 2 & -3 \\ 4 & -7 & -5 \end{bmatrix}$.

Steps	Working
1 Write the elements twice and put in the diagonal lines.	$\begin{matrix} 3 & 5 & -2 & 3 & 5 & -2 \\ -4 & 2 & -3 & -4 & 2 & -3 \\ 4 & -7 & -5 & 4 & -7 & -5 \end{matrix}$
2 Multiply and add forward.	$3 \times 2 \times (-5) + 5 \times (-3) \times 4 + (-2) \times (-4) \times (-7)$ $= -30 + (-60) + (-56)$ $= -146$
3 Multiply and add backward.	$3 \times (-3) \times (-7) + 5 \times (-4) \times (-5) + (-2) \times 2 \times 4$ $= 63 + 100 + (-16)$ $= 147$
4 Subtract the answers.	$\lvert G \rvert = -146 - 147 = -293$

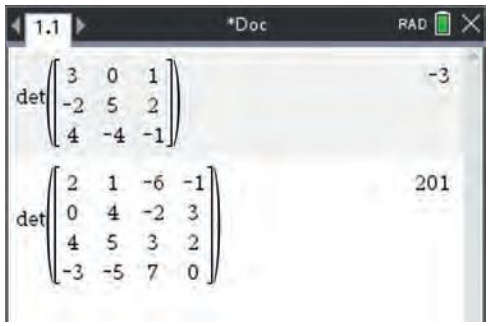
If a matrix is non-singular, we can use determinants to find the inverse, but the process is quite long and messy. Even finding the determinant for matrices with $n = 3$ is prone to calculation errors.

USING CAS 2 Determinants

Find the determinants of each matrix.

a $\begin{bmatrix} 3 & 0 & 1 \\ -2 & 5 & 2 \\ 4 & -4 & -1 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & 1 & -6 & -1 \\ 0 & 4 & -2 & 3 \\ 4 & 5 & 3 & 2 \\ -3 & -5 & 7 & 0 \end{bmatrix}$

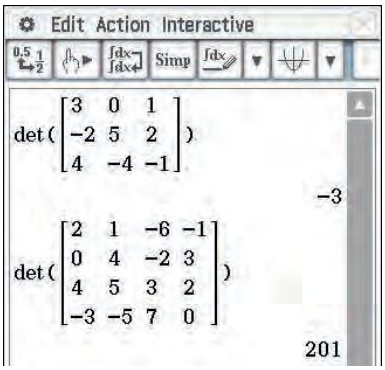
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Press **menu > Matrix & Vector > determinant** and enter the matrix.

Alternatively, store the matrix and then find the determinant.

ClassPad



Enter and highlight the matrix then tap **Interactive > Matrix > Calculation > det**.

Alternatively, store the matrix and then find the determinant.

EXERCISE 5.2 Determinants

ANSWERS p. 579

Recap

1 Given $A = \begin{bmatrix} 2 & -4 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & -2 \\ 2 & 4 \end{bmatrix}$, $2A + B - I =$

A $\begin{bmatrix} -3 & -12 \\ 10 & 17 \end{bmatrix}$ **B** $\begin{bmatrix} -1 & -11 \\ 9 & 17 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & -11 \\ 7 & 13 \end{bmatrix}$ **D** $\begin{bmatrix} 0 & -10 \\ 8 & 13 \end{bmatrix}$ **E** $\begin{bmatrix} 2 & -10 \\ 8 & 19 \end{bmatrix}$

2 $\begin{bmatrix} -3 & 2 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 5 & -2 \end{bmatrix} =$

A $\begin{bmatrix} -6 & 9 \\ 4 & -6 \end{bmatrix}$ **B** $\begin{bmatrix} 4 & 5 \\ -16 & 2 \end{bmatrix}$ **C** $\begin{bmatrix} -6 & -6 \\ 10 & 8 \end{bmatrix}$ **D** $\begin{bmatrix} -12 & -19 \\ 16 & 18 \end{bmatrix}$ **E** $\begin{bmatrix} 2 & 3 \\ 18 & 12 \end{bmatrix}$

Mastery

3 WORKED EXAMPLE 8 Find the inverse of each matrix, if it exists.

a $\begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$ **b** $\begin{bmatrix} -4 & 0 \\ 0 & 3 \end{bmatrix}$ **c** $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{5} \end{bmatrix}$ **d** $\begin{bmatrix} 2.5 & 0 \\ 0 & 0.8 \end{bmatrix}$

e $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ **f** $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ **g** $\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **h** $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$

4 WORKED EXAMPLE 9 Find the determinant of each matrix.

a $\begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix}$ **b** $\begin{bmatrix} 0.6 & 0 \\ 0 & -0.4 \end{bmatrix}$ **c** $\begin{bmatrix} 5 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$ **d** $\begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$

e $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & -4 \end{bmatrix}$ **f** $\begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -12 \end{bmatrix}$ **g** $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ **h** $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$

i $\begin{bmatrix} -3 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & -6 \end{bmatrix}$ **j** $\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 2.5 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix}$

5 WORKED EXAMPLE 10 Find the inverse of each matrix, if it exists.

a $\begin{bmatrix} -1 & 6 \\ 0 & 4 \end{bmatrix}$ **b** $\begin{bmatrix} 2 & 0 \\ -6 & 1 \end{bmatrix}$ **c** $\begin{bmatrix} 5 & 0 \\ -8 & -4 \end{bmatrix}$ **d** $\begin{bmatrix} 3 & -10 \\ 0 & \frac{1}{5} \end{bmatrix}$

e $\begin{bmatrix} 1 & -3 & 5 \\ 0 & 4 & 8 \\ 0 & 0 & 2 \end{bmatrix}$ **f** $\begin{bmatrix} 1 & 0 & 0 \\ 4 & -2 & 0 \\ -6 & 8 & -1 \end{bmatrix}$ **g** $\begin{bmatrix} \frac{1}{3} & 8 & -\frac{1}{2} \\ 0 & \frac{1}{4} & 2 \\ 0 & 0 & 6 \end{bmatrix}$ **h** $\begin{bmatrix} 4 & 0 & 0 \\ 3 & 0 & 0 \\ 16 & 8 & -2 \end{bmatrix}$

6 WORKED EXAMPLE 11 Find the determinant of each matrix.

a $\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$ **b** $\begin{bmatrix} -0.6 & 0 \\ 2.1 & 0.8 \end{bmatrix}$ **c** $\begin{bmatrix} 4 & 3 & -5 \\ 0 & -2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ **d** $\begin{bmatrix} -2 & 1 & 4 \\ 0 & 3 & 7 \\ 0 & 0 & -1 \end{bmatrix}$

e $\begin{bmatrix} 2 & 0 & 0 \\ 6 & \frac{1}{3} & 0 \\ \frac{2}{3} & \frac{4}{7} & -\frac{1}{5} \end{bmatrix}$ **f** $\begin{bmatrix} -2 & 0 & 0 \\ -1 & 4 & 0 \\ -3 & 1 & 5 \end{bmatrix}$ **g** $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -5 & 3 & 4 & 0 \\ 6 & -2 & 3 & 6 \end{bmatrix}$ **h** $\begin{bmatrix} 2 & 3 & 5 & -7 \\ 0 & -4 & 2 & 5 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

i $\begin{bmatrix} 2 & -2 & 1.7 & 4.6 & 0 \\ 0 & -0.6 & 2.1 & 2.4 & 0 \\ 0 & 0 & 4.8 & -1.3 & 2.4 \\ 0 & 0 & 0 & 5 & 3.5 \\ 0 & 0 & 0 & 0 & 1.3 \end{bmatrix}$ **j** $\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 & 0 \\ 1 & 5 & 6 & 0 & 0 \\ 6 & 7 & 4 & 0.1 & 0 \\ 3 & -9 & 2 & 7 & -5 \end{bmatrix}$

7 WORKED EXAMPLE 12 Find the determinant and inverse of each matrix, if they exist.

a $\begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & -1 \\ -5 & 4 \end{bmatrix}$ **c** $\begin{bmatrix} 6 & 5 \\ 4 & 3 \end{bmatrix}$ **d** $\begin{bmatrix} -3 & 5 \\ 2 & -3 \end{bmatrix}$

e $\begin{bmatrix} 10 & -4 \\ 5 & -2 \end{bmatrix}$ **f** $\begin{bmatrix} -0.5 & 2 \\ 3 & 0.4 \end{bmatrix}$ **g** $\begin{bmatrix} 2 & 3 \\ -2 & 7 \end{bmatrix}$ **h** $\begin{bmatrix} -6 & -4 \\ 3 & 2 \end{bmatrix}$

i $\begin{bmatrix} 2 & -4 \\ 5 & 3 \end{bmatrix}$ **j** $\begin{bmatrix} 1 & 3 \\ 5 & -2 \end{bmatrix}$

8 WORKED EXAMPLE 13 Find the determinant of each matrix.

a $\begin{bmatrix} 3 & 0 & -2 \\ -2 & 2 & 4 \\ -1 & 0 & 5 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & -3 & -2 \\ 1 & 4 & 5 \\ 4 & 0 & 0 \end{bmatrix}$ **c** $\begin{bmatrix} 2 & 4 & 3 \\ 1 & -1 & 0 \\ 3 & 5 & -2 \end{bmatrix}$ **d** $\begin{bmatrix} 2 & 3 & -4 \\ 1 & 2 & -3 \\ 0 & -4 & -5 \end{bmatrix}$

e $\begin{bmatrix} 3 & -4 & 1 \\ -4 & 5 & -2 \\ -3 & 4 & -1 \end{bmatrix}$ **f** $\begin{bmatrix} 1 & -1 & 2 \\ 3 & -4 & -3 \\ 4 & 5 & -2 \end{bmatrix}$ **g** $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 6 & 0 \\ 1 & -3 & 3 & 1 \\ -2 & 1 & 5 & -1 \end{bmatrix}$

9 WORKED EXAMPLE 14 Use the shortcut to find the determinant of each matrix.

a $\begin{bmatrix} 2 & -5 & -4 \\ -4 & -3 & -4 \\ 5 & 3 & 4 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & -1 & -1 \\ -2 & -2 & 2 \\ -4 & -2 & 3 \end{bmatrix}$ **c** $\begin{bmatrix} -1 & -3 & 1 \\ 3 & 4 & -1 \\ -5 & 3 & -4 \end{bmatrix}$ **d** $\begin{bmatrix} 3 & -2 & 3 \\ 5 & 5 & -1 \\ 4 & -1 & 4 \end{bmatrix}$

e $\begin{bmatrix} -2 & 2 & -3 \\ -1 & 8 & 8 \\ 2 & -4 & 3 \end{bmatrix}$ **f** $\begin{bmatrix} -6 & -5 & 5 \\ 8 & -1 & 2 \\ 5 & -2 & 2 \end{bmatrix}$ **g** $\begin{bmatrix} -3 & 3 & -7 \\ -2 & 2 & 1 \\ 7 & -7 & 3 \end{bmatrix}$ **h** $\begin{bmatrix} 8 & 5 & 5 \\ -7 & -3 & 7 \\ 1 & -6 & -4 \end{bmatrix}$

10 Using CAS 2 Find the determinant of each matrix.

a $\begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 2 & 7 \\ -3 & 5 & 2 \\ 4 & -5 & -3 \end{bmatrix}$ **c** $\begin{bmatrix} 8 & 2 & 5 \\ 0 & -3 & -2 \\ -4 & 5 & -6 \end{bmatrix}$ **d** $\begin{bmatrix} 5 & -4 & 3 \\ 6 & -6 & 0 \\ 0 & 2 & -4 \end{bmatrix}$

e $\begin{bmatrix} -3 & 1 & 4 \\ 4 & 0.5 & 2.5 \\ -2 & -0.4 & -1.3 \end{bmatrix}$ **f** $\begin{bmatrix} -0.1 & 3.5 & 1.8 \\ 2.4 & -4.2 & 2.5 \\ 1.6 & -5.1 & 0 \end{bmatrix}$ **g** $\begin{bmatrix} 0 & 2 & 8 & 1 \\ 1 & -3 & -4 & 3 \\ 3 & 4 & 0 & 5 \\ 7 & 5 & 4 & 7 \end{bmatrix}$

Exam practice

11 TECH-FREE (2 marks) Find the determinant of each matrix.

a $\begin{bmatrix} 2 & -1 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 4 \\ -2 & -5 \end{bmatrix}$

12 TECH-FREE (2 marks) Find the determinant of $\begin{bmatrix} 2 & 0 & 1 \\ -3 & 3 & -5 \\ -4 & 2 & 4 \end{bmatrix}$.

13 The determinant of $\begin{bmatrix} 3 & 0 & 0 \\ 1 & -5 & 0 \\ -4 & 6 & 2 \end{bmatrix}$ is

A -30 **B** -14 **C** 0 **D** 13 **E** 20

14 $\begin{vmatrix} 4 & 5 \\ 3 & 2 \end{vmatrix} =$

A -8 **B** -7 **C** 7 **D** 8 **E** 23

15 (6 marks)

a $A = \begin{bmatrix} 5 & 0 \\ 0 & -4 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 0 \\ 0 & 2 \end{bmatrix}$. Show that $|AB| = |A||B|$. 2 marks

b Generalise the result from part **a**. 1 mark

c Prove your generalisation. 3 marks

Video playlist
Matrix equations

5.3 Matrix equations

We can solve the matrix equation $2X - \begin{bmatrix} 1 & -5 \\ 11 & 3 \end{bmatrix} = \begin{bmatrix} 7 & -9 \\ 6 & 1 \end{bmatrix}$ using only matrix addition and scalar multiplication. 'Undo' the operations on the unknown, just as you would for simple linear equations.

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WORKED EXAMPLE 15 Simple matrix equations

Solve each matrix equation.

a $2X - \begin{bmatrix} 1 & -5 \\ 11 & 3 \end{bmatrix} = \begin{bmatrix} 7 & -9 \\ 6 & 1 \end{bmatrix}$

b $5Y + \begin{bmatrix} 6 & 5 \\ -2 & 0 \\ 4 & 7 \end{bmatrix} = 3\left(\begin{bmatrix} -2 & 0 \\ 8 & 5 \\ -3 & 6 \end{bmatrix} - Y \right)$

Steps	Working
a 1 Add $\begin{bmatrix} 1 & -5 \\ 11 & 3 \end{bmatrix}$ to both sides and simplify.	$2X - \begin{bmatrix} 1 & -5 \\ 11 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -5 \\ 11 & 3 \end{bmatrix} = \begin{bmatrix} 7 & -9 \\ 6 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -5 \\ 11 & 3 \end{bmatrix}$
2 Multiply both sides by $\frac{1}{2}$ and simplify.	$\frac{1}{2} \times 2X = \frac{1}{2}\begin{bmatrix} 8 & -14 \\ 17 & 4 \end{bmatrix}$ $X = \begin{bmatrix} 4 & -7 \\ \frac{17}{2} & 2 \end{bmatrix}$
b 1 Multiply out the brackets and simplify.	$5Y + \begin{bmatrix} 6 & 5 \\ -2 & 0 \\ 4 & 7 \end{bmatrix} = 3\begin{bmatrix} -2 & 0 \\ 8 & 5 \\ -3 & 6 \end{bmatrix} - 3Y$
Division of a matrix is not defined, so multiply by the reciprocal.	$5Y + \begin{bmatrix} 6 & 5 \\ -2 & 0 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 0 \\ 24 & 15 \\ -9 & 18 \end{bmatrix} - 3Y$
2 Isolate Y and simplify.	$5Y + 3Y = \begin{bmatrix} -6 & 0 \\ 24 & 15 \\ -9 & 18 \end{bmatrix} - \begin{bmatrix} 6 & 5 \\ -2 & 0 \\ 4 & 7 \end{bmatrix}$ $8Y = \begin{bmatrix} -12 & -5 \\ 26 & 15 \\ -13 & 11 \end{bmatrix}$
3 Multiply both sides by $\frac{1}{8}$ and simplify.	$Y = \frac{1}{8}\begin{bmatrix} -12 & -5 \\ 26 & 15 \\ -13 & 11 \end{bmatrix}$ $Y = \begin{bmatrix} -\frac{3}{2} & -\frac{5}{8} \\ \frac{13}{4} & \frac{15}{8} \\ -\frac{13}{8} & \frac{11}{8} \end{bmatrix}$

Exam hack

Unless you are asked for a specific format, put the answer in the format that is quickest to complete (probably the improper fraction – but do cancel by common factors).

It is very easy to make arithmetic errors with matrices, so CAS is useful.

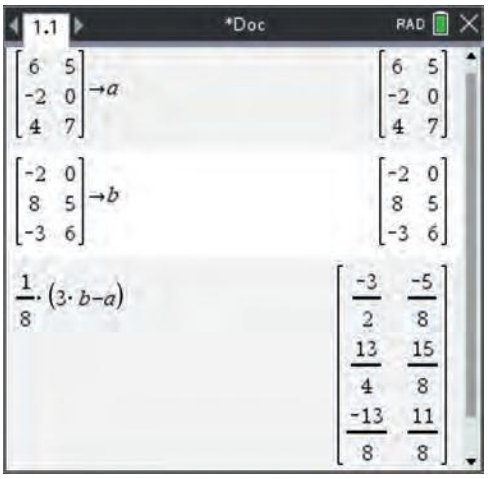
USING CAS 3 Simple matrix equations

Solve $5Y + \begin{bmatrix} 6 & 5 \\ -2 & 0 \\ 4 & 7 \end{bmatrix} = 3\left(\begin{bmatrix} -2 & 0 \\ 8 & 5 \\ -3 & 6 \end{bmatrix} - Y\right)$.

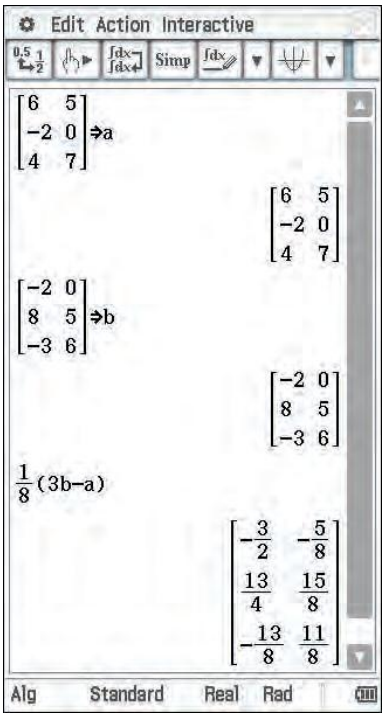
Replace matrices with variables and 'solve' the equation in terms of the variables.

$5Y + A = 3(B - Y) \Rightarrow 5Y + A = 3B - 3Y$
$\Rightarrow 5Y + 3Y = 3B - A \Rightarrow 8Y = 3B - A$
$\Rightarrow Y = \frac{1}{8}(3B - A)$

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1 Enter the matrices as **a** and **b**.
2 Enter the solution formula.

Inverses and matrix equations

We cannot divide by $\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$ to solve $\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} X = \begin{bmatrix} 2 & 7 \\ 0 & -9 \end{bmatrix}$.

To find X, we can multiply both sides by its inverse, if it exists.

The determinant $\begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} = 5$, so the matrix is non-singular and the inverse exists.

However, the determinants and inverses of higher order matrices are tedious to find.

Multiplying both sides of the matrix equation $\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} X = \begin{bmatrix} 2 & 7 \\ 0 & -9 \end{bmatrix}$ by the inverse of $\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$ on the left side gives its solution.

For equations with non-singular 2 × 2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

the inverse is given by the formula $\frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

For higher order matrices, we will usually use CAS.

Because multiplication is not commutative, we must multiply both sides of the equation on the left or both sides of the equation on the right.

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WORKED EXAMPLE 16 Solution of matrix equations involving 2 × 2 matrices

Solve each matrix equation.

a $\begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix} X = \begin{bmatrix} 4 & -10 \\ 0 & 7 \end{bmatrix}$ **b** $X \begin{bmatrix} -2 & -5 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ -6 & 10 \end{bmatrix}$

c $\begin{bmatrix} 2 & -1 \\ 4 & -3 \end{bmatrix} X + \begin{bmatrix} 3 & -4 \\ -5 & 2 \end{bmatrix} = 2X - \begin{bmatrix} 6 & -3 \\ 1 & 5 \end{bmatrix}$

Steps	Working
a 1 Find the inverse of the coefficient matrix.	$\begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix}^{-1} = \frac{1}{10-12}\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} -1 & -\frac{3}{2} \\ -2 & -\frac{5}{2} \end{bmatrix}$
2 Multiply both sides of the equation by the inverse on the left side to find the identity.	$\begin{bmatrix} -1 & -\frac{3}{2} \\ -2 & -\frac{5}{2} \end{bmatrix}\begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix} X = \begin{bmatrix} -1 & -\frac{3}{2} \\ -2 & -\frac{5}{2} \end{bmatrix}\begin{bmatrix} 4 & -10 \\ 0 & 7 \end{bmatrix}$
3 Simplify and write the answer.	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X = \begin{bmatrix} -4 & -\frac{1}{2} \\ -8 & \frac{5}{2} \end{bmatrix}$ Actually do the multiplication that should give I to check the inverse is correct. $X = \begin{bmatrix} -4 & -\frac{1}{2} \\ -8 & \frac{5}{2} \end{bmatrix}$
b 1 Find the inverse of the coefficient matrix.	$\begin{bmatrix} -2 & -5 \\ 4 & 7 \end{bmatrix}^{-1} = \frac{1}{-14-(-20)}\begin{bmatrix} 7 & 5 \\ -4 & -2 \end{bmatrix} = \begin{bmatrix} \frac{7}{6} & \frac{5}{6} \\ -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}$
2 Multiply both sides of the equation by the inverse on the right side.	$X\begin{bmatrix} -2 & -5 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} \frac{7}{6} & \frac{5}{6} \\ -\frac{2}{3} & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ -6 & 10 \end{bmatrix}\begin{bmatrix} \frac{7}{6} & \frac{5}{6} \\ -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}$
3 Simplify and write the answer.	$X\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -\frac{41}{3} & -\frac{25}{3} \end{bmatrix}$ $X = \begin{bmatrix} 2 & 1 \\ -\frac{41}{3} & -\frac{25}{3} \end{bmatrix}$
c 1 Move variables to the LHS and constant matrices to the RHS.	$\begin{bmatrix} 2 & -1 \\ 4 & -3 \end{bmatrix} X - 2X = -\begin{bmatrix} 6 & -3 \\ 1 & 5 \end{bmatrix} - \begin{bmatrix} 3 & -4 \\ -5 & 2 \end{bmatrix}$
2 Insert I on the same side of X as the other coefficient matrix.	$\begin{bmatrix} 2 & -1 \\ 4 & -3 \end{bmatrix} X - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X = \begin{bmatrix} -6 & 3 \\ -1 & -5 \end{bmatrix} + \begin{bmatrix} -3 & 4 \\ 5 & -2 \end{bmatrix}$

3 Factorise and simplify.

$$\left(\begin{bmatrix} 2 & -1 \\ 4 & -3 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) X = \begin{bmatrix} -9 & 7 \\ 4 & -7 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 4 & -5 \end{bmatrix} X = \begin{bmatrix} -9 & 7 \\ 4 & -7 \end{bmatrix}$$

4 Find the inverse of the coefficient matrix.

$$\begin{bmatrix} 0 & -1 \\ 4 & -5 \end{bmatrix}^{-1} = \frac{1}{0-(-4)} \begin{bmatrix} -5 & 1 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} \frac{-5}{4} & \frac{1}{4} \\ -1 & 0 \end{bmatrix}$$

5 Multiply both sides by the inverse on the left side.

$$\begin{bmatrix} \frac{-5}{4} & \frac{1}{4} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 4 & -5 \end{bmatrix} X = \begin{bmatrix} \frac{-5}{4} & \frac{1}{4} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -9 & 7 \\ 4 & -7 \end{bmatrix}$$

6 Simplify and write the answer.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X = \begin{bmatrix} \frac{49}{4} & -\frac{21}{2} \\ 9 & -7 \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{49}{4} & -\frac{21}{2} \\ 9 & -7 \end{bmatrix}$$

In many cases, the equation $AX = B$ has column matrices for X and A, such as

$$\begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

In this case, we still need to multiply both sides by the inverse of A on the left side. This is called *left multiplying*.

WORKED EXAMPLE 17 Solution of matrix equations with column matrices

Solve $\begin{bmatrix} 2 & 6 \\ -3 & -7 \end{bmatrix} X = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$.

Steps	Working
1 Find the inverse of the coefficient matrix.	$\begin{bmatrix} 2 & 6 \\ -3 & -7 \end{bmatrix}^{-1} = \frac{1}{2\times(-7)-6\times(-3)} \begin{bmatrix} -7 & -6 \\ 3 & 2 \end{bmatrix}$ $= \begin{bmatrix} -\frac{7}{4} & -\frac{3}{2} \\ \frac{3}{4} & \frac{1}{2} \end{bmatrix}$
2 Left multiply both sides by the inverse.	$\begin{bmatrix} -\frac{7}{4} & -\frac{3}{2} \\ \frac{3}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 6 \\ -3 & -7 \end{bmatrix} X = \begin{bmatrix} -\frac{7}{4} & -\frac{3}{2} \\ \frac{3}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 \\ -6 \end{bmatrix}$
3 Simplify and write the answer.	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ $X = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

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USING CAS 4 Matrix equations

Solve

a $X\begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix} - \begin{bmatrix} -4 & 5 \\ 2 & 7 \end{bmatrix} = 5X + \begin{bmatrix} 2 & 3 \\ -5 & 6 \end{bmatrix}$ **b** $\begin{bmatrix} 3 & 0 & 1 \\ -2 & 5 & 2 \\ 4 & -4 & -1 \end{bmatrix} X = \begin{bmatrix} -2 & 3 & 0 \\ 4 & 2 & 1 \\ -1 & 5 & -3 \end{bmatrix}$

a Replace matrices by variables and 'solve' equation in terms of variables. Remember to put I on the same side of X in $5X$ as the variable matrix in the other term with X.

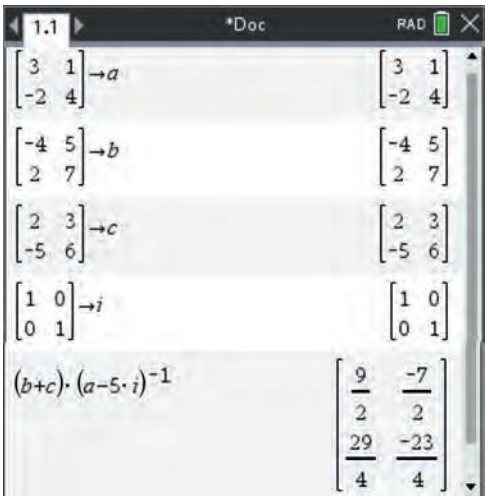
$$XA - B = 5XI + C$$
$$XA - 5XI = B + C$$
$$X(A - 5I) = B + C$$
$$X(A - 5I)(A - 5I)^{-1} = (B + C)(A - 5I)^{-1}$$
$$X = (B + C)(A - 5I)^{-1}$$

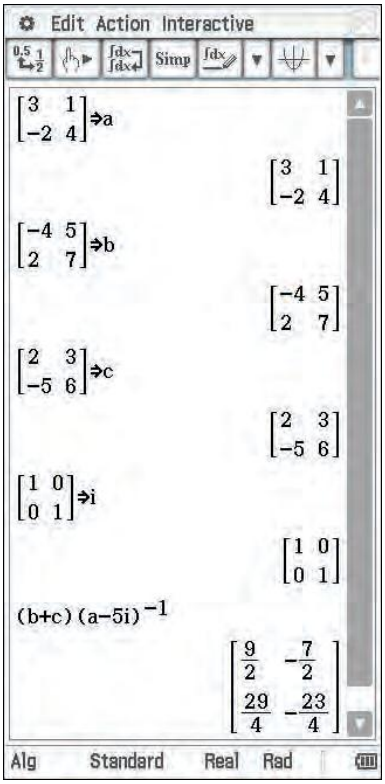
b Replace matrices by variables and 'solve' equation in terms of variables.

$$AX = B$$
$$X = A^{-1}B$$

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a

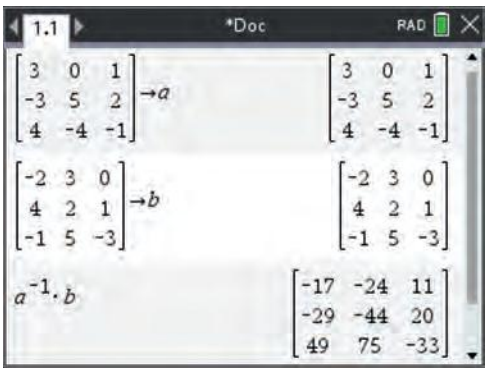


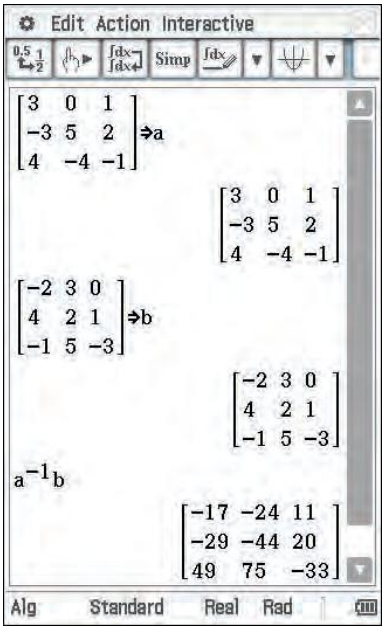


1 Enter the matrices as **a**, **b**, **c** and **i**.

2 Enter the solution formula.

b





1 Enter the matrices as **a** and **b**.

2 Enter the solution formula.

9780170448338

VCE QUESTION ANALYSIS

A VCE-style question **(11 marks)**

Consider the matrices $A = \begin{bmatrix} 3 & 4 \\ -2 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 5 \\ 2 & 4 \end{bmatrix}$.

a Find AB. 2 marks

b Find $|A|$, $|B|$ and $|AB|$. 3 marks

c What do you notice about $|A|$, $|B|$ and $|AB|$? 1 mark

d Generalise your finding above. 1 mark

e Show your generalisation for 2×2 matrices. 4 marks

Reading the question

- The product is needed.
- The determinants are needed.
- Compare the determinants.
- Make the statement for 2×2 matrices to generalise.
- Work from the LHS to the RHS to show the generalisation.

Thinking about the question

- Multiply row by column and add.
- Use $ad - bc$ for the determinants.
- Use general 2×2 matrices (a_{ij}) and (b_{ij}) to show the generalisation.

Worked solution (✓ = 1 mark)

a $AB = \begin{bmatrix} 3 & 4 \\ -2 & -5 \end{bmatrix} \begin{bmatrix} 4 & 5 \\ 2 & 4 \end{bmatrix}$ ✓ $= \begin{bmatrix} \mathbf{20} & \mathbf{31} \\ \mathbf{-18} & \mathbf{-30} \end{bmatrix}$ ✓

b $|A| = \mathbf{-7}$ ✓, $|B| = \mathbf{6}$ ✓ and $|AB| = \mathbf{-42}$ ✓

c $\boldsymbol{|AB| = |A||B|}$ ✓

d $\boldsymbol{|AB| = |A||B|}$ **for all 2×2 matrices.** ✓

e Let $A = (a_{ij})$ and $B = (b_{ij})$ for $i, j \in \{1, 2\}$ and $a_{ij}, b_{ij} \in R$

$$AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} \checkmark$$

$|A| = a_{11}a_{22} - a_{12}a_{21}$

$|B| = b_{11}b_{22} - b_{12}b_{21}$

$|A||B| = (a_{11}a_{22} - a_{12}a_{21})(b_{11}b_{22} - b_{12}b_{21})$

$= a_{11}a_{22}b_{11}b_{22} - a_{11}a_{22}b_{12}b_{21} - a_{12}a_{21}b_{11}b_{22} + a_{12}a_{21}b_{12}b_{21}$ ✓

$|AB| = (a_{11}b_{11} + a_{12}b_{21})(a_{21}b_{12} + a_{22}b_{22}) - (a_{11}b_{12} + a_{12}b_{22})(a_{21}b_{11} + a_{22}b_{21})$ ✓

$= a_{11}b_{11}a_{21}b_{12} + a_{11}b_{11}a_{22}b_{22} + a_{12}b_{21}a_{21}b_{12} + a_{12}b_{21}a_{22}b_{22}$
$- a_{11}b_{12}a_{21}b_{11} - a_{11}b_{12}a_{22}b_{21} - a_{12}b_{22}a_{21}b_{11} - a_{12}b_{22}a_{22}b_{21}$

$= a_{11}b_{11}a_{22}b_{22} - a_{11}b_{12}a_{22}b_{21} - a_{12}b_{22}a_{21}b_{11} + a_{12}b_{21}a_{21}b_{12}$
$+ a_{11}b_{11}a_{21}b_{12} + a_{12}b_{21}a_{22}b_{22} - a_{11}b_{12}a_{21}b_{11} - a_{12}b_{22}a_{22}b_{21}$

$= \boldsymbol{a_{11}a_{22}b_{11}b_{22} - a_{11}a_{22}b_{12}b_{21} - a_{12}a_{21}b_{11}b_{22} + a_{12}a_{21}b_{12}b_{21}}$
$\boldsymbol{+ a_{11}b_{11}a_{21}b_{12} - a_{11}b_{11}a_{21}b_{12} + a_{12}b_{21}a_{22}b_{22} - a_{12}b_{21}a_{22}b_{22}}$

$= \boldsymbol{a_{11}a_{22}b_{11}b_{22} - a_{11}a_{22}b_{12}b_{21} - a_{12}a_{21}b_{11}b_{22} + a_{12}a_{21}b_{12}b_{21}}$

$= \boldsymbol{|A||B|}$ ✓

Video
VCE question analysis: Matrices

EXERCISE 5.3 Matrix equations

ANSWERS p. 580

Recap

1 The determinant of $\begin{bmatrix} 3 & -5 \\ 2 & 4 \end{bmatrix}$ is

A -23 **B** -7 **C** -2 **D** 2 **E** 22

2 The determinant of $\begin{bmatrix} 9 & 2 & 1 \\ 5 & 3 & 0 \\ -3 & 4 & -2 \end{bmatrix}$ is

A -54 **B** -5 **C** 23 **D** 29 **E** 64

Mastery

3 WORKED EXAMPLE 15 Solve each matrix equation.

a $3X + \begin{bmatrix} -5 & 5 \\ 7 & -1 \end{bmatrix} = \begin{bmatrix} -8 & -8 \\ 6 & -5 \end{bmatrix}$

b $\begin{bmatrix} 6 & 2 \\ -1 & 4 \end{bmatrix} - 2Y = Y - \begin{bmatrix} -8 & 7 \\ 8 & 2 \end{bmatrix}$

c $4Z - \begin{bmatrix} 5 & 8 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -6 & 8 \end{bmatrix} + 2Z$

d $A + \begin{bmatrix} 4 & -6 \\ -5 & 6 \end{bmatrix} = 5A - \begin{bmatrix} -2 & -2 \\ 1 & 5 \end{bmatrix}$

e $6P + \begin{bmatrix} 4 & 5 & -4 \\ 4 & 6 & 5 \\ 6 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 6 & 3 \\ 2 & -1 & -6 \\ 1 & 5 & 2 \end{bmatrix}$

f $\begin{bmatrix} -3 & -6 & -3 \\ -4 & 6 & 0 \\ 8 & -5 & 8 \end{bmatrix} - T = 3\left(T - \begin{bmatrix} -1 & 9 & -6 \\ 9 & -8 & -4 \\ 7 & -6 & 5 \end{bmatrix}\right)$

g $-7\begin{bmatrix} -7 & 7 \\ 5 & -4 \\ 5 & -5 \end{bmatrix} - 5X = 5X + \begin{bmatrix} 7 & 9 \\ 3 & 5 \\ -4 & -3 \end{bmatrix}$

h $4\begin{bmatrix} 8 & 6 & 4 \\ 1 & 8 & 4 \end{bmatrix} + K = \begin{bmatrix} 0 & 5 & 1 \\ 4 & 9 & 3 \end{bmatrix} + 3K$

i $7Y + 2\begin{bmatrix} 9 & 9 \\ 5 & -9 \\ -3 & 8 \end{bmatrix} = 2Y + 3\begin{bmatrix} -8 & -1 \\ -7 & 0 \\ -3 & 8 \end{bmatrix}$

j $2\left(M - \begin{bmatrix} -4 & -3 & -3 \\ -1 & -7 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 2 & -2 \\ -1 & 0 & -5 \end{bmatrix} - M$

4 Using CAS 3 Solve each matrix equation.

a $3X + \begin{bmatrix} 9 & 5 \\ -5 & -7 \end{bmatrix} = \begin{bmatrix} -5 & 7 \\ 6 & 9 \end{bmatrix}$

b $5Y - \begin{bmatrix} -2 & 7 \\ -6 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 5 \\ 3 & 0 \end{bmatrix} - Y$

c $\begin{bmatrix} 2 & 9 \\ 8 & -4 \end{bmatrix} - 3Z = Z + 4\begin{bmatrix} 1 & 1 \\ 2 & -5 \end{bmatrix}$

d $4D + \begin{bmatrix} 0 & 1 & -5 \\ 1 & -9 & -6 \\ 2 & -4 & 0 \end{bmatrix} = 2\begin{bmatrix} -1 & -7 & 2 \\ 0 & 6 & -3 \\ -9 & 4 & -4 \end{bmatrix}$

e $M + 3\begin{bmatrix} 7 & -7 \\ 5 & -7 \\ -9 & -3 \end{bmatrix} = 4M - \begin{bmatrix} 3 & 2 \\ 1 & 0 \\ 8 & -7 \end{bmatrix}$

f $V - 3\begin{bmatrix} -9 & 2 & -3 \\ -3 & 6 & 7 \end{bmatrix} = 2\begin{bmatrix} 3 & -7 & 0 \\ -2 & 2 & -6 \end{bmatrix} + 5V$

g $\begin{bmatrix} -7 & 5 & 9 \\ -9 & -6 & 7 \\ -8 & -6 & 8 \end{bmatrix} - 3F = 2F - 3\begin{bmatrix} 1 & 5 & -5 \\ -4 & -9 & 6 \\ 1 & 0 & -1 \end{bmatrix}$

h $3\begin{bmatrix} 5 & 3 & 8 & 7 \\ -9 & 4 & -7 & 2 \\ -5 & 9 & 4 & -5 \end{bmatrix} - 5T = \begin{bmatrix} 3 & 1 & 0 & 8 \\ 6 & -3 & -7 & 1 \\ 8 & -9 & 8 & 6 \end{bmatrix}$

5 WORKED EXAMPLE 16 Solve each matrix equation.

a $\begin{bmatrix} 2 & -3 \\ 4 & -4 \end{bmatrix} P = \begin{bmatrix} 7 & -4 \\ 3 & -5 \end{bmatrix}$

b $R\begin{bmatrix} -2 & 5 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ 7 & 4 \end{bmatrix}$

c $M\begin{bmatrix} -2 & -7 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 0 & 3 \end{bmatrix}$

d $\begin{bmatrix} 0 & -1 \\ 6 & 3 \end{bmatrix} Z = \begin{bmatrix} 5 & -1 \\ -6 & 4 \end{bmatrix}$

e $2M + \begin{bmatrix} 2 & 4 \\ 3 & -8 \end{bmatrix} = M\begin{bmatrix} 8 & 6 \\ 3 & 3 \end{bmatrix} - \begin{bmatrix} -9 & 1 \\ -5 & 0 \end{bmatrix}$

f $\begin{bmatrix} -8 & -6 \\ -1 & -6 \end{bmatrix} X - 3\begin{bmatrix} 7 & -4 \\ 8 & -4 \end{bmatrix} = 4X + \begin{bmatrix} 6 & -5 \\ 3 & -5 \end{bmatrix}$

g $Z\begin{bmatrix} 8 & 3 \\ -3 & 8 \end{bmatrix} + \begin{bmatrix} -3 & 8 \\ 0 & 8 \end{bmatrix} = 3Z + \begin{bmatrix} 0 & -2 \\ 7 & -2 \end{bmatrix}$

h $B\begin{bmatrix} 2 & -7 \\ -1 & -8 \end{bmatrix} + \begin{bmatrix} -9 & 1 \\ 2 & 3 \end{bmatrix} = B\begin{bmatrix} 1 & 3 \\ -9 & 7 \end{bmatrix} - \begin{bmatrix} 3 & -8 \\ 0 & 2 \end{bmatrix}$

i $X\begin{bmatrix} -5 & -3 \\ 8 & -4 \end{bmatrix} + \begin{bmatrix} 8 & 0 \\ 2 & -5 \end{bmatrix} = 2X + \begin{bmatrix} 8 & 7 \\ 8 & -2 \end{bmatrix} - X\begin{bmatrix} -2 & -1 \\ -9 & -1 \end{bmatrix}$

j $\begin{bmatrix} 3 & -5 \\ 2 & 0 \end{bmatrix} Y + Y - \begin{bmatrix} 0 & -1 \\ -8 & -7 \end{bmatrix} = \begin{bmatrix} -6 & 8 \\ -9 & 7 \end{bmatrix} Y - 2\begin{bmatrix} -4 & 3 \\ 6 & 6 \end{bmatrix}$

6 WORKED EXAMPLE 17 Solve each matrix equation.

a $\begin{bmatrix} -3 & -3 \\ -9 & -8 \end{bmatrix} C = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$ **b** $\begin{bmatrix} -1 & 7 \\ 2 & 0 \end{bmatrix} D = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$ **c** $\begin{bmatrix} 2 & -8 \\ 1 & 2 \end{bmatrix} A = \begin{bmatrix} 0 \\ -9 \end{bmatrix}$

d $\begin{bmatrix} 4 & 0 \\ -6 & -5 \end{bmatrix} B = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ **e** $\begin{bmatrix} -3 & 0 \\ 1 & 2 \end{bmatrix} T = \begin{bmatrix} 8 \\ 7 \end{bmatrix}$ **f** $\begin{bmatrix} -3 & -6 \\ -4 & 0 \end{bmatrix} Y = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

g $\begin{bmatrix} -1 & -4 \\ 9 & -6 \end{bmatrix} X = \begin{bmatrix} -9 \\ 7 \end{bmatrix}$ **h** $\begin{bmatrix} 6 & 1 \\ -5 & -8 \end{bmatrix} T = \begin{bmatrix} -9 \\ 0 \end{bmatrix}$ **i** $\begin{bmatrix} 8 & -4 \\ -1 & -3 \end{bmatrix} X = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

7 Using CAS 4 Solve each matrix equation.

a $M\begin{bmatrix} -4 & 4 \\ 9 & -6 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 4 & 8 \end{bmatrix}$

b $\begin{bmatrix} 1 & -8 \\ -6 & -3 \end{bmatrix} X = \begin{bmatrix} 6 & -4 \\ -3 & 8 \end{bmatrix}$

c $\begin{bmatrix} 5 & 6 & 0 \\ 4 & -8 & -6 \\ 2 & 9 & -6 \end{bmatrix} Y = \begin{bmatrix} 5 & 0 & -9 \\ 6 & 3 & -9 \\ 1 & -7 & 4 \end{bmatrix}$

d $N\begin{bmatrix} 0 & 2 & 0 \\ -7 & 9 & 4 \\ -1 & -5 & -1 \end{bmatrix} = \begin{bmatrix} 8 & 8 & -5 \\ -6 & 4 & 0 \\ -5 & -5 & 9 \end{bmatrix}$

e $\begin{bmatrix} 6 & -1 \\ -1 & -3 \end{bmatrix} A + \begin{bmatrix} 7 & 8 \\ 6 & -4 \end{bmatrix} = 3A + \begin{bmatrix} 0 & 8 \\ -6 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 5 & -4 \end{bmatrix} A$

f $B\begin{bmatrix} -4 & 5 \\ -7 & -3 \end{bmatrix} + 5B - \begin{bmatrix} 0 & 7 \\ -1 & -1 \end{bmatrix} = B\begin{bmatrix} 1 & 8 \\ 7 & -3 \end{bmatrix} - \begin{bmatrix} 6 & 3 \\ 1 & -7 \end{bmatrix}$

g $\begin{bmatrix} -7 & 9 \\ -1 & -5 \end{bmatrix} X + \begin{bmatrix} 4 & -6 \\ -1 & -5 \end{bmatrix} - 4X = 3\begin{bmatrix} 0 & -1 \\ 9 & 5 \end{bmatrix} + \begin{bmatrix} -3 & 6 \\ 1 & -4 \end{bmatrix} X$

h $\begin{bmatrix} 6 & -1 & -4 \\ -2 & 0 & 0 \\ -2 & 0 & -4 \end{bmatrix} P - \begin{bmatrix} -3 & -4 & -3 \\ 3 & 1 & 0 \\ -3 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 5 \\ -1 & -4 & 3 \\ 1 & -5 & 0 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 3 \\ -4 & -2 & 0 \\ 1 & 3 & 1 \end{bmatrix} P$

i $T\begin{bmatrix} -3 & -3 & 4 \\ 0 & 4 & 3 \\ 2 & -4 & 4 \end{bmatrix} + 2T + \begin{bmatrix} 1 & -4 & -2 \\ -4 & -2 & -4 \\ 0 & -2 & 2 \end{bmatrix} = T\begin{bmatrix} -4 & -2 & -5 \\ 0 & -1 & 2 \\ -1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 5 & -3 & 3 \\ -1 & 5 & -2 \\ 0 & -3 & 2 \end{bmatrix}$

j $\begin{bmatrix} -5 & 8 & 0 \\ -1 & -7 & 3 \\ 5 & 0 & -2 \end{bmatrix} X - \begin{bmatrix} -6 & 8 & 6 \\ 6 & -8 & -5 \\ -2 & 0 & -8 \end{bmatrix} = 2X + \begin{bmatrix} -4 & -6 & 2 \\ 4 & 8 & -3 \\ 1 & -7 & 2 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 4 \\ -4 & 0 & -6 \\ 3 & 8 & 8 \end{bmatrix} X$

Exam practice

8 TECH-FREE (2 marks) Solve the matrix equation $6T - \begin{bmatrix} 1 & -6 & 0 \\ -3 & 7 & 5 \\ 9 & 0 & -1 \end{bmatrix} = T + \begin{bmatrix} -8 & 8 & -7 \\ -5 & 4 & 0 \\ -7 & 2 & -9 \end{bmatrix}$.

9 TECH-FREE (3 marks) Solve the matrix equation $\begin{bmatrix} 2 & -4 \\ -4 & 4 \end{bmatrix} X + \begin{bmatrix} -4 & 3 \\ 9 & -8 \end{bmatrix} = \begin{bmatrix} 0 & -7 \\ 6 & 5 \end{bmatrix} - X$.

10 The inverse of $\begin{bmatrix} 5 & 2 \\ -6 & -2 \end{bmatrix}$ is

A 6 **B** $\begin{bmatrix} -2 & -6 \\ 2 & 5 \end{bmatrix}$ **C** $\begin{bmatrix} -1 & 1 \\ -3 & \frac{5}{2} \end{bmatrix}$ **D** $\begin{bmatrix} -1 & -1 \\ 3 & -\frac{5}{2} \end{bmatrix}$ **E** $\begin{bmatrix} \frac{5}{2} & -3 \\ 1 & -1 \end{bmatrix}$

11 The solution of $\begin{bmatrix} -8 & 4 \\ 5 & -2 \end{bmatrix} X = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$ is

A $\begin{bmatrix} -2 & -4 \\ -5 & -8 \end{bmatrix}$ **B** $\begin{bmatrix} 0 \\ -2 \end{bmatrix}$ **C** $\begin{bmatrix} 80 \\ -48 \end{bmatrix}$ **D** $\begin{bmatrix} \frac{1}{2} & -1 \\ -\frac{5}{4} & 2 \end{bmatrix}$ **E** $\begin{bmatrix} 64 \\ 20 \end{bmatrix}$

12 The solution of $M\begin{bmatrix} 2 & 0 \\ -4 & -2 \end{bmatrix} + \begin{bmatrix} -6 & 2 \\ 0 & -2 \end{bmatrix} = 3M + \begin{bmatrix} -6 & 6 \\ -4 & -2 \end{bmatrix} - M\begin{bmatrix} 7 & -2 \\ -3 & 8 \end{bmatrix}$ is

A $\begin{bmatrix} 7 & 6 \\ -3 & -2 \end{bmatrix}$ **B** $\begin{bmatrix} -11 & 4 \\ -27 & 8 \end{bmatrix}$ **C** $\begin{bmatrix} -2 & 1 \\ -3 & \frac{1}{2} \end{bmatrix}$ **D** $\begin{bmatrix} 5 & 6 \\ -10 & -8 \end{bmatrix}$ **E** $\begin{bmatrix} \frac{23}{4} & \frac{1}{2} \\ \frac{3}{2} & 1 \end{bmatrix}$

5 Chapter summary

- A **matrix** is a rectangular array of numbers or symbols enclosed by brackets, symbolised by a capital letter, like $A = \begin{bmatrix} 2 & -4 & 60 & 1 \\ 5 & 2.8 & -70 & 0 \end{bmatrix} = (a_{ij})$.

  The **dimensions** of the matrix are the number of rows m and columns n, shown as $m \times n$.

  The **elements** are the numbers in the matrix.

  The element in row i and column j of the matrix A is shown as a_{ij}.
- A **square matrix** with n rows and n columns has dimension n.
- A **column matrix** is $(n \times 1)$: with only one column. A **row matrix** is $(1 \times n)$: it has only one row.

Matrix addition, subtraction and multiplication by a scalar

- For addition and subtraction, matrices must be the same dimensions.
  - To **add matrices**, add the corresponding elements: $A + B = (a_{ij} + b_{ij})$.
  - To **subtract matrices**, subtract the corresponding elements: $A - B = (a_{ij} - b_{ij})$.
  - The **zero matrix**, 0, has every element 0.
  - The **negative of a matrix** has every element of opposite sign: $-A = (-a_{ij})$ is the **additive inverse** of the matrix: $A + (-A) = 0$.
  - A **scalar multiple** has every element multiplied by a real number: $cA = (ca_{ij})$.
- Matrix addition is **commutative**: the order of addition of 2 matrices doesn't matter: $A + B = B + A$.
- It is associative: the order of addition of 3 matrices doesn't matter: $(A + B) + C = A + (B + C)$.
- Scalar multiplication is also commutative and associative: $xA = Ax$, $x(yA) = (xy)A$ and $(xA)\,B = x(AB)$.
- Scalar multiplication is distributive over addition: $x(A + B) = xA + xB$ for all $x \in R$.

Matrix multiplication, identity and inverses

- Matrices A and B are **conformable** if the number of columns in A is the same as number of rows in B: they can be multiplied.
- To multiply the $m \times n$ matrix A by the $n \times p$ matrix B gives $AB = C$. Multiply each element of row i in A by the elements of each column of B and then add them together.
- Matrix multiplication is not commutative. In general, $AB \neq BA$.
- Matrix multiplication is associative: $(AB)C = A(BC)$.
- Matrix multiplication is distributive over addition (left and right): $A(B + C) = AB + AC$ and $(A + B)\,C = AC + BC$.
- The identity for multiplication of square matrices of dimension n is I_n, which has 1s along its leading diagonal and 0s elsewhere; for example, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- The multiplicative inverse of a (square) matrix A, if it exists, is the matrix B such that the product of A and B is the identity: $AB = BA = I$.
- When it exists, the multiplicative inverse of A is written as A^{-1}. It is usually just called the inverse.
- A matrix that has no inverse is singular.
- A matrix that has an inverse is invertible or non-singular.

Determinants

- A **diagonal matrix** is a square matrix A with zeros everywhere except the leading diagonal: $a_{ij} = 0$ for $i \neq j$.
- For a diagonal matrix, the product $a_1a_2a_3 \ldots a_n$ is called the **determinant** of A and is written as $|A|$ or $\det(A)$.
- If $A = \begin{bmatrix} a_1 & 0 & 0.. \\ 0 & a_2 & 0.. \\ 0 & 0 & \ddots \\ \vdots & \vdots & \end{bmatrix}$ is non-singular, then $A^{-1} = \begin{bmatrix} a_1^{-1} & 0 & 0.. \\ 0 & a_2^{-1} & 0.. \\ 0 & 0 & \ddots \\ \vdots & \vdots & \end{bmatrix}$.
- All the elements on one side of a **triangular** matrix A are zero: one of $a_{ij} = 0$ for $i > j$ or $a_{ij} = 0$ for $a < j$ is true, so $\begin{bmatrix} 2 & 4 & -3 \\ 0 & -1 & 2 \\ 0 & 0 & 5 \end{bmatrix}$ is triangular. For a triangular matrix $A = (a_{ij})$, the determinant is $\det(A) = a_{11}a_{22}a_{33} \ldots a_n$.
- A triangular matrix is invertible if and only if $|A| = \det(A) \neq 0$.
- For the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is $|A| = ad - bc$.
- If $|A| \neq 0$, the inverse is given by $\frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
- The determinant of a 1×1 matrix $A = (a_{11})$ is element: $|A| = a_{11}$.
- The minor M_{ij} of the matrix $n \times n$ matrix A is the $(n - 1) \times (n - 1)$ matrix obtained by omitting the ith row and jth column.
- $A = (a_{ij}) \Rightarrow M_{ij} = (a_{pq})$, $p \in \{1, 2 \ldots i - 1, i + 1 \ldots n\}$, $q \in \{1, 2 \ldots j - 1, j + 1 \ldots n\}$.
- The cofactor A_{ij} is the signed determinant of M_{ij}: $A_{ij} = (-1)^{i+j}|M_{ij}|$.
- The determinant of A, $|A| = \det(A) = \sum_{j=1}^{n} a_{ij}A_{ij}$ (for row i) or $\sum_{i=1}^{n} a_{ij}A_{ij}$ (for column j).
- The value of the determinant is independent of the row or column chosen for the expansion.
- The 'shortcut' method for the determinant of a 3×3 matrix:

  1 For $M = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}$, write the elements *twice*:

  $$\begin{matrix} a & b & c & a & b & c \\ p & q & r & p & q & r \\ x & y & z & x & y & z \end{matrix}.$$

  2 Starting with the first a, b and c, multiply diagonally and add: $aqz + brx + cpy$.

  3 With the other a, b and c, multiply back diagonally and add: $ary + bpz + cqx$.

  4 Subtract the answers to get the determinant: $|M| = (aqz + brx + cpy) - (ary + bpz + cqx)$.

Matrix equations

- Any matrix equation that does not involve matrix multiplication is solvable: undo the operations on the unknown, just as we would for simple linear equations.
- Matrix equations involving multiplication require finding the inverse of a coefficient matrix. For a solution to exist, it must be non-singular.

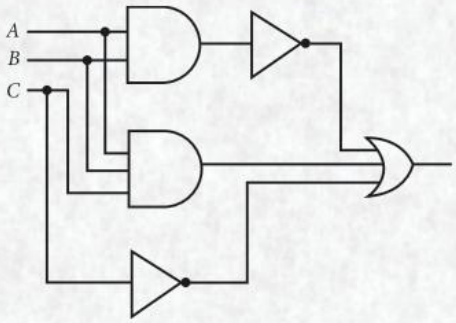
Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 (2 marks) Minimise the sum of products $\overline{A}BC + ABC + AC + \overline{C}$.

2 (3 marks) Find the simplified output of the logic circuit below.



3 (2 marks) Solve $\begin{bmatrix} 6 & -4 \\ 4 & -3 \end{bmatrix} X = \begin{bmatrix} 5 & 8 \\ -6 & -7 \end{bmatrix}$.

4 (3 marks) Find the determinant of $\begin{bmatrix} 2 & -3 & -1 \\ 0 & 4 & -3 \\ 1 & 2 & -5 \end{bmatrix}$.

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

A	B	C	
0	0	0	1
1	0	0	1
0	1	0	1
0	0	1	0
1	1	0	1
1	0	1	0
0	1	1	0
1	1	1	0

1 The truth table above shows

A $\overline{AB} + C$ **B** $A + B + C$ **C** $\overline{A} + BC$

D $AB + \overline{C}$ **E** $AB\overline{C}$

2 What does the following pseudocode print?

```
N ← 6
I ← 1
S ← 0
DO
S ← I^3 + S
I ← I + 1
WHILE I < N
ENDWHILE
PRINT S
```

A 125 **B** 216 **C** 225 **D** 441 **E** 784

3 The solution of $\begin{bmatrix} 4 & -4 \\ 7 & -7 \end{bmatrix} - X = \begin{bmatrix} 8 & -8 \\ 11 & -3 \end{bmatrix} + 3X$ is given by

A $\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$ **B** $\begin{bmatrix} 3 & -3 \\ 4.5 & -2.5 \end{bmatrix}$ **C** $\begin{bmatrix} 6 & -6 \\ 9 & -5 \end{bmatrix}$

D $\begin{bmatrix} 3 & 1 \\ -1 & -2.5 \end{bmatrix}$ **E** It is not solvable since $\begin{bmatrix} 4 & -4 \\ 7 & -7 \end{bmatrix}$ is singular.

9780170448338

4 What is the solution of $3X + \begin{bmatrix} 2 & -5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix} - X\begin{bmatrix} -1 & -5 \\ -1 & 0 \end{bmatrix}$?

A $\begin{bmatrix} -0.4 & -2.6 \\ 1 & -9 \end{bmatrix}$ **B** $\begin{bmatrix} 11 & 19 \\ 19 & 30 \end{bmatrix}$ **C** $\begin{bmatrix} -8 & 5 \\ 1 & -1.4 \end{bmatrix}$

D $\begin{bmatrix} 49 & -19 \\ 19 & -8 \end{bmatrix}$ **E** $\begin{bmatrix} -1 & 1 \\ -\frac{29}{7} & \frac{60}{7} \end{bmatrix}$

5 The determinant of $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ -4 & 3 & -1 & 0 \\ 8 & 2 & 6 & -3 \end{bmatrix}$ is

A -24 **B** -8 **C** 0 **D** 8 **E** 24

Section B 2 questions

15 marks

1 (7 marks)

a Construct a flowchart for an algorithm to find the highest common divisor of 2 positive integers input by a user. 4 marks

b Write pseudocode for your flowchart. 3 marks

2 (8 marks) Consider the matrices $A = \begin{bmatrix} -1 & -9 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 4 \\ -6 & 5 \end{bmatrix}$.

a Find A^2 and B^2. 2 marks

b Describe A and B in terms of identities and inverses. 1 mark

c Find conditions for the matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to have the same property. 5 marks

CHAPTER

6 COUNTING TECHNIQUES

Study Design coverage

Nelson MindTap chapter resources

6.1 The addition and multiplication principles
- The addition principle
- The multiplication principle
- Tree diagrams

6.2 Permutations
- **Using CAS 1:** Permutations
- Factorials
- **Using CAS 2:** Factorials
- Calculating ${}^nP_r$
- Arrangements with restrictions
- Arrangements with identical objects
- Arrangements with groups

6.3 Combinations
- Calculating ${}^nC_r$
- Pascal's triangle and combinations
- **Using CAS 3:** Combinations
- Combinations with restrictions
- Pascal's identity
- Interpreting Pascal's identity
- Combinations taken any number at a time

6.4 The pigeon-hole principle

6.5 The inclusion-exclusion principle
- Basic set theory
- The inclusion-exclusion principle with 2 sets
- The inclusion-exclusion principle with 3 sets

VCE question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2

9780170448338

Study Design coverage

UNIT 1, AREA OF STUDY 2: DISCRETE MATHEMATICS

Combinatorics

- the pigeon-hole principle and its use in solving problems and proving results
- the inclusion-exclusion principle for the union of two sets and the union of three sets
- permutations and combinations and their use in solving problems involving arrangements and selections with or without repeated elements
- derivation and application of simple combinatorial identities.

VCE Mathematics Study Design 2023–2027 p. 60, © VCAA 2022

Video playlists (6):

6.1 The addition and multiplication principles
6.2 Permutations
6.3 Combinations
6.4 The pigeon-hole principle
6.5 The inclusion-exclusion principle
VCE question analysis Counting techniques

Worksheets (12):

6.2 Factorial notation • Permutation calculations • Counting techniques • Permutations with repetitions
6.3 Combinations calculations • Combinations problems • Combinations assignment • Combinations • Permutations and combinations • Pascal's triangle 1 • Pascal's triangle 2
6.5 The inclusion-exclusion principle

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

6.1 The addition and multiplication principles

Video playlist
The addition and multiplication principles

The number of possibilities in an event or set is often too large to list and count. Counting techniques provide many different methods that can be used to find the number of possible outcomes in the sample space.

The addition principle

Maddi is going to watch a movie tonight and she has a number of different choices. She can choose from 3 horror movies, 4 romance movies or 5 science fiction movies.

The total number of possibilities is $3 + 4 + 5 = 12$.

The addition principle

In an event in which we can choose a of one type OR b of another type OR c of another type, the total number of choices can be calculated by adding.

The total number of choices = $a + b + c$.

Exam hack

Generally, adding the number of possibilities is associated with the word 'OR' in a problem.

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WORKED EXAMPLE 1 Applying the addition principle

For breakfast, Elijah can choose only 1 food item from: 3 types of cereals; toasted white or brown bread; or fried, poached or scrambled eggs. How many choices does Elijah have for breakfast?

Steps	Working
1 Elijah can choose only 1 item and in each case the word OR can be used between each of the choices. The word OR between the choices indicates the addition principle is used.	Elijah can choose between 3 cereals OR 2 toasted breads OR 3 styles of eggs.
2 Calculate the possibilities by adding the different options.	number of choices = 3 + 2 + 3 = 8

The multiplication principle

The multiplication principle

When there are 2 or more successive selections to make and we must choose from a of one type AND then from b of another type, the total number of choices can be found by multiplying.

Total number of choices = $a \times b$.

Exam hack

Generally, multiplying the number of possibilities is associated with the word 'AND' in a problem.

WB
p. 129

WORKED EXAMPLE 2 Applying the multiplication principle

Freda must do 1 Mathematics problem and write 1 English essay for homework. She is told to select the Mathematics problem from 8 different problems and the English essay from 4 possible essays. How many different possibilities does she have?

Steps	Working
Freda must choose 1 Mathematics problem from 8 AND 1 English essay from 4. AND indicates the multiplication principle is used.	number of possibilities = 8 × 4 = 32

WB
p. 130

WORKED EXAMPLE 3 Using the addition and multiplication principles

Dominic is ordering lunch at an Italian restaurant. He will order pizza or pasta.

The menu has pizzas with a choice of 3 different bases and 10 different toppings. The menu also has a choice of 4 different pastas with 7 different sauces. How many choices does Dominic have?

Steps	Working
1 Find the number of pizza choices. There are 3 different bases AND 10 different toppings. AND uses the multiplication principle.	pizza choices = 3 × 10 = 30
2 Find the number of pasta choices. There are 4 different pastas AND 7 different sauces.	pasta choices = 4 × 7 = 28
3 Find the total number of choices. Dominic can have pizza OR pasta. OR uses the addition principle.	total number of choices = 30 + 28 = 58

Tree diagrams

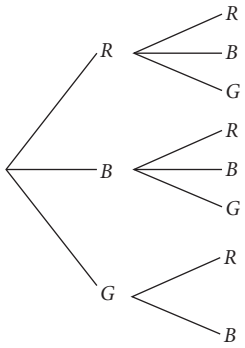
A **compound event** consists of 2 or more simple events that occur together or one after another. Tossing a coin 3 times is an example of a compound event. A **tree diagram** is a useful way of representing a compound event. Each simple event is represented by another stage of branches with the number of branches in each set equal to the number of possible outcomes for the event.

WORKED EXAMPLE 4 Applying tree diagrams

A box contains 3 red marbles, 2 blue marbles and 1 green marble. Two marbles are removed one at a time and placed in a row.

a List the possible arrangements.

b How many of these arrangements have a green marble?

Steps	Working
a 1 Draw a tree diagram with 2 sets of branches representing the 2 selections. Once a green marble has been removed it cannot appear again.	
2 Each path in the tree from left to right represents a possible arrangement.	The 8 possible arrangements are RR, RB, RG, BR, BB, BG, GR and GB.
b The arrangements with a green marble are RG, BG, GR, GB. Count the arrangements with a green marble.	There are 4 arrangements with a green marble.

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EXERCISE 6.1 The addition and multiplication principles

ANSWERS p. 581

Mastery

1 WORKED EXAMPLE 1 Mitchell is going to read a book during his holidays. He can choose from 6 thrillers, 5 science fiction books, 4 biographies and 10 detective novels. How many choices does Mitchell have?

2 WORKED EXAMPLE 2 The menu in a restaurant has 4 choices for entrée, 7 main courses and 3 desserts. How many choices are possible if all 3 courses are ordered?

3 WORKED EXAMPLE 3 Samantha must choose one first semester unit from 3 units of Mathematics and 2 units of Science and one second semester unit from 2 units of History, 2 units of Geography and 3 units of Technology. How many different 2-unit courses are possible?

4 WORKED EXAMPLE 4 A box contains 4 red cubes, 1 blue cube and 1 yellow cube. Two cubes are removed one at a time and placed in a row.

a List the different possible arrangements.

b How many of these arrangements have a yellow cube?

Exam practice

5 TECH-FREE (2 marks) Gretta is travelling from Melbourne to Mildura. She can take 2 different buses from Melbourne to Mildura or 2 trains from Melbourne to Bendigo and 3 connecting buses from Bendigo to Mildura or she can drive from Melbourne to Mildura. How many ways can Gretta travel from Melbourne to Mildura?

6 TECH-FREE (2 marks) A group of 5 Year 11 students are introduced and shake hands with a group of 4 Year 12 students. How many handshakes are there?

7 TECH-FREE (2 marks) A box contains 3 red cubes, 1 blue cube and 1 yellow cube. Three cubes are removed one at a time and placed in a row. List the different possible arrangements.

8 Bunroeun must travel from Melbourne to Darwin. He has a choice of 3 flights from Melbourne to Sydney and 2 direct connecting flights from Sydney to Darwin or 4 different direct flights from Melbourne to Darwin or 3 different buses from Melbourne to Darwin. The total number of choices Bunroeun has is

A 12 **B** 13 **C** 18 **D** 27 **E** 72

9 Lucy is purchasing a laptop computer. She can purchase a brand of computer that has a choice of 3 different hard drives, 3 different CPUs and 2 sizes of RAM or she can choose from 8 other computers with set components. The number of choices for computers Lucy has is

A 16 **B** 19 **C** 25 **D** 26 **E** 144

10 Vehicle number plates can consist of 3 letters and 3 numbers. The number of different number plates with 'T' as the first letter is

A $2 \times 26 + 3 \times 10$ **B** $3 \times 26 + 3 \times 10$ **C** $26^2 \times 10^3$
D $26^2 \times 9^3$ **E** $26^3 \times 10^3$

Use the following information to answer Questions 11 and 12.

Mario's Pizza Shop sells customised pizzas. Patrons can choose from 3 different crusts, 3 different types of cheese and 5 different combinations of toppings. Mario's Pizza Shop also has 10 gourmet pizzas on their menu.

11 The number of possible customised pizzas is

A 14 **B** 18 **C** 30 **D** 40 **E** 45

12 The number of possible pizza choices on Mario's Pizza Shop menu is

A 24 **B** 28 **C** 40 **D** 55 **E** 450

13 Emma is completing an English exam that has 3 sections and must answer 1 question in each section of the paper. Section A has 5 questions, Section B has 2 questions and Section C has 4 questions. The number of possible combinations of questions is

A 1 **B** 3 **C** 11 **D** 14 **E** 40

6.2 Permutations

The **permutations** of a group of objects are the different ways the objects can be arranged in order.

Consider a race of 6 people where the first place and second place are recorded. How many permutations are possible?

1 First place and second place can be represented by 2 boxes.
2 Place a 6 in the first box since there are 6 ways to fill first place.
3 Once first place has been filled there are only 5 ways to fill the second place.
4 Place a 5 in the second box.
5 Use the multiplication principle.
6 The total number of possible arrangements is $6 \times 5 = 30$.

6	
6	5

Video playlist
Permutations

Permutations ${}^nP_r$

The number of **permutations** or arrangements of r objects chosen from n objects is ${}^nP_r$.
For example, the number of arrangements of 2 runners chosen from 6 runners is ${}^6P_2$.

WB
p. 132

WORKED EXAMPLE 5 Finding the number of permutations

Three family photos are displayed on a wall. In how many ways can this be done if there are 8 photos to choose from?

Steps	Working
1 There are 3 places to fill so draw 3 boxes.	☐ ☐ ☐
2 There are 8 ways to fill the first position.	8 ☐ ☐
3 Once the first place is filled, there are 7 photos left to fill the second position.	8 7 ☐
4 Once the second position has been filled, there are 6 photos left to put in the third position.	8 7 6
5 Use the multiplication principle to calculate the number of possible arrangements.	total permutations $= 8 \times 7 \times 6$ $= 336$

USING CAS 1 Permutations

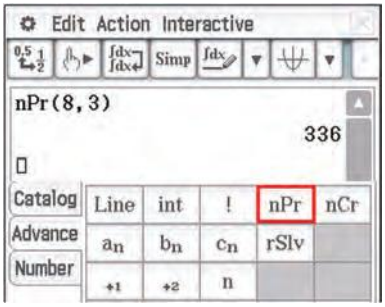
Verify the solution to the above example by calculating ${}^8P_3$.

TI-Nspire



1 Press **menu > Probability > Permutations**.
2 Enter **8,3**.

ClassPad



1 Open the **Keyboard > Advance**.
2 Tap **nPr**.
3 Enter **8,3**.

Worksheet
Factorial notation

Factorials

Factorial notation is a convenient way of expressing the product of consecutive natural numbers.

Factorials *n*!

The mathematical notation for n factorial is $n!$ where $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$

$n!$ is the number of arrangements of n different objects, and by definition, $0! = 1$.

WB
p. 133

WORKED EXAMPLE 6 Applying factorials

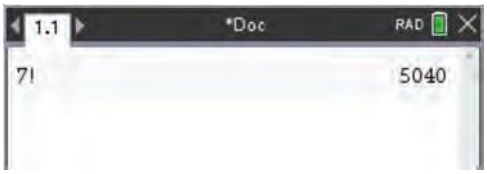
A group of 7 musicians are arranged in a line for a photo. In how many ways can this be done?

Steps	Working
We are required to arrange 7 people in a line. This can be done 7! ways.	number of arrangements $= 7!$ $= 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ $= 5040$

USING CAS 2 Factorials

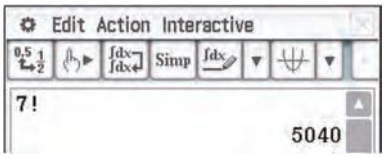
Calculate 7!.

TI-Nspire



1 Enter **7**.
2 Press **menu > Probability > Factorial(!)**.

ClassPad



1 Enter **7**.
2 Open the **Keyboard > Advance**.
3 Tap **!**

Worksheet
Permutation calculations

Calculating ${}^nP_r$

We can use CAS to calculate ${}^nP_r$, but we can also use factorials to write a formula for ${}^nP_r$.

Consider the value of ${}^8P_3$.

${}^8P_3 = 8 \times 7 \times 6$

This provides a shortcut for calculating ${}^nP_r$: find the product of r descending consecutive numbers starting with n. For example, ${}^9P_4 = 9 \times 8 \times 7 \times 6 = 3024$. This method is useful for counting permutations without using a calculator.

But we can also write ${}^8P_3$ like this:

$$\begin{aligned} {}^8P_3 &= 8 \times 7 \times 6 \\ &= 8 \times 7 \times 6 \times \frac{5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} \\ &= \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} \\ &= \frac{8!}{5!} \\ &= \frac{8!}{(8-3)!} \end{aligned}$$

Permutation formula

$$ {}^nP_r = \frac{n!}{(n-r)!} $$

It is useful to know ${}^nP_0 = 1$, ${}^nP_1 = n$, ${}^nP_n = n!$

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To calculate ${}^6P_4$ using the permutation formula, substitute $n = 6$ and $r = 4$.

$${}^6P_4 = \frac{6!}{(6-4)!} = \frac{6!}{2!} = 360$$

Or, notice that:

$${}^6P_4 = \frac{6!}{2!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = 6 \times 5 \times 4 \times 3 = 360$$

Also, $0! = 1! = 1$. There is just one way to take no objects from a group, so 0! is defined to be 1 to make ${}^nP_0 = 1$.

WORKED EXAMPLE 7 Calculating permutations

Use the permutation formula to find the value of **a** ${}^6P_2$ **b** ${}^{10}P_3$

Steps	Working
a Substitute $n = 6$ and $r = 2$ into the formula: ${}^nP_r = \frac{n!}{(n-r)!}$	${}^6P_2 = \frac{6!}{(6-2)!} = \frac{6!}{4!} = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1} = 6 \times 5 = 30$
b Substitute into the formula and calculate as the product of 3 descending consecutive numbers starting with 10.	${}^{10}P_3 = \frac{10!}{(10-3)!} = \frac{10!}{7!} = 10 \times 9 \times 8 = 720$

WB p. 134

Arrangements with restrictions

Some permutation problems have a condition or restriction placed on the arrangement.

Worksheet Counting techniques

Arrangements with restrictions

When counting arrangements with restrictions:

1. determine the number of places to be filled and draw the same number of boxes
2. fill all the spaces where the restriction is involved
3. fill the rest of the spaces from left to right.

WORKED EXAMPLE 8 Applying permutations with a restriction

A mother, father, 3 sisters and a brother are seated at a football match. How many possible seating arrangements are there if the children must be seated between the parents.

Steps	Working
1 There are 6 people to arrange so draw 6 boxes.	☐ ☐ ☐ ☐ ☐ ☐
2 The restriction requires the parents to be seated at either end. The first and last positions are filled first. The first position can be filled in 2 ways since either the mother or father can sit in this position. This leaves one way to fill the last position.	2 ☐ ☐ ☐ ☐ 1
3 The 4 children can now be seated. There are 4 ways to fill the second position. Once that is filled, there are 3 ways to fill the third position. Continue in the same way until all the positions are filled.	2 4 3 2 1 1
4 Use the multiplication principle to calculate the number of arrangements.	number of seating arrangements $= 2 \times 4 \times 3 \times 2 \times 1 \times 1$ $= 48$

WB p. 135

Exam hack

Permutations with restrictions *cannot* be calculated using the permutation formula, ${}^nP_r$. Each problem must be treated as a special case and solved by filling spaces and applying the multiplication principle.

Worksheet
Permutations with repetitions

Arrangements with identical objects

Counting permutations of identical objects

The number of permutations of n objects of which there are a identical objects of one type, b identical objects of a second type and c identical objects of a third type is given by

$$\frac{n!}{a! \times b! \times c!}$$

WB p. 136

WORKED EXAMPLE 9 Counting arrangements with identical numbers

How many 7-digit numbers can be made using four 8s and three 6s?

Steps	Working
There are 7 digits to arrange of which 4 digits are of one type (8s) and 3 digits of a second type (6s). Use the formula $\frac{n!}{a! \times b!}$, where $n = 7$, $a = 4$ and $b = 3$.	number of possible numbers $= \frac{7!}{4! \times 3!} = \frac{7 \times 6 \times 5 \times 4!}{4! \times 3!} = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35$

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WORKED EXAMPLE 10 Counting arrangements with identical letters

How many arrangements of the letters in the word WOOLLOOMOOLOO are possible?

Steps	Working
There are 13 letters to arrange of which 8 letter Os and 3 letter Ls. Use the formula $\frac{n!}{a! \times b!}$, where $n = 13$, $a = 8$ and $b = 3$	number of possible arrangements $= \frac{13!}{8! \times 3!} = 25\,740$

Arrangements with groups

In some arrangements, we want particular groups of objects kept together.

Arrangements with groups

When counting arrangements with groups:

1. put all the objects that must be kept together in a group and count this group as one object
2. determine the number of arrangements of the objects plus the group
3. multiply this by the number of ways of arranging the objects within the group.

WORKED EXAMPLE 11 Applying permutations with groups

A mother, father, 3 sisters and a brother are seated at a football match. How many possible seating arrangements are there if the sisters must sit together?

Steps	Working
1 Put the 3 sisters in a group and count as one object.  There are 4 objects to arrange.	number of ways of arranging the group and the 3 other family members = 4!
2 Determine the number of ways of arranging the sisters in the group.	number of arrangements of the 3 sisters = 3!
3 Determine the total number of possible arrangements by applying the multiplication principle.	total number of arrangements = 4! × 3! = 144

WB p. 138

WORKED EXAMPLE 12 Permutations with 2 restrictions

A 4-digit number is made from the set of digits {2, 3, 5, 6, 8}. How many odd numbers greater than 6000 are possible if no repetitions are allowed?

Steps	Working
1 Draw 4 boxes to represent the 4-digit number.	☐ ☐ ☐ ☐
2 Fill the boxes for the restriction first. The first box can be filled in 2 ways with the digits 6 or 8. This will produce a number greater than 6000. The last box can be filled 2 ways with the digits 3 or 5. This will produce an odd number.	2 ☐ ☐ 2
3 Once these positions are filled, there are 3 numbers remaining. Fill the second box with 3 possible digits and the third box with 2 possible digits.	2 3 2 2
4 Use the multiplication principle to determine the number of possible digits.	number of odd numbers = 2 × 3 × 2 × 2 = 24

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EXERCISE 6.2 Permutations

ANSWERS p. 581

Recap

1 Vasya orders tea and cake at a café. The teas on the menu are peppermint, chamomile, lemongrass and English breakfast, and the available cakes are tea cake, chocolate mudcake, carrot cake, orange cake and lumberjake cake. The number of different orders Vasya can order is

A 1 **B** 4 **C** 9 **D** 20 **E** 9!

2 A bag contains 2 blue, 1 green and 1 red token. Two tokens are removed and placed in a row. The number of different arrangements is

A 2 **B** 4 **C** 7 **D** 12 **E** 24

Mastery

3 WORKED EXAMPLE 5 Six people compete in a hurdles race. How many different ways can first, second and third placings be filled?

4 Using CAS 1 Find the value of

a ${}^{12}P_5$ **b** ${}^{15}P_4$ **c** ${}^{6}P_5$

5 WORKED EXAMPLE 6 A 5-digit access code is made using the digits {1, 2, 3, 4, 5}. How many codes are possible if no repetitions are allowed?

6 Using CAS 2 Find the value of

a 8! **b** 12! ÷ 7! **c** 14! ÷ (7! × 7!)

7 WORKED EXAMPLE 7 TECH-FREE Find the value of

a ${}^{12}P_2$ **b** ${}^{15}P_1$ **c** ${}^{20}P_0$

8 WORKED EXAMPLE 8 Three boys and 4 girls sit on a bench. How many different seating arrangements are possible if the girls and boys must alternate?

9 WORKED EXAMPLE 9 How many 9-digit numbers can be made from five 1s and four 2s?

10 WORKED EXAMPLE 10 How many code words can be made by rearranging the letters of the word TATTLETALE?

11 WORKED EXAMPLE 11 The letters of the word HUMBLE are arranged. How many ways can this be done if the vowels must be together?

12 WORKED EXAMPLE 12 A 4-digit number is made from the set of digits {1, 2, 4, 5, 6, 7, 8}. Find the number of possible numbers greater than 6000 if no repetitions are allowed.

13 Find the value of

a ${}^{8}P_2$ **b** ${}^{11}P_1$ **c** ${}^{6}P_3$ **d** ${}^{35}P_0$ **e** ${}^{4}P_4$

Exam practice

14 TECH-FREE (1 mark) How many arrangements of the letters of the word TENNESSEE are possible?

15 TECH-FREE (2 marks) How many arrangements of the letters of the South Australian town ORRORROO are possible?

16 TECH-FREE (2 marks) How many 6-digit numbers can be made from four 5s and two 9s?

17 TECH-FREE (1 mark) How many 7-digit numbers can be made from the digits in the set {1, 1, 1, 3, 3, 4, 4}?

18 TECH-FREE (3 marks) Three students are to be selected from 4 males and 5 females in the leadership team to fill the roles of school captain, deputy captain and school council representative. Find the number of ways this can be done

a if there is no restriction 1 mark

b in which a female captain and a male deputy captain are chosen 1 mark

c if the roles are filled with students of the same gender. 1 mark

19 TECH-FREE (4 marks) At the senior school sports carnival, the egg-and-spoon race has 3 Year 11 boys, 2 Year 11 girls, 2 Year 12 boys and 1 Year 12 girl. How many different finishing positions are there in which

a the girls occupy the first 3 placings 1 mark

b the Year 11 students occupy the first 5 placings 1 mark

c Year 12 students occupy first and last places 1 mark

d a Year 11 girl does not come last? 1 mark

20 The number of ways 4 Christmas cards can be arranged on a shelf is

A 3 **B** 4 **C** 12 **D** 15 **E** 24

21 Amy has 6 netball trophies that were won in 2014, 2013, 2010, 2009, 2007 and 2006. If she arranges 4 of these on the mantelpiece with 2006 on the left and 2014 on the right, the number of possible arrangements is

A 6 **B** 12 **C** 4! **D** ${}^6P_4$ **E** 6!

22 The letters of the word PEACH are arranged in a row. The number of arrangements that start with a vowel and end with a vowel is

A 5 **B** 12 **C** 24 **D** 48 **E** 120

23 Julie and Andrea compete in the 50-m freestyle race with 6 other swimmers. The number of starting positions where Julie and Andrea race in adjacent lanes is

A 6 **B** 24 **C** 120 **D** 240 **E** 10 080

24 A number is made from the digits {1, 3, 4, 7, 8, 9}. The number of possible numbers greater than 7000 is

A 6 **B** 240 **C** 360 **D** 720 **E** 1620

25 A group of 4 girls and 4 boys stand in a line to be photographed. The number of possible arrangements where the girls are together is

A 5! **B** 4! × 4! **C** ${}^8P_4$ **D** 5! × 4! **E** 8!

6.3 Combinations

Video playlist
Combinations

The **combinations** of a group of objects are the different ways the objects can be selected *where the order of selection is not important.*

Consider a basketball team of 8 players from which a coach must select 5 at any time to play on the court during a match. How many combinations are possible?

8				
8	7	6	5	4

- The 5 places can be represented by 5 boxes.
- Place an 8 in the first box since there are 8 players that can be selected first.
- Fill the other 4 places available on the team.
- ${}^8P_5 = 8 \times 7 \times 6 \times 5 \times 4$ is the number of permutations where the order of selection is important.
- However, with combinations the order is *not* important.
- For each permutation or arrangement of the team, each player could swap places with another player and it would still be the same team.
- The number of arrangements within the team is $5! = 5 \times 4 \times 3 \times 2 \times 1$, so we need to divide ${}^8P_5$ by 5! to find the total number of combinations.
- The number of combinations of 5 from 8 is written as ${}^8C_5 = \dfrac{{}^8P_5}{5!} = \dfrac{8 \times 7 \times 6 \times 5 \times 4}{5 \times 4 \times 3 \times 2 \times 1} = 56$.

Worksheets
Combinations calculations
Combinations problems
Combinations assignment

Calculating ${}^nC_r$

The formula for ${}^nC_r$

The number of combinations or selections of r objects chosen from n objects is ${}^nC_r$,

where ${}^nC_r = \frac{{}^nP_r}{r!} = \frac{n!}{(n-r)!\,r!}$.

${}^nC_r$ is sometimes called 'n choose r'.

To calculate ${}^8C_5$ using the combinations formula, substitute $n = 8$ and $r = 5$.

$${}^8C_5 = \frac{8!}{(8-5)!\,3!} = \frac{8!}{3! \times 5!} = 56$$

Or notice that:

$${}^8C_5 = \frac{8!}{3! \times 5!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1) \times (5 \times 4 \times 3 \times 2 \times 1)} = \frac{8 \times 7 \times 6 \times 5 \times 4}{5 \times 4 \times 3 \times 2 \times 1}$$

We can use the shortcut for finding permutations when calculating combinations.

The numerator is the product of r descending consecutive numbers starting with n and the denominator is $r!$.

For example, ${}^6C_3 = \frac{6 \times 5 \times 4}{3 \times 2 \times 1}$.

Note that the number of terms in the numerator and the denominator are the same. This method is useful for counting combinations without a calculator.

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WORKED EXAMPLE 13 Calculating combinations

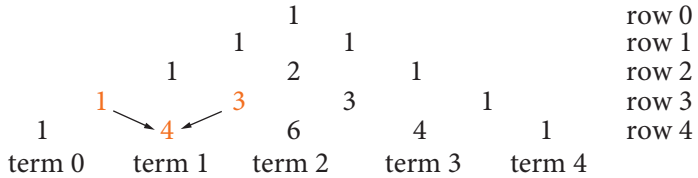
Evaluate ${}^7C_3$.

Steps	Working
1 In the combination, the numerator is the product of 3 descending numbers starting with 7. The denominator is 3!.	${}^7C_3 = \frac{7 \times 6 \times 5}{3 \times 2 \times 1}$
2 Cancel the factors of 6 and evaluate.	${}^7C_3 = \frac{7 \times 5}{1}$ $= 35$

Worksheets
Combinations
Permutations and combinations
Pascal's triangle 1
Pascal's triangle 2

Pascal's triangle and combinations

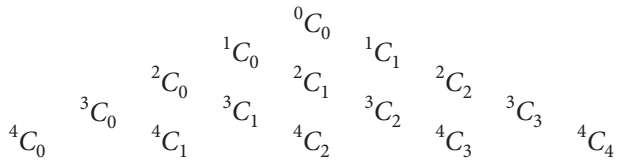
Pascal's triangle



1 + 3 = 4

Add the 2 numbers in the row above to get any number in the next row.

Pascal's triangle can also be written using combinations:



Note that if we number the rows and terms from row 0 and term 0 then ${}^4C_3$ is the number in row 4 term 3 so, in general, ${}^nC_r$ is the number in row n term r.

The symmetry in Pascal's triangle

The symmetry in Pascal's triangle gives ${}^nC_r = {}^nC_{n-r}$.

This can be useful if r is a large number and $n - r$ is smaller.

${}^nC_0 = {}^nC_n = 1$ and ${}^nC_1 = {}^nC_{n-1} = n$

WORKED EXAMPLE 14 Calculating combinations by symmetry

Evaluate ${}^{10}C_7$.

Steps	Working
1 By symmetry, ${}^{10}C_7 = {}^{10}C_3$.	${}^{10}C_7 = {}^{10}C_3$
2 Evaluate ${}^{10}C_3$.	${}^{10}C_3 = \frac{10 \times 9 \times 8}{3 \times 2 \times 1}$ $= \frac{720}{6}$ $= 120$

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USING CAS 3 Combinations

Calculate ${}^{10}C_7$.

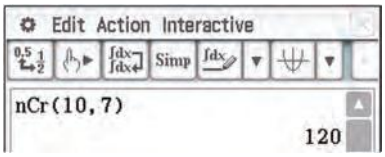
TI-Nspire



1 Press **menu > Probability > Combinations**.

2 Enter **10,7**.

ClassPad



1 Open the **Keyboard > Advance**.

2 Tap **nCr**.

3 Enter **10,7**.

WORKED EXAMPLE 15 Applying combinations

How many different hands of 5 cards can be dealt from a pack of 52 playing cards?

Steps	Working
The number of different hands of 5 cards is a combination because the order the cards are dealt does not change the hand. We are choosing 5 cards from 52 cards, which can be done ${}^{52}C_5$ ways.	number of hands $= {}^{52}C_5$ $= 2\,598\,960$

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Combinations with restrictions

In certain problems, the population is made up of subgroups and selections must be made from each subgroup. The multiplication principle is used with the number of combinations obtained by selecting from each group.

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WORKED EXAMPLE 16 Applying combinations with restrictions

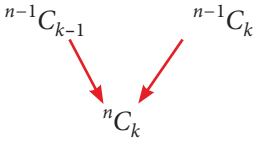
A mixed volleyball team of 9 players is selected from a group of 8 girls and 5 boys. How many different teams are possible if the team contains 5 girls and 4 boys?

Steps	Working
1 The number of different teams is a combination because the order of selection is not important.	number of ways to choose the girls = ${}^8C_5$ = 56
We are choosing 5 girls from 8 girls AND 4 boys from 5 boys.	number of ways to choose the boys = ${}^5C_4$ = 5
2 Use the multiplication principle to find the total number of teams.	number of possible teams = 56 × 5 = 280

Pascal's identity

Remember that each number in Pascal's triangle is the sum of the 2 numbers in the row above it.

This can be written algebraically as ${}^nC_k = {}^{n-1}C_{k-1} + {}^{n-1}C_k$.
This is called **Pascal's identity**.



Pascal's identity

$${}^nC_k = {}^{n-1}C_{k-1} + {}^{n-1}C_k$$

Interpreting Pascal's identity

Consider the situation where a team of 3 managers must be chosen from 12 people, 1 of whom is named Tom, and 11 of his colleagues. The number of possible teams is ${}^{12}C_3$.

The combination of teams can be divided into those that contain Tom and the others that do not contain Tom.

For teams containing Tom, there are ${}^1C_1 \times {}^{11}C_2$ combinations: the number of ways of selecting Tom × number of ways of selecting the remaining 2 from 11.

For teams not containing Tom, there are ${}^{11}C_3$ combinations: the number of ways of selecting 3 from the remaining 11.

It follows that ${}^{12}C_3 = {}^1C_1 \times {}^{11}C_2 + {}^{11}C_3$

or ${}^{12}C_3 = {}^{11}C_2 + {}^{11}C_3$

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WORKED EXAMPLE 17 Using Pascal's identity

Use Pascal's identity to find possible values of x and y if ${}^{15}C_4 = {}^{14}C_x + {}^{14}C_y$.

Steps	Working
1 Find the values of n and k in Pascal's identity ${}^nC_k = {}^{n-1}C_{k-1} + {}^{n-1}C_k$	${}^{15}C_4 = {}^{14}C_x + {}^{14}C_y$ $n = 15$ and $k = 4$
2 Rewrite Pascal's identity for $n = 15$ and $k = 4$.	${}^{15}C_4 = {}^{14}C_3 + {}^{14}C_4$
3 Write other equivalent expressions for the combinations found using ${}^nC_r = {}^nC_{n-r}$.	${}^{14}C_3 = {}^{14}C_{11}$ and ${}^{14}C_4 = {}^{14}C_{10}$
4 Write the possible solutions.	$x = \{3, 11\}$, $y = \{4, 10\}$ or $x = \{4, 10\}$, $y = \{3, 11\}$

Combinations taken any number at a time

Suppose Damien visits the library. There is a limit of 5 books that can be borrowed at any time, so Damien could borrow 0, 1, 2, 3, 4 or 5 books. How many different combinations of books borrowed are possible?

This can be done in ${}^5C_0$, ${}^5C_1$, ${}^5C_2$, ${}^5C_3$, ${}^5C_4$ or ${}^5C_5$ ways, so the answer is ${}^5C_0 + {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$.

However, for each of the 5 books there are only 2 possibilities: it will either be borrowed or it will not. So the answer is also $2 \times 2 \times 2 \times 2 \times 2 = 2^5$. It follows that

$${}^5C_0 + {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5 = 2^5$$

Formula for combinations taken any number at a time

In general,

$${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_{n-1} + {}^nC_n = 2^n$$

This formula also gives the sum of the numbers in row n of Pascal's triangle.

and

$${}^nC_1 + {}^nC_2 + \ldots + {}^nC_{n-1} + {}^nC_n = 2^n - 1$$

WORKED EXAMPLE 18 Applying combinations taken any number at a time

Adele is watching up to 6 different movies at home this weekend. How many choices does she have if she must choose at least one?

Steps	Working
Use the rule ${}^nC_1 + {}^nC_2 + \ldots + {}^nC_{n-1} + {}^nC_n = 2^n - 1$.	The number of choices is $2^6 - 1 = 63$.

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EXERCISE 6.3 Combinations

ANSWERS p. 581

Recap

1 A 3-digit odd number is made from the digits {1, 3, 4, 6, 8}. The number of possible numbers greater than 600 is

A 3 **B** 6 **C** 12 **D** 24 **E** 120

2 The number of arrangements of the letters of the NSW city WAGGA WAGGA is

A 10 **B** 30 **C** 3150 **D** 6300 **E** 10!

Mastery

3 WORKED EXAMPLE 13 Evaluate without a calculator.

a ${}^5C_2$ **b** ${}^7C_3$ **c** ${}^4C_1$

4 WORKED EXAMPLE 14 TECH-FREE Find the value of

a ${}^{11}C_{10}$ **b** ${}^{12}C_{10}$ **c** ${}^{100}C_{99}$

5 Using CAS 3 Find the value of

a ${}^{20}C_4$ **b** ${}^{100}C_{95}$ **c** ${}^{900}C_{899}$

6 WORKED EXAMPLE 15 A relay team of 4 runners is selected from a group of 8 athletes. How many different teams are possible?

7 WORKED EXAMPLE 16 A mixed basketball team of 5 players is selected from 7 females and 5 males. How many teams contain 3 female players and 2 male players?

8 WORKED EXAMPLE 17 Use Pascal's identity to find possible values of x and y if ${}^9C_3 = {}^8C_x + {}^8C_y$.

9 WORKED EXAMPLE 18 Charlotte is going to the library to get some holiday reading. She can borrow a maximum of 4 books. How many different choices does she have if she can borrow any number at a time?

Exam practice

10 TECH-FREE (2 marks) In the first round of a tennis tournament, each competitor plays every other competitor once.

a How many games are played in the first round if there are 8 entries in the tournament? 1 mark

b How many people entered the tennis tournament if there are 78 games played in the first round? 1 mark

11 TECH-FREE (4 marks) Four students are chosen from a group of 8 students to represent the school at the regional maths games.

a How many different teams are possible? 1 mark

b Aura and Vitma are 2 of the 8 students available for selection at the maths games. How many teams are possible if both Aura and Vitma are selected? 1 mark

c How many teams are possible if only either Aura or Vitma are selected? 1 mark

d How many teams are possible if neither Aura nor Vitma are selected? 1 mark

12 TECH-FREE (3 marks) At a party, every person shakes hands with every other person.

a If there are 12 people at the party, how many handshakes are there? 1 mark

b If there are 15 people at the party, how many handshakes are there? 1 mark

c If there are 210 handshakes, how many people were at the party? 1 mark

13 TECH-FREE (4 marks) Use Pascal's identity to find possible values of x and y if ${}^{x}C_{y} = {}^{14}C_{5} + {}^{14}C_{6}$.

14 The number of ways a committee of 4 can be selected from 10 people is

A 120 **B** 210 **C** 5040 **D** 10! **E** 10! × 3!

15 Veni eats 3 lollies selected from a mixed bag of 12 lollies. The number of possible combinations of lollies is

A 12 **B** 24 **C** 36 **D** 66 **E** 220

16 In a game of mini lotto, 2 numbers are selected from the numbers 1, 2, 3, 4, 5 and 6. The number of different pairs of numbers is

A 2! **B** 15 **C** 30 **D** 6! **E** 6! × 2!

17 A committee of 4 people is selected from 6 men and 4 women. The number of different committees that contain 3 men and 1 woman is

A ${}^{6}C_{3}$ **B** ${}^{6}C_{1} \times {}^{4}C_{3}$ **C** ${}^{6}C_{3} \times {}^{4}C_{1}$ **D** ${}^{10}C_{4}$ **E** 6! × 4!

18 At Kathy's Café, customers can choose sandwiches with lettuce, carrot, beetroot, cucumber and tomato as fillings. If any number of fillings can be used, the number of different sandwiches is

A 15 **B** 31 **C** 32 **D** 63 **E** 64

19 Pascal's identity produces the equation ${}^{11}C_{5} = {}^{10}C_{x} + {}^{10}C_{y}$. The possible values of x and y are

A $x = 5$ and $y = 5$ **B** $x = 3$ and $y = 4$ **C** $x = \{4, 6\}$ and $y = 5$

D $x = 6$ and $y = 7$ **E** $x = \{4, 6\}$ and $y = \{4, 6\}$

20 Using Pascal's identity, the combination ${}^{n+2}C_{k-2}$ is equivalent to

A ${}^{n+3}C_{k-3} + {}^{n+3}C_{k-2}$ **B** ${}^{n+3}C_{k-1} + {}^{n+3}C_{k-2}$ **C** ${}^{n+1}C_{k-1} + {}^{n+1}C_{k-2}$

D ${}^{n+1}C_{k-3} + {}^{n+1}C_{k-2}$ **E** ${}^{n}C_{k-1} + {}^{n}C_{k}$

21 (3 marks) Lucile's Fruit Bar has a selection of apples, bananas, peaches, cherries and strawberries. Customers select a variety of fruits in their fruit salad bowls. How many combinations of fruits are possible if

a 4 fruits are selected? 1 mark

b 4 fruits are selected and strawberries are included? 1 mark

c any number of fruits can be chosen? 1 mark

22 (4 marks) Adam, Luke, Michael, Ben, Anthony and Pat all play for the Melbourne Tigers Basketball Club. Their coach must pick 3 players for their 3-on-3 competition. How many teams are possible if

a there is no restriction? 1 mark

b Michael must be selected? 1 mark

c Pat cannot be selected? 1 mark

d one player starting with the letter A must be selected? 1 mark

6.4 The pigeon-hole principle

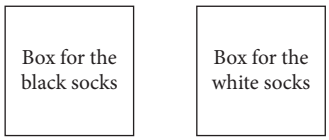
Video playlist
The pigeon-hole principle

Suppose socks are selected at random from a drawer that contains 2 black socks and 3 white socks.

The **pigeon-hole principle** helps us determine the smallest number of socks that must be selected to get a matching pair.

Socks = {B, B, W, W, W}

After each sock is selected, it is placed in the correct box. It will only require 3 socks to be selected to get a matching pair.



The pigeon-hole principle

The pigeon-hole principle states:

If $n + 1$ objects are placed into n holes, then at least 1 hole must contain 2 objects.

WORKED EXAMPLE 19 Applying the pigeon-hole principle

Editha selects socks at random from a drawer that contains 5 white, 4 black and 6 blue socks. What is the smallest number of socks that must be selected to get a matching pair?

Steps	Working
The socks are sorted into colours after selection. There are 3 colours, which correspond to 3 pigeon holes.	The number of choices is $3 + 1 = 4$.
The pigeon-hole principle states there must be $n + 1$ selections for n categories.	

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The generalised pigeon-hole principle

If n objects are placed in k holes, where $n > k$, then there is at least 1 hole that contains at least $\frac{n}{k}$ objects.

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p. 147

WORKED EXAMPLE 20 Applying the generalised pigeon-hole principle 1

Fifty apples of mixed sizes are sorted into baskets as large, medium or small. Find the value of x if at least 1 basket has at least x apples.

Steps	Working
There are 50 apples and 3 baskets. At least 1 basket has at least $\frac{50}{3}$ apples.	$x \geq \frac{50}{3}$ $x \geq 16\frac{2}{3}$ x is at least 17. There are at least 17 apples in at least 1 basket.

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WORKED EXAMPLE 21 Applying the generalised pigeon-hole principle 2

Twenty natural numbers are divided by 6 and the remainder is recorded. Find the value of x if at least x of the numbers have the same remainder.

Steps	Working
1 Determine the possible remainders when a natural number is divided by 6. Each remainder represents a pigeon hole.	The possible remainders are 0, 1, 2, 3, 4 and 5.
2 Divide the number of objects (20) by the number of holes (6).	$x \geq \frac{20}{6}$ $x \geq 3\frac{1}{3}$ x is at least 4. At least 4 of the numbers will have the same remainder.

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WORKED EXAMPLE 22 Proofs using the generalised pigeon-hole principle

Prove that if 6 different numbers are selected from the set {1, 2, 3, 4, 5, 6, 7, 8, 9}, then 2 of these numbers will have a sum of 10.

Steps	Working
1 Determine the categories for the selected numbers.	The pairs of different numbers in the set that add to 10 are: {1, 9} {2, 8} {3, 7} and {4, 6}. If 5 is selected it will have its own category as there is not a different number in the set that can pair with 5 to give a sum of 10. The 5 pigeon holes k, for our pigeon-hole principle are: {1, 9} {2, 8} {3, 7} {4, 6} and {5}.
2 The pigeon-hole principle states: If n objects are placed in k holes, where $n > k$, then there is at least 1 hole that contains at least $\frac{n}{k}$ objects. Divide the number of selected objects (6) by the number of holes (5).	The pigeons n, will be the 6 selected numbers. $n = 6$ and $k = 5$. There is at least 1 pigeon hole with at least $\frac{6}{5}$ different numbers. This means 1 of the pigeon holes must have 2 numbers and the 2 numbers in the defined pigeon holes add to 10.

EXERCISE 6.4 The pigeon-hole principle

ANSWERS p. 582

Recap

1 The number of ways a team of 3 can be selected from 9 people is

A 6 **B** 27 **C** 84 **D** 9! **E** 9! × 3!

2 Using Pascal's identity, the combination ${}^{n+4}C_{k-3}$ is equivalent to

A ${}^{n+3}C_{k-2} + {}^{n+3}C_{k-3}$ **B** ${}^{n+3}C_{k-4} + {}^{n+3}C_{k-3}$ **C** ${}^{n+5}C_{k-4} + {}^{n+5}C_{k-3}$

D ${}^{n-1}C_{k-1} + {}^{n-1}C_{k}$ **E** ${}^{n}C_{k-1} + {}^{n}C_{k}$

Mastery

3 WORKED EXAMPLE 19 Angie selects socks at random from a drawer that contains 8 white, 3 black, 5 red and 7 blue socks. What is the smallest number of socks that must be selected to get a matching pair?

4 WORKED EXAMPLE 20 There are 46 apples of mixed sizes sorted into baskets as large, medium or small. Find the value of x if at least 1 basket has at least x apples.

5 WORKED EXAMPLE 21 Twenty-two natural numbers are divided by 5 and the remainder is recorded. Find the value of x if at least x of the numbers have the same remainder.

6 WORKED EXAMPLE 22 Prove that if 7 different numbers are selected from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, then 2 of these numbers will have a sum of 12.

Exam practice

7 TECH-FREE (2 marks) Ribbons are randomly selected from a box that contains 6 yellow ribbons, 9 red ribbons and 11 pink ribbons. Find the smallest number of ribbons that must be selected to get at least 2 ribbons the same colour.

8 TECH-FREE (2 marks) There are 110 pears of mixed sizes sorted into baskets by size and any damaged fruit are rejected. The baskets are labelled large, medium, small and rejects. Find the value of x if at least 1 basket has at least x pears.

9 TECH-FREE (2 marks) Find the minimum number of people that must be in a room so that at least 2 people have names that begin with the same letter.

10 TECH-FREE (4 marks) Find the minimum number of people that must be in a room so that at least 2 people have birthdays

a in the same month 2 marks

b on the same date (excluding Feb 29). 2 marks

11 (2 marks) Five parents are running for one position on the school council. There are 1052 parents who vote. Find the minimum number of votes needed for someone to win the election.

12 (2 marks) There are 7 different coloured pairs of socks in a drawer. If the 14 socks are unpaired, how many socks must be removed to be guaranteed a matching pair?

13 (4 marks) In a particular group of 20 people, friendship is mutual. That is, if A is friends with B then B is also friends with A.

a Explain why it is not possible for someone in the group to have no friends and another person to have 19 friends. 2 marks

b Use the pigeon-hole principle to show that there exists 2 people in the group who have the same number of friends. 2 marks

14 A drawer contains 12 purple, 10 orange, 8 blue and 14 yellow unpaired socks. The minimum number of socks that needs to be selected to get at least one pair is

A 2 **B** 4 **C** 5 **D** 8 **E** 9

15 Natural numbers are randomly generated and divided by 7 and the remainder is recorded. The number of random numbers that needs to be generated to get at least one number with the same remainder is

A 2 **B** 3 **C** 6 **D** 7 **E** 8

16 Fifty-five objects are placed in 6 boxes. If there must be at m objects in at least one box, the value of m is

A 6 **B** 7 **C** 9 **D** 10 **E** 11

17 Thirty-three natural numbers are divided by 8 and the remainder is recorded. If at least x of the numbers have the same remainder, the value of x is

A 3 **B** 4 **C** 5 **D** 8 **E** 9

18 Three Year 12 students are running for school captain. There are 202 senior students who vote. The minimum number of votes needed for someone to win the election is

A 67 **B** 68 **C** 69 **D** 101 **E** 102

Video playlist
The inclusion-exclusion principle

Worksheet
The inclusion-exclusion principle

6.5 The inclusion–exclusion principle

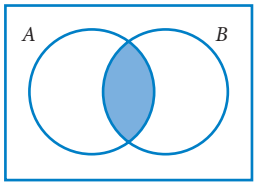
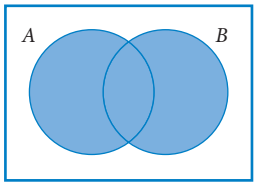
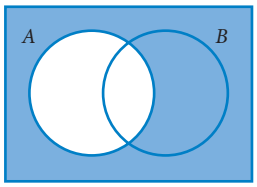
Basic set theory

The **sample space** in a probability experiment is a set that contains every possible outcome.

Each outcome is called an **element** of the set.

A **Venn diagram** represents the sets by showing each set as a circle or oval inside a rectangle that represents the universal set (all outcomes). It is a useful tool for analysing simple event probabilities where the sample space contains different groups with elements in common.

Intersection, union and complement in set theory

Intersection $A \cap B$	The intersection of sets A and B is a set that contains the elements that are in both sets.	
Union $A \cup B$	The union of sets A and B is obtained by combining or uniting both sets. The elements in the union are in set A or set B or both.	
Complement A'	The complement of set A is a set that contains all the elements that are not in set A.	

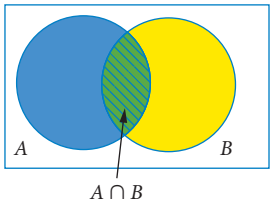
Inclusion–exclusion principle with 2 sets

The mathematical symbol for the number of elements in set A is $|A|$ or $n(A)$.

When set A is added to set B, the intersection of the 2 sets $A \cap B$ is added twice.

The inclusion–exclusion principle for 2 sets

$$|A \cup B| = |A| + |B| - |A \cap B|$$



WORKED EXAMPLE 23 Applying the inclusion–exclusion principle for 2 sets

In a group of 50 students, 32 have sunglasses on and 30 are wearing hats. If every student has either sunglasses or a hat on, find the number of students who are wearing sunglasses and a hat.

Steps	Working
1 Let S represent the set of students wearing sunglasses and H represent the set of students wearing hats.	$\|S\| = 32$ $\|H\| = 30$
Write the given information in set notation.	$\|S \cup H\| = 50$
2 Substitute into the inclusion–exclusion formula.	$\|S \cup H\| = \|S\| + \|H\| - \|S \cap H\|$ $50 = 32 + 30 - \|S \cap H\|$ $\|S \cap H\| = 32 + 30 - 50$ $\|S \cap H\| = 12$
3 Write the answer.	The number of students wearing sunglasses and a hat is 12.

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WORKED EXAMPLE 24 Finding multiples of 2 numbers in an interval

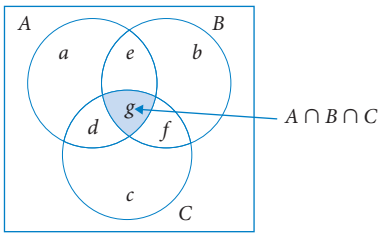
How many integers between 1 and 300 are multiples of 3 or 10?

Steps	Working
1 Let A be the set of numbers that are multiples of 3 and B be the set of numbers that are multiples of 10.	$\|A\| = 300 \div 3 = 100$ $\|B\| = 300 \div 10 = 30$
$A \cap B$ represents integers that are multiples of 3 and 10; that is, multiples of 30. Find the number of elements in each of these sets.	$\|A \cap B\| = 300 \div 30 = 10$
2 Substitute into the inclusion–exclusion formula.	$\|A \cup B\| = \|A\| + \|B\| - \|A \cap B\|$ $\|A \cup B\| = 100 + 30 - 10$ $\|A \cup B\| = 120$
3 Write the answer.	There are 120 integers that are multiples of 3 or 10.

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Inclusion–exclusion principle with 3 sets

The number of elements in each part of the 3 intersecting sets in the diagram is represented by a variable.



When the number of elements in sets A, B and C are added, we obtain:

$|A| + |B| + |C| = a + b + c + 2e + 2d + 2f + 3g$

When the number of elements in the intersections of the paired sets of A, B and C are subtracted, we obtain:

$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| = a + b + c + 2e + 2d + 2f + 3g - (e + g) - (d + g) - (f + g)$

$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| = a + b + c + 2e + 2d + 2f + 3g - e - d - f - 3g$

$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| = a + b + c + e + d + f$

This equation will represent the union of A, B and C if we add the intersection of the 3 sets.

$|A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| = a + b + c + e + d + f + g$

The inclusion–exclusion principle for 3 sets

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

WB

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WORKED EXAMPLE 25 Applying the inclusion-exclusion principle for 3 sets

At a recent swimming carnival, the list of entries showed that

- 14 entered backstroke
- 22 entered freestyle
- 17 entered butterfly
- 7 entered backstroke and freestyle
- 9 entered freestyle and butterfly
- 6 entered backstroke and butterfly
- 5 entered all 3 events.

Find the number of competitors who entered backstroke, freestyle or butterfly.

Steps	Working
1 Let B be the set of backstroke competitors, F be the set of freestyle competitors and Y be the set of butterfly competitors. Write the information in set notation.	$\|B\| = 14, \|F\| = 22, \|Y\| = 17,$ $\|B \cap F\| = 7, \|F \cap Y\| = 9, \|B \cap Y\| = 6,$ $\|B \cap F \cap Y\| = 5$
2 Substitute into the inclusion-exclusion formula and answer the question.	$\|B \cup F \cup Y\|$ $= \|B\| + \|F\| + \|Y\| - \|B \cap F\| - \|B \cap Y\| - \|F \cap Y\| + \|B \cap F \cap Y\|$ $= 14 + 22 + 17 - 7 - 9 - 6 + 5$ $= 36$ There are 36 competitors who entered backstroke, freestyle or butterfly.

WORKED EXAMPLE 26 Finding multiples of 3 numbers in an interval

How many integers between 1 and 300 are multiples of 2, 3 or 5?

Steps	Working
1 Let A be the set of numbers that are multiples of 2, B be the set of numbers that are multiples of 3 and C be the set of numbers that are multiples of 5.	$\lvert A \rvert = 300 \div 2 = 150$ $\lvert B \rvert = 300 \div 3 = 100$ $\lvert C \rvert = 300 \div 5 = 60$
$A \cap B$ are multiples of 2 and 3; that is, multiples of 6.	$\lvert A \cap B \rvert = 300 \div 6 = 50$
$A \cap C$ are multiples of 2 and 5; that is, multiples of 10.	$\lvert A \cap C \rvert = 300 \div 10 = 30$
$B \cap C$ are multiples of 3 and 5; that is, multiples of 15.	$\lvert B \cap C \rvert = 300 \div 15 = 20$
$A \cap B \cap C$ are multiples of 2, 3 and 5; that is, multiples of 30.	$\lvert A \cap B \cap C \rvert = 300 \div 30 = 10$
Find the number of elements in each set.	
2 Substitute into the inclusion-exclusion formula and answer the question.	$\lvert A \cup B \cup C \rvert$ $= \lvert A \rvert + \lvert B \rvert + \lvert C \rvert - \lvert A \cap B \rvert - \lvert A \cap C \rvert - \lvert B \cap C \rvert + \lvert A \cap B \cap C \rvert$ $= 150 + 100 + 60 - 50 - 30 - 20 + 10$ $= 220$ There are 220 integers that are multiples of 2, 3 or 5.

WB
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VCE QUESTION ANALYSIS

A VCE-style question (10 marks)

Leigh is an avid collector of blocks, which are all sorted into different bags in his playroom.

He picks one bag which has 5 blocks numbered 1, 2, 4, 7 and 8.

a How many different numbers can Leigh make with these blocks if he uses

- **i** all 5 blocks — 1 mark
- **ii** exactly 3 blocks — 1 mark
- **iii** any number of blocks. — 2 marks

b How many odd numbers can Leigh make using the numbered blocks? — 2 marks

Another bag has blocks labelled with a letter of the alphabet. Leigh takes 8 blocks from the bag and gets the letters A, A, A, A, C, G, E, E.

c How many different ways can these blocks be arranged in a line? — 1 mark

d In how many of these arrangements are the vowels grouped together? — 1 mark

e In how many of these arrangements does the line start and end with a consonant? — 1 mark

Leigh decides to sort a third bag of blocks into different piles so that each pile has blocks of the same colour. The bag contains 8 red, 10 yellow, 12 purple, 15 blue and 9 green blocks.

f Find the smallest number of blocks that must be selected to get at least 2 blocks of the same colour. — 1 mark

Video
VCE question analysis: Counting techniques

Reading the question

- Note that the first section of the question is about permutations, some of which involve restrictions.
- Highlight 'any number' in part **a iii** since this indicates the addition principle is used when considering different numbers of blocks.
- Highlight the lettered blocks that are being arranged and take note of the identical blocks and how this affects the arrangements.

Thinking about the question

- You will need to know how to count permutations for different numbers of objects and apply these principles.
- You will need to know how to count permutations where some items are kept in groups and also apply permutations when arranging identical objects.
- You will need to know how to apply the pigeon-hole principle.

Worked solution (✓ = 1 mark)

a i number of arrangements with 5 different blocks = 5! = **120** ✓

ii number of arrangements with 3 different blocks = 5 × 4 × 3 = **60** ✓

iii number of arrangements with 5 different blocks = 5! = **120** ✓

number of arrangements with 4 different blocks = ${}^5P_4 = 120$

number of arrangements with 3 different blocks = ${}^5P_3 = 60$

number of arrangements with 2 different blocks = ${}^5P_2 = 20$

number of arrangements with 1 block = ${}^5P_1 = 5$

total number of possible arrangements = 120 + 120 + 60 + 20 + 5 = **325** ✓

b number of 5-digit numbers = 2 × 4! = **48** ✓

number of 4-digit numbers = $2 \times {}^4P_3 = 48$

number of 3-digit numbers = $2 \times {}^4P_2 = 24$

number of 2-digit numbers = $2 \times {}^4P_1 = 8$

number of 1-digit numbers = 2

total number of possible numbers = 48 + 48 + 24 + 8 + 2 = **130** ✓

c number of arrangements = $\frac{8!}{4! \times 2!}$ = **840** ✓

d A, A, A, A, E, E are kept in a group and arranged with C and G.

number of arrangements of A, A, A, A, E, E = $\frac{6!}{4! \times 2!}$ = 15

total number of arrangements = 3! × 15 = **90** ✓

e number of arrangements that start and end with a consonant = 2 × 15 = **30** ✓

f Use the pigeon-hole principle with 5 colours to sort the smallest number of blocks = 5 + 1 = **6** ✓

EXERCISE 6.5 The inclusion-exclusion principle

ANSWERS p. 582

Recap

1 A drawer contains 8 black, 15 green, 7 striped and 6 white socks. The minimum number of socks that needs to be selected to get at least one pair is

A 2 **B** 4 **C** 5 **D** 8 **E** 9

2 Forty-five natural numbers are divided by 6 and the remainder is recorded. If at least x of the numbers have the same remainder, the value of x is

A 3 **B** 4 **C** 5 **D** 8 **E** 9

Mastery

3 WORKED EXAMPLE 23 In a survey about how people keep up to date with the daily news, it is found that 83 people are informed by reading the newspaper and 69 by watching TV. If 45 are informed of the daily news by watching TV and by reading the newspaper, find the number of people that are informed of the news by reading the newspaper or by watching TV.

4 WORKED EXAMPLE 24 How many integers between 1 and 210 are multiples of 3 or 7?

5 WORKED EXAMPLE 25 At a recent track and field competition, the list of entries showed that

- 53 entered the 100 m sprint
- 40 entered javelin
- 33 entered high jump
- 8 entered all 3 events
- 20 entered the 100 m sprint and javelin
- 21 entered the 100 m sprint and high jump
- 13 entered javelin and high jump.

Find the number of competitors who entered the 100 m sprint, javelin or high jump.

6 WORKED EXAMPLE 26 How many integers between 1 and 140 are multiples of 2, 5 or 7?

Exam practice

7 TECH-FREE (4 marks) How many integers between 1 and 660 are multiples of 2, 3 or 11?

8 TECH-FREE (5 marks) A survey of students about the genre of books they prefer finds that

- 53 like detective novels
- 43 like romance novels
- 54 like science fiction novels
- 13 like detective and romance novels
- 11 like romance and science fiction novels
- 26 like detective and science fiction novels
- 8 like all 3 genres of novel.

Find the number of students who like

a detective or science fiction novels 2 marks

b romance or science fiction novels 1 mark

c detective, romance or science fiction novels. 2 marks

9 TECH-FREE (2 marks) A survey is conducted out of 100 people at an ice-cream parlour to find the flavour of ice cream they prefer. They find that 72 people like vanilla, 52 like strawberry, 73 like chocolate and 21 like all 3 flavours. Find the number of people who prefer exactly 2 of the 3 flavours.

10 A group of students are surveyed about the radio stations they listen to. It is found that 46 listen to *Triple K*, 40 listen to *NPR* and 14 listen to *NPR* and *Triple K*. The number of students who listen to *NPR* or *Triple K* is

A 40 **B** 58 **C** 72 **D** 86 **E** 100

11 If $|X| = 55$, $|Y| = 44$ and $|X \cup Y| = 70$, then $|X \cap Y|$ is equal to

A 29 **B** 59 **C** 81 **D** 99 **E** 169

12 A survey at a pizza restaurant finds that on a busy Saturday, 60 people order pizza and 45 people order pasta. If 15 people ordered both pizza and pasta, then the number of people who ordered pizza or pasta was

A 30 **B** 90 **C** 105 **D** 120 **E** 150

13 The number of integers between 1 and 700 that are multiples of 5 or 7 is

A 20 **B** 200 **C** 220 **D** 240 **E** 260

Use the following information to answer Questions 14 and 15.

In a group of 100 Year 11 students who study either Mathematical Methods, Specialist Mathematics or Chemistry, it is found that

- 60 study Specialist Mathematics
- 51 study Mathematical Methods
- 71 study Chemistry
- 30 study Specialist Mathematics and Mathematical Methods
- 41 study Specialist Mathematics and Chemistry
- 28 study Mathematical Methods and Chemistry.

14 The number of students who study Specialist Mathematics or Mathematical Methods is

A 30 **B** 81 **C** 90 **D** 100 **E** 111

15 The number of students who study all 3 subjects is

A 10 **B** 17 **C** 23 **D** 83 **E** 182

6 Chapter summary

The addition principle

- In an event in which we can choose a of one type OR b of another type OR c of another type, the total number of choices can be calculated using the **addition principle**.
- The total number of possibilities = $a + b + c$.
- The addition principle is associated with the word 'OR' in a problem.

The multiplication principle

- When there are 2 or more successive selections to make and we must choose from a of one type AND then from b of another type, the total number of choices can be found by using the **multiplication principle**.
- The total number of possibilities = $a \times b$.
- The multiplication principle is associated with the word 'AND' in a problem.

Factorials

- $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$
- $n!$ is the number of arrangements of n different objects.

Permutations

- The number of **permutations** or arrangements of r objects chosen from n objects is ${}^nP_r = \dfrac{n!}{(n-r)!}$.

Arrangement with restrictions

When counting permutations with restrictions:

1 determine the number of places to be filled and draw the same number of boxes

2 fill all the spaces where the restriction is involved

3 fill the rest of the spaces from left to right.

Arrangements with groups

When counting the number of arrangements with groups:

1 put all the objects that must be kept together in a group and count this group as one object

2 determine the number of arrangements of the objects plus the group

3 multiply this by the number of ways of arranging the objects within the group.

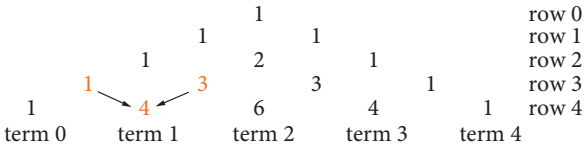
Combinations

- The number of **combinations** or selections of r objects chosen from n objects, where the order of selection is NOT important, is ${}^nC_r = \dfrac{n!}{(n-r)!r!}$.
- To calculate ${}^nC_r$, use CAS or evaluate a fraction where the numerator is the product of r descending consecutive numbers starting with n and the denominator is $r!$.

  For example, ${}^6C_3 = \dfrac{6 \times 5 \times 4}{3 \times 2 \times 1}$.
- Using symmetry:

  ${}^nC_r = {}^nC_{n-r}$, ${}^nC_0 = {}^nC_n = 1$ and ${}^nC_1 = n$

Pascal's triangle



Add the 2 numbers in the row above to get the numbers in the next row.

- **Pascal's identity** is ${}^{n}C_{k} = {}^{n-1}C_{k-1} + {}^{n-1}C_{k}$.
- Combinations taken any number at a time:

  ${}^{n}C_{0} + {}^{n}C_{1} + {}^{n}C_{2} + \ldots + {}^{n}C_{n-1} + {}^{n}C_{n} = 2^{n}$ and ${}^{n}C_{1} + {}^{n}C_{2} + \ldots + {}^{n}C_{n-1} + {}^{n}C_{n} = 2^{n} - 1$

The pigeon-hole principle

The **pigeon-hole principle** states:

- if $n + 1$ objects are placed into n holes, then at least 1 hole must contain 2 objects
- if n objects are sitting in k holes, where $n > k$, then there is at least 1 hole with at least $\frac{n}{k}$ objects.

The inclusion–exclusion principle

- The **inclusion-exclusion principle** for 2 sets is

  $|A \cup B| = |A| + |B| - |A \cap B|$
- The inclusion-exclusion principle for 3 sets is

  $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 (3 marks) Three sisters and 2 of their friends queue for tickets for a soccer game.

a How many different arrangements of the 5 people are possible? 1 mark

b How many arrangements are possible if the 2 friends never stand together? 2 marks

2 (2 marks) Find the number of integers between 1 and 400 that are multiples of 5 or 8.

3 (2 marks) Find the number of edges in the complete graph K_9 and the number of edges that must be removed to change this graph into a tree.

4 (3 marks) A geometric sequence has a 4th term of 135 and a 5th term of 45.

Find t_1, r and S_∞.

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 Standard vehicle licence plates consist of a single-digit number and 2 letters followed by another single-digit number and 2 letters. The number of different licence plates with '1' as the first number and 'Z' as the last letter is

A $26^3 \times 10$ **B** $26^4 \times 10^2$ **C** $26^3 \times 9$ **D** $3 \times 26 + 10$ **E** $4 \times 26 + 2 \times 10$

2 A 3-digit number is made from the digits {1, 3, 4, 7, 8, 9}. The number of odd numbers greater than 900 is

A 6 **B** 12 **C** 16 **D** 120 **E** 720

3 The number of distinct prime factors of 660 is

A 3 **B** 4 **C** 5 **D** 7 **E** 12

4 Using statement A, 'the sky is not blue' and statement B, 'there are no rain clouds', then $\neg(\neg A \wedge B)$ means

A The sky is not blue or there are rain clouds.

B The sky is not blue because there are rain clouds.

C If the sky is not blue, there are rain clouds.

D The sky is blue or there are no rain clouds.

E If the sky is blue, there are no rain clouds.

5 There are 135 peaches sorted into baskets by size and the damaged ones are rejected. The baskets are labelled large, medium, small and rejects. If at least one basket has at least m peaches, the smallest value of m is

A 5 **B** 6 **C** 33 **D** 34 **E** 45

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Section B 2 questions

15 marks

1 (9 marks) A group of physical education students were surveyed about the types of exercise they did to maintain their fitness. The most common types of exercises were running, swimming and cycling. It was found that each of the students surveyed did at least one of these common types of exercise. The results of the survey were summarised for these three categories:

- 117 students exercise by running
- 111 students exercise by swimming
- 87 students exercise by cycling
- 52 did running and swimming
- 47 did running and cycling
- 56 did swimming and cycling
- 40 did all three.

Let R = {physical education students who exercise by running}

S = {physical education students who exercise by swimming} and

C = {physical education students who exercise by cycling}.

Find

a **i** $|R \cup S|$ 1 mark

ii $|R \cup C|$ 1 mark

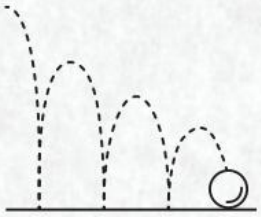
iii $|C \cup S|$ 1 mark

iv $|R \cup S \cup C|$ 2 marks

b the number of students who exercise by running and swimming but do not cycle 2 marks

c the number of students who exercise by doing exactly two of the three types of exercise. 2 marks

2 (6 marks) A ball is dropped from a height of 144 cm and it rebounds to two-thirds of its original height each time.



If the initial height of the ball is h_1 and the maximum height of the ball after the first bounce is h_2, find

a the value of h_2 1 mark

b the value of h_4 1 mark

c the value of h_n, in terms of n, where n is the bounce number 1 mark

d the value of $h_1 + h_2 + h_3 + \ldots$ 1 mark

e the total distance the ball travels if it continues to bounce indefinitely. 2 marks

CHAPTER

7 TRIGONOMETRIC IDENTITIES

Study Design coverage

Nelson MindTap chapter resources

7.1 Arc length, sectors and segments

7.2 The sine rule

The ambiguous case

Using CAS 1: The sine rule

7.3 The cosine rule

Using CAS 2: The cosine rule

7.4 Applications of the sine and cosine rules

7.5 The Pythagorean identities

7.6 The sum and difference identities

Using CAS 3: The sum and difference identities

7.7 The double angle identities

7.8 The products of sines and cosines

7.9 The subsidiary angle identities

7.10 Proving trigonometric identities

Using CAS 4: Verifying trigonometric identities

Using trigonometric identities to solve equations

VCE question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2

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Study Design coverage

UNIT 2, AREA OF STUDY 2: SPACE AND MEASUREMENT

Trigonometry

- radian measure, arc length, sectors and segments
- the sine rule and cosine rule applied to two and three-dimensional situations, including problems involving angles between planes
- compound and double angle formulas for sine, cosine and tangent and the identities: $\sec^2(x) = 1 + \tan^2(x)$ and $\operatorname{cosec}^2(x) = 1 + \cot^2(x)$
- proof and application of identities between $a\sin(x) + b\cos(x)$ and $r\sin(x \pm \alpha)$ or $r\cos(x \pm \alpha)$ where α is in the first quadrant, the identities for products of sines and cosines expressed as sums and differences, and the identities for addition and differences of sines and cosines expressed as products.

VCE Mathematics Study Design 2023–2027 p. 66, © VCAA 2022

Video playlists (11):

7.1 Arc length, sectors and segments
7.2 The sine rule
7.3 The cosine rule
7.4 Applications of the sine and cosine rules
7.5 The Pythagorean identities
7.6 The sum and difference identities
7.7 The double angle identities
7.8 The products of sines and cosines
7.9 The subsidiary angle identities
7.10 Proving trigonometric identities
VCE question analysis Trigonometric identities

Worksheets (11):

7.2 The sine rule • Sine rule problems • Discovering the sine rule
7.3 The cosine rule • Cosine rule problems • Sine and cosine rules problems • Sine and cosine rules assignment
7.6 Simplifying periodic functions
7.7 Trigonometric identities
7.8 Products to sums and differences • Further trigonometric identities

Puzzles (2):

7.3 The sine and cosine rules
7.7 Trigonometric identities

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

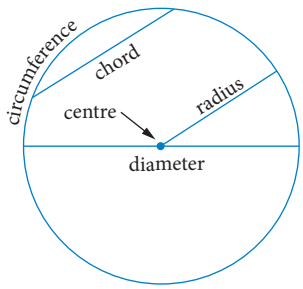
7.1 Arc length, sectors and segments

Video playlist
Arc length, sectors and segments

A **radius** joins the centre of a circle to the **circumference**.

A **chord** joins 2 points on the circumference of a circle.

A **diameter** is a chord that passes through the centre of a circle.



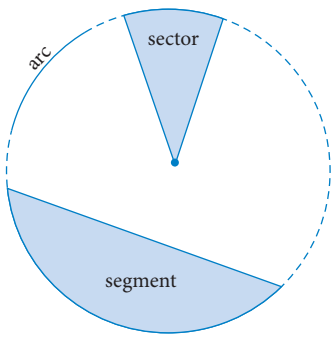
An **arc** is part of the circumference.

A **sector** is a region of a circle cut off by 2 radii.

A **segment** is a region of a circle cut off by a chord.

A **major arc**, **sector** or **segment** is more than half of the circle.

A **minor arc**, **sector** or **segment** is less than half of the circle.



Arc length of a circle

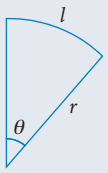
The **arc length** of a circle is

$$l = r\theta$$

where

r = radius of the circle

θ = angle at the centre of the circle in radians.



Area of a sector

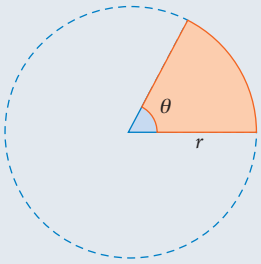
The **area of a sector** of a circle is

$$A = \frac{1}{2}r^2\theta$$

where

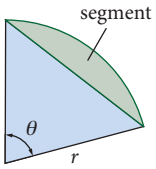
r = radius of the circle

θ = angle at the centre of the circle in radians.



The area of a segment can be considered as the area of the sector less the triangle, as shown in the diagram.

A = area of sector − area of triangle

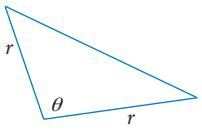


$$\text{Area of triangle} = \frac{1}{2} \times 2 \text{ sides} \times \sin(\text{included angle})$$

$$= \frac{1}{2}r^2 \sin(\theta)$$

$$= \frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin(\theta)$$

$$A = \frac{1}{2}r^2(\theta - \sin(\theta))$$



Area of a segment

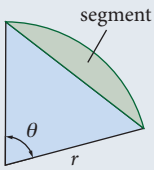
The **area of a segment** of a circle is

$$A = \frac{1}{2}r^2(\theta - \sin(\theta))$$

where

r = radius of the circle

θ = angle at the centre of the circle in radians.



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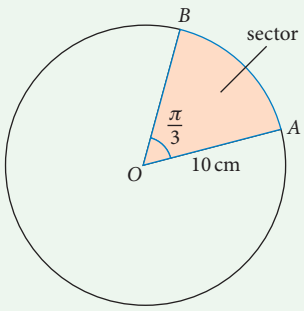
WORKED EXAMPLE 1 Arc length, sector and segment

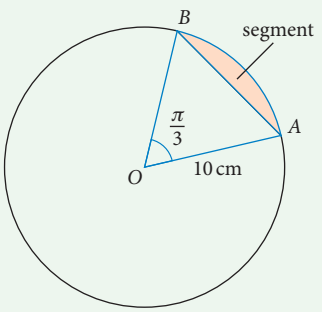
For the sector of a circle with radius 10 cm forming an angle of $\frac{\pi}{3}$ radians at the centre of the circle, O, find correct to 2 decimal places

a the arc length AB

b the area of the sector AOB

c the area of the segment.





Steps	Working
a Use the formula $l = r\theta$.	$l = 10 \times \frac{\pi}{3}$ $= \frac{10\pi}{3}$ ≈ 10.47 units
b Use the formula $A = \frac{1}{2}r^2\theta$.	$A = \frac{1}{2}10^2 \times \frac{\pi}{3}$ $= \frac{50\pi}{3}$ ≈ 52.36 units^2
c Use the formula $A = \frac{1}{2}r^2(\theta - \sin(\theta))$ An exact area is preferred but a decimal approximation helps to compare values.	$A = \frac{1}{2}10^2\left(\frac{\pi}{3} - \sin\left(\frac{\pi}{3}\right)\right)$ $= 50\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$ ≈ 9.06 units^2

Note: If an angle is given in degrees, you need to change that angle to radians to use in these formulas. CAS can convert radians to degrees and vice-versa.

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EXERCISE 7.1 Arc length, sectors and segments

ANSWERS p. 582

Mastery

1 WORKED EXAMPLE 1 Find the arc length in the circle of radius 5 cm subtended by an angle of $\frac{2\pi}{3}$

a in exact form **b** correct to 2 decimal places.

2 Find the area of the sector in the circle with radius 5 cm subtended by an angle of $\frac{5\pi}{3}$

a in exact form **b** correct to 2 decimal places.

3 Find the area of the segment in a circle with radius 5 cm subtended by an angle of $\frac{2\pi}{3}$

a in exact form **b** correct to 2 decimal places.

4 Find the radian measure for each angle.

a 60° **b** 30° **c** 20° **d** 210°

5 A major sector is bigger than a semicircle while a minor sector is a smaller than a semicircle. State whether each angle subtends a major or minor sector.

a $\frac{2\pi}{3}$ **b** $\frac{11\pi}{6}$ **c** $\frac{8\pi}{9}$ **d** $\frac{7\pi}{6}$

Exam practice

6 For the sector of a circle with radius of 8 mm, subtended by the angle of 45°, the arc length is

A $\frac{8\pi}{9}$ mm **B** 2π mm **C** π mm **D** 360° **E** 720°

7 For the sector of a circle with radius of 9 mm, subtended by the angle of 45°, the area of the sector in mm^2 is

A $\frac{81\pi}{8}$ **B** 2π **C** $\frac{\pi}{9}$ **D** 360° **E** 405°

8 For the sector of a circle with radius of 10 mm, subtended by the angle of 45°, the area of the segment is closest to

A 450 mm^2 **B** 4500 mm^2 **C** 2214.6 mm^2 **D** 3.91 mm^2 **E** 405°

9 For the sector of a circle with radius of 10 cm, subtended by the angle of $\frac{9\pi}{8}$ radians, the arc length in cm, is

A $\frac{8\pi}{9}$ **B** π **C** $\frac{9\pi}{8}$ **D** 2π **E** $\frac{45\pi}{4}$

10 For the sector of a circle with radius of 10 cm, subtended by the angle of $\frac{9\pi}{8}$ radians, the percentage that the arc length is of the circumference of the circle is closest to

A $\frac{9\pi}{8}$% **B** 2.5% **C** 5.6% **D** $\frac{45\pi}{4}$% **E** 56%

11 For the minor sector of a circle with radius of 3 mm, subtended by the angle of 45°, the area of the major sector, in mm^2, is closest to

A 2.025 **B** 3.5 **C** 7.07 **D** 67.5 **E** 202.5

12 For the minor sector of a circle with radius of 10 mm, subtended by the angle of $\frac{\pi}{4}$ radians, the area of the major segment, in mm^2, is closest to

A 3.9 **B** 39.3 **C** 274.9 **D** 310.2 **E** 1893.3

7.2 The sine rule

Video playlist
The sine rule

Worksheets
The sine rule

Sine rule problems

Discovering the sine rule

In geometry, **solving** a triangle means finding the sizes of all unknown sides and angles. We already know how to solve a **right-angled triangle** for its sides and angles.

The sine and cosine rules can be used to find the sides and angles in a **non-right-angled triangle**.

In any triangle ABC with angles labelled A, B, C, the opposite side lengths are labelled a, b, c respectively.

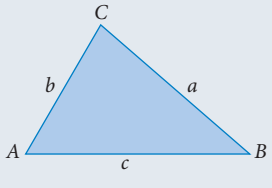
The **sine rule** relates each side of a triangle to its opposite angle. The longest side is opposite the largest angle, and the shortest side is opposite the smallest angle.

The sine rule

In any triangle ABC:

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

or $$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c}$$

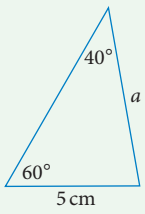


Select any 2 of the pairs to solve for length of side or angle.

The **sine rule** is used when the triangle problem gives 2 sides and 1 non-included angle, or 2 angles and 1 side.

WORKED EXAMPLE 2 The sine rule 1

For this triangle, find the unknown side a correct to 2 decimal places.



Steps	Working
1 Look for 2 pairs of sides and their opposite angles. Put the unknown variable in the numerator. $\frac{a}{\sin A} = \frac{b}{\sin B}$	$\frac{a}{\sin 60°} = \frac{5}{\sin 40°}$
2 Solve for a.	$a = \frac{5 \sin 60°}{\sin 40°}$ $\approx 6.74\,\text{cm}$

Exam hack

From the diagram, an answer of $a \approx 6.74$ cm looks reasonable. Side a is opposite the larger angle (60°), so it should be longer than 5 cm.

WB p. 155

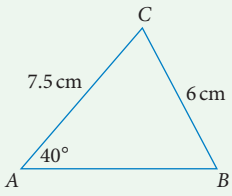
The ambiguous case

An angle θ in a triangle could be **acute**, **right** or **obtuse**, so it has the range $0° < \theta < 180°$.

This is important when finding the size of an unknown angle in a triangle using the sine rule, when it is possible to have 2 answers: 1 acute, 1 obtuse.

WORKED EXAMPLE 3 The sine rule for angles 1

For this triangle, find the possible angles B correct to 2 decimal places.



Steps	Working
1 Look for 2 pairs of angles and their opposite sides. Put the unknown variable in the numerator. $\frac{\sin(A)}{a} = \frac{\sin(B)}{b}$	$\frac{\sin(B)}{7.5} = \frac{\sin(40^\circ)}{6}$ $\sin(B) = \frac{7.5 \sin(40^\circ)}{6}$

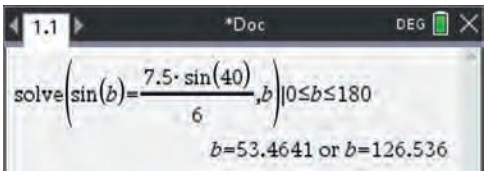
WB p. 156

2 Solve for B using solve or $\sin^{-1}$.

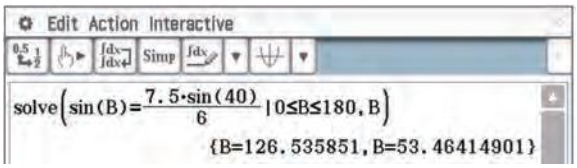
There are 2 solutions to the sine equation that are supplementary (add up to 180°). This is because $\sin(x)$ is positive in the 1st and 2nd quadrants.

From the triangle, angle B is opposite the longer side (7.5), so it should be greater than 40°.

TI-Nspire



ClassPad



$B \approx 53.46°$ or $126.54°$

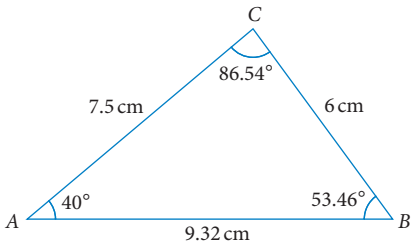
3 Test whether the *obtuse* value can be an angle in a triangle, using the *known* angle 40°.

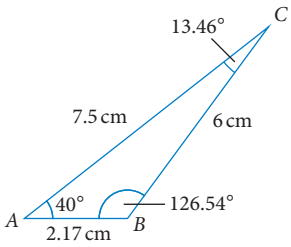
The known angle is 40°.

B can be obtuse because $40° + 126.54°$
$= 166.54° < 180°$ (the angle sum of a triangle).

$B \approx 53.46°$ or $126.54°$

The 2 triangles show the 2 possible solutions to the above example.





This situation is called the **ambiguous case**, where 2 solutions for an unknown angle are possible.

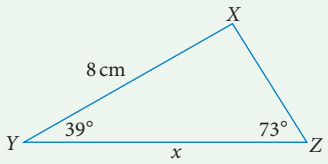
In some cases, both the acute and obtuse angles are correct, but in other cases, only 1 angle is correct for that triangle. It depends on the size of the given angle.

The ambiguous case only happens when we are finding an angle, not a side.

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WORKED EXAMPLE 4 The sine rule 2

Find x, correct to 1 decimal place.

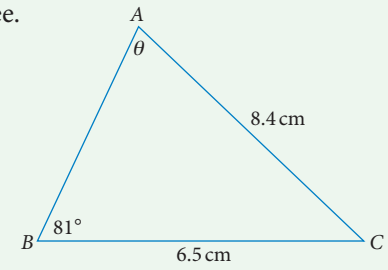


Steps	Working
1 Find $\angle X$.	$\angle X = 180° - 39° - 73°$ (angle sum of a triangle) $= 68°$
2 Find 2 pairs of matching sides and opposite angles with the unknown on top.	$\frac{x}{\sin(68°)} = \frac{8}{\sin(73°)}$
3 Solve for x.	$x = \frac{8\sin(68°)}{\sin(73°)}$
4 Evaluate and round appropriately.	$x = 7.7563\ldots$ ≈ 7.8 cm

9780170448338

WORKED EXAMPLE 5 The sine rule for angles 2

Find θ, correct to the nearest degree.

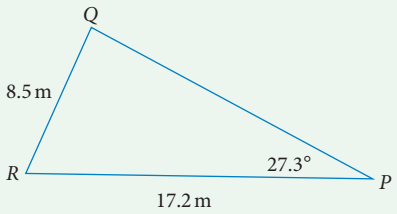


Steps	Working
1 Find 2 pairs of matching angles and opposite sides with unknown on top.	$\frac{\sin(\theta)}{6.5} = \frac{\sin(81°)}{8.4}$
2 Solve for $\sin(\theta)$.	$\sin(\theta) = \frac{6.5\sin(81°)}{8.4}$ $\sin(\theta) = 0.7642\ldots$
3 Use $\sin^{-1}$ to find the acute value of θ.	$\theta = 49.8422\ldots$
4 Find the possible obtuse angle.	$\theta' \approx 180° - 49.8422\ldots°$ $= 130.1578\ldots°$
5 Check the obtuse angle using the known angle 81°.	Sum of angles $= 81° + 130.1578\ldots° > 180°$
6 Sum > 180° means triangle cannot exist for obtuse θ.	$\theta = 49.8422\ldots°$ only $\approx 50°$

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WORKED EXAMPLE 6 The sine rule for angles 3

Find Q, correct to the nearest 0.1°.



Steps	Working
1 Look for 2 pairs of matching angles and opposite sides with the unknown on top.	$\frac{\sin(Q)}{17.2} = \frac{\sin(27.3°)}{8.5}$
2 Solve for $\sin(Q)$.	$\sin(Q) = \frac{17.2\sin(27.3°)}{8.5}$ $= 0.9280\ldots$
3 Use $\sin^{-1}$ to find the acute value of Q.	$Q = 68.1391\ldots$
4 Find the possible obtuse angle.	$Q' \approx 180° - 68.1391\ldots° = 111.8609\ldots°$
5 Check the obtuse angle using the known angle 27.3°.	Sum of angles $= 27.3° + 111.8609\ldots° < 180°$
6 Sum < 180° means triangle does exist for obtuse Q.	$Q \approx 68.1°$ or $111.9°$

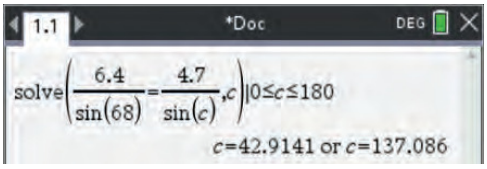
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There are 2 ways to find an angle using the sine rule in CAS: solving a sine rule equation for values between 0° and 180° (shown below), or using $\sin^{-1}$ to find the acute angle first.

USING CAS 1 The sine rule

Find the angle, C, for $\triangle ABC$, with $b = 6.4$ m, $c = 4.7$ m and $B = 68°$.

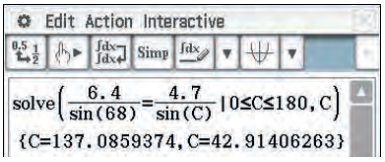
TI-Nspire



1 Set the mode to **DEG**.

2 Solve the sine rule equation by restricting the domain to $|\ \mathbf{0 \le c \le 180}$.

ClassPad



1 Set the mode to **Deg**.

2 Solve the sine rule equation by restricting the domain to $|\ \mathbf{0 \le C \le 180}$.

Consider which of the 2 solutions is the suitable angle. In this case, it is $C = 42.91°$.

The angles in the sine rule can be in degrees or radians, the sides can be exact or approximate.

EXERCISE 7.2 The sine rule

ANSWERS p. 582

Recap

1 TECH-FREE For a sector in a circle with radius 10 cm subtended by an angle of $\frac{\pi}{3}$, find

a the arc length **b** the area of the sector **c** the area of the minor segment.

2 For a sector of a circle with radius 1 cm and an angle of 120° at the centre, calculate correct to 2 decimal places

a the arc length **b** the area of the sector **c** the area of the segment.

Mastery

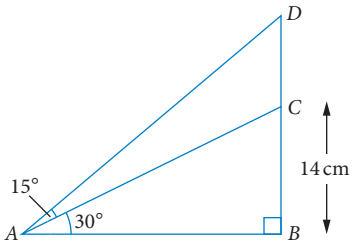
3 Find each length, correct to 1 decimal place.

a AC

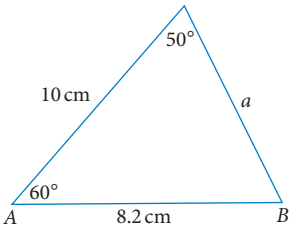
b AB

c AD

d CD

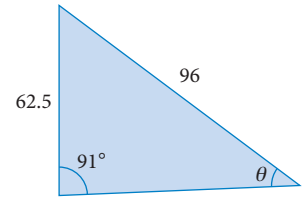


4 WORKED EXAMPLE 2 Find the unknown angle and side, correct to 2 decimal places.

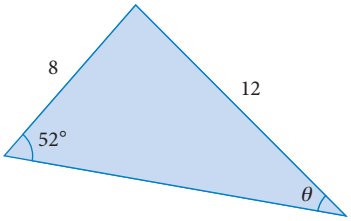


5 WORKED EXAMPLE 3 Find the angle θ, to the nearest degree, in each triangle.

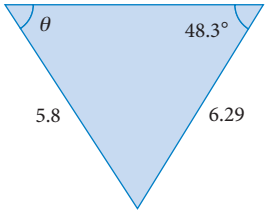
a



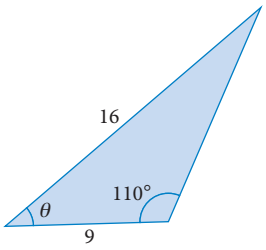
b



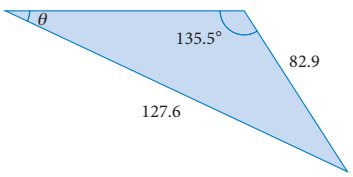
c



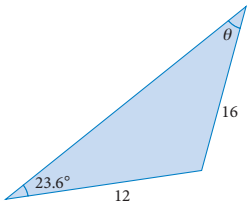
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e

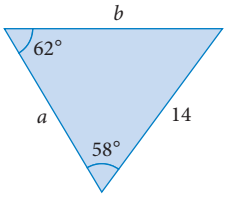


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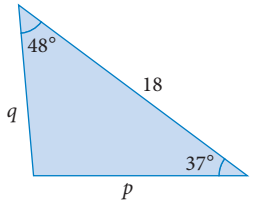


6 WORKED EXAMPLE 4 Find the unknown lengths to 1 decimal place, in each triangle.

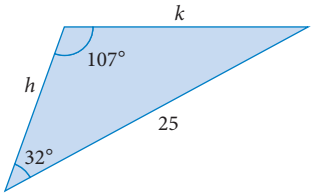
a



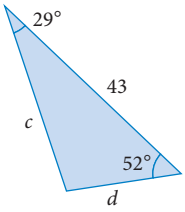
b



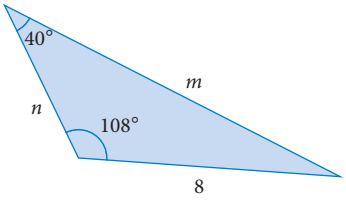
c



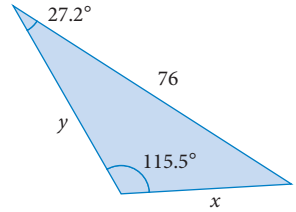
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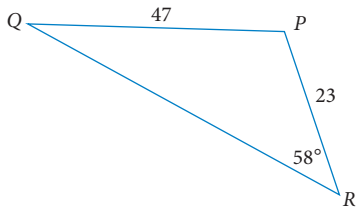


7 WORKED EXAMPLE 5 Using CAS 1 Solve each triangle for all unknown angles and sides, to the nearest integer.

a ΔXYZ where $Z = 65°$, $y = 11$ m and $z = 16$ m **b** ΔPQR where $P = 35°$, $Q = 53°$ and $q = 68$ cm

Exam practice

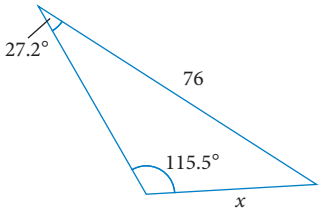
8 For this triangle, $\sin(Q) =$



A $\frac{47}{23\sin(58°)}$ **B** $\frac{23}{47\sin(58°)}$ **C** $\frac{47}{23}\sin(58°)$

D $\frac{23\sin(58°)}{47}$ **E** $\frac{58\sin(23°)}{\sin(47°)}$

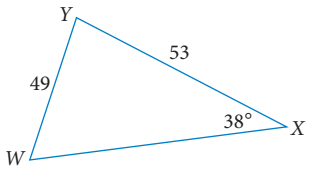
9 In the triangle, $x \approx$



A 25.3 **B** 29.2 **C** 38.5°

D 54.3 **E** 150.1

10 WORKED EXAMPLE 6 In the triangle, $W \approx$

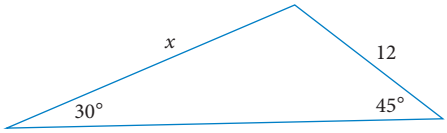


A 41.8° or 138.2° **B** 34.7° or 145.3° **C** 41.8° only

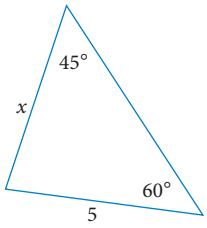
D 34.7° only **E** 38.3° only

11 (4 marks) Find the value x, to the nearest whole number, in each diagram.

a



b

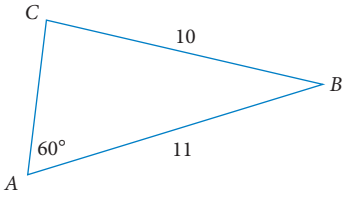


12 (2 marks) Find, to the nearest millimetre, the length of the longest side of ΔPQR if $P = 32°$, $Q = 29°$ and $PR = 42$ mm.

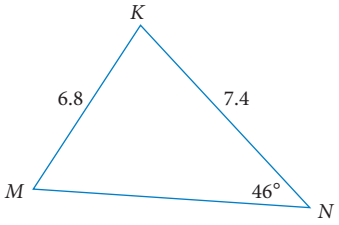
13 (2 marks) Solve the triangle ΔXYZ, where $Z = 65°$, $y = 11$ m and $z = 16.2$ m, to the nearest whole number.

14 (4 marks) Solve each triangle, considering the ambiguous case, giving answers correct to 1 decimal place.

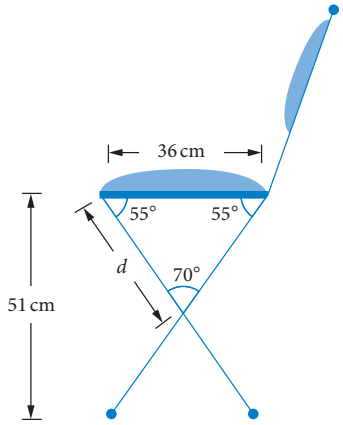
a



b



15 (2 marks) The folding chair shown has a seat that is 36 cm deep. The legs of the chair are inclined at an angle of 55° to the seat and joint at an angle of 70°. The seat is 51 cm above the floor. Find the distance from the seat, d, at which the legs are joined.

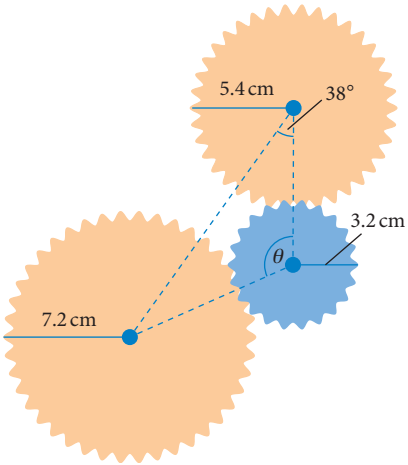


16 (4 marks) ΔABC is an isosceles with $\angle BAC = 52°$ between the equal sides. BC is extended to D so that $AC = CD$. $AB = 12.3$ cm.

a Find the length of AD, correct to 1 decimal place. 2 marks

b Find the length of BD, correct to 1 decimal place. 2 marks

17 (3 marks) Three gears are arranged as shown. The radii of the gears are 7.2 cm, 5.4 cm and 3.2 cm. Calculate θ, correct to 1 decimal place.



7.3 The cosine rule

The **cosine rule** relates 1 side of a triangle to the other 2 sides and its opposite angle. It is an extension of Pythagoras' theorem, $c^2 = a^2 + b^2$.

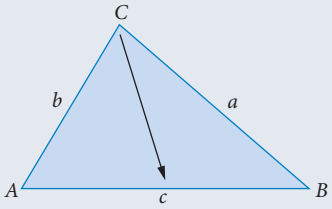
The cosine rule

In any triangle ΔABC,

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

or

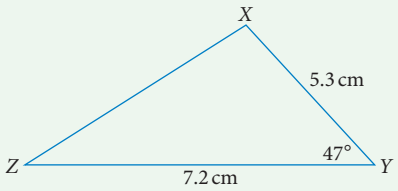
$$\cos(C) = \frac{a^2 + b^2 - c^2}{2ab}$$



The **cosine rule** is used for triangle problems involving 2 sides and the included angle, or all 3 sides.

WORKED EXAMPLE 7 The cosine rule

Find the length of XZ, correct to 2 decimal places.



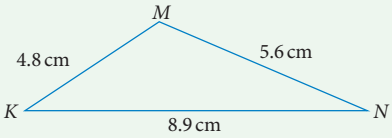
Steps	Working
1 Use the cosine rule of the form $c^2 = a^2 + b^2 - 2ab\cos C$, where c is the unknown length and C is the known opposite angle.	$XZ^2 = 5.3^2 + 7.2^2 - 2(5.3)(7.2)\cos(47°)$
2 Solve for XZ, and take the positive solution.	$XZ = 5.2801\ldots$
3 Round the answer.	$XZ \approx 5.28$ cm

Exam hack

From the diagram, the answer looks reasonable.

WORKED EXAMPLE 8 The cosine rule for angles

Find the angle M, to the nearest degree.



Remember to start the cosine rule with the side opposite the angle we know or want to find out. Here, we start with side $m = 8.9$.

Steps	Working
1 Use the cosine rule of the form $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$, where C is the unknown angle and c is the known opposite side.	$\cos M = \frac{4.8^2 + 5.6^2 - 8.9^2}{2 \times 4.8 \times 5.6}$
2 Solve for M using CAS by solving the cosine rule equation or using $\cos^{-1}$, in the range $0° < M < 180°$.	$M = 117.4836\ldots°$
Note: The ambiguous case does not apply to the cosine rule.	
3 Round the answer.	$M \approx 117°$

Exam hack

From the diagram, the answer looks reasonable.

Video playlist
The cosine rule

Worksheets
The cosine rule

Cosine rule problems

Sine and cosine rules problems

Sine and cosine rules assignment

Puzzle
The sine and cosine rules

WB p. 160

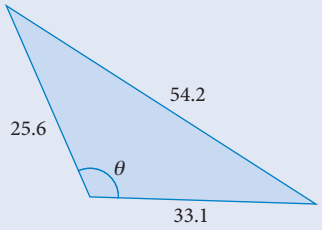
WB p. 161

Solving problems with the sine and cosine rules:

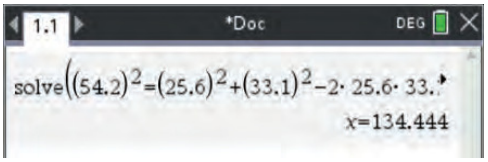
- use the **sine rule** for problems involving 2 sides and 2 angles
- use the **cosine rule** for problems involving 2 sides and the included angle or 3 sides
- use a combination of the **sine rule** and **cosine rule** in the one problem, if suitable, remembering the ambiguous case in the sine rule for angles
- consider larger/smaller side lengths that match larger/smaller opposite angles.

USING CAS 2 The cosine rule

Find the angle θ, to the nearest degree.



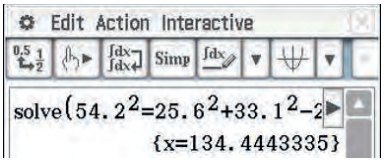
TI-Nspire



1 Set the mode to **DEG**.

2 Solve the cosine rule equation by restricting the domain to $|\ \mathbf{0 \le x \le 180}$.

ClassPad



1 Set the mode to **Deg**.

2 Solve the cosine rule equation by restricting the domain to $|\ \mathbf{0 \le x \le 180}$.

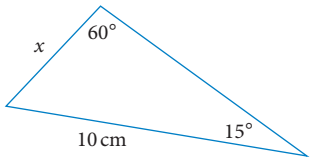
The angle is $x = 134.44°$.

EXERCISE 7.3 The cosine rule

ANSWERS p. 582

Recap

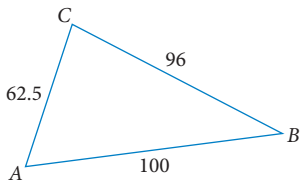
1 Use the sine rule in the triangle below to find the missing side, x, to the nearest whole number.



2 In the triangle, $B \approx$

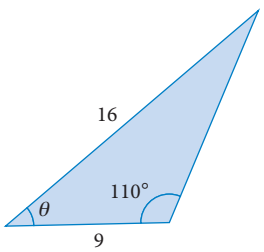
A 37.1° or 142.9° **B** 96.8° or 83.2° **C** 37.1° only

D 38.9° or 141.1° **E** 38.9° only



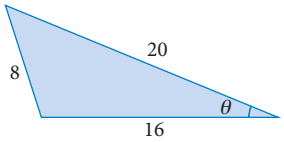
Mastery

3 WORKED EXAMPLE 7 Find the unknown side in this triangle, correct to 1 decimal place.

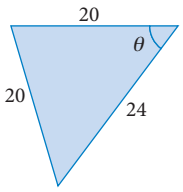


4 WORKED EXAMPLE 8 Using CAS 2 Find the angle(s) θ, to the nearest degree, in each triangle.

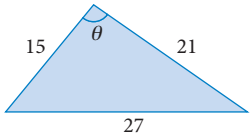
a



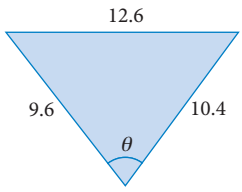
b



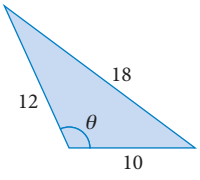
c



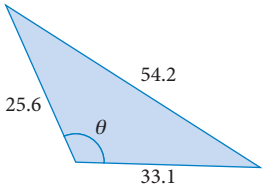
d



e



f



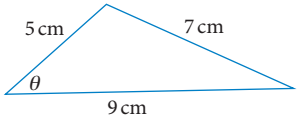
5 Find the unknown side in each triangle to the nearest integer.

a ΔXYZ, where $X = 65°$, $y = 11$ and $z = 16$

b ΔPQR, where $P = 35°$, $r = 53$ and $q = 68$

Exam practice

6 Which statement is true about this triangle?



A $\cos(\theta) = \dfrac{5^2 + 7^2 - 9^2}{2 \times 7 \times 9}$

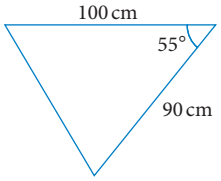
B $\cos(\theta) = \dfrac{7^2 + 9^2 - 5^2}{2 \times 7 \times 9}$

C $\cos(\theta) = \dfrac{5^2 + 7^2 - 9^2}{2 \times 7 \times 9}$

D $\cos(\theta) = \dfrac{5^2 + 7^2 - 9^2}{2 \times 5 \times 7}$

E $\cos(\theta) = \dfrac{5^2 + 9^2 - 7^2}{2 \times 5 \times 9}$

7 The unknown side in this triangle is approximately



A 58 cm

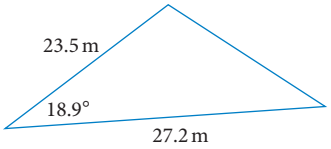
B 88 cm

C 113 cm

D 114 cm

E 169 cm

8 The unknown side in this triangle is approximately



A 9.1 m

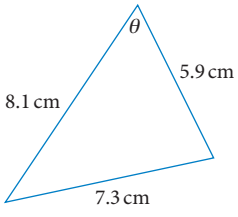
B 10.9 m

C 12.4 m

D 29.6 m

E 50.0 m

9 The value of θ in this triangle is approximately



A 9.5°

B 44.7°

C 60.5°

D 74.9°

E 79.0°

10 (4 marks) Find the unknown side, to the nearest whole number, in each triangle.

a ΔMNO, where $M = 41.1°$, $n = 27.8$ and $o = 39.2$ — 1 mark

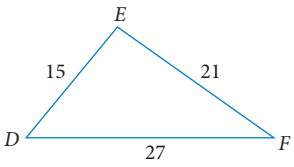
b ΔXYZ, where $Z = 28.3°$, $y = 571$ and $x = 421$ — 1 mark

c ΔDEF, where $F = 45.6°$, $d = 72.3$ and $e = 89.4$ — 1 mark

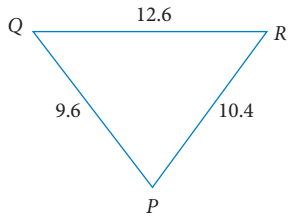
d ΔGHI, where $G = 67.3°$, $h = 37.9$ and $i = 40.8$ — 1 mark

11 (8 marks) Calculate all angles in each triangle, correct to 1 decimal place.

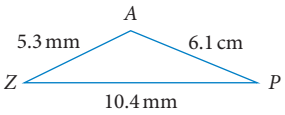
a



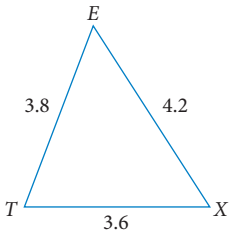
b



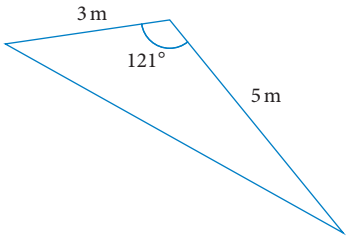
c



d



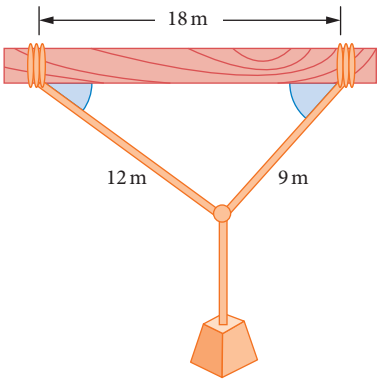
12 (2 marks) Calculate the length of the unknown side to the nearest metre.



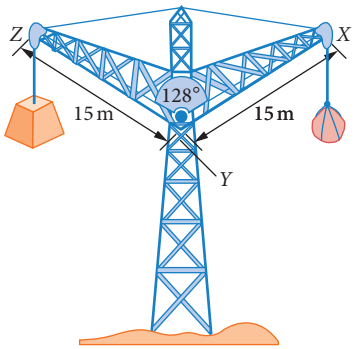
13 (2 marks) A weight is supported by cables attached to opposite sides of an overhead beam 18 m apart.

One cable is 12 m long and the other is 9 m long.

What angles do each of the cables make with the ceiling, correct to 1 decimal place?



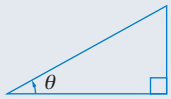
14 (2 marks) What is the horizontal distance between Z and X on the crane shown here, correct to 2 decimal places?



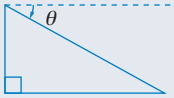
7.4 Applications of the sine and cosine rules

Angles of elevation and depression

The **angle of elevation** of an object is the angle measured from the horizontal (ground) when looking up to an object above.



The **angle of depression** of an object is the angle measured from the horizontal when looking down to an object below.



Bearings are measured clockwise from the north position.

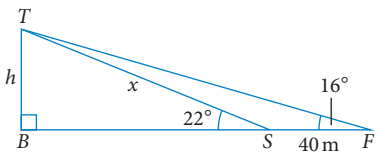
Video playlist
Applications of the sine and cosine rules

Make sure you draw a diagram in applications and identify the important triangles. In most problems, we should leave calculations until the final step.

WB
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WORKED EXAMPLE 9 Angles of elevation

Jermaine sees that the top of a sand dune at the end of a beach is at an angle of elevation of 16°.
On walking 40 metres closer, she finds that the angle of elevation increases to 22°.
What is the approximate height of the sand dune, correct to 1 decimal place?

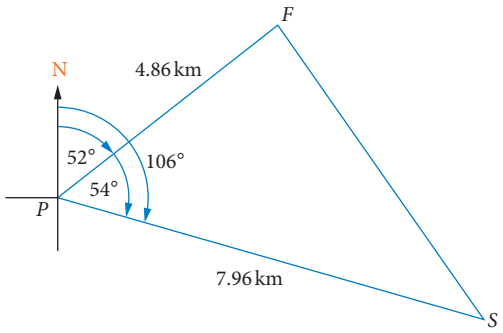
Steps	Working
1 Draw a diagram and label the sides and angles.	
2 Use the exterior angle of ΔSBT to find $\angle STF$.	$\angle STF = 22° - 16° = 6°$
3 Use the sine rule.	$\frac{x}{\sin(16°)} = \frac{40}{\sin(6°)}$
4 Solve for x.	$x = \frac{40\sin(16°)}{\sin(6°)}$
5 Now use ΔSBT to find h from x.	$\sin(22°) = \frac{h}{x}$
6 Solve for h.	$h = x\sin(22°)$
7 Substitute the expression for x and calculate the answer.	$= \frac{40\sin(16°)\sin(22°)}{\sin(6°)}$ $= 39.5129\ldots$
8 Round appropriately and write the answer.	The dune is about 39.5 m high.

WB

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WORKED EXAMPLE 10 Bearings

Two fire towers in the direction N 52° E and at a bearing of 106° are at ranges of 4.86 km and 7.96 km respectively. Find the distance between the towers.

Steps	Working
1 Draw and label a diagram showing the information. The angle between the directions of the towers is 54°.	
2 Write the cosine rule for the desired side.	$p^2 = f^2 + s^2 - 2fs\cos(P)$
3 Substitute values.	$= 7.96^2 + 4.86^2 - 2 \times 7.96 \times 4.86\cos(54°)$
4 Calculate and find p.	$p^2 = 41.5035\ldots$ $p = 6.4423\ldots$
5 Round appropriately and write the answer.	The towers are about 6.44 km apart.

We can use right-angled triangles and the sine and cosine rules in non-right-angled triangles to solve problems in 3 dimensions.

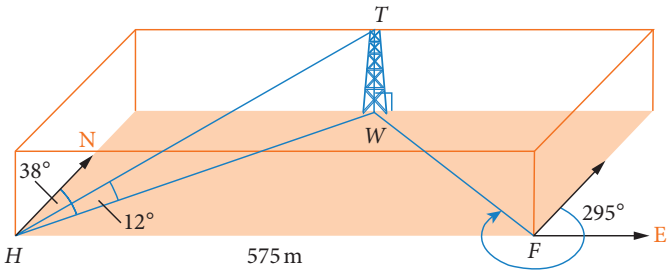
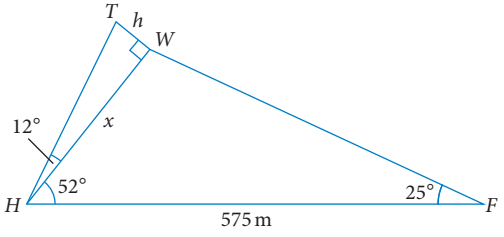
WB

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WORKED EXAMPLE 11 Bearings with angles of elevation 1

Hassan observes that the top of a transmission tower at a bearing of 038° is at an angle of elevation of 12°. Fatima is 575 metres due east of Hassan, and she says the bearing of the tower is 295°.

Find, correct to the nearest metre, the height of the tower.

Steps	Working
1 Draw a diagram in a sketch of a box to make it look 3D and label the sides and angles.	
2 Find $\angle FHW$ and $\angle WFH$.	$\angle FHW = 90° - 38° = 52°$ $\angle WFH = 295° - 270° = 25°$
3 'Unfold' and draw the triangles flat.	
4 Find $\angle HWF$.	$\angle HWF = 180° - 52° - 25° = 103°$

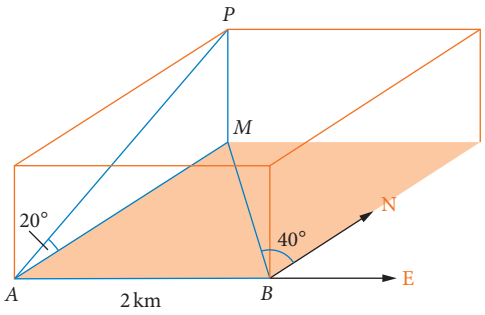
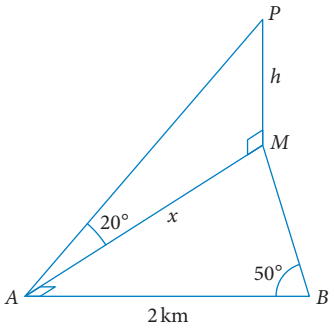
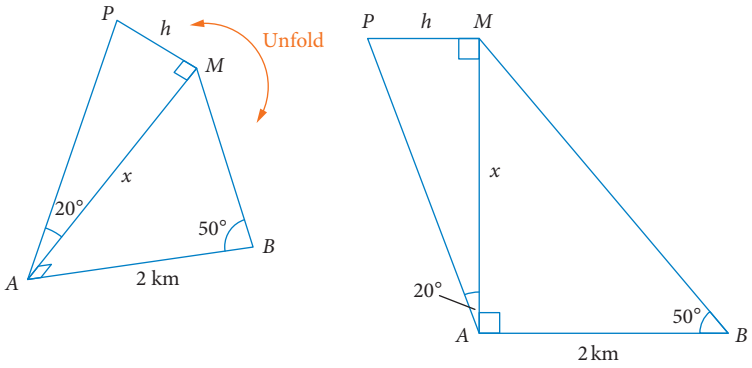
9780170448338

5	Use the sine rule to find x.	$\frac{x}{\sin(25°)} = \frac{575}{\sin(103°)}$
6	Solve for x.	$x = \frac{575\sin(25°)}{\sin(103°)}$
7	Now use ΔSBT to find h from x.	$\tan(12°) = \frac{h}{x}$
8	Solve for h.	$h = x\tan(12°)$
9	Substitute the expressions for x.	$= \frac{575\sin(25°)\tan(12°)}{\sin(103°)} = 53.0110\ldots$
10	Round appropriately and write the answer.	The tower is about 53 m high.

WORKED EXAMPLE 12 Bearings with angles of elevation 2

WB p. 165

From a certain point, A, a mountain peak due north has an angle of elevation of 20°. From another point, B, 2 km east of A and on the same level as A, the bearing of the peak is N 40° W. Find the height of the peak above the level of A and B.

Steps	Working
1 Draw a diagram in a sketch of a box to make it look 3D. Call the foot of the mountain M and the peak P. Show directions of north and east.	
2 Investigate known angles.	$\angle MBN = 40°$ and $\angle ABN = 90°$, so $\angle ABM = 50°$.
3 Call the height of the mountain h and the distance from A to the foot of the mountain x. Redraw the diagram with the sides and angles marked, ignoring the parts of the box you don't need.	
4 'Unfold' and draw the triangles flat. In this case, they are joined along AM.	

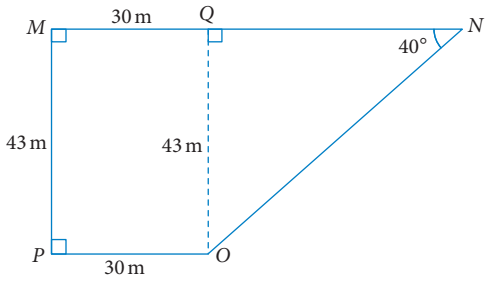
5	Use tan in $\triangle MAB$ to find x and rearrange.	$\tan 50° = \frac{x}{2} \therefore x = 2\tan(50°)$
6	Use tan in $\triangle PAM$ to find h.	$\tan 20° = \frac{h}{x} \therefore h = x\tan(20°)$
7	Substitute x.	$h = 2\tan(50°)\tan(20°)$
8	Calculate the answer.	$h = 0.8675\ldots$ km
9	Round appropriately and write the answer.	The peak is about 868 metres high.

EXERCISE 7.4 Applications of the sine and cosine rules

ANSWERS p. 583

Recap

1 In the diagram, find the length ON and MN, correct to 1 decimal place.



2 In a triangle ABC, where side lengths are $a = 5$ m, $b = 7.1$ m and $c = 8.1$ m, angle B, correct to the nearest degree, is

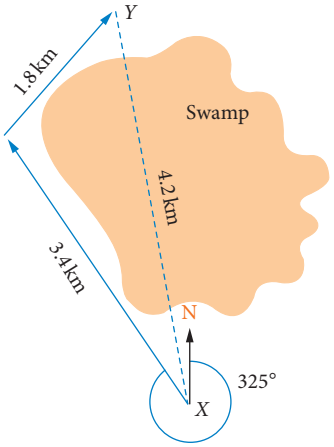
A 38° **B** 60° **C** 82° **D** 98° **E** 120°

Mastery

3 WORKED EXAMPLE 9 A child lying on the ground looks up and sees a ladder of length 14 metres just reaching to the top of a brick wall at an angle of elevation from him of 49°. Find the height of the brick wall, to the nearest whole number.

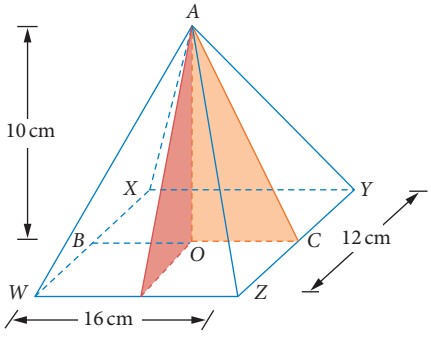
4 WORKED EXAMPLE 10 A hiker encounters a swamp at point X, as shown on the right. To travel around the swamp, the hiker walks 3.4 km from X on a bearing of 325° and then walks 1.8 km to point Y on the other side of the swamp, 4.2 km from X.

Find the bearing of X from Y, expressed to the nearest degree.



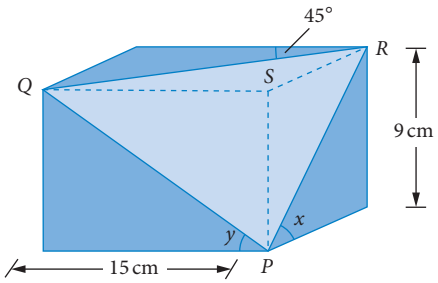
5 A rectangular-based pyramid with a base measuring 16 cm by 12 cm and a height of 10 cm is to be constructed using perspex sheeting.

Calculate the angle that the triangular faces AYZ and AZW make with the horizontal base.



9780170448338

6 A sculptor has removed a section from a copper block with dimensions. as shown in the diagram. The base of the block is square. Calculate:



a QR

b x

c y

d the angle that the triangular face PQR makes with the horizontal base.

Exam practice

7 (3 marks) From the top of a vertical building, the angle of depression of a sports car on the road is 30° and the angle of depression of a truck parked behind the sports car is 60°.

Show that the distance of the sports car from the base of the building is three times the truck's distance from the building.

8 (3 marks) A helicopter is sighted at the same time by 2 ground observers who are 2.4 km apart in the same direction from the helicopter. The angles of elevation from the observers to the helicopter (measured to the horizontal) are 18.4° and 31.6°. How high is the helicopter, correct to 2 decimal places?

9 (3 marks) A crane has 2 arms, each 15 m long and sloping upwards. The arms of the crane make an angle of 128°. The load is attached on the end of one arm point and the counterweight at the end of the other arm.

Find the distance between the attachment points, correct to 2 decimal places.

10 (6 marks) Sophie and Will set out from point A at the same time. Sophie travels at 30 km/h along a straight road in the direction 042°. Will travels at 24 km/h along another straight road in the direction 142°. After 4 hours, to 1 decimal place,

a what is the distance between Will and Sophie? 3 marks

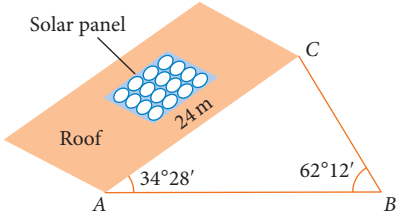
b what is the direction Sophie would have to go in a straight line to reach Will? 3 marks

11 (6 marks) Two flying foxes going in different directions start at the same point on the top of a cliff and finish on the level plain below. One has an angle of depression of 17° and the other 20°. The flying foxes finish 103 metres apart.

a Find the height of the cliff to the nearest metre. 3 marks

b Find the horizontal distance of the ends of each cable from the point directly below the starting point to the nearest metre. 3 marks

12 (6 marks) The roof of a new home is designed so that the solar panel can collect the maximum energy from the sun.



Calculate, to the nearest whole number,

a the height of the roof 3 marks

b the length of the beam AB. 3 marks

13 (9 marks) A flagpole 12 metres high is temporarily erected for an Australia Day ceremony. It is held in place by two wires: one 15 metres long fastened due west of the pole and the other 17 metres long fastened due south of the pole. Both supporting wires are attached to the top of the flagpole. Calculate

a the distance that each wire is from the pole at the point where it meets the ground — 3 marks

b the angle of inclination that each wire makes with the horizontal — 2 marks

c the distance between the points where the wires meet the ground — 2 marks

d the angle between the wires. — 2 marks

14 (4 marks) From a lighthouse 150 metres above sea level, the lighthouse keeper observes a boat due east at an angle of depression of 25° and another boat due south at an angle of depression of 32°.

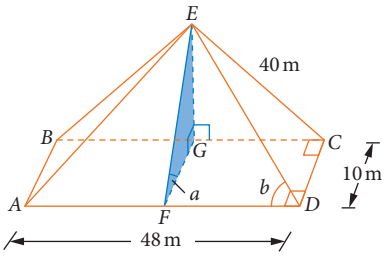
a Find the distance that separates the boats. — 2 marks

b Find the bearing of the second boat from the first. — 2 marks

15 WORKED EXAMPLE 11 (3 marks) An aircraft flying due east at 385 km/h at a constant height of 1500 metres passes over a village at a particular instant. One minute later, a person who is 1.6 kilometres south of the village sees the plane.

In what direction and at what angle of elevation does the person see the plane?

16 (4 marks) A large tent is constructed in the shape of a rectangular-based pyramid *ABCDE* with the vertex *E* vertically above *G*, the midpoint of *BC*, as shown in the diagram.



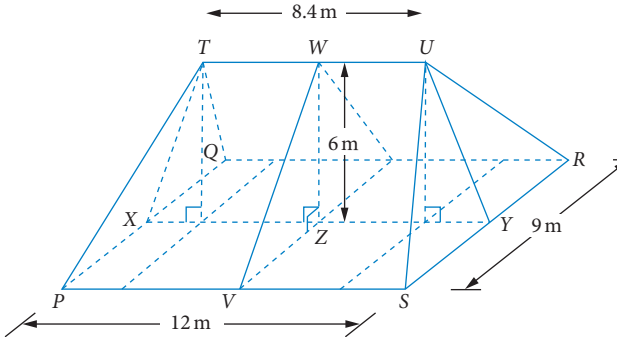
Calculate, to the nearest degree, the angle

a that the triangular face *AED* makes with the base of the pyramid — 2 marks

b between the lines *DE* and *AD*. — 2 marks

17 (3 marks) A straight stick 2 metres long is stuck in the ground and is inclined at 60° to the horizontal, pointing in an easterly direction. Calculate the length and direction of the shadow of the stick when the sun is due north at an elevation of 30°. (Hint: Imagine a piece of string hanging vertically from the top of the stick to the ground. Consider the shadow of the string and its relation to the shadow of the stick.)

18 (10 marks) The diagram shows the frame for a double-hipped roof.



Calculate

a the angle that the triangular face *RSU* makes with the horizontal — 2 marks

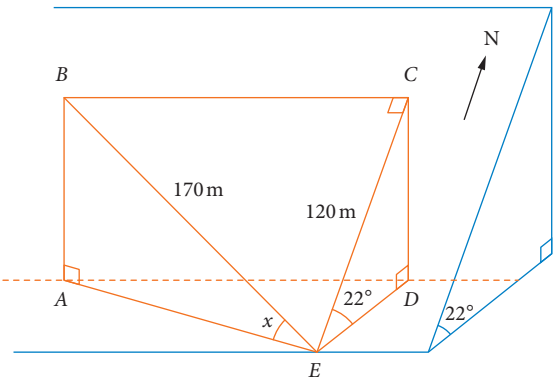
b the length of the rafter *VW* — 2 marks

c the angle the sloping roof *PTUS* makes with the horizontal — 2 marks

d the length of the rafter *UY* — 2 marks

e the length of the rafter *UR*. — 2 marks

19 (3 marks) A hillside is considered to have a uniform slope of 22° with the horizontal, as shown in the diagram. A hiker, starting from point *E*, walks due north up the hillside for 120 metres to point *C*, then walks due west to point *B*. The hiker then walks 170 metres back to the starting point. Calculate the angle of inclination of *BE* to the horizontal and the total distance covered by the hiker.



20 WORKED EXAMPLE 12 (3 marks) Two groups of people set off to spotlight possums at night. One group spots a possum. The tree that has the possum is at a bearing of N 28° W. The angle of elevation of their spotlight beam is 38°. At this instant the other group, which is 100 metres west of the first group, sights the same tree at a bearing of N 29° E. How high up the tree is the possum?

21 (3 marks) Elena sees that the angle of elevation to the top of a tower, 80 metres high, which is due south of her position, is 37°. The top of another tower, 60 metres high, which is due west of her, is at an angle of elevation of 28°. Calculate the horizontal distance between the 2 towers.

7.5 The Pythagorean identities

Video playlist
The Pythagorean identities

The reciprocal circular functions

$$\operatorname{cosec}(x) = \frac{1}{\sin(x)} \qquad \sec(x) = \frac{1}{\cos(x)} \qquad \cot(x) = \frac{1}{\tan(x)}$$

WORKED EXAMPLE 13 Exact values 1

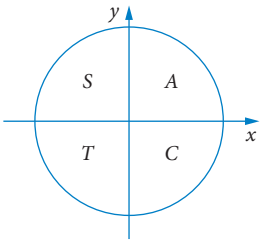
Find the exact value of each trigonometric ratio.

a $\cot(30°)$ **b** $\sec(45°)$ **c** $\operatorname{cosec}(60°)$

Steps	Working
a Write the reciprocal function cot, in terms of the original function tan, evaluate and simplify.	$\cot(30°) = \frac{1}{\tan(30°)} = \frac{1}{\frac{1}{\sqrt{3}}}$ $\therefore \cot(30°) = \sqrt{3}$
b Write the reciprocal function sec, in terms of the original function cos, evaluate and simplify.	$\sec(45°) = \frac{1}{\cos(45°)} = \frac{1}{\frac{1}{\sqrt{2}}}$ $\therefore \sec(45°) = \sqrt{2}$
c Write the reciprocal function cosec, in terms of the original function sin, evaluate and simplify.	$\operatorname{cosec}(60°) = \frac{1}{\sin(60°)} = \frac{1}{\frac{\sqrt{3}}{2}}$ $\therefore \operatorname{cosec}(60°) = \frac{2}{\sqrt{3}}$

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Whenever sine, cosine or tangent are positive or negative in the 4 quadrants (using the ASTC diagram), their reciprocals are also positive or negative in the same 4 quadrants.



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WORKED EXAMPLE 14 Exact values 2

Evaluate $\operatorname{cosec}\left(\frac{5\pi}{3}\right)$.

Steps	Working
1 cosec is the reciprocal of sin.	$\operatorname{cosec}\left(\frac{5\pi}{3}\right) = \frac{1}{\sin\left(\frac{5\pi}{3}\right)}$
2 $\frac{5\pi}{3}$ is in the 4th quadrant, where sin is negative. Its reference angle is $\frac{\pi}{3}$.	$\frac{1}{\sin\left(\frac{5\pi}{3}\right)} = \frac{1}{-\sin\left(\frac{\pi}{3}\right)} = \frac{1}{-\frac{\sqrt{3}}{2}}$ $\therefore \operatorname{cosec}\left(\frac{5\pi}{3}\right) = -\frac{2}{\sqrt{3}}$

Trigonometric functions are called **circular functions** because they are based on the unit circle with equation $x^2 + y^2 = 1$, centre $(0, 0)$ and radius 1.

For any point $P(x, y)$ on the circle:

$\sin(\theta) = y$

$\cos(\theta) = x$

From the right-angled triangle in the diagram, using Pythagoras' theorem, we get $x^2 + y^2 = 1$, the equation of the circle.

Therefore, $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$, which is usually written $\mathbf{\cos^2(\theta) + \sin^2(\theta) = 1}$.

This is called an **identity** because it is true for all angles θ and a **Pythagorean identity** because it is based on Pythagoras' theorem.

Dividing both sides of the identity $\cos^2(\theta) + \sin^2(\theta) \equiv 1$ by $\cos^2(\theta)$ gives:

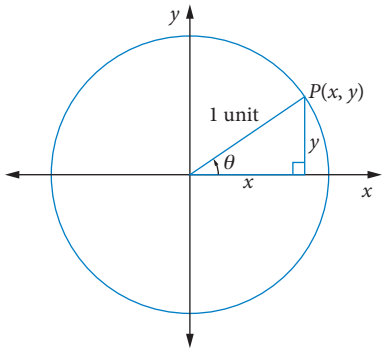
$$\frac{\cos^2(\theta)}{\cos^2(\theta)} + \frac{\sin^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$$

giving $1 + \tan^2(\theta) = \sec^2(\theta)$.

Dividing both sides of the identity $\cos^2(\theta) + \sin^2(\theta) \equiv 1$ by $\sin^2(\theta)$ gives:

$$\frac{\cos^2(\theta)}{\sin^2(\theta)} + \frac{\sin^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)}$$

giving $\cot^2(\theta) + 1 = \operatorname{cosec}^2(\theta)$.



The triple bar symbol '≡' is often used in identities, meaning that the statement is true for every value of θ. For example, we could write $\cos^2(\theta) + \sin^2(\theta) \equiv 1$.

The Pythagorean identities

$$\cos^2(\theta) + \sin^2(\theta) = 1$$
$$1 + \tan^2(\theta) = \sec^2(\theta)$$
$$\cot^2(\theta) + 1 = \operatorname{cosec}^2(\theta)$$

Exam hack

The 3rd identity is the 2nd identity with each function replaced by its complementary ('co-') function.

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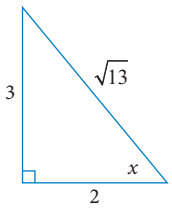
WORKED EXAMPLE 15 Pythagorean identities

Use a trigonometric identity to find $\sec(x)$ for $\frac{\pi}{2} \leq x \leq \pi$ if $\tan(x) = -\frac{3}{2}$.

Steps	Working
1 Use the trigonometric identity $1 + \tan^2(\theta) = \sec^2(\theta)$ and solve for $\sec^2(\theta)$.	$1 + \tan^2(x) = \sec^2(x)$ $\therefore 1 + \left(-\frac{3}{2}\right)^2 = \sec^2(x)$ $\text{LHS} = 1 + \frac{9}{4} = \frac{13}{4}$ $\therefore \sec(x) = \pm\sqrt{\frac{13}{4}} = \pm\frac{\sqrt{13}}{2}$
2 Given $\frac{\pi}{2} \leq x \leq \pi$ in the 2nd quadrant, cos is negative and so is sec.	$\sec(x) = -\frac{\sqrt{13}}{2}$

Alternative method

1 Draw a right-angled triangle using $\tan(x) = \frac{3}{2}$, using Pythagoras' theorem to find the hypotenuse.



> Don't consider the negative at this stage since this is irrelevant for the sides of a triangle.

2 Find $\sec(x)$.

$\sec(x) = \frac{1}{\cos(x)} = \frac{1}{\frac{2}{\sqrt{13}}}$

$\therefore \sec(x) = \frac{\sqrt{13}}{2}$

3 But consider the quadrant.

Given $\frac{\pi}{2} \leq x \leq \pi$ in the 2nd quadrant, cos is negative and so is sec.

$\therefore \sec(x) = -\frac{\sqrt{13}}{2}$

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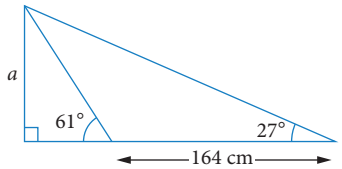
EXERCISE 7.5 The Pythagorean identities

ANSWERS p. 583

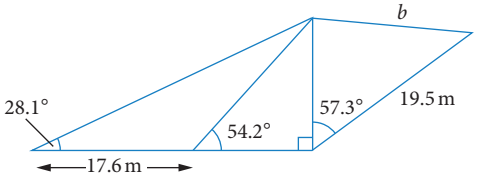
Recap

1 Find the lengths a to the nearest metre and b to the nearest metre, in the triangles below.

a



b



2 (3 marks) Miguel is 90 metres due west of Consuella on a level road. He sees that the angle of elevation of a kite flown by their son is 30°, while Consuella sees that it is due north at an angle of elevation of 38°. What is the height of the kite?

Mastery

3 WORKED EXAMPLE 13 TECH-FREE State the exact value of each trigonometric ratio.

a $\sec(30°)$ **b** $\cot(45°)$ **c** $\operatorname{cosec}\left(\frac{\pi}{4}\right)$

4 WORKED EXAMPLE 14 TECH-FREE Evaluate

a $\sec\left(\frac{2\pi}{3}\right)$ **b** $\cot\left(\frac{2\pi}{3}\right)$

5 WORKED EXAMPLE 15 TECH-FREE Find $\operatorname{cosec}(x)$ for $\frac{\pi}{2} \leq x \leq \pi$ if we know that $\cot(x) = -3$.

6 $\operatorname{cosec}\left(\frac{\pi}{4}\right)$ equals

A $\frac{1}{\sqrt{3}}$ **B** $\frac{1}{\sqrt{2}}$ **C** 1 **D** $\sqrt{2}$ **E** $\sqrt{3}$

7 An equivalent expression for $\sin\left(\frac{7\pi}{6}\right)$ is

A $\sin\left(\frac{\pi}{6}\right)$ **B** $\sin\left(\frac{5\pi}{6}\right)$ **C** $\cos\left(\frac{\pi}{6}\right)$ **D** $\cos\left(\frac{\pi}{3}\right)$ **E** $-\sin\left(\frac{\pi}{6}\right)$

8 The value of $\sin(x)$ for $\frac{3\pi}{2} \leq x \leq 2\pi$, given that $\operatorname{cosec}(x) = -2$, is

A $-\frac{4\sqrt{2}}{3}$ **B** $-\frac{1}{2}$ **C** $\frac{1}{2}$ **D** $\frac{2\sqrt{2}}{3}$ **E** $\frac{3\sqrt{2}}{4}$

Exam practice

80–100% 60–79% 0–59%

9 © VCAA 2008 1Q4 MODIFIED TECH-FREE (2 marks) Given that $\sin\left(\frac{\pi}{10}\right) = \frac{\sqrt{5}-1}{4}$, find $\operatorname{cosec}\left(\frac{\pi}{10}\right)$ in the form $a\sqrt{5} + b$, where $a, b \in R$.

10 TECH-FREE (2 marks) Given that $\sin\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$, find $\sec\left(\frac{5\pi}{4}\right)$.

11 TECH-FREE (2 marks) Find $\operatorname{cosec}(x)$ for $0 \leq x \leq \pi$ given that $\cos(x) = -\frac{2}{3}$.

12 The values of $\cot(60°)$ and $\sec(30°)$ are respectively

A $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}$ **B** $\frac{1}{\sqrt{3}}, \sqrt{2}$ **C** $\sqrt{3}, \frac{3}{\sqrt{2}}$ **D** $\frac{1}{2}, \frac{1}{\sqrt{2}}$ **E** $\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}$

13 $\operatorname{cosec}(2\pi - \theta)$ is equal to

A $\operatorname{cosec}(\theta)$ **B** $-\operatorname{cosec}(\theta)$ **C** $\sec(\theta)$ **D** $-\sec(\theta)$ **E** $\cos(\theta)$

14 The value of $\operatorname{cosec}(x)$ for $\frac{3\pi}{2} \leq x \leq 2\pi$, given that $\cos(x) = \frac{1}{3}$, is

A $-\frac{4\sqrt{2}}{3}$ **B** $-\frac{3\sqrt{2}}{4}$ **C** $-\frac{2\sqrt{2}}{3}$ **D** $\frac{2\sqrt{2}}{3}$ **E** $\frac{3\sqrt{2}}{4}$

15 $\text{cosec}\left(\frac{7\pi}{2}\right)$ equals

A -2 **B** -1 **C** 0 **D** $\frac{1}{2}$ **E** 1

16 $\cos\left(\frac{7\pi}{2}\right)$ equals

A -2 **B** -1 **C** 0 **D** $\frac{1}{2}$ **E** 1

17 The exact value for $\sec(60°)$ is

A $\frac{1}{2}$ **B** $\frac{1}{\sqrt{3}}$ **C** $\frac{\sqrt{3}}{2}$ **D** $\frac{2}{\sqrt{3}}$ **E** 2

18 The exact value for $\text{cosec}(60°)$ is

A $\frac{1}{2}$ **B** $\frac{1}{\sqrt{3}}$ **C** $\frac{\sqrt{3}}{2}$ **D** $\frac{2}{\sqrt{3}}$ **E** 2

19 $\text{cosec}(2\pi + \theta)$ is equal to

A $\text{cosec}(\theta)$ **B** $-\text{cosec}(\theta)$ **C** $\sec(\theta)$ **D** $-\sec(\theta)$ **E** $\cos(\theta)$

20 $\sec(2\pi - \theta)$ is equal to

A $\text{cosec}(\theta)$ **B** $-\text{cosec}(\theta)$ **C** $\sec(\theta)$ **D** $-\sec(\theta)$ **E** $\cos(\theta)$

21 © VCAA 2002 1IQ4 **57%** Let $f: \left(0, \frac{\pi}{2}\right) \to R$, where $f(x) = \sec^2(x) + \text{cosec}^2(x)$.

Which one of the following statements is true?

A f is identical to the function $h: \left(0, \frac{\pi}{2}\right) \to R$, where $h(x) = \sec^2(x)\,\text{cosec}^2(x)$

B f is identical to the function $g: \left(0, \frac{\pi}{2}\right) \to R$, where $g(x) = \tan^2(x) + \cot^2(x)$

C f is equal to 1 for all values of x in $\left(0, \frac{\pi}{2}\right)$

D f has range $[1, \infty)$

E f has range $[2, \infty)$

22 $\cot\left(\frac{\pi}{3}\right)$ equals

A $\frac{1}{\sqrt{3}}$ **B** $\frac{1}{\sqrt{2}}$ **C** 1 **D** $\sqrt{2}$ **E** $\sqrt{3}$

23 $\sec\left(\frac{\pi}{6}\right)$ equals

A $\frac{1}{\sqrt{3}}$ **B** $\frac{1}{\sqrt{2}}$ **C** $\frac{2}{\sqrt{3}}$ **D** $\frac{\sqrt{3}}{2}$ **E** $\sqrt{3}$

24 $\operatorname{cosec}(90°)$ equals

A 0 **B** $\frac{1}{\sqrt{2}}$ **C** 1 **D** $\frac{2}{\sqrt{3}}$ **E** undefined

25 $\cot(45°)$ equals

A $\frac{1}{\sqrt{3}}$ **B** $\frac{1}{\sqrt{2}}$ **C** 1 **D** $\sqrt{2}$ **E** $\sqrt{3}$

26 © VCAA 2018N 2AQ1 Let $f(x) = \operatorname{cosec}(x)$. The graph of f is transformed by:

- a dilation by a factor of 3 from the x-axis, followed by
- a translation of 1 unit horizontally to the right, followed by
- a dilation by a factor of $\frac{1}{2}$ from the y-axis.

The rule of the transformed graph is

A $g(x) = 2\operatorname{cosec}(3x + 1)$ **B** $g(x) = 3\operatorname{cosec}(2x - 1)$ **C** $g(x) = 3\operatorname{cosec}(2(x - 1))$

D $g(x) = 2\operatorname{cosec}\left(\frac{x}{3} - 1\right)$ **E** $g(x) = 3\operatorname{cosec}\left(\frac{x - 1}{2}\right)$

7.6 The sum and difference identities

Video playlist
The sum and difference identities

Worksheet
Simplifying periodic functions

The angle **sum and difference identities** are formulas for finding the trigonometric ratios of a sum or difference of angles, commonly called **compound angle formulas**.

The sum and difference identities

$$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

$$\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$$

$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

$$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$$

$$\tan(x + y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$$

$$\tan(x - y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$$

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WORKED EXAMPLE 16 Compound angle formulas 1

Expand $\cos(3x + 2y)$.

Steps	Working
Use the trigonometric identity $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$.	$\cos(3x + 2y) = \cos(3x)\cos(2y) - \sin(3x)\sin(2y)$

Compound angle formulas can help us find the *exact* trigonometric values of some angles that are the sum or difference of the exact value angles, such as $\frac{\pi}{12}$.

WORKED EXAMPLE 17 Compound angle formulas 2

Evaluate $\sin\left(\frac{\pi}{12}\right)$.

Steps	Working
1 Consider a suitable way of splitting the angle $\frac{\pi}{12}$ into a sum or difference of exact value angles. $\frac{\pi}{12} = \frac{4\pi}{12} - \frac{3\pi}{12}$	$\therefore \sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$
2 Use the trigonometric identity $\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$.	$= \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$
3 Evaluate using exact values.	$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} - \frac{1}{2} \times \frac{\sqrt{2}}{2}$ $= \frac{\sqrt{6} - \sqrt{2}}{4}$

WB p. 171

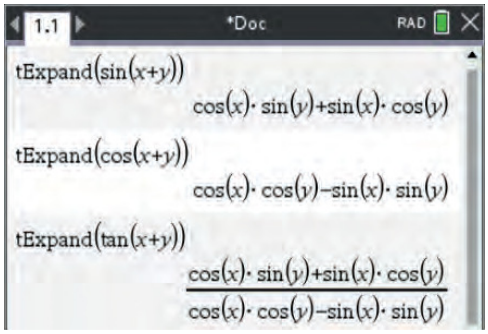
USING CAS 3 The sum and difference identities

a Verify the trigonometry angle sum and difference identities.

b Use a suitable identity to confirm $\sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$.

TI-Nspire

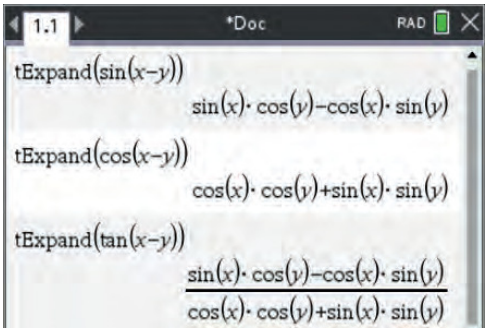
a



1 Press **menu** > **Algebra** > **Trigonometry** > **Expand**.

2 Enter the **sin(x+y)** expression and press **enter**.

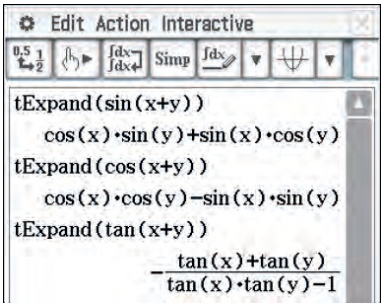
3 Repeat for the other 2 sum expressions.



4 Repeat for the difference expressions.

Note: TI-Nspire simplifies all trigonometric expressions to sin and cos, not tan.

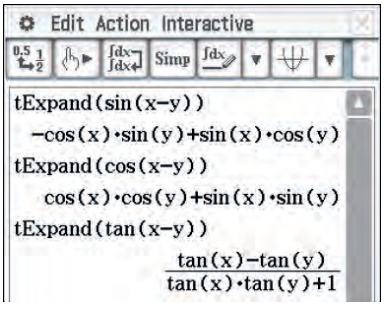
ClassPad



1 Enter and highlight the expression **sin(x + y)**.

2 Tap **Interactive** > **Transformation** > **tExpand**.

3 Repeat for the other 2 sum expressions.

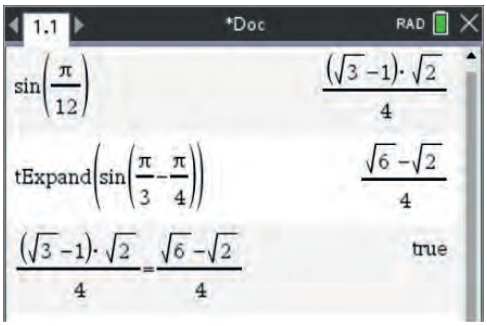


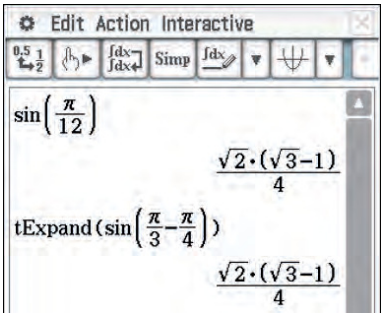
4 Repeat for the difference expressions.

Transformation, and using either **tCollect** or **tExpand** are specifically written for trig expressions.

TI-Nspire | **ClassPad**

b





1 Evaluate $\sin\left(\frac{\pi}{12}\right)$.

2 Apply the **tExpand** function to $\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)$.

3 Equate the 2 expressions to confirm they are equal.

In this case, **tCollect** or **tExpand** gives the same output.

We can use the compound angle formulas to prove simplifications of **trigonometric identities** that we already know.

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WORKED EXAMPLE 18 Compound angle formulas 3

Expand and simplify $\cos\left(\frac{\pi}{2}-x\right)$.

Steps	Working
1 Use the trigonometric identity $\cos(x-y)=\cos(x)\cos(y)+\sin(x)\sin(y)$.	$\cos\left(\frac{\pi}{2}-x\right)=\cos\left(\frac{\pi}{2}\right)\cos(x)+\sin\left(\frac{\pi}{2}\right)\sin(x)$
2 Evaluate and simplify.	$\cos\left(\frac{\pi}{2}-x\right)=0\times\cos(x)+1\times\sin(x)$
We also know that $\cos\left(\frac{\pi}{2}-x\right)=\sin(x)$ is true because sine and cosine are complementary functions.	$\therefore\ \cos\left(\frac{\pi}{2}-x\right)=\sin(x)$

EXERCISE 7.6 The sum and difference identities

ANSWERS p. 583

Recap

1 TECH-FREE Find $\sec(x)$ for $\frac{3\pi}{2}\le x\le 2\pi$ if we know that $\cot(x)=-1$.

2 The value of $\sin(x)$ for $\frac{3\pi}{2}\le x\le 2\pi$, given that $\sec(x)=4$, is

A $-\frac{\sqrt{15}}{4}$ **B** $-\frac{1}{4}$ **C** $\frac{1}{4}$ **D** $\frac{4}{\sqrt{15}}$ **E** $\frac{\sqrt{15}}{4}$

Mastery

3 WORKED EXAMPLE 16 TECH-FREE Expand $\sin(x - 2y)$.

4 WORKED EXAMPLE 17 TECH-FREE Evaluate $\sin\left(\frac{5\pi}{12}\right)$.

5 WORKED EXAMPLE 18 TECH-FREE Use a suitable compound angle formula to find an expansion of $\tan\left(\frac{\pi}{2} - x\right)$.

6 Using CAS 3 $\sin\left(\frac{\pi}{2} - \frac{\pi}{6}\right)$ equals

A $-\frac{\sqrt{3}}{2}$ **B** $-\frac{1}{2}$ **C** $\frac{1}{2}$ **D** $\frac{\sqrt{3}}{2}$ **E** 1

7 An equivalent expression for $\cos\left(\frac{11\pi}{6}\right)$ is

A $\sin\left(\frac{\pi}{6}\right)$ **B** $\sin\left(\frac{5\pi}{6}\right)$ **C** $\cos\left(\frac{\pi}{6}\right)$ **D** $-\cos\left(\frac{\pi}{6}\right)$ **E** $-\sin\left(\frac{\pi}{6}\right)$

8 TECH-FREE Expand each expression.

a $\sin(2x + y)$ **b** $\sin(3p - 5q)$ **c** $\tan(2e - f)$

d $\cos(3\alpha + 2\beta)$ **e** $\tan(w + 3x)$ **f** $\cos(P - 4Q)$

9 TECH-FREE Simplify each expression.

a $\sin(\pi - x)$ **b** $\cos(\pi + x)$ **c** $\tan\left(\frac{\pi}{2} - x\right)$

d $\operatorname{cosec}\left(x + \frac{\pi}{2}\right)$ **e** $\cot\left(\frac{3\pi}{2} - x\right)$ **f** $\sec\left(x + \frac{\pi}{2}\right)$

g $\sin(2\pi - x)$ **h** $\cos\left(\frac{3\pi}{2} - x\right)$ **i** $\tan(x - \pi)$

Exam practice

80–100% 60–79% 0–59%

10 TECH-FREE (4 marks) Use the expansion of $\cos(x - y)$ and $\sin(x - y)$ to prove that $\cos\left(\frac{\pi}{2} - x\right) = \sin(x)$ and $\sin\left(\frac{\pi}{2} - x\right) = \cos(x)$.

11 TECH-FREE (2 marks) Use the expansion of $\tan\left(\frac{\pi}{2} + x\right)$ to prove that $\tan\left(\frac{\pi}{2} + x\right) = -\cot(x)$.

12 TECH-FREE (3 marks) Prove that $\tan(x - y) + \tan(y - z) + \tan(z - x) = \tan(x - y)\tan(y - z)\tan(z - x)$.

13 © VCAA 2003 2Q2a 76% TECH-FREE (3 marks) Using the fact that $\frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}$, show that $\cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{6} + \sqrt{2}}{4}$ and $\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{6} - \sqrt{2}}{4}$.

14 © VCAA 2019N 1Q9a **TECH-FREE** (2 marks) Show that $\tan\left(\frac{5\pi}{12}\right) = \sqrt{3} + 2$.

15 The expression $\cos(x - 2y)$ can be written as

A $\cos(x)\cos(2y) - \sin(x)\sin(2y)$
B $\cos(x)\cos(2y) + \sin(x)\sin(2y)$
C $\cos(2x)\cos(y) + \sin(2x)\sin(y)$
D $\cos(x)\sin(2y) - \sin(x)\cos(2y)$
E $\cos(x)\sin(2y) + \sin(x)\cos(2y)$

16 $\cos\left(\frac{7\pi}{12}\right)$ is equal to

A $\cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$
B $\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$
C $\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$
D $\cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$
E $\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$

17 $\sin\left(\frac{\pi}{2} + x\right)$ is equivalent to

A $\cos\left(\frac{\pi}{2}\right)$ **B** $\sin\left(\frac{\pi}{2}\right)$ **C** $\cos(x)$ **D** $\sin(x)$ **E** $-\cos(x)$

7.7 The double angle identities

Video playlist
The double angle identities

Worksheet
Trigonometric identities

Puzzle
Trigonometric identities

The **double angle identities** come from the sum identities involving $(x + y)$, letting $y = x$.

Consider the compound angle formula $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$.

By replacing $(x + y)$ with $(x + x)$, we get $\sin(x + x) = \sin(x)\cos(x) + \cos(x)\sin(x)$.

This simplifies to $\sin(2x) = 2\sin(x)\cos(x)$, one of the double angle formulas.

The double angle identities

$$\sin(2x) = 2\sin(x)\cos(x)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$$

$$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$$

$\cos(2x)$ has 3 double angle identities. Choose the identity most suitable for the problem.

Consider the identity $\cos(2x) = 2\cos^2(x) - 1$.

It is also true that $\cos(4x) = 2\cos^2(2x) - 1$ and $\cos(6x) = 2\cos^2(3x) - 1$ and so on.

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WORKED EXAMPLE 19 Double angle formula 1

Use a double angle formula to find an expression for $\tan(6x)$ in terms of $3x$.

Steps	Working
1 Express $6x$ as $3x + 3x$.	$\tan(6x) = \tan(3x + 3x)$
2 Use the identity $\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$ to find an expression for $\tan(6x)$.	$\tan(6x) = \frac{2\tan(3x)}{1 - \tan^2(3x)}$

The double angle formulas can also be used to evaluate numerical angles.

Consider the angle $\frac{\pi}{6} = \frac{\pi}{12} + \frac{\pi}{12}$.

WORKED EXAMPLE 20 Double angle formula 2

Use a suitable double angle formula to find an expression for $\sin\left(\frac{\pi}{12}\right)$.

Steps	Working
1 Find an expression for a double angle.	$\frac{\pi}{6} = \frac{\pi}{12} + \frac{\pi}{12}$ $\therefore \frac{\pi}{6} = 2 \times \frac{\pi}{12}$
2 Use the identity $\cos(2x) = 1 - 2\sin^2(x)$ using $\frac{\pi}{6} = 2 \times \frac{\pi}{12}$.	$\cos\left(2 \times \frac{\pi}{12}\right) = 1 - 2\sin^2\left(\frac{\pi}{12}\right)$ $\therefore \cos\left(\frac{\pi}{6}\right) = 1 - 2\sin^2\left(\frac{\pi}{12}\right)$
3 Rearrange to get an expression for $\sin\left(\frac{\pi}{12}\right)$.	Rearrange to get $2\sin^2\left(\frac{\pi}{12}\right) = 1 - \cos\left(\frac{\pi}{6}\right)$ giving $\sin^2\left(\frac{\pi}{12}\right) = \frac{1}{2}\left(1 - \frac{\sqrt{3}}{2}\right)$ $\therefore \sin\left(\frac{\pi}{12}\right) = \pm\sqrt{\frac{2-\sqrt{3}}{4}}$ As $\left(\frac{\pi}{12}\right)$ is in the 1st quadrant, $\sin\left(\frac{\pi}{12}\right)$ is positive. $\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{2-\sqrt{3}}}{2}$

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EXERCISE 7.7 The double angle identities

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Recap

1 $\cos\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$ equals

A $-\frac{\sqrt{3}}{2}$ **B** $-\frac{1}{2}$ **C** $\frac{1}{2}$ **D** 1 **E** $\frac{\sqrt{3}}{2}$

2 The expression $\sin(3x + y)$ can be written as

A $\cos(3x)\cos(y) - \sin(3x)\sin(y)$

B $\cos(3x)\cos(y) + \sin(3x)\sin(y)$

C $\sin(3x)\cos(y) - \cos(3x)\sin(y)$

D $\cos(3x)\sin(y) - \sin(3x)\cos(y)$

E $\cos(3x)\sin(y) + \sin(3x)\cos(y)$

Mastery

3 WORKED EXAMPLE 19 TECH-FREE Use a double angle formula to find an expression for $\sin(6x)$ in terms of $3x$.

4 WORKED EXAMPLE 20 TECH-FREE Use a suitable double angle formula to find an expression for $\cos\left(\frac{\pi}{8}\right)$.

5 $\cos(8x)$ equals

A $2\sin(4x)\cos(4x)$ **B** $\cos^2(4x) - \sin^2(4x)$ **C** $2\cos^2(8x) - 1$

D $1 - 2\sin^2(16x)$ **E** $4\cos^2(4x) - 1$

6 $\sin\left(\frac{\pi}{4}\right)$ is the same as

A $2\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)$ **B** $\cos^2\left(\frac{\pi}{8}\right) - \sin^2\left(\frac{\pi}{8}\right)$ **C** $2\cos^2\left(\frac{\pi}{8}\right) - 1$

D $1 - 2\sin^2\left(\frac{\pi}{8}\right)$ **E** $4\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)$

Exam practice

80–100% 60–79% 0–59%

7 © VCAA 2018 1Q7 81% TECH-FREE (3 marks) Given that $\cot(2x) + \frac{1}{2}\tan(x) = a\cot(x)$, use a suitable double angle formula to find the value of a, $a \in R$.

8 © VCAA 2014 1Q5a 81% TECH-FREE (1 mark) For the function with rule $f(x) = 96\cos(3x)\sin(3x)$, find the value of a such that $f(x) = a\sin(6x)$.

9 © VCAA 2008 1Q4 40% TECH-FREE (3 marks) Given that $\sin\left(\frac{\pi}{10}\right) = \frac{\sqrt{5}-1}{4}$, find $\sec\left(\frac{\pi}{5}\right)$ in the form $a\sqrt{5} + b$, where $a, b \in R$.

10 © VCAA 2007 1Q10 39% TECH-FREE (3 marks) Given that $\tan(2x) = \frac{4\sqrt{2}}{7}$, where $x \in \left[0, \frac{\pi}{4}\right)$, find the exact value of $\sin(x)$.

11 © VCAA 2006 1Q5a 32% TECH-FREE (4 marks) Show that $\tan\left(\frac{\pi}{8}\right) = \sqrt{2} - 1$.

12 TECH-FREE (4 marks) By first writing $\sin(3x)$ as $\sin(2x + x)$ and expanding, then writing the expression in terms of sine only, show that $\sin(3x) = 3\sin(x) - 4\sin^3(x)$.

13 TECH-FREE (4 marks) Use a suitable double angle formula to expand each expression.

a $\sin(8x)$ **b** $\cos(6x)$ **c** $\tan(4x)$ **d** $\cos(x)$

14 TECH-FREE (10 marks) Prove that

a $\cos(2x) = 1 - 2\sin^2(x)$ **b** $2\cos^2(x) - 1 = 1 - 2\sin^2(x)$

c $\cos^2(y) - \sin^2(y) = 1 - 2\sin^2(y)$ **d** $\sin(8x) = 2\sin(4x)\cos(4x)$

e $\cos(5x) = 1 - 2\sin^2\left(\frac{5x}{2}\right)$

15 An expression for $\sin(4x)$ is

A $\sin(2x)\cos(2x)$ **B** $2\sin(4x)$ **C** $2\sin(2x)\cos(2x)$

D $2\sin(4x)\cos(4x)$ **E** $2\sin(8x)\cos(8x)$

16 The expression $\cos(x - 2y)$ can be written as

A $\cos(x)\cos(2y) - \sin(x)\sin(2y)$ **B** $\cos(x)\cos(2y) + \sin(x)\sin(2y)$

C $\cos(2x)\cos(y) + \sin(2x)\sin(y)$ **D** $\cos(x)\sin(2y) - \sin(x)\cos(2y)$

E $\cos(x)\sin(2y) + \sin(x)\cos(2y)$

17 © VCAA 2004 2AQ3 80% Which expression is **not** equal to $\tan\left(\frac{\pi}{5}\right)$?

A $\dfrac{\sin\left(\frac{\pi}{5}\right)}{\cos\left(\frac{\pi}{5}\right)}$ **B** $\dfrac{1}{\cot\left(\frac{\pi}{5}\right)}$ **C** $\cot\left(\frac{3\pi}{10}\right)$ **D** $\dfrac{2\tan\left(\frac{\pi}{10}\right)}{1-\tan^2\left(\frac{\pi}{10}\right)}$ **E** $\dfrac{2\tan\left(\frac{2\pi}{5}\right)}{1-\tan^2\left(\frac{2\pi}{5}\right)}$

18 © VCAA 2007 2AQ4 71% Which equation is **not** true?

A $\sec^2(4\theta) - 1 = \tan^2(4\theta)$ **B** $\cot^2(4\theta) - \operatorname{cosec}^2(4\theta) = 1$

C $\cos(4\theta) = \cos^2(2\theta) - \sin^2(2\theta)$ **D** $\sin(4\theta) = 2\sin(2\theta)\cos(2\theta)$

E $\cos(4\theta) + 4\sin^2(\theta)\cos^2(\theta) = \cos^2(2\theta)$

19 © VCAA 2012 2BQ2a 67% (2 marks) Given that $\cos\left(\frac{\pi}{12}\right) = \frac{\sqrt{\sqrt{3}+2}}{2}$, show that $\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{2-\sqrt{3}}}{2}$.

20 © VCAA 2006 2BQ5bc (5 marks)

a 45% Use a double angle formula to show that the exact value of $\cos\left(\frac{\pi}{8}\right) = \frac{\sqrt{2+\sqrt{2}}}{2}$.

Explain why any values are rejected. 3 marks

b 44% Hence show that the exact value of $\sin\left(\frac{\pi}{8}\right) = \frac{\sqrt{2-\sqrt{2}}}{2}$. 2 marks

7.8 The products of sines and cosines

Products of sines and cosines can be converted into sums and differences. The following product identities can be proved using the sum and difference identities.

Products as sums and differences

$$\cos(x)\cos(y) = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$$

$$\sin(x)\sin(y) = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$$

$$\sin(x)\cos(y) = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$

$$\cos(x)\sin(y) = \frac{1}{2}[\sin(x+y) - \sin(x-y)]$$

You will find these formulas useful when rearranging exact angle expressions.

Video playlist
The products of sines and cosines

Worksheets
Products to sums and differences

Further trigonometric identities

WB
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WORKED EXAMPLE 21 Product of sines 1

Prove that $\sin(x)\sin(y) = \frac{1}{2}[\cos(x - y) - \cos(x + y)]$.

Steps	Working
1 Expand and simplify the RHS using the identities for $\cos(x + y)$ and $\cos(x - y)$.	$\text{RHS} = \frac{1}{2}[\cos(x - y) - \cos(x + y)]$ $= \frac{1}{2}[(\cos(x)\cos(y) + \sin(x)\sin(y)) - (\cos(x)\cos(y) - \sin(x)\sin(y))]$ $= \frac{1}{2}[(\cos(x)\cos(y) + \sin(x)\sin(y)) - \cos(x)\cos(y) + \sin(x)\sin(y)]$ $= \frac{1}{2}[\sin(x)\sin(y) + \sin(x)\sin(y)]$ $= \frac{1}{2}[2\sin(x)\sin(y)]$ $= \sin(x)\sin(y)$ $= \text{LHS}$
2 Write the conclusion.	$\therefore \sin(x)\sin(y) = \frac{1}{2}[\cos(x - y) - \cos(x + y)]$

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WORKED EXAMPLE 22 Product of sines 2

Express $\sin(4x)\sin(7x)$ as a sum or a difference of trigonometric functions.

Steps	Working
1 Use the formula $\sin(x)\sin(y) = \frac{1}{2}[\cos(x - y) - \cos(x + y)]$.	$\sin(4x)\sin(7x) = \frac{1}{2}[\cos(4x - 7x) - \cos(4x + 7x)]$
2 Collect terms and simplify.	$= \frac{1}{2}[\cos(-3x) - \cos(11x)]$
3 Remember that $\cos(-\theta) = \cos(\theta)$.	$= \frac{1}{2}[\cos(3x) - \cos(11x)]$
4 Write the result.	$\therefore \sin(4x)\sin(7x) = \frac{1}{2}[\cos(3x) - \cos(11x)]$

EXERCISE 7.8 The products of sines and cosines

ANSWERS p. 583

Recap

1 TECH-FREE

a Find $\sec(x)$ for $\frac{\pi}{2} \leq x \leq \pi$ if we know that $\cot(x) = -2$.

b Use a suitable double angle formula to find an expression for $\sin\left(\frac{\pi}{8}\right)$.

c Find an expression for $\cos\left(\frac{3\pi}{8}\right)$.

2 $\sec\left(\frac{\pi}{12}\right)$ equals

A $\frac{2\sqrt{2}}{\sqrt{3}+1}$ **B** $-\frac{2\sqrt{2}}{\sqrt{3}+1}$ **C** $\frac{\sqrt{3}+1}{2\sqrt{2}}$ **D** $\frac{2\sqrt{2}}{\sqrt{3}-1}$ **E** $\frac{4\sqrt{3}}{3}$

Mastery

3 WORKED EXAMPLE 21 TECH-FREE Prove that $\cos(x)\sin(y) = \frac{1}{2}[\sin(x+y) - \sin(x-y)]$.

4 WORKED EXAMPLE 22 TECH-FREE Express $\cos(4x)\sin(3x)$ as a sum or a difference of trigonometric functions.

5 $\sin(3y)\cos(4y)$ can be expressed as

A $\frac{1}{2}[\sin(7y) + \sin(y)]$ **B** $\sin(7y) + \sin(-y)$ **C** $\frac{1}{2}[\sin(7y) + \sin(-y)]$

D $\sin(7y) - \sin(y)$ **E** $\frac{1}{2}[\sin(-7y) + \sin(y)]$

6 $\sin(4x)\sin(2x)$ can be expressed as

A $\frac{1}{2}[\cos(6x) - \cos(2x)]$ **B** $\frac{1}{2}[\cos(2x) - \sin(6x)]$ **C** $\frac{1}{2}[\sin(2x) - \sin(6x)]$

D $\frac{1}{2}[\cos(2x) + \cos(6x)]$ **E** $\frac{1}{2}[\cos(2x) - \cos(6x)]$

Exam practice

7 TECH-FREE (3 marks) Show that $\cos(x)\sin(y) = \frac{1}{2}[\sin(x+y) - \sin(x-y)]$.

8 TECH-FREE (4 marks) Given that $\sin(50°) \approx 0.766\,04$, use an appropriate trigonometric products identity to evaluate the expression $\cos(40°)\sin(10°)$. Give your answer correct to 3 decimal places.

9 TECH-FREE (8 marks) Given that $\sin(40°) \approx 0.6428$ and $\cos(40°) \approx 0.7660$, use an appropriate products identity to evaluate each expression, correct to 3 decimal places.

a $\cos(10°)\cos(50°)$ **b** $\sin(10°)\sin(50°)$ **c** $\sin(10°)\cos(50°)$ **d** $\cos(10°)\sin(50°)$

10 TECH-FREE (18 marks) Express each expression as a sum or a difference of trigonometric functions.

a $2\sin(5x)\cos(7x)$ **b** $8\sin(3x)\sin(9x)$ **c** $6\cos(4x)\cos(6x)$

d $4\cos(5x)\sin(6x)$ **e** $\sin(6x)\cos(10x)$ **f** $3\cos(10x)\cos(14x)$

g $5\sin(2x)\sin(3x)$ **h** $7\cos(9x)\cos(11x)$ **i** $\cos(8x)\sin(x)$

11 TECH-FREE (4 marks) Express each expression as a sum or a difference of trigonometric functions and hence find its exact value.

a $\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{12}\right)$ **b** $\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{12}\right)$ **c** $\sin\left(\frac{\pi}{12}\right)\sin\left(\frac{\pi}{4}\right)$ **d** $2\sin\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{4}\right)$

Video playlist
The subsidiary angle identities

7.9 The subsidiary angle identities

When applying trigonometric functions, if we come across a function of the form $a\cos(x) + b\sin(x)$, we can simplify it to a single function involving just sine or cosine, but with an extra constant angle α. We do this by using the sum and difference identities in reverse.

This is sometimes called the **subsidiary angle method** or **auxiliary angle method**, where subsidiary and auxiliary mean 'additional support'.

The subsidiary angle identities

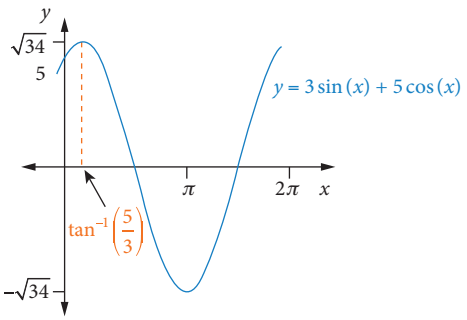
$$a\cos(x) + b\sin(x) = r\cos(x - \alpha)$$
$$a\cos(x) - b\sin(x) = r\cos(x + \alpha)$$
$$a\sin(x) + b\cos(x) = r\sin(x + \alpha)$$
$$a\sin(x) - b\cos(x) = r\sin(x - \alpha)$$

where a and b are positive, $r = \sqrt{a^2 + b^2}$ and $\tan(\alpha) = \frac{b}{a}$ where α is in the 1st quadrant.

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WORKED EXAMPLE 23 Graphing $y = a\cos(x) + b\sin(x)$

Express $3\sin(x) + 5\cos(x)$ in the form $r\sin(x + \alpha)$ and sketch the graph of $y = 3\sin(x) + 5\cos(x)$ for $x = 0$ to 2π.

Steps	Working
1 Expand the RHS using the sum identity.	$r\sin(x + \alpha) = r[\sin(x)\cos(\alpha) + \cos(x)\sin(\alpha)]$ $= r\sin(x)\cos(\alpha) + r\cos(x)\sin(\alpha)$
2 Let $3\sin(x) + 5\cos(x) = r\sin(x + \alpha)$.	$3\sin(x) + 5\cos(x) = r\sin(x)\cos(\alpha) + r\cos(x)\sin(\alpha)$
3 Compare the LHS and RHS. These are simultaneous equations in r and α.	$r\cos(\alpha) = 3 \quad \ldots [1]$ $r\sin(\alpha) = 5 \quad \ldots [2]$
4 To find r, square both equations and add them.	$r^2\cos^2(\alpha) + r^2\sin^2(\alpha) = 3^2 + 5^2$ $r^2[\cos^2(\alpha) + \sin^2(\alpha)] = 34$ $r^2 = 34$ as $\cos^2(\alpha) + \sin^2(\alpha) = 1$ $r = \sqrt{34}$ as $r > 0$
5 To find α, divide [2] by [1].	$\frac{r\sin\alpha}{r\cos\alpha} = \frac{5}{3}$ $\tan\alpha = \frac{5}{3}$ $\alpha = \tan^{-1}\left(\frac{5}{3}\right)$
6 Write the result.	$3\sin(x) + 5\cos(x) = \sqrt{34}\sin\left(x + \tan^{-1}\left(\frac{5}{3}\right)\right)$
7 Sketch a sine graph with amplitude $\sqrt{34}$, period 2π and a y-intercept at $3\sin(0) + 5\cos(0) = 5$.	

WORKED EXAMPLE 24 Solving $a\cos(x) - b\sin(x) = c$

Express $\sqrt{3}\cos(x) - \sin(x)$ in the form $r\cos(x + \alpha)$ and hence solve the equation $\sqrt{3}\cos(x) - \sin(x) = 1$ for $x = 0$ to 2π.

Steps	Working
1 Find $r = \sqrt{a^2 + b^2}$.	$a = \sqrt{3}, b = 1$ $r = \sqrt{(\sqrt{3})^2 + (1)^2}$ $= \sqrt{4}$ $= 2$
2 Use $\tan(\alpha) = \frac{b}{a}$ to find α.	$\tan(\alpha) = \frac{1}{\sqrt{3}}$ $\therefore \alpha = \frac{\pi}{6}$
3 Write $\sqrt{3}\cos(x) - \sin(x)$ in the form $r\cos(x + \alpha)$.	$\sqrt{3}\cos(x) - \sin(x) = 2\cos\left(x + \frac{\pi}{6}\right)$
4 Solve the equation $\sqrt{3}\cos(x) - \sin(x) = 1$.	$\sqrt{3}\cos(x) - \sin(x) = 1$ $2\cos\left(x + \frac{\pi}{6}\right) = 1$ $\therefore \cos\left(x + \frac{\pi}{6}\right) = \frac{1}{2}$ $x + \frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3}$ where $x \in [0, 2\pi]$ $x = \frac{\pi}{3} - \frac{\pi}{6}, \frac{5\pi}{3} - \frac{\pi}{6}$ $\therefore x = \frac{\pi}{6}, \frac{3\pi}{2}$

Exam hack

You can always check your solutions by substituting them into the original equation.

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EXERCISE 7.9 The subsidiary angle identities

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Recap

1 TECH-FREE

a Prove that $\cos(x)\cos(y) = \frac{1}{2}[\cos(x - y) + \cos(x + y)]$.

b Show that $\sin(30°)\sin(45°) = \frac{1}{2}[\cos(15°) - \cos(75°)]$.

c Express $\cos(x)\sin(2x)$ as a sum or a difference of trigonometric functions.

2 $\sin(4x)\cos(3x)$ can be expressed as

A $\frac{1}{2}[\sin(7x) + \sin(x)]$ **B** $\sin(7x) + \sin(x)$ **C** $\frac{1}{2}[\sin(7x) + \sin(-x)]$

D $\frac{1}{2}[\cos(7x) + \cos(x)]$ **E** $\frac{1}{2}[\sin(7x) - \sin(x)]$

3 The expression $\cos(6x)$ can be written as

A $2\cos^2(12x) + 1$ **B** $2\cos^2(3x) - 1$ **C** $1 + 2\sin^2(3x)$

D $\cos^2(3x) + 1$ **E** $\cos^2(12x) - \sin^2(12x)$

Mastery

4 WORKED EXAMPLE 23 **TECH-FREE** Express $7\sin(x) + 3\cos(x)$ in the form $r\sin(x + \alpha)$.

5 **a** WORKED EXAMPLE 24 **TECH-FREE** Express $\cos(x) - \sqrt{3}\sin(x)$ in the form $r\cos(x + \alpha)$.

b **TECH-FREE** Hence, solve the equation $\cos(x) - \sqrt{3}\sin(x) = 1$, $0 \leq x \leq 2\pi$.

6 $4\sin(x) + 3\cos(x)$ can be expressed as

A $5\sin\left(x + \tan^{-1}\left(\frac{3}{4}\right)\right)$ **B** $5\sin\left(x + \tan^{-1}\left(\frac{4}{3}\right)\right)$ **C** $\sqrt{7}\sin\left(x + \tan^{-1}\left(\frac{3}{4}\right)\right)$

D $\sqrt{7}\sin\left(x + \tan^{-1}\left(\frac{4}{3}\right)\right)$ **E** $5\sin\left(x + \frac{\pi}{4}\right)$

7 Express $\sqrt{2}\cos(x) + 2\sin(x)$ in the form $r\cos(x + \alpha)$.

A $2\cos(x + \tan^{-1}(\sqrt{2}))$ **B** $\sqrt{2}\cos(x + \tan^{-1}(\sqrt{2}))$ **C** $\sqrt{6}\cos(x - \tan^{-1}(\sqrt{2}))$

D $\sqrt{6}\cos(x + \tan^{-1}(2))$ **E** $\sqrt{6}\cos(x + \tan^{-1}(\sqrt{2}))$

Exam practice

8 **TECH-FREE** (2 marks) Show that if $a\cos(x) + b\sin(x) = r\cos(x - \alpha)$ where $a, b > 0$ and α is acute, then $r = \sqrt{a^2 + b^2}$ and $\tan(\alpha) = \frac{b}{a}$.

9 **TECH-FREE** (2 marks) Show that $2\sin(x) - 2\cos(x) = 2\sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$.

10 **TECH-FREE** (2 marks) Express $\sin(x) + \cos(x)$ in the form $r\sin(x + \alpha)$.

11 **TECH-FREE** (2 marks) Express $2\sin(x) + 4\cos(x)$ in the form $r\sin(x + \alpha)$.

12 **TECH-FREE** (4 marks)

a Express $6\sin(x) + 8\cos(x)$ in the form $r\sin(x + \alpha)$. 2 marks

b Hence, sketch the graph of $y = 6\sin(x) + 8\cos(x)$ for $x = 0$ to 2π. 2 marks

13 **TECH-FREE** (4 marks)

a Express $\cos(x) + \sin(x)$ in the form $r\cos(x - \theta)$. 2 marks

b Hence sketch the graph of $\cos(x) + \sin(x)$ for $x = 0$ to 2π. 2 marks

14 **TECH-FREE** (5 marks)

a Express $5\cos(x) - 12\sin(x)$ in the form $r\cos(x + \theta)$. 2 marks

b Hence, solve the equation $5\cos(x) - 12\sin(x) = -13$ for $0 \leq x \leq 2\pi$.
Write the solutions in terms of π and θ. 3 marks

15 **TECH-FREE** (3 marks) Solve $\cos(x) + \sqrt{3}\sin(x) = -\sqrt{3}$ for $0 \leq x \leq 2\pi$.

16 **TECH-FREE** (3 marks) Solve $\cos(x) - \sin(x) = 1$ for $0 \leq x \leq 2\pi$.

7.10 Proving trigonometric identities

Exam hack

When proving trigonometric identities, start with the side that is more complicated.

WORKED EXAMPLE 25 Proving trigonometric identities 1

Prove the identity $2\sin^2(\theta)\cos^2(\theta) = \frac{1}{2}\sin^2(2\theta)$.

Steps	Working
Start with the RHS and use the double angle formula $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$.	$\text{RHS} = \frac{1}{2}\sin^2(2\theta)$ $= \frac{1}{2}(2\sin(\theta)\cos(\theta))^2$ $= \frac{1}{2} \times (4\sin^2(\theta)\cos^2(\theta))$ $= 2\sin^2(\theta)\cos^2(\theta)$ $= \text{LHS}$

Video playlist
Proving trigonometric identities

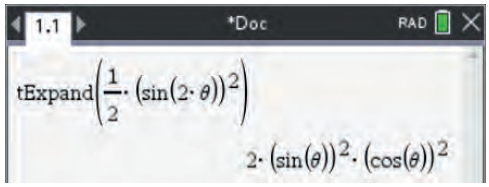
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CAS can be used to verify trigonometric identities, such as the above example.

USING CAS 4 Verifying trigonometric identities

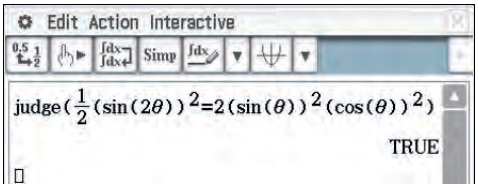
Use CAS to verify the identity $\frac{1}{2}\sin^2(2\theta) = 2\sin^2(\theta)\cos^2(\theta)$.

TI-Nspire



Use the **trigExpand** function to expand the expression $\frac{1}{2}(\sin(2\theta))^2$. Note that when using CAS, you must enter $\sin^2(2\theta)$ as $(\sin(2\theta))^2$.

ClassPad



1. Open the **Keyboard** > **Catalog**.
2. Tap **J** then scroll down to select the **judge** function.
3. Enter the equation. Enter $\sin^2(\theta)$ as $(\sin(\theta))^2$.
4. The **TRUE** result verifies the expressions are equal.

To prove a trigonometric identity, sometimes we need to simplify each side separately.

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WORKED EXAMPLE 26 Proving trigonometric identities 2

Prove the identity $\dfrac{1}{\operatorname{cosec}(x)+\cot(x)} = \dfrac{\frac{1}{2}[1-\cos(2x)]}{\tan(x)\cos(x)+\frac{1}{2}\sin(2x)}$.

Steps	Working
1 Start with the LHS by simplifying the fraction.	$\text{LHS} = \dfrac{1}{\operatorname{cosec}(x)+\cot(x)}$ $= \dfrac{1}{\frac{1}{\sin(x)}+\frac{1}{\tan(x)}}$ $= \dfrac{\sin(x)}{1+\cos(x)}$
2 Now, simplify the RHS.	$\text{RHS} = \dfrac{\frac{1}{2}[1-\cos(2x)]}{\tan(x)\cos(x)+\frac{1}{2}\sin(2x)}$ $= \dfrac{\frac{1}{2}\left(1-\left[1-2\sin^2(x)\right]\right)}{\frac{\sin(x)}{\cos(x)}\cos(x)+\frac{1}{2}\times 2\sin(x)\cos(x)}$ $= \dfrac{\sin^2(x)}{\sin(x)+\sin(x)\cos(x)}$
3 Take out $\sin(x)$ as a common factor and cancel.	$= \dfrac{\sin(x)}{\sin(x)}\left(\dfrac{\sin(x)}{1+\cos(x)}\right)$ $= \dfrac{\sin(x)}{1+\cos(x)}$
4 Compare LHS and RHS.	We have $\dfrac{\sin(x)}{1+\cos(x)} = \dfrac{\sin(x)}{1+\cos(x)}$. $\therefore$ LHS = RHS

Using trigonometric identities to solve equations

Consider the trigonometric equation $\sin(2x) - \sin(x) = 0$ for $x \in [0, 2\pi]$.

$2\sin(x)\cos(x) - \sin(x) = 0$

$\sin(x)(2\cos(x) - 1) = 0$

$\sin(x) = 0 \quad \text{or} \quad 2\cos(x) - 1 = 0$

$\sin(x) = 0 \quad \text{or} \quad \cos(x) = \dfrac{1}{2}$

Solutions are $x = 0, \pi, 2\pi$ and $x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$.

Exam hack

Do not divide both sides of the equation by $\sin(x)$ to simplify the equation. This is because $\sin(x)$ could be 0, and you cannot divide an equation by 0. If you do this, you will miss 3 of the solutions.

WORKED EXAMPLE 27 Solving trigonometric equations

Use a double angle identity to solve the equation $\cos(2x) = \cos^2(x)$ for $x \in [0, 2\pi]$.

Steps	Working
1 Simplify the LHS using the identity $\cos(2x) = 2\cos^2(x) - 1$.	$2\cos^2(x) - 1 = \cos^2(x)$
2 Take $\cos^2(x)$ to the LHS and simplify.	$\cos^2(x) - 1 = 0$
3 Solve the equation for $x \in [0, 2\pi]$.	$\cos^2(x) = 1$ $\cos(x) = \pm 1$ $x = 0, \pi, 2\pi$

VCE QUESTION ANALYSIS

A VCE-style question (9 marks)

Consider the function $f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R, f(x) = a\tan(x) + b\sec(x)$, where a and b are positive real numbers and $a > b$.

a Prove that the graph of f has at least one x-intercept. 2 marks

b Hence, solve $2\tan(x) + \sqrt{2}\sec(x) = 0$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. 1 mark

c Show that the equation $a\tan(x) + b\sec(x) = k$, where k is a real number and $\cos(x) \neq 0$, can be written as $k\cos(x) - a\sin(x) = b$. 1 mark

d If $k\cos(x) - a\sin(x) = r\cos(x + \alpha)$, where k and a are positive and α is acute, show that $k\cos(x) - a\sin(x) = b$ can be simplified to $\cos(x + \alpha) = \frac{b}{\sqrt{k^2 + a^2}}$. 1 mark

e Explain why $0 < \frac{b}{\sqrt{k^2 + a^2}} < 1$. 1 mark

f Hence, explain why $a\tan(x) + b\sec(x) = k$ has a solution for all values of k. 1 mark

g If $\sin(2x) = -\frac{2\sqrt{2}}{3}$ and $x \in \left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$, prove that $f(x) = b\sqrt{3} - a\sqrt{2}$. 2 marks

Reading the question

- The question is about trigonometric functions.
- $a > b > 0$
- Algebra, equations and trigonometric graphs are included.
- Trigonometric identities and the subsidiary angle method will be needed.

Thinking about the question

- Express $\tan(x)$ as $\frac{\sin(x)}{\cos(x)}$ and $\sec(x)$ as $\frac{1}{\cos(x)}$.
- One x-intercept means one solution to the equation.
- Part **a** requires students to find x-intercepts of a graph without actual given values of a and b. One approach is to substitute a range of values, with $a > b$, using CAS to sketch the graph. Another approach is to consider the general nature of the graph.
- Part **b** asks students to use their answer to part **a** to solve a trigonometric equation within a domain.
- Part **g** uses trigonometric identities or, alternatively, the use of a 'right-angled triangle' with the 3 trigonometric ratios sin, cos and tan.

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Video VCE question analysis: Trigonometric identities

Worked solution (✓ = 1 mark)

a Solve $f(x) = a\tan(x) + b\sec(x) = 0$ for x-intercepts.

$$\frac{a\sin(x)}{\cos(x)} + \frac{b}{\cos(x)} = 0$$

$$\frac{a\sin(x) + b}{\cos(x)} = 0$$

$$a\sin(x) + b = 0, \text{ where } \cos(x) \neq 0$$

$$\sin(x) = -\frac{b}{a}$$

Because $a > b > 0$, $-\frac{b}{a}$ must be a negative proper fraction.

$-1 < -\frac{b}{a} < 0$ ✓

$\therefore$ The equation $\sin(x) = -\frac{b}{a}$ has at least 1 real solution for $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

sin is negative in the 4th quadrant.

$\therefore$ **The graph of f has at least 1 x-intercept.** ✓

b $2\tan(x) + \sqrt{2}\sec(x) = 0$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$a = 2, b = \sqrt{2}$

From part **a**, the solution is $\sin(x) = -\frac{b}{a}$

$$\sin(x) = -\frac{\sqrt{2}}{2}$$

$$\boldsymbol{x = -\frac{\pi}{4} \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$$ ✓

c $a\tan(x) + b\sec(x) = k$

$$\frac{a\sin(x)}{\cos(x)} + \frac{b}{\cos(x)} = k$$

$$\frac{a\sin(x) + b}{\cos(x)} = k$$

$$a\sin(x) + b = k\cos(x), \text{ where } \cos(x) \neq 0$$

$$\boldsymbol{k\cos(x) - a\sin(x) = b}$$ ✓

d Let $k\cos(x) - a\sin(x) = r\cos(x + \alpha)$, where $r = \sqrt{k^2 + a^2}$ and $\alpha = \tan^{-1}\left(\frac{a}{k}\right)$.

$\therefore k\cos(x) - a\sin(x) = b$ becomes

$\sqrt{k^2 + a^2}\cos(x + \alpha) = b$

$$\therefore \boldsymbol{\cos(x + \alpha) = \frac{b}{\sqrt{k^2 + a^2}}}$$ ✓

e Because $a > b > 0$, $\sqrt{k^2 + a^2} > b$ and $\frac{b}{\sqrt{k^2 + a^2}}$ must be a positive proper fraction.

$$\boldsymbol{0 < \frac{b}{\sqrt{k^2 + a^2}} < 1}$$ ✓

f Therefore, the equation $\cos(x+\alpha) = \dfrac{b}{\sqrt{k^2+a^2}}$ must have a real solution for $x+\alpha \in \left(0, \dfrac{\pi}{2}\right)$.

So we want to find the (maximum) range of $x \in \left(0-\alpha, \dfrac{\pi}{2}-\alpha\right)$ given that α is acute.

cos is positive in the 1st quadrant.

The lowest value of $0-\alpha$ is when $\alpha = \dfrac{\pi}{2}$. The highest value of $\dfrac{\pi}{2}-\alpha$ is when $\alpha = 0$.

So $x \in \left(0-\dfrac{\pi}{2}, \dfrac{\pi}{2}-0\right) = \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

Hence, $k\cos(x) - a\sin(x) = b$ has a real solution for $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$. ✓

g If $f(x) = a\tan(x) + b\sec(x) = b\sqrt{3} - a\sqrt{2}$, then we need to prove that $\sec(x) = \sqrt{3}$ and $\tan(x) = -\sqrt{2}$.

$\sin(2x) = -\dfrac{2\sqrt{2}}{3}$ and $x \in \left(-\dfrac{\pi}{2}, -\dfrac{\pi}{4}\right)$, which is in the 4th quadrant.

First find $\cos(2x)$ to then find $\cos(x)$ and $\sec(x)$.

$$\cos^2(2x) = 1 - \sin^2(2x)$$
$$= 1 - \left(-\frac{2\sqrt{2}}{3}\right)^2$$
$$= 1 - \frac{8}{9}$$
$$= \frac{1}{9}$$

$\cos(2x) = -\dfrac{1}{3}$ Negative because $2x \in \left(-\pi, -\dfrac{\pi}{2}\right)$, which is in the 3rd quadrant.

Now $\cos(2x) = 2\cos^2(x) - 1 = -\dfrac{1}{3}$

$$2\cos^2(x) = \frac{2}{3}$$
$$\cos^2(x) = \frac{1}{3}$$

$\cos(x) = \dfrac{1}{\sqrt{3}}$ Positive because $x \in \left(-\dfrac{\pi}{2}, -\dfrac{\pi}{4}\right)$, which is in the 4th quadrant.

$$\therefore \sec(x) = \frac{1}{\cos(x)} = \frac{1}{\frac{1}{\sqrt{3}}} = \sqrt{3} \quad ✓$$

Now find $\tan(x)$.

$$1 + \tan^2(x) = \sec^2(x)$$
$$1 + \tan^2(x) = (\sqrt{3})^2$$
$$= 3$$
$$\tan^2(x) = 2$$

$\tan(x) = -\sqrt{2}$ Negative because $x \in \left(-\dfrac{\pi}{2}, -\dfrac{\pi}{4}\right)$, which is in the 4th quadrant.

$\therefore$ **$f(x) = a\tan(x) + b\sec(x) = -a\sqrt{2} + b\sqrt{3} = b\sqrt{3} - a\sqrt{2}$** ✓

EXERCISE 7.10 Proving trigonometric identities

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Recap

1 TECH-FREE

a Express $\sin(2y)\cos(4y)$ as a sum or a difference of trigonometric functions.

b Use a compound angle formula to find an expression for $\cot\left(\frac{11\pi}{12}\right)$.

c Rewrite the expression $3\sin(2x)\cos(x) + \sin(x)\cos(2x)$ in terms of sine terms only.

2 $\cos(20°)\cos(60°)$ is equal to

A $\frac{1}{2}\left[\cos(40°) - \cos(80°)\right]$ **B** $\cos(120°)$ **C** $\frac{1}{2}\left[\cos(40°) + \cos(80°)\right]$

D $\frac{1}{2}\left[\sin(-40°) + \sin(80°)\right]$ **E** $\frac{1}{2}\left[\sin(-40°) + \cos(80°)\right]$

Mastery

3 TECH-FREE Prove that $\cos(x)\sin(y) = \frac{1}{2}\left[\sin(x + y) - \sin(x - y)\right]$.

4 TECH-FREE Show that $\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{6}\right) = \frac{1}{2}\left[\cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{2\pi}{3}\right)\right]$.

5 WORKED EXAMPLE 25 Prove the identity $[1 - \cos^2(\theta)][1 + \cot^2(\theta)] = 1$.

6 WORKED EXAMPLE 26 Prove that $\frac{1}{\sin(x) + \operatorname{cosec}(x)} = \frac{\frac{1}{2}\sec(x)\sin(2x)}{1 - \cos^2(x) + \operatorname{cosec}^2(x) - \cot^2(x)}$.

7 Using CAS 4 Show that $\frac{1}{8}\sin^2(4x) = \frac{1}{2}\sin^2(2x)\cos^2(2x)$.

8 Prove that $\frac{1}{\operatorname{cosec}(x) + 1} + \frac{1}{\operatorname{cosec}(x) - 1} = 2\tan(x)\sec(x)$.

9 WORKED EXAMPLE 27 Use a suitable double angle identity to solve the equation $\sec^2(x) = \cos^2(x)$ for $x \in [0, 2\pi]$.

10 The expression $2\sin^2(x)\cos(2x)$, written in terms of cosine terms only, could be

A $\sin^2(x)\cos(2x)$ **B** $2\cos^2(x)\cos(2x)$ **C** $\frac{1}{2}(2\cos(2x) - \cos(4x) - 1)$

D $2\cos(2x) - \cos(4x) - 1$ **E** $2\cos(2x) - \cos(4x) - 4$

11 The expression $1 + \cos(2x)$ can be written as

A $2\cos^2(x)$ **B** $2\sin^2(x)$ **C** $2\cos^2(x) - 1$

D $1 - 2\cos^2(x)$ **E** $\cos^2(x)$

Exam practice

80–100% 60–79% 0–59%

7.10

12 © VCAA 2011 1Q8 **46%** **TECH-FREE** (3 marks) Find the coordinates of the points of intersection of the graph of the relation $y = \text{cosec}^2\left(\frac{\pi x}{6}\right)$ with the line $y = \frac{4}{3}$, for $0 < x < 12$.

13 © VCAA 2015 1Q7 **46%** **TECH-FREE** (5 marks)

a **58%** Solve $\sin(2x) = \sin(x)$, $x \in [0, 2\pi]$. 3 marks

b **26%** Find $\{x: \text{cosec}(2x) < \text{cosec}(x),\ x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)\}$. 2 marks

14 © VCAA 2017N 1Q6 **TECH-FREE** (5 marks) Find all real solutions of $\tan(2x) = -\tan(x)$.

15 © VCAA 2012 1Q2 **28%** **TECH-FREE** (3 marks) Find all real solutions of $2\cos(x) = \sqrt{3}\cot(x)$.

16 (3 marks) Prove the identity $\frac{1}{1 - \cos(\theta)} + \frac{1}{1 + \cos(\theta)} = 2\,\text{cosec}^2(\theta)$.

17 (3 marks) Prove the identity $\sec(x) + \tan(x) = \frac{\cos(x)}{1 - \sin(x)}$.

18 (3 marks) Prove the identity $\tan(2x) = \frac{2\sin(x)\cos(x)}{\cos^2(x) - \sin^2(x)}$.

19 (3 marks) Show that $2\sin^2(6x) = 8[\sin^2(3x) - \sin^4(3x)]$.

20 (6 marks) Prove each identity.

a $2\sin(x) - 2\sin^2(x) - \cos^2(x) = -[\sin(x) - 1]^2$ 3 marks

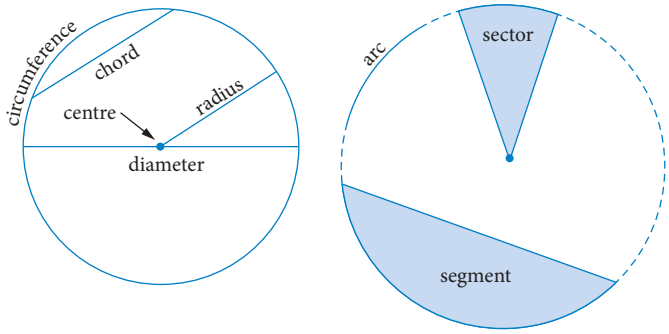
b $\frac{1}{2}\left[\cos(x) + \cos\left(\frac{\pi}{3}\right)\right] = \cos\left(\frac{x}{2} + \frac{\pi}{6}\right)\cos\left(\frac{x}{2} - \frac{\pi}{6}\right)$ 3 marks

21 (3 marks) Prove that $\cot(x) + \cot(y) = \frac{\text{cosec}(x)\,\text{cosec}(y)}{\text{cosec}(x + y)}$.

22 (3 marks) Write $\frac{1}{3}\sin(3x) + \cos(2x)$ in terms of sine terms only.

7 Chapter summary

Parts of a circle

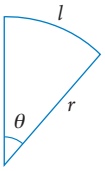


Arc length of a circle

$l = r\theta$

where r = radius of the circle

θ = angle at the centre of the circle in radians

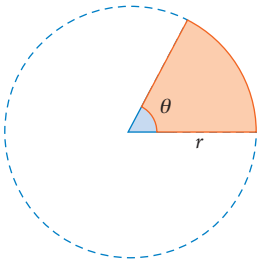


Area of a sector

$A = \frac{1}{2}r^2\theta$

where r = radius of the circle

θ = angle at the centre of the circle in radians

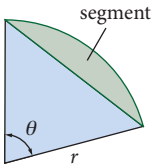


Area of a segment

$A = \frac{1}{2}r^2(\theta - \sin(\theta))$

where r = radius of the circle

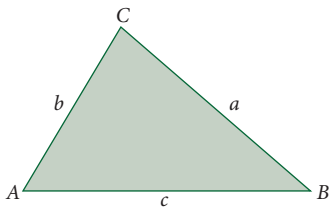
θ = angle at the centre of the circle in radians



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The sine rule

In any triangle ABC with angles labelled A, B, C, the opposite side lengths are labelled a, b, c respectively.



The **sine rule** relates each side of a triangle to its opposite angle.

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$$

or $\dfrac{\sin(A)}{a} = \dfrac{\sin(B)}{b} = \dfrac{\sin(C)}{c}$

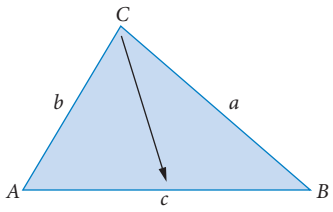
The sine rule is used for triangle problems involving 2 sides and 1 non-included angle, or 2 angles and 1 side.

To find an unknown side, look for 2 matching pairs of sides and opposite angles, and place the unknown side in the numerator.

To find an unknown angle, look for 2 matching pairs of angles and opposite sides, and place the unknown angle in the numerator. There may be 2 supplementary solutions for the unknown angle (acute and obtuse, the ambiguous case), and you must test the obtuse angle using the given angle and the angle sum of a triangle.

The cosine rule

The **cosine rule** relates 1 side of a triangle to the other 2 sides and its opposite angle. It is an extension of Pythagoras' theorem, $c^2 = a^2 + b^2$.



$c^2 = a^2 + b^2 - 2ab\cos(C)$

or $\cos(C) = \dfrac{a^2 + b^2 - c^2}{2ab}$

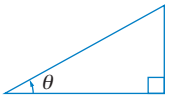
The cosine rule is used for triangle problems involving 2 sides and the included angle, or all 3 sides.

To find an unknown side, let c be the side and C be the opposite angle.

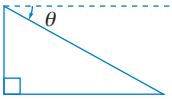
To find an unknown angle, let C be the angle and c be the opposite side.

Angles of elevation and depression

The **angle of elevation** of an object is the angle measured from the horizontal (ground) when looking up to the top of an object.



The **angle of depression** of an object is the angle measured from the horizontal when looking down to an object below.



Bearings

Bearings are measured clockwise from the north position.

Reciprocal trigonometric functions

- $\operatorname{cosec}(x) = \dfrac{1}{\sin(x)}$
- $\sec(x) = \dfrac{1}{\cos(x)}$
- $\cot(x) = \dfrac{1}{\tan(x)}$

Pythagorean identities

- $\sin^2(\theta) + \cos^2(\theta) = 1$
- $\sec^2(\theta) = \tan^2(\theta) + 1$
- $\text{cosec}^2(\theta) = \cot^2(\theta) + 1$

The sum and difference identities

- $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$
- $\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$
- $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$
- $\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$
- $\tan(x + y) = \dfrac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$
- $\tan(x - y) = \dfrac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$

The double angle identities

- $\sin(2x) = 2\sin(x)\cos(x)$
- $\cos(2x) = \cos^2(x) - \sin^2(x)$ or $2\cos^2(x) - 1$ or $1 - 2\sin^2(x)$
- $\tan(2x) = \dfrac{2\tan(x)}{1 - \tan^2(x)}$

Products of sines and cosines

- $\cos(x)\cos(y) = \dfrac{1}{2}\left[\cos(x - y) + \cos(x + y)\right]$
- $\sin(x)\sin(y) = \dfrac{1}{2}\left[\cos(x - y) - \cos(x + y)\right]$
- $\sin(x)\cos(y) = \dfrac{1}{2}\left[\sin(x + y) + \sin(x - y)\right]$
- $\cos(x)\sin(y) = \dfrac{1}{2}\left[\sin(x + y) - \sin(x - y)\right]$

The subsidiary angle identities

- $a\cos(x) + b\sin(x) = r\cos(x - \alpha)$
- $a\cos(x) - b\sin(x) = r\cos(x + \alpha)$
- $a\sin(x) + b\cos(x) = r\sin(x + \alpha)$
- $a\sin(x) - b\cos(x) = r\sin(x - \alpha)$

where a and b are positive, $r = \sqrt{a^2 + b^2}$ and $\tan(\alpha) = \dfrac{b}{a}$, where α is in the 1st quadrant.

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 (4 marks)

a Use Boolean algebra to simplify $\left(\overline{\overline{A} + \overline{B}}\right)(A + B)$. 2 marks

b Display the answer as a truth table. 2 marks

2 (2 marks) The second term of an arithmetic sequence is –96 and the seventh term is –76. State the value of the tenth term.

3 (2 marks) Six students wait in the foyer of a cinema to see the latest movie. They find out there are only 4 seats available in the cinema.

a In how many ways can the 6 students seat four of themselves in a row in the last 4 seats available? 1 mark

b In how many ways can the 6 students seat four of themselves in the row of 4 seats, given that one particular student wants to sit on the leftmost seat of the row? 1 mark

4 (2 marks) Prove the identity $\sin^2(3x)\cos^2(3x) = \frac{1}{4}(1 - \cos^2(6x))$.

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 A 10-sided die is rolled 3 times. The number of elements in the event space is

A 10 **B** 27 **C** 30 **D** 1000 **E** 3000

2 Which one of the following statements is true for the equation $\sin^2(2x) = \pm 1$ for $x \in \left(0, \frac{\pi}{2}\right)$?

A The only solutions are $x = \pm\frac{\pi}{4}$.

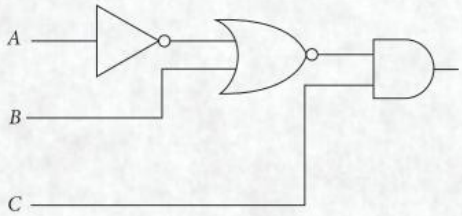
B The only solution is $x = \frac{\pi}{4}$.

C The graph of $y = \sin^2(2x)$ will intersect the graph of $y = 1$ twice.

D The graph of $y = \sin^2(2x)$ will intersect the graph of $y = 1$ three times.

E There are no solutions.

3 The output function for the circuit shown is



A $(\overline{\bar{A} + B})C$ **B** $(\bar{A} + B)\bar{C}$ **C** $(\overline{A\bar{B}})C$ **D** $(\overline{\bar{A} + \bar{B}})\bar{C}$ **E** $\bar{A}BC$

4 A regular graph has 36 edges and 24 vertices. Each vertex is of degree

A 1 **B** 2 **C** 3 **D** 4 **E** 5

5 The graph of $y = 3\sec(x)$ has a local minimum point at

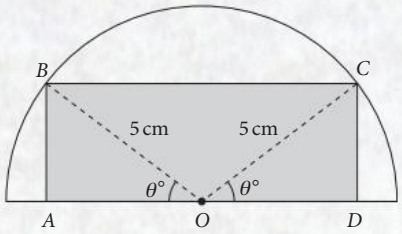
A $x = 3$ **B** $(3, 0)$ **C** $(2\pi, 3)$ **D** $(0, -3)$ **E** $(\pi, 3)$

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Section B 2 questions

15 marks

1 (9 marks) The diagram shows a rectangle $ABCD$ inside a semicircle, centre O and radius 5 cm, with $\angle BOA = \angle COD = \theta°$.



a Show that the perimeter, P cm, of the rectangle is given by $P = 20\cos(\theta) + 10\sin(\theta)$. 3 marks

b Express P in the form $r\cos(\theta - \alpha)$ and hence find the value of θ for which $P = 16$. 2 marks

c Find the value of k for which the area of the rectangle is $k\sin(2\theta)$ cm^2. 2 marks

d Find the value of θ for which the area is a maximum. 2 marks

2 (6 marks)

a Using a suitable compound angle formula, evaluate $\sin\left(\frac{\pi}{12}\right)$. 2 marks

b Using a suitable double angle formula, evaluate $\sin\left(\frac{\pi}{12}\right)$. 2 marks

c Hence find an expression for $4\,\text{cis}\left(\frac{\pi}{12}\right)$, written in the form $x + yi$, where x and y are real numbers. 2 marks

CHAPTER

8 GRAPHING FUNCTIONS AND RELATIONS

Study Design coverage

Nelson MindTap chapter resources

8.1 Functions and relations
- Transformations of functions

8.2 Rational functions
- Coefficients of polynomials
- Coefficients of rational functions
- Partial fractions
- **Using CAS 1:** Partial fractions

8.3 Reciprocal functions
- Reciprocal linear functions
- Reciprocal quadratic functions
- Reciprocal polynomial functions
- General reciprocal functions

8.4 Reciprocal circular functions
- Basic graphs
- Transformations of graphs

8.5 Inverse circular functions
- Graphs of inverse circular functions

8.6 The absolute value function
- Sketching absolute value functions

8.7 Locus definitions of curves and conic sections
- Conic sections
- Circles
- Parabolas
- Ellipses
- Hyperbolas

8.8 Polar and parametric forms of the conic sections
- Ellipses
- Hyperbolas
- Parabolas
- Circles

VCE question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2

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Study Design coverage

UNIT 2, AREA OF STUDY 4: FUNCTIONS, RELATIONS AND GRAPHS

- identities from equating coefficients of polynomials, rational functions and their decompositions into partial fractions with denominators expressed as products of linear and irreducible quadratic terms
- graphs of simple reciprocal functions, including graphs of the reciprocal circular functions cosecant, secant and cotangent, and simple transformations of these
- graphs of the restricted circular functions of sine, cosine and tangent over principal domains and their respective inverse functions $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ (students should be familiar with alternative notations, arcsin, arccos and arctan), and simple transformations of these graphs
- locus definition and construction in the plane of lines, parabolas, circles, ellipses and hyperbolas and their cartesian, polar and parametric forms and graphs
- the absolute value function, its graph and simple transformations of its graph.

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Video playlists (9):

8.1 Functions and relations
8.2 Rational functions
8.3 Reciprocal functions
8.4 Reciprocal circular functions
8.5 Inverse circular functions
8.6 The absolute value function
8.7 Locus definitions of curves and conic sections
8.8 Polar and parametric forms of the conic sections
VCE question analysis Graphing functions and relations

Worksheets (7):

8.2 Partial fractions
8.3 Reciprocal functions
8.5 Inverse trigonometric functions
8.6 Absolute values • Absolute value equations and inequalities • Absolute value inequalities • Absolute value functions

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

8.1 Functions and relations

Video playlist
Functions and relations

A **relation** R is a set of ordered pairs (x, y), where $x \in X$ and $y \in Y$. We can show that an ordered pair belongs to a relation R by writing $(x, y) \in R$ or $y = R(x)$. The set of values of the **independent variable** x is called the **domain** and is often the whole set X. The set Y is called the **co-domain** and the set of values of the **dependent variable** y is called the **range**. We can show a relation as a rule, a list of ordered pairs, a table, a mapping or a graph.

A **function** $y = f(x)$ is a relation such that each x value in the domain has only one ordered pair (x, y); in other words, each x value matches with only one y value. A function may be shown by a rule like $f(x) = 6\sin(3x + 5)$. The largest set of real numbers for which a function rule is defined is called the **maximal domain** (also called **natural** or **implied domain**).

WORKED EXAMPLE 1 Relations, functions, domain and range

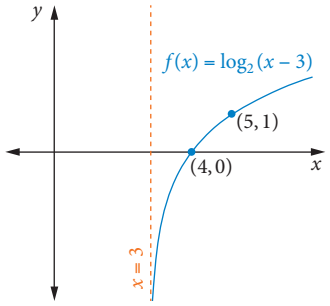
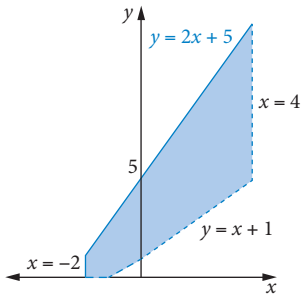
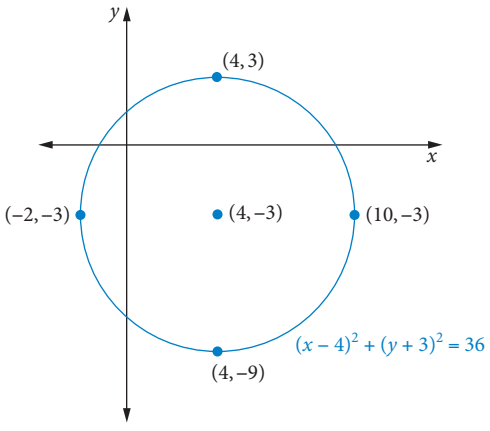
State whether each of the following is a function and state the domains, co-domains and ranges.

a $f: R \to R, f(x) = \log_2(x - 3)$

b $g: [-2, 4) \to R^+, x + 1 < y \leq 2x + 5$

c $(x - 4)^2 + (y + 3)^2 = 36$

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Steps	Working
a 1 Every x value has only one y value.	$f: R \to R, f(x) = \log_2(x-3)$ is a function.
2 Sketch the graph.	
3 What values can x take?	$x > 3$, so the domain is $(3, \infty)$.
4 What are allowable y values?	The stated co-domain is R.
5 What are the actual y values?	The range is R.
b 1 Find the endpoints of the lines. Sketch the graph, the area above $y = x + 1$ and under or on $y = 2x + 5$ between $x = -2$ and $x = 4$ inclusive of -2. Show inclusion of boundaries by solid lines and exclusion by dashed lines.	For $x \in [-2, 4)$, $y = x + 1 \in [-1, 5)$ and $y = 2x + 5 \in [1, 13)$, but $y \in R^+$, so $y > 0$. 
2 Is it a function?	$(0, 2)$ and $(0, 5)$ have the same x value, so it is not a function.
3 What values can x be?	The domain is $[-2, 4)$.
4 What are allowable y values?	The co-domain is R^+.
5 What are the actual y values?	The range is $(0, 13]$.
c 1 State the shape.	The shape is a circle of radius 6 and centre $(4, -3)$.
2 Sketch the graph.	
3 Is it a function?	$(4, 3)$ and $(4, -9)$ have the same x value, so it is not a function.
4 What values can x be?	The domain is $[-2, 10]$.
5 What are allowable y values?	The co-domain is R.
6 What are the actual y values?	The range is $[-9, 3]$.

Transformations of functions

We have studied **transformations** of functions in Mathematical Methods.

Transformations of functions

The graph of $f(x)$ is **transformed** to the graph of $af(n(x + b)) + c$ in the order **dilations**, **reflections** and **translations** (DRT or Dr T). The formats $a(f(n(x + b)) + c)$, $af(nx + b) + c$ and $a(f(nx + b) + c)$ have transformations in other orders.

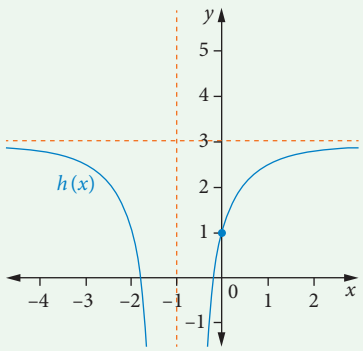
- n and b produce the transformations in the direction of the x-axis:
  - dilation/reflection by the factor $\frac{1}{n}$ and then translation b to the left or right gives $f(n(x + b))$
  - translation first and then dilation/reflection gives $f(nx + b)$.
- a and c produce the transformations in the direction of the y-axis:
  - dilation/reflection by the factor a and then translation c up or down gives $af(n(x + b)) + c$ or $af(nx + b) + c$
  - translation and then dilation/reflection gives $a(f(n(x + b)) + c)$ or $a(f(nx + b) + c)$.

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WORKED EXAMPLE 2 Transformations of functions

State the basic function and transformations for each function. State the equation for part **c**.

a $f(x) = -4 \times 3^{x+1} + 2$ **b** $g(x) = 3\cos(2x) - 1$ **c**



Steps	Working
a 1 State the basic function.	The basic function is $y = 3^x$.
2 State the transformations by comparing with $af(n(x + b)) + c$.	$n = 1$, $b = 1$, $a = -4$ and $c = 2$. It is dilated from the x-axis by a factor of 4, reflected in the x-axis and translated 1 unit left and 2 units up.
b 1 State the basic function.	The basic function is $y = \cos(x)$.
2 State the transformations by comparing with $af(n(x + b)) + c$.	$n = 2$, $b = 0$, $a = 3$ and $c = -1$. It is dilated parallel to the x-axis by a factor of $\frac{1}{2}$, dilated from the x-axis by a factor of 3 and translated 1 unit down.
c 1 State the basic function.	Graph looks most like the truncus, $y = \frac{1}{x^2}$.
2 State the obvious transformations.	It has been translated 1 unit to the left, reflected in the x-axis and translated 3 units up.
3 Use the y-intercept $f(0) = 1$ to check for a dilation.	For $f(x) = -\frac{a}{(x+1)^2} + 3$, $f(0) = -a + 3 = 1$, so $a = 2$, a dilation from the x-axis.
4 State all the transformations and equation.	$y = \frac{1}{x^2}$ has been dilated from the x-axis by a factor of 2, reflected in the x-axis and translated 1 unit left and 3 units up to give $f(x) = -\frac{2}{(x+1)^2} + 3$.

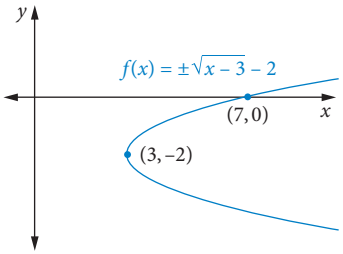
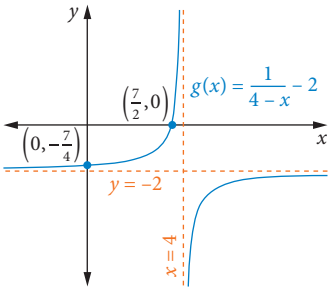
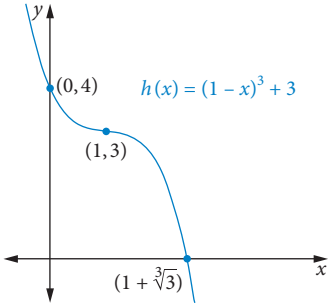
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WORKED EXAMPLE 3 Sketching transformed functions and relations

Sketch each relation.

a $f(x) = \pm\sqrt{x-3} - 2$ **b** $g(x) = \dfrac{1}{4-x} - 2$ **c** $h(x) = (1-x)^3 + 3$

Steps	Working
a 1 State the basic relation and transformations by comparing with $af(n(x+b)) + c$.	The basic relation is $y = \pm\sqrt{x}$ and it has been translated 3 units to the right and 2 units down. It is the inverse of $y = (x+2)^2 + 3$, so it is like a parabola turned on its side.
2 State any important points.	There are no y-intercepts. The zero is given by $\pm\sqrt{x-3} = 2$, $x - 3 = 4$, so $x = 7$. The turning point is $(3, -2)$.
3 Sketch the graph, marking the important points.	
b 1 State the basic relation and transformations by comparing with $af(n(x+b)) + c$.	$g(x) = \dfrac{1}{4-x} - 2 = \dfrac{-1}{x-4} - 2$ The basic function is $y = \dfrac{1}{x}$, reflected in the x-axis and translated 4 units to the right and 2 units down.
2 State any important points.	The asymptotes are $x = 4$, $y = -2$, the y-intercept is $-\dfrac{7}{4}$. The x-intercept is given by $\dfrac{-1}{x-4} - 2 = 0$, $x = \dfrac{7}{2}$.
3 Sketch the graph, marking the important points.	
c 1 State the basic relation and transformations by comparing with $af(n(x+b)) + c$.	The basic function is $y = x^3$ with a reflection in the y-axis and a translation of 1 unit to the right and 3 units up.
2 State any important points.	The y-intercept is 4, the zero is at $x = 1 + \sqrt[3]{3}$ and the stationary point of inflection is at $(1, 3)$.
3 Sketch the graph, marking the important points.	

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EXERCISE 8.1 Functions and relations

ANSWERS p. 585

Mastery

1 WORKED EXAMPLE 1 State whether each relation is a function and state its domain, co-domain and range.

a $f: [-2, 4] \to R, f(x) = 4 - 3^{x-1}$

b $g: R \to R, y = (x^2 - 9)^2$

c $h: [-2\pi, 2\pi] \to R, h(x) = \tan(2x)$

d $f: R \to R, x^2 + y^2 = 25$

e $g: R \to R^+, x^2 + y^2 = 25$

f $h: [-3, 4) \to R, x^2 + y^2 = 25$

g $\frac{(x+2)^2}{9} + \frac{(y-3)^2}{16} = 1$

h $xy = 1$

2 WORKED EXAMPLE 2 State the basic function or relation and the transformations for each of these.

a $f(x) = \log_3(4 - x) + 5$

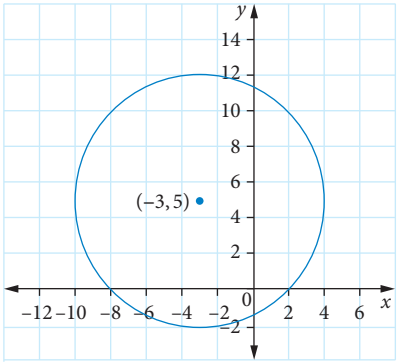
b $g(x) = \frac{1}{2x+6} + 4$

c $\frac{(x+2)^2}{9} + \frac{(y-3)^2}{16} = 1$

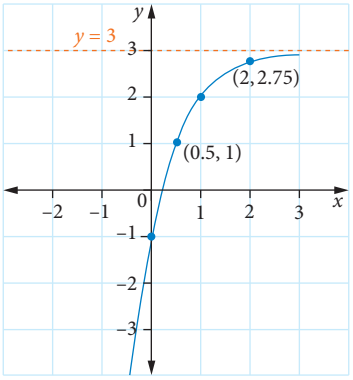
d $(x - 3)(y + 4) = 2$

3 State the basic function or relation, transformations and equation for each graph.

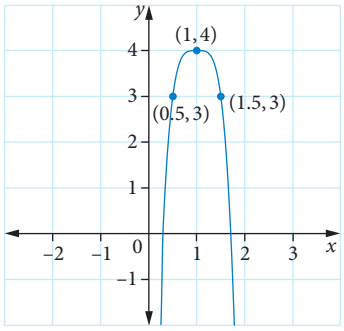
a



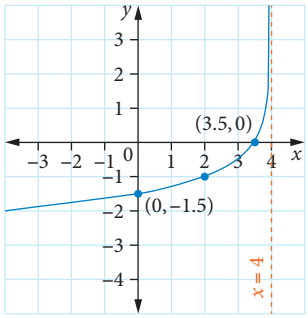
b



c



d



4 WORKED EXAMPLE 3 Sketch each relation.

a $f(x) = 10 - (x + 2)^2$

b $g(x) = \frac{1}{2}(2 - x)(x + 1)^2(x - 3)$

c $h(x) = 3\sin(4x) + 1$

d $m(x) = 4 - 2^{-x-3}$

e $p(x) = (x + 2)(x - 3)(x - 5)$

f $(x + 4)^2 + (y - 1)^2 = 9$

g $y + 3 = \sqrt[3]{2x - 1}$

h $(y - 4)^2 = 2x + 3$

Exam practice

5 The maximal domain of $2 - \log_3(x + 4)$ is

A $[4, \infty)$ **B** $[-4, \infty)$ **C** $(0, \infty)$ **D** $(4, \infty)$ **E** $(-4, \infty)$

6 The range of $f: R \to R, f(x) = 3 - 2^{1-x}$ is

A $(-\infty, 3)$ **B** $(-\infty, 3]$ **C** $(0, 3)$ **D** $[1, \infty)$ **E** $(-\infty, 1]$

7 The function $y = 4^x$ is dilated by a factor of $\frac{1}{2}$ from the y-axis, by a factor of 3 from the x-axis, reflected in the y-axis and translated 2 units to the right and 3 units up. The resulting function is

A $y = 3 \times 4^{-0.5x-2} + 3$ **B** $y = 3 \times 4^{-2x+4} + 9$ **C** $y = -2 \times 4^{3x+2} + 3$

D $y = 3 \times 4^{-2x-4} + 3$ **E** $y = 3 \times 4^{1-2x} + 3$

Video playlist
Rational functions

8.2 Rational functions

Coefficients of polynomials

For the polynomial $p(x) = 3x^3 - x^2 - 7x - 6$, $p(2) = 0$. From Mathematical Methods, we know this means that $x - 2$ is a factor of $p(x)$ and the other factor is a quadratic.

One way to find the quadratic is to write $3x^3 - x^2 - 7x - 6 = (x - 2)(ax^2 + bx + c)$ and expand the brackets. This is actually an **identity**, which is true for all values of the variable x. The symbol $\equiv$ means **identically equal**, and this is written instead of the equal sign to show the identity: $3x^3 - x^2 - 7x - 6 \equiv (x - 2)(ax^2 + bx + c)$.

This also means that the coefficients of terms of the same power are the same on each side of the identity.

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WORKED EXAMPLE 4 Identities and polynomial factors

$x - 2$ is a factor of $3x^3 - x^2 - 7x - 6$. Use an identity to find the other factor.

Steps	Working
1 Write the identity for the factor $x - 2$.	$3x^3 - x^2 - 7x - 6 \equiv (x - 2)(ax^2 + bx + c)$
2 Expand the brackets.	$\equiv ax^3 + (b - 2a)x^2 + (c - 2b)x - 2c$
3 Equate the coefficients of the polynomial.	$3 = a, -1 = b - 2a, -7 = c - 2b, -6 = -2c$
4 Solve for a, b and c.	$a = 3, b = 2a - 1 = 5, c = 3$
Use the 3rd equation to check the values: $c - 2b = 3 - 10 = -7$ ✓	
5 Write the other factor.	The other factor is $3x^2 + 5x + 3$.

The leading term of the polynomial is always the product of the leading terms of the factors. This is the same for the constant term, so you can find the values of a and c above by simple inspection.

Coefficients of rational functions

A **rational function** is of the form $f(x) = \dfrac{p(x)}{q(x)}$, where p and q are polynomials and q is not the zero polynomial. If the degree of the numerator is less than the degree of the denominator, it is called a **simple rational function**.

A **quotient function** has a numerator of degree equal to or higher than the degree of the denominator.

The difference of their degrees is the degree of the quotient (the result of the division). $\frac{4x^2 + 3x - 1}{x - 3}$ has a quotient of degree 1.

We can use identities to equate coefficients to express rational functions as sums of simpler functions.

WORKED EXAMPLE 5 Identities and rational functions

Use an identity to express $\frac{4x^2 + 3x - 1}{x - 3}$ in the form $\frac{A}{x - 3} + Bx + C$.

Steps	Working
1 Write the desired form.	$\frac{A}{x - 3} + Bx + C$
2 Change to a common denominator.	$= \frac{A}{x - 3} + \frac{(x - 3)(Bx + C)}{x - 3}$
3 Simplify the numerator.	$= \frac{A + Bx^2 + Cx - 3Bx - 3C}{x - 3}$ $= \frac{Bx^2 + (C - 3B)x + (A - 3C)}{x - 3}$
4 Write the identity.	$\frac{4x^2 + 3x - 1}{x - 3} \equiv \frac{Bx^2 + (C - 3B)x + (A - 3C)}{x - 3}$
5 Equate coefficients of the numerator.	$B = 4, C - 3B = 3, A - 3C = -1$
6 Solve for A, B and C.	$C - 12 = 3 \Rightarrow C = 15$ $A - 3 \times 15 = -1 \Rightarrow A = 44$
7 Write in the desired form.	$\frac{4x^2 + 3x - 1}{x - 3} = \frac{44}{x - 3} + 4x + 15$

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Some simple rational functions have a denominator that is a product of linear factors. We can always express these as fraction sums with simple linear denominators. We can usually use the identity for the numerators and choose suitable values of x to find the numerators.

WORKED EXAMPLE 6 Rational functions with simple denominators

Express $\frac{3x + 2}{(x + 2)(2x - 3)}$ in the form $\frac{A}{x + 2} + \frac{B}{2x - 3}$.

Steps	Working
1 Write the desired form.	$\frac{A}{x + 2} + \frac{B}{2x - 3}$
2 Change to a common denominator.	$= \frac{A(2x - 3) + B(x + 2)}{(x + 2)(2x - 3)}$
3 Write the numerator identity.	$3x + 2 \equiv A(2x - 3) + B(x + 2)$
4 Choose $x = -2$ and solve for A.	$3 \times (-2) + 2 = A(2 \times (-2) - 3) + B \times 0$ $-4 = -7A \Rightarrow A = \frac{4}{7}$
5 Choose $x = \frac{3}{2}$ and solve for B.	$3 \times \frac{3}{2} + 2 = A \times 0 + B\left(\frac{3}{2} + 2\right)$ $\frac{9 + 4}{2} = \frac{3 + 4}{2}B \Rightarrow B = \frac{13}{7}$

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6 Write the desired form.	$\frac{3x+2}{(x+2)(2x-3)} = \frac{\frac{4}{7}}{x+2} + \frac{\frac{13}{7}}{2x-3}$ $= \frac{4}{7(x+2)} + \frac{13}{7(2x-3)}$
7 If you have time, check the result by expressing with a common denominator.	$\frac{4}{7(x+2)} + \frac{13}{7(2x-3)} = \frac{4(2x-3)+13(x+2)}{7(x+2)(2x-3)}$ $= \frac{8x-12+13x+26}{7(x+2)(2x-3)}$ $= \frac{21x+14}{7(x+2)(2x-3)}$ $= \frac{7(3x+2)}{7(x+2)(2x-3)}$ $= \frac{3x+2}{(x+2)(2x-3)}$ ✓

Worksheet
Partial fractions

Partial fractions

The expression $\frac{4}{7(x+2)} + \frac{13}{7(2x-3)}$ is called the **partial fraction** decomposition of the rational expression $\frac{3x+2}{(x+2)(2x-3)}$. It is often used for simple rational functions.

A denominator with a power of a linear expression can always be decomposed to the sum of simple fractions of powers of the linear factor.

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WORKED EXAMPLE 7 Partial fractions with decreasing powers

Write $\frac{2x^2-5x+1}{(x-1)^3}$ as a sum of partial fractions.

Steps	Working
1 Write as a sum of partial fractions with decreasing powers.	$\frac{2x^2-5x+1}{(x-1)^3} \equiv \frac{A}{(x-1)^3} + \frac{B}{(x-1)^2} + \frac{C}{x-1}$
2 Express with a common denominator and simplify.	$= \frac{A+B(x-1)+C(x-1)^2}{(x-1)^3}$ $= \frac{A+Bx-B+Cx^2-2Cx+C}{(x-1)^3}$ $= \frac{Cx^2+(B-2C)x+A-B+C}{(x-1)^3}$
3 Equate coefficients of the numerator and solve for A, B, C.	$C = 2$ $B - 2C = -5$ $B - 4 = -5 \Rightarrow B = -1$ $A - B + C = 1 \Rightarrow A = -2$
4 Write as partial fractions.	$\frac{2x^2-5x+1}{(x-1)^3} = \frac{-2}{(x-1)^3} + \frac{-1}{(x-1)^2} + \frac{2}{x-1}$ $= \frac{2}{x-1} - \frac{1}{(x-1)^2} - \frac{2}{(x-1)^3}$

An **irreducible quadratic** expression has no real factors. Its discriminant is negative. $2x^2 - 3x + 2$ has discriminant $\Delta = b^2 - 4ac = (-3)^2 - 4 \times 2 \times 2 = -7$, so it is an irreducible quadratic.

A rational function with a factor $ax^2 + bx + c$ in the denominator can be written with the term $\frac{Ax + B}{ax^2 + bx + c}$ as one of its partial fractions.

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WORKED EXAMPLE 8 Partial fractions with a quadratic denominator

Write $\frac{x^2 - 14x - 11}{(x^2 + 5x + 6)(x - 1)}$ in the form $\frac{Ax + B}{x^2 + 5x + 6} + \frac{C}{x - 1}$.

Steps	Working
1 Write the desired form.	$\frac{Ax + B}{x^2 + 5x + 6} + \frac{C}{x - 1}$
2 Change to a common denominator.	$= \frac{(x - 1)(Ax + B) + C(x^2 + 5x + 6)}{(x^2 + 5x + 6)(x - 1)}$
3 Simplify the numerator.	$= \frac{Ax^2 - Ax + Bx - B + Cx^2 + 5Cx + 6C}{(x^2 + 5x + 6)(x - 1)}$ $= \frac{(A + C)x^2 + (B - A + 5C)x + 6C - B}{(x^2 + 5x + 6)(x - 1)}$
4 Write the identity.	$\frac{x^2 - 14x - 11}{(x^2 + 5x + 6)(x - 1)}$ $\equiv \frac{(A + C)x^2 + (B - A + 5C)x + 6C - B}{(x^2 + 5x + 6)(x - 1)}$
5 Equate coefficients of the numerator and solve for A, B and C.	$A + C = 1, B - A + 5C = -14, 6C - B = 11$ $6C - B = -11$ $+ \quad B - A + 5C = -14$ $11C - A = -25$ $+ \quad A + C = 1$ $12C = -24$ $\Rightarrow C = -2, A = 3, B = -1$
6 Write the desired form.	$\frac{x^2 - 14x - 11}{(x^2 + 5x + 6)(x - 1)} = \frac{3x - 1}{x^2 + 5x + 6} - \frac{2}{x - 1}$

In the above worked example, $x^2 + 5x + 6$ factorises to $(x + 2)(x + 3)$. This means it would be better to write $\frac{x^2 - 14x - 11}{(x^2 + 5x + 6)(x - 1)} = -\frac{2}{x - 1} - \frac{7}{x + 2} + \frac{10}{x + 3}$. It is preferred because it has simpler fractions. They can also be more easily found.

Partial fractions

A simple rational function with linear and irreducible quadratic factors in the denominator can be decomposed to a sum of partial fractions. The partial fractions are

- $\frac{A}{ax + b}$ for a simple factor $ax + b$
- $\frac{A}{(ax + b)^2} + \frac{B}{ax + b}$ for a square linear factor $(ax + b)^2$
- $\frac{A}{(ax + b)^3} + \frac{A}{(ax + b)^2} + \frac{B}{ax + b}$ for a cubic linear factor $(ax + b)^3$
- $\frac{Ax + B}{ax^2 + bx + c}$ for an irreducible quadratic factor $ax^2 + bx + c$.

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WORKED EXAMPLE 9 Partial fractions of rational functions

Write each of the following expression as partial fractions, without finding the numerators.

a $\dfrac{3x^3 - 4x + 2}{(3x-1)(x+2)^2(2x+3)}$ **b** $\dfrac{3x^2 + 2}{(x+3)(3x^2+2x+1)(2x+5)}$ **c** $\dfrac{4x^2 + 3x - 1}{(x-3)(2x+5)}$

Steps	Working
a The denominator has 2 simple linear factors and a square linear factor.	$\dfrac{3x^3 - 4x + 2}{(3x-1)(x+2)^2(2x+3)}$ $= \dfrac{A}{3x-1} + \dfrac{B}{(x+2)^2} + \dfrac{C}{x+2} + \dfrac{D}{2x+3}$
b 1 Check the quadratic in the denominator.	For $3x^2 + 2x + 1$, $\Delta = 4 - 12 = -8$, so the quadratic is irreducible.
2 There are also 2 simple linear factors in the denominator.	$\dfrac{3x^2 + 2}{(x+3)(3x^2+2x+1)(2x+5)}$ $= \dfrac{A}{x+3} + \dfrac{Bx + C}{3x^2+2x+1} + \dfrac{D}{2x+5}$
Parts **a** and **b** have denominators with degree 4 and numerators with a lower degree, so they are simple rational expressions. There is no quotient.	
c 1 Check the degrees of the denominator and numerator.	The denominator and numerator have the same degree so there will be a quotient of degree 0.
2 The denominator has 2 simple linear factors.	$\dfrac{4x^2 + 3x - 1}{(x-3)(2x+5)} = A + \dfrac{B}{x-3} + \dfrac{C}{2x+5}$

Where there is a quotient, we can divide the leading term of the numerator by the leading term of the denominator to find the leading term of the quotient. If the quotient is only a constant, we can still use substitution in the numerator to easily find other values.

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WORKED EXAMPLE 10 Mixed partial fractions

Write $\dfrac{6x^3 - 17x^2 + 9x + 7}{(x-2)^2(3x-1)}$ as partial fractions.

Steps	Working
1 There are 2 linear factors in the denominator, one squared. There could be a quotient of degree 0, as the top and bottom are both degree 3.	$\dfrac{6x^3 - 17x^2 + 9x + 7}{(x-2)^2(3x-1)} \equiv \dfrac{A}{(x-2)^2} + \dfrac{B}{(x-2)} + \dfrac{C}{3x-1} + D$
2 Find D from the top and bottom.	The leading terms are $6x^3$ and $3x^3$, so $D = 2$.
3 Change to a common denominator.	$\dfrac{A}{(x-2)^2} + \dfrac{B}{(x-2)} + \dfrac{C}{3x-1} + D$ $= \dfrac{A(3x-1) + B(x-2)(3x-1) + C(x-2)^2 + D(x-2)^2(3x-1)}{(x-2)^2(3x-1)}$
4 Numerators are identically equal.	$6x^3 - 17x^2 + 9x + 7$ $\equiv A(3x-1) + B(x-2)(3x-1) + C(x-2)^2 + D(x-2)^2(3x-1)$
5 Choose $x = 2$ and solve for A.	$6 \times 8 - 17 \times 4 + 9 \times 2 + 7 = A(3 \times 2 - 1) + 0 + 0 + 0$ $5A = 48 - 68 + 18 + 7 = 5$ $\Rightarrow A = 1$

6 Choose $x = \frac{1}{3}$ and solve for C.

$$6 \times \frac{1}{27} - 17 \times \frac{1}{9} + 9 \times \frac{1}{3} + 7 = 0 + 0 + C\left(\frac{1}{3} - 2\right)^2 + 0$$
$$\frac{2}{9} - \frac{17}{9} + 3 + 7 = \frac{25C}{9}$$
$$25C = 90 + 2 - 17$$
$$= 75 \Rightarrow C = 3$$

7 Choose $x = 0$ and substitute A, C and D to find B.

$$0 - 0 + 0 + 7$$
$$= 1 \times (-1) + B \times (-2) \times (-1) + 3 \times (-2)^2 + 2 \times (-2)^2 \times (-1)$$
$$= -1 + 2B + 12 - 8$$
$$\Rightarrow 7 = 2B + 3 \Rightarrow B = 2$$

8 Write the final expression.

$$\frac{6x^3 - 17x^2 + 9x + 7}{(x-2)^2(3x-1)} = \frac{1}{(x-2)^2} + \frac{2}{(x-2)} + \frac{3}{3x-1} + 2$$

When changing to partial fractions, choose the best method to suit the problem.

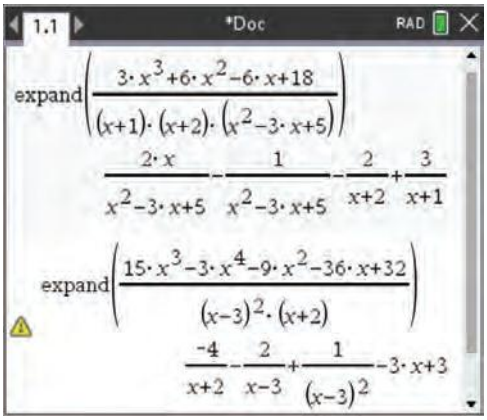
USING CAS 1 Partial fractions

Express each rational expression as a partial fraction.

a $\dfrac{3x^3 + 6x^2 - 6x + 18}{(x+1)(x+2)(x^2 - 3x + 5)}$

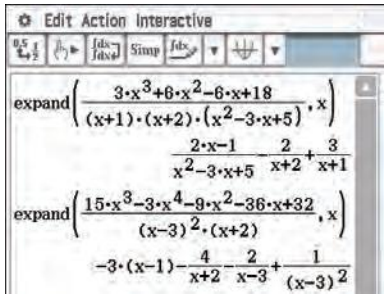
b $\dfrac{15x^3 - 3x^4 - 9x^2 - 36x + 32}{(x-3)^2(x+2)}$

TI-Nspire



1 Press **menu > Algebra > Expand**.
2 Enter the expression.
3 Press **enter**.

ClassPad



1 Enter and highlight the expression.
2 Tap **Interactive > Transformation > expand**.
3 In the dialogue box, tap **Partial Fraction**.
4 In the **Variable:** field, enter **x**.
5 Tap **OK**.

a $\dfrac{3x^3 + 6x^2 - 6x + 18}{(x+1)(x+2)(x^2 - 3x + 5)} = \dfrac{3}{x+1} - \dfrac{2}{x-3} + \dfrac{2x-1}{x^2 - 3x + 5}$

b $\dfrac{15x^3 - 3x^4 - 9x^2 - 36x + 32}{(x-3)^2(x+2)} = \dfrac{1}{(x-3)^2} - \dfrac{2}{x-3} - \dfrac{4}{x+2} - 3x + 3$

In the CAS example above, we need to reformat the answer for either calculator to write it in the usual form. It doesn't matter in which order the terms are given.

EXERCISE 8.2 Rational functions

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Recap

1 $f(x) = \log_3(x + 4)$. The maximal domain and range of $f(x)$ are

A domain = $R\backslash\{0\}$ and range = $(-4, \infty)$

B domain = R and range = $[0, \infty)$

C domain = $[3, \infty)$ and range = $[0, \infty)$

D domain = $(-4, \infty)$ and range = R

E domain = $[-3, \infty)$ and range = $[0\,\infty)$

2 The function $y = 5^x$ is dilated by the factor 3 from the x-axis and 2 from the y-axis, reflected in the x-axis and translated 1 unit up and 4 units to the left. The resulting function is

A $y = 4 - 5^{0.5x+1}$

B $y = 1 - 3 \times 5^{0.5x+2}$

C $y = 1 - 3 \times 5^{0.5x+4}$

D $y = 1 - 3 \times 5^{2x+4}$

E $y = 1 - 3 \times 5^{2x+8}$

Mastery

3 WORKED EXAMPLE 4 Use identities to find the remaining factor in each of the following.

a $(x + 2)$ is a factor of $3x^3 + x^2 - 4x + 12$

b $(x - 3)$ is a factor of $2x^4 - 5x^3 - 4x^2 - 4x + 21$

c $(2x + 3)$ is a factor of $2x^3 + 5x^2 + 11x + 12$

d $(3x - 4)$ is a factor of $12x^4 - 16x^3 + 9x^2 - 21x + 12$

e $(x^2 + x + 3)$ is a factor of $3x^3 - 2x^2 + 4x - 15$

f $(2x^2 - 3x + 4)$ is a factor of $2x^4 + 3x^3 + 7x^2 - 6x + 24$

4 WORKED EXAMPLES 5, 6 Use identities to express

a $\dfrac{14 - 5x - 2x^2}{x + 4}$ in the form $\dfrac{A}{x + 4} + Bx + C$

b $\dfrac{x^2 - 3x + 5}{x - 1}$ in the form $\dfrac{A}{x - 1} + Bx + C$

c $\dfrac{3x^3 - 5x^2 + 11x - 16}{x^2 + 3}$ in the form $\dfrac{Ax + B}{x^2 + 3} + Cx + D$

d $\dfrac{4x^3 - 10x^2 + 7x - 4}{x - 2}$ in the form $\dfrac{A}{x - 2} + Bx^2 + Cx + D$

e $\dfrac{5x + 7}{(x + 3)(x - 1)}$ in the form $\dfrac{A}{x + 3} + \dfrac{B}{x - 1}$

f $\dfrac{6x + 21}{(x - 4)(2x + 1)}$ in the form $\dfrac{A}{x - 4} + \dfrac{B}{2x - 1}$

g $\dfrac{4x^2 - 24x - 94}{(x + 2)(x + 3)(x - 4)}$ in the form $\dfrac{A}{x + 2} + \dfrac{B}{x + 3} + \dfrac{C}{x - 4}$

h $\dfrac{27 - 4x^2 - 14x}{(x - 1)(2x + 1)(3x - 4)}$ in the form $\dfrac{A}{x - 1} + \dfrac{B}{2x + 1} + \dfrac{C}{3x - 4}$.

5 WORKED EXAMPLE 7 Use identities to write each fraction as partial fractions.

a $\dfrac{x+2}{(x-1)^2}$ **b** $\dfrac{2x+3}{(x+4)^2}$ **c** $\dfrac{7x-3}{(3x+5)^2}$

d $\dfrac{2x^2+3x-2}{(x-4)^3}$ **e** $\dfrac{8x^2-24x+15}{(2x-3)^3}$ **f** $\dfrac{3x-2}{(3x+1)^3}$

6 WORKED EXAMPLE 8 Express each fraction as partial fractions.

a $\dfrac{3x^2+2x+6}{(x^2+4)(x-3)}$ **b** $\dfrac{x^2+12x+9}{(x^2+x+3)(x+3)}$

c $\dfrac{14x^2-11x-5}{(3x^2-4x+4)(4x-3)}$ **d** $\dfrac{x^2-12x+12}{(2x^2-3x+2)(3x+1)}$

7 WORKED EXAMPLE 9 Write these expressions as partial fractions with constants A, B, C without finding the numerators.

a $\dfrac{4x^2-3x+1}{(x+1)^2(3x-5)}$ **b** $\dfrac{x^3-2x^2-3x-4}{(2x-5)^2(3x^2+6x+5)}$

c $\dfrac{5x^3+2x^2+x+6}{(x-3)^2(x+4)}$ **d** $\dfrac{3x^3-x^2+4x-5}{(x+2)(x-3)}$

e $\dfrac{x^4+x^2-2x+3}{(x-4)^2(x^2-2x+3)}$ **f** $\dfrac{6x^4-15x^3-32x+17}{(x+5)^2(2x-3)(2x^2-5x+4)}$

g $\dfrac{7x^4-x^2-3x-8}{(5x-4)(x^2-x+2)}$ **h** $\dfrac{5x^4-3x^3+x^2-10x-13}{(2x-3)^2(4-5x)}$

8 WORKED EXAMPLE 10 Write each fraction as partial fractions.

a $\dfrac{7x+4}{(2x-7)(3x-1)}$ **b** $\dfrac{13x+11}{3x^2-8x-3}$

c $\dfrac{9x^2+5x-16}{(x+5)(3x^2-3x+2)}$ **d** $\dfrac{64x^2-6x^3-212x+217}{(x-1)^2(2x-5)^2}$

e $\dfrac{3x^4-17x^3+30x^2-11x+1}{(x-1)^2(x^2-4x+5)(2x+1)}$ **f** $\dfrac{5x^3+3x^2-57x+80}{(x-2)^2(x+4)}$

g $\dfrac{6x^4-39x^3+77x^2-27x+66}{(x-4)^2(3x^2-x+2)}$ **h** $\dfrac{2x^4-2x^3-43x^2+x+25}{x^2-x-20}$

i $\dfrac{x^5-6x^4+8x^3-16x^2+5x-18}{(x^2-2x+3)(x-5)}$ **j** $\dfrac{12x^4-4x^5+35x^3-38x^2-100x-41}{(2x+3)^2(x-2)}$

k $\dfrac{4x^3-2x^2+3x+4}{(x-3)(3x+1)}$ **l** $\dfrac{3x^4-x^2+3x-4}{(2x-5)(x^2+1)}$

9 Using CAS 1 Express each fraction as partial fractions.

a $\dfrac{43 - 32x - 6x^2}{(x+3)(x-2)(4x-5)}$

b $\dfrac{3x^2 - 64x + 108}{(4x+3)(3x-5)^2}$

c $\dfrac{3x^3 + 53x^2 + 151x + 136}{(x-1)(2x+5)(3x+4)^2}$

d $\dfrac{29x^2 - x^4 - 4x^3 + 71x - 221}{(x-3)(x+5)^2}$

e $\dfrac{x^2 + 30x - 10}{(x^2 - 3x + 5)(4x+1)}$

f $\dfrac{48x^4 + 120x^3 + 139x^2 + 101x + 37}{(4x^2 + 3x + 2)(4x+3)}$

g $\dfrac{57x^4 - 10x^5 - 93x^3 + 63x^2 - 12x - 1}{(5x^2 - x + 1)(2x-3)}$

h $\dfrac{255x^3 - 162x^4 + 82x^2 - 277x + 100}{(6x^2 - 10x + 5)(x+4)(3x-2)^2}$

Exam practice

10 © VCAA 2020 2AQ7 MODIFIED For non-zero real constants a and b, where $b > 0$, the expression $\dfrac{1}{(x+a)^2(x^2+b)}$ in partial fraction form with linear denominators, where A, B, C and D are real constants, is

A $\dfrac{Ax+B}{(x+a)^2} + \dfrac{Cx+D}{x^2+b}$

B $\dfrac{Ax+B}{(x+a)^2} + \dfrac{C}{x+\sqrt{b}} + \dfrac{D}{x-\sqrt{b}}$

C $\dfrac{A}{(x+a)^2} + \dfrac{B}{x+a} + \dfrac{C}{x+\sqrt{b}} + \dfrac{D}{x-\sqrt{b}}$

D $\dfrac{A}{(x+a)^2} + \dfrac{B}{x+a} + \dfrac{Cx+D}{x^2+b}$

E $\dfrac{A}{x^2} + \dfrac{B}{x+a} + \dfrac{C}{x+\sqrt{b}} + \dfrac{D}{x-\sqrt{b}}$

11 © VCAA 2021N 2AQ4 The expression $\dfrac{ax+b}{(2x-1)^2(x+1)}$ has partial fraction form $\dfrac{1}{(x+1)} - \dfrac{2}{(2x-1)} - \dfrac{1}{(2x-1)^2}$.

The values of a and b, where a and b are non-zero real constants, are respectively

A 12 and 21 **B** 7 and 16 **C** −5 and 4 **D** −7 and 2 **E** 3 and 6

12 © VCAA VCAA 2017N 2AQ5 Given that A, B, C and D are non-zero rational numbers, the expression $\dfrac{3x+1}{x(x-2)^2}$ can be represented in partial fraction form as

A $\dfrac{A}{x} + \dfrac{B}{(x-2)}$

B $\dfrac{A}{x} + \dfrac{B}{(x-2)^2}$

C $\dfrac{A}{x} + \dfrac{B}{(x-2)} + \dfrac{C}{(x-2)^2}$

D $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{(x-2)}$

E $\dfrac{A}{x} + \dfrac{Bx}{(x-2)} + \dfrac{Cx+D}{(x-2)^2}$

13 © VCAA 2018 2AQ3 MODIFIED Which one of the following, where A, B, C and D are non-zero real numbers, is the partial fraction form for the expression $\dfrac{x^2 - 2x - 3}{(3x-2)^3(x^2-1)}$?

A $\dfrac{A}{3x-2} + \dfrac{B}{x+1} + \dfrac{C}{x-1}$

B $\dfrac{A}{(3x-2)^2} + \dfrac{B}{3x-2} + \dfrac{C}{x+1} + \dfrac{D}{x-1}$

C $\dfrac{A}{(3x-2)^3} + \dfrac{B}{(3x-2)^2} + \dfrac{C}{3x-2} + \dfrac{D}{x-1}$

D $\dfrac{A}{(3x-2)^3} + \dfrac{B}{(3x-2)^2} + \dfrac{C}{3x-2} + \dfrac{D}{x^2-1}$

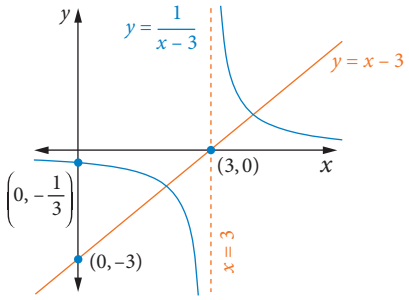
E $\dfrac{Ax+B}{(3x-2)^3} + \dfrac{C}{3x-2} + \dfrac{D}{x-1}$

8.3 Reciprocal functions

Reciprocal linear functions

Consider the linear function $f(x) = x - 3$. It has a zero at $x = 3$.

The reciprocal function is $g(x) = \dfrac{1}{x-3}$. This is a rectangular **hyperbola**.



It has a vertical asymptote at $x = 3$.

For $x < 3$:

$f(x) < 0$ and $g(x) < 0$ and

as $x \to 3$, $f(x) \to 0$ from below and $g(x) \to -\infty$.

For $x > 3$:

$f(x) > 0$ and $g(x) > 0$ and

as $x \to 3$, $f(x) \to 0$ from above and $g(x) \to \infty$.

Considering any function $f(x)$ and its reciprocal $g(x) = \dfrac{1}{f(x)}$,

- the signs of $f(x)$ and $g(x)$ are the same
- the zeros of $f(x)$ correspond to vertical asymptotes of $g(x)$
- as $f(x) \to \pm\infty$, $g(x) \to 0$
- as $f(x) \to 0$, $g(x) \to \pm\infty$.

We can use $f(x) \uparrow 0$ or $f(x) \to 0^-$ to mean $f(x)$ approaches 0 from the negative side.

Similarly, $f(x) \downarrow 0$ and $f(x) \to 0^+$ both mean that $f(x)$ approaches 0 from the positive side.

Video playlist Reciprocal functions

Worksheet Reciprocal functions

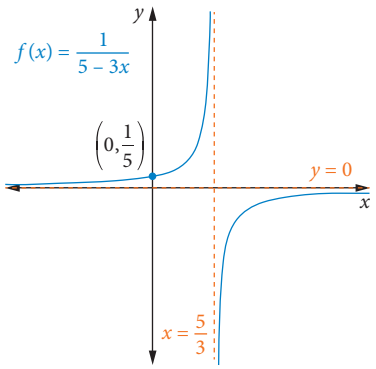
WORKED EXAMPLE 11 Reciprocal linear functions

Sketch the graph of $f(x) = \dfrac{1}{5-3x}$.

Steps	Working
1 Consider where the linear denominator $5 - 3x$ is zero.	$5 - 3x = 0$ at $x = \dfrac{5}{3}$, so $f(x)$ has a vertical asymptote at $x = \dfrac{5}{3}$.
2 Consider the signs of $5 - 3x$.	$5 - 3x$ and $f(x) < 0$ for $x > \dfrac{5}{3}$ $5 - 3x$ and $f(x) > 0$ for $x < \dfrac{5}{3}$
3 Find the y-intercept.	$f(0) = \dfrac{1}{5}$
4 Consider the behaviour as $x \to \pm\infty$.	As $x \to \infty$, $5 - 3x \to -\infty$, so $f(x) \to 0^-$. As $x \to -\infty$, $5 - 3x \to \infty$, so $f(x) \to 0^+$. The x-axis is a horizontal asymptote.

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5 Sketch the graph.



Reciprocal quadratic functions

Quadratic functions have a maximum or minimum. The reciprocal quadratic function will have a corresponding minimum or maximum respectively.

The exact nature of a reciprocal quadratic function depends on the nature of the quadratic function.

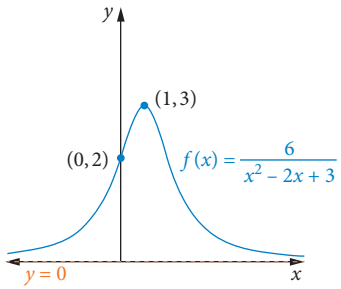
For quadratics that have no real zeros, the reciprocal quadratic function will have no vertical asymptotes. A positive number instead of 1 in the numerator of a reciprocal function makes no difference to asymptotes. It does change the value of a minimum or maximum and the y-intercept.

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WORKED EXAMPLE 12 Reciprocal quadratic functions with no real zeros

Sketch the graph of $f(x) = \dfrac{6}{x^2 - 2x + 3}$.

Steps	Working
1 Consider whether the quadratic denominator $x^2 - 2x + 3$ has zeros.	For $x^2 - 2x + 3$, $\Delta = b^2 - 4ac = -8 < 0$, so there are no real zeros.
2 Find the minimum of $x^2 - 2x + 3$ by completing the square or using $x = \dfrac{-b}{2a}$.	$x^2 - 2x + 3 = (x^2 - 2x + 1) + 2$ $= (x - 1)^2 + 2$, so the denominator has a minimum at $(1, 2)$.
3 Multiply the numerator of $f(x)$ by the y value of the reciprocal of the minimum of the denominator to get the maximum.	$f(x)$ has a maximum at $\left(1, 6 \times \dfrac{1}{2}\right) = (1, 3)$.
4 Find the y-intercept of $f(x)$.	$f(0) = 2$
5 Consider the behaviour as $x \to \pm\infty$.	As $x \to \pm\infty$, $x^2 - 2x + 3 \to \infty$, so $f(x) \to 0^+$. $y = 0$ (the x-axis) is a horizontal asymptote.
6 Sketch the graph.	

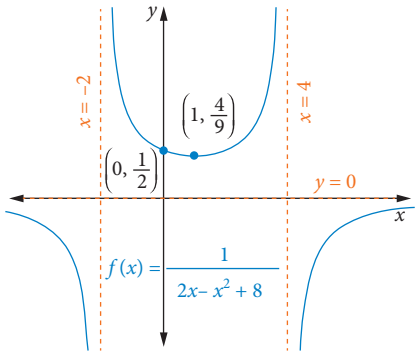
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For quadratics that have real zeros, the reciprocal quadratic functions have vertical asymptotes.

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WORKED EXAMPLE 13 Reciprocal quadratic functions with real zeros

Sketch the graph of $f(x) = \dfrac{4}{2x - x^2 + 8}$.

Steps	Working
1 Consider whether the quadratic denominator $2x - x^2 + 8$ has zeros.	$\Delta = b^2 - 4ac = 36 > 0$, so there are zeros. $2x - x^2 + 8 = -(x + 2)(x - 4)$, so $f(x)$ has vertical asymptotes at $x = -2$ and $x = 4$.
2 Consider the sign(s).	$2x - x^2 + 8$ and $f(x) > 0$ for $-2 < x < 4$ $2x - x^2 + 8$ and $f(x) < 0$ for $x < -2$ or $x > 4$
3 Find the maximum of $2x - x^2 + 8$ by completing the square or using $x = \dfrac{-b}{2a}$.	$\dfrac{-b}{2a} = \dfrac{-2}{2 \times -1} = 1$, $f(1) = 9$, so $2x - x^2 + 8$ has a maximum at $(1, 9)$.
4 Multiply the numerator of $f(x)$ by the y value of the reciprocal of the maximum of the denominator to get the minimum.	$f(x)$ has a minimum at $\left(1, 4 \times \dfrac{1}{9}\right) = \left(1, \dfrac{4}{9}\right)$.
5 Find the y-intercept of $f(x)$.	$f(0) = \dfrac{1}{2}$
6 Consider the behaviour as $x \to \pm\infty$.	As $x \to -\infty$, $2x - x^2 + 8 \to -\infty$, so $f(x) \uparrow 0$. As $x \to \infty$, $2x - x^2 + 8 \to -\infty$, so $f(x) \uparrow 0$. The x-axis is a horizontal asymptote.
7 Sketch the graph.	

Reciprocal polynomial functions

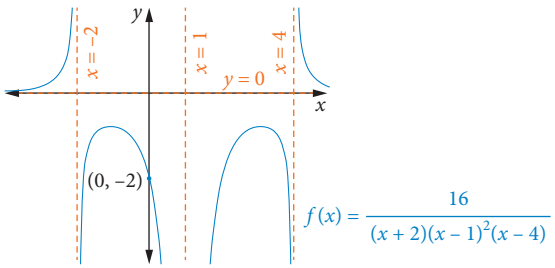
To sketch reciprocal polynomial functions, use the same methods as for reciprocal quadratic functions. We will rarely be able to find the maxima or minima.

WB

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WORKED EXAMPLE 14 Reciprocal polynomial functions 1

Sketch the graph of $f(x) = \dfrac{16}{(x+2)(x-1)^2(x-4)}$.

Steps	Working
1 Consider whether the polynomial denominator has zeros.	$(x+2)(x-1)^2(x-4)$ has zeros and $f(x)$ has vertical asymptotes at $x = -2$, $x = 1$ and $x = 4$.
2 Consider the sign(s).	$(x+2)(x-1)^2(x-4)$ and $f(x) > 0$ for $x < -2$ and $x > 4$ $(x+2)(x-1)^2(x-4)$ and $f(x) < 0$ for $-2 < x < 1$ $(x+2)(x-1)^2(x-4)$ and $f(x) < 0$ for $1 < x < 4$
3 Consider the behaviour as $x \to \pm\infty$.	As $x \to -\infty$, $(x+2)(x-1)^2(x-4) \to \infty$, so $f(x) \to 0^+$. As $x \to \infty$, $(x+2)(x-1)^2(x-4) \to \infty$, so $f(x) \to 0^+$. The x-axis is a horizontal asymptote.
4 Find the y-intercept.	$f(0) = -2$
5 Sketch the graph.	

General reciprocal functions

We can sketch a reciprocal function directly from the graph of the original function.

General reciprocal functions

For any function $f(x)$ and its reciprocal $g(x) = \dfrac{1}{f(x)}$,

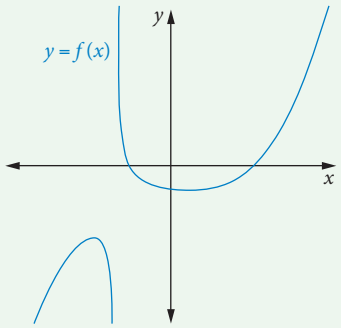
- the signs of $f(x)$ and $g(x)$ are the same
- the zeros of $f(x)$ correspond to vertical asymptotes of $g(x)$
- vertical asymptotes of $f(x)$ correspond to zeros of $g(x)$
- as $f(x) \to \pm\infty$, $g(x) \to 0$
- as $f(x) \to 0$, $g(x) \to \pm\infty$
- maxima and minima of $f(x)$ correspond to minima and maxima of $g(x)$ respectively
- the y-intercept of $g(x)$ is the reciprocal of the y-intercept of $f(x)$
- for $f(x)$ concave up, $g(x)$ is concave down
- for $f(x)$ concave down, $g(x)$ is concave up
- the domain of $g(x)$ is the same as that of $f(x)$, except that zeros of $f(x)$ are excluded from the domain of $g(x)$.

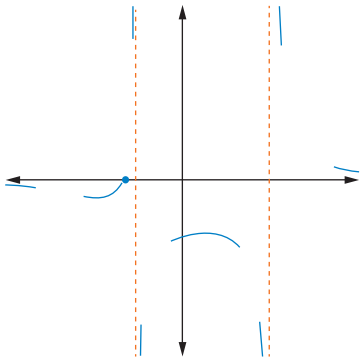
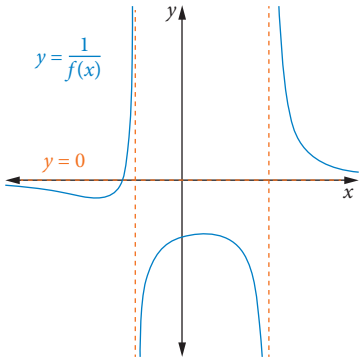
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WORKED EXAMPLE 15 Reciprocal polynomial functions 2

The graph of $f(x)$ is shown on the right.

Sketch the graph of $\frac{1}{f(x)}$.



Steps	Working
1 Look for the relevant features of $f(x)$.	$f(x)$ has 2 zeros (x-intercepts). $f(x)$ appears to have a vertical asymptote a little before the first zero. $f(x)$ has a local maximum and a local minimum. As $x \to \infty, f(x) \to \infty$; as $x \to -\infty, f(x) \to -\infty$. As x approaches the asymptote, $f(x) \to \pm\infty$.
2 Put vertical asymptotes on the new graph instead of the zeros, a zero instead of the vertical asymptote, a minimum instead of a maximum, and a maximum instead of a minimum on the graph of $\frac{1}{f(x)}$. Show $\frac{1}{f(x)} \to 0$ with the same sign where $f(x) \to \pm\infty$.	
3 Join up the lines, making sure the new graph is concave up where the original is concave down and vice versa. Make sure that the signs are the same. Write the equation of the horizontal asymptote. Write the function name, $y = \frac{1}{f(x)}$.	

EXERCISE 8.3 Reciprocal functions

ANSWERS p. 587

Recap

1 The expression $\frac{3x^4 + 2x^3 - 5x + 1}{(x+3)(x^2-1)}$ rewritten as partial fractions is

A $\frac{A}{x+3} + \frac{Bx+c}{x^2-1} + Dx + E$

B $\frac{A}{x+3} + \frac{B}{x^2-1} + \frac{C}{x-1} + D$

C $\frac{A}{x+3} + \frac{B}{x+1} + \frac{C}{x-1} + D$

D $\frac{A}{x+3} + \frac{B}{x+1} + \frac{C}{x-1} + Dx + E$

E $\frac{A}{x+3} + \frac{B}{x+1} + \frac{C}{x-1} + Dx^2 + Ex + F$

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2 The expression $\frac{ax^2 + bx + c}{(x-3)^2(x^2+3)}$ has partial fraction form $\frac{1}{(x-3)^2} + \frac{1}{x-3} - \frac{x+3}{x^2+3}$.
The values of a, b and c are, respectively

A 1, 12 and −33 **B** 13, −60 and 75 **C** −13, 75 and 30
D −1, 11 and −18 **E** −1, −12 and 33

Mastery

3 WORKED EXAMPLE 11 Sketch the graph of each function.

a $f(x) = \frac{1}{x+2}$ **b** $f(x) = \frac{1}{4-x}$ **c** $f(x) = \frac{1}{2x-3}$ **d** $f(x) = \frac{1}{4-3x}$

e $f(x) = \frac{10}{2x+5}$ **f** $f(x) = \frac{4}{4-3x}$ **g** $f(x) = \frac{6}{2x-5}$ **h** $f(x) = \frac{3}{1-4x}$

4 WORKED EXAMPLES 12, 13 Sketch the graph of each function.

a $f(x) = \frac{1}{x^2+6x+11}$ **b** $f(x) = \frac{1}{6x-x^2-5}$ **c** $f(x) = \frac{36}{2x^2-5x-12}$

d $f(x) = \frac{18}{2x^2-4x+9}$ **e** $f(x) = \frac{5}{6x-3x^2+18}$ **f** $f(x) = \frac{5}{3x^2+x-2}$

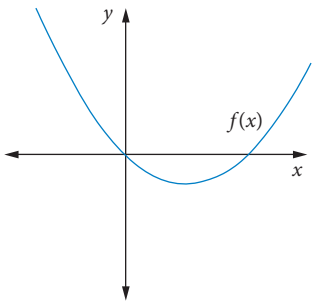
5 WORKED EXAMPLE 14 Sketch the graph of each function.

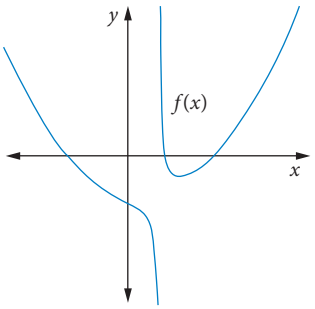
a $f(x) = \frac{1}{(x+5)(x+2)(x-3)}$ **b** $f(x) = \frac{1}{(3-x)(x+2)(x+3)}$

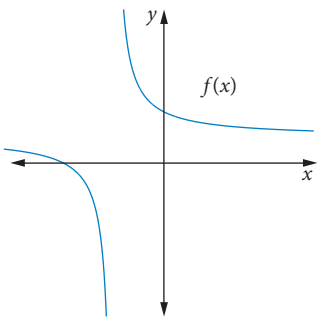
c $f(x) = \frac{45}{(x-5)(x+1)(x+3)}$ **d** $f(x) = \frac{15}{(x+1)^2(x-1)(x-3)}$

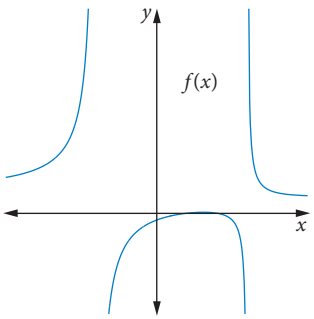
e $f(x) = \frac{18}{(1-x^2)(1+x)(4-x)}$ **f** $f(x) = \frac{15}{(x+3)^2(1-x)(5-2x)}$

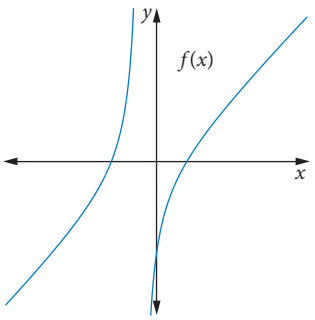
6 WORKED EXAMPLE 15 Sketch the graph of the reciprocal of each function.

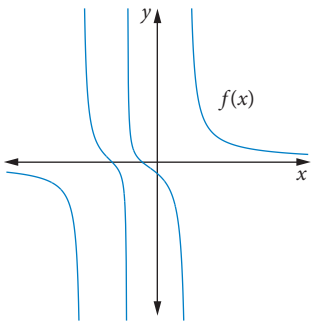
a


b


c


d


e


f


Exam practice

80–100% 60–79% 0–59%

7 © VCAA 2019N 2BQ2a TECH-FREE (2 marks) Consider the function f with rule $f(x) = \dfrac{x^2 + x + 1}{x^2 - 1}$. State the equations of the asymptotes of the graph of f.

8 © VCAA 2006 1Q3a 62% TECH-FREE (3 marks) Sketch the graph with equation $y = \dfrac{36}{2x^2 - 18}$, clearly indicating the location of any asymptotes and intercepts with the axes.

9 © VCAA 2019N 2AQ1 The graph of which one of the following relations does **not** have a vertical asymptote?

A $y = \dfrac{x^3 - 1}{x}$ **B** $y = \dfrac{5x^2 + 2}{x^2 + 1}$ **C** $y = \dfrac{x^4 - 3}{x^2}$ **D** $y = \dfrac{1}{x^2 + 4x}$ **E** $y = \dfrac{x - 1}{\sqrt{x + 2}}$

10 © VCAA 2006 2AQ2 92% The graph of the function with rule $f(x) = \dfrac{1}{(x - 4)(x + 2)}$ over its maximal domain has

A asymptotes $x = -4$ and $x = 2$ and a turning point at $(1, -5)$

B asymptotes $x = -4$ and $x = 2$ and a turning point at $\left(-1, -\dfrac{1}{5}\right)$

C asymptotes $x = 4$ and $x = 2$ and a turning point at $\left(1, -\dfrac{1}{9}\right)$

D asymptotes $x = 4$ and $x = -2$ and a turning point at $(1, -9)$

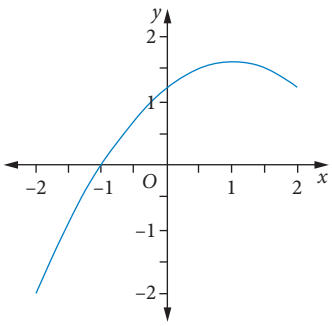
E asymptotes $x = 4$ and $x = -2$ and a turning point at $\left(1, -\dfrac{1}{9}\right)$

11 © VCAA 2012 2AQ1 85% The graph with equation $y = \dfrac{1}{2x^2 - x - 6}$ has asymptotes given by

A $x = -\dfrac{3}{2}$, $x = 2$ and $y = 1$ **B** $x = -\dfrac{3}{2}$ and $x = 2$ only **C** $x = \dfrac{3}{2}$, $x = -2$ and $y = 0$

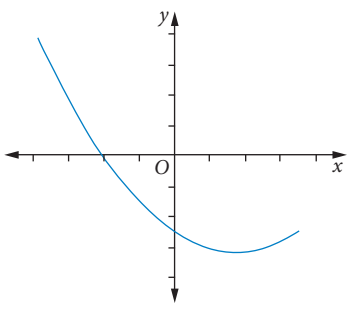
D $x = -\dfrac{3}{2}$, $x = 2$ and $y = 0$ **E** $x = \dfrac{3}{2}$ and $x = -2$ only

12 © VCAA 2012 2AQ3 66% The graph of $y = f(x)$ is shown on the right.

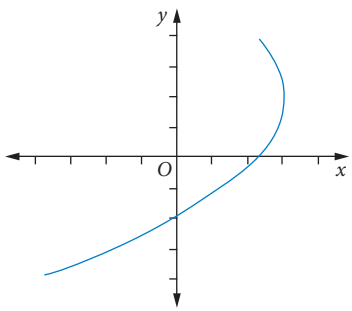


The graph of $y = \dfrac{1}{f(x)}$ is best represented by

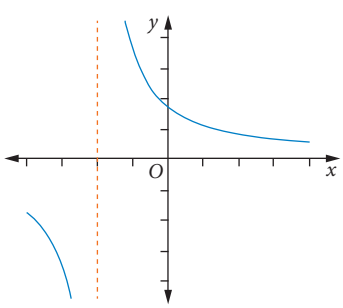
A



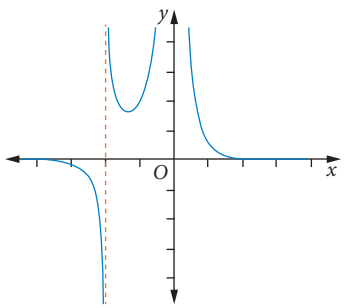
B



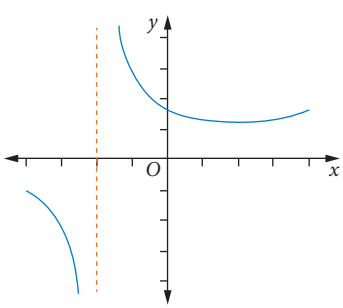
C



D



E



Video playlist
Reciprocal circular functions

8.4 Reciprocal circular functions

The sine, cosine and tangent functions have matching reciprocal functions called **secant**, **cosecant** and **cotangent**.

Reciprocal circular functions

The **secant** function is defined as

$$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{1}{x}$$

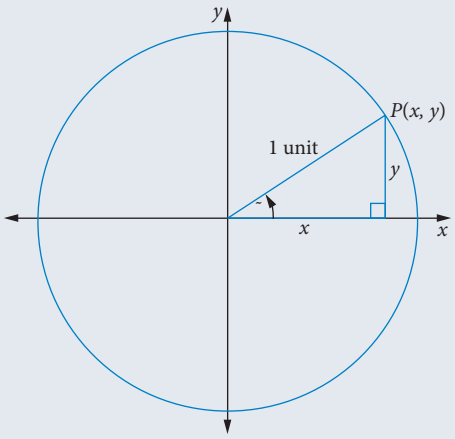
The **cosecant** function is defined as

$$\operatorname{cosec}(\theta) = \frac{1}{\sin(\theta)} = \frac{1}{y}$$

The **cotangent** function is defined as

$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\cos(\theta)}{\sin(\theta)} = \frac{x}{y}$$

where x and y are the coordinates of the point P on the unit circle wth angle θ, as shown on the right.

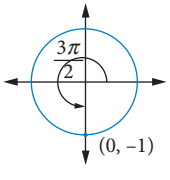


We can work out the values of $\sec(\theta)$, $\operatorname{cosec}(\theta)$ and $\cot(\theta)$ from the corresponding values of $\cos(\theta)$, $\sin(\theta)$ and $\tan(\theta)$. We can find exact values for some angles but, in most cases, we need to use approximations.

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WORKED EXAMPLE 16 Exact values of sec, cosec and cot

Find the values of **a** $\sec\left(\frac{\pi}{3}\right)$ **b** $\operatorname{cosec}\left(\frac{7\pi}{4}\right)$ **c** $\cot\left(\frac{3\pi}{2}\right)$

Steps	Working
a 1 State the value of $\cos\left(\frac{\pi}{3}\right)$.	$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$
2 Find the reciprocal.	$\sec\left(\frac{\pi}{3}\right) = \frac{1}{\cos\left(\frac{\pi}{3}\right)} = \frac{1}{0.5} = 2$
b 1 Use ASTC for $\sin\left(\frac{7\pi}{4}\right)$.	$\sin\left(\frac{7\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right)$ (4th quadrant)
2 Find the value of $\sin\left(\frac{7\pi}{4}\right)$.	$= -\frac{1}{\sqrt{2}}$
3 Find the reciprocal.	$\operatorname{cosec}\left(\frac{7\pi}{4}\right) = \frac{1}{-\frac{1}{\sqrt{2}}} = -\sqrt{2}$
c 1 For multiples of $\frac{\pi}{2}$, find the coordinates on the unit circle.	In the unit circle, $x = 0, y = -1$ 
2 Use the definition.	$\cot\left(\frac{3\pi}{2}\right) = \frac{x}{y} = \frac{0}{-1} = 0$

Dividing Pythagorean identity by $\sin^2(\theta)$ or $\cos^2(\theta)$ gives extra identities.

Pythagorean identities

For any angle θ:

- $\sin^2(\theta) + \cos^2(\theta) = 1$
- $\tan^2(\theta) + 1 = \sec^2(\theta)$
- $1 + \cot^2(\theta) = \operatorname{cosec}^2(\theta)$

WORKED EXAMPLE 17 Exact values using identities

Given $\sec(x) = \sqrt{6}$ and $\frac{3\pi}{2} < x < 2\pi$, find $\operatorname{cosec}(x)$ and $\cot(x)$.

Steps	Working
1 Use a Pythagorean identity to find $\tan(x)$.	$\tan^2(x) + 1 = \sec^2(x)$ $\tan^2(x) + 1 = 6$ $\tan(x) = -\sqrt{5}$ (4th quadrant)
2 Now find $\cot(x)$.	$\cot(x) = \frac{1}{\tan(x)} = -\frac{1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$
3 Use another identity to find $\operatorname{cosec}(x)$.	$1 + \cot^2(x) = \operatorname{cosec}^2(x)$ $1 + \frac{1}{5} = \operatorname{cosec}^2(x)$ $\operatorname{cosec}(x) = -\frac{\sqrt{6}}{\sqrt{5}} = -\frac{\sqrt{30}}{5}$ (4th quadrant)
4 Write the answer.	$\operatorname{cosec}(x) = -\frac{\sqrt{30}}{5}$ and $\cot(x) = -\frac{\sqrt{5}}{5}$

Exam hack

You can use the Pythagorean identities to find any trigonometric value from a given value for the same angle, provided that you know the quadrant, otherwise the sign will be unknown.

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Basic graphs

We can sketch the graphs of $\operatorname{cosec}(x)$, $\sec(x)$ and $\cot(x)$ as reciprocals.

$\sin(x)$ has *zeros* at $-\pi$, 0, π, 2π, 3π etc., so $\operatorname{cosec}(x) = \frac{1}{\sin(x)}$ has *vertical asymptotes* at $n\pi$ for $n \in Z$.

$\sin(x)$ has *maxima* of $(x, 1)$ at $-\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}$ etc., so $\operatorname{cosec}(x)$ has *minima* of $(x, 1)$ at $x = (4n + 1)\frac{\pi}{2}$ for $n \in Z$.

Minima of $\sin(x)$ correspond to *maxima* of $\operatorname{cosec}(x)$.

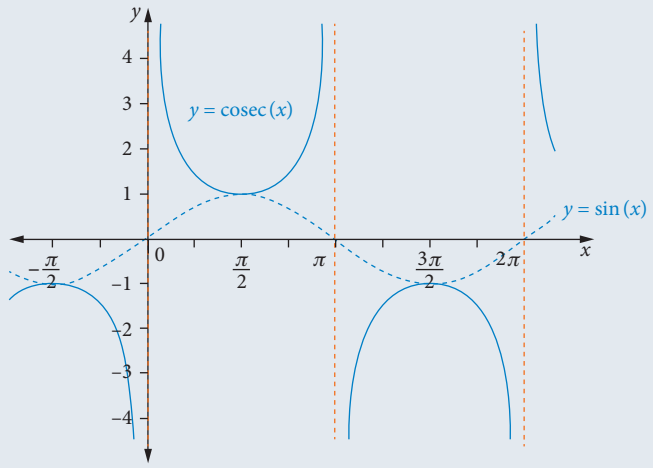
The signs of $\sin(x)$ and $\operatorname{cosec}(x)$ are the same for all values of x (except for zeros/asymptotes).

The same reasoning applies to the graphs of $\cos(x)$/$\sec(x)$ and $\tan(x)$/$\cot(x)$.

Additionally, where $\tan(x)$ has *vertical asymptotes*, $\cot(x)$ has *zeros*.

Properties and graphs of the reciprocal circular functions

$y = \text{cosec}(x)$



Period: 2π

Domain: $R\backslash\{n\pi, n \in Z\}$

Range: $(-\infty,-1] \cup [1,\infty)$

Asymptotes: $x = n\pi, n \in Z$

No zeros

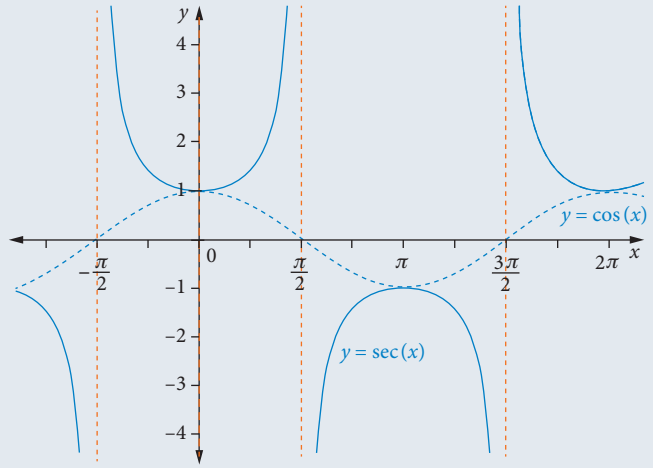
Minima: $-\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2} \ldots$

or $\left((4n+1)\frac{\pi}{2}, 1\right), n \in Z$

Maxima: $-\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2} \ldots$

or $\left((4n-1)\frac{\pi}{2}, -1\right), n \in Z$

$y = \sec(x)$



Period: 2π

Domain: $R\backslash\left\{(2n+1)\frac{\pi}{2} : n \in Z\right\}$

Range: $(-\infty,-1] \cup [1,\infty)$

Asymptotes: $-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2} \ldots$

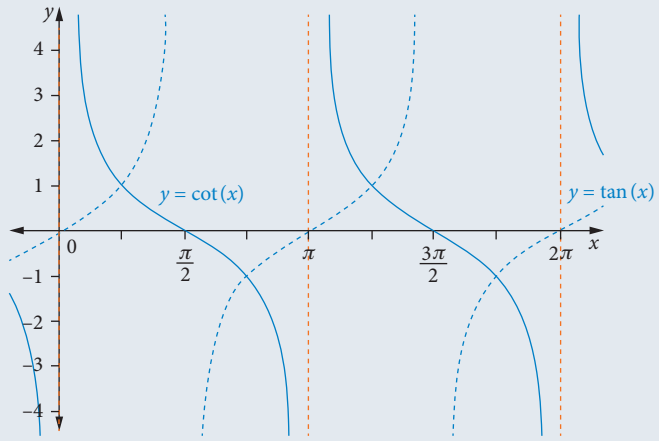
or $(2n+1)\frac{\pi}{2}, n \in Z$

No zeros

Minima: $-2\pi, 0, 2\pi, 4\pi \ldots$
or $(2n\pi, 1), n \in Z$

Maxima: $-\pi, \pi, 3\pi \ldots$
or $((2n+1)\pi, -1), n \in Z$

$y = \cot(x)$



Period: π

Domain: $R\backslash\{n\pi : n \in Z\}$

Range: R

Asymptotes: $-\pi, 0, \pi, 2\pi, 3\pi \ldots$
or $n\pi, n \in Z$

Zeros (centres): $\frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots$

$x = (2n+1)\frac{\pi}{2}, n \in Z$

Other: $y = 1$ at $\frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots$

or $\left((4n+1)\frac{\pi}{4}, 1\right), n \in Z$

$y = -1$ at $\frac{-\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4} \ldots$

or $\left((4n-1)\frac{\pi}{4}, -1\right), n \in Z$

Transformations of graphs

We can sketch graphs of more complex reciprocal trigonometric functions in different ways.

We can do them as reciprocals of the corresponding $\sin(x)$, $\cos(x)$ or $\tan(x)$ graphs, but it is generally easier to use the basic reciprocal function graphs.

To use transformations, we should express them in the form $af(n(x+b))+c$, where $f(x)$ is $\operatorname{cosec}(x)$, $\sec(x)$ or $\cot(x)$. Then transform the graphs in the order of dilations from the x and y axes by the factors a and $\frac{1}{n}$ respectively, followed by reflections and then translations by $-b$ and c parallel to the x and y axes.

For circular functions:

- the **period** is $\frac{2\pi}{n}$ for sin, cos, sec and cosec functions; and $\frac{\pi}{n}$ for tan and cot functions
- a is the **amplitude** for sin and cos; $(x, 1)$ becomes (x, a) for all the functions
- b is the **phase shift**; earlier (to the left) for $b > 0$, later (to the right) for $b < 0$
- c is the **mean (central) value**; (x, a) becomes $(x, a + c)$ for all the functions.

We can use the transformations to find the important features to sketch the graph.

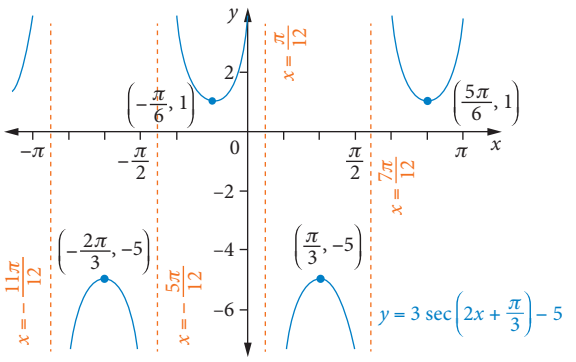
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WORKED EXAMPLE 18 Sketching a graph as a reciprocal function

Sketch the graph of $f(x) = 3\sec\left(2x + \frac{\pi}{3}\right) - 2$ for $-\pi \le x \le \pi$.

Steps	Working
1 At the asymptotes, $\cos\left(2x + \frac{\pi}{3}\right) = 0$.	For asymptotes, $2x + \frac{\pi}{3} = (2n+1)\frac{\pi}{2}$ $2x = (2n+1)\frac{\pi}{2} - \frac{\pi}{3}$ $x = (2n+1)\frac{\pi}{4} - \frac{\pi}{6}$
2 Restrict to $-\pi \le x \le \pi$.	Asymptotes are at $x = -\frac{11\pi}{12}, -\frac{5\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}$.
3 The minima of f are at the maxima of $\cos(x)$, so $\cos\left(2x + \frac{\pi}{3}\right) = 1$.	Minima are at $2x + \frac{\pi}{3} = 2n\pi$ $2x = 2n\pi - \frac{\pi}{3}$ $x = n\pi - \frac{\pi}{6}$
4 Restrict to $-\pi \le x \le \pi$.	Minima are at $-\frac{\pi}{6}, \frac{5\pi}{6}$, with values $3 \times 1 - 2 = 1$, so minima are $\left(-\frac{\pi}{6}, 1\right), \left(\frac{5\pi}{6}, 1\right)$.
5 The maxima of f are at the minima of $\cos(x)$: $\cos\left(2x + \frac{\pi}{3}\right) = -1$.	Maxima are at $2x + \frac{\pi}{3} = (2n+1)\pi$ $2x = (2n+1)\pi - \frac{\pi}{3}$ $x = (2n+1)\frac{\pi}{2} - \frac{\pi}{6}$
6 Restrict to $-\pi \le x \le \pi$.	Maxima are at $-\frac{2\pi}{3}, \frac{\pi}{3}$, with values $3 \times -1 - 2 = -5$, so maxima are $\left(-\frac{2\pi}{3}, -5\right), \left(\frac{\pi}{3}, -5\right)$.

7 Place the important features and complete the sketch.



Exam hack

Put the exact coordinates and equations of important features on a sketch graph unless you are asked for approximations.

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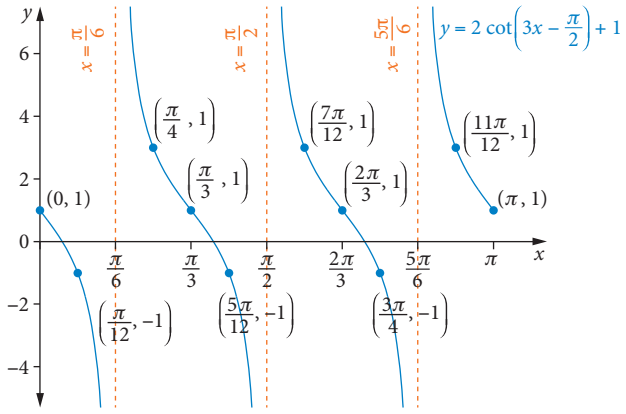
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WORKED EXAMPLE 19 Sketching a graph using the basic function

Sketch the graph of $y = 2\cot\left(3x - \frac{\pi}{2}\right) + 1$ for $0 \le x \le \pi$.

Steps	Working
1 Identify the basic function.	The basic function is $y = \cot(x)$.
2 Write as $af(n(x + b)) + c$.	$y = 2\cot\left(3\left(x - \frac{\pi}{6}\right)\right) + 1$
3 State the transformations.	The basic function is dilated from the y-axis by the factor $\frac{1}{3}$, so the period is $\frac{\pi}{3}$ and from the x-axis by the factor 2 and translated $\frac{\pi}{6}$ units to the right and 1 unit up.
4 Find the asymptotes, working from the easiest one first.	There is an asymptote at $0 \times \frac{1}{3} + \frac{\pi}{6} = \frac{\pi}{6}$ and others are $n\frac{\pi}{3}$ (multiples of the period) away, at $\frac{\pi}{6}$, $\frac{\pi}{2}$ and $\frac{5\pi}{6}$ for $0 \le x \le \pi$.
5 Find the centres, working from the easiest zero first.	There is a zero at $\frac{\pi}{2} \times \frac{1}{3} + \frac{\pi}{6} = \frac{\pi}{3}$ and others are $n\frac{\pi}{3}$ (multiples of the period) away, at 0, $\frac{\pi}{3}$, $\frac{2\pi}{3}$ and π for $0 \le x \le \pi$. The y values are changed to $0 \times 2 + 1 = 1$, so the centres are $(0, 1)$, $\left(\frac{\pi}{3}, 1\right)$, $\left(\frac{2\pi}{3}, 1\right)$ and $(\pi, 1)$.
6 Find what becomes of the points where the y values are ± 1 on the basic function, working from the easiest one first.	$\left(\frac{\pi}{4}, 1\right) \to \left(\frac{\pi}{4} \times \frac{1}{3} + \frac{\pi}{6}, 2 \times 1 + 1\right) = \left(\frac{\pi}{4}, 3\right)$ $\left(\frac{3\pi}{4}, -1\right) \to \left(\frac{3\pi}{4} \times \frac{1}{3} + \frac{\pi}{6}, 2 \times -1 + 1\right) = \left(\frac{5\pi}{12}, -1\right)$ Others are points $n\frac{\pi}{3}$ away, at $\left(\frac{\pi}{4}, 3\right)$, $\left(\frac{7\pi}{12}, 3\right)$, $\left(\frac{11\pi}{12}, 3\right)$, $\left(\frac{\pi}{12}, -1\right)$, $\left(\frac{5\pi}{12}, -1\right)$ and $\left(\frac{3\pi}{4}, -1\right)$.

7 Sketch the graph, including asymptote equations and important coordinates.

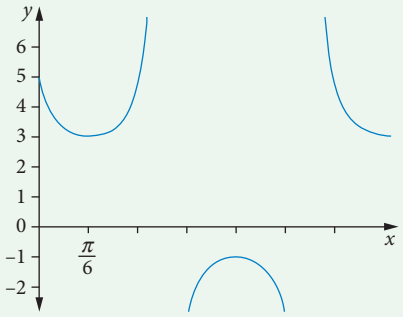


If asked for them, write the x-intercepts. In this case, they are not exact since they involve $\cot\left(\frac{1}{2}\right)$.

WORKED EXAMPLE 20 Identifying the equation of a graph

Part of the graph of $y = f(x)$ is shown.

State a possible equation for the graph.



Steps	Working
1 Identify the basic function.	The basic function is most like $y = \text{cosec}(x)$.
2 Identify the period and dilation from the y-axis.	The period is $6 \times \frac{\pi}{6} = \pi = \frac{2\pi}{2}$, so there is a dilation from the y-axis by a factor of $\frac{1}{2}$ with $n = 2$.
3 Identify the centre line to find any translation parallel to the y-axis.	The centre line is at $y = 1$, so there is a translation of 1 unit up with $c = 1$.
4 Use the distance of the minima or maxima.	The minima and maxima are 2 units from the centre line, so there is a dilation from the x-axis of factor 2 with $a = 2$.
5 Use a maximum or minimum to find the phase shift.	The first minimum is at $\frac{\pi}{6}$, not $\frac{\pi}{2} \div 2 = \frac{\pi}{4}$, so the phase shift is $\frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}$ to the left; $b = \frac{\pi}{12}$.
6 Write the equation.	A possible equation is $y = 2\,\text{cosec}\left(2\left(x + \frac{\pi}{12}\right)\right) + 1$ $= 2\,\text{cosec}\left(2x + \frac{\pi}{6}\right) + 1$.

Exam hack

In Examination 2, you should check your answer using CAS. For multiple-choice questions, try some values from the graph in the equations.

Notice that there are other possible answers for the example above, with different phase shifts and using $\sec(x)$, such as $2\,\text{cosec}\left(2x - \frac{11\pi}{6}\right) + 1$ and $2\sec\left(2x - \frac{\pi}{3}\right) + 1$.

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EXERCISE 8.4 Reciprocal circular functions

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Recap

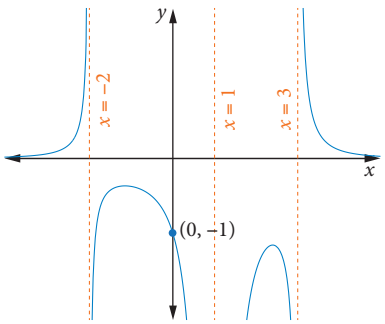
1 The asymptotes of the function $f(x) = \dfrac{1}{2x^2 - 5x - 3}$ are

A $x = \dfrac{1}{2}$, $x = -3$ and $y = 1$ **B** $x = \dfrac{1}{2}$, $x = -3$ and $y = 0$ **C** $x = -\dfrac{1}{2}$, $x = 3$ and $y = 1$

D $x = -\dfrac{1}{2}$, $x = 3$ and $y = 0$ **E** $x = -\dfrac{3}{2}$, $x = 1$ and $y = 0$

2 The graph of $f(x)$ is shown.

Its equation could be



A $f(x) = -\dfrac{1}{(x-2)(x+1)(x+3)}$

B $f(x) = -\dfrac{6}{(x-2)(x+1)^2(x+3)}$

C $f(x) = \dfrac{6}{(x+2)(x-1)^2(x-3)}$

D $f(x) = \dfrac{6}{(x+2)(x-1)(x-3)}$

E $f(x) = \dfrac{1}{(x+2)(x-1)^2(x-3)}$

Mastery

3 WORKED EXAMPLE 16 Find the value of

a $\sec\left(\dfrac{\pi}{6}\right)$ **b** $\cot\left(-\dfrac{\pi}{4}\right)$ **c** $\operatorname{cosec}\left(\dfrac{\pi}{2}\right)$ **d** $\sec(-\pi)$

e $\cot\left(\dfrac{3\pi}{4}\right)$ **f** $\operatorname{cosec}\left(-\dfrac{2\pi}{3}\right)$ **g** $\operatorname{cosec}\left(\dfrac{7\pi}{6}\right)$ **h** $\sec\left(-\dfrac{\pi}{3}\right)$

i $\cot\left(\dfrac{5\pi}{6}\right)$ **j** $\operatorname{cosec}\left(-\dfrac{3\pi}{4}\right)$ **k** $\sec\left(\dfrac{5\pi}{3}\right)$ **l** $\cot\left(-\dfrac{\pi}{2}\right)$

4 WORKED EXAMPLE 17

a $\operatorname{cosec}(x) = -3$ and $-\dfrac{\pi}{2} < x < 0$. Find $\sec(x)$ and $\cot(x)$.

b $\cot(x) = 5$ and $\pi < x < \dfrac{3\pi}{2}$. Find $\sec(x)$ and $\sin(x)$.

c $\tan(x) = \sqrt{3}$ and $-\pi < x < -\dfrac{\pi}{2}$. Find $\sec(x)$ and $\operatorname{cosec}(x)$.

d $\sin(x) = -\dfrac{1}{\sqrt{6}}$ and $\dfrac{3\pi}{2} < x < 2\pi$. Find $\sec(x)$ and $\cot(x)$.

e $\sec(x) = -\sqrt{2}$ and $-2\pi < x < -\pi$. Find $\tan(x)$ and $\operatorname{cosec}(x)$.

f $\cos(x) = -\dfrac{\sqrt{2}}{3}$ and $\pi < x < 2\pi$. Find $\operatorname{cosec}(x)$ and $\cot(x)$.

g $\operatorname{cosec}(x) = 2$ and $\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$. Find $\sec(x)$ and $\tan(x)$.

h $\sec(x) = 3$ and $\pi < x < 2\pi$. Find $\cot(x)$ and $\operatorname{cosec}(x)$.

5 WORKED EXAMPLES 18, 19 Sketch each graph for the given domain.

a $y = 4\operatorname{cosec}(3x) + 2$ for $0 < x < \pi$

b $y = 3\sec(4x) + 1$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

c $f(x) = 3\cot\left(\frac{x}{2}\right) - 1$ for $0 < x < 3\pi$

d $f(x) = 0.25\sec(2x) - 1$ for $-\pi < x < \pi$

e $f(x) = 0.5\cot(\pi x)$ for $0 < x < 2$

f $y = 3\operatorname{cosec}\left(\frac{2\pi x}{3}\right) + 2$ for $-2 < x < 2$

g $f(x) = 0.4\cot\left(\frac{x}{3}\right) - 1$ for $0 < x < 4\pi$

h $y = 2\operatorname{cosec}\left(x + \frac{\pi}{3}\right) - 3$ for $0 < x < 2\pi$

i $y = 2\operatorname{cosec}\left(2x - \frac{\pi}{3}\right) - 3$ for $0 < x < \pi$

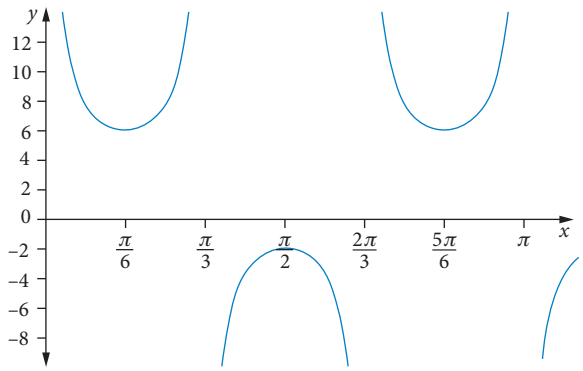
j $f(x) = 2\cot\left(\frac{x}{3} + \frac{\pi}{6}\right) + 4$ for $-2\pi < x < 2\pi$

k $y = 5\sec\left(3x + \frac{\pi}{6}\right) + 4$ for $0 < x < \pi$

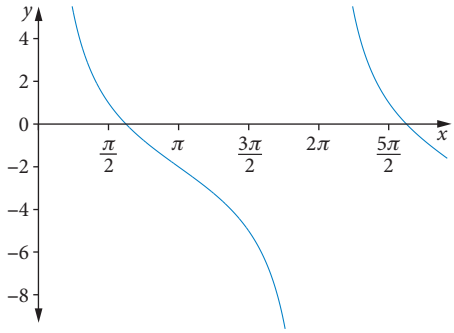
l $f(x) = 2\sec\left(2x - \frac{\pi}{4}\right) + 4$ for $0 < x < \pi$

6 WORKED EXAMPLE 20 State a possible equation for each graph.

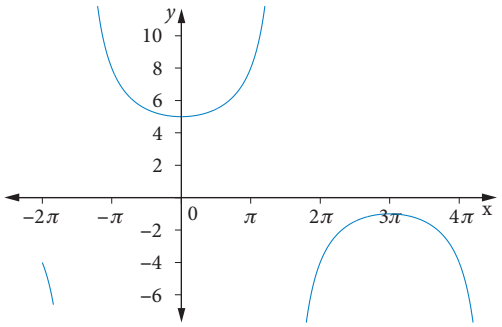
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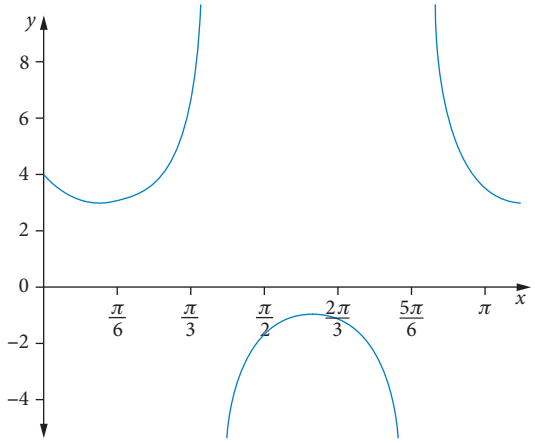
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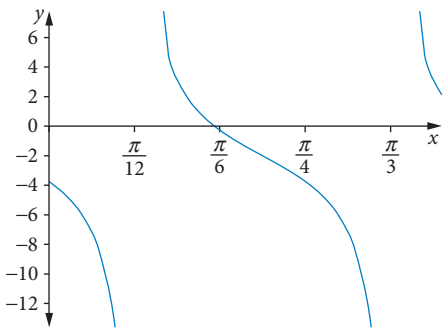
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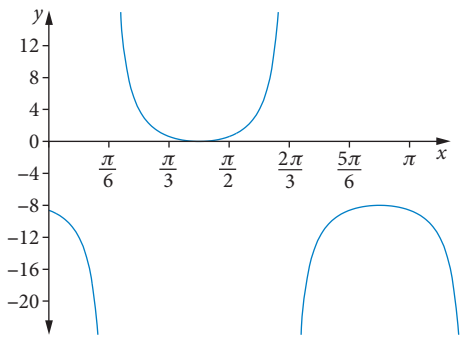
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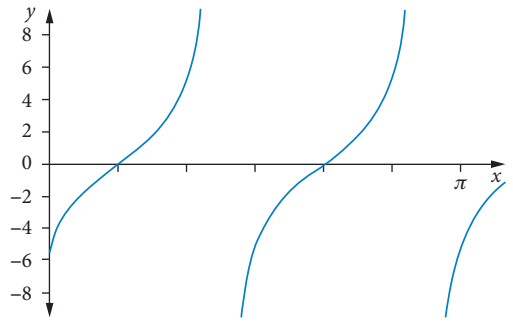


f

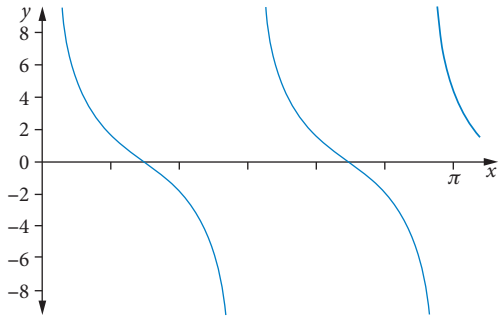


7 Which of these is the graph of $y = 3\cot\left(2x - \frac{\pi}{3}\right)$?

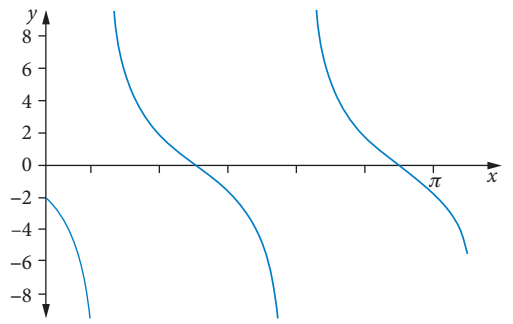
A



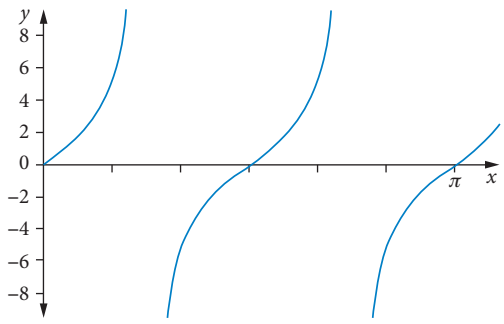
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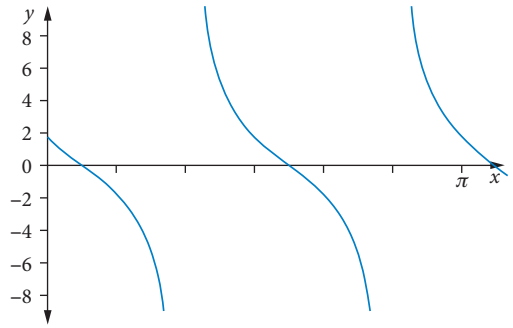
C



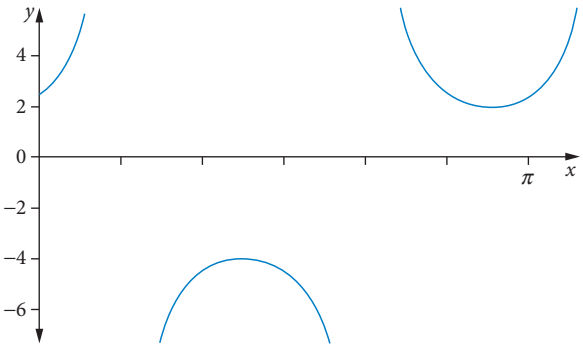
D



E



8 Which of these could be the equation of the graph?



A $f(x) = 3\sec\left(2x + \frac{\pi}{3}\right) - 1$

B $f(x) = 3\operatorname{cosec}\left(2x + \frac{\pi}{6}\right) - 1$

C $f(x) = 3\sec\left(2x - \frac{\pi}{6}\right) - 1$

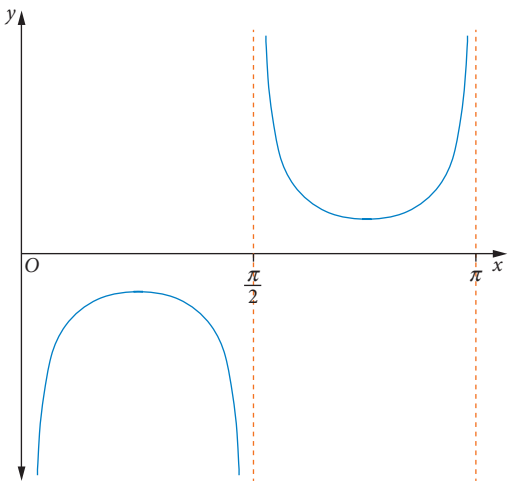
D $f(x) = 3\operatorname{cosec}\left(2x - \frac{\pi}{6}\right) - 1$

E $f(x) = 3\sec\left(2x + \frac{\pi}{6}\right) - 1$

Exam practice

9 © VCAA 2013S 2AQ4 The graph of $y = -\sec(a(x - b))$ is shown for $0 \leq x \leq \pi$.

The values of a and b could be



A $a = 1, b = \frac{\pi}{2}$

B $a = 1, b = \frac{\pi}{4}$

C $a = 2, b = \frac{\pi}{2}$

D $a = 2, b = \frac{\pi}{4}$

E $a = 2, b = -\frac{\pi}{4}$

10 © VCAA 2019 2AQ3 MODIFIED The implied domain of the function with the rule $f(x) = 1 - \text{cosec}\left(x - \frac{\pi}{4}\right)$ is

A R

B $[0, 2]$

C $R \setminus \left\{\frac{(4n-1)\pi}{4}\right\}, n \in Z$

D $R \setminus \left\{\frac{(4n+1)\pi}{4}\right\}, n \in Z$

E $R \setminus \left\{\frac{(2n-1)\pi}{2}\right\}, n \in Z$

11 © VCAA 2013S 2AQ3 MODIFIED Which of the following is **not** equal to $\cot\left(\frac{\pi}{5}\right)$?

A $\frac{\cos\left(\frac{\pi}{5}\right)}{\sin\left(\frac{\pi}{5}\right)}$

B $\frac{1}{\tan\left(\frac{\pi}{5}\right)}$

C $\sqrt{\sec^2\left(\frac{\pi}{5}\right) - 1}$

D $\sqrt{\text{cosec}^2\left(\frac{\pi}{5}\right) - 1}$

E $\frac{1}{\sqrt{\sec^2\left(\frac{\pi}{5}\right) - 1}}$

12 © VCAA 2017N 1Q6 MODIFIED The solutions to $\cot(x) = \cot(2x)$ are

A $x = n\pi, \pm\frac{\pi}{3} + n\pi, n \in Z$

B $x = (2n+1)\frac{\pi}{2}, n \in Z$

C no real solutions

D $x = (4n+1)\frac{\pi}{4}, n \in Z$

E $x = (4n+3)\frac{\pi}{4}, n \in Z$

Video playlist
Inverse circular functions

Worksheet
Inverse trigonometric functions

8.5 Inverse circular functions

For $x = -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6} \ldots \sin(x) = \frac{1}{2}$. The sine function is not one-to-one. Neither are the cosine or tangent functions. They do not have inverses unless the domains are restricted to make them one-to-one.

Principal domains of circular functions

The principal domain of

- $\sin(x)$ is $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
- $\cos(x)$ is $0 \leq x \leq \pi$
- $\tan(x)$ is $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

In their principal domains, sine, cosine and tangent are one-to-one functions. On their principal domains, they do have inverse functions.

The ranges of $\sin(x)$ and $\cos(x)$ on their principal domains are both $[-1, 1]$.

The range of $\tan(x)$ on its principal domain is $(-\infty, \infty)$.

$\sin(x)$, $\cos(x)$ and $\tan(x)$ take all possible values on their principal domains.

Inverse circular functions

The **inverse sine**, **inverse cosine** and **inverse tangent** functions are defined as

- $y = \sin^{-1}(x)$ if and only if $\sin(y) = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
- $y = \cos^{-1}(x)$ if and only if $\cos(y) = x$ and $0 \leq y \leq \pi$
- $y = \tan^{-1}(x)$ if and only if $\tan(y) = x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

$\sin^{-1}(x)$, $\cos^{-1}(x)$ and $\tan^{-1}(x)$ are also called the **arcsine** (**arcsin**), **arccosine** (**arccos**) and **arctangent** (**arctan**).

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WORKED EXAMPLE 21 Values of inverse circular functions

Find the values of

a $\cos^{-1}\left(-\frac{1}{2}\right)$ **b** $\arctan(\sqrt{3})$ **c** $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Steps	Working
a 1 Find $0 \leq x \leq \pi$ such that $\cos(x) = -\frac{1}{2}$.	$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$
2 Write the answer.	$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$
b 1 Find $-\frac{\pi}{2} < x < \frac{\pi}{2}$ such that $\tan(x) = \sqrt{3}$.	$\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$
2 Write the answer.	$\arctan(\sqrt{3}) = \frac{\pi}{3}$

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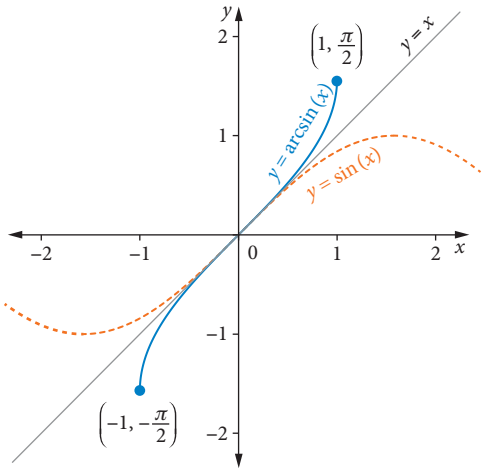
c **1** Find $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ such that $\sin(x) = -\frac{1}{\sqrt{2}}$. $\sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

2 Write the answer. $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$

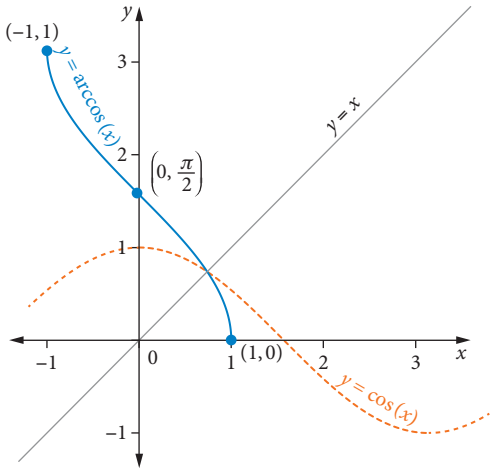
The values of the inverse circular functions are *only* in the principal domains of the circular functions.

Graphs of inverse circular functions

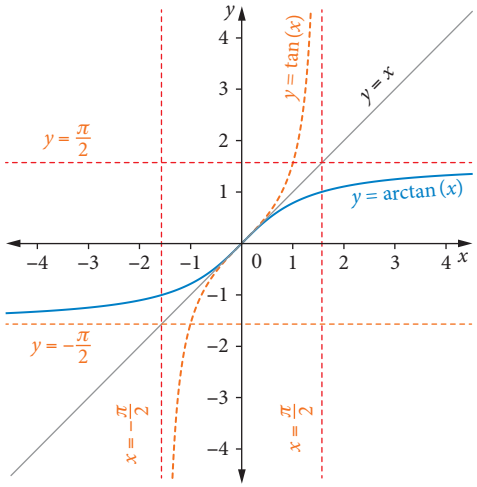
The basic graphs of the inverse circular functions are reflections in the line $y = x$ of the circular functions on their principal domains.



$y = \sin^{-1}(x)$: domain $[-1, 1]$, range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



$y = \cos^{-1}(x)$: domain $[-1, 1]$, range $[0, \pi]$



$y = \tan^{-1}(x)$: domain R, range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, asymptotes $y = -\frac{\pi}{2}$, $y = \frac{\pi}{2}$

We can use transformations to sketch other inverse circular functions. We might find it easier to transform the points with x values -1, 0 and 1, and join these, as well as the asymptotes for $\tan^{-1}(x)$.

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WORKED EXAMPLE 22 Domain and range of inverse circular functions

What are the implied domain and range of these functions?

a $f(x) = 3\arcsin\left(\frac{x+1}{2}\right) + 4$ **b** $g(x) = 0.5\tan^{-1}(x-4) - 2$

Steps	Working
a 1 What do the domain ends −1 and 1 of $\arcsin(x)$ become?	$\left(\frac{x+1}{2}\right) = -1$ at $x = -3$ and $\frac{x+1}{2} = 1$ at $x = 1$, so $[-1, 1] \to [-3, 1]$.
2 What is the range?	$f(-3) = 3 \times -\frac{\pi}{2} + 4 = 4 - \frac{3\pi}{2}$ $f(1) = 3 \times \frac{\pi}{2} + 4 = 4 + \frac{3\pi}{2}$
3 Write the answer.	The implied domain and range are $[-3, 1]$ and $\left[4 - \frac{3\pi}{2}, 4 + \frac{3\pi}{2}\right]$ respectively.
b 1 What is the implied domain?	$(x - 4) \in R$ for $x \in R$, so the implied domain is R.
2 What happens to the y values of $\tan^{-1}(x)$?	They are multiplied by 0.5 and moved 2 units down. $-\frac{\pi}{2} \to -\frac{\pi}{4} - 2$, $\frac{\pi}{2} \to \frac{\pi}{4} - 2$
3 Write the answer.	The implied domain is R and the range is $\left(-\frac{\pi}{4} - 2, \frac{\pi}{4} - 2\right)$.

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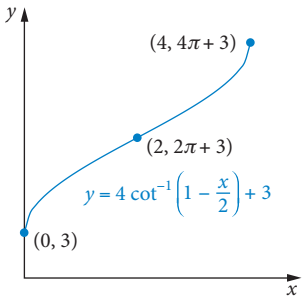
WORKED EXAMPLE 23 Sketching an arccosine function

Sketch the graph of $f(x) = 4\cos^{-1}\left(1 - \frac{x}{2}\right) + 3$.

Steps	Working
Method 1: Find important points using transformations	
1 Write as $af(n(x + b)) + c$.	$f(x) = 4\cos^{-1}\left(-\frac{x-2}{2}\right) + 3$
2 State the transformations.	The graph is reflected in the y-axis, dilated by the factor 2 from the y-axis and moved 2 units to the right; dilated by the factor 4 from the x-axis and moved 3 units up.
3 Find the endpoints and centre.	$(-1, \pi) \to (-1 \times (-2) + 2, 4 \times \pi + 3) = (4, 4\pi + 3)$ $\left(0, \frac{\pi}{2}\right) \to \left(0 \times (-2) + 2, 4 \times \frac{\pi}{2} + 3\right) = (2, 2\pi + 3)$ $(1, 0) \to (1 \times (-2) + 2, 4 \times 0\pi + 3) = (0, 3)$
Method 2: Find important points by analysis	
1 Find where $\cos^{-1}(x) = 1$, 0 and −1.	$\left(1 - \frac{x}{2}\right) = 1$ at $x = 0$, $\left(1 - \frac{x}{2}\right) = 0$ at $x = 2$, $\left(1 - \frac{x}{2}\right) = -1$ at $x = 4$
2 Substitute to find points.	$f(4) = 4 \times \cos^{-1}(-1) + 3 = 4 \times \pi + 3 = 4\pi + 3$ $f(2) = 2\pi + 3$, $f(0) = 3$, so the endpoints and centre are $(4, 4\pi + 3)$, $(2, 2\pi + 3)$ and $(0, 3)$.

Both methods

3 Put the important points on the graph and join to sketch the graph.

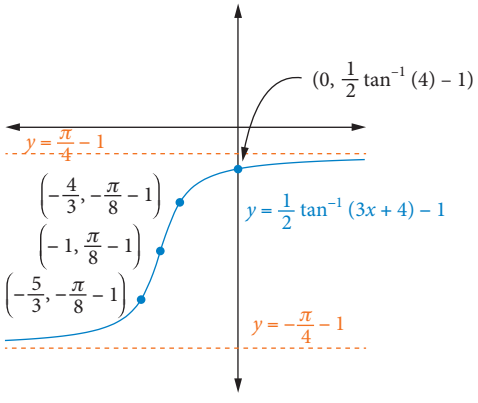


When we sketch an arctangent function, we must also put in the asymptotes. The important points for these functions are those that correspond to $\tan^{-1}(-1) = -\frac{\pi}{4}$, $\tan^{-1}(0) = 0$ and $\tan^{-1}(1) = \frac{\pi}{4}$.

We should also find any simple zeros when sketching inverse circular functions.

WORKED EXAMPLE 24 Sketching an arctangent function

Sketch the graph of $f(x) = \frac{1}{2}\tan^{-1}(3x + 4) - 1$.

Steps	Working
1 Write as $af(n(x + b)) + c$.	$f(x) = \frac{1}{2}\tan^{-1}\left(3\left(x + \frac{4}{3}\right)\right) - 1$
2 State the transformations.	Dilated by the factor $\frac{1}{3}$ from the y-axis and moved $\frac{4}{3}$ units to the left; dilated by the factor $\frac{1}{2}$ from the x-axis and moved 1 unit down.
3 Find the important points.	$\left(-1, -\frac{\pi}{4}\right) \to \left(-1 \times \frac{1}{3} - \frac{4}{3}, -\frac{\pi}{4} \times \frac{1}{2} - 1\right) = \left(-\frac{5}{3}, -\frac{\pi}{8} - 1\right)$ $(0, 0) \to \left(0 \times \frac{1}{3} - \frac{4}{3}, 0 \times \frac{1}{2} - 1\right) = \left(-\frac{4}{3}, -1\right)$ $3\left(x + \frac{4}{3}\right) = 1$ at $x = -1, f(-1) = \frac{\pi}{8} - 1$, point $\left(-1, \frac{\pi}{8} - 1\right)$
4 State the asymptotes.	$-\frac{\pi}{2} \to -\frac{\pi}{2} \times \frac{1}{2} - 1 = -\frac{\pi}{4} - 1$, $\frac{\pi}{2} \to \frac{\pi}{2} \times \frac{1}{2} - 1 = \frac{\pi}{4} - 1$, so the asymptotes are at $y = \pm\frac{\pi}{4} - 1$
5 Find the y-intercept.	$f(0) = \frac{1}{2}\tan^{-1}(4) - 1 \approx -0.3371$
6 Put in the important points and equations and sketch the graph.	

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EXERCISE 8.5 Inverse circular functions

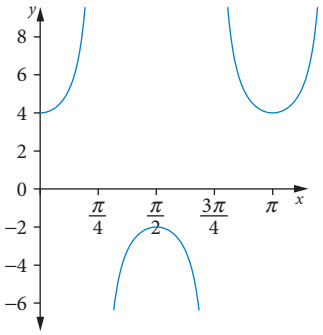
ANSWERS p. 591

Recap

1 The value of $\cot\left(\frac{2\pi}{3}\right)$ is

A $-\sqrt{3}$ **B** -1 **C** $-\frac{1}{\sqrt{3}}$ **D** $\frac{1}{\sqrt{3}}$ **E** 1

2 A possible equation for this graph is



A $y = 3\operatorname{cosec}\left(2x - \frac{\pi}{4}\right) + 1$

B $y = -3\operatorname{cosec}\left(2x + \frac{\pi}{4}\right) + 1$

C $y = 3\operatorname{cosec}\left(2x - \frac{\pi}{4}\right) + 2$

D $y = 3\sec(2x) + 1$

E $y = -3\sec(2x) + 1$

Mastery

3 WORKED EXAMPLE 21 Find the value of

a $\cos^{-1}(-1)$ **b** $\tan^{-1}(1)$ **c** $\sin^{-1}(-1)$ **d** $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

e $\cos^{-1}\left(-\frac{1}{2}\right)$ **f** $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ **g** $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ **h** $\tan^{-1}(-\sqrt{3})$

i $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ **j** $\tan^{-1}(-1)$

4 WORKED EXAMPLE 22 What are the implied domain and range of each function?

a $f(x) = 2\sin^{-1}(x - 3) + 1$ **b** $y = \tan^{-1}(3x + 1) + 2$

c $f(x) = 3\tan^{-1}(2x - 1) - 4$ **d** $y = \frac{1}{2}\cos^{-1}(0.25x + 1) - 2$

e $f(x) = 2\cos^{-1}(0.5x - 1) + 3$ **f** $y = \frac{1}{4}\sin^{-1}\left(\frac{x}{3} + 2\right) - 2$

5 WORKED EXAMPLES 23, 24 Sketch the graph of each function.

a $f(x) = 2\sin^{-1}(x + 3) + 1$ **b** $y = \cos^{-1}(x - 2) + 1$

c $f(x) = -\tan^{-1}(x + 4) + 2$ **d** $y = -3\arctan(2x)$

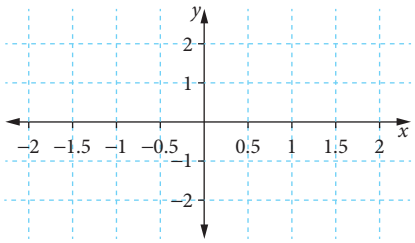
e $f(x) = 3\arcsin(0.5x) - 4$ **f** $y = 0.5\cos^{-1}(1 - 0.5x)$

g $f(x) = 2\cos^{-1}(0.25x - 3) + 1$ **h** $y = \frac{1}{3}\tan^{-1}(3 - 2x) + 1$

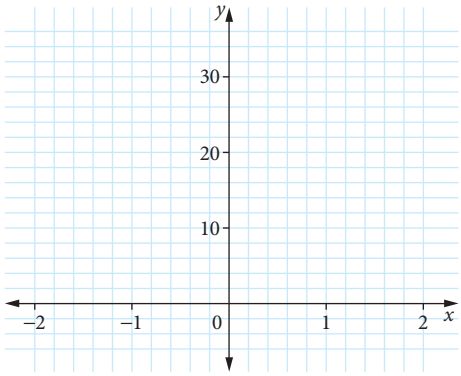
i $f(x) = 3 - 2\sin^{-1}(2 - 0.5x)$ **j** $y = 1 - 2\arccos\left(\frac{x - 2}{3}\right)$

Exam practice

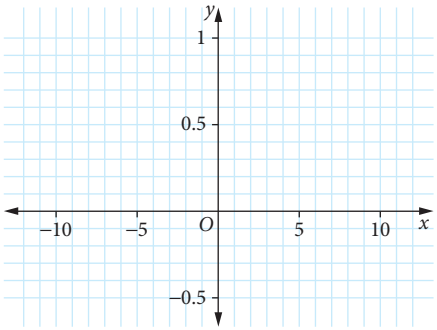
6 © VCAA 2021N 1Q4a **TECH-FREE** (2 marks) Consider the function $f: [-1, 1] \to R, f(x) = \arccos(x) - \frac{\pi}{2}$.
Copy the axes below and sketch the graph of f, labelling the endpoints with their coordinates.



7 © VCAA 2018N 2BQ1a **TECH-FREE** (3 marks) Consider the function f with rule $f(x) = 10\arccos(2 - 2x)$.
Copy the axes below and on them sketch the graph of f over its maximal domain. Label the endpoints with their coordinates.



8 © VCAA 2017N 2BQ1b **TECH-FREE** (3 marks) Consider $f: [\sqrt{3}, 6 + 3\sqrt{3}] \to R, f(x) = \arctan\left(\frac{x}{3}\right) - \frac{\pi}{6}$.
Copy the axes below and on them sketch the graph of f, labelling the endpoints with their coordinates.



9 © VCAA 2017N 2AQ3 For the function $f: R \to R, f(x) = k\arctan(ax - b) + c$, where $k > 0$, $c > 0$ and a, $b \in R, f(x) > 0$ if

A $c < \frac{k\pi}{2}$ **B** $c \geq \frac{k\pi}{2}$ **C** $x > \frac{b}{a}$ **D** $c + k > \frac{\pi}{2}$ **E** $c \geq \frac{\pi}{2}$

10 © VCAA 2018N 2AQ3 The implied domain of the function with rule $f(x) = \frac{3x}{\frac{\pi}{2} - \arccos(2 - x)}$ is

A $[1, 3]$ **B** $[-1, 1]$ **C** $[0, 1) \cup (1, 2]$
D $[-1, 0) \cup (0, 1]$ **E** $[1, 2) \cup (2, 3]$

11 © VCAA 2019N 2AQ3 The maximal domain and range of the function $f(x) = a\cos^{-1}(bx) + c$, where a, b and c are real constants with $a > 0$, $b < 0$ and $c > 0$, are respectively

A $[0, \pi]$ and $[-a, a]$

B $[0, \pi]$ and $[-a + c, a + c]$

C $\left[-\frac{1}{b}, \frac{1}{b}\right]$ and $[c, a\pi + c]$

D $\left[\frac{1}{b}, -\frac{1}{b}\right]$ and $[c, a\pi + c]$

E $\left[\frac{1}{b}, -\frac{1}{b}\right]$ and $[-a\pi + c, a\pi + c]$

Video playlist
The absolute value function

Worksheets
Absolute values

Absolute value equations and inequalities

Absolute value inequalities

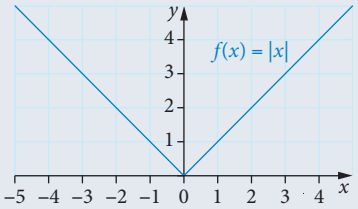
8.6 The absolute value function

The absolute value function

The **absolute value** function, $f(x) = |x|$ is defined by

$$f(x) = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

The graph of the absolute value function is on the right. It is the combination of $y = -x$ and $y = x$ for the 2 parts of the domain.



Notice the graph has a sharp point (vertex) at $(0, 0)$.

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WORKED EXAMPLE 25 Absolute values

Evaluate each expression.

a $|-6|$ **b** $|7|$ **c** $|5 - 8|$

Steps	Working
a Use the absolute value function.	$\|-6\| = -(-6)$ $= 6$
b Use the absolute value function.	$\|7\| = 7$
c Simplify and use the absolute value function.	$\|5 - 8\| = \|-3\|$ $= -(-3)$ $= 3$

From the above example, other ways to state the absolute value are 'the distance from 0' or 'the size without the sign'.

We can easily see that $|x| = 5 \Rightarrow x = -5$ or $x = 5$.

Similarly, $|x| = a \Rightarrow x = -a$ or $x = a$.

We can solve any other equation involving the absolute value as a pair of equations.

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WORKED EXAMPLE 26 Equations with absolute values

Solve $|5 - 4m| = 3m + 9$.

Steps	Working		
1 Split into 2 equations.	$5 - 4m = 3m + 9$	or	$5 - 4m = -(3m + 9)$
2 Solve each equation.	$5 - 9 = 3m + 4m$	or	$5 - 4m = -3m - 9$
	$7m = -4$	or	$5 + 9 = -3m + 4m$
	$m = -\frac{4}{7}$	or	$m = 14$

For inequalities, it is obvious that $|x| < 5 \Rightarrow x > -5$ AND $x < 5$ (also written $-5 < x < 5$).

It should also be clear that $|x| > 5 \Rightarrow x < -5$ OR $x > 5$.

Similarly $|x| < a \Rightarrow x > -a$ AND $x < a$, and $|x| > a \Rightarrow x < -a$ OR $x > a$.

Any inequality involving the absolute value is solved as a pair of inequalities.

WORKED EXAMPLE 27 Inequalities with absolute values

Solve

a $|5x - 4| < 7$ **b** $|7 - 4x| \geq 13$ **c** $|3x - 6| > 5x + 2$

d $|2 - 6x| \leq 6x - 8$ **e** $|5x - 3| > 7x + 9$

Steps	Working
a 1 Split into 2 inequalities.	$5x - 4 > -7$ and $5x - 4 < 7$
2 Solve each inequality.	$5x > -3$ and $5x < 11$
	$x > -\frac{3}{5}$ and $x < \frac{11}{5}$
3 Write the solution.	$-\frac{3}{5} < x < \frac{11}{5}$
b 1 Split into 2 inequalities.	$7 - 4x \leq -13$ or $7 - 4x \geq 13$
2 Solve each inequality.	$4x \geq 20$ or $4x \leq -6$
	$x \geq 5$ or $x \leq -\frac{3}{2}$
c 1 Split into 2 inequalities.	$3x - 6 < -(5x + 2)$ or $3x - 6 > 5x + 2$
2 Solve each inequality.	$3x - 6 < -5x - 2$ or $-6 - 2 > 5x - 3x$
	$3x + 5x < -2 + 6$ or $-8 > 2x$
	$8x < 4$ or $2x > -8$
	$x < \frac{1}{2}$ or $x > -4$
3 Write the solution.	$-4 < x < \frac{1}{2}$
d 1 Split into 2 inequalities.	$2 - 6x \geq -(6x - 8)$ and $2 - 6x \leq 6x - 8$
2 Solve each inequality.	$2 - 6x \geq -6x + 8$ and $2 + 8 \leq 12x$
	$2 \geq 8$ and $12x \geq 10$
	But $2 \geq 8$ is not true for any values of x.
3 Write the solution.	There is no solution.
e 1 Split into 2 inequalities.	$5x - 3 < -(7x + 9)$ or $5x - 3 > 7x + 9$
2 Solve each inequation.	$5x - 3 < -7x - 9$ or $-3 - 9 > 7x - 5x$
	$12x < -6$ or $2x < -12$
	$x < -\frac{1}{2}$ or $x < -12$
3 Write the solution.	$x < -\frac{1}{2}$ (which includes $x < -12$)

Part **d** above shows that you can have no solution to an inequality or equation involving absolute value.

Part **e** shows that the solution may come from only one of the pair of inequalities.

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Sketching absolute value functions

Transformations of $f(x) = |x|$ give functions such as $f(x) = -2|x| - 1$, $y = |x - 3| + 2$.

Reflection in the y-axis leaves $|x|$ unchanged because $|x| = |-x|$.

It is useful to show the sharp 'point' (vertex), y-intercept and zeros or another point to show the slope.

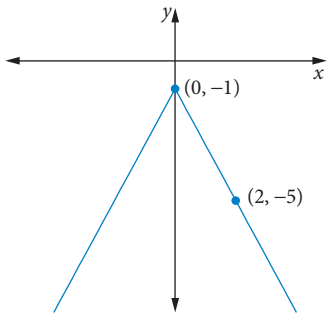
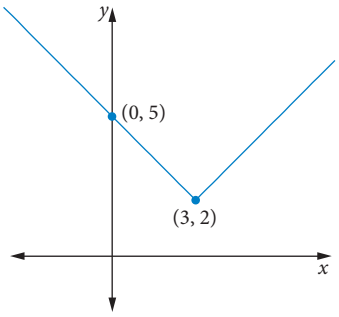
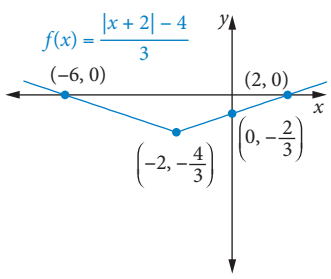
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Worksheet
Absolute value functions

WORKED EXAMPLE 28 Transformations of the absolute value function

Sketch each absolute value function for the stated domain.

a $f(x) = -2|x| - 1, -8 < x < 8$ **b** $y = |x - 3| + 2, -3 < x < 8$ **c** $f(x) = \dfrac{|x + 2| - 4}{3}, -7 < x < 3$

Steps	Working
a 1 State the transformations.	$y = \|x\|$ is dilated from the x-axis by the factor 2, reflected in the x-axis and translated 1 unit down.
2 State the positions of some points.	Vertex $(0, -1)$, another point $(2, -5)$.
3 Sketch the graph.	
b 1 State the transformations.	$y = \|x\|$ is translated 3 units to the right and 2 units up.
2 State the positions of some points.	Vertex at $(3, 2)$ and the y-intercept is $(0, 5)$.
3 Sketch the graph.	
c 1 Write as $af(n(x + b)) + c$.	$f(x) = \dfrac{1}{3}\|x + 2\| - \dfrac{4}{3}$
2 State the transformations.	$y = \|x\|$ is dilated by the factor $\dfrac{1}{3}$ from the x-axis and translated 2 units to the left and $\dfrac{4}{3}$ units down.
3 State the positions of some points.	Vertex at $\left(-2, -\dfrac{4}{3}\right)$, the y-intercept is $\left(0, -\dfrac{2}{3}\right)$ and zeros are at $(-6, 0)$ and $(2, 0)$.
4 Sketch the graph.	

EXERCISE 8.6 The absolute value function

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Recap

1 The implied domain of the function $f(x) = 2\arctan(2x + 1) - 2$ is

A $[-1, 3]$ **B** $(-1, 3)$ **C** $(-1, 5)$ **D** $(-1, 0)$ **E** R

2 The graph on the right could be the graph of the function

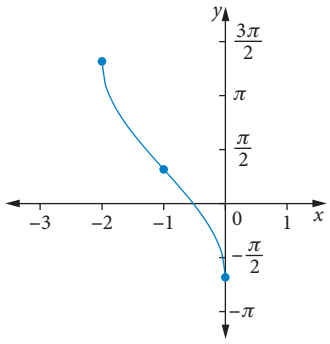
A $2\arcsin(-x - 1) + 1$

B $2\arccos(x - 1) + 1$

C $2\arccos(x + 1) + 1$

D $2\arcsin(x - 1) + 1$

E $2\arcsin(1 - x) + 1$



Mastery

3 WORKED EXAMPLE 25 Simplify each expression, writing without | | signs.

a $|-9|$ **b** $|3|$ **c** $\left|-\frac{3}{5}\right|$

d $\left|\frac{9}{7}\right|$ **e** $|19 - 14|$ **f** $|17 - 29|$

g $|x^2|, x \in R$ **h** $|x - y|, x > y, x, y \in R$ **i** $|y - x|, x > y, x, y \in R$

4 WORKED EXAMPLE 26 Solve each equation.

a $|3x - 2| = 2x + 5$ **b** $|4 - 7y| = 3y + 4$ **c** $|2m + 3| = 2m + 6$

d $|x^2 - 4x + 5| = 1$ **e** $|3v + 9| = |6v + 8|$ **f** $|9b + 6| = |8 - 3b|$

5 WORKED EXAMPLE 27 Solve each inequality.

a $|3m + 10| < 7 - 2m$ **b** $|3c + 9| \leq 7c - 11$ **c** $|2x - 5| > 8x + 9$

d $|6p + 2| < 6p + 8$ **e** $|3 - 4w| \geq 4w + 11$ **f** $|6t + 6| < 6t - 12$

g $|7d - 3| > |4d + 12|$ **h** $|4g + 8| \leq |8g - 4|$ **i** $|6q + 9| \geq |q + 1|$

6 WORKED EXAMPLE 28 Use axes with $-8 \leq x \leq 8$ and $-8 \leq y \leq 8$ to sketch each function, showing the coordinates of any intercepts.

a $f(x) = |x - 2| + 3$ **b** $y = 4 - |x + 3|$

c $f(x) = |x + 3| - 2$ **d** $y = -|x - 1| - 2$

e $f(x) = 4 - 2|3 - x|$ **f** $y = |3x + 4| - 5$

g $f(x) = 6 - \frac{|x - 5|}{3}$ **h** $y = \left|\frac{x}{2} + 2\right| + 1$

i $f(x) = |2x + 3| - |x - 2| - 2$ **j** $y = -\left|\frac{x}{2} - 3\right| - \frac{3}{2}|x - 2| + 10$

Exam practice

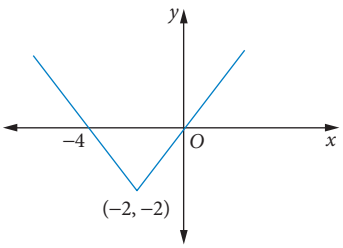
80–100% 60–79% 0–59%

7 © VCAA 2019 1Q2 MODIFIED TECH-FREE (3 marks) Find all values of x for which $|3x + 4| = \frac{x}{3} + 6$.

8 © VCAA 2020 1Q4 MODIFIED TECH-FREE (4 marks) Solve the inequality $x + 1 \leq \frac{1}{|x + 2|}$ for x, expressing your answer in interval notation.

9 © VCAA 2008 Mathematical Methods 2AQ2 93%

The rule of the function whose graph is shown is



A $y = |x| - 4$

B $y = |x - 2| + 2$

C $y = |x + 2| - 2$

D $y = |2 - x| - 2$

E $y = |2 + x| + 2$

10 © VCAA 2006 Mathematical Methods 2AQ11 75% The value(s) of k for which $|2k + 1| = k + 1$ are

A 0 only **B** $-\frac{2}{3}$ only **C** 0 or $\frac{2}{3}$ **D** 0 or $-\frac{2}{3}$ **E** $-\frac{1}{2}$ or 0

Video playlist
Locus definitions of curves and conic sections

8.7 Locus definitions of curves and conic sections

Locus

A **locus** is a set of points that satisfies a given condition. It is the path traced out by a point that moves according to the condition.

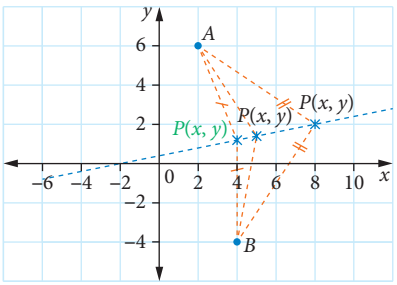
To find the equation of a locus, put the point $P(x, y)$ into the condition and simplify.

Distance is often used to define a locus. Hence, the distance formula is often key to finding the equation of a locus.

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WORKED EXAMPLE 29 A simple locus

Find the equation of the locus of points equidistant from $A(2, 6)$ and $B(4, -4)$ and describe the locus.

Steps	Working
1 Sketch A, B and some possible points $P(x, y)$ on the locus.	
2 Write the condition.	$\|PA\| = \|PB\|$

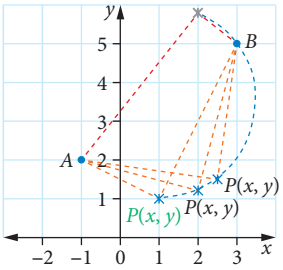
3 Use the distance formula.	$\sqrt{(x-2)^2+(y-6)^2} = \sqrt{(x-4)^2+(y-(-4))^2}$
4 Simplify.	$x^2-4x+4+y^2-12y+36 = x^2-8x+16+y^2+8y+16$ $4x+8=20y$ $x-5y+2=0$ or $y=0.2x+0.4$
5 It looks perpendicular to AB. Calculate gradients.	gradient of locus $= 0.2$ gradient of $AB = \dfrac{6-(-4)}{2-4} = -5$
6 Find product m_1m_2.	$0.2 \times -5 = -1$
7 Find midpoint of AB and check it is on the locus.	midpoint $AB = \left(\dfrac{2+4}{2}, \dfrac{6+(-4)}{2}\right) = (3,1)$ LHS $= x-5y+2 = 3-5\times 1+2 = 0 =$ RHS
8 Write the answer.	The locus is the perpendicular bisector of AB, with equation $x-5y+2=0$.

The conditions for a locus can involve points, lines, distances, angles and so on.

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WORKED EXAMPLE 30 A locus involving angles

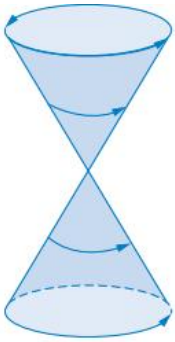
A and B have coordinates $(-1,2)$ and $(3,5)$ respectively. Find the equation of the locus of points P such that $\angle APB = 90°$ and describe the locus.

Steps	Working
1 Sketch A, B and some possible points $P(x,y)$ on the locus. The points look like they could make a circle.	
2 Write the condition.	$\angle APB = 90°$
3 Use gradients with $P(x,y)$.	$\dfrac{y-2}{x+1} \times \dfrac{y-5}{x-3} = -1$
4 Simplify.	$(y-2)(y-5) = -(x+1)(x-3)$ $y^2-7y+10 = -x^2+2x+3$ $y^2-7y+x^2-2x = -7$
5 Complete squares.	$x^2-2x+1+y^2-2\times 3.5y+3.5^2 = -7+3.5^2+1$ $(x-1)^2+(y-3.5)^2 = -7+12.25+1 = 6.25$ $(x-1)^2+(y-3.5)^2 = 2.5^2$
6 Consider the equation.	The equation is that of a circle with centre $(1,3.5)$ and radius 2.5. The centre is the midpoint of AB and the radius is half the distance AB.
7 Consider the endpoints.	The points A and B cannot be included because if P is at A or B then APB does not form an angle.
8 Write the answer.	The locus forms a circle with AB as a diameter, except the points A and B.

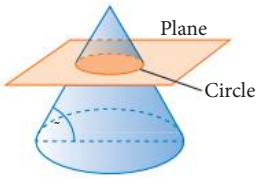
Conic sections

A **conic section** is the shape made when a plane cuts through a cone. There are 4 conic sections: the circle, the ellipse, the parabola and the hyperbola.

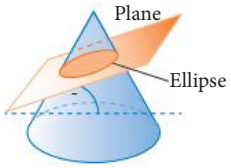
Consider a cone with slant angle α and a plane at angle β to the base.



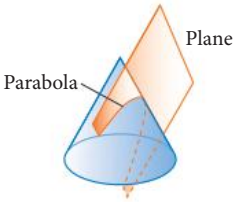
A **circle has** $\beta = 0$ ($\perp$ to axis).



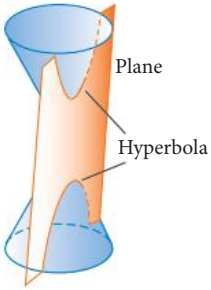
An **ellipse** has $0 < \beta < \alpha$.



A **parabola** has $\beta = \alpha$.



A **hyperbola** has $\beta > \alpha$.



For a hyperbola, the plane intersects both parts of the cone, so a hyperbola has 2 **branches**.

For circles, ellipses and parabolas, the plane intersects only half of the full cone.

Circles

A circle is the locus of points at the distance r from the centre (h, k).

The equation of a circle

The equation of a circle with centre (h, k) and radius r is

$$(x - h)^2 + (y - k)^2 = r^2$$

WORKED EXAMPLE 31 Centre and radius of a circle

Find the centre and radius of the circle with equation $4x^2 + 28x + 4y^2 - 40y = 43$.

Steps	Working
1 Make the coefficients of x^2 and y^2 both 1.	$x^2 + 7x + y^2 - 10y = 10\frac{3}{4}$
2 Complete the squares.	$x^2 + 7x + 12\frac{1}{4} + y^2 - 10y + 25 = 10\frac{3}{4} + 12\frac{1}{4} + 25$
3 Write as squares.	$\left(x + 3\frac{1}{2}\right)^2 + (y - 5)^2 = 48$
4 Write in standard form.	$\left(x + 3\frac{1}{2}\right)^2 + (y - 5)^2 = (4\sqrt{3})^2$
5 Write the answer.	The centre is $\left(-3\frac{1}{2}, 5\right)$ and the radius is $4\sqrt{3}$.

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Parabolas

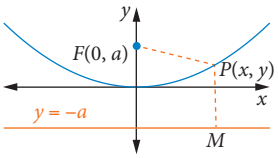
A parabola is the locus of points with equal distances from a fixed point and a line.

The point is called the **focus** and the line is called the **directrix**.

Choose the origin midway between the focus and directrix.

Choose the directrix in the same direction as the x-axis.

Then write the focus as $F(0, a)$ and directrix as $y = -a$.



For a point $P(x, y)$ in the locus, $PF = PM$.

$PF = \sqrt{(x - 0)^2 + (y - a)^2}$ and $PM = y + a$.

$$\begin{aligned}\sqrt{(x - 0)^2 + (y - a)^2} &= y + a \\ x^2 + (y - a)^2 &= (y + a)^2 \\ x^2 + y^2 - 2ay + a^2 &= y^2 + 2ay + a^2 \\ x^2 &= 4ay\end{aligned}$$

The vertex is at $(0, 0)$ and the y-axis is the **axis of symmetry**.

If the vertex is translated to (h, k), the equation becomes $(x - h)^2 = 4a(y - k)$ with the focus $(h, a + k)$ and the directrix $y = -a + k$.

If the direction is chosen with the y-axis parallel to the directrix, then x and y are interchanged in the equation of the 'horizontal' or 'sideways' parabola. In this case, the axis of symmetry is horizontal.

Parabola with centre at (h, k)

	Vertical parabola	Horizontal parabola
Vertex	(h, k)	(h, k)
Equation	$(x - h)^2 = 4a(y - k)$	$(y - k)^2 = 4a(x - h)$
Focus	$(h, a + k)$	$(a + h, k)$
Directrix	$y = -a + k$	$x = -a + h$
Axis of symmetry	$x = h$	$y = k$

A 'vertical' parabola has its axis of symmetry parallel to the y-axis.

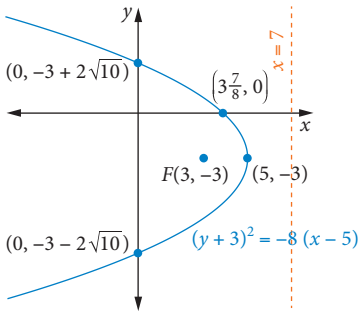
A 'horizontal' parabola has its axis of symmetry parallel to the x-axis.

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WORKED EXAMPLE 32 Horizontal parabola

Sketch the graph of $(y + 3)^2 = -8(x - 5)$.

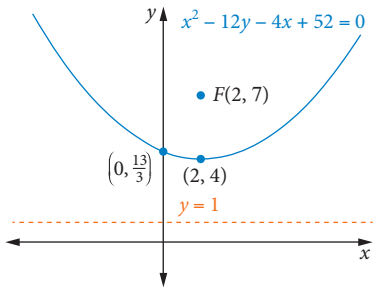
Steps	Working
1 Write the vertex.	$h = 5$, $k = -3$, so the vertex is at $(5, -3)$.
2 Write the other characteristics.	It is a horizontal parabola. $4a = -8$, so $a = -2$, and the focus is at $x = -2 + 5 = 3$, $y = -3$ and the directrix is at $x = 2 + 5 = 7$. The axis of symmetry is horizontal, $y = -3$.
3 Find any intercepts.	The y-intercepts ($x = 0$) are at $y = -3 \pm 2\sqrt{10}$. The x-intercept ($y = 0$) is $\left(3\frac{7}{8}, 0\right)$.
4 Sketch the graph.	

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WORKED EXAMPLE 33 Vertical parabola

Sketch the graph of $x^2 - 12y - 4x + 52 = 0$.

Steps	Working
1 Express in standard form by completing the square.	$x^2 - 4x + 4 = 12y - 48$ $(x - 2)^2 = 12(y - 4)$ is a vertical parabola.
2 Write the vertex.	$h = 2$, $k = 4$, so the vertex is at $(2, 4)$.
3 Write the other characteristics.	$a = 3$, so the focus is at $x = 2$, $y = 3 + 4 = 7$ and the directrix is $y = -3 + 4 = 1$. The axis of symmetry is vertical, $x = 2$.
4 Find any intercepts.	The y-intercept ($x = 0$) is $\left(0, \frac{13}{3}\right)$. There are no x-intercepts.
5 Sketch the graph.	

We don't normally show the focus and directrix when sketching a parabola, unless the question specifically asks for them.

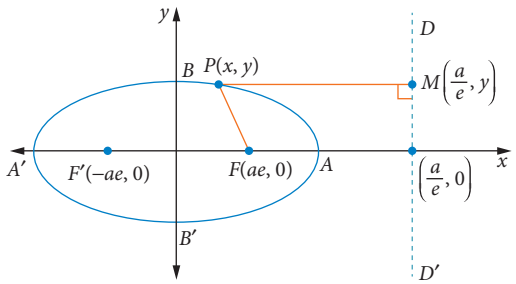
Ellipses

An **ellipse** is the locus of points such that the distance from $P(x, y)$ to the focus is e times the distance from P to the directrix, where $0 < e < 1$. e is called the **eccentricity** of the conic section.

Choose the axes so the focus is on the x-axis and the directrix is parallel to the y-axis.

Then choose a so the distance from the focus to the directrix is $a\left(\frac{1}{e} - e\right)$.

Then the focus is $F(ae, 0)$ and the directrix is $x = \frac{a}{e}$, as shown in the diagram.



Then $PF = \sqrt{(x - ae)^2 + (y - 0)^2}$ and $PM = \frac{a}{e} - x$.

$$
\begin{aligned}
PF &= ePM \\
\sqrt{(x - ae)^2 + (y - 0)^2} &= a - ex \\
(x - ae)^2 + y^2 &= (a - ex)^2 \\
x^2 - 2aex + a^2e^2 + y^2 &= a^2 - 2aex + e^2x^2 \\
x^2(1 - e^2) + y^2 &= a^2(1 - e^2) \\
\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} &= 1
\end{aligned}
$$

Writing $b^2 = a^2(1 - e^2)$ gives the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

By symmetry, there must also be a second focus $F'(-ae, 0)$ and directrix $x = -\frac{a}{e}$.

The **centre** of the ellipse is $(0, 0)$ and the **vertices** $A(a, 0)$ and $A'(-a, 0)$. The y-intercepts are $B(0, b)$ and $B'(0, -b)$.

If the centre is translated to (h, k), the equation becomes $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$.

The horizontal axis $A'A$ is called the **major axis** and $B'B$ is the **minor axis**.

We can show that the sum of the distances from P to the foci, $PF + PF' = 2a$.

A vertical ellipse has a horizontal directrix and the equation $\frac{(y - k)^2}{a^2} + \frac{(x - h)^2}{b^2} = 1$.

Ellipses

An ellipse with centre at (h, k) has the following characteristics.

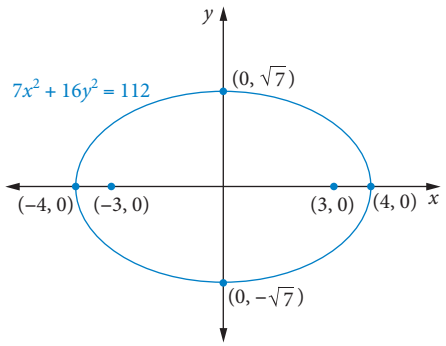
	Major axis horizontal	Major axis vertical
Centre	(h, k)	(h, k)
Equation	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$
Eccentricity	$e^2 = 1 - \frac{b^2}{a^2}$	$e^2 = 1 - \frac{b^2}{a^2}$
Foci	$(h \pm ae, k)$	$(h, k \pm ae)$
Directrices	$x = h \pm \frac{a}{e}$	$y = k \pm \frac{a}{e}$
Vertices	$(h \pm a, k), (h, k \pm b)$	$(h \pm b, k), (h, k \pm a)$

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WORKED EXAMPLE 34 Horizontal ellipse

Sketch the graph of $7x^2 + 16y^2 = 112$.

Steps	Working
1 Write in standard form.	$\frac{7x^2}{112} + \frac{16y^2}{112} = 1$ $\frac{x^2}{16} + \frac{y^2}{7} = 1$
2 Write a and b.	$a = 4, b = \sqrt{7}$
3 Find e.	$e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{7}{16} = \frac{9}{16}$ $e = \frac{3}{4}$ $(0 < e < 1)$
4 Find the foci.	The foci are $(\pm ae, 0) = (\pm 3, 0)$.
5 Sketch the graph, showing the intercepts and foci.	

It is usual to show the foci when you sketch an ellipse.

Hyperbolas

A hyperbola is the locus such that the distance from $P(x, y)$ to the focus is e times the distance from P to the directrix, where the eccentricity is $e > 1$.

Like an ellipse, there are 2 foci, $F(\pm ae, 0)$ (for a horizontal hyperbola) and 2 directrices, $x = \pm\frac{a}{e}$.

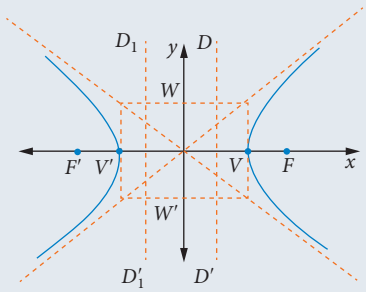
For a vertical hyperbola, they are $F(0, \pm ae)$ and $y = \pm\frac{a}{e}$.

The derivation of the equation, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, is similar to those of the parabola and ellipse, but a hyperbola has 2 **branches**.

Rewriting the equation, $y = \pm b\sqrt{\frac{x^2}{a^2} - 1}$ as $x \to \pm\infty$, $y \to \pm b\sqrt{\frac{x^2}{a^2}} = \pm\frac{b}{a}x$, so there are 2 asymptotes, $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$.

Features of a hyperbola

The equation of a horizontal hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.



The **vertices** and x-intercepts are $(\pm a, 0)$.

The **transverse axis** of the hyperbola is $V'V$ of length $2a$.

The **eccentricity** is given by $b^2 = a^2(e^2 - 1)$ or $e^2 = 1 + \frac{b^2}{a^2}$.

The **directrices**, $x = \pm\frac{a}{e}$, are inside the hyperbola.

The **foci**, $(\pm ae, 0)$, are outside the hyperbola.

There are 2 **branches**, one on each side.

The **auxiliary rectangle** passes through $V(a, 0)$, $V'(-a, 0)$, $W(0, b)$ and $W'(0, -b)$.

The **asymptotes**, $y = \pm\frac{b}{a}x$, pass through the corners of the auxiliary rectangle.

The **difference of the distances from the foci** of a point on the hyperbola is always $2a$.

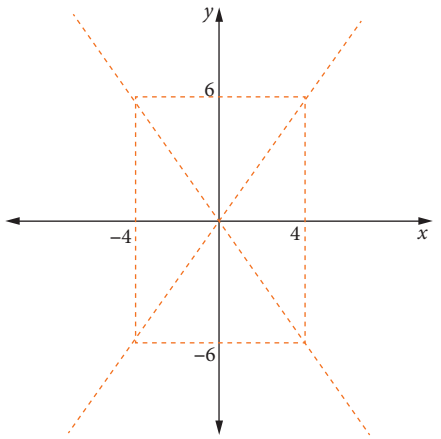
WORKED EXAMPLE 35 Horizontal hyperbola

Sketch the graph of $\frac{x^2}{16} - \frac{y^2}{36} = 1$.

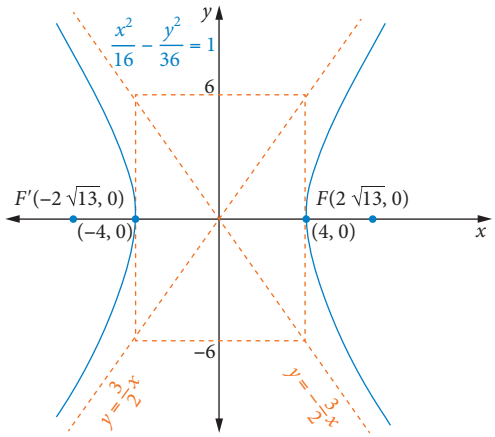
Steps	Working
1 Find the values of a and b.	$a = \sqrt{16} = 4$, $b = \sqrt{36} = 6$
2 Find e and the foci.	$e^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{36}{16} = \frac{52}{16}$, $e = \frac{\sqrt{13}}{2}$
	$ae = 2\sqrt{13}$

WB
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3 Sketch the auxiliary rectangle and use the corners to sketch the asymptotes.



4 Use the asymptotes and auxiliary rectangle to sketch the hyperbola and mark the vertices and foci.



Unlike an ellipse, a does not have to be greater than b in a hyperbola.

Like an ellipse, however, we can have a vertical hyperbola, and the centre can be translated to (h, k).

Hyperbolas

A hyperbola with centre at (h, k) has the following characteristics.

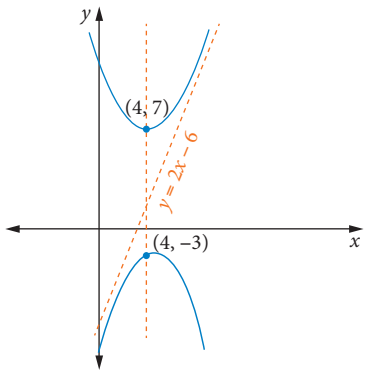
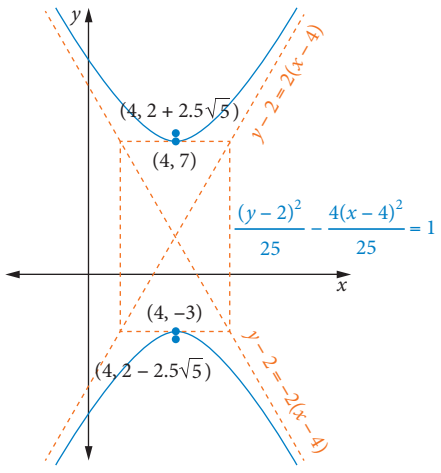
	Transverse axis horizontal	Transverse axis vertical
Centre	(h, k)	(h, k)
Equation	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Eccentricity	$e^2 = 1 + \frac{b^2}{a^2}$	$e^2 = 1 + \frac{b^2}{a^2}$
Foci	$(h \pm ae, k)$	$(h, k \pm ae)$
Directrices	$x = h \pm \frac{a}{e}$	$y = k \pm \frac{a}{e}$
Vertices	$(h \pm a, k)$	$(h, k \pm a)$
Asymptotes	$y - k = \pm\frac{b}{a}(x - h)$	$y - k = \pm\frac{a}{b}(x - h)$

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WORKED EXAMPLE 36 Vertical hyperbola

Find the equation and sketch the graph of a hyperbola with vertices at $(4, 7)$ and $(4, -3)$ and an asymptote given by $y = 2x - 6$.

Steps	Working
1 Make a rough sketch, showing the vertices and the asymptote. The vertices are in a vertical line, so the transverse axis must be vertical.	
2 Find the centre, a, b, e and the foci.	centre (h, k) = midpoint of $(4, 7)$ and $(4, -3)$ = $(4, 2)$ a = distance from centre to vertex = 5 gradient of asymptote = $2 = \frac{a}{b}$, so $b = 2.5$ $e^2 = 1 + \frac{b^2}{a^2} = 1 + \left(\frac{2.5}{5}\right)^2 = \frac{5}{4}$, $e = \frac{\sqrt{5}}{2}$ $ae = 2.5\sqrt{5}$
3 Substitute a, b, h, k for the equation and simplify.	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ $\frac{(y-2)^2}{5^2} - \frac{(x-4)^2}{(2.5)^2} = 1$ $\frac{(y-2)^2}{25} - \frac{4(x-4)^2}{25} = 1$
4 Sketch the graph.	

We normally show the vertices, curve and asymptotes in the final sketch. Unless specifically asked, do not show the foci or directrices of a hyperbola. We can use the auxiliary rectangle as an aid to sketching the curve. Find the intercepts if requested.

EXERCISE 8.7 Locus definitions of curves and conic sections

ANSWERS p. 594

Recap

1 The solution of $|3x - 8| \geq 4x - 6$ is

A $x \leq 2$ **B** $x \leq -2$ **C** $x \geq -2$ **D** $-2 \leq x \leq 2$ **E** $-14 \leq x \leq -2$

2 A possible equation for the graph on the right is

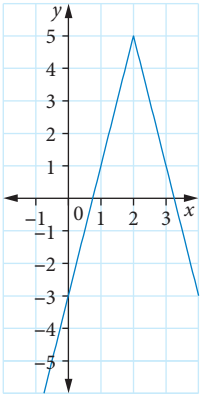
A $y = 2|x - 2| + 5$

B $y = 4|x + 2| + 5$

C $y = -2|x - 2| + 5$

D $y = -2|2x - 4| + 5$

E $y = -4|x + 2| + 5$



Mastery

3 WORKED EXAMPLES 29, 30 Find the equation of each locus and describe the locus.

a Points equidistant from $A(3, 4)$ and $B(9, 0)$.

b Points equidistant from $A(2, -2)$ and $B(5, -6)$.

c The distance from P to the x-axis is 4 times its distance to the y-axis.

d For $A(1, 3)$ and $B(-2, -5)$, $PA^2 - PB^2 = 17$.

e The centres of circles that pass through both $(0, 2)$ and $(5, -3)$.

4 Find the equation of each locus and describe the locus.

a Points a distance of 5 from $(3, 1)$.

b For $N(-5, 7)$ and $M(3, 4)$, points P such that $PN \perp PM$.

c For $S(3, 2)$ and $T(2, -1)$, points P such that $PS^2 = 2PT^2$.

d The distance from P to $A(-2, 12)$ is twice its distance from $B(4, -6)$.

e Points such that the sum of the squares of their distances from $A(5, 3)$ and $B(1, 3)$ is 25 units.

5 WORKED EXAMPLE 31

a Find the centre and radius of the circle with equation $x^2 + y^2 - 6x + 8y = 56$.

b Find the centre and radius of the circle with equation $4x^2 + 28x + 4y^2 - 40y = 43$.

c Write the equation of a circle with centre $(5, -3)$ and radius 7.

d Write the equation of a circle with centre $(-2, 4)$ and radius 4.

e Find the equation of a circle that just fits in the square $A(3, 5)$, $B(8, 9)$, $C(12, 4)$, $D(7, 0)$.

f Find the centre, radius and equation of a circle in which the x-axis is a tangent at $(4, 0)$ and with y-intercepts at $(0, 2)$ and $(0, 6)$.

g An isosceles triangle has vertices $(1, 0)$, $(-5, 2)$ and $(-1, -6)$. Find the equation of the circle that passes through its vertices.

h Find the equation of a circle that just fits inside the parallelogram $ABCD$, where A, B, C and D have coordinates $(3, -2)$, $(5, -8)$, $(7, -6)$ and $(5, 0)$ respectively.

6 WORKED EXAMPLES 32, 33

a Sketch the graph of

i $(x+2)^2 = 12(y-4)$ **ii** $(y-5)^2 = 6(1-x)$

iii $x^2 - 3y - 9 = 0$ **iv** $x + 4 = y^2 + 12y + 36$

v $5x^2 + 40x + y + 78 = 0$ **vi** $y^2 - 18y - 16x + 45 = 0$

b Write the equation and sketch the graph of a parabola

i with vertex $(3,5)$ and focus $(3,9)$

ii with directrix $x = -2$ and vertex $(3,2)$.

7 WORKED EXAMPLE 34

a Sketch the graph of

i $\frac{(y+2)^2}{16} + \frac{(x-3)^2}{9} = 1$ **ii** $\frac{(x+5)^2}{25} + \frac{(y-4)^2}{4} = 1$

iii $\frac{(x+3)^2}{8} + \frac{(y+2)^2}{12} = 1$ **iv** $\frac{(x-5)^2}{36} + \frac{(y-4)^2}{18} = 1$

v $2x^2 + 3y^2 + 6y = 12x + 3$ **vi** $2x^2 + y^2 + 57 = 10y - 24x$

b Write the equation and sketch the graph of

i an ellipse with major axis of length 12 and a minor axis of length 8 with the left end at $(4,3)$

ii an ellipse with foci at $(7,4)$ and $(-3,4)$ and semi-major axis of length 3

iii an ellipse with centre $(1,4)$, focus $(1,5)$ and vertex $(1,9)$

iv the locus of a point that moves such that the sum of its distances from the points $(3,4)$ and $(3,12)$ is 10.

8 WORKED EXAMPLES 35, 36

a Sketch the graph of each equation. Show asymptotes and intercepts.

i $\frac{(y-3)^2}{25} - \frac{(x-4)^2}{9} = 1$ **ii** $\frac{(x+2)^2}{4} - \frac{(y-5)^2}{16} = 1$

iii $\frac{(x+5)^2}{36} - \frac{(y+3)^2}{24} = 1$ **iv** $\frac{(y+2)^2}{20} - \frac{(x-1)^2}{45} = 1$

v $6(x+3)^2 = (y-4)^2 + 6$ **vi** $8y^2 + 32y = 5x^2 + 30x + 53$

b Find the equation and sketch the graph of

i a hyperbola with a focus at $(5,1)$, a vertex at $(4,1)$ and centre at $(1,1)$

ii a hyperbola with asymptotes $y = 2x - 2$, $y = -2x + 8$ and a vertex at $(4,3)$

iii a hyperbola with foci at $(1,6)$ and $(1,-8)$ if one vertex is at $(1,3)$

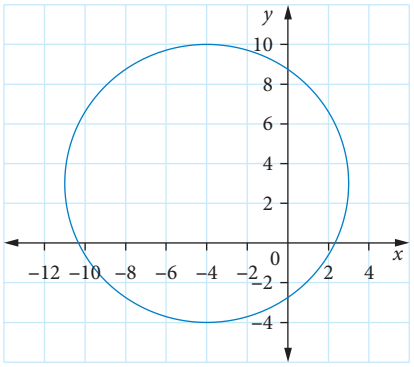
iv the locus of a point that moves such that the difference of its distances from the points $(3,3)$ and $(3,9)$ is 2.

9 Three of the vertices of a rectangle are $A(-3,2)$, $B(1,2)$ and $C(1,9)$. Find the equation of an ellipse that just fits in the rectangle.

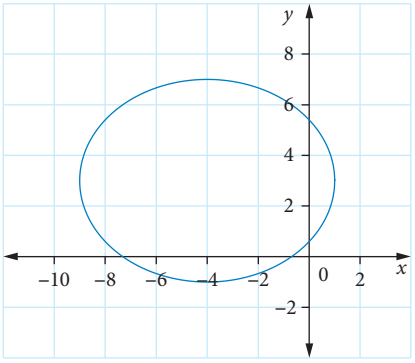
10 A hyperbola with a transverse axis parallel to the x-axis passes through the rectangle with vertices $(1,-3)$, $(1,5)$, $(-9,5)$ and $(-9,-3)$. Write possible equations of the hyperbola in terms of the eccentricity, e.

11 These are the graphs of conic sections. Write possible equations for each one.

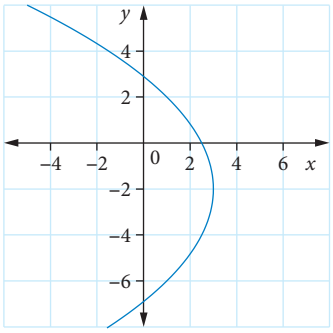
a



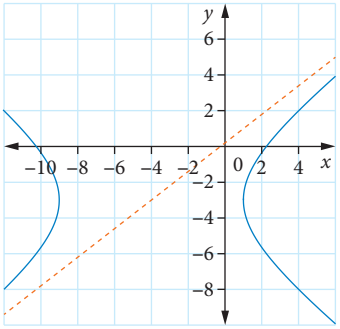
b



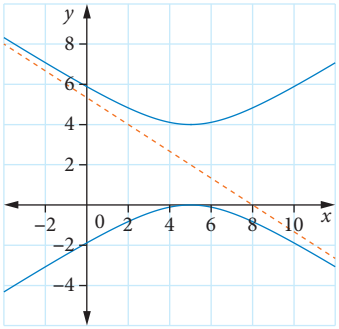
c



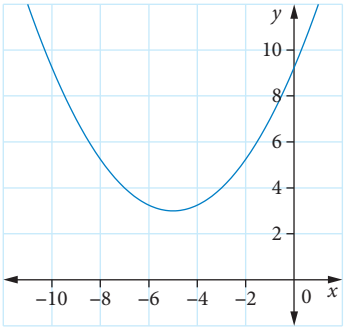
d



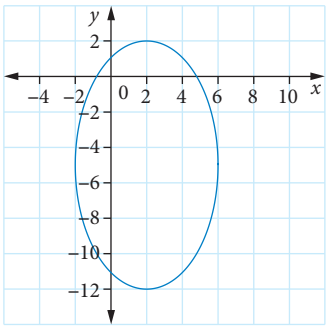
e



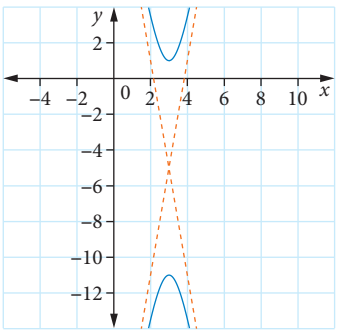
f



g



h



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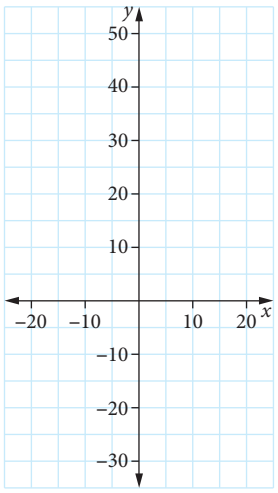
Exam practice

80–100% 60–79% 0–59%

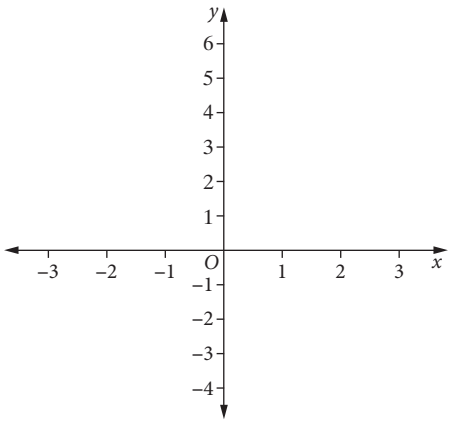
12 © VCAA 2011 2BQ3a **90%** (4 marks)

Sketch the ellipse with the equation $\frac{x^2}{400} + \frac{(y-10)^2}{900} = 1$ on a copy of the axes shown on the right.

Write the intercepts with the x-axis.



13 © VCAA 2010 1Q9a **63%** TECH-FREE (3 marks) Copy the axes shown on the right, and on them sketch the graph with equation $x^2 - \frac{(y-2)^2}{4} = 1$. State all intercepts with the coordinate axes and give the equations of any asymptotes.



14 © VCAA 2012 2AQ2 **85%** A rectangle is drawn so that its sides lie on the lines with equations $x = -2$, $x = 4$, $y = -1$ and $y = 7$.

An ellipse is drawn inside the rectangle so that it just touches each side of the rectangle.

The equation of the ellipse could be

A $\frac{x^2}{9} + \frac{y^2}{16} = 1$ **B** $\frac{(x+1)^2}{9} + \frac{(y+3)^2}{16} = 1$ **C** $\frac{(x-1)^2}{9} + \frac{(y-3)^2}{16} = 1$

D $\frac{(x+1)^2}{36} + \frac{(y+3)^2}{64} = 1$ **E** $\frac{(x-1)^2}{36} + \frac{(y-3)^2}{64} = 1$

15 © VCAA 2014 2AQ1 **85%** The asymptotes of the hyperbola given by $\frac{(x-3)^2}{9} - \frac{y^2}{4} = 1$ intersect the coordinate axes at

A $(0, -4.5), (0, 4.5), (-3, 0)$ **B** $(0, 2), (0, -2), (3, 0)$ **C** $(0, 2), (0, -2), (-3, 0)$

D $(0, -4.5), (0, 4.5), (3, 0)$ **E** $(2, 0), (-2, 0), (0, -3)$

16 © VCAA 2010 2AQ1 **83%** An ellipse has a horizontal semi-axis length of 1 and a vertical semi-axis length of 2. The centre of the ellipse is at the point with coordinates $(3, -5)$.

A possible equation for the ellipse is

A $(x+3)^2 + 4(y-5)^2 = 4$ **B** $(x-3)^2 + 4(y+5)^2 = 4$ **C** $4(x+3)^2 + (y-5)^2 = 4$

D $4(x-3)^2 + (y+5)^2 = 4$ **E** $4(x-3)^2 + (y+5)^2 = 1$

17 © VCAA 2011 2AQ2 **75%** A circle with centre $(a, -2)$ and radius 5 units has equation $x^2 - 6y + y^2 + 4y = b$, where a and b are constants.

The values of a and b are respectively

A -3 and 38 **B** 3 and 12 **C** -3 and -8 **D** -3 and 0 **E** 3 and 18

18 © VCAA 2014 2AQ2 **71%** The ellipse given by $x^2 - 6x + 2y^2 + 8y + 16 = 0$ has centre, length of horizontal semi-axis and length of vertical semi-axis respectively of

A $(-3, 2)$, 1, 2 **B** $(-2, 3)$, 1, $\frac{1}{\sqrt{2}}$ **C** $(3, -2)$, $\frac{1}{2}$, 1

D $(-3, 2)$, $\frac{1}{2}$, 1 **E** $(3, -2)$, 1, $\frac{1}{\sqrt{2}}$

19 © VCAA 2011 2AQ4 **70%** The hyperbola $\frac{(y+3)^2}{9} - \frac{(x-6)^2}{4} = 1$ has asymptotes given by

A $y = \pm\frac{3}{2}x$ **B** $y = \frac{2}{3}x - 7$ and $y = -\frac{2}{3}x - 7$

C $y = \frac{2}{3}x - 7$ and $y = -\frac{2}{3}x + 1$ **D** $y = \frac{3}{2}x - 12$ and $y = -\frac{3}{2}x + 6$

E $y = \frac{9}{4}x - \frac{33}{2}$ and $y = -\frac{9}{4}x + \frac{21}{2}$

20 © VCAA 2010 2AQ2 **60%** Each of the following equations represents a hyperbola.

Which hyperbola **does not** have perpendicular asymptotes?

A $(x-1)^2 - (y+2)^2 = 1$ **B** $x^2 - 2x - y^2 + 4y = 4$ **C** $(x-1)^2 - (y+2)^2 = 9$

D $(y-1)^2 - (x+2)^2 = 1$ **E** $2x^2 - 4x - y^2 - 4y = 4$

Video playlist
Polar and parametric forms of the conic sections

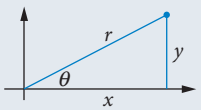
8.8 Polar and parametric forms of the conic sections

Polar and parametric equations

Polar coordinates in the plane express the position of a point in terms of the distance r of a point from the origin and the anticlockwise angle θ from the x-axis.

Conversions between (x, y) and (r, θ):

$\tan(\theta) = \frac{y}{x}$, $r^2 = x^2 + y^2$



$x = r\cos(\theta)$, $y = r\sin(\theta)$

A **polar form** of the equation of a graph has the equation expressed in terms of r and θ.

A **parametric form** of the equation of a graph has 2 equations expressing x and y in terms of an independent variable such as t or θ. The independent variable is called the **parameter**.

Ellipses

The polar form of the equation of an ellipse is derived in the same way as the Cartesian equation.

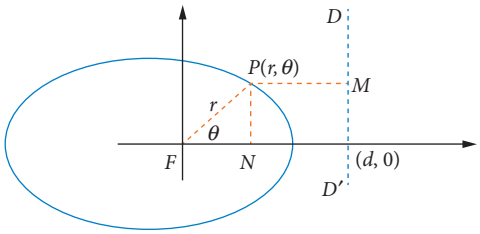
A focus is placed at the origin. A directrix is placed perpendicular to an axis at a distance d from the origin. The diagram shows the situation when the directrix passes through $(d, 0)$.

For an ellipse, $0 < e < 1$ and $\frac{PF}{PM} = e$, so $PF = ePM$.

Now $PF = r$ and $PM = d - r\cos(\theta)$, thus $r = e(d - r\cos(\theta))$.

This gives $r = \frac{de}{1 + e\cos(\theta)}$.

If the directrix is on the other side, we get $r = \frac{de}{1 - e\cos(\theta)}$.



If the ellipse is vertical, the directrix is above or below, so we get $\sin(\theta)$ instead of $\cos(\theta)$.

Note that $d = \frac{a}{e} - ae = \frac{a(1 - e^2)}{e}$ as d is the distance between a directrix and focus.

The polar form of an ellipse

The **polar form of an ellipse** is given by one of

$$r = \frac{de}{1 \pm e\cos(\theta)} \text{ or } r = \frac{de}{1 \pm e\sin(\theta)}.$$

One of the foci is at $(0, 0)$.

A horizontal ellipse has $\cos(\theta)$ and a vertical ellipse has $\sin(\theta)$.

For a horizontal ellipse, the vertices at the ends of the major axis are at $\theta = 0$ and $\theta = \pi$.

For a vertical ellipse, the vertices at the ends of the major axis are at $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$.

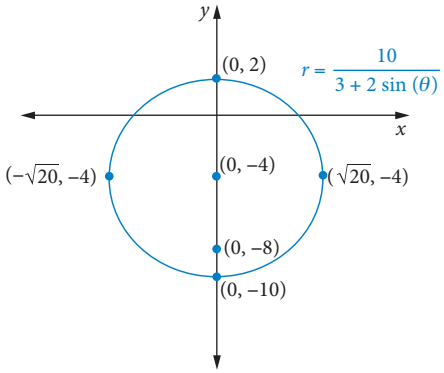
We can change from the polar to the **Cartesian forms** of a horizontal or vertical ellipse. Find the centre and the values of a and b from e and the vertices.

WORKED EXAMPLE 37 Polar to Cartesian form of an ellipse

Find the equation of $r = \frac{10}{3 + 2\sin(\theta)}$ in Cartesian form and sketch the graph.

Steps	Working
1 Write the polar equation in standard form to find e.	$r = \frac{10}{3 + 2\sin(\theta)} = \frac{\frac{10}{3}}{1 + \frac{2}{3}\sin(\theta)}$, so $e = \frac{2}{3}$
2 State the type of conic.	$e < 1$, so it is an ellipse. It is vertical because of $\sin(\theta)$.
3 Substitute $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$ to find the major vertices.	The major vertices are at $r\left(\frac{\pi}{2}\right) = \frac{10}{3 + 2\sin\left(\frac{\pi}{2}\right)} = \frac{10}{5} = 2$ and at $r\left(\frac{3\pi}{2}\right) = \frac{10}{3 + 2\sin\left(\frac{3\pi}{2}\right)} = \frac{10}{1} = 10.$
4 Change to Cartesian form and find a.	$\left(\frac{\pi}{2}, 2\right) = (0, 2)$ and $\left(\frac{3\pi}{2}, 10\right) = (0, -10)$ $\Rightarrow (h, k) = (0, -4)$ $2a = 2 - (-10) \Rightarrow a = 6$

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5 Find the value of b.	$b^2 = 6^2\left(1 - \left(\frac{2}{3}\right)^2\right) = \frac{36 \times 5}{9} \Rightarrow b = 2\sqrt{5}$
6 Write the standard form of the vertical Cartesian equation.	$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$
7 Substitute values, taking account of the translation of the centre.	$\frac{(x-0)^2}{(2\sqrt{5})^2} + \frac{(y-(-4))^2}{6^2} = 1$
8 Simplify to write the answer.	The Cartesian form is $\frac{x^2}{20} + \frac{(y+4)^2}{36} = 1$.
9 Sketch the graph.	

We can change from the Cartesian to the polar forms of a horizontal or vertical ellipse, provided one focus is at $(0, 0)$. Find the values of e from a and b. d is the distance from the focus at $(0, 0)$ to the directrix, so $d = \frac{a}{e} - ae$.

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WORKED EXAMPLE 38 Cartesian to polar form of an ellipse

Express $\frac{(x - 3\sqrt{3})^2}{36} + \frac{y^2}{9} = 1$ in polar form.

Steps	Working
1 State the type of conic.	$a = 6 > 3$ is under the x^2 so it is a horizontal ellipse.
2 Find the value of e.	$e^2 = \left(1 - \left(\frac{3}{6}\right)^2\right) = \frac{3}{4}$, so $e = \frac{\sqrt{3}}{2}$.
3 Check that a focus is at $(0, 0)$.	$h - ae = 3\sqrt{3} - 6 \times \frac{\sqrt{3}}{2} = 0$ ✓ (left)
4 Find the value of d.	$d = \frac{a}{e} - ae = 6 \div \frac{\sqrt{3}}{2} - 6 \times \frac{\sqrt{3}}{2} = \sqrt{3}$
5 Decide on the equation form.	The ellipse is horizontal with the left focus at $(0, 0)$, so it has $-e\cos(\theta)$.
6 Write the general polar equation.	$r = \frac{de}{1 - e\cos(\theta)}$
7 Substitute values and simplify.	$r = \frac{\sqrt{3} \times \frac{\sqrt{3}}{2}}{1 - \frac{\sqrt{3}}{2}\cos(\theta)} = \frac{3}{2 - \sqrt{3}\cos(\theta)}$
8 Write the answer.	The polar form is $r = \frac{3}{2 - \sqrt{3}\cos(\theta)}$.

The parametric form of an ellipse

The horizontal ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$, has **parametric form** $\begin{cases} x = a\cos(\theta) \\ y = b\sin(\theta) \end{cases}$ for $0 \le \theta < 2\pi$, where $a > b$.

The vertical ellipse, $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where $a > b$, has parametric form $\begin{cases} x = b\cos(\theta) \\ y = a\sin(\theta) \end{cases}$ for $0 \le \theta < 2\pi$.

WORKED EXAMPLE 39 Cartesian to parametric form of an ellipse

Write the equation of $\frac{(x-3)^2}{25} + \frac{(y+2)^2}{16} = 1$ in parametric form.

Steps	Working
1 State the type of conic.	It is an ellipse; $a = 5 > 4$ is under the x^2 so it is horizontal.
2 Write the values of a and b.	$a = 5$ and $b = 4$
3 Write the parametric form, including an adjustment for the translation.	The parametric form is $\begin{cases} x = 5\cos(\theta) + 3 \\ y = 4\sin(\theta) - 2 \end{cases}$ for $0 \le \theta < 2\pi$.

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Hyperbolas

The polar form of a hyperbola

The polar form of a hyperbola is given by one of

$$r = \frac{de}{1 \pm e\cos(\theta)} \text{ or } r = \frac{de}{1 \pm e\sin(\theta)}.$$

One of the foci is at $(0, 0)$.

A horizontal hyperbola has $\cos(\theta)$ and a vertical hyperbola has $\sin(\theta)$.

For a horizontal hyperbola, the vertices are at $\theta = 0$ and $\theta = \pi$.

For a vertical hyperbola, the vertices are at $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$.

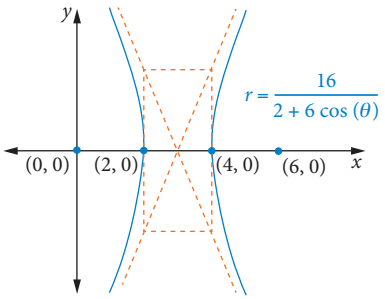
Notice that the polar forms of a hyperbola and an ellipse *are the same*. The value of e determines if it is an ellipse or hyperbola.

WORKED EXAMPLE 40 Polar to Cartesian form of a hyperbola

Write the equation of $r = \frac{16}{2 + 6\cos(\theta)}$ in Cartesian form and sketch the graph.

Steps	Working
1 Write in standard form to find the value of e.	$r = \frac{16}{2 + 6\cos(\theta)} = \frac{8}{1 + 3\cos(\theta)}$.
2 Use the value e to identify the conic.	$e = 3$, so it is a (horizontal) hyperbola.
3 Find the vertices.	The vertices are at $x = r(0) = 2$ and $r(\pi) = -4$, so are $(0, 2)$ and $(\pi, -4)$.
4 Write the vertices in Cartesian form.	$(0, 2) \to (2, 0)$ and $(\pi, -4) \to (4, 0)$

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5 Find the centre (the midpoint of VV') and the value of a ($VV' = 2a$).	The centre is at $(3, 0)$. $2a = 4 - 2 \Rightarrow a = 1$
6 Find the value of b.	$b^2 = a^2(e^2 + 1) = 1(9 - 1) \Rightarrow b = 2\sqrt{2}$
7 Write the standard equation of the horizontal hyperbola.	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$
8 Substitute values.	$\frac{(x-3)^2}{1} - \frac{(y-0)^2}{(2\sqrt{2})^2} = 1$
9 Simplify.	$(x-3)^2 - \frac{y^2}{8} = 1$
10 Sketch the graph.	

To change from the Cartesian to the polar form, one focus must be at $(0, 0)$. The distance from that focus to the directrix will be $d = ae - \frac{a}{e}$.

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WORKED EXAMPLE 41 Cartesian to polar form of a hyperbola

Write the equation of $\frac{(3y-20)^2}{81} - \frac{x^2}{16} = 1$ in polar form.

Steps	Working
1 Write in standard form to find the values of a and b.	$\frac{\left(y - \frac{20}{3}\right)^2}{9} - \frac{x^2}{16} = 1$ is a vertical hyperbola.
2 Write the values of a and b.	$a = 4$ and $b = 3$
3 Find the value of e.	$e^2 = 1 + \left(\frac{b}{a}\right)^2 = 1 + \left(\frac{4}{3}\right)^2 = \frac{25}{9}$, $e = \frac{5}{3}$
4 State the centre.	The centre is $\left(0, \frac{20}{3}\right)$.
5 Check that a focus is at $(0, 0)$.	$k - ae = \frac{20}{3} - 4 \times \frac{5}{3} = 0$ ✓ (lower)
6 Find the value of d.	$d = ae - \frac{a}{e} = \frac{20}{3} - \frac{12}{5} = \frac{64}{15}$
7 Write the standard polar form for a vertical hyperbola with the lower focus at $(0, 0)$.	$r = \frac{de}{1 + e\sin(\theta)} = \frac{\frac{64}{15} \times \frac{5}{3}}{1 + \frac{5}{3}\sin(\theta)} = \frac{64}{9 + 15\sin(\theta)}$
8 Write the answer.	The polar form is $r = \frac{64}{9 + 15\sin(\theta)}$.

The parametric form of a hyperbola

The horizontal hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, is $\begin{cases} x = a\sec(\theta) \\ y = b\tan(\theta) \end{cases}$ for $0 \leq \theta < 2\pi \left(\theta \neq \frac{\pi}{2}, \frac{3\pi}{2}\right)$

or $\begin{cases} x = a\operatorname{cosec}(\theta) \\ y = b\cot(\theta) \end{cases}$ for $0 < \theta < 2\pi\ (\theta \neq \pi)$.

To get the vertical hyperbola, interchange x and y.

The vertical hyperbola, $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$, is $\begin{cases} y = a\sec(\theta) \\ x = b\tan(\theta) \end{cases}$ for $0 \leq \theta < 2\pi \left(\theta \neq \frac{\pi}{2}, \frac{3\pi}{2}\right)$

or $\begin{cases} y = a\operatorname{cosec}(\theta) \\ x = b\cot(\theta) \end{cases}$ for $0 < \theta < 2\pi\ (\theta \neq \pi)$.

The parametric forms are easily shown to be correct using the identities $\tan^2(\theta) + 1 = \sec^2(\theta)$ or $1 + \operatorname{cosec}^2(\theta) = \cot^2(\theta)$.

WORKED EXAMPLE 42 Parametric to Cartesian form of a hyperbola

Write the pair of equations $\begin{cases} x = \dfrac{4}{\cos(\theta)} \\ y = 6\tan(\theta) \end{cases}$ in Cartesian form.

Steps	Working
1 State the type of conic.	It is a horizontal hyperbola.
2 Write the values of a and b.	$a = 4$ and $b = 6$
3 Write the standard Cartesian form.	$\frac{x^2}{4^2} - \frac{y^2}{6^2} = 1$
4 Simplify and write the answer.	The Cartesian form is $\frac{x^2}{16} - \frac{y^2}{36} = 1$.

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Parabolas

The equation of the polar form of a parabola is the same as those of the ellipse and hyperbola. It is distinguished by having $e = 1$.

The polar form of a parabola

The **polar form of a parabola** is given by one of

$$r = \frac{de}{1 \pm e\cos(\theta)} \text{ or } r = \frac{de}{1 \pm e\sin(\theta)}, \text{ where } e = 1.$$

The focus is at $(0, 0)$.

A horizontal parabola has $\cos(\theta)$ and a vertical parabola has $\sin(\theta)$.

For a horizontal hyperbola, the vertex is at $\theta = 0$ for $\cos(\theta)$ and $\theta = \pi$ for $-\cos(\theta)$.

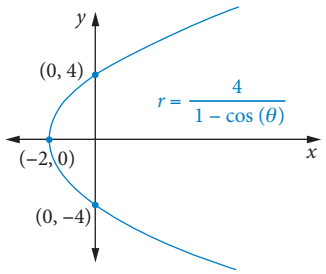
For a vertical parabola, the vertex is at $\theta = \frac{\pi}{2}$ for $\sin(\theta)$ and $\theta = \frac{3\pi}{2}$ for $-\sin(\theta)$.

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WORKED EXAMPLE 43 Polar to Cartesian form of a parabola

Write the equation of $r = \dfrac{4}{1 - \cos(\theta)}$ in Cartesian form and sketch the graph.

Steps	Working
1 State the type of conic.	$e = 1$ and it has $\cos(\theta)$, so it is a horizontal parabola.
2 Find the vertex and other characteristics.	$r(\pi) = 2$, so the vertex is at $(2, \pi)$. The Cartesian coordinates are $(-2, 0)$, so the vertex has been translated 2 units to the left.
3 Find the value of a.	The focus is $(0, 0)$, so $a = 2$.
4 Write the equation.	$(y - k)^2 = 4a(x - h)$
5 Substitute values.	$y^2 = 8(x + 2)$
6 Find the intercepts.	The intercepts are $(-2, 0)$, $(0, 4)$ and $(0, -4)$.
7 Sketch the graph.	

The parametric form of a parabola

The horizontal parabola, $y^2 = 4ax$, has **parametric form** $\begin{cases} x = at^2 \\ y = 2at \end{cases}$ for $t \in R$.

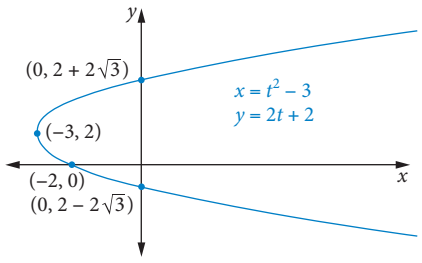
The vertical hyperbola, $x^2 = 4ay$, has **parametric form** $\begin{cases} x = 2at \\ y = at^2 \end{cases}$ for $t \in R$.

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WORKED EXAMPLE 44 Parametric to Cartesian form of a parabola

Write the equation of $\begin{cases} x = t^2 - 3 \\ y = 2t + 2 \end{cases}$ for $t \in R$ in Cartesian form and sketch the graph.

Steps	Working
1 State the type of function.	It is a horizontal parabola with $a = 1$.
2 What other characteristics does it have?	It has been translated 3 units to the left and 2 units up.
3 Write the equation in Cartesian form.	$(y - 2)^2 = 4(x + 3)$
4 Find the y-intercepts.	At $x = 0$, $y = 2 \pm 2\sqrt{3}$
5 Sketch the graph.	

Circles

The equation of the polar form of a circle has 2 forms.

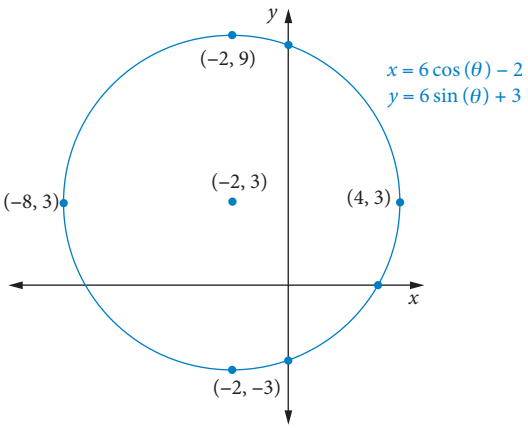
The polar and parametric forms of a circle

The **polar form of a circle** is given by one of $r = 2a\cos(\theta)$ or $r = 2a\sin(\theta)$ for $0 \leq \theta < 2\pi$ with radius $2a$ and centre $(a, 0)$ or $(0, a)$ or $r = a$ with centre $(0, 0)$.

The parametric form of a circle is $\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$ for $0 \leq \theta < 2\pi$.

WORKED EXAMPLE 45 Parametric to Cartesian form of a circle

Write the equation of $\begin{cases} x = 6\cos(\theta) - 2 \\ y = 6\sin(\theta) + 3 \end{cases}$ for $0 \leq \theta < 2\pi$ in Cartesian form and sketch the graph.

Steps	Working
1 State the type of function.	It is a circle with $r = 6$.
2 Write the centre.	The centre is $(-2, 3)$.
3 Write the equation.	$(x + 2)^2 + (y - 3) = 36$
4 Sketch the graph.	

VCE QUESTION ANALYSIS

© VCAA 2019 2BQ1 MODIFIED 2019 Examination 2 Section B Question 1 modified (8 marks)

A curve is defined parametrically by $x = \sec(t) - 2$, $y = \tan(t)$, where $t \in \left[0, \frac{\pi}{2}\right)$.

a Show that the curve can be represented by the rule $f(x) = \sqrt{x^2 - 4x + 3}$. 2 marks

b State the domain and range of $f(x)$. 2 marks

c Sketch the curve, showing any asymptotes and intercepts with the axes. 4 marks

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Video VCE question analysis: Graphing functions and relations

Reading the question

- The curve is a hyperbola.
- In part **a**, the curve is transformed from the standard Cartesian equation.
- Asymptotes will be those of the hyperbola.

Thinking about the question

- Only a quarter of the hyperbola is given as for the whole curve $t \in [0, 2\pi)$, missing some values.
- x is a function of $\sec(t)$, so it is horizontal, a and b are both 1.
- The centre is $(-2, 0)$.
- For domain and range, check the original and transformed curve.
- Use the auxiliary rectangle to help sketch.

Worked solution (✓ = 1 mark)

a $x = \sec(t) - 2$

$\sec(t) = x + 2$

$y = \tan(t)$

$\tan^2(t) + 1 = \sec^2(t)$

$y^2 + 1 = (x + 2)^2$

$y^2 = (x + 2)^2 - 1$

$y^2 = x^2 + 4x + 4 - 1$

$y^2 = x^2 + 4x + 3$

or

Equation is $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$.

Substitute $(h, k) = (-2, 0)$, $a = 1$, $b = 1$, so $(x + 2)^2 - y^2 = 1$ ✓

$y^2 = x^2 + 4x + 4 - 1$

$= x^2 + 4x + 3$

$\boldsymbol{y = \sqrt{x^2 + 4x + 3}}$ ✓

b The hyperbola is only in the 1st quadrant from the vertex $(-1, 0)$, so **domain** $= [-1, \infty)$ ✓, **range** $= [0, \infty)$. ✓

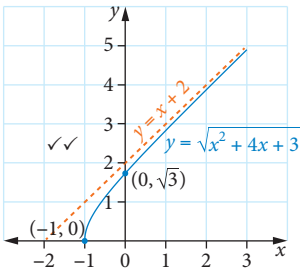
c The asymptote is given by $y - k = \pm\dfrac{b}{a}(x - h)$.

Gradient is positive, so after substituting for h, k, a, b:

$\boldsymbol{y = x + 2}$ ✓

The x-intercept is $\boldsymbol{(-1, 0)}$.

The y-intercept is $\boldsymbol{(0, \sqrt{3})}$. ✓



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EXERCISE 8.8 Polar and parametric forms of the conic sections

ANSWERS p. 597

Recap

1 What is the relation given by $9x^2 + 4y^2 + 109 = 32y + 54x$?

A vertical ellipse

B horizontal ellipse

C vertical hyperbola

D horizontal hyperbola

E none of the above

2 $\frac{4(y-2)^2}{9} - \frac{(x+5)^2}{4} = 1$ has an asymptote with the equation

A $y = \frac{9}{4}x + \frac{53}{4}$ **B** $y = \frac{3}{4}x + \frac{23}{4}$ **C** $y = \frac{3}{2}x + \frac{19}{2}$

D $y = -\frac{9}{4}x - \frac{37}{4}$ **E** $y = -\frac{3}{4}x + \frac{23}{4}$

Mastery

3 WORKED EXAMPLES 37, 40, 43 Change each equation to Cartesian form and sketch the graph.

a $r = \frac{6}{1 + 2\cos(\theta)}$ **b** $r = \frac{9}{4 - 4\cos(\theta)}$ **c** $r = \frac{12}{3 - 5\sin(\theta)}$

d $r = \frac{15}{5 + 4\cos(\theta)}$ **e** $r = \frac{3}{2 + 2\sin(\theta)}$ **f** $r = \frac{4}{5 - 3\sin(\theta)}$

g $r = 6\cos(\theta)$

4 WORKED EXAMPLES 38, 41 Change each equation to polar form.

a $x^2 = 20(y + 5)$ **b** $\frac{(x+4)^2}{25} + \frac{y^2}{9} = 1$ **c** $\frac{(2x-13)^2}{144} - \frac{4y^2}{25} = 1$

d $\frac{x^2}{16} + \frac{(y-3)^2}{25} = 1$ **e** $\frac{(y+5)^2}{9} - \frac{x^2}{16} = 1$ **f** $y^2 = -16(x - 4)$

g $x^2 + (y - 8)^2 = 64$

5 WORKED EXAMPLE 39 Change each equation to parametric form.

a $(x + 4)^2 + (y - 2)^2 = 25$ **b** $\frac{(x-2)^2}{25} - \frac{(y+3)^2}{16} = 1$ **c** $\frac{(x+4)^2}{9} + \frac{(y-5)^2}{10} = 1$

d $(x + 1)^2 = 2(y + 4)$ **e** $\frac{(x+3)^2}{49} + \frac{(y-1)^2}{25} = 1$ **f** $\frac{(y-6)^2}{20} - \frac{(x-4)^2}{36} = 1$

g $(y + 3)^2 = 24(x - 7)$

6 WORKED EXAMPLES 42, 44, 45 Change each pair of equations to Cartesian form and sketch the graph.

a $x = 5\sec(\theta) - 2, y = 3\tan(\theta) + 4$ **b** $x = 7\cos(\theta) + 6, y = 7\sin(\theta) - 5$

c $x = 4\cos(\theta) + 2, y = 5\sin(\theta) + 3$ **d** $x = 3t^2 - 2, y = 6t - 4$

e $x = 5\tan(\theta) + 4, y = 4\sec(\theta) - 1$ **f** $x = 12t + 2, y = 6t^2 + 5$

g $x = 6\cos(\theta) - 3, y = 3\sin(\theta) - 4$

7 Change each equation to parametric form.

a $r = \dfrac{8}{3 - \cos(\theta)}$ **b** $r = \dfrac{12}{1 - \sin(\theta)}$ **c** $r = \dfrac{21}{2 - 5\sin(\theta)}$

d $r = \dfrac{24}{1 + 3\cos(\theta)}$ **e** $r = \dfrac{24}{4 + 3\sin(\theta)}$ **f** $r = 10\cos(\theta)$

g $r = \dfrac{10}{1 + \cos(\theta)}$

8 Change each pair of equations to polar form.

a $x = 2\sec(\theta) + 2, y = 4\tan(\theta)$ **b** $x = 3\cos(\theta), y = 6\sin(\theta) + 1.5\sqrt{3}$

c $x = 4\cot(\theta), y = 3\operatorname{cosec}(\theta) - 13\frac{1}{3}$ **d** $x = -3t^2 + 3, y = 6t$

e $x = 4\cos(\theta) + 2, y = 4\sin(\theta)$ **f** $x = 5\cos(\theta), y = 4\sin(\theta)$

g $x = 4t, y = t^2 + 4$

Exam practice

80–100% 60–79% 0–59%

9 A curve is defined by the polar relation $r = \dfrac{12}{5 - 6\sin(\theta)}$. It is a

A vertical hyperbola **B** vertical ellipse **C** horizontal parabola

D horizontal hyperbola **E** vertical ellipse

10 © VCAA 2015 2AQ1 84% The ellipse $\dfrac{(x-2)^2}{9} + \dfrac{(y-3)^2}{4} = 1$ can be expressed in parametric form as

A $x = 2 + 3t$ and $y = 3 + 2\sqrt{1 + t^2}$ **B** $x = 2 + 3\sec(t)$ and $y = 3 + 2\tan(t)$

C $x = 2 + 9\cos(t)$ and $y = 3 + 4\sin(t)$ **D** $x = 3 + 2\cos(t)$ and $y = 2 + 3\sin(t)$

E $x = 2 + 3\cos(t)$ and $y = 3 + 2\sin(t)$

11 © VCAA 2021 2AQ7 39% A relation is defined parametrically by

$x(t) = 5\cos(2t) + 1, y(t) = 5\sin(2t) - 1.$

If $A(6, -1)$ and $B(1, 4)$ are two points that lie on the graph of the relation, then the shortest distance along the graph from A to B is

A $\dfrac{\pi}{4}$ **B** $\dfrac{\pi}{2}$ **C** π **D** $\dfrac{5\pi}{4}$ **E** $\dfrac{5\pi}{2}$

12 © VCAA 2013 2BQ1 91% TECH-FREE (2 marks) A curve is defined by the parametric equations

$x = 1 + 3\cos(t)$

$y = -2 + 2\sin(t)$ for $t \in [0, 2\pi]$.

Find the Cartesian equation of the curve.

8 Chapter summary

A **function** $y = f(x)$ is a relation such that each x value in the domain has only one ordered pair (x, y); each x value matches with only one y value.

The largest set of real numbers for which a function rule is defined is called the **maximal domain**, **natural** or **implied domain**.

The graph of $f(x)$ is **transformed** to the graph of $af(n(x + b)) + c$.

- n and b produce the transformations in the direction of the x-axis:
  - dilation/reflection by the factor $\frac{1}{n}$ and then translation b to the left or right gives $f(n(x + b))$
  - translation first and then dilation/reflection gives $f(nx + b)$.
- a and c produce the transformations in the direction of the y-axis:
  - dilation/reflection by the factor a and then translation c up or down gives $af(n(x + b)) + c$ or $f(nx + b) + c$
  - translation and then dilation/reflection gives $a(f(n(x + b)) + c)$ or $a(f(nx + b) + c)$.

Coefficients of polynomials

We can find missing values of polynomials by equating coefficients of the same power in an **identity**. For example, $3x^3 - x^2 - 7x - 6 \equiv (x - 2)(ax^2 + bx + c) = ax^3 + (b - 2a)x^2 + (c - 2b)x - 2c$, so $a = 3$, $b = 5$ and $c = 3$.

Rational functions

A **rational function** is of the form $f(x) = \frac{p(x)}{q(x)}$, where p and q are polynomials and q is not the zero polynomial. If the degree of the numerator is less than the degree of the denominator, it is called a **simple rational function**.

A **quotient function** has a numerator of degree equal to or higher than the degree of the denominator.

We can use identities to equate coefficients to express rational functions as sums of simpler functions.

Partial fractions

The expression $\frac{4}{7(x + 2)} + \frac{13}{7(2x - 3)}$ is called the **partial fraction decomposition** of the rational expression $\frac{3x + 2}{(x + 2(2x - 3)}$.

An **irreducible quadratic** expression has no real factors. Its discriminant is negative.

A simple rational function with linear and irreducible quadratic factors in the denominator can be decomposed to a sum of partial fractions. The partial fractions are

- $\frac{A}{ax + b}$ for a simple factor $ax + b$
- $\frac{A}{(ax + b)^2} + \frac{B}{ax + b}$ for a square linear factor $(ax + b)^2$
- $\frac{A}{(ax + b)^3} + \frac{A}{(ax + b)^2} + \frac{B}{ax + b}$ for a cubic linear factor $(ax + b)^3$
- $\frac{Ax + B}{ax^2 + bx + c}$ for an irreducible quadratic factor $ax^2 + bx + c$.

Reciprocal functions

For any function $f(x)$ and its reciprocal $g(x) = \dfrac{1}{f(x)}$:

- the signs of $f(x)$ and $g(x)$ are the same
- the zeros of $f(x)$ correspond to vertical asymptotes of $g(x)$
- vertical asymptotes of $f(x)$ correspond to zeros of $g(x)$
- as $f(x) \to \pm\infty$, $g(x) \to 0$
- as $f(x) \to 0$, $g(x) \to \pm\infty$
- maxima and minima of $f(x)$ correspond to minima and maxima of $g(x)$
- the y-intercept of $g(x)$ is the reciprocal of the y-intercept of $f(x)$
- for $f(x)$ concave up, $g(x)$ is concave down
- for $f(x)$ concave down, $g(x)$ is concave up.

The domain of $g(x)$ is the same as that of $f(x)$, except that zeros of $f(x)$ are excluded from the domain of $g(x)$.

Reciprocal circular functions

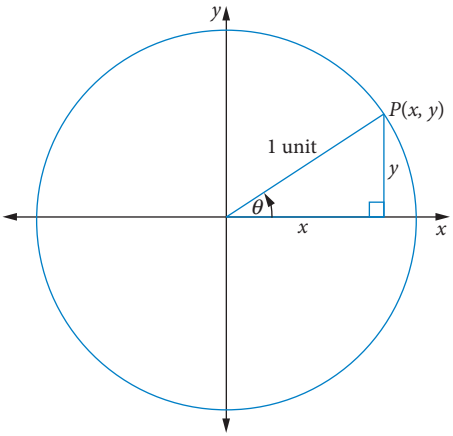
The **secant** function is defined as $\sec(\theta) = \dfrac{1}{\cos(\theta)} = \dfrac{1}{x}$

The **cosecant** function is defined as $\operatorname{cosec}(\theta) = \dfrac{1}{\sin(\theta)} = \dfrac{1}{y}$

The **cotangent** function is defined as

$\cot(\theta) = \dfrac{1}{\tan(\theta)} = \dfrac{\cos(\theta)}{\sin(\theta)} = \dfrac{x}{y}$

where x and y are the coordinates of the point P on the unit circle with angle θ, as shown on the right.



We can sketch the graphs of $\operatorname{cosec}(x)$, $\sec(x)$ and $\cot(x)$ as reciprocals.

$\sin(x)$ has *zeros* at $-\pi$, 0, π, 2π, 3π etc., so $\operatorname{cosec}(x) = \dfrac{1}{\sin(x)}$ has *vertical asymptotes* at $n\pi$ for $n \in Z$.

$\sin(x)$ has *maxima* of $(x, 1)$ at $-\dfrac{3\pi}{2}, \dfrac{\pi}{2}, \dfrac{5\pi}{2}$ etc., so $\operatorname{cosec}(x)$ has *minima* of $(x, 1)$ at $x = (4n + 1)\dfrac{\pi}{2}$ for $n \in Z$.

Minima of $\sin(x)$ correspond to *maxima* of $\operatorname{cosec}(x)$.

The signs of $\sin(x)$ and $\operatorname{cosec}(x)$ are the same for all values of x (except for zeros/asymptotes).

The same reasoning applies to the graphs of $\cos(x)/\sec(x)$ and $\tan(x)/\cot(x)$.

Additionally, where $\tan(x)$ has *vertical asymptotes*, $\cot(x)$ has *zeros*.

$y = \operatorname{cosec}(x)$

Period: 2π

Domain: $R\backslash\{n\pi, n \in Z\}$

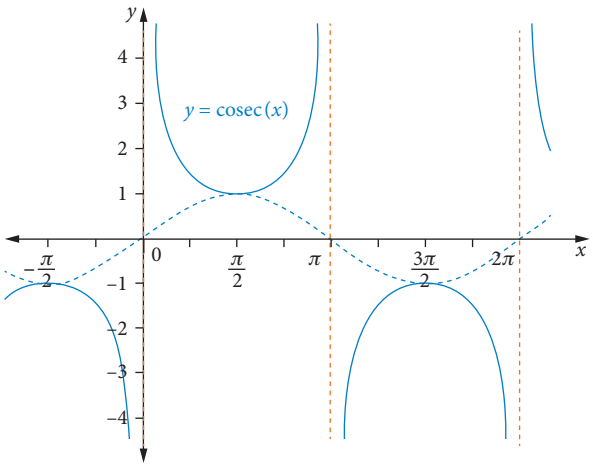
Range: $(-\infty, -1] \cup [1, \infty)$

Asymptotes: $x = n\pi$, $n \in Z$

No zeros

Minima: $-\dfrac{3\pi}{2}, \dfrac{\pi}{2}, \dfrac{5\pi}{2} \ldots$ or $\left((4n + 1)\dfrac{\pi}{2}, 1\right)$, $n \in Z$

Maxima: $-\dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{7\pi}{2} \ldots$ or $\left((4n - 1)\dfrac{\pi}{2}, -1\right)$, $n \in Z$



$y = \sec(x)$

Period: 2π

Domain: $R\backslash\left\{(2n+1)\frac{\pi}{2} : n \in Z\right\}$

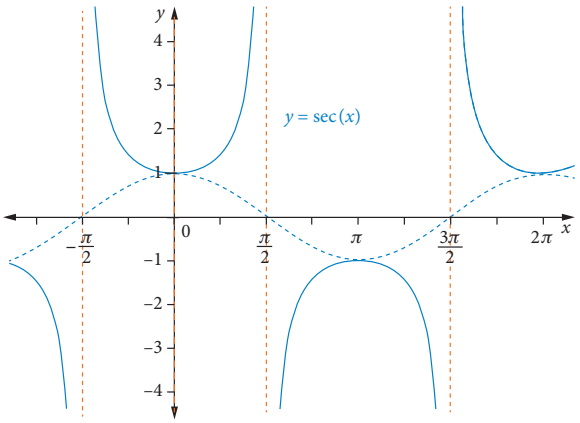
Range: $(-\infty, -1] \cup [1, \infty)$

Asymptotes: $-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2} \ldots$ or $(2n+1)\frac{\pi}{2}, n \in Z$

No zeros

Minima: $-2\pi, 0, 2\pi, 4\pi \ldots$ or $(2n\pi, 1), n \in Z$

Maxima: $-\pi, \pi, 3\pi \ldots$ or $(2n+1)\pi, -1), n \in Z$



$y = \cot(x)$

Period: π

Domain: $R\backslash\{n\pi : n \in Z\}$

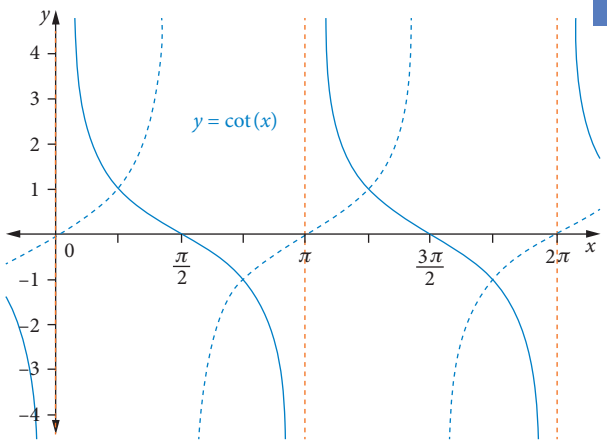
Range: R

Asymptotes: $-\pi, 0, \pi, 2\pi, 3\pi \ldots$ or $n\pi, n \in Z$

Zeros (centres): $\frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots$ or $(2n+1)\frac{\pi}{2}, n \in Z$

$y = 1$ at $\frac{-3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}$ or $\left((4n+1)\frac{\pi}{4}, 1\right), n \in Z$

$y = -1$ at $\frac{-\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4} \ldots$ or $\left((4n-1)\frac{\pi}{4}, -1\right), n \in Z$



Period, amplitude, phase shift and mean value

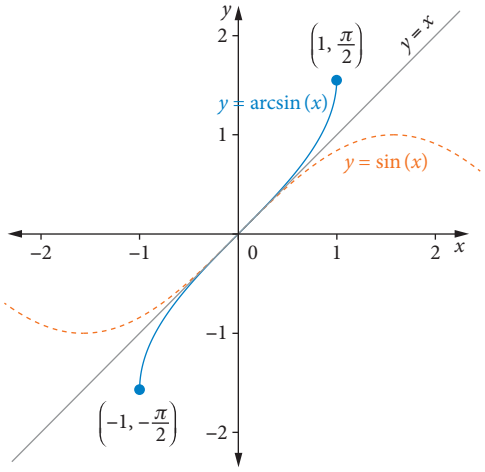
- The **period** is $\frac{2\pi}{n}$ for sin, cos, sec and cosec; and $\frac{\pi}{n}$ for tan and cot.
- a is the **amplitude** for sin and cos; $(x, 1)$ becomes (x, a) for all the functions.
- b is the **phase shift**; earlier (to the left) for $b > 0$, later (to the right) for $b < 0$.
- c is the **mean (central) value**; (x, a) becomes $(x, a + c)$ for all the functions.

Inverse circular functions

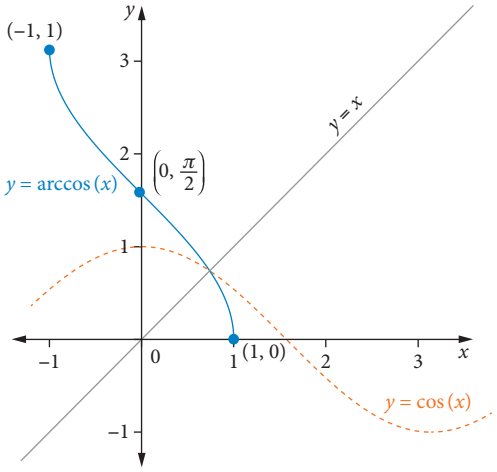
- $y = \sin^{-1}(x)$ if and only if $\sin(y) = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.
- $y = \cos^{-1}(x)$ if and only if $\cos(y) = x$ and $0 \leq y \leq \pi$.
- $y = \tan^{-1}(x)$ if and only if $\tan(y) = x$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

$\sin^{-1}(x)$, $\cos^{-1}(x)$ and $\tan^{-1}(x)$ are also called arcsine (**arcsin**), arccosine (**arccos**) and arctangent (**arctan**).

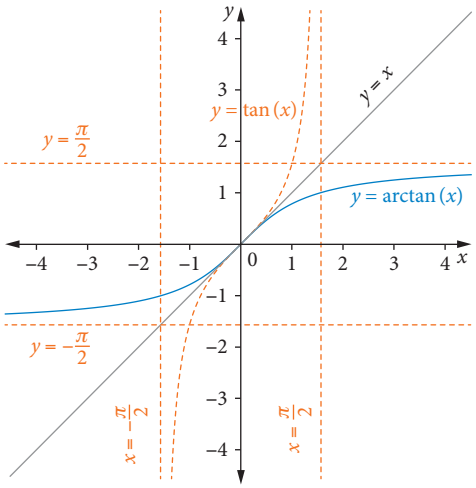
The basic graphs of the inverse circular functions are reflections in the line $y = x$ of the circular functions on their principal domains.



$y = \sin^{-1}(x)$: domain $[-1, 1]$, range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



$y = \cos^{-1}(x)$: domain $[-1, 1]$, range $[0, \pi]$



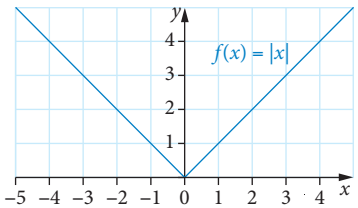
$y = \tan^{-1}(x)$: domain R, range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, asymptotes $y = -\frac{\pi}{2}$, $y = \frac{\pi}{2}$

We can use transformations to sketch other inverse trigonometric functions. We might find it easier to transform the points with x values -1, 0 and 1 and join these, as well as the asymptotes for $\tan^{-1}(x)$.

The absolute value function

The absolute value function, $f(x) = |x|$ is defined by

$$f(x) = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$



$|x| = a \Rightarrow x = -a$ or $x = a$

$|x| < 5 \Rightarrow x > -5$ AND $x < 5$ $(-5 < x < 5)$

$|x| > 5 \Rightarrow x < -5$ OR $x > 5$.

Any inequality involving the absolute value is solved as a pair of inequalities.

Locus, conic sections

A **locus** is a set of points that satisfies a given condition. It is the path traced out by a point that moves according to the condition.

To find the equation of a locus, put the point $P(x, y)$ into the condition and simplify.

There are 4 conic sections: the **circle**, the **ellipse**, the **parabola** and the **hyperbola**.

A circle is the locus of points at the distance r from the centre (h, k).

Its equation is $(x - h)^2 + (y - k)^2 = r^2$.

A parabola is the locus of points with equal distances from a fixed point and a line.

Parabolas	Vertical parabola	Horizontal parabola
Vertex	(h, k)	(h, k)
Equation	$(x - h)^2 = 4a(y - k)$	$(y - k)^2 = 4a(x - h)$
Focus	$(h, a + k)$	$(a + h, k)$
Directrix	$y = -a + k$	$x = -a + h$
Axis of symmetry	$x = h$	$y = k$

An ellipse is the locus of points such that the distance from $P(x, y)$ to the focus is e times the distance from P to the directrix, where $0 < e < 1$. e is called the **eccentricity** of the conic section.

The long axis of an ellipse is called the **major axis** and the other is the **minor axis**.

Ellipses	Major axis horizontal	Major axis vertical
Centre	(h, k)	(h, k)
Equation	$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$	$\frac{(y - k)^2}{a^2} + \frac{(x - h)^2}{b^2} = 1$
Eccentricity	$e^2 = 1 - \frac{b^2}{a^2}$	$e^2 = 1 - \frac{b^2}{a^2}$
Foci	$(h \pm ae, k)$	$(h, k \pm ae)$
Directrices	$x = h \pm \frac{a}{e}$	$y = k \pm \frac{a}{e}$
Vertices	$(h \pm a, k)$, $(h, k \pm b)$	$(h \pm b, k)$, $(h, k \pm a)$

A hyperbola is the locus such that the distance from $P(x, y)$ to the focus is e times the distance from P to the directrix, where the eccentricity $e > 1$.

Hyperbolas	Transverse axis horizontal	Transverse axis vertical
Centre	(h, k)	(h, k)
Equation	$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$	$\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$
Eccentricity	$e^2 = 1 + \frac{b^2}{a^2}$	$e^2 = 1 + \frac{b^2}{a^2}$
Foci	$(h \pm ae, k)$	$(h, k \pm ae)$
Directrices	$x = h \pm \frac{a}{e}$	$y = k \pm \frac{a}{e}$
Vertices	$(h \pm a, k)$	$(h, k \pm a)$
Asymptotes	$y - k = \pm\frac{b}{a}(x - h)$	$y - k = \pm\frac{a}{b}(x - h)$

Polar and parametric equations

$\tan(\theta) = \frac{y}{x}$, $r^2 = x^2 + y^2$

$x = r\cos(\theta)$, $y = r\sin(\theta)$

A **polar form** of the equation of a graph has the equation expressed in terms of r and θ.

A **parametric form** of the equation of a graph has 2 equations expressing x and y in terms of an independent variable such as t or θ. The independent variable is called the **parameter**.

The **polar forms** of parabolas, ellipse and hyperbolas all have the equations $r = \frac{de}{1 \pm e\cos(\theta)}$ or $r = \frac{de}{1 \pm e\sin(\theta)}$, where a focus is placed at the origin.

They are distinguished by the value of e. d is the distance between the focus and directrix.

Horizontal conic sections involve cosine and vertical conic sections involve sine.

Equations of conic sections: circle, ellipse, hyperbola, parabola			
Shape	**Cartesian**	**Parametric**	**Polar**
Ellipses	$a > b$	$a > b$	$0 < e < 1$
Horizontal ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\begin{cases} x = a\cos(\theta) \\ y = b\sin(\theta) \end{cases}$ for $0 \le \theta < 2\pi$	$r = \frac{de}{1 \pm e\cos(\theta)}$
Vertical ellipse	$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$	$\begin{cases} x = b\cos(\theta) \\ y = a\sin(\theta) \end{cases}$ for $0 \le \theta < 2\pi$	$r = \frac{de}{1 \pm e\sin(\theta)}$
Hyperbolas			$e > 1$
Horizontal hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\begin{cases} x = a\sec(\theta) \\ y = b\tan(\theta) \end{cases}$, $0 \le \theta < 2\pi \left(\theta \ne \frac{\pi}{2}, \frac{3\pi}{2}\right)$ or $\begin{cases} x = a\operatorname{cosec}(\theta) \\ y = b\cot(\theta) \end{cases}$, $0 < \theta < 2\pi\ (\theta \ne \pi)$	$r = \frac{de}{1 \pm e\cos(\theta)}$
Vertical hyperbola	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$	$\begin{cases} y = a\sec(\theta) \\ x = b\tan(\theta) \end{cases}$, $0 \le \theta < 2\pi$ or $\begin{cases} y = a\operatorname{cosec}(\theta) \\ x = b\cot(\theta) \end{cases}$, $0 < \theta < 2\pi\ (\theta \ne \pi)$	$r = \frac{de}{1 \pm e\sin(\theta)}$
Parabolas			$e = 1$
Horizontal parabola	$y^2 = 4ax$	$\begin{cases} x = at^2 \\ y = 2at \end{cases}$ for $t \in R$	$r = \frac{de}{1 \pm e\cos(\theta)}$
Vertical parabola	$x^2 = 4ay$	$\begin{cases} x = 2at \\ y = at^2 \end{cases}$ for $t \in R$	$r = \frac{de}{1 \pm e\sin(\theta)}$
Circles	$x^2 + y^2 = r^2$	$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$ for $0 \le \theta < 2\pi$	$r = 2a\cos(\theta)$ or $r = 2a\sin(\theta)$

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 (5 marks) The number of combinations of choosing two people from a group of N people is 55.

a Prove that ${}^{N}C_{2} = \dfrac{N(N-1)}{2}$ 2 marks

b Hence, show that $N^2 - N - 110 = 0$. 2 marks

c Hence, find the number of people in the group. 1 mark

2 © VCAA 2021N 1Q4 MODIFIED (2 marks) Consider the function $f\colon [-1, 1] \to R, f(x) = \arcsin(x) - \dfrac{\pi}{2}$. Sketch the graph of f, labelling the endpoints with their coordinates.

3 © VCAA 2019 1Q2 MODIFIED (3 marks) Find all values of x for which $|2x - 3| = \dfrac{x}{3} + 4$.

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 The difference between the 3rd and 7th terms of an arithmetic sequence is 10 and their average is 6. The 10th term is

A –4 **B** 18.5 **C** 21 **D** 22.5 **E** 29

2 Consider the statement 'If $n \in N$ then $n^2 + 1$ is odd'.
Which of the following statements is false?

A $n = 5$ is a counterexample.

B The statement is false because it is not true for $n = 1$.

C The contrapositive statement is 'If $n \notin N$, then $n^2 + 1$ is even'.

D The contrapositive statement is 'If $n^2 + 1$ is even, then $n \notin N$'.

E $n = 11$ is a counterexample.

3 © VCAA 2012 2AQ1 MODIFIED The graph with equation $y = \dfrac{1}{3x^2 + 2x - 8}$ has asymptotes given by

A $x = -\dfrac{4}{3}$, $x = 2$ and $y = 1$ **B** $x = -\dfrac{4}{3}$, $x = 2$ and $y = 0$

C $x = \dfrac{4}{3}$, $x = -2$ and $y = \dfrac{1}{3}$ **D** $x = -\dfrac{4}{3}$, $x = 2$ and $y = \dfrac{1}{3}$

E $x = \dfrac{4}{3}$, $x = -2$ and $y = 0$

4 © VCAA 2018 2AQ3 MODIFIED Which one of the following, where A, B, C, D and E are non-zero real numbers, is the partial fraction form for the expression $\dfrac{x^2 - 2x - 3}{(x + 2)^2(x^2 - 1)}$?

A $\dfrac{A}{x - 2} + \dfrac{B}{x - 1} + \dfrac{C}{x + 1}$ **B** $\dfrac{A}{x - 2} + \dfrac{B}{(x - 2)^2} + \dfrac{C}{x - 1} + \dfrac{D}{x + 1}$

C $\dfrac{Ax + B}{(x - 2)^2} + \dfrac{C}{x - 1} + \dfrac{D}{x + 1}$ **D** $\dfrac{Ax + B}{(x - 2)^2} + \dfrac{Cx + D}{x^2 - 1}$

E $\dfrac{Ax + B}{(x - 2)^2} + \dfrac{C}{x - 2} + \dfrac{D}{x - 1} + \dfrac{E}{x + 1}$

5 © VCAA 2019 2AQ3 MODIFIED The implied domain of the function with the rule $f(x) = 1 + \text{cosec}\left(x + \dfrac{\pi}{4}\right)$ is

A R **B** $[0, 2]$ **C** $R \setminus \left\{\dfrac{(4n - 1)\pi}{4}\right\}, n \in Z$

D $R \setminus \left\{\dfrac{(4n + 1)\pi}{4}\right\}, n \in Z$ **E** $R \setminus \left\{\dfrac{(2n - 1)\pi}{2}\right\}, n \in Z$

Section B 3 questions

15 marks

1 (4 marks) Andrea is left $12 000 by her great aunt and puts it into an investment fund that gives an average of 4% p.a. interest each year. Every year after that, she puts another $1500 into the fund.

a Write a recurrence relation that shows the a_{n+1} in the fund after $n + 1$ years in terms of the amount in the fund after n years, where $a_0 = 12\,000$. 1 mark

b Prove that the amount in the fund after n years is

$$a_n = 12\,000(1.04^n) + 1500\left(\frac{1.04^n - 1}{0.04}\right)$$ 2 marks

c How much does Andrea have in the fund after 10 years, to the nearest cent? 1 mark

2 (8 marks) A curve is represented by the polar equation $r = \dfrac{12}{2 + \sin(\theta)}$.

a Express as an equation in Cartesian form. 3 marks

b Sketch the graph, including coordinates of intercepts. 3 marks

c Express the curve in parametric form. 2 marks

3 (3 marks)

a Use the sequence $t_0 = a$, $t_n = rt_{n-1}$ to write down the first 5 terms. 1 mark

b If $|r| < 1$, give the expression for the infinite sum of terms of the sequence in part **a**. 2 marks

CHAPTER

9 COMPLEX NUMBERS

Study Design coverage

Nelson MindTap chapter resources

9.1 Complex numbers
- **Using CAS 1:** Imaginary numbers
- **Using CAS 2:** Real and imaginary parts of a complex number
- Adding and subtracting complex numbers
- Multiplying complex numbers
- Complex conjugates
- Dividing complex numbers

9.2 Quadratic equations with complex roots
- The discriminant
- Solving quadratic equations with complex roots

9.3 The Argand diagram
- **Using CAS 3:** Modulus of a complex number
- Multiplication by *i*

9.4 Polar form of a complex number
- Changing between Cartesian form and polar form
- **Using CAS 4:** Convert between Cartesian and polar form
- Multiplication and division

9.5 Graphs and regions in the complex plane
- Lines and rays
- Circles and ellipses
- Regions in the complex plane

VCE question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2

9780170448338

Study Design coverage

UNIT 2, AREA OF STUDY 3: ALGEBRA, NUMBER AND STRUCTURE

Complex numbers

- definition and properties of the complex numbers, C, arithmetic, modulus of a complex number, and the representation of complex numbers on an Argand diagram
- general solution of quadratic equations (with real coefficients) of a single variable over C, and conjugate roots
- lines, rays, circles and ellipses
- regions defined in the complex plane using combinations of the above
- use of the modulus of a complex number and the argument of a non-zero complex number to prove basic identities
- conversion between cartesian and polar form of complex numbers
- multiplication, division, and powers of complex numbers in polar form and their geometric interpretation.

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Video playlists (6):
9.1 Complex numbers
9.2 Quadratic equations with complex roots
9.3 The Argand diagram
9.4 Polar form of a complex number
9.5 Graphs and regions in the complex plane
VCE question analysis Complex numbers

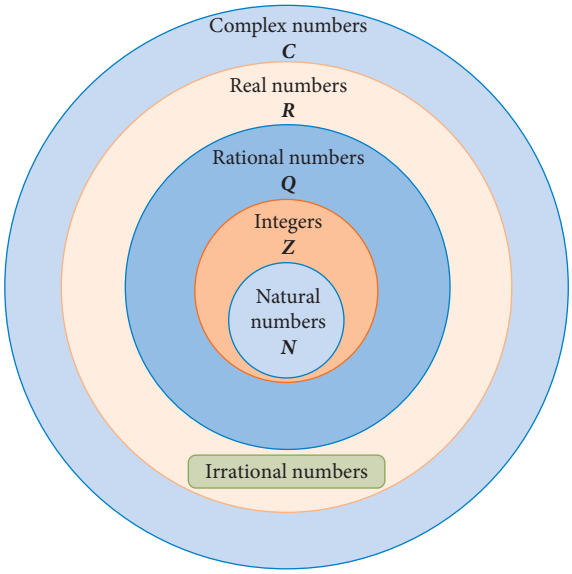
Worksheets (13):
9.1 Complex number operations • Complex numbers • Complex conjugates
9.3 The complex plane • Addition and subtraction in the plane
9.4 Modulus and argument • Complex number conversions • Evaluating the modulus 1 • Evaluating the modulus 2 • Multiplication in the plane • Division in the plane • Polar complex number operations
9.5 Complex plane graphs

Nelson MindTap
To access resources above, visit **cengage.com.au/nelsonmindtap**

9.1 Complex numbers

Video playlist
Complex numbers

In Chapter 2, *Proof and number*, we saw that the different sets of numbers could be organised in a diagram as shown on the right.



The set of **real numbers**, R, consists of **rational numbers and irrational numbers** and thus contains as subsets all the natural numbers and integers.

The set of **complex numbers**, C, are numbers that can be written in the form $a + bi$ where $i = \sqrt{-1}$ or $i^2 = -1$. It includes all of the **real numbers** and thus contains all the numbers mentioned above.

A complex number $z = a + bi$ has a **real part**, written as $\mathbf{Re}(z) = a$, and an **imaginary part**, written as $\mathbf{Im}(z) = b$. For $z = 5 - 2i$, $\text{Re}(z) = 5$ and $\text{Im}(z) = -2$.

If $a = 0$, z is **purely imaginary**, or just **imaginary**; for example, $z = 3i$.

If $b = 0$, z is **purely real**, or just **real**; for example, $z = 2$.

WB

p. 232

WORKED EXAMPLE 1 Finding the square root of a negative number

Use the fact that $i = \sqrt{-1}$ to simplify each root in terms of i.

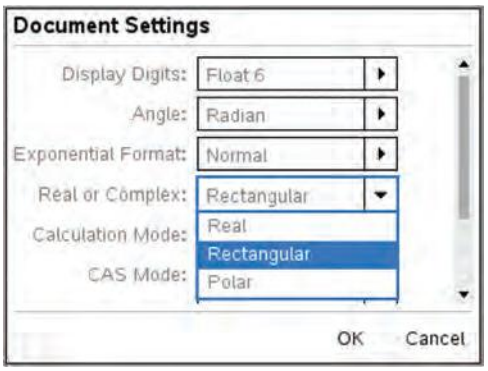
a $\sqrt{-4}$ **b** $\sqrt{-9}$ **c** $\sqrt{-75}$

Steps	Working
a Write -4 as a product of 4 and -1.	$\sqrt{-4} = \sqrt{4} \times \sqrt{-1}$ $= 2i$
b Write -9 as a product of 9 and -1.	$\sqrt{-9} = \sqrt{9} \times \sqrt{-1}$ $= 3i$
c Write -75 as a product of 25, 3 and -1, then simplify.	$\sqrt{-75} = \sqrt{25 \times 3 \times (-1)}$ $= 5 \times \sqrt{3} \times i$ $= 5\sqrt{3}i$

USING CAS 1 Imaginary numbers

Confirm the answers to Worked example 1 by expressing $\sqrt{-4}$, $\sqrt{-9}$ and $\sqrt{-75}$ as imaginary numbers.

TI-Nspire

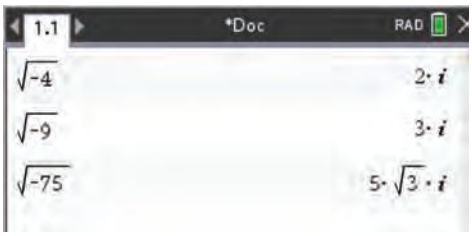


1 Add a **Calculator** page.

2 Press **doc** > **Settings & Status** > **Document Settings**.

3 In the **Real or Complex:** field, select **Rectangular**.

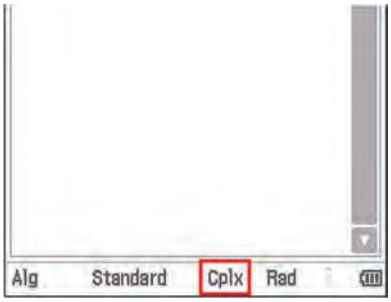
4 Click **OK**.



5 Enter numbers as shown above.

6 The square root of the negative numbers will be displayed as imaginary numbers.

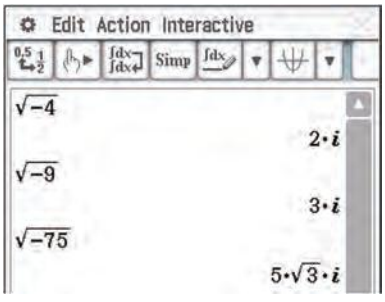
ClassPad



1 Tap **Main**.

2 At the bottom of the screen, tap **Real** to toggle to **Cplx**.

Note: Always ensure that the mode is set to **Cplx** when working with complex numbers.



3 Enter the numbers as shown above.

4 The square root of the negative numbers will be displayed as imaginary numbers.

WORKED EXAMPLE 2 Simplify expressions involving powers of i

Use $i^2 = -1$ to simplify each expression.

a $i^2 - i^3$ **b** $\dfrac{3i^2 + 6i^5}{3i^2}$ **c** $\dfrac{\sqrt{i} \times \sqrt{i^{2n}}}{i^n}$

Steps	Working
a Factorise i^2 in the second term to make it -1.	$i^2 - i^3 = -1 - i^2 i$ $= -1 - (-1)i$ $= -1 + i$
b Factorise the numerator and simplify.	$\dfrac{3i^2 + 6i^5}{3i^2} = \dfrac{3i^2(1 + 2i^3)}{3i^2}$ $= 1 + 2i^3$ $= 1 + 2i(i^2)$ $= 1 + 2i \times (-1)$ $= 1 - 2i$
c **1** Rewrite in index form.	$\dfrac{\sqrt{i} \times \sqrt{i^{2n}}}{i^n} = \dfrac{i^{\frac{1}{2}} \times (i^{2n})^{\frac{1}{2}}}{i^n}$
2 Simplify using index laws.	$= \dfrac{i^{\frac{1}{2}} \times i^n}{i^n}$ $= i^{\frac{1}{2}}$ $= \sqrt{i}$

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WORKED EXAMPLE 3 The real part and the imaginary part of a complex number

Find $\text{Re}(z)$ and $\text{Im}(z)$ for each complex number.

a $z = 5 - 4i$ **b** $z = \sqrt{2}i - 9$ **c** $z = 2x + y + (x - y)i$ **d** $z = \dfrac{x - yi}{x^2 + y^2}$

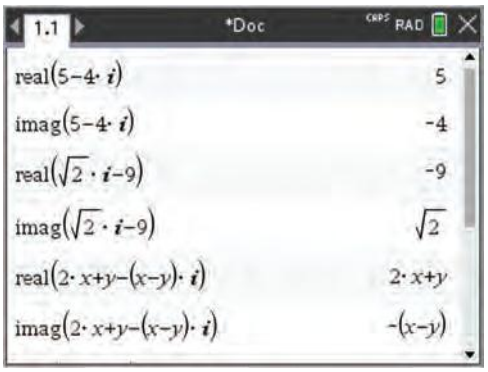
Steps	Working
a For $z = 5 - 4i$, the real part is the number 5. The coefficient of i is -4.	$\text{Re}(z) = 5$ $\text{Im}(z) = -4$
b $z = \sqrt{2}i - 9$ is the same as $z = -9 + \sqrt{2}i$.	$\text{Re}(z) = -9$ $\text{Im}(z) = \sqrt{2}$
c For $z = 2x + y + (x - y)i$, all the terms without an i attached form the real part.	$\text{Re}(z) = 2x + y$ $\text{Im}(z) = x - y$
d Write $z = \dfrac{x - yi}{x^2 + y^2}$ as $z = \dfrac{x}{x^2 + y^2} + \left(\dfrac{-y}{x^2 + y^2}\right)i$.	$\text{Re}(z) = \dfrac{x}{x^2 + y^2}$ $\text{Im}(z) = \dfrac{-y}{x^2 + y^2}$

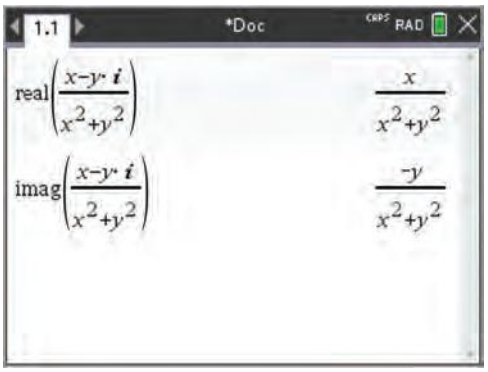
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USING CAS 2 Real and imaginary parts of a complex number

Confirm the answers to Worked example 3 by finding the Re (*z*) and Im (*z*) for each complex number.

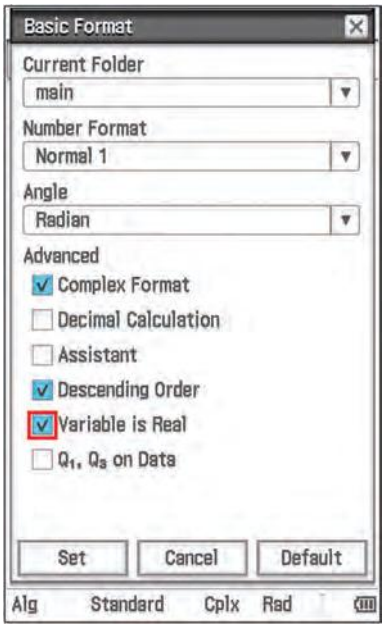
TI-Nspire

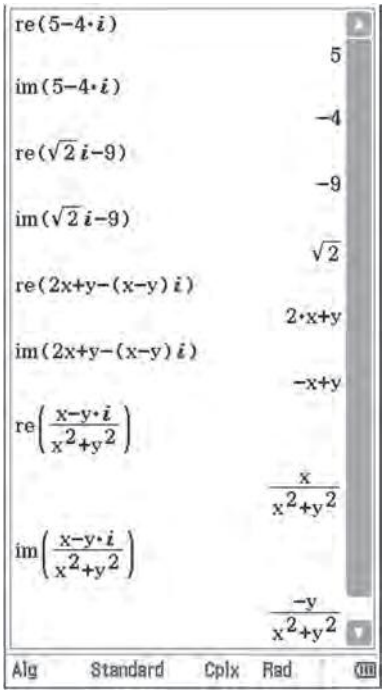




1 Press **menu** > **Number** > **Complex Number Tools** > **Real Part**.
2 Enter the first complex number. Press ***π*** to access the mini-palette for the ***i*** symbol.
3 Press **menu** > **Number** > **Complex Number Tools** > **Imaginary Part**, then copy or enter the first complex number again.
4 Repeat for the other complex numbers.
5 The real and imaginary parts of the complex numbers will be displayed.

ClassPad





1 Tap **Settings** > **Basic Format**.
2 Tap to tick the **Variable is Real** option.
3 Tap **Set**.
4 Enter and highlight the first complex number. Open the **Keyboard** > **Math2** or **Math3** for the ***i*** symbol.
5 Tap **Interactive** > **Complex** > **re**.
6 Copy and highlight the first complex number, then tap **Interactive** > **Complex** > **im**.
7 Repeat for the other complex numbers.
8 The real and imaginary parts of the complex numbers will be displayed.

Adding and subtracting complex numbers

To simplify an algebraic expression such as $3a + 4b + 2a - b$, we join 'like terms' to get $(3a + 2a) + (4b - b) = 5a + 3b$. In a similar way, addition and subtraction of complex numbers involves combining the real parts and the imaginary parts separately.

WORKED EXAMPLE 4 Adding and subtracting complex numbers

Simplify each expression and write the result in the form $z = a + bi$.

a $4 + 5i - 2 - i$ **b** $i(1 - i) - \frac{1}{2}(2 + 8i)$ **c** $\frac{3}{i} + i$

Steps	Working
a 1 Group the real and imaginary parts.	$4 + 5i - 2 - i = (4 - 2) + (5i - i)$
2 Simplify.	$= 2 + 4i$
b Expand and simplify.	$i(1 - i) - \frac{1}{2}(2 + 8i) = i - i^2 - 1 - 4i$ $= i + 1 - 1 - 4i$ $= -3i$
c Simplify the fraction by multiplying numerator and denominator by i. This is called **realising the denominator**: making the denominator a real number.	$\frac{3}{i} + i = \frac{3 \times i}{i \times i} + i$ $= \frac{3i}{-1} + i$ $= -3i + i$ $= -2i$

Worksheets
Complex number operations
Complex numbers

WB p. 235

Multiplying complex numbers

We use the distributive law to expand and simplify the product of 2 complex numbers.

WORKED EXAMPLE 5 Multiplying complex numbers

Simplify each expression.

a $(1 + 2i)(3 - i)$ **b** $(4 - 5i)^2 + 40i$ **c** $4i(1 + i)(1 - i)$

Steps	Working
a 1 Expand the binomial product.	$(1 + 2i)(3 - i) = 3 - i + 6i - 2i^2$
2 Simplify using $i^2 = -1$.	$= 3 - i + 6i + 2$ $= 5 + 5i$
b Expand the perfect square and simplify.	$(4 - 5i)^2 + 40i = 16 - 40i + 25i^2 + 40i$ $= 16 - 25$ $= -9$
c Expand and simplify, using difference of two squares (DOTS).	$4i(1 + i)(1 - i) = 4i(1 - i^2)$ $= 4i(1 + 1)$ $= 8i$

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Complex conjugates

The **complex conjugate** of $z = a + bi$ is $\overline{z} = a - bi$. Notice that

- $\text{Re}(\overline{z}) = \text{Re}(z)$
- $\text{Im}(\overline{z}) = -\text{Im}(z)$
- $z + \overline{z} = (a + bi) + (a - bi) = 2a$, a purely real number
- $z - \overline{z} = (a + bi) - (a - bi) = 2bi$, a purely imaginary number
- $z\overline{z} = (a + bi)(a - bi) = a^2 + b^2$, a purely real number.

Worksheet
Complex conjugates

WB

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WORKED EXAMPLE 6 Finding the complex conjugate

Find the conjugate of each complex number.

a $z = -3 - 4i$ **b** $z = 2i(5 - 6i)$ **c** $z = i + \dfrac{2 - i^3}{5(2 + i)}$

Steps	Working
a State the conjugate.	$z = -3 - 4i$ $\overline{z} = -3 + 4i$
b Expand and simplify.	$z = 2i(5 - 6i)$ $= 10i - 12i^2$ $= 10i + 12$ $= 12 + 10i$ $\overline{z} = 12 - 10i$
c 1 Simplify the algebraic fraction and write in the form $z = a + bi$.	$z = i + \dfrac{2 - i^3}{5(2 + i)}$ $= i + \dfrac{2 + i}{5(2 + i)}$ $= i + \dfrac{1}{5}$ $= \dfrac{1}{5} + i$
2 State the conjugate.	$\overline{z} = \dfrac{1}{5} - i$

Dividing complex numbers

Let z and w be 2 complex numbers. To find $z \div w = \dfrac{z}{w}$, multiply the numerator and denominator by the conjugate of w. This is done to **realise the denominator**, removing i from the denominator.

This is because the product of a complex number with its conjugate is the sum of the squares of the real and imaginary parts.

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WORKED EXAMPLE 7 Dividing complex numbers

Simplify each expression.

a $\dfrac{2 + 5i}{1 + i}$ **b** $\dfrac{4i}{1 - 2i}$ **c** $\dfrac{8i - 3}{6i}$

Steps	Working
a 1 Multiply the numerator and denominator by the conjugate of the denominator, making the denominator a difference of two squares.	$\dfrac{2 + 5i}{1 + i} \times \dfrac{1 - i}{1 - i} = \dfrac{2 - 2i + 5i - 5i^2}{1 - i^2}$
2 Simplify the numerator and denominator, making the denominator real.	$= \dfrac{2 - 2i + 5i + 5}{1 + 1}$ $= \dfrac{7 - 3i}{2}$ $= \dfrac{7}{2} - \dfrac{3i}{2}$

b **1** Multiply the numerator and denominator by the conjugate of the denominator, making the denominator a difference of two squares.

$$\frac{4i}{1-2i} \times \frac{1+2i}{1+2i} = \frac{4i+8i^2}{1-4i^2}$$

2 Simplify the numerator and denominator.

$$= \frac{4i-8}{1+4}$$
$$= \frac{-8+4i}{5}$$
$$= -\frac{8}{5} + \frac{4i}{5}$$

c **1** Multiply the numerator and denominator by the conjugate of the denominator (multiplying by just i also works).

$$\frac{8i-3}{6i} \times \frac{-6i}{-6i} = \frac{-48i^2+18i}{-36i^2}$$

2 Simplify.

$$= \frac{48+18i}{36}$$
$$= \frac{4}{3} + \frac{1}{2}i$$

Operations with complex numbers

- To add or subtract complex numbers, treat the real and imaginary parts separately.
- To multiply complex numbers, expand and use $i^2 = -1$.
- To divide complex numbers such as $\frac{z}{w}$, **realise the denominator** by multiplying by $\frac{\overline{w}}{\overline{w}}$.

WORKED EXAMPLE 8 Simplifying expressions involving 2 complex numbers

If $z = -1 - 4i$ and $w = 2 - 3i$, evaluate each expression.

a $2z - 3w$ **b** zw **c** $\frac{z}{w}$

Steps	Working
a Substitute the complex numbers into the given expression and simplify.	$2z - 3w = 2(-1-4i) - 3(2-3i)$ $= -2 - 8i - 6 + 9i$ $= -8 + i$
b Substitute the complex numbers into the given expression, expand and simplify.	$zw = (-1-4i)(2-3i)$ $= -2 + 3i - 8i + 12i^2$ $= -2 - 5i - 12$ $= -14 - 5i$
c **1** Substitute the complex numbers into the given expression. **2** Realise the denominator and write in the form $z = x + yi$.	$\frac{z}{w} = \frac{-1-4i}{2-3i}$ $= \frac{-1-4i}{2-3i} \times \frac{2+3i}{2+3i}$ $= \frac{-2-3i-8i-12i^2}{4-9i^2}$ $= \frac{-2-11i+12}{4+9}$ $= \frac{10-11i}{13}$ $= \frac{10}{13} - \frac{11}{13}i$

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EXERCISE 9.1 Complex numbers

ANSWERS p. 600

Mastery

1 WORKED EXAMPLE 1 Simplify each expression in terms of i.

a $\sqrt{-24}$ **b** $\sqrt{-121}$ **c** $\sqrt{-12} + \sqrt{-48}$

2 Using CAS 1 Simplify the expression $3\sqrt{-9} - \frac{5}{2}\sqrt{\frac{-16}{25}} + \sqrt{-6.25}$.

3 Using CAS 2 Simplify the expression $\text{Re}\left(\frac{(1-3i)^2}{1+i}\right) + \text{Im}\left(\frac{2-3i}{2+3i}\right)$.

4 WORKED EXAMPLE 2 Use $i^2 = -1$ to simplify each expression.

a $i(i^3 - i)$ **b** $\sqrt{i^6 - i^2}$ **c** $(i+1)^3$

5 WORKED EXAMPLE 3 Find $\text{Re}(z)$ and $\text{Im}(z)$ for each complex number, z.

a $z = 4\left(x - \frac{1}{2}yi\right)$ **b** $z = \frac{3i-12}{3}$ **c** $z = i(-2 + 5i)$

6 WORKED EXAMPLE 4 Simplify each expression and write the result in the form $z = a + bi$.

a $4i + 7 - 3i + 1$ **b** $\frac{3}{4}(12 - 24i) + (2 + i)i$ **c** $\frac{3}{-i} - i + 1$

7 WORKED EXAMPLE 5 Simplify each expression.

a $(3 - i)(2 + i)$ **b** $8i - (1 + 4i)^2$ **c** $i(2i - 1)(1 + 2i)$

8 WORKED EXAMPLE 6 Find the conjugate of each complex number.

a $z = 8 - 11i$ **b** $z = -i(2 + i) - 3i$ **c** $z = \frac{2-i}{i}$

9 WORKED EXAMPLE 7 Simplify each expression.

a $\frac{3 + 2i}{2 - i}$ **b** $\frac{7i}{4 - i}$ **c** $\frac{5i + 1}{2i}$

10 WORKED EXAMPLE 8 If $z = 3 + 2i$ and $w = -1 + 5i$, evaluate each expression, writing your answer in the form $a + bi$, where a and b are real numbers.

a $4w - 3z$ **b** $2zw$ **c** w^{-1}

Exam practice

80–100% 60–79% 0–59%

11 TECH-FREE (2 marks) Express $\frac{3 - 2i}{1 + i}$ in the form $a + bi$, where a and b are real numbers.

12 TECH-FREE (2 marks) Find $\left(\text{Re}(1 - i)\right)^2 + \text{Im}(1 - i)^2$.

13 TECH-FREE (3 marks) If $z = 1 - i$, find $z + \frac{1}{z}$ in the form $a + bi$.

14 TECH-FREE (2 marks) Show that $z = 1 + 3i$ is a solution to $z^2 - 2z + 10 = 0$.

15 TECH-FREE (3 marks) Simplify $(2 - i)^3$.

16 Choose the correct statement.

A The conjugate of $2(3 - 2i)$ is $6 + 2i$ **B** $\text{Re}[i(i^2 - 1)] = 0$

C $C \subset R$ **D** $1 + i^2 + i^4 + i^6 = -1$

E $z - \overline{z} = 2\text{Re}(z)$

17 If $z = x + yi$, then $\dfrac{\text{Re}(z) + \text{Im}(z)}{z\overline{z} - 2[\text{Im}(z)]^2}$ is

A $\dfrac{1}{x + y}$ **B** $x - y$ **C** $x + y$ **D** $\dfrac{1}{x - y}$ **E** $\dfrac{x + y}{x - y}$

18 The sum of n terms of the geometric sequence $a, ar, ar^2, ar^3 \ldots$ is $\dfrac{a(1 - r^n)}{1 - r}$. Given that the number of terms, n, of the geometric sequence $1, i^2, i^4, i^6 \ldots$ is divisible by 4, the sum of n terms is

A $\dfrac{1}{2}i^n$ **B** $1 + i^n$ **C** 0 **D** $i^n - 1$ **E** $1 - i$

19 For $z = (2 + 3i)^2$, $\text{Re}(z)$ is

A -6 **B** -5 **C** 4 **D** 5 **E** 6

20 If $z = 6 - i$ and $w = -8 + 2i$, then $\text{Im}(z - 3w)$ is

A $-7i$ **B** -7 **C** -3 **D** 0 **E** $2i$

21 The value of y in the equation $(3 - 5i)(4 - 7i) = -23 + yi$ must be

A -41 **B** -12 **C** 14 **D** 32 **E** 35

22 If $zw = 1$ and $z = -1 - 7i$, then $50w$ must be

A $-1 + 7i$ **B** $50 + 7i$ **C** $-3 - 7i$ **D** $5 - i$ **E** $-20 + 50i$

23 © VCAA 2004 1IQ5 **77%** If $z = x + yi$, where x and y are non-zero real numbers, which one of the following is a real number?

A $\dfrac{1}{z}$ **B** $\dfrac{1}{\overline{z}}$ **C** $\dfrac{1}{z - \overline{z}}$ **D** $\dfrac{1}{z} - \dfrac{1}{\overline{z}}$ **E** $\dfrac{1}{z} + \dfrac{1}{\overline{z}}$

24 © VCAA 2007 2AQ7 **76%** If $z = 3 - 4i$, then $\dfrac{1}{1 - z}$ is equal to

A $\dfrac{-1 - 2i}{10}$ **B** $\dfrac{-1 + 2i}{10}$ **C** $\dfrac{-1 - i}{6}$ **D** $\dfrac{-1 + i}{6}$ **E** $\dfrac{1 - i}{6}$

25 © VCAA 2002 1IQ6 **71%** If $z = x + yi$, where x and y are non-zero real numbers, which one of the following must be a pure imaginary number (that is, a complex number with real part zero)?

A $z - \overline{z}$ **B** $z + \overline{z}$ **C** iz **D** $i\overline{z}$ **E** $\text{Im}(\overline{z})$

26 © VCAA 2014 2AQ6 **65%** Given that $i^n = p$ and $i^2 = -1$, then i^{2n+3} in terms of p is equal to

A $p^2 - i$ **B** $p^2 + i$ **C** $-p^2$ **D** $-ip^2$ **E** ip^2

Use the index law $a^{b+c} = a^b \times a^c$ and $a^{2b} = a^b \times a^b$.

27 © VCAA 2008 2AQ6 **56%** If $z = \dfrac{3+4i}{1+2i}$, the imaginary part of z is

A -2 **B** $-\dfrac{2}{5}i$ **C** $-\dfrac{2}{5}$ **D** $-2i$ **E** 2

To find the real and imaginary parts, the expression must be in the form $a + bi$, with a, b real numbers.

28 © VCAA 2009 2AQ8 **47%** Given that $(1+i)^n = ai$, where a is a non-zero real constant, then $(1+i)^{2n+2}$ simplifies to

A a^4 **B** $2a^2i$ **C** 0 **D** $1 + a^2i$ **E** $-2a^2i$

29 (3 marks) If $z = x + yi$, with x, y real, and $\dfrac{z^2 - z}{z - 1} = z$, state the restrictions on the values of x and y.

30 (4 marks) Prove each identity for $z = x + yi$.

a $z - \overline{z} = 2\text{Im}(z)$ 2 marks

b $z\overline{z} = [\text{Re}(z)]^2 + [\text{Im}(z)]^2$ 2 marks

31 (4 marks) Given the complex numbers $z = a + bi$ and $w = c + di$, prove each identity.

a $\overline{z + w} = \overline{z} + \overline{w}$ 2 marks

b $\overline{zw} = \overline{z} \times \overline{w}$ 2 marks

32 (3 marks) Given $z = a + bi$ $(b > 0)$, for what values of a and b will $\overline{z} = z^2$?

Video playlist
Quadratic equations with complex roots

9.2 Quadratic equations with complex roots

The discriminant

The quadratic equation $ax^2 + bx + c = 0$, where $a, b, c \in R$ and $a \neq 0$, can be solved using the **quadratic formula**, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. In the formula, the expression under the square root symbol, $\Delta = b^2 - 4ac$, is the **discriminant** and gives information about the number and type of roots in the quadratic equation.

For example, to solve $x^2 + 3x + 2 = 0$ we use $a = 1$, $b = 3$ and $c = 2$ to obtain $x = \dfrac{-3 \pm \sqrt{3^2 - 4 \times 1 \times 2}}{2 \times 1}$, so $x = -1$ and $x = -2$.

The value of the discriminant determines the type of solution.

The discriminant

The **discriminant** for any quadratic equation $ax^2 + bx + c = 0$ is $\Delta = b^2 - 4ac$.

- If $\Delta > 0$, the quadratic equation has 2 real roots. For example, $x^2 - 3x + 2 = 0$ has solutions $x = 1, x = 2$.
- If $\Delta = 0$, the quadratic equation has 1 real root, which is rational. For example, $x^2 - 2x + 1 = 0$ has solution $x = 1$.
- If $\Delta < 0$, the quadratic equation has 2 complex roots (that are not real), each of the form $p \pm qi$. For example, $x^2 - 2x + 2 = 0$ has solutions $x = 1 - i, x = 1 + i$.
- If $\Delta < 0$ and $b = 0$, the quadratic equation has 2 (purely) imaginary roots, each of the form $x = \pm qi$. For example, $x^2 + 64 = 0$ has solutions $x = -8i, 8i$.

Even though the set of complex numbers include all the real numbers, the phrase 'complex roots' usually means 'non-real' roots of the form $a + bi$ where $b \neq 0$. 'Imaginary roots' are of the form bi.

Solving quadratic equations with complex roots

A quadratic equation can be solved by factorising, completing the square or by using the quadratic formula. However, if $\Delta < 0$ and the solutions are complex, the quadratic formula should be used.

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WORKED EXAMPLE 9 Factorising a quadratic expression in the set of complex numbers

Solve $x^2 + 2x + 3 = 0$ and then factorise $x^2 + 2x + 3$.

Steps	Working
1 Evaluate the discriminant to decide on the type of solution.	$a = 1, b = 2, c = 3$. $b^2 - 4ac = 2^2 - 4 \times 1 \times 3 = -8 < 0$ The solutions are complex (non-real).
2 Use the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.	$x = \dfrac{-2 \pm \sqrt{2^2 - 12}}{2}$ $= \dfrac{-2 \pm \sqrt{-8}}{2}$ $= \dfrac{-2 \pm \sqrt{8}\sqrt{-1}}{2}$ $= \dfrac{-2 \pm 2\sqrt{2}i}{2}$ $= -1 \pm \sqrt{2}i$
3 To factorise, write the solutions in the form $a + bi$.	Let x_1 and x_2 be the 2 solutions. Then $x_1 = -1 + \sqrt{2}i, x_2 = -1 - \sqrt{2}i$
4 Write the expression in the form $(x - x_1)(x - x_2)$.	$x^2 + 2x + 3 = [x - (-1 + \sqrt{2}i)][x - (-1 - \sqrt{2}i)]$ $= (x + 1 - \sqrt{2}i)(x + 1 + \sqrt{2}i)$

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WORKED EXAMPLE 10 Classifying the type of solution of a quadratic equation

Solve each quadratic equation and classify the roots as real, purely imaginary or complex.

a $x^2 + 16 = 0$ **b** $x^2 - 6x + 7 = 0$ **c** $x^2 - 6x + 10 = 0$

Steps	Working
a Make x^2 the subject and solve. Notice that the roots are complex conjugates.	$x^2 + 16 = 0$ $x^2 = -16$ $x = \pm\sqrt{-16}$ $= \pm\sqrt{16}\sqrt{-1}$ $= \pm 4i$ The roots are purely imaginary.
b $x^2 - 6x + 7 = 0$ does not factorise. Use the quadratic formula.	$x^2 - 6x + 7 = 0$ $x = \dfrac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(7)}}{2(1)}$ $= \dfrac{6 \pm \sqrt{8}}{2}$ $= \dfrac{6 \pm 2\sqrt{2}}{2}$ $= 3 \pm \sqrt{2}$ The roots are real.
c $x^2 - 6x + 10 = 0$ does not factorise. Use the quadratic formula. Notice that the roots are complex conjugates.	$x^2 - 6x + 10 = 0$ $x = \dfrac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(10)}}{2(1)}$ $= \dfrac{6 \pm \sqrt{-4}}{2}$ $= \dfrac{6 \pm 2i}{2}$ $= 3 \pm i$ The roots are complex.

To solve $z^2 + 9 = 0$ by CAS with TI-Nspire, use **cSolve($z^2 + 9 = 0, z$)**.

For ClassPad, use **solve($z^2 + 9 = 0, z$)**, remembering to set the number option to complex numbers at the bottom of the main screen.

(Note the variable z, which is often used when referring to equations with complex roots.)

The solutions found in the previous example demonstrate that complex roots of $az^2 + bz + c = 0$, with $a, b, c \in R$ and $a \neq 0$, occur as **conjugate pairs**. For example, if $2i$ is a solution to $z^2 - 4 = 0$, then its conjugate, $-2i$, is also a solution.

Complex roots of a quadratic equation

If a quadratic equation with **real** coefficients has complex roots, they are **complex conjugates**.

Suppose z_1 and z_2 are solutions to $z^2 + bz + c = 0$. Then $z^2 + bz + c = (z - z_1)(z - z_2)$.

Expanding gives $z^2 + bz + c = z^2 - (z_1 + z_2)z + z_1z_2$.

Comparing (equating) coefficients of powers of z gives $z_1 + z_2 = -b$ and $z_1z_2 = c$.

This is useful when z_1 and z_2 are known and we wish to find the quadratic expression.

WORKED EXAMPLE 11 Finding the quadratic equation given a complex root

If $z = 1 + 3i$ is a solution to $z^2 + bz + c = 0$, determine the values of b and c.

Steps	Working
1 Find the conjugate, which is the other solution.	Let $z_1 = 1 + 3i$ Let $z_2 = \overline{z}_1 = 1 - 3i$
2 The sum of the roots is $-b$.	$z_1 + z_2 = -b$ $(1 + 3i) + (1 - 3i) = -b$ $b = -2$
3 The product of the roots is c.	$z_1 z_2 = c$ $(1 + 3i)(1 - 3i) = c$ $1 - 9i^2 = c$ $1 + 9 = c$ $c = 10$
4 State the values found.	$b = -2$ and $c = 10$

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EXERCISE 9.2 Quadratic equations with complex roots

ANSWERS p. 600

Recap

1 For $z = (3 - i)^2$, $\text{Re}(z) + \text{Im}(z)$ is

A 0 **B** 1 **C** 2 **D** 3 **E** 4

2 Given $z = 2 + 3i$ and $w = 1 - 2i$, then $\text{Im}\left(\frac{w - z}{i}\right)$ is

A -1 **B** 0 **C** 1 **D** 2 **E** 3

Mastery

3 WORKED EXAMPLE 9 Solve $x^2 - 2x + 8 = 0$ and then factorise $x^2 - 2x + 8$.

4 WORKED EXAMPLE 10 Solve each equation and classify the roots as real, imaginary or complex.

a $2x^2 + 32 = 0$ **b** $x^2 - 8x + 16 = 0$ **c** $x^2 + 6x + 10 = 0$

5 Show that $z = -5 + 2i$ is a solution to $z^2 + 10z + 29 = 0$.

6 WORKED EXAMPLE 11 If $z = -3 - 2i$ is a solution to $z^2 + bz + c = 0$, determine the values of b and c.

Exam practice

7 TECH-FREE (4 marks) Find the solutions to each equation and state the type of roots obtained.

a $(x^2 + 2)(x^2 + 3) = 0$ 2 marks

b $x^2 - 3x + 10 = 0$ 2 marks

8 TECH-FREE (2 marks) For $z = x + yi$, show that $\frac{z\overline{z}}{|z||\overline{z}|} = 1$.

9 TECH-FREE (2 marks) Solve $\frac{1}{2}z^2 + 2 = 0$.

10 TECH-FREE (3 marks) Factorise $w^4 + 29w^2 + 100$.

Hint: Let $u = w^2$ and obtain a quadratic expression that can be factorised.

11 If $3i - 5$ is a solution to a quadratic equation, another solution will be

A $3i + 5$ **B** $-3i - 5$ **C** $5 - 3i$ **D** $5i - 3$ **E** $3 - 5i$

12 The equation $27 + 3z^2 = 0$ has exactly

A 1 solution, imaginary
B 1 solution, real
C 1 solution, complex
D 2 solutions, imaginary
E 2 solutions, real

13 The quadratic equation $4x^2 - 4x + k = 0$ has 1 real solution. The value of k is

A -4 **B** -1 **C** 0 **D** 1 **E** 4

14 The solutions to $z^2 + bz + c$ are $-6 + i$ and $-6 - i$. The values of b and c are

A $b = 12, c = 37$ **B** $b = -37, c = 12$ **C** $b = -12, c = 37$
D $b = 37, c = -12$ **E** $b = 12, c = -37$

15 (3 marks) Solve $(x + 1)^2 = 1 - 2(x + 3)^2$.

16 (3 marks) Show that if $w = x + yi$ is a solution to $z^2 + bz + c = 0$, then $z^2 + bz + c = z^2 - 2xz + x^2 + y^2$.

17 (4 marks)

a Find the values of x and y in $w = x + yi$ if $w\overline{w} = 13$ and $w + \overline{w} = -6$. 2 marks

b w and $\overline{w}$ are solutions of $(z - w)(z - \overline{w}) = 0$. Write $(z - w)(z - \overline{w})$ in the form $az^2 + bz + c$. 2 marks

18 (3 marks) Prove that $(z - k)^2 + kz + k$ has complex solutions for all $k > 0$ and $k < -\frac{4}{3}$.



Exam hack

If the number or the type of solutions of a quadratic equation is given, work with its discriminant.

19 (3 marks) For the function $f(z) = z^2 + 2az + b$, $f(4 - 5i) = 0$. Find the values of a and b.

20 (2 marks) Let $f(z) = az^2 + bz + c$, where a, b and c are real constants.

If the solutions to $f(z) = 0$ are purely imaginary, what range of values can a, b and c take?

21 (3 marks) One of the roots of a quadratic equation is $\frac{4 + i}{\sqrt{2} - i}$. Determine the product of its solutions.

22 (3 marks) Let $(z - z_1)(z - z_2) = 0$, where $z_1 = a + bi$ and $z_2 = c + di$, with a, b, c, d real values.

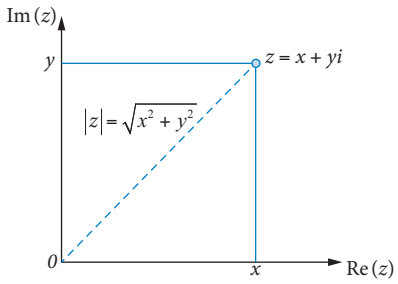
Prove that z_1 and z_2 are conjugates of each other.

Hint: Remember that the coefficients of powers of z in $f(z)$ are real.

9.3 The Argand diagram

We are familiar with specifying the position of a point on a number plane (Cartesian plane) in the form (x, y), where x and y is the point's horizontal distance and vertical distance from the origin respectively.

In a similar way, complex numbers can be represented on a number plane called the **Argand diagram** (or **complex plane**).



The Argand diagram

- The complex number $z = x + yi$ is represented by the point (x, y) on the Argand diagram.
- The **real axis** is horizontal and labelled **Re (z)** or x.
- The **imaginary axis** is vertical and labelled **Im (z)** or iy.

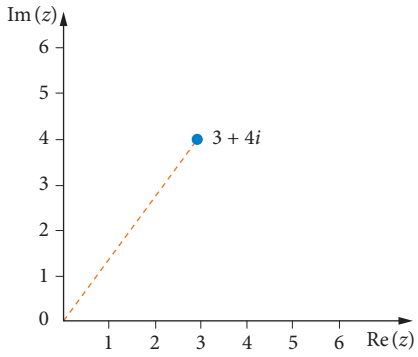
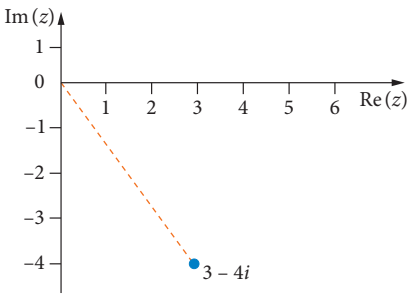
For a complex number z, $|z|$ is the **modulus** of z and is the distance from its position on the Argand diagram to the origin. By applying Pythagoras' theorem in the diagram above, $|z| = \sqrt{x^2 + y^2}$.

It was stated earlier in this chapter that $z\overline{z} = x^2 + y^2$, so $z\overline{z} = |z|^2$.

WORKED EXAMPLE 12 Find the modulus from the graph

Graph each complex number on an Argand diagram and find its modulus.

a $3 + 4i$ **b** $\overline{3 + 4i}$ **c** 3 **d** $4i$

Steps	Working
a 1 Graph $3 + 4i$ on an Argand diagram.	$3 + 4i$ is represented by the point $(3, 4)$. 
2 Use Pythagoras' theorem to find its modulus.	$\lvert 3 + 4i \rvert = \sqrt{3^2 + 4^2} = 5$
b 1 Find the conjugate of $3 + 4i$.	$\overline{3 + 4i} = 3 - 4i$, represented by $(3, -4)$.
2 Graph the point on an Argand diagram and use Pythagoras' theorem to calculate its modulus.	 $\lvert 3 + 4i \rvert = \sqrt{3^2 + (-4)^2} = 5$

Note that complex conjugates are reflections of each other in the real axis.

Video playlist The Argand diagram

Worksheets The complex plane

Addition and subtraction in the plane

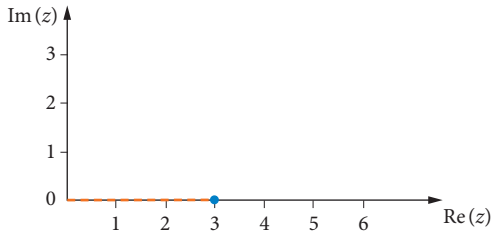
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c 1 Graph 3 on an Argand diagram.

Write 3 as $3 + 0i$, represented by $(3, 0)$.

2 3 is 3 units from the origin.

Purely real numbers lie on the real axis.

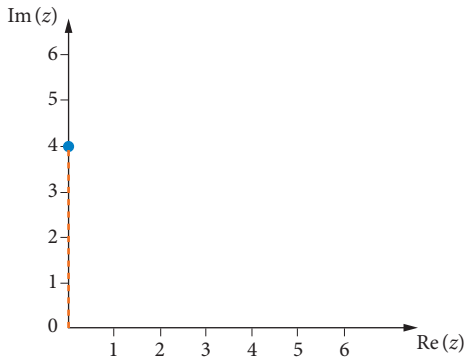


$|3| = 3$

d 1 Graph $4i$ on an Argand diagram.

Write 4i as $0 + 4i$, represented by $(0, 4)$.

Purely imaginary numbers lie on the imaginary axis.



2 $4i$ is 4 units from the origin.

$|4i| = 4$

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WORKED EXAMPLE 13 Calculating the modulus

Find $|z|$ for each complex number.

a $z = 1 - \sqrt{3}i$ **b** $z = -2 + 2i$ **c** $z = \dfrac{1+i}{\sqrt{2}}$

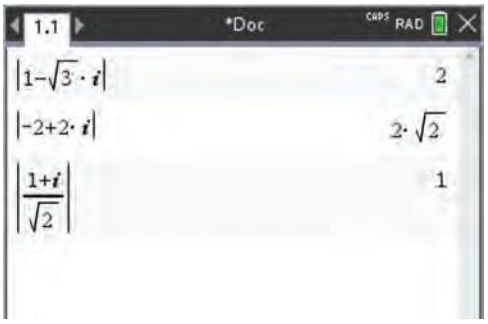
Steps	Working
a Use Pythagoras' theorem to find its modulus.	$\|z\| = \sqrt{(1)^2 + (-\sqrt{3})^2}$ $= \sqrt{4}$ $= 2$
b Use Pythagoras' theorem to find its modulus.	$\|z\| = \sqrt{(-2)^2 + (2)^2}$ $= \sqrt{8}$ $= 2\sqrt{2}$
c Separate into real and imaginary parts and use Pythagoras' theorem to find its modulus.	$\|z\| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$ $= \sqrt{\frac{1}{2} + \frac{1}{2}}$ $= \sqrt{1}$ $= 1$

9780170448338

USING CAS 3 Modulus of a complex number

Use CAS for Worked example 13 to find the modulus of $z = 1 - \sqrt{3}i$, $z = -2 + 2i$ and $z = \dfrac{1+i}{\sqrt{2}}$.

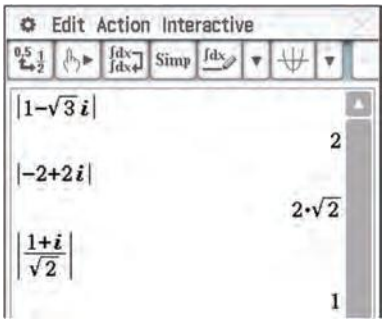
TI-Nspire



1 Press **template** to access the **absolute value** template, which is also known as the **modulus** template.

2 Enter the complex numbers as shown above.

ClassPad



1 Open the **Keyboard** > **Math1** or **Math2** to access the **absolute value** template, which is also known as the **modulus** template.

2 Enter the complex numbers as shown above.

Multiplication by i

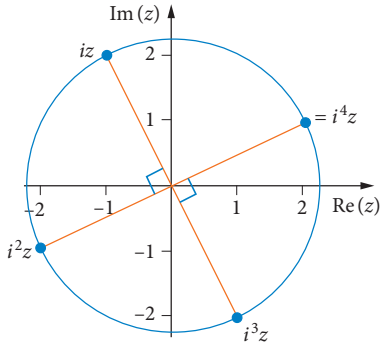
Consider $z = 2 + i$ multiplied by i, i^2, i^3 and i^4.

$i \times z = iz = -1 + 2i$

$i^2 \times z = -z = -2 - i$

$i^3 \times z = -iz = 1 - 2i$

$i^4 \times z = z = 2 + i$



Each product is shown on the Argand diagram.

Notice that:

- z rotates anticlockwise by 90° for each multiplication of i
- the centre of rotation is at the origin
- the points z, zi, zi^2 and zi^3 lie on the circumference of a circle of radius $\sqrt{2^2 + 1^2} = \sqrt{5}$ units.

Multiplication by powers of i

The point $z = x + yi$ lies on the circle of radius $\sqrt{x^2 + y^2}$.

$i^n z$ results in a $90n°$ rotation in an anticlockwise direction of z on the circumference of the circle.

- Multiplying z by i gives a 90° rotation and the product $iz = -y + xi$ is represented by the point $(-y, x)$.
- Multiplying z by i^2 gives a 180° rotation and the product $-z = -x - yi$ is represented by the point $(-x, -y)$.
- Multiplying z by i^3 gives a 270° rotation and the product $-iz = y - xi$ is represented by the point $(y, -x)$.

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WORKED EXAMPLE 14 Rotation of a complex number

Find the complex number, w, formed when $z = 3 - 5i$ is rotated on the Argand diagram by

a 270° anticlockwise **b** 180° clockwise.

Steps	Working
a 1 Decide on the transformation involved.	A 270° rotation to $z = x + yi$ gives $w = y - xi$.
2 Identify the values of x and y and state w.	$x = 3$ and $y = -5$. Therefore, $w = -5 - 3i$.
b 1 Decide on the transformation involved.	A 180° rotation clockwise is equivalent to a 180° rotation anticlockwise. Therefore, $w = -x - yi$.
2 Identify the values of x and y and state w.	$x = 3$ and $y = -5$. Therefore, $w = -3 + 5i$.

EXERCISE 9.3 The Argand diagram

ANSWERS p. 601

Recap

1 If one solution to the quadratic equation $f(z) = 0$ is $\sqrt{a}i$, then $f(z)$ is

A $z - a^2$ **B** $z^2 + a$ **C** $z^2 - a$ **D** $z + \sqrt{a}$ **E** $z^2 - \sqrt{a}$

2 If $z = x + yi$ and w is the conjugate of z, then $(zw)(z + w) - 2|w|^2$ simplifies to

A $(x + 1)(x^2 - y^2)$ **B** $2(x - 1)(x^2 - y^2)$ **C** $2(x + 1)(x^2 - y^2)$

D $(x + 1)(x^2 + y^2)$ **E** $2(x - 1)(x^2 + y^2)$

Mastery

3 WORKED EXAMPLE 12 Graph each complex number on an Argand diagram and find its modulus.

a $20 + 21i$ **b** $\overline{-12 + 35i}$

4 WORKED EXAMPLE 13 Using CAS 3 Find $|z|$ for each complex number.

a $z = \frac{1}{3}\left(\sqrt{2} - i\right)$ **b** $z = -\frac{\sqrt{3}}{2} + \sqrt{2}i$

5 WORKED EXAMPLE 14 Find the complex number, w, formed when $z = 8 + 13i$ is rotated on the Argand diagram

a 180° anticlockwise **b** 90° clockwise.

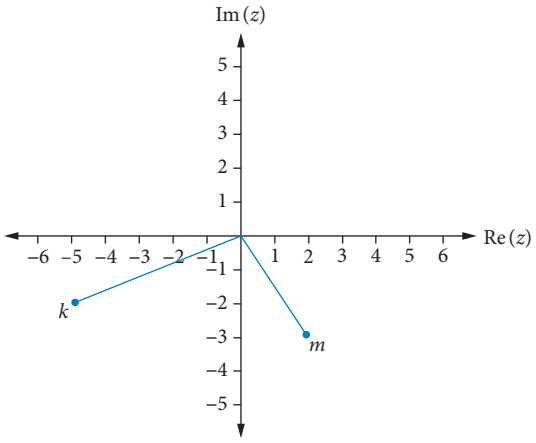
Exam practice

80–100% 60–79% 0–59%

6 TECH-FREE (2 marks) The complex number $z = 2 - 13i$ is rotated on the Argand diagram 270° anticlockwise to obtain $w = a + bi$. What are the values of a and b?

7 TECH-FREE (2 marks) Let $z = a + bi$ and $w = a + 2 + 3bi$, where a and b are positive whole numbers. Determine the values of a and b if the magnitude of z is 5 and the magnitude of w is 13.

8 TECH-FREE (6 marks) Two complex numbers are shown in the Argand diagram.



a State the complex number representing k and m. 2 marks

b State the complex conjugate of k and m. 2 marks

c Copy the Argand diagram above and plot on it $\overline{k}$ and $\overline{m}$. 2 marks

9 TECH-FREE (7 marks) If $z = 6 - 3i$, plot each complex number on an Argand diagram.

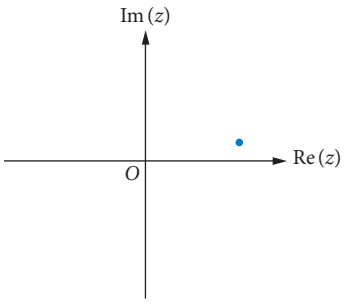
a z 1 mark

b $z + \overline{z}$ 2 marks

c $z - \overline{z}$ 2 marks

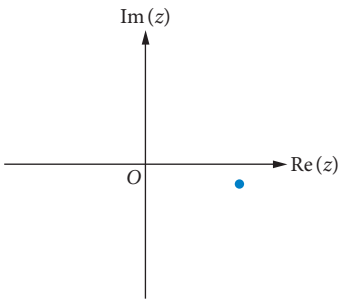
d $(z - 1)(\overline{z} - 4)$ 2 marks

10 © VCAA 2006 2AQ4 74% The complex number $a + bi$, where a and b are real constants, is represented in the diagram on the right.

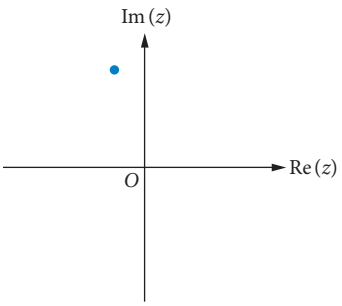


The complex number $-i(a + bi)$ could be represented by

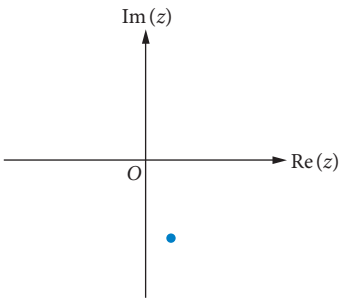
A



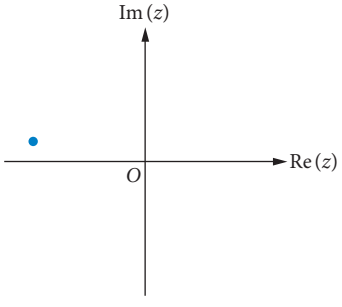
B



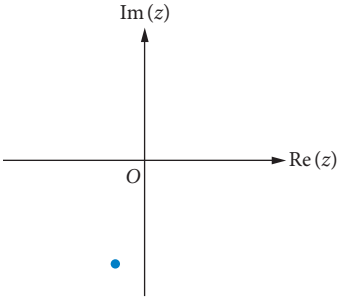
C



D

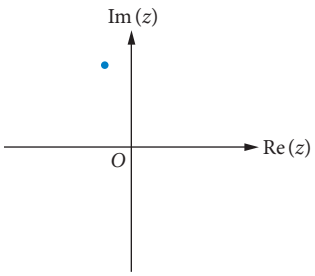


E

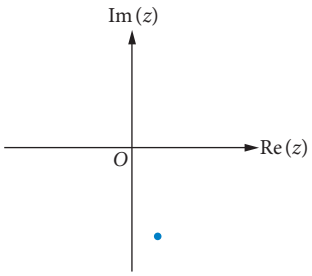


11 © VCAA 2010 2AQ7 **72%** A particular complex number z is represented by the point on the Argand diagram on the right.

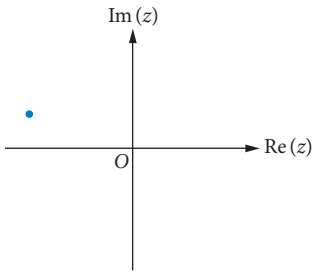
The complex number $i\overline{z}$ is best represented by



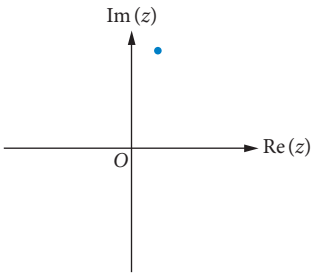
A



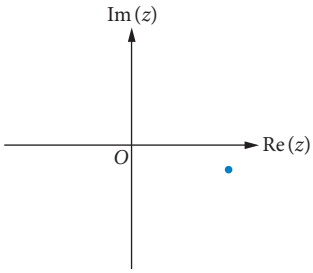
B



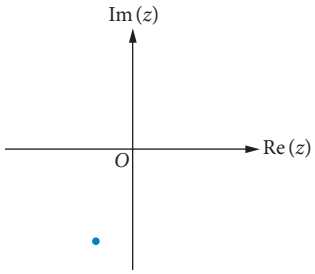
C



D



E



12 $z = -4 - 9i$ is rotated 90° anticlockwise to a point with coordinates (a, b). The values of a and b are

A $a = -4, b = 9$ **B** $a = 9, b = 4$ **C** $a = 4, b = -9$

D $a = 9, b = -4$ **E** $a = -9, b = -4$

13 The complex number $z = a + bi$ after multiplication by a power of i becomes $w = -a - bi$.

The angle of rotation of the point represented by z must be

A 180° anticlockwise **B** 90° anticlockwise **C** 270° anticlockwise

D 360° clockwise **E** 270° clockwise

14 The point represented by z is rotated 450° clockwise. This is the same as a rotation resulting from finding

A i^3z **B** i^2z **C** iz **D** $-iz$ **E** $-i^2z$

15 (2 marks) If $z = a + bi$, state the range of values of a and b so that $w = i^3z$ lies on the imaginary axis.

16 (3 marks) Find all complex numbers of the form $z = a + bi$, $a < 0$, $b > 0$, such that $|z| = 13$ and $\text{Im}(z) = \text{Re}(z) + 17$. Hint: Obtain a quadratic equation involving a or b and solve.

17 (2 marks) Determine the angle that the line connecting the origin to the point $z = 1 + \sqrt{3}i$ makes with the positive real axis.

18 (2 marks) Find the values of x in $z = x + 12i$, given that $|z| = 13$.

19 (3 marks) Point $z_1 = a + bi$ moves to point z_2 after reflection in the real axis and then to point z_3 after a 90° clockwise rotation. Show that $|z_2 + z_3| = \sqrt{2(a^2 + b^2)}$.

20 (3 marks) Point $z = a + bi$ lies in the 1st quadrant and is rotated $\frac{3\pi}{2}$ radians anticlockwise to position w. Find the values of a and b if $zw = 4\sqrt{3} - i$.

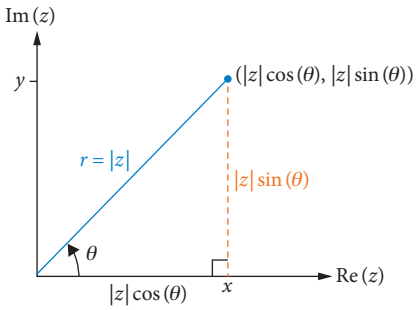
9.4 Polar form of a complex number

The point $z = x + yi$ on an Argand diagram has coordinates (x, y) consisting of a real part, x, and an imaginary part, y.

Consider a straight line from the origin to the point, z, that makes an angle of θ with the positive real axis.

Applying basic trigonometry to the right-angled triangle in the diagram gives $x = |z|\cos(\theta)$ and $y = |z|\sin(\theta)$.

All points, z, lie on a circle of radius r, so we can also write $x = r\cos(\theta)$ and $y = r\sin(\theta)$.



Hence, we have the **polar form** of a complex number:

$z = r\cos(\theta) + r\sin(\theta)i$
$= r(\cos(\theta) + i\sin(\theta))$

$z = r(\cos(\theta) + i\sin(\theta))$ is often written in abbreviated form as $\boldsymbol{z = r\,\mathrm{cis}\,(\theta)}$.

Changing between Cartesian form and polar form

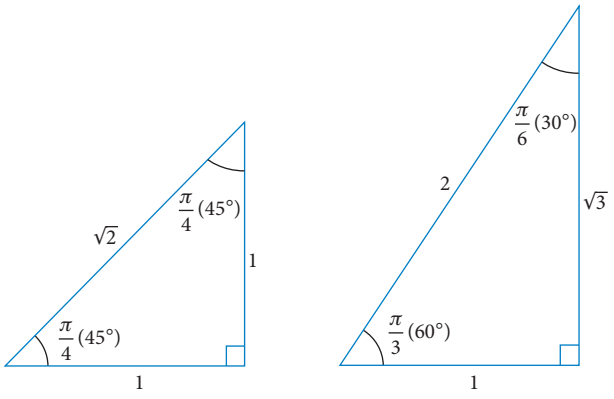
To convert the Cartesian form of a complex number, $z = x + yi$, to polar form, $z = r(\cos(\theta) + i\sin(\theta))$, requires values for r and θ.

The value of r is the magnitude of z, so $r = \sqrt{x^2 + y^2}$.

Angle θ is found using $x = r\cos(\theta)$ and $y = r\sin(\theta)$. We have the ratio $\dfrac{y}{x} = \dfrac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)$, so $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$.

The angle, θ, is the **argument of z**, written $\arg(z) = \theta$. If we restrict the angle to be in the interval $-\pi < \theta \leq \pi$, then we write it as $\mathrm{Arg}(z)$ called the **principal argument**.

Note that the angle can be expressed in degrees or radians. Knowledge of the reference triangles for the exact trigonometric ratios can be useful.



Cartesian and polar forms of a complex number

Cartesian form	$z = x + yi$
Polar form	$z = r(\cos(\theta) + i\sin(\theta))$ or $z = r\,\mathrm{cis}\,(\theta)$, where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$.

9.4

Video playlist
Polar form of a complex number

Worksheets
Modulus and argument
Complex number conversions
Evaluating the modulus 1
Evaluating the modulus 2

WB

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WORKED EXAMPLE 15 Converting from Cartesian to polar forms

Write each complex number in polar form, expressing the angle in radians.

a $z = 1 + i$ **b** $z = -1 + \sqrt{3}i$ **c** $z = -2\sqrt{3} - 2i$

Steps	Working
a 1 Evaluate r.	$x = 1, y = 1$ $r = \sqrt{x^2 + y^2}$ $r = \sqrt{1^2 + 1^2} = \sqrt{2}$
2 Find the angle θ.	$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ $= \tan^{-1}(1)$ $= \frac{\pi}{4}$
3 Write the complex number in polar form.	$z = r(\cos(\theta) + i\sin(\theta))$ $= \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$ $= \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$
b 1 Evaluate r.	$x = -1, y = \sqrt{3}$ $r = \sqrt{x^2 + y^2}$ $r = \sqrt{(-1)^2 + \sqrt{3}^2} = 2$
2 Find the principal argument.	$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ $= \tan^{-1}\left(-\sqrt{3}\right)$ $= -\frac{\pi}{3}$
3 Write the complex number in polar form.	$z = r(\cos(\theta) + i\sin(\theta))$ $= 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$ $= 2\,\text{cis}\left(-\frac{\pi}{3}\right)$
c 1 Evaluate r.	$x = -2\sqrt{3}, y = -2$ $r = \sqrt{(-2\sqrt{3})^2 + (-2)^2} = \sqrt{16} = 4$
2 Find the principal argument.	$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ $= \tan^{-1}\left(\frac{-2}{-2\sqrt{3}}\right)$ $= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$ $= \frac{\pi}{6}$ But z is in the 3rd quadrant, so $\text{Arg}\, z = -\left(\pi - \frac{\pi}{6}\right)$ $= -\frac{5\pi}{6}$
3 Write the complex number in polar form.	$z = 4\,\text{cis}\left(-\frac{5\pi}{6}\right)$

WORKED EXAMPLE 16 Converting from polar to Cartesian forms

Write each complex number in Cartesian form.

a $z = 4\cos\left(\frac{3\pi}{2}\right) + 4i\sin\left(\frac{3\pi}{2}\right)$ **b** $z = \sqrt{2}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$ **c** $z = 2\,\text{cis}\,(30°)$

Steps	Working
a 1 Evaluate $\text{Re}(z)$ and $\text{Im}(z)$.	$4\cos\left(\frac{3\pi}{2}\right) = 4 \times 0 = 0$ $4\sin\left(\frac{3\pi}{2}\right) = 4 \times (-1) = -4$
2 Express z in Cartesian form.	$z = 4 - 4i$
b 1 Evaluate $\text{Re}(z)$ and $\text{Im}(z)$.	$\sqrt{2}\cos\left(-\frac{\pi}{6}\right) = \sqrt{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{6}}{2}$ $\sqrt{2}\sin\left(-\frac{\pi}{6}\right) = -\sqrt{2} \times \frac{1}{2} = -\frac{\sqrt{2}}{2}$
2 Express z in Cartesian form.	$z = \frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2}i$ $= \frac{1}{2}(\sqrt{6} - \sqrt{2}i)$
c 1 Write the expression in expanded form.	$z = 2\cos(30°) + 2i\sin(30°)$
2 Evaluate $\text{Re}(z)$ and $\text{Im}(z)$.	$2\cos(30°) = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$ $2\sin(30°) = 2 \times \frac{1}{2} = 1$
3 Express z in Cartesian form.	$z = \sqrt{3} + i$

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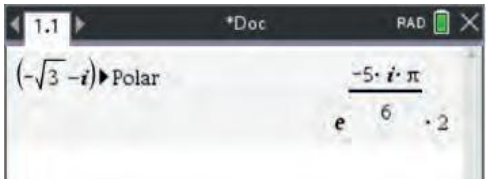
USING CAS 4 Convert between Cartesian and polar form

a Convert $-\sqrt{3} - i$ to polar form, expressing the angle in radians.

b Write $z = 3\cos(-60°) + 3i\sin(-60°)$ in Cartesian form.

TI-Nspire

a



1 Enter the complex number $-\sqrt{3} - i$.

2 Press the **π** key to access **i**. Do not use the letter i.

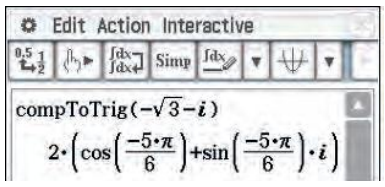
3 Press **menu** > **Number** > **Complex Number Tools** > **Convert to Polar** and press **enter**.

The polar form will appear as $e^{\frac{-5i\pi}{6}} \cdot 2$.

Write this CAS answer as

$$2\cos\left(-\frac{5\pi}{6}\right) + 2i\sin\left(-\frac{5\pi}{6}\right).$$

ClassPad



1 Ensure the modes at the bottom of the screen are set to **Cplx** and **Rad**.

2 Enter the complex number $-\sqrt{3} - i$.

3 Tap **Keyboard** > **Math2** or **Math3** to access **i**. Do not use the letter i.

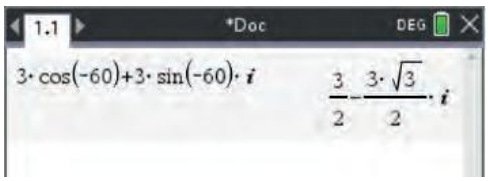
4 Tap **Interactive** > **Complex** > **compToTrig**.

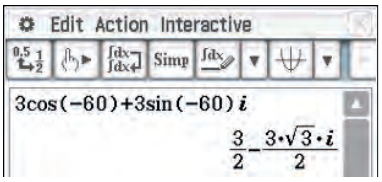
5 Expand the answer to get

$$2\cos\left(-\frac{5\pi}{6}\right) + 2i\sin\left(-\frac{5\pi}{6}\right).$$

TI-Nspire | **ClassPad**

b





1 Change the mode to **DEG**.
2 Enter the polar form number and press **enter**.

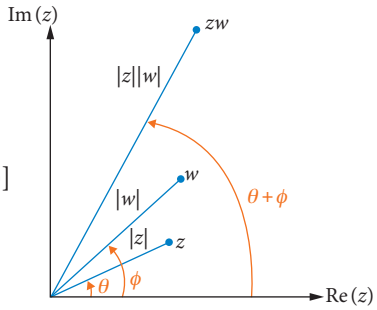
1 Change the mode to **Deg**.
2 Enter the polar form number and press **EXE**.

$3\cos(-60°) + 3\sin(-60°) = \frac{3}{2} - \frac{3\sqrt{2}}{2}i$

Worksheets
Multiplication in the plane
Division in the plane
Polar complex number operations

Multiplication and division

Multiplication can be carried out using the polar form of each complex number.



Let $z = r_1(\cos(\theta) + i\sin(\theta))$ and $w = r_2(\cos(\phi) + i\sin(\phi))$

$$\begin{aligned} zw &= r_1(\cos(\theta) + i\sin(\theta)) \times r_2(\cos(\phi) + i\sin(\phi)) \\ &= r_1r_2[\cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi) + (\sin(\theta)\cos(\phi) + \cos(\theta)\, i\sin(\phi))] \\ &= r_1r_2[\cos(\theta+\phi) + (i\sin(\theta+\phi))]^* \\ &= |z||w|[\cos(\theta+\phi) + (i\sin(\theta+\phi))] \end{aligned}$$

*Using the sum and difference identities from Chapter 7:

$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$,

$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$.

The product of two complex numbers

The product of z with w is a complex number with magnitude $|z| \times |w|$ rotated by an angle of $(\theta + \phi)$ anticlockwise from the positive real axis.

Division of one complex number by another complex number can be treated in a similar way.

$$\frac{z}{w} = \frac{|z|(\cos(\theta) + i\sin(\theta))}{|w|(\cos(\phi) + i\sin(\phi))}$$

Realise the denominator by multiplying the denominator by its conjugate.

$$\begin{aligned} \frac{z}{w} &= \frac{|z|}{|w|} \times \frac{\cos(\theta) + i\sin(\theta)}{\cos(\phi) + i\sin(\phi)} \times \frac{\cos(\phi) - i\sin(\phi)}{\cos(\phi) - i\sin(\phi)} \\ &= \frac{|z|}{|w|} \times \frac{(\cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi)) + i(\sin(\theta)\cos(\phi) - \cos(\theta)\sin(\phi))}{\cos^2(\phi) + \sin^2(\phi)} \\ &= \frac{|z|}{|w|} \times \frac{\cos(\theta-\phi) + i\sin(\theta-\phi)^{\#}}{1} \\ &= \frac{|z|}{|w|} \times \cos(\theta-\phi) + i\sin(\theta-\phi) \end{aligned}$$

Using the identities $\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$, $\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$ and $\cos^2(x) + \sin^2(x) = 1$.

The quotient of 2 complex numbers

The quotient $\frac{z}{w}$ is a complex number with magnitude $\frac{|z|}{|w|}$ rotated by an angle of $(\theta - \phi)$ anticlockwise from the positive real axis.

9780170448338

WORKED EXAMPLE 17 Multiplication and division of complex numbers

a Write $(3 - i)(4 + 6i)$ in the form $a + bi$.

b Given $z = 16\,\text{cis}\left(\frac{3\pi}{2}\right)$ and $w = 4\,\text{cis}\left(\frac{2\pi}{3}\right)$, write $\frac{z}{w}$ in Cartesian form.

c Write z^2 in the form $a + bi$ given that $z = 2\,\text{cis}\,(65°)$. Write a and b correct to 2 decimal places.

Steps	Working
a 1 Expand brackets.	$(3 - i)(4 + 6i) = 12 + 18i - 4i - 6i^2$
2 Simplify and express the answer in the required form.	$= 12 + 14i - 4i + 6\ (i^2 = -1)$ $= 18 + 10i$
b 1 Express $\frac{z}{w}$ in polar form.	$\frac{z}{w} = \frac{16\,\text{cis}\left(\frac{3\pi}{2}\right)}{4\,\text{cis}\left(\frac{2\pi}{3}\right)}$ $= 4\,\text{cis}\left(\frac{3\pi}{2} - \frac{2\pi}{3}\right)$ $= 4\,\text{cis}\left(\frac{5\pi}{6}\right)$
2 Write in Cartesian form and simplify.	$\frac{z}{w} = 4\cos\left(\frac{5\pi}{6}\right) + 4i\sin\left(\frac{5\pi}{6}\right)$ $= 4 \times -\frac{\sqrt{3}}{2} + 2i$ $= -2\sqrt{3} + 2i$
c 1 Write z^2 in polar form.	$z^2 = 2^2\,\text{cis}\,(65° + 65°)$ $= 4\,\text{cis}\,(130°)$
2 Write in Cartesian form, writing answers to the required accuracy.	$= 4\cos(130°) + 4i\sin(130°)$ $= -2.57 + 3.06i$

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EXERCISE 9.4 Polar form of a complex number

ANSWERS p. 601

Recap

1 If $z = 8 - 2i$ and $w = 4 - 3i$, then $|2z - 3w|$ is

A $\sqrt{17}$ **B** $\sqrt{34}$ **C** $\sqrt{41}$ **D** 17 **E** 41

2 If $z = a + bi$, then $\text{Re}(\overline{z + \bar{z}}) + \text{Im}(\overline{z - \bar{z}})$ is

A $3a$ **B** $a + b$ **C** $2a - b$ **D** $2a - 2b$ **E** $a - b$

Mastery

3 If $z = \sqrt{2}(1 + i)$ and $w = 2(1 + \sqrt{3}i)$, then $\text{Arg}\left(\frac{z}{w}\right)$ is

A $\frac{5\pi}{12}$ **B** $\frac{2\pi}{3}$ **C** $-\frac{3\pi}{4}$ **D** $\frac{\pi}{3}$ **E** $-\frac{\pi}{12}$

4 If $z = a + 3i$ and $|z| = \sqrt{13}$, then $\text{Re}\,(z)$ could be

A 1 **B** 2 **C** 3 **D** 4 **E** 5

5 WORKED EXAMPLE 15 Write each complex number in polar form, expressing the angle in radians.

a $z = 4 - 4\sqrt{3}i$ **b** $z = \frac{1}{2}(-1 - i)$

6 WORKED EXAMPLE 16 Write each complex number in Cartesian form.

a $z = 5\cos\left(-\frac{3\pi}{4}\right) + 5i\sin\left(-\frac{3\pi}{4}\right)$ **b** $z = 2\sqrt{3}\,\text{cis}(-60°)$

7 WORKED EXAMPLE 17

a Expand and simplify $2(4 + 3i)(-1 + 5i)$.

b Given $z = \sqrt{3}\,\text{cis}\left(\frac{\pi}{6}\right)$, write z^2 in Cartesian form.

8 Using CAS 4

a Convert $z = 3\sqrt{2}(-1 - i)$ to polar form, expressing the angle in degrees.

b Express $z = 2\sqrt{2}\cos\left(-\frac{\pi}{6}\right) - 2\sqrt{2}i\sin\left(-\frac{\pi}{6}\right)$ in Cartesian form.

Exam practice

80–100% 60–79% 0–59%

9 TECH-FREE (2 marks) Write $z = -4 - 4\sqrt{3}i$ in polar form and with a principal argument.

10 TECH-FREE (3 marks) State, in polar form, all solutions to $z^2 = -25i$.

11 TECH-FREE (2 marks) z and w are points on the circumference of a circle of radius r such that the distance between them is the maximum possible.

a Show that $zw = r^2\,\text{cis}(2\theta)$, where $\text{Arg}(z) = \theta$. 1 mark

b Verify the result using $z = -1 + i$, $w = -1 - i$. 1 mark

12 In polar form, $z = -2(1 - \sqrt{3}i)$ is

A $z = 2\,\text{cis}\left(\frac{\pi}{3}\right)$ **B** $z = 4\,\text{cis}\left(-\frac{\pi}{3}\right)$ **C** $z = 4\,\text{cis}\left(\frac{2\pi}{3}\right)$

D $z = 2\,\text{cis}\left(-\frac{2\pi}{3}\right)$ **E** $z = 4\,\text{cis}\left(\frac{\pi}{6}\right)$

13 © VCAA 2016 2AQ5 72% If $\text{Arg}(-1 + ai) = \frac{2\pi}{3}$, then the real number a is

A $-\sqrt{3}$ **B** $-\frac{\sqrt{3}}{2}$ **C** $-\frac{1}{\sqrt{3}}$ **D** $\frac{1}{\sqrt{3}}$ **E** $\sqrt{3}$

14 (2 marks) If $r\,\text{cis}\,\theta° = 10 + 24i$, determine the values of r and $\theta°$. Write $\theta°$ to the nearest degree.

15 (3 marks) The complex number $z = x + yi$, with $(y < 0)$ lies on the imaginary axis. Write z in polar form, expressing the angle in radians.

16 (3 marks) For complex numbers z and w, $|z| = |w|$, $zw = 9\,\text{cis}\left(\frac{\pi}{2}\right)$ and $\text{Arg}(w) = \frac{\pi}{3}$. Write z in Cartesian form.

17 (3 marks)

a Write $z = 1 - i$ and $w = \sqrt{3} + i$ in polar form. 2 marks

b Find $\text{Arg}(zw)$. 1 mark

18 (3 marks) The complex number $z = 1 + \sqrt{3}i$ is divided by the complex number w so that $\frac{|z|}{|w|}$ is 3 and $\text{Arg}\left(\frac{z}{w}\right) = \frac{7\pi}{12}$. Write w in Cartesian form.

19 (3 marks) By using the property that for $z = r\,\text{cis}(\theta)$, then $z^2 = r^2\text{cis}(\theta + \theta)$, show that $z^3 = r^3\,\text{cis}(3\theta)$.

20 (6 marks) Let $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$.

a Use the identity $\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$ to find an expression for $\sin\left(\frac{\pi}{12}\right)$. 2 marks

b Use the identity $\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$ to find an expression for $\cos\left(\frac{\pi}{12}\right)$. 2 marks

c Show that $\text{Arg}(z) = 2 - \sqrt{3}$, where $z = \text{cis}\left(\frac{\pi}{12}\right)$. 2 marks

9.5 Graphs and regions in the complex plane

Video playlist
Graphs and regions in the complex plane

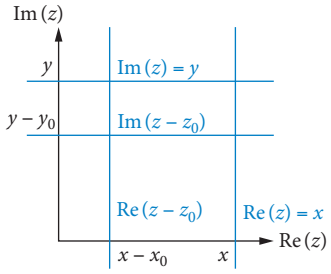
Worksheet
Complex plane graphs

Lines and rays

Lines

On the Cartesian plane, the graph of the equations $x = k$ and $y = k$ is a vertical line and a horizontal line respectively. In the complex plane, if $z = x + yi$ then we have $x = \text{Re}(z)$ and $y = \text{Im}(z)$.

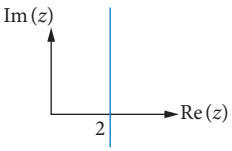
Similarly, if $z_0 = x_0 + y_0 i$, then $\text{Re}(z - z_0) = x - x_0$ and $\text{Im}(z - z_0) = y - y_0$ are, respectively, a horizontal and a vertical translation of x_0 from x and y_0 from y.



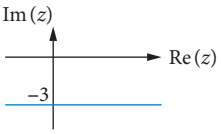
WORKED EXAMPLE 18 Vertical and horizontal lines

a Use $z = 3 + 4i$ and $z_0 = 1 + 2i$ to draw $\text{Re}(z - z_0)$.

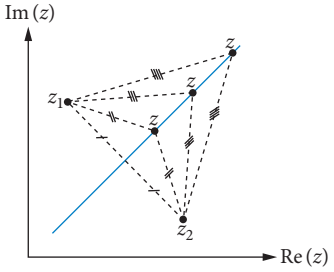
b Use $z = -2 + i$ and $z_0 = -5 + 4i$ to draw $\text{Im}(z - z_0)$.

Steps	Working
a **1** Write $z - z_0$ in the form $a + bi$.	$z - z_0 = (3 + 4i) - (1 + 2i) = 2 + 2i$
2 State $\text{Re}(z - z_0)$.	$\text{Re}(z - z_0) = 2$
3 Show $\text{Re}(z - z_0)$ on the complex number plane.	The graph is a vertical line passing through 2 on the real axis. 

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b 1	Write $z - z_0$ in the form $a + bi$.	$z - z_0 = (-2 + i) - (-5 + 4i) = 3 - 3i$
2	State $\text{Im}(z - z_0)$.	$\text{Im}(z - z_0) = -3$
3	Show $\text{Im}(z - z_0)$ on the complex number plane.	The graph is a horizontal line passing through -3 on the imaginary axis. 

A line forming an oblique angle with the positive real axis is described by $|z - z_1| = |z - z_2|$. The equation can be thought of as the **locus** of points, z, that are equidistant from the points z_1 and z_2.



We can express $|z - z_1| = |z - z_2|$ in Cartesian form by letting $z = x + yi$, $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$.

Then $\sqrt{(x - x_1)^2 + (y - y_1)^2} = \sqrt{(x - x_2)^2 + (y - y_2)^2}$ or $(x - x_1)^2 + (y - y_1)^2 = (x - x_2)^2 + (y - y_2)^2$.

Expanding and simplifying the equation gives $2(x_1 - x_2)x + 2(y_1 - y_2)y = x_1^2 - x_2^2 + y_1^2 - y_2^2$, which is a linear equation of the form $Ax + By = C$.

The angle the line makes with the positive real axis is found using $\theta = \tan^{-1}\left(-\frac{A}{B}\right)$.

If z_1 and z_2 are real, then $2(x_1 - x_2)x = x_1^2 - x_2^2$, or $x = \frac{1}{2}(x_1 + x_2)$, which is a vertical line.

This can also be written as $x = \frac{1}{2}(z_1 + z_2)$.

WB

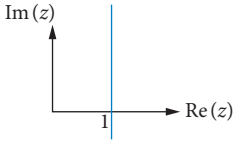
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WORKED EXAMPLE 19 Lines in the complex plane

Sketch each line in the complex plane.

For part **b**, find the angle, to the nearest degree, the line makes with the positive real axis.

a $|z - 4| = |z + 2|$ **b** $|z - 1 + i| = |z - 3|$

Steps	Working
a 1 Decide if z_1 and z_2 are real.	$z_1 = 4$ and $z_2 = -2$
2 Use the appropriate form of the solution.	$z_1 = 4$ and $z_2 = -2$ are real, so $x = \frac{1}{2}(z_1 + z_2)$. $x = \frac{1}{2}(4 + (-2))$ $= 1$
3 Describe the solution.	The solution is the locus of points equidistant from $z = -2$ and $z = 4$ on the real axis. The graph is a vertical line passing through 1 on the real axis. 

b **1** Substitute $z = x + yi$ into the equation.

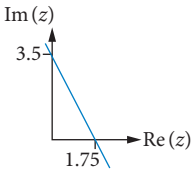
$$|(x-1)+(y+1)i| = |(x-3)+yi|$$

2 Write the expression in linear form.

$$\sqrt{(x-1)^2+(y+1)^2} = \sqrt{(x-3)^2+y^2}$$
$$(x-1)^2+(y+1)^2 = (x-3)^2+y^2$$
$$x^2-2x+1+y^2+2y+1 = x^2-6x+9+y^2$$
$$4x+2y = 7$$

3 Sketch the straight line.

The x-intercept is 1.75 and the y-intercept is 3.5.



4 Calculate the angle the line makes with the positive real axis.

$$4x+2y = 7$$
$$y = -2x+3.5$$
$$\theta = \tan^{-1}(-2) = 117°$$

Rays

A line from the origin to the point $z = x + yi$ has argument $\arg(z) = \theta$, where $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ is the anticlockwise angle the line (**ray**) makes from the positive real axis. Now suppose that the ray starts at $z = z_0$, rather than at the origin. Then $\arg(z) = \theta$ becomes $\arg(z - z_0) = \theta$.

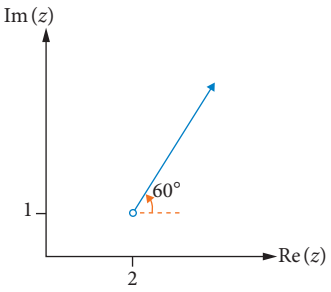
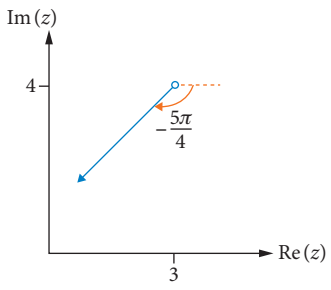
The ray consists of the locus of points on the ray with angle θ anticlockwise from the horizontal. Since the magnitude of a ray cannot be zero, the point z_0 is not part of the ray and is shown as an open circle.

To construct a negative angle, start at 0° and rotate clockwise an amount that is the magnitude of the given angle.

WORKED EXAMPLE 20 Sketching rays

a Use $z_0 = 2 + i$ to show $\arg(z - z_0) = 60°$.

b Use $z_0 = 3 + 4i$ to show $\arg(z - z_0) = -\frac{5\pi}{4}$.

Steps	Working	
a Start at z_0 and draw a line that has been rotated anticlockwise by the given number of degrees.	Locate the point $2 + i$, mark off an angle of 60° anticlockwise and draw the ray.	
b Start at z_0 and draw a line that has been rotated clockwise by the given number of radians.	Locate the point $3 + 4i$, mark off an angle of $-\frac{5\pi}{4}$ clockwise and draw the ray.	

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Circles and ellipses

Circles

Consider $|z - z_0| = r$, where r is a real number, $z = x + yi$ and $z_0 = x_0 + y_0 i$.

Then $|z - z_0| = r$ becomes $|(x + yi) - (x_0 + y_0 i)| = r$, which simplifies to $|(x - x_0) + (y - y_0)i| = r$.

Using the definition of modulus, we obtain $\sqrt{(x - x_0)^2 + (y - y_0)^2} = r$ or $(x - x_0)^2 + (y - y_0)^2 = r^2$.

This equation describes a circle of radius r, whose centre is at (x_0, y_0).

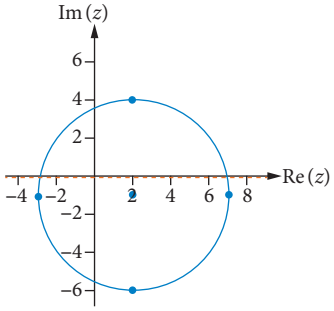
Hence, $|z - z_0| = r$ describes the locus of points whose distance from z_0 is r.

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WORKED EXAMPLE 21 The Cartesian equation of a circle

a Sketch the graph of $|z - 2 + i| = 5$.

b Express $|z - 2 + i| = 5$ in Cartesian form and state the radius of the circle and the coordinates of its centre.

Steps	Working
a 1 Determine the radius of the circle and the coordinates of its centre.	$\|z - 2 + i\| = 5$ is $\|z - (2 - i)\| = 5$. The radius is 5. The centre is at $(x_0, y_0) = (2, -1)$.
2 Mark the points $(x_0 + r, y_0)$, $(x_0, y_0 - r)$, $(x_0 - r, y_0)$ and $(x_0, y_0 + r)$.	$(x_0 + r, y_0) = (2 + 5, -1) = (7, -1)$ $(x_0, y_0 - r) = (2 , -1 - 5) = (2, -6)$ $(x_0 - r, y_0) = (2 - 5, -1) = (-3, -1)$ $(x_0, y_0 + r) = (2, -1 + 5) = (2, 4)$
3 Sketch the circle using the 4 points as a guide.	
b 1 Substitute for z and express in the right form.	Let $z = x + yi$. $\|z - 2 + i\| = \| x + yi - 2 + i \|$ $= \|(x - 2) + (y + 1)i\|$
2 Find the magnitude.	$\|(x - 2) + (y + 1)i\| = \sqrt{(x - 2)^2 + (y + 1)^2}$ $= 5$
3 Express the equation in the form $(x - x_0)^2 + (y - y_0)^2 = r^2$	Square both sides of the equation: $(x - 2)^2 + (y + 1)^5 = 5^2$
4 State the radius and the centre.	The circle has radius 5 and its centre is at $(2, -1)$.

WORKED EXAMPLE 22 Converting the Cartesian equation of a circle to complex form

Write each equation in the form $|z - z_0| = r$.

a $(x-1)^2 + (y+3)^2 = 49$ **b** $x^2 + y^2 - 6x - 8y - 11 = 0$

Steps	Working
a 1 Find the centre and the radius.	The centre is at $(1, -3)$, so $z_0 = 1 - 3i$. $49 = 7^2$, so the radius is 7.
2 Write the equation in the form $\|z - z_0\| = r$.	$\|z - z_0\| = r$ $\|z - (1 - 3i)\| = 7$
b 1 Complete the square. Apply 'complete the square' to the x terms and to the y terms.	$x^2 + y^2 - 6x - 8y - 11$ $= (x^2 - 6x) + (y^2 - 8y) - 11$ $= (x^2 - 6x + 9) - 9 + (y^2 - 8y + 16) - 16 - 11$ $= (x-3)^2 + (y-4)^2 - 36$
2 Write the expression in the form $(x - x_0)^2 + (y - y_0)^2 = r^2$	So $(x-3)^2 + (y-4)^2 - 36 = 0$ $(x-3)^2 + (y-4)^2 = 6^2$
3 Find the centre and the radius.	The centre is at $(3, 4)$, so $z_0 = 3 + 4i$. $36 = 6^2$, so the radius is 6.
4 Write the equation in the form $\|z - z_0\| = r$.	$\|z - (3 + 4i)\| = 6$

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Ellipses

In this section, we will deal only with ellipses whose 2 **foci** lie on the real axis or on the imaginary axis.

Suppose the foci are z_1 and z_2. The equation of an ellipse can be given as

- $|z - z_1| + |z - z_2| = x_{\text{width}}$ if the foci lie on the real axis, where x_{width} is the length of the ellipse parallel to the real axis. The locus of points is values of z whose sum of the distance from z_1 to z and from z_2 to z is x_{width}.
- $|z - z_1| + |z - z_2| = y_{\text{width}}$ if the foci lie on the imaginary axis where y_{width} is the length of the ellipse parallel to the imaginary axis. The locus of points is values of z whose sum of the distance from z_1 to z and from z_2 to z is y_{width}.
- The centre is the midpoint, M, equidistant from each focus and has coordinates $\left(\frac{\text{Re}(z_1 + z_2)}{2}, \frac{\text{Im}(z_1 + z_2)}{2}\right)$.

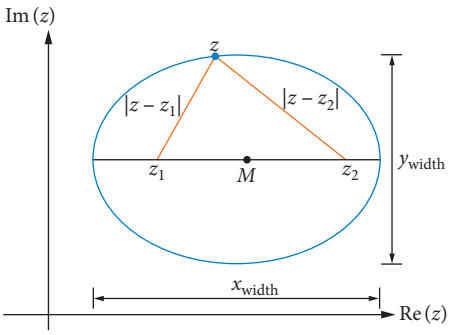


Diagram 1

Diagram 1 shows the situation when both foci lie on the real axis.

To determine the length of y_{width}, consider the case when z is equidistant from z_1 and from z_2 (Diagram 2).

The length from z_1 to M is $\dfrac{\text{Re}(z_1 + z_2)}{2} - \text{Re}(z_1) = \dfrac{\text{Re}(z_2 - z_1)}{2}$.

Now apply Pythagoras' theorem to the right-angled triangle with vertices at z, z_1 and M. This gives the length from M to z as

$$\sqrt{\left(\frac{1}{2}x_{\text{width}}\right)^2 - \left(\frac{\text{Re}(z_2 - z_1)}{2}\right)^2} = \frac{1}{2}\sqrt{(x_{\text{width}})^2 - (\text{Re}(z_2 - z_1))^2}$$

Hence $y_{\text{width}} = \sqrt{(x_{\text{width}})^2 - (\text{Re}(z_2 - z_1))^2}$.

Note: If both foci are imaginary, then we use

$x_{\text{width}} = \sqrt{(y_{\text{width}})^2 - (\text{Im}(z_2 - z_1))^2}$

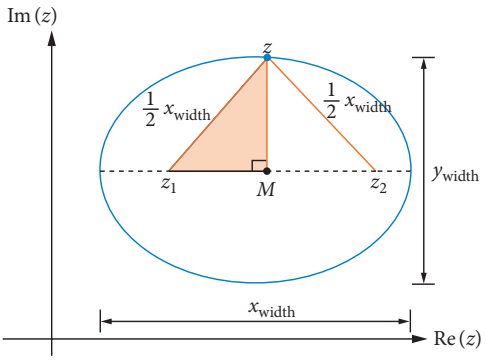


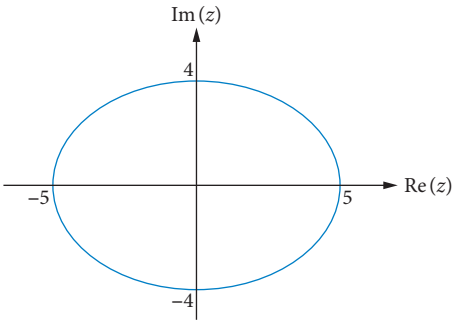
Diagram 2

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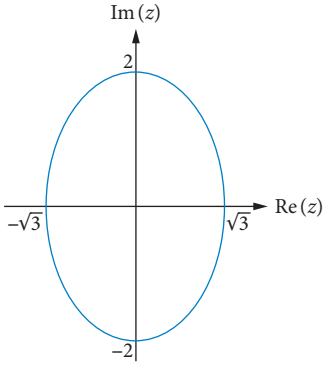
WORKED EXAMPLE 23 Sketching an ellipse with real foci

Draw the ellipse $|z + 3| + |z - 3| = 10$.

Steps	Working
1 State x_{width}.	The width is 10.
2 Identify the foci.	$z_1 = -3$ $z_2 = 3$
3 Find the centre of the ellipse.	$\left(\dfrac{\text{Re}(z_1 + z_2)}{2}, \dfrac{\text{Im}(z_1 + z_2)}{2}\right)$ $= \left(\dfrac{\text{Re}(-3 + 3)}{2}, \dfrac{\text{Im}(0 + 0)}{2}\right)$ $= (0, 0)$
4 Find y_{width}.	$y_{\text{width}} = \sqrt{(x_{\text{width}})^2 - (\text{Re}(z_2 - z_1))^2}$ $= \sqrt{10^2 - 6^2}$ $= 8$
5 Use the centre as reference to locate the endpoints of x_{width} and y_{width}.	x_{width} starts at $\dfrac{0-10}{2} = -5$ and ends at $\dfrac{0+10}{2} = 5$. y_{width} starts at $\dfrac{0-8}{2} = -4$ and ends at $\dfrac{0+8}{2} = 4$.
6 Sketch the graph.	

WORKED EXAMPLE 24 Sketching an ellipse with imaginary foci

Draw the ellipse $|z + i| + |z - i| = 4$.

Steps	Working
1 State y_{width}.	The width is 4.
2 Identify the foci.	$z_1 = -i$ $z_2 = i$
3 Find the centre of the ellipse.	$\left(\frac{\text{Re}(z_1 + z_2)}{2}, \frac{\text{Im}(z_1 + z_2)}{2}\right)$ $= \left(\frac{\text{Re}(0 + 0)}{2}, \frac{\text{Im}(-i + i)}{2}\right)$ $= (0, 0)$
4 Find x_{width}.	$x_{\text{width}} = \sqrt{(y_{\text{width}})^2 - (\text{Im}(z_2 - z_1))^2}$ $= \sqrt{4^2 - (2)^2}$ $= 2\sqrt{3}$
5 Use the centre as reference to locate the endpoints of x_{width} and y_{width}.	x_{width} starts at $\frac{0 - 2\sqrt{3}}{2} = -\sqrt{3}$ and ends at $\frac{0 + 2\sqrt{3}}{2} = \sqrt{3}$. y_{width} starts at $\frac{0-4}{2} = -2$ and ends at $\frac{0+4}{2} = 2$.
6 Sketch the graph.	

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Regions in the complex plane

When sketching graphs of inequalities in the Cartesian plane, points satisfying the inequalities are shown as regions. A similar situation applies to graphs of inequalities in the complex plane. In this book, we will adopt the convention of shading in the region that contains points that satisfy the inequality.

If the inequality is ≤, shade the interior (or exterior for ≥). For < or >, draw the perimeter as a dotted shape.

Straight line inequalities involving $\text{Re}(z - z_0)$, $\text{Im}(z - z_0)$ and $|z - z_1| = |z - z_2|$ usually require shading half of the complex plane.

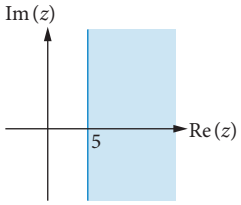
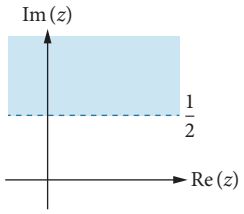
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WORKED EXAMPLE 25 Sketching regions involving straight lines

Show each region in the complex plane.

a $\text{Re}(z - 3 - i) \geq 2$ **b** $|z - 2i| < |z + i|$

Steps	Working
a 1 Describe the shape.	Let $\text{Re}(z - 3 - i) = 2$. $x - 3 = 2$ so $x = 5$. The shape is a vertical line passing through $x = 5$.
2 Decide what region is to be shaded.	$\text{Re}(z - 3 - i) \geq 2$ means $x \geq 5$. Include the line $x = 5$ and shade in the region to the right of the line.
3 Show the shaded region.	
b 1 Find the Cartesian equation.	$\|z - 2i\| < \|z + i\|$ $x^2 + (y - 2)^2 < x^2 + (y + 1)^2$ $-4y + 4 < 2y + 1$ $y > \frac{1}{2}$
2 Draw the shape and shade in the required region.	Shade the region above the horizontal line. Use a dotted line to indicate that the line $y = \frac{1}{2}$ is not a part of the region. 

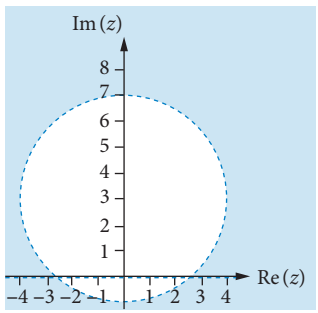
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WORKED EXAMPLE 26 Sketching regions in circles and ellipses

Sketch each region in the complex plane.

a $|z - 3i| > 4$ **b** $|z - 1| + |z + 1| \leq 4$

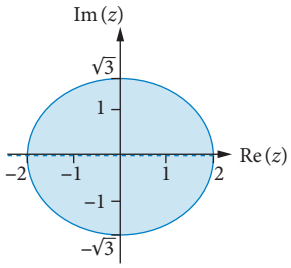
Steps	Working
a 1 Describe the shape and the required region.	The region is the exterior of the circle $\|z - 3i\| = 4$, excluding its circumference. The circle has radius 4 and its centre is at (0, 3).
2 Draw the shape and shade in the required region.	

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b 1 Describe the shape and the region to be shaded.

The region is the interior of an ellipse, including its perimeter. The centre is at $(0, 0)$, the x_{width} is 4 and the y_{width} is $2\sqrt{3}$.

2 Draw the shape and shade in the required region.



Recall that $\arg(z - z_0) = \theta$ is a ray starting at z_0 that makes an anticlockwise angle of θ with the positive real axis. To represent a region, the principal argument, $\text{Arg}(z)$ is used; that is, $-\pi < \theta \leq \pi$.

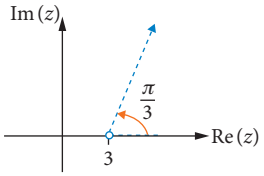
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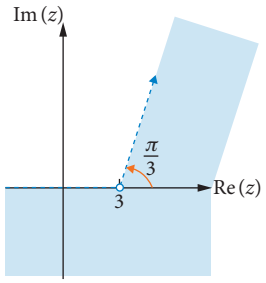
WORKED EXAMPLE 27 Sketching inequalities involving rays

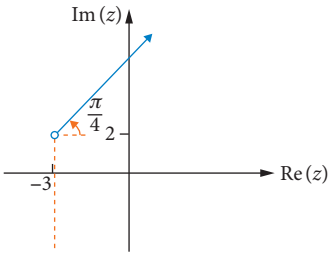
Show each region in the complex plane.

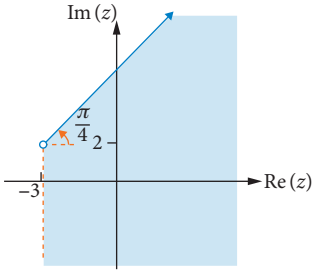
a $\text{Arg}(z - 3) < \frac{\pi}{3}$ **b** $-\frac{\pi}{2} < \text{Arg}(z + 3 - 2i) \leq \frac{\pi}{4}$

Steps	Working
a 1 Draw the ray $\text{Arg}(z - z_0) = \theta$.	$z_0 = 3, \theta = \frac{\pi}{3}$ Draw the ray using a dotted line.
2 Work out the region to be shaded.	We want all angles clockwise from (but not including) $\frac{\pi}{3}$ up to (but not including) π.
b 1 Draw the 2 rays $\text{Arg}(z - z_0) = \theta_1$ and $\text{Arg}(z - z_0) = \theta_2$.	Draw $\text{Arg}(z - z_0) = -\frac{\pi}{2}$, $\text{Arg}(z - z_0) = \frac{\pi}{4}$ $z_0 = -3 + 2i$ Draw the rays using a dotted line for $\text{Arg}(z - z_0) = -\frac{\pi}{2}$.
2 Shade the appropriate region.	The required region contains all angles between $-\frac{\pi}{2}$ and $\frac{\pi}{4}$.









Video
VCE question analysis: Complex numbers

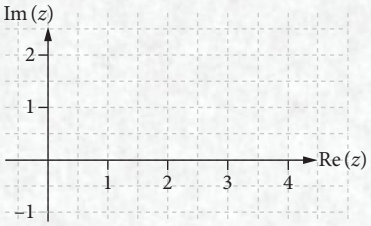
VCE QUESTION ANALYSIS

A VCE-style question (10 marks)

a One root of the quadratic equation $z^2 + Az + B = 0$ is $z_1 = 3 + i$, where A and B are real constants.
State the other root, z_2. 1 mark

b Find the values of A and B. 1 mark

c Copy the complex plane below and plot and label the two roots. 1 mark



d **i** Find, in complex number form, the equation of the circle that passes through z_1 and z_2 and whose centre, z_c, is the midpoint of the line joining the two roots. 1 mark

ii On the complex plane that was used for part **c**, draw the circle and shade its interior. 1 mark

e On the same complex number plane, graph $\text{Arg}(z - z_c + 1) = \frac{\pi}{4}$. 1 mark

f Using technology or otherwise, find the point(s) where $\text{Arg}(z - z_c + 1) = \frac{\pi}{4}$ intersects the circle. 2 marks

g Calculate the exact area of the segment enclosed by $\text{Arg}(z - z_c + 1) \geq \frac{\pi}{4}$ and the circle. 2 marks

Reading the question

- Parts **a** and **b** require you to use the relationship between the coefficients of z in the quadratic and its roots.
- Understand important keywords and terms such as 'midpoint', 'segment' and 'subtended at the centre'.
- Decide which parts can be done more efficiently using technology, such as part **f**.
- Provide answers in the required format. For instance, ensure that in part **d**, the equation of the circle is given in complex number form and not as a Cartesian equation.

Thinking about the question

- The whole question has a common theme in solving equations and showing their solutions on the complex number plane.
- Think about the relationship between the Cartesian form and the complex number form in describing a circle.
- Be accurate with using correct notation. For example, if $z = 2$, don't write $x = 2$. Correctly displayed working will improve readability, assist in picking up errors and enhance your chances of gaining higher marks.
- Ensure that shaded regions are accurate. For instance, in part **g**, the principal value of the argument is implied.

Worked solution (✓ = 1 mark)

a The roots of a quadratic with real coefficients occur as conjugate pairs.

The conjugate of $3 + i$ is $3 - i$.

So $z_2 = \mathbf{3 - i}$. ✓

b $(z - z_1)(z - z_2) = z^2 - (z_1 + z_2)z + z_1z_2$

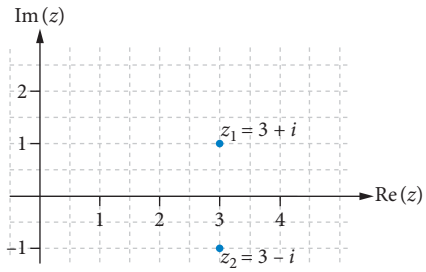
For $z^2 + Az + B$

$A = -(z_1 + z_2)$ $\quad$ $B = z_1z_2$

$= -(3 + i + 3 - i)$ $\quad$ $= (3 + i)(3 - i)$

$= \mathbf{-6}$ $\quad$ $= \mathbf{10}$ ✓ for correct values of A and B.

c



✓ for correct position and labelled z_1, z_2.

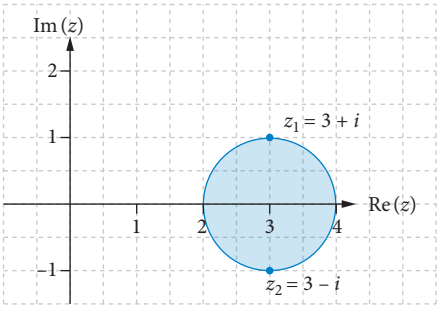
d **i** The midpoint of z_1 and z_2 is at $(3, 0)$.

Hence, the centre of the circle is at $(3, 0)$.

The diameter of the circle is 2, so the radius is 1.

$\mathbf{|z - 3| = 1}$ ✓

ii Draw a circle of radius 1 with centre at $(3, 0)$ and shade the interior.

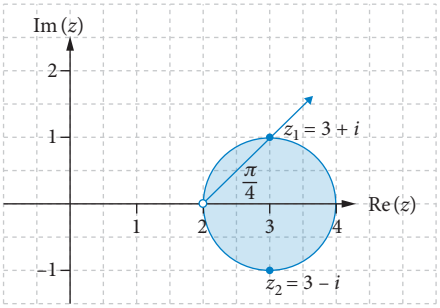


✓

e $\text{Arg}(z - z_c + 1) = \theta$ is a ray starting at (but excluding) z_c in a direction θ measured anticlockwise from the horizontal.

The centre of the circle is at $(3, 0)$ so $z_c = 3$ and $\text{Arg}(z - z_c + 1) = \text{Arg}(z - 2)$.

$\text{Arg}(z - 2) = \dfrac{\pi}{4}$ is a ray starting at $\text{Re}(z) = 2$ whose angle to the horizontal is $\dfrac{\pi}{4}$ (or gradient is $\tan\left(\dfrac{\pi}{4}\right) = 1$).



✓

f The Cartesian equation of the ray is $y = mx + c$ where $m = \tan\left(\frac{\pi}{4}\right) = 1$. Substitute $(2, 0)$ to get $c = -2$.

Hence, the equation of the line is $y = x - 2$.

The Cartesian equation of the circle is $\boldsymbol{(x-3)^2 + y^2 = 1}$. ✓

Solving the 2 equations simultaneously by substitution y gives

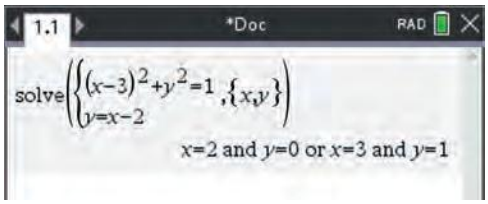
$(x-3)^2 + (x-2)^2 = 1$

$x^2 - 5x + 6 = 0$

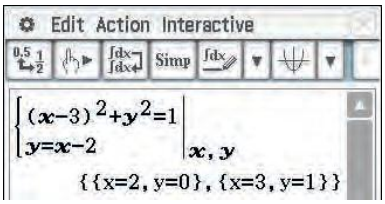
$(x-2)(x-3) = 0$

The solutions are $x = 2$ and $x = 3$, but the ray does not exist at $x = 2$, so the solution is $x = 3$, with corresponding $y = 1$ (which agrees with the graph), so the point of intersection is $\boldsymbol{(3, 1)}$. ✓

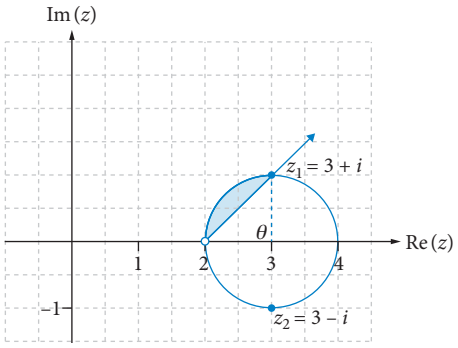
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ClassPad



g Shading $\text{Arg}(z - z_c + 1) \geq \frac{\pi}{4}$ means the required segment is as shown on the right.



$\text{area} = \frac{1}{2}r^2(\theta - \sin(\theta))$ ✓

$\theta = \frac{\pi}{2}, r = 1$

$\text{area} = \frac{1}{2} \times 1^2\left(\frac{\pi}{2} - \sin\left(\frac{\pi}{2}\right)\right)$

$= \boldsymbol{\frac{1}{2}\left(\frac{\pi}{2} - 1\right)\text{units}^2}$ ✓

EXERCISE 9.5 Graphs and regions in the complex plane

ANSWERS p. 602

Recap

1 $\text{Arg}\left(\frac{2(1-i)}{3i}\right)$ is

A $-\frac{3\pi}{4}$ **B** $-\frac{\pi}{4}$ **C** $\frac{\pi}{4}$ **D** $\frac{\pi}{2}$ **E** $\frac{3\pi}{2}$

2 In polar form, $\frac{2(1-\sqrt{3}i)}{\sqrt{3}+i}$ is

A $2\text{cis}\left(-\frac{3\pi}{4}\right)$ **B** $2\text{cis}(-\pi)$ **C** $2\text{cis}\left(-\frac{\pi}{2}\right)$ **D** $2\text{cis}\left(\frac{\pi}{4}\right)$ **E** $2\text{cis}\left(\frac{\pi}{2}\right)$

Mastery

3 WORKED EXAMPLE 18 If $z_2 = -5 - 12i$ and $z_0 = -1 - i$, draw $\text{Re}(z_2 - z_0)$ and $\text{Im}(z_2 - z_0)$ in the same complex plane.

4 WORKED EXAMPLE 19 Sketch $|z + 2| = |z - 3 + 4i|$ in the complex plane.

5 WORKED EXAMPLE 20 Sketch each ray described.

a $\arg(z - 1 + 2i) = 45°$ **b** $\arg(z - i) = -\frac{5\pi}{6}$

6 WORKED EXAMPLE 21

a Sketch the graph of $|z - 1 - i| = 2$.

b Express $|z - 1 - i| = 2$ in Cartesian form and state the radius of the circle and the coordinates of its centre.

7 WORKED EXAMPLE 22 Write each equation in the form $|z - z_0| = r$.

a $(x + 2)^2 + (y + 4)^2 = 9$ **b** $x^2 + y^2 - 6x - 10y - 30 = 0$

8 WORKED EXAMPLE 23 TECH-FREE Sketch the ellipse $|z - 1| + |z + 1| = 3$.

9 WORKED EXAMPLE 24 Sketch the ellipse $|z - 6i| + |z + 2i| = 10$.

10 WORKED EXAMPLE 25 TECH-FREE Sketch each region described below.

a $\text{Re}(z + i) < 3$ **b** $|z - i| \geq |z + 2i|$

11 WORKED EXAMPLE 26 Sketch each region in the complex plane.

a $|z + 2i| < 2$ **b** $|z + 2| + |z - 2| \geq 5$

12 WORKED EXAMPLE 27 Sketch each region in the complex plane.

a $\text{Arg}(z + 1) \geq \frac{2\pi}{3}$ **b** $-\frac{\pi}{4} < \text{Arg}(z - 1 - i) \leq \frac{3\pi}{4}$

Exam practice

80–100% 60–79% 0–59%

13 TECH-FREE (3 marks) Show that the equation of the straight line $|z - ai| = |z - bi|$ (a, b real numbers) is $y = \frac{1}{2}(a + b)$.

14 TECH-FREE (4 marks) In Cartesian form, the equation of an ellipse with centre at $(0, 0)$ is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a and b is the semi-major axis and semi-minor axis respectively.

a Show that for the ellipse $|z + p| + |z - p| = k$ (where p and k are real numbers), $a = \frac{1}{2}k$ and $b = \frac{1}{2}\sqrt{k^2 - 4p^2}$. 2 marks

b Write $|z + 5| + |z - 5| = 12$ in Cartesian form. 2 marks

15 © VCAA 2013 2AQ5 69% The region in the complex plane that is **outside** the circle of radius b centred at the origin is given by the set of points z, where $z \in C$, such that

A $|z| < b$ **B** $|z| > b$ **C** $|z| > b^2$ **D** $|z| = b$ **E** $|z| < b^2$

16 © VCAA 2015 2AQ8 **57%** A relation that does **not** represent a circle in the complex plane is

A $z\overline{z} = 4$ **B** $|z + 3i| = 2|z - i|$ **C** $|z - i| = 2|z + 2|$

D $|z - i + 1| = 4$ **E** $|z| + 2|\overline{z}| = 4$

17 © VCAA 2014 2AQ9 **56%** The circle $|z - 3 - 2i| = 2$ is intersected exactly twice by the line given by

A $|z - i| = 2|z + 1|$ **B** $|z - 3 - 2i| = |z - 5|$ **C** $|z - 3 - 2i| = |z - 10i|$

D $\text{Im}(z) = 0$ **E** $\text{Re}(z) = 5$

18 © VCAA 2011 2AQ7 **54%** In the complex plane, the circle with equation $|z - (2 + 3i)| = 1$ is intersected exactly twice by the curve with equation

A $|z - 3i| = 1$ **B** $|z + 3| = |z - 3i|$ **C** $|z - 3| = |z - 3i|$

D $\text{Im}(z) = 4$ **E** $\text{Re}(z) = 3$

19 (3 marks) Sketch $2 < |z - 1 + i| \leq 3$.

20 (3 marks) Calculate the area of the region defined by $3 \leq |z + 5 + 12i| \leq \sqrt{10}$.

21 (4 marks) Calculate the area of the unshaded shape formed when the following regions are shaded in the same complex plane.

$\text{Arg}(z - 1 + i) \leq \frac{\pi}{3}$, $\text{Arg}(z - 5 + i) \geq \frac{2\pi}{3}$, $\text{Im}(z) \geq -1$

7 Chapter summary

Complex numbers

- A complex number, $z \in C$, is of the form $z = a + bi$, $a, b \in R$, where $i^2 = -1$ (or $i = \sqrt{-1}$).
- The **real part of z** is written as $\text{Re}(z) = a$ and the **imaginary part of z** is $\text{Im}(z) = b$.
- The set of real numbers is a subset of the set of complex numbers ($R \subset C$).
- The **complex conjugate** of $z = a + bi$ is $\overline{z} = a - bi$.
- $z + \overline{z} = 2a$, a real number
- $z - \overline{z} = 2bi$, a purely imaginary number
- $z\overline{z} = a^2 + b^2$, a real number
- $z\overline{z} = |z|^2$, where $|z| = \sqrt{a^2 + b^2}$ is the **modulus** of z
- To add or subtract complex numbers, treat the real and imaginary parts separately.
- To multiply complex numbers, expand and use $i^2 = -1$.
- To divide complex numbers such as $\frac{z}{w}$, **realise the denominator** by multiplying by $\frac{\overline{w}}{\overline{w}}$.

Quadratic equations with complex roots

- The quadratic equation, $ax^2 + bx + c = 0$ ($a, b, c \in R$ and $a \neq 0$), has solutions $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- The type of roots depends on the value of the **discriminant**, $\Delta = b^2 - 4ac$.

  $\Delta < 0$: 2 complex roots (that are not real)

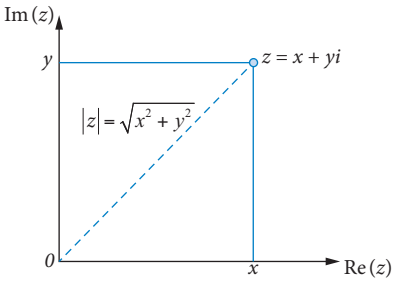
  $\Delta = 0$: 1 real root, which is rational

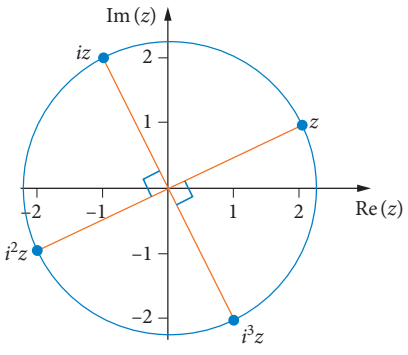
  $\Delta > 0$: 2 real roots
- Complex roots occur as **conjugate pairs**, so that $a(z - z_0)(z - \overline{z}_0) = 0$.
- $z^2 + bz + c = (z - z_1)(z - z_2)$ means $z_1 + z_2 = -b$ and $z_1 z_2 = c$.

The Argand diagram

A complex number, $z = x + yi$, is represented by the point (x, y) on a complex number plane called the **Argand diagram**.

The **modulus** (magnitude) of z is $|z| = \sqrt{x^2 + y^2}$.





- Multiplication of z by i^n results in a $90n°$ rotation about the origin in an anticlockwise direction on the circumference of the circle.
- Multiplying z by i gives a 90° rotation and the product, $iz = -y + xi$, represented by the point $(-y, x)$.
- Multiplying z by i^2 gives a 180° rotation and the product, $-z = -x - yi$, represented by the point $(-x, -y)$.
- Multiplying z by i^3 gives a 270° rotation and the product, $-iz = y - xi$, represented by the point $(y, -x)$.

Polar form of a complex number

- The complex number $z = x + yi$ is expressed in polar form as $z = r(\cos(\theta) + \sin(\theta))$ or $z = r\,\text{cis}(\theta)$, where $r = \sqrt{x^2 + y^2}$ is the magnitude of z and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.
- Angle θ is the **argument of z**, written as $\arg(z) = \theta$. When $-\pi < \theta \leq \pi$ we have the **principal argument**, $\text{Arg}(z)$, and $-\pi < \text{Arg}(z) \leq \pi$.

Multiplication and division of complex numbers

- The product of the complex numbers $z = r_1(\text{cis}(\theta))$ and $w = r_2(\text{cis}(\phi))$ has magnitude $|z| \times |w|$ and an angle of rotation of $(\theta + \phi)$ anticlockwise from the positive real axis.
- The quotient $\frac{z}{w}$ has magnitude $\frac{|z|}{|w|}$ rotated by an angle of $(\theta - \phi)$ anticlockwise from the positive real axis.

Lines and rays

- $x = \text{Re}(z)$ is a vertical line passing through x.
- $y = \text{Im}(z)$ is a horizontal line passing through y.
- $\text{Re}(z - z_0) = x - x_0$ and $\text{Im}(z - z_0) = y - y_0$ are, respectively, a horizontal and a vertical translation of x_0 from x and y_0 from y.
- $|z - z_1| = |z - z_2|$ is a line given by the locus of points, z, equidistant from the numbers z_1 and z_2.
- $\arg(z - z_0) = \theta$ is a ray starting at (but not including) z_0 and consists of the locus of points that make angle θ anticlockwise from the horizontal.
- Use $\text{Arg}(z - z_0) = \theta$ for $-\pi < \theta \leq \pi$.

Circle

- $|z - z_0| = r$ describes a circle of radius r with centre at $z = z_0$.

Ellipse

- $|z - z_1| + |z - z_2| = x_{\text{width}}$ is the equation of an ellipse, where the **foci** z_1 and z_2 lie on the real axis and x_{width} is the length of the ellipse parallel to the real axis.
- The sum of the distance from z_1 to z and from z_2 to z is x_{width}.
- The centre has coordinates $\left(\frac{\text{Re}(z_1 + z_2)}{2}, \frac{\text{Im}(z_1 + z_2)}{2}\right)$.
- The width is $y_{\text{width}} = \sqrt{(x_{\text{width}})^2 - (\text{Re}(z_2 - z_1))^2}$.
  If both foci are imaginary, $x_{\text{width}} = \sqrt{(y_{\text{width}})^2 - (\text{Im}(z_2 - z_1))^2}$.

Regions in the complex plane

- A region is specified using the inequalities $<$, $\leq$, $>$ and $\geq$.
- To sketch a region, draw the shape using $=$ in the given equation and then decide which section of the boundary to shade. For example, to sketch $\text{Re}(z) > 2$, draw a vertical dotted line at $x = 2$ and shade the region to the right of the line.
- For $<$ and $>$, use a dotted line.
- Straight line inequalities involving $\text{Re}(z - z_0)$, $\text{Im}(z - z_0)$ and $|z - z_1| = |z - z_2|$ usually require shading half of the complex plane.
- For $\text{Arg}(z - z_0) < \theta$ and $\text{Arg}(z - z_0) \leq \theta$, the maximum shading is up to $\theta = \pi$.

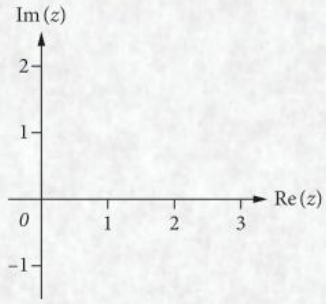
Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 (2 marks)

a Copy the complex plane below and on it sketch the graph of $|z - 2 + i| \leq 1$. 1 mark



b Write the equation of the graph in Cartesian form. 1 mark

2 © VCAA 2017N 2BQ2a (3 marks) One root of a quadratic equation with real coefficients is $\sqrt{3} + i$.

a Write down the other root of the quadratic equation. 1 mark

b Hence determine the quadratic equation, writing it in the form $z^2 + bz + c = 0$. 2 marks

3 © VCAA 2018 1Q6 MODIFIED (5 marks) $f(x) = x^2 - 5x - 6$.

Sketch the graph of $g(x) = \dfrac{1}{f(x)}$, showing the coordinates of any turning points, intercepts with the axis and equations of any asymptotes.

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 © VCAA 2018 2AQ5 Let $z = a + bi$, where $a, b \in R\backslash\{0\}$.

If $z + \frac{1}{z} \in R$, which one of the following must be **true**?

A $\text{Arg}(z) = \frac{\pi}{4}$ **B** $a = -b$ **C** $a = b$ **D** $|z| = 1$ **E** $z^2 = 1$

2 © VCAA 2016 2AQ5 If $\text{Arg}(-1 + ai) = -\frac{2\pi}{3}$, then the real number a is

A $-\sqrt{3}$ **B** $-\frac{\sqrt{3}}{2}$ **C** $-\frac{1}{\sqrt{3}}$ **D** $\frac{1}{\sqrt{3}}$ **E** $\sqrt{3}$

3 The complex numbers z, iz and $z + iz$, where $z \in C\backslash\{0\}$, are plotted in the Argand diagram, forming the vertices of a triangle.

The area of this triangle is given by

A $|z|$ **B** $|z| + |z|^2$ **C** $\frac{|z|^2}{2}$ **D** $|z|^2$ **E** $\frac{\sqrt{3}|z|^2}{2}$

4 © VCAA 2021N 2AQ4 MODIFIED The expression $\frac{3x^2 - 2}{(2x + 1)(x^2 - 1)}$ has partial fraction form

A $\frac{A}{2x+1} + \frac{Bx + C}{x^2 + 1}$ **B** $\frac{Ax + B}{2x+1} + \frac{Cx + D}{x^2 + 1}$ **C** $\frac{A}{2x+1} + \frac{B}{x+1} + \frac{C}{x-1}$

D $\frac{A}{2x+1} + \frac{Bx + C}{x^2 + 1} + \frac{C}{x+1}$ **E** $\frac{A}{2x+1} + \frac{Bx + C}{x^2 + 1} + \frac{C}{x+1} + \frac{D}{x-1}$

5 © VCAA 2011 2AQ4 MODIFIED The hyperbola $\frac{(y+3)^2}{4} - \frac{(x-6)^2}{9} = 1$ has asymptotes

A $y = \pm\frac{2}{3}x$

B $y = \frac{2}{3}x - 7$ and $y = -\frac{2}{3}x - 7$

C $y = \frac{2}{3}x - 7$ and $y = -\frac{2}{3}x + 1$

D $y = \frac{3}{2}x - 12$ and $y = -\frac{3}{2}x + 6$

E $y = \frac{9}{4}x - \frac{33}{2}$ and $y = -\frac{9}{4}x + \frac{21}{2}$

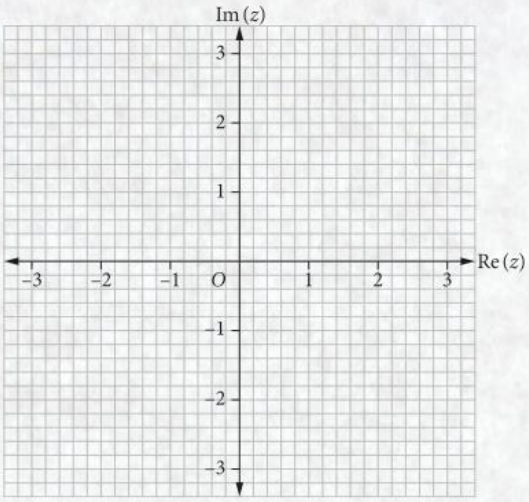
Section B 3 questions

15 marks

1 © VCAA 2019 2BQ2ab (5 marks)

a **i** Show that the solutions to $2z^2 + 4z + 5 = 0$, where $z \in C$, are $z = -1 \pm \frac{\sqrt{6}}{2}i$. 1 mark

ii Copy the Argand diagram and on it plot the solutions of $2z^2 + 4z + 5 = 0$. 1 mark



Let $|z + m| = n$, where $m, n \in R$, represent the circle of minimum radius that passes through the solutions of $2z^2 + 4z + 5 = 0$.

b **i** Find m and n. 2 marks

ii Find the Cartesian equation of the circle $|z + m| = n$. 1 mark

2 (2 marks) Let $z_1 = 2 - \sqrt{3}i$ and $z_2 = 2 + \sqrt{3}i$.

a Find $z_1 + z_2$ and $z_1 \times z_2$. 1 mark

b Hence, state the values of A and B if $f(z) = z^2 + Az + B$ and $f(z_1) = 0, f(z_2) = 0$. 1 mark

3 © VCAA 2019 2BQ1 MODIFIED (8 marks)

A curve is defined parametrically by $x = \text{cosec}(t) + 1$, $y = 2\cot(t) - 3$, where $t \in \left(0, \frac{\pi}{2}\right)$.

a Show that the curve can be represented by $f(x) = 2\sqrt{x^2 - 2x} - 3$. 2 marks

b State the domain and range of $f(x)$. 2 marks

c Sketch the curve, showing any asymptotes and intercepts with the axes. 4 marks

CHAPTER

10 VECTORS

Study Design coverage

Nelson MindTap chapter resources

10.1 Vectors
- Vector notation
- Position vectors

10.2 Operations with vectors
- Column vectors
- Magnitude and direction of a plane vector
- Orthogonal unit vectors
- Unit vectors
- Scalar multiplication
- Adding and subtracting vectors
- **Using CAS 1:** Adding and subtracting vectors
- Equivalent vectors

10.3 Scalar product of vectors
- **Using CAS 2:** The scalar product
- **Using CAS 3:** Scalar product and angle between vectors
- Angle between 2 vectors

10.4 Parallel and perpendicular vectors

10.5 Projection of vectors

10.6 Applying vectors

10.7 Vector proofs of geometric results

VCE question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2

Study Design coverage

UNIT 2, AREA OF STUDY 2: SPACE AND MEASUREMENT

Vectors in the plane

- the representation of plane vectors as directed line segments, magnitude and direction of a plane vector, and unit vectors
- geometric representation of addition, subtraction (triangle and/or parallelogram rules), scalar multiple of a vector and linear combination of plane vectors
- the representation of a plane vector as an ordered pair (a, b) in the form

$$a\underset{\sim}{\mathrm{i}} + b\underset{\sim}{\mathrm{j}} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

- simple vector algebra (addition, subtraction, multiplication by a scalar, linear combination) using these forms
- scalar (dot) product of two plane vectors, perpendicular and parallel vectors, projection of one vector onto another vector, and angle between two vectors
- geometric proofs with vectors
- application of vectors to displacement, velocity, resultant velocity, relative velocity, statics and motion under a constant force.

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10.1

Video playlists (8):

10.1 Vectors
10.2 Operations with vectors
10.3 Scalar product of vectors
10.4 Parallel and perpendicular vectors
10.5 Projection of vectors
10.6 Applying vectors
10.7 Vector proofs of geometric results
VCE question analysis Vectors

Worksheets (17):

10.1 Vectors in the Cartesian plane • Representing vectors
10.2 Magnitude and direction of a vector • Components of vectors • Unit vectors • Scalar multiplication • Triangle addition of vectors • Parallelogram rule for vector addition • Addition of vectors using components
10.3 Scalar product • Component form of the dot product • Properties of the dot product
10.4 Parallel and perpendicular vectors
10.5 Projections
10.6 Vector applications
10.7 Geometric proofs using vectors • Proofs using vectors

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

10.1 Vectors

A **vector** is a quantity that has both **magnitude** (size) and **direction**. A quantity that has magnitude only is called a **scalar** (number). Some examples of scalar quantities are listed below.

- The length of Liam's ring finger is 8 cm.
- 40 minutes is the time for Tanishka to jog around a lake with a circumference of 6 km.

But a **vector quantity** needs both **magnitude and direction**.

- The stadium's position is 18 km north of my house.
- Tina is driving at a **velocity** of 100 km/h on a bearing of 065° T.

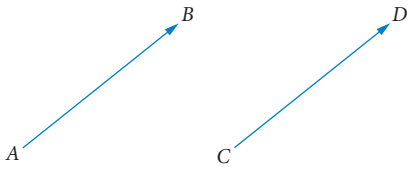
Video playlist
Vectors

Worksheets
Vectors in the Cartesian plane
Representing vectors

Vectors can be represented by an arrow that points in the direction indicated and the length gives the magnitude of the quantity shown.

The diagram shows two parallel vectors.

By looking at their length and direction, vector $\overrightarrow{CD}$ is equivalent to vector $\overrightarrow{AB}$.

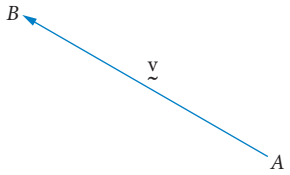


Vector notation

Often it is easier to write a vector as a single letter using a tilde ~.

The vector shown can be written as $\overrightarrow{AB}$ or $\underset{\sim}{v}$.

Vectors can also be written in bold, so $\underset{\sim}{v}$ can be written as $\boldsymbol{v}$.

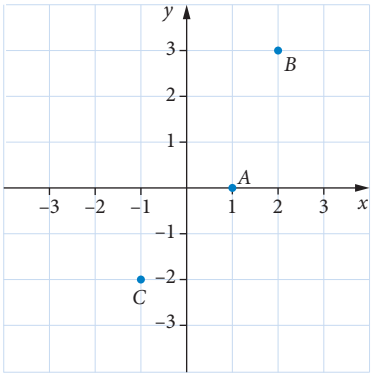
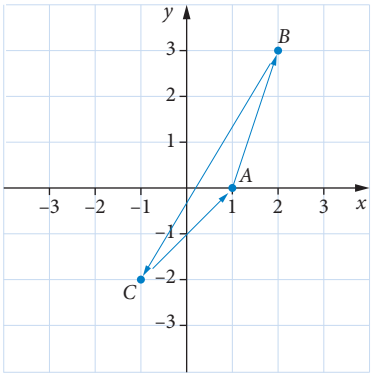


WB
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WORKED EXAMPLE 1 Vector notation

a On a set of coordinate axes, plot the points $A(1, 0)$, $B(2, 3)$ and $C(-1, -2)$.

b Sketch the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$.

Steps	Working
a Set up coordinate axes and plot and label the given points.	
b Sketch the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$.	

Position vectors

We can identify the origin, O, of the Cartesian plane as the starting point for a vector and this gives the **position** of the vector.

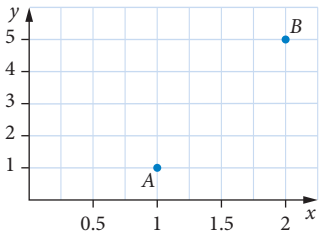
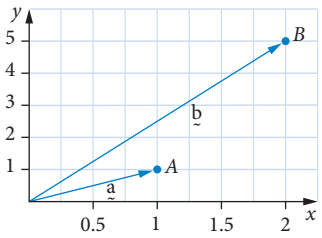
For the points A, B and C, the **position** vectors are $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$.

It is helpful to write these in tilde form as $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{OC} = \underset{\sim}{c}$.

9780170448338

WORKED EXAMPLE 2 Position vectors

On a set of coordinate axes, sketch the position vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$, where the points are $A(1,1)$ and $B(2,5)$. Label the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

Steps	Working
1 Set up coordinate axes and plot and label given points A and B.	
2 Sketch and label the vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.	

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EXERCISE 10.1 Vectors

ANSWERS p. 603

Mastery

1 WORKED EXAMPLE 1 TECH-FREE

a On a set of coordinate axes, plot the points $A(0,1)$, $B(3,4)$ and $C(-2,-5)$.

b Sketch the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$.

2 WORKED EXAMPLE 2 TECH-FREE On a set of coordinate axes, sketch the position vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$, where the points are $A(1,2)$ and $B(3,-5)$. Label the vectors $\underset{\sim}{r}$ and $\underset{\sim}{s}$.

3 TECH-FREE

a On a set of coordinate axes, plot the points $A(2,0)$, $B(3,3)$ and $C(-2,-2)$.

b Sketch the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CA}$.

4 TECH-FREE On a set of coordinate axes, sketch the position vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$, where the points are $A(1,2)$ and $B(-3,5)$. Label the vectors $\underset{\sim}{r}$ and $\underset{\sim}{s}$.

5 On the diagram shown, the vector $\underset{\sim}{e}$ is likely to end at the point

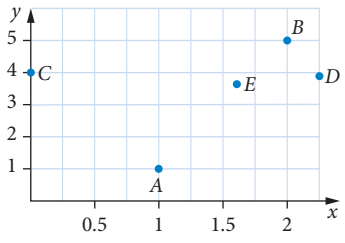
A $(0,4)$

B $(1,1)$

C $(2,5)$

D $(2,3)$

E $(1.6,3.8)$



6 On the diagram shown, the vector $\underset{\sim}{a}$ is likely to end at the point

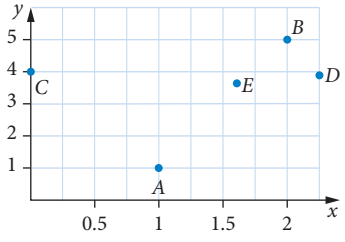
A $(0,4)$

B $(1,1)$

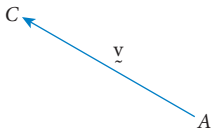
C $(2,5)$

D $(2,3)$

E $(1.6,3.8)$



7 The vector shown can be written as



A $\overrightarrow{CA}$ **B** $\overrightarrow{CV}$ **C** $\underset{\sim}{v}$

D $\underset{\sim}{a}$ **E** $\underset{\sim}{b}$

8 Which of the following is **not** a vector quantity?

A displacement **B** speed **C** acceleration **D** force **E** velocity

Exam practice

9 TECH-FREE (6 marks) If $A = (1,3)$, $B = (0,5)$, $C = (2,-6)$ and O is the origin, sketch the following vectors on a suitable Cartesian plane.

a $\overrightarrow{OA}$ **b** $\overrightarrow{OB}$ **c** $\overrightarrow{OC}$ **d** $\overrightarrow{AB}$ **e** $\overrightarrow{AC}$ **f** $\overrightarrow{BC}$

10 TECH-FREE (6 marks) If $A = (1,3)$, $B = (0,5)$, $C = (2,-6)$ and O is the origin, sketch the following vectors on a suitable Cartesian plane.

a $\overrightarrow{AO}$ **b** $\overrightarrow{BO}$ **c** $\overrightarrow{CO}$ **d** $\overrightarrow{BA}$ **e** $\overrightarrow{CA}$ **f** $\overrightarrow{CB}$

11 TECH-FREE (5 marks) On a suitable Cartesian plane, sketch the vectors joining pairs of points as listed in the direction indicated.

a $(1,1) \rightarrow (4,1)$ **b** $(1,0) \rightarrow (0,1)$ **c** $(-1,-1) \rightarrow (-3,5)$

d $(0,1) \rightarrow (5,12)$ **e** $(-3,-10) \rightarrow (3,3)$

Video playlist
Operations with vectors

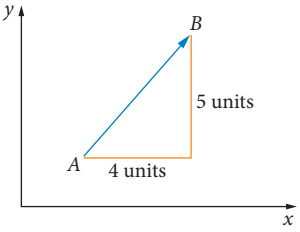
Worksheets
Magnitude and direction of a vector
Components of vectors

10.2 Operations with vectors

Column vectors

A plane vector can be written as an ordered pair (a,b) or as a column matrix $\begin{bmatrix} a \\ b \end{bmatrix}$.

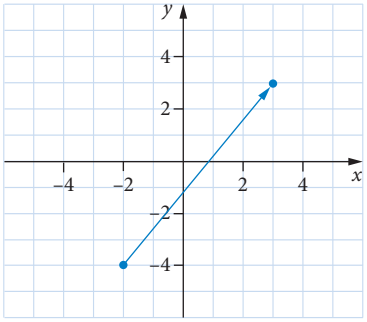
This matrix $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$ tells us to go across 4 units and up 5 units. If we start at (0, 0), this gets us to the point (4, 5). This is called the directed line segment from A to B.



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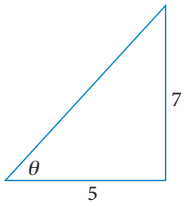
WORKED EXAMPLE 3 Directed line segments

Draw a directed line segment corresponding to the column matrix $\begin{bmatrix} 5 \\ 7 \end{bmatrix}$.

Steps	Working
The vector $\begin{bmatrix} 5 \\ 7 \end{bmatrix}$ means across 5 units and up 7 units.	
Note this arrow goes from $(-2,-4) \rightarrow (3,3)$. It could have started at any point as long as it goes across 5 units and up 7 units.	

Magnitude and direction of a plane vector

In the previous example, the column vector $\begin{bmatrix} 5 \\ 7 \end{bmatrix}$ is the hypotenuse of a right-angled triangle of base 5 units and height 7 units.



We find the length of this vector using Pythagoras' theorem.

$$\text{length} = \sqrt{5^2 + 7^2} = \sqrt{74} \text{ units}$$

This is called the **magnitude** of the vector.

The **direction**, θ, of the vector can be found using trigonometry.

In the diagram, $\tan(\theta) = \frac{7}{5}$.

The direction of the vector is $\theta = \tan^{-1}\left(\frac{7}{5}\right) \approx 54.5°$ from the positive direction of the x-axis.

Magnitude and direction of a vector

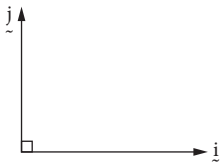
The **magnitude** of the vector $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j}$ is equal to $\sqrt{x^2 + y^2}$.

The **direction** of the vector θ is given by $\tan(\theta) = \frac{y}{x}$.

Orthogonal unit vectors

To 'fix' a position vector, consider the x-axis as the direction where a vector of length 1 unit is placed along it, called $\underset{\sim}{i}$.

Consider the y-axis as the direction where a vector of length 1 unit is placed along it, called $\underset{\sim}{j}$.



The vector (a, b) can be represented in the form

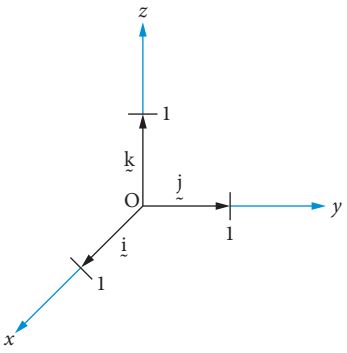
$a\underset{\sim}{i} + b\underset{\sim}{j} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$, where $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are standard **orthogonal unit vectors**. Orthogonal means 'perpendicular' and 'unit' means 1, so $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are perpendicular vectors with magnitude 1.

Moving from the 2D plane to the 3D space, a vector in 3 dimensions can be represented in the form $a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$.

$\underset{\sim}{i}$ is the unit vector in the x direction.

$\underset{\sim}{j}$ is the unit vector in the y direction.

$\underset{\sim}{k}$ is the unit vector in the z direction.



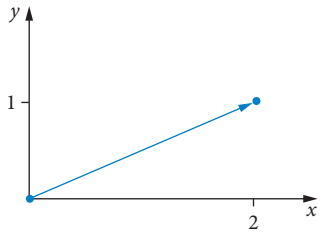
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WORKED EXAMPLE 4 Orthogonal unit vectors

a Write and sketch the directed line segment corresponding to the column matrix $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ in orthogonal unit form.

b Find the magnitude and direction of the column matrix $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Steps	Working
a The vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ means across 2 units and up 1 unit. Write in $\underset{\sim}{i}$ and $\underset{\sim}{j}$ form.	 In orthogonal unit form, this is $2\underset{\sim}{i} + \underset{\sim}{j}$.
b Find the magnitude and direction.	magnitude $= \sqrt{2^2 + 1^2} = \sqrt{5}$ units Direction $= \tan^{-1}\left(\frac{1}{2}\right) \approx 26.6°$ from the positive direction of the x-axis.

Worksheet
Unit vectors

Unit vectors

A **unit vector** is any vector that has a magnitude of 1.

The unit vector heading in the direction of $\underset{\sim}{a}$ has the notation $\hat{\underset{\sim}{a}}$.

Because $\underset{\sim}{a}$ has magnitude $|\underset{\sim}{a}|$, we divide $\underset{\sim}{a}$ by $|\underset{\sim}{a}|$ to find the unit vector $\hat{\underset{\sim}{a}}$.

Consider the vector $\underset{\sim}{a} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} = 4\underset{\sim}{i} + \underset{\sim}{j}$ (4 across and 1 up), then the unit vector in the direction of $\underset{\sim}{a}$ equals $\hat{\underset{\sim}{a}} = \dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$.

So $\hat{\underset{\sim}{a}} = \dfrac{4\underset{\sim}{i} + \underset{\sim}{j}}{|4\underset{\sim}{i} + \underset{\sim}{j}|} = \dfrac{4\underset{\sim}{i} + \underset{\sim}{j}}{\sqrt{4^2 + 1^2}} = \dfrac{1}{\sqrt{17}}(4\underset{\sim}{i} + \underset{\sim}{j})$.

Unit vectors

$\hat{\underset{\sim}{a}} = \dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$

If $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j}$ then $\hat{\underset{\sim}{a}} = \dfrac{x\underset{\sim}{i} + y\underset{\sim}{j}}{\sqrt{x^2 + y^2}}$.

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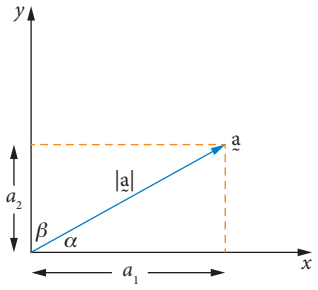
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WORKED EXAMPLE 5 Unit vectors

For the vector $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$, find the magnitude of $\underset{\sim}{a}$ and the unit vector in the direction of $\underset{\sim}{a}$.

Steps	Working
1 Find the magnitude of the vector $\underset{\sim}{a}$.	$\|\underset{\sim}{a}\| = \sqrt{3^2 + 2^2} = \sqrt{13}$
2 Find the unit vector, using $\hat{\underset{\sim}{a}} = \dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$.	$\hat{\underset{\sim}{a}} = \dfrac{3\underset{\sim}{i} + 2\underset{\sim}{j}}{\sqrt{13}} = \dfrac{1}{\sqrt{13}}(3\underset{\sim}{i} + 2\underset{\sim}{j})$

Let α be the angle between $\underset{\sim}{a}$ and the x-axis and β be the angle between $\underset{\sim}{a}$ and the y-axis.



Direction cosines

If $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j}$ then

$$\cos(\alpha) = \frac{a_1}{|\underset{\sim}{a}|}$$

$$\cos(\beta) = \frac{a_2}{|\underset{\sim}{a}|}$$

These are called **direction cosines**.

WORKED EXAMPLE 6 Angles to axes

For the vector $\underset{\sim}{a} = 2\underset{\sim}{i} + \underset{\sim}{j}$, find, correct to one decimal place, α and β, the size of the angle between the vector $\underset{\sim}{a}$ and the x-axis and y-axis respectively.

Steps	Working
1 Find the magnitude of the vector $\underset{\sim}{a}$.	$\|\underset{\sim}{a}\| = \sqrt{2^2 + 1^2} = \sqrt{5}$
2 The vector is in the form $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j}$, where $a_1 = 2$ and $a_2 = 1$. Use these values to find α and β.	$\cos(\alpha) = \frac{2}{\sqrt{5}}$, so $\alpha = \cos^{-1}\left(\frac{2}{\sqrt{5}}\right) \approx 26.6°$
	$\cos(\beta) = \frac{1}{\sqrt{5}}$, so $\beta = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 63.4°$

Note that $\alpha \approx 26.6°$ is the same as in Worked example 4, where direction $\approx 26.6°$, giving 2 different methods for the same answer.

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Scalar multiplication

The magnitude of the vector $\underset{\sim}{u} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ can be changed by multiplying it by a scalar. This does not change its direction.

So, $2\underset{\sim}{u} = 2\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$.

This can be written as $2\underset{\sim}{u} = 2(2\underset{\sim}{i} + 5\underset{\sim}{j}) = 4\underset{\sim}{i} + 10\underset{\sim}{j}$.

This means that the vector $2\underset{\sim}{u}$ is a vector double the magnitude of $\underset{\sim}{u}$, but pointing in the same direction as $\underset{\sim}{u}$.

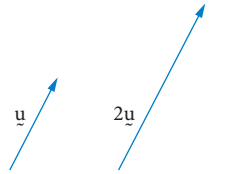
Multiplication by a real number (scalar) changes the magnitude, but not the direction of the vector.

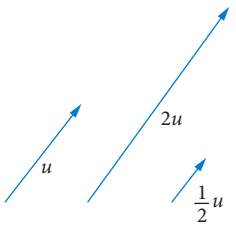
Consider the vector $\underset{\sim}{u} = \underset{\sim}{i} + \underset{\sim}{j}$:

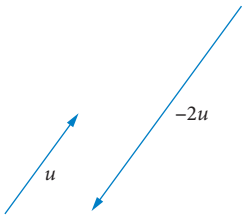
$2\underset{\sim}{u}$ is twice the length of $\underset{\sim}{u}$.

$\frac{1}{2}\underset{\sim}{u}$ is half the length of $\underset{\sim}{u}$.

When $\underset{\sim}{u}$ is multiplied by -2, the vector $\underset{\sim}{u}$ has double the length, but goes in the opposite direction.







Worksheet
Scalar multiplication

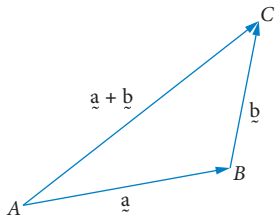
Worksheets
Triangle addition of vector
Parallelogram rule for vector addition
Addition of vectors using components

Adding and subtracting vectors

Triangle rule for adding vectors

If 2 vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are placed head-to-tail to form 2 sides of a triangle, as shown in this diagram, then $\underset{\sim}{a} + \underset{\sim}{b}$ is defined by the triangle of vectors.

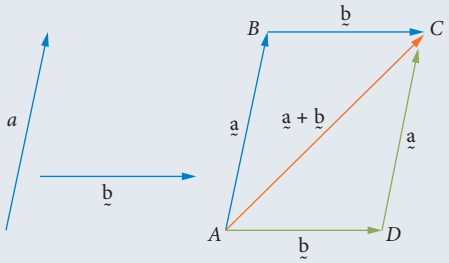
The sum $\underset{\sim}{a} + \underset{\sim}{b}$ is also called the **resultant** of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.



Parallelogram rule for adding vectors

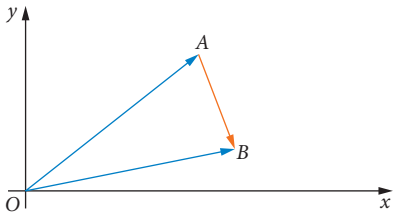
If we form the **parallelogram** $ABCD$ using $\underset{\sim}{a}$ and $\underset{\sim}{b}$ as the sides, the diagonal AC is the resultant vector $\underset{\sim}{a} + \underset{\sim}{b}$.

Note also that $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{b} + \underset{\sim}{a} = \overrightarrow{AC}$.



In the diagram

- $\overrightarrow{OA}$ is the position of point A relative to O
- $\overrightarrow{OB}$ is the position of point B relative to O
- $\overrightarrow{AB}$ is the position of point B relative to A.



Using vector addition, we get

$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$ (Note the common letter A.)

We can reverse this operation to perform vector subtraction:

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

Subtracting vectors

The product $(-1)\underset{\sim}{a}$ is normally written as $-\underset{\sim}{a}$.

The difference of vectors is found by addition of the negative $\underset{\sim}{a} - \underset{\sim}{b} = \underset{\sim}{a} + (-\underset{\sim}{b})$.

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WORKED EXAMPLE 7 Operations with vectors

If $\underset{\sim}{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\underset{\sim}{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, find

a $\underset{\sim}{u} + \underset{\sim}{v}$ **b** $\underset{\sim}{u} - \underset{\sim}{v}$ **c** $3\underset{\sim}{u} + 2\underset{\sim}{v}$

Steps	Working
a Evaluate $\underset{\sim}{u} + \underset{\sim}{v}$.	$\underset{\sim}{u} + \underset{\sim}{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$
b Evaluate $\underset{\sim}{u} - \underset{\sim}{v}$.	$\underset{\sim}{u} - \underset{\sim}{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$
c Evaluate $3\underset{\sim}{u} + 2\underset{\sim}{v}$.	$3\underset{\sim}{u} + 2\underset{\sim}{v} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ 6 \end{bmatrix}$ $\therefore 3\underset{\sim}{u} + 2\underset{\sim}{v} = \begin{bmatrix} 1 \\ 12 \end{bmatrix}$

9780170448338

USING CAS 1 Adding and subtracting vectors

a If $\underset{\sim}{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\underset{\sim}{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, find $\underset{\sim}{u} + \underset{\sim}{v}$, $\underset{\sim}{u} - \underset{\sim}{v}$ and $3(\underset{\sim}{u} + \underset{\sim}{v})$.

b Given the 3 position vectors: $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$, $\underset{\sim}{b} = \underset{\sim}{i} - 4\underset{\sim}{j}$, $\underset{\sim}{c} = -3\underset{\sim}{i} + \underset{\sim}{j}$, find

i $\underset{\sim}{a} + \underset{\sim}{b}$ and $2\underset{\sim}{a} - 3\underset{\sim}{c}$ **ii** the unit vector $\hat{\underset{\sim}{c}}$.

Note: vectors can be stored in either row or column matrices but keep the same format throughout.

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a





1 Enter the matrix $\begin{bmatrix} \mathbf{1} \\ \mathbf{2} \end{bmatrix}$ and **store** as **u**.

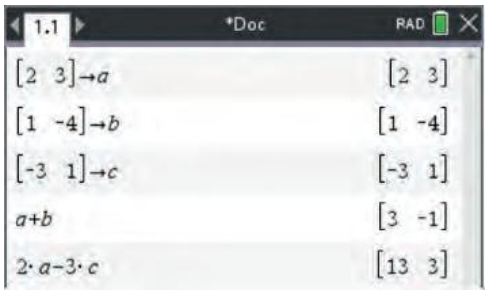
2 Enter the matrix $\begin{bmatrix} \mathbf{-1} \\ \mathbf{3} \end{bmatrix}$ and **store** as **v**.

3 Enter **u+v**.

4 Enter **u–v**.

5 Enter **3(u+v).**

b i



1 Enter the matrix $\begin{bmatrix} \mathbf{2} & \mathbf{3} \end{bmatrix}$ and store as **a**.

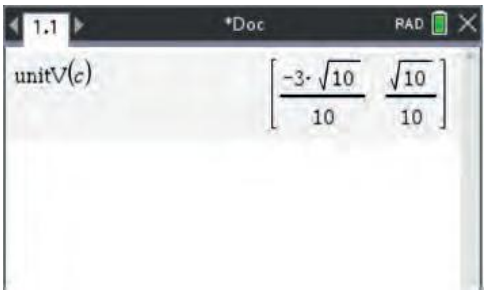
2 Enter the matrix $\begin{bmatrix} \mathbf{1} & \mathbf{-4} \end{bmatrix}$ and store as **b**.

3 Enter the matrix $\begin{bmatrix} \mathbf{-3} & \mathbf{1} \end{bmatrix}$ and store as **c**.

4 Enter **a+b**.

5 Enter **2a–3c**.

ii



1 Press **menu > Matrix & Vector > Vector > Unit Vector**.

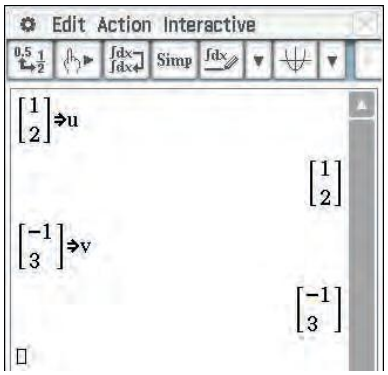
2 Enter **c**.

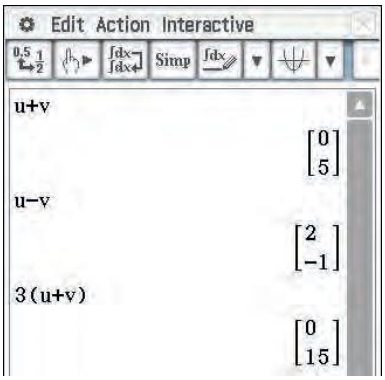
a $\underset{\sim}{u} + \underset{\sim}{v} = 5\underset{\sim}{j}$, $\underset{\sim}{u} - \underset{\sim}{v} = 2\underset{\sim}{i} - \underset{\sim}{j}$, $3(\underset{\sim}{u} + \underset{\sim}{v}) = 15\underset{\sim}{j}$

b **i** $\underset{\sim}{a} + \underset{\sim}{b} = 3\underset{\sim}{i} - \underset{\sim}{j}$, $2\underset{\sim}{a} - 3\underset{\sim}{c} = 13\underset{\sim}{i} + 3\underset{\sim}{j}$ **ii** $\hat{\underset{\sim}{c}} = -\frac{3\sqrt{10}}{10}\underset{\sim}{i} + \frac{\sqrt{10}}{10}\underset{\sim}{j}$

ClassPad

a





1 Enter the matrix $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and store as **u**.

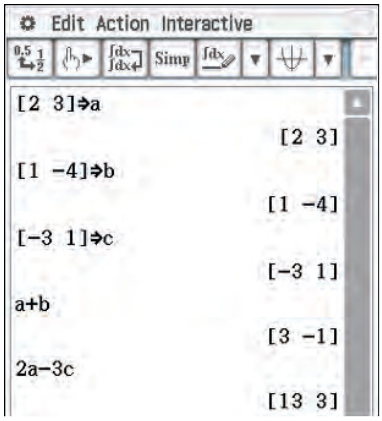
2 Enter the matrix $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and store as **v**.

3 Enter **u+v**.

4 Enter **u–v**.

5 Enter **3(u+v)**.

b i



1 Enter the matrix [**2** **3**] and store as **a**.

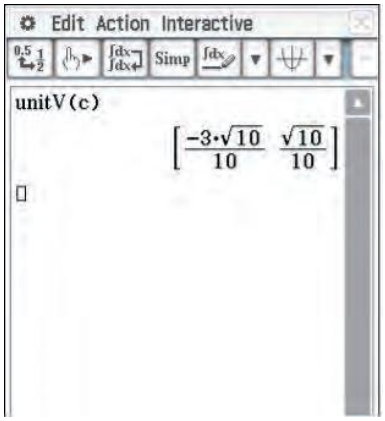
2 Enter the matrix [**1** **−4**] and store as **b**.

3 Enter the matrix [**−3** **1**] and store as **c**.

4 Enter **a+b**.

5 Enter **2a–3c**.

ii



1 Enter and highlight **c**.

2 Tap **Interactive > Vector > UnitV**.

a $\underset{\sim}{u}+\underset{\sim}{v}=5\underset{\sim}{j}, \underset{\sim}{u}-\underset{\sim}{v}=2\underset{\sim}{i}-\underset{\sim}{j}, 3\left(\underset{\sim}{u}+\underset{\sim}{v}\right)=15\underset{\sim}{j}$

b i $\underset{\sim}{a}+\underset{\sim}{b}=3\underset{\sim}{i}-\underset{\sim}{j}, 2\underset{\sim}{a}-3\underset{\sim}{c}=13\underset{\sim}{i}+3\underset{\sim}{j}$

ii $\hat{\underset{\sim}{c}}=-\frac{3\sqrt{10}}{10}\underset{\sim}{i}+\frac{\sqrt{10}}{10}\underset{\sim}{j}$

Equivalent vectors

For 2 vectors, we can equate the coefficients of the vectors, so if $m\underset{\sim}{a} + n\underset{\sim}{b} = p\underset{\sim}{a} + q\underset{\sim}{b}$, then $m = p$ and $n = q$.

Consider the vector equality $(3m + 1)\underset{\sim}{a} + (1 - 2n)\underset{\sim}{b} = 2m\underset{\sim}{a} + (4n + 2)\underset{\sim}{b}$, giving $3m + 1 = 2m$ and $1 - 2n = 4n + 2$ to get $m = -1$ and $n = -\frac{1}{6}$.

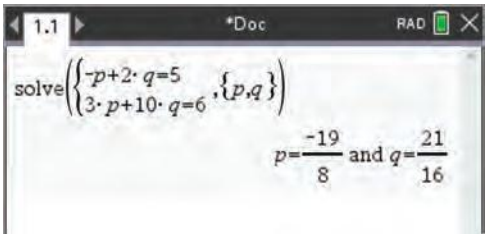
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WORKED EXAMPLE 8 Equivalent vectors

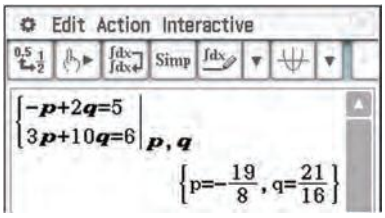
Find values of p and q such that $p\begin{bmatrix} -1 \\ 3 \end{bmatrix} + 2q\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.

Steps	Working
1 Simplify the LHS using scalar multiplication.	$\text{LHS} = p\begin{bmatrix} -1 \\ 3 \end{bmatrix} + 2q\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ $= \begin{bmatrix} -p \\ 3p \end{bmatrix} + \begin{bmatrix} 2q \\ 10q \end{bmatrix}$
2 Add the column vectors.	$= \begin{bmatrix} -p + 2q \\ 3p + 10q \end{bmatrix}$
3 Equate the components of the vectors.	$= \begin{bmatrix} 5 \\ 6 \end{bmatrix}$
4 Solve the simultaneous equations for p and q.	$\therefore -p + 2q = 5$ and $3p + 10q = 6$

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5 Write the answer.	$p = -\frac{19}{8}, q = \frac{21}{16}$

EXERCISE 10.2 Operations with vectors

ANSWERS p. 605

Recap

1 TECH-FREE

a On a set of coordinate axes, plot the points $A(4, 1)$ and $B(-6, -4)$.

b Sketch the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{AB}$.

c If $\overrightarrow{OA} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\overrightarrow{AB} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$, find $\overrightarrow{OA} + \overrightarrow{AB}$.

2 If $\overrightarrow{OA} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\overrightarrow{AB} = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$, then $3\overrightarrow{OA} + 4\overrightarrow{AB}$ equals

A $\begin{bmatrix} -3 \\ 6 \end{bmatrix}$ **B** $\begin{bmatrix} 24 \\ -12 \end{bmatrix}$ **C** $\begin{bmatrix} 5 \\ -1 \end{bmatrix}$ **D** $\begin{bmatrix} 7 \\ 5 \end{bmatrix}$ **E** $\begin{bmatrix} 21 \\ -6 \end{bmatrix}$

Mastery

3 WORKED EXAMPLE 3 **TECH-FREE** Draw a directed line segment corresponding to the column matrix $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$.

4 WORKED EXAMPLE 4 **TECH-FREE** Find the magnitude and direction of the column matrix $\begin{bmatrix} 3 \\ \sqrt{3} \end{bmatrix}$.

5 WORKED EXAMPLE 5 **TECH-FREE** For the vector $\underset{\sim}{a} = -2\underset{\sim}{i} + 2\underset{\sim}{j}$, find the magnitude of $\underset{\sim}{a}$ and the unit vector in the direction of $\underset{\sim}{a}$.

6 WORKED EXAMPLE 6 For the vector $\underset{\sim}{a} = 3\underset{\sim}{i} + 5\underset{\sim}{j}$, the angle between the vector $\underset{\sim}{a}$ and the x-axis is, to the nearest degree

A 30° **B** 31° **C** 59° **D** 60° **E** 80°

7 WORKED EXAMPLE 7 If $\underset{\sim}{u} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ and $\underset{\sim}{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, then $2\underset{\sim}{u} - \underset{\sim}{v}$ equals

A $\begin{bmatrix} 0 \\ 4 \end{bmatrix}$ **B** $\begin{bmatrix} 2 \\ -4 \end{bmatrix}$ **C** $\begin{bmatrix} -2 \\ 6 \end{bmatrix}$ **D** $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ **E** $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$

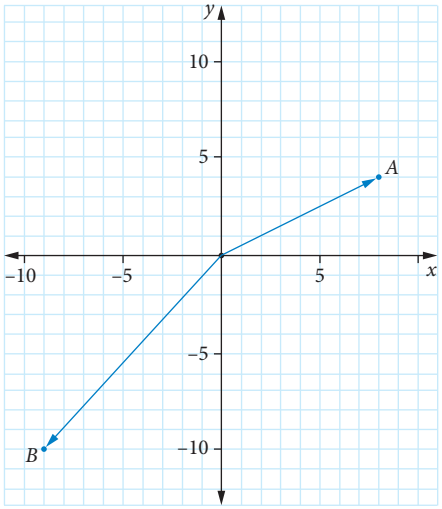
8 WORKED EXAMPLE 8 Given $\begin{bmatrix} -p \\ 3q \end{bmatrix} + \begin{bmatrix} 3p+1 \\ q-5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, the values of p and q equal

A $p = -\frac{1}{2}, q = \frac{3}{2}$ **B** $p = 0, q = \frac{1}{3}$ **C** $p = \frac{3}{2}, q = -\frac{1}{2}$

D $p = -\frac{1}{3}, q = 6$ **E** $p = -\frac{1}{4}, q = \frac{2}{3}$

9 Using CAS 1 In the diagram shown, the vector $\overrightarrow{OA} + \overrightarrow{OB}$ is likely to end at the point

A (8, 4)
B (−2, −6)
C (−9, −10)
D (17, 14)
E (0, 0)



10 **TECH-FREE** Sketch the position vector of each point on the same set of axes.

a (4, 6) **b** (−3, −5) **c** (4, −4) **d** (−2, 6)

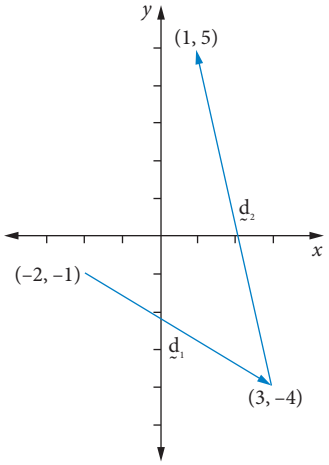
Exam practice

80–100% 60–79% 0–59%

11 TECH-FREE (2 marks) If $\underset{\sim}{a} = (x_1, y_1)$ and $c < 0$, prove that $|c\underset{\sim}{a}| = -c|\underset{\sim}{a}|$.

12 TECH-FREE (2 marks) If $\underset{\sim}{a} = (x_1, y_1)$, $\underset{\sim}{b} = (x_2, y_2)$ and c is a scalar, prove that $c(\underset{\sim}{a} + \underset{\sim}{b}) = c\underset{\sim}{a} + c\underset{\sim}{b}$.

13 TECH-FREE (4 marks)



a Write the vector $\underset{\sim}{d}_1$ in the form of $\underset{\sim}{d}_1 = a\underset{\sim}{i} + b\underset{\sim}{j}$. Hence, find an expression for $3\underset{\sim}{d}_1$. 2 marks

b Write the vector $\underset{\sim}{d}_2$ in the form of $\underset{\sim}{d}_2 = a\underset{\sim}{i} + b\underset{\sim}{j}$. Hence, find an expression for $-2\underset{\sim}{d}_2$. 2 marks

14 TECH-FREE (4 marks) Calculate the direction, in degrees correct to 1 decimal place, of the position vector of each point.

a (3, 4) **b** (−5, 12) **c** (24, −7) **d** (−1, −1)

15 TECH-FREE (3 marks) Find the column vector that describes each of the following movements.

a $A(3, 4)$ to $B(5, 11)$ **b** $R(-2, 1)$ to $B(-7, -2)$ **c** $M(-3, -8)$ to $B(-9, 1)$

16 TECH-FREE (4 marks) Find the sum of each pair of vectors.

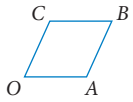
a (−4, 3) and (−9, 10) **b** (7, −9) and (11, −5)

c (−13, 6) and (8, −7) **d** (3.8, −4.5) and (6.2, −7.3)

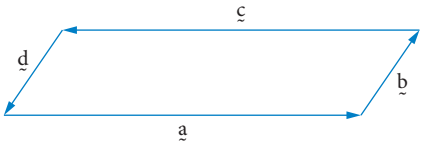
17 TECH-FREE (3 marks) Find the sum of each pair of vectors.

a $5\underset{\sim}{i} - 6\underset{\sim}{j}$ and $2\underset{\sim}{i} - 7\underset{\sim}{j}$ **b** $-8\underset{\sim}{i} + 7\underset{\sim}{j}$ and $-9\underset{\sim}{i} + 3\underset{\sim}{j}$ **c** $-11\underset{\sim}{i} - 4\underset{\sim}{j}$ and $6\underset{\sim}{i} + 9\underset{\sim}{j}$

18 © VCAA 2015 1Q1a 71% TECH-FREE (1 mark) Consider the rhombus $OABC$ shown, where $\overrightarrow{OA} = a\underset{\sim}{i}$ and $\overrightarrow{OC} = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$, and a is a positive real constant. Find a.



19 © VCAA 2006 2AQ15 74% In the parallelogram shown, $|\underset{\sim}{a}| = 2|\underset{\sim}{b}|$.



Which one of the following statements is true?

A $\underset{\sim}{a} = 2\underset{\sim}{b}$ **B** $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{c} + \underset{\sim}{d}$ **C** $\underset{\sim}{b} - \underset{\sim}{d} = \underset{\sim}{0}$

D $\underset{\sim}{a} + \underset{\sim}{c} = \underset{\sim}{0}$ **E** $\underset{\sim}{a} - \underset{\sim}{b} = \underset{\sim}{c} - \underset{\sim}{d}$

20 The magnitude of the position vector of $P(5, 0)$ is

A 0 **B** 1 **C** 2.5 **D** 3 **E** 5

21 A directed line segment corresponding to the column matrix $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ goes from

A $(-2,-4) \rightarrow (-1,6)$ **B** $(2,4) \rightarrow (1,6)$ **C** $(2,4) \rightarrow (6,1)$

D $(1, 6) \rightarrow (3, 3)$ **E** $(1, 6) \rightarrow (2, 4)$

22 The magnitude and direction, respectively, of the column matrix $\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$ are

A 1 and 1° **B** 1 and 45° **C** 2 and 30° **D** 2 and 60° **E** $\sqrt{3}$ and 60°

23 For the vector $\underset{\sim}{a} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$, the unit vector in the direction of $\underset{\sim}{a}$ is

A $\hat{\underset{\sim}{a}} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$ **B** $\hat{\underset{\sim}{a}} = \frac{1}{5}(3\underset{\sim}{i} + 4\underset{\sim}{j})$ **C** $\hat{\underset{\sim}{a}} = \frac{1}{\sqrt{5}}(3\underset{\sim}{i} + 4\underset{\sim}{j})$

D $\hat{\underset{\sim}{a}} = \frac{1}{\sqrt{5}}(\sqrt{5}\underset{\sim}{i} + \sqrt{5}\underset{\sim}{j})$ **E** $\hat{\underset{\sim}{a}} = \sqrt{5}\underset{\sim}{i} + \sqrt{5}\underset{\sim}{j}$

24 For the vector $\underset{\sim}{a} = 3\underset{\sim}{i} + 5\underset{\sim}{j}$, the angle between the vector and the y-axis is closest to

A 30° **B** 31° **C** 53° **D** 59° **E** 60°

25 The value of q such that $p\begin{bmatrix} 0 \\ 3 \end{bmatrix} - q\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ is

A -5 **B** $-\frac{4}{3}$ **C** -1.3 **D** 0 **E** 5

26 $\overrightarrow{OA} + \overrightarrow{AB} =$

A $\overrightarrow{OA}$ **B** $\overrightarrow{OB}$ **C** $\overrightarrow{OA} - \overrightarrow{OB}$ **D** $\overrightarrow{AB}$ **E** $\overrightarrow{BA}$

27 $\overrightarrow{AB} + \overrightarrow{AB} =$

A $2\overrightarrow{BA}$ **B** $2\overrightarrow{OB}$ **C** $\overrightarrow{OA} + \overrightarrow{OB}$ **D** $2\overrightarrow{AB}$ **E** $\overrightarrow{BA}$

28 For $\underset{\sim}{a} = 4\underset{\sim}{i} + m\underset{\sim}{j}$ and $\underset{\sim}{b} = 8\underset{\sim}{i} - 4\underset{\sim}{j}$, $\underset{\sim}{b} = 2\underset{\sim}{a}$ for

A $m = 4$ **B** $m = -4$ **C** $m = -2$ **D** $m = 2$ **E** $\underset{\sim}{a} = 2$

29 Let $\underset{\sim}{b} = -2\underset{\sim}{i} + 7\underset{\sim}{j}$. $|\underset{\sim}{b}|$ equals

A 45 **B** 53 **C** $3\sqrt{5}$ **D** $\sqrt{53}$ **E** $\sqrt{10}$

30 Let $\underset{\sim}{b} = -2\underset{\sim}{i} + n\underset{\sim}{j}$. $|\underset{\sim}{b}|$ equals

A $n^2 - 2$ **B** $\sqrt{n^2 + 4}$ **C** $\sqrt{n^2 + 2}$ **D** $n^2 + 4$ **E** $\sqrt{n^2 - 4}$

10.3 Scalar product of vectors

There are 2 ways of multiplying 2 vectors, giving 2 different types of answers. Now, we will learn about the **scalar product** (also called the **dot product**). In Year 12, we will learn about the **vector product** (also called the **cross product**). The **scalar product** has that name because the answer is a scalar, a number, and not a vector. It is also called the dot product because the symbol used is a dot ·.

Properties of the scalar product

1 The scalar product is a real number, **not** a vector.

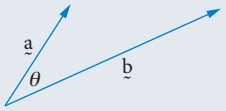
2 The scalar product is commutative; for example, $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{a}$.

3 The scalar product is distributive over vector addition; for example, $\underset{\sim}{a} \cdot (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c}$.

4 The scalar product is found by the formula

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos(\theta)$$

where θ is the angle between the 2 'outgoing' vectors.



5 If $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j} + a_3\underset{\sim}{k}$ and $\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j} + b_3\underset{\sim}{k}$, the scalar product is found by the formula

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

6 $m(\underset{\sim}{a} \cdot \underset{\sim}{b}) = (m\underset{\sim}{a}) \cdot \underset{\sim}{b} = \underset{\sim}{a} \cdot (m\underset{\sim}{b})$ for all $m \in R$.

7 $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0 \Rightarrow \underset{\sim}{a} = \underset{\sim}{0}$ or $\underset{\sim}{b} = \underset{\sim}{0}$ or importantly $\Rightarrow \underset{\sim}{a}$ **is perpendicular to** $\underset{\sim}{b}$.

8 $\underset{\sim}{a} \cdot \underset{\sim}{b} = \pm |\underset{\sim}{a}||\underset{\sim}{b}| \Rightarrow \underset{\sim}{a}$ **is parallel to** $\underset{\sim}{b}$.

9 $\underset{\sim}{i} \cdot \underset{\sim}{i} = \underset{\sim}{j} \cdot \underset{\sim}{j} = \underset{\sim}{k} \cdot \underset{\sim}{k} = 1$

10 $\underset{\sim}{i} \cdot \underset{\sim}{j} = \underset{\sim}{j} \cdot \underset{\sim}{k} = \underset{\sim}{k} \cdot \underset{\sim}{i} = 0$

The following strategies are useful when finding the dot product of 2 vectors.

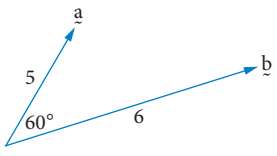
- Select the easiest component formula, $\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$, when both vectors are given.
- Use the formal version, $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos(\theta)$, when there is an angle given or an angle to be found.
- The rule for perpendicular vectors, $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$, is the rule most often used in a variety of contexts.
- The rule for parallel vectors, $\underset{\sim}{a} \cdot \underset{\sim}{b} = \pm|\underset{\sim}{a}||\underset{\sim}{b}|$, has 2 options:
  - positive $|\underset{\sim}{a}||\underset{\sim}{b}|$ is for when $\cos(\theta) = 1$, meaning parallel in the same direction.
  - negative $|\underset{\sim}{a}||\underset{\sim}{b}|$ is for when $\cos(\theta) = -1$, meaning parallel in the opposite direction.

Don't forget to use this most helpful rule.

Consider the scalar product of vectors of magnitudes 5 and 6 at an angle of 60° to each other.

This question has an angle involved, so use the formula $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos(\theta)$.

Substituting the values gives the scalar product $\underset{\sim}{a} \cdot \underset{\sim}{b} = 5 \times 6 \times \cos(60°) = 15$.



Video playlist
Scalar product of vectors

Worksheets
Scalar product

Component form of the dot product

Properties of the dot product

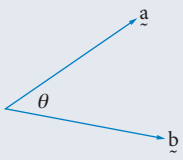
The scalar product

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos(\theta),$$

where θ is the angle between outgoing vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

Alternatively, if $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are expressed in component form:

$\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j}$ and $\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j}$, then $\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1 b_1 + a_2 b_2$.



Make sure that you always use the dot · for scalar product, not the cross ×.

Exam hack

You need to be able to select when and where to apply which dot product formula in suitable questions.

WB

p. 270

WORKED EXAMPLE 9 Scalar product

If $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{b} = \underset{\sim}{i} - 3\underset{\sim}{j}$, find $\underset{\sim}{a} \cdot \underset{\sim}{b}$.

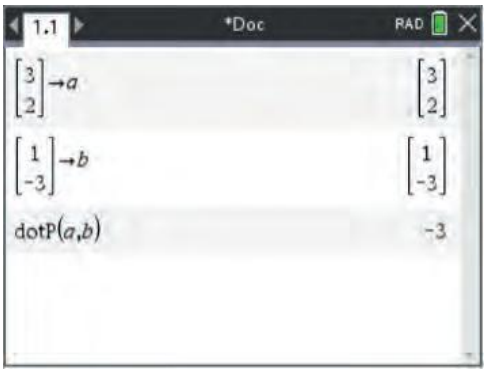
Steps	Working
Use the component form formula: $\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1 b_1 + a_2 b_2$	$\underset{\sim}{a} \cdot \underset{\sim}{b} = 3 \times 1 + 2 \times (-3)$ $= 3 - 6$ $\therefore\ \underset{\sim}{a} \cdot \underset{\sim}{b} = -3$

CAS will perform operations with vectors.

USING CAS 2 The scalar product

If $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{b} = \underset{\sim}{i} - 3\underset{\sim}{j}$, find $\underset{\sim}{a} \cdot \underset{\sim}{b}$.

TI-Nspire



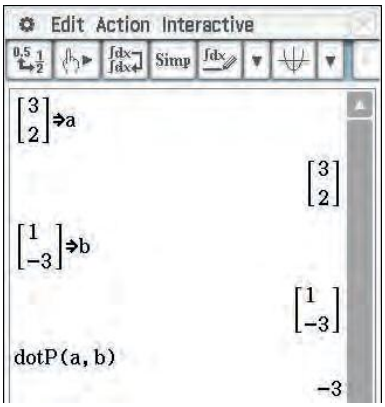
1 Enter the matrix $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and **store** as **a**.

2 Enter the matrix $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and **store** as **b**.

3 Press **menu > Matrix & Vector > Vector > Dot Product.**

4 Enter **a,b**.

ClassPad



1 Enter the matrix $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and **store** as **a**.

2 Enter the matrix $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and **store** as **b**.

3 Tap **Interactive > Vector > dotP**.

4 In the dialogue box, for **Vector**: enter **a** and for **Another:** enter **b**.

$\underset{\sim}{a} \cdot \underset{\sim}{b} = -3$

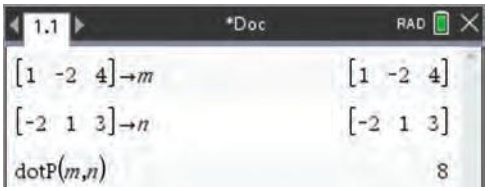
USING CAS 3 Scalar product and angle between vectors

For the vectors $\underset{\sim}{m} = \underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}$ and $\underset{\sim}{n} = -2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$, find the

a dot product **b** angle between the vectors.

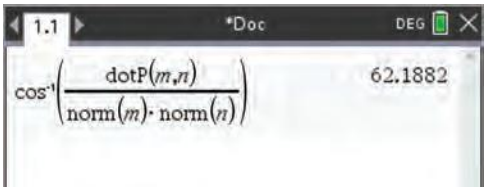
TI-Nspire

a



1. Create a **1x3** matrix and store it into **m**, as shown above.
2. Repeat for matrix **n**.
3. Press **menu** > **Matrix & Vector** > **Vector** > **Dot Product**.
4. Enter **m,n** to find the dot product.

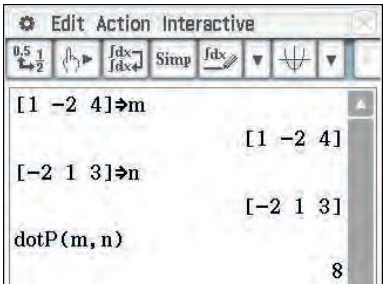
b



1. Set the mode to **DEG**.
2. Press **trig** > **cos^{-1}**.
3. Enter **dotP(m,n)** divided by **norm(m)** times **norm(n)**, as shown above.
4. Press **ctrl** + = to find the approximate angle between the 2 vectors.

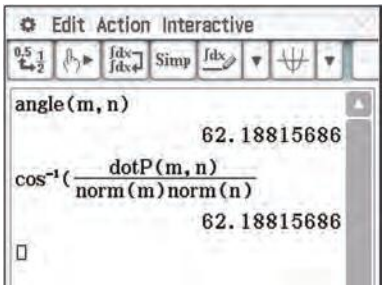
ClassPad

a



1. Create a **1x3** matrix and store using ⇒ as **m**, as shown above.
2. Repeat for matrix **n**.
3. Tap **Interactive** > **Vector** > **dotP**.
4. In the dialogue box, in the **Vector:** field enter **m** and in the **Another:** field, enter **n**.

b



1. Set the modes to **Decimal** and **Deg**.
2. Press **Interactive** > **Vector** > **angle**.
3. In the dialogue box, in the **Vector:** field enter **m** and in the **Another:** field, enter **n**.
4. To confirm, find the **cos^{-1}** of **dotP(m,n)** divided by **norm(m)** times **norm(n)**, as shown above (optional).

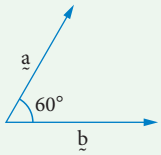
a $\underset{\sim}{m} \cdot \underset{\sim}{n} = 8$

b Angle between $\underset{\sim}{m}$ and $\underset{\sim}{n} \approx 62°$

Alternatively, when a dot product question includes an angle, it is more suitable to use the formula $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos(\theta)$.

WORKED EXAMPLE 10 Angles using dot product

If $|\underset{\sim}{a}| = \sqrt{7}$ and $|\underset{\sim}{b}| = 4$ and the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is 60°, find $\underset{\sim}{a} \cdot \underset{\sim}{b}$.



Steps	Working
Substitute into the formula: $\underset{\sim}{a} \cdot \underset{\sim}{b} = \|\underset{\sim}{a}\|\|\underset{\sim}{b}\|\cos(\theta)$.	$\underset{\sim}{a} \cdot \underset{\sim}{b} = \sqrt{7} \times 4\cos(60°)$ $= 4\sqrt{7} \times \frac{1}{2}$ $\therefore \underset{\sim}{a} \cdot \underset{\sim}{b} = 2\sqrt{7}$

WB
p. 271

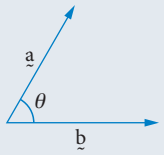
Angle between 2 vectors

When a dot product question includes an angle, but the question asks to find, rather than substitute the angle, we can rearrange the dot product formula to get $\cos(\theta) = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$.

This finds the angle between 2 'outgoing' vectors, the tails of which point out to form the angle.

The angle between two vectors

$$\cos(\theta) = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$



Exam hack

Remember that the angle is between 2 'outgoing' vectors.

If $\cos(\theta)$ is positive, the angle between the vectors is acute.
If $\cos(\theta)$ is negative, the angle between the vectors is obtuse.

WB
p. 272

WORKED EXAMPLE 11 Angle between 2 vectors

If $\underset{\sim}{a} = -\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{b} = \underset{\sim}{i} - 3\underset{\sim}{j}$, find the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ correct to the nearest degree.

Steps	Working
1 Find $\underset{\sim}{a} \cdot \underset{\sim}{b}$ and $\lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert$.	$\underset{\sim}{a} \cdot \underset{\sim}{b} = (-1 \times 1) + (2 \times -3) = -7$ $\lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert = \sqrt{5}\sqrt{10} = 5\sqrt{2}$
2 Substitute into the formula $\cos(\theta) = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert}$ and solve for θ. Note $\cos(\theta)$ is negative, so the angle between the outgoing vectors is obtuse.	$\cos(\theta) = \frac{-7}{5\sqrt{2}}$ $\theta = \cos^{-1}\left(\frac{-7}{5\sqrt{2}}\right)$ $\therefore \theta \approx 172°$

EXERCISE 10.3 Scalar product of vectors

ANSWERS p. 605

Recap

1 TECH-FREE

a On a set of coordinate axes, plot the points $A(3, 2)$, $B(-1, 3)$ and $C(-4, -4)$.

b Sketch the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$.

c On a set of coordinate axes, sketch the position vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$, where the points are $A(2, 12)$ and $B(-9, -3)$. Label the vectors $\underset{\sim}{r}$ and $\underset{\sim}{s}$.

2 TECH-FREE

a If $\overrightarrow{OA} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $\overrightarrow{AB} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, find $\overrightarrow{OA} + \overrightarrow{AB}$.

b If $\overrightarrow{OA} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $\overrightarrow{AB} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, find $\overrightarrow{OA} - \overrightarrow{AB}$.

Mastery

3 WORKED EXAMPLE 9 TECH-FREE If $\underset{\sim}{a} = \underset{\sim}{i} - 2\underset{\sim}{j}$ and $\underset{\sim}{b} = \underset{\sim}{i} + 3\underset{\sim}{j}$, find $\underset{\sim}{a} \cdot \underset{\sim}{b}$.

4 WORKED EXAMPLE 10 TECH-FREE If $|\underset{\sim}{a}| = \sqrt{2}$ and $|\underset{\sim}{b}| = 3$ and the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is 45°, find $\underset{\sim}{a} \cdot \underset{\sim}{b}$.

5 WORKED EXAMPLE 11 **TECH-FREE** If $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} + \underset{\sim}{j}$, find the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

6 **TECH-FREE** If $\underset{\sim}{a} = 5\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{b} = 10\underset{\sim}{i} - 2\underset{\sim}{j}$, evaluate $\underset{\sim}{a} \cdot \underset{\sim}{b}$.

7 **TECH-FREE** If $|\underset{\sim}{p}| = 2$ and $|\underset{\sim}{q}| = 5$ and the angle between $\underset{\sim}{p}$ and $\underset{\sim}{q}$ is 60°, then evaluate $\underset{\sim}{p} \cdot \underset{\sim}{q}$.

8 Using CAS 2 If $\underset{\sim}{b} = -2\underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{a} = -2\underset{\sim}{i}$, $\underset{\sim}{a} \cdot \underset{\sim}{b} =$

A -4 **B** 4 **C** 5 **D** $-4\underset{\sim}{i}$ **E** $-4\underset{\sim}{i} + \underset{\sim}{j}$

9 Using CAS 3 If $\underset{\sim}{a} = \underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - 3\underset{\sim}{j}$, the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ correct to the nearest degree is

A 15° **B** 52° **C** 53° **D** 127° **E** 128°

10 Consider the vectors $\underset{\sim}{a} = \sqrt{5}\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{b} = \sqrt{5}\underset{\sim}{i} - 2\underset{\sim}{j}$. A **true** statement is

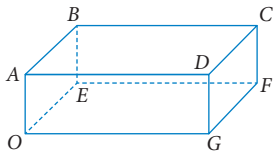
A $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$ **B** $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|$ **C** $|\underset{\sim}{a}| = |\underset{\sim}{b}|$

D $|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 = 0$ **E** $\underset{\sim}{a} \cdot \underset{\sim}{b} = 2\sqrt{5} + 5$

11 A unit vector in the direction of $\underset{\sim}{i} - 2\underset{\sim}{j}$ is

A $\underset{\sim}{i} - 2\underset{\sim}{j}$ **B** $\frac{1}{3}(\underset{\sim}{i} - 2\underset{\sim}{j})$ **C** $2\underset{\sim}{j} - \underset{\sim}{i}$ **D** $\frac{1}{\sqrt{5}}(\underset{\sim}{i} - 2\underset{\sim}{j})$ **E** $\frac{1}{5}(\underset{\sim}{i} - 2\underset{\sim}{j})$

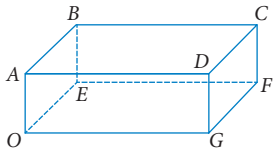
12 In the diagram, $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OE} = \underset{\sim}{e}$ and $\overrightarrow{OG} = \underset{\sim}{g}$.
Vector $\overrightarrow{OF}$ can be represented by



A $\underset{\sim}{a} + \underset{\sim}{e} + \underset{\sim}{g}$ **B** $\underset{\sim}{a} + \underset{\sim}{e} - \underset{\sim}{g}$ **C** $\underset{\sim}{a} - \underset{\sim}{e} - \underset{\sim}{g}$

D $\underset{\sim}{e} + \underset{\sim}{g}$ **E** $\underset{\sim}{a} + \underset{\sim}{e}$

13 In the diagram, $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OE} = \underset{\sim}{e}$ and $\overrightarrow{OG} = \underset{\sim}{g}$.
Vector $\overrightarrow{OC}$ can be represented by



A $\underset{\sim}{a} + \underset{\sim}{e} + \underset{\sim}{g}$ **B** $\underset{\sim}{a} + \underset{\sim}{e} - \underset{\sim}{g}$ **C** $\underset{\sim}{a} - \underset{\sim}{e} - \underset{\sim}{g}$

D $\underset{\sim}{e} + \underset{\sim}{g}$ **E** $\underset{\sim}{a} + \underset{\sim}{e}$

14 If $\underset{\sim}{a} = -2\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{b} = 7\underset{\sim}{i} + 3\underset{\sim}{j}$, find the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ correct to the nearest degree.

Exam practice

80–100% 60–79% 0–59%

15 © VCAA 2017 1Q5 45% **TECH-FREE** (4 marks) Relative to a fixed origin, the points B, C and D are defined respectively by the position vectors $\underset{\sim}{b} = \underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{c} = 2\underset{\sim}{i} - \underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{d} = a\underset{\sim}{i} - 2\underset{\sim}{j}$, where a is a real constant. Given that the magnitude of angle BCD is $\frac{\pi}{3}$, find a.

16 © VCAA 2017N 1Q10 **TECH-FREE** (4 marks) Consider the vectors $\underset{\sim}{a} = -\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} + c\underset{\sim}{j} + \underset{\sim}{k}$.
Find the value of c, $c \in R$, if the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is $\frac{\pi}{3}$.

17 Let O denote the point $(0, 0)$, R the point $(-2, 1)$, and S the point $(-c, 2c)$ where $c \in R$.
The scalar product $\overrightarrow{OS} \cdot \overrightarrow{RS}$ is equal to

A $4c - 5$ **B** $5 - 4c$ **C** $4c - 5c^2$

D $5c^2 - 4c$ **E** $(c^2 - 2c)\underset{\sim}{i} + (4c^2 - 2c)\underset{\sim}{j}$

Video playlist
Parallel and perpendicular vectors

Worksheet
Parallel and perpendicular vectors

10.4 Parallel and perpendicular vectors

The scalar product formula $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos(\theta)$ produces the following results where we know the special cases of $\cos(90°) = 0$, $\cos(0°) = 1$ and $\cos(180°) = -1$.

When $\theta = 90°$ we get $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos(90°)$ $\therefore \underset{\sim}{a} \cdot \underset{\sim}{b} = 0$.

When $\theta = 0°$ we get $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos(0°)$ $\therefore \underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|$.

When $\theta = 180°$ we get $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}| \cos(180°)$ $\therefore \underset{\sim}{a} \cdot \underset{\sim}{b} = -|\underset{\sim}{a}||\underset{\sim}{b}|$.

These significant results solve a variety of problems.

Parallel and perpendicular vectors

- If $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$, then vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular.
- If $\underset{\sim}{a} \cdot \underset{\sim}{b} = \pm|\underset{\sim}{a}||\underset{\sim}{b}|$, then vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel.
- If $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|$, then vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel in the same direction.
- If $\underset{\sim}{a} \cdot \underset{\sim}{b} = -|\underset{\sim}{a}||\underset{\sim}{b}|$, then vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel in the opposite direction.

WB
p. 273

WORKED EXAMPLE 12 Perpendicular vectors

a Show that the vectors $\underset{\sim}{a} = \sqrt{5}\underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{b} = \sqrt{5}\underset{\sim}{i} - 5\underset{\sim}{j}$ are perpendicular.

b For the perpendicular vectors $\underset{\sim}{u} = a\underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{v} = 5\underset{\sim}{i} - b\underset{\sim}{j}$, it is known that $|\underset{\sim}{v}| = \sqrt{26}$. Find the value(s) of a and b.

Steps	Working
a 1 Find $\underset{\sim}{a} \cdot \underset{\sim}{b}$.	$\underset{\sim}{a} \cdot \underset{\sim}{b} = (\sqrt{5} \times \sqrt{5}) + (1 \times -5) = 5 - 5$ $\therefore \underset{\sim}{a} \cdot \underset{\sim}{b} = 0$
2 Use the formula $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$.	So $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$.
b 1 Find $\underset{\sim}{u} \cdot \underset{\sim}{v}$.	$\underset{\sim}{u} \cdot \underset{\sim}{v} = (a \times 5) + (1 \times -b) = 5a - b$
2 Use the formula $\underset{\sim}{u} \cdot \underset{\sim}{v} = 0$ and $\|\underset{\sim}{v}\| = \sqrt{26}$.	$5a - b = 0 \Rightarrow a = \frac{1}{5}b$ and $\sqrt{5^2 + b^2} = \sqrt{26} \Rightarrow 5^2 + b^2 = 26$
3 Solve the equations simultaneously to find a and b.	$\sqrt{5^2 + b^2} = \sqrt{26} \Rightarrow b^2 = 1 \Rightarrow b = \pm 1$ Substitute into $a = \frac{1}{5}b$. When $b = 1$, $a = \frac{1}{5}$. When $b = -1$, $a = -\frac{1}{5}$.

9780170448338

We can combine our knowledge of unit vectors and parallel vectors.

WB p. 274

WORKED EXAMPLE 13 Parallel unit vectors

Find the unit vector parallel to the vector $\underset{\sim}{a} = \sqrt{5}\underset{\sim}{i} + 2\underset{\sim}{j}$.

Steps	Working
1 Find the length of $\underset{\sim}{a}$.	$\lvert\underset{\sim}{a}\rvert = \sqrt{(\sqrt{5})^2 + 2^2} = \sqrt{5+4} = \sqrt{9}$ $\therefore \lvert\underset{\sim}{a}\rvert = 3$
2 Use the formula: $\hat{\underset{\sim}{a}} = \frac{\underset{\sim}{a}}{\lvert\underset{\sim}{a}\rvert}$.	$\hat{\underset{\sim}{a}} = \frac{1}{3}(\sqrt{5}\underset{\sim}{i} + 2\underset{\sim}{j})$

Parallel vectors can occur in the same or opposite direction.

WB p. 275

WORKED EXAMPLE 14 Parallel vectors

Show that the vectors $\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{j}$ and $\underset{\sim}{b} = -6\underset{\sim}{i} + 4\underset{\sim}{j}$ are parallel in the opposite direction.

Steps	Working
1 Find $\lvert\underset{\sim}{a}\rvert$ and $\lvert\underset{\sim}{b}\rvert$.	$\lvert\underset{\sim}{a}\rvert = \sqrt{3^2 + (-2)^2} = \sqrt{9+4} = \sqrt{13}$ $\lvert\underset{\sim}{b}\rvert = \sqrt{(-6)^2 + 4^2} = \sqrt{36+16} = 2\sqrt{13}$
2 Use the formula for parallel vectors: $\underset{\sim}{a} \cdot \underset{\sim}{b} = \pm\lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert$	LHS $= \underset{\sim}{a} \cdot \underset{\sim}{b} = (3 \times -6) + (-2 \times 4) = -18 - 8 = -26$ RHS $= \pm\lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert = \pm\sqrt{13} \times 2\sqrt{13} = \pm 26$ LHS $= -$RHS
3 Write the conclusion.	So vectors are parallel in the opposite direction.

EXERCISE 10.4 Parallel and perpendicular vectors

ANSWERS p. 606

Recap

1 TECH-FREE

a If $\overrightarrow{OA} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$ and $\overrightarrow{AB} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, find $\overrightarrow{OB}$.

b If $\overrightarrow{OA} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $\overrightarrow{AB} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, find $\overrightarrow{BO}$.

c If $\underset{\sim}{a} = 2\underset{\sim}{i} - 2\underset{\sim}{j}$ and $\underset{\sim}{b} = -3\underset{\sim}{i} + 4\underset{\sim}{j}$, find $\underset{\sim}{a} + \underset{\sim}{b}$.

d If $\underset{\sim}{a} = 2\underset{\sim}{i} - 2\underset{\sim}{j}$ and $\underset{\sim}{b} = -3\underset{\sim}{i} + 4\underset{\sim}{j}$, find $\underset{\sim}{a} \cdot \underset{\sim}{b}$.

2 TECH-FREE If $\underset{\sim}{a} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{b} = -\underset{\sim}{i} + 3\underset{\sim}{j}$, find to the nearest degree, the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

Mastery

3 TECH-FREE Show that the vectors $\underset{\sim}{a} = 3\underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{b} = -\underset{\sim}{i} + 3\underset{\sim}{j}$ are perpendicular.

4 WORKED EXAMPLE 12 TECH-FREE For the perpendicular vectors $\underset{\sim}{u} = m\underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{v} = \underset{\sim}{i} - b\underset{\sim}{j}$, it is known that $\lvert\underset{\sim}{v}\rvert = \sqrt{5}$. Find the values of m and b.

5 WORKED EXAMPLE 13 TECH-FREE Find the unit vector parallel to the vector $\underset{\sim}{a} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$.

6 TECH-FREE Find the unit vector parallel and opposite to $\underset{\sim}{a} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$.

7 WORKED EXAMPLE 14 TECH-FREE Show that the vectors $\underset{\sim}{a} = \underset{\sim}{i} - 2\underset{\sim}{j}$ and $\underset{\sim}{b} = -2\underset{\sim}{i} + 4\underset{\sim}{j}$ are parallel in the opposite direction.

8 TECH-FREE Show that the vectors $\underset{\sim}{a} = 4\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{b} = -2\underset{\sim}{i} + 4\underset{\sim}{j}$ are perpendicular.

9 If $\underset{\sim}{b} = -2\underset{\sim}{i} + 15\underset{\sim}{j}$ and $\underset{\sim}{a} = \frac{1}{2}\underset{\sim}{j}$, $\underset{\sim}{a} \cdot \underset{\sim}{b} =$

A $\frac{1}{2}\underset{\sim}{j}$ **B** $\frac{15}{2}\underset{\sim}{j}$ **C** $\frac{15}{2}$ **D** $\frac{11}{2}$ **E** $\frac{11}{2}\underset{\sim}{j}$

10 Consider the vectors $\underset{\sim}{a} = \frac{3}{2}\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\underset{\sim}{b} = -\frac{1}{2}\underset{\sim}{i} - 2\underset{\sim}{j}$. A **true** statement is

A $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$ **B** $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|$ **C** $|\underset{\sim}{a}| = |\underset{\sim}{b}|$

D $\underset{\sim}{a} + \underset{\sim}{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **E** $\underset{\sim}{a} \cdot \underset{\sim}{b} = \frac{19}{4}$

11 A unit vector in the direction of $3\underset{\sim}{i} - 4\underset{\sim}{j}$ is

A $\frac{1}{5}$ **B** $\frac{1}{5}(3\underset{\sim}{i} - 4\underset{\sim}{j})$ **C** $3\underset{\sim}{i} - 4\underset{\sim}{j}$ **D** $\frac{1}{\sqrt{5}}(3\underset{\sim}{i} - 4\underset{\sim}{j})$ **E** $\frac{1}{5}(\underset{\sim}{i} - 2\underset{\sim}{j})$

12 If $\underset{\sim}{a} = \underset{\sim}{i} - 2\underset{\sim}{j}$ and $\underset{\sim}{b} = -\underset{\sim}{i} + 4\underset{\sim}{j}$, $2(\underset{\sim}{a} \cdot \underset{\sim}{b}) + (2\underset{\sim}{a} \cdot \underset{\sim}{b})$ equals

A −36 **B** −18 **C** −9 **D** 9 **E** 18

Exam practice

80–100% 60–79% 0–59%

13 TECH-FREE (1 mark) Find a unit vector that is opposite in direction to $\underset{\sim}{m} = 3\underset{\sim}{i} - 4\underset{\sim}{j}$.

14 TECH-FREE (1 mark) Find a vector of magnitude 6 that has the same direction as $\underset{\sim}{n} = 4\underset{\sim}{i} - 7\underset{\sim}{j}$.

15 TECH-FREE (2 marks) Let $\underset{\sim}{u} = 3\underset{\sim}{i} + 4\underset{\sim}{j}$. Find a vector perpendicular to $\underset{\sim}{u}$ with a magnitude of 5 units.

16 © VCAA 2014 1Q1 TECH-FREE (5 marks) Consider the vector $\underset{\sim}{a} = \sqrt{3}\underset{\sim}{i} - \underset{\sim}{j} - \sqrt{2}\underset{\sim}{k}$, where $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors in the positive directions of the x, y and z axes respectively.

a 79% Find the unit vector in the direction of $\underset{\sim}{a}$. 1 mark

b 47% Find the acute angle that $\underset{\sim}{a}$ makes with the positive direction of the x-axis. 2 marks

c 84% The vector $\underset{\sim}{b} = 2\sqrt{3}\underset{\sim}{i} + m\underset{\sim}{j} - 5\underset{\sim}{k}$. Given that $\underset{\sim}{b}$ is perpendicular to $\underset{\sim}{a}$, find the value of m. 2 marks

17 For the perpendicular vectors $\underset{\sim}{u} = a\underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{v} = \underset{\sim}{i} + b\underset{\sim}{j}$, it is known that $|\underset{\sim}{u}| = \sqrt{2}$.
Possible values of a and b are

A $a = -1$ and $b = -1$ **B** $a = 1$ and $b = 1$ **C** $a = 1$ and $b = 0$

D $a = 1$ and $b = -1$ **E** $a = 1$ and $b = 2$

18 A unit vector parallel and in the same direction to the vector $4\underset{\sim}{i} + 3\underset{\sim}{j}$ is

A $4\underset{\sim}{i} + 3\underset{\sim}{j}$ **B** $\frac{1}{25}(4\underset{\sim}{i} + 3\underset{\sim}{j})$ **C** $\frac{1}{\sqrt{5}}(4\underset{\sim}{i} + 3\underset{\sim}{j})$

D $-\frac{1}{5}(4\underset{\sim}{i} + 3\underset{\sim}{j})$ **E** $\frac{1}{5}(4\underset{\sim}{i} + 3\underset{\sim}{j})$

19 In the Cartesian plane, a vector perpendicular to the line $3x + 2y + 1 = 0$ is

A $3\underset{\sim}{i} + 2\underset{\sim}{j}$ **B** $-\frac{1}{2}\underset{\sim}{i} + \frac{1}{3}\underset{\sim}{j}$ **C** $2\underset{\sim}{i} - 3\underset{\sim}{j}$ **D** $\frac{1}{2}\underset{\sim}{i} - \frac{1}{3}\underset{\sim}{j}$ **E** $2\underset{\sim}{i} + 3\underset{\sim}{j}$

10.5 Projection of vectors

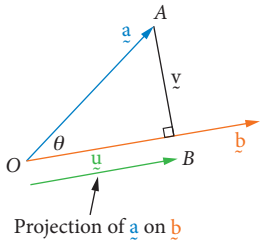
We have already seen a vector written in terms of its horizontal and vertical components.

Any vector $\underset{\sim}{a}$ can also be written in terms of another vector $\underset{\sim}{b}$, as the sum of one vector parallel to $\underset{\sim}{b}$ and another vector perpendicular to $\underset{\sim}{b}$.

The diagram shows 2 vectors, $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

Vector $\underset{\sim}{u}$ has the same direction as $\underset{\sim}{b}$ and is called the **vector projection** or **vector resolute** of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ (or in the direction of $\underset{\sim}{b}$). It is like the shadow of $\underset{\sim}{a}$ on $\underset{\sim}{b}$.

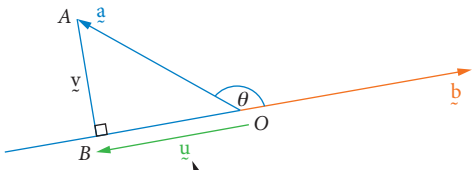
Vector $\underset{\sim}{v}$ is the **vector projection** or **vector resolute** of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$.



Projection of $\underset{\sim}{a}$ on $\underset{\sim}{b}$

The **scalar projection** or **scalar resolute** of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ is the **magnitude** of the **vector projection** of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$.

If the angle between the 2 vectors is obtuse, then the projection of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ is in the opposite direction of $\underset{\sim}{b}$ and will be negative.



Projection of $\underset{\sim}{a}$ on $\underset{\sim}{b}$

Video playlist
Projection of vectors

Worksheet
Projections

Scalar and vector projections

The **scalar projection** of $\underset{\sim}{a}$ on $\underset{\sim}{b}$, is $\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}$.

The **vector projection** of $\underset{\sim}{a}$ on $\underset{\sim}{b}$, is $(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$.

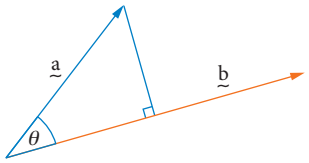
The **vector projection** of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$, is $\underset{\sim}{a} - (\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$.

In the diagram above, we know that
$\underset{\sim}{a} = \underset{\sim}{u} + \underset{\sim}{v} \therefore \underset{\sim}{v} = \underset{\sim}{a} - \underset{\sim}{u}$.
So the perpendicular component,
$\underset{\sim}{v}$ = original $\underset{\sim}{a}$ – parallel component, $\underset{\sim}{u}$,
giving $\underset{\sim}{a} - (\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$.

Every vector can be written as 2 components perpendicular to each other.

In many real-life circumstances, this is used in reference to a particular vector.

Imagine a plane flying in the air and the wind picks up, affecting the plane's trajectory. This wind can be represented as $\underset{\sim}{a}$ and the plane as $\underset{\sim}{b}$. How much of the wind affects the course of the plane, both in magnitude and direction?



$\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}$ is the scalar projection of the wind affecting the plane.

$(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$ is the vector projection of the wind affecting the plane.

$\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}$	Scalar projection of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$	(**or** scalar projection of $\underset{\sim}{a}$ **on** $\underset{\sim}{b}$).
$(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$	Vector projection of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$	(**or** vector projection of $\underset{\sim}{a}$ **on** $\underset{\sim}{b}$).
$\underset{\sim}{a} - (\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$	Vector projection of $\underset{\sim}{a}$ perpendicular to the direction of $\underset{\sim}{b}$.	

Consider the vectors $\underset{\sim}{a} = \underset{\sim}{i} + 5\underset{\sim}{j}$ and $\underset{\sim}{b} = 4\underset{\sim}{i} + 3\underset{\sim}{j}$. We want to consider that vector $\underset{\sim}{b}$ is representing the movement of a particle. When vector $\underset{\sim}{a}$ acts on vector $\underset{\sim}{b}$, it changes its direction and magnitude. Let's express this effect of vector $\underset{\sim}{a}$ on vector $\underset{\sim}{b}$.

Firstly, we need $\hat{\underset{\sim}{b}}$ as the vector being **acted on**.

$|\underset{\sim}{b}| = 5 \therefore \hat{\underset{\sim}{b}} = \frac{1}{5}(4\underset{\sim}{i} + 3\underset{\sim}{j})$

Now we split vector $\underset{\sim}{a}$ into 2 components, one acting **parallel to** $\underset{\sim}{b}$ and one acting **perpendicular** to $\underset{\sim}{b}$.

Scalar projection of $\underset{\sim}{a}$ on $\underset{\sim}{b} = \underset{\sim}{a} \cdot \hat{\underset{\sim}{b}} = \frac{1}{5}(4 + 15) = \frac{19}{5}$ (**a scalar**).

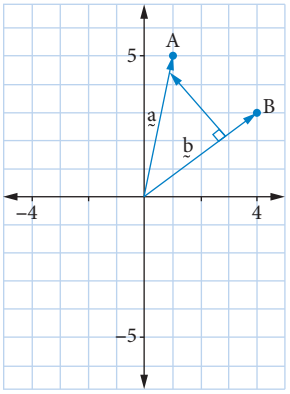
Vector projection of $\underset{\sim}{a}$ on $\underset{\sim}{b} = (\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}} = \frac{19}{5} \times \frac{1}{5}(4\underset{\sim}{i} + 3\underset{\sim}{j}) = \frac{19}{25}(4\underset{\sim}{i} + 3\underset{\sim}{j})$ (**a vector**).

Vector projection of $\underset{\sim}{a}$ perpendicular to the direction of $\underset{\sim}{b} = \underset{\sim}{a} - (\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}} = (\underset{\sim}{i} + 5\underset{\sim}{j}) - \frac{19}{25}(4\underset{\sim}{i} + 3\underset{\sim}{j})$.

$\therefore$ Vector projection of $\underset{\sim}{a}$ perpendicular to the direction of $\underset{\sim}{b} = \frac{1}{25}(-51\underset{\sim}{i} + 68\underset{\sim}{j})$.

To check whether vector $\underset{\sim}{a}$ has been split into 2 components, add them:

$\frac{19}{25}(4\underset{\sim}{i} + 3\underset{\sim}{j}) + \frac{1}{25}(-51\underset{\sim}{i} + 68\underset{\sim}{j}) = \underset{\sim}{i} + 5\underset{\sim}{j} = \underset{\sim}{a}$, as expected.



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WORKED EXAMPLE 15 Vector resolutes

Find the parallel and perpendicular components of $\underset{\sim}{a} = -\underset{\sim}{i} + 3\underset{\sim}{j}$ in the direction of $\underset{\sim}{b} = 2\underset{\sim}{i} + 2\underset{\sim}{j}$.

Steps	Working	
1 First find $\hat{\underset{\sim}{b}}$.	$\hat{\underset{\sim}{b}} = \frac{1}{\sqrt{8}}(2\underset{\sim}{i} + 2\underset{\sim}{j})$	It is useful to leave $\sqrt{8}$ as an unsimplified surd since it is simplified later.
2 Find the vector projection of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ using $(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$.	$(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}) = \frac{1}{\sqrt{8}}(-2 + 6) = \frac{4}{\sqrt{8}}$ $(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}} = \frac{4}{\sqrt{8}} \times \frac{1}{\sqrt{8}}(2\underset{\sim}{i} + 2\underset{\sim}{j})$ $= \frac{4}{8}(2\underset{\sim}{i} + 2\underset{\sim}{j}) = \underset{\sim}{i} + \underset{\sim}{j}$	vector projection of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$
3 Find the vector projection of $\underset{\sim}{a}$ perpendicular to the $\underset{\sim}{b}$ using $\underset{\sim}{a} - (\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$.	$\underset{\sim}{a} - (\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}} = (-\underset{\sim}{i} + 3\underset{\sim}{j}) - (\underset{\sim}{i} + \underset{\sim}{j})$ $= -2\underset{\sim}{i} + 2\underset{\sim}{j}$	vector projection of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$
4 Check the answer: the parallel and perpendicular components of $\underset{\sim}{a}$ should sum to $\underset{\sim}{a}$.	$(\underset{\sim}{i} + \underset{\sim}{j}) + (-2\underset{\sim}{i} + 2\underset{\sim}{j}) = -\underset{\sim}{i} + 3\underset{\sim}{j} = \underset{\sim}{a}$, as expected.	

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WORKED EXAMPLE 16 Vector projections

Find the vector projections of $\underset{\sim}{a} = 3\underset{\sim}{i} + 6\underset{\sim}{j} - 6\underset{\sim}{k}$ parallel and perpendicular to $\underset{\sim}{b} = 4\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$.

Steps	Working
1 Find $\hat{\underset{\sim}{b}} = \dfrac{\underset{\sim}{b}}{\lvert\underset{\sim}{b}\rvert}$.	$\lvert\underset{\sim}{b}\rvert = \sqrt{4^2 + 2^2 + (-2)^2}$ $= 2\sqrt{6}$ $\hat{\underset{\sim}{b}} = \dfrac{1}{2\sqrt{6}}(4\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k})$ $= \dfrac{1}{\sqrt{6}}(2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k})$
2 Calculate the vector projection of $\underset{\sim}{a}$ on $\underset{\sim}{b}$.	$(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}} = \left[(3\underset{\sim}{i} + 6\underset{\sim}{j} - 6\underset{\sim}{k}) \cdot \dfrac{1}{\sqrt{6}}(2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k})\right]\hat{\underset{\sim}{b}}$ $= \dfrac{18}{\sqrt{6}}\dfrac{1}{\sqrt{6}}(2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k})$ $= 6\underset{\sim}{i} + 3\underset{\sim}{j} - 3\underset{\sim}{k}$
3 Find the vector projection of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$.	$\underset{\sim}{a} - (6\underset{\sim}{i} + 3\underset{\sim}{j} - 3\underset{\sim}{k}) = -3\underset{\sim}{i} + 3\underset{\sim}{j} - 3\underset{\sim}{k}$

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EXERCISE 10.5 Projection of vectors

ANSWERS p. 606

Recap

1 The value of m such that the vector $\underset{\sim}{a} = \dfrac{1}{m}(-2\underset{\sim}{i} + 5\underset{\sim}{j})$ is a unit vector is

A 29 **B** $\sqrt{7}$ **C** $\dfrac{1}{\sqrt{29}}$ **D** $\sqrt{29}$ **E** $-\sqrt{29}$

2 The scalar product of the vectors $\underset{\sim}{a} = 3\underset{\sim}{i} - 4\sqrt{3}\underset{\sim}{j}$ and $\underset{\sim}{b} = \sqrt{3}\underset{\sim}{i} + 2\underset{\sim}{j}$ is

A $3\sqrt{3}$ **B** -1 **C** $-5\sqrt{3}$ **D** $5\sqrt{3}$ **E** $\sqrt{3}$

Mastery

3 WORKED EXAMPLE 15 TECH-FREE Find the scalar projection of $\underset{\sim}{a} = \underset{\sim}{i} + 3\underset{\sim}{j}$ on $\underset{\sim}{b} = 2\underset{\sim}{i} - 2\underset{\sim}{j}$.

4 TECH-FREE Find the vector projection of $\underset{\sim}{a} = \underset{\sim}{i} + 3\underset{\sim}{j}$ on $\underset{\sim}{b} = 2\underset{\sim}{i} - 2\underset{\sim}{j}$.

5 TECH-FREE Find the vector component of $\underset{\sim}{a} = \underset{\sim}{i} + 3\underset{\sim}{j}$ perpendicular to $\underset{\sim}{b} = 2\underset{\sim}{i} - 2\underset{\sim}{j}$.

6 For the vectors $\underset{\sim}{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\underset{\sim}{v} = \begin{bmatrix} 1 \\ -2 \\ -6 \end{bmatrix}$, the scalar projection of $\underset{\sim}{u}$ on $\underset{\sim}{v}$ is

A $\begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$ **B** $\begin{bmatrix} 0 \\ 4 \\ 9 \end{bmatrix}$ **C** -21 **D** $\dfrac{21}{\sqrt{41}}$ **E** $-\dfrac{21\sqrt{41}}{41}$

7 WORKED EXAMPLE 16 For the vectors $\underset{\sim}{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\underset{\sim}{v} = \begin{bmatrix} 1 \\ -2 \\ -6 \end{bmatrix}$, the vector projection of $\underset{\sim}{v}$ on $\underset{\sim}{u}$ is

A $\begin{bmatrix} -1.5 \\ -3 \\ -4.5 \end{bmatrix}$ **B** $\begin{bmatrix} 0 \\ 4 \\ 9 \end{bmatrix}$ **C** $-\dfrac{3}{2}$ **D** $\begin{bmatrix} -1.5 \\ 3 \\ 9 \end{bmatrix}$ **E** $-\dfrac{3\sqrt{14}}{2}$

Exam practice

80–100% 60–79% 0–59%

8 © VCAA 2020 1Q5 TECH-FREE (4 marks) Let $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} + m\underset{\sim}{j} - \underset{\sim}{k}$, where m is an integer.
The vector resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is $-\frac{11}{18}\left(\underset{\sim}{i} + m\underset{\sim}{j} - \underset{\sim}{k}\right)$.

a 66% Find the value of m. 3 marks

b 28% Find the component of $\underset{\sim}{a}$ that is perpendicular to $\underset{\sim}{b}$. 1 mark

9 TECH-FREE (3 marks) Consider the vectors $\underset{\sim}{a} = -\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} + c\underset{\sim}{j} + \underset{\sim}{k}$.
Find the value of c, $c \in R$, if the scalar resolute of $\underset{\sim}{b}$ on $\underset{\sim}{a}$ is $\sqrt{14}$.

10 © VCAA 2018N 1Q2 TECH-FREE (3 marks) Let $\underset{\sim}{a} = 3\underset{\sim}{i} - 2\underset{\sim}{j} + m\underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - \underset{\sim}{j} + 3\underset{\sim}{k}$, where $m \in R$.
Find the value(s) of m such that the magnitude of the vector resolute of $\underset{\sim}{a}$ parallel to $\underset{\sim}{b}$ is equal to $\sqrt{14}$.

11 © VCAA 2006 2AQ17 85% Let $\underset{\sim}{u} = \underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{v} = \underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k}$. The angle between the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ is

A 0° **B** 45° **C** 30° **D** 22.5° **E** 90°

12 © VCAA 2018 2AQ11 80% Consider the vectors given by $\underset{\sim}{a} = m\underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{b} = \underset{\sim}{i} + m\underset{\sim}{j}$, where $m \in R$.
If the acute angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is 30°, then m equals

A $\sqrt{2} \pm 1$ **B** $2 \pm \sqrt{3}$ **C** $\sqrt{3}, \frac{1}{\sqrt{3}}$ **D** $\frac{\sqrt{3}}{4 - \sqrt{3}}$ **E** $\frac{\sqrt{39}}{13}$

13 © VCAA 2014 2AQ16 77% Two vectors are given by $\underset{\sim}{a} = 4\underset{\sim}{i} + m\underset{\sim}{j} - 3\underset{\sim}{k}$ and $\underset{\sim}{b} = -2\underset{\sim}{i} + n\underset{\sim}{j} - \underset{\sim}{k}$, where $m, n \in R^+$. If $|\underset{\sim}{a}| = 10$ and $\underset{\sim}{a}$ is perpendicular to $\underset{\sim}{b}$, them m and n respectively are

A $5\sqrt{3}, \frac{\sqrt{3}}{3}$ **B** $5\sqrt{3}, \sqrt{3}$ **C** $-5\sqrt{3}, \sqrt{3}$ **D** $\sqrt{93}, \frac{5\sqrt{93}}{93}$ **E** 5, 1

14 © VCAA 2014 2AQ15 69% If θ is the angle between $\underset{\sim}{a} = \sqrt{3}\underset{\sim}{i} + 4\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} - 4\underset{\sim}{j} + \sqrt{3}\underset{\sim}{k}$, then $\cos(2\theta)$ is

A $-\frac{4}{5}$ **B** $\frac{7}{25}$ **C** $-\frac{7}{25}$ **D** $\frac{14}{25}$ **E** $-\frac{24}{25}$

15 © VCAA 2006 2AQ16 49% A unit vector perpendicular to $5\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$ is

A $\frac{1}{4}(5\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k})$ **B** $2\underset{\sim}{i} - 4\underset{\sim}{j} + 3\underset{\sim}{k}$ **C** $\frac{1}{29}(2\underset{\sim}{i} - 4\underset{\sim}{j} + 3\underset{\sim}{k})$

D $\frac{1}{\sqrt{29}}(2\underset{\sim}{i} - 4\underset{\sim}{j} + 3\underset{\sim}{k})$ **E** $\frac{1}{\sqrt{30}}(5\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k})$

16 © VCAA 2019N 2AQ12 Given that θ is the acute angle between $\underset{\sim}{a} = -2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{b} = -4\underset{\sim}{i} + 4\underset{\sim}{j} + 7\underset{\sim}{k}$, then $\sin(2\theta)$ is equal to

A $2\sqrt{2}$ **B** $\frac{4\sqrt{2}}{9}$ **C** $\frac{2\sqrt{2}}{9}$ **D** $\frac{2\sqrt{2}}{3}$ **E** $\frac{4\sqrt{2}}{3}$

17 © VCAA 2021N 2AQ13

The scalar resolute of $\underset{\sim}{a} = \underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ in the direction of $\underset{\sim}{b} = 4\underset{\sim}{i} - 4\underset{\sim}{j} + 7\underset{\sim}{k}$ is

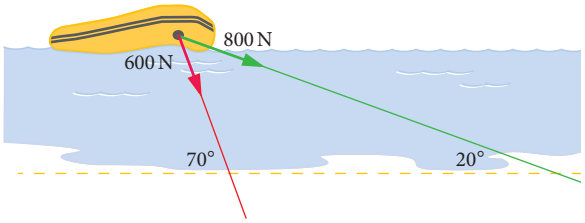
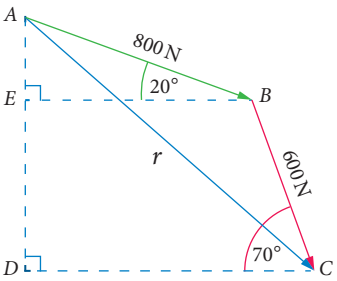
A $-\frac{3}{2}$ **B** $\frac{1}{3}$ **C** $\frac{1}{2}$ **D** $\frac{19}{9}$ **E** $\frac{19}{6}$

10.6 Applying vectors

Many real-life situations such as motion, **displacement**, velocity, resultant velocity, relative velocity, statics and motion under a constant force involve vector quantities.

WORKED EXAMPLE 17 Statics

Two ropes are tied to the prow of a rubber boat and are pulling the boat towards the beach. The first rope is being pulled by two people at an angle of 20° to the shoreline, with a force of 800 N (Newtons). The second rope is being pulled in the same general direction by a lifeguard, with a force of 600 N at an angle of 70° to the shoreline. What is the total force acting on the boat (to the nearest Newton) and in what direction is it acting (in degrees correct to one decimal place)?

Steps	Working
1 Draw a diagram to show the forces.	
2 Draw the forces head-to-tail, and complete the triangle with the resultant. Since $EB \parallel DC$, $\angle EBC = 180° - 70° = 110°$, so $\angle ABC = 20° + 110° = 130°$.	
3 Use the cosine rule on triangle ABC to find r. Keep more decimal places in your calculator while you're working.	$r^2 = AB^2 + BC^2 - 2 \times AB \times BC\cos(B)$ $= 800^2 + 600^2 - 2 \times 800 \times 600\cos(130°)$ $= 1\,617\,076.105\ldots$ Giving $r \approx 1271.643\ldots$ N
4 Use the sine rule on $\triangle ABC$ to find $\angle BCA$.	$\frac{\sin(C)}{AB} = \frac{\sin(B)}{r}$ $\frac{\sin(C)}{800} = \frac{\sin(130°)}{1271.643\ldots}$ $\sin(C) = \frac{800\sin(130°)}{1271.643\ldots} = 0.481\,924\ldots$ $\therefore C \approx 28.811\ldots°$
5 Calculate the angle of the resultant with the shoreline.	Resultant angle $\approx 70° - 28.811\ldots°$ $= 41.188\ldots°$
6 State the answer in an appropriate form.	The total force acting on the boat is about 1272 N, at an angle of 41.2° to the shoreline.

Video playlist
Applying vectors

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Worksheet
Vector applications

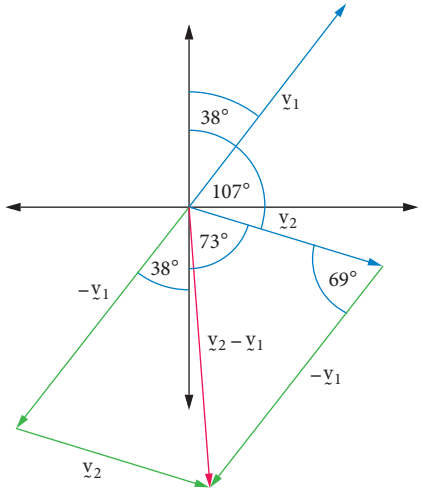
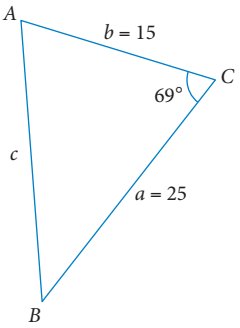
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WORKED EXAMPLE 18 Vectors and velocity

The velocity of a boat changes from 25 knots at 038° to 15 knots at a bearing of 107°.

What is the change in velocity? Give the magnitude and direction correct to one decimal place.

Steps	Working
1 Sketch a diagram. Let $\underset{\sim}{v}_1$ be the velocity 25 knots at 038°. Let $\underset{\sim}{v}_2$ be the velocity 15 knots at 107°. Label $-\underset{\sim}{v}_1$ on the diagram. Rearrange $-\underset{\sim}{v}_1$ and $\underset{\sim}{v}_2$ on the diagram so that the change of velocity can be found. Find the angle between the vectors. The obtuse angle in the parallelogram is 38° + 73° = 111°, so the acute angle is 180° − 111° = 69°.	 Change in velocity = $\underset{\sim}{v}_2 - \underset{\sim}{v}_1 = \underset{\sim}{v}_2 + (-\underset{\sim}{v}_1)$
2 Draw a diagram to find $\underset{\sim}{v}_2 - \underset{\sim}{v}_1$. Use the cosine rule to find a, the magnitude of $\underset{\sim}{v}_2 - \underset{\sim}{v}_1$. Keep more decimal places in your calculator while you're working.	 $c^2 = a^2 + b^2 - 2ab\cos(C)$ $= 25^2 + 15^2 - 2 \times 25 \times 15 \times \cos(69°)$ $c = 24.108\ldots$
3 Use the sine rule to find A, which will help us to find the direction of $\underset{\sim}{v}_2 - \underset{\sim}{v}_1$.	$\frac{\sin(A)}{25} = \frac{\sin(69°)}{24.108\ldots}$ $\sin(A) = \frac{25\sin(69°)}{24.108\ldots} = 0.9680\ldots$ $A \approx 75.5°$
4 Calculate the bearing.	bearing $\approx 107° + 75.5° = 182.5°$
5 Answer the question.	The change in velocity is about 24.1 knots at a bearing of 182.5°.

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EXERCISE 10.6 Applying vectors

ANSWERS p. 606

Recap

1 TECH-FREE Given $A(1,5)$, $B(-4,-3)$ and $C(5,1)$, find the angle between the given vector and the x-axis to the nearest degree.

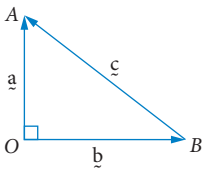
a $\overrightarrow{OA}$ **b** $\overrightarrow{OB}$ **c** $\overrightarrow{OC}$

2 For the vectors $\underset{\sim}{a} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, the vector projection $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is

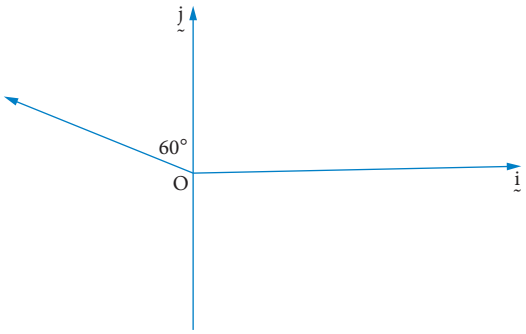
A $\begin{bmatrix} -\frac{7}{5} \\ \frac{14}{5} \end{bmatrix}$ **B** $\begin{bmatrix} \frac{7}{10} \\ \frac{21}{10} \end{bmatrix}$ **C** -7 **D** $\frac{-7}{\sqrt{5}}$ **E** $\frac{-7}{\sqrt{10}}$

Mastery

3 TECH-FREE Use the vectors in the diagram to prove Pythagoras' theorem in a right-angled triangle.



4 WORKED EXAMPLE 17 The diagram shows a straight shoreline with a town at O. The shoreline is at an angle of 60° to the north direction. Let $\underset{\sim}{i}$ and $\underset{\sim}{j}$ be unit vectors in the east and north directions respectively. Displacements are measured in metres.



At 9 am, a fishing boat is at $F(2,6)$ and a catamaran is at $C(8,-4)$.

a State $\overrightarrow{OF}$ and $\overrightarrow{OC}$ in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$ and hence find the scalar product $\overrightarrow{OF} \cdot \overrightarrow{OC}$.

b Find the angle, to the nearest degree between $\overrightarrow{OF}$ and $\overrightarrow{OC}$.

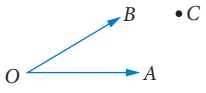
c Find $|\overrightarrow{OF}|$ and $|\overrightarrow{OC}|$.

d Hence, determine the distance between the fishing boat and catamaran at 9 am. Give your answer correct to the nearest tenth of a kilometre.

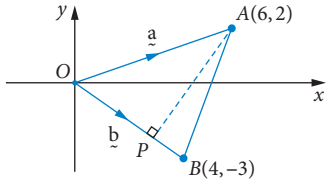
Exam practice

80–100% 60–79% 0–59%

5 © VCAA 2013S 1Q2 TECH-FREE (4 marks) $OACB$ is a parallelogram with $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$. Given that $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{OC}$, prove that $OACB$ is a rhombus.



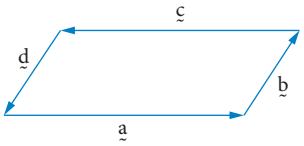
6 © VCAA 2004 1IIQ4 **57%** TECH-FREE (4 marks) Points $O(0, 0)$, $A(6, 2)$ and $B(4, -3)$ form the vertices of a triangle as shown in the diagram below. The position vectors $\overrightarrow{OA} = \underset{\sim}{a} = 6\underset{\sim}{i} + 2\underset{\sim}{j}$ and $\overrightarrow{OB} = \underset{\sim}{b} = 4\underset{\sim}{i} - 3\underset{\sim}{j}$ are indicated. AP is the altitude of triangle OAB.



a **73%** Find the scalar resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$. 2 marks

b **41%** **Hence** find the length of the altitude AP. 2 marks

7 © VCAA 2006 2AQ15 **74%** In the parallelogram shown, $|\underset{\sim}{a}| = 2|\underset{\sim}{b}|$.

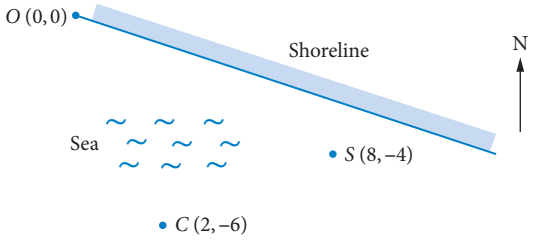


Which one of the following statements is true?

A $\underset{\sim}{a} = 2\underset{\sim}{b}$ **B** $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{c} + \underset{\sim}{d}$ **C** $\underset{\sim}{b} - \underset{\sim}{d} = \underset{\sim}{0}$

D $\underset{\sim}{a} + \underset{\sim}{c} = \underset{\sim}{0}$ **E** $\underset{\sim}{a} - \underset{\sim}{b} = \underset{\sim}{c} - \underset{\sim}{d}$

8 WORKED EXAMPLE 18 © VCAA 2002 2Q1abi (4 marks) The diagram shows a straight shoreline with a town at O. Let $\underset{\sim}{i}$ and $\underset{\sim}{j}$ be unit vectors in the east and north directions respectively. Displacements are measured in kilometres.



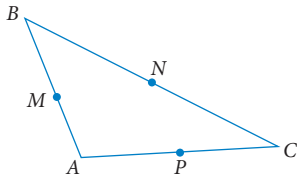
At 12 midday, a cargo ship is at $C(2, -6)$ and a sailing ship is at $S(8, -4)$.

a **77%** Find $\overrightarrow{CS}$ in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$ and hence determine the distance between the cargo ship and the sailing ship at 12 midday. Give your answer correct to the nearest tenth of a kilometre. 2 marks

Let $P(6m, -2m)$ denote a point on the shoreline, where $m > 0$.

b **60%** Express the scalar product $\overrightarrow{OP} \cdot \overrightarrow{PS}$ in terms of m. 2 marks

9 © VCAA 2013 2BQ4e **63%** (4 marks) For the triangle ABC shown below, the midpoints of the sides are the points M, N and P.



Let $\overrightarrow{AC} = \underset{\sim}{u}$ and $\overrightarrow{CB} = \underset{\sim}{v}$.

a **92%** Express $\overrightarrow{AN}$ in terms of $\underset{\sim}{u}$ and $\underset{\sim}{v}$. 1 mark

b **75%** Express $\overrightarrow{CM}$ and $\overrightarrow{BP}$ in terms of $\underset{\sim}{u}$ and $\underset{\sim}{v}$. 2 marks

c **10%** **Hence** simplify the expression $\overrightarrow{AN} + \overrightarrow{CM} + \overrightarrow{BP}$. 1 mark

10 (5 marks) A yacht leaves port and heads north-east with constant speed, v_1 km/h, for t_1 hours. It then changes direction and heads south-east with constant speed, v_2 km/h, for t_2 hours.

a State the displacement vector, $\underset{\sim}{d}_1$, of the yacht during its course north-east and the displacement vector, $\underset{\sim}{d}_2$, of the yacht during its course south-east. 2 marks

b Show that the distance of the yacht from port during the second leg of the journey is $\sqrt{(v_1 t_1)^2 + (v_2 t_2)^2}$. 2 marks

c Hence, given $v_1 = 12$ km/h and $v_2 = 5$ km/h, find the distance of the ship from port if each leg of the journey took 2 hours. 1 mark

10.7 Vector proofs of geometric results

We are required to use vectors to carry out proofs of simple geometric results. This could be as simple as using vectors to prove Pythagoras' theorem, or to show that the diagonals of a rhombus are **perpendicular**, the medians of a triangle are **concurrent**, the angle in a semicircle is a **right angle**, or that the **angle** subtended by a diameter in a circle is a **right angle**.

In the topic of vectors, keywords recognised here are

- perpendicular
- concurrent
- angle
- right angle.

From these keywords, we can see the importance of the dot product in these proofs.

Video playlist Vector proofs of geometric results

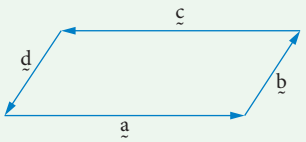
Worksheets Geometric proofs using vectors

Proofs using vectors

WB p. 280

WORKED EXAMPLE 19 Vector proof 1

© VCAA 2006 2AQ15 **74%** In the parallelogram shown, $|\underset{\sim}{a}| = 2|\underset{\sim}{b}|$.



Which one of the following statements is true?

A $\underset{\sim}{a} = 2\underset{\sim}{b}$ **B** $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{c} + \underset{\sim}{d}$ **C** $\underset{\sim}{b} - \underset{\sim}{d} = 0$

D $\underset{\sim}{a} + \underset{\sim}{c} = 0$ **E** $\underset{\sim}{a} - \underset{\sim}{b} = \underset{\sim}{c} - \underset{\sim}{d}$

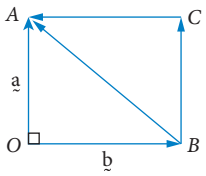
Steps	Working
1 Consider what is given.	$\lvert\underset{\sim}{a}\rvert = 2\lvert\underset{\sim}{b}\rvert$ Meaning also $\lvert\underset{\sim}{c}\rvert = 2\lvert\underset{\sim}{d}\rvert$.
2 Consider what we know.	$\underset{\sim}{a} + \underset{\sim}{b} + \underset{\sim}{c} + \underset{\sim}{d} = 0$ This is none of the options.
3 Consider what is clearly wrong.	Option **A**: $\underset{\sim}{a} = 2\underset{\sim}{b}$ confuses vector with only its magnitude. Option **B**: $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{c} + \underset{\sim}{d}$ confuses with Step 2.
4 Consider vector operations.	Parallel and opposite vectors. Option **D**: $\underset{\sim}{a} + \underset{\sim}{c} = 0$ is correct.

WB

p. 281

WORKED EXAMPLE 20 Vector proof 2

Use vectors to prove that the diagonals of a square intersect at right angles.

Steps	Working
1 Sketch a diagram and set up the proof.	
Exam hack A vector proof is often asked in an exam, not as a question on its own, but rather a small section as part of a broader question.	Let $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$.
2 State what needs to be proved.	To prove: $\overrightarrow{BA} \cdot \overrightarrow{OC} = 0$.
3 Express $\overrightarrow{BA}$ and $\overrightarrow{OC}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.	$\overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA} = \underset{\sim}{a} - \underset{\sim}{b}$ $\quad \overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC} = \underset{\sim}{a} + \underset{\sim}{b}$
4 Express $\overrightarrow{BA} \cdot \overrightarrow{OC}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.	$\overrightarrow{BA} \cdot \overrightarrow{OC} = (\underset{\sim}{a} - \underset{\sim}{b}) \cdot (\underset{\sim}{a} + \underset{\sim}{b})$ $= \underset{\sim}{a} \cdot \underset{\sim}{a} - \underset{\sim}{b} \cdot \underset{\sim}{b}$
5 Use $\underset{\sim}{a} \cdot \underset{\sim}{a} = \lvert\underset{\sim}{a}\rvert^2$.	$= \lvert\underset{\sim}{a}\rvert^2 - \lvert\underset{\sim}{b}\rvert^2$ $= 0$, since $OABC$ is a square and $\lvert\underset{\sim}{a}\rvert^2 = \lvert\underset{\sim}{b}\rvert^2$.
6 State the conclusion.	Hence, $\overrightarrow{BA} \cdot \overrightarrow{OC} = 0$.

Video
VCE question analysis: Vectors

VCE QUESTION ANALYSIS

© VCAA 2006 2BQ2 2006 Examination 2 Section B Question 2 **(10 marks)**

Point A has position vector $\underset{\sim}{a} = -\underset{\sim}{i} - 4\underset{\sim}{j}$, point B has position vector $\underset{\sim}{b} = 2\underset{\sim}{i} - 5\underset{\sim}{j}$, point C has position vector $\underset{\sim}{c} = 5\underset{\sim}{i} - 4\underset{\sim}{j}$, and point D has position vector $\underset{\sim}{d} = 2\underset{\sim}{i} + 5\underset{\sim}{j}$, relative to the origin O.

a Show that $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are perpendicular. 2 marks

b Use a vector method to find the cosine of $\angle ADC$, the angle between $\overrightarrow{DA}$ and $\overrightarrow{DC}$. 3 marks

c Find the cosine of $\angle ABC$, and **hence** show that $\angle ADC$ and $\angle ABC$ are supplementary. 2 marks

Point P has position vector $\underset{\sim}{p} = 2\underset{\sim}{i}$.

d Use the cosine of $\angle APC$ and an appropriate trigonometric formula to prove that $\angle APC = 2\angle ADC$. 3 marks

Reading the question

- The question is about vectors in a 4-sided shape with vertices $ABCD$.
- The question tests the idea of position vectors relative to the origin O.
- It is necessary to understand and use the two formulas for scalar (dot) product.
- The question asks, step-by-step, about perpendicular vectors and then angles between outgoing vectors.

Thinking about the question

- Understand that the vectors $\overrightarrow{AC}$ and $\overrightarrow{BD}$ require the use of vector addition $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$ with reference to the origin O.
- Remember that if the scalar product = 0, the vectors are perpendicular.
- Remember the scalar product formula $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos(\theta)$ and also the component scalar product formula.

Worked solution (✓ = 1 mark)

a Using $\underset{\sim}{a} = -\underset{\sim}{i} - 4\underset{\sim}{j}$, $\underset{\sim}{b} = 2\underset{\sim}{i} - 5\underset{\sim}{j}$, $\underset{\sim}{c} = 5\underset{\sim}{i} - 4\underset{\sim}{j}$, $\underset{\sim}{d} = 2\underset{\sim}{i} + 5\underset{\sim}{j}$.

Showing $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are perpendicular:

$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$ and $\overrightarrow{BD} = \overrightarrow{BO} + \overrightarrow{OD}$

$\therefore \overrightarrow{AC} = 6\underset{\sim}{i}$ and $\overrightarrow{BD} = 10\underset{\sim}{j}$ ✓

$\therefore \overrightarrow{AC} \cdot \overrightarrow{BD} = 0$

Therefore, $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are perpendicular. ✓

b Finding the cosine of $\angle ADC$, the angle between $\overrightarrow{DA}$ and $\overrightarrow{DC}$:

Use dot product $\overrightarrow{DA} \cdot \overrightarrow{DC} = |\overrightarrow{DA}| \cdot |\overrightarrow{DC}| \cos(\theta)$ ✓

where $\overrightarrow{DA} = -3\underset{\sim}{i} - 9\underset{\sim}{j}$ and $\overrightarrow{DC} = 3\underset{\sim}{i} - 9\underset{\sim}{j}$:

$\therefore \mathbf{72 = \sqrt{(-3)^2 + (-9)^2} \times \sqrt{3^2 + (-9)^2} \cos(\theta)}$ ✓

$\therefore 72 = 3\sqrt{10} \times 3\sqrt{10} \cos(\theta)$

$\therefore \cos \angle ADC = \cos(\theta)$

$$= \frac{72}{90}$$

$$= \frac{\mathbf{4}}{\mathbf{5}}$$ ✓

c Finding the cosine of $\angle ABC$, and hence showing that $\angle ADC$ and $\angle ABC$ are supplementary:

Use dot product $\overrightarrow{BA} \cdot \overrightarrow{BC} = |\overrightarrow{BA}| \cdot |\overrightarrow{BC}| \cos(\alpha)$, where $\overrightarrow{BA} = -3\underset{\sim}{i} + \underset{\sim}{j}$ and $\overrightarrow{BC} = 3\underset{\sim}{i} + \underset{\sim}{j}$.

$\therefore -8 = \sqrt{(-3)^2 + 1^2} \times \sqrt{3^2 + 1^2} \cos(\alpha)$

$= \sqrt{10} \times \sqrt{10} \cos(\alpha)$

$\therefore \cos(\alpha) = \cos \angle ABC = \dfrac{-8}{10} = -\dfrac{\mathbf{4}}{\mathbf{5}}$ ✓

We know that $\cos \angle ADC = \dfrac{4}{5}$.

$\cos \angle ABC = -\dfrac{4}{5}$ so $\angle ABC$ is obtuse.

Also $\cos \angle ABC = -\cos \angle ADC$

so $\angle ABC = 180° - \angle ADC$

$\angle ABC + \angle ADC = 180°$

Therefore, $\angle ADC$ and $\angle ABC$ are supplementary. ✓

d Point P has position vector $\underset{\sim}{p} = 2\underset{\sim}{i}$.

Finding $\cos(\angle APC)$ using $\overrightarrow{PA} \cdot \overrightarrow{PC} = |\overrightarrow{PA}| |\overrightarrow{PC}| \cos(\angle APC)$, where $\overrightarrow{PA} = -3\underset{\sim}{i} - 4\underset{\sim}{j}$ and $\overrightarrow{PC} = 3\underset{\sim}{i} - 4\underset{\sim}{j}$:

$\therefore \mathbf{7 = \sqrt{(-3)^2 + (-4)^2} \times \sqrt{3^2 + (-4)^2} \cos(\angle APC)}$ ✓

$= 5 \times 5 \cos(\angle APC)$

$= 25 \cos(\angle APC)$

$\therefore \mathbf{\cos(\angle APC) = \dfrac{7}{25}}$ ✓

$\cos(2\angle ADC) = 2\cos^2(\angle ADC) - 1 \quad (\cos(2x) = 2\cos^2(x) - 1)$

$$= 2 \times \left(\frac{4}{5}\right)^2 - 1$$

$$= \frac{7}{25}$$

$\therefore \mathbf{\angle APC = 2\angle ADC}$ ✓

Student performance

80–100% 60–79% 0–59%

a 77% When students were correct, they wrote $\overrightarrow{AC} \cdot \overrightarrow{BD} = 0$. A common error was for students to write $\overrightarrow{AC} = \underset{\sim}{a} - \underset{\sim}{c}$ instead of $\overrightarrow{AC} = \underset{\sim}{c} - \underset{\sim}{a}$. Also wrongly written was $\overrightarrow{AC} = \overrightarrow{OA} + \overrightarrow{OC}$ instead of $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$.

b 74% This question was reasonably well done except for those who went on to find the angle rather than leaving the answer as $\cos \angle ADC = \frac{4}{5}$.

c 35% Few students were able to prove supplementary angles, as well as the same issues as in part **b**.

d 33% This question gives the hint that 'an appropriate trigonometric formula' is used and the student needs to work out that the suitable formula is the cosine double angle formula. Only a minority of the students used the best double angle formula $\cos(2x) = 2\cos^2(x) - 1$.

EXERCISE 10.7 Vector proofs of geometric results

ANSWERS p. 606

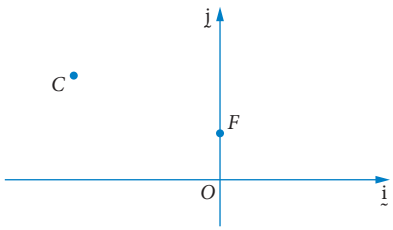
Recap

1 For the vectors $\underset{\sim}{a} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ and $\underset{\sim}{b} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$, the scalar projection of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is

A $\begin{bmatrix} \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{3} \\ -\frac{\sqrt{6}}{6} \end{bmatrix}$ **B** $\begin{bmatrix} -\frac{4}{3} \\ \frac{8}{3} \\ \frac{4}{3} \end{bmatrix}$ **C** $\begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$ **D** $\frac{-4\sqrt{6}}{3}$ **E** $\frac{-4}{\sqrt{6}}$

2 The diagram shows a town at O. The shoreline is along the x-axis due east. Let $\underset{\sim}{i}$ and $\underset{\sim}{j}$ be unit vectors in the east and north directions respectively. Displacements are measured in metres.

At 6 am, a fishing boat is at $F(0, 1)$ and a catamaran is at $C(-4, 2)$.

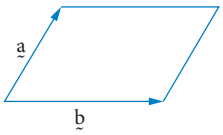


a State $\overrightarrow{OF}$ and $\overrightarrow{OC}$ in terms of $\underset{\sim}{i}$ and $\underset{\sim}{j}$ and hence find the scalar product $\overrightarrow{OF} \cdot \overrightarrow{OC}$.

b Find the angle between $\overrightarrow{OF}$ and $\overrightarrow{OC}$, correct to the nearest degree.

Mastery

3 WORKED EXAMPLE 19 © VCAA 2011 2AQ10 58% The diagram shows a rhombus, spanned by the two vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$. It follows that



A $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$ **B** $\underset{\sim}{a} = \underset{\sim}{b}$ **C** $(\underset{\sim}{a} + \underset{\sim}{b}) \cdot (\underset{\sim}{a} - \underset{\sim}{b}) = 0$

D $|\underset{\sim}{a} + \underset{\sim}{b}| = |\underset{\sim}{a} - \underset{\sim}{b}|$ **E** $2\underset{\sim}{a} + 2\underset{\sim}{b} = 0$

4 WORKED EXAMPLE 20 The median of a triangle is a vector from a vertex to the midpoint of the opposite side. Show that the sum of the medians of a triangle equal zero.

5 Show that the midpoint of the hypotenuse of a right-angled triangle is equidistant from the vertices of the triangle.

6 If $ABCD$ is a parallelogram and M is the midpoint of AC, prove that M is also the midpoint of BD.

Exam practice

80–100% 60–79% 0–59%

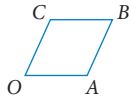
7 © VCAA 2013 1Q3 TECH-FREE (4 marks) The coordinates of three points are $A(-1, 2, 4)$, $B(1, 0, 5)$ and $C(3, 5, 2)$.

a 83% Find $\overrightarrow{AB}$. 1 mark

b 78% The points A, B and C are the vertices of a triangle. Prove that the triangle has a right angle at A. 2 marks

c 72% Find the length of the hypotenuse of the triangle. 1 mark

8 © VCAA 2015 1Q1 TECH-FREE (3 marks) Consider the rhombus $OABC$ shown, where $\overrightarrow{OA} = a\underset{\sim}{i}$ and $\overrightarrow{OC} = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$, and a is a positive real constant.



a 71% Find a. 1 mark

b 67% Show that the diagonals of the rhombus $OABC$ are perpendicular. 2 marks

9 © VCAA 2021N 2AQ11 Let $\underset{\sim}{a}$ and $\underset{\sim}{b}$ be arbitrary non-zero vectors.

Which one of the following statements is always true?

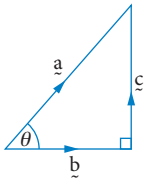
A $|\underset{\sim}{a} + \underset{\sim}{b}| \geq |\underset{\sim}{a} - \underset{\sim}{b}|$ **B** $|\underset{\sim}{a} - \underset{\sim}{b}| \leq |\underset{\sim}{a}| + |\underset{\sim}{b}|$ **C** $|\underset{\sim}{a} - \underset{\sim}{b}| < |\underset{\sim}{a}| - |\underset{\sim}{b}|$

D $|\underset{\sim}{a} - \underset{\sim}{b}| < |\underset{\sim}{a}| + |\underset{\sim}{b}|$ **E** $|\underset{\sim}{a} - \underset{\sim}{b}| > |\underset{\sim}{a}| - |\underset{\sim}{b}|$

10 © VCAA 2021N 2AQ13 The scalar resolute of $\underset{\sim}{a} = \underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$ in the direction of $\underset{\sim}{b} = 4\underset{\sim}{i} - 4\underset{\sim}{j} + 7\underset{\sim}{k}$ is

A $-\frac{3}{2}$ **B** $\frac{1}{3}$ **C** $\frac{1}{2}$ **D** $\frac{19}{9}$ **E** $\frac{19}{6}$

11 © VCAA 2004 1IQ19 50% The right-angled triangle shown has sides represented by the vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$.



Which one of the following statements is **false**?

A $|\underset{\sim}{b}|^2 + |\underset{\sim}{c}|^2 = |\underset{\sim}{a}|^2$ **B** $\underset{\sim}{b} \cdot (\underset{\sim}{a} - \underset{\sim}{c}) = |\underset{\sim}{b}|^2$

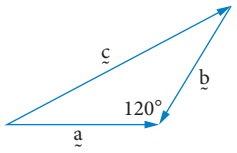
C $\underset{\sim}{b} \cdot (\underset{\sim}{a} - \underset{\sim}{b}) = |\underset{\sim}{b}||\underset{\sim}{c}|$ **D** $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos(\theta)$

E $\underset{\sim}{a} \cdot \underset{\sim}{c} = |\underset{\sim}{a}||\underset{\sim}{c}|\sin(\theta)$

12 © VCAA 2008 2AQ17 43% P, Q and R are three collinear points with position vectors $\underset{\sim}{p}$, $\underset{\sim}{q}$ and $\underset{\sim}{r}$ respectively, where Q lies between P and R. If $|\overrightarrow{QR}| = \frac{1}{2}|\overrightarrow{PQ}|$, then $\underset{\sim}{r}$ is equal to

A $\frac{3}{2}\underset{\sim}{q} - \frac{1}{2}\underset{\sim}{p}$ **B** $\frac{3}{2}\underset{\sim}{p} - \frac{1}{2}\underset{\sim}{q}$ **C** $\frac{3}{2}\underset{\sim}{q} - \frac{3}{2}\underset{\sim}{p}$ **D** $\frac{1}{2}\underset{\sim}{p} - \frac{3}{2}\underset{\sim}{q}$ **E** $\frac{3}{2}\underset{\sim}{p} - \frac{3}{2}\underset{\sim}{q}$

13 © VCAA 2009 2AQ17 30% Vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are shown on the right.



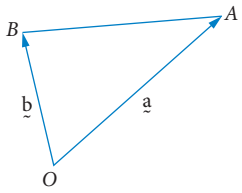
From the diagram it follows that

A $|\underset{\sim}{c}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2$ **B** $|\underset{\sim}{c}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - |\underset{\sim}{a}||\underset{\sim}{b}|$

C $|\underset{\sim}{c}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 + |\underset{\sim}{a} \cdot \underset{\sim}{b}|$ **D** $|\underset{\sim}{c}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 + |\underset{\sim}{a}||\underset{\sim}{b}|$

E $|\underset{\sim}{c}|^2 = |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 - |\underset{\sim}{a} \cdot \underset{\sim}{b}|$

14 © VCAA 2010 2BQ1 **30%** (11 marks) The diagram on the right shows a triangle with vertices O, A and B. Let O be the origin, with vectors $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.



a Find the following vectors in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

i **96%** $\overrightarrow{MA}$, where M is the midpoint of the line segment OA 1 mark

ii **94%** $\overrightarrow{BA}$ 1 mark

iii **80%** $\overrightarrow{AQ}$, where Q is the midpoint of the line segment AB. 1 mark

b **59%** Let N be the midpoint of the line segment OB. Use a vector method to prove that the quadrilateral $MNQA$ is a parallelogram. 3 marks

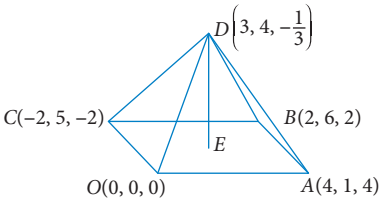
Now consider the **particular** triangle OAB with $\overrightarrow{OA} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + \sqrt{3}\underset{\sim}{k}$ and $\overrightarrow{OB} = \alpha\underset{\sim}{i}$, where α, which is greater than zero, is chosen so that triangle OAB is isosceles, with $|\overrightarrow{OB}| = |\overrightarrow{OA}|$.

c **87%** Show that $\alpha = 4$. 1 mark

d **i** **67%** Find $\overrightarrow{OQ}$, where Q is the midpoint of the line segment AB. 1 mark

ii **61%** Use a vector method to show that $\overrightarrow{OQ}$ is perpendicular to $\overrightarrow{AB}$. 3 marks

15 © VCAA 2003 2Q3 (13 marks) $OABCD$ is a pyramid. The height of the pyramid is the length of DE where E is the point on the base $OABC$ such that DE is perpendicular to the base.



a **48%** Show that the base $OABC$ is a rhombus. 3 marks

b **65%** Use a vector method to find $\angle AOC$ correct to the nearest tenth of a degree. 3 marks

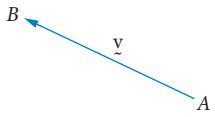
The unit vector $p\underset{\sim}{i} + q\underset{\sim}{j} + r\underset{\sim}{k}$, $p > 0$, is perpendicular to both $\overrightarrow{OA}$ and $\overrightarrow{OC}$.

c **i** **35%** Show that $q = 0$ and find the **exact** values of p and r. 4 marks

ii **3%** Hence find the **exact** height of the pyramid. 3 marks

10 Chapter summary

- A **vector** is a quantity with both **magnitude** (size) and **direction**. A quantity that has magnitude only is called a **scalar** (number).
- The vector shown can be written as $\overrightarrow{AB}$ or $\underset{\sim}{v}$.
- A plane vector can be written as an ordered pair (a, b) or as a column matrix $\begin{bmatrix} a \\ b \end{bmatrix}$.



- The **magnitude** of vector $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j}$ is equal to $\sqrt{x^2 = y^2}$.
- Its direction, θ, relative to the positive direction of the x-axis is given by $\tan(\theta) = \frac{y}{x}$.

Orthogonal unit vectors

- A **unit vector** has a magnitude of 1.
- The vector (a, b) can be represented in the form $a\underset{\sim}{i} + b\underset{\sim}{j} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$, where $\underset{\sim}{i}$ and $\underset{\sim}{j}$ are the standard orthogonal unit vectors.

Unit vectors

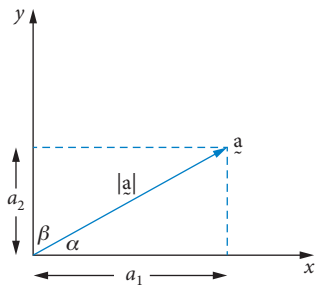
- The unit vector heading in the direction of $\underset{\sim}{a}$ is written as $\hat{\underset{\sim}{a}}$.

$$\hat{\underset{\sim}{a}} = \frac{\underset{\sim}{a}}{|\underset{\sim}{a}|} = \frac{x\underset{\sim}{i} + y\underset{\sim}{j}}{\sqrt{x^2 + y^2}}$$

- If α is the angle between $\underset{\sim}{a}$ and the x-axis, β is the angle between $\underset{\sim}{a}$ and the y-axis and $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j}$, then the **direction cosines** are

$$\cos(\alpha) = \frac{a_1}{|\underset{\sim}{a}|}$$

$$\cos(\beta) = \frac{a_2}{|\underset{\sim}{a}|}$$

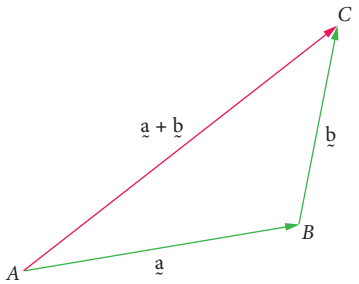


Scalar multiplication

If $\underset{\sim}{u} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$, then $2\underset{\sim}{u} = 2\begin{bmatrix} 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \\ -10 \end{bmatrix}$.

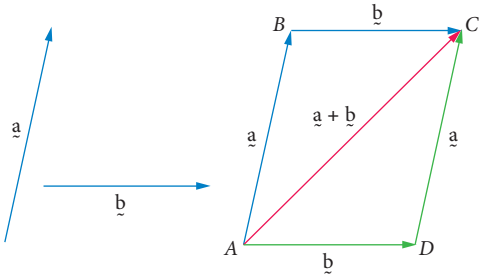
Triangle rule for adding vectors

- If 2 vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are placed head-to-tail to form 2 sides of a triangle, then $\underset{\sim}{a} + \underset{\sim}{b}$ is defined by the triangle of vectors.



Parallelogram rule for adding vectors

- Consider vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ as shown in the diagram. If we form the parallelogram, *ABCD*, using $\underset{\sim}{a}$ and $\underset{\sim}{b}$ as the sides, the diagonal *AC* is the resultant of $\underset{\sim}{a} + \underset{\sim}{b}$.



Subtracting vectors

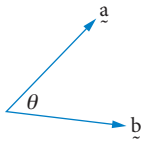
$\underset{\sim}{a} - \underset{\sim}{b} = \underset{\sim}{a} + (-\underset{\sim}{b})$

Equivalent vectors

For 2 vectors, if $m\underset{\sim}{a} + n\underset{\sim}{b} = p\underset{\sim}{a} + p\underset{\sim}{b}$ then $m = p$ and $n = q$, so we can equate the coefficients of the vectors.

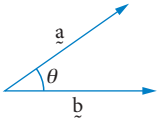
Scalar product

- The **scalar product** (or **dot product**) of 2 vectors is a **scalar**, a real number, not a vector.
- $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos(\theta)$, where θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$.
- If $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are expressed in component form: $\underset{\sim}{a} = a_1\underset{\sim}{i} + a_2\underset{\sim}{j}$ and $\underset{\sim}{b} = b_1\underset{\sim}{i} + b_2\underset{\sim}{j}$, then $\underset{\sim}{a} \cdot \underset{\sim}{b} = a_1 b_1 + a_2 b_2$.



Angle between 2 vectors

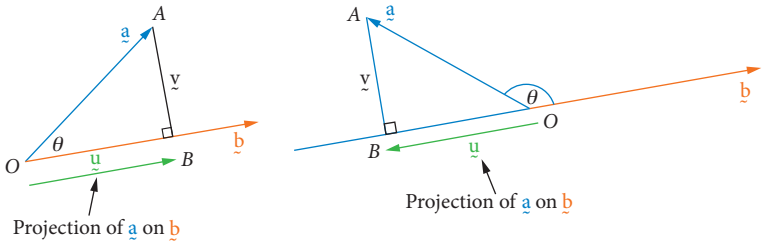
$$\cos(\theta) = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$



- If $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$, then vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are perpendicular.
- If $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|$, then vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel in the same direction.
- If $\underset{\sim}{a} \cdot \underset{\sim}{b} = -|\underset{\sim}{a}||\underset{\sim}{b}|$, then vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel in the opposite direction.

Projection of vectors

- Vector $\underset{\sim}{v}$ is the **vector projection** or **vector resolute** of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$.
- The **scalar projection** or **scalar resolute** of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ is the **magnitude** of the **vector projection** of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$.



- The scalar projection or scalar resolute of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ is $\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}$.
- The **vector projection** or **vector resolute** of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ is $(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$.
- The vector projection of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$ is $\underset{\sim}{a} - (\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$.

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 (3 marks) Given $z_1 = 2 + 3i$ and $z_2 = 3 - 2i$, simplify the following.

a $z_1 + z_2$ 1 mark

b $z_1 - 2z_2$ 1 mark

c $|z_1 z_2|$ 1 mark

2 (5 marks) $\underset{\sim}{a} = \underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{b} = \underset{\sim}{i} - \underset{\sim}{j}$.

a Find $\underset{\sim}{a} + \underset{\sim}{b}$. 1 mark

b Find $\underset{\sim}{a} \cdot \underset{\sim}{b}$. 1 mark

c Find the scalar projection of $\underset{\sim}{a}$ on $\underset{\sim}{b}$. 1 mark

d Find the vector projection of $\underset{\sim}{a}$ on $\underset{\sim}{b}$. 1 mark

e Find the vector projection of $\underset{\sim}{b}$ on $\underset{\sim}{a}$. 1 mark

3 (2 marks) Sketch the graph of $y = \frac{1}{4}\sec(x)$ for $x \in \left(-\frac{3\pi}{2}, \pi\right)$.

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 6 multiple-choice questions

6 marks

1 If $z_1 = 2 + 3i$ and $z_2 = 3 + 2i$, $\overline{z_1 z_2}$ equals

A $5 + 5i$ **B** $-1 + i$ **C** $12 + 5i$ **D** $13i$ **E** $-13i$

2 © VCAA 2017 2AQ5 On an Argand diagram, a point that lies on the path defined by $|z - 2 + i| = |z - 4|$ is

A $\left(3, -\frac{1}{2}\right)$ **B** $\left(-3, -\frac{1}{2}\right)$ **C** $\left(-3, \frac{3}{2}\right)$ **D** $\left(3, \frac{1}{2}\right)$ **E** $\left(3, -\frac{3}{2}\right)$

3 A unit vector perpendicular to $3\underset{\sim}{i} + 5\underset{\sim}{j} - 4\underset{\sim}{k}$ is

A $2\underset{\sim}{i} - 6\underset{\sim}{j} - 6\underset{\sim}{k}$ **B** $\frac{1}{2}\left(\underset{\sim}{i} - 3\underset{\sim}{j} - 3\underset{\sim}{k}\right)$ **C** $\frac{1}{36}\left(4\underset{\sim}{i} - 4\underset{\sim}{j} - 2\underset{\sim}{k}\right)$

D $\frac{1}{3}\left(2\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}\right)$ **E** $\frac{1}{\sqrt{76}}\left(\underset{\sim}{i} - 3\underset{\sim}{j} - 3\underset{\sim}{k}\right)$

4 Let $\underset{\sim}{u} = 4\underset{\sim}{i} - 3\underset{\sim}{j} + 5\underset{\sim}{k}$ and $\underset{\sim}{v} = \underset{\sim}{i} - 7\underset{\sim}{j} - 7\underset{\sim}{k}$.

$|\underset{\sim}{u} - \underset{\sim}{v}| =$

A $\sqrt{97}$ **B** $\sqrt{129}$ **C** 13 **D** $\sqrt{253}$ **E** 169

5 $\cos\left(\frac{\pi}{3}\right)$ is equivalent to

A $2\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right)$ **B** $\sin^2\left(\frac{\pi}{6}\right) - \cos^2\left(\frac{\pi}{6}\right)$ **C** $\cos^2\left(\frac{\pi}{6}\right) - \sin^2\left(\frac{\pi}{6}\right)$

D $1 - 2\cos^2\left(\frac{\pi}{6}\right)$ **E** $2\tan^2\left(\frac{\pi}{6}\right)$

6 Let $\underset{\sim}{a} = 7\underset{\sim}{i} - 24\underset{\sim}{j}$ and $\underset{\sim}{b} = \sqrt{3}\underset{\sim}{i} + \underset{\sim}{j}$. The scalar resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$ is

A $\frac{7\sqrt{3} - 24}{2}$ **B** $7\sqrt{3}$ **C** $7\sqrt{3} - 24$

D $\frac{7\sqrt{3} - 24}{25}$ **E** $\frac{7\sqrt{3} - 24}{4}$

Section B 2 questions

14 marks

1 (8 marks) Two ships, A and B, are observed from a lighthouse at origin O. Relative to O, their position vectors at time t hours after midday are given by

$$\underset{\sim}{r}_A = 5(1-t)\underset{\sim}{i} + 3(1+t)\underset{\sim}{j}$$
$$\underset{\sim}{r}_B = 4(t-2)\underset{\sim}{i} + (5t-2)\underset{\sim}{j}$$

where displacements are measured in kilometres.

a Find the position of ship A at $t = 0$. 1 mark

b Find the position of ship B at $t = 0$. 1 mark

c Find the scalar product of $\underset{\sim}{r}_A$ and $\underset{\sim}{r}_B$, in terms of t. 1 mark

d Hence, find the t value(s) in hours after midday, correct to one decimal place, for which $\underset{\sim}{r}_A$ is perpendicular to $\underset{\sim}{r}_B$. 1 mark

The distance between two vectors is given by $|\underset{\sim}{r}_A - \underset{\sim}{r}_B|$.

e Find, in terms of t, the distance between vectors $\underset{\sim}{r}_A$ and $\underset{\sim}{r}_B$. 2 marks

f Hence, find the time, in hours, correct to two decimal places, when the distance between ships A and B is 20 kilometres. 2 marks

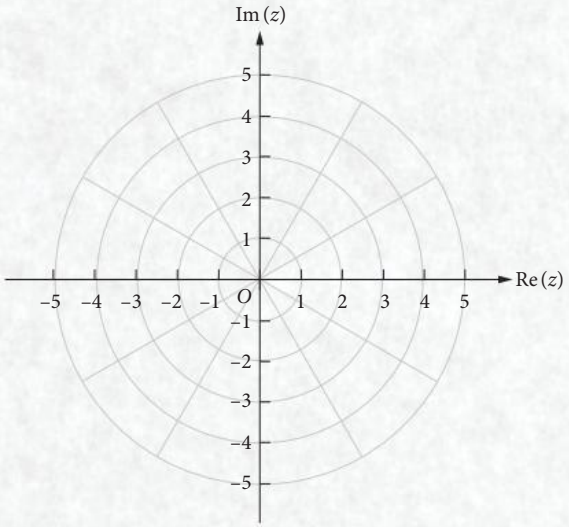
2 © VCAA 2017 2BQ4abde (6 marks)

a Express $-2 - 2\sqrt{3}\,i$ in polar form. 1 mark

b Show that the roots of $z^2 + 4z + 16 = 0$ are $z = -2 - 2\sqrt{3}\,i$ and $z = -2 + 2\sqrt{3}\,i$. 1 mark

c Show that the Cartesian form of the relation $|z| = |z - (2 - 2\sqrt{3}\,i)|$ is $x - \sqrt{3}\,y - 4 = 0$. 2 marks

d Copy the Argand diagram below and on it sketch the line represented by $x - \sqrt{3}y - 4 = 0$ and plot the roots of $z^2 + 4z + 16 = 0$. 2 marks



CHAPTER

11 SAMPLING DISTRIBUTIONS

Study Design coverage
Nelson MindTap chapter resources

11.1 Discrete random probability distributions
- Discrete random probability distributions
- The mean of a discrete random variable
- The variance and standard deviation
- **Using CAS 1:** Expected value, variance and standard deviation

11.2 Linear combinations of random variables
- The distribution of $2X$
- The sum of 2 identically distributed independent random variables

11.3 Populations and samples
- Populations and samples
- Choosing samples
- Random samples
- **Using CAS 2:** Generating random numbers
- Population parameters and sample statistics

11.4 Simulation
- Random experiments, events and event spaces
- Simulation methods

11.5 The distribution of the sample means
- The sampling distribution of the sample means
- Normal distributions
- The sampling distribution of the sample means by simulation
- **Using CAS 3:** The distribution of sample means by simulation
- Larger samples and the sampling distribution of the sample means
- The standard error for sample means

VCE question analysis
Chapter summary
Cumulative examination 1
Cumulative examination 2

9780170448338

Study Design coverage

UNIT 2, AREA OF STUDY 1: DATA ANALYSIS, PROBABILITY AND STATISTICS

Simulation, sampling and sampling distributions

Distribution of sums of discrete random variables

- the mean, variance and standard deviation of a discrete random variable X
- the distribution of the sum of n identically distributed independent discrete random variables
- comparison of the distribution of $2X$, where X is a discrete random variable and the sum of 2 independent discrete random variables that are each identically distributed as X.

Simulation

- random experiments, events and event spaces
- use of simulation to generate a random sample.

Sampling distributions

- the distinction between a population parameter and a sample statistic and the use of a sample statistic (sample mean $\overline{x}$) as an estimate of the associated population parameter (mean μ)
- the concept of a sampling distribution
- the distribution of sample means considered empirically including comparing the distributions of different size samples from the same population in terms of centre and spread
- display of variation in sample means through dot plots and other displays and considering the centre and spread of these distributions
- consideration of the mean and standard deviation of the distribution of sample means and the effect of taking larger samples.

VCE Mathematics Study Design 2023–2027 p. 65, © VCAA 2022

Video playlists (6):

11.1 Discrete random probability distributions
11.2 Linear combinations of random variables
11.3 Populations and samples
11.4 Simulation
11.5 The distribution of the sample means
VCE question analysis Sampling distributions

Worksheets (3):

11.3 Sampling methods • Sample proportions • Sample proportion calculations

Nelson MindTap

To access resources above, visit **cengage.com.au/nelsonmindtap**

Video playlist
Discrete random probability distributions

11.1 Discrete random probability distributions

A **random variable** is a variable whose possible values are the numerical result of some random process.

A **discrete random variable** can only take a countable number of values; for example, the number of pets in a household.

An uppercase letter is used to describe the variable and a lowercase letter is used to represent the values the variable can take.

If a coin is tossed 3 times and the number of heads is recorded, then the discrete random variable X represents the number of heads in 3 tosses.

The list of possible outcomes is HHH, HHT, HTH, HTT, THH, THT, TTH and TTT.

The smallest number of heads is 0 and the largest number of heads is 3. This is represented by x, where $x \in \{0, 1, 2, 3\}$.

Discrete random probability distributions

The probability distribution of a discrete random variable shows all the possible values of the variable and the probabilities associated with those values. This is normally shown as a table.

Let $p(x)$ or $\Pr(X = x)$ be the probability of obtaining x heads in 3 tosses of a coin.

The probability distribution of X is

x	0	1	2	3
$\Pr(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Properties of a probability distribution

- All probabilities must be between 0 and 1 inclusive: $0 \leq p(x) \leq 1$.
- The sum of all probabilities must equal 1.

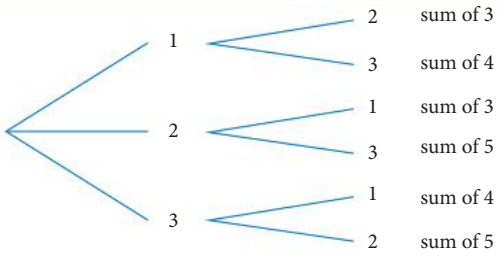
WB
p. 282

WORKED EXAMPLE 1 Finding the probability distribution of a discrete random variable

A bag contains 3 discs numbered {1, 2, 3}. Two discs are taken from the bag, without replacement. If X represents the sum of the 2 numbers obtained, list the probability distribution of X.

Steps | **Working**

1 Draw a tree diagram to illustrate the problem.



2 Draw a probability distribution table.
X = the sum of the numbers and $x \in \{3, 4, 5\}$.

x	3	4	5
$\Pr(X = x)$			

3 Use the tree diagram to find the probabilities.

$\Pr(\text{sum of } 3) = \frac{2}{6} = \frac{1}{3}$

$\Pr(\text{sum of } 4) = \frac{2}{6} = \frac{1}{3}$

$\Pr(\text{sum of } 5) = \frac{2}{6} = \frac{1}{3}$

4 Write the probabilities in the table.

The probability distribution of X is

x	3	4	5
$\Pr(X = x)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

The mean of a discrete random variable

The mean of a probability distribution is called the **expected value**. For a discrete random variable X, the expected value is written in mathematical notation as $E(X)$ or μ.

The expected value is found by summing the product of x and $\Pr(X = x)$ for all possible values of x. The mathematical notation for this sum is $\Sigma x \cdot p(x)$.

Remember that $p(x)$ is a short way of writing $\Pr(X = x)$.

Expected value of a discrete probability distribution

$$E(X) = \Sigma x \cdot p(x)$$

WORKED EXAMPLE 2 Finding the expected value of a discrete probability distribution

The probability distribution of a discrete random variable, X, is shown below.

x	0	1	2	3
$\Pr(X = x)$	0.25	0.3	0.2	0.25

Find the mean of X.

Steps

Rewrite the table with rows as columns and add a column headed $x \times p(x)$. Calculate the product of the x values and their probabilities.

The sum of the $x \times p(x)$ column is the expected value of X or $E(X)$.

Working

x	$p(x)$	$x \times p(x)$
0	0.25	0
1	0.3	0.3
2	0.2	0.4
3	0.25	0.75
Total		1.45

$E(X) = \Sigma x \cdot p(x)$

The mean or $E(X) = 1.45$.

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The variance and standard deviation

The most useful **measures of spread** for a probability distribution are the **variance** and the **standard deviation**.

The variance and standard deviation of a discrete random variable, X, measure the spread of the variable about its mean.

A variance of 0 indicates all the values in the distribution are identical. A small variance indicates the values in the distribution are very close to the mean and each other, while a large variance indicates the values in the distribution are spread out from the mean and each other.

$\text{Var}(X) = \Sigma(x - \mu)^2 \cdot p(x)$

$\text{SD}(X) = \sqrt{\text{Var}(X)}$

The variance formula

A simpler formula for the **variance** is

$$\text{Var}(X) = \text{E}(X^2) - \mu^2$$

where $\text{E}(X^2) = \Sigma x^2 \cdot p(x)$ and $\mu = \text{E}(X)$.

The **standard deviation** $\text{SD}(X) = \sqrt{\text{Var}(X)}$

The symbols for the variance and standard deviation are $\sigma^2 = \text{Var}(X)$ and $\sigma = \text{SD}(X)$.

Exam hack

The variance formula can be given as

$$\text{Var}(X) = \sigma^2 = \text{E}(x - \mu)^2$$

Always use the computational formula $\text{Var}(X) = \text{E}(X^2) - \mu^2$ to calculate the variance of a discrete distribution.

WB

p. 284

WORKED EXAMPLE 3 Finding the variance and standard deviation of a discrete probability distribution

For the probability distribution below, find

a the expected value

b the variance

c the standard deviation, correct to 3 decimal places.

x	0	1	2	3	4
$p(x)$	0.4	0.1	0.1	0.2	0.2

Steps

a Add 2 extra columns headed $x \times p(x)$ and $x^2 \times p(x)$ for calculating $\text{E}(X)$ and $\text{E}(X^2)$ respectively.

Multiply x by $p(x)$, enter the results in the $x \times p(x)$ column and find the total.

Multiply x by $x \times p(x)$, enter the results in the $x^2 \times p(x)$ column and find the total.

The total of the $x \times p(x)$ column is $\text{E}(X)$.

Working

x	$p(x)$	$x \times p(x)$	$x^2 \times p(x)$
0	0.4	0	0
1	0.1	0.1	0.1
2	0.1	0.2	0.4
3	0.2	0.6	1.8
4	0.2	0.8	3.2
Total	1.0	$\text{E}(X) = 1.7$	$\text{E}(X^2) = 5.5$

$\text{E}(X) = \Sigma x \cdot p(x) = 1.7$

b The total of the $x^2 \times p(x)$ column is $\text{E}(X^2)$.

Use the computational formula to find $\text{Var}(X)$.

$\text{E}(X^2) = 5.5$

$\text{Var}(X) = \text{E}(X^2) - \mu^2$

$\text{Var}(X) = 5.5 - 1.7^2$

$= 2.61$

c Find the standard deviation using the formula $\text{SD}(X) = \sqrt{\text{Var}(X)}$.

$\text{SD}(X) = \sqrt{2.61} \approx 1.616$

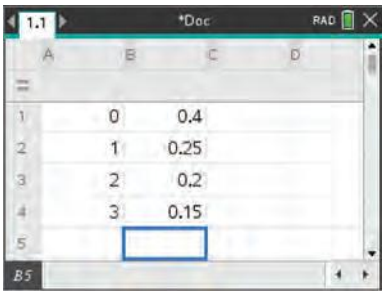
CAS can be used to find the expected value, the variance and the standard deviation of a probability distribution.

USING CAS 1 Expected value, variance and standard deviation

Using the results from the following table, find the mean, variance and standard deviation of X.

x	0	1	2	3
$\Pr(X = x)$	0.4	0.25	0.2	0.15

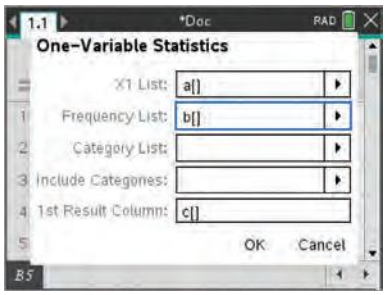
TI-Nspire



1 Add a **List & Spreadsheet** page.

2 Enter the data from the table in columns **A** and **B**, as shown above.

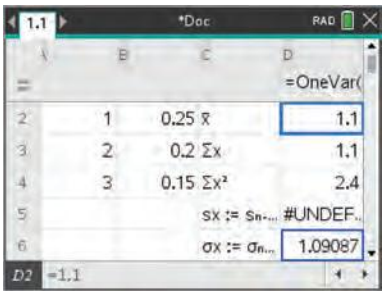
3 Press **menu > Statistics > Stat Calculations > One-Variable Statistics**.



4 On the next screen, keep the **Num of Lists:** default setting of **1**.

5 On the next screen, in the **X1 List:** field, enter **a[ ]**.

6 In the **Frequency List:** field, enter **b[ ]**.

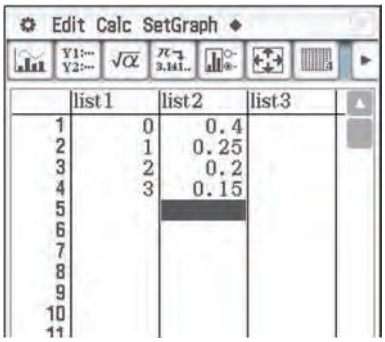


7 The labels and corresponding values will be displayed in columns **C** and **D**.

8 The mean and standard deviation values are highlighted in **blue**.

9 To calculate the variation, square the standard deviation.

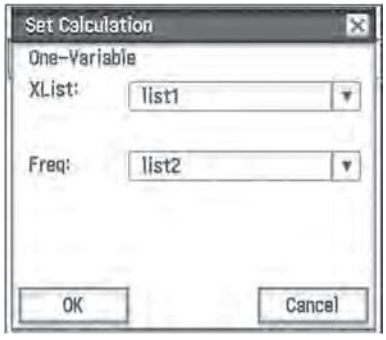
ClassPad



1 Tap **Menu > Statistics**.

2 Clear all lists.

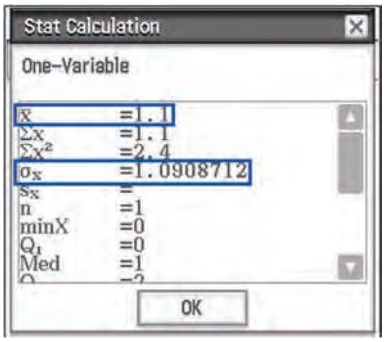
3 Enter the values from the table into **list1** and **list2**, as shown above.



4 Tap **Calc > One-Variable**.

5 Keep the **XList:** field set to **list1**.

6 Change the **Freq:** field to **list2**.



7 The labels and corresponding values will be displayed.

8 The mean and standard deviation values are highlighted in **blue**.

9 To calculate the variation, square the standard deviation.

Mean = 1.1, standard deviation = 1.091, variance = $2.4 - 1.1^2 = 1.19$

EXERCISE 11.1 Discrete random probability distributions

ANSWERS p. 607

Mastery

1 WORKED EXAMPLE 1 A bag contains 4 discs numbered {1, 2, 3, 4}. Two discs are taken from the bag, without replacement. If X represents the sum of the 2 numbers obtained, list the probability distribution of X.

2 WORKED EXAMPLE 2 The probability distribution of a discrete random variable X is shown below. Find the mean of X.

x	1	2	3	4
$\Pr(X = x)$	0.2	0.35	0.18	0.27

3 WORKED EXAMPLE 3 For the probability distribution below, find

a the expected value

b the variance, correct to 3 decimal places

c the standard deviation, correct to 3 decimal places.

x	0	1	2	3	4
$p(x)$	0.33	0.15	0.07	0.25	0.2

4 Using CAS 1 Using the results in the table, find the mean, variance and standard deviation of X. Express answers correct to 3 decimal places.

x	5	10	15	20	25	30
$\Pr(X = x)$	0.18	0.21	0.2	0.1	0.09	0.22

Exam practice

5 TECH-FREE (2 marks) A disc is removed at random from a bag that contains 20 numbered discs. The numbers on the discs are shown in the set {1, 1, 1, 2, 3, 3, 3, 3, 3, 4, 4, 5, 5, 5, 5, 5, 5, 6, 6, 6}. The random variable N represents the number chosen. List the probability distribution of N.

6 TECH-FREE (2 marks) A discrete random variable X represents the number of girls in a family of 3 children. Assume that the probability of a family having a boy child is equal to the probability of a girl child, and the gender of the second or third child is independent of the gender of the previous child. List the probability distribution of X.

7 TECH-FREE (2 marks) Find the expected value of the random variable X.

x	1	2	3	4
$p(x)$	0.1	0.4	0.1	0.4

8 TECH-FREE (3 marks) The probability distribution of a discrete random variable X is shown below.

x	−1	0	1	2
$\Pr(X = x)$	0.2	0.3	0.4	0.1

a Find the mean of X. 1 mark

b Find the variance of X. 2 marks

9 (4 marks) Over a 3-day long weekend, the probability of rain on any day is 0.3. The random variable Y represents the number of days on which rain occurs. Assume the probability of rain on any day over the long weekend is independent. Find the probability distribution of Y.

10 (10 marks) Yulia and Victor play 3 games of the hand game *Rock-Paper-Scissors*. If both players choose the same hand sign, the game is drawn. Assume that, at each play, they make their choices with equal probability $\left(\frac{1}{3}\right)$ for each of the 3 moves, independently of any previous play.

a Find the probability that a game is drawn. 2 marks

b If X represents the number of drawn games in the 3 games played, list the probability distribution of X 4 marks

c Find

i the expected value of X 2 marks

ii the variance of X. 2 marks

11 (6 marks) The probability distribution of the discrete random variable X is defined as

$$\Pr(X = x) = \frac{x(x+1)}{40}, x \in \{1, 2, 3, 4\}$$

a List the probability distribution of X. 2 marks

b Find

i the mean of X 2 marks

ii the variance of X. 2 marks

12 (7 marks) The probability distribution of the discrete random variable X is defined as

$$\Pr(X = x) = \frac{x^2 + 4x + 3}{85}, x \in \{0, 1, 2, 3, 4\}$$

a List the probability distribution of X. 2 marks

b Find, correct to 2 decimal places

i the mean of X 2 marks

ii the variance of X 2 marks

iii the standard deviation of X. 1 mark

Video playlist
Linear combinations of random variables

11.2 Linear combinations of random variables

The distribution of $2X$

At the school fete, Hien plays a skill game called 'the ball and bucket toss'. The aim of the game is to toss 3 balls into a bucket from a distance, one at a time. Hien finds that his probability of getting a ball in a bucket is 0.5 and the results from his throws are independent. The discrete random variable, N, represents the number of balls that Hien successfully throws into a bucket.



Alamy Stock Photo/allanbellimages

The probability distribution of N is

n	0	1	2	3
$\Pr(N = n)$	0.125	0.375	0.375	0.125

mean = 1.5, variance = 0.75

The prize (P) for tossing a ball into a bucket is \$2 for every successful toss. The random variable $P = 2N$ represents the total prize from 3 tosses.

The probability distribution P is

p	0	2	4	6
$\Pr(P = p)$	0.125	0.375	0.375	0.125

mean = 3, variance = 3

The connection between the mean and variance of N and P is shown below.

	N	P
Mean	$\mathrm{E}(N) = \frac{3}{2}$	$\mathrm{E}(P) = 2 \times \frac{3}{2} = 3$
Variance	$\mathrm{Var}(N) = \frac{3}{4}$	$\mathrm{Var}(P) = 2^2 \times \frac{3}{4} = 3$

The mean and variance of $2X$

$$\mathrm{E}(2X) = 2\,\mathrm{E}(X)$$
$$\mathrm{Var}(2X) = 4\,\mathrm{Var}(X)$$

9780170448338

WORKED EXAMPLE 4 Finding the mean and variance of $2X$ from a probability distribution

The probability distribution of a discrete random variable X is shown.

x	10	20	30	40
$p(x)$	0.1	0.3	0.5	0.1

a List the probability distribution of N where $N = 2X$.

b Hence find

i $E(N)$

ii $\text{Var}(N)$

Steps	Working
a Complete a probability distribution table. Calculate the values of n by multiplying the x values by 2. The probabilities in the table are unchanged.	

n	20	40	60	80
$p(n)$	0.1	0.3	0.5	0.1

b i Add 2 extra columns headed $n \times p(n)$ and $n^2 \times p(n)$. Complete the table calculations.

Find $E(N)$ by calculating the sum of the $n \times p(n)$ column.

n	$p(n)$	$n \times p(n)$	$n^2 \times p(n)$
20	0.1	2	40
40	0.3	12	480
60	0.5	30	1800
80	0.1	8	640
Total	1.0	$E(N) = 52$	$E(N^2) = 2960$

$E(N) = \Sigma n \cdot p(n) = 52$

ii Find $E(N^2)$ by finding the sum of the $n^2 \times p(n)$ column.

Use the computational formula $\text{Var}(X) = E(X^2) - (E(X))^2$ to find $\text{Var}(N)$.

$E(N^2) = \Sigma n^2 \cdot p(n) = 2960$

$\text{Var}(N) = E(N^2) - (E(N))^2$

$\text{Var}(N) = 2960 - 52^2$

$= 256$

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WORKED EXAMPLE 5 Finding the mean and variance of $2X$ given the mean and variance of X

A discrete random variable Y has a mean of 4.5 and a variance of 4.

a Find the mean of $2Y$.

b Find the variance of $2Y$.

c Find the standard deviation of $2Y$.

Steps	Working
a Substitute $E(Y) = 4.5$ into $E(2Y) = 2 \times E(Y)$.	The mean of $2Y = E(2Y)$. $E(2Y) = 2 \times 4.5 = 9$
b Substitute $\text{Var}(Y) = 4$ into $\text{Var}(2Y) = 4 \times \text{Var}(Y)$.	$\text{Var}(2Y) = 2^2 \times 4 = 16$
c Substitute in $\text{SD}(X) = \sqrt{\text{Var}(X)}$.	$\text{SD}(2Y) = \sqrt{16} = 4$

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The sum of 2 identically distributed independent random variables

There is a difference between the sum of 2 identically distributed random variables and the distribution of a random variable $2X$, where X represents a discrete random variable.

Consider the situation where an arrow is fired at a target where the score, X, can be 1, 2, 3, 4 or 5.



Shutterstock.com/NikWB

An archer has 2 choices.

The first choice is to shoot 2 arrows and get the sum of the scores from 2 arrows. If x_1 is the score on the first shot and x_2 is the score on the second shot, then the sample space $x_1 + x_2$ is $\{2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

The second choice is to shoot one arrow but double the points in each region of the target. The sample space for $2X$ is $\{2, 4, 6, 8, 10\}$.

Both ways produce the same minimum score, maximum score and mean score. The variance of the first method, however, is smaller than the variance of the second method.

The mean and variance of the sum of 2 identically distributed random variables

$\mathrm{E}(x_1 + x_2) = 2 \times \mathrm{E}(X)$

$\mathrm{E}(x_1 + x_2 + x_3 + \ldots + x_k) = k \times \mathrm{E}(X)$

$\mathrm{Var}(x_1 + x_2) = 2 \times \mathrm{Var}(X)$

$\mathrm{Var}(x_1 + x_2 + x_3 + \ldots + x_k) = k \times \mathrm{Var}(X)$

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WORKED EXAMPLE 6 Finding the mean and variance of the sum of 2 identically distributed random variables 1

A chicken named Molly is an unreliable layer. On any day, Molly lays either 0, 1 or 2 eggs. The discrete random variable X, representing the number of eggs Molly lays on any day, is given by the distribution below.

x	0	1	2
$\Pr(X = x)$	0.1	0.8	0.1

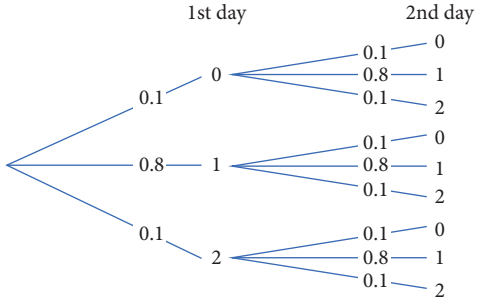
a Find the mean and variance of X.

b The eggs laid by Molly are only collected every 2 days. Let T represent the total number of eggs collected over 2 days.

Illustrate this situation with a tree diagram, if the number of eggs Molly lays on the first day is independent of the number of eggs she lays on the second day, and hence write the probability distribution for T.

c Find the mean and variance of T.

d Verify the results found in part **c** using the formula for the mean and variance of the sum of identical independent random variables.

Steps	Working
a Calculate the mean and variance of X from the probability distribution.	$\mathrm{E}(X) = 1.0$ $\mathrm{E}(X^2) = 1.2$ $\mathrm{Var}(X) = 1.2 - 1^2 = 0.2$
b Draw a tree diagram to represent the number of eggs laid on 2 consecutive days.	
c Calculate the probabilities of a total of 0, 1, 2, 3 and 4 eggs. List the probability distribution of T.	$\Pr(T = 0) = 0.1 \times 0.1 = 0.01$ $\Pr(T = 1) = 2 \times 0.1 \times 0.8 = 0.16$ $\Pr(T = 2) = 2 \times 0.1 \times 0.1 + 0.8 \times 0.8 = 0.66$ $\Pr(T = 3) = 2 \times 0.1 \times 0.8 = 0.16$ $\Pr(T = 4) = 0.1 \times 0.1 = 0.01$
d Calculate the mean and variance of the total number of eggs over 2 days, T.	$\mathrm{E}(T) = 2.0$ $\mathrm{E}(T^2) = 4.4$ $\mathrm{Var}(T) = 4.4 - 2^2 = 0.4$
e The total number of eggs is the sum of 2 random values: $x_1 + x_2$. Substitute into the mean and variance formulas: $\mathrm{E}(x_1 + x_2) = 2 \times \mathrm{E}(X)$ $\mathrm{Var}(x_1 + x_2) = 2 \times \mathrm{Var}(X)$	Substitute into $\mathrm{E}(x_1 + x_2) = 2\mathrm{E}(X)$ $\mathrm{Var}(x_1 + x_2) = 2\,\mathrm{Var}(X)$

T	0	1	2	3	4
$\Pr(T = t)$	0.01	0.16	0.66	0.16	0.01

T	$p(t)$	$t \times p(t)$	$t^2 \times p(t)$
0	0.01	0	0
1	0.16	0.16	0.16
2	0.66	1.32	2.64
3	0.16	0.48	1.44
4	0.01	0.04	0.16
Total	1.00	2.0	4.4

	X	T
Mean	$\mathrm{E}(X) = 1$	$\mathrm{E}(T) = 2 \times 1 = 2$
Variance	$\mathrm{Var}(X) = 0.2$	$\mathrm{Var}(T) = 2 \times 0.2 = 0.4$

The formulas verify the results found in the probability distribution of T.

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WORKED EXAMPLE 7 Finding the mean and variance of the sum of 2 identically distributed random variables 2

A probability spinner is divided into 5 equal sections numbered 1, 2, 3, 4 and 5. The discrete random variable X, representing the number obtained on a single spin, has a mean of 3 and a variance of 2. The spinner is spun twice and the sum of the numbers obtained is recorded. The discrete random variable Y represents the total of the numbers obtained after 2 spins.

Find

a the mean of Y

b the variance of Y

c the standard deviation of Y.

Steps	Working
a Substitute $E(X) = 3$ and $k = 2$ into $E(x_1 + x_2) = 2 \times E(X)$.	$E(x_1 + x_2) = 2 \times E(X)$ mean: $E(Y) = 2 \times 3 = 6$
b Substitute $\text{Var}(Y) = 2$ and $k = 2$ into $\text{Var}(x_1 + x_2) = 2 \times \text{Var}(X)$.	$\text{Var}(x_1 + x_2) = 2 \times \text{Var}(X)$ $\text{Var}(Y) = 2 \times 2 = 4$
c Substitute in $SD(X) = \sqrt{\text{Var}(X)}$.	$SD(Y) = \sqrt{4} = 2$

EXERCISE 11.2 Linear combinations of random variables

ANSWERS p. 607

Recap

1 The discrete random variable M, has a probability distribution given by

m	5	6	7	8	9
$\Pr(M = m)$	0.2	0.5	0.1	0.1	0.1

The mean is

A 0.5 **B** 6.4 **C** 6.5 **D** 7 **E** 7.5

2 A discrete random variable X has a mean of 12 and $E(X^2) = 148$. The value of $SD(X)$ is

A 2 **B** $2\sqrt{3}$ **C** $\sqrt{14}$ **D** 4 **E** 14

Mastery

3 WORKED EXAMPLE 4 The probability distribution of a discrete random variable X is shown.

x	100	200	300	400
$p(x)$	0.15	0.55	0.25	0.05

a List the probability distribution of N where $N = 2X$.

b Hence find

i $E(N)$

ii $\text{Var}(N)$

4 WORKED EXAMPLE 5 A discrete random variable H has a mean of 12.25 and a variance of 6. Find

a the mean of $2H$

b the variance of $2H$

c the standard deviation of $2H$.

5 WORKED EXAMPLE 6 Andre plays a game called 'Balloon Pop' at the town fair. He purchases a ticket, which gives him 3 darts to throw at a wall covered in balloons. He will either pop 0, 1, 2 or 3 balloons with each ticket he purchases. The discrete random variable X, representing the number of balloons Andre pops with the purchase of a single ticket, is given by the distribution below.

x	0	1	2	3
$\Pr(X = x)$	0.4	0.3	0.2	0.1

a Find the mean and variance of X.

Andre purchases 2 tickets. Let T represent the total number of balloons popped with 2 tickets.

b Illustrate this situation with a tree diagram, if the number of balloons Andre pops with any ticket is independent, and hence write the probability distribution for T.

c Find the mean and variance of T.

d Verify the results found in part **c** using the formula for the mean and variance of the sum of identical independent random variables.

6 WORKED EXAMPLE 7 The discrete random variable X has a mean of 15 and a variance of 9. The discrete random variable Y represents the sum of 2 values randomly selected from the distribution X, with replacement. Find

a the mean of Y **b** the variance of Y **c** the standard deviation of Y.

Exam practice

7 TECH-FREE (5 marks) The probability distribution of a discrete random variable X is shown.

x	1	2	3	4
$p(x)$	0.6	0.1	0.2	0.1

a List the probability distribution of N where $N = 5X$. 1 mark

b Hence find

i $E(N)$ 2 marks

ii $SD(N)$ 2 marks

8 TECH-FREE (10 marks) A bag contains 10 numbered cubes. Six cubes are numbered 3, three cubes are numbered 2 and one cube is numbered 1. A cube is selected, its number is noted, and it is returned to the bag, then a second cube is selected. The random variable S represents the sum of the numbers obtained.

a List the probability distribution of S. 4 marks

b Find

i $E(S)$ **ii** $\text{Var}(S)$ **iii** $SD(S)$ 6 marks

9 TECH-FREE (12 marks) When a fair coin is tossed, the outcomes, head or tail, are used to define a discrete random variable, X, where $X = \begin{cases} 1 & \text{head} \\ 0 & \text{tail} \end{cases}$

If x_1 is the value of X for the first toss and x_2 is the value of X for the second toss, find

a $E(X)$ 2 marks

b $\text{Var}(X)$ 2 marks

c $E(2X)$ 2 marks

d $\text{Var}(2X)$ 2 marks

e $E(x_1 + x_2)$ 2 marks

f $\text{Var}(x_1 + x_2)$ 2 marks

Use the information below to answer Questions 10 and 11.

The probability distribution of the discrete random variable X is shown.

x	−2	−1	0	1	2
$p(x)$	0.24	0.38	0.16	0.11	0.11

10 The mean of X is

A −4 **B** −0.53 **C** 0 **D** 0.5 **E** 1

11 The value of SD($2X$), correct to 2 decimal places, is

A 0.48 **B** 1.27 **C** 2.54 **D** 5.08 **E** 11.42

12 (12 marks) Francesca is a tennis player who wishes to improve the accuracy of her ground strokes. In one of her drills, she aims for a spot in the far-right corner of the court. The probability of her hitting the spot on her first attempt is 0.3. If she hits the spot on an attempt, the probability of her hitting the spot on her next attempt improves to 0.6; however, if she misses, the probability of hitting the spot reduces to 0.2. Francesca hits 3 balls and records her results. Let the discrete random variable H represent the number of times she hits the spot on 3 attempts.

a List the probability distribution of H. 4 marks

b Hence find, correct to 3 decimal places

i E(H) 2 marks

ii Var (H) 2 marks

c Francesca completes 2 sets of the accuracy drill and records her total score T.

Use the results of part **b** to find, correct to 3 decimal places

i E(T) 2 marks

ii Var (T) 2 marks

13 (16 marks) Hua completes a set of 3 multiple-choice trivia questions in an online quiz. She guesses the answer to each question and is given a score at the completion of the quiz. Let the discrete random variable X represent the score for one completed quiz. Each multiple-choice question has options A, B, C and D.

a List the probability distribution of X. 4 marks

b Find

i E(X) **ii** Var (X) **iii** SD(X) 6 marks

c Hua completes 2 different sets of trivia questions and records her total score, Y.

Find

i E(Y) **ii** Var (Y) **iii** SD(Y) 6 marks

11.3 Populations and samples

Sampling is the process of selecting items from a population and using the information gathered to draw conclusions about the population.

Populations and samples

A Bendigo baker wishes to expand his business. He makes 3 different types of pies and needs to know the proportion of people in Bendigo that prefer each pie. It is too difficult to ask everyone in Bendigo so the baker asks a sample of people which pie they prefer. From this sample, the baker can predict what the results should be for Bendigo.

In this situation, Bendigo is the population and the group of people the baker asks about which pie they prefer is the sample.

The sample can be used to infer information about the population.



Video playlist
Populations and samples

Worksheet
Sampling methods

Populations and samples

The **population** includes all elements of a set of data.

The **sample** consists of one or more elements from the population and is a subset of the population.

Choosing samples

To make valid conclusions about the population from a sample, the sample must represent the population. If the baker only asked the customers who came to his shop which pie they preferred, his sample would not represent the population of Bendigo. He would exclude all the people in Bendigo who do not come to his bakery and the opinions of these people are important for the baker to expand his business.

Choosing samples

To make reliable conclusions about a population, the sample must contain representatives of all the subgroups in the population. These subgroups must appear in the same proportion as exists in the population.

Random samples

One way of selecting a representative sample is to generate a simple random sample.

Methods for generating random samples are covered in detail in this chapter.

Simple random samples

Simple random sampling ensures each element of the population has the same chance of being selected.

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WORKED EXAMPLE 8 Determining if a sample is random

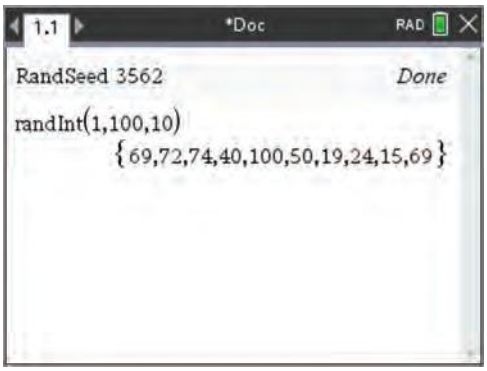
A researcher wants to know how obedient dogs are in a city. She asks a group of people at the dog obedience school to complete a survey she has designed. Will this result in a random sample?

Steps	Working
Simple random sampling ensures each element of the population has the same chance of being selected.	The sample will not be random since the only people surveyed already train their dogs. The researcher has excluded an important subgroup of the population. A better sample would be a random sample of the entire population of the city.

USING CAS 2 Generating random numbers

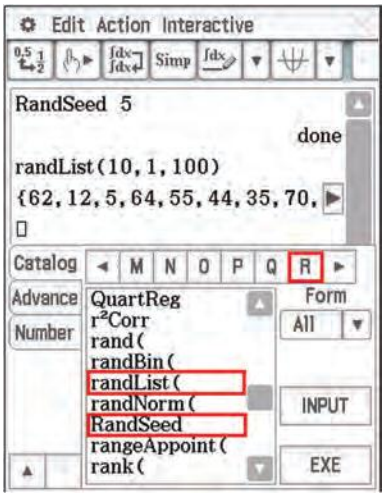
Generate 10 random numbers between 1 and 100.

TI-Nspire



1. Press **menu > Probability > Random > Random Seed***.
2. Enter a number (e.g. last 4 digits of a phone number).
3. Press **menu > Probability > Random > RandInt**.
4. Enter **1,100,10**.

ClassPad



1. Open the **Keyboard > Catalog**.
2. Tap **R** to jump to the functions starting with r.
3. Tap **RandSeed***.
4. Enter a number between 0 and 9.
5. Tap **randList(**.
6. Enter **10,1,100)**.

A list of 10 random numbers between 1 and 100 will be generated. Answers will vary.

*RandSeed ensures that each set of random numbers has a random starting point.

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WORKED EXAMPLE 9 Selecting a random sample

The table shows the highest high tide (HHT) and lowest low tide (LLT) values for a 60-consecutive day period recorded at a place somewhere between Dampier and Darwin in north-western Australia. These tide heights are in metres (m).

Use a random number generator to select a random sample of 10 items from this population.

Number	HHT	LLT
1	10.08	2.90
2	9.39	3.71
3	8.69	4.58
4	8.36	4.92
5	8.70	4.79
6	9.34	4.11
7	10.06	3.34
8	10.69	2.65
9	10.33	2.10
10	11.20	1.68
11	11.59	1.37
12	11.83	1.19
13	11.88	1.19
14	11.70	1.38
15	11.30	1.78
16	10.85	2.35
17	10.21	3.11
18	9.45	4.02
19	8.96	4.18
20	9.24	4.26
21	10.05	3.56
22	10.87	2.63
23	10.59	1.86
24	11.51	1.33
25	11.92	1.04
26	12.09	0.95
27	12.03	1.04
28	11.76	1.30
29	11.37	1.73
30	10.82	2.33

Number	HHT	LLT
31	10.09	3.10
32	9.22	4.01
33	8.30	5.00
34	7.81	5.30
35	8.50	5.06
36	9.50	4.10
37	10.45	3.11
38	10.39	2.26
39	11.25	1.59
40	11.85	1.20
41	12.22	0.83
42	12.33	0.81
43	12.24	1.04
44	11.86	1.49
45	11.28	2.13
46	10.44	2.96
47	9.36	4.03
48	8.34	5.22
49	8.63	4.90
50	9.75	3.99
51	10.76	2.84
52	10.81	1.97
53	11.48	1.41
54	11.91	1.11
55	12.12	1.01
56	12.18	1.06
57	12.04	1.25
58	11.65	1.61
59	11.18	2.13
60	10.49	2.81

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Steps	Working
1 Generate 10 random numbers as shown in Using CAS 2. Include extra data to avoid duplicates. **2** Identify the item numbers in the list that match the random numbers generated.	The 10 ordered random numbers are {10, 16, 28, 30, 31, 40, 41, 48, 49, 50}. The random sample of 10 items is

10	11.20	1.68
16	10.85	2.35
28	11.76	1.30
30	10.82	2.33
31	10.09	3.10
40	11.85	1.20
41	12.22	0.83
48	8.34	5.22
49	8.63	4.90
50	9.75	3.99



Worksheets
Sample proportions

Sample proportion calculations

Population parameters and sample statistics

A local council intends to introduce extended trading hours during summer. They must determine what proportion of the city is in favour of this proposal. It will be too costly and time consuming to survey everyone in the city so the council surveys a random sample of residents.

The proportion of the city is called a **parameter** and it is not known. The proportion of the sample is called a **statistic** and is used to estimate the population parameter.

Statistics and parameters are very similar since they are both numerical characteristics of groups.

The numerical characteristic may be a proportion, a mean, a variance or a standard deviation.

Parameters and statistics

A **statistic** is a numerical characteristic of a sample. The sample statistics vary depending on the sample and the size of the sample.

A **parameter** is a numerical characteristic of a population. Population parameters are fixed because the population data is a defined set.

Population and sample, proportions and means

$$\text{Population proportion } (p) = \frac{\text{Number of items in the population with an attribute}}{\text{Population size}}$$

$$\text{Sample proportion } (\hat{p}) = \frac{\text{Number of items in the sample with an attribute}}{\text{Sample size}}$$

$$\text{Population mean } (\mu) = \frac{\text{Sum of the items in the population}}{\text{Population size}}$$

$$\text{Sample mean } (\bar{x}) = \frac{\text{Sum of the items in the sample}}{\text{Sample size}}$$

WORKED EXAMPLE 10 Finding a sample proportion

Two dice are rolled 10 times and the results below are obtained.

(4, 4) (2, 2) (1, 1) (6, 1) (3, 3) (6, 4) (4, 5) (5, 5) (5, 1) (4, 3)

Find the proportion of sums greater than 9.

Steps	Working
1 Count the number of items in the sample where the sum is greater than 9 and write the number of items in the sample.	The results that have a sum greater than 9 are (6, 4) (5, 5). Number of items greater than 9 = 2 Number of items in the sample = 10
2 Substitute in the formula for sample proportion.	Sample proportion $\hat{p} = \frac{2}{10} = 0.2$ The proportion of tosses where the sum is greater than 9 is 0.2 or 20%.

WORKED EXAMPLE 11 Finding a sample mean

A sample of competitors who completed a marathon was taken by selecting all the people in the race whose surname began with M. Their race times, in minutes, are shown below.

210, 435, 315, 233, 336, 285, 145, 167, 202, 346, 389, 421, 275,
312, 202, 397, 336, 333, 198, 256, 278, 384, 362, 384, 346, 267

a Determine the sample mean $\bar{x}$, correct to 2 decimal places.

b Is the value $\bar{x}$ a statistic or a parameter?

Steps	Working
a 1 Find the sum of the times.	The sum of the times = 7814
2 Substitute into the formula $\bar{x} = \frac{\Sigma x}{n}$.	sample mean = $\frac{7814}{26} \approx 300.54$
b Determine if the mean relates to a sample or the population. Statistics are for samples. Parameters are for populations.	The value $\bar{x}$ is a sample statistic.

EXERCISE 11.3 Populations and samples

ANSWERS p. 608

Recap

1 A number is selected at random from the set {1, 1, 1, 1, 2, 3, 3, 4, 4, 4}.

Let N = the number selected from the set. The mean and variance of $2N$ is

A 2.5 and 3 **B** 2.4 and 1.64 **C** 4.8 and 6.56

D 2.4 and 7.4 **E** 4.8 and 14.8

2 Three numbers are randomly selected, with replacement, from a set of integers. The discrete random variable A, representing the number selected, has a mean of 36 and a variance of 25. If the first number randomly selected from the set is a_1, the second is a_2 and the third is a_3, then $SD(a_1 + a_2 + a_3)$ is equal to

A 5 **B** $5\sqrt{3}$ **C** $6\sqrt{3}$ **D** 18 **E** 75

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Mastery

3 WORKED EXAMPLE 8 A researcher wants to find out the number of hours of exercise teenagers in his area do each day. He conducts a questionnaire at the gym on a Saturday morning and asks all the teenagers that arrive. Will this result in a random sample?

4 Using CAS 2 Generate 8 random numbers between 1 and 50.

5 WORKED EXAMPLE 9 The number of pets per household for a class of 24 students is recorded below. Use a random number generator to select a sample of 5 items.

Number of pets in a household

Item number	Number of pets	Item number	Number of pets
1	4	13	1
2	6	14	1
3	2	15	1
4	1	16	2
5	0	17	0
6	3	18	2
7	1	19	0
8	2	20	2
9	2	21	2
10	3	22	3
11	0	23	0
12	1	24	0

6 WORKED EXAMPLE 10 Two dice are rolled 10 times and the results below are obtained.

$(4,4)$ $(2,2)$ $(1,1)$ $(6,1)$ $(3,3)$ $(6,4)$ $(4,5)$ $(5,5)$ $(5,1)$ $(4,3)$

Find the proportion of sums less than 7.

7 WORKED EXAMPLE 11 The lengths (cm) of babies born in the month of January at the Mercy Hospital are shown in the table below.

50.2	51.3	46.8	47.9	50.9	51.3	52.8	50.6	49.7	49.2	50
48.6	52.3	51.9	53.8	50.8	49.3	51	52.3	51.3	48.9	49.9

a Determine the sample mean $\bar{x}$, correct to 2 decimal places.

b Is the value $\bar{x}$ a statistic or a parameter?

Exam practice

8 TECH-FREE (2 marks) Scientific studies show that 35% of fish have a detectable level of mercury poisoning. In a sample of 40 fishes, 15 test positive for mercury poisoning.

a Find the population parameter (p). 1 mark

b Find the sample statistic ($\hat{p}$). 1 mark

Use the following information to answer Questions 9, 10 and 11.

A researcher is interested in finding out the incomes from part-time work for students at Melbourne University. The first 20 university engineering students leaving a morning lecture were asked to complete a survey. The table below shows their weekly income from part-time work.

304	285	331	270	306	294	299	302	295	299
294	315	291	299	314	270	299	284	270	302

9 Which of the following is true?

A The data are for a population.

B The data are for a sample.

C The data are for a random sample.

D The mean of the data is a population parameter.

E The data are representative of the students at the university.

10 The proportion of the students who have incomes less than $300 is

A 0.35 **B** 0.55 **C** 0.65 **D** 1.35 **E** 13

11 The value of $\bar{x}$ is

A 270 **B** 285.35 **C** 296.15 **D** 299 **E** 301.15

12 At a secondary school, 45% of the students travel to school by bus. In a random sample of 50 students, 20 of the students indicate they travel to school by bus. Which of the following statements is true?

A $p = 0.4$ **B** $\hat{p} = 0.45$ **C** $\bar{x} = 0.45$ **D** $\mu = 0.45$ **E** $p = 0.45$

13 In the city of Ballarat, 60% of the population are in favour of investing in renewable energy. A group of 50 Ballarat residents was selected at random and 35 were in favour of investing in renewable energy. Which of the following statements is true?

A $p = 0.7$ **B** $\hat{p} = 0.7$ **C** $\bar{x} = 0.6$ **D** $\mu = 0.6$ **E** $\hat{p} = 0.6$

14 (5 marks) The heights (cm) of a group of 24 players trying out for a basketball team are listed in the table below.

Height of basketball players

Player number	Height (cm)	Player number	Height (cm)
1	198	13	162
2	182	14	173
3	187	15	184
4	185	16	173
5	178	17	178
6	176	18	182
7	174	19	190
8	163	20	188
9	164	21	175
10	165	22	195
11	189	23	210
12	175	24	200

a Find the proportion, p, of players who are taller than 180 cm. 1 mark

b Find the mean height, μ, of the players trying out for the basketball team. 2 marks

c Use a random number generator to select a sample of 5 players. 1 mark

d Find the proportion, $\hat{p}$, of your sample of players who are taller than 180 cm. 1 mark

15 (5 marks) Breakfast cereals sold by a wholesaler were all tested to determine their nutritional value. A higher rating indicates the cereal has more nutritional value. The nutrition ratings for these breakfast cereals are shown in the table.

Nutrition rating of breakfast cereals

No.	Cereal	Rating	No.	Cereal	Rating
1	100% Bran	68.4	21	Cream of Wheat (Quick)	64.5
2	100% Natural Bran	34.0	22	Crispix	46.9
3	All-Bran	59.4	23	Crispy Wheat & Raisins	36.2
4	All-Bran with Extra Fibre	93.7	24	Double Chex	44.3
5	Almond Delight	34.4	25	Froot Loops	32.2
6	Apple Cinnamon Cheerios	29.5	26	Frosted Flakes	31.4
7	Apple Jacks	33.2	27	Frosted Mini-Wheats	58.3
8	Basic 4	37.0	28	Fruit & Fibre Dates; Walnuts; and Oats	40.9
9	Bran Chex	49.1	29	Fruitful Bran	41.0
10	Bran Flakes	53.3	30	Fruity Pebbles	28.0
11	Cap'n Crunch	18.0	31	Golden Crisp	35.3
12	Cheerios	50.8	32	Golden Grahams	23.8
13	Cinnamon Toast Crunch	19.8	33	Grape Nuts Flakes	52.1
14	Clusters	40.4	34	Grape-Nuts	53.4
15	Cocoa Puffs	22.7	35	Great Grains Pecan	45.8
16	Corn Chex	41.4	36	Honey Graham Ohs	21.9
17	Corn Flakes	45.9	37	Honey Nut Cheerios	31.1
18	Corn Pops	35.8	38	Honey-comb	28.7
19	Count Chocula	22.4	39	Just Right Crunchy Nuggets	28.7
20	Cracklin' Oat Bran	40.4	40	Kix	39.2

a Find the proportion, p, of breakfast cereals that have a rating lower than 35. 1 mark

b Find the mean rating, μ, of the breakfast cereals available at the warehouse. 2 marks

c Use a random number generator to select a sample of 8 breakfast cereals. 1 mark

d Find the proportion, $\hat{p}$, of breakfast cereals that have a rating lower than 35. 1 mark

11.4 Simulation

Random experiments, events and event spaces

In a random experiment, the individual outcomes are not known until the trial is completed.

An example of a random experiment is rolling a die. On any roll, the probability of each outcome is the same and the outcome is unknown until the roll is completed.

The observed outcome of the experiment is called an **event**.

If a die is rolled 5 times, the 5 observed outcomes are the **event space**.

Events and the event space

An event is the observed outcome of an experiment.

The event space is a list of all the possible observed outcomes in the experiment.

Video playlist
Simulation

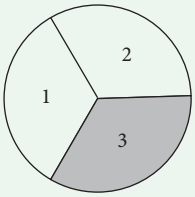
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WORKED EXAMPLE 12 Listing an event and the event space

The spinner shown is spun twice.

a List the event space for this experiment.

b If A is the set of events where a 3 occurs on exactly one spin, list the elements of A.



Steps	Working
1 Draw a lattice with 3 rows and 3 columns. Complete the cells in the matrix.	
2 Each cell represents an event or outcome. Write the outcomes as ordered pairs.	Event space = {(1, 1) (1, 2) (1, 3) (2, 1) (2, 2) (2, 3) (3, 1) (3, 2) (3, 3)}
3 Write the set that contains the events where a 3 occurs only once.	A = {(1, 3) (2, 3) (3, 1) (3, 2)}

	1	2	3
1	(1, 1)	(1, 2)	(1, 3)
2	(2, 1)	(2, 2)	(2, 3)
3	(3, 1)	(3, 2)	(3, 3)

Simulation

Simulation is the use of mathematics to recreate a situation, often repeatedly, so that the likelihood of various situations can be accurately estimated.

Simulations are used in many contexts such as engineering and scientific modelling of natural or human systems. One of the basic tools of simulation involves the generation of random numbers.

Consider the following situation:

Two equally matched people, Ali and Blake, play tennis, and for any set, each player has the same chance of winning.

This situation could be modelled using simulation by tossing a coin.

A head can represent Ali winning a set and a tail can represent Blake winning a set.

We will not know the result of any set until the coin is tossed. The player that wins the match will be the first to record 3 wins.

Simulation methods

A random experiment can be simulated by tossing a coin, rolling a die, drawing from a pack of cards, using a probability spinner, using random number tables or generating random numbers with a calculator or computer.

Developing an effective simulation

- Make sure you understand the situation.
- Develop an effective model so that the simulation method matches the probability of the outcomes.
- Generate enough data. The more simulations, the more accurate the result.
- Perform the simulations and record the results.
- Interpret the results to make conclusions about the situation being investigated.

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WORKED EXAMPLE 13 Applying simulation methods

Two equally matched people, Ali and Blake, play tennis. A probability simulation with the toss of a coin is used to determine the results of 5 matches.

Let a head represent Ali winning the set and a tail represent Blake winning the set.

The winner of a match is the first player to win 3 sets.

Use the simulations below to determine the match results.

H, T, T, T, T, T, T, T, H, H, T, H, T, H, T, T, T, T, H, H, H, H, H

Steps

Draw a table showing the simulations, wins (Ali (A) or Blake (B)) and the progressive match score.

Each time a player reaches 3 wins, they win the match and then a new match begins.

Complete the table and convert the H, T simulation to A wins or B wins.

Complete until 5 match results are obtained. The winner of each completed match is circled in the table.

Working

Simulation	A wins or B wins	Score
H	A	1–0
T	B	1–1
T	B	1–2
T	(B)	1–3
T	B	0–1
T	B	0–2
T	(B)	0–3
T	B	0–1
H	A	1–1
H	A	2–1
T	B	2–2
H	(A)	3–2
T	B	0–1
H	A	1–1
T	B	1–2
T	(B)	1–3
T	B	0–1
T	B	0–2
H	A	1–2
H	A	2–2
H	(A)	3–2
H	A	
H	A	

Simulation result.

Ali wins 2 matches and Blake wins 3 matches.

EXERCISE 11.4 Simulation

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Recap

1 A study of Australian wildlife observed that 45% of Australia's birds are found only in Australia. In a random sample of 50 birds, there were 20 that are found only in Australia.

Which of the following statements is **not** true?

A 0.45 is a population parameter.
B The sample size is 50.
C 0.45 is a statistic.
D 0.4 is the sample proportion.
E 0.4 is a statistic.

2 Nell wants to know how many chocolate chips are in her favourite cookies. She carefully crumbles 20 cookies and counts the number of chocolate chips in each. Her results are listed below.

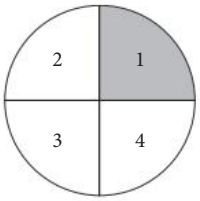
12, 17, 23, 19, 19, 21, 30, 15, 17, 14, 14, 11, 13, 18, 18, 18, 27, 31, 29, 16

If X represents the number of chocolate chips in a cookie and p the proportion of cookies in the sample of 20 that have more than 19 chocolate chips, then the values of $\bar{x}$ and p are

A 19.1, 0.4 **B** 19.1, 0.3 **C** 19.1, 6 **D** 20, 6 **E** 20, 8

Mastery

3 WORKED EXAMPLE 12 The spinner shown is spun twice.



a List the event space for this experiment.

b List the elements of A if A is the set of events where a total of 4 occurs after adding the results of the 2 spins.

4 WORKED EXAMPLE 13 A probability simulation with a random number generator is used to determine the result of a volleyball match.

Let '0' represent team Yellow winning the point and '1' represent team Purple winning the point.

The winner of a match is the first team to reach 11 points; however, the team must have a lead of 2.

The random numbers generated are 0, 0, 1, 1, 0, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 1, 0, 0, 1, 1.

Determine the winner of the match and the final score.

Exam practice

5 TECH-FREE (3 marks) In a game of darts, Kynan has 1 chance in 8 of hitting a bullseye on any throw.

In the probability simulation, random numbers between 1 and 8 are generated:

5, 3, 6, 3, 4, 5, 2, 4, 8, 3, 5, 1, 7, 5, 7, 2, 2, 7, 5, 4, 8, 3, 5, 4, 7

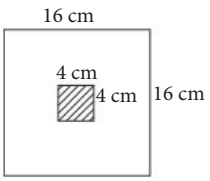
If 8 represents a bullseye and 1 to 7 represent not getting a bullseye, use the simulations above to find

a how many shots are required to hit the first bullseye 1 mark

b the proportion of shots that are bullseyes. 2 marks

6 TECH-FREE (6 marks) A game of chance involves throwing a coin on a table that has the areas shown.

The winning area is in the centre of the board.



a Show the probability of landing a coin in the winning area is $\frac{1}{16}$. 2 marks

A probability simulation is used to determine the length of the average losing streak when the game is played.

b If '16' represents a winning attempt, find the average losing streak in the simulation below. 4 marks

11, 5, 16, 13, 7, 4, 5, 7, 3, 9, 11, 1, 8, 11, 9, 13, 5, 1, 1, 6, 9, 1, 4, 5, 13, 5, 2, 15, 14, 7, 6, 10, 1, 2, 9, 12, 15, 13, 10, 3, 8, 13, 12, 6, 5, 16, 14, 15, 9, 16, 3, 5, 16, 5, 1, 2, 10, 16, 15, 13, 14, 3, 16, 2, 10, 4, 11, 11, 14, 4, 10, 15, 16, 11, 7, 2, 15, 6, 12, 9, 9, 9, 8, 5, 14, 2,3 , 13, 16, 8, 4, 14, 5, 16, 2, 1

7 Two 6-sided dice are rolled. If set X represents the elements in the event space where the sum of the 2 numbers is greater than 5, then the number of elements in set X is

A 10 **B** 15 **C** 21 **D** 26 **E** 36

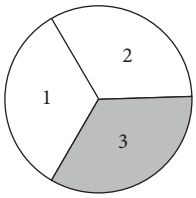
8 A card is drawn from a pack of 52 playing cards. The set A is the elements in the event space where a card drawn is a heart and a king. The number of elements in set A is

A 1 **B** 4 **C** 13 **D** 17 **E** 26

9 The spinner shown is spun 3 times.

The number of elements in the event space is

A 3 **B** 6 **C** 8 **D** 9 **E** 27



The following information refers to Questions 10 and 11

A probability simulation is used to find the result of a set of tennis between 2 people, Xander and Youssef.

Xander wins a game if the numbers 0, 1, 2, 3, 4 or 5 are generated.

Yousef wins a game if the numbers 6, 7, 8 or 9 are generated.

The first person to win 6 games with a lead of 2 is the winner (no tiebreakers).

The random numbers generated are

8, 9, 7, 2, 0, 3, 2, 1, 0, 7, 9, 9, 2, 4, 0, 8, 9, 5, 5, 5.

10 The set score of Xander vs Youssef after 6 games is

A 5–1 **B** 4–2 **C** 2–4 **D** 6–0 **E** 3–3

11 The number of games required to win the set is

A 7 **B** 8 **C** 9 **D** 10 **E** 11

12 (10 marks) Manu unfortunately has an infectious disease when he visits a small village on an island. Any person who has this infectious disease is only infectious for one day and after that day is immune.

Manu visits his friend Moana, who lives in the island village. The next day Moana is now infected, and Manu is immune. Moana visits and infects her friend Oliana, so now Oliana is infected and Manu and Moana are immune. On the following day, Oliana visits one more inhabitant of the island village and the process continues until an infected person visits someone who is immune.

Assume there are 25 inhabitants in the island village. Using simulation, generate random numbers between 1 and 25. Each random number represents a different inhabitant of the village who is visited and potentially infected with the disease. Determine the number of people infected in the island village, by generating random numbers until a number is repeated. (If the same number comes up twice in a row, it means the person stayed home and did not visit anybody.)

For example: the random numbers 13, 24, 23, 4, 11, 19, 2, 9, 25, 6, 20, 24 indicate 11 people were infected. The disease can no longer spread because an immune person (24) has been visited.

Repeat this process 10 times and find the average number (correct to the nearest integer) of infected people in a population of 25.

11.5 The distribution of the sample means

In this section, we will examine the properties of the sampling distribution of the sample means. We will refer to the normal distribution, which is studied in detail in Unit 4 of the Mathematical Methods course.

The sampling distribution of the sample means

Consider the situation where we have a population of trout in a lake, and we wish to estimate the mean weight of the trout (μ). We capture a sample of 10 trout and calculate the mean weight of this sample. This mean weight, $\bar{x}_1$, is an estimate of the mean of total population of trout in the lake.

Every time we take a different sample, of size 10, we will get a slightly different sample mean. The set of every possible sample mean is called the **sampling distribution** of the sample means of size 10.

Video playlist The distribution of the sample means

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WORKED EXAMPLE 14 The sampling distribution of sample means

Samples of size 2 are selected, with replacement, from a box that contains balls numbered 1, 2 and 3. Let the random variable $\bar{X}$ represent the means of the samples of size 2.

List the probability distribution of $\bar{X}$.

Steps

1 List all possible samples of size 2 and calculate the mean of each sample.

Working

Sample	Sample mean
1, 1	1
1, 2	1.5
1, 3	2
2, 1	1.5
2, 2	2
2, 3	2.5
3, 1	2
3, 2	2.5
3, 3	3

2 There is 1 of the 9 possible samples with a mean of 1.

$\Pr(\bar{X} = 1) = \frac{1}{9}$

Complete a probability distribution table that shows the different possible values of $\bar{X}$ and their probabilities.

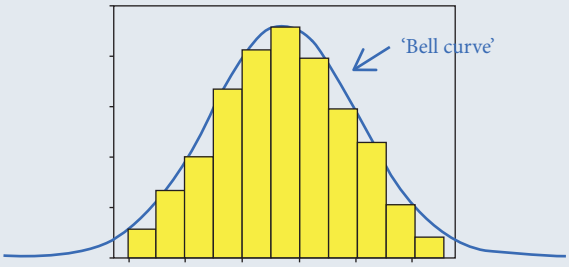
$\bar{x}$	$\Pr(\bar{X} = \bar{x})$
1	$\frac{1}{9}$
1.5	$\frac{2}{9}$
2	$\frac{1}{3}$
2.5	$\frac{2}{9}$
3	$\frac{1}{9}$

Normal distributions

The normal distribution

The **normal distribution** is a continuous bell-shaped distribution.

The data in a normal distribution is symmetrically spread about the mean. The mean, mode and median are all at the centre of the distribution.



The normal distribution is a frequently occurring distribution in nature and can be found in data sets such as the height of children of a certain age, the mass of Granny Smith apples from an orchard or the resting pulse of a group of athletes.

A variable that is normally distributed with a mean of μ and a standard deviation of σ can be written mathematically as $N \sim (\mu, \sigma^2)$.

The sampling distribution of the sample means by simulation

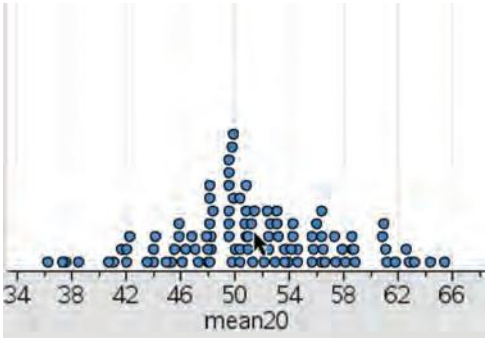
One hundred random samples of 20 integers are taken from the set {1, 2, 3 … 100} and the mean of each sample is calculated. The mean of the population is 50.5.

Each of the 100 samples has a different mean and these are shown in the dot plot.

The means of the samples are symmetrically spread about the centre, which is the mean of all the sample means.

In this simulation, the mean of the sample means is 51.2, which is approximately equal to the mean of the population.

In the simulations that follow, samples will be taken from a population which is normally distributed and whose population mean is known.

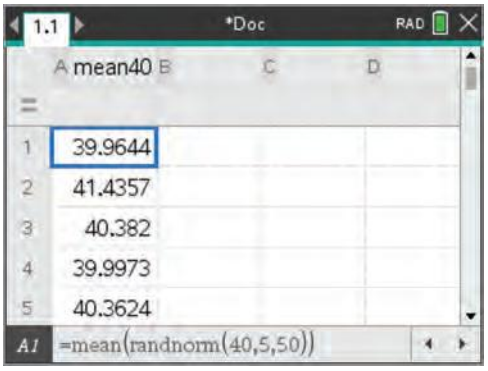


USING CAS 3 The distribution of sample means by simulation

Generate 100 samples of size 50 from a population that is normally distributed with a mean of 40 and a standard deviation of 5.

Calculate the mean of each sample and plot the distribution of the sample means.

TI-Nspire



1 Add a **Lists & Spreadsheet** page.

2 Label column **A** with a heading such as **mean40**.

3 Place the cursor in cell **A1**.

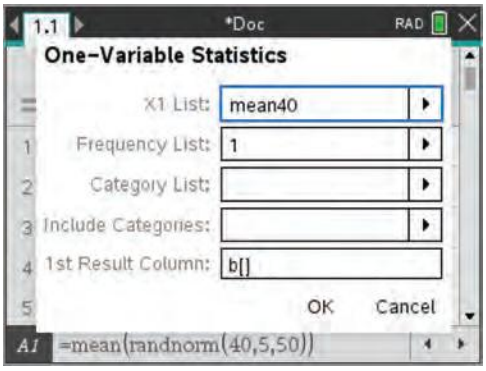
4 Press **menu > Data > List Math > Mean**.

5 Press **menu > Data > Random > Normal**.

6 Enter **40, 5, 50** as shown above.

7 Press **menu > Data > Fill**.

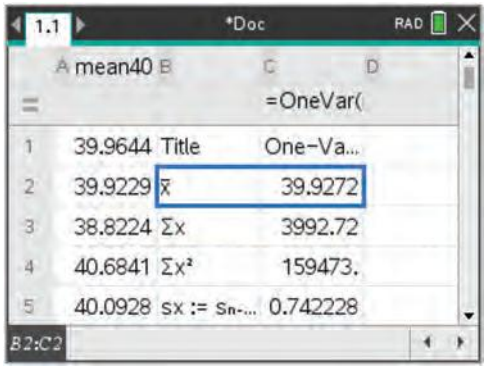
8 Press the **down arrow** to fill column **A** with 100 values.



9 Press **menu > Statistics > Stat Calculations > One-Variable Statistics**.

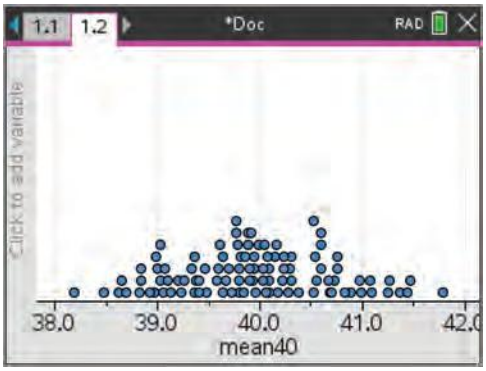
10 On the next screen, keep the **Num of Lists:** equal to **1**.

11 In the **One-Variable Statistics** screen for the **X1 List:** field, select **mean40**, as shown above.



12 The one-variable labels and statistics will be displayed in columns **B** and **C**.

13 Note the $\bar{x}$ value, which should be very close to the mean of 40. Answers will vary.

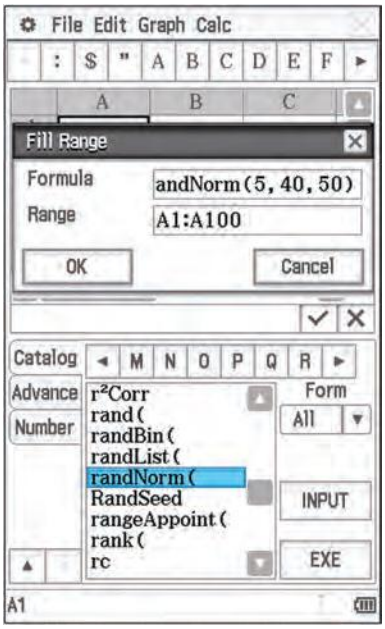


14 Add a **Data & Statistics** page.

15 For the horizontal axis, select **mean40.**

16 A dot plot of the sample means will be displayed.

ClassPad



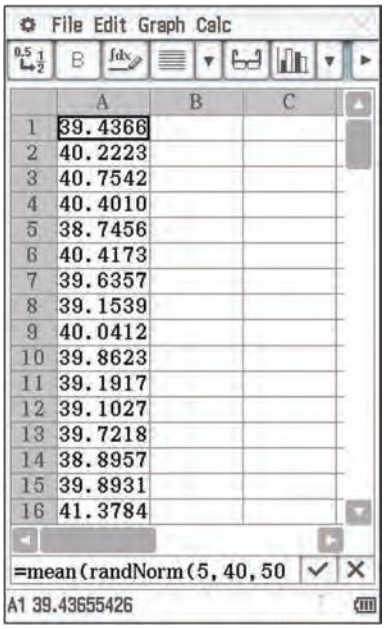
1 **Tap Menu > Spreadsheet**.

2 Tap **Edit > Fill > Fill Range**.

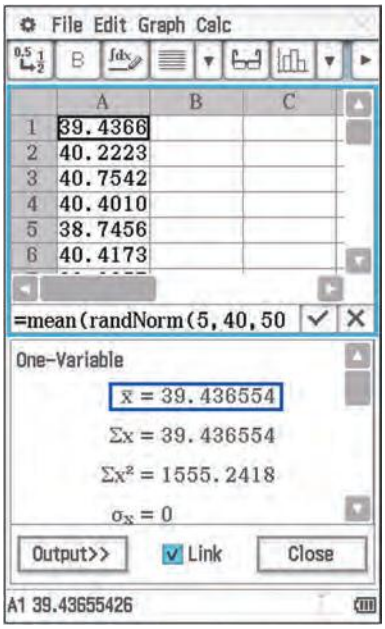
3 In the **Fill Range** dialogue box, enter the following:

Formula mean(randNorm(5,40,50)}
Range A1:A100

Open the **Keyboard > Catalog** to access the **mean** and **randNorm** functions.



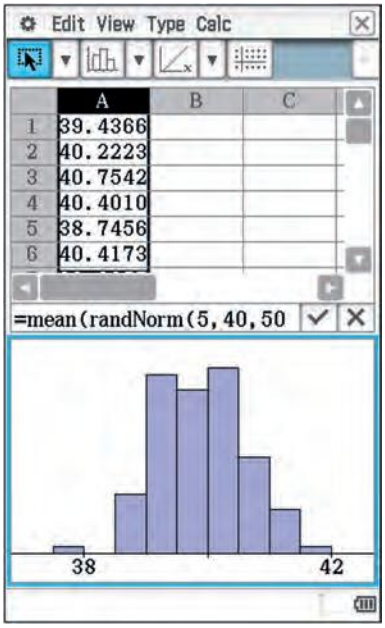
4 The means of the random norms will be displayed in column **A**.



5 Tap **Calc > One-Variable**.

6 The one-variable statistics will be displayed in the lower window.

7 Note the $\bar{x}$ value, which should be very close to the mean of 40. Answers will vary.



8 Tap **A** in the upper window to highlight all the values in the column.

9 Tap **Graph > Histogram**.

10 A histogram of the sample means will be displayed.

CAS syntax for generating normally distributed random variables

The calculator syntax for generating normally distributed random variables is:

Ti Nspire

randnorm(mean, standard deviation, sample size)

ClassPad

randNorm(standard deviation, mean, sample size)

The mean of the sample means

The sampling distribution of the sample means contains the means $(\bar{x})$ of all possible samples of size n from a population.

The mean of the sample means $(\mu_{\bar{x}})$ is equal to the mean of the population (μ).

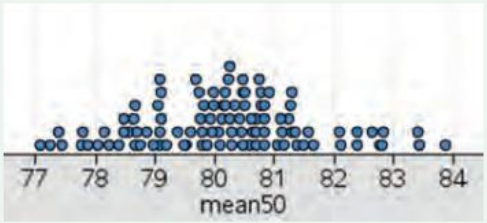
$$\mu_{\bar{x}} = \mu$$

There is some variability in the sample means and the standard deviation is a measure of the variability.

The standard deviation of the sample means is the **standard error** of the sample mean.

WORKED EXAMPLE 15 Finding probability from a simulation

A random sample is taken from a population that is normally distributed with a mean of 80 and a standard deviation of 10. Using simulation, 100 of these random samples are generated and the results are shown in the dot plot.



Find the probability a sample contains a mean greater than 82.

Steps	Working
Count the number of sample means greater than 82. The probability is a fraction of the total number of samples.	10 of the samples have sample means greater than 82. Probability of a mean greater than 82 $= \frac{10}{100} = \frac{1}{10}$

WB
p. 297

WB

p. 298

WORKED EXAMPLE 16 Finding the average of the sample means by simulation

The amount of time spent in the supermarket shopping for groceries is normally distributed with a mean of 30 minutes and a standard deviation of 6 minutes.

a Simulate 50 samples of size 40 and calculate the sample means for each sample.

b Find the average of the sample means for the samples of size 40.

c Display the results as a dot plot or histogram.

Steps

a Generate the means of the samples using the randNorm command (see pp. 509–510).

TI-Nspire

A1 cell formula **=mean(randnorm(30,6,40)**

Copy for 50 cells.

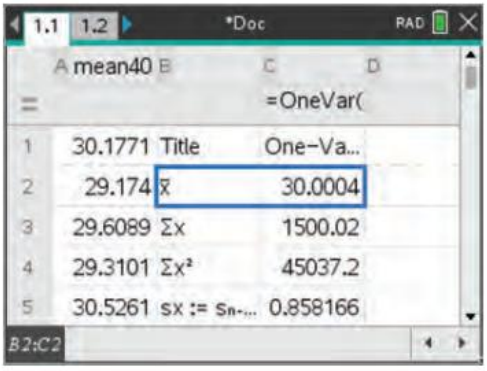
ClassPad

Formula **=mean(randNorm(6,30,40)**

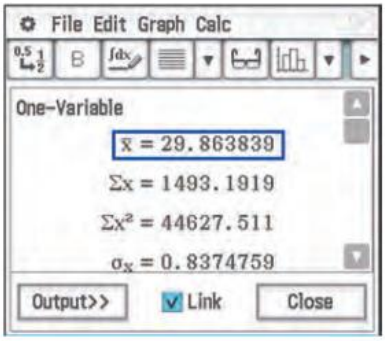
Range **A1:A50**

b Calculate one-variable statistics for the data in column A.

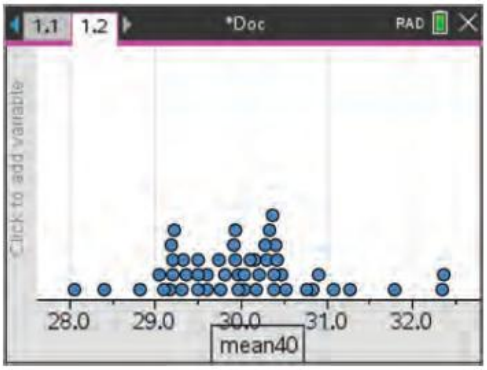
Mean of $\bar{x}$ = 30.00

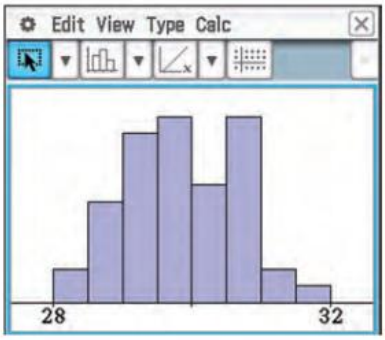


Mean of $\bar{x}$ = 29.86



c Graph the data.



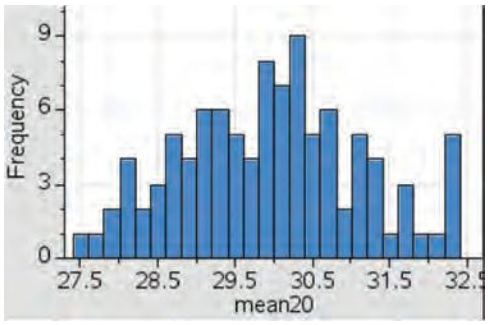


Larger samples and the sampling distribution of the sample means

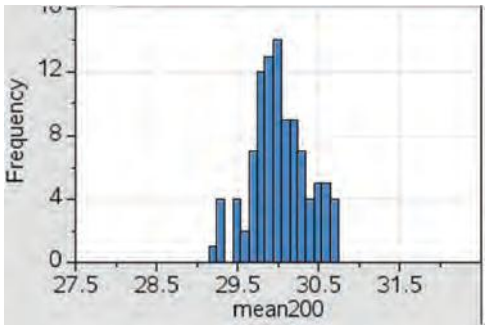
When a random sample is taken from a population, the reliability is influenced by the sample size. This can be seen when the mean is calculated with samples of different sizes.

For example, in the situation below, samples are taken from a normally distributed population with a mean of 30 and a standard deviation of 5.

Sample means with a sample size of 20



Sample means with a sample size of 200



Sample size	Mean of $\bar{x}$	Standard deviation of $\bar{x}$
20	29.96	1.174
200	30.01	0.346

The effect of a larger sample size on the sample means

- The sample mean is closer to the population mean.
- There is less variability in the means.

A larger sample size produces less error and a more reliable estimate of the population mean.

WB
p. 299

WORKED EXAMPLE 17 Comparing the standard deviation of the sample means for 2 different sample sizes

Random samples of size n are taken from a population that is normally distributed with a mean of 160 and a standard deviation of 20. Complete 100 simulations for sample sizes of 30 and 150 and determine the mean and standard deviation of the sample means.

Summarise this information in a copy of the table below.

Sample size	Mean of $\bar{x}$	Standard deviation of $\bar{x}$
30		
150		

Steps

Generate the means of the samples by CAS using the randNorm command.

Cell formula:

Use $n = 30$ and $n = 150$ in the formula.

TI-Nspire =mean(randnorm(160,20,*n*)
ClassPad =mean(randNorm(20,160,*n*)

Copy for 100 cells.

Working

Sample size	Mean of $\bar{x}$	Standard deviation of $\bar{x}$
30	159.50	3.385
150	159.77	1.621

Answers may vary.

The standard error for sample means

The standard deviation of the sample means is also called the **standard error** of a sample mean. This standard error of $\bar{x}$ is written as $\text{SE}(\bar{x})$ or $\sigma_{\bar{x}}$. The standard error indicates how much the sample mean, $\bar{x}$, for a particular sample size, deviates from the population mean, μ.

The standard error of a sample mean

The standard error for a sample mean $\text{SE}(\bar{x}) = \dfrac{\sigma}{\sqrt{n}}$, where σ is the population standard deviation and n is the sample size.

WB
p. 300

WORKED EXAMPLE 18 Finding the standard error of a sample mean

A random sample of 200 pears is taken at a fruit cannery. The weight of the pears is normally distributed with a mean weight of 70 g and a standard deviation of 8 g. Find the standard error in the sample mean.

Steps	Working
Use the formula $\text{SE}(\bar{x}) = \dfrac{\sigma}{\sqrt{n}}$.	$\text{SE}(\bar{x}) = \dfrac{8}{\sqrt{200}} \approx 0.566$

Video
VCE question analysis: Sampling distributions

VCE QUESTION ANALYSIS

A VCE-style question **(11 marks)**

Roberto Industrial manufactures three types of components for robots: sensors, motors and controllers. The manufacturing processes for each type of component are independent. It is known that 25% of all of the sensors manufactured are defective.

Sensors are sold in boxes of 3. Let the discrete random variable D represent the number of defective sensors in a box.

a List the probability distribution of D. 2 marks

b Find the mean and variance of D, correct to three decimal places. 2 marks

The weight of motors is normally distributed with a mean of 1.5 kg and a standard deviation of 0.1 kg.

For quality control, the mean of random samples of 5 motors is measured.

The weights of motors for one of the samples is

1.439 kg, 1.502 kg, 1.496 kg, 1.375 kg, 1.395 kg.

c Calculate the mean weight of this sample, correct to three decimal places. 1 mark

d Use simulation to randomly generate the means of 100 samples of size 5, correct to three decimal places, from a population which is normally distributed with a mean of 1.5 kg and a standard deviation of 0.1 kg. 1 mark

Let $\bar{X}$ = the sample mean of the weight of 5 motors.

e Find

i the percentage of samples that have a mean weight greater than 1.6 kg 1 mark

ii the percentage of samples that have a mean weight less than 1.4 kg. 1 mark

f Sort the sampling distribution of the 100 sample means and, by counting five values in from each end, find the lower limit a and upper limit b such that $\Pr(a < \bar{x} < b) \approx 0.90$. 2 marks

The interval (a, b) is called the 90% confidence interval for the sample means for samples of size 5.

g Find the approximate 90% confidence interval for the mean weight of motors, correct to three decimal places, using samples of size 5. 1 mark

Reading the question

- Highlight the information necessary to determine a discrete probability distribution.
- The question uses a simulation method to estimate the properties of a sampling distribution. Highlight the mean, standard deviation, sample size and the number of samples.
- Highlight the type and accuracy of answers required in each part. Full marks will not be awarded for answers if the required accuracy is not given.
- Highlight the information in part **f** that is used to find the approximate limits for a confidence interval by simulation.

Thinking about the question

- The question requires a knowledge of discrete probability distributions.
- You will need to be able to use CAS to calculate the mean and variance of a discrete random variable.
- You will need to know how to use CAS to generate and sort a list of sample means.
- The question requires an understanding of simulation and its use in finding probabilities.

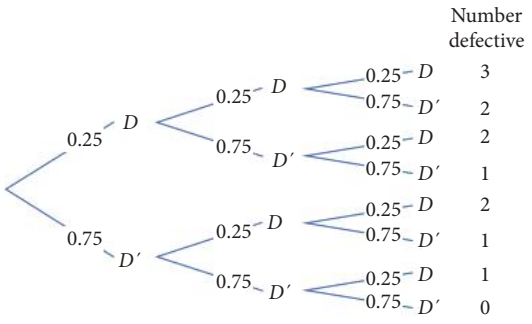
Worked solution (✓ = 1 mark)

a $\Pr(D = 0) = \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{27}{64}$ ✓

$\Pr(D = 1) = 3 \times \frac{1}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{27}{64}$

$\Pr(D = 2) = 3 \times \frac{1}{4} \times \frac{1}{4} \times \frac{3}{4} = \frac{9}{64}$

$\Pr(D = 3) = \frac{1}{4} \times \frac{1}{4} \times \frac{1}{4} = \frac{1}{64}$



d	0	1	2	3
$\Pr(D = d)$	$\frac{27}{64}$	$\frac{27}{64}$	$\frac{9}{64}$	$\frac{1}{64}$

✓

You can use either fractions or decimals in this problem.

b $E(D) = 0 \times \frac{27}{64} + 1 \times \frac{27}{64} + 2 \times \frac{9}{64} + 3 \times \frac{1}{64}$

$= \frac{3}{4}$

$= \mathbf{0.75}$ ✓

$\text{Var}(D) = E(D^2) - \mu^2$

$= 0^2 \times \frac{27}{64} + 1^2 \times \frac{27}{64} + 2^2 \times \frac{9}{64} + 3^2 \times \frac{1}{64} - \left(\frac{3}{4}\right)^2$

$= \frac{9}{16}$

$= 0.5625$

$\approx \mathbf{0.563}$ ✓

c mean = $(1.439 + 1.502 + 1.496 + 1.375 + 1.395) \div 5 = \mathbf{1.441\ kg}$ ✓

d The sorted means of 100 samples of size 5, randomly selected from a normally distributed population with mean 1.5 kg and standard deviation 0.1 kg: Simulated answers will vary.

1.384	1.391	1.395	1.396	1.407	1.410	1.413	1.414	1.415	1.420
1.422	1.426	1.427	1.429	1.437	1.442	1.445	1.445	1.447	1.450
1.451	1.454	1.454	1.456	1.456	1.458	1.458	1.462	1.463	1.464
1.466	1.468	1.469	1.471	1.472	1.473	1.473	1.475	1.476	1.476
1.478	1.479	1.479	1.479	1.480	1.482	1.484	1.485	1.486	1.486
1.487	1.490	1.492	1.492	1.493	1.495	1.496	1.501	1.501	1.502
1.502	1.503	1.504	1.505	1.505	1.506	1.506	1.507	1.508	1.508
1.509	1.510	1.512	1.514	1.514	1.516	1.517	1.517	1.518	1.522
1.528	1.528	1.529	1.529	1.531	1.534	1.540	1.548	1.551	1.557
1.573	1.576	1.580	1.586	1.597	1.598	1.601	1.601	1.604	1.617

✓

e **i** There are 4 sample means out of the 100 sample means in the table that are greater than 1.6. **4%** of the samples have a mean weight greater than 1.6 kg. ✓

ii There are 4 sample means out of the 100 sample means in the table that are less than 1.4. **4%** of the samples have a mean weight less than 1.4 kg. ✓

f $a = 1.407$ ✓

$b = 1.598$ ✓

g The 90% confidence interval for the mean weight of motors is **(1.407, 1.598)**. ✓

EXERCISE 11.5 The distribution of the sample means

ANSWERS p. 608

Recap

Use the information below to answer Questions 1 and 2.

A gardener purchases a bag that contains a mixture of different coloured bulbs from the nursery. The owner of the nursery tells the gardener that 40% of their original stock of bulbs had yellow flowers. The gardener plants 20 of the bulbs in a row.

Using simulation, the 20 random numbers generated are:

9, 9, 1, 5, 4, 7, 0, 3, 9, 2, 7, 9, 2, 3, 0, 9, 1, 0, 5, 8

where the numbers 0, 1, 2 and 3 represent a bulb producing a yellow flower.

1 The most yellow flowers in a row is

A 1 **B** 2 **C** 3 **D** 5 **E** 9

2 The total number of yellow flowering bulbs is

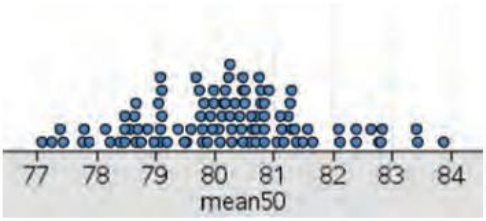
A 1 **B** 2 **C** 3 **D** 5 **E** 9

Mastery

3 WORKED EXAMPLE 14 Samples of size 2 are selected, with replacement, from a box that contains balls numbered 1, 2, 3 and 4. Let the random variable $\overline{X}$ represent the means of the samples of size 2.
List the probability distribution of $\overline{X}$.

4 Using CAS 3 Use CAS to generate 120 samples of size 80, from a population that is normally distributed with a mean of 7.5 and a standard deviation of 1.5. Find the mean and standard deviation of the sample means generated.

5 WORKED EXAMPLE 15 A random sample is taken from a population that is normally distributed with a mean of 80 and a standard deviation of 10. Using simulation, 100 of these random samples are generated and the results are shown in the dot plot.



Find the probability a sample contains a mean less than 78.

6 WORKED EXAMPLE 16 The amount of time a person spends standing, in a normal working day, is normally distributed with a mean of 400 minutes and a standard deviation of 30 minutes.

a Simulate 50 samples of size 80 and calculate the sample means for each sample. Display the results as a dot plot or histogram.

b Find the average of the sample means for the samples of size 80.

7 WORKED EXAMPLE 17 Random samples of size n are taken from a population that is normally distributed with a mean of 250 and a standard deviation of 50. Complete 100 simulations for sample sizes of 20 and 200 and determine the mean and standard deviation of the sample means.
Summarise this information in the table below.

Sample size	Mean of $\overline{x}$	Standard deviation of $\overline{x}$
20		
200		

8 WORKED EXAMPLE 18 A random sample of 100 eggs is taken at a free range farm. The weight of the eggs is normally distributed with a mean weight of 12 grams and a standard deviation of 1.5 grams. Find the standard error in the sample mean.

Exam practice

9 TECH-FREE (10 marks) Let $N = \{1, 2, 3, 4, 5\}$. Samples of size 2 are randomly selected, with replacement, from the set N, and the mean of each pair of numbers is calculated. Let the random variable $\bar{X}$ represent the means of the samples of size 2.

a Find

i $E(N)$ 1 mark

ii $\text{Var}(N)$ 1 mark

iii $SD(N)$ 1 mark

b List the probability distribution of $\bar{X}$. 2 marks

c Find

i $E(\bar{X})$ 1 mark

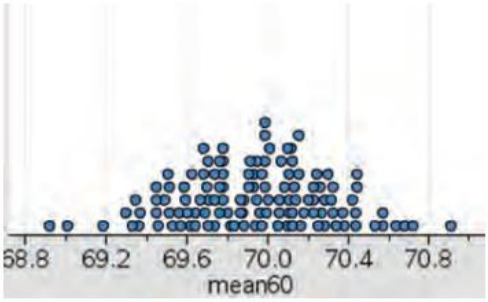
ii $\text{Var}(\bar{X})$ 1 mark

iii $SD(\bar{X})$ 1 mark

d Hence, verify the result for part **c iii** using the standard error formula $SE(\bar{x}) = \frac{\sigma}{\sqrt{n}}$. 2 marks

Use the following information to answer Questions 10 and 11.

The weight of hatched chicks is normally distributed with a mean weight of 70 grams and a standard deviation of 3 grams. A simulation of 120 samples of size 60 is completed and the sample means are shown.



10 Estimated from the graph, the probability of a sample having a sample mean less than 69.2 grams is

A $\frac{1}{40}$ **B** $\frac{3}{100}$ **C** $\frac{1}{20}$ **D** $\frac{3}{70}$ **E** $\frac{1}{3}$

11 The probability of a sample having a sample mean greater than x is $\frac{1}{10}$. The value of x is

A 70 **B** 70.2 **C** 70.4 **D** 70.6 **E** 70.8

Use the following information to answer Questions 12, 13 and 14.

Random samples of size 50 are taken from a population that is normally distributed with a mean of a and a standard deviation of b, and the mean of the samples is calculated. This process is repeated by simulation x times and the one-variable statistics for the sample means are shown.

One-Variable
$\bar{x} = 749.26918$
$\Sigma x = 59941.535$
$\Sigma x^2 = 44913324$
$\sigma_x = 3.4981891$
$s_x = 3.5202599$
$n = 80$
$\min X = 740.92766$
$Q_1 = 746.84473$
$\text{Med} = 749.29794$
$Q_3 = 751.3998$

12 The number of simulations is

A 50 **B** 80 **C** 100 **D** 150 **E** 750

13 The value of the population mean, a, is approximately

A 3.5 **B** 50 **C** 80 **D** 741 **E** 750

14 The standard deviation of the population is approximately

A 3.5 **B** 3.52 **C** 20 **D** 25 **E** 31.5

15 A random sample of size 64 is taken from a normally distributed population with a mean of 40 and a standard deviation of 8. The standard error for the sample means is

A 1 **B** 2 **C** 4 **D** 5 **E** 8

16 (3 marks) A random sample of 50 items is taken from a normally distributed population.
The sample mean is 29.95 and the standard deviation of the sample means of size 50 is 0.566.
Find to the nearest integer

a the population mean 1 mark

b the population standard deviation. 2 marks

17 (3 marks) The birth weight of babies is normally distributed with a mean of μ and a standard deviation of σ. A random sample of 100 babies is taken and the birth weights are recorded.
If the sample mean is 3.49 kg and the standard deviation of the sample means of size 100 is 0.05 kg, find, correct to 1 decimal place

a the population mean 1 mark

b the population standard deviation. 2 marks

18 (6 marks) The volume of soft drink in a can is normally distributed with a mean of 345 mL and a standard deviation of 2 mL.

a Simulate 50 sample means by taking random sample of size 60. Present the summary as a dot plot or histogram. 4 marks

b Find the mean of the sample means and the standard deviation of the sample means. 2 marks

11 Chapter summary

Discrete random probability distributions

The probability distribution of a discrete random variable shows all the possible values of the variable and the probabilities associated with those values. This is normally shown as a table.

Expected value of a discrete probability distribution

The expected value or mean of a discrete distribution $E(X)$ = the sum of $x \times \Pr(X = x)$.

The variance formula

A simpler formula for the variance is

$$\text{Var}(X) = E(X^2) - \mu^2$$

where $E(X^2) = \Sigma x^2 \cdot p(x)$ and $\mu = E(X)$.

The **standard deviation**

$$SD(X) = \sqrt{\text{Var}(X)}$$

The **mean and variance** of $2X$

$$E(2X) = 2E(X)$$

$$\text{Var}(2X) = 4\,\text{Var}(X)$$

The sum of two identically distributed independent random variables

$$E(x_1 + x_2) = 2E(X)$$

$$\text{Var}(x_1 + x_2) = 2\,\text{Var}(X)$$

Populations and samples

- The **population** includes all elements of a set of data.
- A **sample** consists of one or more elements from the population and is a subset of the population.
- To make reliable conclusions about a population, the sample must contain representatives of all the subgroups in the population.
- These subgroups must appear in the same proportion as exists in the population.
- **Simple random sampling** ensures that each element of the population has the same chance of being selected.

Population parameters and sample statistics

- A **statistic** is a numerical characteristic of a sample. The sample statistics vary depending on the sample and the size of the sample.
- A **parameter** is a numerical characteristic of a population. Population parameters are fixed since the population data is a defined set.
- Population proportion $(p) = \dfrac{\text{Number of items in the population with an attribute}}{\text{Population size}}$
- Sample proportion $(\hat{p}) = \dfrac{\text{Number of items in the sample with an attribute}}{\text{Sample size}}$
- Population mean $(\mu) = \dfrac{\text{Sum of the items in the population}}{\text{Population size}}$
- Sample mean $(\bar{x}) = \dfrac{\text{Sum of the items in the sample}}{\text{Sample size}}$

Random experiments, events and event spaces

- An **event** is the observed outcome of an experiment.
- The **event space** is a list of all the possible observed outcomes in the experiment.

Simulation methods

To develop an effective simulation:

1 Make sure that you understand the situation.

2 Develop an effective model so that the simulation outcomes match the probability of the real outcomes.

3 Generate enough data: the more simulations, the more accurate the result.

4 Perform the simulations and record the results.

5 Interpret the results to make conclusions about the situation being investigated.

Sampling distributions

- If all possible samples of size n are drawn from a population and a particular statistic is calculated (the mean, proportion or standard deviation), the probability distribution of the statistic is called a **sampling distribution**.
- The standard deviation of the statistic is the **standard error**.

Normal distributions

- The **normal distribution** is a continuous bell-shaped distribution.
- The data in a normal distribution is symmetrically spread about the mean.
- The mean, mode and median are all at the centre of the distribution.

The sampling distribution of sample means

- The sampling distribution of the sample means contains the means $(\bar{x})$ of all possible samples of size n taken from a population.
- The mean of the sample means $(\mu_{\bar{x}})$ is equal to the mean of the population (μ).

$$\mu_{\bar{x}} = \mu$$

- There is some variability in the sample means and the standard deviation is a measure of the variability.
- The standard deviation of the sample means is the standard error of the sample means.

Larger samples and the distribution of sample means

The effect of a larger sample size:

– the sample mean is closer to the population mean.

– there is less variability in the means.

– a larger sample size produces less error and a more reliable estimate of the population mean.

– the standard error for a sample mean is $\mathrm{SE}(\bar{x}) = \dfrac{\sigma}{\sqrt{n}}$, where σ is the population standard deviation and n is the sample size.

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 (2 marks) The probability distribution of a discrete random variable X is shown below.

x	1	3	4	5
$\Pr(X = x)$	0.5	0.1	0.3	0.1

Find

a the mean of X 1 mark

b the variance of X. 1 mark

2 (3 marks) A fruit orchard finds that 25% of apricots harvested have blemishes. Let B represent the number of blemished apricots found when two randomly selected apricots, from the days harvest, are checked.

a List the probability distribution of B. 2 marks

b Find $\Pr(B > 0)$. 1 mark

3 (3 marks) Find the centre and radius of the circle with equation $x^2 + 6x + y^2 - 2y = 26$.

4 (2 marks) In an arithmetic sequence, the 3rd term is 41 and the 5th term is 51.

Find the sum of the first 5 terms.

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 Eli purchases 25 Plymouth rock chickens and finds the mean weight of his chickens is 3.4 kg. The weight of Plymouth rock chickens is normally distributed with a mean of 3.5 kg and a standard deviation of 0.2 kg.

Let $\overline{X}$ = the mean weight of a sample of Plymouth rock chickens.

The values of $\overline{x}$, $\mathrm{E}(\overline{X})$ and $\mathrm{SE}(\overline{x})$ are

A $\overline{x} = 3.5$, $\mathrm{E}(\overline{X}) = 3.5$, $\mathrm{SE}(\overline{x}) = 0.04$

B $\overline{x} = 3.4$, $\mathrm{E}(\overline{X}) = 3.5$, $\mathrm{SE}(\overline{x}) = 0.04$

C $\overline{x} = 3.4$, $\mathrm{E}(\overline{X}) = 3.5$, $\mathrm{SE}(\overline{x}) = 0.2$

D $\overline{x} = 3.4$, $\mathrm{E}(\overline{X}) = 3.4$, $\mathrm{SE}(\overline{x}) = 1$

E $\overline{x} = 3.5$, $\mathrm{E}(\overline{X}) = 3.4$, $\mathrm{SE}(\overline{x}) = 0.04$

2 The mean of a random sample of 25 loaves of sourdough bread taken from High-Rise Bakery is 850 grams. The weight of loaves of sourdough bread is normally distributed with a mean weight of 900 grams and a standard deviation of 20 grams. The standard error in the sample mean is

A 4 g **B** 5 g **C** 20 g **D** 36 g **E** 50 g

3 Given $z = -2 + 2\sqrt{3}i$, the equivalent polar form is

A $4\,\mathrm{cis}\left(\frac{\pi}{3}\right)$ **B** $4\sqrt{3}\,\mathrm{cis}\left(-\frac{\pi}{3}\right)$ **C** $4\,\mathrm{cis}\left(-\frac{\pi}{3}\right)$

D $4\,\mathrm{cis}\left(\frac{2\pi}{3}\right)$ **E** $4\,\mathrm{cis}\left(-\frac{2\pi}{3}\right)$

4 The vertex of the graph of $y = |5 - 3x| - 2$ is

A $\left(\frac{3}{5}, -2\right)$ **B** $\left(-\frac{3}{5}, -2\right)$ **C** $\left(-\frac{5}{3}, -2\right)$ **D** $\left(-\frac{5}{3}, 2\right)$ **E** $\left(\frac{5}{3}, -2\right)$

5 When written in terms of i, $\sqrt{-12} - \sqrt{-48}$ is

A $-6\sqrt{3}i$ **B** $-2\sqrt{3}i$ **C** $6\sqrt{3}i$ **D** $-2\sqrt{3}$ **E** $2\sqrt{3} - 4\sqrt{3}i$

Section B 2 questions

15 marks

1 (6 marks) Celia has a small orchard on her property where she grows mangoes. The weights of mangoes are found to be normally distributed with a mean of 0.150 kg and a standard deviation of 0.015 kg. Each day Celia picks and weighs 40 mangoes, then records their mean weight in her journal. The mean weights for 20 weeks are shown in the table below.

0.1431	0.1468	0.1468	0.1473	0.1483	0.1487	0.1488	0.1491	0.1492	0.1493
0.1502	0.1503	0.1504	0.1507	0.1510	0.1513	0.1514	0.1516	0.1528	0.1536

Let $\bar{M}$ = the mean weight of a sample of 40 mangoes.

a Find, correct to 4 decimal places, the mean and standard deviation of the mean weights for 20 weeks. 2 marks

b Find, correct to 4 decimal places, the standard error for the sample means of size 40 using the parameters for the normal distribution. 1 mark

c Use the results of the table to find

i $\Pr(\bar{M} > \mu)$ 1 mark

ii $\Pr(\bar{M} < 0.151 \mid \bar{M} > 0.147)$ 1 mark

iii the smallest value of a, correct to 4 decimal places, such that $\Pr(\bar{M} > a) = 0.75$. 1 mark

2 (9 marks) Consider the vectors $\underset{\sim}{a} = 3\underset{\sim}{i} + 5\underset{\sim}{j} - 2\underset{\sim}{k}$, $\underset{\sim}{b} = \underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$ and $\underset{\sim}{c} = \underset{\sim}{i} + d\underset{\sim}{k}$, where d is a real constant.

a Find $|\underset{\sim}{b}|$. 1 mark

b Find $\hat{\underset{\sim}{b}}$. 1 mark

c Find the vector resolute of $\underset{\sim}{a}$ in the direction of $\underset{\sim}{b}$. 2 marks

d If $\underset{\sim}{c} = m\underset{\sim}{a} + n\underset{\sim}{b}$

i write 2 equations in terms of m and n using the coefficients of $\underset{\sim}{i}$ and $\underset{\sim}{j}$. 2 marks

ii find the values of m and n. 2 marks

iii find the value of d. 1 mark

CHAPTER

12 TRANSFORMATIONS AND MATRICES

Study Design coverage

Nelson MindTap chapter resources

12.1 Translations and linear transformations
- Translations
- Linear transformations
- **Using CAS 1:** Linear transformation of a shape
- Matrix representation
- **Using CAS 2:** Matrix representation
- Dilations
- Rotations
- **Using CAS 3:** Rotations
- Reflections

12.2 Inverse transformations
- Inverse linear transformations
- **Using CAS 4:** Inverse transformations

12.3 Composite transformations
- Composition using matrices
- **Using CAS 5:** Composition using matrices

12.4 Properties of transformations
- Invariance properties
- Transformations of shapes
- Other properties

12.5 Equivalent transformations

VCE question analysis

Chapter summary

Cumulative examination 1

Cumulative examination 2

9780170448338

Study Design coverage

UNIT 2, AREA OF STUDY 2: SPACE AND MEASUREMENT

Transformations

- points in the plane, coordinates and their representation as 2 × 1 matrices (column vectors)
- translations of the plane $T(x, y) \rightarrow (x + a, y + b)$
- linear transformations of the plane $T(x, y) \rightarrow (ax + by, cx + dy)$ and $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ as a map of the plane onto itself, rotations about the origin and reflection in a line through the origin
- the effect of translation, linear transformations and their inverse transformations, and compositions of these transformations on subsets of the plane such as points, lines, shapes and graphs of functions and relations
- invariance of properties under transformation, and the relationship between the determinant of a transformation matrix and the effect of the linear transformation on the area of a bounded region of the plane.

VCE Mathematics Study Design 2023–2027 p. 66, © VCAA 2022

Video playlists (6):

12.1 Translations and linear transformations
12.2 Inverse transformations
12.3 Composite transformations
12.4 Properties of transformations
12.5 Equivalent transformations
VCE question analysis Transformations and matrices

Worksheets (5):

12.1 Linear transformations • Matrix multiplications • Rotations • Reflections
12.2 Inverse and determinant of a matrix

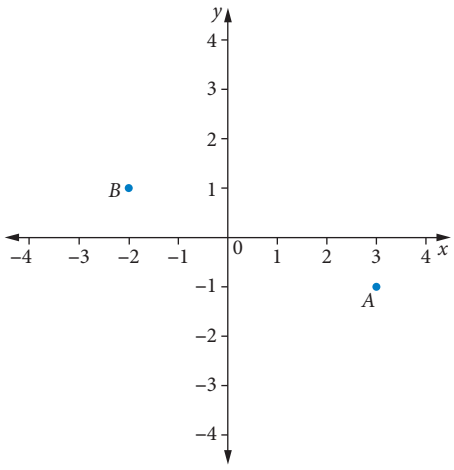
Nelson MindTap

To access resources above, visit
cengage.com.au/nelsonmindtap

12.1 Translations and linear transformations

Translations

We can show the points in the plane by coordinates, column or row vectors. For example, we can show A and B on the diagram as $A(3,-1)$ and $B(-2,1)$, $\begin{bmatrix} 3 & -1 \end{bmatrix}$ and $\begin{bmatrix} -2 & 1 \end{bmatrix}$, or $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.



For some of this work, column vectors (matrices) will be the most convenient form. We have already seen some **transformations of the plane**, such as translations and reflections, in previous years. Transformations change all the points in the plane to other points.

Video playlist
Translations and linear transformations

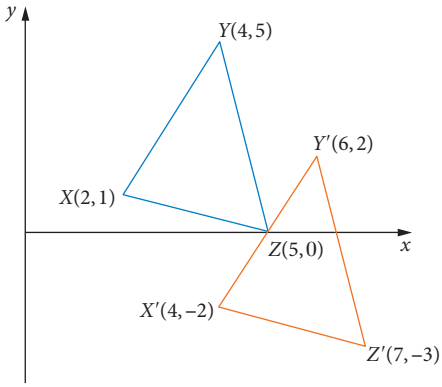
Translations of the plane

A translation of the plane changes the point $P(x, y)$ to $P'(x + a, y + b)$, where a and b are constants $(a, b \in R)$. Using R^2 as a symbol for the plane and T as the symbol for the translation, it can be written formally as $T: R^2 \rightarrow R^2$, $T(x, y) = (x + a, y + b)$ or $T(x, y) \rightarrow (x + a, y + b)$.

Translations do not change shapes, but slide them to another part of the plane. The translated shape is often called the **image** and the original shape is called the **object** of the translation.

WORKED EXAMPLE 1 Translation of a shape

Find the image of the triangle $X(2,1)$ $Y(4,5)$ $Z(5,0)$ under the translation $P(x, y) \rightarrow P'(x + 2, y - 3)$. What is the shape of the image?

Steps	Working
1 Find the image points by substituting the coordinates into the transformation.	$X(2,1) \rightarrow X'(2 + 2, 1 - 3) = X'(4, -2)$ $Y(4,5) \rightarrow Y'(4 + 2, 5 - 3) = Y'(6, 2)$ $Z(5,0) \rightarrow Z'(5 + 2, 0 - 3) = Z'(7, -3)$
2 Draw the original and image.	
3 Write the answer.	The image is $X'(4, -2)$ $Y'(6, 2)$ $Z'(7, -3)$. It is a triangle identical to the original.

WB
p. 301

Worksheet
Linear transformations

Linear transformations

Linear transformations of the plane

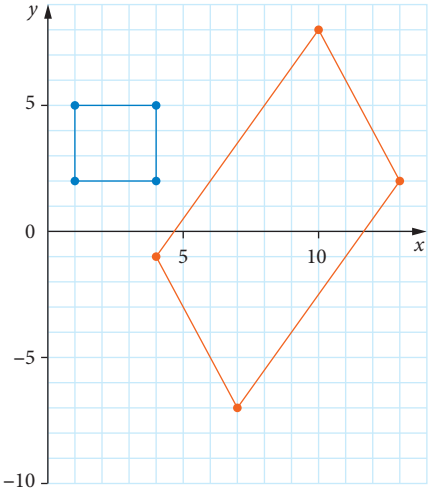
A **linear transformation of the plane** changes the point $P(x, y)$ to $P'(ax + by, cx + dy)$, where a, b, c and d are constants (real numbers). We can use R^2 as a symbol for the plane, so if the transformation above is called T, it can be formally written as $T: R^2 \to R^2$, $T(x, y) = (ax + by, cx + dy)$.

Linear transformations do change shape, but straight lines remain straight or shrink to a single point.

WB
p. 302

WORKED EXAMPLE 2 Linear transformation of a shape

Find the image of the square with vertices $A(1, 2)$ $B(1, 5)$ $C(4, 5)$ $D(4, 2)$ under the transformation $P(x, y) \to P'(2x + y, 3x - 2y)$. What is the shape of the image?

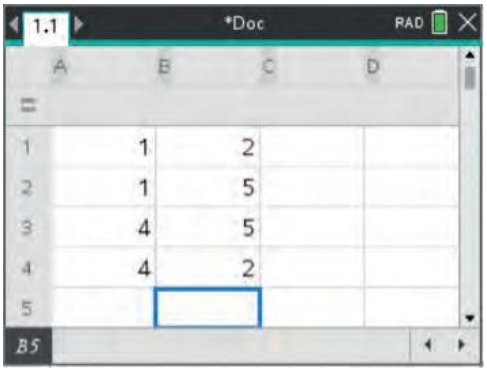
Steps	Working
1 Find the image points by substituting the coordinates into the transformation.	$A(1, 2) \to A'(2 \times 1 + 2, 3 \times 1 - 2 \times 2) = A'(4, -1)$ $B(1, 5) \to B'(2 \times 1 + 5, 3 \times 1 - 2 \times 5) = B'(7, -7)$ $C(4, 5) \to C'(2 \times 4 + 5, 3 \times 4 - 2 \times 5) = C'(13, 2)$ $D(4, 2) \to D'(2 \times 4 + 2, 3 \times 4 - 2 \times 2) = D'(10, 8)$
2 Draw the image.	
3 It looks like a parallelogram. Check the gradients to confirm this.	$m_{A'B'} = \frac{-6}{3} = -2$, $m_{C'D'} = \frac{6}{-3} = -2$, so $A'B' \parallel C'D'$; $m_{B'C'} = \frac{9}{6} = \frac{3}{2}$, $m_{A'D'} = \frac{9}{6} = \frac{3}{2}$, so $B'C' \parallel A'D'$.
4 Write the answer.	The image is $A'(4, -1)$ $B'(7, -7)$ $C'(13, 2)$ $D'(10, 8)$. It is a parallelogram.

9780170448338

USING CAS 1 Linear transformation of a shape

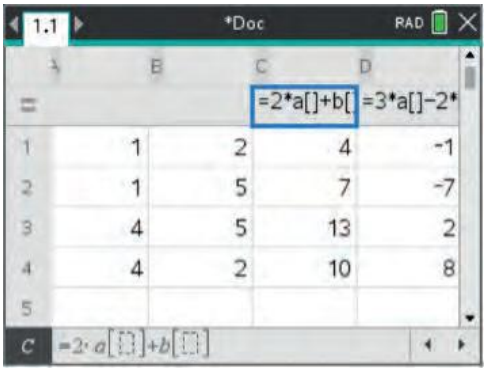
Find the image of the square with vertices $A(1, 2)$ $B(1, 5)$ $C(4, 5)$ $D(4, 2)$ under the transformation $P(x, y) \rightarrow P'(2x + y, 3x - 2y)$.

TI-Nspire



1 Add a **Lists & Spreadsheet** page.

2 In columns **A** and **B**, enter the coordinates of the 4 initial points as shown above.

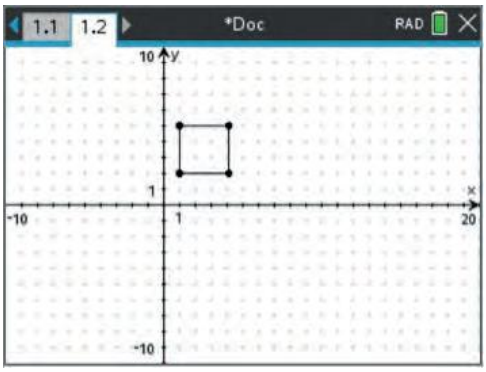


3 Place the cursor in the cell immediately under **C**.

4 Enter the formula **=2a+b**.

5 Repeat for column **D** by entering the formula **=3a–2b**.

6 The coordinates of the transformed image will be displayed in columns **C** and **D**.



7 Add a **Graphs** page.

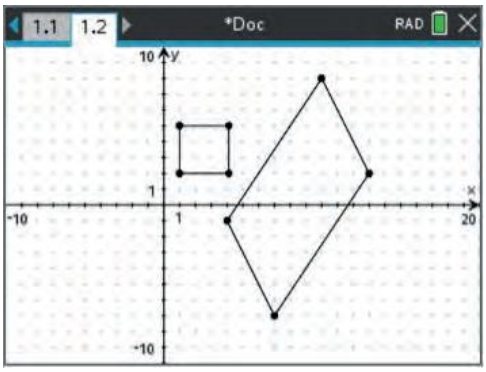
8 Press **esc** to hide the Graph Entry Line.

9 Change the window settings to those shown above.

10 Press **menu > View > Grid > Dot Grid**.

11 Press **menu > Geometry > Shapes > Rectangle**.

12 Draw a square using the initial coordinates.

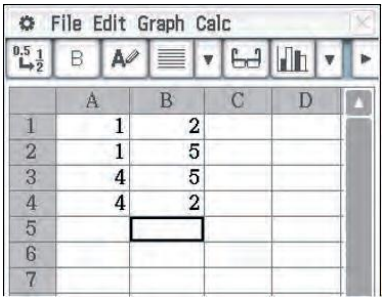


13 Press **menu > Geometry > Shapes > Polygon**.

14 Draw a polygon using the coordinates of the transformed image. Be sure to return to the starting point to complete the polygon.

15 Note that the transformed image is a parallelogram.

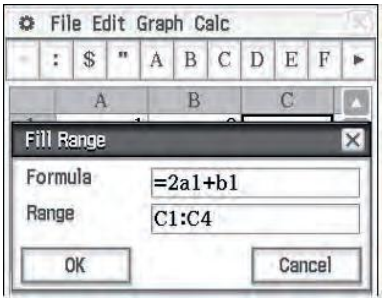
ClassPad



1 Tap **Menu > Spreadsheet**.

2 In columns **A** and **B**, enter the coordinates of the 4 initial points as shown above.

3 Place the cursor in cell C1.

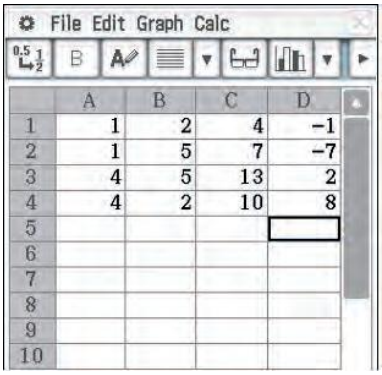


4 Tap **Edit > Fill > Fill Range**.

5 In the **Formula**: field, enter **=2a1+b1**.

6 In the **Range**: field, enter **C1:C4** then tap **OK**.

7 Repeat to insert the formula **=3a1–2b1** for the range **D1:D4**.



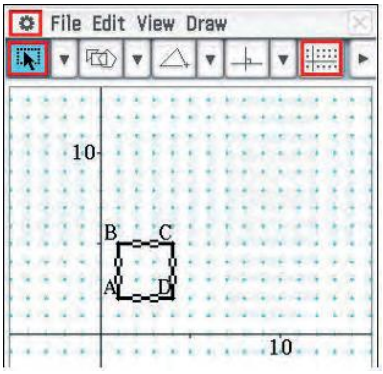
The coordinates of the transformed image will be displayed in columns **C** and **D**.

8 Tap **Menu > Geometry**.

9 Tap ⚙ **> View Window**.

10 Set **xmin:** to **–5**, **xmax:** to **15** and **ymid:** to **0**.

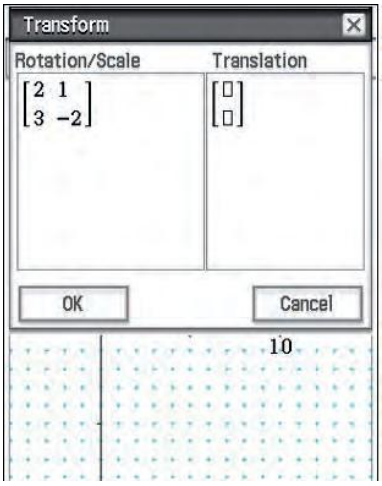
11 Tap the **Grid** tool until the dot grid is displayed.



12 Tap **Draw > Basic Object > Polygon**.

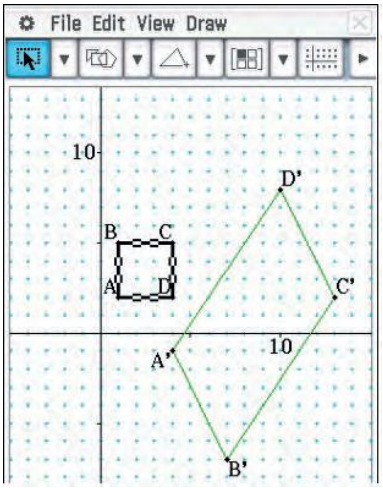
13 Draw a square using the initial coordinates, ensuring you return to tap the first point again.

14 Tap the **Select** tool then tap to highlight each side of the square.



15 Tap **Draw > Construct > General Transform**.

16 Complete the **Rotation/Scale** matrix, as shown above, then tap **OK**.



17 The transformed image, which is a parallelogram, will be displayed.

18 Alternatively, tap **Draw > Basic Object > Polygon** and draw the transformed image using the new coordinates.

Matrix representation

We can represent a translation as matrix addition of column matrices.

The translation T: $(x, y) \rightarrow (x + a, y + b)$ can be shown as $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$.

As matrix addition is commutative and associative, we can combine 2 translations by simply adding the column matrices.

WORKED EXAMPLE 3 Translations using matrices

a Find the image of the triangle $X(2, 1)$ $Y(4, 5)$ $Z(5, 0)$ under the translation $P(x, y) \rightarrow P'(x + 2, y - 3)$ using matrices.

b What is the image of the point $(3, 5)$ after translation 2 units down and 4 units to the right, followed by translation 3 units up and 1 unit to the right?

Steps	Working
a 1 Find the image points by adding matrices.	$\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 5 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$
2 Write the answer.	The image is $X'(4, -2)$ $Y'(6, 2)$ $Z'(7, -3)$.
b 1 Add the translations.	$\begin{bmatrix} 4 \\ -2 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$
2 Find the image point.	$\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \end{bmatrix}$
3 Write the answer.	The image point is $(10, 4)$.

We can represent a linear transformation as a 2×2 **square matrix** using the 4 numbers that define the transformation.

The linear transformation $P(x, y) \rightarrow P'(ax + by, cx + dy)$ is shown as $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Matrix multiplication with a column matrix reproduces the transformation of a point.

Remember that when we multiply matrices, we multiply the elements in each row of the first matrix by the elements in each column of the second matrix and add them together.

Worksheet
Matrix multiplications

WB
p. 303

WB

p. 304

WORKED EXAMPLE 4 Linear transformation using a matrix

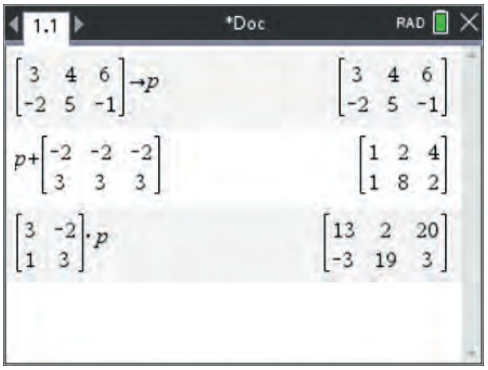
Apply matrix multiplication to the image of the square with vertices $A(1,2)$ $B(1,5)$ $C(4,5)$ $D(4,2)$ under the transformation $P(x,y) \rightarrow P'(2x+y, 3x-2y)$.

Steps	Working
1 Write the matrix.	$P = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix}$
2 Multiply the matrix by $A(1,2)$ as a column vector.	$A' = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2\times1+1\times2 \\ 3\times1+(-2)\times2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$
3 Multiply the matrix by $B(1,5)$ as a column vector.	$B' = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 2\times1+1\times5 \\ 3\times1+(-2)\times5 \end{bmatrix} = \begin{bmatrix} 7 \\ -7 \end{bmatrix}$
4 Multiply the matrix by $C(4,5)$ as a column vector.	$C' = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 2\times4+1\times5 \\ 3\times4+(-2)\times5 \end{bmatrix} = \begin{bmatrix} 13 \\ 2 \end{bmatrix}$
5 Multiply the matrix by $D(4,2)$ as a column vector.	$D' = \begin{bmatrix} 2 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2\times4+1\times2 \\ 3\times4+(-2)\times2 \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \end{bmatrix}$
6 Write the answer.	The image is $A'(4,-1)$ $B'(7,-7)$ $C'(13,2)$ $D'(10,8)$.

USING CAS 2 Matrix representation

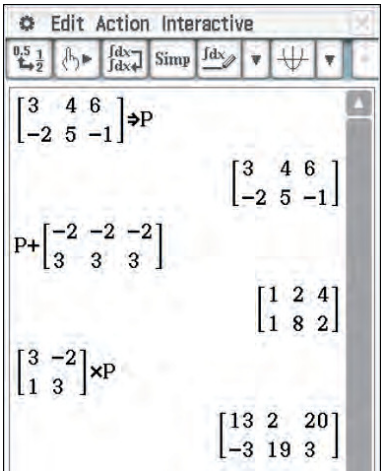
Use matrices to transform the image of $P(3,-2)$ $Q(4,5)$ $R(6,-1)$ under the transformations $(x,y) \rightarrow (x-2, y+3)$ and $(x,y) \rightarrow (3x-2y, x+3y)$.

TI-Nspire



1 Press the **template** key to access the matrix templates.
2 Enter the coordinates of P as a **2 × 3** matrix and **store** the matrix in **p**.
3 For the first transformation, add the **2 × 3** translation matrix to **p**, as shown above.
4 For the second transformation, multiply the **2 × 2** matrix by **p**, as shown above.
5 The coordinates of the images will be displayed in resulting matrices.

ClassPad



1 Open the **Keyboard** > **Math2** to access the matrix templates.
2 Enter the coordinates of P as a **2 × 3** matrix and **store** the matrix in **P**.
3 For the first transformation, add the **2 × 3** translation matrix to **P**, as shown above.
4 For the second transformation, multiply the **2 × 2** matrix by **P**, as shown above.
5 The coordinates of the images will be displayed in resulting matrices.

The coordinates of the transformed image are $P'(1,1)$ $Q'(2,8)$ $R'(4,2)$ and $P'(13,-3)$ $Q'(2,19)$ $R'(20,3)$.

Dilations

From Mathematical Methods Unit 1, Chapter 4 we know that a **dilation** by a factor of 2 **from the *y*-axis** changes the points $(-3, 5)$ and $(2, -1)$ to $(-6, 5)$ and $(4, -1)$. We can show this as $(x, y) \rightarrow (2x, y)$.

Similarly, a dilation by a factor of 3 **from the *x*-axis** changes the points $(-3, 5)$ and $(2, -1)$ to $(-3, 15)$ and $(2, -3)$. We can show this as $(x, y) \rightarrow (x, 3y)$.

Dilations

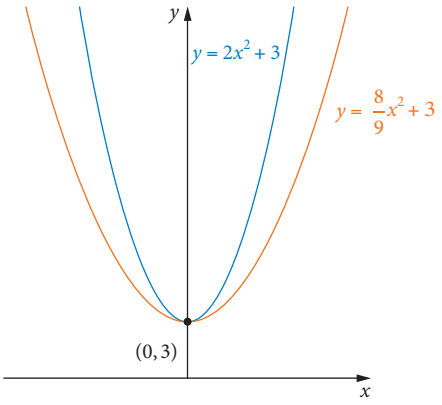
A **dilation** by a factor of a **from the *y*-axis** (or parallel to the x-axis) changes the point $P(x, y)$ to the point (ax, y). The matrix for a dilation from the y-axis is $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$.

A dilation by a factor of b from the x-axis (or parallel to the y-axis) changes the point $P(x, y)$ to the point (x, by). The matrix for a dilation from the x-axis is $\begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}$.

WB
p. 305

WORKED EXAMPLE 5 Dilation of a function

Find the image of the curve $y = 2x^2 + 3$ under dilation by a factor of 1.5 parallel to the x-axis.
Sketch the curves and comment on the transformation.

Steps	Working
1 Write the transformation.	$(x, y) \rightarrow (x', y') = (1.5x, y)$
2 Express x and y in terms of x' and y'.	$x' = \frac{3}{2}x$ and $y' = y$, so $x = \frac{2}{3}x'$ and $y = y'$.
3 Substitute into the equation of the line and express in the normal form.	$y' = 2\left(\frac{2}{3}x'\right)^2 + 3$ So $y' = \frac{8}{9}(x')^2 + 3$.
4 Sketch the parabolas.	
5 Write the answer.	The image is $y = \frac{8}{9}x^2 + 3$; it is 1.5 times wider than $y = 2x^2 + 3$.

We can use CAS to perform dilations of shapes using matrices in the same way as for linear transformations on the previous page.

Remember that $f(x) \rightarrow af(n(x + b)) + c$ follows the order of dilation, reflection and translation ('Dr T').

- a dilates from the x-axis by the factor a and c translates c in the direction of the y-axis.

  If $a < 0$, there is a reflection in the x-axis.

- n dilates from the y-axis by the factor $\frac{1}{n}$ and b translates $-b$ in the direction of the x-axis.

  If $n < 0$, there is a reflection in the y-axis.

Worksheet
Rotations

Rotations

Rotation

A **rotation** through an anticlockwise angle θ with centre at the origin changes a point $P(x, y)$ to the point $P'(x\cos(\theta) - y\sin(\theta), x\sin(\theta) + y\cos(\theta))$.

The matrix for a rotation about the origin is $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.

WB
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WORKED EXAMPLE 6 Rotation of a shape

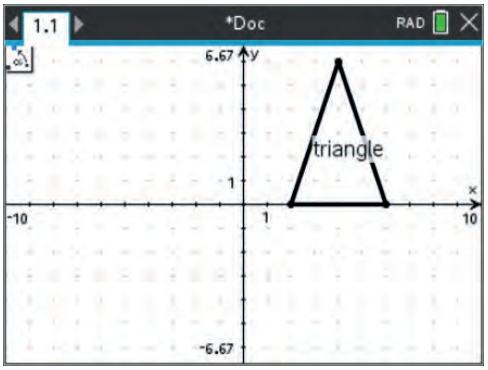
Find the image of the triangle with vertices $A(2,0)$ $B(4,6)$ $C(6,0)$ after rotation through an angle of 120° anticlockwise about the origin.

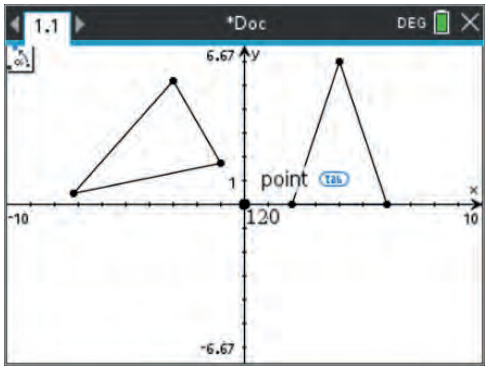
Steps	Working
1 Write the transformation.	$(x, y) \rightarrow (x\cos(120°) - y\sin(120°), x\sin(120°) + y\cos(120°))$
2 Substitute values.	$(x, y) \rightarrow \left(-\frac{1}{2}x - \frac{\sqrt{3}}{2}y, \frac{\sqrt{3}}{2}x - \frac{1}{2}y\right)$
3 Apply the transformation to each point.	$A(2,0) \rightarrow A'\left(-\frac{1}{2} \times 2 - \frac{\sqrt{3}}{2} \times 0, \frac{\sqrt{3}}{2} \times 2 - \frac{1}{2} \times 0\right)$ $B(4,6) \rightarrow B'\left(-\frac{1}{2} \times 4 - \frac{\sqrt{3}}{2} \times 6, \frac{\sqrt{3}}{2} \times 4 - \frac{1}{2} \times 6\right)$ $C(6,0) \rightarrow C'\left(-\frac{1}{2} \times 6 - \frac{\sqrt{3}}{2} \times 0, \frac{\sqrt{3}}{2} \times 6 - \frac{1}{2} \times 0\right)$
4 Simplify.	$A(2,0) \rightarrow A'(-1, \sqrt{3})$ $B(4,6) \rightarrow B'(-2 - 3\sqrt{3}, 2\sqrt{3} - 3)$ $C(6,0) \rightarrow C'(-3, 3\sqrt{3})$
5 Write the answer.	The image is $A'(-1, \sqrt{3})$ $B'(-2 - 3\sqrt{3}, 2\sqrt{3} - 3)$ $C'(-3, 3\sqrt{3})$.

USING CAS 3 Rotations

Show the image of the triangle with vertices $A(2, 0)$ $B(4, 6)$ $C(6, 0)$ after rotation through an angle of 120° about the origin, from the previous example.

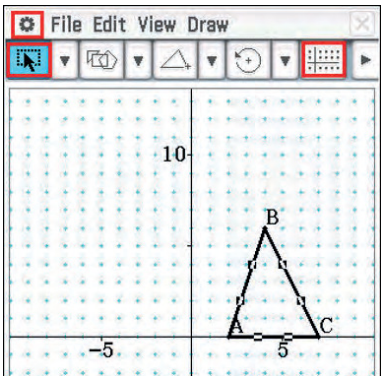
TI-Nspire

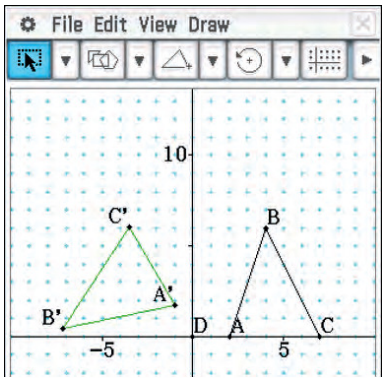




1 Add a **Graphs** page.
2 Press **esc** to hide the **Graph Entry Line**.
3 Press **menu** > **Settings** to ensure the **Graphing Angle** is set to **Degree**.
4 Press **menu** > **View** > **Grid** > **Dot Grid**.
5 Press **menu** > **Geometry** > **Shapes** > **Triangle**.
6 Draw a triangle with vertices **(2, 0)**, **(4, 6)** and **(6, 0)** as shown above.
7 Press **menu** > **Geometry** > **Transformation** > **Rotation**.
8 Move the cursor over the triangle until the word **triangle** appears.
9 Press **enter** to select the triangle.
10 Move the cursor to the origin and press **enter** to select it.
11 Enter **120** on the screen as text.
12 Press enter.
13 The rotated image of the triangle will be displayed.
14 Press **menu** > **Actions** > **Coordinates and Equations**, then select the vertices to find the approximate coordinates (optional).

ClassPad





1 Tap **Menu** > **Geometry**.
2 Tap ⚙ > **View Window** and set **xmin:** to **−10**, **xmax:** to **10** and **ymid**: to **0**.
3 Tap ⚙ > **Geometry Format** to ensure **Measure Angle** is set to **Degree**.
4 Tap the **Grid** tool to display the dot grid.
5 Tap **Draw** > **Basic Object** > **Polygon**.
6 Draw a triangle with vertices **(2, 0)**, **(4, 6)** and **(6, 0)** and back to **(2, 0)** as shown above.
7 Tap the **Select** tool then tap to highlight each side of the triangle.
8 Tap **Draw** > **Construct** > **Rotation**.
9 Tap on the origin.
10 In the dialogue box, enter **120**.
11 The rotated image of the triangle will be displayed.
12 Tap anywhere on the screen to remove the highlighted sides of the original triangle.
13 Tap on the vertices to display the approximate coordinates (optional).

Worksheet
Reflections

Reflections

Reflection

A **reflection** in a line through the origin with inclination θ, $y = x\tan(\theta)$, changes a point $P(x, y)$ to the point $P'(x\cos(2\theta) + y\sin(2\theta), x\sin(2\theta) - y\cos(2\theta))$.

For a line given as $y = mx$, $\theta = \tan^{-1}(m)$, $\cos(2\theta) = p = \dfrac{1 - m^2}{1 + m^2}$ and $\sin(2\theta) = q = \dfrac{2m}{1 + m^2}$.

The matrix for a reflection in a line through the origin is $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} = \begin{bmatrix} p & q \\ q & -p \end{bmatrix}$.

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WORKED EXAMPLE 7 Reflection of a function

Find the image of the line $y = 3x - 2$ after reflection in the line through the origin with inclination 30°.

Steps	Working
1 Write the transformation.	$(x, y) \to (x', y') = (x\cos(60°) + y\sin(60°), x\sin(60°) - y\cos(60°))$
2 Simplify.	$(x', y') = \left(\dfrac{1}{2}x + \dfrac{\sqrt{3}}{2}y, \dfrac{\sqrt{3}}{2}x - \dfrac{1}{2}y\right)$
3 Express x and y in terms of x' and y'.	$x' = \dfrac{1}{2}x + \dfrac{\sqrt{3}}{2}y$ and $y' = \dfrac{\sqrt{3}}{2}x - \dfrac{1}{2}y$
4 Eliminate y by multiplying the 2nd equation by $\sqrt{3}$ and adding it to the 1st equation.	$2x' = x + \sqrt{3}y$ and $2y' = \sqrt{3}x - y$ $2\sqrt{3}y' = 3x - \sqrt{3}y$ $2x' + 2\sqrt{3}y' = x + \sqrt{3}y + 3x - \sqrt{3}y = 4x$
5 Make x the subject.	$x = \dfrac{1}{2}x' + \dfrac{\sqrt{3}}{2}y'$
6 Now eliminate x by multiplying the 1st equation by $\sqrt{3}$, the 2nd equation by 2, and subtracting.	$2\sqrt{3}x' = \sqrt{3}x + 3y$ and $2y' = \sqrt{3}x - y$ $2\sqrt{3}x' - 2y' = \sqrt{3}x + 3y - \sqrt{3}x + y$ $= 4y$
7 Make y the subject.	$y = \dfrac{\sqrt{3}}{2}x' - \dfrac{1}{2}y'$
8 Substitute in $y = 3x - 2$ and simplify.	$\dfrac{\sqrt{3}}{2}x' - \dfrac{1}{2}y' = 3\left(\dfrac{1}{2}x' + \dfrac{\sqrt{3}}{2}y'\right) - 2$ $\sqrt{3}x' - y' = 3x' + 3\sqrt{3}y' - 4$ $(3\sqrt{3} + 1)y' = (\sqrt{3} - 3)x' + 4$ $y' = \dfrac{\sqrt{3} - 3}{3\sqrt{3} + 1}x' + \dfrac{4}{3\sqrt{3} + 1}$ $y' = \dfrac{6 - 5\sqrt{3}}{13}x' + \dfrac{6\sqrt{3} - 2}{13}$
9 Write the answer.	The image of the reflection is the straight line $y = \dfrac{6 - 5\sqrt{3}}{13}x + \dfrac{6\sqrt{3} - 2}{13}$.

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We can use CAS to do reflections of shapes using matrices, as previously shown for linear transformations. We can also show transformations using the Geometry features of CAS.

Exam hack

If you are asked to sketch something, it is not enough to copy from your CAS; you have to label any important points.

EXERCISE 12.1 Translations and linear transformations

ANSWERS p. 609

Mastery

1 WORKED EXAMPLE 1 Find the image of each shape under the given translation.

a $A(3,5)$ $B(5,-3)$ $C(5,-4)$ under $P(x,y) \rightarrow P'(x-2,y-1)$

b $X(-3,-2)$ $Y(0,4)$ $Z(2,-1)$ under $P(x,y) \rightarrow P'(x+5,y+4)$

c $P(-3,-2)$ $Q(3,1)$ $R(1,4)$ $S(-5,1)$ 4 units down and 3 units to the right

d $D(-2,3)$ $E(-1,5)$ $F(0,3)$ $G(2,4)$ $H(1,-3)$ 3 units left and 4 units up

e $K(0,0)$ $L(3,1)$ $M(-1,6)$ under $P(x,y) \rightarrow P'(x+3,y+4)$

2 WORKED EXAMPLE 2 Find the image of each figure under the specified transformation and state the shape of the image.

a The triangle $A(-1,-2)$ $B(-2,1)$ $C(2,-3)$ under the transformation $(x,y) \rightarrow (-2x-2y,3x+y)$

b The square $A(4,-3)$ $B(6,2)$ $C(1,4)$ $D(-1,-1)$ under the transformation $(x,y) \rightarrow (2x-3y,3y-x)$

c The parallelogram $A(4,-3)$ $B(3,-1)$ $C(12,6)$ $D(13,4)$ under $(x,y) \rightarrow (3x-y,4y-2x)$

3 Using CAS 1 Use CAS to find the image of each shape under the specified transformations and state the shape of the image.

a The triangle $A(-1,3)$ $B(2,5)$ $C(3,-3)$ under the transformation T: $(x,y) \rightarrow (2x-4y,4x+3y)$

b The triangle $A(1,2)$ $B(4,4)$ $C(5,-4)$ under the transformation T: $(x,y) \rightarrow (x+y,-x-y)$

c The parallelogram $A(-2,-3)$ $B(-1,-1)$ $C(-3,2)$ $D(-4,0)$ under T: $(x,y) \rightarrow (3y-2x,4x+3y)$

4 WORKED EXAMPLE 3 Using matrices, find the image of each shape under the given translation.

a $A(-1,4)$ $B(2,0)$ $C(6,3)$ under $P(x,y) \rightarrow P'(x-4,y-8)$

b $X(1,2)$ $Y(-1,3)$ $Z(2,-4)$ under $P(x,y) \rightarrow P'(x+2,y+6)$

c $P(2,-5)$ $Q(-2,-5)$ $R(-2,4)$ $S(2,4)$ 2 units right and 3 units down

d $D(-3,3)$ $E(2,3)$ $F(3,-1)$ $G(-1,-3)$ $H(-4,0)$ 3 units left and 4 units up

e $K(3,5)$ $L(5,-3)$ $M(5,-4)$ under $P(x,y) \rightarrow P'(x-3,y-5)$

5 Find the image of each point under the transformations given.

a $(2,-4)$ under $P(x,y) \rightarrow P'(x+1,y-3)$, followed by $P(x,y) \rightarrow P'(x-4,y-5)$

b $(3,6)$ under $P(xy) \rightarrow P'(x-6,y-4)$, followed by 3 units to the right and 7 units up

c $(-3,5)$ under a translation of 2 units up and 4 units to the right, followed by $P(x,y) \rightarrow P'(x+7,y+1)$

d $(1,7)$ under translation of 2 units left and 3 units down, followed by 5 units up and 6 units to the left

e $(-5,0)$ under translation of 6 units up and 4 units to the right, then $P(x,y) \rightarrow P'(x-3,y+2)$

6 WORKED EXAMPLE 4 Express each transformation as a matrix.

a T: $(x,y) \rightarrow (3x,0.5y)$ **b** T: $(x,y) \rightarrow (x+2y,3x)$ **c** T: $(x,y) \rightarrow (y,x)$

d T: $(x,y) \rightarrow (2x-y,x-2y)$ **e** T: $(x,y) \rightarrow (x+3y,y-4x)$

7 Find the image of each equation under the specified transformation.

a $y = 3x - 4$ under the transformation $(x, y) \rightarrow (2x - 3y, y - x)$

b $f(x) = 2x^2 - 4x$ under the transformation $(x, y) \rightarrow (3x, 0.4y)$

c $y = \dfrac{1}{x + 2}$ under the transformation $(x, y) \rightarrow (2x + y, 2x - y)$ using a matrix

d $y^2 - x^2 = 4$ under the transformation $(x, y) \rightarrow (1 - x, y + 3)$ using a matrix

e $x^2 + y^2 = 9$ under the transformation $(x, y) \rightarrow (3x + 4y, 4x - y)$ using a matrix

8 Using CAS 2 Use matrices to find the image of each set of points under the given transformation.

a $P(3, 4)$ $Q(2, 3)$ under the transformation T: $(x, y) \rightarrow (2x + y, x + 2y)$

b $A(-3, 5)$ $B(1, 6)$ $C(2, -4)$ under the transformation T: $(x, y) \rightarrow (x - 3y, x + 3y)$

c $X(1, -2)$ $Y(3, 5)$ $Z(2, -4)$ under the transformation T: $(x, y) \rightarrow (3x, x + y)$

9 WORKED EXAMPLE 5 Find the image of each shape under the given dilation, and state the new shape.

a The rectangle $A(-10, 8)$ $B(-5, 4)$ $C(3, 14)$ $D(-2, 18)$ under a dilation by factor 2 parallel to the x-axis

b The isosceles triangle $A(-5, 4)$ $B(7, 4)$ $C(1, -3)$ under a dilation by factor 0.4 from the x-axis

c The pentagon $A(-2, -3)$ $B(-3, 1)$ $C(0, 3)$ $D(3, 1)$ $E(2, -3)$ under a dilation of factor 2 parallel to the x-axis and 0.5 parallel to the y-axis

10 Express a dilation by the factor 3 parallel to the y-axis as a matrix and find the image of the line $y = 3x - 4$ under the transformation.

11 Express a dilation by the factor 2 from the y-axis as a matrix and find the image of the curve $y = x^3$ under the transformation.

12 Using CAS 3 Find the image of the line segment $A(3, 6)$ $B(5, 2)$ after rotation of $\dfrac{\pi}{3}$ anticlockwise about the origin.

13 WORKED EXAMPLE 6

a Find the image of the line segment $A(3, 6)$ $B(5, 2)$ after rotation through an angle of 45° anticlockwise about the origin.

b Express a rotation of 60° anticlockwise about the origin as a matrix and find the image of the line $y = 2x + 1$ under the rotation.

c Find the image of the triangle $A(2, 4)$ $B(7, 1)$ $C(5, -3)$ after rotation through an angle of 135° anticlockwise about the origin.

d Express a rotation through an angle of 210° anticlockwise about the origin as a matrix and find the image of the square $A(2, 2)$ $B(6, 5)$ $C(9, 1)$ $D(5, -2)$ under the rotation.

14 WORKED EXAMPLE 7

a Express the reflection in a line through the origin with inclination 60° as a matrix and find the reflection of $A(4, 5)$ $B(7, 9)$ $C(11, 6)$ $D(7, 1)$ in the line.

b Find the image of $A(5, 5)$ $B(-5, 0)$ $C(-1, 6)$ after it is reflected in a line through the origin with a slope of $-\dfrac{1}{3}$.

c Express the reflection in a line through the origin with a slope of 3 as a matrix and find the image of the line $y = 5 - 2x$ after the reflection.

Exam practice

15 The image of $(3, -2)$ under the transformation $(x, y) \to (2x - 3y, x + 2y)$ is

A $(0, -1)$ **B** $(12, -1)$ **C** $(6, -4)$ **D** $(0, -4)$ **E** $(6, 6)$

16 The matrix that represents a dilation by a factor of 3 parallel to the x-axis is

A $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$ **C** $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ **D** $\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$ **E** $\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$

17 The image of $(2, 4)$ after rotation through an angle of 60° anticlockwise about the origin is

A $(1 + 2\sqrt{3}, \sqrt{3} - 2)$ **B** $(2\sqrt{3}, 4)$ **C** $(\sqrt{3} - 2, 1 + 2\sqrt{3})$

D $(\sqrt{3} + 2, 1 - 2\sqrt{3})$ **E** $(1 - 2\sqrt{3}, 2 + \sqrt{3})$

18 The matrix that represents reflection in the line $y = 2x$ is

A $\begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$ **B** $\begin{bmatrix} -0.6 & -0.8 \\ -0.8 & 0.6 \end{bmatrix}$ **C** $\begin{bmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{bmatrix}$

D $\begin{bmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{bmatrix}$ **E** $\begin{bmatrix} 0.8 & -0.6 \\ -0.6 & -0.8 \end{bmatrix}$

19 The image of the line $y = 2x + 3$ after a dilation from the y-axis by a factor of 3 is

A $y = \frac{2}{3}x + 1$ **B** $y = 3x + 3$ **C** $y = \frac{2}{3}x + 3$

D $y = 3x + 9$ **E** $y = 3x + 1$

12.2 Inverse transformations

A mathematical **inverse** returns a changed value to the original value, whether this is a number, point or position.

Video playlist
Inverse transformations

Inverse of a dilation

The inverse of a dilation of factor k from an axis is the dilation by the factor $\frac{1}{k}$ from the same axis.

WORKED EXAMPLE 8 Inverse of a dilation

a Find the inverse of a dilation by the factor 0.4 from the x-axis.

b Confirm they are inverses by applying them in succession to a general point.

Steps	Working
a 1 Find the reciprocal of 0.4.	$\frac{1}{0.4} = 2.5$
2 Write the inverse.	The inverse is the dilation by the factor 2.5 from the x-axis.
b 1 Write both transformations.	$(x, y) \to (x, 0.4y)$ and $(x, y) \to (x, 2.5y)$.
2 Apply the first to (a, b).	$(a, b) \to (a, 0.4b)$
3 Now apply the second.	$(a, 0.4b) \to (a, 2.5 \times 0.4b) = (a, b)$
4 State the conclusion.	The dilation by the factor 0.4, dilated by the factor 2.5, gives the original point, so they are inverses.

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We can demonstrate that the dilations in the previous example are inverses using the matrix representations of the dilations.

$\begin{bmatrix} 1 & 0 \\ 0 & 2.5 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ 2.5b \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix}\begin{bmatrix} a \\ 2.5b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$, so the original point is recovered.

Notice that the overall product is $\begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2.5 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$.

The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the **matrix** for the **identity transformation** I: $(x, y) \rightarrow (x, y)$.

The physical effects of rotations and reflections make the inverses obvious.

Inverse of a rotation and reflection

The **inverse of a rotation** through an angle θ about the origin is the rotation through the angle $-\theta$.

The **inverse of a reflection** is the same reflection. It is its own inverse.

Worksheet
Inverse and determinant of a matrix

Inverse linear transformations

The inverse of a general linear transformation is more difficult to work out than the inverse of dilations, rotations and reflections. We can use the formula for the inverse of a 2 × 2 matrix to find the inverse.

Inverse linear transformations

If it exists, the inverse of the 2 × 2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by $\frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

The inverse of the transformation T: $(x, y) \rightarrow (ax + by, cx + dy)$ is thus given by

T^{-1}: $(x, y) \rightarrow \frac{1}{ad - bc}(dx - by, -cx + ay)$, provided $ad - bc \neq 0$.

The value $ad - bc$ is called the **determinant** of the transformation and written as $|T|$.

A linear transformation for which $ad - bc = 0$ is called a **singular transformation** and does not have an inverse.

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WORKED EXAMPLE 9 Inverse of a linear transformation

Find the inverse of the transformation T: $(x, y) \rightarrow (2x + 3y, 4x + 5y)$ and confirm that it is the inverse by using the point $(5, -3)$.

Steps	Working
1 Find the value of the determinant.	$\lvert T\rvert = ad - bc = 2 \times 5 - 3 \times 4 = -2 \neq 0$
2 Write the rule for the inverse.	T^{-1}: $(x, y) \rightarrow \frac{1}{\lvert T\rvert}(dx - by, -cx + ay)$
3 Substitute in the values.	T^{-1}: $(x, y) \rightarrow -0.5(5x - 3y, -4x + 2y)$
4 Simplify.	T^{-1}: $(x, y) \rightarrow (1.5y - 2.5x, 2x - y)$
5 Apply the transformation to $(5, -3)$.	$(5, -3) \rightarrow (2 \times 5 + 3 \times (-3), 4 \times 5 + 5 \times (-3)) = (1, 5)$
6 Apply the inverse to $(1, 5)$.	$(1, 5) \rightarrow (1.5 \times 5 - 2.5 \times 1, 2 \times 1 - 5) = (5, -3)$
7 Write the answer.	The inverse reverses the transformation for $(5, -3)$.

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USING CAS 4 Inverse transformations

Use matrices to find the inverse transformation of

a the linear transformation $(x, y) \rightarrow (2x - 2y, 2x - 3y)$

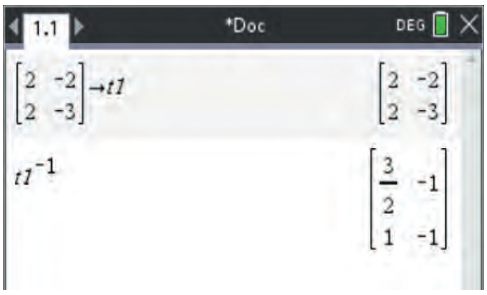
b the dilation of factor 1.5 parallel to the y-axis

c the rotation through an angle of 240° about the origin

d the reflection in the line $y = -3x$.

TI-Nspire

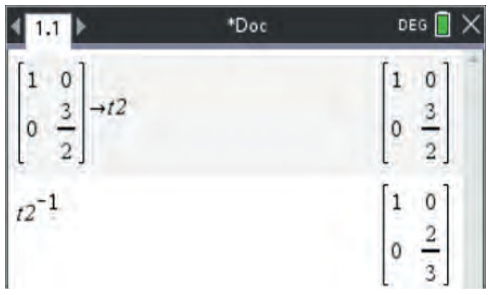
a



1 Create the matrix $\begin{bmatrix} 2 & -2 \\ 2 & -3 \end{bmatrix}$ and **store** it into ***t*1**.

2 Enter ***t*1**$^{-1}$ to find the inverse, which is $(x, y) \rightarrow \left(\frac{3}{2}x - y, x - y\right)$.

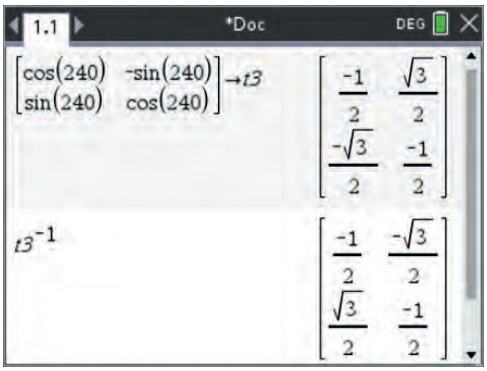
b



1 Create the matrix $\begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{bmatrix}$ and **store** it into ***t*2**.

2 Enter ***t*2**$^{-1}$ to find the inverse, which is a dilation of factor of $\frac{2}{3}$ parallel to the y-axis.

c



1 Set the angle to **DEG**.

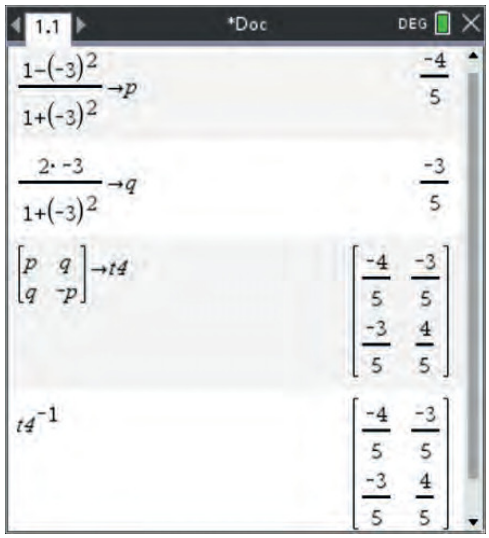
2 Create the matrix

$$\begin{bmatrix} \cos(240°) & -\sin(240°) \\ \sin(240°) & \cos(240°) \end{bmatrix}$$

and **store** it into ***t*3**.

3 Enter ***t*3**$^{-1}$ to find the inverse, which is a rotation through −240° about the origin.

d



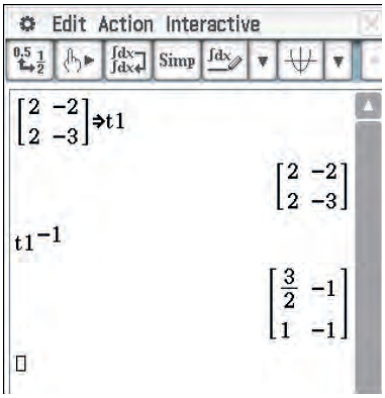
1 For $m = -3$, calculate $p = \frac{1 - m^2}{1 + m^2}$ and $q = \frac{2m}{1 + m^2}$.

2 Create the matrix $\begin{bmatrix} p & q \\ q & -p \end{bmatrix}$ and **store** it into ***t*4**.

3 Enter ***t*4**$^{-1}$ to find the inverse, which means the inverse is a reflection across the line $y = -3x$.

ClassPad

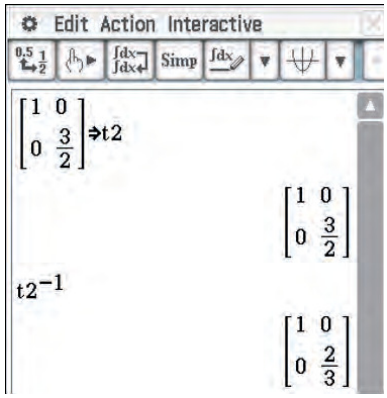
a



1 Create the matrix $\begin{bmatrix} 2 & -2 \\ 2 & -3 \end{bmatrix}$ and **store** it into ***t*1**.

2 Enter ***t*1**$^{-1}$ to find the inverse, which is $(x, y) \to \left(\frac{3}{2}x - y, x - y\right)$.

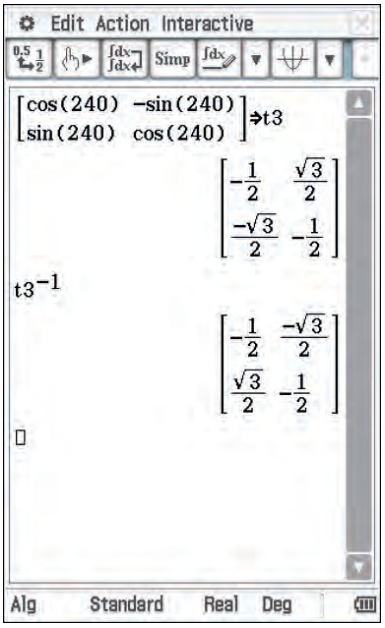
b



1 Create the matrix $\begin{bmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{bmatrix}$ and **store** it into ***t*2**.

2 Enter ***t*2**$^{-1}$ to find the inverse, which is a dilation of a factor of $\frac{2}{3}$ parallel to the y-axis.

c



1 Set the angle to **Deg**.

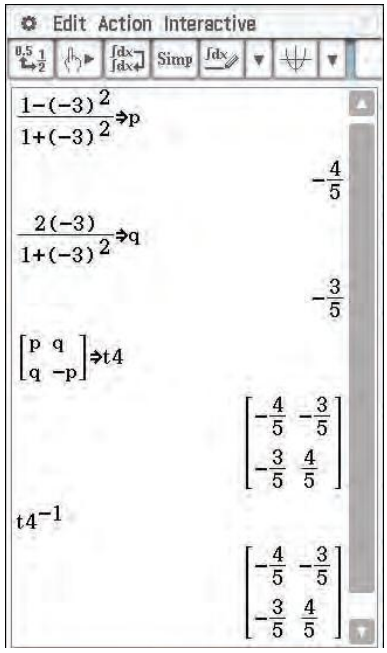
2 Create the matrix

$$\begin{bmatrix} \cos(240) & -\sin(240) \\ \sin(240) & \cos(240) \end{bmatrix}$$

and **store** it into ***t*3**.

3 Enter ***t*3**$^{-1}$ to find the inverse, which is a rotation through −240° about the origin.

d



1 For $m = -3$, calculate $p = \frac{1 - m^2}{1 + m^2}$ and $q = \frac{2m}{1 + m^2}$.

2 Create the matrix $\begin{bmatrix} p & q \\ q & -p \end{bmatrix}$ and **store** it into ***t*4**.

3 Enter ***t*4**$^{-1}$ to find the inverse, which means the inverse is a reflection across the line $y = -3x$.

9780170448338

EXERCISE 12.2 Inverse transformations

ANSWERS p. 610

Recap

1 The image of the point $(3, 2)$ after a reflection in the line $y = \sqrt{3}x$ is

A $\left(\frac{3\sqrt{3}-2}{2}, \frac{2\sqrt{3}+3}{2}\right)$ **B** $(3\sqrt{3}, 2)$ **C** $(3, 2\sqrt{3})$

D $\left(\frac{2\sqrt{3}-3}{2}, \frac{3\sqrt{3}+2}{2}\right)$ **E** $\left(\frac{3\sqrt{3}+2}{2}, \frac{2\sqrt{3}-3}{2}\right)$

2 The matrix $\begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}$ represents the transformation

A rotation through the angle 45° about the origin.

B reflection in the line $y = \sqrt{2}x$.

C dilation from the x-axis by the factor $\sqrt{2}$.

D dilation parallel to the x-axis by the factor $\sqrt{2}$.

E reflection in the line at an inclination of 45° through the origin.

Mastery

3 WORKED EXAMPLE 8 Find the inverse of each transformation. Confirm it is the inverse.

a A dilation of factor 3 from the y-axis

b A dilation of factor 0.5 parallel to the x-axis

c A dilation by the factor $\frac{1}{4}$ parallel to the y-axis

d A dilation by the factor 2 from the x-axis

e A rotation of 50° about the origin

f A rotation of $-\frac{2\pi}{3}$ about the origin

g A reflection in a line through the origin with inclination 75°

h A reflection in a line though the origin with inclination $-\frac{\pi}{4}$

4 WORKED EXAMPLE 9 Find the inverse of each transformation.

a $(x, y) \rightarrow (x + y, x - y)$ **b** $(x, y) \rightarrow (x + 2y, 3x + 2y)$

c $(x, y) \rightarrow (3x - 2y, 4x - 3y)$ **d** $T: (x, y) \rightarrow (3x - 2y, 6x + 4y)$

e $T: (x, y) \rightarrow (x + 2y, 2y - x)$

5 Using CAS 4 Use matrices to find the inverse of each transformation.

a $(x, y) \rightarrow (2x + 4y, x - 3y)$

b $(x, y) \rightarrow (3x - y, x + 2y)$

c A dilation parallel to the x-axis by the factor 3

d A rotation through the angle $\frac{3\pi}{4}$ about the origin

e A reflection in the line $y = 4x$

Exam practice

6 A transformation is represented by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1.25 \end{bmatrix}$.

The inverse transformation is

A the rotation through an angle –1.25.

B the reflection in the line $y = 0.8x$.

C the dilation parallel to the y-axis with factor 0.8.

D the dilation parallel to the x-axis of factor 1.25.

E the transformation $(x, y) \rightarrow (x - 1.25y, 1.25x + y)$.

7 A transformation is represented by the matrix $\begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$. The inverse transformation is

A a rotation of 45° about the origin.

B a rotation of –135° about the origin.

C a reflection in a line of inclination 67.5° through the origin.

D a reflection in a line of inclination –67.5° through the origin.

E a rotation of –45° about the origin.

8 The inverse transformation to a reflection in the line $y = \frac{x}{3}$ is represented by

A $\begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$ **B** $\begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}$ **C** $\begin{bmatrix} \frac{2}{3} & -\frac{8}{9} \\ \frac{8}{9} & \frac{2}{3} \end{bmatrix}$

D $\begin{bmatrix} \frac{8}{9} & \frac{2}{3} \\ \frac{2}{3} & -\frac{8}{9} \end{bmatrix}$ **E** $\begin{bmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{bmatrix}$

9 The inverse transformation for $(x, y) \rightarrow (3x + 7y, x + 2y)$ is

A $(x, y) \rightarrow (7y - 2x, x - 3y)$ **B** $(x, y) \rightarrow (2x - 7y, x - 3y)$

C $(x, y) \rightarrow \frac{1}{19}(2x - 7y, x - 3y)$ **D** $(x, y) \rightarrow \frac{1}{19}(2x - 7y, x - 3y)$

E $(x, y) \rightarrow \frac{1}{11}(2x - 7y, x - 3y)$

10 (3 marks)

a Show that the transformation T: $(x, y) \rightarrow (6x - 8y, 9x - 12y)$ has no inverse. 1 mark

b Find the image of a unit square and hence the image of the plane under the transformation. 2 marks

12.3 Composite transformations

The **composition of two transformations** is the transformation obtained by doing one transformation followed by the other. The **composition** $T \circ S$ means doing S first and then doing T on the result.

Exam hack

The order of compositions is important. Always do the one on the right first.

WORKED EXAMPLE 10 Using a composition of 2 transformations

$T: (x, y) \rightarrow (2x + 3y, 4x + 5y)$, $S: (x, y) \rightarrow (2x, y)$ and $R: (x, y) \rightarrow (x - 1, y + 2)$

Find

a $T \circ S(3, -4)$ **b** $S \circ T(3, -4)$ **c** $T \circ R(3, -4)$ **d** $R \circ T(3, -4)$

Steps	Working
a 1 Perform the transformation S first.	$(3, -4) \rightarrow (2 \times 3, -4) = (6, -4)$
2 Now perform T on the result.	$(6, -4) \rightarrow (2 \times 6 + 3 \times (-4), 4 \times 6 + 5 \times (-4)) = (0, 4)$
3 Write the final answer.	$T \circ S(3, -4) = (0, 4)$
b 1 Perform the transformation T first.	$(3, -4) \rightarrow (2 \times 3 + 3 \times (-4), 4 \times 3 + 5 \times (-4)) = (-6, -8)$
2 Now perform S on the result.	$(-6, -8) \rightarrow (2 \times (-6), -8) = (-12, -8)$
3 Write the final answer.	$S \circ T(3, -4) = (-12, -8)$
c 1 Perform the transformation R first.	$(3, -4) \rightarrow (3 - 1, -4 + 2) = (2, -2)$
2 Now perform T on the result.	$(2, -2) \rightarrow (2 \times 2 + 3 \times (-2), 4 \times 2 + 5 \times (-2)) = (-2, -2)$
3 Write the result.	$T \circ R(3, -4) = (-2, -2)$
d 1 Perform the transformation T first.	$(3, -4) \rightarrow (2 \times 3 + 3 \times (-4), 4 \times 3 + 5 \times (-4)) = (-6, -8)$
2 Now perform R on the result.	$(-6, -8) \rightarrow (-6 - 1, -8 + 2) = (-7, -6)$
3 Write the result.	$R \circ T(3, -4) = (-7, -6)$

Video playlist
Composite transformations

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Notice that if the order is reversed, the compositions above are different. These compositions are **not commutative**. We can work out the *rule* for a composite transformation.

WORKED EXAMPLE 11 Rule for a composition

$T: (x, y) \rightarrow (2x + 3y, 4x + 5y)$ and $S: (x, y) \rightarrow (x - 2y, x + y)$. Find $T \circ S$.

Steps	Working
1 Do S first on a general point.	$(x, y) \rightarrow (x - 2y, x + y)$
2 Now do T on the result. For T, think 2 times the first + 3 times the second, then 4 times the first + 5 times the second.	$(x - 2y, x + y) \rightarrow (2 \times (x - 2y) + 3 \times (x + y), 4 \times (x - 2y) + 5 \times (x + y))$
3 Simplify.	$= (2x - 4y + 3x + 3y, 4x - 8y + 5x + 5y)$ $= (5x - y, 9x - 3y)$
4 Write the overall answer.	$T \circ S: (x, y) \rightarrow (5x - y, 9x - 3y)$

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Your teacher might want you to work out $S \circ T$ and compare your result to $T \circ S$.

Composition using matrices

We can use matrix multiplication to find composite linear transformations, but not translations. Remember that these are represented as 2×1 matrices and have to be added.

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WORKED EXAMPLE 12 Using matrices for a composition

$T: (x, y) \to (2x + 3y, 4x + 5y)$ and $S: (x, y) \to (x - 2y, x + y)$. Use matrices to find $T \circ S$.

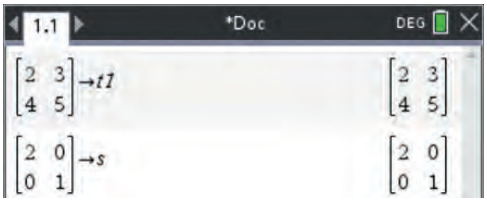
Steps	Working
1 Write S as a matrix.	$S = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$
2 Write T as a matrix.	$T = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$
3 Find $T \circ S$ as the matrix product.	$T \circ S = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 9 & -3 \end{bmatrix}$
4 Write the overall answer.	$T \circ S: (x, y) \to (5x - y, 9x - 3y)$

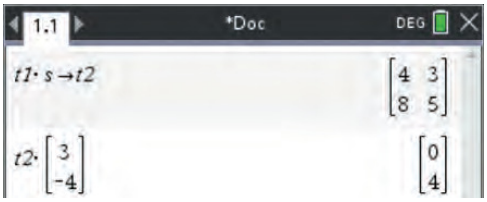
It is easier to compose transformations with matrices than doing it 'by hand'.

USING CAS 5 Composition using matrices

Given $T: (x, y) \to (2x + 3y, 4x + 5y)$ and $S: (x, y) \to (2x, y)$, find $T \circ S$ and $T \circ S(3, -4)$, as in the previous example.

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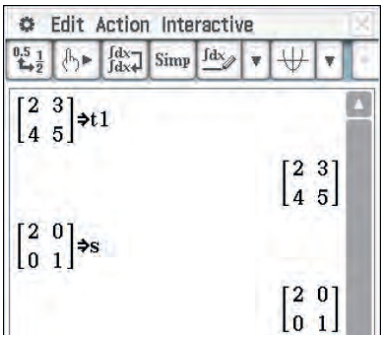
1 Enter the matrix $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ and **store** into ***t1***.

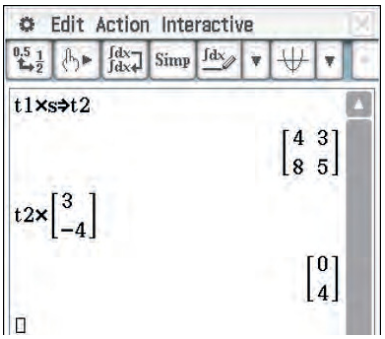
2 Enter the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and **store** it into **s**.

3 Enter ***t1*** × **s** and store in ***t2***.

4 Enter ***t2*** × $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$.

ClassPad





1 Enter the matrix $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ and **store** into ***t1***.

2 Enter the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and **store** it into **s**.

3 Enter ***t1*** × **s** and store in ***t2***.

4 Enter ***t2*** × $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$.

$T \circ S: (x, y) \to (4x + 3y, 8x + 5y)$ and $T \circ S(3, -4) = (0, 4)$

EXERCISE 12.3 Composite transformations

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Recap

1 The inverse of the transformation $(x, y) \to (3x - y, 5x - 2y)$ is

A $(x, y) \to (y - 2x - y, 5x + 3y)$

B $(x, y) \to (2x - y, -5x - 3y)$

C $(x, y) \to \left(\frac{1}{11}y - \frac{2}{11}x, \frac{5}{11}x + \frac{3}{11}y\right)$

D $(x, y) \to \left(\frac{2}{11}x - \frac{1}{11}y, -\frac{5}{11}x - \frac{3}{11}y\right)$

E $(x, y) \to (2x - 5y, -x - 3y)$

2 The inverse of the transformation represented by $T = \begin{bmatrix} \frac{5}{13} & \frac{12}{13} \\ \frac{12}{13} & -\frac{5}{13} \end{bmatrix}$ is

A $(x, y) \to (4x - y, 5x + 2y)$

B $(x, y) \to (4x + 5y, -x + 2y)$

C $(x, y) \to \left(\frac{5}{13}x + \frac{12}{13}y, \frac{12}{13}x - \frac{5}{13}y\right)$

D $(x, y) \to \left(-\frac{5}{13}x - \frac{12}{13}y, -\frac{12}{13}x + \frac{5}{13}y\right)$

E $(x, y) \to \left(\frac{65}{129}x - \frac{155}{129}y, \frac{155}{129}x + \frac{65}{129}y\right)$

Mastery

3 WORKED EXAMPLE 10 $T: (x, y) \to (x - y, x - 2y)$, $S: (x, y) \to (x + 2y, 2x - y)$, $U: (x, y) \to (2x - 3y, 3x - y)$ and $V: (x, y) \to (x + 2, y - 3)$. Find

a $T \circ S(2, 3)$ **b** $S \circ U(-1, 4)$ **c** $U \circ V(3, -3)$ **d** $V \circ U(3, -3)$

e $T \circ T(-5, -1)$ **f** $T \circ V(5, -5)$ **g** $S \circ V(-6, 0)$ **h** $V \circ S(-6, 0)$

i $T \circ S \circ U(3, -5)$ **j** $U \circ S \circ T(3, -5)$

4 WORKED EXAMPLE 11 $T: (x, y) \to (2x + y, 3x - y)$, $S: (x, y) \to (x + y, 2x + 3y)$, $U: (x, y) \to (x - 3, y + 4)$ and $V: (x, y) \to (3x + 4y, 2x + 2y)$. Find

a $T \circ S$ **b** $S \circ T$ **c** $U \circ V$ **d** $V \circ U$

e $S \circ V$ **f** $V \circ S$ **g** $V \circ T$ **h** $T \circ V$

i $U \circ T$ **j** $U \circ S$

5 WORKED EXAMPLE 12 T is a dilation of factor 1.5 from the y-axis, S is a rotation of $-\frac{2\pi}{3}$ about the origin, U is a reflection in the line $y = x\tan(60°)$ and $V: (x, y) \to (x + y, x - 2y)$. Express each composite transformation as a matrix.

a $T \circ V$ **b** $V \circ U$ **c** $S \circ U$ **d** $V \circ T$

e $S \circ V$ **f** $V \circ S$ **g** $S \circ T \circ U$ **h** $T \circ S \circ U$

i $S \circ U \circ T$ **j** $S \circ U \circ V$

6 Using CAS 5 $A: (x, y) \to (2x, x + y)$, $B: (x, y) \to (3x - y, 2y - x)$, C is a rotation about the origin through the angle $\frac{\pi}{6}$, D is a dilation by the factor 3 parallel to the y-axis and a dilation by the factor 2 parallel to the x-axis, and E is a reflection in the line through the origin at the inclination $-\frac{\pi}{6}$. Simplify each composite transformation.

a $A \circ B(2, 4)$ **b** $B \circ C(-3, -4)$ **c** $D \circ E(4, -5)$ **d** $E \circ A$

e $E \circ B \circ D$ **f** $B \circ A \circ D$ **g** $B \circ E \circ E$ **h** $E \circ B \circ E$

Exam Practice

7 S is a rotation through an angle of 60° about the origin and T is a dilation by factor 2 parallel to the x-axis. $S \circ T(3, -4) =$

A $\left(4\sqrt{3} + 3, \frac{3\sqrt{3}}{2} - 2\right)$ **B** $(3\sqrt{3} + 2, 3 - 2\sqrt{3})$ **C** $(2\sqrt{3} + 3, 3\sqrt{3} - 2)$

D $\left(3\sqrt{3} + 4, \frac{3}{2} - 2\sqrt{3}\right)$ **E** $\left(4\sqrt{3} + \frac{3}{2}, \frac{3\sqrt{3}}{2} - 4\right)$

8 S is a rotation through an angle of $\frac{\pi}{3}$ about the origin and T is a reflection in a line through the origin with inclination $\frac{\pi}{3}$. $S \circ T$, in matrix form, is

A $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ **B** $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ **C** $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

D $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ **E** $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$

9 T is a dilation of factor 1.5 from the x-axis and $U: (x, y) \to (x - 3y, y - 3x)$. $T \circ U$ is given by

A $(x, y) \to (x - 3y, 1.5y - 4.5x)$ **B** $(x, y) \to (x - 4.5y, 1.5y - 3x)$

C $(x, y) \to (1.5x - 4.5y, y - 3x)$ **D** $(x, y) \to (1.5x - 3y, y - 4.5x)$

E $(x, y) \to (1.5x - 4.5y, x - 3y)$

10 $P: (x, y) \to (2x - y, x - 3y)$ and $Q: (x, y) \to (3x + 2y, 4x + 3y)$. $P \circ Q^{-1} =$

A $(x, y) \to (5x + y, 5x)$ **B** $(x, y) \to (4x - 5y, 3x - 5y)$

C $(x, y) \to (4x - 3y, 5x - 5y)$ **D** $(x, y) \to (4x + 3y, -5x - 5y)$

E $(x, y) \to (10x - 7y, 15x - 11y)$

11 (3 marks) Show that the composition of a reflection in the line $y = x\tan(60°)$ and a reflection in the line $y = x\tan(30°)$ is the same as a rotation through 60° about the origin.

12.4 Properties of transformations

Invariance properties

A relation or property that doesn't change under a transformation is called an **invariance property**.

We say it is **invariant** under the transformation.

WORKED EXAMPLE 13 Linear transformation of a straight line

Show that the image of a straight line L with slope m is another straight line with slope $\frac{c+md}{a+mb}$ under the non-singular linear transformation T: $(x, y) \to (ax + by, cx + dy)$.

Steps	Working
1 Write the equation of a straight line L.	$y = mx + k$ has slope m and y-intercept k.
2 Find the image of (x, y) under T.	$(x', y') = T(x, y) = (ax + by, cx + dy)$
3 Write x' and y' in terms of x and y.	$x' = ax + by, y' = cx + dy$
4 Express x and y in terms of x' and y'.	$dx' = dax + dby, by' = bcx + bdy$ $dx' - by' = adx - bcx$
T is non-singular, so $ad - bc \neq 0$ and you can divide by it.	$x = \frac{dx' - by'}{ad - bc}$, similarly $y = \frac{ay' - cx'}{ad - bc}$
5 Substitute in the equation of L.	$\frac{ay' - cx'}{ad - bc} = m\frac{dx' - by'}{ad - bc} + k$
6 Simplify and express y' in terms of x'.	$ay' - cx' = mdx' - mby' + k(ad - bc)$ $(a + mb)y' = (c + md)x' + k(ad - bc)$ $y' = \frac{c+md}{a+mb}x' + \frac{k(ad-bc)}{a+mb}$
7 Write the conclusion.	The image of the straight line under the transformation T is a straight line with slope $\frac{c+md}{a+mb}$ $\left(\text{and } y\text{-intercept } \frac{k(ad-bc)}{a+mb}\right)$.

Invariance of straight lines

Straight lines are invariant under non-singular linear transformations: the images of straight lines are themselves straight lines.

For a singular transformation, $ad = bc$, so $\frac{c}{a} = \frac{d}{b} (a, b \neq 0)$.

Writing $\frac{c}{a} = \frac{d}{b} = k$ gives $c = ka$ and $d = kb$, so $(ax + by, cx + dy) = (ax + by, kax + kby)$.

In this case, for $(x, y) \to (ax + by, cx + dy) = (x', y')$, $y' = kax + kby = k(ax + by) = kx'$.

$y' = kx'$ is a line passing through the origin, so a linear transformation with no inverse must transform the plane to a line passing through the origin. In the trivial case, it produces just the origin itself.

Video playlist
Properties of transformations

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Singular transformation of a straight line

The image of a straight line under a **singular transformation** is a straight line passing through the origin.

Since a straight line with slope m is transformed by a linear transformation to a straight line with slope $\frac{c+md}{a+mb}$, another straight line with the same slope will also be transformed to a straight line with slope $\frac{c+md}{a+mb}$.

Straight lines with the same slope are parallel.

Invariance of parallel lines

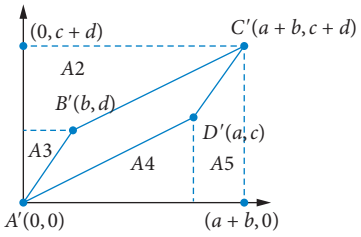
Parallel lines are invariant under non-singular linear transformations: the images of parallel lines are themselves parallel.

Transformations of shapes

We have already seen that shapes can change under a linear transformation. However, since parallel lines remain parallel, parallelograms are transformed to other parallelograms by non-singular transformations. However, angles may be changed by linear transformations, so a rectangle could be changed to a parallelogram.

Rotations and reflections do not change shapes, so shapes are invariant under these linear transformations.

Consider the unit square $A(0,0)$ $B(0,1)$ $C(1,1)$ $D(1,0)$ under the transformation T: $(x,y) \rightarrow (ax+by, cx+dy)$.



The unit square becomes $A'(0,0)$ $B'(b,d)$ $C'(a+b,c+d)$ $D'(a,c)$.

This is a parallelogram, as shown in the above diagram.

The area of the parallelogram is the area $A1$ of the large rectangle formed by $A'(0,0)$, $(0,c+d)$, $C'(a+c,b+d)$, $(a+b,0)$, less the areas of the triangles $A3$ and $A4$ and the trapeziums $A2$ and $A5$.

$$A1 = (a+b)(c+d) = ac + ad + bc + bd$$

$$A2 = \frac{c(b+a+b)}{2} = \frac{1}{2}ac + bc$$

$$A3 = \frac{1}{2}bd$$

$$A4 = \frac{1}{2}ac$$

$$A5 = \frac{b(c+c+d)}{2} = \frac{1}{2}bd + bc$$

so the area of the parallelogram is

$$\begin{aligned} A &= ac + ad + bc + bd - \frac{1}{2}ac - bc - \frac{1}{2}bd - \frac{1}{2}ac - \frac{1}{2}bd - bc \\ &= ad - bc \end{aligned}$$

However, $ad - bc = |T|$, the determinant of the transformation.

We can approximate the area of any shape by counting unit squares that fill the shape.

Each of these will be transformed by T to areas of size $|T|$.

Thus, the transformed shape will have an area of $|T|$ times the area of the original shape.

Linear transformations of shapes

The image of a shape of area A will be a shape of area $|T|A$ under the linear transformation T, where $|T|$ is the determinant of the transformation.

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WORKED EXAMPLE 14 Areas under linear transformations

Show that $A(-1, 3)$ $B(3, -3)$ $C(-3, 1)$ $D(-7, 7)$ is a rhombus and find the area of its image under the transformation T: $(x, y) \rightarrow (2x + y, 3x - 2y)$.

Steps	Working
1 Check that $ABCD$ is a rhombus by finding the lengths of the sides.	$AB = \sqrt{(3-(-1))^2 + (-3-3)^2} = \sqrt{52}$ $BC = \sqrt{(-3-3)^2 + (1-(-3))^2} = \sqrt{52}$ $CD = \sqrt{(-7-(-3))^2 + (7-1)^2} = \sqrt{52}$ $AD = \sqrt{(-7-(-1))^2 + (7-3)^2} = \sqrt{52}$
2 Write the result.	The side lengths are equal, so $ABCD$ is a rhombus.
3 Find the lengths of the diagonals.	$AC = \sqrt{(1-3)^2 + (-3-(-1))^2} = 2\sqrt{2}$ $BD = \sqrt{(7-(-3))^2 + (-7-3)^2} = 10\sqrt{2}$
4 Find the area.	area of $ABCD = \frac{1}{2} AC \times BD$ $= \frac{1}{2} \times 2\sqrt{2} \times 10\sqrt{2}$ $= 20$
5 Find the determinant of T.	$\|T\| = ad - bc$ $= 2 \times (-2) - 1 \times 3$ $= -7$
6 Find the area of the image.	area of image $= \|T\| \times$ area of object $= 7 \times 20$ $= 140$
7 Write the answer.	The area of the image of $ABCD$ under T is 140 square units.

Other properties

We already know that reflections are their own inverses. There are other transformations that are their own inverses.

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WORKED EXAMPLE 15 Transformation that is its own inverse

Show that $T: (x, y) \rightarrow (3x + 4y, -2x - 3y)$ is its own inverse.

Steps	Working
1 Write T as a matrix.	$T = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$
2 Multiply it by itself.	$= \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$ $= \begin{bmatrix} 3 \times 3 + 4 \times (-2) & 3 \times 4 + 4 \times (-3) \\ -2 \times 3 + (-3) \times (-2) & -2 \times 4 + (-3) \times (-3) \end{bmatrix}$ $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
3 Write the conclusion.	The product is the identity, so the transformation is its own inverse.

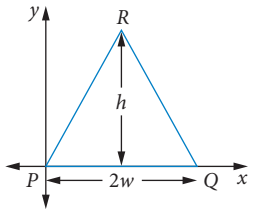
Can a triangle have its properties preserved under a transformation?

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WORKED EXAMPLE 16 Proving a property of a transformation

Show that the image of an isosceles triangle with its base parallel to the x-axis is also isosceles under a dilation from the x-axis.

Steps	Working
1 Choose a general isosceles triangle, ΔPQR.	Let the base length be $2w$ and the height be h. One vertex may be chosen to be the origin with no loss of generality, so the coordinates of the triangle are $P(0, 0)$ $Q(2w, 0)$ $R(w, h)$.
2 Check that it is isosceles.	$PR = \sqrt{(w - 0)^2 + (h - 0)^2} = \sqrt{w^2 + h^2}$ $QR = \sqrt{(w - 2w)^2 + (h - 0)^2} = \sqrt{w^2 + h^2}$ $PR = QR$, so ΔPQR is isosceles.
3 Choose a general dilation from the x-axis.	Choose $T: (x, y) \rightarrow (x, ky)$.
4 Apply the transformation to ΔPQR.	$P(0, 0) \rightarrow P'(0, 0)$ $Q(2w, 0) \rightarrow Q'(2w, 0)$ $R(w, h) \rightarrow R'(w, kh)$
5 Find the lengths of $P'R'$ and $Q'R'$.	$P'R' = \sqrt{(w - 0)^2 + (kh - 0)^2} = \sqrt{w^2 + k^2h^2}$ $Q'R' = \sqrt{(w - 2w)^2 + (kh - 0)^2} = \sqrt{w^2 + k^2h^2}$
6 Write the conclusion.	$P'R' = Q'R'$, so the image is also isosceles under the dilation.



Exam hack

If possible, choose easy general points in a proof. You can often choose the origin as one point and a point on an axis as another.

9780170448338

EXERCISE 12.4 Properties of transformations

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Recap

1 $T: (x, y) \to (2x + 3y, 4y)$ and $U: (x, y) \to (3x, x - y)$. $T \circ U(-4, 1) =$

A $(33, 12)$ **B** $(-9, -15)$ **C** $(-22, -27)$ **D** $(-39, -20)$ **E** $(-32, 8)$

2 $G: (x, y) \to (4x - 5y, 3x + y)$ and $H: (x, y) \to (x + 3y, 2x - y)$. $G \circ H$ is represented by

A $\begin{bmatrix} 13 & 8 \\ 5 & 9 \end{bmatrix}$ **B** $\begin{bmatrix} 14 & 7 \\ 5 & 8 \end{bmatrix}$ **C** $\begin{bmatrix} 13 & -2 \\ 5 & -11 \end{bmatrix}$ **D** $\begin{bmatrix} -6 & 17 \\ 5 & 8 \end{bmatrix}$ **E** $\begin{bmatrix} -6 & 7 \\ 5 & 10 \end{bmatrix}$

Mastery

3 WORKED EXAMPLE 13 Show that addition of position vectors is invariant under linear transformations.

That is, $T(v + u) = T(v) + T(u)$ for any linear transformation T, $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ and $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$.

4 WORKED EXAMPLE 14 Find the area of the triangle $A(1, 3)$ $B(4, 8)$ $C(4, -3)$ under the transformation $T: (x, y) \to (2x + 3y, 4x + 5y)$.

5 WORKED EXAMPLE 15 Show that $T: (x, y) \to (x, bx - y)$ is its own inverse for $b \in R$.

6 Given that $T: (x, y) \to (ax + by, cx + dy)$ and $T \circ T = I$, show that $a^2 = b^2$.

7 WORKED EXAMPLE 16 Show that the image of an equilateral triangle with one side parallel to the y-axis is an isosceles triangle under a dilation parallel to the x-axis.

Exam Practice

8 TECH-FREE (2 marks) Show that $T: (x, y) \to (-5x - 3y, 8x + 5y)$ is its own inverse.

9 The area of the image of the rectangle $A(1, 3)$ $B(5, 3)$ $C(5, -4)$ $D(1, -4)$ under the linear transformation $T: (x, y) \to (3x - y, 2x - 3y)$ is

A 11 square units **B** 28 square units **C** 98 square units
D 196 square units **E** 308 square units

10 Which of the following properties are invariant under non-singular linear transformations?

I parallelism of lines
II the ratio of the areas of 2 plane shapes
III angles between lines

A I only **B** II and III **C** I and II **D** I and III **E** III only

11 The gradient of the image of $y = 3x + 5$ under reflection in the line $y = 2x$ is

A $\frac{1}{3}$ **B** $\frac{5}{9}$ **C** $\frac{9}{13}$ **D** $\frac{13}{9}$ **E** 3

12 (3 marks) Show that non-parallel lines are invariant under a non-singular linear transformation; that is, the images of the lines are also non-parallel.

Video playlist
Equivalent transformations

12.5 Equivalent transformations

Equivalent transformations

The composition of 2 transformations T and S is **commutative** if and only if $T \circ S = S \circ T$.

T and S are non-commutative transformations if $T \circ S \neq S \circ T$.

Equivalent (combinations of) transformations work out to be the same.

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WORKED EXAMPLE 17 Non-commutativity

Show that composition of a dilation and a reflection is not commutative.

Steps	Working
1 Determine what you need to do.	A single example of a dilation and reflection that do not commute is enough (proof by counterexample).
2 Try a dilation and a reflection.	Choose the dilation of factor 3 parallel to the x-axis and a reflection in the line $y = x\tan(15°)$.
3 Write the matrices.	$D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ and $R = \begin{bmatrix} \cos(30°) & \sin(30°) \\ \sin(30°) & -\cos(30°) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$
4 Find $D \circ R$.	$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{3\sqrt{3}}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$
5 Find $R \circ D$.	$\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3\sqrt{3}}{2} & \frac{1}{2} \\ \frac{3}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$
6 State the result.	The compositions in different orders are not equivalent, so the composition of a dilation and a reflection is not commutative.

It is obvious that rotations will commute with each other because rotation by one angle and then another amounts to a single rotation of the sum of the angles. The composition of rotations of θ and φ about the origin is equivalent to a single rotation of $\theta + \varphi$.

Dilations will also commute with each other, but other combinations of linear transformations do not usually commute.

WORKED EXAMPLE 18 Equivalence of transformations

Find the composition of reflection in the line $y = x\tan(15°)$ and then in $y = 2x$, and show that it is equivalent to a rotation.

Steps	Working
1 Write the matrix for the first reflection.	$R_1 = \begin{bmatrix} \cos(30°) & \sin(30°) \\ \sin(30°) & -\cos(30°) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$
2 Find p and q for the second reflection.	$p = \frac{1-m^2}{1+m^2} = \frac{1-4}{1+4} = -\frac{3}{5}$ $q = \frac{2m}{1+m^2} = \frac{2\times 2}{1+4} = \frac{4}{5}$
3 Write the matrix for the second reflection.	$R_2 = \begin{bmatrix} p & q \\ q & -p \end{bmatrix} = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}$
4 Find the composition. Make sure you use the correct order.	$R_2 \circ R_1 = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$
5 Simplify.	$= \begin{bmatrix} \frac{4-3\sqrt{3}}{10} & -\frac{3+4\sqrt{3}}{10} \\ \frac{3+4\sqrt{3}}{10} & \frac{4-3\sqrt{3}}{10} \end{bmatrix}$
6 Check that the values could be sine and cosine of the same angle.	$\left(\frac{4-3\sqrt{3}}{10}\right)^2 + \left(\frac{3+4\sqrt{3}}{10}\right)^2$ $= \frac{16-24\sqrt{3}+27}{100} + \frac{9+24\sqrt{3}+48}{100}$ $= \frac{43-24\sqrt{3}+57+24\sqrt{3}}{100}$ $= \frac{100}{100}$ $= 1$
7 State the values of $\sin(\theta)$ and $\cos(\theta)$.	$\sin(\theta) = \frac{3+4\sqrt{3}}{10} > 0, \cos(\theta) = \frac{4-3\sqrt{3}}{10} < 0$
8 Write the product matrix in terms of $\sin(\theta)$ and $\cos(\theta)$, taking into account the signs of $\sin(\theta)$ and $\cos(\theta)$.	$R_2 \circ R_1 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$, where $\theta = 180° + \tan^{-1}\left(\frac{3+4\sqrt{3}}{4-3\sqrt{3}}\right)$.
9 Write the answer.	The composition of reflection in the line $y = \tan(15°)$ and then in $y = 2x$ and is equivalent to a rotation through the angle $180° + \tan^{-1}\left(\frac{3+4\sqrt{3}}{4-3\sqrt{3}}\right)$.

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The result of the above example applies generally to reflections.

It is obvious that we can add the angles of the composition of 2 rotations.

Composition of 2 reflections

The **composition of 2 reflections** in lines that pass through the origin is equivalent to a rotation.

It can be shown that reflection in the line $y = x\tan(\theta)$, followed by reflection in the line $y = x\tan(\varphi)$, is equivalent to a rotation about the origin through the angle $2\theta - 2\varphi$.

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WORKED EXAMPLE 19 Equivalent transformations

Use equivalent rotations to show that $\sin(x + y) = \sin(x)\cos(y) + \sin(y)\cos(x)$.

Steps	Working
1 Write the matrices for rotations about the origin through angles x and y.	$R_x = \begin{bmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{bmatrix}$, $R_y = \begin{bmatrix} \cos(y) & -\sin(y) \\ \sin(y) & \cos(y) \end{bmatrix}$
2 Find the composition of rotations about the origin through angles x and y.	$R_x \circ R_y$ $= \begin{bmatrix} \cos(x) & -\sin(x) \\ \sin(x) & \cos(x) \end{bmatrix} \begin{bmatrix} \cos(y) & -\sin(y) \\ \sin(y) & \cos(y) \end{bmatrix}$
3 Simplify.	$= \begin{bmatrix} \cos(x)\cos(y) - \sin(x)\sin(y) & -\cos(x)\sin(y) - \sin(x)\cos(y) \\ \sin(x)\cos(y) + \cos(x)\sin(y) & -\sin(x)\sin(y) + \cos(x)\cos(y) \end{bmatrix}$
4 Write the equivalence.	$R_x \circ R_y = R_{x+y}$
5 Write the matrix for rotation about the origin through angle $x + y$.	$R_{x+y} = \begin{bmatrix} \cos(x + y) & -\sin(x + y) \\ \sin(x + y) & \cos(x + y) \end{bmatrix}$
6 Equate the sine terms.	From $R_x \circ R_y$ and R_{x+y}, $\sin(x + y) = \sin(x)\cos(y) + \sin(y)\cos(x)$.
	State which matrix elements you are using.

VCE QUESTION ANALYSIS

Video
VCE question analysis: Transformations and matrices

A VCE-style question (13 marks)

A trapezium has vertices $A(-2, -1)$ $B(-1, -2)$ $C(2, 1)$ $D(2, 3)$.

a Find the area of $ABCD$. 3 marks

Consider the transformations

$$T: (x, y) \to (5x - 0.5y, y - 4x) \text{ and } S: (x, y) \to (x + y, 3x + 5y)$$

b Write the matrix representations of T and S. 2 marks

c **i** Find $T \circ S$ and $S \circ T$. 2 marks

ii Interpret the result. 1 mark

d **i** Find the areas of $ABCD$ under the transformations $T \circ S$ and $S \circ T$. 1 mark

ii Interpret the result. 1 mark

e Generalise and prove the result of part **d** for any non-singular transformations. 3 marks

Reading the question

- A trapezium has 2 parallel sides.
- Use the area formula for a trapezium.
- Composition of transformations must be done in opposite orders.
- Make a rule for all.

Thinking about the question

- Area formula for trapezium: $A = \left(\frac{a+b}{2}\right)h$.
- Matrices are 2 × 2.
- Are the transformations commutative?
- Areas use $|T|$.

Worked solution (✓ = 1 mark)

a $m_{AD} = \frac{3-(-1)}{2-(-2)} = 1$

$m_{AB} = \frac{-2-(-1)}{-1-(-2)} = -1$

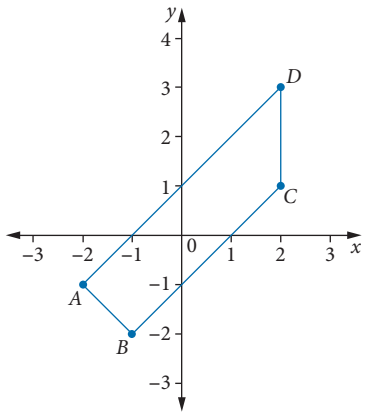
$m_{AD} \times m_{AB} = -1$, so $AD \perp AB$.

$h = |AB| = \sqrt{1^2 + 1^2} = \sqrt{2}$ ✓

$a = |AD| = \sqrt{4^2 + 4^2} = 4\sqrt{2}$

$b = |BC| = \sqrt{3^2 + 3^2} = 3\sqrt{2}$ ✓

Area $= \frac{4\sqrt{2} + 3\sqrt{2}}{2} \times \sqrt{2} =$ **7 square units** ✓



b $T = \begin{bmatrix} \mathbf{5} & \mathbf{-0.5} \\ \mathbf{-4} & \mathbf{1} \end{bmatrix}$ ✓, $S = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{3} & \mathbf{5} \end{bmatrix}$ ✓

c **i** $T \circ S = \begin{bmatrix} 5 & -0.5 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} \mathbf{3.5} & \mathbf{2.5} \\ \mathbf{-1} & \mathbf{1} \end{bmatrix}$ ✓

$S \circ T = \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 5 & -0.5 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0.5} \\ \mathbf{-5} & \mathbf{3.5} \end{bmatrix}$ ✓

ii Multiplication of T and S **is not commutable**. ✓

d **i** $|T \circ S| = 3.5 - (-2.5) = 6$, so area of $ABCD$ under $T \circ S = 6 \times 7 =$ **42 square units**.

$|S \circ T| = 3.5 - (-2.5) = 6$, so area of $ABCD$ under $S \circ T = 6 \times 7 =$ **42 square units**. ✓

ii The areas under $T \circ S$ and $S \circ T$ are the same. ✓

e The area of a shape under a composition is **invariant under commutation** of the composition. ✓

Proof:

- Suppose S and T are non-singular transformations and the area of a shape is A.
- area under $T \circ S = |T \circ S|A = \boldsymbol{|T||S|A}$
- area under $S \circ T = |S \circ T|A = \boldsymbol{|S||T|A}$ ✓
- But $|T|, |S|, A \in R$, and multiplication is commutable for the real numbers
- $\Rightarrow |T||S|A = |S||T|A$
- $\Rightarrow$ **area under $T \circ S$ = area under $S \circ T$** ✓

EXERCISE 12.5 Equivalent transformations

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Recap

1 The area of the triangle $A(-2,1)$ $B(4,1)$ $C(1,4)$ under the transformation $(x,y) \to (x-4y, 3y-2x)$ is

A 9 **B** 18 **C** 45 **D** 70 **E** 126

2 Which of the following transformations is its own inverse?

A $(x,y) \to (3x-5y, 2x-3y)$ **B** $(x,y) \to (5y-3x, 3y-2x)$

C $(x,y) \to (5x-3y, 5x-8y)$ **D** $(x,y) \to (3y-5x, 5y-8x)$

E $(x,y) \to (3x-5y, 5x-8y)$

Mastery

3 WORKED EXAMPLE 17 Show that each composition of transformations does not commute.

a Reflection in the line $y = 2x$ and reflection in the line $y = 3x$

b $T: (x,y) \to (3x+2y, 4x+3y)$ and $U: (x,y) \to (4x+2y, 2x+3y)$

c Dilation parallel to the x-axis by the factor 2 and reflection in the line $y = -x$

d $V: (x,y) \to (x, x+y)$ and $W: (x,y) \to (x+y, y)$

e Rotation through the angle $\frac{\pi}{4}$ about the origin and reflection in the line $y = x\tan\left(\frac{\pi}{6}\right)$

4 Show that each composition of transformations commutes.

a Dilation parallel to the x-axis by the factor 3 and dilation parallel to the x-axis by the factor 2

b $T: (x,y) \to (2x+5y, x+3y)$ and $U: (x,y) \to (6x-10y, 2x+4y)$

c Dilation parallel to the x-axis by the factor 2 and a dilation parallel to the y-axis by the factor 4

d $V: (x,y) \to (2x+4y, 3x+7y)$ and $W: (x,y) \to (4y-4x, 3x+y)$

e Any 2 dilations parallel to the x-axis

5 WORKED EXAMPLE 18 Find a single rotation that is equivalent to

a a reflection in $y = x\tan(30°)$, followed by a reflection in $y = x\tan(45°)$

b a reflection in $y = \frac{1}{3}x$, followed by a reflection in $y = 4x$.

6 WORKED EXAMPLE 19 Use equivalent rotations to show that $\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$.

7 **a** Show that composition of 2 dilations is commutative.

b Hence, show that composition of any 3 dilations commute in any order.

Exam practice

8 TECH-FREE (3 marks) Find a rotation that is equivalent to a reflection in the line $y = \frac{1}{2}x$, followed by a reflection in the line $y = x\tan(30°)$.

9 A rotation about the origin of 30°, followed by a rotation about the origin of 40° is equivalent to a rotation about the origin of

A 10° **B** 20° **C** 50° **D** 70° **E** 140°

10 S_1 and S_2 are rotations about the origin, T_1 and T_2 are reflections in lines through the origin and U_1 and U_2 are dilations from the y-axis. Which of the following statements are **true**?

I $S_1 \circ S_2$ is equivalent to $S_2 \circ S_1$

II $T_1 \circ T_2$ is equivalent to $T_2 \circ T_1$

III $U_1 \circ U_2$ is equivalent to $U_2 \circ U_1$

A I and II **B** II and III **C** I and III **D** I only **E** I, II and III

11 T is a reflection in the line $y = x\tan(60°)$ and S is a reflection in the line $y = x\tan(15°)$.

$S \circ T$ is equivalent to

A $(x, y) \to (y, -x)$

B $(x, y) \to \left(\frac{\sqrt{2}}{2}x, \frac{\sqrt{2}}{2}y\right)$

C $(x, y) \to \left(\frac{1}{2}x - \frac{\sqrt{3}}{2}y, \frac{\sqrt{3}}{2}x + \frac{1}{2}y\right)$

D $(x, y) \to \left(\frac{1}{2}x + \frac{\sqrt{3}}{2}y, \frac{\sqrt{3}}{2}x - \frac{1}{2}y\right)$

E $(x, y) \to \left(\frac{\sqrt{6}+\sqrt{2}}{4}x + \frac{\sqrt{6}-\sqrt{2}}{4}y, \frac{\sqrt{6}-\sqrt{2}}{4}x - \frac{\sqrt{6}+\sqrt{2}}{4}y\right)$

12 (3 marks) Use equivalent rotations to show that $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$.

12 Chapter summary

Linear transformations

- A **translation** of the plane changes the point $P(x, y)$ to $P(x + a, y + b)$, where a and b are constants $(a, b \in R)$.
- A **linear transformation of the plane** changes the point $P(x, y)$ to $P'(ax + by, cx + dy)$, where a, b, c and d are constants (real numbers).
- R^2 is a symbol for the plane, so a linear transformation of the plane is written as $T: R^2 \rightarrow R^2$, $T(x, y) = (ax + by, cx + dy)$.
- The translation $T: (x, y) \rightarrow (x + a, y + b)$ is shown by matrix addition as $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} a \\ b \end{bmatrix}$.
- The linear transformation $P(x, y) \rightarrow P'(ax + by, cx + dy)$ is shown as the 2×2 square matrix $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
- Matrix multiplication with a column vector reproduces the transformation of a point.
- A **dilation** by a factor of a from the y-axis (or parallel to the x-axis or horizontally) changes the point $P(x, y)$ to the point $P'(ax, y)$.
- The matrix for a dilation from the y-axis is $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$.
- A dilation by a factor of b from the x-axis (or parallel to the y-axis or vertically) changes the point $P(x, y)$ to the point $P'(x, by)$.
- The matrix for a dilation from the x-axis is $\begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix}$.
- A **rotation** through an anticlockwise angle θ with centre at the origin changes a point $P(x, y)$ to the point $P'(x\cos(\theta) - y\sin(\theta), x\sin(\theta) + y\cos(\theta))$.
- The matrix for a rotation about the origin is $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.
- A **reflection** in a line through the origin with inclination θ, $y = x\tan(\theta)$, changes a point $P(x, y)$ to the point $P'(x\cos(2\theta) + y\sin(2\theta), x\sin(2\theta) - y\cos(2\theta))$.
- For a line given as $y = mx$, $\theta = \tan^{-1}(m)$, $\cos(2\theta) = p = \dfrac{1 - m^2}{1 + m^2}$ and $\sin(2\theta) = q = \dfrac{2m}{1 + m^2}$.
- The matrix for a reflection in a line through the origin is $\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} = \begin{bmatrix} p & q \\ q & -p \end{bmatrix}$.

Inverse transformations

- The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ represents the **identity transformation** I: $(x, y) \rightarrow (x, y)$.
- A mathematical **inverse** returns a changed value to the original value, whether this is a number, point or position.
- The **inverse of a dilation** of factor k parallel to or from an axis is the dilation by the factor $\frac{1}{k}$ parallel to or from the same axis.
- The **inverse of a rotation** through an angle θ about the origin is the rotation through the angle $-\theta$.
- The **inverse of a reflection** is the same reflection. It is its own inverse.
- The inverse of the transformation T: $(x, y) \rightarrow (ax + by, cx + dy)$ is given by

  T^{-1}: $(x, y) \rightarrow \frac{1}{ad - bc}(dx - by, cx + ay)$, provided $ad - bc \neq 0$,

  represented by the inverse matrix $\frac{1}{ad - bc}\begin{bmatrix} d & -b \\ c & a \end{bmatrix}$.
- The value $ad - bc$ is called the **determinant** of the transformation and written as $|T|$.
- A linear transformation for which $ad - bc = 0$ is called a **singular transformation** and does not have an inverse.

Composition of linear transformations

- The **composition of 2 transformations** $T \circ S$ is the transformation obtained by doing the transformation S followed by the transformation T.
- Composition is not usually commutative.
- We can use matrix multiplication to do the composition of linear transformations.

Properties of transformations

- A relation or property that doesn't change under a transformation is called **invariant** under the transformation.
- **Straight lines** are invariant under **non-singular linear transformations**: the images of straight lines are themselves straight lines.
- The image of a **straight line** under a **singular transformation** is a straight line passing through the origin.
- **Parallel lines** are invariant under non-singular linear transformations: the images of parallel lines are themselves parallel.
- The image of a shape of **area** A will be a shape of area $|T|A$ under the linear transformation T, where $|T|$ is the determinant of the transformation.

Equivalent transformations

- The composition of 2 transformations T and S is **commutative** if and only if $T \circ S = S \circ T$.
- T and S are **non-commutative** transformations if $T \circ S \neq S \circ T$.
- **Equivalent** (combinations of) transformations work out to be the same.
- **Rotations commute** with each other. The composition of rotations of θ and φ about the origin is equivalent to a single rotation of $\theta + \varphi$.
- **Dilations commute** with each other.
- Reflections do not commute with each other.
- The **composition of 2 reflections** in lines that pass through the origin is equivalent to a rotation.
- A reflection in the line $y = x\tan(\theta)$ followed by reflection in the line $y = x\tan(\varphi)$ is equivalent to a rotation about the origin through the angle $2\theta - 2\varphi$.

Cumulative examination 1

Total number of marks: 10 Reading time: 4 minutes Writing time: 15 minutes

TECH-FREE *Technology is NOT permitted.*

1 (2 marks) Given $z_1 = 2\text{cis}\left(-\frac{\pi}{6}\right)$ and $z_2 = 3\text{cis}\left(\frac{2\pi}{3}\right)$, find

a $z_1 z_2$ 1 mark

b $\frac{\overline{z_1}}{z_2}$ 1 mark

2 (2 marks) Given $\underset{\sim}{a} = -2\underset{\sim}{i} + 2\sqrt{3}\underset{\sim}{j}$ and $\underset{\sim}{b} = -3\underset{\sim}{i} - 3\underset{\sim}{j}$, find

a $\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}}$ 1 mark

b $(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}}$ 1 mark

3 (1 mark) Given $A = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $B = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$, find $A - 2B$.

4 (5 marks)

a Express the transformation T: $(x, y) \rightarrow (x + 4y, 2y - 3x)$ in matrix form. 1 mark

b The triangle $A(3, 5)$ $B(8, 9)$ $C(10, 5)$ is transformed by T. What are the coordinates of the image? 2 marks

c Find the area of the image of ΔABC. 2 marks

Cumulative examination 2

Total number of marks: 20 Reading time: 4 minutes Writing time: 30 minutes

Approved technology is permitted.

Section A 5 multiple-choice questions

5 marks

1 Given $z = -2 + 2\sqrt{3}i$, the equivalent polar form is

A $4\,\text{cis}\left(\frac{\pi}{3}\right)$ **B** $4\sqrt{3}\,\text{cis}\left(-\frac{\pi}{3}\right)$ **C** $4\,\text{cis}\left(-\frac{\pi}{3}\right)$

D $4\,\text{cis}\left(\frac{2\pi}{3}\right)$ **E** $4\,\text{cis}\left(-\frac{2\pi}{3}\right)$

2 If $\underset{\sim}{a} = -2\underset{\sim}{i} + 6\underset{\sim}{j}$ and $\underset{\sim}{b} = \underset{\sim}{i} - 3\underset{\sim}{j}$, then $\underset{\sim}{a} \cdot 2\underset{\sim}{b}$ equals

A −40 **B** −20 **C** −3 **D** 16 **E** 38

3 © VCAA 2015 2AQ1 MODIFIED The ellipse $\frac{(x+1)^2}{16} + \frac{(y-2)^2}{9} = 1$ can be expressed in parametric form as

A $x = 4t - 1$ and $y = 2 + 3\sqrt{1 + t^2}$

B $x = 4\sec(t) - 1$ and $y = 2 + 3\tan(t)$

C $x = 4\sin(t) - 1$ and $y = 3\cos(t) + 2$

D $x = 4\cos(t) - 1$ and $y = 2 + 3\sin(t)$

E $x = 1 + 4\cos(t)$ and $y = 3 + 3\sin(t) - 2$

4 The inverse of a reflection in the line $y = 2x$ is

A a reflection in the line $y = -2x$.

B a reflection in the line $y = 2x$.

C a reflection in the line $y = 0.5x$.

D a reflection in the line $y = -0.5x$.

E rotation through the angle π.

5 $T: (x, y) \rightarrow (x + 2y, 3x + 5y)$ and $U: (x, y) \rightarrow (2y - 3x, x + 3y)$

$T \circ U =$

A $(x, y) \rightarrow (4y - 2x, 4x + 8y)$

B $(x, y) \rightarrow (4x + 3y, 11x + 6y)$

C $(x, y) \rightarrow (-7x - 11y, 10x + 17y)$

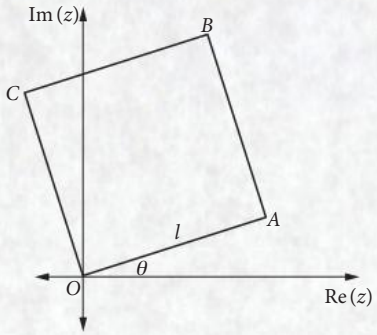
D $(x, y) \rightarrow (3x + 4y, 10x + 17y)$

E $(x, y) \rightarrow (8y - x, 21y - 4x)$

Section B 2 questions

15 marks

1 © VCAA 2021N 2BQ2 MODIFIED (8 marks) Points O, A, B and C in the complex plane represent complex numbers 0, a, b and c respectively, where $\text{Arg}(a) = \theta$. When plotted on an Argand diagram, the points are the vertices of a square of side length l, as shown below.



a Express b and c in polar form in terms of l and θ. 4 marks

b Point P represents the complex number p and is the midpoint of B and C.

Express p in polar form in terms of l and w. 2 marks

c Show that $\frac{b}{a} = 1 + i$. 2 marks

2 (7 marks) The transformation $T: (x, y) \rightarrow (2y, 3x)$ is applied to the shape $A(3, 5)$ $B(6, 5)$ $C(7, 3)$ $D(4, 3)$.

a Show that T is the composition of dilations parallel to the x and y axes by factors 2 and 3 respectively, followed by reflection in the line $y = -x$. 3 marks

b What is the shape of $ABCD$? 1 mark

c What is the shape to which it is transformed? 1 mark

d What does this illustrate about a composition of dilations and reflections in a line through the origin? 2 marks

Answers

CHAPTER 1

EXERCISE 1.1

1 **a** $t_n = 3t_{n-1}$ **b** $t_n = t_{n-1} - 3$ **c** $t_n = -2t_{n-1}$
d $t_n = 4t_{n-1}$ **e** $t_n = -4t_{n-1}$

2 0, 2, 4, 6, 8 **3** 0, 6, 16, 30

4 A **5** C

6 **a** 1, 2, 3, 4 **b** 5, 9, 13, 17
c −2, −6, −10, −14 **d** 7, 22, 67, 202

7 **a** 3, 6, 9, 12 **b** 3, 12, 27, 48
c −1, 0, 3, 8 **d** 2, 4, 8, 16

8 **a** 4, 7, 10, 13, 16 **b** 2, 6, 12, 20, 30
c 9, 27, 81, 243, 729

9 **a**

n	a_n
1	4
2	13
3	40
4	121
5	364

b

n	a_n
1	10
2	101
3	10202
4	1.0e+8
5	1e+16

c

n	a_n
1	−6
2	−11
3	−16
4	−21
5	−26

10 13, 27, 55, 111, 223

11 **a**

Square number	1	2	3	4	5	6	7	8	9
Number of rice grains	1	3	5	7	9	11	13	15	17

b 23 **c** 2048, will not fit

12 B **13** A **14** D **15** E

16 C **17** B

EXERCISE 1.2

1 5, 7, 9, 11, 13

2 −28, −37, −46

3 **a** yes, $1 + 4\sqrt{2}$, $1 + 5\sqrt{2}$
b no **c** yes, −17, −23

4 −53 **5** 6th term **6** E **7** D

8 C

9 **a** 3, 8, 13 **b** 6, 3, 0 **c** −5, −7, −9

10 **a** 37 **b** −41 **c** −2.3

11 **a** 20 **b** 32 **c** 61

12 **a** $t_n = 2n + 5$ **b** $t_n = 4n + 3$
c $t_n = -5.6n + 8.6$

13 43 **14** $a + 117d$ **15** 65

16 E **17** E **18** B **19** D

20 height of first floor, $a = 5$ m, height of each floor, $d = 3$ m

EXERCISE 1.3

1 C **2** A **3** 165 **4** 7

5 **a** $t_1 = 28$, $d = -2$ **b** 190

6 41, 51 **7** −40, −50, −60

8 D **9** C **10** D **11** A

12 C **13** 21

14 **a** 28 875 **b** 3276 **c** −1419
d 6426 **e** 5126

15 $a = 14$, $d = 4$ **16** 12

17 $n = 2$, $n = 11$ **18** B

19 **a** 65 mm **b** 80 mm **c** 15 **d** 765 mm

20 **a** 180 cm **b** yes, 760 cm > 600 cm

EXERCISE 1.4

1 −3.5, −4.5 **2** B **3** 80, −160, 320

4 $-\frac{1}{3}$ **5** $\frac{1}{6561}$ **6** 8th term

7 $-\frac{1}{524\,288}$

8 C **9** C **10** D **11** A

12 **a** 2 **b** 3 **c** −3

13 **a** $r = 3$; terms 27, 81 **b** $r = -2$; terms −24, 48
c $r = 4$; terms 32, 128 **d** $r = -1$; terms −13, 13

14 **a** $x = \pm 10$ **b** $x = \pm 2$ **c** $x = \frac{5}{4}$ **d** $x = 9$

15 2560 **16** 1.055 **17** 10 **18** 6

19 5 **20** 11

21 $t_1 = a = 2$, $r = \pm 3$ **22** $x = \pm 15$

23 139 968 **24** $r = \pm 4$

25 **a** converge **b** diverge
c converge **d** diverge

26 B **27** D **28** D

EXERCISE 1.5

1 $\frac{1}{16}, \frac{1}{32}, \frac{1}{64}$ **2** 125, −625

3 2186 **4** 6 **5** $t_1 = 3$ **6** 5465

7 6.655 **8** $\frac{81}{16}$ **9** $\frac{2059}{24}$ **10** 1792

11 8 **12** 75, −25

13 217, −77 **14** D

15 D **16** B **17** 10 **18** $\frac{2728}{27}$

19 **a** $2552.56 **b** $13 603.93

20 −56 **21** 14

EXERCISE 1.6

1 −521 **2** D **3** 6 **4** $\frac{1}{33}$

5 D **6** B

7 **a** yes **b** no **c** yes **d** no
e yes **f** yes

8 **a** $\frac{4}{9}$ **b** $\frac{2}{3}$ **c** $\frac{25}{99}$

9 4.5

10 **a** $a = -729, r = -\frac{1}{3}$ or $a = 729, r = \frac{1}{3}$
b $S_\infty = -546.75$ or $S_\infty = 1093.5$

11 **a** 80 **b** $\frac{1280}{3}$ **c** $\frac{200}{3}$ **d** 12

12 **a** $-1 < k < 1$ **b** $-\frac{2}{5}$ **c** $\frac{3}{4}$

13 $r = \frac{2}{3}, S_\infty = 162$ **14** $\frac{2}{5}$ **15** $\frac{29}{5}$

16 $t_1 = 192, r = -\frac{1}{4}, S_\infty = \frac{768}{5}$

17 $\frac{21}{32}$

EXERCISE 1.7

1 D **2** B **3** $t_n = 4n - 5$

4 10, 15, 25, 45, 85

5 **a** $\frac{31}{6} \times 3^{n-1} - \frac{7}{6}$ **b** $1.4 \times (-4)^{n-1} + 0.6$
c $\frac{11}{3} - (-2)^{n-1} - \frac{1}{3}$ **d** $\frac{3}{4} - \frac{31}{4} \times 5^{n-1}$
e $\left(\frac{1}{2}\right)^{n-2} + 8$ **f** $7.2 \times \left(-\frac{2}{3}\right)^{n-1} - 1.2$
g $\frac{17}{8} \times \left(-\frac{3}{5}\right)^{n-1} - \frac{15}{8}$ **h** $9 \times \left(\frac{3}{4}\right)^{n-1} - 4$

6 converge, 15 **7** diverge

8 **a** $(-2)^{n-1} \times 3 + 1$ **b** $0.6^{n-1} \times 9 - 5$
c $-\frac{75}{13}(-0.3)^{n-1} + \frac{10}{13}$ **d** $5^{n-1} \times 4 + 2$
e $\frac{0.15^{n-1}}{17} + \frac{16}{17}$ **f** $(-3.2)^{n-1} \times \frac{76}{21} + \frac{8}{21}$
g $12 - 14.5 \times 0.9^{n-1}$ **h** $(1.8)^{n-1} \times \frac{9}{4} - \frac{9}{4}$

9 **a** arithmetic **b** geometric, diverges
c geometric, converges

10 **a** converges, 1.25 **b** diverges
c converges, –2.5 **d** diverges
e diverges **f** converges, –3
g converges, $\frac{36}{7}$ **h** diverges

11 $t_n = 5 \times (-2)^n$ **12** $t_{n+1} = 4t_n + 1, t_1 = 1$

13 $2.75 \times (-3)^{n-1} + 1.25$ **14** $t_n = 25 - 20 \times 0.8^{n-1}$, 25

15 C **16** B

EXERCISE 1.8

1 D **2** E

3 **a** $A_{n+1} = 1.09A_n$ **b** $A_8 = 6000(1.09^8)$
c \$11 955

4 **a** $A_{n+1} = 1.115A_n$ **b** $A_6 = 2500(1.115^6)$
c \$4804

5 **a** $A_{n+1} = 1.01A_n$ **b** $A_{240} = 4000(1.01^{240})$
c \$43 570

6 **a** $A_{92} = 8000(1.0075^{92})$
b \$15 908. The investment almost doubles in 7 years 8 months

7 **a** $A_{n+1} = 1.1A_n + 1400$ **b** $A_n = 1400\left(\frac{1.1^n - 1}{0.1}\right)$
c \$44 481

8 **a** $1.008A_n + 300$ **b** $A_n = 300\left(\frac{1.008^{120} - 1}{0.008}\right)$
c \$60 065

9 **a** $A_{n+1} = 1.11A_n - 42\,000$
b $A_n = 350\,000(1.11^n) - 42\,000 \times \left(\frac{1.11^n - 1}{0.11}\right)$
c \$291 473 **d** \$125 290

10 **a** $A_{n+1} = 1.009A_n - 6000$
b $A_n = 600\,000(1.009^n) - 6000\left(\frac{1.009^n - 1}{0.009}\right)$
c \$471 301 **d** \$94 149

11 **a** $340\,000\left(\frac{0.009}{1 - 1.009^{-30}}\right)$ **b** \$992 831
c \$652 831

12 **a** $460\,000\left(\frac{0.007}{1 - 1.007^{-240}}\right)$ **b** \$951 101
c \$491 101

13 **a** 20×1.25^n **b** 61, 186, 568

14 **a** 1000×1.3^n
b The population would be over 8000/20 ha = 400/ha, so it would be damaged.

15 **a** $A_{n+1} = 1.07A_n$ **b** $9000(1.07^{10})$

16 C **17** D **18** B **19** C **20** C

CUMULATIVE EXAMINATION 1

1 **a** yes **b** no

2 3.6

3 **a** 81 **b** 121

4 **a** –7 **b** –15

5 –6 **6** $2000\left(\frac{1.08^{25} - 1}{0.08}\right)$

CUMULATIVE EXAMINATION 2

Section A

1 A **2** C **3** B **4** C

5 C

Section B

1 **a** 54 m **b** $10\frac{2}{3}$ m **c** $362\frac{1}{3}$ m **d** 405 m

2 **a** $d = 5$ **b** $r = 1.1$
c Bill = 20 km, Ben = 12.1 km
d 95 km **e** 55 km **f** 23.6 km

9780170448338

CHAPTER 2

EXERCISE 2.1

1 **a** $4\frac{65}{99}$ **b** $\frac{19}{74}$

2 **a** $\frac{1}{9}$ **b** $\frac{7}{12}$ **c** $\frac{3097}{990}$

d $\frac{12}{7}$ **e** $\frac{3}{2}$

3 **a** Z **b** Q **c** N **d** Q

4 $\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd} = \frac{m}{n}$, $m, n \in Z, n \neq 0$

$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} = \frac{m}{n}$, $m, n \in Z, n \neq 0$

5 $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$

6 $2\frac{1}{30}$

7 Each recurring decimal can be written as a fraction, so the sum or difference will be a fraction.

8 A **9** D **10** B

11 **a** yes

b No. If $n = 2$, then $m = \frac{3}{2}$.

c No. $(-3)^2 = 9$ but $n = -3$.

d yes

12 **a** $n = 4k, k \in N$ **b** $n = 4k - 2, k \in N$

13 **a** For example, $\sqrt{2} \times \sqrt{2} = 2, \sqrt{2} \times \sqrt{18} = \sqrt{36} = 6$.

b The product of a surd $\sqrt{x}$ by a natural number, n, is the same as the finding the sum of n number of $\sqrt{x}$s. That is, $n\sqrt{x} = \sqrt{x} + \sqrt{x} + \sqrt{x} + \ldots$ (up to $n\sqrt{x}$s)

This is the sum of n surds of the form $\sqrt{x}$, which is an irrational number.

14 Let $r_1 = \frac{a}{b}, r_2 = \frac{c}{d}$

$r_1 + r_2 = \frac{ad + bc}{bd}$

$r_1 + r_2 \in N$ if $ad + bc = k(bd), k \in N$ (that is, the numerator is a multiple of the denominator)

15 $a^2 - b^2x = k^2$

$x = \frac{a - k^2}{b^2}$

Require $a^2 < k^2$ or $-k < a < k$

16 For example, $(3 + \sqrt{5}) + (3 - \sqrt{5}) = 6$

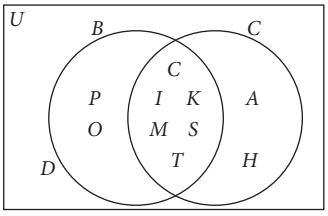
EXERCISE 2.2

1 A **2** D

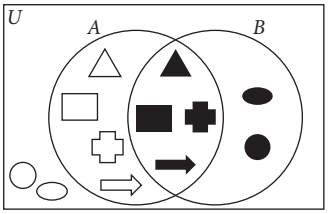
3 **a** $\varnothing$ **b** {5, 6, 19}

c {10, 20} **d** 10

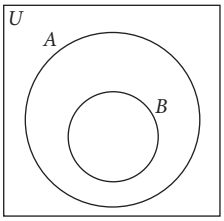
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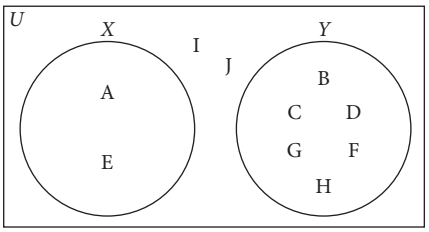
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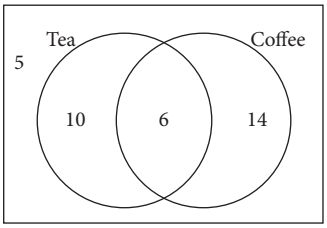
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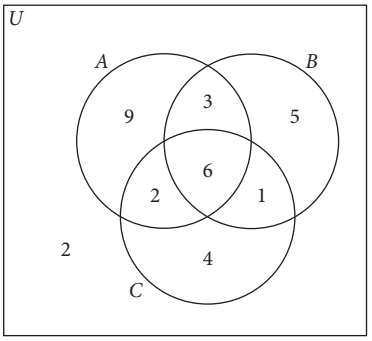
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8



9

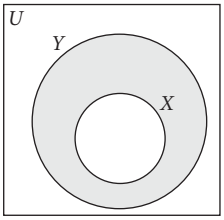


10 11

11 **a** {2, 3, 5, 7} **b** {3, 5, 7}

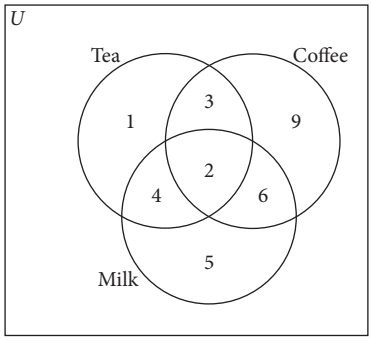
c {15} **d** {12, 14, 16, 18, 20}

12

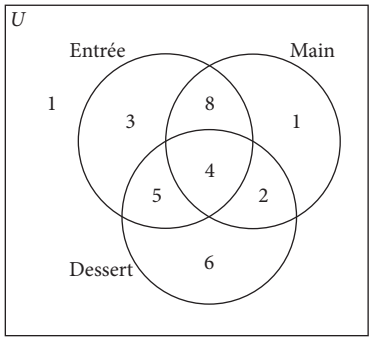


13 C **14** C

15

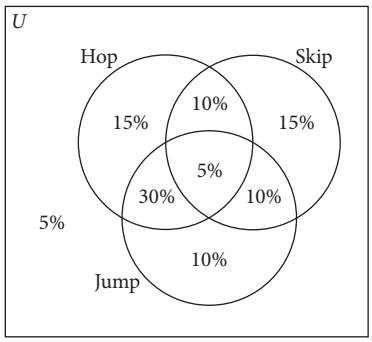


16 a



b 3 **c** 2 **d** 17 **e** 1

17 a



b 50% **c** 5%

EXERCISE 2.3

1 C **2** E **3** B

4 1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, 36, 48, 54, 72, 108, 144, 216, 432

5 $525 = 3 \times 5^2 \times 7$, divisors 3, 5, 7, 15, 21, 25, 35, 75, 105, 175, 525

6 a $1764 = 2^2 \times 3^2 \times 7^2$

$\sqrt{1764} = \sqrt{2^2 \times 3^2 \times 7^2} = 2 \times 3 \times 7 = 42$

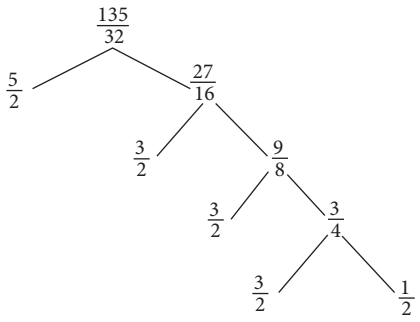
b $1350 = 2 \times 3^3 \times 5^2$

$\sqrt{1350} = \sqrt{2 \times 3^3 \times 5^2} = \sqrt{2 \times 3 \times 3^2 \times 5^2}$

$= 3 \times 5\sqrt{2 \times 3}$

$= 15\sqrt{6}$

7



$$\sqrt{4\frac{7}{32}} = \sqrt{\left(\frac{3}{2}\right)^2 \times \left(\frac{1}{2}\right)^2 \times \frac{3}{2} \times 5}$$

$$= \frac{3}{2} \times \frac{1}{2}\sqrt{\frac{15}{2}} = \frac{3}{4}\sqrt{\frac{15}{2}}$$

8 n is odd, so by using the property that the product of two odd numbers is odd, $n^2 = n \times n$ is odd, $n^3 = n^2 \times n$ is odd, and so on up to $n^n = n^{n-1} \times n$ is odd, the product of n numbers is odd.

9 a When p is divided by 6, the remainder can't be 0, 2 or 4, since this would mean p is even and not prime. The remainder can't be 3 either since this would mean p is divisible by 3 and not prime.

This means the remainder is either 1 or 5.

If the remainder is 1, p can be written as $6k + 1$.

If the remainder is 5, p can be written as $6k + 5$, but also as $6(k - 1) + 5 = 6k - 6 + 5 = 6k - 1$.

So p can be written as $6k \pm 1$.

b $p = 6k \pm 1$

$p^2 = (6k \pm 1)^2$

$= 36k^2 \pm 12k + 1$

$= 12k(3k \pm 1) + 1$

k is a natural number, so $12k(3k \pm 1)$ is a natural number.

12 is even, so $12k(3k \pm 1)$ is also even because any multiple of an even number is even.

So $12k(3k \pm 1) + 1$ is odd.

So p^2 is odd.

10 For $p > 2$, p is odd, and the product of n odd numbers is odd. Show this by considering 2 numbers at a time.

11 All prime numbers greater than 2 are odd. So the product of 2 such prime numbers is odd.

12 Let $m = 2a$, $n = 2b + 1$, $m, n \in N$

$m \times n = 2(2ab + a) = 2k$, $k \in N$

13 $n^2 + 3n = n(n + 3)$

For n even, $n + 3$ is odd, hence $n(n + 3)$ is even × odd = even.

For n odd, $n + 3$ is even, hence $n(n + 3)$ is odd × even = even.

14 a $m, n \in N$; $(2m)(2n) = 4mn$ is divisible by 2.

b $2^n = 2(2^{n-1})$, which is divisible by 2.

c $2^n + 1$ is odd, so we have the product of two odd numbers, which is odd.

15 D **16** A $384 = 2^7 \times 3$

EXERCISE 2.4

1 A **2** E

3 a A rectangle has 4 sides but is not a square.

b The sum of 2 odd numbers produces an even number, such as 3 + 3 = 6.

c $n = \sqrt{2}$; $n^2 = \frac{2}{1}$ is rational, but n is irrational.

4 n^2 is odd and $3n$ is odd (product of 2 odd numbers).

So $n^2 + 3n$ is even (sum of 2 odd numbers).

Hence $(n^2 + 3n) + 5$ is odd (sum of even and odd numbers).

9780170448338

5 Proof: see worked solutions

6 **a**, **b** $P = 2 \times 3 \times 5 \times 7 \times 11 \times 13 + 1 = 30\,031 = 59 \times 509$

c When P is composite, the prime factorisation will have at least one prime number that is not already included in our assumed list of finite primes, $p_1 \times p_2 \times p_3 \times \ldots \times p_n$. This then adds another prime number, say, p_{n+1} to the list. This process can continue indefinitely to produce infinitely many primes.

7 Proof: see worked solutions

8 Suppose n is even. We show that $n^2 - 6n + 5$ is odd.

Let $n = 2m$, $m \in Z$

$n^2 - 6n + 5 = 2(2m^2 - 6m + 2) + 1 = 2k + 1$, $k \in Z$

9 D **10** D **11** C

12 **a** **i** $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ **ii** $\frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$ **iii** $\frac{\frac{1}{2}}{\frac{1}{4}} = 2$

b $\frac{1}{a} \times \frac{1}{b} = \frac{1}{ab}$, which is of the form $\frac{1}{x}$.

13 **a** $x = -3$ **b** not true for $x = 3$

c In 3D space, skew lines are not parallel and do not meet.

d octopus, some crustaceans

14 $F_5 = 2^{32} + 1 = 4\,294\,967\,297 = 641 \times 6\,700\,417$, a composite number.

15 **a** True **b** False, let $x = -3$.

c True **d** False, the number 2 is prime.

16 Proof: see worked solutions

17 Assume that $n^2 - 2n + 7$ and n are even.

Since n is even, $n = 2a$ for some $a \in Z$.

$n^2 - 2n + 7 = 2(2a^2 - 2a + 3) + 1$, so $n^2 - 2n + 7$ must be odd.

This is a contradiction, because we assumed $n^2 - 2n + 7$ to be even.

18 Assume that there exist integers a and b for which $18a + 6b = 1$.

Then $2(9a) + 2(3b) = 1$. This is a contradiction because it implies that the sum of two even numbers is 1.

19 Proof: see worked solutions

EXERCISE 2.5

1 E **2** B

Partial proofs shown below: For full induction proofs, see worked solutions

3 True for $n = 3$. Assume true for $n = k$. Look at $n = k + 1$

Let $P(k) = k - 2$ and hence
$P(k + 1) = (k + 1) - 2 = (k - 2) + 1 = P(k) + 1$

This shows that for each vertex added, one extra triangle is required.

For each vertex added, one extra triangle is required. Hence, if the number of triangles required for $n = k$ is $P(k)$, the number required for $n = k + 1$ is $P(k) + 1 = P(k + 1)$.

4 True for $n = 1$ LHS is $P(1) = 4$

RHS is $P(1) = \frac{1}{2} \times (5 + 3) = 4$

Assume true for $n = k$.

Show $4 + 9 + 14 + \ldots + (5k - 1) + [5(k + 1) - 1]$

$= \frac{k + 1}{2}(5(k + 1) + 3) = \frac{1}{2}(k + 1)(5k + 8)$

$[4 + 9 + 14 + \ldots + (5k - 1)] + [5(k + 1) - 1]$

$= \frac{k}{2}(5k + 3) + 5k + 4$

$= \frac{1}{2}(k(5k + 3) + 10k + 8) = \frac{1}{2}(5k^2 + 13k + 8)$

$= \frac{1}{2}(k + 1)(5k + 8)$

5 True for $n = 1$ LHS is 2 RHS is $1(1 + 1) = 2$

Assume true for $n = k$.

$2 + 4 + 6 + \ldots + (2k) + [2(k + 1)] = k(k + 1) + 2(k + 1)$
$= (k + 1)(k + 2)$
$= P(k + 1)$

6 True for $n = 1$ LHS is 1 RHS is $1^2 = 1$

Assume true for $n = k$.

$1 + 3 + 5 + \ldots + (2k - 1) + [2(k + 1) - 1] = k^2 + 2(k + 1) - 1$
$= k^2 + 2k + 1$
$= (k + 1)^2$
$= P(k + 1)$

7 Let $P(n) = 9^n - 1$

True for $n = 1$ $P(1) = 9 - 1 = 8$ is divisible by 8.

Assume true for $n = k$, so $P(k) = 9^k - 1$ is divisible by 8.

$P(k + 1) = 9^{k+1} - 1$
$= 9 \times 9^k - 9 + 8$
$= 9 \times (9^k - 1) + 8$
$= 9 \times P(k) + 8$

$P(k)$ and 8 are both divisible by 8, so $9 \times P(k) + 8$ is divisible by 8.

8 Let $P(n) = 11^n - 6$

True for $n = 1$ $P(1) = 11^1 - 6 = 5$ is divisible by 5.

Assume true for $n = k$, so $P(k) = 11^k - 6$ is divisible by 5.

$P(k + 1) = 11^{k+1} - 6$
$= 11(11^k - 6) + 60$
$= 11P(k) + 60$

$P(k)$ and 60 are both divisible by 5, so $11P(k) + 6$ is divisible by 5.

9 True for $n = 1$ LHS is 1 RHS is $\frac{1(3 - 1)}{2} = 1$

Assume true for $n = k$.

$[1 + 4 + 7 + \ldots + (3k - 2)] + [3(k + 1) - 2]$

$= \frac{k(3k - 1)}{2} + 3(k + 1) - 2$

$= \frac{1}{2}(3k^2 + 5k + 2)$

$= \frac{1}{2}(3k + 2)(k + 1)$

$= P(k + 1)$

10 Let $P(n) = 3^n - 1$

True for $n = 1$ $P(1) = 3^1 - 1 = 2$ is divisible by 2.

Assume true for $n = k$, so $P(k) = 3^k - 1$ is divisible by 2.

$$\begin{aligned}P(k+1) &= 3^{k+1} - 1\\ &= 3(3^k - 1) + 2\\ &= 3P(k) + 2\end{aligned}$$

$P(k)$ and 2 are both divisible by 2, so $3P(k) + 2$ is divisible by 2.

11 Let $P(n) = n^3 - n$

True for $n = 2$ $P(2) = 2^3 - 2 = 6$ is divisible by 3.

Assume true for $n = k$, so $P(k) = k^3 - k$ is divisible by 3.

$$\begin{aligned}P(k+1) &= (k+1)^3 - (k+1)\\ &= k^3 + 3k^2 + 3k + 1 - k - 1\\ &= (k^3 - k) + 3k(k+1)\\ &= P(k) + 3k(k+1)\end{aligned}$$

$P(k)$ and $3k(k + 1)$ are both divisible by 3, so $P(k) + 3k(k + 1)$ is divisible by 3.

12 True for $n = 1$ LHS is 1 RHS is $\frac{1}{6}(2)(3) = 1$

Assume true for $n = k$.

$$\begin{aligned}&(1^2 + 2^2 + 3^2 + \ldots + k^2) + [(k+1)^2]\\ &= \frac{k}{6}(k+1)(2k+1) + (k+1)^2\\ &= \frac{k+1}{6}(k(2k+1) + 6(k+1))\\ &= \frac{k+1}{6}(2k^2 + 7k + 6)\\ &= \frac{k+1}{6}[(k+1)+1][(2(k+1)+1)]\\ &= P(k+1)\end{aligned}$$

13 True for $n = 1$ LHS is $2^0 = 1$ RHS is $2^1 - 1 = 1$

Assume true for $n = k$.

$$\begin{aligned}[2^0 + 2^1 + 2^2 + \ldots + 2^{k-1}] + 2^{k+1-1} &= 2^k - 1 + 2^k\\ &= 2 \times 2^k - 1\\ &= 2^{k+1} - 1\\ &= P(k+1)\end{aligned}$$

14 Let $P(n) = 7^n - 4^n$

True for $n = 1$ $P(1) = 7^1 - 4^1 = 3$ is a multiple of 3.

Assume true for $n = k$, so $P(k) = 7^k - 4^k$ is a multiple of 3.

$$\begin{aligned}P(k+1) &= 7^{k+1} - 4^{k+1}\\ &= 7(7^k) - 4(4^k)\\ &= (7^k - 4^k) + (6 \times 7^k - 3 \times 4^k)\\ &= P(k) + 3(2 \times 7^k - 4^k)\end{aligned}$$

$P(k)$ and $3(2 \times 7^k - 4^k)$ are both multiples of 3, so $P(k) + 3(2 \times 7^k - 4^k)$ is a multiple of 3.

15 Let $P(n) = 3^{2n} + 11$

True for $n = 1$ $P(1) = 3^2 + 11 = 20$ is divisible by 4.

Assume true for $n = k$, so $P(k) = 3^{2k} + 11$ is divisible by 4.

$$\begin{aligned}P^{(k+1)} &= 3^{2k+2} + 11\\ &= 9(3^{2k}) + 11\\ &= (3^{2k} + 11) + 8(3^{2k})\\ &= P(k) + 8(3^{2k})\end{aligned}$$

$P(k)$ and $8(3^{2k})$ are both divisible by 4, so $P(k) + 8(3^{2k})$ is divisible by 4.

16 Let $P(n) = n^3 - 7n + 9$

True for $n = 1$ $P(1) = 1 - 7 + 9 = 3$ is divisible by 3.

Assume true for $n = k$, so $P(n) = n^3 - 7n + 9$ is divisible by 3.

$$\begin{aligned}P(k+1) &= (k+1)^3 - 7(k+1) + 9\\ &= k^3 + 3k^2 - 4k + 3\\ &= (k^3 - 7k + 9) + 3k^2 + 3k - 6\\ &= (k^3 - 7k + 9) + 3(k^2 + k - 2)\\ &= P(k) + 3K,\ K = k^2 + k - 2\end{aligned}$$

$P(k)$ and $3K$ are both divisible by 3, so $(k^3 - 7k + 9) + 3(k^2 + k - 2)$ is divisible by 3.

17 Let $P(n) = 11^{n+1} + 12^{2n-1}$

True for $n = 1$ $P(1) = 11^2 + 12^1 = 133$ is divisible by 133.

Assume true for $n = k$, so $P(k) = 11^{k+1} + 12^{2k-1}$ is divisible by 133.

$$\begin{aligned}P(k+1) &= 11^{k+2} + 12^{2k+1}\\ &= 11(11^{k+1}) + 144(12^{2k-1})\\ &= 11 \times 11^{k+1} + 144 \times 12^{2k-1})\\ &= 11(11^{k+1} + 12^{2k-1}) + 133 \times 12^{2k-1}\\ &= 11P(k) + 133 \times 12^{2k-1}\end{aligned}$$

$P(k)$ is divisible by 133, so $11P(k) + 133 \times 12^{2k-1}$ is divisible by 133.

18 True for $n = 1$ LHS is $\frac{1}{1^2} = 1$ RHS is $2 - \frac{1}{1} = 1$

Assume true for $n = k$. That is,

$$\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots + \frac{1}{k^2}\right) \leq 2 - \frac{1}{k}$$

$$\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots + \frac{1}{k^2}\right) + \frac{1}{(k+1)^2}$$

$$\leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \text{ from the assumption.}$$

$$\begin{aligned}&= 2 - \left(\frac{1}{k} - \frac{1}{(k+1)^2}\right)\\ &= 2 - \frac{k^2 + k + 1}{k(k+1)^2}\\ &= 2 - \frac{k(k+1) + 1}{k(k+1)^2}\\ &= 2 - \frac{1}{k+1} - \frac{1}{k(k+1)^2}\\ &< 2 - \frac{1}{k+1}\end{aligned}$$

So $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1}$

19 True for $n = 1$ $s_1 = 1^2 = 1$.

Assume true for $n = k$. That is, $s_k = k^2$.

$$\begin{aligned}s_{k+1} &= (a_1 + a_2 + a_3 + \ldots + a_k) + a_{k+1}\\ &= k^2 + 2(k+1) - 1\\ &= k^2 + 2k + 1\\ &= (k+1)^2\end{aligned}$$

CUMULATIVE EXAMINATION 1

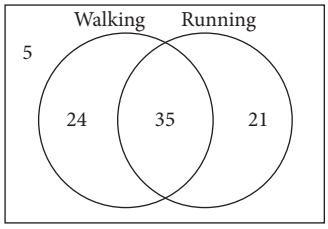
1 $(a + bi)^3 = (a^3 - 3ab^2) + (3a^2b - b^3)i$

Need $3a^2b - b^3 = 0$, so $b^2 = 3a^2$.

2 **a** $2 \times 5 \times 7 \times 7$ or $2 \times 5 \times 7^2$

b 2, 5, 7, 10, 14, 35, 49, 70, 98, 245

3



4 900 cm (9 m)

5 $a = 2, d = 3$

CUMULATIVE EXAMINATION 2

Section A

1 E **2** C **3** D **4** D **5** C

Section B

1 For $a, b > 0$, assume $\frac{a}{b} + \frac{b}{a} < 2$. Then

$$\frac{a}{b} + \frac{b}{a} < 2 \Rightarrow a^2 + b^2 < 2ab$$
$$\Rightarrow a^2 + b^2 - 2ab < 0$$
$$\Rightarrow (a - b)^2 < 0$$

This implies that the square of a number is negative, which is not possible.

2
$$2016 = 2^5 \times 3^2 \times 7$$
$$\sqrt{2016} = \sqrt{2^4 \times 3^2 \times 2 \times 7}$$
$$= 2^2 \times 3\sqrt{2 \times 7}$$
$$= 12\sqrt{14}$$

3 Let $p > 2$ be a prime number, so it is divisible only by 1 and p. Hence it is not divisible by 2. Hence p is not even.

4 All recurring decimals are rational numbers.

Let $r_1 = \frac{a}{b}$, $r_2 = \frac{c}{d}$, $a, b, c, d \in Z$, where $b, d \neq 0$

$r_1 r_2 = \frac{ac}{bd} = \frac{e}{f}$, $e, f \in Z$, where $f \neq 0$

5 $a = 5, n = 6$

6 **a** $P(t) = 5000 \times 1.1^t$

b April 2027 (Note: 3.27 months means after the end of March)

CHAPTER 3

EXERCISE 3.1

1 **a** vertex E **b** vertex C

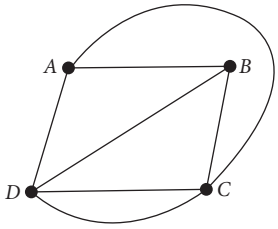
c vertices B, C and D **d** vertices B and D

2 vertices = {A, B, C, D, E}
edges = {AB, BC, BC, CC, CD, CD}

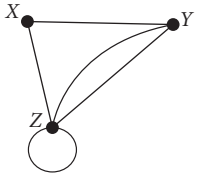
3

	A	B	C	D	E
A	0	1	0	1	0
B	1	0	1	1	1
C	0	1	0	1	0
D	1	1	1	0	0
E	0	1	0	0	0

4



5



6 deg (A) = 3, deg (B) = 3, deg (C) = 4, deg (D) = 6

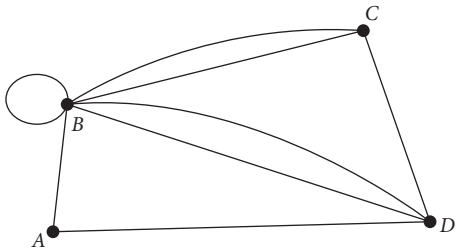
7 deg (A) = 4

8 **a** vertices = {$A, B, C. D$}
edges = {$AB, AD, BB, BC, BC, BD, BD, CD$}

b

	A	B	C	D
A	0	1	0	1
B	1	1	2	2
C	0	2	0	1
D	1	2	1	0

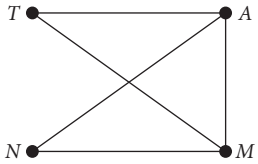
c



9 **a** A zero means the two people are not able to communicate.

b $x = 1, y = 1$

10 **a**



b deg (N) = 2. Nick plays games against 2 other people.

11 B **12** A **13** B **14** D **15** D

EXERCISE 3.2

1 D **2** E

3 **a** simple, completed, connected

b connected, planar

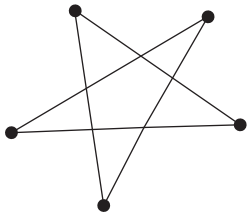
c bipartite, simple

4 **a** 15 edges **b** 66 edges **c** 21 edges

5 15 vertices, 105 edges

6 A **7** 16 edges

8 Complement of G



ANSWERS

9 **a** simple, connected, planar

b simple, connected, complete

c simple

10 **a** 12 edges **b** $n = 7$

11 degree sum = 10

12 C **13** C **14** C **15** E

16 B **17** B

EXERCISE 3.3

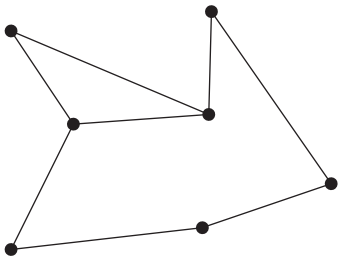
1 D **2** C

3 **a** planar **b** non-planar

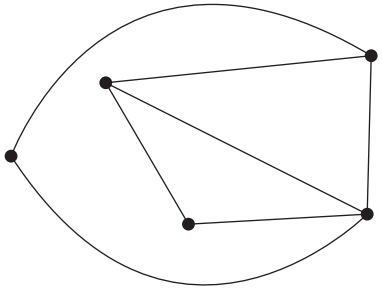
4 **a** $f = 3$ **b** $f = 5$

5 **a** 5 faces **b** 10 edges

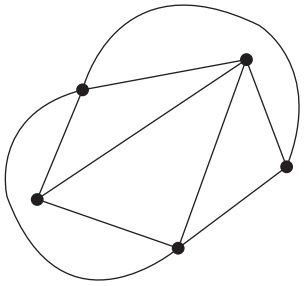
6 **a**



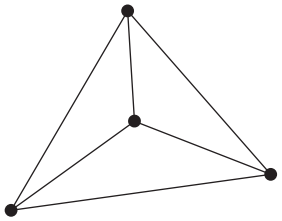
b



c



d



There are many different representations of these graphs.

7 B **8** A **9** A **10** B

11 C **12** B **13** B **14** D

EXERCISE 3.4

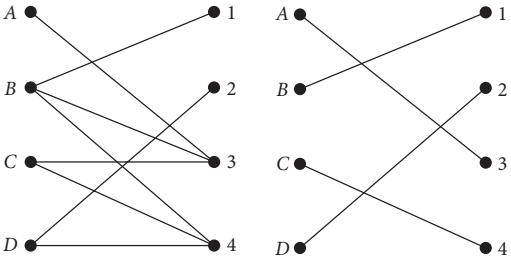
1 C **2** E **3** C **4** 25

5 a and b are trees.

6 **a** 11 edges **b** 21 vertices

7 42

8



Allocation *A*3, *B*1, *C*4, *D*2

9 **a** Town *A* **b** 4 roads **c** 3 **d** 3

10 **a** tree **b** 17 vertices **c** 16 edges

11 E **12** B **13** A **14** C **15** C

EXERCISE 3.5

1 B **2** D

3 **a** A closed walk, a circuit and a cycle

b An open walk and a trail

c An open walk, a trail and a path

4 **a** vertex *C*

b Eulerian trail *B*–*A*–*E*–*B*–*C*–*E*–*D*–*C* (more trails may exist)

5 **a** 1–2–7–4–6–5–3 (more paths exist)

b 1–2–7–4–6–5–3–1 (more cycles exist)

6 **a** *A*–*E*–*B*–*C*–*D*–*A*–*B* (other solutions may exist)

b *A*–*E*–*B*–*C*–*D* (other solutions may exist)

c *A*–*E*–*B*–*C*–*D*–*A* (other solutions may exist)

7 **a** *T*–*S*–*R*–*Q*–*P*–*T*–*Q*–*R* (other valid trails exist)

b an Eulerian trail

c between town *T* and town *R*

8 **a** Hamiltonian path **b** Hamiltonian circuit

9 **a** 4 ways

b It is missing another edge *AB*.

c Start at *C* and finish at *B* or start at *B* and finish at *C*.

10 C **11** B **12** B **13** E

14 D **15** B

CUMULATIVE EXAMINATION 1

1 **a** 10 **b** 45

2 **a** vertices = {*A*, *B*, *C*, *D*}
edges = {*AB*, *AD*, *AD*, *BC*, *CC*, *CD*, *CD*, *DD*}

b

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 2 & 0 & 2 & 1 \end{bmatrix} \end{array}$$

3 vertices = {*A*, *B*, *C*, *D*}
edges = {*AA*, *AB*, *AB*, *AB*, *AC*, *AC*, *AD*, *AD*, *BC*, *BD*, *CC*, *CD*, *CD*, *DD*}

4 $\frac{2}{11}$ **5** 16

9780170448338

CUMULATIVE EXAMINATION 2

Section A

1 C **2** 58% A **3** 88% D

4 C **5** B

Section B

1 a 96% F **b** 54% 2

c 62% There are numerous possibilities. Each Hamiltonian path starts at F, includes all 10 vertices and has 9 edges.

A possible answer is $F–G–J–I–H–E–A–B–C–D$

2 a 2, 5, 11 **b** 383 km **c** 3.125 km **d** 757 km

e 796.875 km **f** week 5

g Jen by 39.875 km **h** $t_{n+1} = \frac{1}{2}t_n, t_1 = 400$

CHAPTER 4

EXERCISE 4.1

1 a 3 and 7 **b** 4 and 5 **c** 1 and 6 **d** 3

2 a If a number is even, then it is divisible by 2.

b If a number is divisible by 2, then it is either the sum of two even numbers or the sum of two odd numbers.

c If a number is not odd and not even, then it is not divisible by 2.

3 a $Y \rightarrow Z$

b $\neg Y \rightarrow \neg Z$

c $\neg Z \vee \neg X \rightarrow \neg Y$

4 B **5** C **6** E **7** E **8** D

9 a For example: Let P be 'Fish have gills'. Then $\neg P$ is 'Fish do not have gills' and $\neg(\neg P)$ means 'It is not true that fish do not have gills', or 'It is true that fish have gills'.

b For example: Let P be 'I am sleeping'. Then $\neg P$ is 'I am not sleeping' $P \vee \neg P$ is 'I am sleeping or I am not sleeping'.

10 a $A \rightarrow B$ $\neg A \rightarrow B$ $A \rightarrow \neg B$ $\neg A \rightarrow \neg B$
$B \rightarrow A$ $\neg B \rightarrow A$ $B \rightarrow \neg A$ $\neg B \rightarrow \neg A$

b $\neg A \rightarrow \neg B$ 'if you don't exercise you won't live longer'
$\neg B \rightarrow \neg A$ 'not living longer means no exercise'

11 a no; not all numbers that are divisible by 2 are divisible by 6

b no; breathing does not mean you are sleeping

X: I am sleeping Y: I am breathing

c no: a car with good tyres but no brakes still shouldn't be driven; X: car with a flat tyre Y: the car is not driven

d yes; both statements are always true

12 a $\neg P$ **b** $\neg(B \vee E)$

c $\neg H \rightarrow \neg B$ **d** $C \wedge F$

e $R \rightarrow (W \wedge \neg F \wedge P)$

f $(B \wedge \neg E) \leftrightarrow (P \vee \neg R)$ or $(P \vee \neg R) \leftrightarrow (B \wedge \neg E)$

13 Sample answer: A: rich, B: happy,
$\neg(A \vee B)$: not (rich or happy).
$\neg A \wedge \neg B$: not rich and not happy.

EXERCISE 4.2

1 C **2** D **3** E

4

A	B	$\neg B$	$A \wedge \neg B$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

5

A	B	$\neg A$	$\neg B$	$\neg A \wedge \neg B$	$\neg(\neg A \wedge \neg B)$	$A \vee B$
T	T	F	F	F	T	T
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	T	F	F

6

P	Q	$\neg P$	$\neg Q$	$\neg P \rightarrow \neg Q$
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

7

A	B	$A \rightarrow B$	$\neg B$	$(A \rightarrow B) \wedge \neg B$	$\neg A$	$((A \rightarrow B) \wedge \neg B) \rightarrow \neg A$
T	T	T	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

8

P	Q	$P \vee Q$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$	$(P \vee Q) \wedge (\neg P \wedge \neg Q)$
T	T	T	F	F	F	F
T	F	T	F	T	F	F
F	T	T	T	F	F	F
F	F	F	T	T	T	F

9 E **10** A **11** E **12** C

13

A	B	$\neg A$	$\neg A \vee B$	$A \rightarrow B$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

14

A	B	C	$\overline{A}$	$\overline{B}$	$\overline{C}$	$\overline{ABC}$	$\overline{A} + \overline{B} + \overline{C}$
F	F	F	T	T	T	T	T
F	F	T	T	T	F	T	T
F	T	F	T	F	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	F	F	T	T
T	F	T	F	T	F	T	T
T	T	F	F	F	T	T	T
T	T	T	F	F	F	F	F

15 Proof: see worked solutions

16

A	B	$A \rightarrow B$	$\neg B$	$(A \rightarrow B) \wedge \neg B$	$\neg A$	$((A \rightarrow B) \wedge \neg B) \rightarrow \neg A$
T	T	T	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

17, 18 Proof: see worked solutions

ANSWERS

19

P	Q	R	$P \to Q$	$P \to R$	$Q \to R$	$(P \to Q) \wedge (Q \to R)$	$(P \to Q) \wedge (Q \to R) \to (P \to R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	F	T	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

The last column is true for all values of P, Q and R, so it is a tautology.

20 Proof: see worked solutions

EXERCISE 4.3

1 C **2** E

3 **a** $100\,001\,000\,111_2$ **b** 102_{10}

4 **a** $100\,011_2$ **b** 29_{10}

5 **a** $10\,000_2$ **b** 10_2

6 **a** 1010_2 **b** 10_{10}

7 1001_2

8 **a** 23.375_{10} **b** 101.01_2

c $0.01\,001\,100\,110\,011\ldots_2$

9 $1\,000\,010_2$ **10** $11\,100_2$

11 **a** 8.44 pm **b** 00 001 010 : 00 101 111

12 **a** $abc_2 = a \times 2^2 + b \times 2^1 + c \times 2^0 = 4a + 2b + c$

$de_2 = d \times 2^{-1} + e \times 2^{-2}$

$= \frac{d}{2} + \frac{e}{4} = \frac{1}{4}(2d + e)$

b $a + b = 2 \quad a = 1, b = 1$

$d + e = 2 \quad d = 1, e = 1$

$a > c \quad c = 0$

$abc_2 = 4 \times 1 + 2 \times 1 + 0 = 6$

$de_2 = \frac{1}{4}(2 \times 1 + 1) = \frac{3}{4}$

$101_2 = 6 + \frac{3}{4} = 6\frac{3}{4}$

13 **a** 16 **b** 9

14 The maximum value will come from the binary number consisting of n 1's, that is, 111 111…

The value is $1 + 2 + 4 + 8 + \ldots = \frac{1 - 2^n}{1 - 2} = 2^n - 1$,

(a geometric series, common ratio: 2)

15 0000 1110 1001 0010 1111

16 D **17** E

18 $n_{10} = 8a + 4b + 2 + 1$

$m_{10} = 8 + 4 + 2a + b$

$m_{10} = n_{10} + 2 \Rightarrow 8 + 4 + 2a + b = 8a + 4b + 2 + 1 + 2$

$2a + b + 12 = 8a + 4b + 5$

$6a + 3b = 7$

19 The sequence is 2, 8, 14 … which after division by 3 leaves remainder 2.

After division by 2, the remainder is 0.

EXERCISE 4.4

1 A **2** C

3 $A(\overline{A} + B) + \overline{A}B = A\overline{A} + AB + \overline{A}B$

$= 0 + B(A + \overline{A})$

$= B$

4 $\overline{A} + \overline{B}$

5 Proof: see worked solutions

6 B

7 **a** 'blue cars' OR 'white cars'

b trees NOT deciduous

c ('apple trees' AND ('red apples' OR 'green apples')) OR ('pear trees' AND 'grafted' AND 'green pears' AND 'yellow pears')

8

	A	B	C	D	E	F
1	Adelaide	Hobart	Sydney	0	0	FALSE
2	Perth	Perth	Perth	1	1	FALSE
3	Melbourne	Darwin	Darwin	0	0	FALSE
4	Sydney	Sydney	Sydney	1	1	FALSE
5	Brisbane	Canberra	Brisbane	0	1	TRUE

9, 10 Proofs: see worked solutions

11 C **12** C

13 Proof: see worked solutions

14 D **15** A

16 Proof: see worked solutions

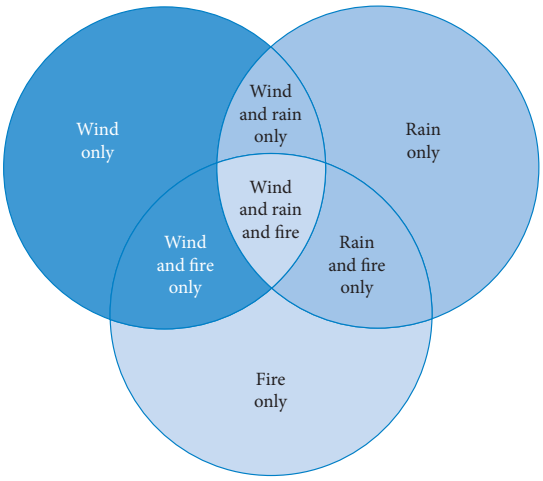
17 $\overline{A} + \overline{B} + \overline{C}$

18 $(A + B)^N = A^N + B^N + (N - 2)$ terms containing powers of AB.

Using the property $Y \cdot Y = Y$, all terms containing AB simplify to AB.

19 For a conjunction, the only combination that produces 1 for the product is when all variables are 1. The product sum is $A_1A_2A_3A_4 \ldots A_N$ since all other products are 0 and are not used in the calculation.

20



EXERCISE 4.5

1 D **2** B

3 $AB + AD + \overline{B}\overline{C}D + BCD$ **4** $\overline{B}C + A\overline{B} + AC + \overline{A}B\overline{C}$

5 **a** $\overline{D}$ **b** $\overline{D}$

9780170448338

6 $\overline{A}\overline{B} + \overline{A}B + A\overline{B} + AB = \overline{A}(\overline{B} + B) + A(\overline{B} + B)$
$= \overline{A} + A = 1$

7

AB \ CD	00	01	11	10
00	1	0	0	1
01	1	1	0	0
11	0	1	1	0
10	0	0	1	1

This gives SOP = $\overline{A}\overline{B}\overline{D} + \overline{A}B\overline{C} + A\overline{B}C + ABD$

AB \ CD	00	01	11	10
00	1	0	0	1
01	1	1	0	0
11	0	1	1	0
10	0	0	1	1

This gives SOP = $\overline{A}C\overline{D} + B\overline{C}D + AC\overline{D} + \overline{B}C\overline{D}$

8 C **9** A **10** A **11** C **12** D

13 None of the 4 variables remains constant as we move left to right and from top to bottom. Further, the group is not a rectangle, and it contains a zero.

14

A \ BC	00	01	11	10
0	0	0	1	0
1	0	1	1	0

$AC + BC$

15 1

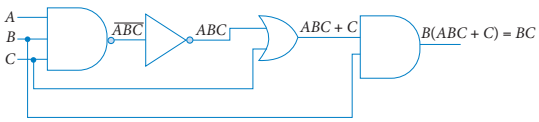
EXERCISE 4.6

1 B **2** D

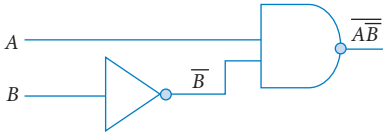
3

A	B	C	$D = \overline{ABC}$
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

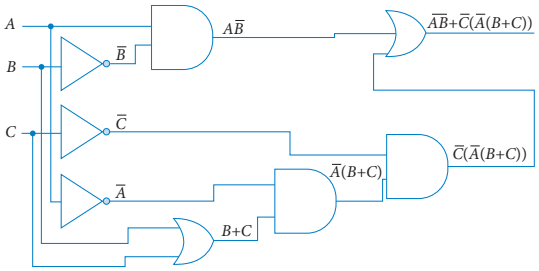
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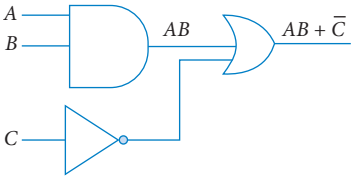
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6

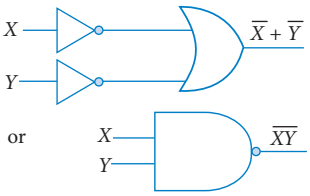


7



8 **a** $Q = \overline{XY}$ OR $Q = \overline{X} + \overline{Y}$

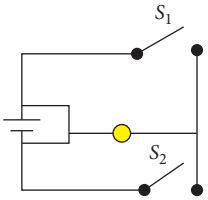
b



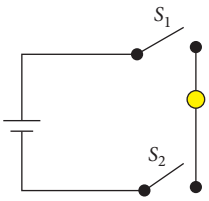
9 Suppose A is 0 and the value entering the OR gate as output from the inverter is also 0. The 0 OR 0 is 0, which is inverted to 1. The value of Q is now 1. This value returns to the OR gate. The value of A (0) is ORed with 1 and inverted to become 0. The value of Q is now 0. This is an infinite cycle, where the value of Q alternates between 0 and 1.

10 E **11** C **12** D **13** A **14** A

15 **a** Other circuits are possible.



b Other circuits are possible.



16 **a** L_1 **b** L_3 **c** L_2

17 **a** $\overline{V_1V_2V_3 \ldots V_N} = \overline{V_1(V_2V_3 \ldots V_N)}$
$= \overline{V}_1 + \overline{V_2(V_3 \ldots V_N)}$
$= \overline{V}_1 + \overline{V}_2 + \overline{(V_3 \ldots V_N)}$
$\vdots$
$= \overline{V}_1 + \overline{V}_2 + \ldots + \overline{V}_N$

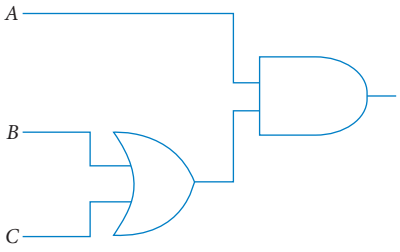
b $\overline{V_1V_2V_3 \ldots V_N}$ has one AND gate with N inputs. The output is NOTed to obtain $\overline{V_1V_2V_3 \ldots V_N}$. There are 2 gates needed in total. $\overline{V}_1 + \overline{V}_2 + \overline{V}_3 + \ldots + \overline{V}_N$ requires N NOT gates followed by one OR gate. This is $N + 1$ gates in total. The difference is $N + 1 - 2 = N - 1$ gates.

18 **a** $A\overline{B} + \overline{A}B$

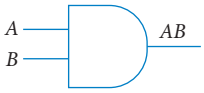
b

A	B	$Z = A\overline{B} + \overline{A}B$
0	0	0
0	1	1
1	0	1
1	1	0

c A XOR B

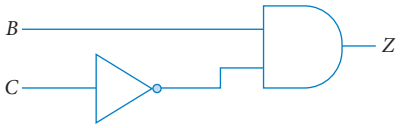
19 

20 $\overline{A\overline{B} + \overline{A}B} = \overline{A\overline{B}} + AB$ so Z corresponds to AB.



21 a $Z = \overline{A}B\overline{C}\overline{D} + \overline{A}B\overline{C}D + AB\overline{C}\overline{D} + AB\overline{C}D$

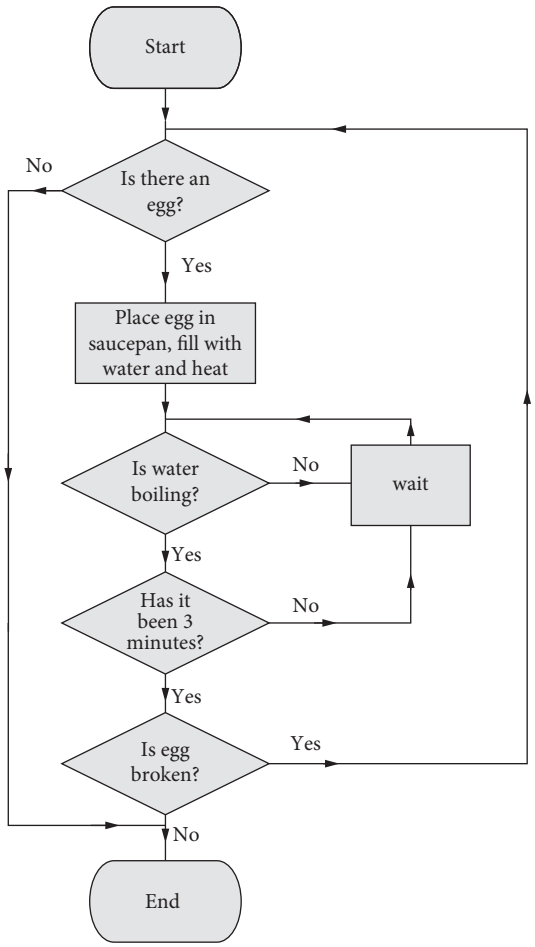
b $Z = B\overline{C}$



EXERCISE 4.7

1 E **2** C

3



4 D

5 a count = 4
new product = 4
count = 3
new product = 12
count = 2
new product = 24
count = 1
new product = 24
count = 0

b The algorithm finds $N!$, the factorial of N.

6 C

7 a A loop is created that produces the length, width and height of a rectangular prism. The volume is then calculated using these values.

b 2, 32, 162

8
```
sum ← 0
FOR loop = 1 TO 5
   number ← RAND(9)
   sum ← sum + number^2
ENDFOR
PRINT sum
```

9
```
INTEGER even[5]
INTEGER odd[5]
FOR loop = 1 TO 5
   REPEAT
      even[loop] ← RAND(20)
   UNTIL even[loop] MOD2 = 0
   REPEAT
      odd[loop] ← RAND(20)
   UNTIL odd[loop] MOD2 = 1
ENDFOR
FOR loop = 1 TO 5
      PRINT even[loop]
      PRINT odd[loop]
ENDFOR
```

10 B **11** A

12
```
INTEGER array[50]
INTEGER reverse[50]
count ← 0
FOR loop = 1 TO 50
   IF loop MOD3 = 0
      count ← count + 1
      array[count] ← loop
   ENDIF
ENDFOR
FOR loop = 1 TO count
   reverse[loop] ← array[count + 1 – loop]
ENDFOR
or:
INTEGER reverse[50]
count ← 0
FOR loop = 1 TO 50
   n ← 51 – loop
   IF n MOD3 = 0
      count ← count + 1
      reverse[count] ← loop
   ENDIF
ENDFOR
```

13
```
INTEGER array[10]
INTEGER ratio[10]
FOR loop = 1 TO 10
   INPUT next Fibonacci number
   array[loop] ← next Fibonacci number
ENDFOR
FOR loop = 1 TO 9
   ratio[loop] ← array[loop] / array[loop + 1]
ENDFOR
```

14
```
REPEAT
number1 ← RAND(10)
IF number1 MOD2 = 1
    THEN number2 ← RAND(2)
ENDIF
UNTIL number1 MOD2 = 0 or number2 = 1
```

15
```
INPUT a, b, c
IF a = 0
  THEN
ENDIF
IF b^2 – 4ac < 0
  THEN
  PRINT "No real solutions"
ENDIF
IF b^2 – 4ac = 0
  THEN
  PRINT "One rational solution"
  PRINT –b/(2*a)
ENDIF
IF b^2 – 4ac > 0
  THEN
  PRINT "Two solutions"
  PRINT (–b + (b^2 – 4ac)^0.5)/(2*a)
  PRINT (–b – (b^2 – 4ac)^0.5)/(2*a)
ENDIF
```

16 3, 4, 2, 1
2, 4, 3, 1
1, 4, 3, 2
1, 3, 4, 2
1, 2, 4, 3
1, 2, 3, 4

17 a P – 2

b I = 1 and I = P are already known to be divisors of P.

c The count remains zero until a divisor is found. If the count remains zero, the number is prime, and if the count is more than zero, it is composite.

d If P is 2, Line 3 requires performing a loop "from 2 to 1". We have not defined what it means to do a FOR loop where the second parameter is smaller than the first. It might result in the FOR loop not executing at all, in which case count is never incremented so we will correctly conclude that 2 is prime. But it might also result in the FOR loop executing once (with I = 2) or even twice (with I = 2 and I = 1). Because both 1 and 2 divide 2, either of these cases will result in count being incremented at least once, leading to the false conclusion that 2 is not prime.

18 a
```
READ x1,x2, f(x)
READ accuracy value
DO
  Xaverage ← (x1+x2)/2
  IF f(Xaverage) > 0
    THEN x1 ← Xaverage
  ELSE x2 ← Xaverage
  ENDIF
WHILE |x1 – x2| ≥ A
ENDWHILE
```

b $x1=2.5$, $x2=4$
$x1=2.5$, $x2=3.25$
$x1=2.875$, $x2=3.25$
$x1=2.875$, $x2=3.0625$
$x1=2.96875$, $x2=3.0625$

CUMULATIVE EXAMINATION 1

1 a 'If I am thirsty, I drink' and 'If I drink, I am thirsty'.

b 'If I am not thirsty, I do not drink.'

2

A	B	$\neg A$	$\neg A \rightarrow B$
F	F	T	F
F	T	T	T
T	F	F	T
T	T	F	T

3 $A(\overline{A + \overline{B}}) = A(\overline{A} \bullet B)$
$= A\overline{A} \bullet B$
$= 0 \bullet B$
$= 0$

4 a 4 **b** 4

5 a The graph is connected and every degree is of even order.

b A–C–D–E–C–B–E–F–B–A (other answers are possible)

CUMULATIVE EXAMINATION 2

Section A

1 C **2** C **3** C **4** B **5** D

Section B

1 a
```
INPUT t
H ← T DIV 60
T ← T MOD 60
PRINT H
PRINT T
```

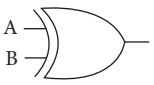
b 142 DIV 60 = 2
142 MOD 60 = 22
142 minutes = 2 hours 22 minutes

2 a

A	B	AB	$\overline{AB}$	$A + B$	$(A + B)\overline{AB}$
0	0	0	1	0	0
0	1	0	1	1	1
1	0	0	1	1	1
1	1	1	0	1	0

b $\overline{A}B + A\overline{B}$

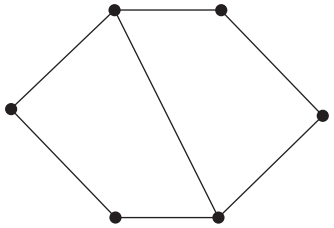
c



XOR

3 $(m + 1)^2 - 1 = m^2 + 2m + 1 - 1$
$= 5(5k^2 + 2k)$, $m = 5k$
$= 5p$, $p \in N$ is divisible by 5.

4 $v = 6$, $f = 3$, $e = 7$. Other graphs are possible.



5 Janet $22 000, Marty $21 829.

CHAPTER 5

EXERCISE 5.1

1 **a** 2×3 **b** 3×2 **c** 2×1 **d** 3×4
e 1×3 **f** 0 **g** 2 **h** -3
i 1 **j** 0 **k** -3 **l** -4

2 **a** $\begin{bmatrix} -2 & -1 & 3 \\ 7 & 1 & 2 \end{bmatrix}$ **b** $\begin{bmatrix} 0 & 12 & -15 \\ -18 & -3 & -12 \end{bmatrix}$

c $\begin{bmatrix} 4 & -8 & 5 \\ 4 & 2 & 1 \end{bmatrix}$ **d** $\begin{bmatrix} -4 & -6 & 11 \\ 20 & 3 & 8 \end{bmatrix}$

e $\begin{bmatrix} 2 & 7 & -12 \\ -13 & -1 & -13 \end{bmatrix}$ **f** $\begin{bmatrix} 8 & -28 & 22 \\ 32 & 10 & 4 \end{bmatrix}$

g $\begin{bmatrix} 6 & -27 & 24 \\ 33 & 9 & 9 \end{bmatrix}$ **h** $\begin{bmatrix} 6 & -27 & 24 \\ 33 & 9 & 9 \end{bmatrix}$

i $\begin{bmatrix} 2 & 5 & -11 \\ -7 & 1 & -16 \end{bmatrix}$ **j** $\begin{bmatrix} 2 & 5 & -11 \\ -7 & 1 & -16 \end{bmatrix}$

k $\begin{bmatrix} 0 & -6 & 6 \\ 12 & 3 & 1 \end{bmatrix}$ **l** $\begin{bmatrix} 0 & -6 & 6 \\ 12 & 3 & 1 \end{bmatrix}$

m $\begin{bmatrix} 2 & -17 & 15 \\ 29 & 8 & 1 \end{bmatrix}$ **n** $\begin{bmatrix} 2 & -13 & 10 \\ 23 & 7 & -3 \end{bmatrix}$

o $\begin{bmatrix} 8 & 4 & -18 \\ -16 & 2 & -28 \end{bmatrix}$ **p** $\begin{bmatrix} 2 & 3 & -7 \\ -7 & 0 & -9 \end{bmatrix}$

3 **a** $\begin{bmatrix} -6 & 5 \\ -2 & 7 \end{bmatrix}$ **b** $\begin{bmatrix} -11 & -7 \\ -5 & 12 \end{bmatrix}$

c No product, *rows* of $R \neq$ *columns* of Q

d No product, *rows* of $P \neq$ *columns* of S

e $\begin{bmatrix} 7 & 9 & 14 \\ -11 & -9 & -16 \end{bmatrix}$

f No product, *rows* of $T \neq$ *columns* of Q

g No product, *rows* of $T \neq$ *columns* of R

h No product, *rows* of $R \neq$ *columns* of T

i $\begin{bmatrix} -15 & 22 \end{bmatrix}$

j $\begin{bmatrix} -22 & -3 & 4 \\ 32 & 9 & -14 \end{bmatrix}$ **k** $\begin{bmatrix} 10 \\ -2 \end{bmatrix}$

l No product, *rows* of $S \neq$ *columns* of Q

m $\begin{bmatrix} -4 & 18 \\ 20 & 2 \end{bmatrix}$ **n** $\begin{bmatrix} -23 & -7 & 11 \\ 20 & 12 & -20 \\ 3 & -5 & 9 \end{bmatrix}$

o $\begin{bmatrix} 15 & -16 \\ -12 & 16 \\ -3 & 0 \end{bmatrix}$

p No product, *rows* of $T \neq$ *columns* of U

4 **a** $\begin{bmatrix} -20 & 40 \\ 26 & -52 \end{bmatrix}$ **b** $\begin{bmatrix} -40 & -20 \\ -80 & -40 \end{bmatrix}$

c $\begin{bmatrix} 22 & 20 \\ -44 & -40 \end{bmatrix}$ **d** $\begin{bmatrix} -12 & 2 \\ -28 & -10 \end{bmatrix}$

e $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **f** $\begin{bmatrix} -12 & 24 \\ -24 & 48 \end{bmatrix}$

g $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$!! **h** $\begin{bmatrix} 19 & 18 \\ 45 & 46 \end{bmatrix}$

i $\begin{bmatrix} -194 & -188 \\ 94 & 92 \end{bmatrix}$ **j** $\begin{bmatrix} -194 & -188 \\ 94 & 92 \end{bmatrix}$

k $\begin{bmatrix} -58 & -44 \\ -62 & -44 \end{bmatrix}$ **l** $\begin{bmatrix} 16 & -32 \\ 96 & -192 \end{bmatrix}$

m $\begin{bmatrix} -6 & -3 \\ -66 & -37 \end{bmatrix}$ **n** $\begin{bmatrix} -48 & 58 \\ 62 & -68 \end{bmatrix}$

o $\begin{bmatrix} -36 & 72 \\ 78 & -156 \end{bmatrix}$ **p** $\begin{bmatrix} -24 & -84 \\ -48 & -168 \end{bmatrix}$

5 **a** $\begin{bmatrix} -7 \\ 8 \end{bmatrix}$ **b** $\begin{bmatrix} 6 \\ 0 \end{bmatrix}$ **c** $\begin{bmatrix} -25 \\ 73 \end{bmatrix}$

d No product, *rows* of $F \neq$ *columns* of $D + 3E$

e No product, *rows* of $F \neq$ *columns* of D

f $\begin{bmatrix} -32 \\ 76 \end{bmatrix}$ **g** $\begin{bmatrix} -10 \\ -19 \end{bmatrix}$

h No product, *rows* of $C \neq$ *columns* of D

i $\begin{bmatrix} -106 & -69 \\ -22 & 102 \end{bmatrix}$ **j** $\begin{bmatrix} -69 & -12 \\ -43 & 65 \end{bmatrix}$

k $\begin{bmatrix} -1 \\ 48 \end{bmatrix}$ **l** $\begin{bmatrix} 13 \\ -8 \end{bmatrix}$

m No sum, dimensions of A and BC are different.

n $\begin{bmatrix} -5 \\ 39 \end{bmatrix}$ **o** $\begin{bmatrix} -26 \\ -29 \end{bmatrix}$ **p** $\begin{bmatrix} 51 & 75 \\ -5 & -9 \end{bmatrix}$

6 **a** $\begin{bmatrix} -43 & 45 \\ -18 & 19 \end{bmatrix}$ **b** $\begin{bmatrix} -11 & 15 \\ 10 & -13 \end{bmatrix}$

c $\begin{bmatrix} 3 & -13 \\ 26 & -10 \end{bmatrix}$ **d** $\begin{bmatrix} 3 & -13 \\ 26 & -10 \end{bmatrix}$

e $\begin{bmatrix} 18 & -22 \\ 44 & -4 \end{bmatrix}$ **f** $\begin{bmatrix} 6 & -13 \\ 26 & -7 \end{bmatrix}$

g $\begin{bmatrix} 34 & -33 \\ 6 & 10 \end{bmatrix}$ **h** $\begin{bmatrix} 6 & -21 \\ 42 & -15 \end{bmatrix}$

i $\begin{bmatrix} 30 & 18 & 10 \\ -10 & -16 & -27 \\ -19 & 2 & 26 \end{bmatrix}$ **j** $\begin{bmatrix} 12 & 19 & 8 \\ 12 & 21 & 6 \\ -3 & -8 & 6 \end{bmatrix}$

k $\begin{bmatrix} -4 & 36 & 96 \\ -116 & -112 & -114 \\ -358 & -96 & 154 \end{bmatrix}$ **l** $\begin{bmatrix} -50 & -18 & 11 \\ 36 & 69 & 49 \\ -53 & -55 & -16 \end{bmatrix}$

m $\begin{bmatrix} -14 & -20 & 15 \\ -8 & -5 & -17 \\ -85 & -35 & 22 \end{bmatrix}$ **n** $\begin{bmatrix} 12 & 30 & -79 \\ -12 & -29 & -3 \\ 6 & 31 & 46 \end{bmatrix}$

o $\begin{bmatrix} 12 & 21 & 27 \\ 15 & 12 & -30 \\ -9 & -16 & 73 \end{bmatrix}$ **p** $\begin{bmatrix} 37 & 5 & 16 \\ -51 & -30 & 39 \\ 46 & -19 & 52 \end{bmatrix}$

7 **a** $7A + AB + 2BA$ **b** $8XY - 5X + 4YX$
c $2AB + 4B + 5A - BA$ **d** $2PQ + 3P + QP$
e $10P - 7Q$ **f** $6PQ + 2QP - 2P$
g $8A + 4B - 10AB$ **h** $G - F$
i $4H + HK - 3K$ **j** $5G + 5P - GP$
k $3AY + Y$ **l** $6MN - NM$
m $3PS + SP - S$ **n** $8V + 2VW - 3WV$
o $4T - 2TU + 3U$

8 **a** $12A - 15B$ **b** $2A^2 + 6AB$
c $2M^2 - 2MN + 6M$ **d** $2X - 6Y + 4I + 2XY$
e $4M^2N + 4MN^2$
f $9H^2 + 12HG - 12GH - 16G^2$
g $4A^2 - 10AB - 10BA + 25B^2$
h $2P^2Q - 4PQ^2$ **i** $A^2 + 2AB - BA + 2B^2$
j $G^2 + H^2$
k $2DK + 2DM - 3KM - 3KD$
l $M^2 - M - 6I$ **m** $6X^2 - X - 2I$
n $X^3 - X^2Y + XY^2 + YX^2 - YXY + Y^3$
o $2X^2 - 2XY - 9Y^2$

9 **a** $X(3I + Y)$ **b** $(2X - 3I)Y$
c Cannot be factorised
d $2(4P - 3R)Q$ or $(4P - 3R)2Q$
e $M(2N + 5P)$ **f** $3A(A + B) + (A + B)B$
g $(S + T)(3S - 2T)$ **h** $M(5M + 2N + I)$
i $(I + 2B + 3BA)A$ **j** $(A - I)^2$
k $(3X + I)(X + 2I)$
l $3A(A + 2B) + (A + 2B)B$
m $P(P - 2M) - M(M - 2P)$
n $(2M - P)(M - 2P)$
o $A^2(A + 2B) + (A + 2B)B^2$

10 $\begin{bmatrix} -14 & 16 \\ -3 & 6 \end{bmatrix}$

11 Show that $\begin{bmatrix} 1 & 3 & -5 \\ 0 & 5 & -8 \\ 2 & -2 & 3 \end{bmatrix}\begin{bmatrix} -1 & 1 & 1 \\ -16 & 13 & 8 \\ -10 & 8 & 5 \end{bmatrix} = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ by showing the product for every element of I: $1 \times (-1) + 3 \times -16 + (-5) \times (-10) = 1$, etc. See worked solutions.

12 A **13** B **14** E **15** C

EXERCISE 5.2

1 D **2** B

3 **a** $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 4 \end{bmatrix}$ **b** $\begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$
c $\begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}$ **d** $\begin{bmatrix} 0.4 & 0 \\ 0 & 1.25 \end{bmatrix}$
e $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$ **f** no inverse
g $\begin{bmatrix} 3 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **h** $\begin{bmatrix} -0.5 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 10 \end{bmatrix}$

4 **a** -8 **b** -0.24 **c** $-\frac{5}{2}$ **d** 12
e -7.2 **f** -8 **g** -24 **h** 0
i -45 **j** -67.5

5 **a** $\begin{bmatrix} -1 & 1.5 \\ 0 & 0.25 \end{bmatrix}$ **b** $\begin{bmatrix} 0.5 & 0 \\ 3 & 1 \end{bmatrix}$
c $\begin{bmatrix} 0.2 & 0 \\ -0.4 & -0.25 \end{bmatrix}$ **d** $\begin{bmatrix} \frac{1}{3} & 16\frac{2}{3} \\ 0 & 5 \end{bmatrix}$
e $\begin{bmatrix} 1 & 0.75 & -5.5 \\ 0 & 0.25 & -1 \\ 0 & 0 & 0.5 \end{bmatrix}$ **f** $\begin{bmatrix} 1 & 0 & 0 \\ 2 & -0.5 & 0 \\ 10 & -4 & -1 \end{bmatrix}$
g $\begin{bmatrix} 3 & -96 & \frac{129}{4} \\ 0 & 4 & -1\frac{1}{3} \\ 0 & 0 & \frac{1}{6} \end{bmatrix}$ **h** no inverse

6 **a** -2 **b** -0.48 **c** 0 **d** 6
e $-\frac{2}{15}$ **f** -40 **g** 48 **h** 0
i -37.44 **j** -9

7 **a** $2, \begin{bmatrix} 2 & 1 \\ -1.5 & -0.5 \end{bmatrix}$
b $7, \frac{1}{7}\begin{bmatrix} 4 & 1 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} \frac{4}{7} & \frac{1}{7} \\ \frac{5}{7} & \frac{3}{7} \end{bmatrix}$
c $-2, \begin{bmatrix} -1.5 & 2.5 \\ 2 & -3 \end{bmatrix}$ **d** $-1, \begin{bmatrix} 3 & 5 \\ 2 & 3 \end{bmatrix}$
e 0, singular
f $-6.2, -\frac{1}{62}\begin{bmatrix} 4 & -20 \\ -30 & -5 \end{bmatrix} = \begin{bmatrix} -\frac{2}{31} & \frac{10}{31} \\ \frac{15}{31} & \frac{5}{62} \end{bmatrix}$
g $20, \begin{bmatrix} 0.35 & -0.15 \\ 0.1 & 0.1 \end{bmatrix}$
h 0, singular
i $26, \frac{1}{26}\begin{bmatrix} 3 & 4 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{26} & \frac{2}{13} \\ -\frac{5}{26} & \frac{1}{13} \end{bmatrix}$
j $-17, -\frac{1}{17}\begin{bmatrix} -2 & -3 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{17} & \frac{3}{17} \\ \frac{5}{17} & -\frac{1}{17} \end{bmatrix}$

8 **a** 26 **b** -28 **c** 36 **d** -13
e 0 **f** 91 **g** 0
9 **a** 8 **b** 0 **c** -9 **d** 30
e -38 **f** -37 **g** 0 **h** 552
10 **a** 22 **b** -42 **c** 180 **d** 60
e -3.25 **f** 2.789 **g** -648
11 **a** 24 **b** -7
12 50 **13** A **14** B

15 a LHS $= |AB| = \begin{vmatrix} -20 & 0 \\ 0 & -8 \end{vmatrix} = 160$

RHS $= |A||B| = -20 \times -8 = 160$

b For diagonal matrices A, B of the same size, $|AB| = |A||B|$

c Use $|A| = a_{11}a_{22} \ldots a_{nn}$ etc.

EXERCISE 5.3

1 E

2 B

3 a $\begin{bmatrix} -1 & -\frac{13}{3} \\ -\frac{1}{3} & -\frac{4}{3} \end{bmatrix}$

b $\begin{bmatrix} -\frac{2}{3} & 3 \\ \frac{7}{3} & 2 \end{bmatrix}$

c $\begin{bmatrix} 5 & 3.5 \\ -4 & 6.5 \end{bmatrix}$

d $\begin{bmatrix} 0.5 & -2 \\ -1 & 2.75 \end{bmatrix}$

e $\begin{bmatrix} \frac{1}{3} & \frac{1}{6} & \frac{7}{6} \\ -\frac{1}{3} & -\frac{7}{6} & -\frac{11}{6} \\ -\frac{5}{6} & \frac{1}{2} & -\frac{1}{6} \end{bmatrix}$

f $\begin{bmatrix} -1.5 & 5.25 & -5.25 \\ 5.75 & -4.5 & -3 \\ 7.25 & -5.75 & 5.75 \end{bmatrix}$

g $\begin{bmatrix} 4.2 & -5.8 \\ -3.8 & 2.3 \\ -3.1 & 3.8 \end{bmatrix}$

h $\begin{bmatrix} 16 & 9.5 & 7.5 \\ 0 & 11.5 & 6.5 \end{bmatrix}$

i $\begin{bmatrix} -8.4 & -4.2 \\ -6.2 & 3.6 \\ -0.6 & 1.6 \end{bmatrix}$

j $\begin{bmatrix} -\frac{8}{3} & -\frac{4}{3} & -\frac{8}{3} \\ -1 & -\frac{14}{3} & -\frac{5}{3} \end{bmatrix}$

4 a $\begin{bmatrix} -\frac{14}{3} & \frac{2}{3} \\ \frac{11}{3} & \frac{16}{3} \end{bmatrix}$

b $\begin{bmatrix} -\frac{1}{3} & 2 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$

c $\begin{bmatrix} -0.5 & 1.25 \\ 0 & 4 \end{bmatrix}$

d $\begin{bmatrix} -0.5 & -3.75 & -2.25 \\ -0.25 & 5.25 & 0 \\ -5 & 3 & -2 \end{bmatrix}$

e $\begin{bmatrix} 8 & -\frac{19}{3} \\ \frac{16}{3} & -7 \\ -\frac{19}{3} & -\frac{16}{3} \end{bmatrix}$

f $\begin{bmatrix} 5.25 & 2 & 2.25 \\ 3.25 & -5.5 & -2.25 \end{bmatrix}$

g $\begin{bmatrix} -0.8 & 4 & -1.2 \\ -4.2 & -6.6 & 5 \\ -1 & -1.2 & 1 \end{bmatrix}$

h $\begin{bmatrix} 2.4 & 1.6 & 4.8 & 2.6 \\ -6.6 & 3 & -2.8 & 1 \\ -4.6 & 7.2 & 0.8 & -4.2 \end{bmatrix}$

5 a $\begin{bmatrix} -4.75 & 0.25 \\ -5.5 & 1.5 \end{bmatrix}$

b $\begin{bmatrix} \frac{24}{13} & \frac{55}{13} \\ \frac{19}{13} & \frac{43}{13} \end{bmatrix}$

c $\begin{bmatrix} \frac{11}{17} & \frac{30}{17} \\ -\frac{9}{17} & -\frac{6}{17} \end{bmatrix}$

d $\begin{bmatrix} 1.5 & \frac{1}{6} \\ -5 & 1 \end{bmatrix}$

e $\begin{bmatrix} \frac{11}{6} & -6 \\ -\frac{11}{6} & 3 \end{bmatrix}$

f $\begin{bmatrix} -\frac{18}{19} & \frac{34}{57} \\ -\frac{99}{38} & \frac{187}{114} \end{bmatrix}$

g $\begin{bmatrix} -\frac{15}{34} & -\frac{59}{34} \\ \frac{5}{34} & -\frac{71}{34} \end{bmatrix}$

h $\begin{bmatrix} -\frac{146}{65} & \frac{67}{65} \\ \frac{14}{13} & -\frac{5}{13} \end{bmatrix}$

i $\begin{bmatrix} \frac{7}{59} & -\frac{63}{59} \\ -\frac{39}{59} & -\frac{3}{59} \end{bmatrix}$

j $\begin{bmatrix} -\frac{308}{83} & -\frac{205}{83} \\ -\frac{288}{83} & -\frac{113}{83} \end{bmatrix}$

6 a $\begin{bmatrix} -19 \\ 21 \end{bmatrix}$

b $\begin{bmatrix} -2 \\ -\frac{1}{7} \end{bmatrix}$

c $\begin{bmatrix} -6 \\ -\frac{3}{2} \end{bmatrix}$

d $\begin{bmatrix} 0.75 \\ -2.3 \end{bmatrix}$

e $\begin{bmatrix} -\frac{8}{3} \\ \frac{29}{6} \end{bmatrix}$

f $\begin{bmatrix} -0.25 \\ \frac{11}{24} \end{bmatrix}$

g $\begin{bmatrix} \frac{41}{21} \\ \frac{37}{21} \end{bmatrix}$

h $\begin{bmatrix} -\frac{72}{43} \\ \frac{45}{43} \end{bmatrix}$

i $\begin{bmatrix} -\frac{3}{28} \\ -\frac{1}{28} \end{bmatrix}$

7 a $\begin{bmatrix} 7 & \frac{11}{3} \\ 8 & 4 \end{bmatrix}$

b $\begin{bmatrix} \frac{14}{17} & -\frac{76}{51} \\ -\frac{11}{17} & \frac{16}{51} \end{bmatrix}$

c $\begin{bmatrix} \frac{115}{97} & \frac{60}{97} & -\frac{231}{97} \\ -\frac{15}{97} & -\frac{50}{97} & \frac{47}{97} \\ -\frac{1}{291} & \frac{349}{582} & -\frac{427}{582} \end{bmatrix}$

d $\begin{bmatrix} 10 & -\frac{13}{11} & \frac{3}{11} \\ 5 & \frac{6}{11} & \frac{24}{11} \\ -18 & \frac{14}{11} & -\frac{43}{11} \end{bmatrix}$

e $\begin{bmatrix} -1.625 & 0.8125 \\ 0.55 & -0.975 \end{bmatrix}$

f $\begin{bmatrix} -\frac{13}{21} & \frac{3}{7} \\ -\frac{37}{21} & \frac{1}{7} \end{bmatrix}$

g $\begin{bmatrix} -\frac{32}{23} & -\frac{75}{23} \\ -\frac{116}{23} & -\frac{77}{23} \end{bmatrix}$

h $\begin{bmatrix} 0.8 & 7.9 & -1.4 \\ 0.2 & -9.4 & 2.9 \\ -0.2 & 2.9 & -0.9 \end{bmatrix}$

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i $\begin{bmatrix} \frac{5}{23} & \frac{56}{69} & \frac{77}{69} \\ -\frac{11}{46} & \frac{77}{46} & \frac{57}{46} \\ \frac{1}{46} & -\frac{7}{46} & -\frac{1}{46} \end{bmatrix}$ **j** $\begin{bmatrix} 1 & -1 & -\frac{14}{9} \\ -\frac{11}{4} & \frac{17}{12} & \frac{197}{54} \\ \frac{13}{4} & -\frac{31}{12} & -\frac{307}{54} \end{bmatrix}$

8 $\begin{bmatrix} -1.4 & 0.4 & -1.4 \\ -1.6 & 2.2 & 1 \\ 0.4 & 0.4 & -2 \end{bmatrix}$ **9** $\begin{bmatrix} -8 & -2 \\ -7 & 1 \end{bmatrix}$

10 D **11** B **12** A

CUMULATIVE EXAMINATION 1

1 $\overline{C}$ **2** $AB + \overline{C}$

3 $\begin{bmatrix} \frac{39}{2} & 26 \\ 28 & 37 \end{bmatrix}$ **4** −15

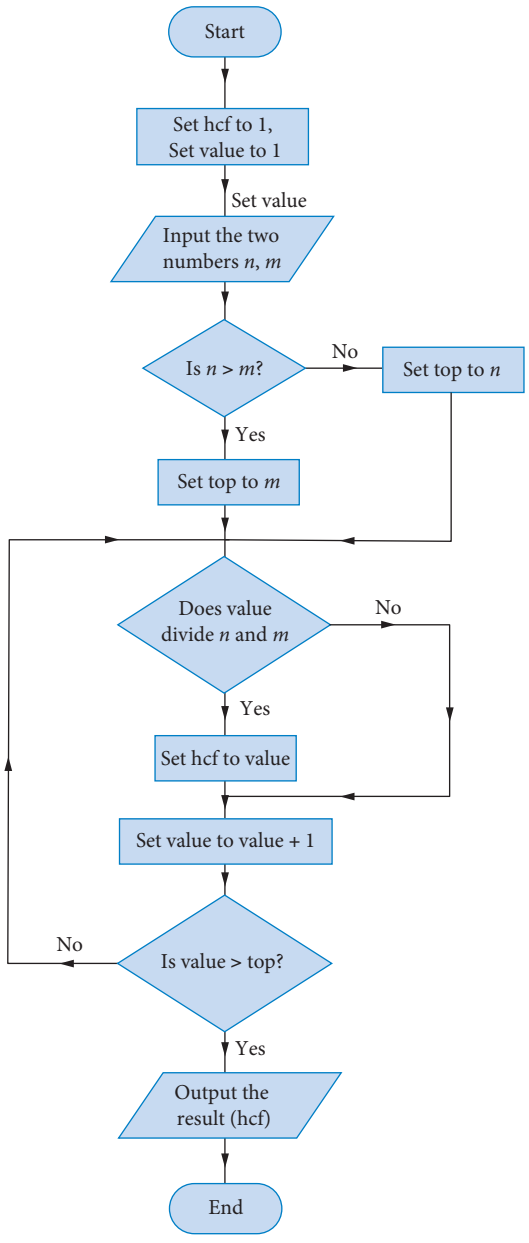
CUMULATIVE EXAMINATION 2

Section A

1 D **2** D **3** A **4** B **5** E

Section B

1



b
```
H ← 1
INPUT N, M
IF N > M
THEN T ← M
ELSE T ← N
ENDIF
FOR V = 1 TO T
IF V divides N AND V divides M
   THEN H ← V
ENDIF
ENDFOR
PRINT H
```

2 a $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b They are their own inverses.

c $a = \pm 1$, $b = c = 0$ and $d = \pm 1$ or $a = \pm 1$, $b = 0$ and $d = -a$ or $a = \pm 1$, $c = 0$ and $d = -a$ or $d = -a$, $c = \frac{1 - a^2}{b}$. That is, $M = \begin{bmatrix} \pm 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \pm 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} \pm 1 & b \\ 0 & \mp 1 \end{bmatrix}, \begin{bmatrix} \pm 1 & 0 \\ c & \mp 1 \end{bmatrix}$ or $\begin{bmatrix} a & b \\ \frac{1 - a^2}{b} & -a \end{bmatrix}$.

CHAPTER 6

EXERCISE 6.1

1 25 **2** 84 **3** 35

4 a RR, RB, RY, BR, BY, YR, YB **b** 4

5 9 **6** 20

7 RRR, RRB, RRY, RBR, RBY, RYR, RYB, BRR, BYR, BRY, YRR, YRB, YBR

8 B **9** D **10** C **11** E

12 D **13** E

EXERCISE 6.2

1 D **2** C **3** 120

4 a 95 040 **b** 32 760 **c** 720

5 120

6 a 40 320 **b** 95 040 **c** 3432

7 a 132 **b** 15 **c** 1

8 144 **9** 126 **10** 18 900 **11** 240

12 360

13 a 56 **b** 11 **c** 120

d 1 **e** 24

14 3780 **15** 70 **16** 15 **17** 210

18 a 504 **b** 140 **c** 84

19 a 6 **b** 120 **c** 4320 **d** 30 240

20 E **21** B **22** B **23** D

24 E **25** D

EXERCISE 6.3

1 C **2** C

3 a 10 **b** 35 **c** 4

4 a 11 **b** 66 **c** 100

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5 **a** 4845 **b** 75 287 520 **c** 900

6 70 **7** 350

8 $x = \{2, 6\}$ and $y = \{3, 5\}$ **9** 15

10 **a** 28 **b** 13

11 **a** 70 **b** 15 **c** 40 **d** 15

12 **a** 66 **b** 105 **c** 21

13 $x = 15$, $y = \{6, 9\}$ **14** B

15 E **16** C **17** C **18** C

19 C **20** D

21 **a** 5 **b** 4 **c** 31

22 **a** ${}^6C_3 = 20$ **b** ${}^1C_1 \times {}^5C_2 = 10$

c ${}^5C_3 = 10$ **d** ${}^2C_1 \times {}^4C_2 = 12$

EXERCISE 6.4

1 C **2** B **3** 5 socks

4 16 **5** 5

6 Proof: see worked solutions

7 4 **8** 28 **9** 27

10 **a** 13 **b** 366

11 211 **12** 8

13 **a** It is not possible to be friends with yourself, so the maximum number of friends in a group of 20 people is 19. If someone in the group has 19 friends, then they are friends with everyone, which means there cannot be anyone in the group with 0 friends.

b The number of possible friends {1, 2, 3 … 19} = 19, number of people = 20. Therefore, there must be at least 2 people who have the same number of friends.

14 C **15** E **16** D **17** C **18** B

EXERCISE 6.5

1 C **2** D **3** 107 **4** 90

5 80 **6** 92 **7** 460

8 **a** 81 **b** 86 **c** 108

9 55 **10** C **11** A **12** B

13 C **14** B **15** B

CUMULATIVE EXAMINATION 1

1 **a** 120 **b** 72

2 120 **3** 36, 28

4 $t_1 = 3645$, $r = \frac{1}{3}$, $S_\infty = \frac{10\,935}{2}$

CUMULATIVE EXAMINATION 2

Section A

1 A **2** B **3** B **4** A **5** D

Section B

1 **a** **i** 176 **ii** 157

iii 142 **iv** 200

b 12 **c** 35

2 **a** 96 **b** $\frac{128}{3}$ **c** $144\left(\frac{2}{3}\right)^{n-1}$

d 432 **e** 720 m

CHAPTER 7

EXERCISE 7.1

1 **a** $\frac{10\pi}{3}$ cm **b** 10.47 cm

2 **a** $\frac{125\pi}{6}$ cm^2 **b** 65.45 cm^2

3 **a** $\frac{25\pi}{3} - \frac{25\sqrt{3}}{4}$ cm **b** 15.35 cm

4 **a** $\frac{\pi}{3}$ **b** $\frac{\pi}{6}$ **c** $\frac{\pi}{9}$ **d** $\frac{7\pi}{6}$

5 **a** minor **b** major **c** minor **d** major

6 B **7** A **8** D **9** E

10 E **11** B **12** D

EXERCISE 7.2

1 **a** $\frac{10\pi}{3}$ cm **b** $\frac{50\pi}{3}$ cm^2

c $50\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)$ cm^2

2 **a** 2.09 cm **b** 1.05 cm^2 **c** 0.61 cm^2

3 **a** 28 cm **b** 24.2 cm **c** 34.3 cm **d** 10.2 cm

4 $a = 9.27$ cm, $B = 70°$

5 **a** 41° **b** 32° **c** 54° or 126°

d 38° **e** 27° **f** 17°

6 **a** $a = 13.7$, $b = 13.4$ **b** $p = 13.4$, $q = 10.9$

c $h = 17.2$, $k = 13.9$ **d** $c = 34.3$, $d = 21.1$

e $m = 11.8$, $n = 6.6$ **f** $x = 38.5$, $y = 51.0$

7 **a** $Y = 39°$, $x = 17$ m, $X = 76°$

b $p = 49$ cm, $r = 85$ cm, $R = 92°$

8 D **9** C **10** A

11 **a** $x = 17$ **b** $x = 6$

12 76 mm

13 $Y = 38°$, $X = 77°$, $x = 17$ m

14 **a** $B = 47.7°$, $C = 72.3°$, $b = 8.5$ or $B = 12.3°$, $C = 107.7°$, $b = 2.5$

b $K = 82.5°$, $M = 51.5°$, $k = 9.4$ or $K = 5.5°$, $M = 128.5°$, $k = 0.9$

15 31.38 cm, to 2 decimal places

16 **a** 20.9 cm **b** 23.1 cm

17 $\theta = 111.4°$

EXERCISE 7.3

1 3 cm **2** C **3** 10.5

4 **a** $\theta = 22°$ **b** $\theta = 53°$ **c** $\theta = 96°$

d $\theta = 78°$ **e** $\theta = 109°$ **f** $\theta = 134°$

5 **a** $x = 15$ **b** $p = 39$

6 E **7** B **8** A **9** C

10 **a** $m = 26$ **b** $z = 283$ **c** $f = 65$ **d** $g = 44$

11 **a** $D = 50.7°$, $E = 95.7°$, $F = 33.6°$

b $P = 78.0°$, $Q = 53.8°$, $R = 48.2°$

c $P = 22.4°$, $A = 131.5°$, $Z = 26.1°$

d $E = 53.2°$, $T = 69.1°$, $X = 57.7°$

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12 7 m **13** 36.3° and 26.4°

14 26.96 m

EXERCISE 7.4

1 $ON = 66.9$, $MN = 81.2$

2 B

3 11 m **4** 170°

5 AYZ 51.34°, AZW 59.04°

6 **a** 21.21 cm **b** 30.97° **c** 30.97° **d** 40.32°

7 Proof: see worked solutions

8 1.74 km **9** 26.96 m

10 **a** 166.2 km **b** 187.3° T

11 **a** 197 m **b** 541 m and 644 m

12 **a** 14 m **b** 27 m

13 **a** 12.04 m (south), 9 m (west)

b 44.9° (south), 53.13° (west)

c 15.03 m **d** 55.62°

14 **a** 401 m **b** 233.27°T or S53.27° W

15 76° T or N 76° E at 12.78°

16 **a** 73° **b** 54°

17 3.16 m, 161.57°T or S 18.43°E

18 **a** 73.3° **b** 7.5 m **c** 53.13°

d 6.3 m **e** 7.7 m

19 15.3°, 410.4 m

20 81.5 m **21** 155 m

EXERCISE 7.5

1 **a** $a = 116$ cm **b** $b = 77$ m

2 77.1 m

3 **a** $\frac{2}{\sqrt{3}}$ **b** 1 **c** $\sqrt{2}$

4 **a** -2 **b** $-\frac{\sqrt{3}}{3}$

5 $\sqrt{10}$ in 2nd quadrant

6 D **7** E **8** B **9** $\sqrt{5}+1$

10 $-\sqrt{2}$ **11** $\frac{3\sqrt{5}}{5}$ **12** E **13** B

14 B **15** B **16** C **17** E

18 D **19** A **20** C **21** A

22 A **23** C **24** C **25** C

26 B

EXERCISE 7.6

1 $\sqrt{2}$ in 4th quadrant **2** A

3 $\sin(x)\cos(2y) - \cos(x)\sin(2y)$

4 $\frac{\sqrt{6}+\sqrt{2}}{4}$ **5** $\cot(x)$

6 D **7** C

8 **a** $\sin(2x)\cos(y) + \cos(2x)\sin(y)$

b $\sin(3p)\cos(5q) - \cos(3p)\sin(5q)$

c $\frac{\tan(2e) - \tan(f)}{1 + \tan(2e)\tan(f)}$

d $\cos(3\alpha)\cos(2\beta) - \sin(3\alpha)\sin(2\beta)$

e $\frac{\tan(w) + \tan(3x)}{1 - \tan(w)\tan(3x)}$

f $\cos(P)\cos(4Q) + \sin(P)\sin(4Q)$

9 **a** $\sin(x)$ **b** $-\cos(x)$ **c** $\cot(x)$

d $\sec(x)$ **e** $\tan(x)$ **f** $-\text{cosec}(x)$

g $-\sin(x)$ **h** $-\sin(x)$ **i** $\tan(x)$

10–14 Proof: see worked solutions

15 B **16** A **17** C

EXERCISE 7.7

1 E **2** E

3 $\sin(6x) = 2\sin(3x)\cos(3x)$

4 $\frac{\sqrt{2+\sqrt{2}}}{2}$ **5** B **6** A

7 $a = \frac{1}{2}$ **8** $a = 48$ **9** $\sqrt{5} - 1$

10 $\sin(x) = \frac{1}{3}$

11, 12 Proof: see worked solutions

13 **a** $\sin(8x) = 2\sin(4x)\cos(4x)$

b $\cos(6x) = \cos^2(3x) - \sin^2(3x)$
or $2\cos^2(3x) - 1$
or $1 - 2\sin^2(3x)$

c $\tan(4x) = \frac{2\tan(2x)}{1 - \tan^2(2x)}$

d $\cos(x) = 2\cos^2\left(\frac{1}{2}x\right) - 1$

14 Proofs: see worked solutions

15 C **16** B **17** E **18** B

19 Proof: see worked solutions

20 **a** Proof: see worked solutions. Negative value rejected because acute angle $\left(\frac{\pi}{8}\right)$ is in 1st quadrant.

b Proof: see worked solutions

EXERCISE 7.8

1 **a** $-\frac{\sqrt{5}}{2}$ in 2nd quadrant

b $\frac{\sqrt{2-\sqrt{2}}}{2}$ **c** $\frac{\sqrt{2-\sqrt{2}}}{2}$

2 A

3 Proof: see worked solutions

4 $\cos(4x)\sin(3x) = \frac{1}{2}[\sin(7x) - \sin(x)]$

5 C **6** E

7 Proof: see worked solutions

8 0.133

9 **a** 0.633 **b** 0.133 **c** 0.112 **d** 0.754

10 **a** $\sin(12x) - \sin(2x)$

b $4[\cos(6x) - \cos(12x)]$

c $3[\cos(2x) + \cos(10x)]$

d $2[\sin(11x) - \sin(-x)] = 2[\sin(11x) + \sin(x)]$

e $\frac{1}{2}[\sin(-4x) + \sin(16x)] = \frac{1}{2}[\sin(16x) - \sin(4x)]$

f $\frac{3}{2}[\cos(4x) + \cos(24x)]$

g $\frac{5}{2}[\cos(x) - \cos(5x)]$

h $\frac{7}{2}[\cos(2x) + \cos(20x)]$

i $\frac{1}{2}[\sin(9x) - \sin(7x)]$

11 a $\frac{1}{2}\left[\cos\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{3}\right)\right] = \frac{(1+\sqrt{3})}{4}$

b $\frac{1}{2}\left[\sin\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{6}\right)\right] = \frac{(1+\sqrt{3})}{4}$

c $\frac{1}{2}\left[\cos\left(\frac{\pi}{6}\right) - \cos\left(\frac{\pi}{3}\right)\right] = \frac{(\sqrt{3}-1)}{4}$

d $\sin\left(\frac{\pi}{3}\right) + \sin\left(-\frac{\pi}{6}\right) = \frac{(\sqrt{3}-1)}{2}$

EXERCISE 7.9

1 a, b Proof: see worked solutions

c $\cos(x)\sin(2x) = \frac{1}{2}[\sin(3x) - \sin(-x)]$

$= \frac{1}{2}[\sin(3x) + \sin(x)]$

2 A **3** B

4 $\sqrt{58}\sin\left(x + \tan^{-1}\left(\frac{3}{7}\right)\right)$

5 a $2\cos\left(x + \frac{\pi}{3}\right)$ **b** $x = 0, \frac{4\pi}{3}, 2\pi$

6 A **7** E

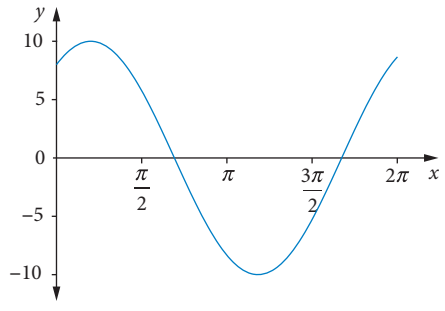
8, 9 Proof: see worked solutions

10 $\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$

11 $2\sqrt{5}\sin(x + \tan^{-1}(2))$

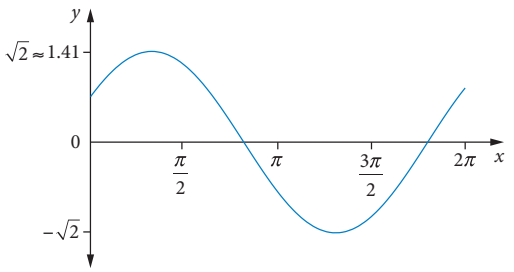
12 a $10\sin\left(x + \tan^{-1}\left(\frac{4}{3}\right)\right)$

b



13 a $\sqrt{2}\cos\left(x - \frac{\pi}{4}\right)$

b



14 a $13\cos\left(x + \tan^{-1}\left(\frac{12}{5}\right)\right)$ **b** $x = \pi - \tan^{-1}\left(\frac{12}{5}\right)$

15 $x = \frac{3\pi}{2}, \frac{7\pi}{6}$ **16** $x = 0, \frac{3\pi}{2}, 2\pi$

EXERCISE 7.10

1 a $\frac{1}{2}[\sin(6y) + \sin(-2y)]$ **b** $\frac{1}{\sqrt{3}-2}$

c $\sin(x) + 2\sin(3x)$ (because $\sin(3x) = 3\sin(x) - 4\sin^3(x)$, other equivalent answers are possible.)

2 C

3–8 Proofs: see worked solutions

9 $x = 0, \pi, 2\pi$

10 C **11** A

12 $\left(2, \frac{4}{3}\right), \left(4, \frac{4}{3}\right), \left(8, \frac{4}{3}\right), \left(10, \frac{4}{3}\right)$

13 a $x = 0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$ **b** $x \in \left(0, \frac{\pi}{3}\right) \cup \left(\frac{\pi}{2}, \pi\right)$

14 $x = \pi k, 2\pi k \pm \frac{2\pi}{3}, 2\pi k \pm \frac{\pi}{3}, k \in Z$

15 $x = 2\pi k + \frac{\pi}{3}, 2\pi k + \frac{2\pi}{3}, \frac{(2k+1)\pi}{2}, k \in Z$

16–21 Proof: see worked solutions

22 $\frac{1}{3}(\sin(3x) + 3 - 6\sin^2(x))$

CUMULATIVE EXAMINATION 1

1 a AB

b

A	B	AB
F	F	F
F	T	F
T	F	F
T	T	T

2 −64

3 a 360 **b** 60

4 Proof: see worked solutions

CUMULATIVE EXAMINATION 2

Section A

1 D **2** B **3** A **4** C **5** C

Section B

1 a Proof: see worked solutions

b $P = 10\sqrt{5}\cos(\theta - \alpha)$ where

$\alpha = \tan^{-1}\left(\frac{1}{2}\right)$; $\theta = 70.88°$

c $k = 25$ **d** $\theta = 45°$

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2 a $\frac{\sqrt{6}-\sqrt{2}}{4}$ **b** $\frac{\sqrt{6}-\sqrt{2}}{4}$

c $(\sqrt{6}+\sqrt{2})+(\sqrt{6}-\sqrt{2})i$

CHAPTER 8

EXERCISE 8.1

1 a function, domain $[-2,4]$, co-domain R, range $\left[-23, 3\frac{26}{27}\right]$

b function, domain R, co-domain $[0,\infty)$, range $[81,\infty)$

c function, domain $[-2\pi,2\pi]/\left\{\frac{n\pi}{4}, n \text{ odd}\right\}$, co-domain R, range R

d not a function, domain $[-5,5]$, co-domain R, range $[-5,5]$

e function, domain $[-5,5]$, co-domain R^+, range $(0,5]$

f not a function, domain $[-3,4)$, co-domain R, range $[-5,-3)\cup(3,5]$

g not a function, domain $[-5,1]$, co-domain R, range $[-1,7]$

h function, domain $R\backslash\{0\}$, co-domain R, range $R\backslash\{0\}$

2 a basic function $y=\log_3(x)$, reflection in the y-axis, translation 4 units left and translation 5 units up OR translation 4 units right, reflection in the y-axis and translation 5 units up

b basic function $y=\frac{1}{x}$, translation 3 units left, dilation parallel to the y-axis by the factor 0.5 and translation 4 units up OR dilation parallel to the y-axis by the factor 0.5, translation 6 units left and 4 units up

c basic relation $\frac{x^2}{9}+\frac{y^2}{16}=1$, translation 2 units to the left and 3 units up OR basic relation $x^2+y^2=1$, dilation parallel to the x-axis by the factor 3, dilation parallel to the y-axis by the factor 4, translation 2 units to the left and 3 units up

d basic function $y=\frac{1}{x}$, dilation parallel to the y-axis by the factor 2, translation 3 units to the right and 4 units down

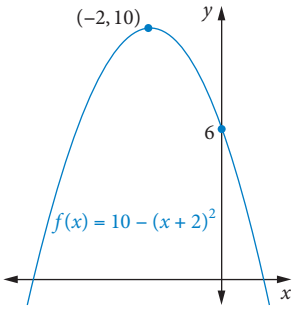
3 a basic relation $x^2+y^2=49$, translation 3 units left and 5 units up, equation $(x+3)^2+(y-5)^2=49$ OR basic relation $x^2+y^2=1$, dilation parallel to the both axes by the factor 7, translation 3 units left and 5 units up, equation $(x+3)^2+(y-5)^2=49$

b basic function $y=4^x$, translation 1 unit right, reflection both axes, translation 3 units up, equation $y=3-4^{1-x}$

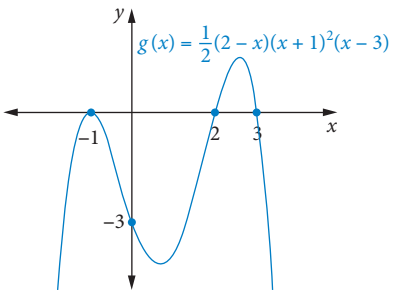
c basic function $y=x^4$, reflection in the x-axis, dilation by the factor 0.5 parallel to the x-axis, translation 1 unit to the right and 4 units up, equation $y=-(0.5(x-1))^4+4$

d basic function $y=\log_4(x)$, reflection in both axes, dilation by the factor 2 parallel to the x-axis and translation 4 units to the right and 2 units down, equation $y=-\log_4(-0.5(x-4))-2$

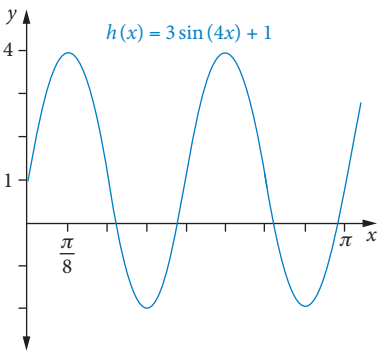
4 a



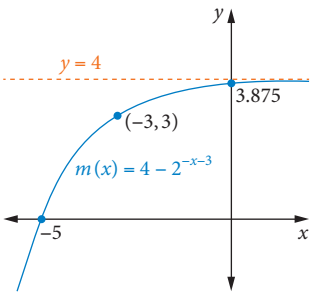
b



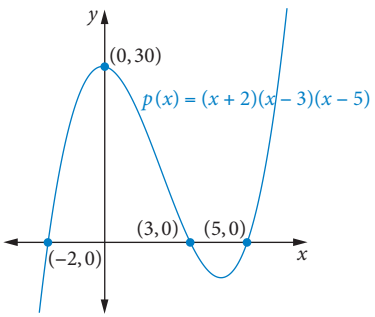
c



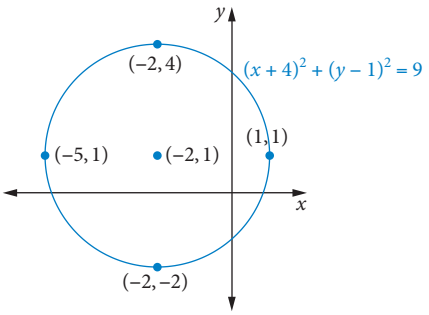
d



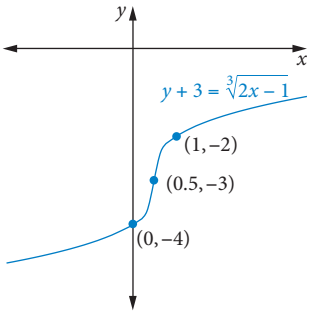
e



f

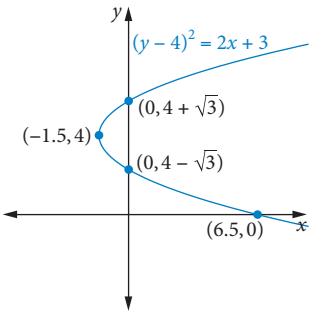


g



Zero at $(14, 0)$

h



5 E **6** A **7** B

EXERCISE 8.2

1 D **2** B

3 **a** $3x^2 - 5x + 6$ **b** $2x^3 + x^2 - x - 7$ **c** $x^2 + x + 4$
d $4x^3 + 3x - 3$ **e** $3x - 5$ **f** $x^2 + 3x + 6$

4 **a** $\frac{2}{x+4} - 2x + 3$ **b** $\frac{3}{x-1} + x - 2$
c $\frac{2x-1}{x^2+3} + 3x - 5$ **d** $\frac{2}{x-2} + 4x^2 - 2x + 3$
e $\frac{2}{x+3} - \frac{3}{x-1}$ **f** $\frac{5}{x-4} - \frac{4}{2x+1}$
g $\frac{2}{x+3} - \frac{3}{x-4} + \frac{5}{x+2}$
h $\frac{4}{2x+1} + \frac{1}{3x-4} - \frac{3}{x-1}$

5 **a** $\frac{3}{(x-1)^2} + \frac{1}{x-1}$ **b** $\frac{2}{x+4} - \frac{5}{(x+4)^2}$
c $\frac{7}{3(3x+5)} - \frac{44}{3(3x+5)^2}$
d $\frac{2}{x-4} + \frac{19}{(x-4)^2} + \frac{42}{(x-4)^3}$
e $\frac{2}{2x-3} - \frac{3}{(2x-3)^3}$
f $\frac{1}{(3x+1)^2} - \frac{3}{(3x+1)^3}$

6 **a** $\frac{3}{x-2} + \frac{2}{x^2+4}$
b $\frac{3x+5}{x^2+x+3} - \frac{2}{x+3}$
c $\frac{5x-1}{3x^2-4x+4} - \frac{2}{4x-3}$
d $\frac{2-3x}{2x^2-3x+2} + \frac{5}{3x+1}$

7 **a** $\frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{C}{3x-5}$
b $\frac{A}{(2x-5)^2} + \frac{B}{2x-5} + \frac{Cx+D}{3x^2+6x+5}$
c $A + \frac{B}{(x-3)^2} + \frac{C}{x-3} + \frac{D}{x+4}$
d $\frac{A}{x+2} + \frac{B}{x-3} + Cx + D$
e $\frac{A}{(x-4)^2} + \frac{B}{x-4} + \frac{Cx+D}{x^2-2x+3} + E$
f $\frac{A}{(x+5)^2} + \frac{B}{x+5} + \frac{C}{2x-3} + \frac{Dx+E}{2x^2-5x+4}$
g $Ax + B + \frac{C}{5x-4} + \frac{Dx+E}{x^2-x+2}$
h $\frac{A}{(2x-3)^2} + \frac{B}{2x-3} + \frac{C}{5x-4} + Dx + E$

8 **a** $\frac{3}{2x-7} - \frac{1}{3x-1}$ **b** $\frac{5}{x-3} - \frac{2}{3x+1}$
c $\frac{3x-4}{3x^2-3x+2} + \frac{2}{x+5}$
d $\frac{7}{(x-1)^2} - \frac{2}{x-1} - \frac{3}{(2x-5)^2} + \frac{1}{2x-5}$
e $\frac{1}{(x-1)^2} + \frac{2}{x-1} + \frac{1}{2x+1} - \frac{x-1}{x^2-4x+5}$
f $\frac{3}{(x-2)^2} + \frac{2}{x-2} + \frac{1}{x+4} + 5$
g $\frac{5}{(x-4)^2} + \frac{3}{x-4} - \frac{2x+1}{3x^2-x+2} + 2$
h $\frac{3}{x+4} - \frac{5}{x-5} + 2x^2 - 3$
i $\frac{x-3}{x^2-2x+3} - \frac{1}{x-5} + x^2 + x + 2$
j $\frac{1}{(2x+3)^2} + \frac{2}{2x+3} - \frac{1}{x-2} - x^2 + 4x + 1$
k $\frac{103}{10(x-3)} - \frac{71}{90(3x+1)} + \frac{4}{3}x + \frac{26}{9}$
l $\frac{6-15x}{29(x^2+1)} + \frac{1831}{116(2x-5)} + \frac{3}{2}x + \frac{15}{4}$

9 **a** $\frac{1}{x+3} - \frac{3}{x-2} + \frac{2}{4x-5}$
b $\frac{3}{4x+3} + \frac{1}{(3x-5)^2} - \frac{2}{3x-5}$
c $\frac{1}{x-1} - \frac{1}{2x+5} - \frac{4}{(3x+4)^2} - \frac{1}{3x+4}$
d $\frac{1}{x-3} - \frac{3}{(x+5)^2} + \frac{2}{x+5} - x + 3$
e $\frac{x+5}{x^2-3x+5} - \frac{3}{4x+1}$

f $\dfrac{2x+5}{4x^2+3x+2}+\dfrac{2}{4x+3}+3x+3$

g $\dfrac{2}{2x-3}-\dfrac{2x-3}{5x^2-x+1}-x^2+4x-2$

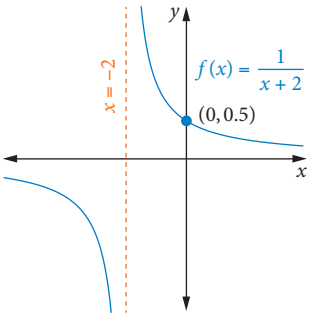
h $\dfrac{5-2x}{6x^2-10x+5}-\dfrac{2}{x+4}-\dfrac{1}{(3x-2)^2}-\dfrac{2}{3x-2}$

10 D **11** D **12** C **13** C

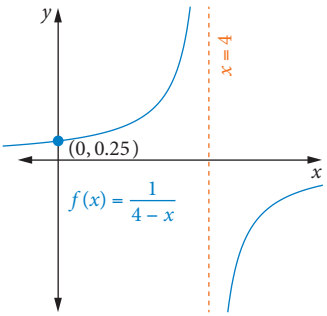
EXERCISE 8.3

1 D **2** A

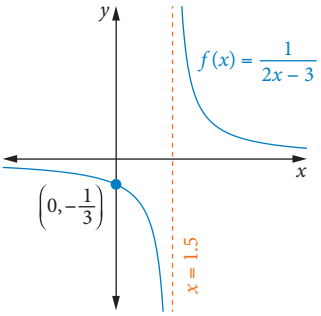
3 a



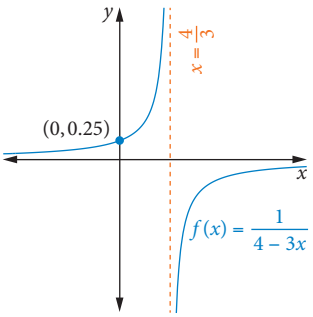
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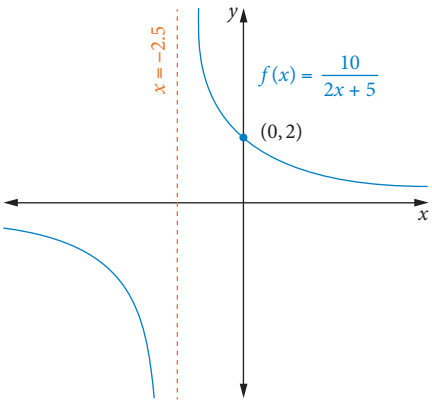
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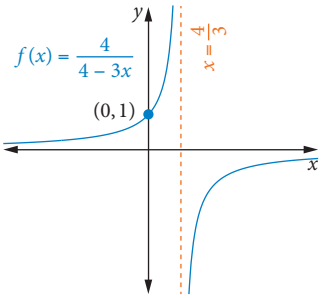
d



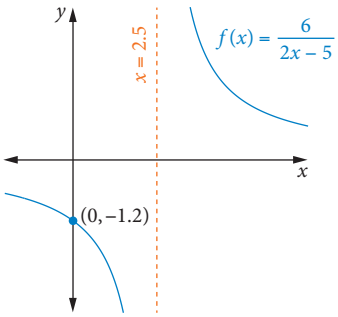
e



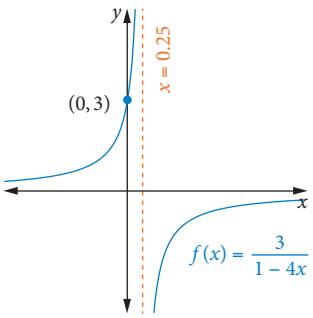
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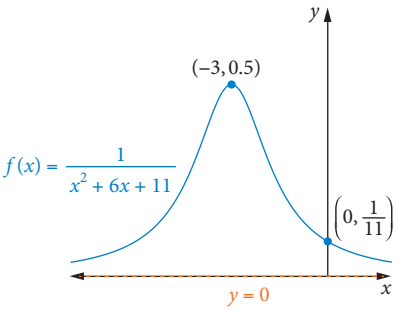
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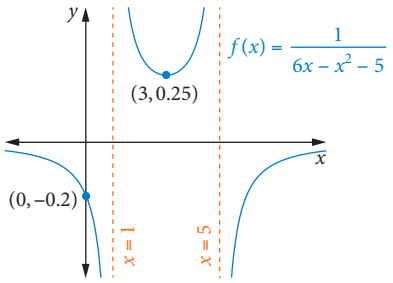


4 a

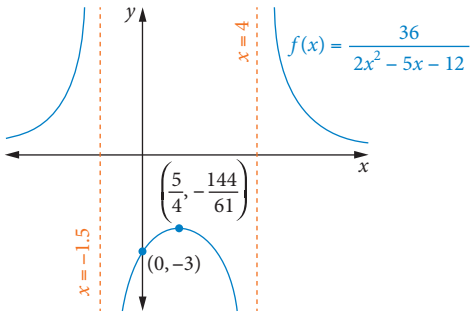


ANSWERS

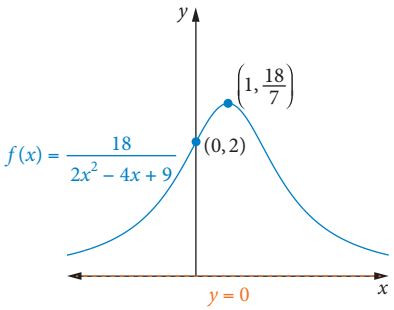
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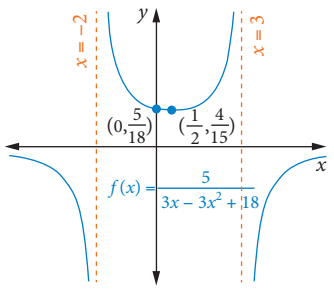
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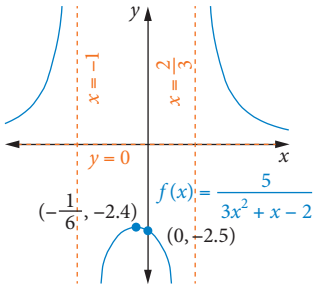
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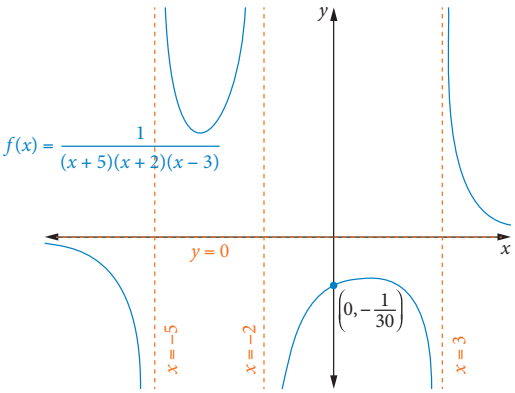
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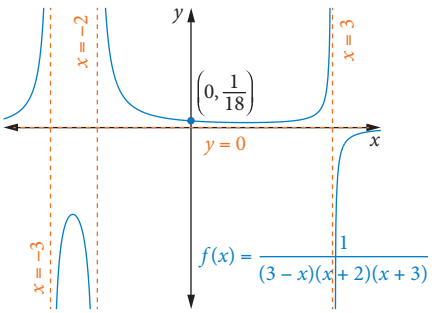
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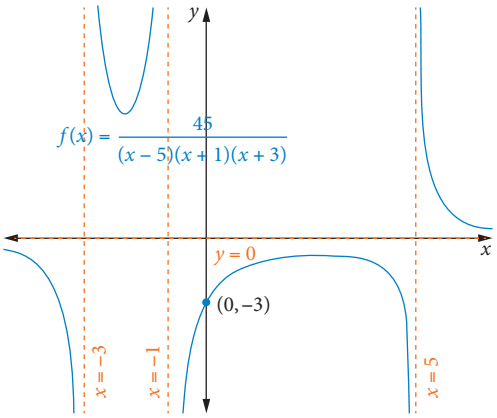
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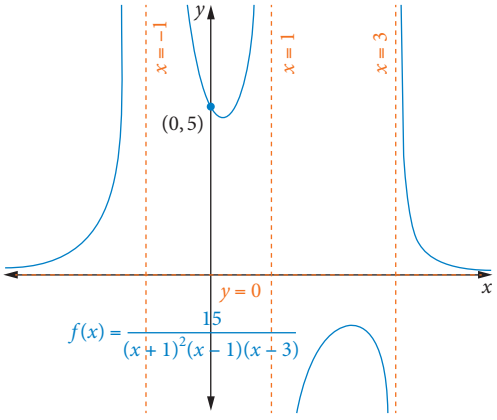
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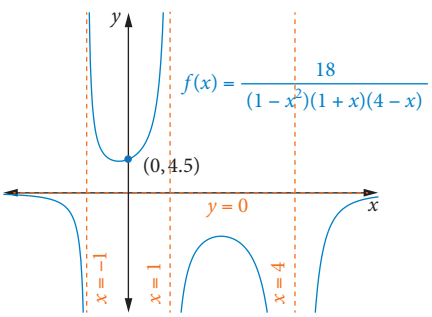
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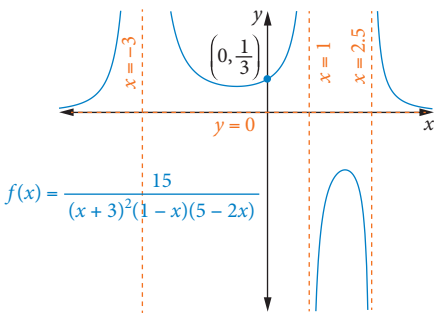


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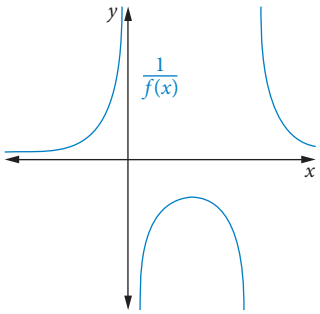


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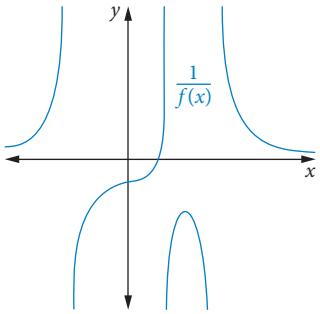
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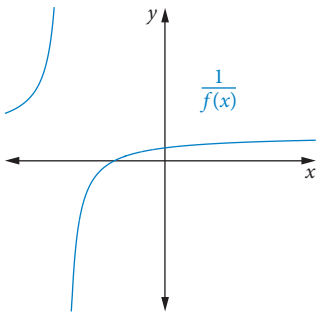
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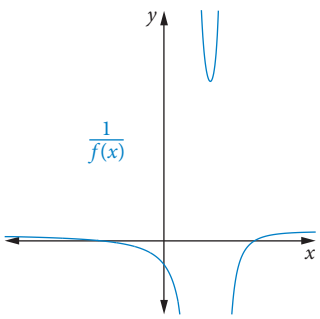
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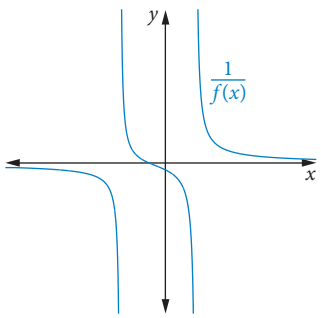
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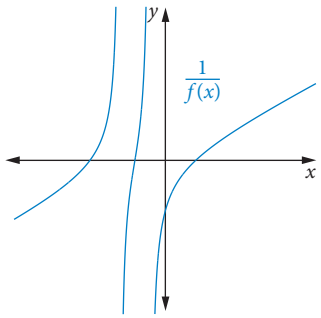
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e

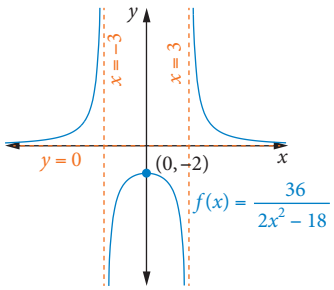


f



7 $x = -1, x = 1, y = 1$

8



9 B **10** E **11** D **12** E

EXERCISE 8.4

1 D **2** C

3 **a** $\frac{2\sqrt{3}}{3}$ **b** -1 **c** 1

d undefined **e** -1 **f** $-\frac{2\sqrt{3}}{3}$

g -2 **h** 2 **i** $-\sqrt{3}$

j $-\sqrt{2}$ **k** 2 **l** 0

4 **a** $\sec(x) = \frac{3\sqrt{2}}{4}, \cot(x) = -2\sqrt{2}$

b $\sec(x) = -\frac{\sqrt{26}}{5}, \sin(x) = -\frac{\sqrt{26}}{26}$

c $\sec(x) = -2, \operatorname{cosec}(x) = -\frac{2\sqrt{3}}{3}$

d $\sec(x) = \frac{\sqrt{30}}{5}, \cot(x) = -\frac{\sqrt{5}}{5}$

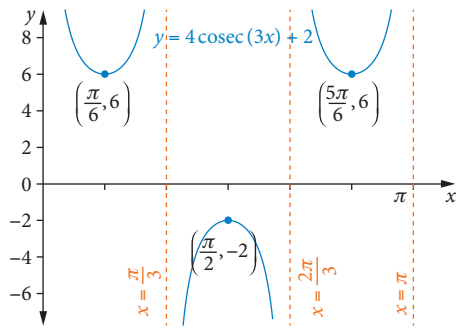
e $\tan(x) = -1, \operatorname{cosec}(x) = \sqrt{2}$

f $\operatorname{cosec}(x) = -\frac{3\sqrt{7}}{7}, \cot(x) = \frac{\sqrt{14}}{7}$

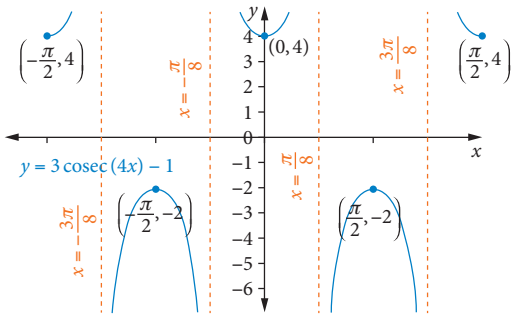
g $\sec(x) = -\frac{2\sqrt{3}}{3}, \tan(x) = -\frac{\sqrt{3}}{3}$

h $\cot(x) = -\frac{\sqrt{2}}{4}, \operatorname{cosec}(x) = -\frac{3\sqrt{2}}{4}$

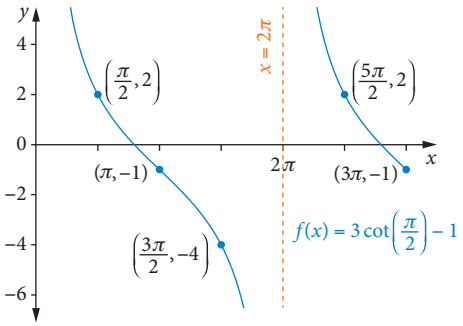
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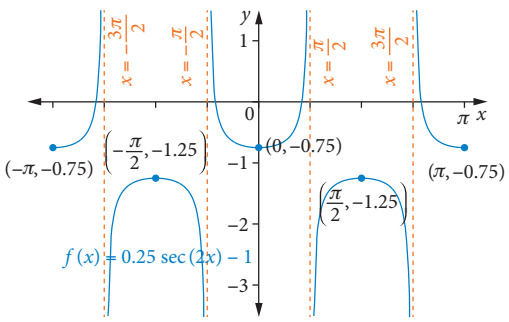
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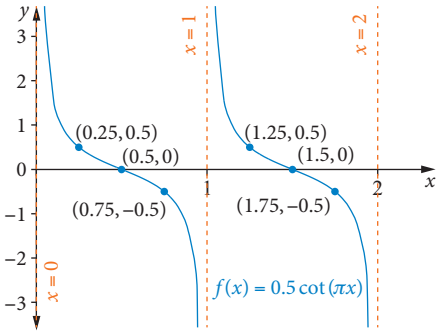
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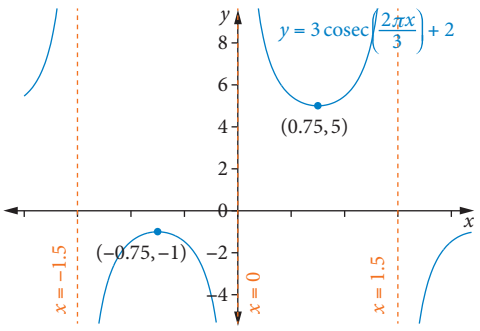
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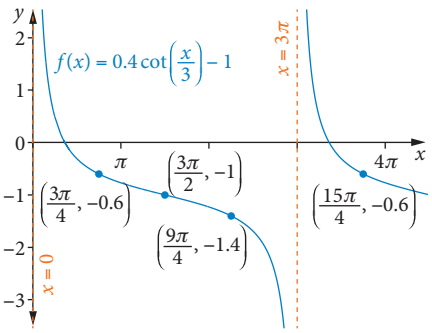
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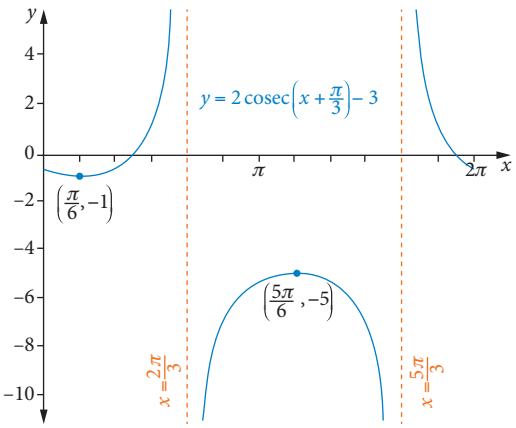
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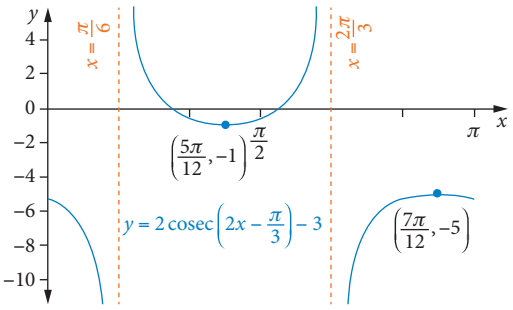
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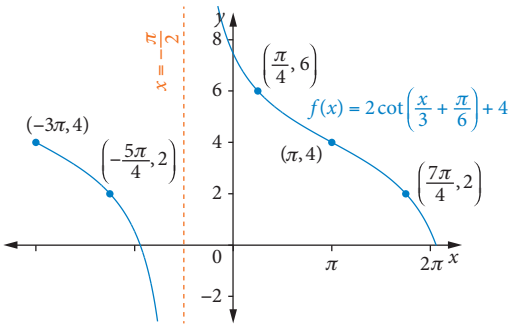
h



i

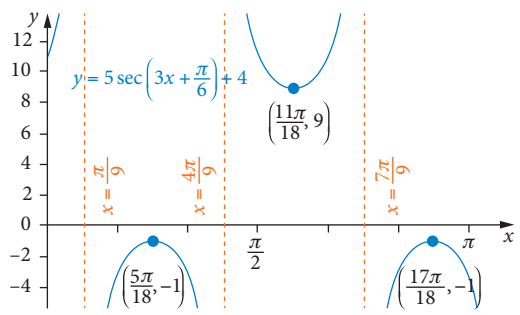


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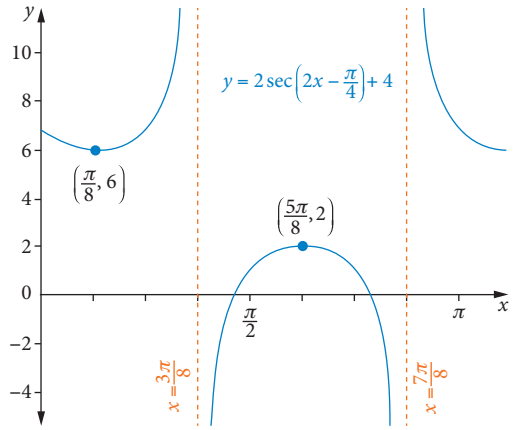


9780170448338

k



l



6 a $f(x) = 4\operatorname{cosec}(3x) + 2$

b $f(x) = 3\cot\left(\frac{x}{2}\right) - 2$

c $f(x) = 3\sec\left(\frac{x}{3}\right) + 2$

d $f(x) = 2\sec\left(2x - \frac{\pi}{4}\right) + 1$ or $y = 2\operatorname{cosec}\left(2x + \frac{\pi}{4}\right) + 1$

e $f(x) = 3\cot\left(4x + \frac{\pi}{6}\right) - 2$

f $f(x) = 4\operatorname{cosec}\left(2x - \frac{\pi}{6}\right) - 4$

7 C **8** E **9** E

10 D **11** C **12** A

EXERCISE 8.5

1 C **2** D

3 a π **b** $\frac{\pi}{4}$ **c** $-\frac{\pi}{2}$

d $-\frac{\pi}{6}$ **e** $\frac{2\pi}{3}$ **f** $\frac{5\pi}{6}$

g $-\frac{\pi}{3}$ **h** $-\frac{\pi}{3}$ **i** $\frac{\pi}{4}$

j $-\frac{\pi}{4}$

4 a domain $= [2, 4]$, range $= [-\pi + 1, \pi + 1]$

b domain $= R$, range $= \left[-\frac{\pi}{2} + 2, \frac{\pi}{2} + 2\right]$

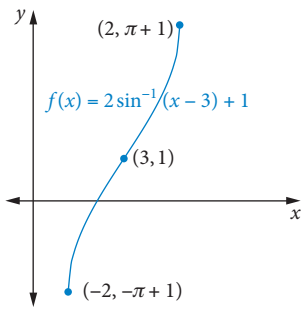
c domain $= R$, range $= \left[-\frac{3\pi}{2} - 4, \frac{3\pi}{2} - 4\right]$

d domain $= [-8, 0]$, range $= \left[-2, \frac{\pi}{2} - 2\right]$

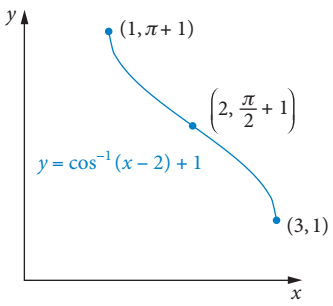
e domain $= [0, 4]$, range $= [3, 2\pi + 3]$

f domain $= [-9, -3]$, range $= \left[-\frac{\pi}{8} - 2, \frac{\pi}{8} - 2\right]$

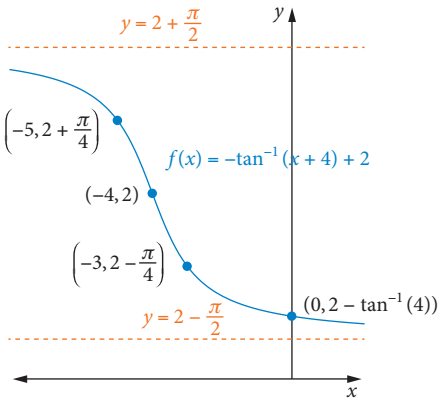
5 a



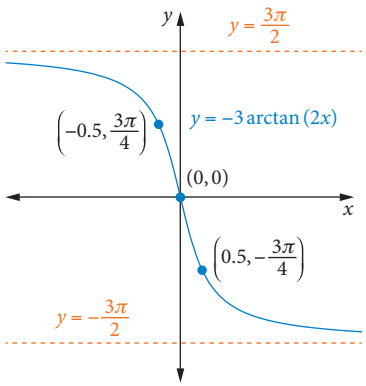
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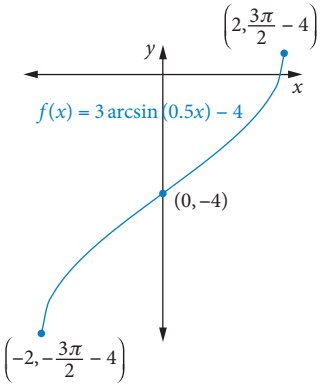
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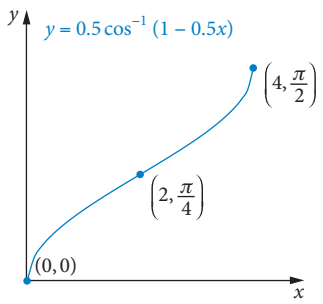
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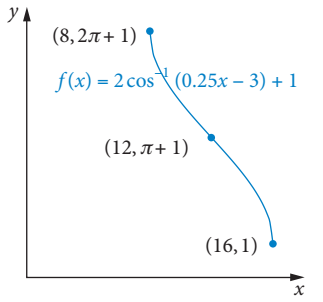
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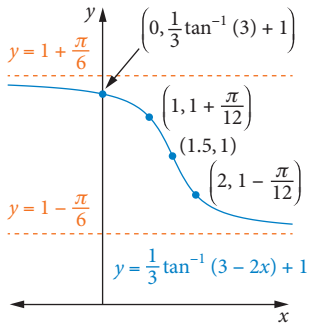
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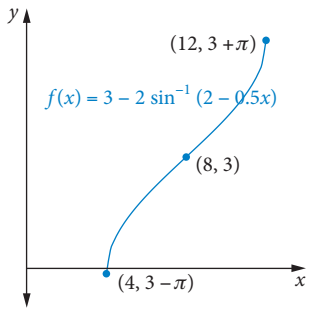
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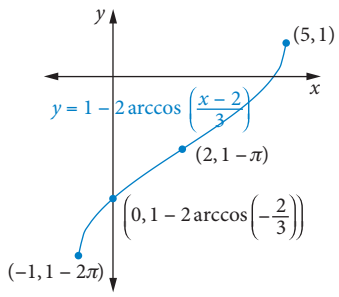
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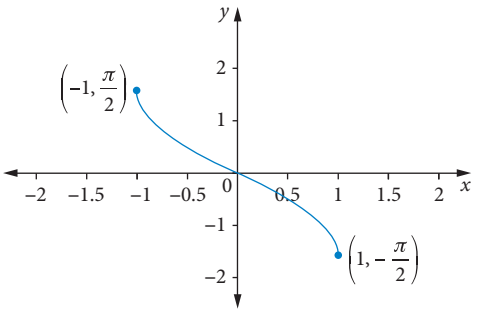
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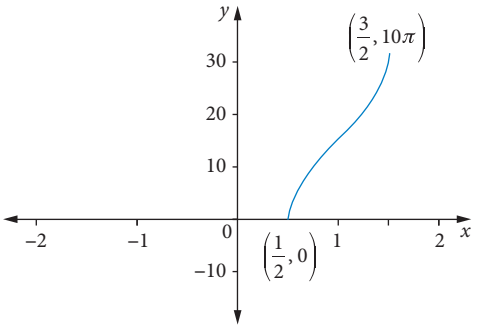
j



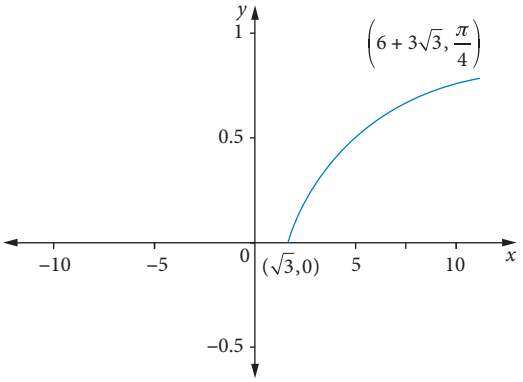
6



7



8



9 B **10** E **11** D

EXERCISE 8.6

1 E **2** A

3 **a** 9 **b** 3 **c** $\frac{3}{5}$ **d** $\frac{9}{7}$

e 5 **f** 12 **g** x^2 **h** $x - y$

i $y - x$

4 **a** $x = -\frac{3}{5}, 7$ **b** $y = 0, 2$ **c** $m = -\frac{9}{4}$

d $x = 2$ **e** $v = -\frac{17}{9}, \frac{1}{3}$ **f** $b = -\frac{7}{3}, \frac{1}{6}$

5 **a** $-17 < m < -\frac{3}{5}$ **b** $c \geq 5$

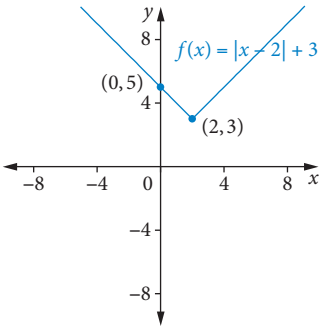
c $x < -\frac{2}{5}$ **d** $p > -\frac{5}{6}$

e $w \leq -1$ **f** no solution

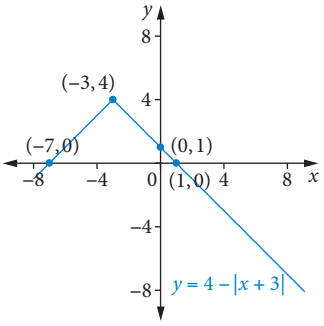
g $d < -\frac{9}{11}$ or $d > 5$ **h** $g \leq -\frac{1}{3}$ or $g \geq 3$

i $q \leq -\frac{8}{5}$ or $q \geq -\frac{10}{7}$

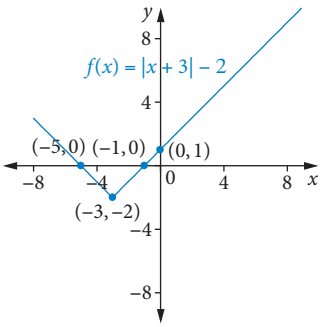
6 a



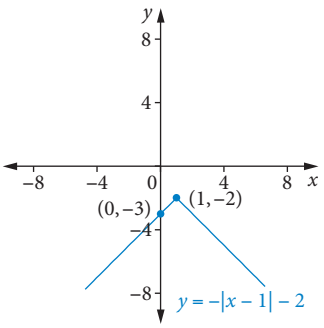
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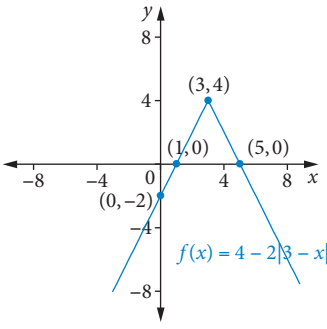
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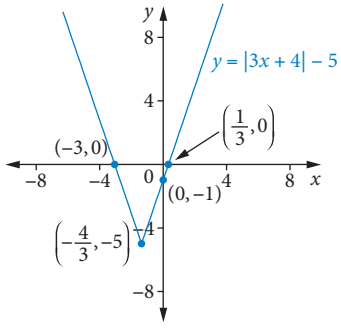
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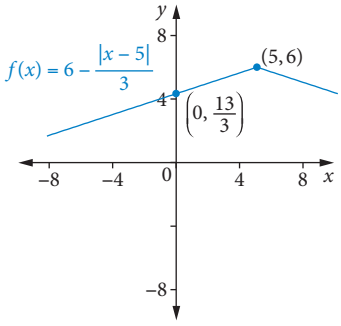
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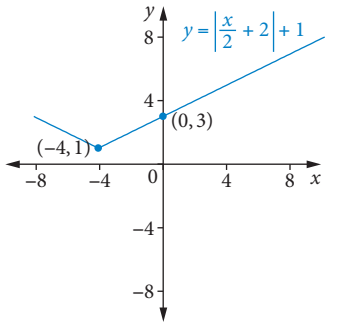
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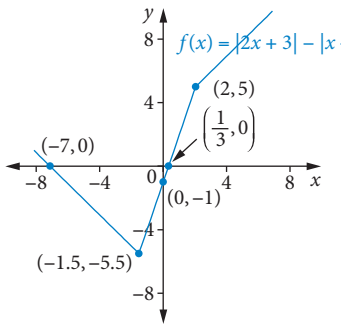
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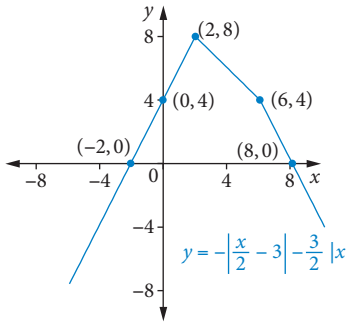
h



i



j



7 $x = -3, \frac{3}{4}$

8 $\left(-\infty, \frac{\sqrt{5}-3}{2}\right]$

9 C

10 D

EXERCISE 8.7

1 A **2** D

3 **a** $3x - 2y - 14 = 0$, perpendicular bisector of the given points

b $6x - 8y - 53 = 0$, perpendicular bisector of the given points

c $4x \pm y = 0$, straight lines of slope ± 4 through the origin

d $3x + 8y + 18 = 0$, straight line

e $x - y - 3 = 0$, a straight line

4 **a** $(x-3)^2 + (y-1)^2 = 25$, circle with centre $(3, 1)$, radius 5

b $(x+1)^2 + \left(y - 5\frac{1}{2}\right)^2 = \frac{73}{4}$, circle with centre $\left(-1, 5\frac{1}{2}\right)$, radius $\frac{\sqrt{73}}{2}$

c $(x-1)^2 + (y+4)^2 = 20$, circle with centre $(-1, -4)$, radius $2\sqrt{5}$

d $(x-8)^2 + (y-30)^2 = 720$, a circle with centre $(-8, 30)$ and $r = 12\sqrt{5}$

e $(x-3)^2 + (y-3)^2 = \frac{17}{2}$, circle with centre $(3, 3)$, radius $\frac{\sqrt{34}}{2}$

5 **a** $(3, -4)$, 9

b $\left(-\frac{7}{2}, 5\right)$, $4\sqrt{3}$

c $(x-5)^2 + (y+3)^2 = 49$

d $(x+2)^2 + (y-4)^2 = 16$

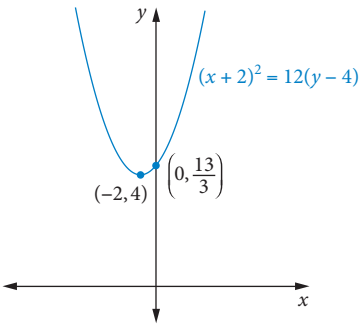
e $\left(x - 7\frac{1}{2}\right)^2 + \left(y - 4\frac{1}{2}\right)^2 = 41$

f Centre $(3.5, 4)$, radius $\frac{\sqrt{65}}{2}$, $(x-3.5)^2 + (y-4)^2 = \frac{65}{4}$

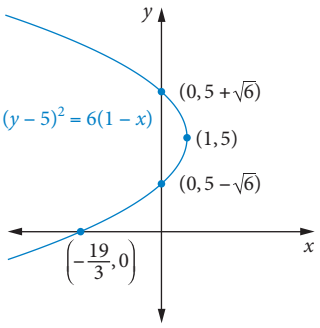
g $(x+3)^2 + (y+2)^2 = 20$

h $(x-5)^2 + (y+4)^2 = \frac{5}{2}$

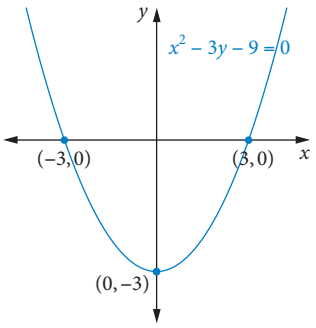
6 **a** **i**



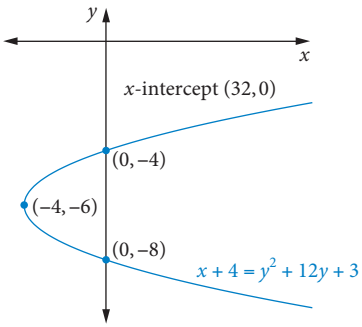
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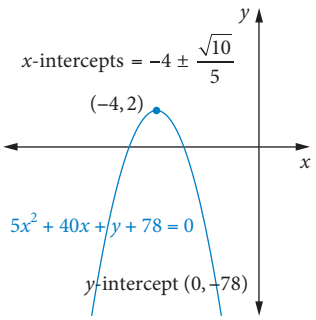
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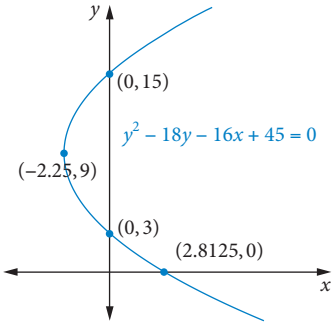
iv



v

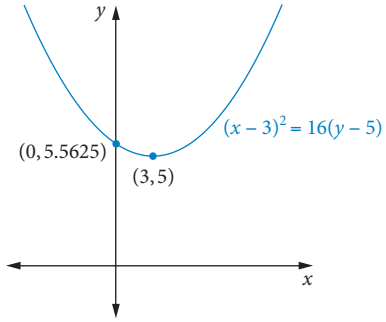


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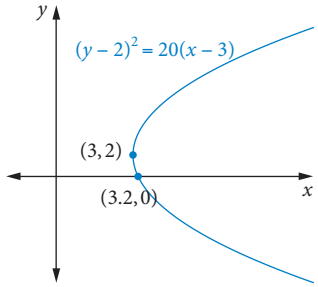


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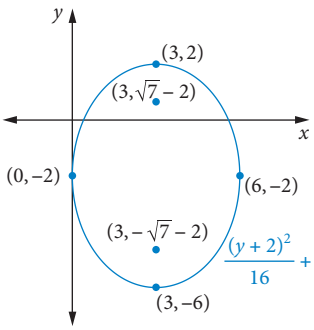
b **i** $(x-3)^2 = 16(y-5)$



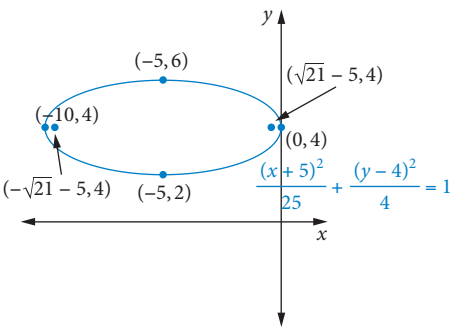
ii $(y-2)^2 = 20(x-3)$



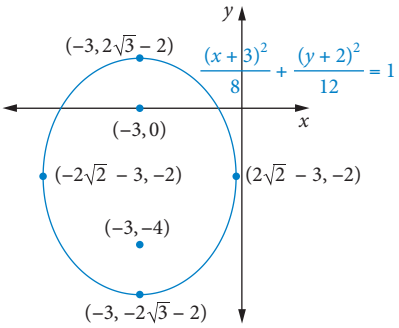
7 **a** **i**



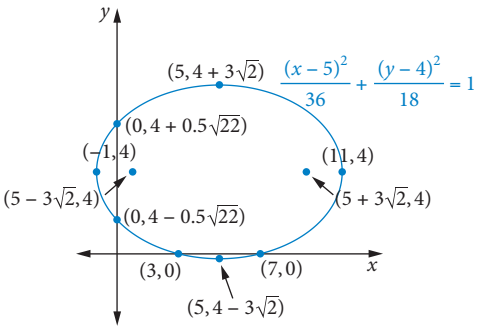
ii



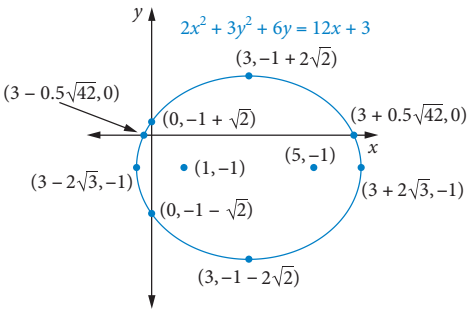
iii



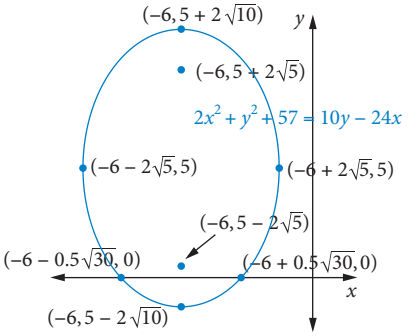
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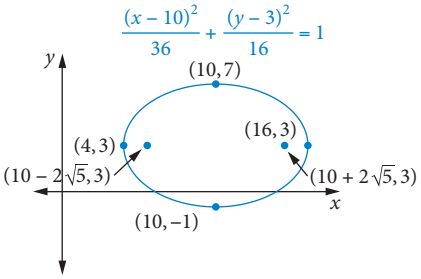
v



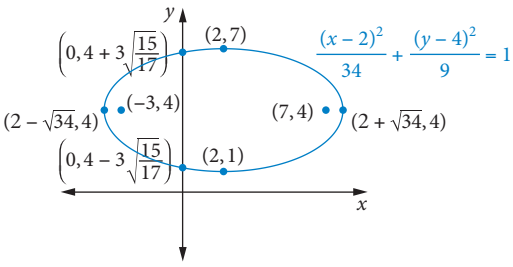
vi



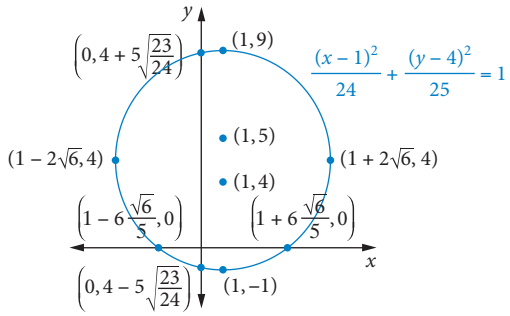
b **i** $\frac{(x-10)^2}{36} + \frac{(y-3)^2}{16} = 1$



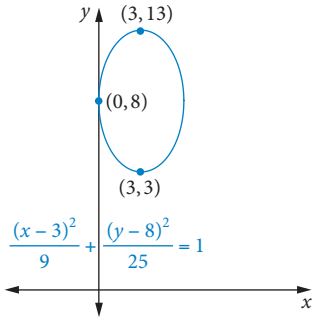
ii $\frac{(x-2)^2}{34} + \frac{(y-4)^2}{9} = 1$



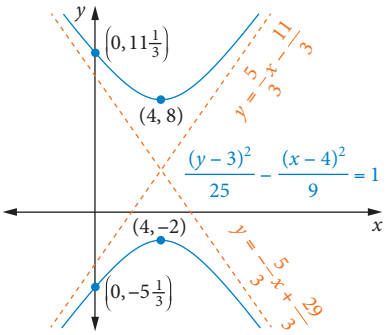
iii $\dfrac{(x-1)^2}{24}+\dfrac{(y-4)^2}{25}=1$



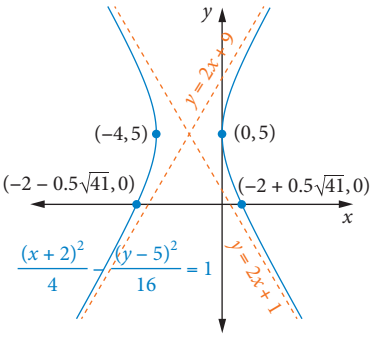
iv $\dfrac{(x-3)^2}{9}+\dfrac{(y-8)^2}{25}=1$



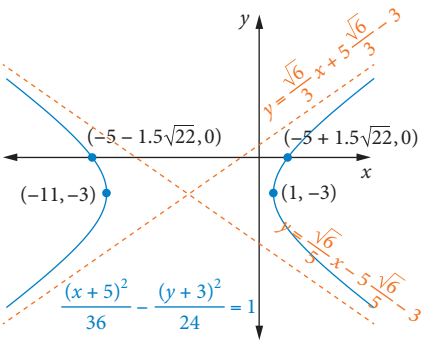
8 a i



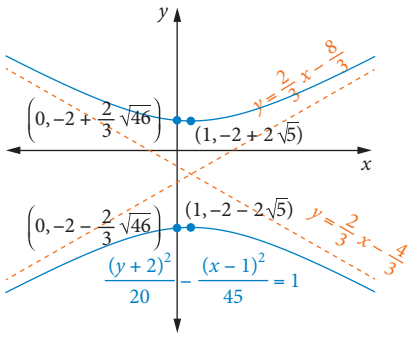
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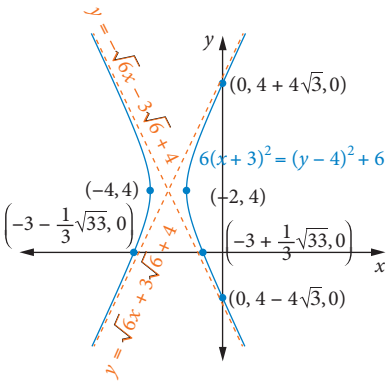
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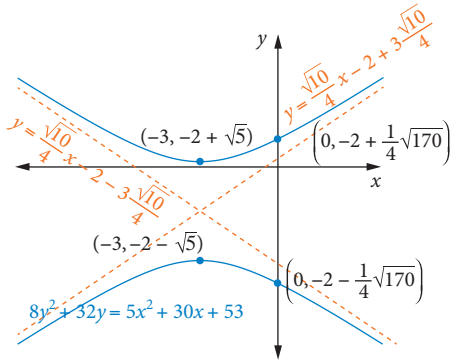
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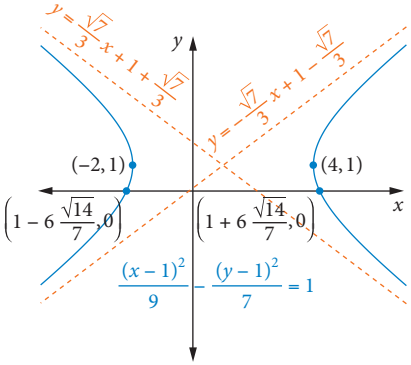
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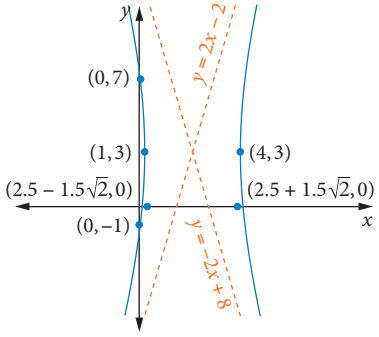
vi



b i $\dfrac{(x-1)^2}{9}-\dfrac{(y-1)^2}{7}=1$

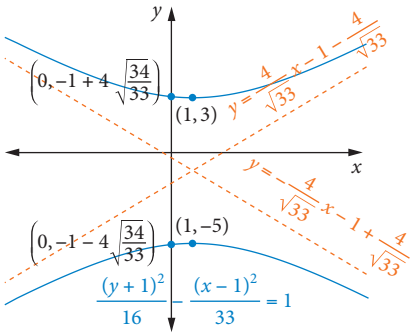


ii $\dfrac{(x-2.5)^2}{2.25}-\dfrac{(y-3)^2}{9}=1$

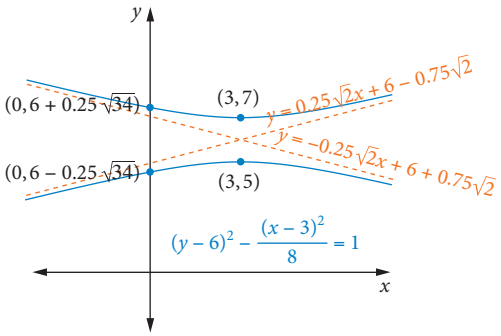


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iii $\frac{(y+1)^2}{16} - \frac{(x-1)^2}{33} = 1$



iv $(y-6)^2 - \frac{(x-3)^2}{8} = 1$



9 $\frac{(x+1)^2}{4} + \frac{4(y-5.5)^2}{49} = 1$

10 $\frac{(e^2-1)(x+4)^2}{25e^2-41} - \frac{(y-1)^2}{25e^2-41} = 1$

11 **a** $(x+4)^2 + (y-3)^2 = 49$

b $\frac{(x+4)^2}{25} + \frac{(y-3)^2}{16} = 1$

c $x - 3 = -\frac{3}{25}(y+2)^2$

d $\frac{(x+4)^2}{25} - \frac{(y+3)^2}{16} = 1$

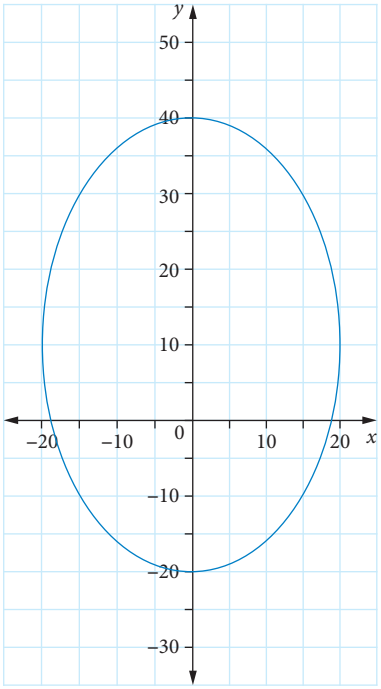
e $\frac{(y-2)^2}{4} - \frac{(x-5)^2}{9} = 1$

f $y - 3 = \frac{6}{25}(x+5)^2$

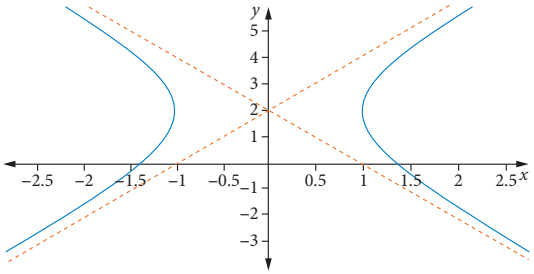
g $\frac{(x-2)^2}{16} + \frac{(y+5)^2}{49} = 1$

h $\frac{(y+5)^2}{36} - (x-3)^2 = 1$

12 Intercepts with x-axis are $\pm\frac{40\sqrt{2}}{3}$.



13



Asymptotes at $y = 2x + 2$ and $y = -2x + 2$

Intercepts at $(\sqrt{2}, 0)$ and $(-\sqrt{2}, 0)$

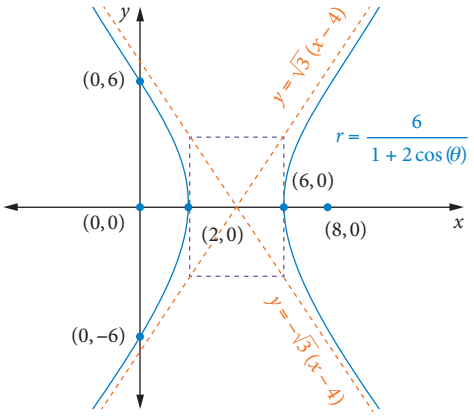
14 C **15** B **16** D **17** B

18 E **19** D **20** E

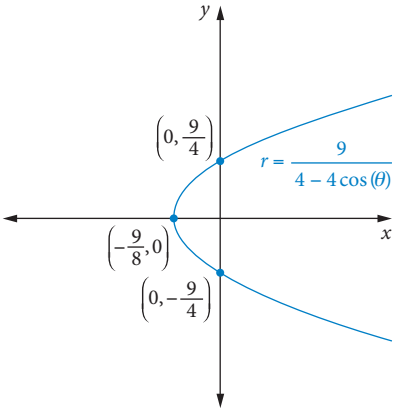
EXERCISE 8.8

1 A **2** B

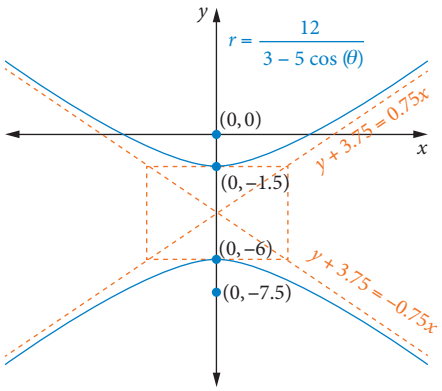
3 **a** $\frac{(x-4)^2}{4} - \frac{y^2}{12} = 1$



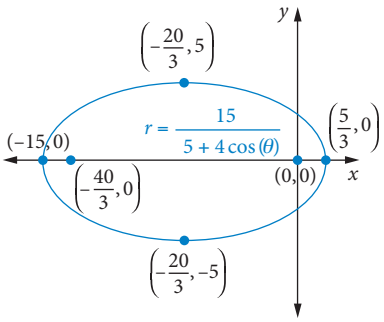
b $y^2 = \frac{9}{2}\left(x + \frac{9}{8}\right)$



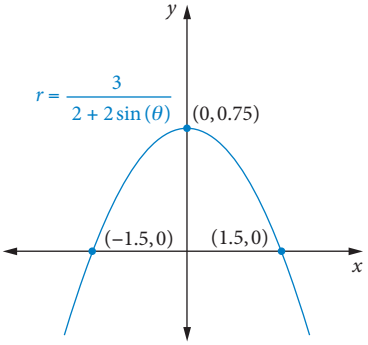
c $\frac{(4y + 15)^2}{81} - \frac{x^2}{9} = 1$



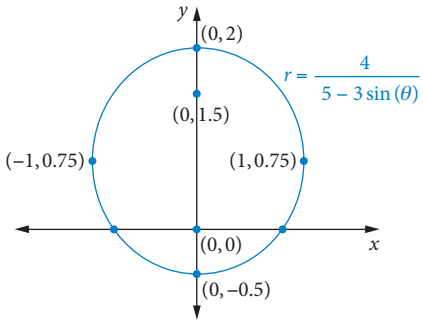
d $\frac{(3x + 20)^2}{625} + \frac{y^2}{25} = 1$



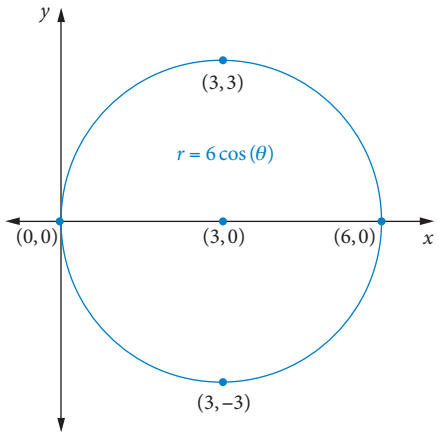
e $x^2 = -3\left(y - \frac{3}{4}\right)$



f $\frac{(4y - 3)^2}{25} + x^2 = 1$



g $(x - 3)^2 + y^2 = 9$



4 **a** $r = \frac{10}{1 - \sin(\theta)}$ **b** $r = \frac{9}{5 + 4\cos(\theta)}$

c $r = \frac{25}{24 + 26\cos(\theta)}$ **d** $r = \frac{16}{5 - 3\sin(\theta)}$

e $r = \frac{16}{3 - 5\sin(\theta)}$ **f** $r = \frac{8}{1 + \cos(\theta)}$

g $r = 16\sin(\theta)$

5 **a** $x = 5\cos(\theta) - 4, y = 5\sin(\theta) + 2$

b $x = 5\sec(\theta) + 2, y = 4\tan(\theta) - 3$

c $x = 3\cos(\theta) - 4, y = \sqrt{10}\sin(\theta) + 5$

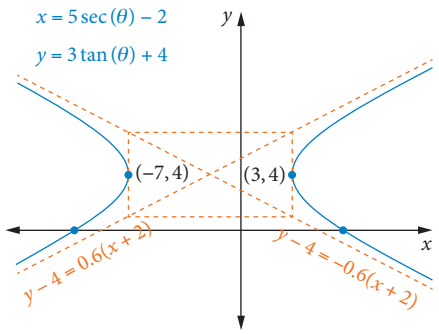
d $x = \frac{1}{2}t - 1, y = \frac{1}{2}t^2 - 4$

e $x = 7\cos(\theta) - 3, y = 5\sin(\theta) + 1$

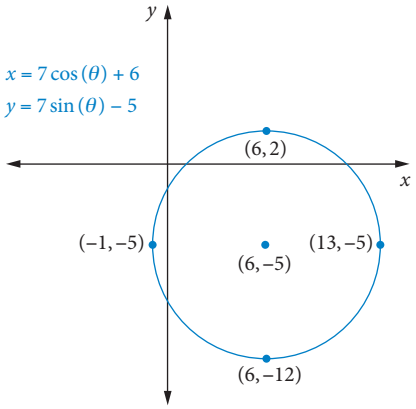
f $y = 2\sqrt{5}\sec(\theta) + 6, x = 6\tan(\theta) + 4$

g $x = 6t^2 + 7, y = 6t^2 - 3$

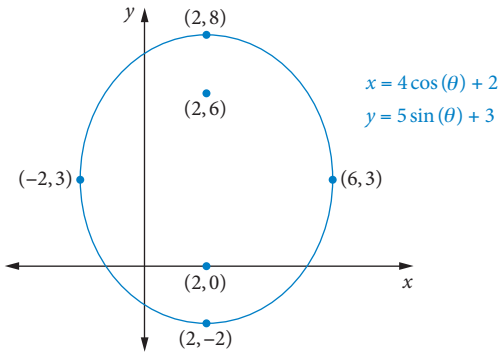
6 **a** $\frac{(x + 2)^2}{25} - \frac{(y - 4)^2}{9} = 1$



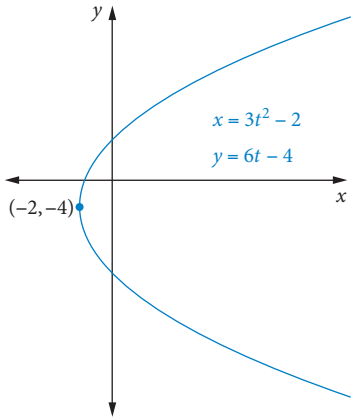
b $(x-6)^2+(y+5)^2=49$



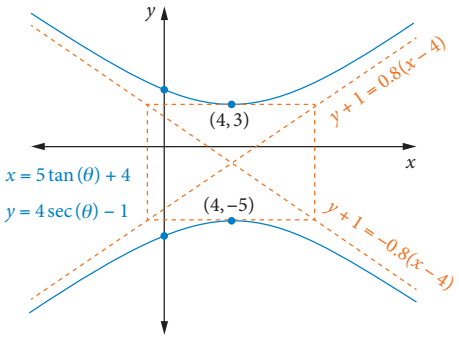
c $\frac{(x-2)^2}{16}+\frac{(y-3)^2}{25}=1$



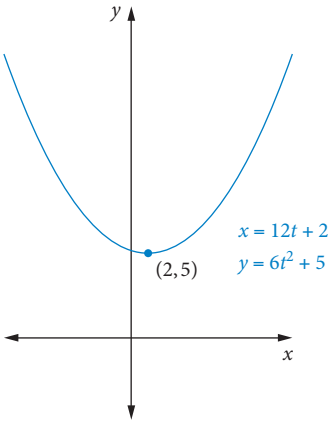
d $(y+4)^2=12(x+2)$



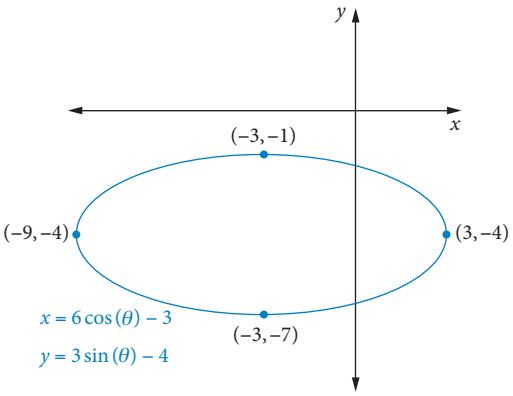
e $\frac{(y+1)^2}{16}-\frac{(x-4)^2}{25}=1$



f $(x-2)^2=24(y-5)$



g $\frac{(x+3)^2}{36}+\frac{(y+4)^2}{9}=1$



7 **a** $x = 3\cos(\theta)+1,\ y = 2\sqrt{2}\sin(\theta)$

b $y = 6t^2-6,\ x = 12t$

c $y = \sqrt{21}\sec(\theta),\ x = 2\tan(\theta)-5$
or $y = \sqrt{21}\operatorname{cosec}(\theta),\ x = 2\cot(\theta)-5$

d $x = 3\sec(\theta)+9,\ y = 6\sqrt{2}\tan(\theta)$
or $x = 3\operatorname{cosec}(\theta)+9,\ y = 6\sqrt{2}\cot(\theta)$

e $y = 3\cos(\theta)+1,\ x = 2\sqrt{2}\sin(\theta)$

f $x = 5\cos(\theta)+5,\ y = 5\sin(\theta)$

g $x = -4t^2+5,\ y = -8t$

8 **a** $r=\dfrac{8}{\sqrt{5}+5\cos(\theta)}$

b $r=\dfrac{\frac{3}{4}}{1-0.5\sqrt{3}\sin(\theta)}=\dfrac{3}{4-2\sqrt{3}\sin(\theta)}$

c $r=\dfrac{16}{3-5\sin(\theta)}$

d $r=\dfrac{6}{1+\cos(\theta)}$

e $r = 8\cos(\theta)$ or $r = 8\sin(\theta)$

f $r=\dfrac{16}{5-3\cos(\theta)}$

g $r=\dfrac{8}{1-\sin(\theta)}$

9 A **10** E **11** E

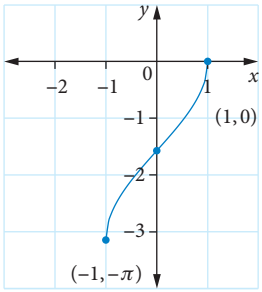
12 $\frac{(x-1)^2}{9}+\frac{(y+2)^2}{4}=1$

CUMULATIVE EXAMINATION 1

1 **a, b** Proofs: see worked solutions

c 11

2



3 $x = \frac{21}{5}, -\frac{3}{7}$

CUMULATIVE EXAMINATION 2

Section A

1 B **2** C **3** E **4** B **5** C

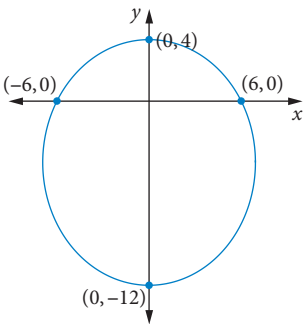
Section B

1 **a** $a_{n+1} = 1.04a_n + 1500$

b Proofs: see worked solutions. **c** $35 772

2 **a** $\frac{(y+4)^2}{64} + \frac{x^2}{48} = 1$

b



c $x = 4\sqrt{3}\cos(\theta), y = 8\sin(\theta) - 4, 0 \le \theta < 2\pi$

3 **a** $ar, ar^2, ar^3, ar^4, ar^5$ **b** $\frac{ar}{1-r}$

CHAPTER 9

EXERCISE 9.1

1 **a** $2\sqrt{6}i$ **b** $11i$ **c** $6\sqrt{3}i$

2 $9.5i$ **3** $-\frac{103}{13}$

4 **a** 2 **b** 0 **c** $-2 + 2i$

5 **a** $4x, -2y$ **b** $-4, 1$ **c** $-5, -2$

6 **a** $8 + i$ **b** $8 - 16i$ **c** $1 + 2i$

7 **a** $7 + i$ **b** 15 **c** $-5i$

8 **a** $8 + 11i$ **b** $1 + 5i$ **c** $-1 + 2i$

9 **a** $\frac{4}{5} + \frac{7}{5}i$ **b** $-\frac{7}{17} + \frac{28}{17}i$ **c** $\frac{5}{2} - \frac{1}{2}i$

10 **a** $-13 + 14i$ **b** $-26 + 26i$ **c** $-\frac{1}{26} - \frac{5}{26}i$

11 $0.5 - 2.5i$ **12** -1 **13** $\frac{3}{2} - \frac{1}{2}i$

14 See worked solutions; Substitute for z in the equation and show that the left side is zero.

15 $2 - 11i$ **16** B **17** D **18** C

19 B **20** B **21** A **22** A

23 E **24** A **25** A **26** D

27 C **28** E **29** $x \neq 1, y \in R\backslash\{0\}$

30, 31 Proofs: see worked solutions

32 $a = -\frac{1}{2}, b = \frac{\sqrt{3}}{2}$

EXERCISE 9.2

1 C **2** C

3 $x = 1 \pm \sqrt{7}i, (x - 1 + \sqrt{7}i)(x - 1 - \sqrt{7}i)$

4 **a** $x = \pm 4i$, purely imaginary

b $x = 4$, real **c** $x = -3 \pm i$, complex

5 $(-5 + 2i)^2 + 10(-5 + 2i) + 29$
$= 25 - 20i - 4 - 50 + 20i + 29 = 0$

6 $b = 6, c = 13$

7 **a** $x = \pm\sqrt{2}i, x = \pm\sqrt{3}i$ purely imaginary

b $x = \frac{3 \pm \sqrt{31}i}{2}$ complex

8 Proof: see worked solutions

9 $z = \pm 2i$

10 $(w + 5i)(w - 5i)(w + 2i)(w - 2i)$

11 B **12** D **13** D **14** C

15 $\frac{-7}{3} - \frac{\sqrt{5}}{3}i, \frac{-7}{3} + \frac{\sqrt{5}}{3}i$

16 $b = -[(x + yi) + (x - yi)] = -2x$
$c = (x + yi)(x - yi) = x^2 + y^2$
$z^2 + bz + c = z^2 - 2xz + x^2 + y^2$

17 **a** $x = \pm 3, y = \pm 2$

b $(z - 3 + 2i)(z - 3 - 2i) = z^2 - 6z + 13$

18 Require the discriminant to be less than 0.

$3k^2 + 4k > 0$, so $k > 0$ and $k < -\frac{4}{3}$

19 $a = -4, b = 41$

20 $a < 0, b = 0, c < 0$ or $a > 0, b = 0, c > 0$.

21 $z_1 = -\frac{4 + i}{\sqrt{2} - i} = \frac{(4\sqrt{2} - 1) + (4 + \sqrt{2})i}{3}$

$z_2 = \frac{(4\sqrt{2} - 1) - (4 + \sqrt{2})i}{3}$

$z_1 z_2 = \frac{((4\sqrt{2} - 1) + (4 + \sqrt{2})i)}{3}$
$\frac{((4\sqrt{2} - 1) - (4 + \sqrt{2})i)}{3} = \frac{17}{3}$

22 $(z - z_1)(z - z_2) = z^2 - (z_1 + z_2)z + z_1 z_2$

$z_1 + z_2 = (a + c) + (b + d)i$ is real, so $b = -d$

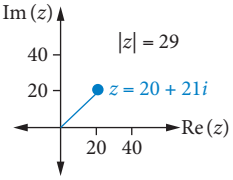
$z_1 z_2 = (ac - bd) + (ad + bc)i$

$z_1 z_2$ is real, so $ad + bc = 0$ or $ab + bc = 0$, hence $a = -c$, (using $b = -d$)

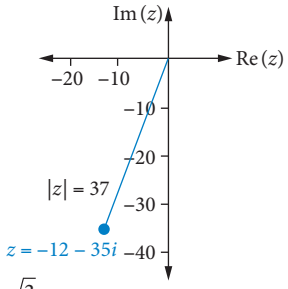
EXERCISE 9.3

1 B **2** E

3 **a**



b



4 **a** $\frac{\sqrt{3}}{3}$ **b** $\frac{\sqrt{11}}{2}$

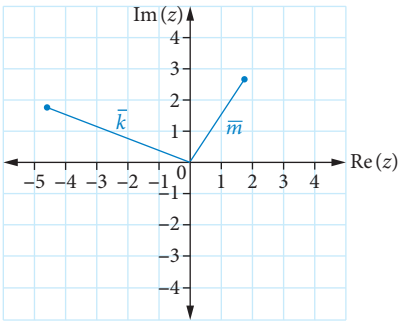
5 **a** $z = -8 - 13i$ **b** $z = 13 - 8i$

6 $a = -13, b = -2$ **7** $a = 3, b = 4$

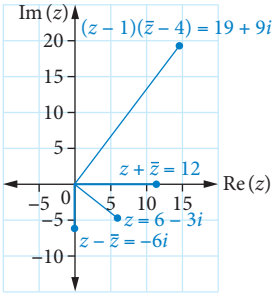
8 **a** $k = -5 - 2i, m = 2 - 3i$

b $\overline{k} = -5 + 2i, \overline{m} = 2 + 3i$

c



9 **a** $6 - 3i$ **b** 12 **c** $-6i$ **d** $19 + 9i$



10 C **11** D **12** D **13** A

14 A

15 $a \in R, b = 0$

16 $z = -12 + 5i, z = -5 + 12i$

17 60° **18** $x = \pm 5$

19 $z_2 = a - bi$, $z_3 = -iz_2 = -b - ai$

$|z_2 + z_3| = \sqrt{(a-b)^2 + (a+b)^2} = \sqrt{2(a^2 + b^2)}$

20 $z = a + bi, w = b - ai$

$zw = 2ab + (b^2 - a^2)i$

Hence $2ab = 4\sqrt{3} \Rightarrow ab = 2\sqrt{3}$ and $b^2 - a^2 = -1$.

Solving gives $a = 2$, $b = \sqrt{3}$

EXERCISE 9.4

1 C **2** D **3** E **4** B

5 **a** $z = 8\,\text{cis}\left(-\frac{\pi}{3}\right)$ **b** $z = \frac{\sqrt{2}}{2}\,\text{cis}\left(-\frac{3\pi}{4}\right)$

6 **a** $z = -\frac{5\sqrt{2}}{2}(1 + i)$ **b** $z = \sqrt{3} - 3i$

7 **a** $z = -38 + 34i$ **b** $z = \frac{3}{2} + \frac{3\sqrt{3}}{2}i$

8 **a** $z = 6\,\text{cis}\,(-135°)$

b $z = \sqrt{6} + \sqrt{2}i$

9 $z = 8\,\text{cis}\left(-\frac{2\pi}{3}\right)$

10 $z = 5\,\text{cis}\left(\frac{(4n-1)\pi}{4}\right), n \in Z$

11 **a** $z = r\,\text{cis}\,(\theta), w = r\,\text{cis}\,(-\pi + \theta)$

$zw = r^2\,\text{cis}\,(\theta - \pi + \theta)$ $zw = r^2\,\text{cis}\,(2\theta)$

b $zw = (-1 + i)(-1 - i) = -2i$

$z = \sqrt{2}\,\text{cis}\left(\frac{3\pi}{4}\right)$ $w = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)$

$zw = \sqrt{2}^2\,\text{cis}\left(\frac{3\pi}{4} - \frac{\pi}{4}\right) = 2\,\text{cis}\left(\frac{\pi}{2}\right) = -2i$

12 C **13** A

14 $r = \sqrt{10^2 + 24^2} = 26, \theta = \tan^{-1}\left(\frac{24}{10}\right) = 67°$

15 $z = |y|\left(\cos\left(-\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)i\right)$ or $z = |y|\,\text{cis}\left(-\frac{\pi}{2}\right)$

16 $|z||w| = 9, |z| = 3$

$\text{Arg}\,(z) + \frac{\pi}{3} = \frac{\pi}{2} \Rightarrow \text{Arg}\,(z) = \frac{\pi}{6}$

$z = 3\cos\left(\frac{\pi}{6}\right) + 3\sin\left(\frac{\pi}{6}\right)i$

$= \frac{3\sqrt{3}}{2} + \frac{3}{2}i$

17 **a** $z = \sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right), w = 2\,\text{cis}\left(\frac{\pi}{6}\right)$

b $zw = 2\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4} + \frac{\pi}{6}\right) = 2\sqrt{2}\,\text{cis}\left(-\frac{\pi}{12}\right)$

$\text{Arg}\,(z) = -\frac{\pi}{12}$

18 $|z| = 2, \frac{2}{|w|} = 3 \Rightarrow |w| = \frac{2}{3}$

$\text{Arg}\,(z) = \frac{\pi}{3}$

$\frac{\pi}{3} - \text{Arg}\,(w) = \frac{7\pi}{12} \Rightarrow \text{Arg}\,(w) = -\frac{\pi}{12}$

$w = \frac{2}{3}\,\text{cis}\left(-\frac{\pi}{12}\right)$

19 $z^3 = z^2 \times z$

$= r^2\,\text{cis}\,(2\theta) \times r\,\text{cis}\,(\theta)$

$= r^2 \times r\,\text{cis}\,(2\theta + \theta)$

$= r^3\,\text{cis}\,(3\theta)$

20 Let $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$.

a $\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$

$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} - \frac{1}{2} \times \frac{1}{\sqrt{2}}$

$= \frac{\sqrt{3} - 1}{2\sqrt{2}}$

b $\cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{4}\right)$

$= \frac{1}{2} \times \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}$

$= \frac{\sqrt{3} + 1}{2\sqrt{2}}$

c $\text{Arg}\, z = \frac{\sqrt{3} - 1}{2\sqrt{2}} \div \frac{\sqrt{3} + 1}{2\sqrt{2}}$

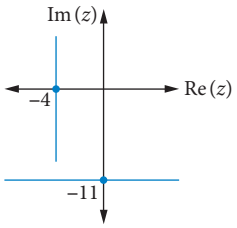
$= \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \times \frac{\sqrt{3} - 1}{\sqrt{3} - 1}$

$= 2 - \sqrt{3}$

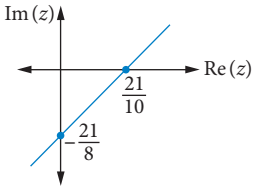
EXERCISE 9.5

1 A **2** C

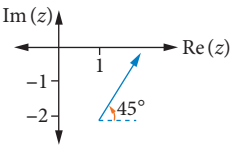
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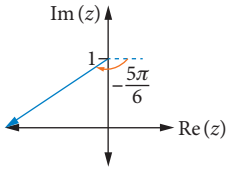
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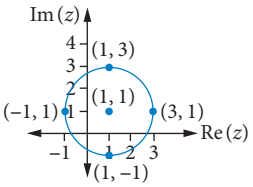
5 a



b



6 a



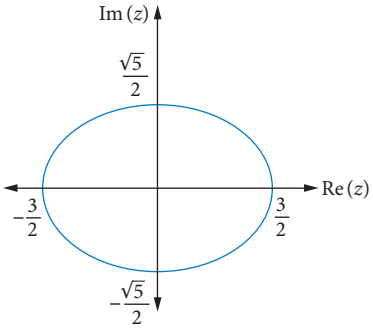
b $(x - 1)^2 + (y - 1)^2 = 2^2$

radius = 2

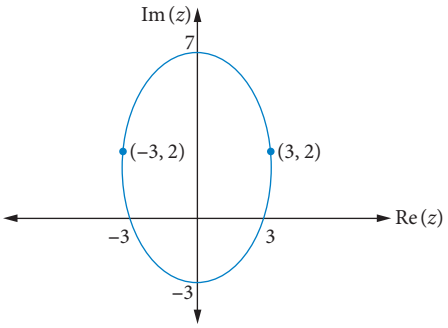
centre (1, 1)

7 a $|z + 2 + 4i| = 3$ **b** $|z - 3 - 5i| = 8$

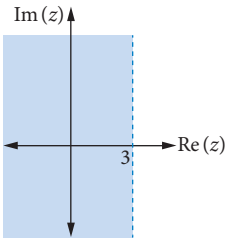
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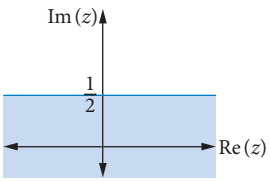
9



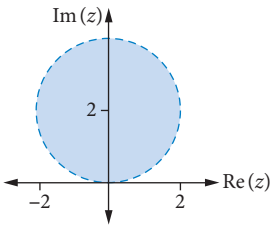
10 a



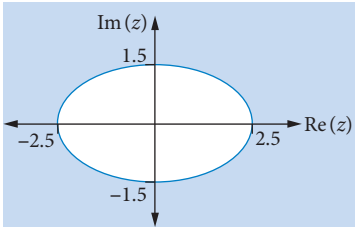
b



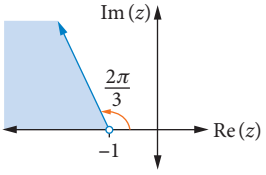
11 a



b

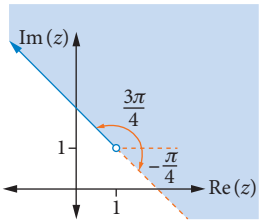


12 a



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b



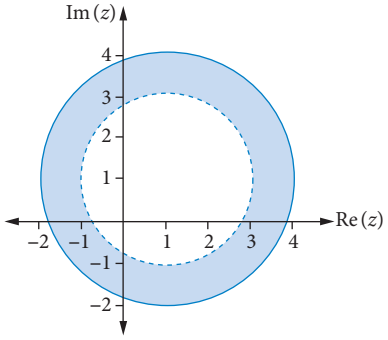
13
$$z = x + yi$$
$$\sqrt{x^2 + (y-a)^2} = \sqrt{x^2 + (y-b)^2}$$
$$x^2 + (y-a)^2 = x^2 + (y-b)^2$$
$$a^2 - 2ay = b^2 - 2by$$
$$y = \frac{1}{2}\left(\frac{b^2 - a^2}{b-a}\right)$$
$$= \frac{1}{2}\left(\frac{(b-a)(b+a)}{b-a}\right)$$
$$y = \frac{1}{2}(a+b)$$

14 a Proof: see worked solutions

b $\frac{x^2}{36} + \frac{y^2}{11} = 1$

15 B **16** C **17** B **18** C

19

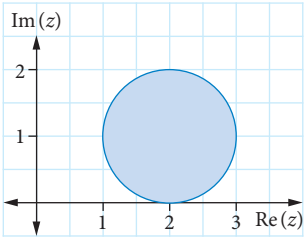


20 π square units

21 $4\sqrt{3}$ square units

CUMULATIVE EXAMINATION 1

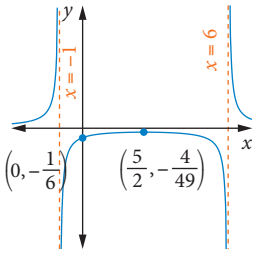
1 a



b $(x-2)^2 + (y+1)^2 = 1$

2 a $\sqrt{3} - i$ **b** $z^2 - 2\sqrt{3}z + 4 = 0$

3



CUMULATIVE EXAMINATION 2

Section A

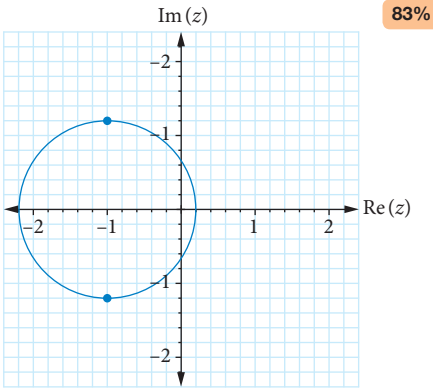
1 D 41% **2** A 72% **3** C **4** C

5 C

Section B

1 a i
$$z = \frac{-4 \pm \sqrt{16 - 4 \times 2 \times 5}}{4}$$
$$= \frac{-4 \pm \sqrt{24}i}{4}$$
$$= -1 \pm \frac{\sqrt{6}i}{2}$$ 68%

ii 83%



b i $m = 1, n = \frac{\sqrt{6}}{2}$ 64%

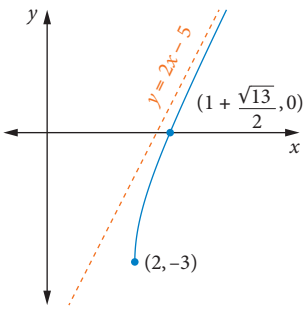
ii $(x+1)^2 + y^2 = \frac{3}{2}$ 59%

2 a 4, 7 **b** $A = -4, B = 7$

3 a Use $1 + \cot^2(t) = \operatorname{cosec}^2(t)$.

b domain: $(2, \infty)$, range: $(-3, \infty)$

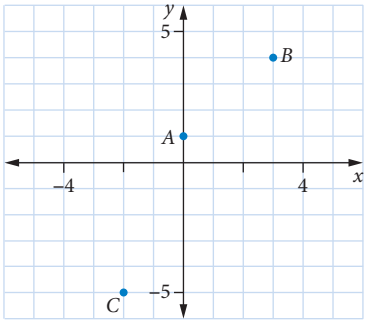
c



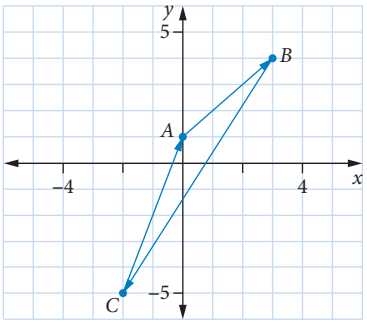
CHAPTER 10

EXERCISE 10.1

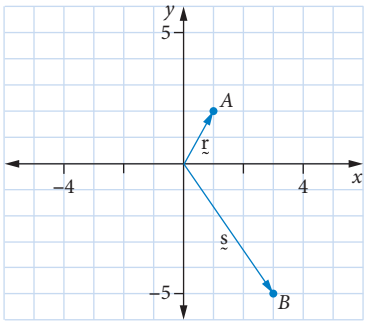
1 a



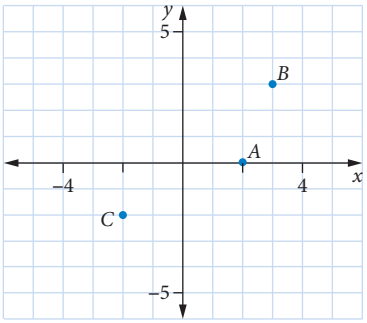
b



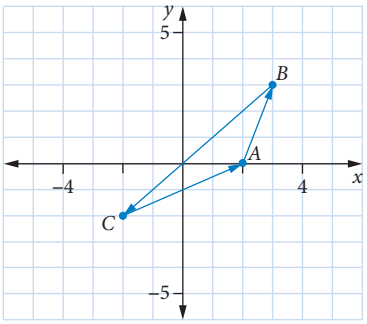
2



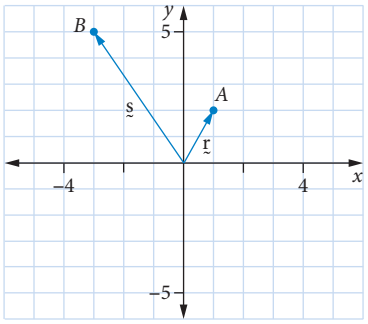
3 a



b

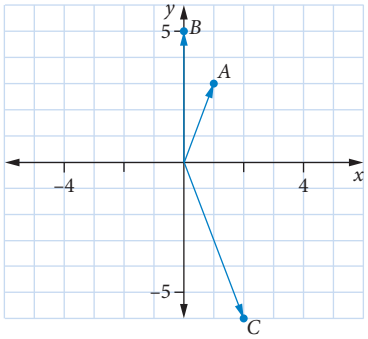


4

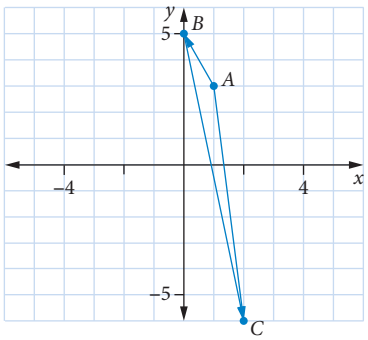


5 E **6** B **7** C **8** B

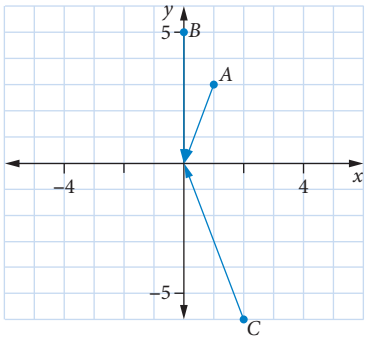
9 a–c



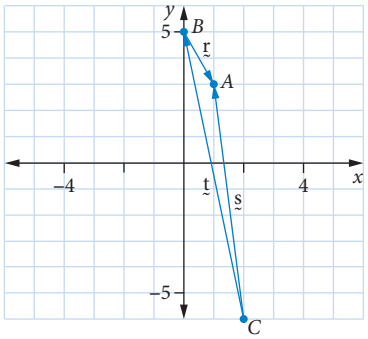
d–f



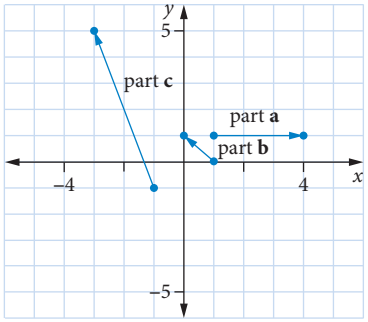
10 a–c



d–f

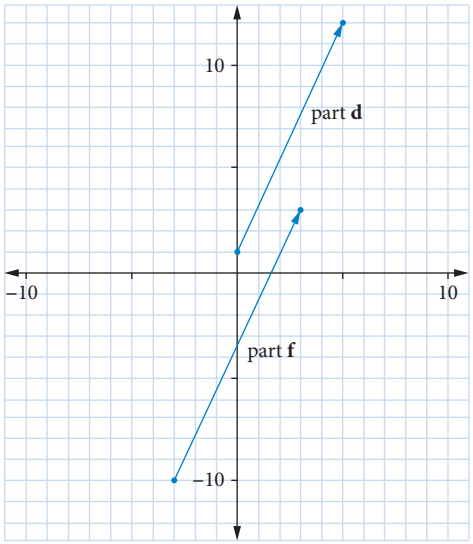


11 a–c



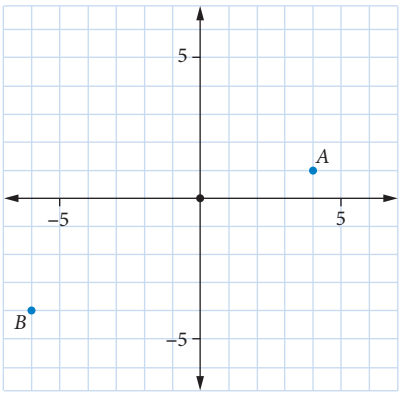
9780170448338

d, e

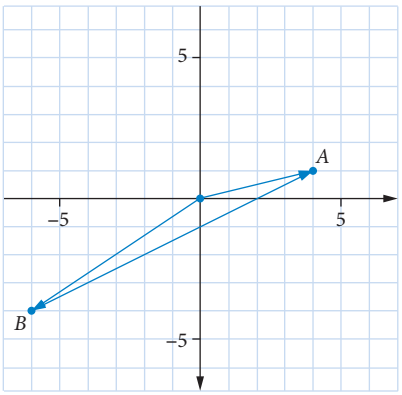


EXERCISE 10.2

1 a



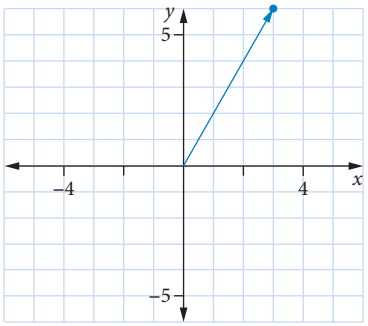
b



c $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$

2 E

3

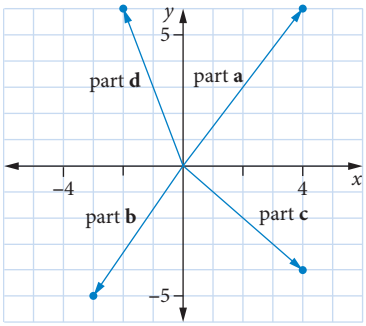


4 magnitude = $2\sqrt{3}$, direction from x-axis = 30°

5 magnitude = $2\sqrt{2}$, $\hat{\underset{\sim}{a}} = \frac{1}{2\sqrt{2}}(-2\underset{\sim}{i} + 2\underset{\sim}{j})$

6 C **7** D **8** A **9** B

10



11, 12 Proofs: see worked solutions

13 a $\underset{\sim}{d}_1 = 5\underset{\sim}{i} - 3\underset{\sim}{j}$, $\therefore 3\underset{\sim}{d}_1 = 15\underset{\sim}{i} - 9\underset{\sim}{j}$

b $\underset{\sim}{d}_2 = -2\underset{\sim}{i} + 9\underset{\sim}{j}$, $\therefore -2\underset{\sim}{d}_2 = 4\underset{\sim}{i} - 18\underset{\sim}{j}$

14 a 53.1° **b** 112.6°

c 163.7° **d** −135°

15 a $\begin{bmatrix} 2 \\ 7 \end{bmatrix}$ **b** $\begin{bmatrix} -5 \\ -3 \end{bmatrix}$ **c** $\begin{bmatrix} -6 \\ 9 \end{bmatrix}$

16 a $\begin{bmatrix} -13 \\ 13 \end{bmatrix}$ **b** $\begin{bmatrix} 18 \\ -14 \end{bmatrix}$

c $\begin{bmatrix} -5 \\ -1 \end{bmatrix}$ **d** $\begin{bmatrix} 10 \\ -11.8 \end{bmatrix}$

17 a $7\underset{\sim}{i} - 13\underset{\sim}{j}$ **b** $-17\underset{\sim}{i} + 10\underset{\sim}{j}$ **c** $-5\underset{\sim}{i} + 5\underset{\sim}{j}$

18 $a = \sqrt{3}$

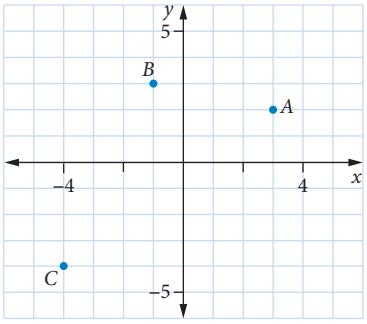
19 D **20** E **21** B **22** D

23 B **24** B **25** A **26** B

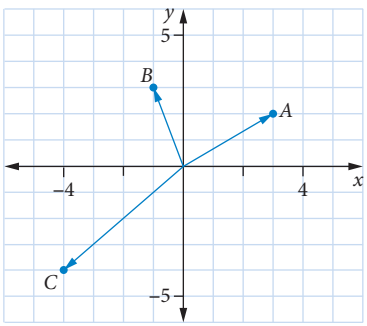
27 D **28** C **29** D **30** B

EXERCISE 10.3

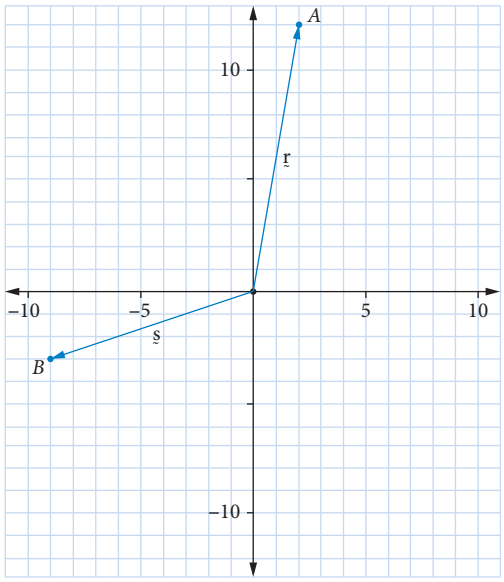
1 a



b



c



2 a $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ **b** $\begin{bmatrix} -4 \\ 4 \end{bmatrix}$

3 $\underset{\sim}{a} \cdot \underset{\sim}{b} = -5$ **4** $\underset{\sim}{a} \cdot \underset{\sim}{b} = 3$ **5** 45° **6** 46

7 5 **8** B **9** E **10** C

11 D **12** D **13** A **14** 101°

15 $a = -2$ **16** $c = -3$ **17** D

EXERCISE 10.4

1 a $\begin{bmatrix} 2 \\ 11 \end{bmatrix}$ **b** $\begin{bmatrix} -2 \\ -2 \end{bmatrix}$ **c** $-\underset{\sim}{i} + 2\underset{\sim}{j}$ **d** −18

2 52°

3 Proof: see worked solutions

4 $m = -2, b = -2$ and $m = 2, b = 2$

5 $\frac{1}{5}(3\underset{\sim}{i} + 4\underset{\sim}{j})$ **6** $-\frac{1}{5}(3\underset{\sim}{i} + 4\underset{\sim}{j})$

7, 8 Proofs: see worked solutions

9 C **10** D **11** B **12** A

13 $\frac{1}{5}(-3\underset{\sim}{i} + 4\underset{\sim}{j})$ **14** $\frac{6}{\sqrt{65}}(4\underset{\sim}{i} - 7\underset{\sim}{j})$

15 $-4\underset{\sim}{i} + 3\underset{\sim}{j}$

16 a $\hat{\underset{\sim}{a}} = \frac{1}{\sqrt{6}}(\sqrt{3}\underset{\sim}{i} - \underset{\sim}{j} - \sqrt{2}\underset{\sim}{k})$ **b** 45°

c $m = 6 + 5\sqrt{2}$

17 D **18** E **19** A

EXERCISE 10.5

1 D **2** C **3** $-\sqrt{2}$

4 $-\underset{\sim}{i} + \underset{\sim}{j}$ **5** $2\underset{\sim}{i} + 2\underset{\sim}{j}$ **6** E

7 A

8 a $m = 4$ **b** $\frac{47}{18}\underset{\sim}{i} - \frac{5}{9}\underset{\sim}{j} + \frac{7}{18}\underset{\sim}{k}$

9 $c = -6.5$ **10** $m = 2, m = -\frac{22}{3}$

11 D **12** C **13** A **14** B

15 D **16** B **17** D

EXERCISE 10.6

1 a 79° **b** −143° **c** 11°

2 A **3** Proofs: see worked solutions

4 a $\overrightarrow{OF} = 2\underset{\sim}{i} + 6\underset{\sim}{j}$, $\overrightarrow{OC} = 8\underset{\sim}{i} - 4\underset{\sim}{j}$, $\overrightarrow{OF} \cdot \overrightarrow{OC} = -8$

b 98°

c $|\overrightarrow{OF}| = 2\sqrt{10}$, $|\overrightarrow{OC}| = 4\sqrt{5}$

d 12 metres

5 Proof: see worked solutions

6 a $\frac{18}{5}$ **b** $AP = 5.2$

7 D

8 a $\overrightarrow{CS} = 6\underset{\sim}{i} + 2\underset{\sim}{j}$; 6.3 km **b** $8m(7 - 5m)$

9 a $\overrightarrow{AN} = \underset{\sim}{u} + \frac{1}{2}\underset{\sim}{v}$

b $\overrightarrow{CM} = \frac{1}{2}\underset{\sim}{v} - \frac{1}{2}\underset{\sim}{u}$ and $\overrightarrow{BP} = -\frac{1}{2}\underset{\sim}{u} - \underset{\sim}{v}$

c $\overrightarrow{AN} + \overrightarrow{CM} + \overrightarrow{BP} = \underset{\sim}{0}$

10 a $\underset{\sim}{d}_1 = \frac{1}{\sqrt{2}}v_1t_1\underset{\sim}{i} + \frac{1}{\sqrt{2}}v_1t_1\underset{\sim}{j}$,

$\underset{\sim}{d}_2 = \frac{1}{\sqrt{2}}(v_1t_1 + v_2t_2)\underset{\sim}{i} + \frac{1}{\sqrt{2}}(v_1t_1 - v_2t_2)\underset{\sim}{j}$

b $\sqrt{\underset{\sim}{d}_2} = \frac{1}{\sqrt{2}}\sqrt{(v_1t_1 + v_2t_2)^2 + (v_1t_1 - v_2t_2)^2}$

$= \sqrt{(v_1t_1)^2 + (v_2t_2)^2}$

c $\sqrt{(v_1t_1)^2 + (v_2t_2)^2} = \sqrt{24^2 + 10^2} = 26$ km

EXERCISE 10.7

1 D

2 a $\overrightarrow{OF} = \underset{\sim}{j}$, $\overrightarrow{OC} = -4\underset{\sim}{i} + 2\underset{\sim}{j}$, $\overrightarrow{OF} \cdot \overrightarrow{OC} = 2$

b 63°

3 C

4, 5, 6 Proofs: see worked solutions

7 a $\overrightarrow{AB} = 2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$

b Proof: see worked solutions **c** $\sqrt{38}$

8 a $a = \sqrt{3}$

b Proof: see worked solutions

9 B **10** D **11** C **12** A **13** D

14 a **i** $\frac{1}{2}a$ **ii** $\underset{\sim}{a} - \underset{\sim}{b}$ **iii** $\frac{1}{2}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a}$

b $\overrightarrow{MN} = \frac{1}{2}\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{a} = \overrightarrow{AQ}$,

$\overrightarrow{NQ} = \frac{1}{2}\underset{\sim}{b} + \frac{1}{2}(\underset{\sim}{a} - \underset{\sim}{b}) = \frac{1}{2}\underset{\sim}{a} = \overrightarrow{MA}$

c $|\overrightarrow{OA}| = \sqrt{9 + 4 + 3} = 4$

d **i** $\overrightarrow{OQ} = \frac{7}{2}\underset{\sim}{i} + \underset{\sim}{j} + \frac{\sqrt{3}}{2}\underset{\sim}{k}$ or $\frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})$

ii $|\overrightarrow{OA}| = |\overrightarrow{OB}|$, $\overrightarrow{OQ} \cdot \overrightarrow{AB} = \frac{7}{2} - 2 - \frac{3}{2} = 0$

or $\overrightarrow{AB} \cdot \overrightarrow{AB} = \frac{1}{2}(\underset{\sim}{b} \cdot \underset{\sim}{b} - \underset{\sim}{a} \cdot \underset{\sim}{a}) = \frac{1}{2}(|\underset{\sim}{b}|^2 - |\underset{\sim}{a}|^2) = 0$

15 **a** Proof: see worked solutions **b** 109.5°

c **i** $p = \frac{1}{\sqrt{2}}, r = -\frac{1}{\sqrt{2}}$ **ii** $\frac{10}{3\sqrt{2}}$ or $\frac{5\sqrt{2}}{3}$

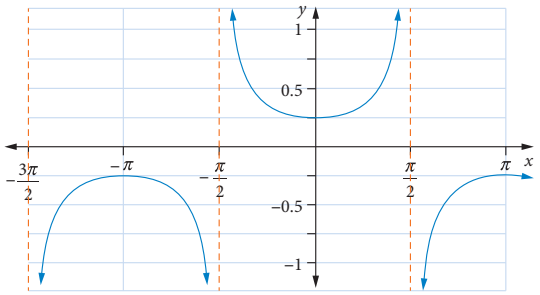
CUMULATIVE EXAMINATION 1

1 **a** $5 + i$ **b** $-4 + 7i$ **c** 13

2 **a** $2\underset{\sim}{i} + 2\underset{\sim}{j}$ **b** -2 **c** $-\sqrt{2}$

d $-\underset{\sim}{i} + \underset{\sim}{j}$ **e** $-\frac{1}{5}(\underset{\sim}{i} + 3\underset{\sim}{j})$

3



CUMULATIVE EXAMINATION 2

Section A

1 E **2** A 75% **3** D

4 B **5** C **6** A

Section B

1 **a** $\underset{\sim}{r}_A(0) = 5\underset{\sim}{i} + 3\underset{\sim}{j}$

b $\underset{\sim}{r}_B(0) = -8\underset{\sim}{i} - 2\underset{\sim}{j}$

c $\underset{\sim}{r}_A \cdot \underset{\sim}{r}_B = 5(1-t)4(t-2) + 3(1+t)(5t-2)$
$= -5t^2 + 69t - 46$

d $\underset{\sim}{r}_A \cdot \underset{\sim}{r}_B = 0$ at $t = 0.7$, $t = 13.1$

e $|\underset{\sim}{r}_A - \underset{\sim}{r}_B| = |(-9t + 13)\underset{\sim}{i} + (-2t + 5)\underset{\sim}{j}|$
$= \sqrt{(-9t + 13)^2 + (-2t + 5)^2}$

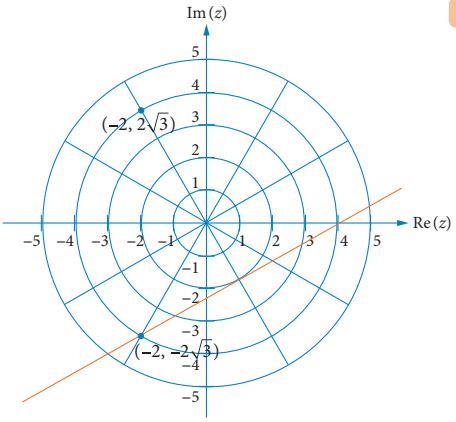
f Solve $\sqrt{(-9t + 13)^2 + (-2t + 5)^2} = 20$, $t = 3.65$ hours

2 **a** $4\,\text{cis}\left(-\frac{2\pi}{3}\right)$ 74%

b Proof: see worked solutions 55%

c Proof: see worked solutions 73%

d 69%



CHAPTER 11

EXERCISE 11.1

1

x	3	4	5	6	7
$\Pr(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$

2 2.52

3 **a** 1.84 **b** 2.494 **c** 1.579

4 mean = 16.850, variance = 80.828, standard deviation = 8.990

5

x	1	2	3	4	5	6
$\Pr(X = x)$	$\frac{3}{20}$	$\frac{1}{20}$	$\frac{1}{4}$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{3}{20}$

6

x	0	1	2	3
$\Pr(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

7 2.8

8 **a** 0.4 **b** 0.84

9

y	0	1	2	3
$\Pr(Y = y)$	0.343	0.441	0.189	0.027

10 **a** $\frac{1}{3}$

b

x	0	1	2	3
$\Pr(X = x)$	$\frac{8}{27}$	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{1}{27}$

c **i** 1 **ii** $\frac{2}{3}$

11 **a**

x	1	2	3	4
$\Pr(X = x)$	$\frac{1}{20}$	$\frac{3}{20}$	$\frac{3}{10}$	$\frac{1}{2}$

b **i** $\frac{13}{4}$ **ii** $\frac{63}{80}$

12 **a**

x	0	1	2	3	4
$\Pr(X = x)$	$\frac{3}{85}$	$\frac{8}{85}$	$\frac{3}{17}$	$\frac{24}{85}$	$\frac{7}{17}$

b **i** 2.94 **ii** 1.28 **iii** 1.13

EXERCISE 11.2

1 B **2** A

3 **a**

x	200	400	600	800
$p(x)$	0.15	0.55	0.25	0.05

b **i** 440 **ii** 22 400

4 **a** 24.5 **b** 24 **c** $2\sqrt{6}$

5 **a** $E(X) = 1$, $\text{Var}(X) = 1$

b

x	0	1	2	3	4	5	6
$\Pr(X = x)$	0.16	0.24	0.25	0.2	0.1	0.04	0.01

c $E(T) = 2$, $\text{Var}(X) = 2$

d $E(T) = 2 \times E(X) = 2$, $\text{Var}(T) = 2 \times \text{Var}(X) = 2$

6 **a** 30 **b** 18 **c** $3\sqrt{2}$

7 **a**

x	5	10	15	20
$p(x)$	0.6	0.1	0.2	0.1

b **i** 9 **ii** $\sqrt{29}$

ANSWERS

8 a

s	2	3	4	5	6
$\Pr(S = s)$	0.01	0.06	0.21	0.36	0.36

b **i** 5 **ii** 0.9 **iii** $\frac{3\sqrt{10}}{10}$

9 a 0.5 **b** 0.25 **c** 1
d 1 **e** 1 **f** 0.5

10 B **11** C

12 a

h	0	1	2	3
$\Pr(H = h)$	0.448	0.264	0.18	0.108

b **i** 0.948 **ii** 1.057
c **i** 1.896 **ii** 2.115

13 a

x	0	1	2	3
$\Pr(X = x)$	$\frac{27}{64}$	$\frac{27}{64}$	$\frac{9}{64}$	$\frac{1}{64}$

b **i** $\frac{3}{4}$ **ii** $\frac{9}{16}$ **iii** $\frac{3}{4}$
c **i** $\frac{3}{2}$ **ii** $\frac{9}{8}$ **iii** $\frac{3\sqrt{2}}{4}$

EXERCISE 11.3

1 C **2** B

3 No. The survey is not a random sample. He will only question teenagers who already exercise and will exclude a proportion of the teenagers from the sample.

4 4, 13, 27, 6, 32, 33, 16, 34. Answers will vary.

5 Answers will vary. **6** 0.4

7 a $\bar{x} = \frac{1110.8}{22} \approx 50.49$ **b** $\bar{x}$ is a sample statistic.

8 a $p = 0.35$ **b** $\hat{p} = 0.375$

9 B **10** C **11** C **12** E **13** B

14 a 0.5 **b** 181.08 **c** Samples will vary.
d Answers will vary but should be close to 0.5.

15 a 0.4 **b** 40.3 **c** Samples will vary.
d Answers will vary but should be close to 0.4.

EXERCISE 11.4

1 C **2** B

3 a {(1,1) (1,2) (1,3) (1,4) (2,1) (2,2) (2,3) (2,4) (3,1) (3,2) (3,3) (3,4) (4,1) (4,2) (4,3) (4,4)}
b A = {(1,3) (2,2) (3,1)}

4 7–11, team yellow wins.

5 a 9 shots **b** 0.08

6 a winning area = 4 × 4 = 16
total area = 16 × 16 = 256
probability = $\frac{16}{256} = \frac{1}{16}$

b losing streaks = {2, 42, 3, 2, 4, 4, 9, 15, 4, 2}
average = 9.22

7 D **8** A **9** E **10** E **11** C

12 average of 7 infected people (Answers may vary.)

EXERCISE 11.5

1 C **2** E

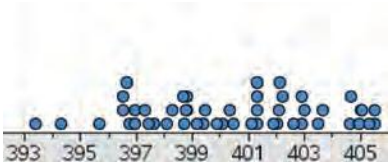
3

x	1	1.5	2	2.5	3	3.5	4
$\Pr(\bar{X} = x)$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{3}{16}$	$\frac{1}{8}$	$\frac{1}{16}$

4 mean = 7.5 standard deviation = 0.17 Answers may vary slightly depending on the data generated.

5 $\frac{7}{100}$

6 a Answers will vary depending on the data generated. Example result:



b Average of the sample means = approximately 400. Answers may vary slightly depending on the data generated.

7

Sample size	Mean of $\bar{x}$	Standard deviation of $\bar{x}$
20	250	11.2
200	250	3.5

Answers may vary slightly depending on the data generated.

8 0.15 g

9 a **i** $E(N) = 3$ **ii** $\text{Var}(N) = 2$ **iii** $\text{SD}(N) = \sqrt{2}$

b

$\bar{x}$	$\Pr(\bar{X} = \bar{x})$
1	$\frac{1}{25}$
1.5	$\frac{2}{25}$
2	$\frac{3}{25}$
2.5	$\frac{4}{25}$
3	$\frac{1}{5}$
3.5	$\frac{4}{25}$
4	$\frac{3}{25}$
4.5	$\frac{2}{25}$
5	$\frac{1}{25}$

c **i** $E(\bar{X}) = 3$ **ii** $\text{Var}(\bar{X}) = 1$
iii $\text{SD}(\bar{X}) = 1$

d $\text{SE}(\bar{x}) = \sqrt{2} \div \sqrt{2} = 1$

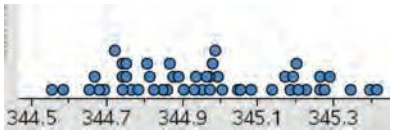
10 A **11** C **12** B **13** E
14 D **15** A

16 a population mean = 30
b population standard deviation = 4

17 a 3.5 kg **b** 0.5 kg

9780170448338

18 a Answers may vary slightly depending on the data generated. Example result:



b Mean of sample means = 350

Standard deviation of the sample means = 0.26

CUMULATIVE EXAMINATION 1

1 a 2.5 **b** 2.45

2 a

b	0	1	2
$\Pr(B = b)$	$\frac{9}{16}$	$\frac{3}{8}$	$\frac{1}{16}$

b $\frac{7}{16}$

3 centre = $(-3, 1)$, radius = 6

4 205

CUMULATIVE EXAMINATION 2

Section A

1 B **2** A **3** D **4** E **5** B

Section B

1 a mean = 0.1495, standard deviation = 0.0024

b 0.0024

c **i** 0.5 **ii** $\frac{11}{17}$ **iii** 0.1483

2 a $\sqrt{14}$

b $\frac{\sqrt{14}}{14}\underset{\sim}{\mathrm{i}} - \frac{\sqrt{14}}{7}\underset{\sim}{\mathrm{j}} + \frac{3\sqrt{14}}{14}\underset{\sim}{\mathrm{k}}$

c $-\frac{13}{14}\underset{\sim}{\mathrm{i}} + \frac{13}{7}\underset{\sim}{\mathrm{j}} - \frac{39}{14}\underset{\sim}{\mathrm{k}}$

d **i** $3m + n = 1$ and $5m - 2n = 0$

ii $m = \frac{2}{11}, n = \frac{5}{11}$ **iii** $d = 1$

CHAPTER 12

EXERCISE 12.1

1 a $A'(1, 4)\ B'(3, -4)\ C'(3, -5)$

b $X'(2, 2)\ Y'(5, 8)\ Z'(7, 3)$

c $P'(0, -6)\ Q'(6, -3)\ R'(4, 0)\ S'(-2, -3)$

d $D'(-5, 7)\ E'(-4, 9)\ F'(-3, 7)\ G'(-1, 8)\ H'(-2, 1)$

e $K'(3, 4)\ L'(6, 5)\ M'(2, 10)$

2 a $A'(6, -5)\ B'(2, -5)\ C'(2, 3)$; triangle

b $A'(17, -13)\ B'(6, 0)\ C'(-10, 11)\ D'(1, -2)$; parallelogram

c $A'(15, -20)\ B'(10, -10)\ C'(30, 0)\ D'(35, -10)$; square

3 a $\begin{bmatrix} 2 & -4 \\ 4 & 3 \end{bmatrix}$, $A'(-14, 5)\ B'(-16, 23)\ C'(18, 3)$; another triangle

b $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$, $A'(3, -3)\ B'(8, -8)\ C'(1, -1)$; a straight line through the origin

c $\begin{bmatrix} -2 & 3 \\ 4 & 3 \end{bmatrix}$, $A'(-5, -17)\ B'(-1, -7)\ C'(12, -6)$ $D'(8, -16)$; another parallelogram

4 a $A'(-5, -4)\ B'(-2, -8)\ C'(2, -5)$

b $X'(3, 8)\ Y'(1, 9)\ Z'(4, 2)$

c $P'(4, -8)\ Q'(0, -8)\ R'(0, 1)\ S'(4, 1)$

d $D'(-6, 7)\ E'(-1, 7)\ F'(0, 3)\ G'(-4, 1)\ H'(-7, 4)$

e $K'(0, 0)\ L'(2, -8)\ M'(2, -9)$

5 a $(-1, -12)$ **b** $(0, 9)$ **c** $(8, 8)$ **d** $(-7, 9)$ **e** $(-4, 8)$

6 a $\begin{bmatrix} 3 & 0 \\ 0 & 0.5 \end{bmatrix}$ **b** $\begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$

c $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ **d** $\begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix}$

e $\begin{bmatrix} 1 & 3 \\ -4 & 1 \end{bmatrix}$

7 a $-x + 7y - 12 = 0$

b $y = \frac{4}{45}x^2 - \frac{8}{15}x$

c $(x + 4)^2 - (y + 4)^2 = 8$

d $(y + 3)^2 - (x - 1)^2 = 4$

e $25y^2 - 16xy + 17x^2 = 3249$

8 a $P'(10, 11)\ Q'(7, 8)$

b $A'(-18, 12)\ B'(-17, 19)\ C'(14, -10)$

c $X'(3, -1)\ Y'(9, 8)\ Z'(6, -2)$

9 a $A'(-20, 8)\ B'(-10, 4)\ C'(6, 14)\ D'(-4, 18)$; a parallelogram

b $A'(-5, 1.6)\ B'(7, 1.6)\ C'(1, -1.2)$; another isosceles triangle

c $A(-4, -0.5)$, $B(-6, 0.5)$, $C(0, 1.5)$, $D(6, 0.5)$, $E(4, -105)$; another pentagon

10 $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, $y = 9x - 12$ **11** $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $y = \frac{x^3}{8}$

12 $A'\left(\frac{3 - 6\sqrt{3}}{2}, \frac{6 + 3\sqrt{3}}{2}\right)\ B'\left(\frac{5 - 2\sqrt{3}}{2}, \frac{2 + 5\sqrt{3}}{2}\right)$

13 a $A'\left(-\frac{3\sqrt{2}}{2}, \frac{9\sqrt{2}}{2}\right)\ B'\left(\frac{3\sqrt{2}}{2}, \frac{7\sqrt{2}}{2}\right)$

b $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$, $y = \frac{-5\sqrt{3} - 11}{11}x - \frac{2(2\sqrt{3} + 1)}{11}$

c $A'(-3\sqrt{2}, -\sqrt{2})\ B'(-4\sqrt{2}, 3\sqrt{2})\ C'(-\sqrt{2}, 4\sqrt{2})$; another triangle

d $\begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$, $A'(1-\sqrt{3}, -1-\sqrt{3})$

$B'\left(\frac{5-6\sqrt{3}}{2}, \frac{-6-5\sqrt{3}}{2}\right)$ $C'\left(\frac{1-9\sqrt{3}}{2}, \frac{-9-\sqrt{3}}{2}\right)$

$D'\left(\frac{-2-5\sqrt{3}}{2}, \frac{-5+2\sqrt{3}}{2}\right)$; another square

14 a $\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$, $A'\left(\frac{-4+5\sqrt{3}}{2}, \frac{5+4\sqrt{3}}{2}\right)$

$B'\left(\frac{-7+9\sqrt{3}}{2}, \frac{9+7\sqrt{3}}{2}\right)$ $C'\left(\frac{-11+6\sqrt{3}}{2}, \frac{6+11\sqrt{3}}{2}\right)$

$D'\left(\frac{-7+\sqrt{3}}{2}, \frac{1+7\sqrt{3}}{2}\right)$

b $A'(1,-7)$ $B'(-4,3)$ $C'\left(-\frac{22}{5}, -\frac{21}{5}\right)$

c $\begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}$, $y = \frac{1}{5}x + \frac{5}{2}$

15 B **16** A **17** E **18** C **19** C

EXERCISE 12.2

1 D **2** D

3 a A dilation of factor $\frac{1}{3}$ from the y-axis

b A dilation of factor 2 parallel to the x-axis

c A dilation by the factor 4 parallel to the y-axis

d A dilation by the factor $\frac{1}{2}$ from the x-axis

e A rotation of $-50°$ about the origin

f A rotation of $\frac{2\pi}{3}$ about the origin

g A reflection in a line through the origin with inclination 75°

h A reflection in a line though the origin with inclination $-\frac{\pi}{4}$

4 a $(x,y) \to (0.5x + 0.5y, 0.5x - 0.5y)$

b $(x,y) \to (-0.5x + 0.5y, 0.75x - 0.25y)$

c $(x,y) \to (3x - 2y, 4x - 3y)$

d T^{-1}: $(x,y) \to \left(\frac{1}{6}x + \frac{1}{12}y, -\frac{1}{4}x + \frac{1}{8}y\right)$

e T^{-1}: $(x,y) \to \left(\frac{1}{2}x - \frac{1}{2}y, \frac{1}{4}x + \frac{1}{4}y\right)$

5 a $\begin{bmatrix} 0.3 & 0.4 \\ 0.1 & -0.2 \end{bmatrix}$, $(x,y) \to (0.3x + 0.3y, 0.1x - 0.2y)$

b $\frac{1}{7}\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$, $(x,y) \to \left(\frac{2}{7}x + \frac{1}{7}y, -\frac{1}{7}x + \frac{3}{7}y\right)$

c $\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$, $(x,y) \to \left(\frac{1}{3}x, y\right)$

d $\begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$,

$(x,y) \to \left(\frac{\sqrt{2}}{2}y - \frac{\sqrt{2}}{2}x, -\frac{\sqrt{2}}{2}x - \frac{\sqrt{2}}{2}y\right)$

e $\begin{bmatrix} -\frac{15}{17} & \frac{8}{17} \\ \frac{8}{17} & \frac{15}{17} \end{bmatrix}$, $(x,y) \to \left(\frac{15}{17}x + \frac{8}{17}y, \frac{8}{17}x + \frac{15}{17}y\right)$

6 C **7** B **8** E **9** A

10 a $|T| = 0$, so there is no inverse.

b $\begin{bmatrix} 6 & -8 \\ 9 & -12 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 6 & -2 & -8 \\ 0 & 9 & -3 & -12 \end{bmatrix}$,

all points on $y = \frac{3}{2}x$, so the image of the plane is the line $y = \frac{3}{2}x$.

EXERCISE 12.3

1 B **2** C

3 a $(7,6)$ **b** $(-28,-21)$ **c** $(28,21)$

d $(17,9)$ **e** $(-1,2)$ **f** $(15,23)$

g $(-10,-5)$ **h** $(-4,-15)$ **i** $(21,-7)$

j $(59,99)$

4 a $(x,y) \to (4x + 5y, x)$

b $(x,y) \to (5x, 13x - y)$

c $(x,y) \to (3x + 4y - 3, 2x + 2y + 4)$

d $(x,y) \to (3x + 4y + 7, 2x + 2y + 2)$

e $(x,y) \to (5x + 6y, 12x + 14y)$

f $(x,y) \to (11x + 15y, 6x + 8y)$

g $(x,y) \to (18x - y, 10x)$

h $(x,y) \to (8x + 10y, 7x + 10y)$

i $(x,y) \to (2x + y - 3, 3x - y + 4)$

j $(x,y) \to (x + y - 3, 2x + 3y + 4)$

5 a $\begin{bmatrix} 1.5 & 1.5 \\ 1 & -2 \end{bmatrix}$ **b** $\begin{bmatrix} \frac{\sqrt{3}-1}{2} & \frac{\sqrt{3}+1}{2} \\ \frac{-2\sqrt{3}-1}{2} & \frac{\sqrt{3}-2}{2} \end{bmatrix}$

c $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ **d** $\begin{bmatrix} 1.5 & 1 \\ 1.5 & -2 \end{bmatrix}$

e $\begin{bmatrix} \frac{\sqrt{3}-1}{2} & -\frac{2\sqrt{3}+1}{2} \\ -\frac{\sqrt{3}+1}{2} & \frac{2-\sqrt{3}}{2} \end{bmatrix}$

f $\begin{bmatrix} -\frac{\sqrt{3}+1}{2} & \frac{\sqrt{3}-1}{2} \\ \frac{2\sqrt{3}-1}{2} & \frac{\sqrt{3}+2}{2} \end{bmatrix}$

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g $\begin{bmatrix} \frac{9}{8} & -\frac{\sqrt{3}}{8} \\ \frac{\sqrt{3}}{8} & -\frac{11}{8} \end{bmatrix}$ **h** $\begin{bmatrix} \frac{3}{2} & 0 \\ 0 & -1 \end{bmatrix}$

i $\begin{bmatrix} \frac{3}{2} & 0 \\ 0 & -1 \end{bmatrix}$ **j** $\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$

6 a $(4,6)$

b $\left(\frac{15-5\sqrt{3}}{2}, -\frac{10+5\sqrt{3}}{2}\right)$ **c** $\left(5\sqrt{3}+4, \frac{15-12\sqrt{3}}{2}\right)$

d $(x,y) \to \left(\frac{2-\sqrt{3}}{2}x - \frac{\sqrt{3}}{2}y, -\frac{2\sqrt{3}+1}{2}x - \frac{1}{2}y\right)$

e $(x,y) \to$
$\left((3+\sqrt{3})x - \frac{3+6\sqrt{3}}{2}y, (1-3\sqrt{3})x + \frac{3\sqrt{3}-6}{2}y\right)$

f $(x,y) \to (10x-3y, -6y)$

g $(x,y) \to (3x-y, 2y-x)$

h $(x,y) \to$
$\left(\frac{9+2\sqrt{3}}{4}x - \frac{2+\sqrt{3}}{4}y, -\frac{2+\sqrt{3}}{4}x + \frac{11-2\sqrt{3}}{4}y\right)$

7 C **8** A **9** A **10** E

11 Show the composition:

$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \cos(60°) & -\sin(60°) \\ \sin(60°) & \cos(60°) \end{bmatrix}$$

EXERCISE 12.4

1 D **2** D

3 Proof: see worked solutions; use $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

4 33 square units

5 Show $T \circ T = I$. See worked solutions.

6 Use the matrix representation of $T \circ T = I$.
See worked solutions.

7 Use, say, $\Delta ABC = A(0,0)\ B(0,b)\ C(c, 0.5b)$.
See worked solutions.

8 Show $T \circ T = I$. See worked solutions.

9 D **10** C **11** D

12 Proof: see worked solutions; use contradiction.

EXERCISE 12.5

1 C **2** D

3 Proof: see worked solutions; show that each composition and its reverse are different.

4 Proof: see worked solutions; show that each composition and its reverse are equal.

5 a Rotation of 30° about the origin.

b Rotation of $\pi - \tan^{-1}\left(\frac{77}{52}\right)$ about the origin.

6 Equate T_{12} of $R_x \circ R_{-y}$ and R_{x-y}.

7 a Show that for, say, $D_1 = \begin{bmatrix} x_1 & 0 \\ 0 & y_1 \end{bmatrix}$ and
$D_2 = \begin{bmatrix} x_2 & 0 \\ 0 & y_2 \end{bmatrix}, D_1 \circ D_2 = D_2 \circ D_1$

b Show $D_1 \circ (D_2 \circ D_3) = D_1 \circ (D_3 \circ D_2)$
$= D_2 \circ (D_1 \circ D_3) = \ldots$

8 rotation of $60° - \left(\tan^{-1}\left(\frac{1}{2}\right)\right)$ about the origin

9 D **10** C **11** A

12 Equate T_{11} of $R_x \circ R_y$ and R_{x+y}.

CUMULATIVE EXAMINATION 1

1 a $z_1 z_2 = 6\,\text{cis}\left(\frac{\pi}{2}\right)$ **b** $\frac{\overline{z_1}}{z_2} = \frac{2}{3}\,\text{cis}\left(-\frac{\pi}{2}\right)$

2 a $\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}} = \frac{2-2\sqrt{3}}{\sqrt{2}}$

b $(\underset{\sim}{a} \cdot \hat{\underset{\sim}{b}})\hat{\underset{\sim}{b}} = (\sqrt{3}-1)(\underset{\sim}{i} + \underset{\sim}{j})$

3 $\begin{bmatrix} 9 \\ -12 \end{bmatrix}$

4 a $T = \begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix}$

b $A'(23,1)\ B'(44,-6)\ C'(30,-20)$, another triangle

c 196 square units

CUMULATIVE EXAMINATION 2

Section A

1 D **2** A **3** D **4** B

5 E

Section B

1 a $b = l\sqrt{2}\,\text{cis}\left(\theta + \frac{\pi}{4}\right), c = l\,cis\left(\theta + \frac{\pi}{2}\right)$

b $p = \frac{\sqrt{5}}{2} l\,\text{cis}\left(\theta + \frac{\pi}{2} - \tan^{-1}(0.5)\right)$

c $\frac{b}{a} = \frac{l\sqrt{2}\,\text{cis}\left(\theta + \frac{\pi}{4}\right)}{l\,\text{cis}(\theta)}$
$= \sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)$
$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$
$= 1 + i$

2 a Proof: see worked solutions; use matrix representations.

b parallelogram

c parallelogram

d invariance of parallel lines under dilation and reflection

Glossary and index

absolute value (or **modulus**) The unsigned magnitude of a real number, or the distance of the number from 0 on the number line. For example, the absolute value of −7, written as |−7|, is 7. Also, |2| = 2. Absolute value can be defined by the piecewise function:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

or by the formula $|x| = \sqrt{x^2}$. (p. 354)

acceleration The rate of change of velocity of a moving object, represented by the function $a = \frac{dv}{dt}$, where v is the velocity ('signed speed'). (p. 442)

additive inverse (matrix) The negative of a matrix. The additive inverse of a matrix A is $-A$, whose element values are the negative of the corresponding element values of A, because $A + (-A)$ gives the zero matrix. (p. 198)

adjacency matrix A square matrix that shows how many edges connect each pair of vertices in a graph. (p. 92)

adjacent vertices Two vertices that are connected by one or more edges. (p. 91)

algorithm A precise list of instructions for completing a procedure. (p. 175)

See also **flowchart** and **pseudocode**.

AND A logical conjunction joining premises. For premises A, B, A AND B is true only when both A and B are true. (p. 131)

angle of depression The angle made between the horizontal and a direction below the horizontal. (p. 279)

angle of elevation The angle made between the horizontal and a direction above the horizontal. (p. 279)

annuity A type of investment where payments into the investment are made at regular intervals and interest is compounded; for example, a superannuation fund. (p. 40)

arc Part of the circumference of a circle. (p. 265)

See **sector** for diagram.

Argand diagram A number plane for representing complex numbers $x + yi$. The x-axis represents the real part of the number while the y-axis represents the imaginary part of the number. (p. 407)

argument The angle that the vector of a complex number makes with the positive direction of the x-axis, written $\theta = \arg(z)$. The **principal argument** is the value of θ in the interval $(-\pi, \pi]$. (p. 131)

arithmetic sequence (or **arithmetic progression [AP]**) A number sequence that increases or decreases by a constant amount, d, called the common difference. (p. 10)

atomic sentence A statement that cannot be broken down into fewer sentences. (p. 131)

bearing An angle that shows the direction of one point from another point. (p. 279)

biconditional statement A statement involving 'if and only if' conditions. $X \leftrightarrow Y$ means 'if X, then Y' and 'if Y, then X'. (p. 131)

binary number A number containing the digits 0 and 1 only, using base 2. Each digit in a binary number is called a **bit**, for **binary digit**. (p. 141)

bipartite graph A graph whose set of vertices can be split into 2 distinct groups, where vertices from one group connect only to vertices in the other group. (p. 99)

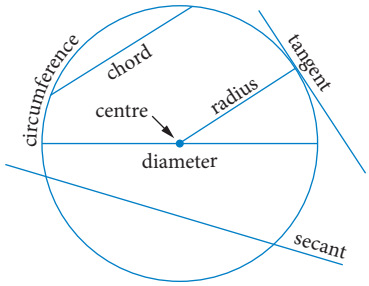
Boolean algebra A branch of algebra where all variables are true or false, represented by 1 and 0. (p. 141)

cardinality The number of elements in a set. (p. 58)

Cartesian form The conventional form of the equation of a graph in which y is written as a function of x. (p. 373)

See also **parametric form** and **polar form**.

chord A line segment that joins 2 points on a circle or curve. (p. 265)



circuit A walk with no repeated edges that starts and finishes at the same vertex. (p. 115)

circumference The perimeter (distance) around a circle. (p. 265)

closure A property of a set of numbers under a specific operation. For example, the set of rational numbers, $\boldsymbol{Q}$, is closed under addition because the sum of 2 rational numbers is another rational number. The set of natural numbers, $\boldsymbol{N}$, is not closed under subtraction because the difference between 2 natural numbers is not necessarily another natural number; for example, 3 − 7 = −4. (p. 56)

co-domain The set of values that the range of a function $f: [0, 7] \to R$, $g(x) = x^2 + 4$ has domain $[0, 7]$, co-domain R and range $[4, \infty)$. (p. 317)

cofactor (of a matrix) The signed determinant of the **minor** M_{ij} of a square matrix, $A_{ij} = (-1)^{i+j}|M_{ij}|$, used to calculate the determinant of a 3 × 3 matrix. (p. 213)

column matrix A column that has only one column. (p. 196)

combination An unordered selection from a group of items, for example, choose 5 players for a basketball team from a group of 12 players. ${}^nC_r$ represents the number of possible combinations of r items chosen from n items.

$${}^nC_r = \frac{n!}{(n-r)!r!}$$ (p. 243)

common difference The constant difference between consecutive terms in an arithmetic sequence. (p. 10)

common ratio The constant quotient or ratio between consecutive terms in a geometric sequence. (p. 18)

complement (of a set) All elements in the universal set other than the elements in the given set. (pp. 59, 252)

complement (or **inverse**) of a graph A graph that has the same vertices as the original graph and all of the edges for the complete graph that are *not* present in the original graph. (p. 99)

complete graph A graph that has every vertex connected to every other vertex. (p. 99)

complex conjugate The complex conjugate of the complex number $x + yi$ is $x - yi$, the same expression with the '+' changed to a '–' (or vice-versa). (p. 397)

complex numbers (set C) Numbers of the form $x + yi$, where x and y are real numbers and i is an imaginary number, where $i = \sqrt{-1}$ includes all real and imaginary numbers. (p. 53)

complex plane *See* **Argand diagram**.

compound angle formulas *See* **sum and difference identities**.

compound event An event consisting of 2 or more simple events that occur in succession, such as tossing a coin 3 times. (p. 235)

compound proposition Two or more atomic propositions connected by logical connectives. (p. 131)

conclusion A proposition that can be logically drawn from the premise. (p. 131)

conditional statement A statement involving an 'if … then' condition. $C \rightarrow D$ means 'if C, then D'. (p. 137)

conformable matrices Two matrices where the number of columns in one is the same as the number of rows in the other, which means that the first matrix can be multiplied by the second matrix. (p. 199)

conic section A shape made when a cone is cut by a plane: circle, ellipse, parabola, hyperbola. (p. 360)

conjecture A statement that seems to be true but has not been proved conclusively. Once it is proved, it becomes a theorem. (p. 70)

conjugate *See* **complex conjugate**.

connected graph A graph where every vertex in the graph can be reached from every other vertex in the graph. (p. 99)

contradiction A proposition that is always false. The opposite of a **tautology**. (p. 72)

See also **proof by contradiction**.

contrapositive The contrapositive of a statement is the reverse negative statement. The contrapositive of *if P then Q* is *if not P then not Q*. (p. 73)

See also **proof by contrapositive**.

converse The converse of a statement is the statement in the reverse direction. The converse of 'If a triangle is right-angled, then the square of the hypotenuse is the sum of the squares of the other 2 sides' is 'If the square of the longest side of a triangle is the sum of the squares of the other 2 sides, then the triangle is right-angled'. In this case, the converse (for Pythagoras' theorem) is also true, but the converse of a true statement is not necessarily true.

cosecant Abbreviation: cosec. The reciprocal of sine.

$$\operatorname{cosec}(\theta) = \frac{1}{\sin(\theta)}$$ (p. 338)

cosine rule In any triangle ABC, $a^2 = b^2 + c^2 - 2bc\cos(A)$. (p. 275)

cotangent Abbreviation: cot. The reciprocal of tangent.

$$\cot(\theta) = \frac{1}{\tan(\theta)}$$ (p. 338)

counterexample An example that is used to disprove a conjecture. (p. 70)

See also **proof by counterexample**.

cycle A walk with no repeated vertices that starts and finishes at the same vertex. (p. 111)

degree (order) of a vertex The number of edges connected to a vertex. (p. 91)

De Morgan's laws The rules $\leftarrow (P \vee Q) \equiv \leftarrow P \wedge \leftarrow Q$ and $\leftarrow (P \wedge Q) \equiv \leftarrow P \vee \leftarrow Q$. (p. 150)

determinant The value $ad - bc$ for the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. (p. 209)

diagonal matrix A square matrix where the only non-zero elements are in the leading diagonal. (p. 209)

diameter The distance from one side of a circle to the other passing through the centre. (p. 265)

dilation The stretching or squashing (compressing) of a graph, either vertically or horizontally. (pp. 319, 533)

dimensions of a matrix The number of rows and columns in a matrix; for example, $\begin{bmatrix} 2 & -4 & 60 & 1 \\ 5 & 2.8 & -70 & 0 \end{bmatrix}$ has dimensions 2×4. (p. 196)

direction (of a vector) The angle a vector makes with the positive direction of the x-axis. (p. 439)

directrix (plural: directrices) The reference line for the locus of a parabola, ellipse and hyperbola. (p. 361)

discrete random variable A random variable that can only take a countable number of values; for example, the number of people visiting a website. (p. 482)

discriminant $\Delta = b^2 - 4ac$, the part of the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ that gives information about the number and type of roots of the quadratic equation, $ax^2 + bx + c = 0$. (p. 402)

displacement The change of position of an object from its start position to its end position (its 'signed distance'), represented by the function $x(t)$, where displacement is a function of time. (p. 465)

displacement vector The vector from one point to another. The change from position A to position B is written as $\overrightarrow{AB}$. (p. 465)

domain The set of values of the independent variable x in a relation or function. (p. 317)

eccentricity, e A value that determines a type of conic section: $0 < e < 1$ for an ellipse, $e = 1$ for a parabola, $e > 1$ for a hyperbola. (p. 363)

edge The lines connecting the vertices of a graph. (p. 91)

element One unit of a set. 'John is an element of the set of boys in the class.' (p. 52)

element (of a matrix) A value in a matrix. (p. 195)

ellipse An oval or egg shape that is one of the conic sections. (p. 363)

Eulerian circuit A walk that includes *every edge* once only and starts and finishes at the *same* vertex. (p. 116)

Eulerian trail A walk that includes *every edge* once only, and starts and finishes at *different* vertices. (p. 116)

Euler's formula A formula that is true for planar graphs: vertices + faces − edges = 2, or $v + f - e = 2$. (p. 107)

even number A number exactly divisible by 2: $\{0, \pm2, \pm4, \pm6 \ldots\}$, with each element, n, given by $n = 2k, k \in Z$. (p. 67)

face A region of a graph that is enclosed by 3 or more edges, or the region outside the graph. (p. 106)

factorial symbol: $n!$ for 'n factorial.' The product of consecutive whole numbers from 1 to n. For example, $4! = 4 \times 3 \times 2 \times 1$. 0! is defined to be equal to 1.

$$n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$$

$n!$ is the number of arrangements of n different objects. (p. 238)

Fibonacci numbers A sequence of numbers beginning with 1, 1, where each new term is the sum of the previous 2 terms; that is, 1, 1, 2, 3, 5, 8, 13, 21 … (p. 38)

flowchart A diagram that shows pathways of instructions for completing a procedure to describe an algorithm. (p. 175) *See also* **algorithm** and **pseudocode**.

focus (Plural: foci) The reference point for the locus of a parabola, ellipse and hyperbola. (p. 361)

function A relation such that each value in the domain has only one ordered pair, each x value matches with only one y value. (p. 317)

fundamental theorem of arithmetic Every integer greater than 1 is either prime or can be written as a unique product of **prime numbers**. (p. 66)

general term The nth term of a sequence, represented by t_n. (p. 11)

geometric sequence (or **geometric progression [GP]**) A number sequence that increases or decreases by a constant multiple, r, called the common ratio. (p. 18)

graph (network diagram) A diagram consisting of vertices that are connected by edges. (p. 91)

Hamiltonian cycle A walk that includes *every vertex* in a graph once only, and starts and finishes at the *same* vertex. (p. 117)

Hamiltonian path A walk that includes *every vertex* in a graph once only, and starts and finishes at *different* vertices. (p. 117)

handshaking lemma The sum of the degrees of the vertices of a graph is equal to twice the number of edges. (p. 95)

hyperbola A graph with 2 separate branches and asymptotes that is a type of conic section. (p. 331)

identity matrix A square matrix where all the elements in the leading diagonal are '1' and the other elements are '0'. (p. 204)

See **multiplicative identity**.

imaginary axis The vertical axis in the complex plane. (p. 407)

imaginary numbers Numbers of the form yi, where i is the imaginary number, $i = \sqrt{-1}$. (p. 53)

Im (z) The real coefficient of the imaginary part of the complex number, $x + yi$. $\text{Im}(z) = y$. (p. 393)

inclusion-exclusion principle For 2 sets, the number of elements in the union of set A and set B is equal to the sum of the elements in A and the elements in B, minus the number of elements in the intersection of A and B:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

For 3 sets,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \quad \text{(p. 253)}$$

infinite face The region outside the graph; a face of a graph that is unbounded. (p. 106)

integers (set Z) The positive and negative whole numbers and zero $\{\ldots -3, -2, -1, 0, 1, 2, 3 \ldots\}$. (p. 52)

intersection The elements common to 2 or more sets. (p. 59)

invariant A relation or property that doesn't change under a transformation is invariant. (p. 549)

inverse (of a matrix) The matrix that, when multiplied by another matrix, results in the identity matrix. (p. 208)

inverse circular functions The functions that are the inverse of the sine, cosine and tangent functions, namely the **inverse sine**, **inverse cosine** and **inverse tangent** functions respectively. (p. 348) *See* below.

inverse cosine (or **arccosine**) Abbreviation: $\cos^{-1}$. The inverse function to the cosine function. $y = \arccos(x)$ if and only if $\cos(y) = x$ and $0 \leq y \leq \pi$. (p. 348)

inverse sine (or **arcsine**) Abbreviation: $\sin^{-1}$. The inverse function to the sine function. $y = \arcsin(x)$ if and only if $\sin(y) = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. (p. 348)

inverse tangent (or **arctangent**) Abbreviation: $\tan^{-1}$. The inverse function to the tangent function. $y = \arctan(x)$ if and only if $\tan(y) = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. (p. 348)

irrational numbers Numbers that cannot be written in fraction form $\frac{p}{q}$ as the ratio of 2 integers, p and q. Includes all surds and special constants like π. (p. 53)

isolated vertex A vertex that is not connected to any other vertex in a graph. (p. 91)

isomorphic graphs Graphs that have the same number of vertices connected by the same number of edges, so they are identical but may be drawn differently. (p. 100)

Karnaugh map (or **K-map**) A table for minimising a Boolean function having many variables. (p. 159)

Truth table

A	B	Product $A + B$
0	0	0
0	1	1
1	0	1
1	1	1

K-map

A \ B	0	1
0	0	1
1	1	1

limiting sum (of an infinite geometric series) The value that the sum of the series approaches, when the common ratio of the series is between –1 and 1. (p. 26)

locus A set of points that satisfies a given condition; for example, equidistant from a point and a line. (p. 358)

logic gate A Boolean function defined by its truth table. (p. 167)



AND NAND (NOT AND)

loop An edge that connects a vertex to itself. (p. 91)

magnitude (of a vector) The length of a vector, found using the distance formula $d = \sqrt{(x_2 + x_1^2) + (y_2 + y_1^2)}$. (p. 439)

major arc, **sector** or **segment** An arc, sector or segment that covers more than half of a circle. (p. 265)

major axis The longer of the 2 axes of an ellipse. (p. 363)

mathematical induction *See* **proof by mathematical induction**.

matrix A rectangular arrangement of numbers organised into rows and columns, usually presented in square brackets. (p. 196)

maximal domain (**natural** or **implied domain**) The largest set of real numbers for which a function rule is defined. (p. 317)

minor (of a square matrix) Denoted by M_{ij}, this is the $n \times n$ matrix with the ith column and jth row removed, which is a smaller square matrix. (p. 213)

minor arc, **sector** or **segment** An arc, sector or segment that covers less than half of a circle. (p. 265)

minor axis The shorter of the 2 axes of an ellipse. (p. 363)

modulus The size of a complex number, represented on an Argand diagram by its distance from the origin, written as $|z| = \sqrt{x^2 + y^2}$. (p. 407)

See **absolute value**.

modulus-argument form *See* **polar form**.

multiple edges Edges that connect the same 2 vertices. (p. 91)

multiplicative identity A square matrix I of dimension n that has 1s along its leading diagonal and 0s elsewhere; for example, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

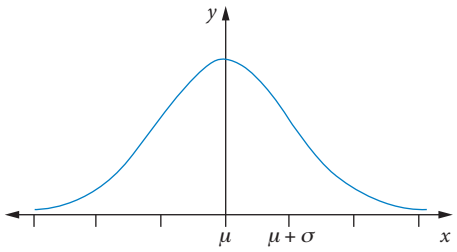
When any square matrix of dimension n is multiplied by I, its values do not change. (p. 204)

multiplicative inverse (of a matrix) The inverse of a square matrix. The multiplicative inverse of a matrix A is A^{-1}, where $A \times A^{-1}$ and $A^{-1} \times A$ both give the identity matrix, I. (p. 208)

See also **multiplicative identity**.

natural numbers (set *N*) The counting numbers or positive whole numbers {1, 2, 3, 4 …}. (p. 52)

normal distribution A continuous probability distribution whose graph is a bell-shaped curve. People's heights or reaction times approximate a normal distribution. (p. 508)



NOT The negation of a statement. (p. 131)

odd number A number that leaves a remainder of 1 when divided by 2: {±1, ±3, ±5 …}, with each element, n, given by $n = 2k + 1$, $k \in Z$. (p. 67)

OR A logical conjunction joining premises. For premises A, B, A OR B is true when either or both A and B are true. (p. 131)

parameter (in algebra) An independent variable such as t or θ that is used in the parametric form of the equation of a graph, upon which the variables x and y are defined. (p. 372)

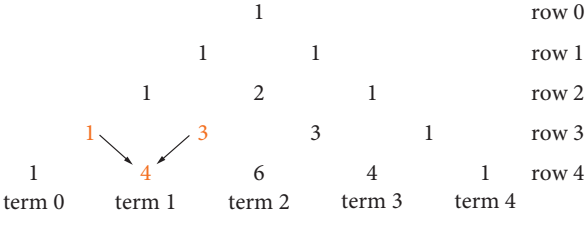
parameter (in statistics) A characteristic value of a particular population, such as the mean. (p. 498)

parametric form A form of the equation of a graph in which there are separate equations for x and y in terms of an independent variable such as t or θ, called the parameter. (p. 372)

partial fraction One of the fractions when a rational function is written as the sum or difference of fractions; for example, in $\frac{5x+1}{(x-1)(x+2)} = \frac{2}{x-1} + \frac{3}{x+2}$ the partial fractions are $\frac{2}{x-1}$ and $\frac{3}{x+2}$. (p. 324)

Pascal's identity The rule about ${}^nC_k$ and the combinations shown in Pascal's triangle: that each number is the sum of the 2 numbers in the row above it. Written algebraically as ${}^nC_k = {}^{n-1}C_{k-1} + {}^{n-1}C_k$. (p. 246)

Pascal's triangle A triangular pattern of numbers where the values in each row are the sums of the 2 numbers in the row above it. (p. 244)



path A walk where there are no repeated vertices. (p. 115)

permutation An ordered arrangement of a group of items; for example, the first 3 placegetters in a running race. ${}^nP_r$ represents the number of possible arrangements of r items chosen from n items. ${}^nP_r = \frac{n!}{(n-r)}$. (p. 237)

pigeon-hole principle If $n + 1$ objects are placed into n holes, at least 1 hole must contain 2 objects. (p. 249)

planar graph A connected graph that can be drawn without any edges crossing over. (p. 100)

plane drawing A graph that has been redrawn so the edges do not cross. (p. 105)

polar form (or **modulus-argument form** or **trigonometric form**) A form of a complex number expressed in terms of its modulus r, the length of its vector in the Argand diagram, and its principal argument θ, the angle the vector makes with the positive direction of the x-axis. (p. 372)

population The whole group from which data could be collected for a statistical study; for example, all high school students if high school students were being surveyed. (p. 495)

premise A statement from which a conclusion can be drawn. (p. 131)

principal argument *See* **argument**.

proof by contradiction A proof that first assumes that the conjecture is false, then reaches a logical conclusion that is impossible and contradicts the original assumption. (p. 72)

proof by contrapositive A form of proof where to prove $P \rightarrow Q$ involves showing $\neg Q \rightarrow \neg P$. (p. 73)

proof by counterexample When a conjecture is proved false by finding one counterexample. An example that contradicts the conjecture shows that it cannot be true. (p. 70)

proof by mathematical induction A form of mathematical proof that involves proving that if a conjecture is true for some integer $n = k$, then it will also be true for the next consecutive number, $n = k + 1$. (p. 76)

propositional constant Letters or symbols representing fixed truth values. (p. 131)

propositional variable A variable that can be either true or false. (p. 131)

pseudocode Specific language containing command words used to describe an algorithm, resembling computer programming code. (p. 175)

See also **algorithm** and **flowchart**.

Pythagorean identities *See* **trigonometric identities**.

quadratic formula The formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ that gives the solutions of the quadratic equation $ax^2 + bx + c = 0$. (p. 402)

quotient function A rational function whose numerator is of degree equal to or higher than the degree of the denominator; for example, $\frac{3x^3 - x^2 + 4x - 1}{9x^2 + 2}$. (p. 322)

radius The distance from the centre of a circle to its circumference. (p. 265)

random variable A variable whose possible values are the numerical result of some random process. (p. 482)

range A measure of the spread of data; the difference between the largest and smallest observations. (p. 317)

rational function A function of the form $f(x) = \frac{p(x)}{q(x)}$, where p and q are polynomials and q is not the zero polynomial. (p. 322)

rational numbers (set Q) Numbers that can be written in fraction form $\frac{p}{q}$, as the ratio of 2 integers, p and q, with $q \neq 0$. Includes all integers and decimals. (p. 52)

ray 'Half' a line, extending indefinitely in only one direction. The starting point appears first in the name, like $\overrightarrow{FG}$. (p. 421)



real axis The horizontal axis in the complex plane. (p. 407)

real numbers (set R) The rational and irrational numbers combined. (p. 53)

realise the denominator To convert a complex expression of the form $\frac{z}{w}$, so that its denominator is real, by multiplying by $\frac{\overline{w}}{\overline{w}}$, where $\overline{w}$ is the conjugate of the denominator. (p. 397)

reciprocal circular functions The functions that are the reciprocals of the sine, cosine and tangent functions, namely the cosecant, secant and cotangent functions respectively. $\text{cosec}(\theta) = \frac{1}{\sin(\theta)}$, $\sec(\theta) = \frac{1}{\cos(\theta)}$, $\cot(\theta) = \frac{1}{\tan(\theta)}$. (p. 285)

recurrence relation (or **recursive formula**) A formula that defines a term in a sequence or series as a rule involving the previous term; for example, $t_n = 3t_{n-1}$. (p. 4)

reflection Mirror-image or 'flipping' of a graph so that it is back-to-front or bottom-to-top. (pp. 319, 536)

regular graph A graph whose vertices all have the same degree. If every vertex has degree r, then the graph is regular of degree r. (p. 100)

relation A set of ordered pairs (x, y) that describe a relationship between the variables x and y. (p. 317)

resultant vector The vector that is the sum of 2 or more vectors. (p. 446)

Re (z) The real part of the complex number $x + yi$. $\text{Re}(z) = x$. (p. 393)

rotation The spinning of a graph about a point by a given angle size. (p. 534)

row matrix A matrix that has just one row. (p. 196)

sample A part of a population selected for statistical study. (p. 495)

sample space The set of all possible outcomes in a probability experiment. (p. 252)

sampling distribution The probability distribution of the statistic such as the mean, proportion or standard deviation. (p. 507)

scalar A quantity that has magnitude (size) but not direction. In comparison, a **vector** has both magnitude and direction. (p. 439)

scalar product (or **dot product**) The product of 2 vectors that gives a real number, not a vector.

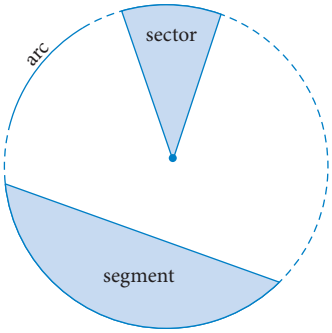
$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos(\theta)$, where θ is the angle between $\underset{\sim}{a}$ and $\underset{\sim}{b}$. (p. 453)

scalar projection or **scalar resolute** The (scalar) magnitude of a vector projection (but not the direction). (p. 461)

secant (in algebra) A line that passes through 2 points on a circle or curve. (p. 338)

secant (in trigonometry) Abbreviation: sec. The reciprocal of cosine. $\sec(\theta) = \dfrac{1}{\cos(\theta)}$. (p. 338)

sector A region of a circle cut off by 2 radii. (p. 265)



segment A region of a circle cut off by a chord. (p. 265)

sequence An ordered list of numbers called terms, where the first term, second term, and nth term are usually written as t_1, t_2 and t_n respectively. (p. 4)

series The sum of the terms in a sequence. The sum to 5 terms is written as S_5 and the sum to n terms is written as S_n. (p. 14)

set A collection of objects. (p. 52)

simple graph A graph with no loops or multiple edges. (p. 99)

simulation The use of a model to artificially recreate a real situation, often repeatedly, so that the probabilities of various outcomes in the situation can be accurately estimated. (p. 503)

sine rule In any triangle ABC, $\dfrac{a}{\sin(A)} = \dfrac{b}{\sin(B)} = \dfrac{b}{\sin(C)}$. (p. 268)

singular matrix A matrix that doesn't have an inverse. (p. 208)

singular transformation A transformation that does not have an inverse, for which the determinant $ad - bc = 0$. (p. 540)

SOP *See* **sum of products**.

square matrix A matrix with the same number of rows as columns. (p. 196)

standard deviation A measure of spread of a distribution, equal to the square root of the variance.

$\sigma = SD(X) = \sqrt{Var(X)}$ (p. 484)

standard error The standard deviation of the sample means. (p. 511)

statement A sentence that is either true or false. (p. 130)

statistic An estimate of a population parameter found from a sample. (p. 498)

subgraph A graph that exists within another graph and contains only vertices and edges from the original graph. (p. 99)

subset A part of a set. (p. 53)

sum and difference identities (or **compound angle formulas**) The formulas for the trigonometric ratios of sums and differences of angles.

$\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$

$\sin(x - y) = \sin(x)\cos(y) - \cos(x)\sin(y)$

$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

$\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y)$

$\tan(x + y) = \dfrac{\tan(x)(x) + \tan(y)}{1 - \tan(x)\tan(y)}$

$\tan(x - y) = \dfrac{\tan(x) - \tan(y)}{1 + \tan(x)\tan(y)}$ (p. 290)

sum of products (SOP) Apply OR to the true products in the **Karnaugh map**. (p. 151)

tautology A proposition that is true for all values of its variables. The opposite of a **contradiction**. (p. 138)

theorem A statement that is always true. (p. 70)

trail A walk where there are no repeated edges. (p. 115)

transformation A change to a function and its graph through **dilation**, **reflection** or **translation**. (p. 319)

translation Horizontal or vertical movement of a graph. (p. 319)

transverse axis The axis of a hyperbola that crosses the hyperbola, usually the x-axis. (p. 365)

tree A connected graph with no loops, multiple edges or cycles. A tree with n vertices contains $n - 1$ edges. (p. 111)

tree diagram A diagram with branches that lists all of the possible outcomes in a multi-stage chance experiment. (p. 235)

triangular matrix A matrix in which all the elements on one side of the diagonal are '0'; for example,

$$\begin{bmatrix} 7 & 1 & -5 \\ 0 & -18 & 4 \\ 0 & 0 & 9 \end{bmatrix}$$ (p. 210)

trigonometric function *See* **circular function**.

trigonometric identities (or **Pythagorean identities**) For any value x:

$\sin^2(x) + \cos^2(x) = 1$

$\tan^2(x) + 1 = \sec^2(x)$

$\cot^2(x) + 1 = \operatorname{cosec}^2(x)$ (p. 292)

truth table A table that displays all possible truth values of the variables involved. (p. 135)

P	P'
T	F
F	T

union The combined elements of 2 or more sets. (pp. 59, 252)

unit vector A vector that has a magnitude of 1. The unit vector heading in the direction of $\underset{\sim}{a}$ has the notation $\hat{\underset{\sim}{a}}$. (p. 443)

universal set A set from which all subsets are defined. (p. 59)

variance A measure of the spread of a distribution, equal to the square of the standard deviation.

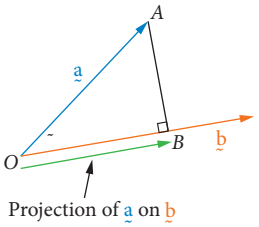
$$\sigma^2 = \text{Var}(X) = \text{E}(X^2) - \mu^2$$

where $\mu = \text{E}(X)$ and $\text{E}(X^2) = \sum x^2 \cdot p(x)$

for a discrete probability distribution. (p. 484)

vector A quantity that has both magnitude (size) and direction. In a comparison, a **scalar** has magnitude only (and is a number). Examples of vector notation are $\underset{\sim}{a}$ or $\overrightarrow{AB}$. (p. 439)

vector projection or **vector resolute** A vector that is the component of a main vector in the direction of another vector. For example, the vector projection of $\underset{\sim}{a}$ on $\underset{\sim}{b}$ (or in the direction of $\underset{\sim}{b}$) is shown on the diagram. It is like the shadow of $\underset{\sim}{a}$ on $\underset{\sim}{b}$. (p. 461)



velocity The 'signed speed' of a moving object, represented by the function $v = \dfrac{dx}{dt}$, the rate of change of the displacement, x. (p. 439)

Venn diagram A diagram consisting of circles and a rectangle used to represent the relationship between sets. (p. 59)

vertex (plural: **vertices**) The points on a graph. (p. 91)

walk Any sequence of vertices of a graph along edges. (p. 115)

XOR A Boolean operator that is true if only one of the variables is true. (p. 154)

zero matrix A matrix where all elements are '0'. (p. 197)

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ISBN 978-0170448338