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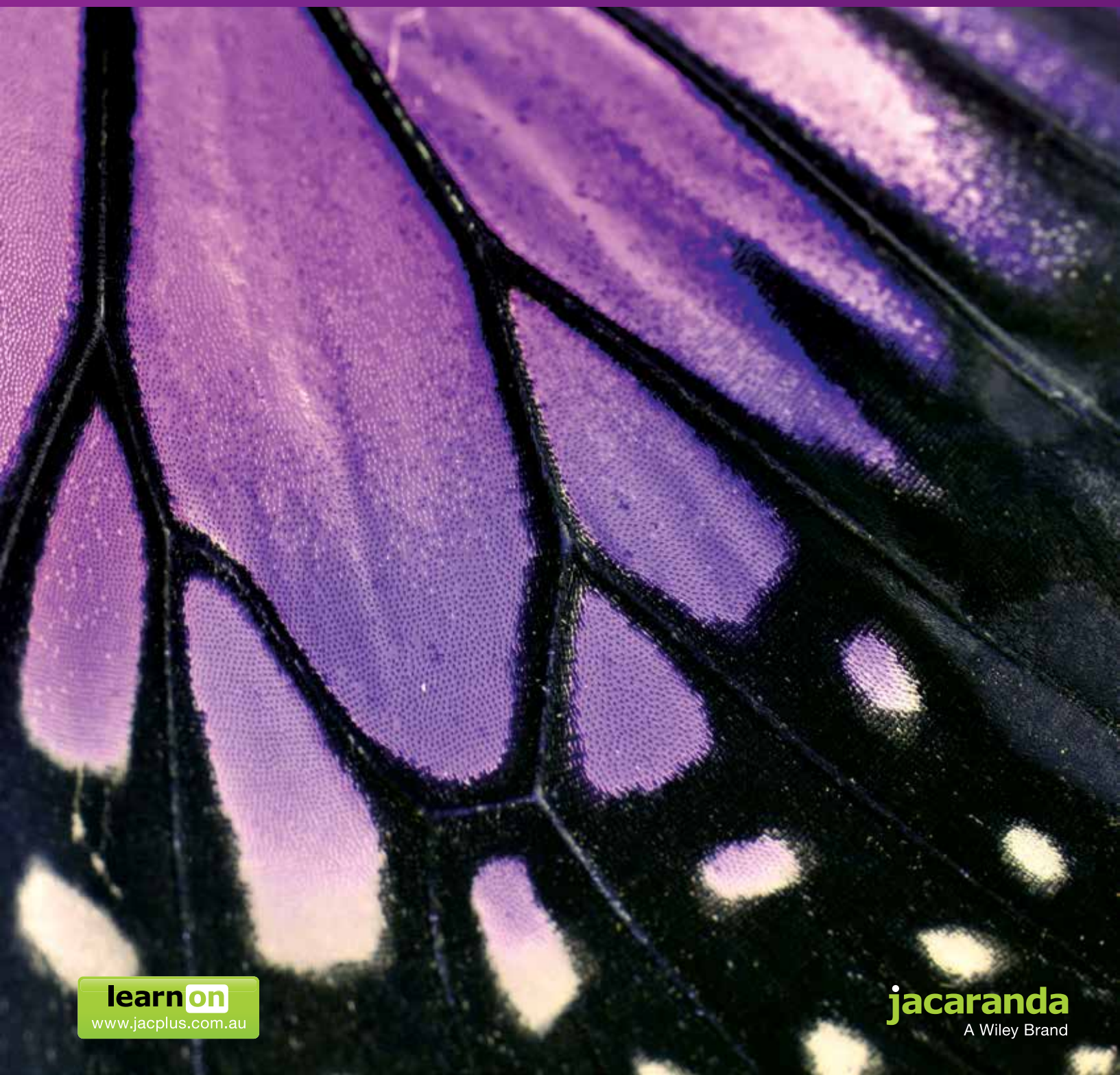

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OVERVIEW

Maths Quest 10 Stage 5 for the NSW Australian curriculum has been completely revised to help teachers and students discover the NSW Mathematics syllabus. All content has been carefully mapped to Stages 5.1, 5.2 and 5.3 of the syllabus to ensure comprehensive coverage. Individual sub-topics are labelled by stage level to assist teachers with lesson planning and their navigation through the NSW Mathematics syllabus.

Maths Quest is designed to cater for all abilities: no student is left behind and none is held back. Written with the specific purpose of helping students deeply understand mathematical concepts, the content is organised around a number of features, in both print and online through Jacaranda's *learnON* platform, to allow for seamless sequencing through material to scaffold every student's learning.

Each topic is mapped to individual stages of the NSW Mathematics Syllabus.

Topic introductions put the topic into a real-world context.

Start each topic with an engaging Discussion point.

The learning sequence is mapped to stages of the NSW Mathematics Syllabus.

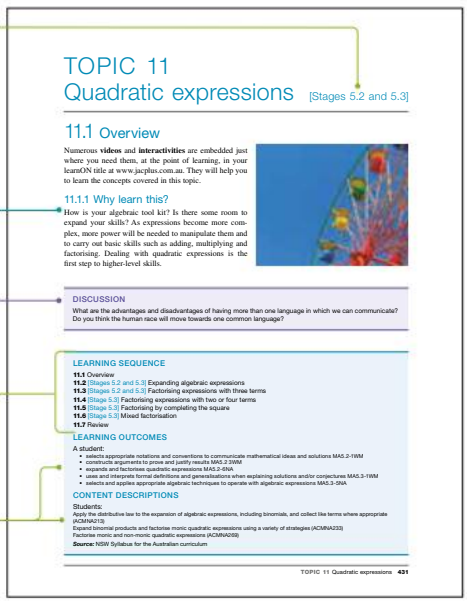
NSW Mathematics Syllabus references are linked to contents of the topic.

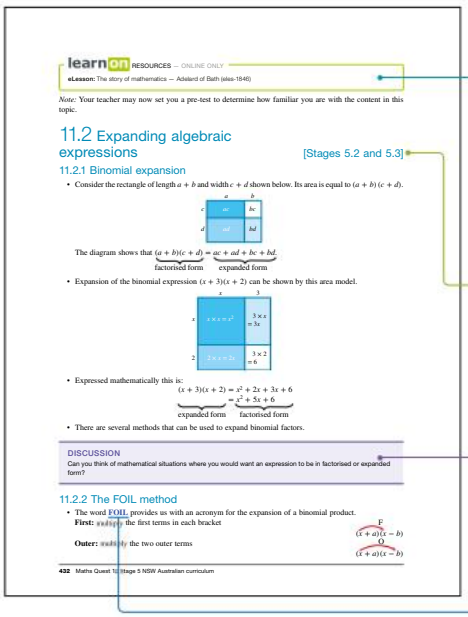
Fully Worked examples throughout the text demonstrate key concepts.

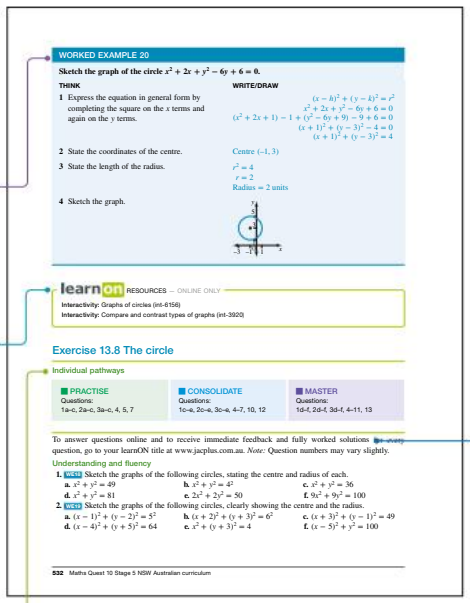
Your FREE online *learnON* resources contain hundreds of videos, interactivities and traditional WorkSHEETs and SkillSHEETs to support and enhance learning.

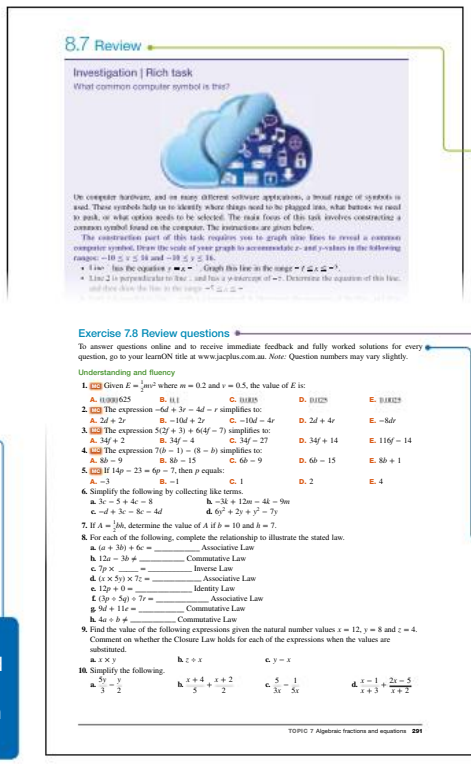
Carefully graded questions cater for all abilities. Question types are classified according to the Working Mathematically strand of the NSW Mathematics Syllabus.

Fully worked solutions to every question are provided online, and answers are provided at the end of each print topic.









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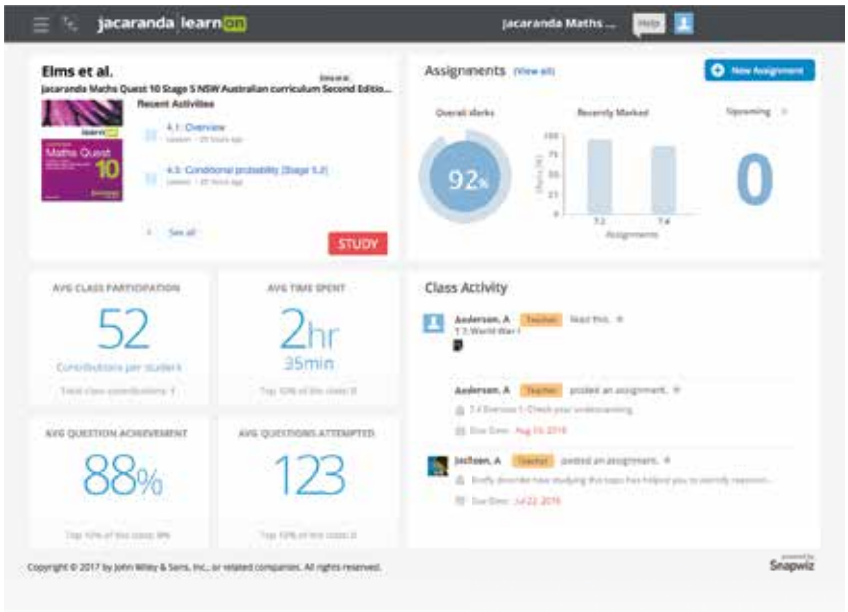
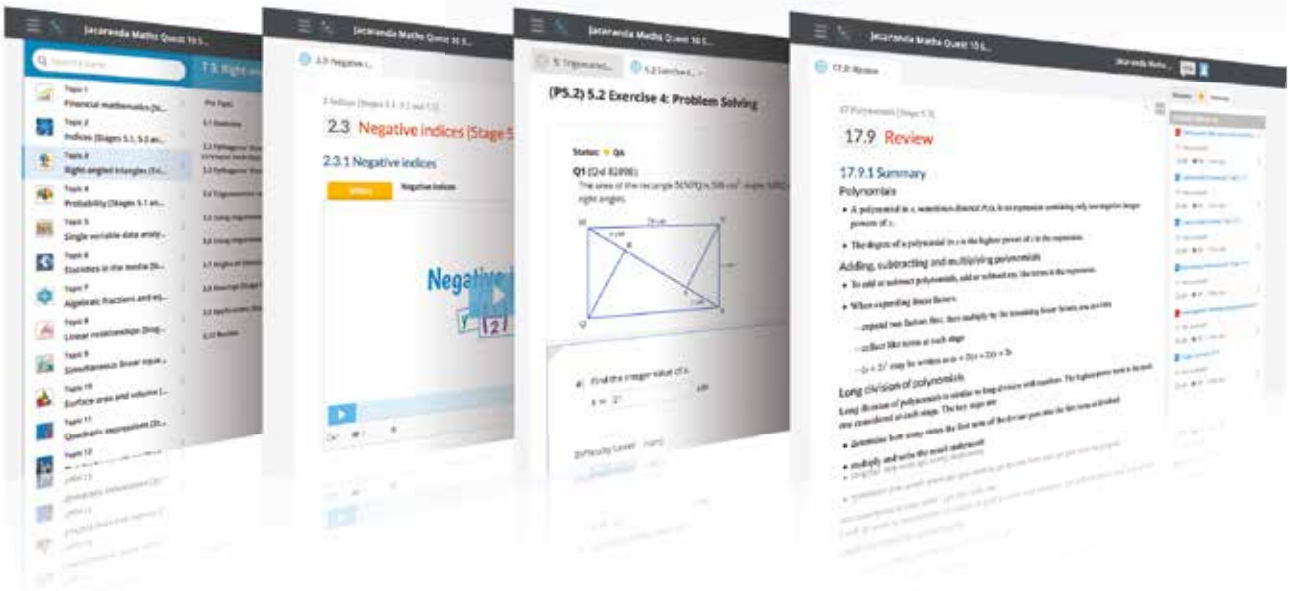
- And much more …





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TOPIC 1
Financial mathematics [Stages 5.1 and 5.2]

1.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the concepts covered in this topic.

1.1.1 Why learn this?

Everyone requires food, housing, clothing and transport, and a fulfilling social life. Money allows us to purchase the things we need and desire. The ability to manage money is key to a financially secure future and a reasonable retirement with some fun along the way. Each individual is responsible for managing his or her own finances; therefore, it is imperative that everyone is financially literate.

DISCUSSION

Recent studies have shown that beyond a certain point, having more money does not lead to increased happiness. What do you think? Does money = happiness?

LEARNING SEQUENCE

1.1 Overview
1.2 [Stage 5.1] Simple interest
1.3 [Stage 5.1] Buying on terms
1.4 [Stage 5.2] Compound interest
1.5 [Stage 5.2] Depreciation
1.6 Review

LEARNING OUTCOMES

A student:

- uses appropriate terminology, diagrams and symbols in mathematical contexts MA5.1-1WM
- selects and uses appropriate strategies to solve problems MA5.1-2WM
- provides reasoning to support conclusions that are appropriate to the context MA5.1-3WM
- solves financial problems involving spending and investing money MA5.1-4NA
- selects appropriate notations and conventions to communicate mathematical ideas and solutions MA5.2-1WM
- interprets mathematical or real-life situations, systematically applying appropriate strategies to solve problems MA5.2-2WM
- solves financial problems involving compound interest MA5.2-4NA

CONTENT DESCRIPTIONS

Students:

Solve problems involving simple interest (ACMNA211)

Connect the compound interest formula to repeated applications of simple interest using appropriate digital technologies (ACMNA229)

Source: NSW Syllabus for the Australian curriculum

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eLesson: The story of mathematics — Money, money, money! (eles-1855)

Note: Your teacher may now set you a pre-test to determine how familiar you are with the content in this topic.

1.2 Simple interest [Stage 5.1]

1.2.1 The simple interest formula

- The **simple interest formula** can be used to calculate the interest charged on borrowed money. The formula is:

$$I = \frac{P \times r \times T}{100}$$

where I is the simple interest
P is the **principal** or amount borrowed or invested
r is the rate of interest per time period
T is the time for which the money is invested or borrowed.
If T is in years, then r is the rate of interest per annum (% p.a.).

WORKED EXAMPLE 1

Find the simple interest on $4000 invested at 4.75% p.a. for 4 years.

THINK	WRITE
1 Write the formula and the known values of the variables.	$I = \frac{P \times r \times T}{100}$, where $P = \$4000$, $r = 4.75\%$, $T = 4$
2 Substitute known values to find I.	$I = \frac{4000 \times 4.75 \times 4}{100}$ $= 760$
3 Write the answer.	The simple interest is $760.

1.2.2 Purchasing goods and simple interest

- There are many different payment options when purchasing major goods, such as flat screen televisions and computers. Payment options include:
  – cash
  – credit cards
  – **lay-by**
  – deferred payment
  – buying on terms
  – loans.
- The cost of purchasing an item can vary depending on the method of payment used.
- Some methods of payment involve borrowing money and, as such, mean that interest is charged on the money borrowed.

- What are the ways of purchasing the item shown in the advertisement below?

120 cm HD TV	
 $800	5-year warranty • High definition • HDMI ports • 16 : 9 aspect ratio • 1080i

1.2.3 Payment options

Cash

- With cash, the marked price is paid on the day of purchase with nothing more to pay.
- A cash-paying customer can often negotiate with the retailer to obtain a lower price for the item.

Lay-by

- With lay-by, the item is held by the retailer while the customer makes regular payments towards paying off the marked price.
- In some cases a small administration fee may be charged.

Credit cards

- With a credit card, the retailer is paid by the credit card provider, generally a financial lender.
- The customer takes immediate possession of the goods.
- The financial lender collates all purchases over a monthly period and bills the customer accordingly. The entire balance shown on the bill can often be paid with no extra charge, but if the balance is not paid in full, interest is charged on the outstanding amount, generally at a very high rate.

WORKED EXAMPLE 2

The ticketed price of a mobile phone is $600. Andrew decides to purchase the phone using his credit card. At the end of 1 month the credit card company charges interest at a rate of 15% p.a. Calculate the amount of interest that Andrew must pay on his credit card after 1 month.

THINK	WRITE
1 Write the formula and the known values of the variables. Remember that 1 month $= \frac{1}{12}$ year.	$I = \frac{P \times r \times T}{100}$ $P = \$600,\ r = 15\%,\ T = \frac{1}{12}$
2 Substitute known values to find I.	$I = \frac{600 \times 15 \times 1}{100 \times 12}$ $= 7.50$
3 Write the answer.	The interest Andrew pays is $7.50.

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Interactivity: Simple interest (int-6074)
Digital doc: SkillSHEET Converting a percentage to a decimal (doc-5345)
Digital doc: SkillSHEET Finding simple interest (doc-5346)

Exercise 1.2 Simple interest

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–3, 5, 7, 10	Questions: 1–4, 6, 8, 10, 11	Questions: 1–3, 5, 7, 9–13

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE1 Find the simple interest payable on a loan of $8000 at 6% p.a. for 5 years.

2. Find the simple interest on each of the following loans.
- **a.** $5000 at 9% p.a. for 4 years
- **b.** $4000 at 7.5% p.a. for 3 years
- **c.** $12 000 at 6.4% p.a. for $2\frac{1}{2}$ years
- **d.** $6000 at 8% p.a. for $1\frac{1}{2}$ years

3. Find the simple interest on each of the following investments.
- **a.** $50 000 at 6% p.a. for 6 months
- **b.** $12 500 at 12% p.a. for 1 month
- **c.** $7500 at 15% p.a. for 3 months
- **d.** $4000 at 18% p.a. for 18 months

4. Calculate the monthly interest charged on each of the following outstanding credit card balances.
- **a.** $1500 at 15% p.a.
- **b.** $4000 at 16.5% p.a.
- **c.** $2750 at 18% p.a.
- **d.** $8594 at 17.5% p.a.
- **e.** $5690 at 21% p.a.

5. WE2 The ticketed price of a mobile phone is $800. Elena decides to purchase the phone using her credit card. After 1 month the credit card company charges interest at a rate of 15% p.a. Calculate the amount of interest that Elena must pay on her credit card after 1 month.

6. Arup decides to purchase a new sound system using her credit card. The ticketed price of the sound system is $900. When Arup's credit card statement arrives, it shows that she will pay no interest if she pays the full amount by the due date.
- **a.** If Arup pays $200 by the due date, what is the balance owing?

b. If the interest rate on the credit card is 18% p.a., how much interest will Arup be charged in the month?

c. What will be the balance that Arup owes at the end of the month?

d. At this time Arup pays another $500 off her credit card. How much interest is Arup then charged for the next month?

e. Arup then pays off the entire remaining balance of her card. What was the true cost of the sound system including all the interest payments?

7. Carly has an outstanding balance of $3000 on her credit card for June and is charged interest at a rate of 21% p.a.

a. Calculate the amount of interest that Carly is charged for June.

b. Carly makes the minimum repayment of $150 and makes no other purchases using the credit card in the next month. Calculate the amount of interest that Carly will be charged for July.

c. If Carly had made a repayment of $1000 at the end of June instead of $150, calculate the amount of interest that Carly would then have been charged for July.

d. How much would Carly save in July had she made the higher repayment at the end of June?

8. Shane buys a new home theatre system using his credit card. The ticketed price of the bundle is $7500. The interest rate that Shane is charged on his credit card is 18% p.a. Shane pays off the credit card at a rate of $1000 each month.

a. Complete the table below.

Month	Balance owing	Interest	Payment	Closing balance
January	$7500.00	$112.50	$1000.00	$6612.50
February	$6612.50	$99.19	$1000.00	
March			$1000.00	
April			$1000.00	
May			$1000.00	
June			$1000.00	
July			$1000.00	
August			$1015.86	$0

b. What is the total amount of interest that Shane pays?

c. What is the total cost of purchasing the home theatre system using his credit card?

Communicating, reasoning and problem solving

9. Design a table that compares the features of each method of payment: cash, lay-by and credit card.

10. Choose the most appropriate method of payment for each of the described scenarios below. Explain your choice.

Scenario 1: Andy has no savings and will not be paid for another two weeks. Andy would like to purchase an HD television and watch tomorrow's football final.

Scenario 2: In September Lena spots on special a home theatre system which she would like to purchase for her family for Christmas.

11. Merchant banks offer simple interest on all investments. Merchant bank A had an investor invest $10 000 for 5 years. Merchant bank B had a different investor invest $15 000 for 3 years. Investor B obtained $2500 more in interest than investor A because the rate of interest per annum she received was 6% greater than the interest obtained by investor A. Find the simple interest and rate of interest for each investor.

12. Compare the following two investments where simple interest is paid.

	Rate	Principal	Time	Interest
Investment A	r_A	\$8000	4 years	SI_A
Investment B	r_B	\$7000	5 years	SI_B

It is known that $r_A : r_B = 2 : 3$ and that investment B earned \$2000 more interest than investment A. Find the values of r_A, r_B, SI_A and SI_B. Give your answers correct to 2 decimal places. (Use unrounded calculations to determine subsequent values.)

13. What can you do to remember the simple interest formula?

1.3 Buying on terms [Stage 5.1]

1.3.1 Buying on terms

- When a customer buys an item on terms:
  - the customer pays a deposit
  - the customer pays off the balance over an agreed period of time with set payments
  - the set payments may be calculated as a stated arbitrary amount or interest rate
  - the total monies paid will exceed the initial cash price.

WORKED EXAMPLE 3

The cash price of a computer is \$2400. It can also be purchased on the following terms: 25% deposit and payments of \$16.73 per week for 3 years. Calculate the total cost of the computer purchased on terms as described.

THINK

1 Calculate the deposit.

WRITE

Deposit = 25% of \$2400
= 0.25 × \$2400
= \$600

2 Calculate the total of the weekly repayments.

Total repayment = \$16.73 × 52 × 3
= \$2609.88

3 Add these two amounts together to find the total cost.

Total cost = \$600 + \$2609.88
= \$3209.88

WORKED EXAMPLE 4

A diamond engagement ring has a purchase price of \$2500. Michael buys the ring on the following terms: 10% deposit with the balance plus simple interest paid monthly at 12% p.a. over 3 years.

a Calculate the amount of the deposit.
b What is the balance owing after the initial deposit?
c Calculate the interest payable.
d What is the total amount to be repaid?
e Find the amount of each monthly repayment.

THINK	WRITE
a Calculate the deposit by finding 10% of $2500.	**a** Deposit = 10% of $2500 = 0.1 × $2500 = $250
b Find the balance owing by subtracting the deposit from the purchase price.	**b** Balance = $2500 − $250 = $2250
c Find the simple interest on $2250 at 12% p.a. for 3 years.	**c** $I = \frac{P \times r \times T}{100}$, where $P = \$2250$, $r = 12\%$, $T = 3$ $I = 2250 \times 0.12 \times 3$ = $810
d Find the total repayment by adding the balance owing with the interest payable.	**d** Total repayment = $2250 + $810 = $3060
e Find the monthly repayment by dividing the total repayment by the number of months over which the ring is to be repaid.	**e** Monthly repayment = $3060 ÷ 36 = $85

ACTIVITY: BUY TODAY, NO MORE TO PAY UNTIL ...

Many large department stores offer white goods and furniture on plans described in terms such as 'Take the product home today and don't pay anything for one to two years.' For many people this is a very tempting offer, as it means they can have the goods they need and defer payment until they have the money.

Working in small groups, use the internet to investigate one of these plans and find out what happens if the customer is unable to pay at the end of the interest-free period. Prepare a report to present to the class.

1.3.2 Loans

- Money can be borrowed from a bank or other financial institution.
- Interest is charged on the amount of money borrowed.
- Both the money borrowed and the interest charged must be paid back.
- The interest rate on a loan is generally lower than the interest rate offered on a credit card or when buying on terms.
- The calculation of loan payments is done in the same way as for buying on terms; that is, calculate the interest and add it to the principal before dividing into equal monthly repayments.

DISCUSSION

If someone defaults on their car loan repayments and the car is repossessed, does this mean that their loan is cancelled? If it is not cancelled, what happens next?

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Digital doc: SkillSHEET Finding a percentage of a quantity (money) (doc-5347)

Digital doc: WorkSHEET Buying on terms (doc-14607)

Exercise 1.3 Buying on terms

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–4, 8, 10, 13–15	Questions: 1–4, 6, 8, 10, 12–15	Questions: 1–5, 8–18

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. Calculate the total cost of a $3000 purchase given the terms described below.
   a. i. 12% deposit and monthly payments of $60 over 5 years
   ii. 20% deposit and weekly payments of $20 over 3 years
   iii. 15% deposit and annual payments of $700 over 5 years
   b. Which of these options is the best deal for a purchaser?

2. Calculate the amount of each repayment for a $5000 purchase given the terms described below.
   a. 10% deposit with the balance plus simple interest paid monthly at 15% p.a. over 5 years
   b. 10% deposit with the balance plus simple interest paid fortnightly at 12% over 5 years
   c. 20% deposit with the balance plus simple interest paid monthly at 10% over 3 years

3. Calculate the total repayment and the amount of each monthly repayment for each of the following loans.
   a. $10 000 at 9% p.a. repaid over 4 years
   b. $25 000 at 12% p.a. repaid over 5 years
   c. $4500 at 7.5% p.a. repaid over 18 months
   d. $50 000 at 6% p.a. repaid over 10 years
   e. $200 000 at 7.2% p.a. repaid over 20 years

4. WE3 The cash price of a bedroom suite is $4200. The bedroom suite can be purchased on the following terms: 20% deposit and weekly repayments of $43.94 for 2 years. Calculate the total cost of the bedroom suite if you buy it on terms.

5. Guy purchases a computer that has a cash price of $3750 on the following terms: $500 deposit with the balance plus interest paid over 2 years at $167.92 per month. What is the total amount that Guy pays for the computer?

6. Dmitry wants to buy a used car with a cash price of $12 600. The dealer offers terms of 10% deposit and monthly repayments of $812.70 for 2 years.
   a. Calculate the amount of the deposit.
   b. Calculate the total amount to be paid in monthly repayments.
   c. What is the total amount Dmitry pays for the car if he buys it on terms?
   d. How much more than the cash price of the car does Dmitry pay? (This is the interest charged by the dealer.)

7. Alja wants to purchase an entertainment system that has a cash price of $5800. She purchases the entertainment system on terms of no deposit and monthly repayments of $233.61 for 3 years.
   a. Calculate the total amount that Alja pays for the entertainment system.
   b. Calculate the amount that Alja pays in interest.
   c. Calculate the amount of interest that Alja pays each year.
   d. Calculate this amount as a percentage of the cash price of the entertainment system.

8. WE4 A used car has a purchase price of $9500. Dayna buys the car on the following terms: 25% deposit with balance plus interest paid at 12% p.a. over 3 years.
   a. Calculate the amount of the deposit.
   b. What is the balance owing?
   c. Calculate the interest payable.
   d. What is the total amount to be repaid?
   e. Find the amount of each monthly repayment.

9. A department store offers the following terms: one-third deposit with the balance plus interest paid in equal, monthly instalments over 18 months. The interest rate charged is 9% p.a. Ming buys a lounge suite with a ticketed price of $6000.
   a. Calculate the amount of the deposit.
   b. What is the balance owing?
   c. Calculate the interest payable.
   d. What is the total amount to be repaid?
   e. Find the amount of each monthly repayment.

10. Calculate the monthly payment on each of the following items bought on terms. (*Hint:* Use the steps shown in question **8**.)
   a. Dining suite: cash price $2700, deposit 10%, interest rate 12% p.a., term 1 year

b. Video camera: cash price $990, deposit 20%, interest rate 15% p.a., term 6 months

c. Car: cash price $16 500, deposit 25%, interest rate 15% p.a., term 5 years

d. Mountain bike: cash price $3200, one-third deposit, interest rate 9% p.a., term $2\frac{1}{2}$ years

e. Watch: cash price $675, no deposit, interest rate 18% p.a., term 9 months

11. Samir wants to purchase his first car. He has saved $1000 as a deposit but the cost of the car is $5000. Samir takes out a loan from the bank to cover the balance of the car plus $600 worth of on-road costs.
- **a.** How much will Samir need to borrow from the bank?
- **b.** Samir takes the loan out over 4 years at 9% p.a. interest. How much interest will Samir need to pay?
- **c.** What will be the amount of each monthly payment that Samir makes?
- **d.** What is the total cost of the car after paying off the loan, including the on-road costs? Give your answer to the nearest $.

12. MC Kelly wants to borrow $12 000 for some home improvements. Which of the following loans will lead to Kelly making the lowest total repayment?

A. Interest rate 6% p.a. over 4 years
B. Interest rate 7% p.a. over 3 years
C. Interest rate 5.5% p.a. over 3 years
D. Interest rate 6.5% p.a. over 5 years
E. Interest rate 7.5% p.a. over 3 years

Communicating, reasoning and problem solving

13. MC Without completing any calculations, explain which of the following loans will be the best value for the borrower.

A. Interest rate 8.2% p.a. over 5 years
B. Interest rate 8.2% over 4 years
C. Interest rate 8% over 6 years
D. Interest rate 8% over 5 years
E. Interest rate 8% over 4 years

14. Explain how, when purchasing an item, making a deposit using existing savings and taking out a loan for the balance can be an advantage.

15. Gavin borrows $18 000 over 5 years from the bank. The loan is charged at 8.4% p.a. flat-rate interest. The loan is to be repaid in equal monthly instalments. Calculate the amount of each monthly repayment.

16. Andrew purchased a new car valued at $32 000. He paid a 10% deposit and was told he could have 4 years to pay off the balance of the car price plus interest. An alternative scheme was also offered to him. It involved paying off the balance of the car price plus interest in 8 years. If he chose the latter scheme, he would end up paying $19 584 more. The interest rate for the 8-year scheme was 1% more than for the 4-year scheme.
- **a.** How much deposit did he pay?
- **b.** What was the balance to be paid on the car?
- **c.** Find the interest rate for each of the two schemes.
- **d.** Find the total amount paid for the car for each of the schemes.
- **e.** What were the monthly repayments for each of the schemes?

17. When buying on terms, what arrangements are the most beneficial to the buyer?

18. Ingrid offered to pay her brother $2 for doing her share of the housework each day, but fined him $5 if he forgot to do it. After 4 weeks, Ingrid discovered that she did not owe her brother any money. For how many days did Ingrid's brother do her share of the housework?

1.4 Compound interest [Stage 5.2]

1.4.1 Compound interest

- Interest on the principal in a savings account or a short-term or long-term deposit is generally calculated using **compound interest** rather than simple interest.

- When interest is added to the principal at regular intervals, increasing the balance of the account, and each successive interest payment is calculated on the new balance, it is called compound interest.
- Compound interest can be calculated by calculating simple interest one period at a time.
- The amount to which the initial investment grows is called the **compounded value** or **future value**.

WORKED EXAMPLE 5

Kyna invests $8000 at 8% p.a. for 3 years with interest paid at the end of each year. Find the compounded value of the investment by calculating the simple interest on each year separately.

THINK	WRITE
1 Write the initial (first year) principal.	Initial principal = $8000
2 Calculate the interest for the first year.	Interest for year 1 = 8% of $8000 = $640
3 Calculate the principal for the second year by adding the first year's interest to the initial principal.	Principal for year 2 = $8000 + $640 = $8640
4 Calculate the interest for the second year.	Interest for year 2 = 8% of $8640 = $691.20
5 Calculate the principal for the third year by adding the second year's interest to the second year's principal.	Principal for year 3 = $8640 + $691.20 = $9331.20
6 Calculate the interest for the third year.	Interest for year 3 = 8% of $9331.20 = $746.50
7 Calculate the future value of the investment by adding the third year's interest to the third year's principal.	Compounded value after 3 years = $9331.20 + $746.50 = $10077.70

- To calculate the actual amount of interest received, we subtract the initial principal from the future value.
- In the example above, compound interest = $10077.70 − $8000
  = $2077.70
- We can compare this with the simple interest earned at the same rate.

$$I = \frac{P \times r \times T}{100}$$
$$= \frac{800 \times 8 \times 3}{100}$$
$$= \$1920$$

- The table below shows a comparison between the total interest earned on an investment of $8000 earning 8% p.a. at both simple interest (I) and compound interest (CI) over an eight-year period.

YEAR	1	2	3	4	5	6	7	8
Total (I)	$640.00	$1280.00	$1920.00	$2560.00	$3200.00	$3840.00	$4480.00	$5120.00
Total (CI)	$640.00	$1331.20	$2077.70	$2883.91	$3754.62	$4694.99	$5710.59	$6807.44

1.4.2 Using the compound interest formula

- We can develop a formula for the future value of an investment rather than do each example by repeated use of simple interest. Consider Worked example 5. Let the compounded value after year n be A_n.
  After 1 year, $A_1 = 8000 \times 1.08$ (increasing \$8000 by 8%)
  After 2 years, $A_2 = A_1 \times 1.08$
  $= 8000 \times 1.08 \times 1.08$ (substituting the value of A_1)
  $= 8000 \times 1.08^2$
  After 3 years, $A_3 = A_2 \times 1.08$
  $= 8000 \times 1.08^2 \times 1.08$ (substituting the value of A_2)
  $= 8000 \times 1.08^3$
- The pattern then continues such that the value of the investment after n years equals
  $\$8000 \times 1.08^n$.
- This can be generalised for any investment:

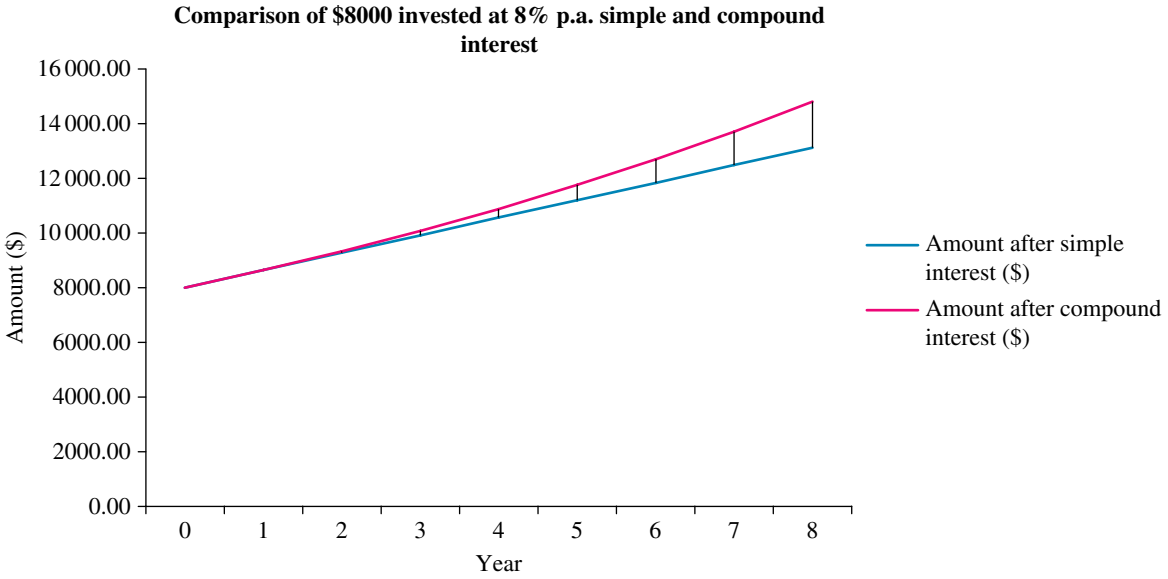
$$\boxed{A = P(1 + R)^n}$$

  where A is the amount (or future value) of the investment
  P is the principal (or present value)
  R is the interest rate per **compounding period** expressed as a decimal
  n is the number of compounding periods.
- To calculate the amount of compound interest (CI) we then use the formula

$$\boxed{CI = A - P}$$

1.4.3 Using technology

- Digital technologies such as spreadsheets can be used to draw graphs in order to compare interest accrued through simple interest and compound interest.



WORKED EXAMPLE 6

William has \$14 000 to invest. He invests the money at 9% p.a. for 5 years with interest compounded annually.

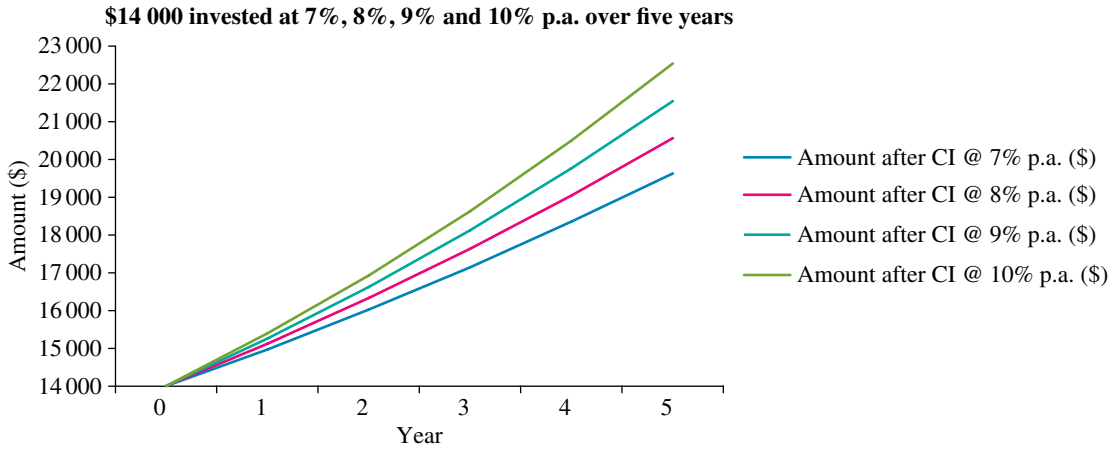
a Use the formula $A = P(1 + R)^n$ to calculate the amount to which this investment will grow.

b Calculate the compound interest earned on the investment.

THINK	WRITE
a 1 Write the compound interest formula.	**a** $A = P(1 + R)^n$
2 Write the values of P, R and n.	$P = \$14000$, $R = 0.09$, $n = 5$
3 Substitute the values into the formula.	$A = 14000 \times 1.09^5$
4 Calculate the amount.	$= 21540.74$ The investment will grow to \$21 540.74.
b Calculate the compound interest earned.	**b** $CI = A - P$ $= 21540.74 - 14000$ $= 7540.74$ The compound interest earned is \$7540.74.

1.4.4 Comparison of fixed principal at various interest rates over a period of time

- It is often helpful to compare the future value (\$A) of the principal at different compounding interest rates over a fixed period of time.
- Spreadsheets are very useful tools for making comparisons. The graph below, generated from a spreadsheet, shows the comparisons for \$14 000 invested for 5 years at 7%, 8%, 9% and 10% compounding annually.
- There is a significant difference in the future value depending on which interest rate is applied.



ACTIVITY: COMPARING INTEREST RATES

Use a spreadsheet to graph and compare the value of an investment of \$20 000 at interest rates of 5%, 6%, 7% and 8% for 8 years, compounding monthly.

1.4.5 Compounding period

- In Worked example 6, interest is paid annually.
- Interest can be paid more regularly — it may be paid six-monthly (twice a year), quarterly (4 times a year), monthly or even daily. This is called the compounding period.
- The time and interest rate on an investment must reflect the compounding period. For example, an investment over 5 years at 6% p.a. compounding *quarterly* will have:

$$n = 20\,(5 \times 4) \text{ and } R = 0.015\,(6\% \div 4)$$

To find n:

$$n = \text{number of years} \times \text{compounding periods per year}$$

To find R:

$$R = \text{interest rate per annum} \div \text{compounding periods per year}$$

WORKED EXAMPLE 7

Calculate the future value of an investment of $4000 at 6% p.a. for 2 years with interest compounded quarterly.

THINK	WRITE
1 Write the compound interest formula.	$A = P(1 + R)^n$
2 Write the values of P, R and n.	$P = \$4000$, $R = 0.015$, $n = 8$
3 Substitute the values into the formula.	$A = 4000 \times 1.015^8$
4 Calculate the future value.	$= \$4505.97$ The future value of the investment is $4505.97.

1.4.6 Guess and refine

- Sometimes it is useful to know approximately how long it will take to reach a particular future value once an investment has been made. Mathematical formulas can be applied to determine when a particular future value will be reached. In this section, a 'guess and refine' method will be shown.
- For example, to determine the number of years required for an investment of $1800 at 9% compounded quarterly to reach a future value of $2500, the following method can be used.
  Let n = the number of compounding periods (quarters) and A = the future value in $.
  Therefore, it will take approximately 15 quarters, or 3 years and 9 months, to reach the desired amount.

n	$A = P\left(1 + \frac{R}{4}\right)^n$	Comment
1	$1840.50	It is useful to know how the principal is growing after 1 quarter, but the amount is quite far from $2500.
3	$1924.25	The amount is closer to $2500 but still a long way off, so jump to a higher value for n.
10	$2248.57	The amount is much closer to $2500.
12	$2350.89	The amount is much closer to $2500.
14	$2457.87	The amount is just below $2500.
15	$2513.17	The amount is just over $2500.

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Interactivity: Compound interest (int-2791)

Interactivity: Compounding periods (int-6186)

Digital doc: WorkSHEET Compound interest (doc-14608)

Digital doc: SpreadSHEET Simple and compound interest (doc-10907)

Exercise 1.4 Compound interest

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–3, 5, 6, 11, 13, 15, 18	Questions: 1–7, 9, 10, 12–16, 18	Questions: 1–8, 11, 12, 14–21

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. Use the formula $A = P(1 + R)^n$ to calculate the amount to which each of the following investments will grow with interest compounded annually.
- **a.** $3000 at 4% p.a. for 2 years
- **b.** $9000 at 5% p.a. for 4 years
- **c.** $16 000 at 9% p.a. for 5 years
- **d.** $12 500 at 5.5% p.a. for 3 years
- **e.** $9750 at 7.25% p.a. for 6 years
- **f.** $100 000 at 3.75% p.a. for 7 years

2. Calculate the compounded value of each of the following investments.
- **a.** $870 for 2 years at 3.50% p.a. with interest compounded six-monthly
- **b.** $9500 for $2\frac{1}{2}$ years at 4.6% p.a. with interest compounded quarterly
- **c.** $148 000 for $3\frac{1}{2}$ years at 9.2% p.a. with interest compounded six-monthly
- **d.** $16 000 for 6 years at 8% p.a. with interest compounded monthly
- **e.** $130 000 for 25 years at 12.95% p.a. with interest compounded quarterly

3. WE5 Danielle invests $6000 at 10% p.a. for 4 years with interest paid at the end of each year. Find the compounded value of the investment by calculating the simple interest on each year separately.

4. Ben is to invest $13 000 for 3 years at 8% p.a. with interest paid annually. Find the amount of interest earned by calculating the simple interest for each year separately.

5. WE6 Simon has $2000 to invest. He invests the money at 6% p.a. for 6 years with interest compounded annually.
- **a.** Use the formula $A = P(1 + R)^n$ to calculate the amount to which this investment will grow.
- **b.** Calculate the compound interest earned on the investment.

6. WE7 Calculate the future value of an investment of $14 000 at 7% p.a. for 3 years with interest compounded quarterly.

7. A passbook savings account pays interest of 0.3% p.a. Jill has $600 in such an account. Calculate the amount in Jill's account after 3 years, if interest is compounded quarterly.

8. Damien is to invest $35 000 at 7.2% p.a. for 6 years with interest compounded six-monthly. Calculate the compound interest earned on the investment. If the account compounded annually, how much less interest would Damien's investment have earned?

9. Sam invests $40 000 in a one-year fixed deposit at an interest rate of 7% p.a. with interest compounding monthly.

a. Convert the interest rate of 7% p.a. to a rate per month.

b. Calculate the value of the investment upon maturity.

10. MC A sum of $7000 is invested for 3 years at the rate of 5.75% p.a., compounded quarterly. The interest paid on this investment, to the nearest dollar, is:

A. $1208 **B.** $1308 **C.** $8208 **D.** $8308 **E.** $8508

11. MC After selling their house and paying off their mortgage, Mr and Mrs Fong have $73 600. They plan to invest it at 7% p.a. with interest compounded annually. The value of their investment will first exceed $110 000 after:

A. 5 years **B.** 6 years **C.** 8 years

D. 10 years **E.** 15 years

12. MC Maureen wishes to invest $15 000 for a period of 7 years. The following investment alternatives are suggested to her. The best investment would be:

A. simple interest at 8% p.a.

B. compound interest at 6.7% p.a. with interest compounded annually

C. compound interest at 6.6% p.a. with interest compounded six-monthly

D. compound interest at 6.5% p.a. with interest compounded quarterly

E. compound interest at 6.4% p.a. with interest compounded monthly

13. MC An amount is to be invested for 5 years and compounded semi-annually at 7% p.a. Which of the following investments will have a future value closest to $10 000?

A. $700 **B.** $6500 **C.** $7400 **D.** $9000 **E.** $9900

14. Jake invests $120 000 at 9% p.a. for a 1-year term. For such large investments interest is compounded daily.

a. Calculate the daily percentage interest rate, correct to 4 decimal places. Use 1 year = 365 days.

b. Hence, calculate the compounded value of Jake's investment on maturity.

c. Calculate the amount of interest paid on this investment.

d. Calculate the extra amount of interest earned compared with the case where the interest is calculated only at the end of the year.

Communicating, reasoning and problem solving

15. Daniel has $15 500 to invest. An investment over a 2-year term will pay interest of 7% p.a.

a. Calculate the compounded value of Daniel's investment if the compounding period is:

i. 1 year **ii.** 6 months **iii.** 3 months **iv.** 1 month.

b. Explain why it is advantageous to have interest compounded on a more frequent basis.

16. Jasmine invests $6000 for 4 years at 8% p.a. simple interest. David also invests $6000 for 4 years, but his interest rate is 7.6% p.a. with interest compounded quarterly.

a. Calculate the value of Jasmine's investment on maturity.

b. Show that the compounded value of David's investment is greater than Jasmine's investment.

c. Explain why David's investment is worth more than Jasmine's investment despite receiving a lower rate of interest.

17. Quan has $20 000 to invest over the next 3 years. He has the choice of investing his money at 6.25% p.a. simple interest or 6% p.a. compound interest.

a. Calculate the amount of interest that Quan will earn if he selects the simple interest option.

b. Calculate the amount of interest that Quan will earn if the interest is compounded:

i. annually **ii.** six-monthly **iii.** quarterly.

c. Clearly Quan's decision will depend on the compounding period. Under what conditions should Quan accept the lower interest rate on the compound interest investment?

d. Consider an investment of $10 000 at 8% p.a. simple interest over 5 years. Use a trial-and-error method to find an equivalent rate of compound interest over the same period.

e. Will this equivalent rate be the same if we change:

i. the amount of the investment **ii.** the period of the investment?

18. A building society advertises investment accounts at the following rates.

a. 3.875% p.a. compounding daily

b. 3.895% p.a. compounding monthly

c. 3.9% p.a. compounding quarterly

Peter thinks the first account is the best one because the interest is calculated more frequently. Paul thinks the last account is the best one because it has the highest interest rate. Explain whether either is correct.

19. Two banks offer the following investment packages.

Bankwest: 7.5% p. a. compounded annually, fixed for 7 years

Bankeast: 5.8% p. a. compounded annually, fixed for 9 years

a. Which bank's package will yield the greater interest?

b. If a customer invests $20 000 with Bankwest, how much would she have to invest with Bankeast to produce the same amount as Bankwest at the end of the investment period?

20. How is compound interest calculated differently to simple interest?

21. How long will it take for a sum of money to double if it is invested at a rate of 15% p.a. compounded monthly?

1.5 Depreciation [Stage 5.2]

- **Depreciation** is the reduction in the value of an item as it ages over a period of time. For example, a car that is purchased new for $45 000 will be worth less than that amount 1 year later and less again each year.
- Depreciation is usually calculated as a percentage of the yearly value of the item.
- To calculate the depreciated value of an item, use the formula

$$A = P(1 - R)^n$$

where A is the depreciated value of the item

P is the initial value of the item

R is the percentage that the item depreciates each year expressed as a decimal

n is the number of years that the item has been depreciating for.

- This formula is almost the same as the compound interest formula except that it subtracts a percentage of the value each year instead of adding.
- In many cases, depreciation can be a tax deduction.
- When the value of an item falls below a certain value, it is said to be *written off*. That is to say, for tax purposes, the item is considered to be worthless.
- Trial-and-error methods can be used to calculate the length of time that the item will take to reduce to this value.

WORKED EXAMPLE 8

A farmer purchases a tractor for $115 000. The value of the tractor depreciates by 12% p.a. Find the value of the tractor after 5 years.

THINK	WRITE
1 Write the depreciation formula.	$A = P(1 - R)^n$
2 Write the values of P, R and n.	$P = \$115\,000$, $R = 0.12$, $n = 5$
3 Substitute the values into the formula.	$A = 115\,000 \times (0.88)^5$
4 Calculate the value of the tractor.	$= 60\,689.17$ The value of the tractor after 5 years is $60 689.17.

WORKED EXAMPLE 9

A truck driver buys a new prime mover for $500 000. The prime mover depreciates at the rate of 15% p.a. and is written off when its value falls below $1 00 000. How long will it take for the prime mover to be written off?

THINK	WRITE
1 Make an estimate of, say, $n = 5$. Use the depreciation formula to find the value of the prime mover after 5 years.	Consider $n = 5$. $A = P(1 - R)^n$ $= 500\,000 \times (0.85)^5$ $= \$221\,852.66$
2 Because the value will still be greater than $100 000, try a larger estimate, say, $n = 10$.	Consider $n = 10$. $A = P(1 - R)^n$ $= 500\,000 \times (0.85)^{10}$ $= \$98\,437.20$
3 As the value is below $100 000, check $n = 9$.	Consider $n = 9$. $A = P(1 - R)^n$ $= 500\,000 \times (0.85)^9$ $= \$115\,808.47$
4 Because $n = 10$ is the first time that the value falls below $100 000, conclude that it takes 10 years to be written off.	The prime mover will be written off in 10 years.

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Interactivity: Different rates of depreciation (int-1155)
Interactivity: Depreciation (int-6187)
eLesson: What is depreciation? (eles-0182)

Exercise 1.5 Depreciation

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1, 2, 7, 9, 12, 14, 16	Questions: 1–3, 5, 7, 9, 11, 12, 14, 16	Questions: 1–8, 11, 12, 14–18

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Understanding and fluency

1. Calculate the depreciated value of an item for the initial values, depreciation rates and times given below.
   a. Initial value of $30 000 depreciating at 16% p.a. over 4 years
   b. Initial value of $5 000 depreciating at 10.5% p.a. over 3 years
   c. Initial value of $12 500 depreciating at 12% p.a. over 5 years

2. WE8 A laundromat installs washing machines and clothes dryers to the value of $54 000. If the value of the equipment depreciates at a rate of 20% p.a., find the value of the equipment after 5 years.

3. A drycleaner purchases a new machine for $38 400. The machine depreciates at 16% p.a.
   a. Calculate the value of the machine after 4 years.
   b. Find the amount by which the machine has depreciated over this period of time.

4. A tradesman values his new tools at $10 200. For tax purposes, their value depreciates at a rate of 15% p.a.
   a. Calculate the value of the tools after 6 years.
   b. Find the amount by which the value of the tools has depreciated over these 6 years.
   c. Calculate the percentage of the initial value that the tools are worth after 6 years.

5. A taxi is purchased for $52 500 with its value depreciating at 18% p.a.
   a. Find the value of the taxi after 10 years.
   b. Calculate the accumulated depreciation over this period.

6. A printer depreciates the value of its printing presses by 25% p.a. Printing presses are purchased new for $2.4 million. What is the value of the printing presses after:
   a. 1 year **b.** 5 years **c.** 10 years?

7. MC A new computer workstation costs $5490. With its value depreciating at 26% p.a., the workstation's value at the end of the third year will be close to:
   A. $1684 **B.** $2225 **C.** $2811 **D.** $3082 **E.** $3213

8. MC The value of a new photocopier is $8894. Its value depreciates by 26% in the first year, 21% in the second year and 16% p.a. in the remaining 7 years. The value of the photocopier after this time, to the nearest dollar, is:
   A. $1534 **B.** $1851 **C.** $2624 **D.** $3000 **E.** $3504

9. MC A company was purchased 8 years ago for $2.6 million. With a depreciation rate of 12% p.a., the total amount by which the company has depreciated is closest to:
   A. $0.6 million **B.** $1.0 million **C.** $1.7 million
   D. $2.0 million **E.** $2.3 million

10. MC Equipment is purchased by a company and is depreciated at the rate of 14% p.a. The number of years that it will take for the equipment to reduce to half of its initial value is:

A. 4 years **B.** 5 years **C.** 6 years
D. 7 years **E.** 8 years

11. MC An asset that was bought for \$12 300 has a value of \$6920 after 5 years. The depreciation rate is close to:

A. 10.87% **B.** 16.76% **C.** 18.67%
D. 21.33% **E.** 27.34%

12. WE9 A farmer buys a light aeroplane for crop dusting. The aeroplane costs \$900 000. The aeroplane depreciates at the rate of 18% p.a. and is written off when its value falls below \$150 000. How long will it take for the aeroplane to be written off? Give your answer in whole years.

13. A commercial airline buys a jumbo jet for \$750 million. The value of this aircraft depreciates at a rate of 12.5% p.a.

a. Find the value of the plane after 5 years, correct to the nearest million dollars.

b. How many years will it take for the value of the jumbo jet to fall below \$100 million?

Communicating, reasoning and problem solving

14. A machine purchased for \$48 000 will have a value of \$3000 in 9 years.

a. Use a trial-and-error method to find the rate at which the machine is depreciating per annum.

b. Consider the equation $x = a^n$, $a = \sqrt[n]{x}$. Verify your answer to part **a** using this relationship.

15. Camera equipment purchased for \$150 000 will have a value of \$9000 in 5 years.

a. Find the rate of annual depreciation using trial and error first and then algebraically with the relationship 'if $x = a^n$, then $a = \sqrt[n]{x}$'.

b. Compare and contrast each method.

16. The value of a new tractor is \$175 000. The value of the tractor depreciates by 22.5% p.a.

a. Find the value of the tractor after 8 years.

b. What percentage of its initial value is the tractor worth after 8 years?

17. Anthony has a home theatre valued at \$$P$. The value of the home theatre depreciates by r% annually over a period of 5 years. At the end of the 5 years, the value of the home theatre has been reduced by $\$\frac{P}{12}$. Find the value of r correct to 3 decimal places.

18. How and why is the formula for depreciation different to compound interest?

1.6 Review [Stage 5.2]

Investigation | Rich task

Consumer price index

The Consumer Price Index (CPI) measures price movements in Australia. Let's investigate this further to gain an understanding of how this index is calculated.

A collection of goods and services is selected as representative of a high proportion of household expenditure. The prices of these goods are recorded each quarter. The collection on which the CPI is based is divided into eight groups, which are further divided into subgroups. The groups are food, clothing, tobacco/alcohol, housing, health/personal care, household equipment, transportation, and recreation/education. Weights are attached to each of these subgroups to reflect the importance of each in relation to the total household expenditure. The table shows the weights of the eight groups.

The weights indicate that a typical Australian household spends 19% of its income on food purchases, 7% on clothing and so on. The CPI is regarded as an indication of the cost of living as it records changes in the level of retail prices from one period to another.

CPI group	Weight (% of total)
Food	19
Clothing	7
Tobacco/alcohol	8.2
Housing	14.1
Health/personal care	5.6
Household equipment	18.3
Transportation	17
Recreation/education	10.8

Consider a simplified example showing how this CPI is calculated and how we are able to compare prices between one period and another. Take three items with prices as follows: a pair of jeans costing $75, a hamburger costing $3.90 and a CD costing $25. Let us say that during the next period of time, the jeans sell for $76, the hamburger for $4.20 and the CD for $29. This can be summarised in the following table.

		Period 1		Period 2	
Item	Weight (W)	Price (P)	W × P	Price (P)	W × P
Jeans	7	$75	525		
Hamburger	19	$3.90	74.1		
CD	10.8	$25	270		
Total			869.1		

In order to calculate the CPI for Period 2, we regard the first period as the base and allocate it an index number of 100 (it is classed as 100%). We compare the second period with the first by expressing it as a percentage of the first period.

$$\text{CPI} = \frac{\text{weighted expenditure for Period 2}}{\text{weighted expenditure for Period 1}} \times 100\%$$

1. Complete the table to determine the total weighted price for Period 2.
2. **a.** Calculate the CPI for the above example, correct to 1 decimal place.
   b. This figure is over 100%. The amount over 100% is known as the inflation factor. What is the inflation factor in this case?
3. Now apply this procedure to a more varied basket of goods. Complete the following table, then calculate the CPI and inflation factor for the second period.

		Period 1		Period 2	
Item	**Weight (W)**	**Price (P)**	**W × P**	**Price (P)**	**W × P**
Bus fare		$4.80		$4.95	
Rent		$220		$240	
Movie ticket		$10.50		$10.80	
Air conditioner		$1200		$1240	
Haircut		$18.50		$21.40	
Bread		$2.95		$3.20	
Shirt		$32.40		$35	
Bottle of scotch		$19.95		$21	
Total					

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Digital doc: Investigation — Consumer price index (doc-15944)
Digital doc: Code puzzle — The longest and shortest gestation periods in mammals (doc-15945)
Digital doc: Summary Topic 1 (doc-22899)
Interactivity: Word search Topic 1 (int-2868)
Interactivity: Crossword Topic 1 (int-2869)
Interactivity: Sudoku Topic 1 (int-3602)

Exercise 1.6 Review questions

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. Calculate the simple interest that is earned on $5000 at 5% p.a. for 4 years.
2. MC Jim invests a sum of money at 9% p.a. Which one of the following statements is true?
   A. Simple interest will earn Jim more money than if compound interest is paid annually.
   B. Jim will earn more money if interest is compounded annually rather than monthly.
   C. Jim will earn more money if interest is compounded quarterly rather than six-monthly.
   D. Jim will earn more money if interest is compounded annually rather than six-monthly.
   E. It does not matter whether simple interest or compound interest is used to calculate the growth of Jim's investment.

3. Benito has a credit card with an outstanding balance of $3600. The interest rate charged on the loan is 18% p.a. Calculate the amount of interest that Benito will be charged on the credit card for the next month.

4. An LCD television has a cash price of $5750. It can be purchased on terms of 20% deposit plus weekly repayments of $42.75 for 3 years. Calculate the total cost of the television if it is purchased on terms.

5. Erin purchases a new entertainment unit that has a cash price of $6400. Erin buys the unit on the following terms: 10% deposit with the balance plus interest to be repaid in equal monthly repayments over 4 years. The simple interest rate charged is 12% p.a.
- **a.** Calculate the amount of the deposit.
- **b.** Calculate the balance owing after the deposit has been paid.
- **c.** Calculate the interest that will be charged.
- **d.** What is the total amount that Erin has to repay?
- **e.** Calculate the amount of each monthly repayment.

6. A new car has a marked price of $40 000. The car can be purchased on terms of 10% deposit and monthly repayments of $1050 for 5 years.
- **a.** Find the total cost of the car if it is purchased on terms.
- **b.** Calculate the amount of interest paid.
- **c.** Calculate the amount of interest paid per year.
- **d.** Calculate the interest rate charged.

7. Ryan invests $12 500 for 3 years at 8% p.a. with interest paid annually. By calculating the amount of simple interest earned each year separately, determine the amount to which the investment will grow.

8. Calculate the compound interest earned on $45 000 at 12% p.a. over 4 years if interest is compounded:
a. annually **b.** six-monthly **c.** quarterly **d.** monthly.

9. A new computer server costs $7290. With 22% p.a. reducing-value depreciation, the server's value at the end of the third year will be close to:
a. $1486 **b.** $2257 **c.** $2721 **d.** $3023 **e.** $3460

10. An asset that was bought for $34 100 has a value of $13 430 after 5 years. The depreciation rate is close to:
a. 11% **b.** 17% **c.** 18% **d.** 21% **e.** 22%

11. The value of a new car depreciates by 15% p.a. Find the value of the car after 5 years if it was purchased for $55 000.

Communicating, reasoning and problem solving

12. Virgin Blue buys a new plane so that extra flights can be arranged between Sydney, Australia and Wellington, New Zealand. The plane costs $1 200 000. It depreciates at a rate of 16.5% p.a. and is written off when its value falls below $150 000. How long can Virgin Blue use this plane before it is written off?

13. Thomas went to an electronics store to buy a flat screen HD TV together with some accessories. The store offered him two different loans to buy the television and equipment.
Loan 1: $7000 for 3 years at 10.5% p.a. compounding yearly
Loan 2: $7000 for 5 years at 8% p.a. compounding yearly
The following agreement was struck with the store.
- Thomas will not be penalised for paying off the loans early.
- Thomas does not have to pay the principal and interest until the end of the loan period.

a. Explain which loan Thomas should choose if he decides to pay off the loan at the end of the first, second or third year.

b. Explain which loan Thomas should choose for these two options.
Paying off loan 1 at term
Paying off loan 2 at the end of four years

c. Thomas considers the option to pay off the loans at the end of their terms. Explain how you can determine the better option without further calculations.

d. Why would Thomas decide to choose loan 2 instead of loan 1 (paying over its full term), even if it cost him more money?

14. Jan bought a computer for her business at a cost of $2500. Her accountant told her that she was entitled to depreciate the cost of the computer over 5 years at 40% per year.

a. How much was the computer worth at the end of the first year?

b. By how much could Jan reduce her taxable income at the end of the first year? (The amount by which Jan can reduce her taxable income is equal to how much value the asset lost from one year to the next.)

c. Explain whether the amount she can deduct from her taxable income will increase or decrease at the end of the second year.

Answers

Topic 1 Financial mathematics

Exercise 1.2 Simple interest

1. $2400

2. a. $1800 **b.** $900 **c.** $1920 **d.** $720

3. a. $1500 **b.** $125 **c.** $281.25 **d.** $1080

4. a. $18.75 **b.** $55.00 **c.** $41.25 **d.** $125.33 **e.** $99.58

5. $10

6. a. $700 **b.** $10.50 **c.** $710.50 **d.** $3.16 **e.** $913.66

7. a. $52.50 **b.** $50.79 **c.** $35.92 **d.** $14.87

8. a.

Month	Balance owing	Interest	Payment	Closing balance
January	$7500.00	$112.50	$1000.00	$6612.50
February	$6612.50	$99.19	$1000.00	$5711.69
March	$5711.69	$85.68	$1000.00	$4797.37
April	$4797.37	$71.96	$1000.00	$3869.33
May	$3869.33	$58.04	$1000.00	$2927.37
June	$2927.37	$43.91	$1000.00	$1971.28
July	$1971.28	$29.57	$1000.00	$1000.85
August	$1000.85	$15.01	$1015.86	$0

b. $515.86

c. $8015.86

9.

Payment option	Immediate payment	Immediate possession	Possible extra cost	Possible price negotiation
Cash	✓	✓		✓
Lay-by	Possible deposit		✓	
Credit card		✓	✓	

Payment option	Payment	Possession	Extra cost	Price
Cash	Immediate	Immediate	Nil	Negotiable
Lay-by	Intervals	Delayed	Limited	–
Credit card	Delayed	Immediate	Possible	–

10. S1: Credit card — payment is delayed, but possession is immediate

S2: Lay-by, or cash if she has savings, would like to negotiate a lower price and has somewhere to store it.

11. $r_A = 4\%$, $r_B = 10\%$, $SI_A = \$2000$ and $SI_B = \$4500$

12. $r_A = 9.76\%$, $r_B = 14.63\%$, $SI_A = \$3121.95$ and $SI_B = \$5121.95$

13. Answers will vary.

Exercise 1.3 Buying on terms

1. a. i. $3960 **ii.** $3720 **iii.** $3950

b. The best deal is the one with the lowest cost — 20% deposit and weekly payments of $20 over 3 years.

2. a. $131.25 **b.** $55.38 **c.** $144.44

3. a. $13 600, $283.33 **b.** $40 000, $666.67 **c.** $5006.25, $278.13

d. $80 000, $666.67 **e.** $488 000, $2033.33

4. $5409.76 **5.** $4530.08

6. a. $1260 **b.** $19 504.80 **c.** $20 764.80 **d.** $8164.80

7. a. $8409.96 **b.** $2609.96 **c.** $869.99 **d.** 15%

8. a. $2375 **b.** $7125 **c.** $2565 **d.** $9690 **e.** $269.17

9. a. $2000 **b.** $4000 **c.** $540 **d.** $4540 **e.** $252.22

10. a. $226.80 **b.** $141.90 **c.** $360.94 **d.** $87.11 **e.** $85.13

11. a. $4600 **b.** $1656 **c.** $130.33 **d.** $7256

12. C

13. E. This option has the lowest interest rate and time frame when compared to all others.

14. The larger the deposit, the smaller the loan and hence the interest charged. Loans generally offer a lower rate than buying on terms.

15. $426

16. a. $3200 **b.** $28 800 **c.** $r_4 = 15\%$, $r_8 = 16\%$

d. 4 years: $49 280, 8 years: $68 864 **e.** 4 years: $960, 8 years: $684

17. The most advantageous terms are those which minimise the total monies paid. This can include accepting a larger deposit to reduce the interest paid.

18. 20 days

Exercise 1.4 Compound interest

1. a. $3244.80 **b.** $10 939.56 **c.** $24 617.98

d. $14 678.02 **e.** $14 838.45 **f.** $129 394.77

2. a. $932.52 **b.** $10 650.81 **c.** $202 760.57 **d.** $25 816.04 **e.** $3 145 511.41

3. $8784.60 **4.** $3376.26

5. a. $2837.04 **b.** $837.04

6. $17 240.15 **7.** $605.42 **8.** $18 503.86; $386.47

9. a. 0.5833% **b.** $42 891.60

10. B **11.** B **12.** C **13.** C

14. a. 0.0247% **b.** ≈ $131 319.80 **c.** ≈ $11 319.80 **d.** ≈ $519.80

15. a. i. $17 745.95 **ii.** $17 786.61 **iii.** $17 807.67 **iv.** $17 821.99

b. The interest added to the principal also earns interest.

16. a. $7920 **b.** David's investment = $8108.46

c. Because David's interest is compounded, the interest is added to the principal each quarter and earns interest itself.

17.a. $3750 interest

b. i. $3820.32 interest **ii.** $3881.05 **iii.** $3912.36

c. Compounding quarterly gives the best return.

d. If we assume that interest is compounded annually, an equivalent return of $R = 7\%$ would be achieved.

e. i. Yes **ii.** No

18. Neither is correct. The best option is to choose 3.895% p.a. compounding monthly.

19. a. Bankeast **b.** $19 976.45

20. Compound interest is added to the total at the end of each compounding period. Simple interest is a fixed amount.

21. 4 years, 8 months

Exercise 1.5 Depreciation

1. a. $14 936.14 **b.** $3584.59 **c.** $6596.65

2. $17 694.72

3. a. $19 118.26 **b.** $19 281.74

4. a. $3846.93 **b.** $6353.07 **c.** 38%

5. a. $7216.02 **b.** $45 283.98

6. a. $1.8 million **b.** $569 531.25 **c.** $135 152.44

7. B **8.** A **9.** C **10.** B **11.** A

12. 10 years

13. a. $385 million **b.** 16 years

14. a. 27%

b.
$$\begin{aligned} A &= P(1-R)^n \\ 3000 &= 48\,000(1-R)^9 \\ 0.0625 &= (1-R)^9 \\ \sqrt[9]{0.0625} &= 1-R \\ R &= 1-\sqrt[9]{0.0625} \\ R &= 0.265132 \\ r &= 26.5\ldots\% \\ r &= 27\% \end{aligned}$$

15. a. Approx. 43%

b. Trial and error: can be time consuming, answer is often an estimate; algebraic solution: correct answer calculated immediately from equation

16. a. $22 774.65 **b.** 13%

17. 1.725%

18. The depreciation formula is different from the compound interest formula in that it has a subtraction sign instead of an addition sign. This is because the value is decreasing, not increasing.

Investigation | Rich task

1.

		Period 1		Period 2	
Item	**Weight (W)**	**Price (P)**	**W × P**	**Price (P)**	**W × P**
Jeans	7	$75	525	$76	532
Hamburger	19	$3.90	74.1	$4.20	79.8
CD	10.8	$25	270	$29	313.2
Total	36.8	$103.90	869.1	$109.20	925

2. a. 106.4% **b.** 6.4%

3. 104%; 4%

		Period 1		Period 2	
Item	**Weight (W)**	**Price (P)**	**W × P**	**Price (P)**	**W × P**
Bus fare	17	$4.80	81.6	$4.95	84.15
Rent	14.1	$220	3102	$240	3384
Movie ticket	10.8	$10.50	113.4	$10.80	116.64
Air conditioner	18.3	$1200	21 960	$1240	22 692
Haircut	5.6	$18.50	103.6	$21.40	119.84
Bread	19	$2.95	56.05	$3.20	60.8
Shirt	7	$32.40	226.8	$35	245
Bottle of Scotch	8.2	$19.95	163.59	$21	172.2
Total	100	$1509.10	25 807.04	$1576.35	26 874.63

Exercise 1.6 Review questions

1. $1000

2. C

3. $54

4. $7819

5. a. $640 **b.** $5760 **c.** $2764.80 **d.** $8524.80 **e.** $177.60

6. a. $67 000 **b.** $27 000 **c.** $5400 **d.** 13.5% p.a.

7. $15 746.40

8. a. $25 808.37 **b.** $26 723.16 **c.** $27 211.79 **d.** $27 550.17

9. E **10.** B **11.** $24 403.79 **12.** 12 years

13. a. Since the interest rate is lower for loan 2 than for loan 1, Thomas should choose loan 2 if he decides to pay the loan off at the end of the first, second or third year.

b. Loan 1 at term (3 years) amounts to $9444.63. Loan 2 at the end of 4 years amounts to $9523.42. Thomas should choose loan 1.

c. Thomas should choose loan 1. At the end of its term (3 years), it amounts to less than loan 2 at 4 years, 1 year before its term is finished.

d. Thomas may not have the money to pay off loan 1 in 3 years. He may need the extra 2 years to accumulate his funds.

14. a. $1500 **b.** $1000

c. Since the depreciation of 40% is on a lower value each year, the amount Jan can deduct from her taxable income decreases every year.

TOPIC 2
Indices

[Stages 5.1, 5.2 and 5.3]

2.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the concepts covered in this topic.

2.1.1 Why learn this?

Don't you wish that your money could grow as quickly as a culture of bacteria? Perhaps it can — both financial investments and a culture of bacteria can grow exponentially, that is, according to the laws of indices. Indices are useful when a number is continually multiplied by itself, becoming very large or perhaps very small.

LEARNING SEQUENCE

2.1 Overview
2.2 [Stage 5.1] Review of index laws
2.3 [Stage 5.2] Negative indices
2.4 [Stage 5.3] Fractional indices
2.5 [Stage 5.3] Combining index laws
2.6 Review

LEARNING OUTCOMES

A student:

- uses appropriate terminology, diagrams and symbols in mathematical contexts MA5.1-1WM
- provides reasoning to support conclusions that are appropriate to the context MA5.1-3WM
- operates with algebraic expressions involving positive-integer and zero indices MA5.1-5NA
- selects appropriate notations and conventions to communicate mathematical ideas and solutions MA5.2-1WM
- constructs arguments to prove and justify results MA5.2-3WM
- applies index laws to operate with algebraic expressions involving integer indices MA5.2-7NA
- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3-1WM
- generalises mathematical ideas and techniques to analyse and solve problems efficiently MA5.3-2WM
- uses deductive reasoning in presenting arguments and formal proofs MA5.3-3WM
- performs operations with indices MA5.3-6NA

CONTENT DESCRIPTIONS

Students:

Extend and apply the index laws to variables, using positive-integer indices and the zero index (ACMNA212)
Simplify algebraic products and quotients using index laws (ACMNA231)
Apply index laws to numerical expressions with integer indices (ACMNA209)
Apply index laws to algebraic expressions involving integer indices (NSW)
Perform operations with fractional indices (ACMNA264)

Source: NSW Syllabus for the Australian curriculum

learnON RESOURCES — ONLINE ONLY

eLesson: The story of mathematics — Leibniz (eles-1840)

Note: Your teacher may now set you a pre-test to determine how familiar you are with the content in this topic.

2.2 Review of index laws [Stage 5.1]

- When a number or pronumeral is repeatedly multiplied by itself, it can be written in a shorter form called index form.
- A number written in index form has two parts, the **base** and the **index**, and is written as:

- Another name for an index is an exponent or a power.
- Performing operations on numbers or pronumerals written in index form requires the application of the index laws.
- **First Index Law:** When terms with the same base are multiplied, the indices are added.

$$a^m \times a^n = a^{m+n}$$

- **Second Index Law:** When terms with the same base are divided, the indices are subtracted.

$$a^m \div a^n = a^{m-n}$$

WORKED EXAMPLE 1

Simplify each of the following.

a $m^4n^3p \times m^2n^5p^3$ **b** $2a^2b^3 \times 3ab^4$ **c** $\dfrac{2x^5y^4}{10x^2y^3}$

THINK

a 1 Write the expression.

2 Multiply the terms with the same base by adding the indices. *Note:* $p = p^1$.

b 1 Write the expression.

2 Simplify by multiplying the coefficients, then multiply the terms with the same base by adding the indices.

c 1 Write the expression.

2 Simplify by dividing both of the coefficients by the same factor, then divide terms with the same base by subtracting the indices.

WRITE

a $m^4n^3p \times m^2n^5p^3$

$= m^{4+2}n^{3+5}p^{1+3}$

$= m^6n^8p^4$

b $2a^2b^3 \times 3ab^4$

$= 2 \times 3 \times a^{2+1} \times b^{3+4}$

$= 6a^3b^7$

c $\dfrac{2x^5y^4}{10x^2y^3}$

$= \dfrac{1x^{5-2}y^{4-3}}{5}$

$= \dfrac{x^3y}{5}$

- **Third Index Law:** Any term (excluding 0) with an index of 0 is equal to 1.

$$a^0 = 1,\ a \neq 0$$

WORKED EXAMPLE 2

Simplify each of the following.
a $(2b^3)^0$ **b** $-4(a^2b^5)^0$

THINK	WRITE
a 1 Write the expression.	**a** $(2b^3)^0$
2 Apply the Third Index Law, which states that any term (excluding 0) with an index of 0 is equal to 1.	$= 1$
b 1 Write the expression.	**b** $-4(a^2b^5)^0$
2 The entire term inside the brackets has an index of 0, so the bracket is equal to 1.	$= -4 \times 1$
3 Simplify.	$= -4$

- **Fourth Index Law:** When a power (a^m) is raised to a power, the indices are multiplied.

$$(a^m)^n = a^{mn}$$

- **Fifth Index Law:** When the base is a product, raise every part of the product to the index outside the brackets.

$$(ab)^m = a^m b^m$$

- **Sixth Index Law:** When the base is a fraction, multiply the indices of both the numerator and denominator by the index outside the brackets.

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

WORKED EXAMPLE 3

Simplify each of the following.
a $(2n^4)^3$ **b** $(3a^2b^7)^3$ **c** $\left(\frac{2x^3}{y^4}\right)^4$ **d** $(-4)^3$

THINK	WRITE
a 1 Write the term.	**a** $(2n^4)^3$
2 Apply the Fourth Index Law and simplify.	$= 2^{1\times3} \times n^{4\times3}$ $= 2^3 n^{12}$ $= 8n^{12}$
b 1 Write the expression.	**b** $(3a^2b^7)^3$
2 Apply the Fifth Index Law and simplify.	$= 3^{1\times3} \times a^{2\times3} \times b^{7\times3}$ $= 3^3 a^6 b^{21}$ $= 27a^6b^{21}$
c 1 Write the expression.	**c** $\left(\frac{2x^3}{y^4}\right)^4$
2 Apply the Sixth Index Law and simplify.	$= \frac{2^{1\times4} \times x^{3\times4}}{y^{4\times4}}$ $= \frac{16x^{12}}{y^{16}}$

d 1 Write the expression. **d** $(-4)^3$

2 Expand the expression. $= -4 \times -4 \times -4$

3 Simplify, taking careful note of the negative sign. $= -64$

learnON RESOURCES — ONLINE ONLY

Interactivity: First Index Law (int-3709)

Interactivity: Second Index Law (int-3711)

Interactivity: Third Index Law (int-3713)

Interactivity: Fourth Index Law (int-3715)

Interactivity: Fourth Index Law — multiplication (int-3716)

Interactivity: Fifth and Sixth Index Laws — multiplication (int-6063)

eLesson: Index laws (eles-1903)

Digital doc: SkillSHEET Index form (doc-5168)

Digital doc: SkillSHEET Using a calculator to evaluate numbers given in index form (doc-5169)

Exercise 2.2 Review of index laws

Individual pathways

PRACTISE
Questions:
1a–f, 2a–f, 3a–f, 4a–f, 6, 7a–f, 9, 10

CONSOLIDATE
Questions:
1d–i, 2d–i, 3a–f, 4e–l, 6, 7a–f, 9–11, 15

MASTER
Questions:
1d–l, 2d–l, 3, 4d–o, 5, 6, 7d–i, 8–18

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE1a, b Simplify each of the following.

a. $a^3 \times a^4$ **b.** $a^2 \times a^3 \times a$ **c.** $b \times b^5 \times b^2$

d. $ab^2 \times a^3b^5$ **e.** $m^2n^6 \times m^3n^7$ **f.** $a^2b^5c \times a^3b^2c^2$

g. $mnp \times m^5n^3p^4$ **h.** $2a \times 3ab$ **i.** $4a^2b^3 \times 5a^2b \times \frac{1}{2}b^5$

j. $3m^3 \times 2mn^2 \times 6m^4n^5$ **k.** $4x^2 \times \frac{1}{2}xy^3 \times 6x^3y^3$ **l.** $2x^3y^2 \times 4x \times \frac{1}{2}x^4y^4$

2. WE1c Simplify each of the following.

a. $a^4 \div a^3$ **b.** $a^7 \div a^2$ **c.** $b^6 \div b^3$

d. $\dfrac{4a^7}{3a^3}$ **e.** $\dfrac{21b^6}{7b^2}$ **f.** $\dfrac{48m^8}{12m^3}$

g. $\dfrac{m^7n^3}{m^4n^2}$ **h.** $\dfrac{2x^4y^3}{4x^4y}$ **i.** $6x^7y \div 8x^4$

j. $7ab^5c^4 \div ab^2c^4$ **k.** $\dfrac{20m^5n^3p^4}{16m^3n^3p^2}$ **l.** $\dfrac{14x^3y^4z^2}{28x^2y^2z^2}$

3. WE2 Simplify each of the following.

a. a^0 **b.** $(2b)^0$ **c.** $(3m^2)^0$ **d.** $3x^0$ **e.** $4b^0$

f. $-3 \times (2n)^0$ **g.** $4a^0 - \left(\frac{a}{4}\right)^0$ **h.** $5y^0 - 12$ **i.** $5x^0 - (5xy^2)^0$

4. WE3 Simplify each of the following.

a. $(a^2)^3$ **b.** $(2a^5)^4$ **c.** $\left(\frac{m^2}{3}\right)^4$ **d.** $\left(\frac{2n^4}{3}\right)^2$ **e.** $(a^2b)^3$

f. $(3a^3b^2)^2$ **g.** $(2m^3n^5)^4$ **h.** $\left(\frac{3m^2n}{4}\right)^3$ **i.** $\left(\frac{a^2}{b^3}\right)^2$ **j.** $\left(\frac{5m^3}{n^2}\right)^4$

k. $\left(\frac{7x}{2y^5}\right)^3$ **l.** $\left(\frac{3a}{5b^3}\right)^4$ **m.** $(-3)^5$ **n.** $(-7)^2$ **o.** $(-2)^5$

5. MC **a.** $2m^{10}n^5$ is the simplified form of:

A. $m^5n^3 \times 2m^4n^2$ **B.** $\frac{6m^{10}n^4}{3n}$ **C.** $(2m^5n^2)^2$ **D.** $2n(m^5)^2 \times n^4$ **E.** $\left(\frac{2m^5}{n^3}\right)^2$

b. The value of $4 - (5a)^0$ is:

A. -1 **B.** 9 **C.** 1 **D.** 3 **E.** 5

6. MC **a.** $4a^3b \times b^4 \times 5a^2b^3$ simplifies to:

A. $9a^5b^8$ **B.** $20a^5b^7$ **C.** $20a^5b^8$ **D.** $9a^5b^7$ **E.** $21a^5b^8$

b. $\frac{15x^9 \times 3x^6}{9x^{10} \times x^4}$ simplifies to:

A. $5x^9$ **B.** $9x$ **C.** $5x^{29}$ **D.** $9x^9$ **E.** $5x$

c. $\frac{3p^7 \times 8q^9}{12p^3 \times 4q^5}$ simplifies to:

A. $2q^4$ **B.** $\frac{p^4q^4}{2}$ **C.** $\frac{q^4}{2}$ **D.** $\frac{p^4q^4}{24}$ **E.** $\frac{q^4}{24}$

d. $\frac{7a^5b^3}{5a^6b^2} \div \frac{7b^3a^2}{5b^5a^4}$ simplifies to:

A. $\frac{49a^3b}{25}$ **B.** $\frac{25a^3b}{49}$ **C.** a^3b **D.** ab^3 **E.** $\frac{25ab^3}{49}$

7. Evaluate each of the following.

a. $2^3 \times 2^2 \times 2$ **b.** $2 \times 3^2 \times 2^2$ **c.** $(5^2)^2$ **d.** $\frac{3^5 \times 4^6}{3^4 \times 4^4}$ **e.** $(2^3 \times 5)^2$

f. $\left(\frac{3}{5}\right)^3$ **g.** $\frac{4^4 \times 5^6}{4^3 \times 5^5}$ **h.** $(3^3 \times 2^4)^0$ **i.** $4(5^2 \times 3^5)^0$

8. Simplify each of the following.

a. $(x^y)^{3z}$ **b.** $a^b \times (p^q)^0$ **c.** $m^a \times n^b \times (mn)^0$

d. $\left(\frac{a^2}{b^3}\right)^x$ **e.** $\frac{n^3m^2}{n^pm^q}$ **f.** $(a^{m+n})^p$

Communicating, reasoning and problem solving

9. Explain why $a^3 \times a^2 = a^5$ and not a^6.

10. Is $2x$ ever the same as x^2? Explain your reasoning using examples.

11. Explain the difference between $3x^0$ and $(3x)^0$.

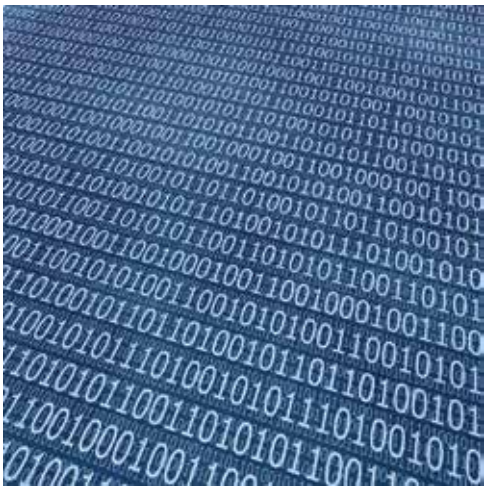
12. **a.** In the following table, enter the values of $3a^2$ and $5a$ when $a = 0$, 1, 2 and 3.

a	0	1	2	3
$3a^2$				
$5a$				
$3a^2 + 5a$				
$3a^2 \times 5a$				

b. Enter the values of $3a^2 + 5a$ and $3a^2 \times 5a$ in the table.

c. What do you think will happen as a becomes very large?

13. Find algebraically the exact value of x if $4^{x+4} = 2^{x^2}$. Justify your answer.



14. Binary numbers (base 2 numbers) are used in computer operations. As the name implies, binary uses only two types of numbers, 0 and 1, to express all numbers.

A binary number such as 101 (read 'one, zero, one') means $(1 \times 2^2) + (0 \times 2^1) + (1 \times 2^0) = 4 + 0 + 1 = 5$ in base 10, the base we are most familiar with.

The number 1010 (read 'one, zero, one, zero') means $(1 \times 2^3) + (0 \times 2^2) + (1 \times 2^1) + (0 \times 2^0) = 8 + 0 + 2 + 0 = 10$.

If we read the binary number from right to left, the index of 2 increases by one each time, beginning with a power of zero.

Using this information, write out the numbers 1 to 10 in binary (base 2) form.

15. Solve the following for x.

a. $\dfrac{7^x \times 7^{1+2x}}{(7^x)^2} = 16807$

b. $2^{2x} - 5(2^x) = -4$

16. For the following:

a. calculate the correct answer

b. identify the error in the solution.

$$\left(\frac{a^2b^3c}{a^2b^2}\right)^3 \times \left(\frac{a^3b^2c^2}{a^2b^3}\right)^2 = \left(\frac{b^3c}{b^2}\right)^3 \times \left(\frac{ab^2c^2}{b^3}\right)^2$$

$$= \left(\frac{bc}{1}\right)^3 \times \left(\frac{ac^2}{b}\right)^2$$

$$= \left(\frac{abc^3}{b}\right)^6$$

$$= \left(\frac{ac^3}{1}\right)^6$$

$$= a^6c^{18}$$

17. Why are these laws called index laws?

18. It was estimated that there were 4×10^{10} locusts in the largest swarm ever seen. If each locust can consume 2 grams of grain in a day, how long would it take the swarm to consume 1 tonne of grain?

2.3 Negative indices [Stage 5.2]

- Consider the expression $\frac{a^3}{a^5}$. This expression can be simplified in two different ways.

  1. Written in expanded form: $\frac{a^3}{a^5} = \frac{a \times a \times a}{a \times a \times a \times a \times a}$

  $$= \frac{1}{a \times a}$$

  $$= \frac{1}{a^2}$$

  2. Using the Second Index Law: $\frac{a^3}{a^5} = a^{3-5}$

  $$= a^{-2}$$

  So, $a^{-2} = \frac{1}{a^2}$.

- In general, $\frac{1}{a^n} = \frac{a^0}{a^n}$ $(1 = a^0)$

  $= a^{0-n}$ (using the Second Index Law)

  $= a^{-n}$

- **Seventh Index Law:** $a^{-n} = \frac{1}{a^n}$

- The convention is that an algebraic term should be written using positive indices and with pronumerals given in alphabetical order.

DISCUSSION

Why do we write algebraic terms in alphabetical order?

WORKED EXAMPLE 4

Express each of the following with positive indices.

a x^{-3} **b** $2m^{-4}n^2$ **c** $\frac{4}{a^{-3}}$

THINK	WRITE
a 1 Write the expression.	**a** x^{-3}
2 Apply the Seventh Index Law.	$= \frac{1}{x^3}$
b 1 Write the expression.	**b** $2m^{-4}n^2$
2 Apply the Seventh Index Law to write the expression with positive indices.	$= \frac{2n^2}{m^4}$
c 1 Write the expression and rewrite the fraction using a division sign.	**c** $\frac{4}{a^{-3}} = 4 \div a^{-3}$

2 Apply the Seventh Index Law to write the expression with positive indices.	$= 4 \div \frac{1}{a^3}$
3 To divide by the fraction, change the fraction division into multiplication.	$= 4 \times \frac{a^3}{1}$ $= 4a^3$

- Part **c** from Worked example 4 demonstrates the **converse** of the Seventh Index Law, $\frac{1}{a^{-n}} = a^n$.

WORKED EXAMPLE 5

Simplify each of the following, expressing the answers with positive indices.

a $a^2b^{-3} \times a^{-5}b$ **b** $\frac{2x^4y^2}{3xy^5}$ **c** $\left(\frac{2m^3}{n^{-2}}\right)^{-2}$

THINK	WRITE
a 1 Write the expression.	**a** $a^2b^{-3} \times a^{-5}b$
2 Apply the First Index Law. Multiply terms with the same base by adding the indices.	$= a^{2+-5}b^{-3+1}$ $= a^{-3}b^{-2}$
3 Express the answer with positive indices.	$= \frac{1}{a^3b^2}$
b 1 Write the expression.	**b** $\frac{2x^4y^2}{3xy^5}$
2 Apply the Second Index Law. Divide terms with the same base by subtracting the indices.	$= \frac{2x^{4-1}y^{2-5}}{3}$ $= \frac{2x^3y^{-3}}{3}$
3 Express the answer with positive indices.	$= \frac{2x^3}{3y^3}$
c 1 Write the expression.	**c** $\left(\frac{2m^3}{n^{-2}}\right)^{-2}$
2 Apply the Sixth Index Law. Multiply the indices of both the numerator and denominator by the index outside the brackets.	$= \frac{2^{-2}m^{-6}}{n^4}$
3 Express all terms with positive indices.	$= \frac{1}{2^2m^6n^4}$
4 Simplify the expression.	$= \frac{1}{4m^6n^4}$

- Numbers in index form can easily be evaluated if they are expressed with positive indices first. Consider the following example.

WORKED EXAMPLE 6

Evaluate 6×3^{-3} without using a calculator

THINK	WRITE
1 Write the multiplication.	6×3^{-3}
2 Apply the Seventh Index Law to write 3^{-3} with a positive index.	$= 6 \times \frac{1}{3^3}$
3 Multiply the numerator of the fraction by the whole number.	$= \frac{6}{3^3}$
4 Evaluate the denominator.	$= \frac{6}{27}$
5 Cancel by dividing both the numerator and denominator by the highest common factor (3).	$= \frac{2}{9}$

learnON RESOURCES — ONLINE ONLY

Interactivity: Colour code breaker (int-2777)
Interactivity: Negative indices (int-6064)
eLesson: Negative indices (eles-1910)

Exercise 2.3 Negative indices

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–i, 2a–i, 3a–f, 4, 5a–e, 6a–b, 8a–c, 9, 11a, 12	Questions: 1a–i, 2a–i, 3c–h, 4, 5a–g, 6, 7, 8b–e, 9, 11a–b, 12, 13, 15, 18	Questions: 1, 2c–o, 3c–l, 4, 5d–j, 6, 7, 8c–f, 9–20

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE4 Express each of the following with positive indices.

a. x^{-5} **b.** y^{-4} **c.** $2a^{-9}$ **d.** $\frac{4}{5}a^{-3}$

e. $3x^2y^{-3}$ **f.** $2^{-2}m^{-3}n^{-4}$ **g.** $6a^3b^{-1}c^{-5}$ **h.** $\frac{1}{a^{-6}}$

i. $\frac{2}{3a^{-4}}$ **j.** $\frac{6a}{3b^{-2}}$ **k.** $\frac{7a^{-4}}{2b^{-3}}$ **l.** $\frac{2m^3n^{-5}}{3a^{-2}b^4}$

2. WE5 Simplify each of the following, expressing the answers with positive indices.

a. $a^3b^{-2} \times a^{-5}b^{-1}$ **b.** $2x^{-2}y \times 3x^{-4}y^{-2}$ **c.** $3m^2n^{-5} \times m^{-2}n^{-3}$ **d.** $4a^3b^2 \div a^5b^7$

e. $2xy^6 \div 3x^2y^5$ **f.** $5x^{-2}y^3 \div 6xy^2$ **g.** $\frac{6m^4n}{2n^3m^6}$ **h.** $\frac{4x^2y^9}{x^7y^{-3}}$

i. $\dfrac{2m^2n^{-4}}{6m^5n^{-1}}$ **j.** $(2a^3m^4)^{-5}$ **k.** $4(p^7q^{-4})^{-2}$ **l.** $3(a^{-2}b^{-3})^4$

m. $\left(\dfrac{2p^2}{3q^3}\right)^{-3}$ **n.** $\left(\dfrac{a^{-4}}{2b^{-3}}\right)^2$ **o.** $\left(\dfrac{6a^2}{3b^{-2}}\right)^{-3}$

3. WE6 Evaluate each of the following without using a calculator.

a. 2^{-3} **b.** 6^{-2} **c.** 3^{-4} **d.** $3^{-2} \times 2^3$

e. $4^{-3} \times 2^2$ **f.** 5×6^{-2} **g.** $\dfrac{6}{2^{-3}}$ **h.** $\dfrac{4 \times 3^{-3}}{2^{-3}}$

i. $\frac{1}{3} \times 5^{-2} \times 3^4$ **j.** $\dfrac{16^0 \times 2^4}{8^2 \times 2^{-4}}$ **k.** $\dfrac{5^3 \times 25^0}{25^2 \times 5^{-4}}$ **l.** $\dfrac{3^4 \times 4^2}{12^3 \times 15^0}$

4. Write each of these numbers as a power of 2.

a. 8 **b.** $\frac{1}{8}$ **c.** 32 **d.** $\frac{1}{64}$

5. Complete each statement by writing the correct index.

a. $125 = 5^{\cdots}$ **b.** $\frac{1}{16} = 4^{\cdots}$ **c.** $\frac{1}{7} = 7^{\cdots}$ **d.** $216 = 6^{\cdots}$

e. $0.01 = 10^{\cdots}$ **f.** $1 = 8^{\cdots}$ **g.** $64 = 4^{\cdots}$ **h.** $\frac{1}{64} = 4^{\cdots}$

i. $\frac{1}{64} = 2^{\cdots}$ **j.** $\frac{1}{64} = 8^{\cdots}$

6. Evaluate the following expressions.

a. $\left(\frac{2}{3}\right)^{-1}$ **b.** $\left(\frac{5}{4}\right)^{-1}$ **c.** $\left(3\frac{1}{2}\right)^{-1}$ **d.** $\left(\frac{1}{5}\right)^{-1}$

7. Write the following expressions with positive indices.

a. $\left(\dfrac{a}{b}\right)^{-1}$ **b.** $\left(\dfrac{a^2}{b^3}\right)^{-1}$ **c.** $\left(\dfrac{a^{-2}}{b^{-3}}\right)^{-1}$ **d.** $\left(\dfrac{m^3}{n^{-2}}\right)^{-1}$

8. Evaluate each of the following using a calculator.

a. 3^{-6} **b.** 12^{-4} **c.** 7^{-5} **d.** $\left(\frac{1}{2}\right)^{-8}$

e. $\left(\frac{3}{4}\right)^{-7}$ **f.** $(0.04)^{-5}$

9. MC **a.** x^{-5} is the same as:

A. $-x^5$ **B.** $-5x$ **C.** $5x$ **D.** $\dfrac{1}{x^5}$ **E.** $\dfrac{1}{x^{-5}}$

b. $\dfrac{1}{a^{-4}}$ is the same as:

A. $4a$ **B.** $-4a$ **C.** a^4 **D.** $\dfrac{1}{a^4}$ **E.** $-a^4$

c. $\frac{1}{8}$ is the same as:

A. 2^3 **B.** 2^{-3} **C.** 3^2 **D.** 3^{-2} **E.** $\dfrac{1}{2^{-3}}$

10. MC **a.** Which of the following, when simplified, gives $\dfrac{3m^4}{4n^2}$?

A. $\dfrac{3m^{-4}n^{-2}}{4}$ **B.** $3 \times 2^{-2} \times m^4 \times n^{-2}$ **C.** $\dfrac{3n^{-2}}{2^{-2}m^{-4}}$

D. $\dfrac{2^2n^{-2}}{3^{-1}m^{-4}}$ **E.** $3m^4 \times 2^2n^{-2}$

b. When simplified, $3a^{-2}b^{-7} \div \frac{3}{4}a^{-4}b^6$ is equal to:

A. $\dfrac{4}{a^6b^{13}}$ **B.** $\dfrac{9b}{4a^6}$ **C.** $\dfrac{9a^2}{4b}$ **D.** $\dfrac{4a^2}{b^{13}}$ **E.** $\dfrac{4a^2}{b}$

c. When $(2x^6y^{-4})^{-3}$ is simplified, it is equal to:

A. $\dfrac{2x^{18}}{y^{12}}$ **B.** $\dfrac{x^{18}}{8y^{12}}$ **C.** $\dfrac{y^{12}}{8x^{18}}$ **D.** $\dfrac{8y^{12}}{x^{18}}$ **E.** $\dfrac{x^{18}}{6y^{12}}$

d. If $\left(\frac{2a^x}{b^y}\right)^3$ is equal to $\frac{8b^9}{a^6}$, then x and y (in that order) are:

A. -3 and -6 **B.** -6 and -3 **C.** -3 and 2 **D.** -3 and -2 **E.** -2 and -3

11. Simplify the following, expressing your answers with positive indices.

a. $\frac{m^{-3}n^{-2}}{m^{-5}n^6}$ **b.** $\frac{(m^3n^{-2})^{-7}}{(m^{-5}n^3)^4}$ **c.** $\frac{5(a^3b^{-3})^2}{(ab^{-4})^{-1}} \div \frac{(5a^{-2}b)^{-1}}{(a^{-4}b)^3}$

12. Simplify the following, expanding any expressions in brackets.

a. $(r^3 + s^3)(r^3 - s^3)$ **b.** $(m^5 + n^5)^2$

c. $\frac{(x^{a+1})^b \times x^{a+b}}{x^{a(b+1)} \times x^{2b}}$ **d.** $\left(\frac{p^{x+1}}{p^{x-1}}\right)^{-4} \times \frac{p^{8(x+1)}}{(p^{2x})^4} \times \frac{p^2}{(p^{12x})^0}$

13. Write $\left(\frac{2^r \times 8^r}{2^{2r} \times 16}\right)$ in the form 2^{ar+b}.

14. Write $2^{-m} \times 3^{-m} \times 6^{2m} \times 3^{2m} \times 2^{2m}$ as a power of 6.

15. Solve for x if $4^x - 4^{x-1} = 48$.

Communicating, reasoning and problem solving

16. Explain why each of these statements is false. Illustrate each answer by substituting a value for the pronumeral.

a. $5x^0 = 1$ **b.** $9x^5 \div 3x^5 = 3x$ **c.** $a^5 \div a^7 = a^2$ **d.** $2c^{-4} = \frac{1}{2c^4}$

17. Solve for x and y if $5^{x-y} = 625$ and $3^{2x} \times 3^y = 243$.

Hence, evaluate $\frac{35^x}{7^{-2y} \times 5^{-3y}}$.

18. Solve for n. Verify your answers.

a. $(2^n)^n \times (2^n)^3 \times 4 = 1$ **b.** $\frac{(3^n)^n \times (3^n)^{-1}}{81} = 1$

19. Are there any index laws from Section 2.2 that do not apply to negative indices?

20. Are x^{-2} and $-2x$ equivalent expressions? Why or why not?

2.4 Fractional indices [Stage 5.3]

- Terms with fractional indices can be written as surds, using the following laws:

  1. $a^{\frac{1}{n}} = \sqrt[n]{a}$

  2. $a^{\frac{m}{n}} = \sqrt[n]{a^m}$
     $= (\sqrt[n]{a})^m$

- To understand how these laws are formed, consider the following numerical examples.

  We know $4^{\frac{1}{2}} \times 4^{\frac{1}{2}} = 4^1$

  and that $\sqrt{4} \times \sqrt{4} = \sqrt{16}$
  $= 4$

  It follows, then, that $4^{\frac{1}{2}} = \sqrt{4}$.

  Similarly, we know that $8^{\frac{1}{3}} \times 8^{\frac{1}{3}} \times 8^{\frac{1}{3}} = 8^1$

  and that $\sqrt[3]{8} \times \sqrt[3]{8} \times \sqrt[3]{8} = \sqrt[3]{512}$
  $= 8$

  It follows, then, that $8^{\frac{1}{3}} = \sqrt[3]{8}$.

This observation can be generalised to $a^{\frac{1}{n}} = \sqrt[n]{a}$.

Now consider: $a^{\frac{m}{n}} = a^{m \times \frac{1}{n}}$ or $a^{\frac{m}{n}} = a^{\frac{1}{n} \times m}$

$= (a^m)^{\frac{1}{n}}$ $= (a^{\frac{1}{n}})^m$

$= \sqrt[n]{a^m}$ $= (\sqrt[n]{a})^m$

- **Eighth Index Law:** $a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

WORKED EXAMPLE 7

Evaluate each of the following without using a calculator.

a $9^{\frac{1}{2}}$ **b** $16^{\frac{3}{2}}$

THINK	WRITE
a 1 Rewrite the number using the Eighth Index Law.	**a** $9^{\frac{1}{2}} = \sqrt{9}$
2 Evaluate the result.	$= 3$
b 1 Rewrite the number using $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$.	**b** $16^{\frac{3}{2}} = (\sqrt{16})^3$
	$= 4^3$
2 Simplify and evaluate the result.	$= 64$

WORKED EXAMPLE 8

Simplify each of the following.

a $m^{\frac{1}{5}} \times m^{\frac{2}{5}}$ **b** $(a^2b^3)^{\frac{1}{6}}$ **c** $\left(\frac{x^{\frac{2}{3}}}{y^{\frac{3}{4}}}\right)^{\frac{1}{2}}$

THINK	WRITE
a 1 Write the expression.	**a** $m^{\frac{1}{5}} \times m^{\frac{2}{5}}$
2 Apply the First Index Law to multiply terms with the same base by adding the indices.	$= m^{\frac{3}{5}}$
b 1 Write the expression.	**b** $(a^2b^3)^{\frac{1}{6}}$
2 Use the Fourth Index Law to multiply each index inside the brackets by the index outside the brackets.	$= a^{\frac{2}{6}}b^{\frac{3}{6}}$
3 Simplify.	$= a^{\frac{1}{3}}b^{\frac{1}{2}}$
c 1 Write the expression.	**c** $\left(\frac{x^{\frac{2}{3}}}{y^{\frac{3}{4}}}\right)^{\frac{1}{2}}$
2 Use the Sixth Index Law to multiply the index in both the numerator and denominator by the index outside the brackets.	$= \frac{x^{\frac{1}{3}}}{y^{\frac{3}{8}}}$

DISCUSSION

Why is finding the square root of an expression the same as raising the expression to the power of a half?

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Interactivity: Fractional indices (int-6107)
eLesson: Fractional indices (eles-1950)
Digital doc: SkillSHEET Addition of fractions (doc-5176)
Digital doc: SkillSHEET Subtraction of fractions (doc-5177)
Digital doc: SkillSHEET Multiplication of fractions (doc-5178)
Digital doc: SkillSHEET Writing roots as fractional indices (doc-5179)
Digital doc: WorkSHEET Fractional indices (doc-5180)

Exercise 2.4 Fractional indices

Individual pathways

PRACTISE
Questions:
1–5, 6a, d, g, 7a, d, 8a, d, g, 9a, d, 10a, d, g, 11a, d, g, 12, 13, 14a, d, g, 15, 16

CONSOLIDATE
Questions:
1–5, 6a, b, e, h, i, 7a, b, c, f, 8a, b, d, e, g, h, 9a, b, d, e, 10b, e, h, 11b, e, h, 12, 13, 14b, e, h, 15, 16, 17

MASTER
Questions:
1–5, 6c, f, i, 7c, f, 8c, f, i, 9b, c, e, f, 10c, f, i, 11c, f, i, 12–20

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE7 Evaluate each of the following without using a calculator.

a. $16^{\frac{1}{2}}$ **b.** $25^{\frac{1}{2}}$ **c.** $81^{\frac{1}{2}}$ **d.** $8^{\frac{1}{3}}$ **e.** $64^{\frac{1}{3}}$ **f.** $81^{\frac{1}{4}}$

2. Write the following in surd form.

a. $15^{\frac{1}{2}}$ **b.** $m^{\frac{1}{4}}$ **c.** $7^{\frac{2}{5}}$ **d.** $7^{\frac{5}{2}}$ **e.** $w^{\frac{3}{8}}$ **f.** $w^{1.25}$

g. $5^{33^{\frac{1}{3}}}$ **h.** $a^{0.3}$

3. Write the following in index form.

a. $\sqrt{t}$ **b.** $\sqrt[4]{5^7}$ **c.** $\sqrt[6]{6^{11}}$ **d.** $\sqrt[7]{x^6}$ **e.** $\sqrt[6]{x^7}$ **f.** $\sqrt[5]{w^{10}}$

g. $\sqrt[5]{w^{10}}$ **h.** $\sqrt[x]{11^n}$

4. Without using a calculator, find the exact value of each of the following.

a. $8^{\frac{2}{3}}$ **b.** $8^{\frac{4}{3}}$ **c.** $32^{\frac{3}{5}}$ **d.** $32^{\frac{4}{5}}$ **e.** $25^{\frac{3}{2}}$ **f.** $27^{\frac{2}{3}}$

g. $27^{\frac{-2}{3}}$ **h.** $81^{\frac{3}{4}}$ **i.** $10^{\frac{6}{2}}$ **j.** $36^{1\frac{1}{2}}$ **k.** $7^{\frac{1}{2}}$ **l.** $12^{\frac{1}{3}}$

5. Using a calculator, evaluate each of the following. Give answers correct to 2 decimal places.

a. $3^{\frac{1}{3}}$ **b.** $5^{\frac{1}{2}}$ **c.** $7^{\frac{1}{5}}$ **d.** $8^{\frac{1}{9}}$ **e.** $12^{\frac{3}{8}}$ **f.** $(0.6)^{\frac{4}{5}}$

g. $\left(\frac{2}{3}\right)^{\frac{3}{2}}$ **h.** $\left(\frac{3}{4}\right)^{\frac{3}{4}}$ **i.** $\left(\frac{4}{5}\right)^{\frac{2}{3}}$

6. WE8a Simplify each of the following.

a. $4^{\frac{3}{5}} \times 4^{\frac{1}{5}}$ **b.** $2^{\frac{1}{8}} \times 2^{\frac{3}{8}}$ **c.** $a^{\frac{1}{2}} \times a^{\frac{1}{3}}$ **d.** $x^{\frac{3}{4}} \times x^{\frac{2}{5}}$ **e.** $5m^{\frac{1}{3}} \times 2m^{\frac{1}{5}}$ **f.** $\frac{1}{2}b^{\frac{3}{7}} \times 4b^{\frac{2}{7}}$

g. $-4y^2 \times y^{\frac{2}{9}}$ **h.** $\frac{2}{5}a^{\frac{3}{8}} \times 0.05a^{\frac{3}{4}}$ **i.** $5x^3 \times x^{\frac{1}{2}}$

7. Simplify each of the following.

a. $a^{\frac{2}{3}}b^{\frac{3}{4}} \times a^{\frac{1}{3}}b^{\frac{3}{4}}$ **b.** $x^{\frac{3}{5}}y^{\frac{2}{9}} \times x^{\frac{1}{5}}y^{\frac{1}{3}}$ **c.** $2ab^{\frac{1}{3}} \times 3a^{\frac{3}{5}}b^{\frac{4}{5}}$

d. $6m^{\frac{3}{7}} \times \frac{1}{3}m^{\frac{1}{4}}n^{\frac{2}{5}}$ **e.** $x^3y^{\frac{1}{2}}z^{\frac{1}{3}} \times x^{\frac{1}{6}}y^{\frac{1}{3}}z^{\frac{1}{2}}$ **f.** $2a^{\frac{2}{5}}b^{\frac{3}{8}}c^{\frac{1}{4}} \times 4b^{\frac{3}{4}}c^{\frac{3}{4}}$

8. Simplify each of the following.

a. $3^{\frac{1}{2}} \div 3^{\frac{1}{3}}$ **b.** $5^{\frac{2}{3}} \div 5^{\frac{1}{4}}$ **c.** $12^2 \div 12^{\frac{3}{2}}$ **d.** $a^{\frac{6}{7}} \div a^{\frac{3}{7}}$ **e.** $x^{\frac{3}{2}} \div x^{\frac{1}{4}}$ **f.** $\dfrac{m^{\frac{4}{5}}}{m^{\frac{5}{9}}}$

g. $\dfrac{2x^{\frac{3}{4}}}{4x^{\frac{3}{5}}}$ **h.** $\dfrac{7n^2}{21n^{\frac{4}{3}}}$ **i.** $\dfrac{25b^{\frac{3}{5}}}{20b^{\frac{1}{4}}}$

9. Simplify each of the following.

a. $x^3y^2 \div x^{\frac{4}{3}}y^{\frac{3}{5}}$ **b.** $a^{\frac{5}{9}}b^{\frac{2}{3}} \div a^{\frac{2}{5}}b^{\frac{2}{5}}$ **c.** $m^{\frac{3}{8}}n^{\frac{4}{7}} \div 3n^{\frac{3}{8}}$

d. $10x^{\frac{4}{5}}y \div 5x^{\frac{2}{3}}y^{\frac{1}{4}}$ **e.** $\dfrac{5a^{\frac{3}{4}}b^{\frac{3}{5}}}{20a^{\frac{1}{5}}b^{\frac{1}{4}}}$ **f.** $\dfrac{p^{\frac{7}{8}}q^{\frac{1}{4}}}{7p^{\frac{2}{3}}q^{\frac{1}{6}}}$

10. Simplify each of the following.

a. $(2^{\frac{3}{4}})^{\frac{3}{5}}$ **b.** $(5^{\frac{2}{3}})^{\frac{1}{4}}$ **c.** $(7^{\frac{1}{5}})^6$ **d.** $(a^3)^{\frac{1}{10}}$ **e.** $(m^{\frac{4}{9}})^{\frac{3}{8}}$ **f.** $(2b^{\frac{1}{2}})^{\frac{1}{3}}$

g. $4(p^{\frac{3}{7}})^{\frac{14}{15}}$ **h.** $(x^{\frac{m}{n}})^{\frac{n}{p}}$ **i.** $(3m^{\frac{a}{b}})^{\frac{b}{c}}$

11. WE8b, c Simplify each of the following.

a. $(a^{\frac{1}{2}}b^{\frac{1}{3}})^{\frac{1}{2}}$ **b.** $(a^4b)^{\frac{3}{4}}$ **c.** $(x^{\frac{3}{5}}y^{\frac{7}{8}})^2$ **d.** $(3a^{\frac{1}{3}}b^{\frac{3}{5}}c^{\frac{3}{4}})^{\frac{1}{3}}$ **e.** $5(x^{\frac{1}{2}}y^{\frac{2}{3}}z^{\frac{2}{5}})^{\frac{1}{2}}$ **f.** $\left(\dfrac{a^{\frac{3}{4}}}{b}\right)^{\frac{2}{3}}$

g. $\left(\dfrac{m^{\frac{4}{5}}}{n^{\frac{7}{8}}}\right)^2$ **h.** $\left(\dfrac{b^{\frac{3}{5}}}{c^{\frac{4}{9}}}\right)^{\frac{1}{2}}$ **i.** $\left(\dfrac{4x^7}{2y^{\frac{3}{4}}}\right)^{\frac{1}{2}}$

12. MC **a.** $y^{\frac{2}{5}}$ is equal to:

A. $(y^{\frac{1}{2}})^5$ **B.** $y \times \dfrac{2}{5}$ **C.** $(y^5)^{\frac{1}{2}}$ **D.** $2\sqrt[5]{y}$ **E.** $(y^{\frac{1}{5}})^2$

b. $k^{\frac{2}{3}}$ is not equal to:

A. $(k^{\frac{1}{3}})^2$ **B.** $\sqrt[3]{k^2}$ **C.** $(k^{\frac{1}{2}})^3$ **D.** $(\sqrt[3]{k})^2$ **E.** $(k^2)^{\frac{1}{3}}$

c. $\dfrac{1}{\sqrt[5]{g^2}}$ is equal to:

A. $g^{\frac{2}{5}}$ **B.** $g^{-\frac{2}{5}}$ **C.** $g^{\frac{5}{2}}$ **D.** $g^{-\frac{5}{2}}$ **E.** $2g^{\frac{1}{5}}$

13. MC **a.** If $(a^{\frac{3}{4}})^{\frac{m}{n}}$ is equal to $a^{\frac{1}{4}}$, then m and n could not be:

A. 1 and 3 **B.** 2 and 6 **C.** 3 and 8 **D.** 4 and 9 **E.** both C and D

b. When simplified, $\left(\dfrac{a^{\frac{m}{n}}}{b^{\frac{n}{p}}}\right)^{\frac{p}{m}}$ is equal to:

A. $\dfrac{a^{\frac{m}{p}}}{b^{\frac{n}{m}}}$ **B.** $\dfrac{a^{\frac{p}{n}}}{b^{\frac{n}{m}}}$ **C.** $\dfrac{a^{\frac{mp}{n}}}{b^{\frac{n}{m}}}$ **D.** $\dfrac{a^p}{b^m}$ **E.** $\dfrac{a^{\frac{m^2}{np}}}{b^{\frac{nm}{p^2}}}$

14. Simplify each of the following.

a. $\sqrt{a^8}$ **b.** $\sqrt[3]{b^9}$ **c.** $\sqrt[4]{m^{16}}$ **d.** $\sqrt{16x^4}$ **e.** $\sqrt[3]{8y^9}$

f. $\sqrt[4]{16x^8y^{12}}$ **g.** $\sqrt[3]{27m^9n^{15}}$ **h.** $\sqrt[5]{32p^5q^{10}}$ **i.** $\sqrt[3]{216a^6b^{18}}$

15. The relationship between the length of a pendulum (L) in a grandfather clock and the time it takes to complete one swing (T) in seconds is given by the following rule. Note that g is the acceleration due to gravity and should be taken as 9.8.

$$T = 2\pi\left(\frac{L}{g}\right)^{\frac{1}{2}}$$

a. Calculate the time it takes a 1-m-long pendulum to complete one swing.

b. Calculate the time it takes the pendulum to complete 10 swings.

c. How many swings will be completed after 10 seconds?

Communicating, reasoning and problem solving

16. Using the index laws, show that $\sqrt[5]{32a^5b^{10}} = 2ab^2$.

17. To rationalise a fraction means to remove all non-rational numbers from the denominator of the fraction. Rationalise $\dfrac{a^2}{3+\sqrt{b^3}}$ by multiplying the numerator and denominator by $3-\sqrt{b^3}$, and then evaluate if $b = a^2$ and $a = 2$. Show all of your working.

18. Simplify $\dfrac{m^{\frac{2}{5}} - 2m^{\frac{1}{5}}n^{\frac{1}{5}} + n^{\frac{2}{5}} - p^{\frac{2}{5}}}{m^{\frac{1}{5}} - n^{\frac{1}{5}} - p^{\frac{1}{5}}}$.

19. A scientist has discovered a piece of paper with a complex formula written on it. She thinks that someone has tried to disguise a simpler formula. The formula is:

$$\frac{\sqrt[4]{a^{13}}a^2\sqrt{b^3}}{\sqrt{a^1b}} \times b^3 \times \left(\frac{\sqrt{a^3b}}{ab^2}\right)^2 \times \left(\frac{b^2}{a^2\sqrt{b}}\right)^3$$

a. Simplify the formula using index laws so that it can be worked with.

b. From your simplified formula, can a take a negative value? Explain.

c. What is the smallest value for a for which the expression will give a rational answer? Consider only integers.

20. Why is it easier to perform operations with fractional indices than with expressions using surds?

2.5 Combining index laws [Stage 5.3]

- When several steps are needed to simplify an expression, expand brackets first.
- When fractions are involved, it is usually easier to carry out all multiplications first, leaving one division as the final process.
- Final answers are conventionally written using positive indices.

DISCUSSION

Do you still need to follow the correct order of operations when simplifying expressions with indices? Use some examples to explain your response.

WORKED EXAMPLE 9

Simplify each of the following.

a $\dfrac{(2a)^4b^4}{6a^3b^2}$ **b** $\dfrac{3^{n-2} \times 9^{n+1}}{81^{n-1}}$

THINK	WRITE
a 1 Write the expression.	**a** $\dfrac{(2a)^4b^4}{6a^3b^2}$
2 Apply the Fourth Index Law to remove the bracket.	$= \dfrac{16a^4b^4}{6a^3b^2}$
3 Apply the Second Index Law for each number and pronumeral to simplify.	$= \dfrac{8a^{4-3}b^{4-2}}{3}$
4 Write the answer.	$= \dfrac{8ab^2}{3}$
b 1 Write the expression.	**b** $\dfrac{3^{n-2} \times 9^{n+1}}{81^{n-1}}$
2 Rewrite each term in the expression so that it has a base of 3.	$= \dfrac{3^{n-2} \times (3^2)^{n+1}}{(3^4)^{n-1}}$
3 Apply the Fourth Index Law to expand the brackets.	$= \dfrac{3^{n-2} \times 3^{2n+2}}{3^{4n-4}}$
4 Apply the First and Second Index Laws to simplify and write your answer.	$= \dfrac{3^{3n}}{3^{4n-4}}$ $= \dfrac{1}{3^{n-4}}$

WORKED EXAMPLE 10

Simplify each of the following.

a $(2a^3b)^4 \times 4a^2b^3$ **b** $\dfrac{7xy^3}{(3x^3y^2)^2}$ **c** $\dfrac{2m^5n \times 3m^7n^4}{7m^3n^3 \times mn^2}$

THINK	WRITE
a 1 Write the expression.	**a** $(2a^3b)^4 \times 4a^2b^3$
2 Apply the Fourth Index Law. Multiply each index inside the brackets by the index outside the brackets.	$= 2^4a^{12}b^4 \times 4a^2b^3$
3 Evaluate the number.	$= 16a^{12}b^4 \times 4a^2b^3$
4 Multiply coefficients and multiply pronumerals. Apply the First Index Law to multiply terms with the same base by adding the indices.	$= 16 \times 4 \times a^{12+2}b^{4+3}$ $= 64a^{14}b^7$
b 1 Write the expression.	**b** $\dfrac{7xy^3}{(3x^3y^2)^2}$
2 Apply the Fourth Index Law in the denominator. Multiply each index inside the brackets by the index outside the brackets.	$= \dfrac{7xy^3}{9x^6y^4}$
3 Apply the Second Index Law. Divide terms with the same base by subtracting the indices.	$= \dfrac{7x^{-5}y^{-1}}{9}$
4 Use $a^{-m} = \dfrac{1}{a^m}$ to express the answer with positive indices.	$= \dfrac{7}{9x^5y}$
c 1 Write the expression.	**c** $\dfrac{2m^5n \times 3m^7n^4}{7m^3n^3 \times mn^2}$
2 Simplify each numerator and denominator by multiplying coefficients and then terms with the same base.	$= \dfrac{6m^{12}n^5}{7m^4n^5}$
3 Apply the Second Index Law. Divide terms with the same base by subtracting the indices.	$= \dfrac{6m^8n^0}{7}$
4 Simplify the numerator using $a^0 = 1$.	$= \dfrac{6m^8 \times 1}{7}$ $= \dfrac{6m^8}{7}$

WORKED EXAMPLE 11

Simplify each of the following.

a $\dfrac{(5a^2b^3)^2}{a^{10}} \times \dfrac{a^2b^5}{(a^3b)^7}$ **b** $\dfrac{8m^3n^{-4}}{(6mn^2)^3} \div \dfrac{4m^{-2}n^{-4}}{6m^{-5}n}$

THINK	WRITE
a 1 Write the expression.	**a** $\dfrac{(5a^2b^3)^2}{a^{10}} \times \dfrac{a^2b^5}{(a^3b)^7}$
2 Remove the brackets in the numerator of the first fraction and in the denominator of the second fraction.	$= \dfrac{25a^4b^6}{a^{10}} \times \dfrac{a^2b^5}{a^{21}b^7}$
3 Multiply the numerators and then multiply the denominators of the fractions. (Simplify across.)	$= \dfrac{25a^6b^{11}}{a^{31}b^7}$

4 Divide terms with the same base by subtracting the indices. (Simplify down.)

$= 25a^{-25}b^4$

5 Express the answer with positive indices.

$= \frac{25b^4}{a^{25}}$

b 1 Write the expression.

b $\frac{8m^3n^{-4}}{(6mn^2)^3} \div \frac{4m^{-2}n^{-4}}{6m^{-5}n}$

2 Remove the brackets.

$= \frac{8m^3n^{-4}}{216m^3n^6} \div \frac{4m^{-2}n^{-4}}{6m^{-5}n}$

3 Change the division to multiplication.

$= \frac{8m^3n^{-4}}{216m^3n^6} \times \frac{6m^{-5}n}{4m^{-2}n^{-4}}$

4 Multiply the numerators and then multiply the denominators. (Simplify across.)

$= \frac{48m^{-2}n^{-3}}{864mn^2}$

5 Cancel common factors and divide pronumerals with the same base. (Simplify down.)

$= \frac{m^{-3}n^{-5}}{18}$

6 Simplify and express the answer with positive indices.

$= \frac{1}{18m^3n^5}$

- Note that the whole numbers in part **b** of Worked example 11 could be cancelled in step 3.

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Interactivity: Combining index laws (int-6108)

Digital doc: WorkSHEET Combining index laws (doc-5181)

Exercise 2.5 Combining index laws

Individual pathways

PRACTISE
Questions:
1a–d, 2a–d, 3a–d, 4a–d, 5a–d, 6, 7, 9, 10, 11a, d, 12

CONSOLIDATE
Questions:
1c–h, 2c–f, 3c–g, 4b–f, 5c–f, 6–10, 11b–e, 12

MASTER
Questions:
1f–j, 2e–i, 3f–i, 4d–f, 5e–h, 6–10, 11c–f, 12–17

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE10a Simplify each of the following.

a. $(3a^2b^2)^3 \times 2a^4b^3$
b. $(4ab^5)^2 \times 3a^3b^6$
c. $2m^3n^{-5} \times (m^2n^{-3})^{-6}$
d. $(2pq^3)^2 \times (5p^2q^4)^3$
e. $(2a^7b^2)^2 \times (3a^3b^3)^2$
f. $5(b^2c^{-2})^3 \times 3(bc^5)^{-4}$
g. $6x^{\frac{1}{2}}y^{\frac{1}{3}} \times (4x^{\frac{3}{4}}y^{\frac{4}{5}})^{\frac{1}{2}}$
h. $(16m^3n^4)^{\frac{3}{4}} \times (m^{\frac{1}{2}}n^{\frac{1}{4}})^3$
i. $2(p^{\frac{2}{3}}q^{\frac{1}{3}})^{-\frac{3}{4}} \times 3(p^{\frac{1}{4}}q^{-\frac{3}{4}})^{-\frac{1}{3}}$
j. $(8p^{\frac{1}{5}}q^{\frac{2}{3}})^{-\frac{1}{3}} \times (64p^{\frac{1}{3}}q^{\frac{3}{4}})^{\frac{2}{3}}$

2. WE10b Simplify each of the following.

a. $\dfrac{5a^2b^3}{(2a^3b)^3}$ **b.** $\dfrac{4x^5y^6}{(2xy^3)^4}$ **c.** $\dfrac{(3m^2n^3)^3}{(2m^5n^5)^7}$ **d.** $\left(\dfrac{4x^3y^{10}}{2x^7y^4}\right)^6$ **e.** $\dfrac{3a^3b^{-5}}{(2a^7b^4)^{-3}}$

f. $\left(\dfrac{3g^2h^5}{2g^4h}\right)^3$ **g.** $\dfrac{(5p^6q^{\frac{1}{3}})^2}{25(p^{\frac{1}{2}}q^{\frac{1}{4}})^{\frac{2}{3}}}$ **h.** $\left(\dfrac{3b^2c^3}{5b^{-3}c^{-4}}\right)^{-4}$ **i.** $\dfrac{(x^{\frac{1}{3}}y^{\frac{1}{4}}z^{\frac{1}{2}})^2}{(x^{\frac{2}{3}}y^{-\frac{1}{4}}z^{\frac{1}{3}})^{-\frac{3}{2}}}$

3. WE10c Simplify each of the following.

a. $\dfrac{2a^2b \times 3a^3b^4}{4a^3b^5}$ **b.** $\dfrac{4m^6n^3 \times 12mn^5}{6m^7n^6}$ **c.** $\dfrac{10m^6n^5 \times 2m^2n^3}{12m^4n \times 5m^2n^3}$

d. $\dfrac{6x^3y^2 \times 4x^6y}{9xy^5 \times 2x^3y^6}$ **e.** $\dfrac{(6x^3y^2)^4}{9x^5y^2 \times 4xy^7}$ **f.** $\dfrac{5x^2y^3 \times 2xy^5}{10x^3y^4 \times x^4y^2}$

g. $\dfrac{a^3b^2 \times 2(ab^5)^3}{6(a^2b^3)^3 \times a^4b}$ **h.** $\dfrac{(p^6q^2)^{-3} \times 3pq}{2p^{-4}q^{-2} \times (5pq^4)^{-2}}$ **i.** $\dfrac{6x^{\frac{3}{2}}y^{\frac{1}{2}} \times x^{\frac{4}{5}}y^{\frac{3}{5}}}{2(x^{\frac{1}{2}}y)^{\frac{1}{5}} \times 3x^{\frac{1}{2}}y^{\frac{1}{5}}}$

4. WE11a Simplify each of the following.

a. $\dfrac{a^3b^2}{5a^4b^7} \times \dfrac{2a^6b}{a^9b^3}$ **b.** $\dfrac{(2a^6)^2}{10a^7b^3} \times \dfrac{4ab^6}{6a^3}$ **c.** $\dfrac{(m^4n^3)^2}{(m^6n)^4} \times \dfrac{(m^3n^3)^3}{(2mn)^2}$

d. $\left(\dfrac{2m^3n^2}{3mn^5}\right)^3 \times \dfrac{6m^2n^4}{4m^3n^{10}}$ **e.** $\left(\dfrac{2xy^2}{3x^3y^5}\right)^4 \times \left(\dfrac{x^3y^9}{2y^{10}}\right)^2$ **f.** $\dfrac{4x^{-5}y^{-3}}{(x^2y^2)^{-2}} \times \dfrac{3x^5y^6}{2^{-2}x^{-7}y}$

g. $\dfrac{5p^6q^{-5}}{3q^{-4}} \times \left(\dfrac{5p^6q^4}{3p^5}\right)^{-2}$ **h.** $\dfrac{2a^{\frac{1}{2}}b^{\frac{1}{3}}}{6a^{\frac{1}{3}}b^{\frac{1}{2}}} \times \dfrac{(4a^{\frac{1}{4}}b)^{\frac{1}{2}}}{b^{\frac{1}{4}}a}$ **i.** $\dfrac{3x^{\frac{2}{3}}y^{\frac{1}{5}}}{9x^{\frac{1}{3}}y^{\frac{1}{4}}} \times \dfrac{4x^{\frac{1}{2}}}{x^{\frac{3}{4}}y}$

5. WE11b Simplify each of the following.

a. $\dfrac{5a^2b^3}{6a^7b^5} \div \dfrac{a^9b^4}{3ab^6}$ **b.** $\dfrac{7a^2b^4}{3a^6b^7} \div \left(\dfrac{3ab}{2a^6b^4}\right)^3$ **c.** $\left(\dfrac{4a^9}{b^6}\right)^3 \div \left(\dfrac{3a^7}{2b^5}\right)^4$

d. $\dfrac{5x^2y^6}{(2x^4y^5)^2} \div \dfrac{(4x^6y)^3}{10xy^3}$ **e.** $\left(\dfrac{x^5y^{-3}}{2xy^5}\right)^{-4} \div \dfrac{4x^6y^{-10}}{(3x^{-2}y^2)^{-3}}$ **f.** $\dfrac{3m^3n^4}{2m^{-6}n^{-5}} \div \left(\dfrac{2m^4n^6}{m^{-1}n}\right)^{-2}$

g. $4m^{\frac{1}{2}}n^{\frac{3}{4}} \div \dfrac{6m^{\frac{1}{3}}n^{\frac{1}{4}}}{8m^{\frac{3}{4}}n^{\frac{1}{2}}}$ **h.** $\left(\dfrac{4b^3c^{\frac{1}{3}}}{6c^{\frac{1}{5}}b}\right)^{\frac{1}{2}} \div (2b^3c^{-\frac{1}{5}})^{-\frac{3}{2}}$

6. Evaluate each of the following.

a. $(5^2 \times 2)^0 \times (5^{-3} \times 2^0)^5 \div (5^6 \times 2^{-1})^{-3}$ **b.** $(2^3 \times 3^3)^{-2} \div \dfrac{(2^6 \times 3^9)^0}{2^6 \times (3^{-2})^{-3}}$

7. Evaluate the following for $x = 8$. (*Hint:* Simplify first.)

$(2x)^{-3} \times \left(\dfrac{x}{2}\right)^2 \div \dfrac{2x}{(2^3)^4}$

8. a. Simplify the following fraction. $\dfrac{a^{2y} \times 9b^y \times (5ab)^y}{(a^y)^3 \times 5(3b^y)^2}$

b. Find the value of y if the fraction is equal to 125.

9. MC Which of the following is not the same as $(4xy)^{\frac{3}{2}}$?

A. $8x^{\frac{3}{2}}y^{\frac{3}{2}}$ **B.** $(\sqrt{4xy})^3$ **C.** $\sqrt{64x^3y^3}$ **D.** $\dfrac{(2x^3y^3)^{\frac{1}{2}}}{(\sqrt{32})^{-1}}$ **E.** $4xy^{\frac{1}{2}} \times (2xy^2)^{\frac{1}{2}}$

10. MC The expression $\frac{x^2y}{(2xy^2)^3} \div \frac{xy}{16x^0}$ is equal to:

A. $\frac{2}{x^2y^6}$ **B.** $\frac{2x^2}{b^6}$ **C.** $2x^2y^6$ **D.** $\frac{2}{xy^6}$ **E.** $\frac{1}{128xy^5}$

11. Simplify the following.

a. $\sqrt[3]{m^2n} \div \sqrt{mn^3}$ **b.** $(g^{-2}h)^3 \times \left(\frac{1}{n^{-3}}\right)^{\frac{1}{2}}$ **c.** $\frac{45^{\frac{1}{3}}}{9^{\frac{3}{4}} \times 15^{\frac{3}{2}}}$

d. $2^{\frac{3}{2}} \times 4^{-\frac{1}{4}} \times 16^{-\frac{3}{4}}$ **e.** $\left(\frac{a^3b^{-2}}{3^{-3}b^{-3}}\right)^{-2} \div \left(\frac{3^{-3}a^{-2}b}{a^4b^{-2}}\right)^2$ **f.** $(\sqrt[5]{d^2})^{\frac{3}{2}} \times (\sqrt[3]{d^5})^{\frac{1}{5}}$

Communicating, reasoning and problem solving

12. In a controlled breeding program at Melbourne Zoo, the population (P) of koalas at t years is modelled by $P = P_0 \times 10^{kt}$. Given $P_0 = 20$ and $k = 0.3$:

a. calculate the number of koalas after 2 years

b. determine when the population will be equal to 1000.

13. The decay of uranium is modelled by $D = D_0 \times 2^{-kt}$. If it takes 6 years for the mass of uranium to halve, find the percentage remaining after:

a. 2 years **b.** 5 years **c.** 10 years.

Give your answers to the nearest whole number.

14. Simplify $\frac{7^{2x+1} - 7^{2x-1} - 48}{36 \times 7^{2x} - 252}$.

15. Simplify $\frac{z^4 + z^{-4} - 3}{z^2 + z^{-2} - 5^{\frac{1}{2}}}$.

16. Do index laws need to be performed in a certain order?

17. Find an expression for x in terms of y, given that $(\sqrt{a^y})^x = a^x \times a^y$.

2.6 Review

Investigation | Rich task

Digital world: 'A bit of this and a byte of that'

'The digital world of today is run by ones and zeros.' What does this mean?

Data is represented on a modern digital computer using a base two (binary) system, that is, using the two digits 1 and 0, thought of as 'on' and 'off'. The smallest unit of data that is transferred on a computer is a **bit** (an abbreviation of binary digit). Computer and storage mechanisms need to hold much larger values than a bit. Units such as bytes, kilobytes (KB), megabytes (MB), gigabytes (GB), and terabytes (TB) are based on the conversion of 8 bits to 1 byte. Your text messages, graphics, music and photos are files stored in sequences of bytes, each byte being 8 bits (8b = 1B).

You may have heard the terms 'meg' and 'gig'. In computer terminology, these refer to gigabytes and megabytes. In the digital world, the prefixes kilo-, mega- and giga- express powers of two, where kilo- means 2^{10}, mega- means $(2^{10})^2$ and so on. Thus the number of bytes in a computer's memory builds in powers of 2, for example 1 kilobyte = 1024 bytes (2^{10} bytes). (This differs from the decimal system, in which the prefixes kilo-, mega- and giga- express powers of ten, with kilo- meaning 10^3, mega- meaning $(10^3)^2$ and so on.)

A byte (8 bits) is used to represent a single character. For example the letter 'A' is represented in binary as 01000001. A book of a thousand pages in print can be stored in millions of bits, but more commonly it would be described as being stored in megabytes with one byte per character.

1. Complete the table below to show the difference in value between the binary and decimal systems.

Unit	Symbol	Power of 2 and value in bytes	Power of 10 and value in bytes
Byte	B	$2^0 = 1$	$10^0 = 1$
Kilobyte	KB	$2^{10} = 1024$	$10^3 = 1000$
Megabyte		$2^{20} =$	
Gigabyte			
Terabyte			

2. The two numbering systems have led to some confusion, with some manufacturers of digital products thinking of a kilobyte as 1000 bytes rather than 1024 bytes. Similar confusion arises with megabytes, gigabytes, terabytes and so on. This means you might not be getting exactly the amount of storage that you think.

If you bought a device quoted as having 16GB memory, what would be the difference in memory storage if the device had been manufactured using the decimal value of GB as opposed to the binary system?

Many devices allow you to check the availability of storage. On one such device, the iPhone, available storage is found by going to 'General' under the heading 'Settings'.

3. How much storage is left in MB on the following iPhone?

4. If each photo uses 3.2 MB of memory, how many photos can be added?

Have you ever wondered about the capacity of our brain to store information and the speed at which information is transmitted inside it?

5. Discuss how the storage and speed of our brains compares to our current ability to send and store information in the digital world. The capacity of the human brain is 10–100 terabytes. On average, 20 million billion bits of information are transmitted within the brain per second.

6. Investigate which country has the fastest internet speed and compare this to Australia.

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Digital doc: Investigation — Digital world: 'A bit of this and a byte of that' (doc-15916)
Digital doc: Code puzzle — What historical event took place in France in 1783? (doc-15917)
Digital doc: Summary Topic 2 (doc-22900)
Interactivity: Word search Topic 2 (int-2826)
Interactivity: Crossword Topic 2 (int-2827)
Interactivity: Sudoku Topic 2 (int-3588)

Exercise 2.6 Review questions

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at http://www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. MC $3d^{10}e^4$ is the simplified form of:

A. $d^6e^2 \times 3d^4e^3$ **B.** $\frac{6d^{10}e^5}{2e^2}$ **C.** $(3d^5e^2)^2$ **D.** $3e\,(d^5)^2 \times e^3$ **E.** $3\left(\frac{d^5}{e^2}\right)^2$

2. MC $8m^3n \times n^4 \times 2m^2n^3$ simplifies to:

A. $10m^5n^8$ **B.** $16m^5n^7$ **C.** $16m^5n^8$ **D.** $10m^5n^7$ **E.** $17m^5n^8$

3. MC $8x^3 \div 4x^{-3}$ is equal to:

A. 2 **B.** $2x^0$ **C.** $2x^6$ **D.** $2x^{-1}$ **E.** $\frac{2}{x^9}$

4. MC $\frac{12x^8 \times 2x^7}{6x^9 \times x^5}$ simplifies to:

A. $4x^5$ **B.** $8x$ **C.** $4x$ **D.** $8x^5$ **E.** $4x^{29}$

5. MC The expression $\frac{(a^2b^3)^5}{(2a^2b)^2}$ is equal to:

A. $\frac{a^6b^{13}}{4}$ **B.** $2a^6b^{13}$ **C.** $\frac{a^3b^6}{2}$ **D.** $\frac{a^6b^{13}}{2}$ **E.** $\frac{a^3b^6}{4}$

6. MC $\frac{(p^2q)^4}{(2p^5q^2)^3} \div \frac{(p^5q^2)^2}{2pq^5}$ can be simplified to:

A. $\frac{1}{4p^{16}q}$ **B.** $\frac{2^2}{p^{16}q}$ **C.** $\frac{1}{4p^8}$ **D.** $\frac{1}{2p^{16}q}$ **E.** $2^2p^{16}q$

7. MC $16^{-\frac{3}{4}} \div 9^{\frac{3}{2}}$ can be simplified to:

A. 2 **B.** $\frac{1}{216}$ **C.** $\frac{8}{27}$ **D.** $3\frac{3}{8}$ **E.** $\frac{1}{2}$

8. MC $\frac{(2l^{\frac{2}{9}}m^{-1})^{-3}}{8\left(\frac{1}{16}lm^{-2}\right)^2}$ can be simplified to:

A. $\frac{8m^7}{l^{\frac{11}{3}}}$ **B.** $\frac{2m^7}{l^{\frac{7}{3}}}$ **C.** $\frac{4m^7}{l^{\frac{8}{3}}}$ **D.** $\frac{16m^7}{l^{\frac{5}{3}}}$ **E.** $\frac{m^7}{2l^{\frac{7}{3}}}$

9. MC $\sqrt[5]{32i^{\frac{10}{7}}j^{\frac{5}{11}}k^2}$ can be simplified to:

A. $\frac{32i^{\frac{2}{7}}j^{\frac{1}{11}}k^{\frac{2}{5}}}{5}$ **B.** $2i^{\frac{2}{7}}j^{\frac{1}{11}}k^{\frac{2}{5}}$ **C.** $\frac{32i^{\frac{10}{7}}j^{\frac{5}{11}}k^2}{5}$ **D.** $2i^{\frac{50}{7}}j^{\frac{25}{11}}k^{10}$ **E.** $\frac{2i^{\frac{2}{7}}j^{\frac{1}{11}}k^{\frac{2}{5}}}{5}$

10. Simplify each of the following.

a. $5x^3 \times 3x^5y^4 \times \frac{3}{5}x^2y^6$ **b.** $\frac{26a^4b^6c^5}{12a^3b^3c^3}$ **c.** $\left(\frac{20m^5n^2}{6}\right)^3$ **d.** $\left(\frac{14p^7}{21q^3}\right)^4$

11. Evaluate each of the following.

a. $5a^0 - \left(\frac{2a}{3}\right)^0 + 12$ **b.** $-(3b)^0 - \frac{(4b)^0}{2}$

12. Simplify each of the following and express your answer with positive indices.

a. $2a^{-5}b^2 \times 4a^{-6}b^{-4}$ **b.** $4x^{-5}y^{-3} \div 20x^{12}y^{-5}$ **c.** $(2m^{-3}n^2)^{-4}$

13. Evaluate each of the following without using a calculator.

a. $\left(\frac{1}{2}\right)^{-3}$ **b.** $2 \times (3)^{-3} \times \left(\frac{9}{2}\right)^2$ **c.** $4^{-3} \times \frac{5}{8^{-2}} - 5$

14. Simplify each of the following.

a. $2a^{\frac{4}{5}}b^{\frac{1}{2}} \times 3a^{\frac{1}{2}}b^{\frac{3}{4}} \times 5a^{\frac{3}{4}}b^{\frac{2}{5}}$ **b.** $\frac{4^3x^{\frac{3}{4}}y^{\frac{1}{9}}}{16x^{\frac{4}{5}}y^{\frac{1}{3}}}$ **c.** $\left(\frac{4a^{\frac{1}{3}}}{b^3}\right)^{\frac{1}{2}}$

15. Evaluate each of the following without using a calculator. Show all working.

a. $\frac{16^{\frac{3}{4}} \times 81^{\frac{1}{4}}}{6 \times 16\frac{1}{2}}$ **b.** $(125^{\frac{2}{3}} - 27^{\frac{2}{3}})^{\frac{1}{2}}$

16. Simplify the following.

a. $\sqrt[3]{a^9} + \sqrt[4]{16a^8b^2}\text{-}3(\sqrt[5]{a})^{15}$ **b.** $\sqrt[5]{32x^5y^{10}} + \sqrt[3]{64x^3y^6}$

17. Simplify the following.

a. $\frac{(5a^{-2}b)^{-3} \times 4a^6b^{-2}}{2a^2b^3 \times 5^{-2}a^{-3}b^{-6}}$ **b.** $\frac{2x^4y^{-5}}{3y^6x^{-2}} \times \left(\frac{4xy^{-2}}{3x^{-6}y^3}\right)^{-3}$ **c.** $\left(\frac{2m^3n^4}{5m^{\frac{1}{2}}n}\right)^{\frac{1}{3}} \div \left(\frac{4m^{\frac{1}{3}}n^{-2}}{5^{-\frac{2}{3}}}\right)^{-\frac{1}{2}}$

18. Simplify each of the following and then evaluate.

a. $(3 \times 5^6)^{\frac{1}{2}} \times 3^{\frac{3}{2}} \times 5^{-2} + (3^6 \times 5^{-\frac{1}{2}})^0$ **b.** $(6 \times 3^{-2})^{-1} \div \frac{(3^{\frac{1}{2}} \times 6^{\frac{1}{3}})^6}{-6^2 \times (3^{-3})^0}$

Communicating, reasoning and problem solving

19. If $m = 2$, determine the value of $\frac{6a^{3m} \times 2b^{2m} \times (3ab)^{-m}}{(4b)^m \times (9a^{4m})^{\frac{1}{2}}}$.

20. Answer the following and explain your reasoning.

a. What is the tens digit of 3^{3^3}?

b. What is the ones digit of 6^{309}?

c. What is the ones digit of 8^{1007}?

21. For the work shown below:

a. calculate the correct answer

b. identify where the student has made mistakes.

$$\left(\frac{3a^3b^5c^3}{5a^2b}\right)^2 \div \left(\frac{2ab}{c}\right) = \frac{3a^6b^{10}c^6}{10a^4b^2} \div \frac{2ab}{c}$$
$$= \frac{3a^6b^{10}c^6}{10a^4b^2} \times \frac{c}{2ab}$$
$$= \frac{3a^6b^{10}c^7}{20a^5b^3}$$
$$= \frac{3ab^7c^7}{20}$$

22. A friend is trying to calculate the volume of water in a reservoir amid fears there may be a severe water shortage. She comes up with the following expression:

$$W = \frac{r^4u^2}{r^{\frac{3}{2}}d^2\sqrt{u}} \times \frac{ru \times d^2}{dr^3u^4},$$

where r is the amount of rain, d is how dry the area is, u is the usage of water by the townsfolk, and W is the volume of water in kL.

a. Help her simplify the expression by simplifying each pronumeral one at a time.

b. Does the final expression contain any potential surds?

c. Express the fraction with a rational denominator.

d. List the requirements for the possible values of r, u and d to give a rational answer.

e. Calculate the volume of water in the reservoir when $r = 4$, $d = 60$ and $u = 9$. Write your answer in:

i. kL **ii.** L **iii.** mL.

f. Does a high value for d mean the area is dry? Explain using working.

23. The speed of a toy plane can be modelled by the equation $S = \dfrac{p^2}{2 + \sqrt{w^3}}$, where

w = wind resistance and

p = battery power (from 0 (empty) to 10 (full)).

a. Rationalise the denominator of the expression.

b. Using your knowledge of perfect squares, estimate the speed of a toy plane with its battery half full and a wind resistance of 2. Check your answer with a calculator.

c. How does the speed of the toy plane change with increasing wind resistance? Explain, providing supportive calculations.

Answers

Topic 2 Indices

Exercise 2.2 Review of index laws

1. a. a^7 **b.** a^6 **c.** b^8 **d.** a^4b^7 **e.** m^5n^{13} **f.** $a^5b^7c^3$
g. $m^6n^4p^5$ **h.** $6a^2b$ **i.** $10a^4b^9$ **j.** $36m^8n^7$ **k.** $12x^6y^6$ **l.** $4x^8y^6$

2. a. a **b.** a^5 **c.** b^3 **d.** $\frac{4}{3}a^4$ **e.** $3b^4$ **f.** $4m^5$
g. m^3n **h.** $\frac{1}{2}y^2$ **i.** $\frac{3}{4}x^3y$ **j.** $7b^3$ **k.** $\frac{5}{4}m^2p^2$ **l.** $\frac{1}{2}xy^2$

3. a. 1 **b.** 1 **c.** 1 **d.** 3 **e.** 4
f. -3 **g.** 3 **h.** -7 **i.** 4

4. a. a^6 **b.** $16a^{20}$ **c.** $\frac{1}{81}m^8$ **d.** $\frac{4}{9}n^8$ **e.** a^6b^3
f. $9a^6b^4$ **g.** $16m^{12}n^{20}$ **h.** $\frac{27}{64}m^6n^3$ **i.** $\frac{a^4}{b^6}$ **j.** $\frac{625m^{12}}{n^8}$
k. $\frac{343x^3}{8y^{15}}$ **l.** $\frac{81a^4}{625b^{12}}$ **m.** -243 **n.** 49 **o.** -32

5. a. D **b.** D

6. a. C **b.** E **c.** B **d.** D

7. a. 64 **b.** 72 **c.** 625 **d.** 48 **e.** 1600
f. $\frac{27}{125}$ **g.** 20 **h.** 1 **i.** 4

8. a. x^{3yz} **b.** a^b **c.** m^an^b **d.** $\frac{a^{2x}}{b^{3x}}$ **e.** $n^{3-p}m^{2-q}$ **f.** a^{mp+np}

9. $a^3 = a \times a \times a$
$a^2 = a \times a$
$a^3 \times a^2 = a \times a \times a \times a \times a$
$= a^5$, not a^6
Explanations will vary.

10. They are equal when $x = 2$. Explanations will vary.

11. $3x^0 = 3$ and $(3x)^0 = 1$. Explanations will vary.

12. a, b

a	0	1	2	3
$3a^2$	0	3	12	27
$5a$	0	5	10	15
$3a^2 + 5a$	0	8	22	42
$3a^2 \times 5a$	0	15	120	405

c. $3a^2 \times 5a$ will become much larger than $3a^2 + 5a$.

13. $x = -2$ or 4

14. $1 \equiv 1$ $2 \equiv 10$ $3 \equiv 11$ $4 \equiv 100$ $5 \equiv 101$
$6 \equiv 110$ $7 \equiv 111$ $8 \equiv 1000$ $9 \equiv 1001$ $10 \equiv 1010$

15. a. $x = 4$ **b.** $x = 0, 2$

16. a. a^2bc^7

b. The student made a mistake when multiplying the two brackets in line 3. Individual brackets should be expanded first.

17. Discuss with your teacher

18. 1.08 seconds

Exercise 2.3 Negative indices

1. a. $\frac{1}{x^5}$ **b.** $\frac{1}{y^4}$ **c.** $\frac{2}{a^9}$ **d.** $\frac{4}{5a^3}$
e. $\frac{3x^2}{y^3}$ **f.** $\frac{1}{4m^3n^4}$ **g.** $\frac{6a^3}{bc^5}$ **h.** a^6
i. $\frac{2a^4}{3}$ **j.** $2ab^2$ **k.** $\frac{7b^3}{2a^4}$ **l.** $\frac{2m^3a^2}{3b^4n^5}$

2. a. $\frac{1}{a^2b^3}$ **b.** $\frac{6}{x^6y}$ **c.** $\frac{3}{n^8}$ **d.** $\frac{4}{a^2b^5}$ **e.** $\frac{2y}{3x}$

f. $\frac{5y}{6x^3}$ **g.** $\frac{3}{m^2n^2}$ **h.** $\frac{4y^{12}}{x^5}$ **i.** $\frac{1}{3m^3n^3}$ **j.** $\frac{1}{32a^{15}m^{20}}$

k. $\frac{4q^8}{p^{14}}$ **l.** $\frac{3}{a^8b^{12}}$ **m.** $\frac{27q^9}{8p^6}$ **n.** $\frac{b^6}{4a^8}$ **o.** $\frac{1}{8a^6b^6}$

3. a. $\frac{1}{8}$ **b.** $\frac{1}{36}$ **c.** $\frac{1}{81}$ **d.** $\frac{8}{9}$ **e.** $\frac{1}{16}$ **f.** $\frac{5}{36}$

g. 48 **h.** $\frac{32}{27}$ **i.** $\frac{27}{25} = 1\frac{2}{25}$ **j.** 4 **k.** 125 **l.** $\frac{3}{4}$

4. a. 2^3 **b.** 2^{-3} **c.** 2^5 **d.** 2^{-6}

5. a. 3 **b.** -2 **c.** -1 **d.** 3 **e.** -2

f. 0 **g.** 3 **h.** -3 **i.** -6 **j.** -2

6. a. $\frac{3}{2}$ **b.** $\frac{4}{5}$ **c.** $\frac{2}{7}$ **d.** 5

7. a. $\frac{b}{a}$ **b.** $\frac{b^3}{a^2}$ **c.** $\frac{a^2}{b^3}$ **d.** $\frac{1}{m^3n^2}$

8. a. $\frac{1}{729}$ **b.** $\frac{1}{20\,736}$ **c.** 0.000 059 499 or $\frac{1}{16\,807}$

d. 256 **e.** $\frac{16\,384}{2187}$ **f.** 9 765 625

9. a. D **b.** C **c.** B

10. a. B **b.** D **c.** C **d.** E

11. a. $\frac{m^2}{n^8}$ **b.** $\frac{n^2}{m}$ **c.** $\frac{25}{a^7b^6}$

12. a. $r^6 - s^6$ **b.** $m^{10} + 2m^5n^5 + n^{10}$

c. 1 **d.** p^2

13. 2^{2r-4} **14.** 6^{3m} **15.** $x = 3$

16. Answers will vary; check with your teacher.

17. $x = 3,\ y = -1; 7$

18. a. $n = -1, -2$ **b.** $n = -1, 4$

19. Discuss with your teacher

20. They are not equivalent expressions. For example, if $x = 3$, then $x^{-2} = 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$, but $-2x = -2 \times 2 = -6$.

Exercise 2.4 Fractional indices

1. a. 4 **b.** 5 **c.** 9 **d.** 2 **e.** 4 **f.** 3

2. a. $\sqrt{15}$ **b.** $\sqrt[4]{m}$ **c.** $\sqrt[5]{7^2}$ **d.** $\sqrt{7^5}$

e. $\sqrt[8]{w^3}$ **f.** $\sqrt[4]{w^5}$ **g.** $\sqrt[3]{5^{10}}$ **h.** $\sqrt[10]{a^3}$

3. a. $t^{\frac{1}{2}}$ **b.** $5^{\frac{7}{4}}$ **c.** $6^{\frac{11}{6}}$ **d.** $x^{\frac{6}{7}}$

e. $x^{\frac{7}{6}}$ **f.** w^2 **g.** $w^{\frac{1}{2}}$ **h.** $11^{\frac{n}{x}}$

4. a. 4 **b.** 16 **c.** 8 **d.** 16 **e.** 125 **f.** 9

g. $\frac{1}{9}$ **h.** 27 **i.** 1000 **j.** 216 **k.** $\sqrt{7}$ **l.** $\sqrt[3]{12}$

5. a. 1.44 **b.** 2.24 **c.** 1.48 **d.** 1.26 **e.** 2.54 **f.** 0.66

g. 0.54 **h.** 0.81 **i.** 0.86

6. a. $4^{\frac{4}{5}}$ **b.** $2^{\frac{1}{2}}$ **c.** $a^{\frac{5}{6}}$ **d.** $x^{\frac{23}{20}}$ **e.** $10m^{\frac{8}{15}}$ **f.** $2b^{\frac{5}{7}}$

g. $-4y^{\frac{20}{9}}$ **h.** $0.02a^{\frac{9}{8}}$ **i.** $5x^{\frac{7}{2}}$

7. a. $ab^{\frac{3}{2}}$ **b.** $x^{\frac{4}{5}}y^{\frac{5}{9}}$ **c.** $6a^{\frac{8}{5}}b^{\frac{17}{15}}$ **d.** $2m^{\frac{19}{28}}n^{\frac{2}{5}}$ **e.** $x^{\frac{19}{6}}y^{\frac{5}{6}}z^{\frac{5}{6}}$ **f.** $8a^{\frac{2}{5}}b^{\frac{9}{8}}c$

8. a. $3^{\frac{1}{6}}$ **b.** $5^{\frac{5}{12}}$ **c.** $12^{\frac{1}{2}}$ **d.** $a^{\frac{3}{7}}$ **e.** $x^{\frac{5}{4}}$ **f.** $m^{\frac{11}{45}}$

g. $\frac{1}{2}x^{\frac{3}{20}}$ **h.** $\frac{1}{3}n^{\frac{2}{3}}$ **i.** $\frac{5}{4}b^{\frac{7}{20}}$

9. a. $x^{\frac{5}{3}}y^{\frac{7}{5}}$ **b.** $a^{\frac{7}{45}}b^{\frac{4}{15}}$ **c.** $\frac{1}{3}m^{\frac{3}{8}}n^{\frac{11}{56}}$ **d.** $2x^{\frac{2}{15}}y^{\frac{3}{4}}$ **e.** $\frac{1}{4}a^{\frac{11}{20}}b^{\frac{7}{20}}$ **f.** $\frac{1}{7}p^{\frac{5}{24}}q^{\frac{1}{12}}$

10. a. $2^{\frac{9}{20}}$ **b.** $5^{\frac{1}{6}}$ **c.** $7^{\frac{6}{5}}$ **d.** $a^{\frac{3}{10}}$ **e.** $m^{\frac{1}{6}}$ **f.** $2^{\frac{1}{3}}b^{\frac{1}{6}}$

g. $4p^{\frac{2}{5}}$ **h.** $x^{\frac{m}{p}}$ **i.** $3^{\frac{b}{c}}m^{\frac{a}{c}}$

11. a. $a^{\frac{1}{4}}b^{\frac{1}{6}}$ **b.** $a^3b^{\frac{3}{4}}$ **c.** $x^{\frac{6}{5}}y^{\frac{7}{4}}$ **d.** $3^{\frac{1}{3}}a^{\frac{1}{9}}b^{\frac{1}{5}}c^{\frac{1}{4}}$ **e.** $5x^{\frac{1}{4}}y^{\frac{1}{3}}z^{\frac{1}{5}}$ **f.** $\dfrac{a^{\frac{1}{2}}}{b^{\frac{2}{3}}}$

g. $\dfrac{m^{\frac{8}{5}}}{n^{\frac{7}{4}}}$ **h.** $\dfrac{b^{\frac{3}{10}}}{c^{\frac{2}{9}}}$ **i.** $\dfrac{2^{\frac{1}{2}}x^{\frac{7}{2}}}{y^{\frac{3}{8}}}$

12. a. E **b.** C **c.** B

13. a. E **b.** B

14. a. a^4 **b.** b^3 **c.** m^4 **d.** $4x^2$ **e.** $2y^3$ **f.** $2x^2y^3$

g. $3m^3n^5$ **h.** $2pq^2$ **i.** $6a^2b^6$

15. a. 2.007 *s* **b.** 20.07 *s* **c.** 4.98 swings

16. $(2^5a^5b^{10})^{\frac{1}{5}} = 2ab^2$ **17.** $\dfrac{a^2(3-\sqrt{b^3})}{9-b^3}; \dfrac{4}{11}$ **18.** $m^{\frac{1}{5}} - n^{\frac{1}{5}} + p^{\frac{1}{5}}$

19. a. $a^{-\frac{1}{4}} \times b^{\frac{13}{2}}$

b. No, because you can't take the fourth root of a negative number.

c. $a = 1$

20. Discuss with your teacher

Exercise 2.5 Combining index laws

1. a. $54a^{10}b^9$ **b.** $48a^5b^{16}$ **c.** $\dfrac{2n^{13}}{m^9}$ **d.** $500p^8q^{18}$ **e.** $36a^{20}b^{10}$ **f.** $\dfrac{15b^2}{c^{26}}$

g. $12x^{\frac{7}{8}}y^{\frac{11}{15}}$ **h.** $8m^{\frac{15}{4}}n^{\frac{15}{4}}$ **i.** $\dfrac{6}{p^{\frac{7}{12}}}$ **j.** $8p^{\frac{7}{45}}q^{\frac{5}{18}}$

2. a. $\dfrac{5}{8a^7}$ **b.** $\dfrac{x}{4y^6}$ **c.** $\dfrac{27}{128m^{29}n^{26}}$ **d.** $\dfrac{64y^{36}}{x^{24}}$ **e.** $24a^{24}b^7$ **f.** $\dfrac{27h^{12}}{8g^6}$

g. $p^{\frac{35}{3}}q^{\frac{1}{2}}$ **h.** $\dfrac{625}{81b^{20}c^{28}}$ **i.** $x^{\frac{5}{3}}y^{\frac{1}{8}}z^{\frac{3}{2}}$

3. a. $\dfrac{3a^2}{2}$ **b.** $8n^2$ **c.** $\dfrac{m^2n^4}{3}$ **d.** $\dfrac{4x^5}{3y^8}$ **e.** $\dfrac{36x^6}{y}$

f. $\dfrac{y^2}{x^4}$ **g.** $\dfrac{b^7}{3a^4}$ **h.** $\dfrac{75q^5}{2p^{11}}$ **i.** $x^{\frac{17}{10}}y^{\frac{7}{10}}$

4. a. $\dfrac{2}{5a^4b^7}$ **b.** $\dfrac{4a^3b^3}{15}$ **c.** $\dfrac{n^9}{4m^9}$ **d.** $\dfrac{4m^5}{9n^{15}}$ **e.** $\dfrac{4}{81x^2y^{14}}$

f. $48x^{11}y^6$ **g.** $\dfrac{3p^4}{5q^9}$ **h.** $\dfrac{2b^{\frac{1}{12}}}{3a^{\frac{17}{24}}}$ **i.** $\dfrac{4x^{\frac{1}{12}}}{3y^{\frac{21}{20}}}$

5. a. $\dfrac{5}{2a^{13}}$ **b.** $\dfrac{56a^{11}b^6}{81}$ **c.** $\dfrac{1024b^2}{81a}$ **d.** $\dfrac{25}{128x^{23}y^4}$ **e.** $\dfrac{4y^{36}}{27x^{16}}$

f. $6m^{19}n^{19}$ **g.** $\dfrac{16m^{\frac{11}{12}}n}{3}$ **h.** $\dfrac{4b^{\frac{11}{2}}}{3^{\frac{1}{2}}c^{\frac{7}{30}}}$

6. a. $\frac{125}{8}$ **b.** 1

7. 1

8. a. 5^{y-1} **b.** $y = 4$

9. E

10. A

11. a. $m^{\frac{1}{6}}n^{-\frac{7}{6}}$ or $\sqrt[6]{\dfrac{m}{n^7}}$ **b.** $g^{-6}h^3n^{\frac{3}{2}}$ **c.** $3^{-\frac{7}{3}} \times 5^{-\frac{7}{6}}$ **d.** 2^{-2} or $\frac{1}{4}$ **e.** a^6b^{-8} or $\dfrac{a^6}{b^8}$ **f.** $d^{\frac{14}{15}}$ or $\sqrt[15]{d^{14}}$

12. a. 80 koalas **b.** During the 6th year

13. a. 79% **b.** 56% **c.** 31%

14. $\frac{4}{21}$

15. $z^2 + z^{-2} + \sqrt{5}$

16. Discuss with your teacher.

17. $x = \frac{2y}{y-2}$

Investigation | Rich task

1.

Unit	Symbol	Power of 2 and value in bytes	Power of 10 and value in bytes
Byte	B	$2^0 = 1$	$10^0 = 1$
Kilobyte	KB	$2^{10} = 1024$	$10^3 = 1000$
Megabyte	MB	$2^{20} = 1048576$	$10^6 = 1000000$
Gigabyte	GB	$2^{30} = 1073741824$	$10^9 = 1000000000$
Terabyte	TB	$2^{40} = 1099511627776$	$10^{12} = 1000000000000$

2. Approximately 1.1 GB

3. 3993.6 MB

4. 1248 photos

5. Discuss with your teacher.

6. Discuss with your teacher. The discussion will depend on the latest information from the internet.

Exercise 2.6 Review questions

1. D **2.** C **3.** C **4.** C **5.** A

6. A **7.** B **8.** C **9.** B

10. a. $9x^{10}y^{10}$ **b.** $\frac{13ab^3c^2}{6}$ **c.** $\frac{1000m^{15}n^6}{27}$ **d.** $\frac{16p^{28}}{81q^{12}}$

11. a. 16 **b.** $-\frac{3}{2}$

12. a. $\frac{8}{a^{11}b^2}$ **b.** $\frac{y^2}{5x^{17}}$ **c.** $\frac{m^{12}}{16n^8}$

13. a. 8 **b.** $\frac{3}{2}$ **c.** 0

14. a. $30a^{\frac{41}{20}}b^{\frac{33}{20}}$ **b.** $\frac{4}{x^{\frac{1}{20}}y^{\frac{2}{9}}}$ **c.** $\frac{2a^{\frac{1}{6}}}{b^{\frac{3}{2}}}$

15. a. 1 **b.** 4

16. a. $-2a^3 + 2a^2b^{\frac{1}{2}}$ **b.** $6xy^2$

17. a. $\frac{2a^{13}}{5b^2}$ **b.** $\frac{9y^4}{32x^{15}}$ **c.** $2^{\frac{4}{3}}m$

18. a. 46 **b.** $-\frac{1}{18}$

19. $\frac{1}{36}$

20. a. 8 **b.** 6 **c.** 2

21. a. $\frac{9ab^7c^7}{50}$

b. The student has made two mistakes when squaring the left-hand bracket in line 1 : $3^2 = 9$, $5^2 = 25$.

22. a. $\frac{\sqrt{r}}{d\sqrt{u^3}}$ **b.** Yes, $\sqrt{r}$, $\sqrt{u^3}$ **c.** $\frac{\sqrt{ru^3}}{du^3}$

d. r should be a perfect square, u should be a perfect cube and d should be a rational number.

e. i. 0.0012346 kL **ii.** 1.2346L **iii.** 1234.6mL

f. A high value for d causes the expression to be smaller, as d only appears on the denominator of the fraction. This means that when d is high there is less water in the reservoir and the area is dry.

23. a. $\dfrac{p^2(2-\sqrt{w^3})}{4-w^3}$

b. Answers will vary; approximately 5.

c. Speed decreases as wind resistance increases.

TOPIC 3
Right-angled triangles (trigonometry)

[Stages 5.2 and 5.3]

3.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the concepts covered in this topic.

3.1.1 Why learn this?

Nearly 2000 years ago, Ptolemy of Alexandria published the first book of trigonometric tables, which he used to chart the heavens and plot the courses of the Moon, stars and planets. He also created geographical charts and provided instructions on how to create maps. Trigonometry is the branch of mathematics that makes the whole universe more easily understood.

DISCUSSION

Hypatia was a famous Greek philosopher and mathematician who was brutally murdered c. 415. Name another influential female mathematician and explain why she made a significant contribution to the study of mathematics.

LEARNING SEQUENCE

3.1 Overview
3.2 [Stage 4] Pythagoras' theorem
3.3 [Stage 5.3] Pythagoras' theorem in three dimensions
3.4 [Stage 5.2] Trigonometric ratios
3.5 [Stage 5.2] Using trigonometry to calculate side lengths
3.6 [Stage 5.2] Using trigonometry to calculate angle size
3.7 [Stage 5.2] Angles of elevation and depression
3.8 [Stage 5.2] Bearings
3.9 [Stages 5.2 and 5.3] Applications
3.10 Review

LEARNING OUTCOMES

A student:

- selects appropriate notations and conventions to communicate mathematical ideas and solutions MA5.2-1WM
- interprets mathematical or real-life situations, systematically applying appropriate strategies to solve problems MA5.2-2WM
- applies trigonometry to solve problems, including problems involving bearings MA5.2-13MG
- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3-1WM

- generalises mathematical ideas and techniques to analyse and solve problems efficiently MA5.3-2WM
- uses deductive reasoning in presenting arguments and formal proofs MA5.3-3WM
- applies Pythagoras' theorem to solve problems, including problems involving three dimensions MA5.3-15MG

CONTENT DESCRIPTIONS

Students:

Use similarity to investigate the constancy of the sine, cosine and tangent ratios for a given angle in right-angled triangles (ACMMG223)

Apply trigonometry to solve right-angled triangle problems (ACMMG224)

Solve right-angled triangle problems, including those involving direction and angles of elevation and depression (ACMMG245)

Apply Pythagoras' theorem and trigonometry to solve three-dimensional problems in right-angled triangles (ACMMG276)

Source: NSW Syllabus for the Australian curriculum

learnon RESOURCES — ONLINE ONLY

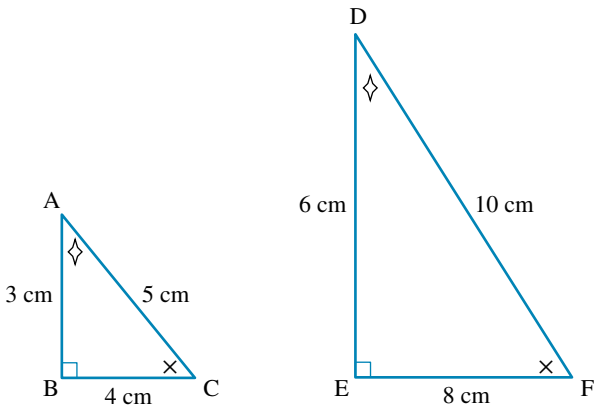
eLesson: The story of mathematics — Hypatia (eles-1844)

Note: Your teacher may now set you a pre-test to determine how familiar you are with the content in this topic.

3.2 Pythagoras' theorem [Stage 4]

3.2.1 Similar right-angled triangles

- In the two similar right-angled triangles shown below, the angles are the same and the corresponding sides are in the same ratio.



- The corresponding sides are in the same ratio.

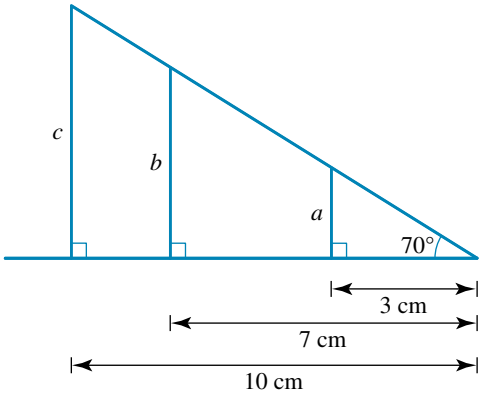
$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$

- To write this using the side lengths of the triangles gives:

$$\frac{AB}{DE} = \frac{3}{6} = \frac{1}{2}$$
$$\frac{AC}{DF} = \frac{5}{10} = \frac{1}{2}$$
$$\frac{BC}{EF} = \frac{4}{8} = \frac{1}{2}$$

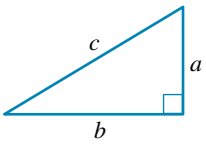
- This means that for right-angled triangles, when the angles are fixed, the ratios of the sides in the triangle are constant.

- We can examine this idea further by completing the following activity.
  Using a protractor and ruler, draw an angle of 70°, measuring horizontal distances of 3 cm, 7 cm and 10 cm as demonstrated in the diagram below.



  Note: Diagram not drawn to scale.
  Measure the perpendicular heights a, $b$and c.
  $a \approx 8.24$ cm $\quad b \approx 19.23$ cm $\quad c \approx 27.47$ cm
- To test if the theory for right-angled triangles is correct (that is, when the angles are fixed, the ratios of the sides in the triangle are constant), calculate the ratios of the side lengths.
  $\frac{a}{3} \approx \frac{8.24}{3} \approx 2.75 \qquad \frac{b}{7} \approx \frac{19.23}{7} \approx 2.75 \qquad \frac{c}{10} \approx \frac{27.47}{10} \approx 2.75$
  The ratios are the same because the triangles are similar.
- This important concept forms the basis of trigonometry.

3.2.2 Review of Pythagoras' theorem



- The **hypotenuse** is the longest side of a right-angled triangle and is always the side that is opposite the right angle.
- Pythagoras' theorem states that in any right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides. The rule is written as $c^2 = a^2 + b^2$, where a and b are the two shorter sides and c is the hypotenuse.
- Pythagoras' theorem gives us a way of finding the length of the third side in a triangle, if we know the lengths of the two other sides.

DISCUSSION

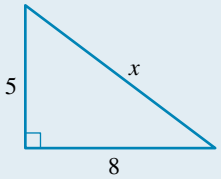
Pythagoras' theorem was known about before the age of Pythagoras. Find out which other civilisations knew about the theory, and construct a timeline for its history.

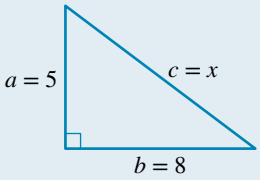
3.2.3 Finding the hypotenuse

- To calculate the length of the hypotenuse when given the length of the two shorter sides, substitute the known values into the formula for Pythagoras' theorem, $c^2 = a^2 + b^2$.

WORKED EXAMPLE 1

For the triangle shown, calculate the length of the hypotenuse, x, correct to 1 decimal place.



THINK	WRITE
1 Copy the diagram and label the sides a, b and c. Remember to label the hypotenuse as c.	
2 Write Pythagoras' theorem.	$c^2 = a^2 + b^2$
3 Substitute the values of a, b and c into this rule and simplify.	$x^2 = 5^2 + 8^2$ $= 25 + 64$ $= 89$
4 Take the square root of both sides. Round the positive answer correct to 1 decimal place, since $x > 0$.	$x = \pm\sqrt{89}$ $x = 9.4$, since $x > 0$ (correct to 1 d.p.)

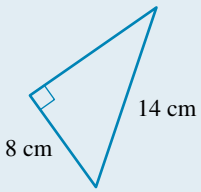
3.2.4 Finding a shorter side

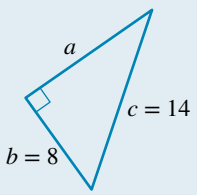
- Sometimes a question will give you the length of the hypotenuse and ask you to find one of the shorter sides. In such examples, we need to rearrange Pythagoras' formula. Given that $c^2 = a^2 + b^2$, we can rewrite this as:

$$a^2 = c^2 - b^2$$
$$\text{or} \quad b^2 = c^2 - a^2.$$

WORKED EXAMPLE 2

Calculate the length, correct to 1 decimal place, of the unmarked side of the triangle shown.



THINK	WRITE/DRAW
1 Copy the diagram and label the sides a, b and c. Remember to label the hypotenuse as c; it does not matter which side is a and which side is b.	
2 Write Pythagoras' theorem.	$c^2 = a^2 + b^2$
3 Substitute the values of a, b and c into this rule and solve for a.	$14^2 = a^2 + 8^2$ $196 = a^2 + 64$ $a^2 = 196 - 64$ $= 132$
4 Find a by taking the square root of both sides and round to 1 decimal place ($a > 0$).	$a = \pm\sqrt{32}$ $a = 11.5$ cm, since $a > 0$ (correct to 1 d.p.)

- Pythagoras' theorem can be used to solve many practical problems.
- First model the problem by drawing a diagram, then use Pythagoras' theorem to solve the right-angled triangle. Use the result to give a worded answer.

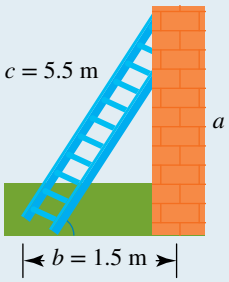
WORKED EXAMPLE 3

A ladder that is 5.5 m long leans up against a vertical wall. The foot of the ladder is 1.5 m from the wall. How far up the wall does the ladder reach? Give your answer correct to 1 decimal place.

THINK

WRITE/DRAW

1 Draw a diagram and label the sides a, b and c. Remember to label the hypotenuse as c.



2 Write Pythagoras' theorem.

$c^2 = a^2 + b^2$

3 Substitute the values of a, b and c into this rule and simplify.

$5.5^2 = a^2 + 1.5^2$
$30.25 = a^2 + 2.25$
$a^2 = 30.25 - 2.25$
$= 28$

4 Find a by taking the square root of 28. Round to 1 decimal place, $a > 0$.

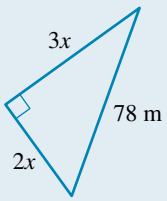
$a = \pm\sqrt{28}$
$= 5.3$, since $a > 0$ (correct to 1 d.p.)

5 Answer the question in a sentence using words.

The ladder reaches 5.3 m up the wall.

WORKED EXAMPLE 4

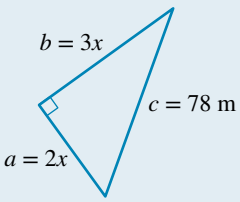
Determine the unknown side lengths of the triangle shown, correct to 2 decimal places.



THINK

WRITE/DRAW

1 Copy the diagram and label the sides a, b and c.



2 Write Pythagoras' theorem.

$c^2 = a^2 + b^2$

3 Substitute the values of a, b and c into this rule and simplify.

$78^2 = (3x)^2 + (2x)^2$
$6084 = 9x^2 + 4x^2$
$6084 = 13x^2$

4 Rearrange the equation so that the pronumeral is on the left-hand side of the equation.

$13x^2 = 6084$

5 Divide both sides of the equation by 13.

$$\frac{13x^2}{13} = \frac{6084}{13}$$
$$x^2 = 468$$

6 Find x by taking the square root of both sides. Round the answer correct to 2 decimal places.

$$x = \pm\sqrt{468}$$
$$\approx 21.6333$$

7 Substitute the value of x into $2x$ and $3x$ to find the length of the unknown sides.

$2x = 43.27$ m (correct to 2 d.p.)
$3x = 64.90$ m (correct to 2 d.p.)

learnON RESOURCES — ONLINE ONLY

Interactivity: Right-angled triangles (int-3843)
Interactivity: Finding the hypotenuse (int-3844)
Interactivity: Finding the shorter side (int-3845)
Digital doc: SkillSHEET Rounding to a given number of decimal places (doc-5224)

Exercise 3.2 Pythagoras' theorem

Individual pathways

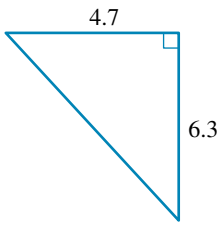
PRACTISE	CONSOLIDATE	MASTER
Questions: 1–4, 6, 12–15, 17, 20	Questions: 1–3, 5–8, 12, 15–18, 20, 22	Questions: 1, 2, 5, 7, 9–11, 19–24

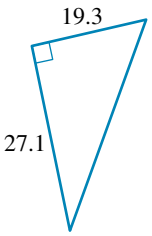
To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

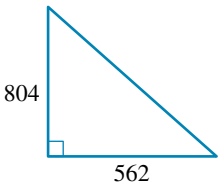
Understanding and fluency

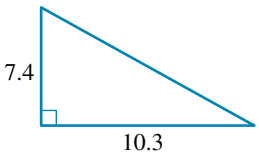
Give your answers to the following questions correct to 2 decimal places unless directed otherwise.

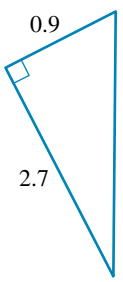
1. WE1 For each of the following triangles, calculate the length of the hypotenuse.

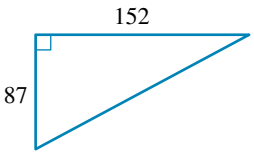
a.


b.


c.


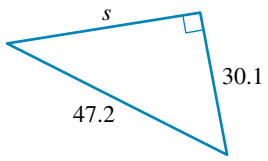
d.


e.


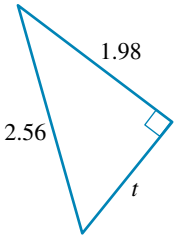
f.


2. WE2 For each of the following triangles, find the value of the pronumeral.

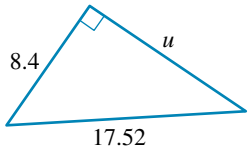
a.



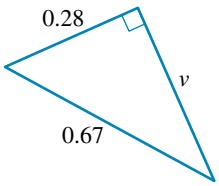
b.



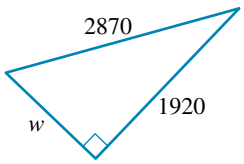
c.



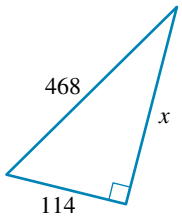
d.



e.



f.



3. WE3 The diagonal of the rectangular sign shown at right is 34 cm. If the height of this sign is 25 cm, find the width.

4. A right-angled triangle has a base of 4 cm and a height of 12 cm. Calculate the length of the hypotenuse.

5. Calculate the lengths of the diagonals of squares that have side lengths of:

a. 10 cm **b.** 17 cm **c.** 3.2 cm.

6. The diagonal of a rectangle is 90 cm. One side has a length of 50 cm. Determine:

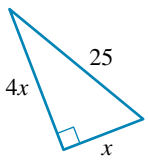
a. the length of the other side

b. the perimeter of the rectangle

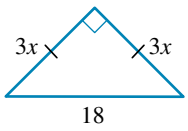
c. the area of the rectangle.

7. WE4 Find the value of the pronumeral for each of the following.

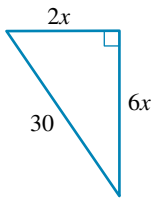
a.



b.



c.



8. An isosceles triangle has a base of 25 cm and a height of 8 cm. Calculate the length of the two unknown equal sides.

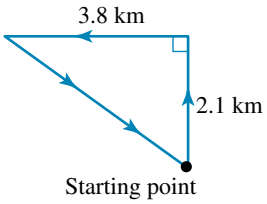
9. An equilateral triangle has sides of length 18 cm. Find the height of the triangle.

10. A right-angled triangle has a height of 17.2 cm, and a base that is half the height. Calculate the length of the hypotenuse.

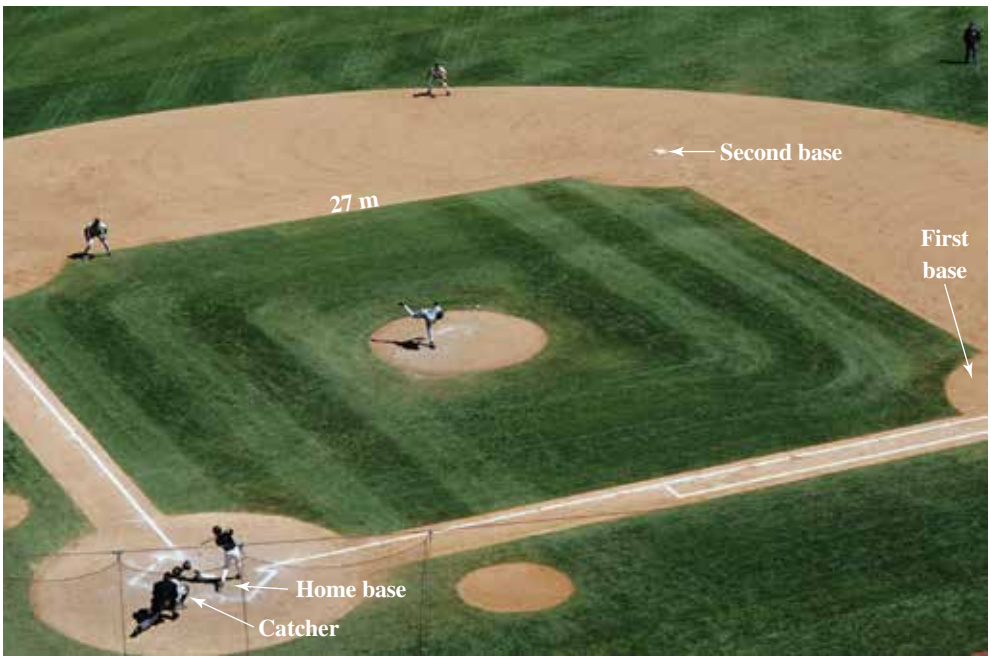
11. The road sign shown at right is based on an equilateral triangle. Find the height of the sign and, hence, find its area.

12. A flagpole, 12 m high, is supported by three wires attached from the top of the pole to the ground. Each wire is pegged into the ground 5 m from the pole. How much wire is needed to support the pole? Give your answer to the nearest whole number.

13. Sarah goes canoeing on a large lake. She paddles 2.1 km to the north, then 3.8 km to the west. Use the triangle shown to find out how far she must then paddle to get back to her starting point in the shortest possible way.



14. A baseball diamond is a square of side length 27 m. When a runner on first base tries to reach second base, the catcher has to throw the ball from home base to second base. How far is that throw? Give your answer correct to 1 decimal place.



15. A rectangle measures 56 mm by 2.9 cm. Calculate the length of its diagonal in millimetres.

16. A rectangular envelope has a length of 24 cm and a diagonal measuring 40 cm. Calculate to the nearest whole number:

a. the width of the envelope

b. the area of the envelope.

17. A swimming pool is 50 m by 25 m. Peter is bored by his usual training routine and decides to swim the diagonal of the pool. How many diagonals must he swim to complete his normal distance of 1500 m?

18. A hiker walks 2.9 km north, then 3.7 km east. How far in metres is she from her starting point?

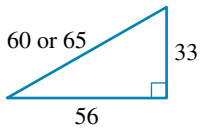
19. A square has a diagonal of 14 cm. What is the length of each side?

Communicating, reasoning and problem solving

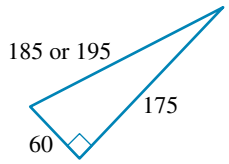
Give your answers to the following questions to 2 decimal places unless directed otherwise.

20. The triangles below are right-angled triangles. Two possible measurements have been suggested for the hypotenuse in each case. For each triangle, complete calculations to determine which of the lengths is correct for the hypotenuse in each case. Show your working.

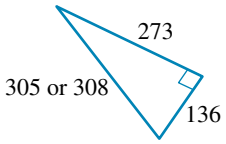
a.



b.



c.



21. Four possible side length measurements are 105, 208, 230 and 233. Three of them together produce a right-angled triangle.

a. Which of the measurements could not be the hypotenuse of the triangle? Explain.

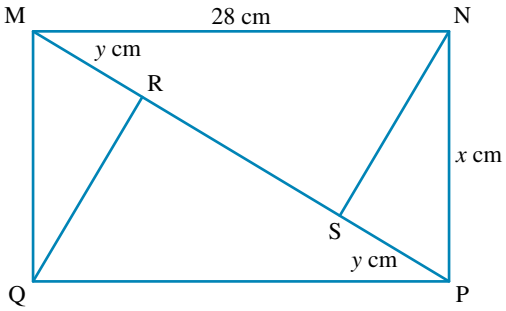
b. Complete as few calculations as possible to calculate which combination of side lengths will produce a right-angled triangle.

22. The area of the rectangle MNPQ is 588 cm^2. Angles MRQ and NSP are right angles. Answer the following correct to 1 decimal place.

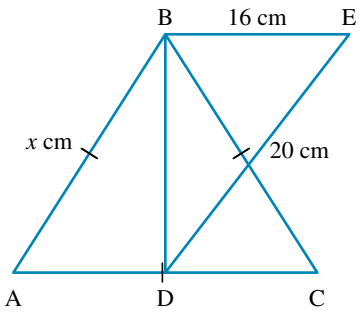
a. Find the integer value of x.

b. Find the length of MP.

c. Find the value of y and hence determine the length of RS.



23. Triangle ABC is an equilateral triangle of side length x cm. Angles ADB and DBE are right angles. Find the value of x.

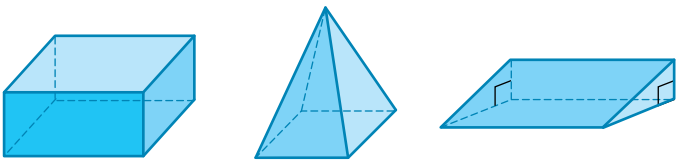


24. The square root of a number usually gives us both a positive and negative answer. Why do we take only the positive answer when using Pythagoras' theorem?

3.3 Pythagoras' theorem in three dimensions

[Stage 5.3]

- Many real-life situations involve **3-dimensional** (3-D) objects: objects with length, width and height. Some common 3-D objects used in this section include cuboids, pyramids and right-angled wedges.

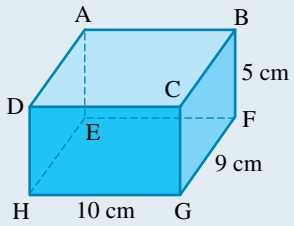


Cuboid | Pyramid | Right-angled wedge

- In diagrams of 3-D objects, right angles may not look like right angles, so it is important to redraw sections of the diagram in two dimensions, where the right angles can be seen accurately.

WORKED EXAMPLE 5

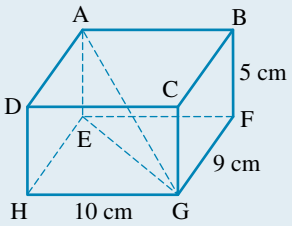
Determine the length AG in this rectangular prism (cuboid), correct to two decimal places.



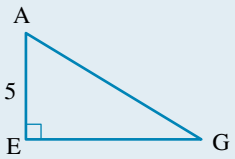
THINK

WRITE/DRAW

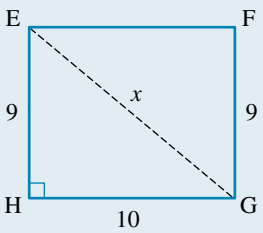
1 Draw the diagram in three dimensions.
Draw the lines AG and EG. ∠AEG is a right angle.



2 Draw ΔAEG, showing the right angle. Only 1 side is known, so EG must be found.



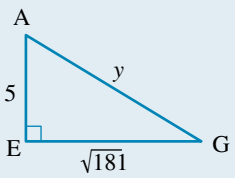
3 Draw EFGH in two dimensions and label the diagonal EG as x.



4 Use Pythagoras' theorem to calculate x. ($c^2 = a^2 + b^2$)

$x^2 = 9^2 + 10^2$
$= 81 + 100$
$= 181$
$x = \sqrt{181}$

5 Place this information on triangle AEG. Label the side AG as y.



6 Use Pythagoras' theorem to find y. ($c^2 = a^2 + b^2$)

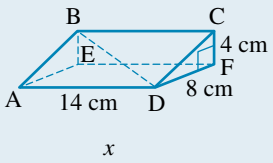
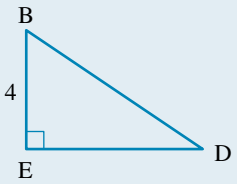
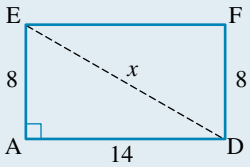
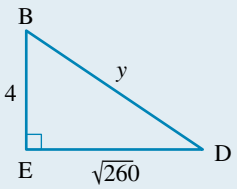
$y^2 = 5^2 + (\sqrt{181})^2$
$= 25 + 181$
$= 206$
$y = \sqrt{206}$
$= 14.35$ (correct to 2 d.p.)

7 Answer the question in a sentence.

The length of AG is 14.35 cm.

WORKED EXAMPLE 6

A piece of cheese in the shape of a right-angled wedge sits on a table. It has a rectangular base measuring 14 cm by 8 cm, and is 4 cm high at the thickest point. An ant crawls diagonally across the sloping face. How far, to the nearest millimetre, does the ant walk?

THINK	WRITE/DRAW
1 Draw a diagram in three dimensions and label the vertices. Mark BD, the path taken by the ant, with a dotted line. ∠BED is a right angle.	
2 Draw ΔBED, showing the right angle. Only one side is known, so ED must be found.	
3 Draw EFDA in two dimensions and label the diagonal ED. Label the side ED as x in both diagrams.	
4 Use Pythagoras' theorem to calculate x.	$c^2 = a^2 + b^2$ $x^2 = 8^2 + 14^2$ $= 64 + 196$ $= 260$ $x = \sqrt{260}$
5 Place this information on triangle BED. Label the side BD as y.	
6 Solve this triangle for y.	$y^2 = 4^2 + (\sqrt{260})^2$ $= 16 + 260$ $= 276$ $y = \sqrt{276}$ ≈ 16.51 cm ≈ 166.1 mm
7 Answer the question in a sentence.	The ant walks 166 mm, correct to the nearest millimetre.

learnON RESOURCES — ONLINE ONLY

Interactivity: Right angles in 3-dimensional objects (int-6132)
eLesson: Pythagoras' theorem in three dimensions (eles-1913)
Digital doc: SkillSHEET Drawing 3-D shapes (doc-5229)
Digital doc: WorkSHEET Pythagoras' theorem (doc-5230)

Exercise 3.3 Pythagoras' theorem in three dimensions

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–b, 2, 6, 7, 8a, 10	Questions: 1, 3, 4, 6, 7, 8, 10, 11, 13, 14	Questions: 1, 3–5, 8–17

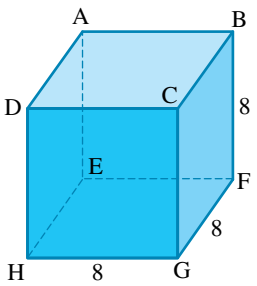
To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

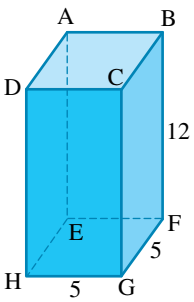
Give your answers to the following questions correct to 2 decimal places unless directed otherwise.

1. WE5 Calculate the length of AG in each of the following figures.

a.



b.



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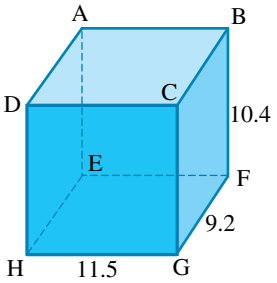
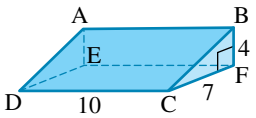
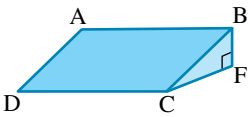


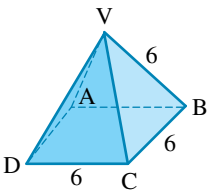
2. Calculate the length of CE in the wedge below and, hence, obtain AC.



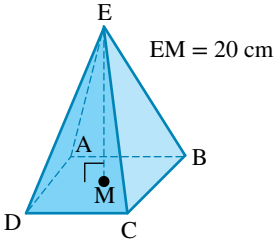
3. If DC = 3.2 m, AC = 5.8 m and CF = 4.5 m in the figure below, calculate the lengths of AD and BF.



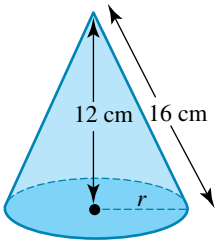
4. Calculate the length of BD and, hence, the height of the pyramid ABCDV.



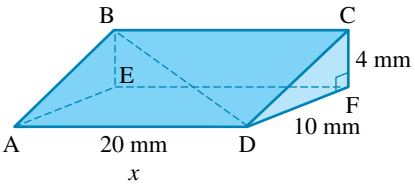
5. The pyramid ABCDE has a square base. The pyramid is 20 cm high. Each sloping edge measures 30 cm. Calculate the length of the sides of the base.



6. The sloping side of a cone is 16 cm and the height is 12 cm. What is the length of the radius of the base?



7. WE6 A piece of cheese in the shape of a right-angled wedge sits on a table. It has a base measuring 20 mm by 10 mm, and is 4 mm high at the thickest point, as shown in the figure. A fly crawls diagonally across the sloping face. How far, to the nearest millimetre, does the fly walk?



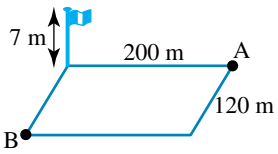
8. A 7-m-high flagpole is in the corner of a rectangular park that measures 200 m by 120 m.

a. Calculate:

i. the length of the diagonal of the park

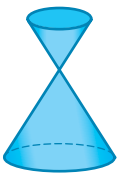
ii. the distance from A to the top of the pole

iii. the distance from B to the top of the pole.

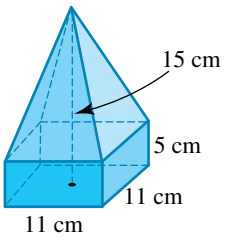


b. A bird flies from the top of the pole to the centre of the park. How far does it fly?

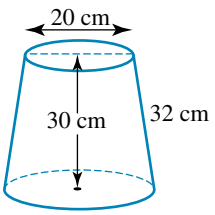
9. A candlestick is in the shape of two cones, joined at the vertices as shown. The smaller cone has a diameter and sloping side of 7 cm, and the larger one has a diameter and sloping side of 10 cm. How tall is the candlestick?



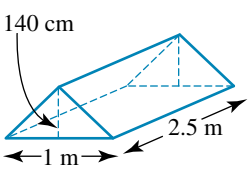
10. The total height of the shape shown is 15 cm. Calculate the length of the sloping side of the pyramid.



11. A sandcastle is in the shape of a truncated cone as shown. Calculate the length of the diameter of the base.



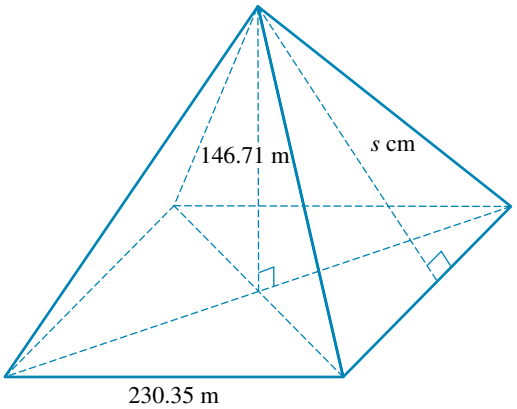
12. A tent is in the shape of a triangular prism, with a height of 140 cm as shown in the diagram. The width across the base of the door is 1 m, and the tent is 2.5 m long. Calculate the length of each sloping side, in metres. Then calculate the area of fabric used in the construction of the sloping rectangles which form the sides.



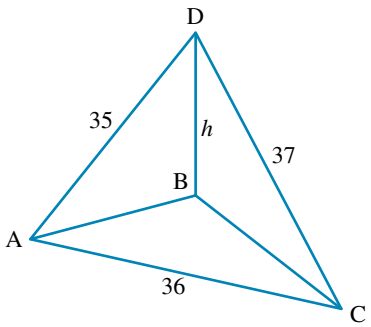
Communicating, reasoning and problem solving

13. Stephano is renovating his apartment, which he accesses through two corridors. The corridors of the apartment building are 2 m wide with 2-m-high ceilings, and the first corridor is at right angles to the second. Show that he can carry lengths of timber up to 6 m long to his apartment.

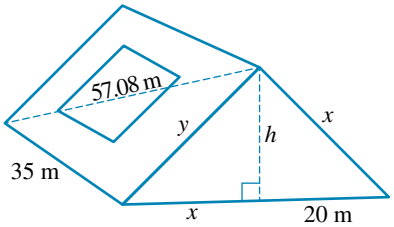
14. The Great Pyramid in Egypt is a square-based pyramid. The square base has a side length of 230.35 metres and the perpendicular height is 146.71 metres, as shown at right. Find the slant height, s cm, of the Great Pyramid. Give your answer correct to 1 decimal place.



15. In the figure shown, angles ABD, CBD and ABC are right angles. Find the value of h, correct to 3 decimal places.



16. The roof of a squash centre is constructed to allow for maximum use of sunlight. Find the value of h, giving your answer correct to 1 decimal place.

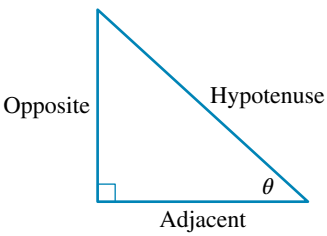


17. The diagonal of a rectangle of dimensions x by y is $\sqrt{x^2 + y^2}$. What would be the rule to find the length of a diagonal across a cuboid of dimensions x by y by z? Use your rule to check your answers to question **1**.

3.4 Trigonometric ratios [Stage 5.2]

3.4.1 Naming the sides in a right-angled triangle

- In a right-angled triangle, the longest side is called the hypotenuse.
- If one of the two acute angles is named (say θ), then the other two sides can also be given names, as shown in the diagram.



3.4.2 Three basic definitions

- Using the diagram at right, the following three **trigonometric ratios** can be defined:
  - the **sine ratio**, $\sin\theta = \dfrac{\text{length of opposite side}}{\text{length of hypotenuse}}$

– the **cosine ratio**, $\text{cosine } \theta = \dfrac{\text{length of adjacent side}}{\text{length of hypotenuse}}$

– the **tangent ratio**, $\text{tangent } \theta = \dfrac{\text{length of opposite side}}{\text{length of adjacent side}}$.

- The names of the three ratios are usually shortened to sin θ, cos θ and tan θ.
- The three ratios are often remembered using the mnemonic SOHCAHTOA, where SOH means Sin θ = Opposite over Hypotenuse and so on.

DISCUSSION

Do you know of any other mnemonics that you can use to help you remember important information?

3.4.3 Finding values using a calculator

- The sine, cosine and tangent of an angle have numerical values that can be found using a calculator.
- Traditionally angles are measured in degrees, minutes and seconds, where 60 seconds = 1 minute and 60 minutes = 1 degree.
- For example, 50°33′48″ means 50 degrees, 33 minutes and 48 seconds.
- In this course, angles will be measured in degrees or in degrees and minutes.

WORKED EXAMPLE 7

Calculate the value of each of the following, correct to 4 decimal places, using a calculator. (Remember to first work to 5 decimal places before rounding.)

a **cos 65°57′**

b **tan 56°45′**

THINK	WRITE
a Write your answer to the required number of decimal places.	**a** $\cos 65°57' = 0.40753\ldots$ $= 0.4075$ (correct to 4 d.p.)
b Write your answer to the correct number of decimal places.	**b** $\tan 56°45' = 1.52525\ldots$ $= 1.5253$ (correct to 4 d.p.)

WORKED EXAMPLE 8

Calculate the size of angle θ, correct to the nearest degree, given $\sin\theta = 0.7854$.

THINK	WRITE
1 Write the given equation.	$\sin\theta = 0.7854$
2 To find the size of the angle, we need to undo sine with its inverse, $\sin^{-1}$. (Ensure your calculator is in degrees mode.)	$\theta = \sin^{-1} 0.7854$ $= 51.8\ldots°$
3 Write your answer to the nearest degree.	$\theta = 52°$ (correct to the nearest degree)

WORKED EXAMPLE 9

Calculate the value of θ:

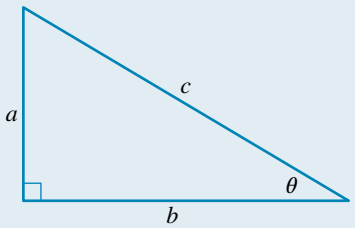
a **correct to the nearest minute, given that $\cos\theta = 0.2547$**

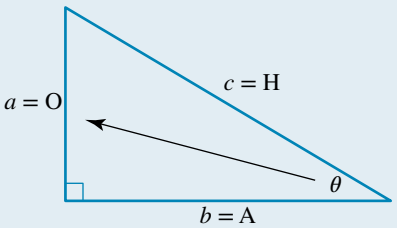
b **correct to the nearest minute, given that $\tan\theta = 2.364$.**

THINK	WRITE
a 1 Write the equation.	**a** $\cos\theta = 0.2547$
2 Using the inverse cos function on your calculator, write your answer in degrees. There are 60 seconds in 1 minute. Multiply 0.2442 by 60. Round to the nearest minute.	$\cos^{-1} 0.2547 = 75.2442\ldots°$ $= 75°15'$ (correct to the nearest minute)
b 1 Write the equation.	**b** $\tan\theta = 2.364$
2 Using the inverse tan function on your calculator, write your answer in degrees. There are 60 seconds in 1 minute. Multiply 0.0711 by 60. Round to the nearest minute.	$\tan^{-1} 2.364 = 67.0711\ldots°$ $= 67°4'$ (correct to the nearest minute)

WORKED EXAMPLE 10

For the triangle shown, write the rules for the sine, cosine and tangent ratios of the given angle.

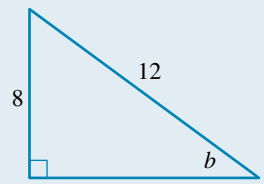


THINK	WRITE/DRAW
1 Label the diagram using the symbols O, A, H with respect to the given angle (angle θ).	
2 From the diagram, identify the values of O (opposite side), A (adjacent side) and H (the hypotenuse).	O = a, A = b, H = c
3 Write the rule for each of the sine, cosine and tangent ratios.	$\sin\theta = \frac{\text{O}}{\text{H}}$, $\cos\theta = \frac{\text{A}}{\text{H}}$, $\tan\theta = \frac{\text{O}}{\text{A}}$
4 Substitute the values of A, O and H into each rule.	$\sin\theta = \frac{a}{c}$, $\cos\theta = \frac{b}{c}$, $\tan\theta = \frac{a}{b}$

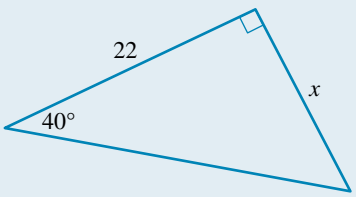
WORKED EXAMPLE 11

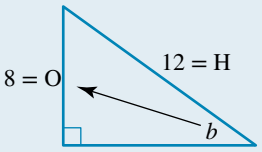
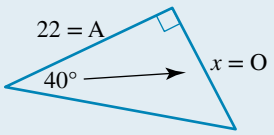
For each of the following triangles, write the equation that relates the two marked sides and the marked angle.

a



b



THINK	WRITE/DRAW
a 1 Label the given sides of the triangle.	**a** 
2 Write the ratio that contains O and H.	$\sin\theta = \frac{O}{H}$
3 Identify the values of the pronumerals.	$O = 8, H = 12$
4 Substitute the values of the pronumerals into the ratio. (Since the given angle is denoted with the letter b, replace θ with b.)	$\sin b = \frac{8}{12} = \frac{2}{3}$
b 1 Label the given sides of the triangle.	**b** 
2 Write the ratio that contains O and A.	$\tan\theta = \frac{O}{A}$
3 Identify the values of the pronumerals.	$O = x, A = 22, \theta = 40°$
4 Substitute the values of the pronumerals into the ratio.	$\tan 40° = \frac{x}{22}$

learnON RESOURCES — ONLINE ONLY

Interactivity: Trigonometric ratios (int-2577)

Digital doc: SkillSHEET Labelling the sides of a right-angled triangle (doc-5226)

Digital doc: SkillSHEET Selecting an appropriate trigonometric ratio based on the given information (doc-5231)

Exercise 3.4 Trigonometric ratios

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1, 3, 6a–f, 7, 8	Questions: 2–4, 6a–f, 7–9, 11	Questions: 2, 3, 4, 5, 6c–l, 7–12

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. Calculate each of the following, correct to 4 decimal places.

a. sin 30° **b.** cos 45° **c.** tan 25° **d.** sin 57° **e.** tan 83° **f.** cos 44°

2. WE7 Calculate each of the following, correct to 4 decimal places.

a. sin 40°30′ **b.** cos 53°57′ **c.** tan 27°34′ **d.** tan 123°40′

e. sin 92°32′ **f.** sin 42°8′ **g.** cos 35°43′ **h.** tan 27°43′

i. $\cos 143°25'$ **j.** $\sin 23°58'$ **k.** $\cos 8°54'$ **l.** $\sin 286°$

m. $\tan 420°$ **n.** $\cos 845°$ **o.** $\sin 367°35'$

3. WE8 Find the size of angle θ, correct to the nearest degree, for each of the following.

a. $\sin\theta = 0.763$ **b.** $\cos\theta = 0.912$ **c.** $\tan\theta = 1.351$

d. $\cos\theta = 0.321$ **e.** $\tan\theta = 12.86$ **f.** $\cos\theta = 0.756$

4. WE9a Find the size of the angle θ, correct to the nearest minute.

a. $\sin\theta = 0.814$ **b.** $\sin\theta = 0.110$ **c.** $\tan\theta = 0.015$

d. $\cos\theta = 0.296$ **e.** $\tan\theta = 0.993$ **f.** $\sin\theta = 0.450$

5. Find the value of each expression, correct to 3 decimal places.

a. $3.8\cos 42°$ **b.** $118\sin 37°$ **c.** $2.5\tan 83°$

d. $\dfrac{2}{\sin 45°}$ **e.** $\dfrac{220}{\cos 14°}$ **f.** $\dfrac{2\cos 23°}{5\sin 18°}$

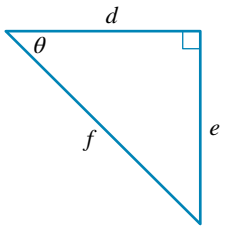
g. $\dfrac{12.8}{\tan 60°32'}$ **h.** $\dfrac{18.7}{\sin 35°26'}$ **i.** $\dfrac{55.7}{\cos 89°21'}$

j. $\dfrac{3.8\tan 1°52'}{4.5\sin 25°45'}$ **k.** $\dfrac{2.5\sin 27°8'}{10.4\cos 83°2'}$ **l.** $\dfrac{3.2\cos 34°52'}{0.8\sin 12°48'}$

6. WE10 For each labelled angle in the following triangles, write an expression for:

i. sine **ii.** cosine **iii.** tangent.

a.



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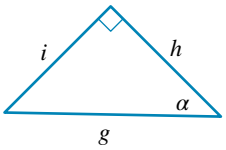

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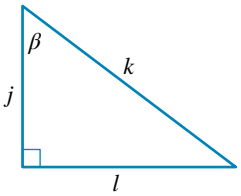

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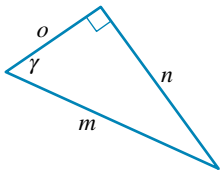
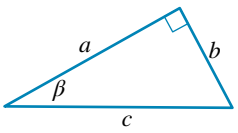
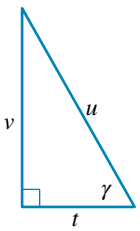


e.

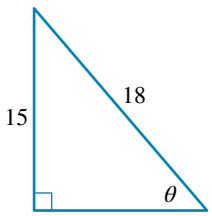


f.



7. WE11 Write the equation that relates the two marked sides and the marked angle in each of the following triangles.

a.



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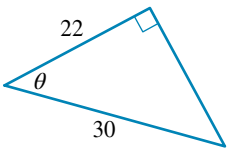

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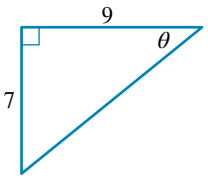

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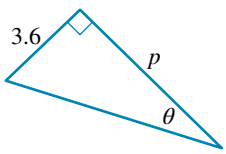
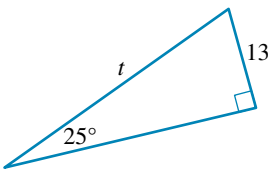


e.



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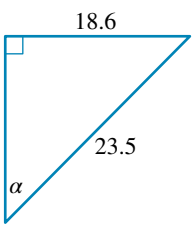
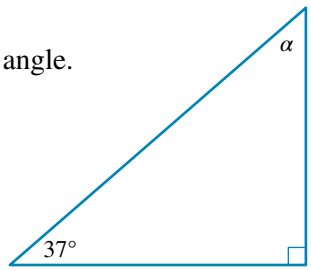


Communicating, reasoning and problem solving

8. Consider the right-angled triangle shown.



a. Label each of the sides using the letters O, A and H with respect to the 37° angle.

b. Determine the value of each trigonometric ratio. (Where applicable, answers should be given correct to 2 decimal places.)

i. sin 37° **ii.** cos 37° **iii.** tan 37°

c. What is the value of the unknown angle, α?

d. Determine the value of each of these trigonometric ratios, correct to 2 decimal places.

i. $\sin\alpha$ **ii.** $\cos\alpha$ **iii.** $\tan\alpha$

(*Hint:* First relabel the sides of the triangle with respect to angle α.)

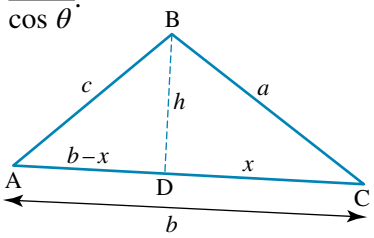
e. What do you notice about the relationship between $\sin 37°$ and $\cos\alpha$?

f. What do you notice about the relationship between $\sin\alpha$ and $\cos 37°$?

g. Make a general statement about the two angles.

9. Using algebra and a triangle labelled with a, h and o, show that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$. (*Hint:* Write all the sides in terms of the hypotenuse.)

10. ABC is a scalene triangle with side lengths a, b and c as shown. Angles BDA and BDC are right angles.



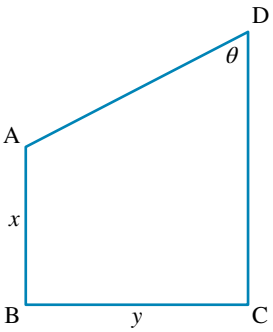
a. Express h^2 in terms of a and x.

b. Express h^2 in terms of b, c and x.

c. Equate the two equations for h^2 to show that $c^2 = a^2 + b^2 - 2bx$.

d. Use your knowledge of trigonometry to produce the equation $c^2 = a^2 + b^2 - 2ab\cos C$, which is known as the cosine rule for non-right-angled triangles.

11. Find the length of the side DC in terms of x, y and θ.



12. How do we determine which of sin, cos or tan to use in a trigonometry question?

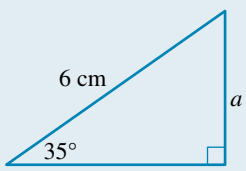
3.5 Using trigonometry to calculate side lengths [Stage 5.2]

- In a right-angled triangle, if one side length and one acute angle are known, the lengths of the other sides can be found by applying trigonometric ratios.

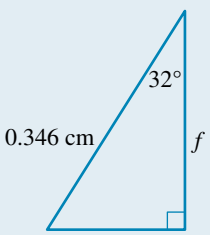
WORKED EXAMPLE 12

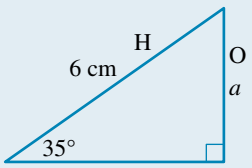
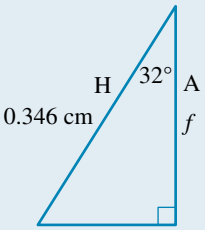
Find the value of each pronumeral in the triangles shown, giving answers correct to 3 decimal places.

a



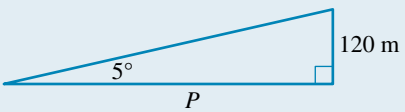
b

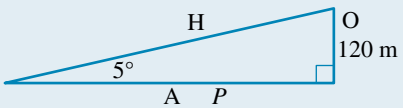


THINK	WRITE/DRAW
a 1 Label the marked sides of the triangle.	**a** 
2 Identify the appropriate trigonometric ratio to use.	$\sin\theta = \frac{O}{H}$
3 Substitute O = a, H = 6 and $\theta = 35°$.	$\sin 35° = \frac{a}{6}$
4 Make a the subject of the equation.	$6 \sin 35° = a$ $a = 6 \sin 35°$ $a = 3.441$ cm (correct to 3 d.p.)
5 Calculate and round the answer, correct to 3 decimal places.	
b 1 Label the marked sides of the triangle.	**b** 
2 Identify the appropriate trigonometric ratio to use.	$\cos\theta = \frac{A}{H}$
3 Substitute A = f, H = 0.346 and $\theta = 32°$.	$\cos 32° = \frac{f}{0.346}$
4 Make f the subject of the equation.	$0.346 \cos 32° = f$ $f = 0.346 \cos 32°$
5 Calculate and round the answer, correct to 3 decimal places.	$= 0.293$ cm (correct to 3 d.p.)

WORKED EXAMPLE 13

Find the value of the pronumeral in the triangle shown. Give the answer correct to 2 decimal places.



THINK	WRITE/DRAW
1 Label the marked sides of the triangle.	
2 Identify the appropriate trigonometric ratio to use.	$\tan\theta = \frac{O}{A}$
3 Substitute O = 120, A = P and $\theta = 35°$.	$\tan 5° = \frac{120}{P}$
4 Make P the subject of the equation. **i** Multiply both sides of the equation by P. **ii** Divide both sides of the equation by tan 5°.	$P \times \tan 5° = 120$ $P = \frac{120}{\tan 5°}$
5 Calculate and round the answer, correct to 2 decimal places.	$P = 1361.61$ m (correct to 2 d.p.)

learnON RESOURCES — ONLINE ONLY

Interactivity: Using trigonometry (int-1146)

Interactivity: Using trigonometry to calculate side lengths (int-6133)

Exercise 3.5 Using trigonometry to calculate side lengths

Individual pathways

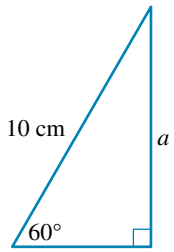
PRACTISE	CONSOLIDATE	MASTER
Questions: 1–5, 8	Questions: 1–6, 8, 9	Questions: 1–11

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

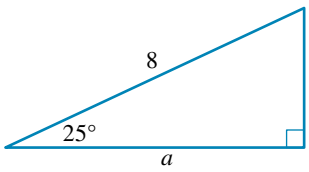
Understanding and fluency

1. WE12 Find the length of the unknown side in each of the following triangles, correct to 3 decimal places.

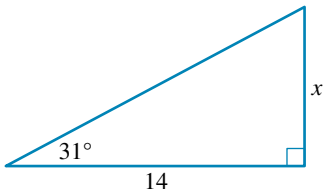
a.



b.

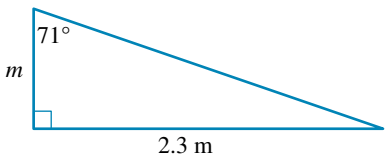


c.

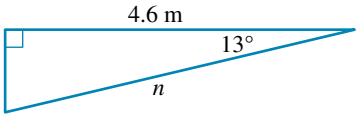


2. WE13 Find the length of the unknown side in each of the following triangles, correct to 2 decimal places.

a.



b.



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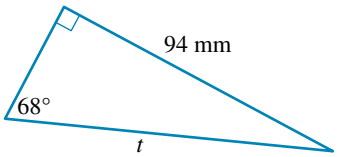
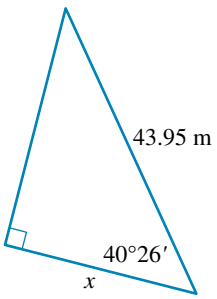
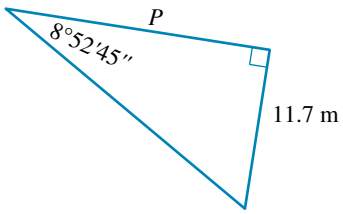


3. Find the length of the unknown side in each of the following, correct to 2 decimal places.

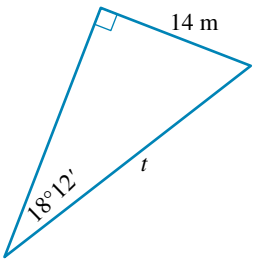
a.



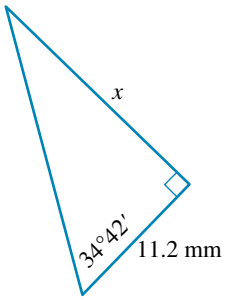
b.



c.



d.



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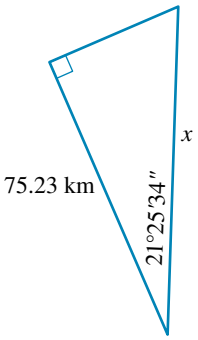

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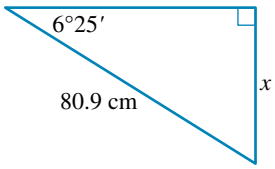
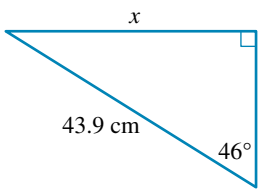
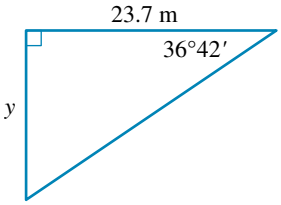


4. Find the value of the pronumeral in each of the following, correct to 2 decimal places.

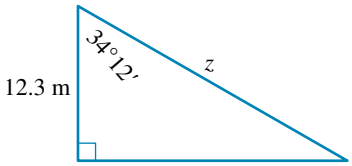
a.



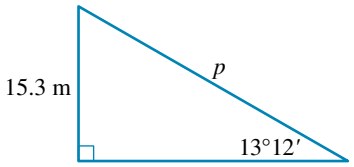
b.



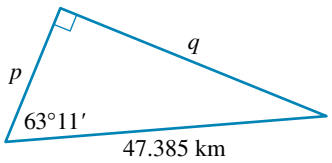
c.



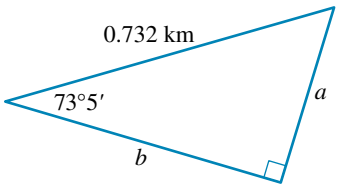
d.



e.



f.



5. Given that the angle θ is 42° and the length of the hypotenuse is 8.95 m in a right-angled triangle, find the length of:

a. the opposite side to θ

b. the adjacent side to θ.

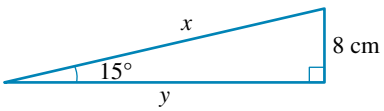
Give each answer correct to 1 decimal point.

6. A ladder rests against a wall. If the angle between the ladder and the ground is 65° and the foot of the ladder is 1.5 m from the wall, how high up the wall does the ladder reach? Give your answer correct to 2 decimal places.

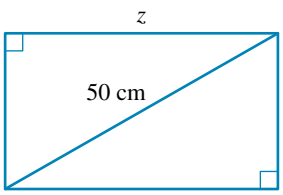
Communicating, reasoning and problem solving

7. Tran is going to construct an enclosed rectangular desktop that is at an incline of 15°. The diagonal length of the desktop is 50 cm. At the high end, the desktop, including top, bottom and sides, will be raised by 8 cm. The desktop will be made of wood. The diagrams at right represent this information.

Side view of the desktop



Top view of the desktop



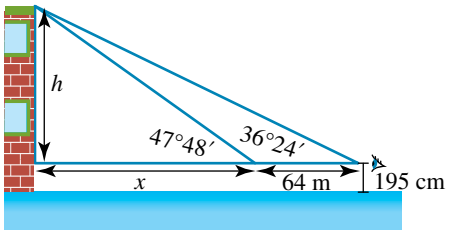
a. Determine the values (in centimetres) of x, y and z of the desktop. Write your answers correct to 2 decimal places.

b. Using your answer from part **a**, determine the area of wood, in cm^2, Tran needs to construct his desktop, including top, bottom and sides. Write your answer correct to 2 decimal places.

8. a. In a right-angled triangle, under what circumstances will the opposite side and the adjacent side have the same length?

b. In a right-angled triangle, for what values of θ (the reference angle) will the adjacent side be longer than the opposite side?

9. A surveyor needs to determine the height of a building. She measures the angle of elevation of the top of the building from two points, 64 m apart. The surveyor's eye level is 195 cm above the ground.

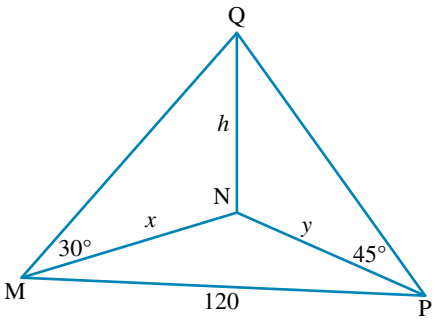


a. Find the expressions for the height of the building, h, in terms of x using the two angles.

b. Solve for x by equating the two expressions obtained in part **a**. Give your answer to 2 decimal places.

c. Find the height of the building correct to 2 decimal places.

10. If angles QNM, QNP and MNP are right angles, find the length of NQ.



11. How does solving a trigonometric equation differ when we are finding the length of the hypotenuse side compared to when finding the length of a shorter side?

3.6 Using trigonometry to calculate angle size

[Stage 5.2]

- Just as inverse operations are used to solve equations, inverse trigonometric ratios are used to solve trigonometric equations for the value of the angle.
  - Inverse sine ($\sin^{-1}$) is the inverse of sine.
  - Inverse cosine ($\cos^{-1}$) is the inverse of cosine.
  - Inverse tangent ($\tan^{-1}$) is the inverse of tangent.

  For example, since $\sin(30°) = 0.5$, then $\sin^{-1}(0.5) = 30°$; this is read as 'inverse sine of 0.5 is 30 degrees'.

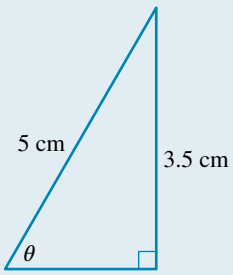
$$\text{If } \sin\theta = a, \text{ then } \sin^{-1} a = \theta.$$
$$\text{If } \cos\theta = a, \text{ then } \cos^{-1} a = \theta.$$
$$\text{If } \tan\theta = a, \text{ then } \tan^{-1} a = \theta.$$

- A calculator can be used to calculate the values of inverse trigonometric ratios.
- The size of any angle in a right-angled triangle can be found if:
  - the lengths of any two sides are known
  - an appropriate trigonometric ratio is identified from the given lengths
  - a calculator is used to evaluate the inverse trigonometric ratio.

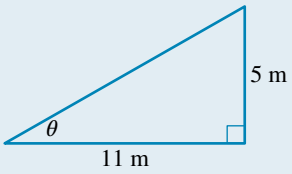
WORKED EXAMPLE 14

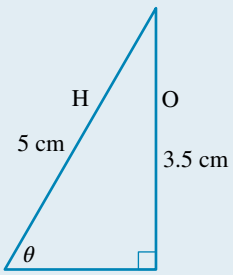
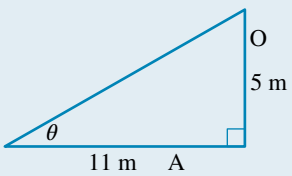
For each of the following, find the size of the angle, θ, correct to the nearest degree.

a



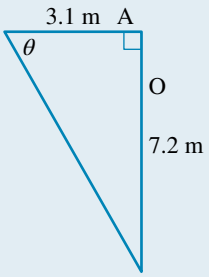
b

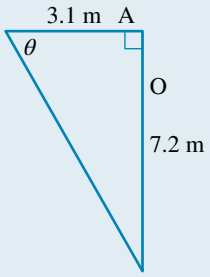


THINK	WRITE/DRAW
a 1 Label the given sides of the triangle.	**a** 
2 Identify the appropriate trigonometric ratio to use. We are given O and H.	$\sin\theta = \frac{O}{H}$
3 Substitute O = 3.5 and H = 5 and evaluate the expression.	$\sin\theta = \frac{3.5}{5}$ $= 0.7$
4 Make θ the subject of the equation using inverse sine.	$\theta = \sin^{-1} 0.7$ $= 44.427\,004°$
5 Evaluate θ and round the answer, correct to the nearest degree.	$\theta = 44°$ (correct to the nearest degree)
b 1 Label the given sides of the triangle.	**b** 
2 Identify the appropriate trigonometric ratio to use. Given O and A.	$\tan\theta = \frac{O}{A}$
3 Substitute O = 5 and A = 11.	$\tan\theta = \frac{5}{11}$
4 Make θ the subject of the equation using inverse tangent.	$\theta = \tan^{-1}\left(\frac{5}{11}\right)$ $= 24.443\,954\,78°$
5 Evaluate θ and round the answer, correct to the nearest degree.	$\theta = 24°$ (correct to the nearest degree)

WORKED EXAMPLE 15

Find the size of the angle θ, correct to the nearest minute.



THINK	WRITE/DRAW
a 1 Label the given sides of the triangle.	**a** 
2 Identify the appropriate trigonometric ratio to use.	$\tan\theta = \frac{\text{O}}{\text{A}}$
3 Substitute O = 7.2 and A = 3.1.	$\tan\theta = \frac{7.2}{3.1}$
4 Make θ the subject of the equation using inverse tangent.	$\theta = \tan^{-1}\left(\frac{7.2}{3.1}\right)$
5 Evaluate θ and write the calculator display.	$\theta = 66.705\,436\,75°$
6 Use the calculator to convert the answer to degrees, minutes and seconds or multiply 0.705 436 75 by 60.	$\theta = 66.705\,436\,75°$ $= 66°42\ 19.5472''$ or $\theta = 66.705\,436\,75°$ $= 66°42.3262'$
7 Round the answer to the nearest minute.	$\theta = 66°42'$ (correct to the nearest minute)

ACTIVITY: IMPORTANT TRIANGLES

There are two important triangles that are commonly used in trigonometry. Complete the following steps to create these triangles.

Triangle 1

1. Sketch an equilateral triangle with side length 2 units.
2. Calculate the sizes of the internal angles.
3. Bisect the triangle to form two right-angled triangles.
4. Redraw one of the triangles formed.
5. Calculate the side lengths of this right-angled triangle as exact values.
6. Fully label your diagram, showing all side lengths and angles.

Triangle 2

1. Draw a right-angled isosceles triangle.
2. Calculate the sizes of the internal angles.
3. Let the sides of equal length be 1 unit long.
4. Calculate the length of the third side.
5. Fully label your diagram, showing all side lengths and angles.

learnON RESOURCES — ONLINE ONLY

Interactivity: Finding the angle when two sides are known (int-6046)

Digital doc: WorkSHEET Using trigonometry (doc-5233)

Exercise 3.6 Using trigonometry to calculate angle size

Individual pathways

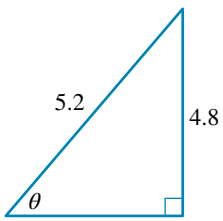
PRACTISE	CONSOLIDATE	MASTER
Questions: 1–3, 6, 8	Questions: 1–6, 8, 10	Questions: 1–11

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

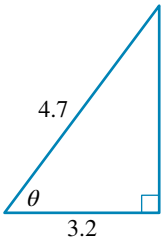
Understanding and fluency

1. WE14 Find the size of the angle, θ, in each of the following. Give your answer correct to the nearest degree.

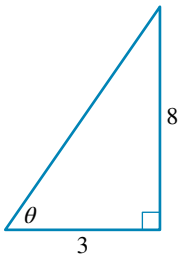
a.



b.

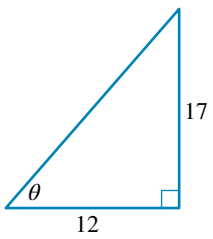


c.

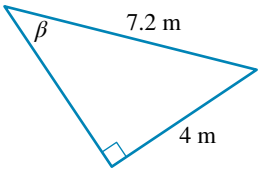


2. WE15 Find the size of the angle marked with the pronumeral in each of the following. Give your answer correct to the nearest minute.

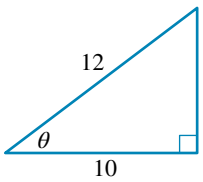
a.



b.

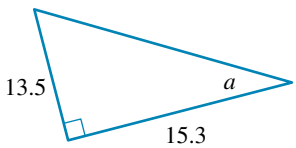


c.

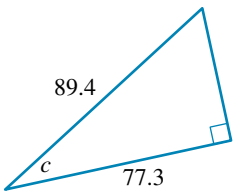


3. Find the size of the angle marked with the pronumeral in each of the following, giving your answer correct to the nearest degree.

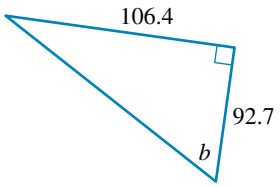
a.



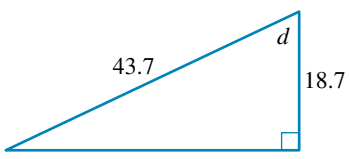
b.



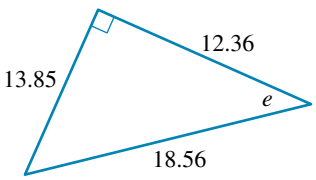
c.



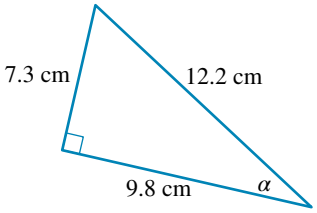
d.



e.

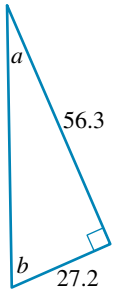


f.

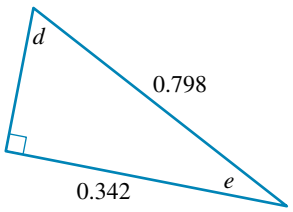


4. Find the size of each of the angles in the following, giving your answers correct to the nearest minute.

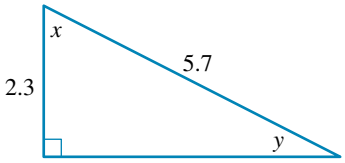
a.



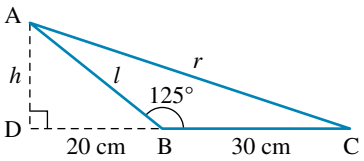
b.



c.



5. Consider the triangle shown.



a. Calculate the length of the sides r, l and h. Write your answers correct to 2 decimal places.

b. Calculate the area of ABC, correct to the nearest square centimetre.

c. Calculate $\angle BCA$.

6. In the sport of air racing, small planes have to travel between two large towers or pylons. The gap between a pair of pylons is smaller than the wing-span of the plane, so the plane has to go through on an angle with one wing 'above' the other. The wing-span of a competition airplane is 8 metres.

a. Determine the angle, correct to 1 decimal place, that the plane has to tilt if the gap between pylons is:

i. 7 metres

ii. 6 metres

iii. 5 metres.

b. Because the plane has rolled away from the horizontal as it travels between the pylons, it loses speed. If the plane's speed is below 96 km/h it will stall and possibly crash. For each degree of 'tilt' the speed of the plane is reduced by 0.98 km/h. What is the minimum speed at which the plane must go between each of the gaps in part **a**? Write your answer correct to 2 decimal places.

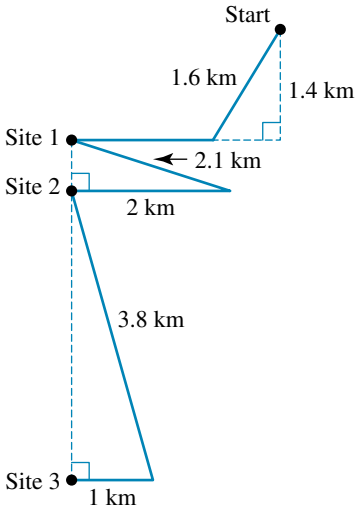
Communicating, reasoning and problem solving

7. a. Use the triangles formed in the activity 'Important triangles' in Section 3.6.1 to calculate exact values for $\sin 30°$, $\cos 30°$ and $\tan 30°$. Justify your answers.

b. Use the exact values for $\sin 30°$, $\cos 30°$ and $\tan 30°$ to show that $\tan 30° = \dfrac{\sin 30°}{\cos 30°}$.

c. Use the formulas $\sin\theta = \dfrac{o}{h}$ and $\cos\theta = \dfrac{a}{h}$ to prove that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

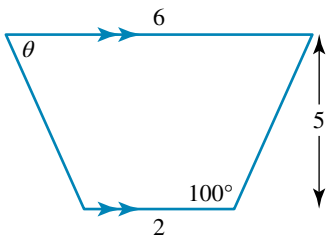
8. During a Science excursion, your class visited an underground cave to observe rock formations. You were required to walk along a series of paths and steps as shown in the diagram below.



a. Calculate the angle of the incline (slope) you have to travel down between each site. Give your answers to the nearest whole number.

b. Determine which path would have been the most challenging; that is, which path had the steepest slope.

9. Find the angle θ in degrees and minutes.

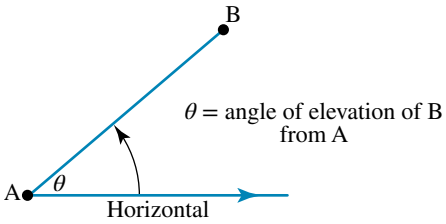


10. How is finding the angle of a right-angled triangle different to finding a side length?

11. At midday, the hour hand and the minute hand on a standard clock are both pointing at the twelve. Calculate the angles the minute hand and the hour hand have moved 24.5 minutes later. Express both answers in degrees and minutes.

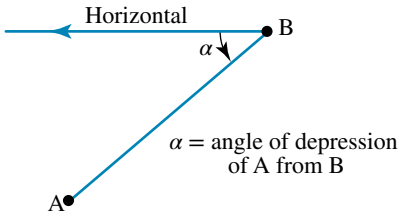
3.7 Angles of elevation and depression [Stage 5.2]

- Consider the points A and B, where B is at a higher elevation compared to A.

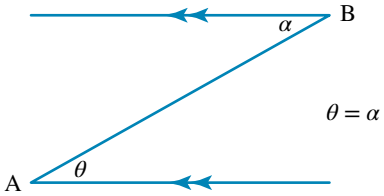


- If a horizontal line is drawn from A as shown, forming the angle θ, then θ is called the **angle of elevation** of B *from* A.

- If a horizontal line is drawn from B, forming the angle α, then α is called the **angle of depression** of A *from* B.

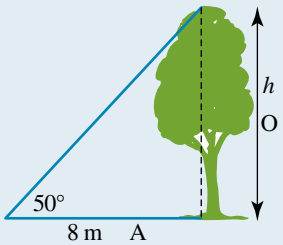


- Because the horizontal lines are parallel, θ and α are the same size (alternate angles).



WORKED EXAMPLE 16

From a point P, on the ground, the angle of elevation of the top of a tree is 50°. If P is 8 metres from the tree, find the height of the tree correct to 2 decimal places.

THINK	WRITE/DRAW
1 Let the height of the tree be h. Sketch a diagram and show the relevant information.	
2 Identify the appropriate trigonometric ratio.	$\tan\theta = \frac{\text{O}}{\text{A}}$
3 Substitute O = h, A = 8 and $\theta = 50°$.	$\tan 50° = \frac{h}{8}$
4 Rearrange to make h the subject.	$h = 8 \tan 50°$
5 Calculate and round the answer to 2 decimal places.	$= 9.53$ (correct to 2 d.p.)
6 Give a worded answer.	The height of the tree is 9.53 m.

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Interactivity: Finding the angle of elevation and angle of depression (int-6047)
eLesson: Height of a satellite (eles-0173)
Digital doc: SkillSHEET Drawing a diagram from given directions (doc-5228)
Digital doc: WorkSHEET Elevation and depression (doc-5234)

Exercise 3.7 Angles of elevation and depression

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–5, 8, 10	Questions: 1–6, 9, 10, 14, 15	Questions: 1–7, 9, 11–17

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

Give your answers to the following questions correct to 2 decimal places unless directed otherwise.

1. WE16 From a point P on the ground, the angle of elevation from an observer to the top of a tree is 54°22′. If the tree is known to be 12.19 m high, how far is P from the tree (measured horizontally)?

2. From the top of a cliff 112 m high, the angle of depression to a boat is 9°15′. How far is the boat from the foot of the cliff? Give your answer correct to 1 decimal place.

3. A person on a ship observes a lighthouse on a cliff that is 830 metres away from the ship. The angle of elevation of the top of the lighthouse is 12°.

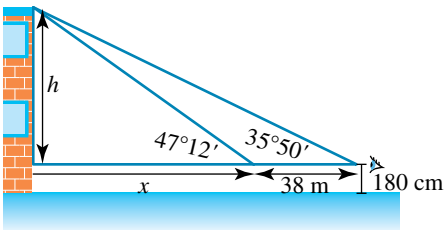
a. How far above sea level is the top of the lighthouse?

b. If the height of the lighthouse is 24 m, how high is the cliff?

4. At a certain time of the day a post that is 4 m tall casts a shadow of 1.8 m. What is the angle of elevation of the sun at that time? Give your answer correct to the nearest minute.

5. An observer who is standing 47 m from a building measures the angle of elevation of the top of the building as 17°. If the observer's eye is 167 cm from the ground, what is the height of the building?

6. A surveyor needs to determine the height of a building. She measures the angle of elevation of the top of the building from two points, 38 m apart. The surveyor's eye level is 180 cm above the ground.

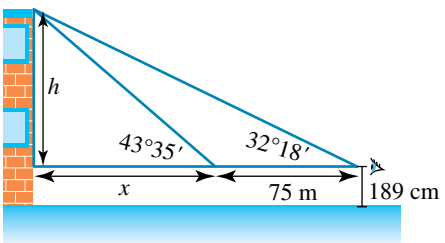


a. Find two expressions for the height of the building, h, in terms of x using the two angles.

b. Solve for x by equating the two expressions obtained in part **a**.

c. Find the height of the building.

7. The height of another building needs to be determined but cannot be found directly. The surveyor decides to measure the angle of elevation of the top of the building from two different sites that are 75 m apart. The surveyor's eye level is 189 cm above the ground.

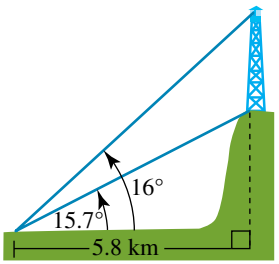


a. Using the two angles, find two expressions for the height of the building above the surveyor's eye level, h, in terms of x.

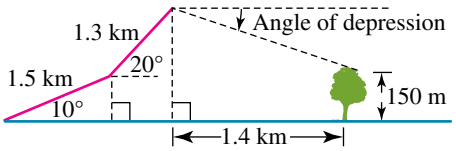
b. Solve for x.

c. Find the height of the building.

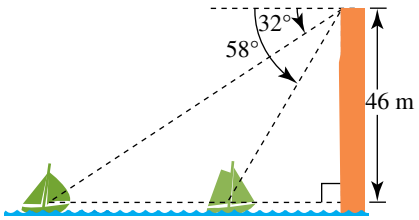
8. A lookout tower has been erected on top of a cliff. At a distance of 5.8 km from the foot of the cliff, the angle of elevation to the base of the tower is 15.7° and to the observation deck at the top of the tower is 16° respectively, as shown in the diagram. How high from the top of the cliff is the observation deck? Give your answer correct to the nearest metre.



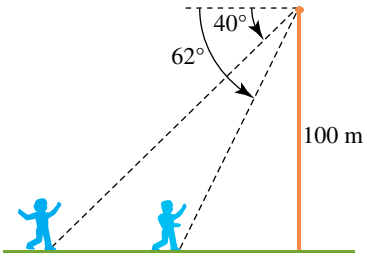
9. Elena and Sonja were on a camping trip in the Grampians, where they spent their first day hiking. They first walked 1.5 km along a path inclined at an angle of 10° to the horizontal. Then they had to follow another path, which was at an angle of 20° to the horizontal. They walked along this path for 1.3 km, which brought them to the edge of the cliff. Here Elena spotted a large gum tree 1.4 km away. If the gum tree is 150 m high, what is the angle of depression from the top of the cliff to the top of the gum tree? Give your answer in degrees, correct to 1 decimal place.



10. From a point on top of a cliff, two boats are observed. If the angles of depression are 58° and 32° and the top of the cliff is 46 m above sea level, how far apart are the boats?



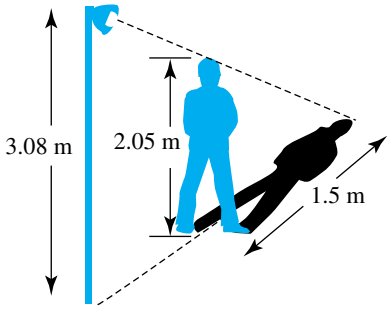
11. The competitors of a cross-country run are nearing the finish line. From a lookout 100 m above the track, the angles of depression to the two leaders, Nathan and Rachel, are 40° and 62° respectively. How far apart are the two competitors? Give your answer correct to the nearest metre.



12. A 2.05 m tall man, standing in front of a street light 3.08 m high, casts a 1.5-m shadow.

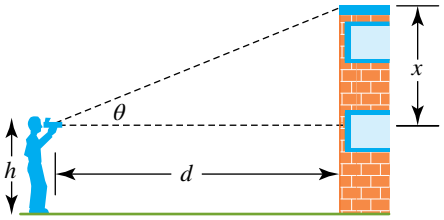
a. What is the angle of elevation from the ground to the source of light? Give your answer correct to the nearest degree.

b. How far is the man from the bottom of the light pole?



Communicating, reasoning and problem solving

13. Joseph is asked to obtain an estimate of the height of his house using any mathematical technique. He decides to use an inclinometer and basic trigonometry. Using the inclinometer, Joseph determines the angle of elevation, θ, from his eye level to the top of his house to be 42°. The point from which Joseph measures the angle of elevation is 15 m away from his house and the distance from Joseph's eyes to the ground is 1.76 m.



a. Fill in the given information on the diagram provided (substitute values for the pronumerals).

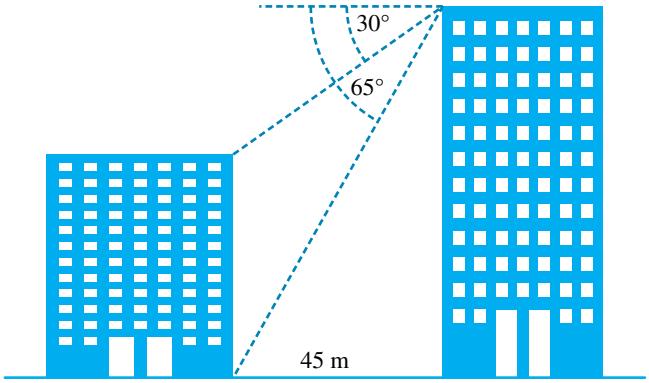
b. Determine the height of Joseph's house.

14. The angle of elevation of a vertically rising hot air balloon changes from 27° at 7.00 am to 61° at 7.03 am, according to an observer who is 300 m away from the take-off point.

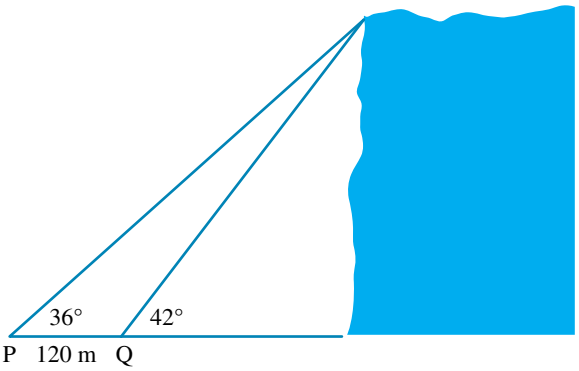
a. Assuming a constant speed, calculate the speed (in m/s and km/h) at which the balloon is rising, correct to 2 decimal places.

b. The balloon then falls 120 metres. What is the angle of elevation now? Write your answer correct to 1 decimal place.

15. The angle of depression from the top of one building to the foot of another building across the same street and 45 metres horizontally away is 65°. The angle of depression to the roof of the same building is 30°. Calculate the height of the shorter building.



16. P and Q are two points on a horizontal line that are 120 metres apart. The angles of elevation from P and Q to the top of a mountain are 36° and 42° respectively. Find the height of the mountain correct to 1 decimal place.

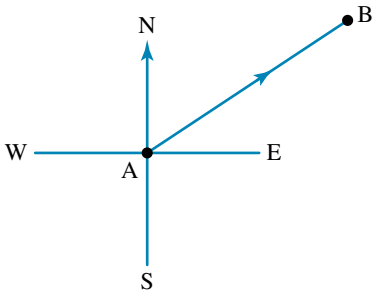


17. What is the difference between an angle of elevation and an angle of depression?

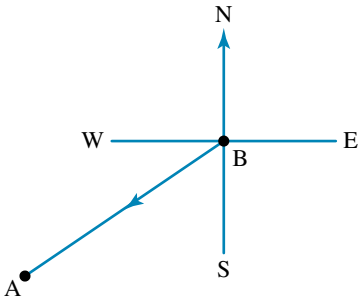
3.8 Bearings [Stage 5.2]

3.8.1 Bearings

- A bearing gives the direction of travel from one point or object to another.
- The bearing of B from A tells how to get to B *from* A. A compass rose would be drawn at A.



- To illustrate the bearing of A *from* B, a compass rose would be drawn at B.

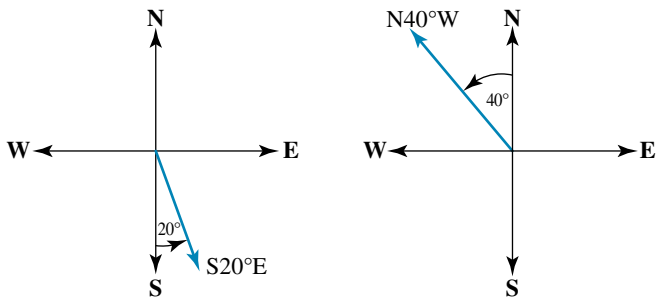


- There are two ways in which bearings are commonly written. They are compass bearings and true bearings.

3.8.2 Compass bearings

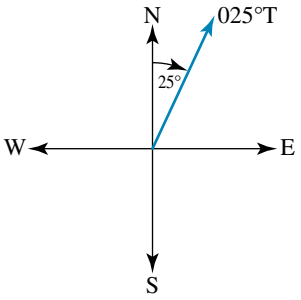
- A **compass bearing** (for example N40°E or S72°W) has three parts.
  - The first part is either N or S (for north or south).
  - The second part is an acute angle.
  - The third part is either E or W (for east or west).
- For example, the compass bearing S20°E means start by facing south and then turn 20° towards the east. This is the direction of travel.

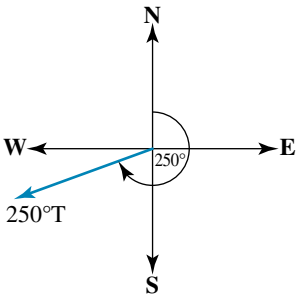
N40°W means start by facing north and then turn 40° towards the west.



3.8.3 True bearings

- **True bearings** are measured from north in a clockwise direction and are expressed in 3 digits.
- The diagrams below show the bearings of 025° true and 250° true respectively. (These true bearings are more commonly written as 025°T and 250°T.)





WORKED EXAMPLE 17

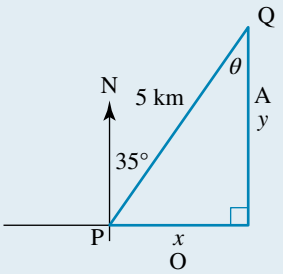
A boat travels a distance of 5 km from P to Q in a direction of 035° T.
a How far east of P is Q? Give your answer correct to 2 decimal places.
b How far north of P is Q? Give your answer correct to 2 decimal places.
c What is the true bearing of P from Q?

THINK

a 1 Draw a diagram showing the distance and bearing of Q from P. Complete a right-angled triangle travelling x km due east from P and then y km due north to Q.

WRITE/DRAW

a



2 To determine how far Q is east of P, we need to find the value of x. We are given the length of the hypotenuse (H) and need to find the length of the opposite side (O). Write the sine ratio.

$$\sin\theta = \frac{\text{O}}{\text{H}}$$

3 Substitute O = x, H = 5 and $\theta = 35°$.

$$\sin 35° = \frac{x}{5}$$

4 Make x the subject of the equation.

$$x = 5\sin 35°$$

5 Evaluate and round the answer, correct to 2 decimal places.

$$= 2.87 \text{ (correct to 2 d.p.)}$$

6 Write the answer in words.

Point Q is 2.87 km east of P.

b 1 To determine how far Q is north of P, we need to find the value of y. This can be done in several ways, for example using the cosine ratio, the tangent ratio or Pythagoras' theorem. Write the cosine ratio.

b

$$\cos\theta = \frac{\text{A}}{\text{H}}$$

2 Substitute A = y, H = 5 and $\theta = 35°$.

$$\cos 35° = \frac{y}{5}$$

3 Make y the subject of the equation.

$$y = 5\cos 35°$$

4 Evaluate and round the answer, correct to 2 decimal places.

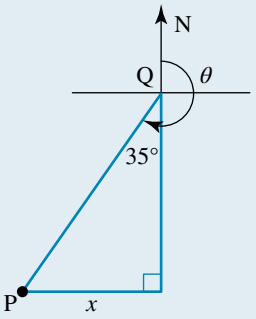
$$= 4.10 \text{ (correct to 2 d.p.)}$$

5 Write the answer in words.

Point B is 4.10 km north of A.

c 1 To find the bearing of P from Q, draw a compass rose at Q. The true bearing is given by θ.

c



2 The value of θ is the sum of 180° (from north to south) and 35°. Write the value of θ.

True bearing $= 180° + \alpha$
$\alpha = 35°$:
True bearing $= 180° + 35°$
$= 215°$

3 Write the answer in words.

The bearing of P from Q is 215°T.

- Sometimes a journey includes a change in directions. In such cases, each section of the journey should be dealt with separately.

WORKED EXAMPLE 18

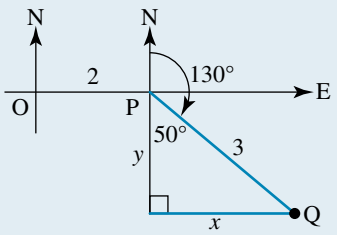
A boy walks 2 km on a true bearing of 090° and then 3 km on a true bearing of 130°.
a How far east of the starting point is the boy at the completion of his walk? (Answer correct to 1 decimal place.)
b How far south of the starting point is the boy at the completion of his walk? (Answer correct to 1 decimal place.)
c To return directly to his starting point, how far must the boy walk and on what bearing?

THINK

a 1 Draw a diagram of the boy's journey.
The first leg of the journey is due east. Label the easterly component x and the southerly component y.

WRITE/DRAW

a



2 Write the ratio to find x.

$\sin\theta = \frac{O}{H}$

3 Substitute O = x, H = 3 and $\theta = 50°$.

$\sin 50° = \frac{x}{3}$

4 Make x the subject of the equation.

$x = 3\sin 50°$

5 Evaluate and round the result correct to 1 decimal place.

$= 2.3$ km (correct to 1 d.p.)

6 Add to this the 2 km east that was walked in the first leg of the journey and give a worded answer.

Total distance east $= 2 + 2.3$
$= 4.3$ km
The boy is 4.3 km east of the starting point.

b 1 To find y (see the diagram in part **a**) we can use Pythagoras' theorem, as we know the lengths of two of the three sides in the right-angled triangle. Round the answer correct to 1 decimal place.
Note: Alternatively, the cosine ratio could have been used.

b Distance south = y km

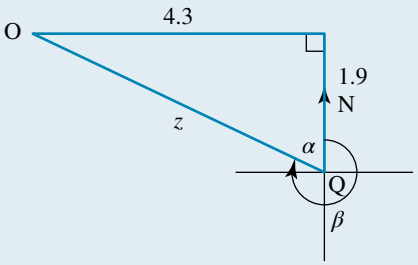
$$\begin{aligned} a^2 &= c^2 - b^2 \\ y^2 &= 3^2 - 2.3^2 \\ &= 9 - 5.29 \\ &= 3.71 \\ y &= \sqrt{3.71} \\ &= 1.9 \text{ km (correct to 1 d.p.)} \end{aligned}$$

2 Write the answer in words.

The boy is 1.9 km south of the starting point.

c 1 Draw a diagram of the journey and write in the results found in parts **a** and **b**. Draw a compass rose at Q.

c



2 Find z using Pythagoras' theorem.

$$\begin{aligned} z^2 &= 1.9^2 + 4.3^2 \\ &= 22.1 \\ z &= \sqrt{22.1} \\ &= 4.7 \text{ km (correct to 1 d.p.)} \end{aligned}$$

3 Find α using trigonometry.

$$\tan\alpha = \frac{4.3}{1.9}$$

4 Make α the subject of the equation using the inverse tangent function.

$$\alpha = \tan^{-1}\left(\frac{4.3}{1.9}\right)$$

5 Evaluate and round to the nearest minute.

$$\begin{aligned} &= 66.16125982° \\ &= 66°9'40.535'' \\ &= 66°10' \end{aligned}$$

6 The angle β gives the bearing.

$$\begin{aligned} \beta &= 360° - 66°10' \\ &= 293°50' \end{aligned}$$

7 Write the answer in words.

The boy travels 4.7 km, correct to 1 decimal place, on a bearing of 293°50′ T.

learnon RESOURCES — ONLINE ONLY

Interactivity: Bearings (int-6481)
eLesson: Bearings (eles-1935)

Exercise 3.8 Bearings

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1, 2, 3a–d, 4a–b, 5–7, 11	Questions: 1, 2, 3, 4a–c, 5–8, 11, 13	Questions: 1–6, 8–16

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

Where appropriate, give answers to the following questions correct to 2 decimal places unless directed otherwise.

1. Change each of the following compass bearings to true bearings.

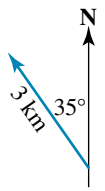
a. N20°E **b.** N20°W **c.** S35°W **d.** S28°E **e.** N34°E **f.** S42°W

2. Change each of the following true bearings to compass bearings.

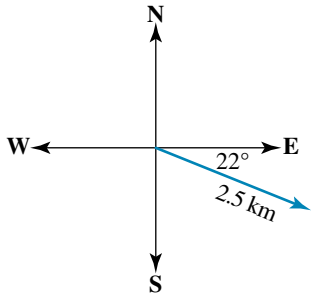
a. 049°T **b.** 132°T **c.** 267°T **d.** 330°T **e.** 086°T **f.** 234°T

3. Describe the following paths using true bearings.

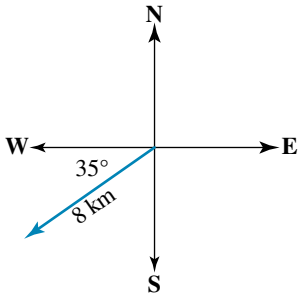
a.



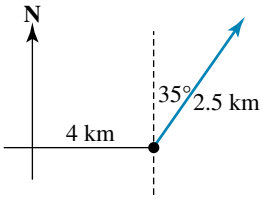
b.



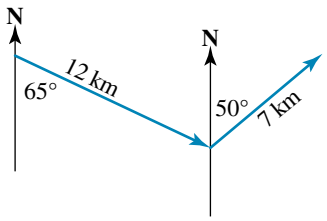
c.



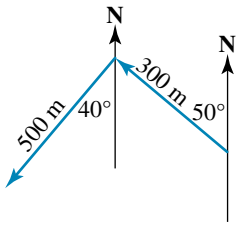
d.



e.



f.



4. Show each of the following journeys as a diagram.

a. A ship travels 040°T for 40 km and then 100°T for 30 km.

b. A plane flies for 230 km in a direction 135°T and a further 140 km in a direction 240°T.

c. A bushwalker travels in a direction 260°T for 0.8 km, then changes direction to 120°T for 1.3 km, and finally travels in a direction of 32° for 2.1 km.

d. A boat travels N40°W for 8 km, then changes direction to S30°W for 5 km and then S50°E for 7 km.

e. A plane travels N20°E for 320 km, N70°E for 180 km and S30°E for 220 km.

5. WE17 **a.** A yacht travels 20 km from A to B on a bearing of 042°T.

i. How far east of A is B?

ii. How far north of A is B?

iii. What is the bearing of A from B?

b. The yacht then sails 80 km from B to C on a bearing of 130°T.

i. Show the journey using a diagram. **ii.** How far south of B is C?

iii. How far east of B is C? **iv.** What is the bearing of B from C?

6. If a farmhouse is situated 220 m N35°E from a shed, what is the true bearing of the shed from the house?

7. A pair of hikers travel 0.7 km on a true bearing of 240° and then 1.3 km on a true bearing of 300°. How far west have they travelled from their starting point? Give your answer correct to 3 decimal places.

8. WE18 A boat travels 6 km on a true bearing of 120° and then 4 km on a true bearing of 080°.

a. How far east is the boat from the starting point on the completion of its journey? Give your answer correct to 3 decimal places.

b. How far south is the boat from the starting point on the completion of its journey? Give your answer correct to 3 decimal places.

c. What is the bearing of the boat from the starting point on the completion of its journey?

9. A plane flies on a true bearing of 320° for 450 km. It then flies on a true bearing of 350° for 130 km and finally on a true bearing of 050° for 330 km. How far north of its starting point is the plane?

Communicating, reasoning and problem solving

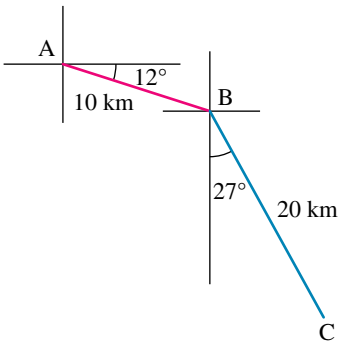
10. A bushwalker leaves her tent and walks due east for 4.12 km, then walks a further 3.31 km on a bearing of N20°E. If she wishes to return directly to her tent, how far must she walk and what bearing should she take? (Answer to the nearest degree.)

11. A car travels due south for 3 km and then due east for 8 km. What is the bearing of the car from its starting point? (Answer to the nearest degree.)

12. If the bearing of A from O is θ°T, then what is the bearing of O from A:

a. if $0° < \theta° < 180°$ **b.** if $180° < \theta° < 360°$?

13. A boat sails on a compass direction of E12°S for 10 km, then changes direction to S27°E for another 20 km. The crew then decide to sail back to the boat's starting point.



a. How far, correct to 2 decimal places, is the boat from its starting point?

b. On what bearing should the boat travel to return to its starting point? Write the angle correct to the nearest degree.

14. Samantha and Tim set off early from the car park of a national park to hike for the day. Initially they walk N60°E for 12 km to see a spectacular waterfall. They then change direction and walk in a south-easterly direction for 6 km, then stop for lunch. Give all answers correct to 2 decimal places.

a. Draw a diagram of the hiking path they completed.

b. How far north of the car park are they at the lunch stop?

c. How far east of the car park are they at the lunch stop?

d. What is the bearing of the lunch stop from the car park?

Samantha and Tim then walk directly back to the car park.

e. Calculate the distance they have covered after lunch.

15. What is the difference between true bearings and compass directions?

16. Starting from their base in the national park, a group of bushwalkers travel 1.5 km at a true bearing of 030°, then 3.5 km at a true bearing of 160°, and then 6.25 km at a true bearing of 300°. How far, and at what true bearing, should the group walk to return to its base?

3.9 Applications [Stages 5.2 and 5.3]

- When applying trigonometry to practical situations, it is essential to draw good mathematical diagrams using points, lines and angles.
- Several diagrams may be required to show all the necessary right-angled triangles.

WORKED EXAMPLE 19

A ladder of length 3 m makes an angle of 32° with the wall. Give all answers correct to 2 decimal places.

a How far is the foot of the ladder from the wall?

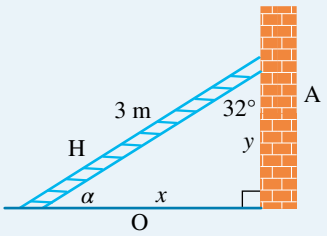
b How far up the wall does the ladder reach?

c What angle does the ladder make with the ground?

THINK

Sketch a diagram and label the sides of the right-angled triangle with respect to the given angle.

WRITE/DRAW



a 1 We need to find the distance of the foot of the ladder from the wall (O) and are given the length of the ladder (H). Write the sine ratio.

a $\sin\theta = \frac{O}{H}$

2 Substitute O = x, H = 3 and $\theta = 32°$.

$\sin 32° = \frac{x}{3}$

3 Make x the subject of the equation.

$x = 3 \sin 32°$

4 Evaluate and round the answer to 2 decimal places.

$= 1.59$ m (correct to 2 d.p.)

5 Write the answer in words.

The foot of the ladder is 1.59 m from the wall.

b 1 We need to find the height the ladder reaches up the wall (A) and are given the hypotenuse (H). Write the cosine ratio.

b $\cos\theta = \frac{A}{H}$

2 Substitute A = y, H = 3 and $\theta = 32°$.

$\cos 32° = \frac{y}{3}$

3 Make y the subject of the equation.

$y = 3 \cos 32°$

4 Evaluate and round the answer to 2 decimal places.

$y = 2.54$ m (correct to 2 d.p.)

5 Write the answer in words.

The ladder reaches 2.54 m up the wall.

c 1 To find the angle that the ladder makes with the ground, we could use any of the trigonometric ratios, as the lengths of all three sides are known. However, it is quicker to use the angle sum of a triangle.

c $\alpha + 90° + 32° = 180°$

$\alpha + 122° = 180°$

$\alpha = 180° - 22°$

$\alpha = 58°$

2 Write the answer in words.

The ladder makes a 58° angle with the ground.

learnON RESOURCES — ONLINE ONLY

Interactivity: Drafting problems (int-2781)

Exercise 3.9 Applications

Individual pathways

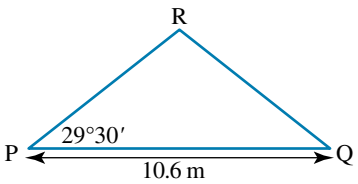
PRACTISE	CONSOLIDATE	MASTER
Questions: 1–4, 8, 10, 15	Questions: 1–5, 8, 11, 13, 14, 16	Questions: 1, 3, 4, 6, 7, 9, 12–18

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

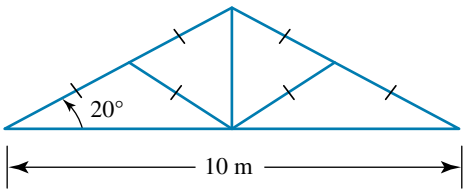
Where appropriate, give answers to the following questions correct to 2 decimal places unless directed otherwise.

1. A carpenter wants to make a roof pitched at 29°30′, as shown in the diagram. How long should he cut the beam PR?



2. The mast of a boat is 7.7 m high. A guy wire from the top of the mast is fixed to the deck 4 m from the base of the mast. Determine the angle the wire makes with the horizontal. Give your answer correct to the nearest minute.

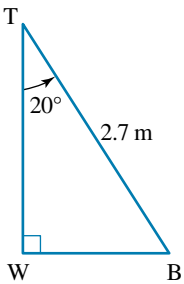
3. A steel roof truss is to be made to the following design.



a. How high is the truss?

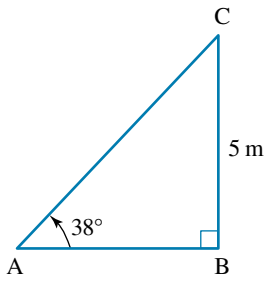
b. What is the total length of steel required to make the truss?

4. WE19 A ladder that is 2.7 m long is leaning against a wall at an angle of 20° as shown.



If the base of the ladder is moved 50 cm further away from the wall, what angle will the ladder make with the wall? Give your answer correct to the nearest minute.

5. A wooden framework is built as shown.



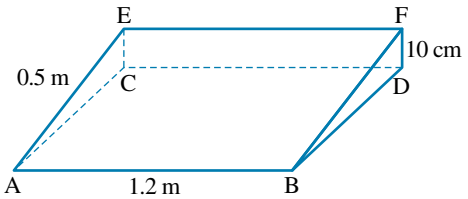
Bella plans to reinforce the framework by adding a strut from C to the midpoint of AB. What will be the length of the strut?

6. Atlanta is standing due south of a 20-m flagpole at a point where the angle of elevation of the top of the pole is 35°. Ginger is standing due east of the flagpole at a point where the angle of elevation of the top of the pole is 27°. How far is Ginger from Atlanta? Give your answer correct to the nearest metre.

7. From a point at ground level, Henry measures the angle of elevation of the top of a tall building to be 41°. After walking directly towards the building, he finds the angle of elevation to be 75°. If the building is 220 m tall, how far did Henry walk between measurements? Give your answer correct to the nearest metre.

8. Sailing towards a mountain peak of height 893 m, Imogen measured the angle of elevation to be 14°. A short time later the angle of elevation was 27°. How far had Imogen sailed in that time?

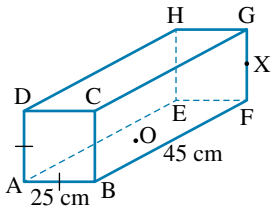
9. A desk top of length 1.2 m and width 0.5 m rises to 10 cm.



Calculate, correct to the nearest minute:

a. ∠DBF **b.** ∠CBE.

10. A cuboid has a square end.



a. If the length of the cuboid is 45 cm and its height and width are 25 cm each, calculate:

i. the length of BD **ii.** the length of BG **iii.** the length of BE

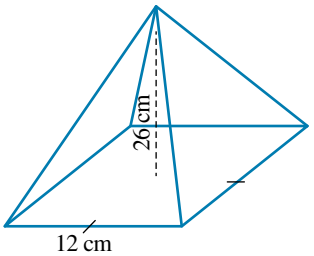
iv. the length of BH **v.** ∠FBG **vi.** ∠EBH.

b. If the midpoint of FG is X and the centre of the rectangle ABFE is O, calculate:

i. the length of OF **ii.** the length of FX

iii. ∠FOX **iv.** the length of OX.

11. In a right square-based pyramid, the length of the side of the base is 12 cm and the height is 26 cm.



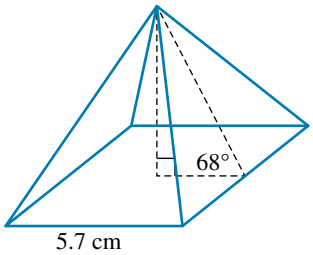
Determine:

a. the angle the triangular face makes with the base

b. the angle the sloping edge makes with the base

c. the length of the sloping edge.

12. In a right square-based pyramid, the length of the side of the square base is 5.7 cm.



If the angle between the triangular face and the base is 68°, determine:

a. the height of the pyramid

b. the angle the sloping edge makes with the base

c. the length of the sloping edge.

13. In a right square-based pyramid, the height is 47 cm. If the angle between a triangular face and the base is 73°, calculate:

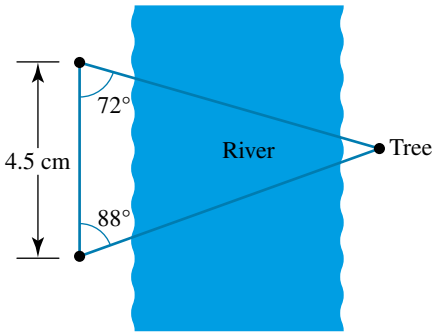
a. the length of the side of the square base

b. the length of the diagonal of the base

c. the angle the sloping edge makes with the base.

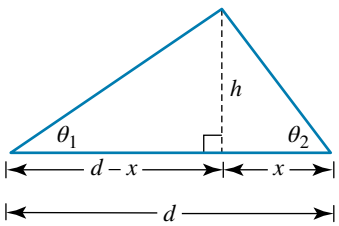
Communicating, reasoning and problem solving

14. Aldo the carpenter is lost in a rainforest. He comes across a large river and he knows that he cannot swim across it. Aldo intends to build a bridge across the river. He draws some plans to calculate the distance across the river as shown in the diagram below.



a. Aldo used a scale of 1 cm to represent 20 m. Find the real-life distance represented by 4.5 cm in Aldo's plans.

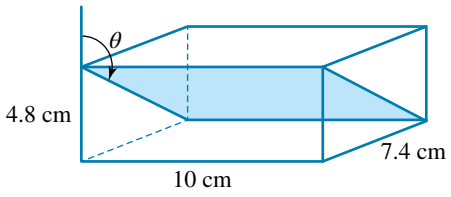
b. Use the diagram below to write an equation for h in terms of d and the two angles.



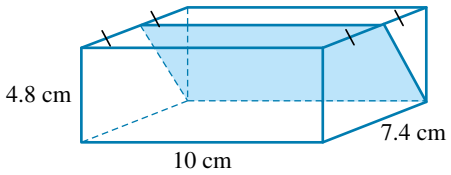
c. Use your equation from part **b** to find the distance across the river, correct to the nearest metre.

15. A block of cheese is in the shape of a rectangular prism as shown. The cheese is to be sliced with a wide blade that can slice it in one go. Calculate the angle (to the vertical) that the blade must be inclined if:

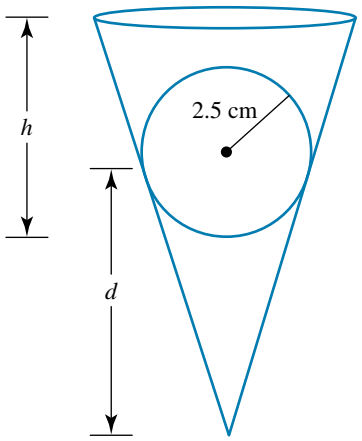
a. the block is to be sliced diagonally into two identical triangular wedges



b. the blade is to be placed in the middle of the block and sliced through to the bottom corner, as shown.



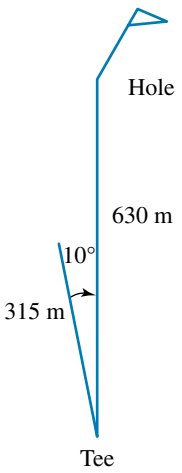
16. A sphere of radius length 2.5 cm rests in a hollow inverted cone as shown. The height of the cone is 12.5 cm and its vertical angle is equal to 36°. Answer the following correct to 2 decimal places.



a. Find the distance, d, from the tip of the cone to the point of contact with the sphere.

b. Find the distance, h, from the open end of the cone to the bottom of the ball.

17. The ninth hole on a municipal golf course is 630 m from the tee. A golfer drives a ball from the tee a distance of 315 m at a 10° angle off the direct line as shown.



Find how far the ball is from the hole and state the angle of the direct line that the ball must be hit along to go directly to the hole. Give your answers correct to 1 decimal place.

18. What are some real-life applications of trigonometry?

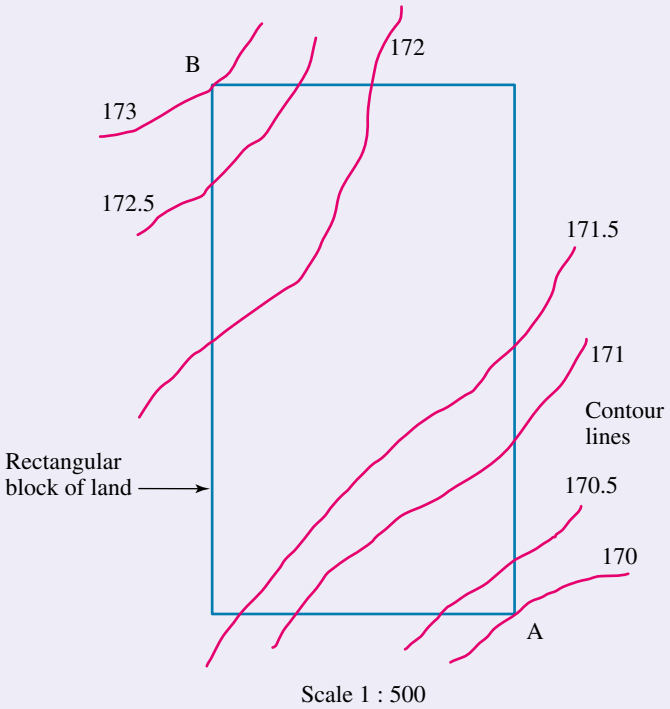
3.10 Review

Investigation | Rich task

How steep is the land?

When we buys a block of land on which to build a house, the slope of the land is often not very obvious. The slab of a house built on the ground must be level, so it is frequently necessary to remove or build up soil to obtain a flat area. The gradient of the land can be determined from a contour map of the area.

Consider the building block shown in the diagram. The contour lines join points that have the same height above sea level. Their measurements are in metres. The plan clearly shows that the land rises from A to B. The task is to determine the angle of this slope.

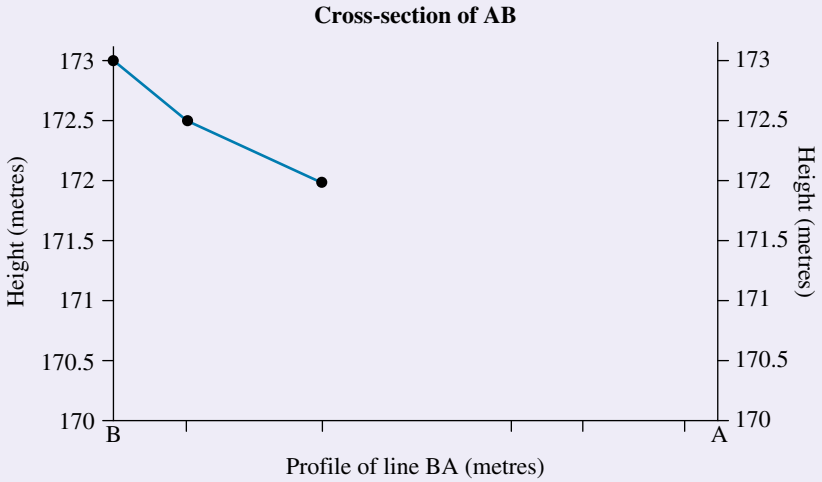


Scale 1 : 500

1. A cross-section shows a profile of the surface of the ground. Let us look at the cross-section of the ground between A and B. The technique used is as follows.

- Place the edge of a piece of paper on the line joining A and B.
- Mark the edge of the paper at the points where the contour lines intersect the paper.
- Transfer this paper edge to the horizontal scale of the profile and mark these points.
- Choose a vertical scale within the range of the heights of the contour lines.
- Plot the height at each point where a contour line crosses the paper.
- Join the points with a smooth curve.

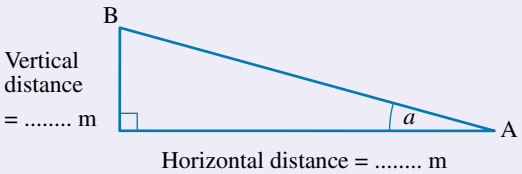
The cross-section has been started for you. Complete the profile of the line AB.
You can now see a visual picture of the profile of the soil between A and B.



2. We now need to determine the horizontal distance between A and B.
   a. Measure the map distance between A and B using a ruler. What is the map length?
   b. Using the scale of 1 : 500, calculate the actual horizontal distance AB (in metres).

3. The vertical difference in height between A and B is indicated by the contour lines. What is this vertical distance?

4. Complete the measurements on this diagram.

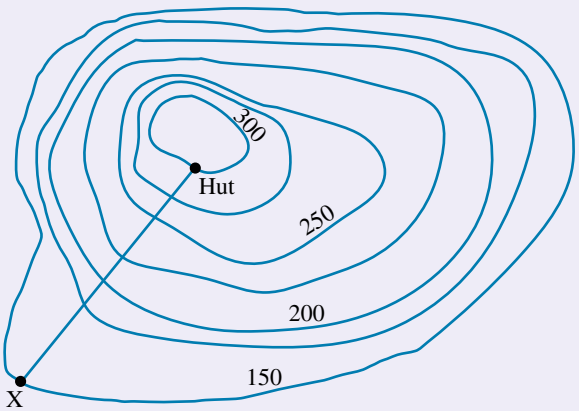


5. The angle *a* represents the angle of the average slope of the land from A to B. Use the tangent ratio to calculate this angle (to the nearest minute).

6. In general terms, an angle less than 5° can be considered a gradual to moderate rise. An angle between 5° and 15° is regarded as moderate to steep, and more than 15° is a steep rise. How would you describe this block of land?

7. Imagine that you are going on a bush walk this weekend with a group of friends. A contour map of the area is shown. Starting at X, the plan is to walk directly to the hut.

Draw a cross-section profile of the walk and calculate the average slope of the land. How would you describe the walk?



Scale 1 : 20 000

learnON RESOURCES — ONLINE ONLY

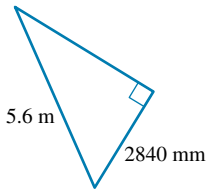
Digital doc: Investigation — How steep is the land? (doc-15924)
Digital doc: Code puzzle — What will Sir have to follow the chicken? (doc-15925)
Digital doc: Summary Topic 3 (doc-22901)
Interactivity: Word search Topic 3 (int-2838)
Interactivity: Crossword Topic 3 (int-2839)
Interactivity: Sudoku Topic 3 (int-3592)

Exercise 3.10 Review questions

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

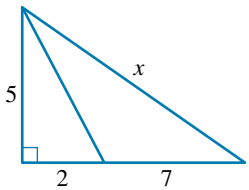
Understanding and fluency

1. MC The most accurate measure for the length of the third side in the triangle below is:



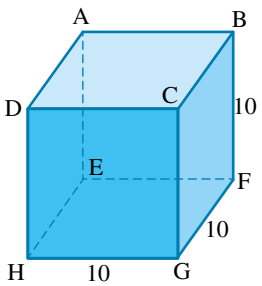
A. 4.83 m **B.** 23.3 cm **C.** 3.94 m **D.** 2330 mm **E.** 4826 mm

2. MC What is the value of x in this figure?



A. 5.4 **B.** 7.5 **C.** 10.1 **D.** 10.3 **E.** 4

3. MC What is the closest length of AG of the cube below?

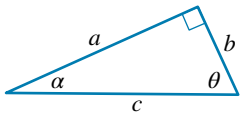


A. 10 **B.** 30 **C.** 20 **D.** 14 **E.** 17

4. MC If $\sin 38° = 0.6157$, which of the following will also give this result?
A. $\sin 218°$ **B.** $\sin 322°$ **C.** $\sin 578°$ **D.** $\sin 682°$ **E.** $\sin 142°$

5. MC The angle $118°52'$ is also equal to:
A. 118.5234° **B.** $118\frac{52°}{34}$ **C.** 118.876° **D.** 118.867° **E.** 118.786°

6. MC Which trigonometric ratio for the triangle shown below is incorrect?

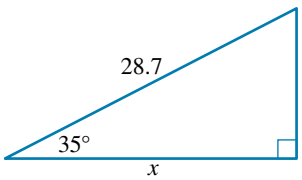


A. $\sin\alpha = \frac{b}{c}$ **B.** $\sin\alpha = \frac{a}{c}$ **C.** $\cos\alpha = \frac{a}{c}$ **D.** $\tan\alpha = \frac{b}{a}$ **E.** $\tan\theta = \frac{a}{b}$

7. MC Which of the following statements is correct?

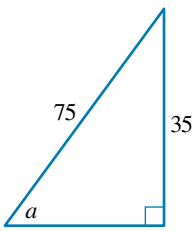
A. $\sin 55° = \cos 55°$
B. $\sin 45° = \cos 35°$
C. $\cos 15° = \sin 85°$
D. $\sin 30° = \sin 60°$
E. $\sin 42° = \cos 48°$

8. MC Which of the following can be used to find the value of x in the diagram below?



A. $28.7\sin 35°$ **B.** $28.7\cos 35°$ **C.** $28.7\tan 35°$ **D.** $\frac{28.7}{\sin 35°}$ **E.** $\frac{28.7}{\cos 35°}$

9. MC Which of the following expressions can be used to find the value of a in the triangle shown?



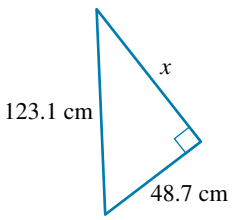
A. $35\sin 75°$ **B.** $\sin^{-1}\frac{35}{75}$ **C.** $\sin^{-1}\frac{75}{35}$ **D.** $\cos^{-1}\frac{35}{75}$ **E.** $\cos^{-1}\frac{75}{35}$

10. MC If a school is 320 m S42°W from the police station, what is the true bearing of the police station from the school?

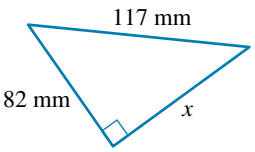
A. 042°T **B.** 048°T **C.** 222°T **D.** 228°T **E.** 312°T

11. Calculate x for the following triangles, correct to 2 decimal places.

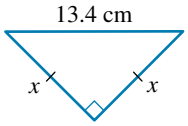
a.



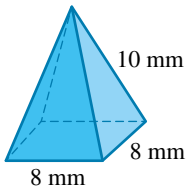
b.



12. Calculate the value of the pronumeral, correct to 2 decimal places.



13. Calculate the height of this pyramid, correct to 2 decimal places.



14. A person standing 23 m away from a tree observes the top of the tree at an angle of elevation of 35°. If the person's eye level is 1.5 m from the ground, what is the height of the tree? Give your answer correct to 1 decimal place.

15. A man with an eye level height of 1.8 m stands at the window of a tall building. He observes his young daughter in the playground below. If the angle of depression from the man to the girl is 47° and the floor on which the man stands is 27 m above the ground, how far from the bottom of the building is the child? Give your answer correct to 2 decimal places.

16. A plane flies 780 km in a direction of 185°T. How far west has it travelled from the starting point? Give your answer correct to 2 decimal places.

17. A hiker travels 3.2 km on a bearing of 250°T and then 1.8 km on a bearing of 320°T. How far west has she travelled from the starting point? Give your answer correct to 2 decimal places.

18. If a 4-m ladder is placed against a wall and the foot of the ladder is 2.6 m from the wall, what angle (in degrees and minutes) does the ladder make with the wall?

Communicating, reasoning and problem solving

19. The height of a right square-based pyramid is 13 cm. If the angle the face makes with the base is 67°, find:

a. the length of the edge of the square base, correct to 2 decimal places

b. the length of the diagonal of the base, correct to 2 decimal places

c. the angle the slanted edge makes with the base in degrees and minutes.

20. A car is travelling northwards on an elevated expressway 6 m above ground at a speed of 72 km/h. At noon another car passes under the expressway at ground level, travelling west, at a speed of 90 km/h.

a. How far apart, in metres, are the two cars 40 seconds after noon?

b. At this time the first car stops, while the second car keeps going. At what time will they be 3.5 km apart? Write your answer correct to the nearest tenth of a second.

21. Two towers face each other separated by a distance, d, of 20 metres. As seen from the top of the first tower, the angle of depression of the second tower's base is 59° and that of the top is 31°. What is the height, in metres correct to 2 decimal places, of each of the towers?

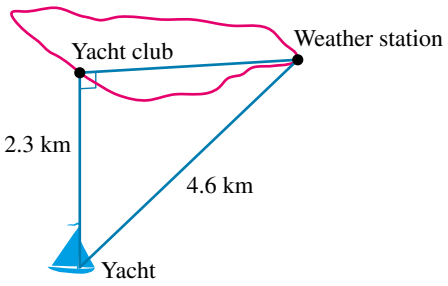
22. A piece of flat pastry is cut in the shape of a right-angled triangle. The longest side is $6b$ cm and the shortest is $2b$ cm

a. Find the length of the third side. Give your answer in exact form.

b. Find the sizes of the angles in the triangle. Give your answers in degrees.

c. Prove that the area of the triangle is equal to $4\sqrt{2}b^2$ cm^2.

23. A yacht is anchored off an island. It is 2.3 km from the yacht club and 4.6 km from a weather station. The three points form a right-angled triangle at the yacht club.



a. Calculate the angle at the yacht between the yacht club and the weather station.

b. Calculate the distance between the yacht club and the weather station, correct to 2 decimal places.

The next day the yacht travels directly towards the yacht club but is prevented from reaching the club because of dense fog. The weather station notifies the yacht that it is now 4.2 km from the station.

c. Calculate the new angle at the yacht between the yacht club and the weather station.

d. Determine how far the yacht is now from the yacht club, correct to 2 decimal places.

24. From a point A, due south of a flagpole on level ground, the angle of elevation to the top of the flagpole is 45°. The flagpole is 80 m tall. The top of the same flagpole is observed with an angle of elevation of 30° from a point B due east of the flagpole. What is the distance from A to B?

Answers

Topic 3 Right-angled triangles (trigonometry)

Exercise 3.2 Pythagoras' theorem

1. a. 7.86 **b.** 33.27 **c.** 980.95 **d.** 12.68 **e.** 2.85 **f.** 175.14

2. a. 36.36 **b.** 1.62 **c.** 15.37 **d.** 0.61 **e.** 2133.19 **f.** 453.90

3. 23.04 cm

4. 12.65 cm

5. a. 14.14 cm **b.** 24.04 cm **c.** 4.53 cm

6. a. 74.83 cm **b.** 249.67 cm **c.** 3741.66 cm^2

7. a. 6.06 **b.** 4.24 **c.** 4.74

8. 14.84 cm

9. 15.59 cm

10. 19.23 cm

11. 72.75 cm; 3055.34 cm^2

12. 39 m

13. 4.34 km

14. 38.2 m

15. 63.06 mm

16. a. 32 cm **b.** 768 cm^2

17. 26.83 diagonals, so would need to complete 27

18. 4701.06 m

19. 9.90 cm

20. a. 65 **b.** 185 **c.** 305

21. a. Neither 105 nor 208 can be the hypotenuse of the triangle, because they are the two smallest values. The other two values could be the hypotenuse if they enable the creation of a right-angled triangle.

b. 105, 208, 233

22. a. 21 cm

b. 35 cm

c. $y = 12.6$ cm and RS = 9.8 cm

23. 13.86 cm

24. Only positive numbers are used in Pythagoras' theorem as it uses length measurements, which are always positive.

Exercise 3.3 Pythagoras' theorem in three dimensions

1. a. 13.86 **b.** 13.93 **c.** 18.03

2. 12.21, 12.85

3. 4.84 m, 1.77 m

4. 8.49, 4.24

5. 31.62 cm

6. 10.58 cm

7. 23 mm

8. a. i. 233.24 m

ii. 200.12 m

iii. 120.20 m

b. 116.83 m

9. 14.72 cm

10. 12.67 cm

11. 42.27 cm

12. 1.49 m, 7.43 m^2

13. Students' own working

14. 186.5 m

15. 25.475

16. 28.6 m

17. $\sqrt{x^2 + y^2 + z^2}$

Exercise 3.4 Trigonometric ratios

1. a. 0.5000 **b.** 0.7071 **c.** 0.4663 **d.** 0.8387 **e.** 8.1443 **f.** 0.7193

2. a. 0.6494 **b.** 0.5885 **c.** 0.5220 **d.** −1.5013 **e.** 0.9990 **f.** 0.6709

g. 0.8120 **h.** 0.5253 **i.** −0.8031 **j.** 0.4063 **k.** 0.9880 **l.** −0.9613

m. 1.7321 **n.** −0.5736 **o.** 0.1320

3. a. 50° **b.** 24° **c.** 53° **d.** 71° **e.** 86° **f.** 41°

4. a. 54°29′ **b.** 6°19′ **c.** 0°52′ **d.** 72°47′ **e.** 44°48′ **f.** 26°45′

5. a. 2.824 **b.** 71.014 **c.** 20.361 **d.** 2.828 **e.** 226.735 **f.** 1.192

g. 7.232 **h.** 32.255 **i.** 4909.913 **j.** 0.063 **k.** 0.904 **l.** 14.814

6. a. i. $\sin(\theta) = \frac{e}{f}$ **ii.** $\cos(\theta) = \frac{d}{f}$ **iii.** $\tan(\theta) = \frac{e}{d}$

b. i. $\sin(\alpha) = \frac{i}{g}$ **ii.** $\cos(\alpha) = \frac{h}{g}$ **iii.** $\tan(\alpha) = \frac{i}{h}$

c. i. $\sin(\beta) = \frac{l}{k}$ **ii.** $\cos(\beta) = \frac{j}{k}$ **iii.** $\tan(\beta) = \frac{l}{j}$

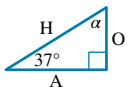
d. i. $\sin(\gamma) = \frac{n}{m}$ **ii.** $\cos(\gamma) = \frac{o}{m}$ **iii.** $\tan(\gamma) = \frac{n}{o}$

e. i. $\sin(\beta) = \frac{b}{c}$ **ii.** $\cos(\beta) = \frac{a}{c}$ **iii.** $\tan(\beta) = \frac{b}{a}$

f. i. $\sin(\gamma) = \frac{v}{u}$ **ii.** $\cos(\gamma) = \frac{t}{u}$ **iii.** $\tan(\gamma) = \frac{v}{t}$

7. a. $\sin(\theta) = \frac{15}{18}$ **b.** $\cos(\theta) = \frac{22}{30}$ **c.** $\tan(\theta) = \frac{7}{9}$ **d.** $\tan(\theta) = \frac{3.6}{p}$ **e.** $\sin(25°) = \frac{13}{t}$ **f.** $\sin(\alpha) = \frac{18.6}{23.5}$

8. a.



b. i. $\sin(37°) = 0.60$

ii. $\cos(37°) = 0.80$

iii. $\tan(37°) = 0.75$

c. $\alpha = 53°$

d. i. $\sin(53°) = 0.80$

ii. $\cos(53°) = 0.60$

iii. $\tan(53°) = 1.33$

e. They are equal.

f. They are equal.

g. The sin of an angle is equal to the cos of its complement angle.

9. $\sin(\theta) = \frac{\text{opp}}{\text{hyp}}, \cos(\theta) = \frac{\text{adj}}{\text{hyp}} \Rightarrow \frac{\sin(\theta)}{\cos(\theta)} = \frac{\text{opp}}{\text{adj}} = \tan(\theta)$

10. a. $h^2 = a^2 - x^2$

b. $h^2 = c^2 - b^2 + 2bx - x^2$

c, d Check with your teacher.

11. $\text{DC} = x + \frac{y}{\tan(\theta)}$

12. Use the mnemonic SOHCAHTOA.

Exercise 3.5 Using trigonometry to calculate side lengths

1. a. 8.660 **b.** 7.250 **c.** 8.412

2. a. 0.79 **b.** 4.72 **c.** 101.38

3. a. 33.45 m **b.** 74.89 m **c.** 44.82 m **d.** 7.76 mm **e.** 80.82 km **f.** 9.04 cm

4. a. $x = 31.58$ cm **b.** $y = 17.67$ m **c.** $z = 14.87$ m

d. $p = 67.00$ m **e.** $p = 21.38$ km, $q = 42.29$ km **f.** $a = 0.70$ km, $b = 0.21$ km

5. a. 6.0 m **b.** 6.7 m

6. 3.22 m

7. a. $x = 30.91$ cm, $y = 29.86$ cm, $z = 39.30$ cm **b.** 2941.54 cm^2

8. a. In an isosceles right-angled triangle **b.** $\theta < 45°$

9. a. $h = \tan(47°48')x$ m
$h = \tan(36°24')(x + 64)$ m **b.** 129.07 m **c.** 142.34 m

10. 60

11. Tangent is not used to calculate the length of the hypotenuse.

Exercise 3.6 Using trigonometry to calculate angle size

1. a. 67° **b.** 47° **c.** 69°

2. a. 54°47′ **b.** 33°45′ **c.** 33°33′

3. a. 41° **b.** 30° **c.** 49° **d.** 65° **e.** 48° **f.** 37°

4. a. $a = 25°47'$, $b = 64°13'$ **b.** $d = 25°23'$, $e = 64°37'$ **c.** $x = 66°12'$, $y = 23°48'$

5. a. $r = 57.58$, $l = 34.87$, $h = 28.56$ **b.** 428 cm^2 **c.** 29.7°

6. a. i. 29.0° **ii.** 41.4° **iii.** 51.3°

b. i. 124.42 km/h **ii.** 136.57 km/h **iii.** 146.27 km/h

7. a $\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{\sqrt{3}}{2}$, $\tan 30° = \frac{\sqrt{3}}{3}$ **b, c** Answers will vary.

8. a. Between start and site 1: 61°
Between site 1 and site 2: 18°
Between site 2 and site 3: 75° **b.** Between site 2 and site 3: 75° slope

9. 58°3′

10. Inverse trigonometric ratios are used

11. 147°0′; 12°15′

Exercise 3.7 Angles of elevation and depression

1. 8.74 m

2. 687.7 m

3. a. 176.42 m **b.** 152.42 m

4. 65°46′

5. 16.04 m

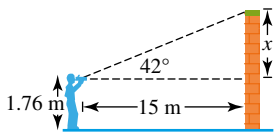
6. a. $h = x\tan(47°12)$ m; $h = (x + 38)\tan(35°50)$ m **b.** $x = 76.69$ m **c.** 84.62 m

7. a. $h = x\tan(43°35')$ m; $h = (x + 75)\tan(32°18')$ m **b.** 148.37 m **c.** 143.10 m

8. 0.033 km or 33 m **9.** 21.6° **10.** 44.87 m **11.** 66 m

12. a. 54° **b.** 0.75 m

13. a.



b. 15.27 m

14. a. 2.16 m/s, 7.7 km/h **b.** 54.5°

15. 70.52 m

16. 451.5 m

17. The angle of elevation is the angle upwards from horizontal and the angle of depression is the angle downwards from horizontal.

Exercise 3.8 Bearings

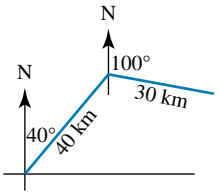
1. a. 020°T **b.** 340°T **c.** 215°T **d.** 152°T **e.** 034°T **f.** 222°T

2. a. N49°E **b.** S48°E **c.** S87°W **d.** N30°W **e.** N86°E **f.** S54°W

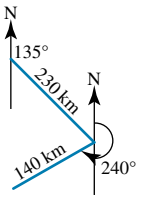
3. a. 3 km 325°T **b.** 2.5 km 112°T **c.** 8 km 235°T

d. 4 km 090°T, then 2.5 km 035°T **e.** 12 km 115°T, then 7 km 050°T **f.** 300 m 310°T, then 500 m 220°T

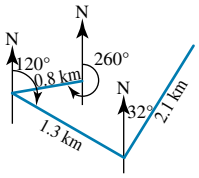
4. a.



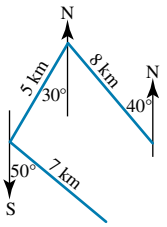
b.



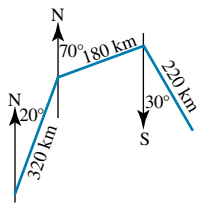
c.



d.

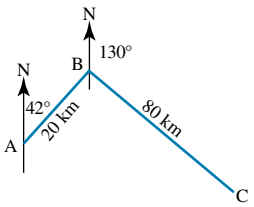


e.



5. a. i. 13.38 km **ii.** 14.86 km **iii.** 222°T

b. i.



ii. 51.42 km **iii.** 61.28 km **iv.** 310°T

6. 215°T

7. 1.732 km

8. a. 9.135 km **b.** 2.305 km **c.** 104°10′T

9. 684.86 km

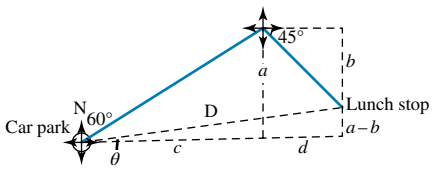
10. 6.10 km and 239°T

11. 111°T

12. a. $(180 + \theta)$°T **b.** $(\theta - 180)$°T

13. a. 27.42 km **b.** N43°W or 317°T

14. a.



b. 1.76 km North **c.** 14.63 km East **d.** N83.15°E **e.** D = 14.74 km

15. True bearings use three digits up to 360° and start an angle from north (0°). Compass directions include N or S, an acute angle and then E or W.

16. 3.65 km on a bearing of 108°T

Exercise 3.9 Applications

1. 6.09 m **2.** 62°33′

3. a. 1.82 m **b.** 27.78 m

4. 31°49′ **5.** 5.94 m **6.** 49 m **7.** 194 m **8.** 1.8 km

9. a. 11°32′ **b.** 4°25′

10. a. i. 35.36 cm **ii.** 51.48 cm **iii.** 51.48 cm **iv.** 57.23 cm **v.** 29°3′ **vi.** 25°54′

b. i. 25.74 cm **ii.** 12.5 cm **iii.** 25°54′ **iv.** 28.61 cm

11. a. 77° **b.** 71°56′ **c.** 27.35 cm

12. a. 7.05 cm **b.** 60°15′ **c.** 8.12 cm

13. a. 28.74 cm **b.** 40.64 cm **c.** 66°37′

14. a. 90 m **b.** $h = \dfrac{d\tan\theta_1}{\tan\theta_1 + \tan\theta_2} \times \tan\theta_2$ **c.** 250 m

15. a. 122.97° **b.** 142.37°

16. a. 7.69 cm **b.** 6.91 cm

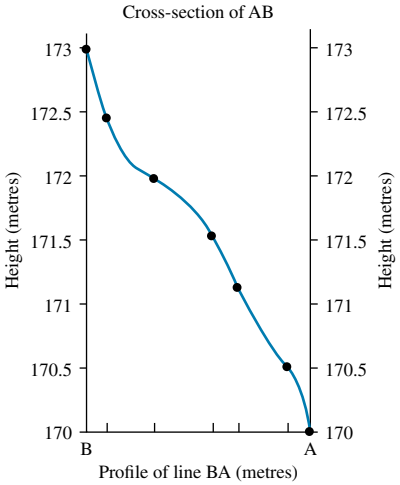
17. The golfer must hit the ball 324.4 m at an angle of 9.7° off the direct line.

18. Discuss with your teacher.

Investigation | Rich task

Answers may vary, depending on physical measurements.

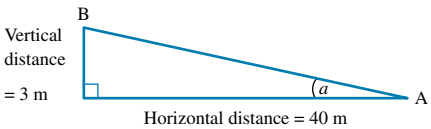
1.



2. a. 8 cm **b.** 40 m

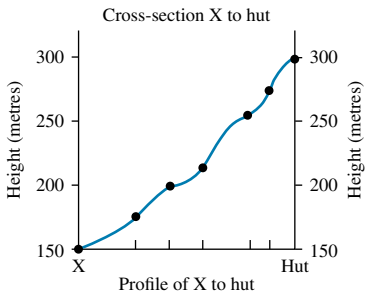
3. 3 m

4.



5. $a = 4°17'$ **6.** Gradual to moderate

7.



The average slope is 12°5′ — moderate to steep.

Exercise 3.10 Review questions

1. E **2.** D **3.** E **4.** E **5.** D

6. B **7.** E **8.** B **9.** B **10.** A

11. a. $x = 113.06$ cm **b.** $x = 83.46$ mm

12. 9.48 cm **13.** 8.25 mm **14.** 17.6 m **15.** 26.86 m **16.** 67.98 km **17.** 4.16 km **18.** 40°32′

19. a. 11.04 cm **b.** 15.61 cm **c.** 59°1′

20. a. 1280.64 m **b.** 12:02:16.3 pm

21. 33.29 m, 21.27 m

22. a. $4\sqrt{2}b$ **b.** 19.5°, 70.5°, 90° **c.** Area $= \frac{1}{2} \times$ base $\times$ height
$= \frac{1}{2} \times 2b \times 4\sqrt{2}b$
$= 4\sqrt{2}b^2$ cm^2.

23. a. 60° **b.** 3.98 km **c.** 71.5° **d.** 1.33 km

24. 160 m

TOPIC 4 Probability

[Stages 5.1 and 5.2]

4.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the concepts covered in this topic.

4.1.1 Why learn this?

Probability is the mathematics of chance. Our entire lives are affected by events that are based on chance. Learning about probability will help you understand how chance is involved in everyday events and in many decisions that you will make in your life.

DISCUSSION

The chance of winning Tattslotto is 1 in 8.5 million, the chance of being killed by a shark is 1 in 300 million, the chance of being killed by a dog is 1 in 20 million, and the chance of guessing the number rolled by a die is 1 in 6. How do you think these probabilities were estimated? Is it possible to estimate a probability for any event, regardless of how unlikely it is to happen?

LEARNING SEQUENCE

4.1 Overview
4.2 [Stage 5.1] Calculating relative frequencies
4.3 [Stage 5.2] Two- and three-step chance experiments
4.4 [Stage 5.2] Independent and dependent events
4.5 [Stage 5.2] Conditional probability
4.6 Review

LEARNING OUTCOMES

A student:

- uses appropriate terminology, diagrams and symbols in mathematical contexts MA5.1-1WM
- selects and uses appropriate strategies to solve problems MA5.1-2WM
- provides reasoning to support conclusions that are appropriate to the context MA5.1-3WM
- selects appropriate notations and conventions to communicate mathematical ideas and solutions MA5.2-1WM
- interprets mathematical or real-life situations, systematically applying appropriate strategies to solve problems MA5.2-2WM
- constructs arguments to prove and justify results MA5.2-3WM
- calculates relative frequencies to estimate probabilities of simple and compound events MA5.1-13SP
- describes and calculates probabilities in multi-step chance experiments MA5.2-17SP

CONTENT DESCRIPTIONS

Students:

Calculate relative frequencies from given or collected data to estimate probabilities of events involving 'and' or 'or' (ACMSP226)
List all outcomes for two-step chance experiments, with and without replacement, using tree diagrams or arrays; assign probabilities to outcomes and determine probabilities for events (ACMSP225)
Describe the results of two- and three-step chance experiments, with and without replacement, assign probabilities to outcomes, and determine probabilities of events; investigate the concept of independence (ACMSP246)

Use the language of 'if ... then', 'given', 'of', 'knowing that' to investigate conditional statements and to identify common mistakes in interpreting such language (ACMSP247)

Source: NSW Syllabus for the Australian curriculum

learnon RESOURCES — ONLINE ONLY

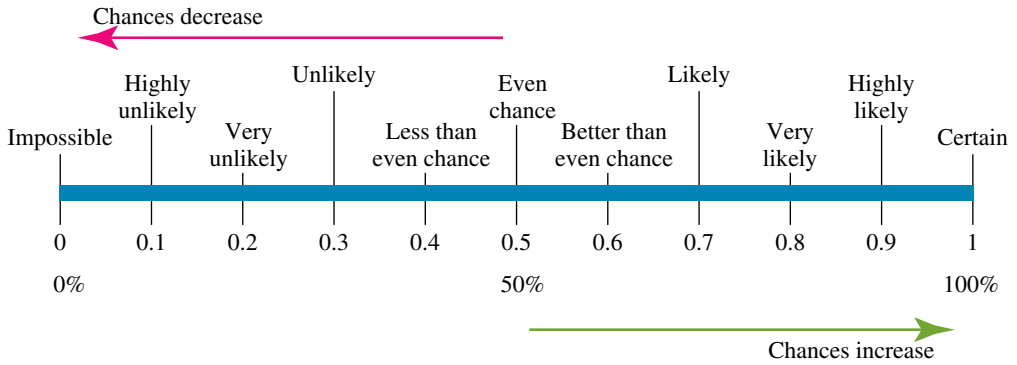
eLesson: The story of mathematics — The mathematics of chance (eles-1851)

Note: Your teacher may now set you a pre-test to determine how familiar you are with the content in this topic.

4.2 Calculating relative frequencies [Stage 5.1]

4.2.1 The language of probability

- **Probability** measures the chance of an event taking place and ranges from zero (0) for an impossible event to one (1) for a certain event.



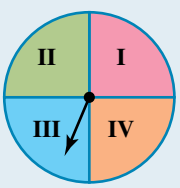
- The **experimental probability** of an event is based on the outcomes of experiments, simulations or surveys.
- A **trial** is a single experiment; for example, a single flip of a coin.

$$\text{Experimental probability} = \frac{\text{number of successful trials}}{\text{total number of trials}}$$

- The **relative frequency** of an event is the same as the experimental probability of that event.
- The list of all possible outcomes of an experiment is known as the **event space** or sample space. For example, when flipping a coin there are two possible outcomes: Heads or Tails. The event space can be written, using set notation, as $\{H, T\}$.

WORKED EXAMPLE 1

The spinner shown here is made up of 4 equal-sized segments. It is known that the probability that the spinner will land on any one of the four segments from one spin is $\frac{1}{4}$. To test if the spinner shown here is fair, a student spun the spinner 20 times and each time recorded the segment in which the spinner stopped. The spinner landed as follows.



Segment	I	II	III	IV
Tally	5	4	8	3

a List the event space.
b Given the experimental results, determine the relative frequency for each segment.
c Compare the results from the experiment with the known probabilities and suggest how the experiment could be changed to ensure that the results give a better estimate of the true probability.

THINK

a The event space lists all possible outcomes from one spin of the spinner. There are four possible outcomes.

b 1 For segment I there were 5 successful trials out of the 20. Substitute these values into the relative frequency formula.

2 Repeat for segments:
- II (4 successes)
- III (8 successes)
- IV (3 successes).

c Compare the relative frequency values with the known value of $\frac{1}{4}$ (0.25). Answer the question.

WRITE

a $\text{Event space} = \{\text{I, II, III, IV}\}$

b $\text{Relative frequency}_{\text{I}} = \dfrac{\text{number of successful trials}}{\text{total number of trials}}$
$= \dfrac{5}{20}$
$= 0.25$

$\text{Relative frequency}_{\text{II}} = \dfrac{4}{20}$
$= 0.2$

$\text{Relative frequency}_{\text{III}} = \dfrac{8}{20}$
$= 0.4$

$\text{Relative frequency}_{\text{IV}} = \dfrac{3}{20}$
$= 0.15$

c The relative frequency of segment I was the only segment that mirrored the known value. To ensure that relative frequency gives a better estimate of the true probability, the spinner should be spun many more times.

4.2.2 Two-way tables

- The sample space can be displayed using a **two-way table**.
- A two-way table represents two of the outcomes of events in a two-dimensional table. A two-way table for the experiment of tossing a coin and rolling a die simultaneously is shown below.

		Die outcomes					
		1	**2**	**3**	**4**	**5**	**6**
Coin outcomes	**H**	H, 1	H, 2	H, 3	H, 4	H, 5	H, 6
	T	T, 1	T, 2	T, 3	T, 4	T, 5	T, 6

WORKED EXAMPLE 2

Two dice are rolled, and the values on the two uppermost faces are multiplied together. Draw a diagram to illustrate the sample space.

THINK

The sample space for rolling 1 die is {1, 2, 3, 4, 5, 6}. When two dice are rolled and the two uppermost faces are multiplied, the sample space is made up of 36 products. This is best represented with the use of a two-way table.

- Draw a 7 × 7 grid.
- In the first row and column list the outcomes of each die.
- At the intersection of a column and row write the product of the relevant die outcomes, as shown in green.

DRAW

FIRST DIE

SECOND DIE ×	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

4.2.3 Theoretical probability

- **Theoretical probability** is the probability of an event occurring, based on the number of possible favourable outcomes, $n(A)$, and the total number of possible outcomes, $n(\xi)$.
- When all outcomes are equally likely, the theoretical probability of an event can be calculated using the formula:

$$\text{P(event)} = \frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}} \text{ or P(event)} = \frac{n(E)}{n(\xi)}$$

where $n(E)$ is the number of favourable events and $n(\xi)$ is the total number of possible outcomes.

DISCUSSION

Explain the difference between experimental probability and theoretical probability.

WORKED EXAMPLE 3

A fair die is rolled and the value of the uppermost side is recorded. Calculate the theoretical probability that a 4 is uppermost.

THINK

1 Write the number of favourable outcomes and the total number of possible outcomes. The number of fours on a fair die is 1. There are 6 possible outcomes.

2 Substitute the values found in part 1 to calculate the probability of the event that a four is uppermost when a die is rolled.

3 Write the answer.

WRITE

$n(E) = 1$
$n(\xi) = 6$

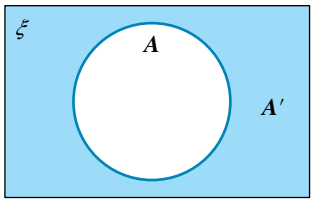
$$\text{P(a four)} = \frac{n(E)}{n(\xi)} = \frac{1}{6}$$

The probability that a 4 is uppermost when a fair die is rolled is $\frac{1}{6}$.

4.2.4 Complementary events

- The **complement** of the set A is the set of all elements that belong to the universal set (ξ) but that do *not* belong to A. For example, the complement of {blue socks in a drawer} is {all socks in the drawer that are not blue}.

- The complement of A is written as A' and is read as 'A dashed' or 'A prime'.
- On a Venn diagram, **complementary events** appear as separate regions that together occupy the whole universal set.



$$\text{Since } n(A) + n(A') = n(\xi),$$
$$\text{P}(A) + P(A') = 1$$

WORKED EXAMPLE 4

A player is chosen from a cricket team. Are the events 'selecting a batsman' and 'selecting a bowler' complementary events if a player can have more than one role? Give a reason for your answer.

THINK

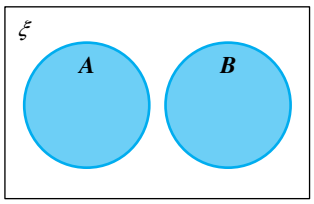
Explain the composition of a cricket team. Players who can bat and bowl are not necessarily the only players in a cricket team. There is a wicket-keeper as well. Some players (all rounders) can bat and bowl.

WRITE

No, the events 'selecting a batsman' and 'selecting a bowler' are not complementary events. These events may have common elements; that is, the all rounders in the team who can bat and bowl. The cricket team also includes a wicket-keeper.

4.2.5 Mutually exclusive events (A or B)

- Two events are **mutually exclusive** if one event happening excludes the other from happening. These events may not encompass all possible events. For example, when selecting a card from a deck of cards, selecting a black card excludes the possibility that the card is a Heart.
- For events that are mutually exclusive: $\text{P}(A \cap B) = 0$.
- On a Venn diagram, mutually exclusive events appear as disjointed sets within the universal set.



A and B are mutually exclusive events.

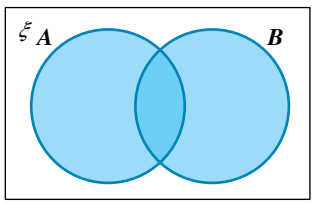
$$\text{P}(A \cup B) = \text{P}(A) + \text{P}(B)$$

4.2.6 Probability of intersecting events (A or B or both)

- When two events have outcomes in common, $\text{P}(A \cup B) \neq \text{P}(A) + \text{P}(B)$, since this would count the outcomes they have in common twice.

$$\text{P}(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A \cap B)$$

- This is known as the **Addition Law** of probability.



WORKED EXAMPLE 5

A card is drawn from a pack of 52 playing cards. What is the probability that the card is a heart or a club?

THINK	WRITE
1 Determine whether the given events are mutually exclusive.	The two events are mutually exclusive as they have no common elements.
2 Determine the probability of drawing a heart and of drawing a club.	$\text{P(heart)} = \frac{13}{52}$ $\text{P(club)} = \frac{13}{52}$ $= \frac{1}{4}$ $= \frac{1}{4}$
3 Write the Addition Law for two mutually exclusive events.	$\text{P}(A \text{ or } B) = \text{P}(A) + P(B)$ where A = drawing a heart and B = drawing a club
4 Substitute the known values into the rule.	$\text{P(heart or club)} = \text{P(heart)} + \text{P(club)}$ $= \frac{1}{4} + \frac{1}{4}$
	$= \frac{2}{4}$
5 Evaluate and simplify.	$= \frac{1}{2}$
6 Write your answer.	The probability of drawing a heart or a club is $\frac{1}{2}$.
Note: Alternatively, we can use the formula for theoretical probability.	$\text{P(heart or club)} = \frac{n(\text{heart or club})}{n(\xi)}$ $= \frac{26}{52}$ $= \frac{1}{2}$

WORKED EXAMPLE 6

A die is rolled. Determine:
a P(an odd number)
b P(a number less than 4)
c P(an odd number or a number less than 4).

THINK	WRITE
a 1 Determine the probability of obtaining an odd number; that is, $\{1, 3, 5\}$.	**a** $\text{P(odd)} = \frac{3}{6}$ $= \frac{1}{2}$
2 Write your answer.	The probability of obtaining an odd number is $\frac{1}{2}$.
b 1 Determine the probability of obtaining a number less than 4; that is, $\{1, 2, 3\}$.	**b** $\text{P(less than 4)} = \frac{3}{6}$ $= \frac{1}{2}$
2 Write your answer.	The probability of obtaining a number less than 4 is $\frac{1}{2}$.
c 1 Determine whether the given events are mutually exclusive.	**c** The two events are not mutually exclusive as they have common elements; that is, 1 and 3.
2 Write the Addition Law for two non-mutually exclusive events.	$\text{P}(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$ where A = selecting an odd number and B = selecting a number less than 4.

3 Substitute the known values into the rule.
Note: $P(A \text{ and } B) = \frac{2}{6}\left(=\frac{1}{3}\right)$ since the events have two elements in common.

$P[\text{odd number} \cup (\text{number} < 4)]$
$= P(\text{odd number}) + P[(\text{number} < 4)] - P[\text{odd number} \cap (\text{number} < 4)]$
$= \frac{1}{2} + \frac{1}{2} - \frac{1}{3}$

4 Evaluate and simplify.

$= \frac{2}{3}$

5 Write your answer.

The probability of obtaining an odd number or a number less than 4 is $\frac{2}{3}$.

WORKED EXAMPLE 7

Given $P(A) = 0.6$, $P(B) = 0.4$ and $P(A \cup B) = 0.9$:
a use the Addition Law of probability to calculate the value of $P(A \cap B)$
b draw a Venn diagram to represent the universal set
c calculate $P(A \cap B')$.

THINK

a 1 Write the Addition Law of probability and substitute given values.

2 Collect like terms and rearrange to make $P(A \cap B)$ the subject. Solve the equation.

b 1 Draw intersecting sets A and B within the universal set and write $P(A \cap B) = 0.1$ inside the overlapping section, as shown in blue.

2 • As $P(A) = 0.6$, 0.1 of this belongs in the overlap, the remainder of set A is $0.5\,(0.6 - 0.1)$.
• Since $P(B) = 0.4$, 0.1 of this belongs in the overlap, the remainder of set B is $0.3\,(0.4 - 0.1)$.

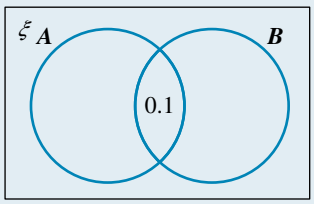
3 The total probability for events A and B is 1. That means $P(A \cup B)' = 0.1$. Write 0.1 outside sets A and B to form the remainder of the universal set.

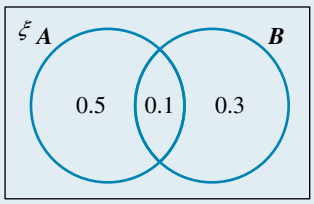
c $P(A \cap B')$. is the overlapping region of $P(A)$ and $P(B')$. Shade the region and write down the corresponding probability value for this area.

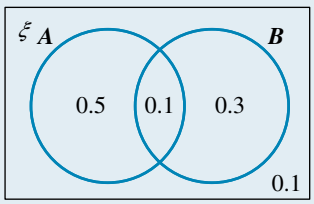
WRITE

a $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$0.9 = 0.6 + 0.4 - P(A \cap B)$
$0.9 = 0.6 + 0.4 - P(A \cap B)$
$0.9 = 1.0 - P(A \cap B)$
$P(A \cap B) = 1.0 - 0.9$
$= 0.1$

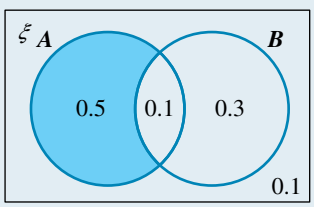
b







c



$P(A \cap B') = 0.5$

WORKED EXAMPLE 8

a Draw a Venn diagram representing the relationship between the following sets. Show the position of all the elements in the Venn diagram.

$\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$

$A = \{3, 6, 9, 12, 15, 18\}$

$B = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$

b Determine:

i $P(A)$ **ii** $P(B)$ **iii** $P(A \cap B)$ **iv** $P(A \cup B)$ **v** $P(A' \cap B')$

THINK

a 1 Draw a rectangle with two partly intersecting circles labelled A and B.

2 Analyse sets A and B and place any common elements in the central overlap.

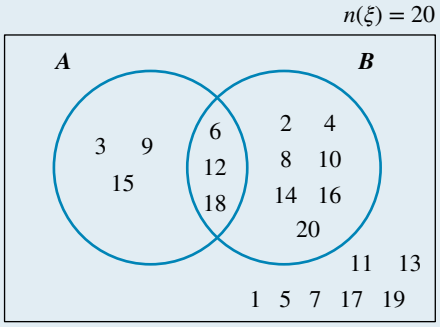
3 Place the remaining elements of set A in circle A.

4 Place the remaining elements of set B in circle B.

5 Place the remaining elements of the universal set ξ in the rectangle.

WRITE/DRAW

a



b i 1 Write the number of elements that belong to set A and the total number of elements.

2 Write the rule for probability.

3 Substitute the known values into the rule.

4 Evaluate and simplify.

b i $n(A) = 6,\ n(\xi) = 20$

$$P(A) = \frac{n(A)}{n(\xi)}$$

$$P(A) = \frac{6}{20}$$

$$= \frac{3}{10}$$

ii 1 Write the number of elements that belong to set B and the total number of elements.

2 Repeat steps 2 to 4 of part **b i**.

ii $n(B) = 10,\ n(\xi) = 20$

$$P(B) = \frac{n(B)}{n(\xi)}$$

$$P(B) = \frac{10}{20}$$

$$= \frac{1}{2}$$

iii 1 Write the number of elements that belong to set $(A \cap B)$ and the total number of elements.

2 Repeat steps 2 to 4 of part **b i**.

iii $n(A \cap B) = 3,\ n(\xi) = 20$

$$P(A \cap B) = \frac{n(A \cap B)}{n(\xi)}$$

$$P(A \cap B) = \frac{3}{20}$$

iv 1 Write the number of elements that belong to set $A \cup B$ and the total number of elements.

iv $n(A \cup B) = 13,\ n(\xi) = 20$

2 Repeat steps 2 to 4 of part **b i**.

$$P(A \cup B) = \frac{n(A \cup B)}{n(\xi)}$$

$$P(A \cup B) = \frac{13}{20}$$

v 1 Write the number of elements that belong to set $A' \cap B'$ and the total number of elements.

v $n(A' \cap B') = 7, n(\xi) = 20$

2 Repeat steps 2 to 4 of part **b i**.

$$P(A' \cap B') = \frac{n(A' \cap B')}{n(\xi)}$$

$$P(A' \cap B') = \frac{7}{20}$$

WORKED EXAMPLE 9

In a class of 35 students, 6 students like all three subjects: PE, Science and Music. Eight of the students like PE and Science, 10 students like PE and Music, and 12 students like Science and Music. Also, 22 students like PE, 18 students like Science and 17 like Music. Two students don't like any of the subjects.

a Display this information on a Venn diagram.

b Determine the probability of selecting a student who:

i likes PE only

ii does not like Music.

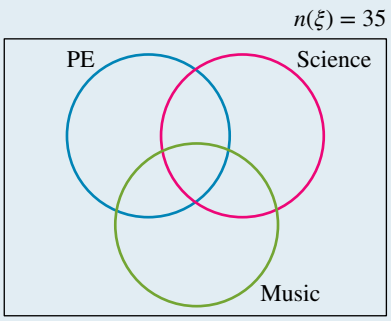
c Find P[(Science ∪ Music) ∩ PE′]

THINK

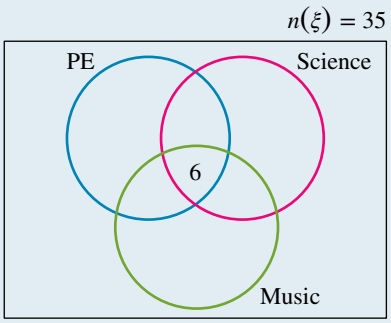
a 1 Draw a rectangle with three partly intersecting circles, labelled PE, Science and Music.

WRITE/DRAW

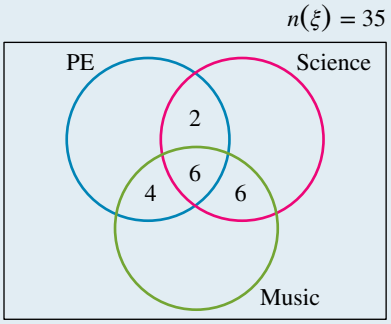
a



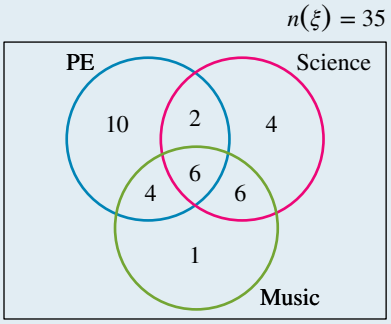
2 Extract the information relating to students liking all three subjects. *Note:* The central overlap is the key to solving these problems. Six students like all three subjects, so place the number 6 into the section corresponding to the intersection of the three circles.



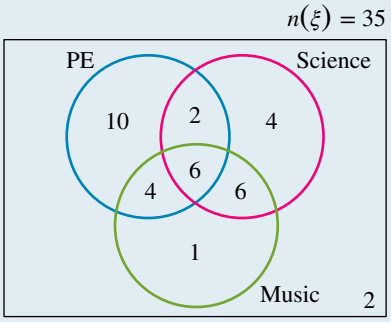
3 Extract the relevant information from the second sentence and place it into the appropriate position.
Note: Eight students like PE and Science; however, 6 of these students have already been accounted for in step 2. Therefore, 2 will fill the intersection of only PE and Science. Similarly, 4 of the 10 who like PE and Music will fill the intersection of only PE and Music, and 6 of the 12 students will fill the intersection of only Science and Music.



4 Extract the relevant information from the third sentence and place it into the appropriate position.
Note: Twenty-two students like PE and 12 have already been accounted for in the set. Therefore, 10 students are needed to fill the circle corresponding to PE only. Similarly, 4 students are needed to fill the circle corresponding to Science only to make a total of 18 for Science. One student is needed to fill the circle corresponding to Music only to make a total of 17 for Music.



5 Extract the relevant information from the final sentence and place it into the appropriate position.
Note: Two students do not like any of the subjects, so they are placed in the rectangle outside the three circles.



6 Check that the total number in all positions is equal to the number in the universal set.
$10 + 2 + 4 + 4 + 6 + 6 + 1 + 2 = 35$

b i 1 Write the number of students who like PE only and the total number of students in the class.

2 Write the rule for probability.

3 Substitute the known values into the rule.

4 Evaluate and simplify.

b i $n(\text{students who like PE only}) = 10$
$n(\xi) = 35$

$$\text{P(likes PE only)} = \frac{n(\text{likes PE only})}{n(\xi)}$$

$$\text{P(likes PE only)} = \frac{10}{35}$$

$$= \frac{2}{7}$$

	5 Write your answer.	The probability of selecting a student who likes PE only is $\frac{2}{7}$.
ii	**1** Write the number of students who do not like Music and the total number of students in the class. *Note:* Add all the values that do not appear in the Music circle as well as the two that sit in the rectangle outside the circles.	**ii** $n(\text{students who do not like PE only}) = 18$ $n(\xi) = 35$
	2 Write the rule for probability.	$\text{P(does not like Music)} = \frac{n(\text{does not like Music})}{n(\xi)}$
	3 Substitute the known values into the rule.	$\text{P(does not like Music)} = \frac{18}{35}$
	4 Write your answer.	The probability of selecting a student who does not like Music is $\frac{18}{35}$.
c	**1** Write the number of students who like Science and Music but not PE. *Note:* Add the values that appear in the Science and Music circles but do not overlap with the PE circle.	**c** $n[(\text{Science} \cup \text{Music}) \cap \text{PE}'] = 11$ $n(\xi) = 35$
	2 Repeat steps 2 to 4 of part **b ii**.	$\text{P}[(\text{Science} \cup \text{Music}) \cap \text{PE}'] = \frac{n[\text{Science} \cup \text{Music}) \cap \text{PE}']}{n(\xi)}$ $\text{P}[(\text{Science} \cup \text{Music}) \cap \text{PE}'] = \frac{11}{35}$ The probability of selecting a student who does not like Music is $\frac{11}{35}$.

learnon RESOURCES — ONLINE ONLY

Interactivity: Experimental probability (int-3825)
Interactivity: Two-way table (int-6082)
Interactivity: Theoretical probability (int-6081)
Interactivity: Venn diagrams (int-3828)
Interactivity: Addition Law of probability (int-6168)
Interactivity: Random number generator (int-0089)
eLesson: Venn diagrams (eles-1934)
Digital doc: SkillSHEET Set notation (doc-5286)
Digital doc: SkillSHEET Simplifying fractions (doc-5287)
Digital doc: SkillSHEET Determining complementary events (doc-5288)
Digital doc: SkillSHEET Addition and subtraction of fractions (doc-5289)
Digital doc: SkillSHEET Working with Venn diagrams (doc-5291)
Digital doc: SkillSHEET Distinguishing between complementary and mutually exclusive events (doc-5294)
Digital doc: WorkSHEET Introducing probability (doc-14590)

Exercise 4.2 Calculating relative frequencies

Individual pathways

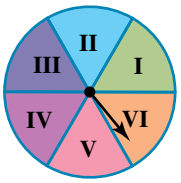
PRACTISE	CONSOLIDATE	MASTER
Questions: 1–8, 10, 12, 14, 16, 18, 20	Questions: 1, 3–6, 8, 9, 11, 13, 15, 17, 19, 21	Questions: 1, 2, 4, 5, 7, 9–13, 15, 17, 19–22

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. Explain the difference between experimental and theoretical probability.

2. WE1 The spinner shown was spun 50 times and the outcome each time was recorded in the table below.



Segment	I	II	III	IV	V	VI
Tally	10	6	8	7	12	7

a. List the event space.

b. Given the experimental results, determine the relative frequency for each segment.

c. The theoretical probability of the spinner landing on any particular segment with one spin is $\frac{1}{6}$. How could the experiment be changed to give a better estimate of the true probabilities?

3. A laptop company conducted a survey to determine the most appealing colours for laptop computers among students 15 to 18 years old. The results were as follows.

Colour	Black Black	Sizzling Silver	Power Pink	Blazing Blue	Gooey Green	Glamour Gold
Number	102	80	52	140	56	70

a. How many students were surveyed?

b. What is the relative frequency of students who found silver the most appealing laptop colour?

c. What is the relative frequency of students who found black and green to be their most appealing colours?

d. Which colour was found to be most appealing?

4. WE2 Two dice are rolled and the values on the two uppermost faces are added together.

a. Construct a table to illustrate the sample space.

b. What is the most likely outcome?

c. What is the least likely outcome?

5. WE7 Given $P(A) = 0.5$, $P(B) = 0.4$ and $P(A \cup B) = 0.8$:

a. use the Addition Law of probability to calculate the value of $P(A \cap B)$

b. draw a Venn diagram to represent the universal set

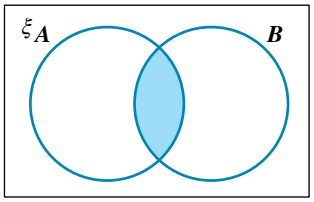
c. calculate $P(A \cap B')$.

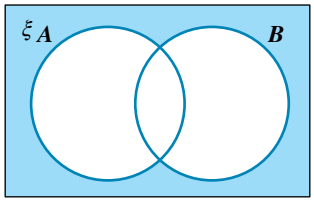
6. Let $P(A) = 0.25$, $P(B) = 0.65$ and $P(A \cap B) = 0.05$.

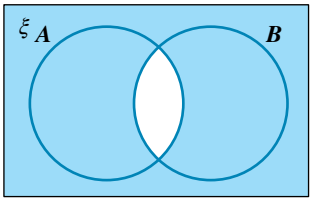
a. Calculate:

i. $P(A \cup B)$ **ii.** $P(A \cap B)'$.

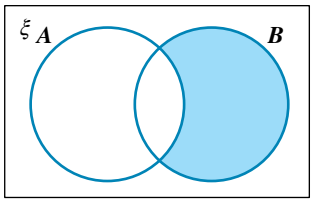
b. MC Which Venn diagram best illustrates $P(A \cap B)'$?

A.


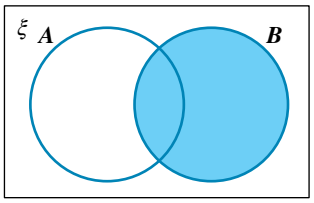
B.


C.


D.



E.



7. WE3 A die is rolled. What is the probability that the outcome is an even number or a 5?

8. WE6 A card is drawn from a well-shuffled pack of 52 playing cards. Calculate:

a. P(a king is drawn) **b.** P(a heart is drawn)

c. P(a king or a heart is drawn).

9. WE4 For each of the following pairs of events:

i. state, giving justification, if the pair are complementary events

ii. alter the statements, where applicable, so that the events become complementary events.

a. Having Weet Bix or having Strawberry Pops for breakfast

b. Walking to a friend's place or driving there

c. Watching TV or reading as a leisure activity

d. Rolling a number less than 5 or rolling a number greater than 5 with a ten-sided die with faces numbered 1 to 10

e. Passing a maths test or failing a maths test

10. a. WE8 Draw a Venn diagram representing the relationship between the following sets. Show the position of all the elements in the Venn diagram.

$\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$

$A = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$

$B = \{1, 4, 9, 16\}$

b. Calculate:

i. $P(A)$ **ii.** $P(B)$ **iii.** $P(A \cap B)$ **iv.** $P(A \cup B)$ **v.** $P(A' \cap B')$.

11. You and a friend are playing a dice game. You have an eight-sided die (with faces numbered 1 to 8 inclusive) and your friend has a six-sided die (with faces numbered 1 to 6 inclusive). You each roll your own die.

a. The person who rolls the number 4 wins. Is this game fair?

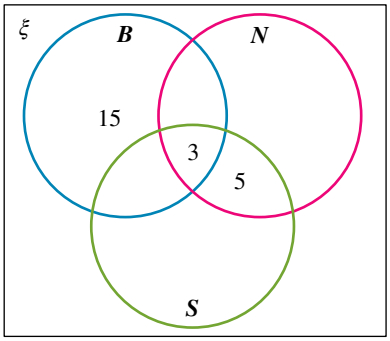
b. The person who rolls an odd number wins. Is this game fair?

12. A six-sided die has three faces numbered 5; the other faces are numbered 6. Are the events 'rolling a 5' and 'rolling a 6' equally likely?

13. Ninety students were asked which lunchtime sports on offer, of basketball, netball and soccer, they had participated in on at least one occasion in the last week. The results are shown in the following table.

Sport	Basketball	Netball	Soccer	Basketball and netball	Basketball and soccer	Netball and soccer	All three
Number of students	35	25	39	5	18	8	3

a. Copy and complete the Venn diagram shown to illustrate the sample space.



b. How many students did not play basketball, netball or soccer at lunchtime?

c. How many students played basketball and/or netball but not soccer?

d. How many students are represented by the region (basketball ∩ not netball ∩ soccer)?

e. Calculate the relative frequency of the region described in part **d** above.

f. Estimate the probability that a student will play 3 of the sports offered.

14. WE5 A card is drawn from a shuffled pack of 52 cards. Find the probability that the card drawn is:

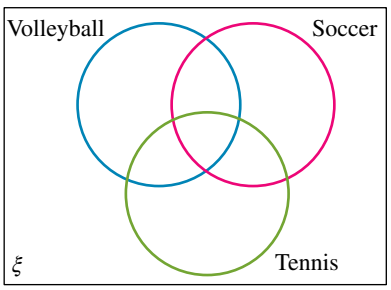
a. an ace **b.** a club **c.** a red card

d. not a jack **e.** a green card **f.** not a red card.

15. A bag contains 4 blue marbles, 7 red marbles and 9 yellow marbles. All marbles are of the same size. A marble is selected at random. What is the probability that the marble is:

a. blue **b.** red **c.** not yellow **d.** black?

16. WE9 Thirty students were asked which lunchtime sports they enjoyed — volleyball, soccer or tennis. Five students chose all three sports. Six students chose volleyball and soccer, 7 students chose volleyball and tennis, and 9 chose soccer and tennis. Fifteen students chose volleyball, 14 students chose soccer and 18 students chose tennis.



a. Copy the Venn diagram shown and enter the given information.

b. If a student is selected at random, determine the probability of selecting a student who:

i. chose volleyball **ii.** chose all three sports

iii. chose both volleyball and soccer but not tennis **iv.** did not choose tennis

v. chose soccer.

c. Determine:

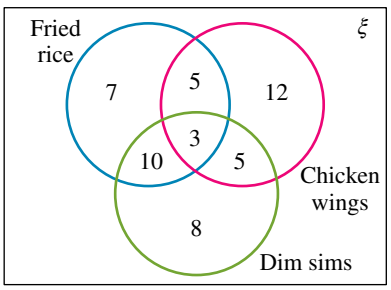
i. P[(soccer ∪ tennis) ∩ volleyball′] **ii.** P[(volleyball ∪ tennis) ∩ soccer′].

Communicating, reasoning and problem solving

17. A six-sided die has three faces numbered 1 and the other three faces numbered 2. Are the events 'rolling a 1' and 'rolling a 2' equally likely?

18. With the use of diagrams, show that $P(A' \cap B') = P(A \cup B)'$.

19. The Venn diagram shows the results of a survey completed by a Chinese restaurateur to find out the food preferences of his regular customers.



a. Determine the number of customers:

i. surveyed

ii. showing a preference for fried rice only

iii. showing a preference for fried rice

iv. showing a preference for chicken wings and dim sims.

b. A customer from this group won the draw for a lucky door prize. Determine the probability that this customer:

i. likes fried rice

ii. likes all three — fried rice, chicken wings and dim sims

iii. prefers chicken wings only.

c. A similar survey was conducted a month later with another group of 50 customers. This survey yielded the following results: 2 customers liked all three foods; 6 preferred fried rice and chicken wings only; 7 preferred chicken wings and dim sims only; 8 preferred fried rice and dim sims only; 22 preferred fried rice; 23 preferred chicken wings; and 24 preferred dim sims.

i. Display this information on a Venn diagram.

ii. What is the probability of selecting a customer who prefers all three foods if a random selection is made?

20. A pair of dice is rolled and the sum of the numbers shown is noted.

a. Show the sample space in a two-way table.

b. In how many different ways can the sum of 7 be obtained?

c. Are all outcomes equally likely?

d. Complete the given table.

Sum	2	3	4	5	6	7	8	9	11	12
Frequency										

e. What are the relative frequencies of the following sums?

i. 2 **ii.** 7 **iii.** 11

f. What is the probability of obtaining the following sums?

i. 2 **ii.** 7 **iii.** 11

g. If a pair of dice is rolled 300 times, how many times do you expect to obtain the sum of 7?

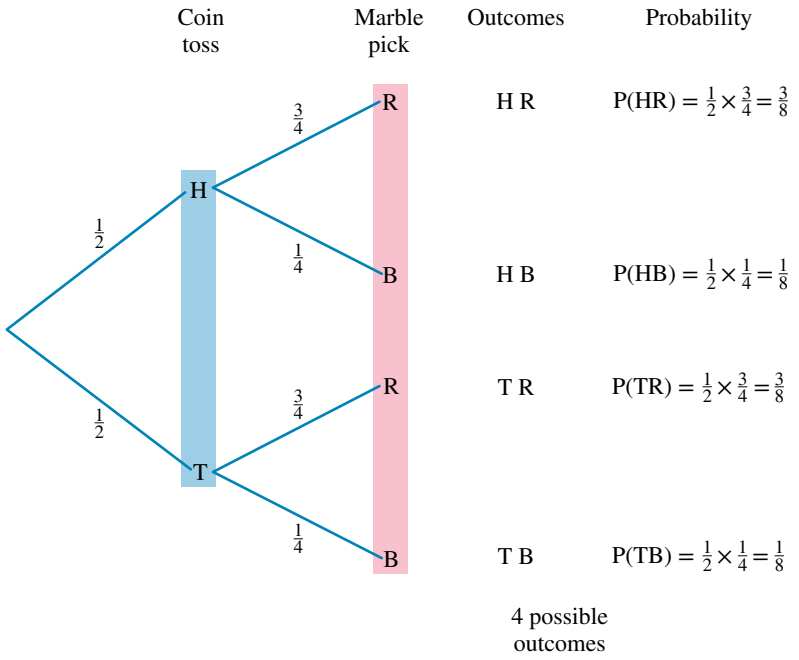
21. What basic formula must be remembered in order to calculate simple probabilities?

22. A drawer contains purple socks and red socks. The chance of obtaining a red sock is 2 in 9. There are 10 red socks in the drawer. What is the smallest number of socks that need to be added to the drawer so that the probability of drawing a red sock increases to 3 in 7?

4.3 Two- and three-step chance experiments [Stage 5.2]

4.3.1 Two-step chance experiments

- In **two-step chance experiments** the result is obtained after performing two trials. Two-step chance experiments are often represented using tree diagrams.
- **Tree diagrams** are used to list all possible outcomes of two or more events that are not necessarily equally likely.
- The probability of obtaining the result for a particular event is listed on the branches.
- The probability for each outcome in the sample space is the product of the probabilities associated with the respective branches. For example, the tree diagram shown here represents the sample space for flipping a coin, then choosing a marble from a bag containing three red marbles and one black marble.



- When added together, all the probabilities for the outcomes should sum to 1. They are complementary events. For example,

$$\begin{aligned} \text{P(HR)} + \text{P(HB)} + \text{P(TR)} + \text{P(TB)} &= \frac{3}{8} + \frac{1}{8} + \frac{3}{8} + \frac{1}{8} \\ &= 1 \end{aligned}$$

- Other probabilities can also be calculated from the tree diagram. For example, the probability of getting an outcome that contains a red marble can be calculated by summing the probabilities of each of the possible outcomes that include a red marble. Outcomes that contain a red marble are HR and TR. Therefore:

$$\begin{aligned} \text{P(red marble)} &= \text{P(HR)} + \text{P(TR)} \\ &= \frac{3}{8} + \frac{3}{8} \\ &= \frac{6}{8} \\ &= \frac{3}{4} \end{aligned}$$

WORKED EXAMPLE 10

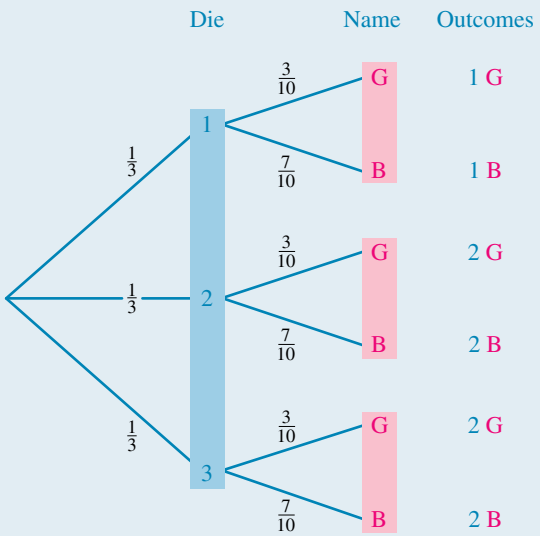
A three-sided die is rolled and a name is picked out of a hat that contains 3 girls' names and 7 boys' names.

a Use a tree diagram to display the sample space.

b Calculate the probability of:

i rolling a 3, then choosing a boy's name

ii choosing a boy's name after rolling an odd number.

THINK	WRITE
a 1 Draw 3 branches from the starting point to show the 3 possible outcomes of rolling a three-sided die (shown in blue), and then draw 2 branches off each of these to show the 2 possible outcomes of choosing a name out of a hat (shown in red).	**a** 
2 Write probabilities on the branches to show the individual probabilities of rolling a 1, 2 or 3 on a three-sided die. As these are equally likely outcomes, $P(1) = P(2) = P(3) = \frac{1}{3}$.	
3 Write probabilities on the branches to show the individual probabilities of choosing a name. Since there are 3 girls' names and 7 boys' names in the hat, $P(G) = \frac{3}{10}$ and $P(B) = \frac{7}{10}$.	
b i 1 Follow the pathway of rolling a 3 [$P(3) = \frac{1}{3}$] and choosing a boy's name [$P(B) = \frac{7}{10}$], and multiply the probabilities.	**b i** $P(3B) = P(3) \times P(B)$ $= \frac{1}{3} \times \frac{7}{10}$ $= \frac{7}{30}$
2 Write the answer.	The probability of rolling a 3, then choosing a boy's name is $\frac{7}{10}$.
ii 1 To roll an odd number (1 or 3) then choose a boy's name: • roll a 1, then choose a boy's name *or* • roll a 3, then choose a boy's name. Find the probability of each of these and add them together to find the total probability.	**ii** $P(\text{odd B}) = P(1B) + P(3B)$ $= P(1) \times P(B) + P(3) \times P(B)$ $= \frac{1}{3} \times \frac{7}{10} + \frac{1}{3} \times \frac{7}{10}$ $= \frac{7}{30} + \frac{7}{30}$ $= \frac{14}{30}$ $= \frac{7}{15}$
2 Write the answer.	The probability of choosing a boy's name after rolling an odd number is $\frac{7}{15}$.

4.3.2 Three-step chance experiments

- Outcomes are often made up of combinations of events. For example, when a coin is flipped three times, three of the possible outcomes are HHT, HTH and THH. These outcomes all contain 2 Heads and 1 Tail.
- The probability of an outcome with a particular order is written such that the order required is shown. For example, HHT is the probability of H on the first die, H on the second die and T on the third die.
- The probability of an outcome with a particular combination of events in which the order is not important is written describing the particular combination required. For example, P(2 Heads and 1 Tail).

WORKED EXAMPLE 11

A coin is biased so that the chance of it falling as a Head when flipped is 0.75.
a Draw a tree diagram to represent the coin being flipped three times.
b Calculate the following probabilities:
i P(HTT) ii P(1H and 2T) iii P(at least 2 Tails).

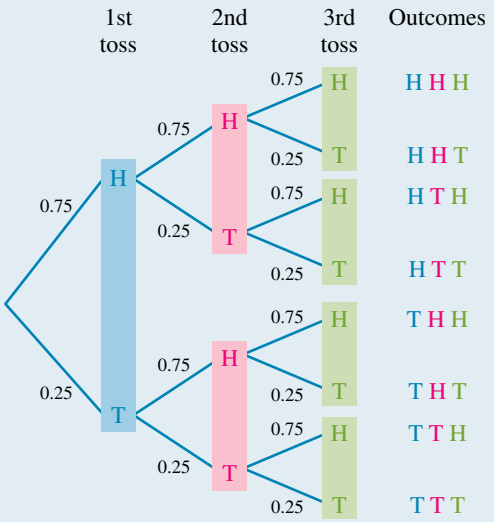
THINK

a 1 Tossing a coin has two outcomes. Draw 2 branches from the starting point to show the first toss, 2 branches off each of these to show the second toss and then 2 branches off each of these to show the third toss.

2 Write probabilities on the branches to show the individual probabilities of tossing a Head (0.75) and a Tail. Because tossing a Head and tossing a Tail are mutually exclusive, P(T) = 1 − P(H) = 0.25.

WRITE

a



b i • P(HTT) implies the order: H(0.75), T(0.25), T(0.25).
• Multiply the probabilities.

ii • P(1H and 2T) implies: P(HTT), P(THT), P(TTH).
• Add these probabilities.

iii • P(at least 2 Tails) implies: P(HTT), P(THT), P(TTH) and P(TTT).
• Add these probabilities.

b i $P(HTT) = P(H) \times P(T) \times P(T)$
$= (0.75) \times (0.25)^2$
$= 0.047$

ii P(1H and 2T)
$= P(HTT) + P(THT) + P(TTH)$
$= 3(0.75 \times 0.25^2)$
$= 0.141$

iii P(at least 2T)
$= P(HTT) + P(THT) + P(TTH) + P(TTT)$
$= 3(0.75 \times 0.25^2) + 0.25^3$
$= 0.156$

learnon RESOURCES — ONLINE ONLY

Interactivity: Tree diagrams (int-6171)
eLesson: Tree diagrams (eles-1894)
eLesson: Games at Wimbledon (eles-1032)
Digital doc: SkillSHEET Multiplying fractions for calculating probabilities (doc-5290)
Digital doc: WorkSHEET Tree diagrams (doc-14591)

Exercise 4.3 Two- and three-step chance experiments

Individual pathways

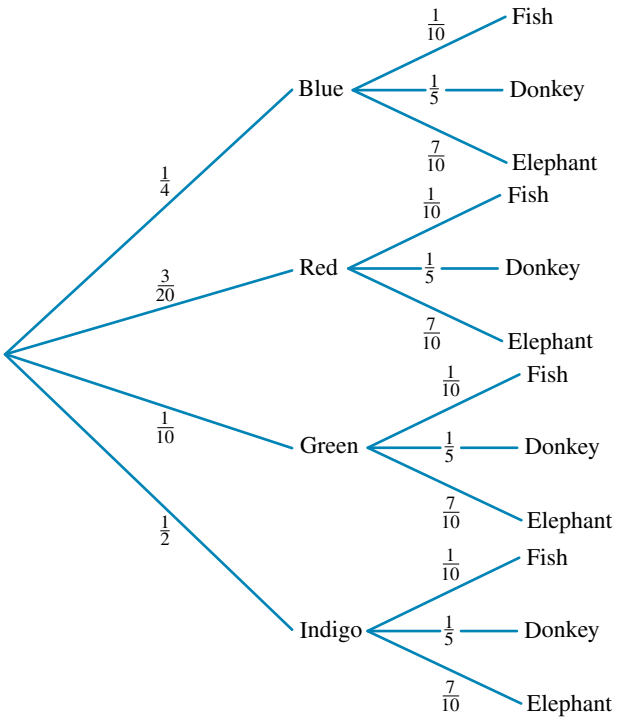
PRACTISE	CONSOLIDATE	MASTER
Questions: 1–7, 9, 11	Questions: 1, 3–6, 8, 9, 11	Questions: 1–13

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. Explain how a tree diagram can be used to calculate probabilities of events that are not equally likely.

2. Use this tree diagram to answer the following questions.



a. How many different outcomes are there?

b. Are all outcomes equally likely? Explain.

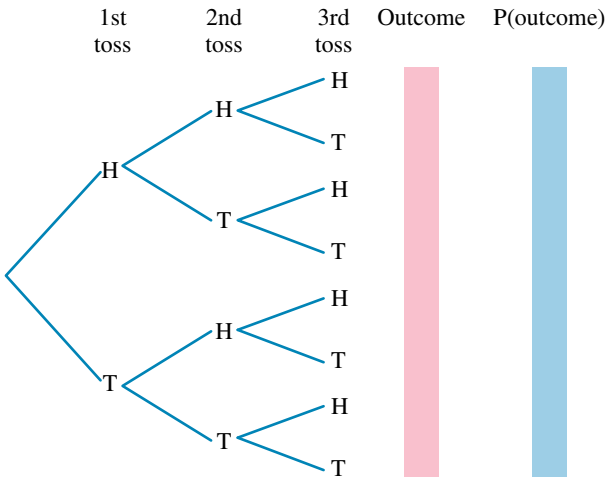
c. Is getting a red fish more, less or equally likely than getting a green elephant?

d. What is the most likely outcome?

e. Calculate the following probabilities.

i. P(blue elephant) **ii.** P(indigo elephant) **iii.** P(donkey)

3. a. Copy the tree diagram shown and complete the labelling for tossing a biased coin three times when the chance of tossing one Head in one toss is 0.7.



b. What is the probability of tossing three Heads?

c. What is the probability of getting at least one Tail?

d. What is the probability of getting exactly two Tails?

4. The following questions relate to rolling a fair die.

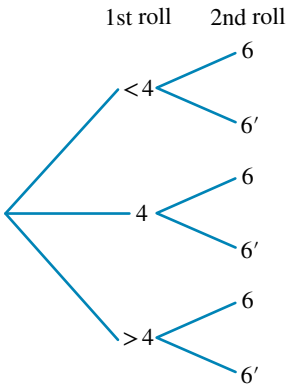
a. What is the probability of each of the following outcomes from one roll of a die?

i. P(rolling number< 4)

ii. P(rolling a 4)

iii. P(rolling a number other than a 6)

b. The tree diagram shown has been condensed to depict rolling a die twice, noting the number relative to 4 on the first roll and 6 on the second. Complete a labelled tree diagram, showing probabilities on the branches and all outcomes, similar to that shown.



c. What is the probability of rolling the following with 2 rolls of the die?

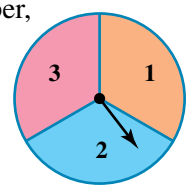
i. P(a 4 then a 6)

ii. P(a number less than 4 then a 6)

iii. P(a 4 then 6′)

iv. P(a number > 4 and then a number < 6)

5. **WE10** The spinner shown is divided into 3 equal-sized wedges labelled 1, 2 and 3. It is spun three times, and it is noted whether the spinner lands on a prime number, $P = \{2, 3\}$ = 'prime', or not a prime number, $P' = \{1\}$ = 'not prime'.



a. Construct a labelled tree diagram for 3 spins of the spinner, showing probabilities on the branches and all possible outcomes.

b. Find the following probabilities.

i. P(3 prime numbers)

ii. P(PPP' in this order)

iii. P(PPP' in any order)

6. **WE11** A coin is biased so that the chance of it falling as a Tail when tossed is 0.2.

a. Draw a tree diagram to represent the coin being tossed three times.

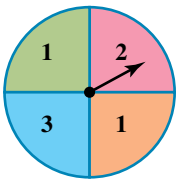
b. What is the probability of getting the same outcome on each toss?

7. A die is tossed twice and each time it is recorded whether or not the number is a multiple of 3. If M = the event of getting a multiple of 3 on any one toss and M' = the event of not getting a multiple of 3 on any one toss:

a. draw a tree diagram to represent the 2 tosses

b. what is the probability of getting two multiples of 3?

8. The biased spinner illustrated is spun three times.



a. Draw a completely labelled tree diagram for 3 spins of the spinner, showing probabilities on the branches and all possible outcomes and associated probabilities.

b. What is the probability of getting exactly 2 *ones*?

c. What is the probability of getting at most 2 *ones*?

Communicating, reasoning and problem solving

9. A restaurant offers its customers a three-course dinner, where they choose between two entrées, three main meals and two desserts. The managers find that 30% choose soup and 70% choose prawn cocktail for the entrée; 20% choose vegetarian, 50% chicken, and the rest have beef for their main meal; and 75% have sticky date pudding while the rest have apple crumble for dessert.

a. Draw a fully labelled tree diagram showing all possible choices.

b. What is the probability that a customer will choose the soup, chicken and sticky date pudding?

c. If there are 210 people booked for the following week at the restaurant, how many would you expect to have the meal combination referred to in part **b**?

10. A bag contains 7 red and 3 white balls. A ball is taken at random, its colour noted and it is then placed back in the bag before a second ball is chosen at random and its colour noted.

a. i. Show the possible outcomes with a fully labelled tree diagram.

ii. As the first ball was chosen, how many balls were in the bag?

iii. As the second ball was chosen, how many balls were in the bag?

iv. Does the probability of choosing a red or white ball change from the first selection to the second? Explain.

v. Calculate the probability of choosing a red ball twice.

b. Suppose that after the first ball had been chosen it was not placed back in the bag.

i. As the second ball is chosen, how many balls are in the bag?

ii. Does the probability of choosing a red or white ball change from the first selection to the second? Explain.

iii. Construct a fully labelled tree diagram to show all possible outcomes.

iv. Calculate the probability of choosing two red balls.

11. An eight-sided die is rolled three times to see whether 5 occurs.

a. Draw a tree diagram to show the sample space.

b. Calculate:

i. P(three 5s)

ii. P(no 5s)

iii. P(two 5s)

iv. P(at least two 5s).

12. A tetrahedral die (four faces labelled 1, 2, 3 and 4) is rolled and a coin is tossed simultaneously.

a. Show all the outcomes on a two-way table.

b. Draw a tree diagram and list all outcomes and their respective probabilities.

c. Calculate the probability of getting a Head on the coin and an even number on the die.

13. What strategies would you use to remember how to construct tree diagrams?

4.4 Independent and dependent events [Stage 5.2]

4.4.1 Independent events

- If a coin is tossed the outcome is a Head or a Tail. The outcome of the first toss does not affect the outcome of the next toss of the coin. The second toss will still yield a Head or a Tail irrespective of the outcome of the first toss. Similarly, the outcome on the roll of a die will not affect the outcome of the next roll.
- If successive events have no effect on each other, they are called **independent events**.
- If events A and B are independent, then the **Multiplication Law of probability** states that: $\text{P}(A \text{ and } B) = \text{P}(A) \times \text{P}(B)$ or $\text{P}(A \cap B) = \text{P}(A) \times \text{P}(B)$.
- The reverse is also true. If: $\text{P}(A \text{ and } B) = \text{P}(A) \times \text{P}(B)$ or $\text{P}(A \cap B) = \text{P}(A) \times \text{P}(B)$ is true, then event A and event B are independent events.

WORKED EXAMPLE 12

Adam is one of the 10 young golfers to represent his state. Paz is one of the 12 netball players to represent her state. All the players in their respective teams have an equal chance of being nominated as captains.

a Are the events 'Adam is nominated as captain' and 'Paz is nominated as captain' independent?
b Determine:
i P(Adam is nominated as captain)
ii P(Paz is nominated as captain).
c What is the probability that both Adam and Paz are nominated as captains of their respective teams?

THINK	WRITE
a Determine whether the given events are independent and write your answer.	**a** Adam's nomination has nothing to do with Paz's nomination and vice versa. Therefore, the events are independent.
b i 1 Determine the probability of Adam being nominated as captain. He is one of 10 players.	**b i** $\text{P(Adam is nominated)} = \text{P}(A) = \frac{n(\text{Adam is nominated})}{n(\xi)}$ $\text{P(Adam is nominated)} = \frac{1}{10}$
2 Write your answer.	The probability that Adam is nominated as captain is $\frac{1}{10}$.
ii 1 Determine the probability of Paz being nominated as captain. She is one of 12 players.	**ii** $\text{P(Paz is nominated)} = \text{P}(A) = \frac{n(\text{Paz is nominated})}{n(\xi)}$ $\text{P(Paz is nominated)} = \frac{1}{12}$
2 Write your answer.	The probability that Paz is nominated as captain is $\frac{1}{12}$.
c 1 Write the Multiplication Law of probability for independent events.	**c** $\text{P}(A \text{ and } P) = \text{P}(A \cap P) = \text{P}(A) \times \text{P}(P)$ $\text{P(Adam and Paz are nominated)} = \text{P(Adam is nominated)} \times \text{P(Paz is nominated)}$
2 Substitute the known values into the rule.	$= \frac{1}{10} \times \frac{1}{12}$
3 Evaluate.	$= \frac{1}{120}$
4 Write your answer.	The probability that both Adam and Paz are nominated as captains is $\frac{1}{120}$.

4.4.2 Dependent events

- Sometimes one event affects the outcome of another. For example, if a card is drawn from a pack of playing cards, the probability that its suit is hearts, P(hearts), is $\frac{13}{52}$ (or $\frac{1}{4}$). If this card is a heart and is not replaced, then this will affect the probability of subsequent draws. The probability that the second card drawn is a heart will be $\frac{12}{51}$, and the probability that the second card is not a heart will be $\frac{39}{51}$.
- When one event affects the occurrence of another, the events are called **dependent events**.
- If two events are dependent, then the probability of occurrence of one event affects that of the subsequent event.

WORKED EXAMPLE 13

A bag contains 5 blue, 6 green and 4 yellow marbles. The marbles are identical in all respects except in their colours. Two marbles are picked in succession without replacement. Determine the probability of picking 2 blue marbles.

THINK

1 Determine the probability of picking the first blue marble.

WRITE

$$\text{P(picking a blue marble)} = \frac{n(B)}{n(\xi)}$$
$$\text{P(picking a blue marble)} = \frac{5}{15}$$
$$= \frac{1}{3}$$

2 Determine the probability of picking the second blue marble.
Note: The two events are dependent since marbles are not being replaced. Since we have picked a blue marble this leaves 4 blue marbles remaining out of a total of 14 marbles.

$$\text{P(picking second blue marble)} = \frac{n(B)}{n(\xi)}$$
$$\text{P(picking second blue marble)} = \frac{4}{14}$$
$$= \frac{2}{7}$$

3 Calculate the probability of obtaining 2 blue marbles.

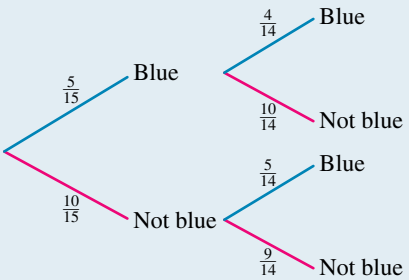
$$\text{P(2 blue marbles)} = \text{P(1st blue)} \times \text{P(2nd blue)}$$
$$= \frac{1}{3} \times \frac{2}{7}$$
$$= \frac{2}{21}$$

4 Write your answer.

The probability of obtaining 2 blue marbles is $\frac{2}{21}$.

Note: Alternatively, a tree diagram could be used to solve this question.

The probability of selecting 2 blue marbles successively can be read directly from the first branch of the tree diagram.



$$\text{P(2 blue marbles)} = \frac{5}{15} \times \frac{4}{14}$$
$$= \frac{1}{3} \times \frac{2}{7}$$
$$= \frac{2}{21}$$

DISCUSSION

Explain the difference between dependent and independent events using examples.

learnon RESOURCES — ONLINE ONLY

Interactivity: Multiplication Law of probability (int-6172)
Interactivity: Dependent events (int-6173)
Interactivity: Independent and dependent events (int-2787)
Interactivity: Random numbers (int-0085)

Exercise 4.4 Independent and dependent events

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–7, 9, 11, 13, 16	Questions: 1, 3–6, 8, 9, 11, 12, 14, 16	Questions: 1–18

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. If A and B are independent events and $P(A) = 0.7$ and $P(B) = 0.4$, calculate:
- **a.** $P(A \text{ and } B)$
- **b.** $P(A' \text{ and } B')$ where A' is the complement of A
- **c.** $P(A' \text{ and } B')$ where B' is the complement of B
- **d.** $P(A' \text{ and } B')$.

2. WE12 A die is rolled and a coin is tossed.
- **a.** Are the outcomes independent?
- **b.** Determine:
  - **i.** P(Head) on the coin
  - **ii.** P(6) on the die.
- **c.** Determine P(6 on the die and Head on the coin).

3. A tetrahedral (4-faced) die and a 10-sided die are rolled simultaneously. What is the probability of getting a 3 on the tetrahedral die and an 8 on the 10-sided die?

4. A blue die and a green die are rolled. What is the probability of getting a 5 on the blue die and not a 5 on the green die?

5. Dean is an archer. The experimental probability that Dean will hit the target is $\frac{4}{5}$.

- **a.** What is the probability that Dean will hit the target on two successive attempts?
- **b.** What is the probability that Dean will hit the target on three successive attempts?
- **c.** What is the probability that Dean will not hit the target on two successive attempts?
- **d.** What is the probability that Dean will hit the target on the first attempt but miss on the second attempt?

6. MC A bag contains 20 apples, of which 5 are bruised. Peter picks an apple and realises that it is bruised. He puts the apple back in the bag and picks another one.

a. The probability that Peter picks 2 bruised apples is:

A. $\frac{1}{4}$ **B.** $\frac{1}{2}$ **C.** $\frac{1}{16}$

D. $\frac{3}{4}$ **E.** $\frac{15}{16}$

b. The probability that Peter picks a bruised apple first but a good one on his second attempt is:

A. $\frac{1}{4}$ **B.** $\frac{1}{2}$ **C.** $\frac{3}{4}$

D. $\frac{3}{16}$ **E.** $\frac{1}{16}$

7. The probability that John will be late for a meeting is $\frac{1}{7}$ and the probability that Phil will be late for a meeting is $\frac{3}{11}$. What is the probability that:

a. John and Phil are both late **b.** neither of them is late

c. John is late but Phil is not late **d.** Phil is late but John is not late?

8. On the roulette wheel at the casino there are 37 numbers, 0 to 36 inclusive. Bidesi puts his chip on number 8 in game 20 and on number 13 in game 21.

a. What is the probability that he will win in game 20?

b. What is the probability that he will win in both games?

c. What is the probability that he wins at least one of the games?

9. Based on her progress through the year, Karen was given a probability of 0.8 of passing the Physics exam. If the probability of passing both Maths and Physics is 0.72, what is her probability of passing the Maths exam?

10. Suresh found that, on average, he is delayed 2 times out of 7 at Melbourne airport. Rakesh made similar observations at Brisbane airport, but found he was delayed 1 out of every 4 times. Find the probability that both Suresh and Rakesh will be delayed if they are flying out of their respective airports.

11. Bronwyn has 3 pairs of Reebok and 2 pairs of Adidas running shoes. She has 2 pairs of Reebok, 3 pairs of Rio and a pair of Red Robin socks. Preparing for an early morning run, she grabs at random for a pair of socks and a pair of shoes. What is the probability that she chooses:
- **a.** Reebok shoes and Reebok socks
- **b.** Rio socks and Adidas shoes
- **c.** Reebok shoes and Red Robin socks
- **d.** Adidas shoes and socks that are not Red Robin?

12. WE13 Two cards are drawn successively and without replacement from a pack of playing cards. Determine the probability of drawing:
- **a.** 2 hearts
- **b.** 2 kings
- **c.** 2 red cards.

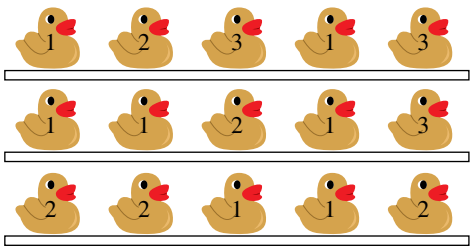
13. In a class of 30 students there are 17 girls. Two students are picked randomly to represent the class in the Student Representative Council. Determine the probability that:
- **a.** both students are boys
- **b.** both students are girls
- **c.** one of the students is a boy.

Communicating, reasoning and problem solving

14. Greg has tossed a tail on each of 9 successive coin tosses. He believes that his chances of tossing a Head on his next toss must be very high. Is Greg correct? Justify your answer.

15. The multiplication law of probability relates to *independent events*. Tree diagrams can illustrate the sample space of successive *dependent events* and the probability of any one combination of events can be calculated by *multiplying* the stated probabilities along the branches. Is this a contradiction to the multiplication law of probability? Explain.

16. A game at a carnival requires blindfolded contestants to throw balls at numbered ducks sitting on 3 shelves. The game ends when 3 ducks have been knocked off the shelves. Assume that the probability of hitting each duck is equal and that a duck is hit with each throw.



- **a.** Are the events described in the game dependent or independent?
- **b.** What are the initial probabilities of hitting each number?
- **c.** Draw a labelled tree diagram to show the possible outcomes for a contestant.
- **d.** Calculate the probabilities of hitting the following:
  - **i.** P(1, 1, 1)
  - **ii.** P(2, 2, 2)
  - **iii.** P(3, 3, 3)
  - **iv.** P(at least one 3).

17. Question **16** describes a game at a carnival. A contestant pays $3 to play and must make 3 direct hits to be eligible for a prize. The numbers on the ducks hit are then summed and the contestant wins a prize according to the following table.

Winners' table	
Total score	**Prize**
9	Major prize ($30 value)
7–8	Minor prize ($10 value)
5–6	$2 prize
3–4	No prize

a. Calculate the probability of winning each prize listed.

b. Suppose 1000 games are played on an average show day. What profit could be expected to be made by the sideshow owner on any average show day?

18. How are dependent events, independent events and the multiplication law of probability reflected on a tree diagram?

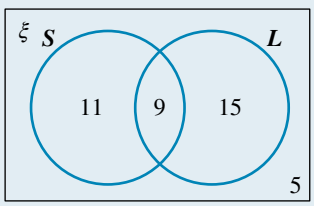
4.5 Conditional probability [Stage 5.2]

4.5.1 Conditional probability

- **Conditional probability** is when the probability of an event is conditional (depends) on another event occurring first.
- Conditional probability can be expressed using a variety of language. Some examples of conditional probability statements follow. The key words to look for in a conditional probability statement have been highlighted in each instance.
  - **If** a student receives a B+ or better in their first Maths test, **then** the chance of them receiving a B+ or better in their second Maths test is 75%.
  - **Given** that a red marble was picked out of the bag with the first pick, the probability of a blue marble being picked out with the second pick is 0.35.
  - **Knowing that** the favourite food of a student is hot chips, the probability of their favourite condiment being tomato sauce is 68%.
- The effect of conditional probability is to reduce the event space and thus increase the probability of the desired outcome.

WORKED EXAMPLE 14

A group of students was asked if they liked, spaghetti (*S*) or lasagne (*L*). The results are illustrated in the Venn diagram. Use the Venn diagram to calculate the following probabilities relating to a student's food preferences.



a What is the probability that a randomly selected student likes spaghetti?

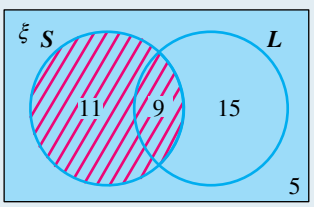
b What is the probability that a randomly selected student likes lasagne given that they also like spaghetti?

THINK

a 1 From 40 students surveyed, shown in blue, 20 liked 'spaghetti' or 'spaghetti and lasagne' as shown in red.

WRITE/DRAW

a



2 The probability that a randomly selected student likes spaghetti is found by substituting these values into the probability formula.

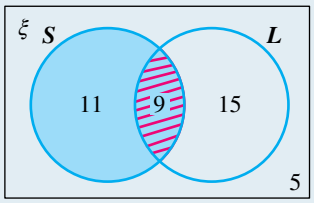
$$\text{P(event)} = \frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}}$$

$$\text{P(spaghetti)} = \frac{20}{40}$$

$$= \frac{1}{2}$$

b 1 The condition imposed 'given they also like spaghetti' alters the sample space to the 20 students who like spaghetti or spaghetti and lasagne described in part **a**, as shaded in blue. Of these 20 students, 9 stated they liked lasagne and spaghetti, as shown in red.

b



2 The probability that a randomly selected student likes lasagne, given that they like spaghetti, is found by substituting these values into the probability formula.

$$\text{P(event)} = \frac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}}$$

$$\text{P(student likes lasagne given that they like spaghetti)} = \frac{9}{20}$$

ACTIVITY: MONSTER VENN DIAGRAM

Equipment: chalk, string or rope

1. Using chalk, string or rope, create a large Venn diagram of two overlapping sets representing students with brown hair and students with blue eyes. Create the diagram on the floor of your classroom or on the ground outside.
2. Each member of the class should stand on the Venn diagram in the appropriate set.
3. As a class:
   a. impose each of the conditions stated in the following table by having students sit down if they do not meet the condition imposed
   b. calculate the probabilities stated.

	Condition	Probability
A	A student has brown hair	P(blue eyes/brown hair)
B	A student has blue eyes	P(brown hair/blue eyes)
C	A student does not have brown hair	P(does not have blue eyes/does not have brown hair)

4. As a class, discuss how the number of people changes if you put a condition on the probability.

4.5.2 Extension: The conditional probability formula

- For two events, A and B, the conditional probability of event B, *given* that event A occurs, is denoted by $\text{P}(B \mid A)$ and can be calculated using the formula:

$$\text{P}(B \mid A) = \frac{\text{P}(A \cap B)}{\text{P}(A)}, \ \text{P}(A) \neq 0$$

WORKED EXAMPLE 15

A die is rolled on a fair six-sided die and the number noted.

a Calculate the probability that the number is a multiple of 3

b Calculate the probability that the number is a multiple of 3, given that the number is less than 5.

THINK	WRITE
a 1 State the probability formula.	**a** $P(\text{event}) = \dfrac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}}$
2 3 and 6 are the only multiples of 3. So there are 2 favourable outcomes and 6 possible outcomes.	$P(\text{number is a multiple of three}) = \dfrac{2}{6} = \frac{1}{3}$
b 1 State the formula for conditional probability. Event B is the number is a multiple of 3 Event A is the number is less than 5.	**b** $\mathrm{P}(B \mid A) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)}, \mathrm{P}(A) \neq 0$ $\mathrm{P}(A \cap B) = \dfrac{1}{6}, \mathrm{P}(A) = \dfrac{4}{6}$
2 Substitute the values given in the question into this formula and simplify.	$\mathrm{P}(B \mid A) = \dfrac{\frac{1}{6}}{\frac{4}{6}} = \frac{1}{4}$

DISCUSSION

In Worked example 15, what is the effect of the given condition in part **b** on the probability of rolling a multiple of 3?

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Interactivity: Conditional probability (int-6085)

eLesson: Conditional probability (eles-1928)

Exercise 4.5 Conditional probability

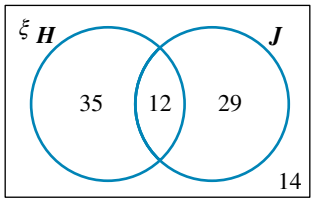
Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–7, 9, 11	Questions: 1, 3–6, 8, 9, 11, 12	Questions: 1–13

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

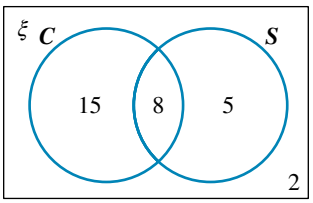
1. WE15 A group of students was asked to nominate their favourite form of dance, hip hop (H) or jazz (J). The results are illustrated in the Venn diagram. Use the Venn diagram given to calculate the following probabilities relating to a student's favourite form of dance.



a. What is the probability that a randomly selected student prefers jazz?

b. What is the probability that a randomly selected student prefers hip hop, given that they prefer jazz?

2. WE15 A group of students was asked which seats they found most comfortable, the seats in the computer lab or the science lab. The results are illustrated in the Venn diagram. Use the Venn diagram given to calculate the following probabilities relating to the most comfortable seats.



a. What is the probability that a randomly selected student prefers the science lab?

b. What is the probability that a randomly selected student prefers the science lab, given that they might prefer the computer lab or the science lab?

3. A medical degree requires applicants to participate in two tests, an aptitude test and an emotional maturity test. This year 52% passed the aptitude test and 30% passed both tests. Calculate the probability that a student who passed the aptitude test also passed the emotional maturity test.

4. At a school classified as a 'Music school for excellence', the probability that a student elects to study Music and Physics is 0.2. The probability that a student takes Music is 0.92. What is the probability that a student takes Physics, given that the student is taking Music?

5. The probability that a student is well and misses a work shift the night before an exam is 0.045, and the probability that a student misses a work shift is 0.05. What is the probability that a student is well, given they miss a work shift the night before an exam?

6. Two marbles are chosen, without replacement, from a jar containing only red and green marbles. The probability of selecting a green marble and then a red marble is 0.67. The probability of selecting a green marble on the first draw is 0.8. What is the probability of selecting a red marble on the second draw, given the first marble drawn was green?

7. MC A group of 80 schoolgirls consists of 54 dancers and 35 singers. Each member of the group is either a dancer or a singer, or both. The probability that a randomly selected student is a singer given that she is a dancer is:

A. 0.17 **B.** 0.44 **C.** 0.68 **D.** 0.11 **E.** 0.78

Communicating, reasoning and problem solving

8. Explain how imposing a condition alters probability calculations.

9. At your neighbouring school, 65% of the students are male and 35% are female. Of the male students, 10% report that dancing is their favourite activity; of the female students, 25% report that dancing is their favourite activity. Find the probability that:
- **a.** a student selected at random prefers dancing and is female
- **b.** a student selected at random prefers dancing and is male.

10. Using the information presented in question **9**, construct a tree diagram. From your diagram, calculate:
- **a.** the probability that a student is male and does not prefer dancing
- **b.** the overall percentage of students who prefer dancing.

11. Two marbles are chosen, without replacement, from a jar containing only red and green marbles. The probability of selecting a green marble and then a red marble is 0.72. The probability of selecting a green marble on the first draw is 0.85. What is the probability of selecting a red marble on the second draw if the first marble drawn was green?

12. When walking home from school during the summer months, Harold buys either an ice-cream or a drink from the corner shop. If Harold bought an ice-cream the previous day, there is a 30% chance that he will buy a drink the next day. If he bought a drink the previous day, there is a 40% chance that he will buy an ice-cream the next day. On Monday, Harold bought an ice-cream. Determine the probability that he buys an ice-cream on Wednesday.

13. Four letters, A, B, C and D, are arranged in a row. What is the probability that A and B will always be together?

4.6 Review

Investigation | Rich task

Tricky dice

Dice games have been played throughout the world for many years. Professional gamblers resort to all types of devious measures in order to win. Often the other players are unaware of the tricks employed.

Imagine you are playing a game that involves rolling two dice. Instead of having each die marked with the numbers 1 to 6, let the first die have only the numbers 1, 2 and 3 (two of each) and the second die the numbers 4, 5 and 6 (two of each). If you were an observer to this game, you would see the numbers 1 to 6 occurring and probably not realise that the dice were not the regular type.

1. Complete the grid to show the sample space on rolling these two dice.

2. How many different outcomes are there? Compare this with the number of different outcomes using two regular dice.

		Die 1					
		1	**2**	**3**	**1**	**2**	**3**
Die 2	**4**						
	5						
	6						
	4						
	5						
	6						

3. What is the chance of rolling a double using these dice?

4. The numbers on the two dice are added after rolling. Complete the table to show the totals possible.

		Die 1					
		1	**2**	**3**	**1**	**2**	**3**
Die 2	**4**						
	5						
	6						
	4						
	5						
	6						

5. How many different totals are possible? What are they?

6. Which total do you have the greatest chance of rolling? Which total do you have the least chance of rolling?

7. If you played a game in which you had to bet on rolling a total of less than 7, equal to 7 or greater than 7, which option would you be best to take? Explain why.

8. If you had to bet on an even-number outcome or an odd-number outcome, which would be the better option? Explain your answer.

9. The rules are changed to subtracting the numbers on the two dice instead of adding them. Complete the following table to show the outcomes possible.

		Die 1					
		1	**2**	**3**	**1**	**2**	**3**
Die 2	**4**						
	5						
	6						
	4						
	5						
	6						

10. How many different outcomes are possible in this case? What are they?

11. What is the most frequently occurring outcome? How many times does it occur?

12. Devise a game of your own using these dice. On a separate sheet of paper, write out rules for your game and provide a solution, indicating the best options for winning.

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Digital doc: Investigation — Tricky dice (doc-15936)
Digital doc: Code puzzle — The study of... (doc-15937)
Digital doc: Summary Topic 4 (doc-22902)
Interactivity: Word search Topic 4 (int-2856)
Interactivity: Crossword Topic 4 (int-2857)
Interactivity: Sudoku Topic 4 (int-3598)

Exercise 4.6 Review questions

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. MC Which of the following is always true for an event, M, and its complementary event, M'?

A. $\mathrm{P}(M) + \mathrm{P}(M') = 1$
B. $\mathrm{P}(M) - \mathrm{P}(M') = 1$
C. $\mathrm{P}(M) + \mathrm{P}(M') = 0$
D. $\mathrm{P}(M) - \mathrm{P}(M') = 0$
E. $\mathrm{P}(M) \times \mathrm{P}(M') = 1$

2. MC A number is chosen from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Which of the following pairs of events is mutually exclusive?

A. $\{2, 4, 6\}$ and $\{1, 2, 3\}$
B. $\{1, 2, 3, 5\}$ and $\{4, 6, 7, 8\}$
C. $\{0, 1, 2, 3\}$ and $\{3, 4, 5, 6\}$
D. {multiples of 2} and {factors of 8}
E. {even numbers} and {multiples of 3}

3. MC Which of the following states the Multiplication Law of probability correctly?

A. $\mathrm{P}(A \cap B) = \mathrm{P}(A) + \mathrm{P}(B)$
B. $\mathrm{P}(A \cap B) = \mathrm{P}(A) \times \mathrm{P}(B)$
C. $\mathrm{P}(A \cup B) = \mathrm{P}(A) \times \mathrm{P}(B)$
D. $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$
E. $\mathrm{P}(A) = \mathrm{P}(A \cup B) + \mathrm{P}(B)$

The following information relates to questions 4–6.

$\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{2, 3, 4\}$ and $B = \{3, 4, 5, 8\}$

4. MC $A \cap B$ equals:

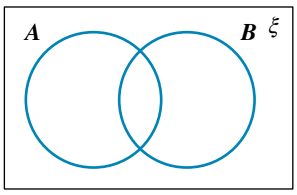
A. $\{2, 3, 3, 4, 4, 5, 8\}$
B. $\{3, 4\}$
C. $\{2, 3, 4\}$
D. $\{2, 3, 4, 5, 8\}$
E. $\{2, 5, 8\}$

5. MC $A \cap B'$ equals:

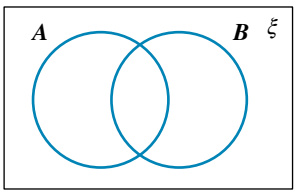
A. $\{3, 4\}$
B. $\{2\}$
C. $\{2, 3, 4, 5, 8\}$
D. $\{2, 3, 4\}$
E. $\{1, 2, 6, 7, 9, 10\}$

6. Shade the region stated for each of the following Venn diagrams.

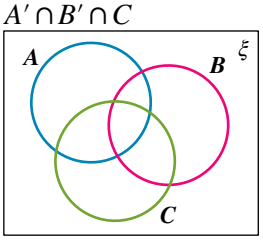
a. $A' \cup B$



b. $A' \cap B'$



c. $A' \cap B' \cap C$



Communicating, reasoning and problem solving

7. MC From past experience, it is concluded that there is a 99% probability that July will be a wet month in Launceston (it has an average rainfall of approximately 80 mm). The probability that July will not be a wet month next year in Launceston is:

A. 99% **B.** 0.99 **C.** $\frac{1}{100}$ **D.** 1 **E.** 0

8. MC A card is drawn from a well-shuffled deck of 52 cards. What is the theoretical probability of not selecting a red card?

A. $\frac{3}{4}$ **B.** $\frac{1}{4}$ **C.** $\frac{1}{13}$ **D.** $\frac{1}{2}$ **E.** 0

9. MC Which of the following events is not equally likely?

A. Obtaining a 5 or obtaining a 1 when a die is rolled

B. Obtaining a club or obtaining a diamond when a card is drawn from a pack of cards

C. Obtaining 2 Heads or obtaining 2 Tails when a coin is tossed twice

D. Obtaining 2 Heads or obtaining 1 Head when a coin is tossed twice

E. Obtaining a 3 or obtaining a 6 when a die is rolled

10. MC The Australian cricket team has won 12 of the last 15 Test matches. What is the experimental probability of Australia not winning its next Test match?

A. $\frac{4}{5}$ **B.** $\frac{1}{5}$ **C.** $\frac{1}{4}$ **D.** $\frac{3}{4}$ **E.** 1

11. A card is drawn from a well-shuffled pack of 52 cards. What is the theoretical probability of drawing:

a. an ace **b.** a spade **c.** a queen or a king **d.** not a heart?

12. A die is rolled five times.

a. What is the probability of rolling five 6s?

b. What is the probability of not rolling five 6s?

13. Alan and Mary own 3 of the 8 dogs in a race. What is the probability that:

a. one of Alan's or Mary's dogs will win

b. none of Alan's or Mary's dogs will win?

14. A die is rolled. Event A is obtaining an even number. Event B is obtaining a 3.

a. Are events A and B mutually exclusive?

b. Calculate $\text{P}(A)$ and $\text{P}(B)$.

c. Calculate$\text{P}(A \cup B)$.

15. A card is drawn from a shuffled pack of 52 playing cards. Event A is drawing a club and event B is drawing an ace.

a. Are events A and B mutually exclusive?

b. Calculate $\text{P}(A)$, $\text{P}(B)$ and $\text{P}(A \cap B)$

c. Calculate $\text{P}(A \cup B)$.

16. A tetrahedral die is numbered 0, 1, 2 and 3. Two of these dice are rolled and the sum of the numbers (the number on the face that the die sits on) is taken.

a. Show the possible outcomes in a two-way table.

b. Are all the outcomes equally likely?

c. Which total has the least chance of being rolled?

d. Which total has the best chance of being rolled?

e. Which sums have the same chance of being rolled?

17. A bag contains 20 pears, of which 5 are bad. Cathy picks 2 pears (without replacement) from the bag. What is the probability that:

a. both pears are bad?

b. both pears are good?

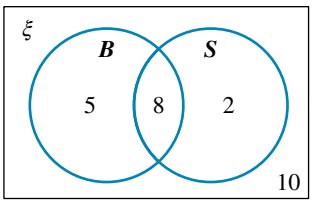
c. one of the two pears is good?

18. Determine the probability of drawing 2 aces from a pack of cards if:

a. the first card is replaced before the second one is drawn

b. the first card drawn is not replaced.

19. On grandparents day at a school, a group of grandparents was asked where they most like to take their grandchildren — the beach (B) or shopping (S). The results are illustrated in the Venn diagram. Use the Venn diagram given to calculate the following probabilities relating to the place grandparents most like to take their grandchildren.



a. What is the probability that a randomly selected grandparent preferred to take their grandchildren to the beach or shopping?

b. What is the probability that a randomly selected grandparent preferred to take their grandchildren to the beach, given that they preferred to take their grandchildren shopping?

20. When all of Saphron's team players turn up for their twice weekly netball training the chance that they then win their Saturday game is 0.65. If not all players are at the training session then the chance of winning their Saturday game is 0.40. Over a four week period, Saphron's players all turn up for training three times.

a. Using a tree diagram, with T to represent all players training and W to represent a win, represent the winning chance of Saphron's netball team.

b. Using the tree diagram you constructed in part **a**, determine the probability of Saphron's team winning their Saturday game. Write your answer correct to 4 decimal places.

c. Determine the exact probability that Saphron's team did not train given that they won their Saturday game.

21. Andrew does not know the answer to two questions on a multiple choice exam. The first question has four choices and the second question he does not know has five choices.

a. What is the probability that he will get both questions wrong?

b. If he is certain that one of the choices cannot be the answer in the first question, how will this change the probability that he will get both questions wrong?

22. Mariah the Mathematics teacher wanted to give her students a chance to win a reward at the end of the term. She placed 20 cards into a box, and wrote the word ON on 16 cards, and OFF on 4 cards. After a student chooses a card, that card is replaced into the box for the next student to draw. If a student chooses an OFF card, then they do not have to attend school on a specified day. If they choose an ON card, then they do not receive a day off.

a. Mick, a student, chose a random card from the box. What is the probability he received a day off?

b. Juanita, a student, chose a random card from the box after Mick. What is the probability that she did not receive a day off?

c. What is the probability that Mick and Juanita both received a day off?

23. In the game of draw poker, a player is dealt 5 cards from a deck of 52. To obtain a flush, all 5 cards must be of the same suit.

a. Determine the probability of getting a diamond flush.

b. Determine the probability of getting any flush.

24. a. A Year 10 boy is talking with a Year 10 girl and asks her if she has any brothers or sisters. She says, 'Yes, I have one'. What is the probability that she has a sister?

b. A Year 10 boy is talking with a Year 10 girl and asks her if she has any brothers or sisters. She says, 'Yes, I have an older one'. What is the probability that she has a sister?

Answers

Topic 4 Probability

Exercise 4.2 Calculating relative frequencies

1. Experimental probability is based on the outcomes of experiments, simulations or surveys. Theoretical probability is based on the number of possible favourable outcomes and the total possible outcomes.

2. a. {I, II, III, IV, V, VI}

b. Frequency for I = 0.2

Frequency for II = 0.12

Frequency for III = 0.16

Frequency for IV = 0.14

Frequency for V = 0.24

Frequency for VI = 0.14

c. The spinner should be spun a larger number of times.

3. a. 500 students

b. Frequency for silver = 0.16

c. Frequency for black and green = 0.316

d. Blazing Blue

4. a.

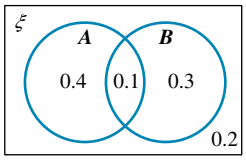
+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

b. The sum of 7

c. The sum of 2 or 12

5. a. $P(A \cap B) = 0.1$

b.



c. $P(A \cap B') = 0.4$

6. a. i. $P(A \cup B) = 0.85$ **ii.** $P(A \cap B)' = 0.95$

b. C

7. $\frac{2}{3}$

8. a. $\frac{1}{13}$ **b.** $\frac{1}{4}$ **c.** $\frac{4}{13}$

9. Answers may vary; check with your teacher.

a. i. No. There are many others foods one could have.

ii. Having Weet Bix and not having Weet Bix

b. i. No. There are other means of transport; for example, catching a bus.

ii. Walking to a friend's place and not walking to a friend's place

c. i. No. There are other possible leisure activities.

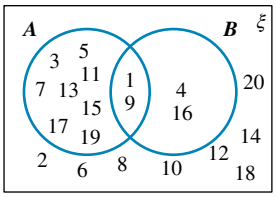
ii. Watching TV and not watching TV

d. i. No. The number 5 can be rolled too.

ii. Rolling a number less than 5 and rolling a number 5 or greater

e. Yes. There are only two possible outcomes; passing or failing.

10. a.



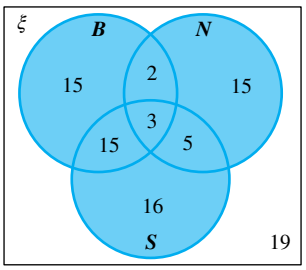
b. i. $\frac{10}{20} = \frac{1}{2}$ **ii.** $\frac{4}{20} = \frac{1}{5}$ **iii.** $\frac{2}{20} = \frac{1}{10}$ **iv.** $\frac{12}{20} = \frac{3}{5}$ **v.** $\frac{8}{20} = \frac{2}{5}$

11. a. No. For a 6-sided die, $P(4) = \frac{1}{6}$; for an 8-sided die, $P(4) = \frac{1}{8}$.

b. Yes; $P(\text{odd}) = \frac{1}{2}$.

12. Yes; $P(5) = \frac{1}{2}$, $P(6) = \frac{1}{2}$.

13. a.



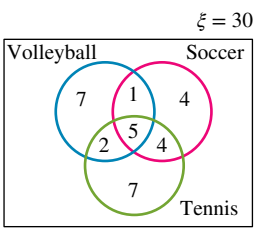
b. 19 students **c.** 32 students **d.** 15 students

e. Frequency = 0.1667 **f.** Probability = $\frac{3}{90}$

14. a. $\frac{1}{13}$ **b.** $\frac{1}{4}$ **c.** $\frac{1}{2}$ **d.** $\frac{12}{13}$ **e.** 0 **f.** $\frac{1}{2}$

15. a. $\frac{1}{5}$ **b.** $\frac{7}{20}$ **c.** $\frac{11}{20}$ **d.** 0

16. a.

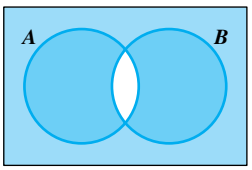


b. i. $\frac{1}{2}$ **ii.** $\frac{1}{6}$ **iii.** $\frac{1}{30}$ **iv.** $\frac{2}{5}$ **v.** $\frac{7}{15}$

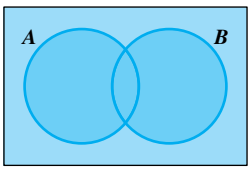
c. i. $\frac{1}{2}$ **ii.** $\frac{8}{15}$

17. Yes. Both have a probability of $\frac{1}{2}$.

18.



Overlaying A' and B' shows $A' \cap B'$ as the area surrounding A and B, shaded light blue.

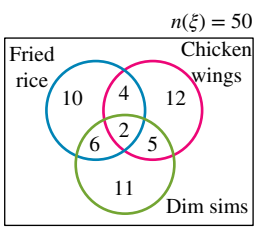


The union of A and B is shown in dark blue, leaving the surrounding area as $(A \cup B)'$, shaded light blue.

19. a. i. 50 **ii.** 7 **iii.** 25 **iv.** 8

b. i. $\frac{1}{2}$ **ii.** $\frac{3}{50}$ **iii.** $\frac{6}{25}$

c. i. **ii.** $\frac{1}{25}$



20. a.

Die 1 outcomes \ Die 2 outcomes	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

b. 6

c. No. The frequency of the numbers is different.

d.

Sum	2	3	4	5	6	7	8	9	10	11	12
Frequency	1	2	3	4	5	6	5	4	3	2	1

e. i. $\frac{1}{36}$ **ii.** $\frac{1}{6}$ **iii.** $\frac{1}{18}$

f. i. $\frac{1}{36}$ **ii.** $\frac{1}{6}$ **iii.** $\frac{1}{18}$

g. 50

21. $\text{P(event)} = \dfrac{\text{number of favourable outcomes}}{\text{total number of possible outcomes}}$ or $\text{P(event)} = \dfrac{n(E)}{n(\xi)}$

22. 11 red and 4 purple, i.e. 15 socks more

Exercise 4.3 Two and three-step chance experiments

1. If the probabilities of 2 events are different, the first column of branches indicates the probabilities for the first event and the second column of branches indicates the probabilities for the second event. The product of each branch gives the probability. All probabilities add to 1.

2. a. 12 different outcomes

b. No. Each branch is a product of different probabilities.

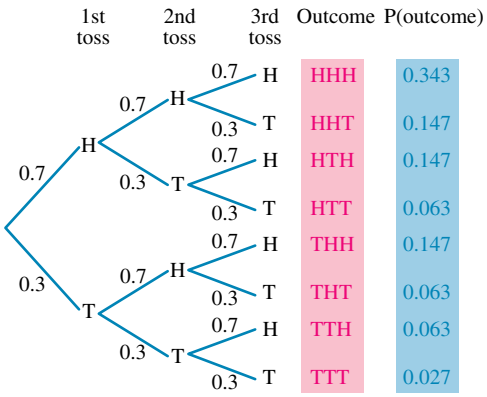
c. Less likely

d. Indigo elephant

e. i. $\text{P(blue elephant)} = \frac{7}{40}$ **ii.** $\text{P(indigo elephant)} = \frac{7}{20}$ **iii.** $\text{P(donkey)} = \frac{1}{5}$

3.

a.



b. P(HHH) = 0.343

c. P(at least 1 Tail) = 0.657

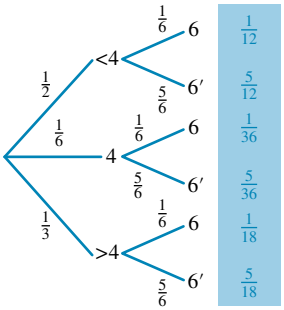
d. P(exactly 2 Tail) = 0.189

4. a. **i.** $\frac{1}{2}$

ii. $\frac{1}{6}$

iii. $\frac{5}{6}$

b.



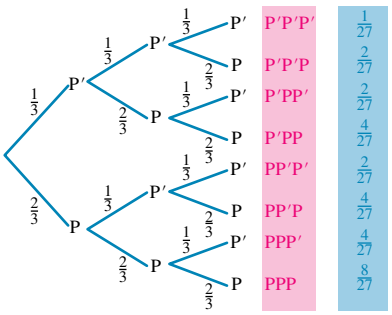
c. **i.** $\frac{1}{36}$

ii. $\frac{1}{12}$

iii. $\frac{5}{36}$

iv. $\frac{5}{18}$

5. a.

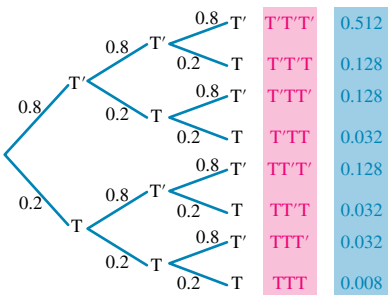


b. **i.** $\frac{8}{27}$

ii. $\frac{4}{27}$

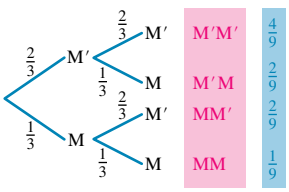
iii. $\frac{12}{27}$

6. a.



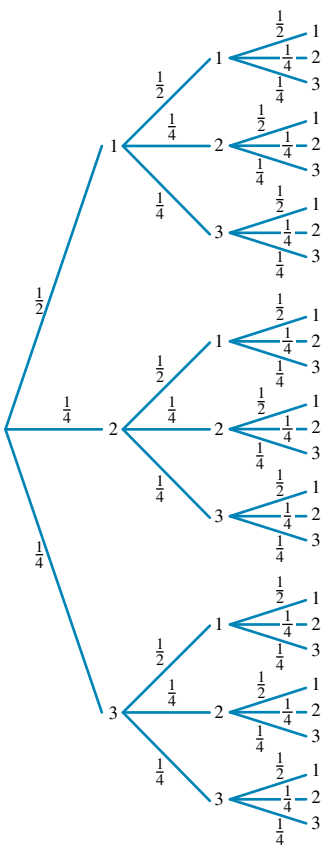
b. 0.520

7. a.



b. $\frac{1}{9}$

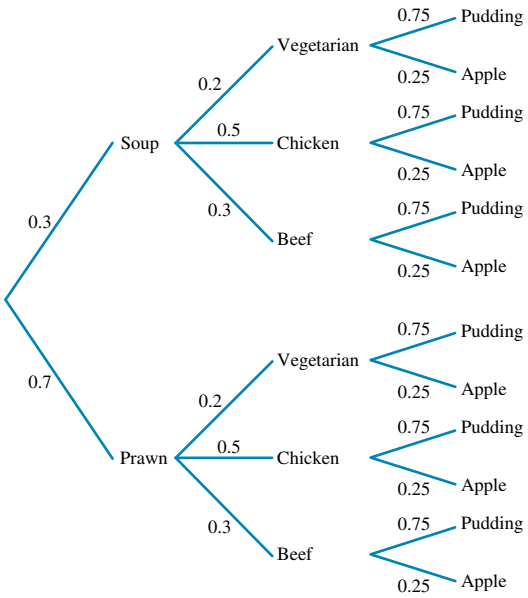
8. a.



b. $\frac{3}{8}$

c. $\frac{7}{8}$

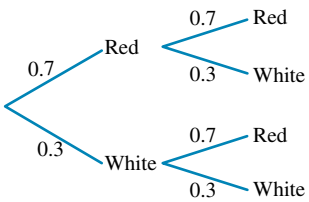
9. a.



b. 0.1125

c. 24 people

10. a. i.



ii. 10 balls

iii. 10 balls

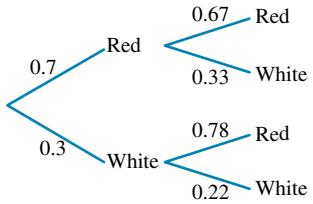
iv. No; the events are independent.

v. P(RR) = 0.49

b. i. 9 balls

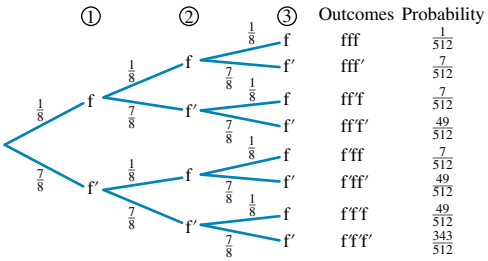
ii. Yes. One ball has been removed from the bag.

iii.



iv. P(RR) = $\frac{7}{15}$ or 0.469 using the rounded values from iii

11. a.

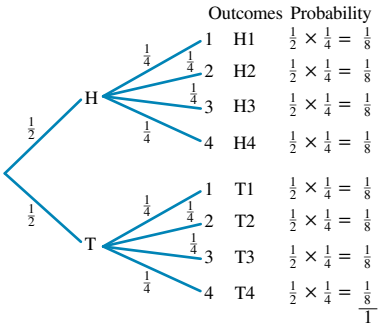


b. i. $\frac{1}{512}$ **ii.** $\frac{343}{512}$ **iii.** $\frac{21}{512}$ **iv.** $\frac{11}{256}$

12. a.

Coin outcomes \ Die outcomes	1	2	3	4
H	(H, 1)	(H, 2)	(H, 3)	(H, 4)
T	(T, 1)	(T, 2)	(T, 3)	(T, 4)

b.



c. $\frac{1}{4}$

13. Discuss with teacher

Exercise 4.4 Independent and dependent events

1. a. 0.28 **b.** 0.12 **c.** 0.42 **d.** 0.18

2. a. Yes

b. i. $\frac{1}{2}$ **ii.** $\frac{1}{6}$

c. $\frac{1}{12}$

3. $\frac{1}{40}$

4. $\frac{5}{36}$

5. a. $\frac{16}{25}$ **b.** $\frac{64}{125}$ **c.** $\frac{1}{25}$ **d.** $\frac{4}{25}$

6. a. C **b.** D

7. a. $\frac{3}{77}$ **b.** $\frac{48}{77}$ **c.** $\frac{8}{77}$ **d.** $\frac{18}{77}$

8. a. $\frac{1}{37}$ **b.** $\frac{1}{1369}$ **c.** $\frac{73}{1369}$

9. 0.9

10. $\frac{1}{14}$

11. a. $\frac{1}{5}$ **b.** $\frac{1}{5}$ **c.** $\frac{1}{10}$ **d.** $\frac{1}{3}$

12. a. $\frac{1}{17}$ **b.** $\frac{1}{221}$ **c.** $\frac{25}{102}$

13. a. $\frac{26}{145}$ **b.** $\frac{136}{435}$ **c.** $\frac{221}{435}$

14. No. Coin tosses are independent events. No one toss affects the outcome of the next. The probability of a Head or Tail on a fair coin is always 0.5. Greg has a 50% chance of tossing a Head on the next coin toss as was the chance in each of the previous 9 tosses.

15. No. As events are illustrated on a tree diagram, the individual probability of each outcome is recorded. The probability of a dependent event is calculated (altered according to the previous event) and can be considered as if it was an independent event. As such, the multiplication law of probability can be applied along the branches to calculate the probability of successive events.

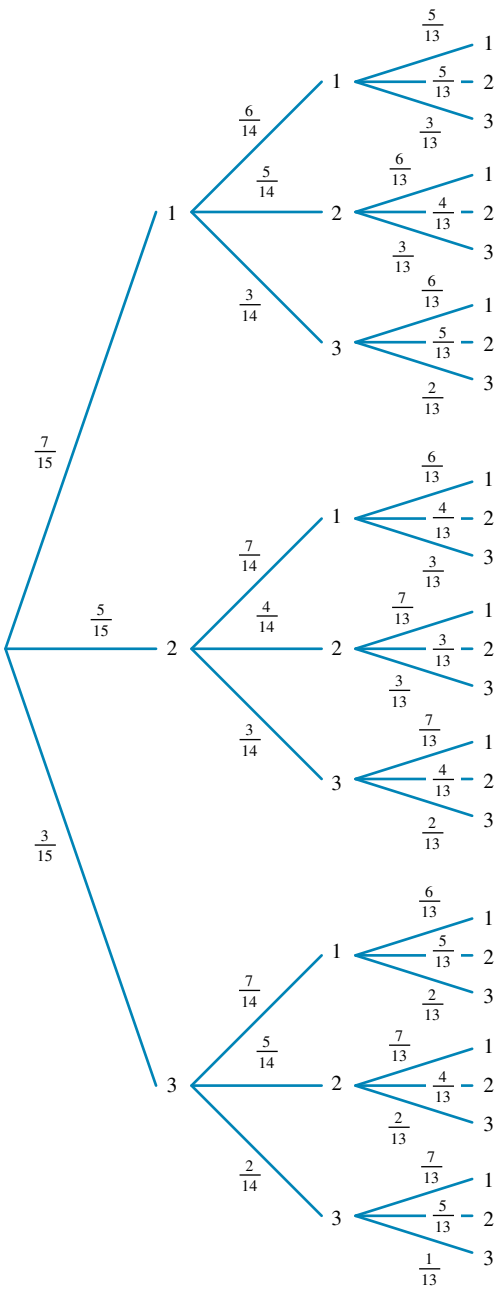
16. a. Dependent

b. $P(1) = \frac{7}{15}$

$P(2) = \frac{5}{15}$

$P(3) = \frac{3}{15}$

c.



d. i. $P(1, 1, 1) = \frac{1}{13}$ **ii.** $P(2, 2, 2) = \frac{2}{91}$ **iii.** $P(3, 3, 3) = \frac{1}{455}$ **iv.** $P(\text{at least one 3}) = \frac{47}{91}$

17. a. $P(9) = \frac{1}{455}$

$P(7–8) = \frac{66}{455}$

$P(5–6) = \frac{248}{455}$

$P(3–4) = \frac{4}{13}$

b. $393.40

18. Discuss with your teacher.

Exercise 4.5 Conditional probability

1. a. $P(J) = \frac{41}{90}$ **b.** $P(H \mid J) = \frac{12}{41}$

2. a. $P(S) = \frac{13}{30}$ **b.** $P(S \mid (CS) = \frac{13}{28}$

3. 0.58 or $\frac{15}{26}$

4. 0.22 or $\frac{5}{23}$

5. 0.9

6. 0.8375

7. A

8. It reduces the event space and increases the probability of a desired outcome

9. a. 0.0875

b. 0.065

10. a. 0.585

b. 0.1525 or 15.25%.

11. 0.847

12. 0.61

13. Probability $= \frac{1}{2}$

Investigation | Rich task

1.

		Die 1					
		1	**2**	**3**	**1**	**2**	**3**
Die 2	**4**	(1, 4)	(2, 4)	(3, 4)	(1, 4)	(2, 4)	(3, 4)
	5	(1, 5)	(2, 5)	(3, 5)	(1, 5)	(2, 5)	(3, 5)
	6	(1, 6)	(2, 6)	(3, 6)	(1, 6)	(2, 6)	(3, 6)
	4	(1, 4)	(2, 4)	(3, 4)	(1, 4)	(2, 4)	(3, 4)
	5	(1, 5)	(2, 5)	(3, 5)	(1, 5)	(2, 5)	(3, 5)
	6	(1, 6)	(2, 6)	(3, 6)	(1, 6)	(2, 6)	(3, 6)

2. 9

3. 0

4.

		Die 1					
		1	**2**	**3**	**1**	**2**	**3**
Die 2	**4**	5	6	7	5	6	7
	5	6	7	8	6	7	8
	6	7	8	9	7	8	9
	4	5	6	7	5	6	7
	5	6	7	8	6	7	8
	6	7	8	9	7	8	9

5. 5; 5, 6, 7, 8, 9

6. 7; 5, 9

7. Equal to 7; probability is the highest.

8. Odd-number outcome; probability is higher.

9.

Coin \ Die	1	2	3	4	5	6	7	8
Head	(H, 1)	(H, 2)	(H, 3)	(H, 4)	(H, 5)	(H, 6)	(H, 7)	(H, 8)
Tail	(T, 1)	(T, 2)	(T, 3)	(T, 4)	(T, 5)	(T, 6)	(T, 7)	(T, 8)

Die 2 \ Die 1	1	2	3	1	2	3
4	3	2	1	3	2	1
5	4	3	2	4	3	2
6	5	4	3	5	4	3
4	3	2	1	3	2	1
5	4	3	2	4	3	2
6	5	4	3	5	4	3

10. 5; 1, 2, 3, 4, 5,

11. 3; 12

12. Answers will vary.

Exercise 4.6 Review questions

1. A

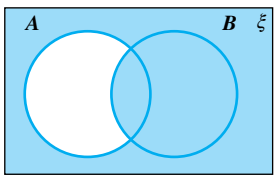
2. B

3. B

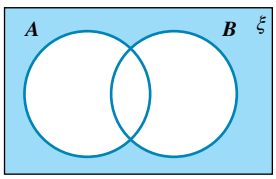
4. B

5. B

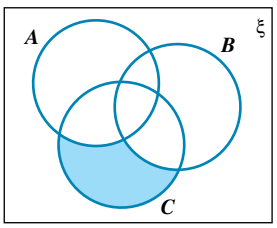
6. a.



b.



c.



7. C

8. D

9. D

10. B

11. a. $\frac{1}{13}$ **b.** $\frac{1}{4}$ **c.** $\frac{2}{13}$ **d.** $\frac{3}{4}$

12. a. $\frac{1}{7776}$ **b.** $\frac{7775}{7776}$

13. a. $\frac{3}{8}$ **b.** $\frac{5}{8}$

14. a. Yes

b. $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{6}$

c. $\frac{2}{3}$

15. a. No

b. $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{13}$, $P(A \cap B) = \frac{1}{52}$

c. $\frac{4}{13}$

16. a.

Die 1 outcomes \ Die 2 outcomes	0	1	2	3
0	(0, 0)	(0, 1)	(0, 2)	(0, 3)
1	(1, 0)	(1, 1)	(1, 2)	(1, 3)
2	(2, 0)	(2, 1)	(2, 2)	(2, 3)
3	(3, 0)	(3, 1)	(3, 2)	(3, 3)

b. No

c. 0 and 6

d. 3

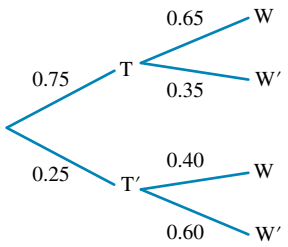
e. 0 and 6, 1 and 5, 2 and 4

17. a. $\frac{1}{19}$ **b.** $\frac{21}{38}$ **c.** $\frac{15}{38}$

18. a. $\frac{1}{169}$ **b.** $\frac{1}{221}$

19. a. $\frac{15}{25} = \frac{3}{5}$ **b.** $\frac{8}{10} = \frac{4}{5}$

20. a.



b. 0.5875

c. $\frac{8}{47}$

21. a. $\frac{3}{5}$ **b.** $\frac{8}{15}$

22. a. $\frac{1}{5}$ **b.** $\frac{4}{5}$ **c.** $\frac{1}{25}$

23. a. 0.000495 **b.** 0.001981

24. a. $\frac{1}{3}$ **b.** $\frac{1}{2}$

TOPIC 5
Single variable data analysis

[Stages 5.1, 5.2 and 5.3]

5.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the concepts covered in this topic.

5.1.1 Why learn this?

According to the novelist Mark Twain, 'There are three kinds of lies: lies, damned lies and statistics.' There is so much information in our lives, increasingly so with the World Wide Web, smart phones and social media tracking our every move and accumulating vast amounts of data about us. The data are used to gather information about our likes and dislikes, our buying habits, our voting preferences and so on. Statistics can easily be used to manipulate people unless they have an understanding of the basic concepts involved.

DISCUSSION

In the modern world, athletes undergo many hours of specialised training, but they also spend many hours analysing their personal data. In what other ways has data changed the sports world, and in what ways do you think it may change the sports world in the future?

LEARNING SEQUENCE

5.1 Overview
5.2 [Stage 5.1] Comparing data displays using mean, median and range
5.3 [Stage 5.2] Quartiles and interquartile range
5.4 [Stage 5.2] Constructing, interpreting and comparing boxplots
5.5 [Stage 5.3] The standard deviation
5.6 [Stage 5.3] Comparing data sets
5.7 Review

LEARNING OUTCOMES

A student:

- uses appropriate terminology, diagrams and symbols in mathematical contexts MA5.1-1WM
- selects and uses appropriate strategies to solve problems MA5.1-2WM
- provides reasoning to support conclusions that are appropriate to the context MA5.1-3WM
- uses statistical displays to compare sets of data MA5.1-12SP
- selects appropriate notations and conventions to communicate mathematical ideas and solutions MA5.2-1WM
- constructs arguments to prove and justify results MA5.2-3WM
- uses quartiles and boxplots to compare sets of data MA5.2-15SP
- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3-1WM

- generalises mathematical ideas and techniques to analyse and solve problems efficiently MA5.3 2WM
- uses deductive reasoning in presenting arguments and formal proofs MA5.3 3WM
- uses standard deviation to analyse data MA5.3 18SP

CONTENT DESCRIPTIONS

Students:

Compare data displays using mean, median and range to describe and interpret numerical data sets in terms of location (centre) and spread (ACMSP283)
Determine quartiles and interquartile range (ACMSP248)
Construct and interpret boxplots and use them to compare data sets (ACMSP249)
Compare shapes of boxplots to corresponding histograms and dot plots (ACMSP250)
Calculate and interpret the mean and standard deviation of data and use these to compare data sets (ACMSP278)

Source: NSW Syllabus for the Australian curriculum

learnon RESOURCES — ONLINE ONLY

eLesson: The story of mathematics — Koby's bid to make the Olympic athletics team (eles-1852)

Note: Your teacher may now set you a pre-test to determine how familiar you are with the content in this topic.

5.2 Comparing data displays using mean, median and range [Stage 5.1]

5.2.1 Mean, median and range

- The **mean** and the **median** are **measures of central tendency**. They measure the middle (or centre) of the data.
  - The mean is the average of all observations in a set of data.
  - The median is the middle observation in an ordered set of data.
- The **range** is the difference between the highest and lowest values in a set of data and hence gives a measure of the spread of the data.
- The mean, median and range can be used to compare and analyse sets of **single variable data**. Single variable data are data with one variable, for example the heights of Year 10 students.

5.2.2 Calculating the mean, median and range

Mean

- To obtain the mean of a set of ungrouped data, all numbers (scores) in the set are added together and then the total is divided by the number of scores in that set.

$$\text{Mean} = \frac{\text{sum of all scores}}{\text{number of scores}}$$

- Symbolically this is written $\bar{x} = \frac{\Sigma x}{n}$.

Median

- The median is the middle value of any set of data arranged in numerical order. In the set of n numbers, the median is located at the $\frac{n+1}{2}$th score. The median is:
  - the middle score for an odd number of scores arranged in numerical order
  - the average of the two middle scores for an even number of scores arranged in numerical order.

Range

- Range = highest value − lowest value
  $R = x_H - x_L$

WORKED EXAMPLE 1

For the data 6, 2, 4, 3, 4, 5, 4, 5, find the:
a mean **b median** **c range**

THINK	WRITE
a 1 Calculate the sum of the scores, that is, Σx.	**a** $\Sigma x = 6 + 2 + 4 + 3 + 4 + 5 + 4 + 5$ $= 33$
2 Count the number of scores, that is, n.	$n = 8$
3 Write the rule for the mean.	$\bar{x} = \frac{\Sigma x}{n}$
4 Substitute the known values into the rule.	$= \frac{33}{8}$
5 Evaluate.	$= 4.125$
6 Answer the question.	The mean is 4.125.
b 1 Write the median scores in ascending numerical order.	**b** 2 3 4 4 4 5 5 6
2 Locate the position of the median using the rule $\frac{n+1}{2}$, where $n = 8$. This places the median as the 4.5th score, that is, between the 4th and 5th score.	Median $= \frac{n+1}{2}$th score $= \frac{8+1}{2}$th score $= 4.5$th score 2 3 4 4 4 5 5 6
3 Obtain the average of the two middle scores.	Median $= \frac{4+4}{2}$ $= \frac{8}{2}$ $= 4$
4 Answer the question.	The median is 4.
c 1 Determine the highest and lowest values	Lowest value = 2 Highest value = 6
2 Write the rule for the range.	$R = x_H - x_L$ $= 6 - 2$ $= 4$
3 Answer the question.	The range is 4.

5.2.3 Calculating the mean, median and range from a frequency distribution table

- If data are presented in a frequency distribution table, the formula used to calculate the mean is $\bar{x} = \frac{\Sigma(f \times x)}{n}$.
- Here, each value (score) in the table is multiplied by its corresponding frequency, then all the $f \times x$ products are added together and the total sum is divided by the number of observations in the set.
- To find the median, find the position of each score from the cumulative frequency column.
- The range is the highest value minus the lowest value.

WORKED EXAMPLE 2

For the table shown, find:
a the mean **b the median** **c the range.**

Score (x)	Frequency (f)
4	**1**
5	**2**
6	**5**
7	**4**
8	**3**
Total	**15**

THINK

1 Rule up a table with four columns titled Score (x), Frequency (f), Frequency × score ($f \times x$) and Cumulative frequency (cf).

2 Enter the data and complete both the $f \times x$ and cumulative frequency columns.

WRITE

Score (x)	Frequency (f)	Frequency × score ($f \times x$)	Cumulative frequency (cf)
4	1	4	1
5	2	10	1 + 2 = 3
6	5	30	3 + 5 = 8
7	4	28	8 + 4 = 12
8	3	24	12 + 3 = 15
	$n = 15$	$\Sigma(f \times x) = 96$	

a 1 Write the rule for the mean.

a $\bar{x} = \frac{\Sigma(f \times x)}{n}$

2 Substitute the known values into the rule and evaluate.

$\bar{x} = \frac{96}{15}$
$= 6.4$

3 Answer the question.

The mean of the data set is 6.4.

b 1 Locate the position of the median using the rule $\frac{n+1}{2}$, where $n = 15$. This places the median as the 8th score.

b The median is the $\frac{15+1}{2}$th or 8th score.

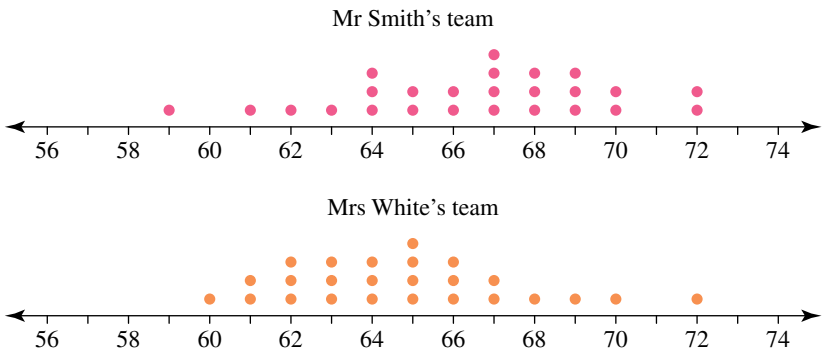
2	Use the cumulative frequency column to find the 8th score and answer the question.	The median of the data set is 6.
c 1	The range is the difference between the highest value and the lowest value.	**c** $R = 8 - 4$ $= 4$
2	Answer the question.	The range of the data set is 4.

5.2.4 Data display: Back-to-back stem-and-leaf plots and parallel dot plots

- **Back-to-back stem-and-leaf plots** and **parallel dot plots** are graphs that allow us to compare two sets of data on the same graph. (For a review of dot plots, see section 5.4.5.)
- A back-to-back stem-and-leaf plot has a central stem with leaves on either side. The stem-and-leaf plot shown gives the age of members of two teams competing in a bowling tournament.

Club A	*Stem*	*Club B*
1	4	
	5	5 7 8 9
	6	0 2 3 5 7
6 5 4 3	7	0 1 2
8 6 5 4 3 2 1	8	8
	9	0

- From a study of the stem-and-leaf plot, it can be seen that Club A has one relatively young member, age 41, but all other members are in their 70s and 80s. Most of the members of Club B are in their 50s and 60s, so it appears that overall Club B has a younger membership.
- Parallel dot plots are drawn one above the other, using the same scale. The set of parallel dot plots shown provides a means of comparing the results of two teams in a basketball tournament.



- Each team has played 25 games (counting up the dots for each team). From a study of the dots plots, it appears that Mr Smith's team outperformed Mrs White's team, as there are more high scores for Mr Smith's team.
- Calculating the mean, median and range for distributions provides us with statistical measures of comparison. These measures are more precise and more useful than the information we can obtain by visually comparing shapes of distributions.

Comparing data displays using mean, median and range

- When the mean and median are the same value, the data are symmetrical around this value. In this case, either the mean or the median can be used to represent the centre of the data. The median is the middle value; hence, there are the same numbers of values above and below it.
- Very large or very small values make smaller differences to the median than the mean.
- The range is an indication of the spread of the data.
- Comparisons can be made between two like sets of data by referring to the mean, median and range.

WORKED EXAMPLE 3

The ages of male and female groups using a ten-pin bowling centre are displayed in the back-to-back stem-and-leaf plot shown.
For each group:
a determine the mean
b determine the median
c determine the range
d use the mean, median and range to compare the two sets of data.
Key: 1 \| 5 = 15

Leaf: female	*Stem*	*Leaf: male*
9 8 7 7 6 6	1	5 5 6 6 6 7 7 7 8 9
9 8 5 3 3 3 2 1 1 0	2	1 1 2 2 2 3 3 5 7 8
8 8 6 4 2 2 1	3	0 0 1 4 5
5 5 3	4	3 6 8
2	5	0 4 7
0	6	5

THINK

a 1 Count the number of males and females and total the ages of males and females.

2 Determine the mean for each of the two groups by applying the formula $\bar{x} = \frac{\Sigma x}{n}$.

3 Write the answer.

WRITE

Number of females = 28
Number of males = 32
Total of female ages = 824
Total of male ages = 923

$\bar{x}_F = \frac{\Sigma x}{n}$
$= \frac{824}{28}$
$= 29.43$
$\bar{x}_M = \frac{\Sigma x}{n}$
$= \frac{923}{32}$
$= 28.84$

The mean of the female group is 29.43.
The mean of the male group is 28.84.

b 1 The median is the middle value of the ordered data set.
For the female group there are 28 values, so the median falls between the 14th and 15th values.
For the male group there are 32 people so the median falls between the 16th and 17th values.

$$\text{Median}_F = \frac{25 + 28}{2} = 26.5$$

$$\text{Median}_M = \frac{23 + 23}{2} = 23$$

2 Write the answer.

The median of the female group is 26.5.
The median of the male group is 23.

c 1 The range is the difference between the highest and lowest value.

$$\text{Range}_F = 60 - 16 = 44$$

$$\text{Range}_M = 65 - 15 = 50$$

2 Write the answer.

The range of the female group is 44.
The range of the male group is 50.

d 1 Compare the two data sets.

There is a more diverse age range of males playing than females as the range in age for males is 50 whereas for females it is 44. The mean age of the females and males are 29.43 and 28.83 respectively, which are very similar, but when comparing the median age, the female median age is 2.5 years older. This reinforces what the stem-and-leaf plot shows: that tenpin bowling is more popular for females in their 20s and 30s and males in their teens and 20s.

learnON RESOURCES — ONLINE ONLY

Interactivity: Mean (int-3818)
Interactivity: Median (int-3819)
eLesson: Mean and median (eles-1905)
Digital doc: SkillSHEET Finding the mean of a small data set (doc-5299)
Digital doc: SkillSHEET Finding the median of a small data set (doc-5300)

Exercise 5.2 Comparing data displays using mean, median and range

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–10, 12	Questions: 1–10, 12, 14	Questions: 1–16

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE1 For each of the following sets of data, find the:

i. mean **ii.** median **iii.** range.

a. 3, 5, 6, 8, 8, 9, 10

b. 4, 6, 7, 4, 8, 9, 7, 10

c. 17, 15, 48, 23, 41, 56, 61, 52

d. 4.5, 4.7, 4.8, 4.8, 4.9, 5.0, 5.3

e. $7\frac{1}{2}$, $10\frac{1}{4}$, 12, $12\frac{1}{4}$, 13, $13\frac{1}{2}$, $13\frac{1}{2}$, 14

2. The following back-to-back stem-and-leaf plot displays the heights of a group of students in a class.
Key: 13 | 7 = 137 cm

Leaf: boys	*Stem*	*Leaf: girls*
9 8	13	7 8
9 8 8 7 6	14	3 5 6
9 8 8	15	1 2 3 7
7 6 6 5	16	3 5 6
8 7 6	17	1

a. The total number of students in the class is:

A. 13 **B.** 17 **C.** 30
D. 36 **E.** 40

b. The range in heights of boys and girls respectively is:

A. 34 and 40 **B.** 41 and 35 **C.** 35 and 41
D. 40 and 34 **E.** 138 and 178

c. The mean height of boys and girls (correct to the nearest whole number) respectively is:

A. 150 and 145 **B.** 134 and 148 **C.** 148 and 134
D. 153 and 158 **E.** 158 and 153

d. The median height of boys and girls respectively is:

A. 158 and 152 **B.** 152 and 158 **C.** 149 and 134
D. 134 and 149 **E.** 150 and 145

3. The following back-to-back stem-and-leaf plot shows the test results of 25 Year 10 students in Mathematics and Science. Find the mean, median and range for each of the two subjects.
Key: 3 | 2 = 32

Leaf: Science	*Stem*	*Leaf: Mathematics*
8 7 3	3	2 9
9 6 2 2 1	4	0 6 8
8 7 6 1 1 0	5	1 3 5
9 7 4 3 2	6	2 6 7 9
8 5 1 0	7	3 6 7 8
7 3	8	0 4 4 6 8 9
	9	2 5 8

4. WE2 For the data shown in each of the following frequency distribution tables, find the:

i. mean
ii. median
iii. range.

a.

Score (x)	Frequency (f)
4	3
5	6
6	9
7	4
8	2
Total	24

b.

Score (x)	Frequency (f)
12	4
13	5
14	10
15	12
16	9
Total	40

5. The following data show the number of bedrooms in each of the 10 houses in a particular neighbourhood: 2, 1, 3, 4, 2, 3, 2, 2, 3, 3.

a. Calculate the mean and median number of bedrooms.

b. A local motel contains 20 rooms. Add this observation to the set of data and recalculate the values of the mean and median.

c. Compare the answers obtained in parts **a** and **b** and complete the following statement: When the set of data contains an unusually large value(s), called an outlier, the _____ (mean/median) is the better measure of central tendency, as it is less affected by this extreme value.

6. A random sample was taken, composed of 30 people shopping at a Coles supermarket on a Tuesday night. The amount of money (to the nearest dollar) spent by each person was recorded as follows:
6, 32, 66, 17, 45, 1, 19, 52, 36, 23, 28, 20, 7, 47, 39
6, 68, 28, 54, 9, 10, 58, 40, 12, 25, 49, 74, 63, 41, 13
Find the mean and median amount of money spent at the checkout by the people in this sample.

7. a. Add one more number to the set of data 3, 4, 4, 6 so that the mean of a new set is equal to its median.

b. Design a set of five numbers so that mean = median = 5.

c. In the set of numbers 2, 5, 8, 10, 15, change one number so that the median remains unchanged while the mean increases by 1.

Communicating, reasoning and problem solving

8. The batting scores for two cricket players over 6 innings are as follows:
Player A: 31, 34, 42, 28, 30, 41
Player B: 0, 0, 1, 0, 250, 0

a. Find the mean score for each player.
b. Which player appears to be better, based upon mean result?
c. Find the median score for each player.
d. Which player appears to be better when the decision is based on the median result?
e. Which player do you think would be the most useful to have in a cricket team and why? How can the mean result sometimes lead to a misleading conclusion?

9. The number of goals scored in football matches by Micky and Yani were recorded as follows.

Micky	0	3	3	0	1	2	0	0	0	1
Yani	1	2	0	1	0	1	2	2	1	1

a. Calculate the mean for each player.
b. Calculate the median for each player.
c. Who is the more consistent performer? Explain.

10. Percentages in a mathematics exam for two classes were as follows.

10A	32	65	60	54	85	73	67	65	49	96	57	68
10B	46	74	62	78	55	73	60	75	73	77	68	81

a. Display the data on a back-to-back stem-and-leaf plot.
b. Comment on the distribution.
c. Calculate the mean percentage for each class.
d. Calculate the median percentage for each class.
e. Compare class performance.

11. MC In a set of data there is one score that is extremely small when compared to all the others. This outlying value is most likely to:
A. have greatest effect upon the mean of the data.
B. have greatest effect upon the median of the data.
C. have greatest effect upon the mode of the data.
D. have very little effect on any of the statistics as we are told that the number is extremely small.
E. have equal effect on the mean, median and mode of the data.

12. The following frequency table gives the number of employees in different salary brackets for a small manufacturing plant.

Position	Salary ($)	Number of employees
Machine operator	18000	50
Machine mechanic	20000	15
Floor steward	24000	10
Manager	62000	4
Chief executive officer	80000	1

a. Workers are arguing for a pay rise but the management of the factory claims that workers are well paid because the mean salary of the factory is $22 100. Are they being honest?
b. Suppose that you were representing the factory workers and had to write a short submission in support of the pay rise. How could you explain the management's claim? Quote some other statistics in favour of your case.

13. Design a set of five numbers with:
- **a.** mean = median
- **b.** mean > median
- **c.** mean < median

14. The numbers 15, a, 17, b, 22, c, 10 and d have a mean of 14. Find the mean of a, b, c and d.

15. The numbers m, n, p, q, r and s have a mean of a, and x, y and z have a mean of b. Find the mean of all nine numbers.

16. Under what circumstances might the median be a more reliable measure of centre than the mean?

5.3 Quartiles and interquartile range [Stage 5.2]

5.3.1 Measures of spread

- **Measures of spread** describe how far data values are spread from the centre or from each other.
- A music store proprietor has stores in Newcastle and Wollongong. The number of CDs sold each day over one week is recorded below.

Newcastle: 45, 60, 50, 55, 48, 40, 52

Wollongong: 20, 85, 50, 15, 30, 60, 90

- In each of these data sets consider the measures of central tendency.

Newcastle:	Mean = 50	Wollongong:	Mean = 50
	Median = 50		Median = 50

- With these measures being the same for both data sets we could come to the conclusion that both data sets are very similar. However, if we look at the data sets, they are very different. We can see that the data for Newcastle are very clustered around the mean, whereas the Wollongong data are spread out more.
- The data from Newcastle are between 45 and 60, whereas the Wollongong data are between 15 and 90.
- Range and **interquartile range (IQR)** are both measures of spread.

5.3.2 Review of the range

- The most basic measure of spread is the range. It is defined as the difference between the highest and the lowest values in the set of data.

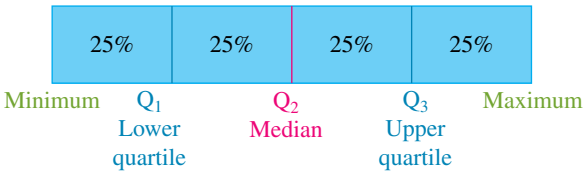
$$\text{Range} = \text{highest score} - \text{lowest score}$$
$$\Rightarrow \text{Range} = X_{\max} - X_{\min}$$

WORKED EXAMPLE 4

Find the range of the given data set: 2.1, 3.5, 3.9, 4.0, 4.7, 4.8, 5.2.

THINK	WRITE
1 Identify the lowest score (X_{min}) of the data set.	Lowest score = 2.1
2 Identify the highest score (X_{max}) of the data set.	Highest score = 5.2
3 Write the rule for the range.	Range = $X_{max} - X_{min}$
4 Substitute the known values into the rule.	$= 5.2 - 2.1$
5 Evaluate.	$= 3.1$

5.3.3 Interquartile range



- The interquartile range (IQR) is the range of the middle 50% of all the scores in an ordered set.
- When calculating the interquartile range, the data are first organised into quartiles, each containing 25% of the data. The word 'quartile' comes from the word 'quarter'.
- Interquartile range = upper quartile − lower quartile
- This can be written as:
  $IQR = Q_{upper} - Q_{lower}$
  or
  $IQR = Q_3 - Q_1$
- The IQR is not affected by extremely large or extremely small data values (**outliers**), so in some circumstances the IQR is a better indicator of the spread of data than the range.

WORKED EXAMPLE 5

Calculate the interquartile range (IQR) of the following set of data: 3, 2, 8, 6, 1, 5, 3, 7, 6.

THINK	WRITE
1 Arrange the scores in order.	123356678
2 Locate the median and use it to divide the data into two halves. *Note:* The median is the 5th score in this data set and should not be included in the lower or upper ends of the data.	1233 5 6678
3 Find Q_1, the median of the lower half of the data.	$Q_1 = \frac{2+3}{2}$ $= \frac{5}{2}$ $= 2.5$
4 Find Q_3, the median of the upper half of the data.	$Q_3 = \frac{6+7}{2}$ $= \frac{13}{2}$ $= 6.5$
5 Calculate the interquartile range.	$IQR = Q_3 - Q_1$ $= 6.5 - 2.5$ $= 4$

DISCUSSION

Both the range and the interquartile range can be used to represent a set of data. In your own words, explain the difference between the two measures and what they indicate about a set of data.

learnON RESOURCES — ONLINE ONLY

Interactivity: Range (int-3822)

Interactivity: The interquartile range (int-4813)

Exercise 5.3 Quartiles and interquartile range

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–7, 10	Questions: 1–8, 10	Questions: 1–10

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE4 Find the range for each of the following sets of data.

a. 4, 3, 9, 12, 8, 17, 2, 16

b. 49.5, 13.7, 12.3, 36.5, 89.4, 27.8, 53.4, 66.8

c. $7\frac{1}{2}$, $12\frac{3}{4}$, $5\frac{1}{4}$, $8\frac{2}{3}$, $9\frac{1}{6}$, $3\frac{3}{4}$

2. WE5 Calculate the interquartile range (IQR) for each of the following sets of data.

a. 3, 5, 8, 9, 12, 14

b. 7, 10, 11, 14, 17, 23

c. 66, 68, 68, 70, 71, 74, 79, 80

d. 19, 25, 72, 44, 68, 24, 51, 59, 36

3. The following stem-and-leaf plot shows the mass of newborn babies (rounded to the nearest 100 g). Find:

Key: 1* | 9 = 1.9 kg

Stem	*Leaf*
1*	9
2	2 4
2*	6 7 8 9
3	0 0 1 2 3 4
3*	5 5 6 7 8 8 8 9
4	0 1 3 4 4
4*	5 6 6 8 9
5	0 1 2 2

a. the range of the data

b. the IQR of the data.

4. MC Calculate the interquartile range of the following data.
17, 18, 18, 19, 20, 21, 21, 23, 25

A. 8 **B.** 18 **C.** 4 **D.** 20 **E.** **16**

5. For each of the following sets of data, state:
i. the range **ii.** the IQR.

a. 6, 9, 12, 13, 20, 22, 26, 29

b. 7, 15, 2, 26, 47, 19, 9, 33, 38

c. 120, 99, 101, 136, 119, 87, 123, 115, 107, 100

Communicating, reasoning and problem solving

6. As newly appointed coach of Terrorolo's Meteors netball team, Kate decided to record each player's statistics for the previous season. The number of goals scored by the leading goal shooter was:
1, 3, 8, 18, 19, 23, 25, 25, 25, 26, 27, 28, 28, 28, 28, 29, 29, 30, 30, 33, 35, 36, 37, 40.

a. Find the mean of the data.

b. Find the median of the data.

c. Find the range of the data.

d. Find the interquartile range of the data.

e. There are three scores that are much lower than most. Explain the effect these scores have on the summary statistics.

7. The following back-to-back stem-and-leaf plot shows the ages of 30 pairs of men and women when entering their first marriage.
Key: 1 | 6 = 16 years old

Leaf: Men	*Stem*	*Leaf: Women*
9 9 8	1	6 7 7 8 9
9 9 8 8 7 6 4 4 3 2 0	2	0 0 1 2 3 4 5 6 7 7 8 9
9 8 8 8 6 5 5 4 3 2	3	0 1 2 2 3 4 7 9
6 3 0 0	4	1 2 4 8
6 0	5	2

a. Find the mean, median, range and interquartile range of each set.

b. Write a short paragraph comparing the two distributions.

8. Find the mean, median, mode, range and IQR of the following data collected when the temperature of the soil around 25 germinating seedlings was recorded.
28.9, 27.4, 23.6, 25.6, 21.1, 22.9, 29.6, 25.7, 27.4, 23.6, 22.4, 24.6, 21.8, 26.4, 24.9, 25.0, 23.5, 26.1, 23.6, 25.3, 29.5, 23.5, 22.0, 27.9, 23.6

9. Four positive numbers a, b, c and d have a mean of 12, a median 9 and a range of 14. Find the values of a, b, c and d.

10. Explain when the range or the interquartile range would be a better measure of spread when comparing sets of data.

5.4 Constructing, interpreting and comparing boxplots [Stage 5.2]

5.4.1 The five-number summary

- A five-number summary is a list consisting of the lowest score, lower quartile, median, upper quartile and greatest score of a set of data.

X_{min}	Q_1	Median (Q_2)	Q_3	X_{max}

WORKED EXAMPLE 6

From the following five-number summary, find:
a the interquartile range
b the range.

X_{min}	Q_1	Median (Q_2)	Q_3	X_{max}
29	37	39	44	48

THINK

a The interquartile range is the difference between the upper and lower quartiles.

b The range is the difference between the greatest score and the lowest score.

WRITE

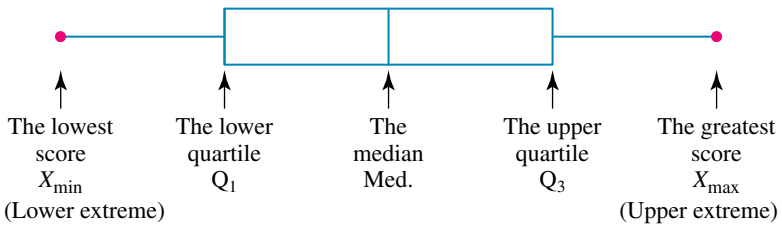
$Q_3 = 44$, $X_{max} = 48$

a $\text{IQR} = Q_3 - Q_1$
$= 44 - 37$
$= 7$

b $\text{Range} = X_{max} - X_{min}$
$= 48 - 29$
$= 19$

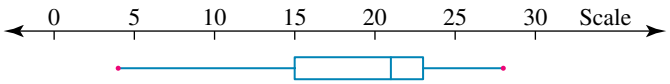
5.4.2 Boxplots

- A **boxplot** (or box-and-whisker plot) is a graph of the five-number summary.
- Boxplots consist of a central divided box with attached whiskers.
- The box spans the interquartile range.
- The median is marked by a vertical line drawn inside the box.
- The whiskers indicate the range of scores.



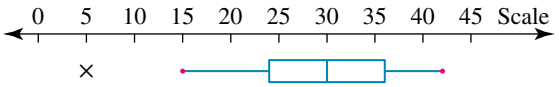
- Box-and-whisker plots are *always drawn to scale*.

- They are presented either with the five-number summary figures attached as labels (as in the diagram above) or with a scale presented alongside the box-and-whisker plot (as in the diagram below). They can also be drawn vertically.



5.4.3 Identification of extreme values

- If an extreme value or outlier occurs in a set of data, it can be denoted by a small cross on the box-and-whisker plot. The whisker is then shortened to the next largest (or smallest) figure.
- The box-and-whisker plot below shows that the lowest score was 5. This was an extreme value as the rest of the scores were located within the range 15 to 42.

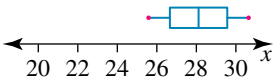


5.4.4 Describing distributions

Symmetry and skewness

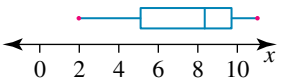
- A **symmetrical** plot has data that are evenly spaced around a central point. Examples of a stem-and-leaf plot and a symmetrical boxplot are shown below.

Stem	*Leaf*
26*	6
27	0 1 3
27*	5 6 8 9
28	0 1 1 1 2 4
28*	5 7 8 8
29	2 2 2
29*	5



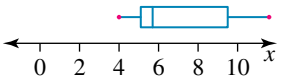
- A **negatively skewed** plot has larger amounts of data at the higher end. This is illustrated by the stem-and-leaf plot below where the leaves increase in length as the data increase in value. It is illustrated on the boxplot when the median is to the right within the box.

Stem	*Leaf*
5	1
6	2 9
7	1 1 2 2
8	1 4 4 5 6 6
9	5 3 4 4 5 6 7 7 7



- A **positively skewed** plot has larger amounts of data at the lower end. This is illustrated on the stem-and-leaf plot below where the leaves increase in length as the data decrease in value. It is illustrated on the boxplot when the median is to the left within the box.

Stem	*Leaf*
5	1 3 4 4 5 6 7 7 7
6	2 4 4 5 6 6
7	1 1 2 2
8	1 6
9	5



WORKED EXAMPLE 7

The stem-and-leaf plot gives the speed of 25 cars caught by a roadside speed camera.

Key: 8 | 2 = 82 km/h, 8* | 6 = 86 km/h

Stem	*Leaf*
8	**2 2 4 4 4 4**
8*	**5 5 6 6 7 9 9 9**
9	**0 1 1 2 4**
9*	**5 6 9**
10	**0 2**
10*	
11	**4**

a Prepare a five-number summary of the data.
b Draw a box-and-whisker plot of the data. (Identify any extreme values.)
c Describe the distribution of the data.

THINK

1 First identify the positions of the median and upper and lower quartiles. There are 25 data values. The median is the $\frac{n+1}{2}$th score. The lower quartile is the median of the lower half of the data. The upper quartile is the median of the upper half of the data (each half contains 12 scores).

2 Mark the positions of the median and upper and lower quartiles on the stem-and-leaf plot.

WRITE

The median is the $\frac{25+1}{2}$th score — that is, the 13th score.

Q_1 is the $\frac{25+1}{2}$th score in the lower half — that is, the 6.5th score. That is, halfway between the 6th and 7th scores.

Q_3 is halfway between the 6th and 7th scores in the upper half of the data.

Key: 8 | 2 = 82 km/h
8* | 6 = 86 km/h

Stem	*Leaf*	
8	2 2 4 4 4 4 1	Q_1
8*	5 5 6 6 7 9 ⑨ 9	Median
9	0 1 1 2 4 1	
9*	5 6 9	
10	0 2	Q_3
10*		
11	4	

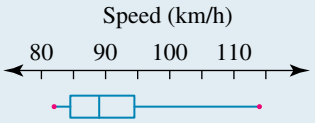
a Write the five-number summary:
The lowest score is 82.
The lower quartile is between 84 and 85; that is, 84.5.
The median is 89.
The upper quartile is between 94 and 95; that is, 94.5.
The greatest score is 114.

a Five-number summary:

X_{min}	Q_1	Median	Q_3	X_{max}
82	84.5	89	94.5	114

b Draw a labelled axis using an appropriate scale. Plot the points from the five-number summary.

b



c Describe the distribution.

c The data are skewed (positively).

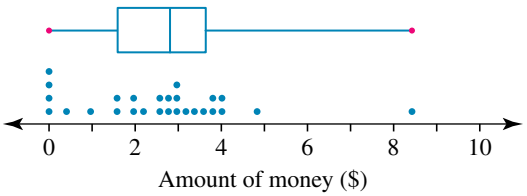
5.4.5 Shapes of graphs

Boxplots and dot plots

- Boxplots are a concise summary of data. A boxplot can be directly related to a **dot plot**.
- Dot plots display each data value represented by a dot placed on a number line.
- The following data are the amount of money (in $) that a group of 27 five-year-olds had with them on a day visiting the zoo with their parents.

0	1.65	0	2.60	3	8.45	4	0.55	4.10	3.35	3.25
2	2.85	2.90	1.70	3.65	1	0	0	2.25	2.05	3
3.80	2.65	4.75	3.90	2.95						

- The dot plot below and its comparative boxplot show the distribution of these data.



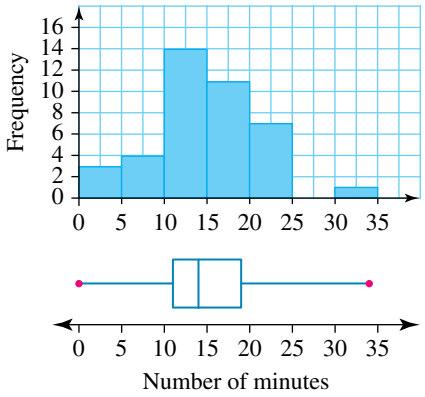
- Both graphs indicate that the data are positively skewed. The dot plot clearly shows the presence of the outlier. This is less obvious with the boxplot. However, the boxplot provides an excellent summary of the centre and spread of the distribution.

Boxplots and histograms

- Histograms are graphs that display continuous numerical variables and do not retain all original data.
- The following data are the number of minutes, rounded to the nearest minute, that forty Year 10 students take to travel to their school on a particular day.

15	22	14	12	21	34	19	11	13	0	16
4	23	8	12	18	24	17	14	3	10	12
9	15	20	5	19	13	17	11	16	19	24
12	7	14	17	10	14	23				

- The data are displayed in the histogram and boxplot shown.



- Both graphs indicate that the data are slightly positively skewed. The histogram clearly shows the frequencies of each class interval. Neither graph displays the original values. The histogram does not give precise information about the centre, but the distribution of the data is visible. However, the boxplot provides an excellent summary of the centre and spread of the distribution.

5.4.6 Parallel boxplots

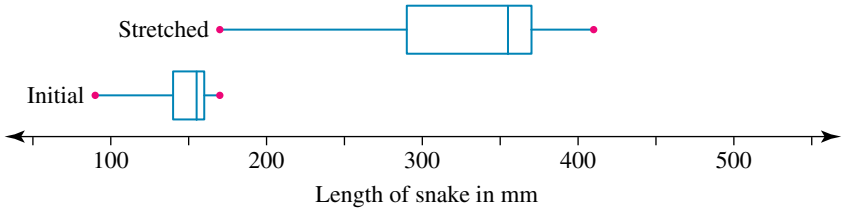
- A major reason for developing statistical skills is to be able to make comparisons between sets of data.
- Consider the following scenario.
  - Each member of a class was given a jelly snake to stretch. They each measured the initial length of their snake to the nearest centimetre and then slowly stretched the snake to make it as long as possible. They then measured the maximum length of the snake by recording how far it had stretched at the time it broke. The results were recorded in the following table.

Initial length (cm)	Stretched length (cm)	Initial length (cm)	Stretched length (cm)
13	29	14	27
14	28	13	27
17	36	15	36
10	24	16	36
14	35	15	36
16	36	16	34
15	37	17	35
16	37	12	27

(*continued*)

Initial length (cm)	Stretched length (cm)	Initial length (cm)	Stretched length (cm)
14	30	9	17
16	33	16	41
17	36	17	38
16	38	16	36
17	38	17	41
14	31	16	33
17	40	11	21

– The data were then displayed on **parallel boxplots**, with the axis displaying in millimetres.
– By drawing the two boxplots on a single axis, it is easy to compare them.



- The change in the length of the snake when stretched is evidenced by the increased median and spread shown on the boxplots. The median snake length before being stretched was 150 mm, but the median snake length after being stretched was 350 mm. The range increased after stretching, as did the IQR.

DISCUSSION

Explain the usefulness of having parallel boxplots to easily compare two sets of data. What information can and can't easily be obtained just from looking at the boxplots compared to studying the data in a histogram?

learnON RESOURCES — ONLINE ONLY

Interactivity: Boxplots (int-6245)
Interactivity: Skewness (int-3823)
Interactivity: Parallel boxplots (int-6248)

Exercise 5.4 Constructing, interpreting and comparing boxplots

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–7, 10, 13, 16, 19	Questions: 1–8, 10–12, 14, 16, 19	Questions: 1–21

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE6 From the following five-number summary, find:

X_{min}	Q_1	Median	Q_3	X_{max}
6	11	13	16	32

a. the interquartile range
b. the range.

2. From the following five-number summary, find:

X_{min}	Q_1	Median	Q_3	X_{max}
101	119	122	125	128

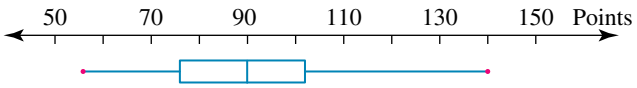
a. the interquartile range
b. the range.

3. From the following five-number summary, find:

X_{min}	Q_1	Median	Q_3	X_{max}
39.2	46.5	49.0	52.3	57.8

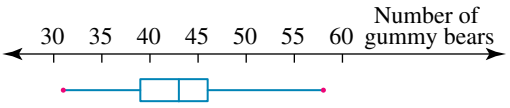
a. the interquartile range
b. the range.

4. The following box-and-whisker plot shows the distribution of final points scored by a football team over a season's roster.



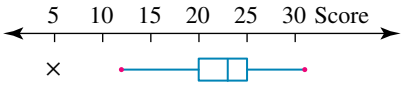
a. What was the team's greatest points score?
b. What was the team's least points score?
c. What was the team's median points score?
d. What was the range of points scored?
e. What was the interquartile range of points scored?

5. The following box-and-whisker plot shows the distribution of data formed by counting the number of gummy bears in each of a large sample of packs.



a. What was the largest number of gummy bears in any pack?
b. What was the smallest number of gummy bears in any pack?
c. What was the median number of gummy bears in any pack?
d. What was the range of numbers of gummy bears per pack?
e. What was the interquartile range of gummy bears per pack?

Questions 6–8 refer to the following box-and-whisker plot.



6. MC The median of the data is:

A. 5 **B.** 20 **C.** 23 **D.** 25 **E.** 31

7. MC The interquartile range of the data is:

A. 5 **B.** 23 **C.** 26 **D.** 20 to 25 **E.** 12 to 31

8. MC Which of the following is not true of the data represented by the box-and-whisker plot?

A. One-quarter of the scores are between 5 and 20.

B. Half of the scores are between 20 and 25.

C. The lowest quarter of the data is spread over a wide range.

D. Most of the data are contained between the scores of 5 and 20.

E. The third quartile is 25.

9. The number of sales made each day by a salesperson is recorded over a 2-week period:

25, 31, 28, 43, 37, 43, 22, 45, 48, 33

a. Prepare a five-number summary of the data. (There is no need to draw a stem-and-leaf plot of the data. Just arrange them in order of size.)

b. Draw a box-and-whisker plot of the data.

10. The data in the table show monthly rainfall in millimetres.

J	F	M	A	M	J	J	A	S	O	N	D
10	12	21	23	39	22	15	11	22	37	45	30

a. Prepare a five-number summary of the data.

b. Draw a box-and-whisker plot of the data.

11. WE7 The stem-and-leaf plot shown details the age of 25 offenders who were caught during random breath testing.

Key: 1 | 8 = 18 years

Stem	*Leaf*
1	8 8 9 9 9
2	0 0 0 1 1 3 4 6 9
3	0 1 2 7
4	2 5
5	3 6 8
6	6
7	4

a. Prepare a five-number summary of the data.

b. Draw a box-and-whisker plot of the data.

c. Describe the distribution of the data.

12. The following stem-and-leaf plot details the price at which 30 blocks of land in a particular suburb sold for.

Key: 12 | 4 = $124 000

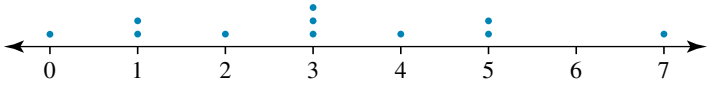
Stem	*Leaf*
12	4 7 9
13	0 0 2 5 5
14	0 0 2 3 5 5 7 9 9
15	0 0 2 3 7 7 8
16	0 2 2 5 8
17	5

a. Prepare a five-number summary of the data.

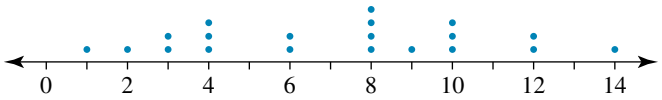
b. Draw a box-and-whisker plot of the data.

13. Prepare comparative boxplots for the following dot plots (using the same axis) and describe what each plot reveals about the data.

a. Number of sick days taken by workers last year at factory A



b. Number of sick days taken by workers last year at factory B



14. An investigation into the transport needs of an outer suburb community recorded the number of passengers boarding a bus during each of its journeys, as follows.
12, 43, 76, 24, 46, 24, 21, 46, 54, 109, 87, 23, 78, 37, 22, 139, 65, 78, 89, 52, 23, 30, 54, 56, 32, 66, 49
Display the data by constructing a histogram using class intervals of 20 and a comparative boxplot on the same axis.

15. At a weight-loss clinic, the following weights (in kilograms) were recorded before and after treatment.

Before	75	80	75	140	77	89	97	123	128	95	152	92
After	69	66	72	118	74	83	89	117	105	81	134	85

Before	85	90	95	132	87	109	87	129	135	85	137	102
After	79	84	90	124	83	102	84	115	125	81	123	94

a. Prepare a five-number summary for weight before and after treatment.
b. Draw parallel boxplots for weight before and after treatment.
c. Comment on the comparison of weights before and after treatment.

Communicating, reasoning and problem solving

16. The following data detail the number of hamburgers sold by a fast food outlet every day over a 4-week period.

M	T	W	T	F	S	S
125	144	132	148	187	172	181
134	157	152	126	155	183	188
131	121	165	129	143	182	181
152	163	150	148	152	179	181

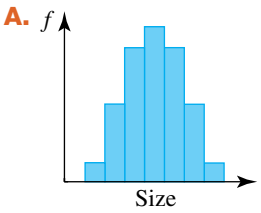
a. Prepare a stem-and-leaf plot of the data. (Use a class size of 10.)
b. Draw a box-and-whisker plot of the data.
c. What do these graphs tell you about hamburger sales?

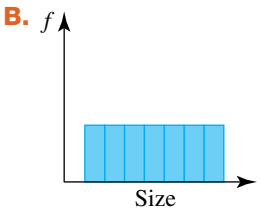
17. The following data show the ages of 30 mothers upon the birth of their first baby.

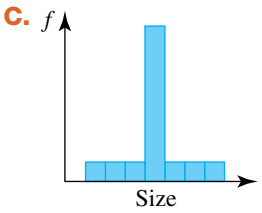
22	21	18	33	17	23	22	24	24	20
25	29	32	18	19	22	23	24	28	20
31	22	19	17	23	48	25	18	23	20

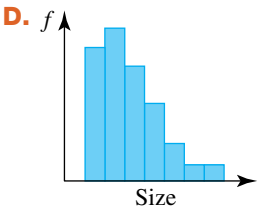
a. Prepare a stem-and-leaf plot of the data. (Use a class size of 5.)
b. Draw a box-and-whisker plot of the data. Indicate any extreme values appropriately.
c. Describe the distribution in words. What does the distribution say about the age that mothers have their first baby?

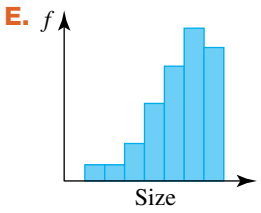
18. MC Match the box-and-whisker plot with its most likely histogram.



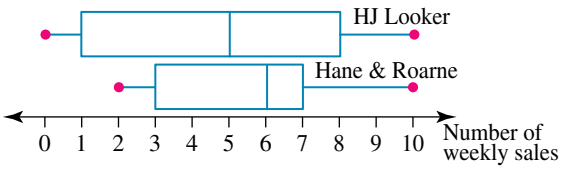






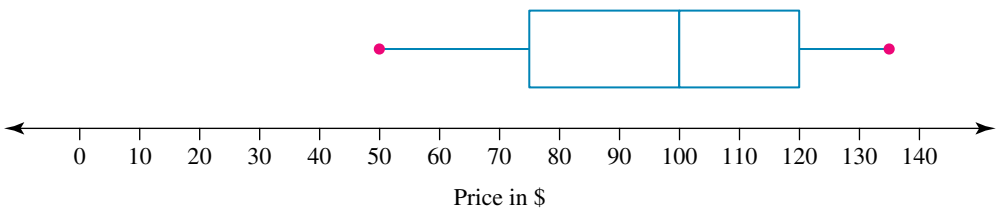


19. Consider the box-and-whisker plot below which shows the number of weekly sales of houses by two real estate agencies.



a. What is the median number of weekly sales for each real estate agency?
b. Which agency had the greater range of sales?
c. Which agency had the greater interquartile range of sales?
d. Which agency performed better? Explain your answer.

20. Fifteen French restaurants were visited by three newspaper restaurant reviewers. The average price of a meal for a single person was investigated. The following box-and-whisker plot shows the results.



a. What was the price of the cheapest meal?
b. What was the price of the most expensive meal?
c. What is the median cost of a meal?
d. What is the interquartile range for the price of a meal?
e. What percentage of the prices were below the median?

21. What advantages and disadvantages do box-and-whisker plots have as a visual form of representing data?

5.5 The standard deviation [Stage 5.3]

5.5.1 Defining the standard deviation

- The **standard deviation** for a set of data is a measure of how far the data values are spread out (how far they deviate) from the mean.
- **Deviation** is the difference between each data value and the mean. The standard deviation is calculated from the square of the deviations.
- Standard deviation is denoted by the Greek letter sigma, σ, and can be calculated by using the formula

$$\sigma = \sqrt{\frac{\Sigma(x - \bar{x})^2}{n}}$$

where $\bar{x}$ is the mean of the data values and n is the number of data values.
- A low standard deviation indicates that the data values tend to be close to the mean.
- A high standard deviation indicates that the data values tend to be spread out over a large range, away from the mean.
- Standard deviation can be calculated using a scientific calculator, or it can be calculated from a frequency table by following the steps below.

Step 1 Calculate the mean.
Step 2 Calculate the deviations.
Step 3 Square each deviation.
Step 4 Sum the squares.
Step 5 Divide the sum of the squares by the number of data values.
Step 6 Take the square root of the result.

WORKED EXAMPLE 8

The number of lollies in each of 8 packets is 11, 12, 13, 14, 16, 17, 18, 19. Calculate the mean and standard deviation correct to 2 decimal places.

THINK

1 Calculate the mean.

WRITE

$$\bar{x} = \frac{11 + 12 + 13 + 14 + 16 + 17 + 18 + 19}{8}$$
$$= \frac{120}{8}$$
$$= 15$$

2 To calculate the deviations $(x - \bar{x})$, set up a frequency table as shown and complete.

No. of lollies (x)	$(x - \bar{x})$
11	$11 - 15 = -4$
12	-3
13	-2
14	-1
16	1
17	2
18	3
19	4
Total	

3 Add another column to the table to calculate the square of the deviations, $(x - \bar{x})^2$.
Then sum the results: $\Sigma(x - \bar{x})^2$.

No. of lollies (x)	$(x - \bar{x})$	$(x - \bar{x})^2$
11	11–15 $= -4$	16
12	-3	9
13	-2	4
14	-1	1
16	1	1
17	2	4
18	3	9
19	4	16
Total		$\Sigma(x - \bar{x})^2 = 60$

4 To calculate the standard deviation, divide the sum of the squares by the number of data values, then take the square root of the result.

$$\sigma = \sqrt{\frac{\Sigma(x - \bar{x})^2}{n}}$$
$$= \sqrt{\frac{60}{8}}$$
≈ 2.74 (correct to 2 decimal places)

5 Check the result using a calculator.

The calculator returns an answer of $\sigma_n = 2.73861$.
Answer confirmed.

6 Interpret the result.

The average (mean) number of lollies in each pack is 15 with a standard deviation of 2.74, which means that the number of lollies in each pack differs from the mean by an average of 2.74.

- When calculating the standard deviation from a frequency table, the frequencies must be taken into account. Therefore, the following formula is used.

$$\sigma = \sqrt{\frac{\Sigma f(x - \bar{x})^2}{n}}$$

WORKED EXAMPLE 9

Lucy's scores in her last 12 games of golf were 87, 88, 88, 89, 90, 90, 90, 92, 93, 93, 95 and 97. Calculate the mean score and the standard deviation correct to 2 decimal places.

THINK

1 To calculate the mean, first set up a frequency table.

WRITE

Golf score (x)	Frequency (f)	fx
87	1	87
88	2	176
89	1	89
90	3	270
92	1	92
93	2	186
95	1	95
97	1	97
Total	$\Sigma f = 12$	$\Sigma fx = 1092$

2 Calculate the mean.

$$\bar{x} = \frac{\Sigma fx}{\Sigma f}$$
$$= \frac{1092}{12}$$
$$= 91$$

3 To calculate the deviations $(x - \bar{x})$, add another column to the frequency table and complete.

Golf score (x)	Frequency (f)	fx	$(x - \bar{x})$
87	1	87	$87 - 91 = -4$
88	2	176	-3
89	1	89	-2
90	3	270	-1
92	1	92	1
93	2	186	2
95	1	95	4
97	1	97	6
Total	$\Sigma f = 12$	$\Sigma fx = 1092$	

4 Add another column to the table and multiply the square of the deviations, $(x - \bar{x})^2$, by the frequency $f(x - \bar{x})^2$. Then sum the results: $\Sigma f(x - \bar{x})^2$.

Golf score (x)	Frequency (f)	fx	$(x - \bar{x})$	$f(x - \bar{x})^2$
87	1	87	$87 - 91 = -4$	$1 \times (-4)^2 = 16$
88	2	176	-3	18
89	1	89	-2	4
90	3	270	-1	3
92	1	92	1	1
93	2	186	2	8
95	1	95	4	16
97	1	97	6	36
Total	$\Sigma f = 12$	$\Sigma fx = 1092$		$\Sigma f(x - \bar{x})^2 = 102$

5 Calculate the standard deviation using the formula.

$$\sigma = \sqrt{\frac{\Sigma f(x - \bar{x})^2}{n}}$$
$$= \sqrt{\frac{120}{12}}$$
$$\approx 2.92$$

(correct to 2 decimal places)

6 Check the result using a calculator.

The calculator returns an answer of $\sigma_n = 2.91548$.
The answer is confirmed.

7 Interpret the result.

The average (mean) score for Lucy is 91 with a standard deviation of 2.92, which means that her score differs from the mean by an average of 2.92.

5.5.2 Why the deviations are squared

- For large data sets that are symmetrically distributed, the sum of the deviations is usually zero, that is, $\Sigma(x - \bar{x}) = 0$. When the mean is greater than the data value ($\bar{x} > x$), the deviation is negative. When the mean is smaller than the data value ($\bar{x} < x$), the deviation is positive. The negative and positive deviations cancel each other out; therefore, calculating the sum and average of the deviations is not useful. This explains why the standard deviation is calculated using the squares of the deviations, $(x - \bar{x})^2$, for all data values.

5.5.3 Standard deviations of populations and samples

- So far we have calculated the standard deviation for a population of data, that is, for complete sets of data. There is another formula for calculating standard deviation for samples of data, that is, data that have been randomly selected from a larger population.
- For example, a sample of 100 Year 10 students from New South Wales is taken to determine the amount of time they spend on their mobile phones. In this case, the standard deviation formula, denoted by s, that would apply is

$$s = \sqrt{\frac{\Sigma f(x - \bar{x})^2}{n - 1}}$$

- The calculator usually displays both values for the standard deviation, so it is important to understand the difference between them. However, in this course we will use the formula for populations, σ.

5.5.4 Effects on the standard deviation

- The standard deviation is affected by extreme values.

WORKED EXAMPLE 10

On a particular day Lucy played golf brilliantly and scored 60. The scores in her previous 12 games of golf were 87, 88, 88, 89, 90, 90, 90, 92, 93, 93, 95 and 97 (see Worked example 9). Comment on the effect this latest score has on the standard deviation.

THINK

1 Use a calculator to calculate the mean and the standard deviation.

2 Interpret the result and compare it to the results found in Worked example 9.

WRITE

$\bar{x} = 88.6154$ $\quad \sigma = 8.7225$
≈ 88.62 $\quad \approx 8.72$

In the first 12 games Lucy's mean score was 91 with a standard deviation of 2.92. This implied that Lucy's scores on average were 2.92 either side of her average of 91.
Lucy's latest performance resulted in a mean score of 88.62 with a standard deviation of 8.72. This indicates a slightly lower mean score, but the much higher standard deviation indicates that the data are now much more spread out and that the extremely good score of 60 is an anomaly.

5.5.5 Properties of the standard deviation

- If a constant c is added to all data values in a set, the deviations $(x - \bar{x})$ will remain unchanged and consequently the standard deviation remains unchanged.
- If all data values in a set are multiplied by a constant k, the deviations $(x - \bar{x})$ will be multiplied by k, that is $k(x - \bar{x})$. Consequently, the standard deviation is increased by a factor of k.
- Standard deviation can be used to measure consistency.
- When the standard deviation is low, we are able to say that the scores in the data set are more consistent with each other.

WORKED EXAMPLE 11

For the data 5, 9, 6, 11, 10, 7:
a calculate the standard deviation
b calculate the standard deviation if 4 is added to each data value. Comment on the effect.
c calculate the standard deviation if all data values are multiplied by 2. Comment on the effect.

THINK

a 1 Calculate the mean.

WRITE

a $\bar{x} = \dfrac{5 + 9 + 6 + 11 + 10 + 7}{6}$

$= 8$

2 Set up a frequency table and enter the squares of the deviations.

x	$x - \bar{x}$	$(x - \bar{x})^2$
5	$5 - 8 = -3$	9
6	-2	4
7	-1	1
9	1	1
10	2	4
11	3	9
Total		$\Sigma(x - \bar{x})^2 = 28$

3 To calculate the standard deviation, apply the formula for standard deviation.

$\sigma = \sqrt{\dfrac{\Sigma(x - \bar{x})^2}{n}}$

$= \sqrt{\dfrac{28}{6}}$

≈ 2.16

(correct to 2 decimal places)

b 1 Add 4 to each data value in the set.

b 9, 13, 10, 15, 14, 11

2 Calculate the mean.

$\bar{x} = \dfrac{9 + 13 + 10 + 15 + 14 + 11}{6}$

$= 12$

3 Set up a frequency table and enter the squares of the deviations.

x	$x - \bar{x}$	$(x - \bar{x})^2$
9	$9 - 12 = -3$	9
10	-2	4
11	-1	1
13	1	1
14	2	4
15	3	9
Total		$\Sigma(x - \bar{x})^2 = 28$

4 To calculate the standard deviation, apply the formula for standard deviation.

$$\sigma = \sqrt{\frac{\Sigma(x - \bar{x})^2}{n}}$$
$$= \sqrt{\frac{28}{6}}$$
$$\approx 2.16$$
(correct to 2 decimal places)

5 Comment on the effect of adding of 4 to each data value.

Adding 4 to each data value increased the mean but had no effect on the standard deviation, which remained at 2.16.

c 1 Multiply each data value in the set by 2.

c 10, 18, 12, 22, 20, 14

2 Calculate the mean.

$$\bar{x} = \frac{10 + 18 + 12 + 22 + 20 + 14}{6}$$
$$= 16$$

3 Set up a frequency table and enter the squares of the deviations.

x	$x - \bar{x}$	$(x - \bar{x})^2$
10	$10 - 16 = -6$	36
12	-4	16
14	-2	4
18	2	4
20	4	16
22	6	36
Total		$\Sigma(x - \bar{x})^2 = 112$

4 To calculate the standard deviation, apply the formula for standard deviation.

$$\sigma = \sqrt{\frac{\Sigma(x - \bar{x})^2}{n}}$$
$$= \sqrt{\frac{112}{6}}$$
$$\approx 4.32$$
(correct to 2 decimal places)

5 Comment on the effect of multiplying each data value by 2.

Multiplying each data value by 2 doubled the mean and doubled the standard deviation, which changed from 2.16 to 4.32.

ACTIVITY: DATA COLLECTION

Working in a small group, collect a set of data of interest about your year level.
For example, the amount of homework completed per week per student or the amount of time spent interacting with social media per week per student. Use a spreadsheet to calculate the standard deviation for this data. A set of at least 30 pieces of data should be collected.

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Interactivity: The standard deviation for a sample (int-4814)

Digital doc: WorkSHEET Univariate data I (doc-14595)

Exercise 5.5 The standard deviation

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–7, 9, 10, 13	Questions: 1–11, 13	Questions: 1–15

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE8 Calculate the standard deviation of each of the following data sets, correct to 2 decimal places.

a. 3, 5, 8, 2, 7, 1, 6, 5

b. 11, 8, 7, 12, 10, 11, 14

c. 25, 15, 78, 35, 56, 41, 17, 24

d. 5.2, 4.7, 5.1, 12.6, 4.8

2. WE9 Calculate the standard deviation of each of the following data sets, correct to 2 decimal places.

a.

Score (x)	Frequency (f)
1	1
2	5
3	9
4	7
5	3

b.

Score (x)	Frequency (f)
16	15
17	24
18	26
19	28
20	27

c.

Score (x)	Frequency (f)
8	15
10	19
12	18
14	7
16	6
18	2

d.

Score (x)	Frequency (f)
65	15
66	15
67	16
68	17
69	16
70	15
71	15
72	12

3. Complete the following frequency distribution table and use the table to calculate the standard deviation of the data set, correct to 2 decimal places.

Class	Class centre (x)	Frequency (f)
1–10		6
11–20		15
21–30		25
31–40		8
41–50		6

4. First-quarter profit increases for 8 leading companies are given below as percentages.

2.3 0.8 1.6 2.1 1.7 1.3 1.4 1.9

Calculate the standard deviation for this set of data and express your answer correct to 2 decimal places.

5. The heights in metres of a group of army recruits are given below.

1.8 1.95 1.87 1.77 1.75 1.79 1.81 1.83 1.76 1.80 1.92 1.87 1.85 1.83

Calculate the standard deviation for this set of data and express your answer correct to 2 decimal places.

6. Times (to the nearest tenth of a second) for the heats in the open 100 m sprint at the school sports are shown in the stem-and-leaf plot. Calculate the standard deviation for this set of data and express your answer correct to 2 decimal places.

Key: 11 | 0 = 11.0 s

Stem	*Leaf*
11	0
11	23
11	445
11	66
11	889
12	01
12	223
12	44
12	6
12	9

7. The number of outgoing phone calls from an office each day over a 4-week period is shown on the following stem plot. Calculate the standard deviation for this set of data and express your answer correct to 2 decimal places.

Key: 1 | 3 = 13 calls

Stem	*Leaf*
0	89
1	3479
2	01377
3	34
4	15678
5	38

8. MC A new legal aid service has been operational for only 5 weeks. The number of people who have made use of the service each day during this period is set out below.

Key: 1|6 = 16 people

Stem	*Leaf*
0	24
0	779
1	014444
1	5667889
2	122333
2	7

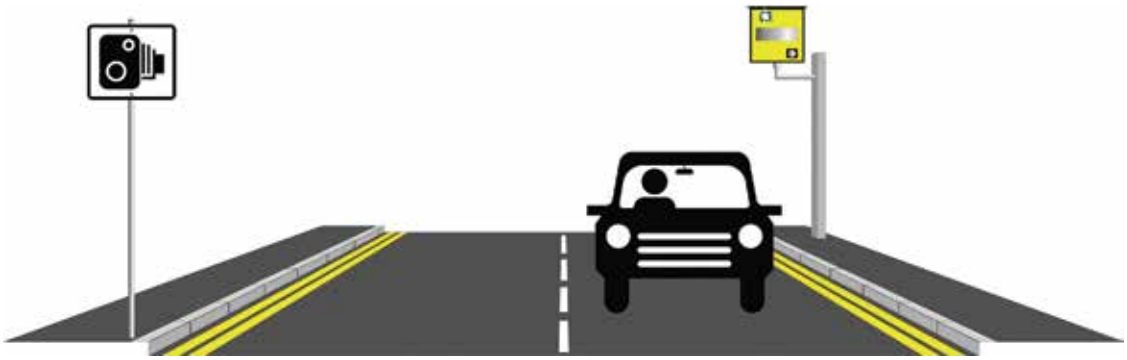
The standard deviation (to 2 decimal places) of these data is:

A. 6.00 **B.** 6.34 **C.** 6.47
D. 15.44 **E.** 16.00

9. WE10 The speeds, in km/h, of the first 25 cars caught by a roadside speed camera on a particular day were:

82, 82, 84, 84, 84, 84, 85, 85, 85, 86, 86, 87, 89, 89, 89, 90, 91, 91, 92, 94, 95, 96, 99, 100, 102

The next car that passed the speed camera was travelling at 140 km/h. Comment on the effect of the speed of this last car on the standard deviation for the data.



Communicating, reasoning and problem solving

10. WE11 For the data 1, 4, 5, 9, 11:

a. calculate the standard deviation

b. calculate the standard deviation if 7 is added to each data value. Comment on the effect.

c. calculate the standard deviation if all data values are multiplied by 3. Comment on the effect.

11. Show using an example the effect, if any, on the standard deviation of adding a data value to a set of data that is equivalent to the mean.

12. If the mean for a set of data is 45 and the standard deviation is 6, how many standard deviations above the mean is a data value of 57?

13. Five numbers, a, b, c, d and e, have a mean of 12 and a standard deviation of 4.

a. If each number is increased by 3, find the new mean and standard deviation in terms of the original mean and standard deviation.

b. If each number is multiplied by 3, find the new mean and standard deviation in terms of the original mean and standard deviation.

14. Twenty-five students sat a test. The results for 24 of the students are given in the following stem-and-leaf plot.

Stem	*Leaf*
0	89
1	123789
2	23568
3	012468
4	02568

a. If the average mark for the test was 27.84, determine the mark obtained by the 25 student.

b. How many students scored higher than the median score?

c. Find the standard deviation of the marks, giving your answer correct to 2 decimal places.

15. What does the standard deviation tell us about a set of data?

5.6 Comparing data sets [Stage 5.3]

- Besides locating the centre of the data, using the mean or median, any analysis of data must measure the extent of the spread of the data (range, interquartile range and standard deviation). Two data sets may have centres that are very similar but be quite differently distributed.
- Decisions need to be made about which measure of centre and which measure of spread to use when analysing and comparing data.
- The mean is calculated using every data value in the set. The median is the middle score of an ordered set of data, so it does not include every individual data value in its calculation. The mode is the most frequently occurring data value, so it also does not include every individual data value in its calculation.
- The range is calculated by finding the difference between the maximum and minimum data values, so it includes outliers. It provides only a rough idea about the spread of the data and is inadequate in providing sufficient detail for analysis. It is useful, however, when we are interested in extreme values such as high and low tides or maximum and minimum temperatures.

  The interquartile range is the difference between the upper and lower quartiles, so it does not include every data value in its calculation, but it will overcome the problem of outliers skewing data.

  The standard deviation is calculated using every data value in the set.

DISCUSSION

Can you think of some examples of data sets for which the mean and median wouldn't be very useful measures of centre?

WORKED EXAMPLE 12

For the two sets of data 6, 7, 8, 9, 10 and 12, 4, 10, 11, 3:
a calculate the mean
b calculate the standard deviation
c comment on the similarities and differences.

THINK	WRITE
a 1 Calculate the mean of the first set of data.	**a** $\bar{x}_1 = \frac{6 + 7 + 8 + 9 + 10}{5}$ $= 8$
2 Calculate the mean of the second set of data.	$\bar{x}_2 = \frac{12 + 4 + 10 + 11 + 3}{5}$ $= 8$
b 1 Calculate the standard deviation of the first set of data.	**b** $\sigma_1 = \sqrt{\frac{(6-8)^2 + (7-8)^2 + (8-8)^2 + (9-8)^2 + (10-8)^2}{5}}$ ≈ 1.41
2 Calculate the standard deviation of the second set of data.	$\sigma_2 = \sqrt{\frac{(12-8)^2 + (4-8)^2 + (10-8)^2 + (11-8)^2 + (3-8)^2}{5}}$ ≈ 3.74
c Comment on the findings.	**c** For both sets of data the mean was the same, 8. However, the standard deviation for the second set (3.74) was much higher than the standard deviation of the first set (1.41), implying that the second set is more widely distributed than the first. This is confirmed by the range, which is $10 - 6 = 4$ for the first set and $12 - 3 = 9$ for the second.

- When multiple data displays are used to display similar sets of data, comparisons and conclusions can then be drawn about the data.
- We can use back-to-back stem-and-leaf plots and multiple or parallel boxplots to help compare statistics such as the median, range and interquartile range.

WORKED EXAMPLE 13

Below are the scores achieved by two students in eight Mathematics tests throughout the year.
John: 45, 62, 64, 55, 58, 51, 59, 62
Penny: 84, 37, 45, 80, 74, 44, 46, 50

a Determine the most appropriate measure of centre and measure of spread to compare the performance of the students.
b Which student had the better overall performance on the eight tests?
c Which student was more consistent over the eight tests?

THINK	WRITE
a In order to include all data values in the calculation of measures of centre and spread, calculate the mean and standard deviation.	**a** John: $\bar{x} = 57$, $\sigma = 6$ Penny: $\bar{x} = 57.5$, $\sigma = 17.4$
b Compare the mean for each student. The student with the higher mean performed better overall.	**b** Penny performed slightly better on average as her mean mark was higher than John's.
c Compare the standard deviation for each student. The student with the lower standard deviation performed more consistently.	**c** John was the more consistent student because his standard deviation was much lower than Penny's. This means that his test results were closes to his mean score than Penny's were to hers.

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Interactivity: Back-to-back stem plots (int-6252)
Digital doc: WorkSHEET Univariate data II (doc-14596)

Exercise 5.6 Comparing data sets

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–7, 9, 10, 12, 14	Questions: 1–12, 15	Questions: 1–16

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE12 For the two sets of data 65, 67, 61, 63, 62, 60 and 56, 70, 65, 72, 60, 55:
a. calculate the mean
b. calculate the standard deviation
c. comment on the similarities and differences.

2. A bank surveys the average morning and afternoon waiting times for customers. The figures were taken each Monday to Friday in the morning and afternoon for one month. The stem-and-leaf plot below shows the results.

Key: 1 | 2 = 1.2 minutes

Leaf: Morning	*Stem*	*Leaf: Afternoon*
7	0	7 8 8
8 6 3 1 1	1	1 1 2 4 4 5 6 6 6 7
9 6 6 6 5 5 4 3 3 1	2	2 5 5 8
9 5 2	3	1 6
5	4	
	5	7

a. Find the median morning waiting time and the median afternoon waiting time.
b. Calculate the range for morning waiting times and the range for afternoon waiting times.
c. What conclusions can be made from the display about the average waiting time at the bank in the morning compared with the afternoon?

3. In a class of 30 students there are 15 boys and 15 girls. Their heights are measured in metres and are listed below.
Boys: 1.65, 1.71, 1.59, 1.74, 1.66, 1.69, 1.72, 1.66, 1.65, 1.64, 1.68, 1.74, 1.57, 1.59, 1.60
Girls: 1.66, 1.69, 1.58, 1.55, 1.51, 1.56, 1.64, 1.69, 1.70, 1.57, 1.52, 1.58, 1.64, 1.68, 1.67
Display this information in a back-to-back stem-and-leaf plot.

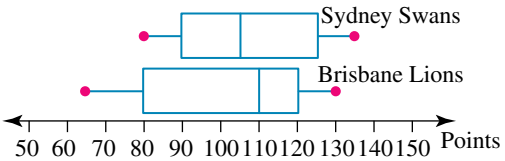
4. The following stem-and-leaf plot is used to display the number of vehicles sold by the Ford and Holden dealerships in a Sydney suburb each week for a three-month period.

Key: 1|5 = 15 vehicles

Leaf: Ford	*Stem*	*Leaf: Holden*
7 4	0	3 9
9 5 2 2 1 0	1	1 1 1 6 6 8
8 5 4 4	2	2 2 7 9
0	3	5

a. State the median of both distributions.
b. Calculate the range of both distributions.
c. Calculate the interquartile range of both distributions.
d. Show both distributions on parallel boxplots.
e. Compare their sales figures over the 3 month period.

5. The box-and-whisker plot drawn below displays statistical data for two AFL teams over a season.



a. Which team had the higher median score?
b. What was the range of scores for each team?
c. For each team calculate the interquartile range.

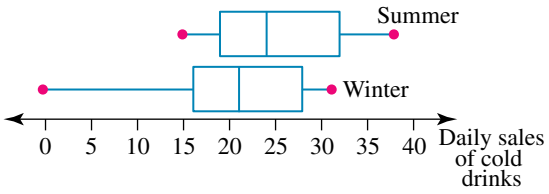
6. Tanya measures the heights (in m) of a group of Year 10 boys and girls and produces the following five-point summaries for each data set.
Boys: 1.45, 1.56, 1.62, 1.70, 1.81
Girls: 1.50, 1.55, 1.62, 1.66, 1.73

a. Draw a box-and-whisker plot for both sets of data and display them on the same scale.
b. What is the median of each distribution?
c. What is the range of each distribution?

d. What is the interquartile range for each distribution?

e. Comment on the spread of the heights among the boys and the girls.

7. The box-and-whisker plots show the average daily sales of cold drinks at the school canteen in summer and winter.

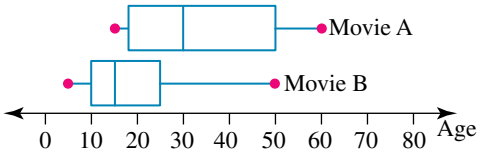


a. Calculate the range of sales in both summer and winter.

b. Calculate the interquartile range of the sales in both summer and winter.

c. Comment on the relationship between the two data sets, both in terms of measures of centre and measures of spread.

8. MC Andrea surveys the age of people at two movies being shown at a local cinema. The box-and-whisker plot shows the results.



Which of the following conclusions could be drawn based on the above information?

A. Movie A attracts an older audience than Movie B.

B. Movie B attracts an older audience than Movie A.

C. Movie A appeals to a wider age group than Movie B.

D. Movie B appeals to a wider age group than Movie A.

E. Both movies attract a similar age group

9. MC The figures below show the age of the first 10 men and women to finish a marathon.

Men: 28, 34, 25, 36, 25, 35, 22, 23, 40, 24

Women: 19, 27, 20, 26, 30, 18, 28, 25, 28, 22

Which of the following statements is correct?

Note: There may be more than one correct answer.

A. The mean age of the men is greater than the mean age of the women.

B. The range is greater among the men than among the women.

C. The interquartile range is greater among the men than among the women.

D. The standard deviation is greater among the men than among the women.

E. None of the above

Communicating, reasoning and problem solving

10. WE13 Cory recorded his marks for each test that he did in English and Science throughout the year.

English: 55, 64, 59, 56, 62, 54, 65, 50

Science: 35, 75, 81, 32, 37, 62, 77, 75

a. In which subject did Cory achieve the better average mark?

b. In which subject was Cory more consistent? Explain your answer.

11. The police set up two radar speed checks on a back street of Sydney and on a main road. In both places the speed limit is 60 km/h. The results of the first 10 cars that have their speed checked are given below.

Back street: 60, 62, 58, 55, 59, 56, 65, 70, 61, 64

Main road: 55, 58, 59, 50, 40, 90, 54, 62, 60, 60

a. Calculate the mean and standard deviation of the readings taken at each point.
b. On which road are drivers generally driving faster?
c. On which road is the spread of the reading taken greater? Justify your answer.

12. Nathan and Timana are wingers in their local rugby league team. The number of tries they have scored in each season are listed below.

Nathan: 25, 23, 13, 36, 1, 8, 0, 9, 16, 20
Timana: 5, 10, 12, 14, 18, 11, 8, 14, 12, 19

a. Calculate the mean number of tries scored by each player.
b. What is the range of tries scored by each player?
c. What is the interquartile range of tries scored by each player?
d. Which player would you consider to be the more consistent player? Justify your answer.

13. In boxes of Smarties it is advertised that there are 50 Smarties in each box. Two machines are used to distribute the Smarties into the boxes. The results from a sample taken from each machine are shown in the stem-and-leaf plot below.

Key: 5 | 1 = 51 5* | 6 = 56

Leaf: Machine A	*Stem*	*Leaf: Machine B*
4	4	
9 9 8 7 7 6 6 5	4*	5 7 8 9 9 9 9 9 9 9
4 3 2 2 2 1 1 1 0 0 0 0 0 0	5	0 0 0 0 0 1 1 1 1 1 2 2 3
5 5	5*	9

a. Display the data from both machines on parallel boxplots.
b. Calculate the mean and standard deviation of the number of Smarties distributed from both machines.
c. Which machine is the more dependable? Justify your answer.

14. A new drug for the relief of cold symptoms has been developed. To test the drug, 40 people were exposed to a cold virus. Twenty patients were then given a dose of the drug while another 20 patients were given a placebo. (In medical tests a control group is often given a *placebo* drug. The subjects in this group believe that they have been given the real drug but in fact their dose contains no drug at all.) All participants were then asked to indicate the time when they first felt relief of symptoms. The number of hours from the time the dose was administered to the time when the patients first felt relief of symptoms are detailed below.

Group A (drug)

25	29	32	45	18	21	37	42	62	13
42	38	44	42	35	47	62	17	34	32

Group B (placebo)

25	17	35	42	35	28	20	32	38	35
34	32	25	18	22	28	21	24	32	36

a. Detail the data on a back-to-back stem-and-leaf plot.
b. Display the data for both groups on a parallel boxplot.
c. Make comparisons of the data. Use statistics in your answer.
d. Does the drug work? Justify your answer.
e. What other considerations should be taken into account when trying to draw conclusions from an experiment of this type?

15. Which data display is best for comparing data sets?

16. A sample of 50 students was surveyed on whether they owned an iPod or a mobile phone. The results showed that 38 per cent of the students owned both. Sixty per cent of the students owned a mobile phone, and there were four students who had an iPod only. What percentage of students did not own a mobile phone or an iPod?

5.7 Review

Investigation | Rich task

Cricket scores

Data are used to predict, analyse, compare and measure many aspects of the game of cricket. Attendance is tallied at every match. Players' scores are analysed to see if they should be kept on the team. Comparisons of bowling and batting averages are used to select winners for awards. Runs made, wickets taken, no-balls bowled, the number of ducks scored in a game as well as the number of 4s and 6s are all counted and analysed after the game. Data of all sorts are gathered and recorded, and measures of central tendency and spread are then calculated and interpreted.

Sets of data have been made available for you to analyse, and decisions based on the resultant measures can be made.

Batting averages

The following table shows the runs scored by four cricketers who are vying for selection to the state team.

Player	Runs in the last 25 matches	Mean	Median	Range	IQR
Allan	13, 18, 23, 21, 9, 12, 31, 21, 20, 18, 14, 16, 28, 17, 10, 14, 9, 23, 12, 24, 0, 18, 14, 14, 20				
Shane	2, 0, 112, 11, 0, 0, 8, 0, 10, 0, 56, 4, 8, 164, 6, 12, 2, 0, 5, 0, 0, 0, 8, 18, 0				
Glenn	12, 0, 45, 23, 0, 8, 21, 32, 6, 0, 8, 14, 1, 27, 23, 43, 7, 45, 2, 32, 0, 6, 11, 21, 32				
Rod	2, 0, 3, 12, 0, 2, 5, 8, 42, 0, 12, 8, 9, 17, 31, 28, 21, 42, 31, 24, 30, 22, 18, 20, 31				

1. Find the mean, median, range and IQR scored for each cricketer.
2. You need to recommend the selection of two of the four cricketers. For each of the four players, write two points as to why you would or would not select them. Use statistics in your comments. Based on your points, recommend two of the cricketers for selection.

Batting averages

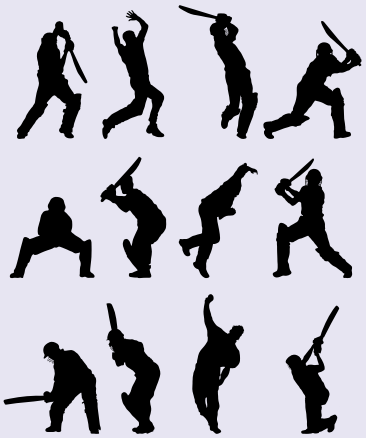
The bowling average is the number of runs per wicket taken.

$$\text{Bowling average} = \frac{\text{no. of runs scored}}{\text{no. of wickets taken}}$$

The smaller the average, the better the bowler has performed.

Brad and Dennis were competing for three bowling awards:

- **Best in semifinal**
- **Best in final**
- **Best overall**



The following table gives their scores.

	Semifinal		Final	
	Runs scored	**Wickets taken**	**Runs scored**	**Wickets taken**
Brad	12	5	28	6
Dennis	10	4	15	3

3. Calculate the bowling averages for the following and fill in the table below.
   a. Semifinal **b.** Final **c.** Overall

	Semifinal average	Final average	Overall average
Brad			
Dennis			

4. Explain how Dennis can have the better overall average when Brad has the better average in both the semifinal and final.

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Digital doc: Investigation — Cricket scores (doc-15938)
Digital doc: Code puzzle — Medical discovery of 1928 (doc-15939)
Digital doc: Summary Topic 5 (doc-22903)
Interactivity: Word search Topic 5 (int-2859)
Interactivity: Crossword Topic 5 (int-2860)
Interactivity: Sudoku Topic 5 (int-3599)

Exercise 5.7 Review questions

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. Find the mean and median for each of the following sets of data:

a. 7, 15, 8, 8, 20, 14, 8, 10, 12, 6, 19

b. Key: 1 | 2 = 12

Stem	*Leaf*
1	2 6
2	1 7 8
3	0 3 3 4 6 8
4	0 1 1 5 9
5	1 3 6

c.

Score (x)	Frequency (f)
70	2
71	6
72	9
73	7
74	4

2. For each of the following data sets, find the range.

a. 4, 3, 6, 7, 2, 5, 8, 4, 3

b.

x	13	14	15	16	17	18	19
f	3	6	7	12	6	7	8

c. Key: 1 | 8 = 18

Stem	*Leaf*
1	7 8 8 9
2	1 2 4 4 5 7 7 7 8 9 9
3	0 0 0 1 3 4 7

3. For each of the following data sets, find the interquartile range.

a. 18, 14, 15, 19, 20, 11, 16, 19, 18, 19

b. Key: 9 | 8 = 9.8

Stem	*Leaf*
8	7 8 8 9
9	0 2 4 4 5 7 7 7 8 9 9
10	0 1 1 1 3

4. The following back-to-back stem-and-leaf plot shows the typing speed in words per minute (wpm) of 30 Year 8 and Year 10 students.

Key: 2 | 6 = 26 wpm

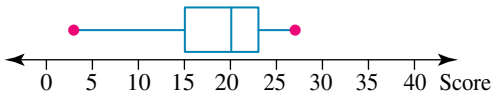
Leaf: Year 8	*Stem*	*Leaf: Year* 10
9 9	0	
9 8 6 5 4 2 0	1	7 9
9 8 8 6 4 2 1 0 0	2	2 3 6 8 9
9 7 7 6 4 1 0	3	0 2 4 5 5 7 8 8
8 6 5 2 0	4	1 2 5 8 8 9 9
	5	0 3 5 7 8
	6	0 0 3

a. Construct a pair of parallel boxplots to represent the two sets of data.

b. Find the mean, median, range, interquartile range and standard deviation of each set.

c. Compare the two distributions, using your answers to parts **a** and **b**.

5. Consider the boxplot drawn below.



a. Find the median.
b. Find the range.
c. Find the interquartile range.

6. The following data give the amount of cut meat (in kg) obtained from each of 20 lambs.

4.5 6.2 5.8 4.7 4.0 3.9 6.2 6.8 5.5 6.1
5.9 5.8 5.0 4.3 4.0 4.6 4.8 5.3 4.2 4.8

a. Detail the data on a stem-and-leaf plot. (Use a stem size of 0.5 kg.)
b. Prepare a five number summary of the data.
c. Draw a boxplot of the data.

7. Find the standard deviation of each of the following data sets correct to one decimal place.

a. 58, 12, 98, 45, 60, 34, 42, 71, 90, 66

b.

x	1	2	3	4	5
f	2	6	12	8	5

c. Key: 1 | 4 = 14

Stem	*Leaf*
0	1 3 4 4 5 7 8
1	0 0 0 1 2 2 4 5 7 8 9
2	0 2 2 3 5 7

8. MC The Millers obtained a number of quotes on the price of having their home painted. The quotes, to the nearest hundred dollars, were:

4200 5100 4700 4600 4800 5000 4700 4900

The standard deviation for this set of data, to the nearest whole dollar, is:

A. 260 **B.** 278 **C.** 324
D. 325 **E.** 800

9. MC The number of Year 12 students who, during semester 2, spent all their spare periods studying in the resource centre is shown on the stem-and-leaf plot below.

Key: 2 | 5 = 25 students

Stem	*Leaf*
0	8
1	
1	5 6 6 7
3	0 2 3 6 9
4	7 9
5	6
6	1

The standard deviation for this set of data, to the nearest whole number is:

A. 12 **B.** 14 **C.** 17
D. 35 **E.** 53

10. Each week, varying amounts of a chemical are added to a filtering system. The amounts required (in mL) over the past 20 weeks are shown in the stem-and-leaf plot below.
Key: 3 | 8 = 0.38 mL

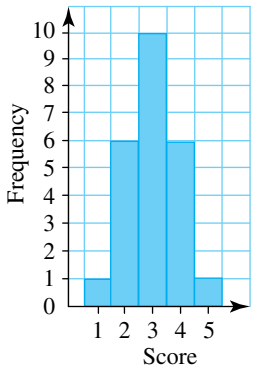
Stem	*Leaf*
2	1
2	2 2
2	4 4 4 5
2	6 6
2	8 8 9 9
3	0
3	2 2
3	4 5
3	6
3	8

Calculate to 2 decimal places the standard deviation of the amounts used.

Communicating, reasoning and problem solving

11. A sample of 30 people was selected at random from those attending a local swimming pool. Their ages (in years) were recorded as follows:
19 7 58 41 17 23 62 55 40 37 32 29 21 18 16 10 40 36 33 59 65 68
15 9 20 29 38 24 10 30
Find the mean and the median age of the people in this sample.

12. Consider the data set represented by the frequency histogram below.



a. Are the data symmetrical?
b. Can the mean and median of the data be seen? If so, what are their values?

13. The table below shows the number of cars that are garaged at each house in a certain street each night.

Number of cars	Frequency
1	9
2	6
3	2
4	1
5	1

a. Show these data in a frequency histogram.
b. Are the data positively or negatively skewed? Justify your answer.

14. Find the mean and median of this data set: 2, 5, 6, 2, 5, 7, 8. Comment on the shape of the distribution.

15. MC A data set has a mean of 75 and a standard deviation of 5. Another score of 50 is added to the data set. Which of the following will occur?

A. The mean will increase and the standard deviation will increase.
B. The mean will increase and the standard deviation will decrease.
C. The mean will decrease and the standard deviation will increase.
D. The mean will decrease and the standard deviation will decrease.
E. There will be no change in the mean and standard deviation.

Note: There may be more than one correct answer.

16. MC A data set has a mean of 60 and a standard deviation of 10. A score of 100 is added to the data set. This score becomes the highest score in the data set. Which of the following will increase?

A. Mean **B.** Standard deviation **C.** Range
D. Interquartile range **E.** Median

17. There are $3m$ values in a data set for which $\bar{x} = m$ and $\sigma = \frac{m}{2}$.

a. Comment on the changes to the mean and standard deviation if each value of the data set is multiplied by m.
b. An additional value is added to the original data set, giving a new mean of $m + 2$. Find the additional value.

18. The following data show the number of pets in each of the 12 houses in Coral Avenue, Rosebud.
2, 3, 3, 2, 2, 3, 2, 4, 3, 1, 1, 0

a. Calculate the mean and median number of pets.
b. The empty block of land at the end of the street was bought by a Cattery and now houses 20 cats. Recalculate the mean and median.
c. Explain why the answers are so different, and which measure of central tendency is best used for certain data.

19. The heights of Year 10 and Year 12 students (to the nearest centimetre) are being investigated. The results of some sample data are shown below.

Year 10	160	154	157	170	167	164	172	158	177	180	175	168	159	155	163	163	169	173	172	170
Year 12	160	172	185	163	177	190	183	181	176	188	168	167	166	177	173	172	179	175	174	180

a. Draw a back-to-back stem-and-leaf plot.
b. Draw parallel boxplots.
c. Comment on what the plots tell you about the heights of Year 10 and Year 12 students.

20. Kloe compares her English and Maths marks. The results of eight tests in each subject are shown below.
English: 76, 64, 90, 67, 83, 60, 85, 37
Maths: 80, 56, 92, 84, 65, 58, 55, 62

a. Calculate Kloe's mean mark in each subject.
b. Calculate the range of marks in each subject.
c. Calculate the standard deviation of marks in each subject.
d. Based on the above data, in which subject would you say that Kloe has performed more consistently?

Answers

Topic 5 Single variable data analysis

Exercise 5.2 Comparing data displays using mean, median and range

1. a. i. 7 **ii.** 8 **iii.** 7

b. i. 6.875 **ii.** 7 **iii.** 6

c. i. 39.125 **ii.** 44.5 **iii.** 46

d. i. 4.857 **ii.** 4.8 **iii.** 0.8

e. i. 12 **ii.** 12.625 **iii.** 6.5

2. a. C **b.** D **c.** E **d.** A

3. Science: mean $= 57.6$, median $= 57$, range $= 54$

Maths: mean $= 69.12$, median $= 73$, range $= 66$

4. a. i. 5.83 **ii.** 6 **iii.** 4

b. i. 14.425 **ii.** 15 **iii.** 4

5. a. mean $= 2.5$, median $= 2.5$ **b.** mean $= 4.09$, median $= 3$ **c.** Median

6. mean $= \$32.93$, median $= \$30$

7. a. 3

b. 4, 5, 5, 5, 6 (one possible solution)

c. One possible solution is to exchange 15 with 20.

8. a. Player A mean $= 34.33$, Player B mean $= 41.83$

b. Player B

c. Player A median $= 32.5$, Player B median $= 0$

d. Player A

e. Player A is more consistent. One large score can distort the mean.

9. a. Mean (Micky): 1, mean (Yani): 1.1

b. Median (Micky): 0.5, median (Yani): 1

c. Both the median and the mean for Yani are slightly higher, which would suggest that Yani is the more consistent player.

10. a. Key $3 \mid 2 = 32$

Leaf: 10A	*Stem*	*Leaf: 10B*
2	3	
9	4	6
74	5	5
87550	6	028
3	7	334578
5	8	1
6	9	

b. The results for 10A are more spread, with a range of 64, whereas the results for 10B are more clustered with a range of 35.

c. Mean(10A) $= 64.25$, mean(10B) $= 68.5$

d. Median(10A) $= 65$, median(10B) $= 73$

e. While one member of class 10A scored the highest with 96, class 10B has consistently better results overall, with a higher mean and median.

11. A

12. Check with your teacher.

13. There are many possible answers; examples are shown below.

a. 3, 4, 5, 5, 8 **b.** 4, 4, 5, 10, 16 **c.** 2, 3, 6, 6, 12

14. 12

15. $\frac{2a + b}{3}$

16. As the mean is the sum of all values divided by the number of values, it can be skewed by outliers (very low or very high values). Median is a better measure of centre when outliers are present.

Exercise 5.3 Quartiles and interquartile range

1. a. 15 **b.** 77.1 **c.** 9

2. a. 7 **b.** 7 **c.** 8.5 **d.** 39

3. a. 3.3 kg **b.** 1.5 kg

4. C

5. a. i. Range = 23 **ii.** IQR = 13.5

b. i. Range = 45 **ii.** IQR = 27.5

c. i. Range = 49 **ii.** IQR = 20

6. a. 25.5 **b.** 28 **c.** 39 **d.** 6

e. The three lower scores affect the mean but not the median or mode.

7. a. Men: mean = 32.3; median = 32.5; range = 38; IQR = 14

Women: mean = 29.13; median = 27.5; range = 36; IQR = 13

b. Typically, women marry younger than men, although the spread of ages is similar.

8. Mean = 25.036, median = 24.9, mode = 23.6, range = 8.5, IQR = 3.4

9. $a = 22, b = 9, c = 9$ and $d = 8$

10. The range gives the difference between the highest and lowest values in a data set,. The IQR provides the difference between the middle 50% of the data. If there are any very large or very small values in the data sets being compared, these will have a big effect on the range but not on the IQR. In such circumstances, the IQR would be a better measure to compare the spread of the data.

Exercise 5.4 Constructing, interpreting and comparing boxplots

1. a. 5 **b.** 26

2. a. 6 **b.** 27

3. a. 5.8 **b.** 18.6

4. a. 140 **b.** 56 **c.** 90 **d.** 84 **e.** 26

5. a. 58 **b.** 31 **c.** 43 **d.** 27 **e.** 7

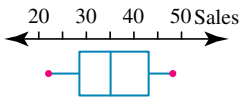
6. C

7. A

8. D

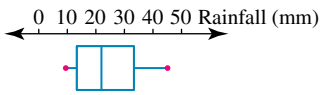
9. a. (22, 28, 35, 43, 48)

b.



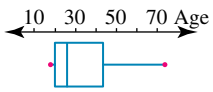
10. a. (10, 13.5, 22, 33.5, 45)

b.



11. a. (18, 20, 26, 43.5, 74)

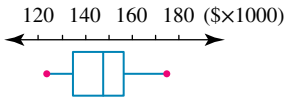
b.



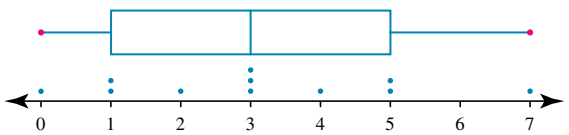
c. The distribution is positively skewed, with most of the offenders being young drivers.

12. a. 124 000, 135 000, 148 000, 157 000, 175 000

b.

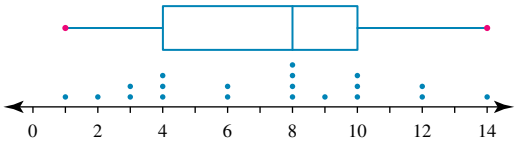


13. a.



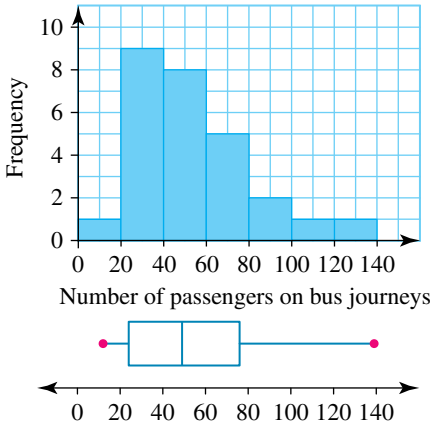
Both graphs indicate that the data is slightly positively skewed. However, the boxplot provides an excellent summary of the centre and spread of the distribution.

b.



Both graphs indicate that the data is slightly negatively skewed. However, the boxplot provides an excellent summary of the centre and spread of the distribution.

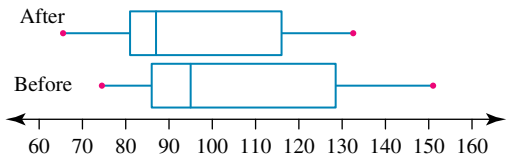
14.



15. a.

	X_{min}	Q_1	Median	Q_3	X_{max}
Before	75	86	95	128.5	152
After	66	81	87	116	134

b.

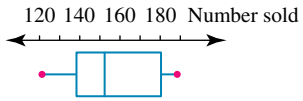


c. As a whole, the program was effective. The median weight dropped from 95kg to 87kg, a loss of 8kg. A noticeable shift in the graph shows that after the program 50% of participants weighed between 66 and 87kg, compared to 25% of participants weighing between 75 and 86kgbefore they started. Before the program the range of weights was 77kg (from 75kg to152kg); after the program the range had decreased to 68kg. The IQR also diminished from 42.5kg to35kg.

16. a. Key: 12 | 1 = 121

Stem	*Leaf*
12	1 5 6 9
13	1 2 4
14	3 4 8 8
15	0 2 2 2 5 7
16	3 5
17	2 9
18	1 1 1 2 3 7 8

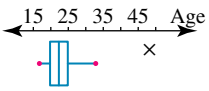
b.



c. On most days the hamburger sales are less than 160. Over the weekend the sales figures spike beyond this.

17. a. Key: 1* | 7 = 17 years

Stem	*Leaf*
1*	7 7 8 8 8 9 9
2	0 0 0 1 2 2 2 2 3 3 3 3 4 4 4
2*	5 5 8 9
3	1 2 3
3*	
4	
4*	8

b. 

c. The distribution is positively skewed, with first-time mothers being under the age of 30. There is one outlier (48) in this group.

18. C

19. a. HJ Looker: median = 5; Hane and Roarne: median = 6

b. HJ Looker

c. HJ Looker

d. Hane and Roarne had a higher median and a lower spread and so they appear to have performed better.

20. a. $50 **b.** $135 **c.** $100 **d.** $45 **e.** 50%

21. Discuss with your teacher.

Exercise 5.5 The standard deviation

1. a. 2.29 **b.** 2.19 **c.** 20.17 **d.** 3.07

2. a. 1.03 **b.** 1.33 **c.** 2.67 **d.** 2.22

3. 10.82 **4.** 0.45% **5.** 0.06 m **6.** 0.49 s **7.** 15.10 calls

8. B

9. The mean of the first 25 cars is 89.24 km/h with a standard deviation of 5.60. The mean of the first 26 cars is 91.19 with a standard deviation of 11.20, indicating that the extreme speed of 140 km/h is an anomaly.

10. a. $\sigma \approx 3.58$

b. The mean is increased by 7 but the standard deviation remains at $\sigma \approx 3.58$.

c. The mean is tripled and the standard deviation is tripled to $\sigma \approx 10.74$.

11. The standard deviation will decrease because the average distance to the mean has decreased.

12. 57 is two standard deviations above the mean.

13. a. New mean is the old mean increased by 3 but no change to the standard deviation.

b. New mean is 3 times the old mean and new standard deviation is 3 times the old standard deviation.

14. a. 43 **b.** 12 **c.** 12.19

15. Standard deviations indicates how far a data set is spread out from the mean.

Exercise 5.6 Comparing data sets

1. a. The mean of the first set is 63. The mean of the second set is 63.

b. The standard deviation of the first set is 2.38. The standard deviation of the second set is 6.53.

c. For both sets of data the mean is the same, 63. However, the standard deviation for the second set (6.53) is much higher than the standard deviation of the first set (2.38), implying that the second set is more widely distributed than the first. This is confirmed by the range, which is $67 - 60 = 7$ for the first set and $72 - 55 = 17$ for the second.

2. a. Morning: median = 2.45; afternoon: median = 1.6

b. Morning: range = 3.8; afternoon: range = 5

c. The waiting time is generally shorter in the afternoon. One outlier in the afternoon data causes the range to be larger. Otherwise the afternoon data are far less spread out.

3. Key: 16 | 1 = 1.61 m

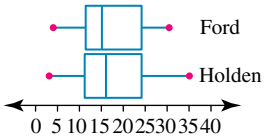
Leaf: Boys	*Stem*	*Leaf: Girls*
9 9 7	15	1 2 5 6 7 8 8
9 8 6 6 5 5 4 0	16	4 4 6 7 8 9 9
4 4 2 1	17	0

4. a. Ford: median = 15; Holden: median = 16

b. Ford: range = 26; Holden: range = 32

c. Ford: IQR = 14; Holden: IQR = 13.5

d.



e. The sales figures for the two dealerships are very similar, their measures of centre and IQR (size of the box) differ by very small amounts. Holden has a larger range, due to a particularly good week, in which they sold 35 cars.

5. a. Brisbane Lions

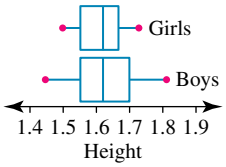
b. Brisbane Lions: range = 63;

Sydney Swans: range = 55

c. Brisbane Lions: IQR = 40;

Sydney Swans: IQR = 35

6. a.



b. Boys: median = 1.62; girls: median = 1.62

c. Boys: range = 0.36; girls: range = 0.23

d. Boys: IQR = 0.14; girls: IQR = 0.11

e. Although boys and girls have the same median height, the spread of heights is greater among boys as shown by the greater range and interquartile range.

7. a. Summer: range = 23; winter: range = 31

b. Summer: IQR = 14; winter: IQR = 11

c. There are generally more cold drinks sold in summer as shown by the higher median. The spread of data is similar as shown by the IQR, although the range in winter is greater.

8. A

9. A, B, C, D

10. a. Cory achieved a better average mark in Science (59.25) than he did in English (58.125).

b. Cory was more consistent in English ($\sigma = 4.9$) than he was in Science ($\sigma = 19.7$)

11. a. Back street: $\bar{x} = 61$, $\sigma = 4.3$; main road: $\bar{x} = 58.8$, $\sigma = 12.1$

b. The drivers are generally driving faster on the back street.

c. The spread of speeds is greater on the main road as indicated by the higher standard deviation.

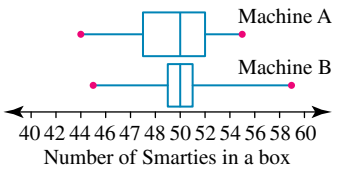
12. a. Nathan: mean = 15.1; Timana: mean = 12.3

b. Nathan: range = 36; Timana: range = 14

c. Nathan: IQR = 15; Timana: IQR = 4

d. Timana's lower range and IQR shows that he is the more consistent player.

13. a.



b. Machine A: mean = 49.88, standard deviation = 2.87;
Machine B: mean = 50.12, standard deviation = 2.44

c. Machine B is more reliable, as shown by the lower standard deviation and IQR. The range is greater on machine B only because of a single outlier.

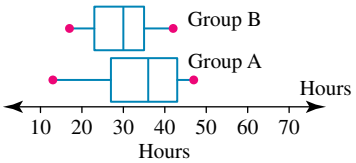
14. a. Key: 2|3 = 2.3 hours

Leaf: *Group A*	*Stem*	*Leaf:* *Group B*
8 7 3	1	7 8
9 5 1	2	0 1 2 4 5 5 8 8
8 7 5 4 2 2	3	2 2 2 4 5 5 5 6 8
7 5 4 2 2 2	4	2
	5	
2 2	6	

b. Five-point summaries:

Group A: 13 27 36 43 62

Group B: 17 23 30 35 42



c. Student comparison:

Statistic	Group A	Group B
Five-number summary	13 27 36 43 62	17 23 30 35 42
$\bar{x}$	35.85 hours	28.95 hours
Range	49 hours	25 hours
IQR	16 hours	12 hours
σ	13 hours	7 hours

d. Student decision, justifying answer

e. Class discussion

15. Discuss with your teacher.

16. 32%

Investigation | Rich task

1.

Player	Runs in the last 25 matches	Mean	Median	Range	IQR
Allan	13, 18, 23, 21, 9, 12, 31, 21, 20, 18, 14, 16, 28, 17, 10, 14, 9, 23, 12, 24, 0, 18, 14, 14, 20	16.76	17	31	8.5
Shane	2, 0, 112, 11, 0, 0, 8, 0, 10, 0, 56, 4, 8, 164, 6, 12, 2, 0, 5, 0, 0, 0, 8, 18, 0	17.04	4	164	10.5
Glenn	12, 0, 45, 23, 0, 8, 21, 32, 6, 0, 8, 14, 1, 27, 23, 43, 7, 45, 2, 32, 0, 6, 11, 21, 32	16.76	12	45	25.5
Rod	2, 0, 3, 12, 0, 2, 5, 8, 42, 0, 12, 8, 9, 17, 31, 28, 21, 42, 31, 24, 30, 22, 18, 20, 31	16.72	17	42	25

2. Selection points:

- Allan has a similar mean and median, which shows he was fairly consistent. The range and IQR values are low, indicating that his scores remain at the lower end with not much deviation for the middle 5%.
- Shane has the best average but a very low median, indicating his scores are not consistent. The range is extremely high and the IQR very low in comparison showing he can score very well at times but is not a consistent scorer.

- Glenn has a similar mean to Allan and Rod but a lower median, indicating his scores are sometimes high but generally are lower than the average. The range and IQR show a consistent batting average and spread with only a few higher scores and some lower ones.
- Rod has a similar mean and median, which shows he was a consistent player. The range and IQR show a consistent batting average and spread.

Players to be selected:

- Recommend Allan if the team needs someone with very consistent batting scores every game but no outstanding runs.
- Recommend Shane if the team needs someone who might score very high occasionally but in general fails to score many runs.
- Recommend Glenn if the team needs someone who is fairly consistent but can score quite well at times and the rest of the time does OK.
- Recommend Rod if the team needs someone who is fairly consistent but can score quite well at times and the rest of the time has a better median than Glenn.

3.

	Semifinal average	Final average	Overall average
Brad	2.4	4.67	3.64
Dennis	2.5	5	3.57

a. Brad was best in the semifinal.

b. Brad was best in the final.

c. Dennis was best overall.

4. In the final, wickets were more costly than in the semifinal. Brad therefore conceded many runs in getting his six wickets. This affected the overall mean. In reality Brad was the most valuable player overall, but this method of combining the data of the two matches led to this unexpected result.

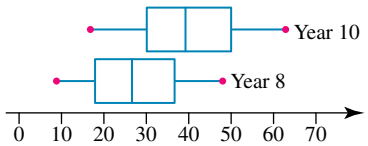
Exercise 5.7 Review questions

1. a. Mean = 11.55, median = 10 **b.** Mean = 36, median = 36 **c.** Mean = 72.18, median = 72

2. a. 6 **b.** 6 **c.** 20

3. a. 4 **b.** 8.5

4. a.



b. Year 8: mean = 26.83, median = 27, range = 39, IQR = 19, $\sigma = 11.45$

Year 10: mean = 40.7, median = 39.5, range = 46, IQR = 20, $\sigma = 12.98$

c. The typing speed of Year 10 students is about 13 to 14 wpm faster than that of Year 8 students. The spread of data in Year 8 is slightly less than in Year 10

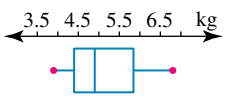
5. a. 20 **b.** 24 **c.** 8

6. a. Key: 3* | 9 = 3.9 kg

Stem	*Leaf*
3*	9
4	0 0 2 3
4*	5 6 7 8 8
5	0 3
5*	5 8 8 9
6	1 2 2
6*	8

b. 3.9, 4.4, 4.9, 5.85, 6.8

c.



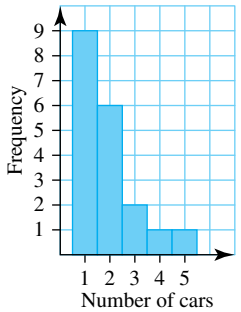
7. a. 24.4 **b.** 1.1 **c.** 7.3

8. A **9.** B **10.** 0.05 mL

11. Mean $= 32.03$, median $= 29.5$

12. a. Yes **b.** Yes. Both are 3.

13. a.



b. Positively skewed — a greater number of scores is distributed at the lower end of the distribution.

14. Mean $= 5$, median $= 5$. The distribution is positively skewed.

15. C

16. A, B and C

17. a. $\bar{x} = m^2, \sigma = \frac{m^2}{2}$ **b.** $7m + 2$

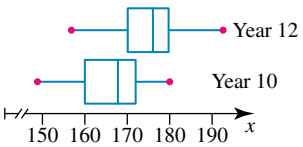
18. a. Mean = 2.17, median = 2 **b.** Mean = 3.54, median = 2

c. The median relies on the middle value of the data and won't change much if an extra value is added. The mean however has increased because this large value will change the average of the numbers. The mean is used as a measure of central tendency if there are no outliers or if the data are symmetrical. The median is used as a measure of central tendency if there are outliers or the data are skewed.

19. a.

Leaf: Year 10	*Stem*	*Leaf: Year 11*
9 8 7 5 4	15	
9 8 7 4 3 3 0	16	0 3 6 7 8
7 5 3 2 2 0 0	17	2 2 3 4 5 6 7 7 9
0	18	0 1 3 5 8
	19	0

b.



c. On average, the Year 11 students are about 6–10 cm taller than the Year 10 students. The heights of the majority of Year 11 students are between 170 cm and 180 cm, whereas the majority of the Year 10 students are between 160 and 172 cm in height.

20. a. English: mean = 70.25; Maths: mean = 69

b. English: range = 53; Maths: range = 37

c. English: $\sigma = 16.1$; Maths: $\sigma = 13.4$

d. Kloe has performed more consistently in Maths, as the range and standard deviation are both lower.

TOPIC 6 Statistics in the media

[Stages 5.1, 5.2 and 5.3]

6.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the concepts covered in this topic.

6.1.1 Why learn this?

Statistics, when used properly, can be an invaluable aid to good decision making. However, deliberate distortion of the data or meaningless pictures can be used to support almost any claim or point of view. Whenever you read an advertisement, hear a news report or are given some data by a friend, you need to have a healthy degree of scepticism about the reliability of the source and nature of the data presented.

DISCUSSION

Do you think the current standards on advertising are too strict or too lenient? How important is it that consumers be given factual information?

LEARNING SEQUENCE

6.1 Overview
6.2 [Stage 5.1] How is data collected?
6.3 [Stages 5.1 and 5.2] Evaluating statistical reports
6.4 [Stages 5.2 and 5.3] Statistical investigations
6.5 [Stages 5.2 and 5.3] Estimating population means and medians
6.6 Review

LEARNING OUTCOMES

A student:

- uses appropriate terminology, diagrams and symbols in mathematical contexts MA5.1–1WM
- selects and uses appropriate strategies to solve problems MA5.1–2WM
- provides reasoning to support conclusions that are appropriate to the context MA5.1–3WM
- evaluates statistical claims made in the media MA5.1–12SP
- selects appropriate notations and conventions to communicate mathematical ideas and solutions MA5.2–1WM
- constructs arguments to prove and justify results MA5.2–3WM
- evaluates sources of data MA5.2-15SP
- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3–1WM
- generalises mathematical ideas and techniques to analyse and solve problems efficiently MA5.3–2WM
- explores how data is used to inform decision-making processes MA5.3–19SP

CONTENT DESCRIPTIONS

Students:

Evaluate statistical reports in the media and other places by linking claims to displays, statistics and representative data (ACMSP253)
Investigate reports of surveys in digital media and elsewhere for information on how data was obtained to estimate population means and medians (ACMSP227)

Source: NSW Syllabus for the Australian curriculum

learnon RESOURCES — ONLINE ONLY

eLesson: The story of mathematics — Behind the advertising (eles-1854)

Note: Your teacher may now set you a pre-test to determine how familiar you are with the content in this topic.

6.2 How is data collected? [Stage 5.1]

6.2.1 Primary data

- **Primary data** collection involves collecting **data** from the source.
- This means that the person who collects the data has ownership of the data, and no one else has access to the data until it is released or published.
- A variety of methods for collecting primary data are used. These include observation, digital footprint, measurement, survey, experiment or simulation.

Observation

- Observation involves recording the behavioural patterns of people, objects and events in a systematic manner.
- Data can be collected by disguised observation (respondents are unaware they are being observed) or undisguised observation (the respondent is aware). The existence of video surveillance cameras, for example, can mean that people know that there is a possibility that their movements are being recorded, but they are not always aware of when the recording takes place.
- Observations can be in a natural environment (for example, small children may be observed at play to assist in understanding their cognitive development) or a contrived environment (for example a food-tasting session for a food company).
- Mechanical devices (for example video cameras, closed circuit television or counting devices across a road) can also be used.

Digital footprint

- Vast amounts of data are collected through our **digital footprints**. Search engines, emails, social media websites, online shopping websites, apps on our smart phones and loyalty cards are just examples of a whole range of means by which data collection agencies are able to learn about individuals. Each click of a mouse or tap on a tablet adds to the trillions of pieces of data that are being generated around the world every second, all of which can potentially be mined and analysed.

DISCUSSION

After reflecting on your internet use both for study and leisure, list some of the information about you that is now available online. Give careful consideration to information that may have been selected as the result of your online activities but does not reflect your intentions.

Measurement

- Measurement involves using some measuring device to collect data.
- This generally involves conducting an experiment of some type.
  - The height of everyone in your class can be measured.
  - The mass of all new-born babies can be collected.
  - A pedometer measures the number of steps the wearer takes.

Survey

- Surveys involve designing a questionnaire to interview people. Often the questionnaire requires many rewrites to obtain one which is clear and unbiased.
- The interview can be in person, for example face to face or by telephone. Advantages of these methods are that you are able to gauge the reactions of those you are interviewing and you can explain particular questions if necessary.
- Email can also be used to survey participants. However, there are advantages and disadvantages to using this type of survey.
  Advantages:
  - Can cover a large number of people or organisations
  - A wide geographic coverage is possible.
  - It avoids embarrassment on the part of the respondent.
  - There is no interviewer bias.
  - The respondent has time to consider responses.
  - It's relatively cheap.

  Disadvantages:
  - The questions have to be relatively simple.
  - The response rate is often quite low (inducements often given as an incentive to return).
  - The reliability of the answers is questionable.
  - There is no control over who actually completes the questionnaire.
  - Participants may return incomplete questionnaires.

Experiment

- Generally, when an experiment is conducted, the data collected are quantitative.
- Particular care should be taken to ensure that the experiment is conducted in a manner that would produce similar results if repeated.
- Care must be taken with the recording of results.
- The results must be in a form that can readily be analysed.
- All results need to be recorded, including any weird or unexpected outcomes.

Simulation

- Experiments such as rolling a die, tossing a coin or drawing a card from a deck may be conducted to model real-life situations.
- Simulations occur in areas such as business, engineering, medical and scientific research.
- They are often used to imitate real-life situations that are dangerous, impractical or too expensive to explore by other means.
- Before any primary data are collected, it must be clear what data are to be collected.

WORKED EXAMPLE 1

It is widely believed that, across a large population, there is equal chance of having a boy or girl with each birth.

a Design an experiment to simulate the chance of giving birth to a boy or a girl.

b Describe how your experiment could be conducted to determine the number of children a couple should have, on average, to ensure they have offspring of both sexes.

THINK

a Use a device that can simulate two outcomes which are equally likely. This could be a random number generator to generate two integers, say a 0 (representing a boy) and 1 (representing a girl). A fair coin could be tossed, such that, a Head represents a boy, and a Tail represents a girl.

b 1 Describe how the experiment will be conducted.

WRITE

a A fair coin will be tossed with a Head representing a boy (B), and a Tail representing a girl (G).

b The experiment will be conducted 50 times, and a record kept of each experiment. For each experiment, the coin will be tossed until both sexes result. This may mean that there could be for example 7 trials in an experiment (GGGGGGB) before both sexes are represented.

2 Display the table of results.

The table below shows the results of the 50 experiments.

Exp. no.	Results	No. of trials	Exp. no.	Results	No. of trials
1	BG	2	26	GGGB	4
2	GGB	3	27	GGGGB	5
3	BG	2	28	GGGB	4
4	GGGGB	5	29	BG	2
5	BBBBBBG	7	30	BBBG	4
6	GGGB	4	31	BG	2
7	BBG	3	32	GB	2
8	BBG	3	33	GGGB	4
9	BBBBG	5	34	BG	2
10	GB	2	35	GGGGGGB	7
11	BG	2	36	BBBBBBG	7
12	GGGB	4	37	GB	2
13	BBG	3	38	BG	2
14	BBG	3	39	GGB	3
15	GB	2	40	GGGGB	5
16	BG	2	41	BBG	3
17	GGB	3	42	BBBBBG	6
18	GB	2	43	GGB	3
19	GGB	3	44	GGB	3
20	BBBG	4	45	BBBG	4
21	BG	2	46	BBG	3
22	GB	2	47	GGGGGGB	7
23	GGGGB	5	48	BG	2
24	BG	2	49	BBG	3
25	GGGGB	5	50	GGGGGB	6
				Total	175

This table shows that 175 trials were undertaken in 50 experiments where each experiment resulted in both sexes.

3 Determine the average number of children required to produce offspring of both sexes.

Average number of children $= \frac{175}{50} = 3.5$

4 Write a conclusion.

The average number of children a couple should have to reach the goal of having both sexes is 4.

WORKED EXAMPLE 2

You have been asked to obtain primary data to determine the methods of transport used to travel to school by the students at your school. The data collected are to provide support for the Student Council's proposal for a school bus.

a What data should be collected?

b Outline possible methods which could be used to collect the data.

c Decide which method you consider to be the best option, and discuss its advantages and disadvantages.

THINK	WRITE
a Outline the various forms of transport available to the students.	**a** The modes of transport available to students at the school are: car, bus, train, bicycle and walking
b Consider all the alternatives for collecting the data.	**b** Several methods could be used to collect the data. • Could stand at the school gate one morning and ask students as they arrive. • An online questionnaire could be designed. • Students could be asked to write their mode of transport on a piece of paper and place in a collection tin.
c 1 Decide on the best option.	**c** The first option of standing at the school gate is very time-consuming, and students could arrive at the back gate. The third option does not seem reliable, as some students may not comply, and other students may place multiple pieces of paper in the collection tin. The second option seems the best of the three.
2 Discuss advantages and disadvantages.	The advantages of an online questionnaire include: • There is a permanent record. • It is not time-consuming to distribute or collect and collate. • Students can complete it at their leisure. Disadvantages include: • Students may not complete it.

- A decision must be made as to the method of collection.
- The advantages and disadvantages of the collection method must be acknowledged.
- The reason for the data collection should be clear from the outset.

Note: This example does not represent the views of all those collecting such data. It merely serves to challenge students to explore and discuss available options.

DISCUSSION

What are the advantages and disadvantages of the different methods of collecting data?

6.2.2 Secondary data

- **Secondary data** are data that have already been collected by someone else.
- The data can come from a variety of sources:
  - Paper — books, journals, magazines, company reports
  - Electronic — online databases, internet, broadcasts, DVDs
  - Government sources — the Australian Bureau of Statistics (ABS) provides a wealth of statistical data.
  - General business sources — academic institutions, stockbroking firms, sporting clubs
  - Media — newspapers, TV reports.
- Secondary data sources often provide data that would not be possible for an individual to collect.

- The data can be qualitative or quantitative.
- The accuracy and reliability of the data sometimes needs to be questioned, depending on its source.
- The age of the data should always be considered.
- Often the data that surrounds us passes by unnoticed.
- It is important to learn the skills to be able to critically analyse secondary data.

WORKED EXAMPLE 3

Bigbite advertises the energy and fat content of some of their rollaways on their placemats.

a What information can you gain from these data?

b Bigbite advertise that they have a range of rollaways with less than 6 grams of fat. Comment on this claim.

c This could be the starting point of a statistical investigation. How could you proceed from here?

d Investigations are not conducted simply for the sake of investigating. Suggest some aims for investigating further.

BIGBITE fresh as meals

Bigbite fresh as rollaways	Energy (kJ)	Fat (g)	Sat. fat (g)
Roasted vegetable	900	3.0	1.0
HAM	1100	6.0	1.4
TURKEY	1140	4.8	1.7
BBQ BEEF	1150	5.0	1.5
Bigbite ribbon	1130	4.8	1.3
TURKEY AND HAM	1250	4.5	1.5
BBQ CHICKEN	1460	4.7	1.2
CHICKEN Tandoori	1110	4.0	1.0
Fresh as Dessert			
Fruit Slices	200	<1	<1

Bigbite rollaways

Regular rollaways include white and/or wholemeal bread, salads and meat.

Nutritional value is changed by adding cheese or sauces.

THINK

a Look at the data on the placemat to gain as much information as possible.

b Examine the data to discover if there is evidence to support the claim. Make further comment.

WRITE

a The placemat reveals the following information:
- A higher energy content of a rollaway does not necessarily mean that its fat content is higher.
- As the fat content of a rollaway increases, generally the saturated fat content also increases.
- The addition of some types of protein (ham, turkey, beef, chicken) increases the energy content of the rollaway.
- These data are only for those rollaways on white or wholemeal bread with salads and meat.
- The addition of condiments (sauces) or cheese will alter these figures.
- A fruit slice has much less energy and fat than a rollaway.

b All the rollaways displayed have less than 6 grams of fat, so Bigbite's claim is true.
It must be remembered that the addition of cheese and sauce to these rollaways would increase their fat content. Also, if the rollaway was on any bread other than white or wholemeal, the fat content could go beyond 6 grams.

c What would be the next step in the investigation?

c The placemat displays a toll-free phone number for further information.
Their website also contains additional detailed information.

d What are some interesting facts which could be revealed through a deeper investigation?

d Suggested aims for investigating further could be:
- How much extra fat is added to a rollaway by the addition of cheese and/or sauce?
- What difference does a different type of bread make to the fat content of the rollaway?
- Which rollaway contains the highest fat content?
- What is the sugar content of the rollaways?

learnON RESOURCES — ONLINE ONLY

Interactivity: Simulation (int-6184)

Exercise 6.2 How is data collected?

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–8, 11, 14	Questions: 1–9, 11, 14	Questions: 1, 3, 6, 7, 10–15

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE1 Devise an experiment to simulate each of the following situations and specify the device used to represent the outcomes.
- **a.** A true/false test in which answers are randomly distributed
- **b.** A casino game with outcomes grouped in colours of either red or black
- **c.** Breakfast cereal boxes containing 4 different types of plastic toys
- **d.** Choosing one person out of a group of six people as the leader.
- **e.** A choice of three main meals on a restaurant's menu, all of which are equally popular
- **f.** Five possible holiday destinations offered by a travel agent; such that all destinations are equally available and equally priced
- **g.** Five types of takeaway fast foods available in one area, where one pizza is twice as popular as each of the other types of takeaway food (the other 4 are equally popular)

2. WE2 You have been asked to obtain primary data from students at your school to determine the extent of internet access students have at home. The data collected are to provide support for opening the computer room for student use at night.

a. What data should be collected?

b. Outline possible methods which could be used to collect these data.

c. Decide which method you consider to be the best option, and discuss its advantages and disadvantages.

3. WE3 This label shows the nutrition information of Brand X rolled oats.

Nutrition information			
Servings per package: 25			Serving size 30 g
	Per serving 30 g	%DI* per serving	Per 100 g
Energy	486 kJ	6%	1620 kJ
Protein	4.3 g	9%	14.3 g
Fat – Total	2.8 g	4%	9.3 g
– Saturated	0.5 g	2%	1.7 g
– Trans	Less than 0.1 g	-	Less than 0.1 g
– Polyunsaturated	1.0 g	-	3.2 g
– Monounsaturated	1.3 g	-	4.4 g
Carbohydrate	16.8 g	5%	56 g
– Sugars	0.9 g	1%	3.0 g
Dietary fibre	3.1 g	10%	10.4 g
Sodium	0.7 mg	0.1%	2 mg

* % DI = Percentage daily intake

a. What information can you gain from the data?

b. This could be the starting point of a statistical investigation. How could you proceed from here?

c. Suggest some aims for investigating further.

4. a. Provide a list of methods you could use to collect primary data.

b. Describe which methods you would use to collect the following primary data.

i. Heights of trees along the footpaths of a tree-lined street

ii. Number of buses that transport students to your school in the morning

iii. Sunrise times during summer

iv. Student opinion regarding length of lessons

For questions **5** and **6**, design an experiment to simulate the situation, carry out the experiment and give the results of the experiment.

5. A mouse in a maze can make left or right turns at each junction. Assuming each turn is equally likely, how many junctions on average must the mouse go through before each type of turn will have been made?

6. A restaurant menu features 4 desserts which are assumed to be equally popular. How many dessert orders must be filled (on average) before the owner can be sure all types will have been ordered?

7. This label shows the nutrition information of Brand Y rolled oats.

Nutrition information		
Servings per package: 30		
Serving size: 30 g		
	Avg. quantity per serving 30 g	Avg. quantity per 100 g
Energy	480 kJ (115 Cal)	1600 kJ (383 Cal)
Protein	3.2 g	10.5 g
Fat, total	2.4 g	8.0 g
– saturated	LESS THAN 1 g	1.5 g
Carbohydrate	18.3 g	61.0 g
– sugars	0.0 g	0.0 g
Dietary Fibre, total	3.3 g	10.0 g
Sodium	LESS THAN 5 mg	LESS THAN 5 mg
Ingredients		
Oats (100%)		
Attention		
THIS PRODUCT CONTAINS GLUTEN.		
Storage		
Store in a cool, dry place.		

Compare the nutrition information for Brand Y with that on the Brand X label in question **3**.

8. Comment on this claim.

> We did a survey on 100 people regarding eating chocolate.
> 60 of these people said they regularly ate chocolate.
> We then measured the heights of all 100 people.
> **** The result ****
> Eating chocolate makes you taller!!

9. Russel operates a computer software sales outlet. He keeps a log of all complaints from customers. Suggest how he could organise his log.

10. The following claim has been made regarding secondary data.
There's a lot more secondary data than primary data, it's a lot cheaper and it's easier to acquire.
Comment on this statement.

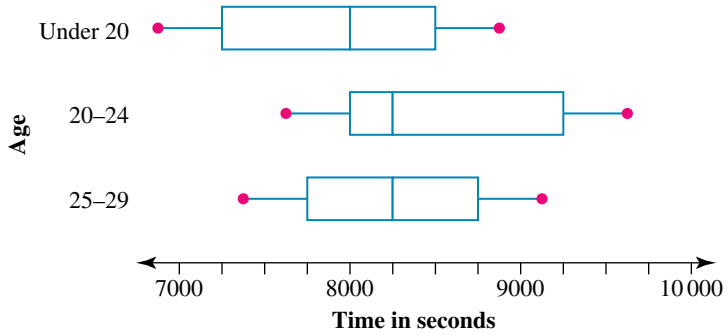
Communicating, reasoning and problem solving

11. The local Bed Barn was having a sale on selected beds by Sealy and Sleepmaker. Four of the beds on sale were:

- Sealy Posturepremier on sale for $1499 a saving of $1000
- Sealy Posturepedic on sale for $2299 a saving of $1600
- Sleepmaker Casablanca on sale for $1199 a saving of $800
- Sleepmaker Umbria on sale for $2499 a saving of $1800.

The store claimed that all these beds had been discounted by 40%. Comment on whether this statement is true, supporting your comments with sound mathematical reasoning.

12. The following boxplots display the data for three different age groups in a triathlon for under thirties.



a. What was the slowest time for the 20–24 year olds?

b. Estimate the difference in time between the fastest triathlete in:

i. the under 20s and the 20–24 group

ii. the under 20s and the 25–29 group

iii. the 20–24 group and the 25–29 group.

c. What conclusion could you draw about the overall performance between the three groups?

13. This comparison of the contents of canned soups is based on a newspaper article.

COMPARISON OF CONTENTS: CANNED SOUPS

Rosella Classic Australian Chicken and Vegetables

Water, Vegetables (35%), Chicken Breast (4%), Thickener, Onion, Chicken Flavour, Salt, Yeast, Milk Powder, Mineral Salt, Pepper, Basil, Colour

The breakdown: 107 calories; 5.8 gm protein; 19 gm carbs; 3.0 gm sugars; 750 mg sodium.

Campbell's Roast Chicken & Winter Vegetable

Vegetables (54%), Chicken Stock, Chicken (4%), Potato Starch, Cream, Modified Tapioca Starch, Salt, Sugar, Vegetable Protein Extract, Garlic, Flavours, Yeast Extract, Flavour Enhancer, Mineral Salts, Soy Protein, Maltodextrin

The breakdown: 134 calories; 5.1 gm protein; 19.2 gm carbs; 4.7 gm sugars; 602 mg sodium.

Campbell's Country Ladle Butternut Pumpkin

Pumpkin (54%), Vegetable Stock, Potato, Modified Tapioca Starch, Cream, Salt, Sugar, Butter, Flavours, Vegetable Protein Extract, Spices, Yeast Extract

The breakdown: 99 calories; 3.3 gm protein; 13.0 gm carbs; 8.2 gm sugars; 673 mg sodium.

Campbell's Café Chicken Mushroom

Chicken Stock, Wholegrain Brown Rice, Great Northern Beans, Wild Rice (4%), Mushrooms (4%), Chicken (3.5%), Thickeners, Vegetable Oil, Salt, Sugar, Cream, Vegetable Protein Extract, Flavours, Soy Protein, Milk Solids, Garlic, Herbs, Thyme, Flavour Enhancers, Mineral Salts, Yeast Extract, Food Colour

The breakdown: 134 calories; 7.0 gm protein; 16.8 gm carbs; 2.0 gm sugars; 675 mg sodium.

Create a table summarising the data provided.

a. Using your table, identify the variation in quantities of:

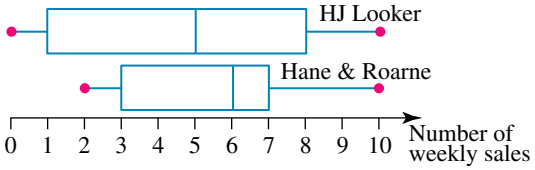
i. calories **ii.** protein **iii.** carbs **iv.** sugar **v.** sodium.

b. Which soups have the highest levels of:

i. calories **ii.** protein **iii.** sodium?

c. What can be noted about Campbell's Country Ladle Butternut Pumpkin soup?

14. These parallel boxplots show the number of weekly house sales by two real estate agencies over a 3-month period.



Prepare a report to compare the performance of the two agencies.

15. When using secondary data from other countries, what different unit classifications could you encounter?

6.3 Evaluating statistical reports [Stages 5.1 and 5.2]

6.3.1 Data collection methods

- Statistical investigations involve collecting data, recording the data, analysing the data and reporting the results.
- Collection methods involve gathering primary data, or using secondary data from stored records.
- Primary data can be collected by observation, digital footprint, measurement, survey, experiment or simulation as discussed above.
- Secondary data can be collected electronically or by reading records.
- It is important to be able to justify the particular method chosen for each of these processes.
- Sometimes alternative methods are just as appropriate.

WORKED EXAMPLE 4

You have been given an assignment to investigate which year level uses the school library, after school, the most.

a Explain whether it is more appropriate to use primary or secondary data in this case. Justify your choice.

b Describe how the data could be collected. Discuss any problems which might be encountered.

c Explain whether an alternative method would be just as appropriate.

THINK	WRITE
a No records have been kept on library use.	**a** Since records are not kept on library use, secondary data is not an option.
b The data can be collected via a questionnaire	**b** A questionnaire could be designed online and sent out to all students. This would be a way of collecting data that could be easily collated. The difficulty would be how to ensure enough people responded.
c The data could be collected in person by standing at the school gate on a designated day.	**c** Observation could be used to personally interview students as they left the school. This would take more time, and could not ensure all students were included.

WORKED EXAMPLE 5

Which method would be the most appropriate to collect the following data? Suggest an alternative method in each case

a The number of cars parked in the staff car park each day
b The mass of books students carry to school each day
c The length a spring stretches when weights are added to it
d The cost of mobile phone plans with various network providers

THINK	WRITE
a Observation	**a** The best way would probably be observation of the staff car park to count the number of cars there. An alternative method would be to conduct a census of all workers to ask if they parked in the staff car park. This is probably not as accurate as the direct observation method.
b Measurement	**b** The mass of the books could be measured by weighing each student's pack on scales. A random sample would probably yield a reasonably accurate result.
c Experiment	**c** Conduct an experiment and measure the extension of the spring with various weights. There is probably no alternative to this method.
d Internet search	**d** An internet search would enable data to be collected. Alternatively, a visit to multiple mobile phone outlets would yield similar results.

6.3.2 Analysing the data

- Once the data have been collected and collated, a decision must be made with regard to the best methods for analysing the data. Possible methods include:
  - a measure of central tendency — mean, median or mode
  - a measure of spread — range, interquartile range or standard deviation
  - an appropriate graph.

Statistical graphs

- Data can be graphed in a variety of ways, such as line graphs, bar graphs, histograms, stem plots and box plots. These have all been discussed in detail previously.
- In media reports it is common to see line and bar graphs.
- Graphs can summarise a lot of data into one visual representation. It can therefore be tempting to not study them in great detail. However, graphs can be misleading.
- It is easy to manipulate a graph to create a biased impression. This is achieved by careful choice of scale on the horizontal and vertical axes.
  - Shortening the horizontal axis tends to highlight the increasing/decreasing nature of the trend of the graph. Lengthening the vertical axis tends to have the same effect.
  - Lengthening the horizontal and shortening the vertical axes tends to level out the trends.

WORKED EXAMPLE 6

This report shows the annual median house prices in a number of suburbs from 2014–15 to 2015–16.

a Draw a bar graph that would give the impression that the percentage annual change was much the same throughout all of the suburbs.

b Draw a bar graph to give the impression that the percentage annual change in Thomastown was far greater than that in the other suburbs.

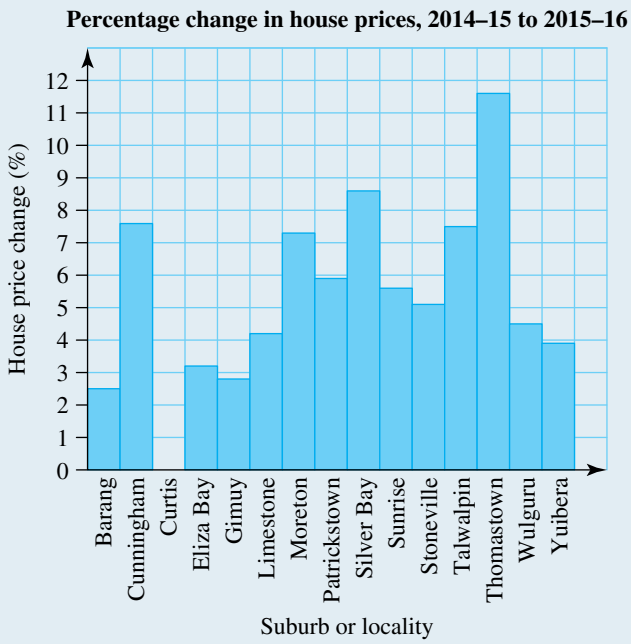
Suburb or locality	Median price in 2014–15	Median price in 2015–16	Change in median price (%)
Barang	$275 000	$282 000	2.5%
Cunningham	$269 000	$289 500	7.6%
Curtis	$370 000	$370 000	0.0%
Eliza Bay	$297 750	$307 400	3.2%
Gimuy	$355 000	$365 000	2.8%
Limestone	$310 000	$323 000	4.2%
Moreton	$372 000	$399 000	7.3%
Patrickstown	$340 000	$360 000	5.9%
Silver Bay	$465 000	$505 000	8.6%
Sunrise	$445 000	$470 000	5.6%
Stoneville	$300 000	$315 250	5.1%
Talwalpin	$435 000	$467 500	7.5%
Thomastown	$475 000	$530 000	11.6%
Wulguru	$359 000	$375 000	4.5%
Yuibera	$383 000	$398 000	3.9%

THINK

a To flatten out trends, lengthen the horizontal axis and shorten the vertical axis.

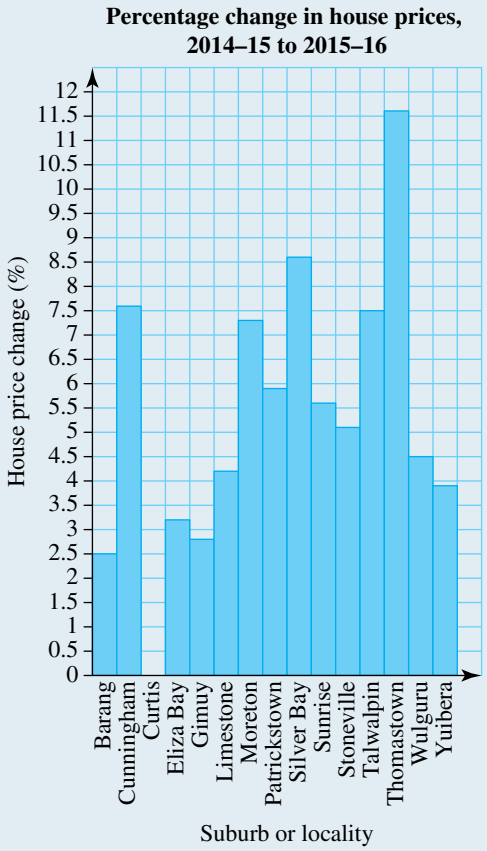
DRAW

a



b To accentuate trends, shorten the horizontal axis and lengthen the vertical axis.

b



WORKED EXAMPLE 7

Consider the data displayed in the table of Worked example 6. Use the data collected for the median house prices in 2015–16.

a Explain whether these data would be classed as primary or secondary data.

b Why do the data show median house prices rather than the mean house price?

c Calculate a measure of central tendency for the data. Explain the reason for this choice.

d Give a measure of spread of the data, giving a reason for the particular choice.

e Display the data in a graphical form, explaining why this particular form was chosen.

THINK	WRITE
a These are data that have been collected by someone else.	**a** These are secondary data because they have been collected by someone else.
b The median is the middle price, the mean is the average price, and the mode is the most frequently occurring price.	**b** The median price is the middle one. It is not affected by outliers as the mean is. The modal house price may only occur for two house sales with the same value. On the other hand, there may not be any mode. The median price is the most appropriate in this case.

c Which measure of central tendency is the most appropriate one?

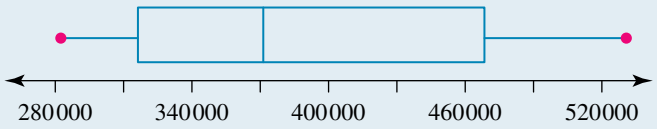
c The measures of central tendency are the mean, median and mode. The mean is affected by high values (e.g. $530 000) and low values (e.g. $282 000). These are not typical values, so the mean would not be appropriate.
There is no modal value, as all the house prices are different.
The median house price is the most suitable measure of central tendency to represent the house prices in the suburbs listed.
The median value is $370 000.

d Consider the range and the interquartile range as measures of spread.

d The five-number summary values are:
Lowest score = $282 000
Lowest quartile = $315 250
Median = $370 000
Upper quartile = $467 500
Highest score = $530 000
Range = $530 000 − $282 000
$248 000
Interquartile range = $467 500 − $315 250
$152 250
The interquartile range is a better measure for the range, as the house prices form a cluster in this region.

e Consider the graphing options.

e Of all the graphing options, the boxplot seems the most appropriate because it shows the spread of the prices, as well as how they are grouped around the median price.



Median house prices 2015–16 ($)

WORKED EXAMPLE 8

The Australian women's national basketball team, the Opals, has 12 team members. Their heights (in metres) have been recorded:

1.85, 1.72, 1.78, 1.93, 1.65, 1.96, 1.85, 1.73, 1.85, 1.88, 2.03, 1.94

Source: http://www.basketball.net.au/ba_player_team/opals/

Provide calculations and explanations as evidence to verify or refute the following statements.

a The mean height of the team is greater than their median height.

b The range of the heights of the 12 players is almost 3 times their interquartile range.

c Given that only 5 players are on the court at any one time, a team of 5 players can be chosen such that their mean, median and modal heights are all the same.

THINK	WRITE
a 1 Calculate the mean height of the 12 players.	**a** Mean $= \frac{\Sigma x}{n} = \frac{22.17}{12} = 1.85$ m

2 Order the heights to determine the median.

The heights of the players, in order, are:
1.65, 1.72, 1.73, 1.78, 1.85, 1.85, 1.88, 1.93, 1.94, 1.96, 2.03
There are 12 scores, so the median is the average of the 6th and 7th scores.

$$\text{Median} = \frac{1.85 + 1.85}{2} = 1.85 \text{ m}$$

3 Comment on the statement.

Both the mean and the median are 1.85 m. This means that the statement is not true.

b 1 Determine the range and the interquartile range of the 12 heights.

b $\text{Range} = 2.03 - 1.65 = 0.38 \text{ m}$
The lower quartile is the average of 3rd and 4th scores.

$$\text{Lower quartile} = \frac{1.73 + 1.78}{2} = 1.755 \text{ m}$$

The upper quartile is average of 3rd and 4th scores from the end.

$$\text{Upper quartile} = \frac{1.93 + 1.94}{2} = 1.935 \text{ m}$$

$\text{Interquartile range} = 1.935 - 1.755 = 0.18 \text{ m}$

2 Compare the two values.

$$\text{Range} = 0.38 \text{ m}$$
$$\text{Interquartile range} = 0.18 \text{ m}$$
$$\frac{\text{Range}}{\text{Interquartile range}} = \frac{0.38}{0.18} = 2.1$$

3 Comment on the statement.

$\text{Range} = 2.11 \times \text{interquartile range}$
This is only 2.11 times, so the statement is not true.

c 1 Choose 5 players whose mean, median and modal heights are all equal. Trial and error is appropriate here. There may be more than one answer.

c Three players have a height of 1.85 m. If a player shorter and one taller are chosen, both the same measurement from 1.85 m, this would make the mean, median and mode all the same.

Choose players with heights:
1.78, 1.85, 1.85, 1.85, 1.93

$$\text{Mean} = \frac{9.26}{5} = 1.852 \text{ m}$$
$\text{Median} = \text{3rd score} = 1.85 \text{ m}$
$\text{Mode} = \text{most frequent score} = 1.85 \text{ m}$

If alternatively the players with heights as follow are chosen:
1.73, 1.85, 1.85, 1.85, 1.96

$$\text{Mean} = \frac{9.24}{5} = 1.848 \text{ m}$$
$\text{Median} = \text{3rd score} = 1.85 \text{ m}$
$\text{Mode} = \text{most frequent score} = 1.85 \text{ m}$

2 Comment on the statement.

In both cases, if the players with heights as listed are chosen and if the mean values are rounded to 2 decimal places, then we will have mean, median and modal heights of 1.85 m.
It is true that a team of 5 such players can be chosen.

Statistical reports

- Reported data must not be simply taken at face value. All reports should be examined with a critical eye.

WORKED EXAMPLE 9

This is an excerpt from an article that appeared in a newspaper on Father's Day. It was reported as a national survey based on the findings of a Gallup Poll of data from 1255 fathers of children aged 17 and under.

THE GREAT AUSSIE DADS SURVEY

Thinking about all aspects of your life, how happy would you say you are?	**%**
I am very happy	**26**
I am fairly happy	**49**
Totally happy	**75**
Some days I'm happy and some days I'm not	**21**
I am fairly unhappy	**3**
I am very unhappy	**1**
Totally unhappy	**4**

How often, if ever, do you regret having children?	
Every day	**1**
Most days	**2**
Some days	**18**
Never	**79**

Which one of these best describes the impact of having children on your relationship with your partner?	
We're closer than ever	**29**
We don't spend as much time together as we should	**40**
We're more like friends now than lovers	**21**
We have drifted apart	**6**
None of the above	**4**

Which one of these best describes the allocation of cooking and cleaning duties in your household?	
My partner does nothing/I do everything	**1**
I do most of it	**11**
We share the cooking and cleaning	**42**
My partner does most of it	**41**
I do nothing/my partner does everything	**4**
None of the above	**1**

Which of these aspects of your children's future do you have concerns about?	
Their safety	**70**
Being exposed to drugs	**67**
Their health	**54**
Bullying or cyber-bullying	**50**
Teenage violence	**50**
Their ability to afford a home	**50**
Alcohol consumption and binge drinking	**47**
Achieving academic success	**47**
Feeling pressured into sex	**41**
Being able to afford the lifestyle they expect to have	**38**
Climate change	**23**
Having them living with you in their mid 20s	**14**
None of the above	**3**

What is the best thing about being a dad?	
The simple pleasures of family life	**61**
Enjoying the successes of your kids	**24**
The unpredictability it brings	**9**
The comfort of knowing that you will be looked after in later life	**3**
None of the above	**3**

Key findings

75% of Aussie dads are totally happy
79% have never regretted having children
67% are worried about their children being exposed to drugs
57% would like more intimacy with their partner

"Work–life balance is definitely an issue for dads in 2010." David Briggs, Galaxy principal

Source: *The Sunday Mail*, 5 September 2010, pp. 14–15.

a Comment on the sample chosen.
b Discuss the percentages displayed.
c Comment on the claim that 57% of dads would like more intimacy with their partner.

THINK	WRITE
a How is the sample chosen? Is it truly representative of the population of Australian dads?	**a** The results of a national survey such as this should reveal the outlook of the whole nation's dads. There is no indication of how the sample was chosen, so without further knowledge we tend to accept that it is representative of the population. A sample of 1255 is probably large enough.
b Look at the percentages in each of the categories.	**b** For the first question regarding happiness, the percentages total more than 100%. It seems logical that, in a question such as this, the respondents would tick only one box, but obviously this has not been the case. In the question regarding aspects of concern of 'your children's future', these percentages also total more than 100%. It seems appropriate here that dads would have more than one concerning area, so it is possible for the percentages to total more than 100%. In each of the other three questions, the percentages total 100%, which is appropriate.
c Look at the tables to try to find the source of this figure.	**c** Examining the reported percentages in the question regarding 'relationship with your partner', there is no indication how a figure of 57% was determined.

Note: Frequently media reports make claims where the reader has no hope of confirming their truth.

WORKED EXAMPLE 10

This article appeared in a newspaper. Read the article, then answer the following questions.

Sydney Metro surveys likened to 'push polling'

Sydney Metro is being accused of 'push polling' the public over the Sydenham to Bankstown section of the $20 billion rail line in telephone surveys critics say omit crucial information about the impact of the project.

Two companies, The Knowledge Warehouse and Newgate Research, have been conducting the surveys to test public attitudes towards the metro, which will replace the Bankstown heavy rail line.

Sydney Metro to deliver new stations

New rail stations will be built at Crows Nest, Victoria Cross, Central, Waterloo, Martin Place, Pitt St and Barangaroo under the Sydney Metro rail project.

The surveys cost $122 500 and involved more than 2800 people who live along the 66 kilometre Sydney Metro route from Rouse Hill in Sydney's northwest, through the city to Bankstown.

After establishing a respondent's attitude, the surveys present a series of statements about the benefits of the Sydney Metro.

They include that Sydney Metro 'will help increase rail capacity across Sydney, allowing the number of trains entering the CBD to go from 120 an hour to 200 during peak times' and highlight 24 hour security at stations and extra car parking in the north-west.

The survey notes that during construction of the city to Bankstown section 'stations between Sydenham and Bankstown will be closed so the line can be converted to Sydney Metro – alternative transport arrangements will be in place during this time'.

However, a community group opposing the Sydenham to Bankstown section argues it fails to mention many negative aspects of the project.

They include a reduction in the number of seats on each train from 896 to 378, the loss of direct access to St Peters, Erskineville, Town Hall, Circular Quay and other City Circle stations and demolition of houses.

'The state government appear to be push-polling the community through this questionnaire, presenting many statements that are at best half-truths,' said Peter Olive, spokesman for the Sydenham to Bankstown Corridor Alliance.

'Shutting down the existing train service between Sydenham and Bankstown and replacing it with another train service, in the same corridor, is a waste of taxpayers' money and just plain stupid. The state government's questionnaire is trying to put spin on this dodgy plan.'

'For commuters between Sydenham and Bankstown the metro will mean fewer seats, loss of access to City Circle Stations, major disruption during construction and the loss of the existing quality service.'

A spokesman for Sydney Metro said community feedback 'is crucial to getting this once-in-a-century project right and shaping it for the future'.

'This questionnaire used industry-standard methods and guidelines to ensure the views of the community were accurately captured to allow planners and designers to take on board the community's feedback,' he said.

'The questionnaire specifically addressed people's concerns including inconvenience caused by construction, quality of public transport, changes to suburbs and the upgrade of two existing railway lines to metro standards.'

Source: The Sydney Morning Herald, 6 February 2017.

a Comment on the sample used in this survey.
b Comment on the claims of the survey.
c Is the heading of the article appropriate?

THINK

a Look at sample size and selection of sample.

WRITE

a The report claims that the sample size was more than 2800 and was selected from people who live along the 66-kilometre Sydney Metro route, from Rouse Hill, in Sydney's northwest, through the city to Bankstown. Two points to keep in mind are whether this sample is truly representative of the population consisting of all people who use the train service, not just those who live along the route, and how many of those sampled gave recordable and worthwhile responses. We have no way of knowing.

b What are the results of the survey?	**b** The stated purpose of the surveys was to test public attitudes towards the Metro, which will replace the Bankstown heavy rail line, but it would appear that the writer of the article focused on the report's statements about the benefits of the project. From this article, we cannot tell what the actual results of the survey were. We can only read that 'After establishing a respondent's attitude, the surveys present a series of statements about the benefits of the Sydney Metro.'
c Examine the heading in the light of the contents of the article.	**c** The heading is sensational, designed to catch the attention of readers. The article focuses on statements made by the spokesman for the Sydenham and Bankstown Corridor Alliance, who alleges that 'The state government appear to be push-polling the community through this questionnaire, presenting many statements that are at best half-truths.' The article details a number of negative aspects of the project that the spokesman says are not mentioned in the report.
d From reading the article, what do you understand by the term 'push polling'?	**d** In broad terms, one meaning could be that the companies have used the poll to achieve a pre-determined outcome that will focus on only the positives of the project.

DISCUSSION

There is much discussion in the media about 'fake news'. How do we determine what is 'fake news' and what is 'authentic news'?

learnON RESOURCES — ONLINE ONLY

Interactivity: Compare statistical reports (int-2790)

Exercise 6.3 Evaluating statistical reports

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–4, 6, 8–10, 12	Questions: 1–4, 7–10, 12	Questions: 1–13

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE4, 5 You have been given an assignment to investigate which year level has the greatest number of students who are driven to school each day by car.
   a. Explain whether it is more appropriate to use primary or secondary data in this case. Justify your choice.
   b. Describe how the data could be collected. Discuss any problems which might be encountered.
   c. Explain whether an alternative method would be just as appropriate.
2. WE6 You run a small company that is listed on the Australian Stock Exchange (ASX). During the past year you have given substantial rises in salary to all your staff. However, profits have not been as spectacular as in the year before. This table gives the figures for the salary and profits for each quarter.

	1st quarter	2nd quarter	3rd quarter	4th quarter
Profits ($'000 000)	6	5.9	6	6.5
Salaries ($'000 000)	4	5	6	7

   Draw two graphs, one showing profits, the other showing salaries, which will show you in the best possible light to your shareholders.
3. WE7 The data below were collected from a real estate agent and show the sale prices of ten blocks of land in a new estate.
   $150 000, $190 000, $175 000, $150 000, $650 000, $150 000, $165 000, $180 000, $160 000, $180 000

   a. Calculate a measure of central tendency for the data. Explain the reason for this choice.
   b. Give a measure of spread of the data, giving a reason for the particular choice.
   c. Display the data in a graphical form, explaining why this particular form was chosen.
   d. The real estate agent advertises the new estate land as:
      Own one of these amazing blocks of land for only $150 000 *(average)!*
      Comment on the agent's claims.
4. WE8 Using the data for the heights of the Opal players in Worked example 8, provide calculations and explanations as evidence to verify or refute the following statements.
   a. The mean height of the team is closer to the lower quartile than it is to the median.
   b. Half the players have a height within the interquartile range.
   c. Which 5 players could be chosen to have the minimum range in heights?

5. The table below shows the number of shoes of each size that were sold over a week at a shoe store.

Size	Number sold
4	5
5	7
6	19
7	24
8	16
9	8
10	7

a. Calculate the mean shoe size sold.
b. Determine the median shoe size sold.
c. Determine the modal shoe size sold.
d. Explain which measure of central tendency has the most meaning to the store proprietor.

6. The batting scores for two cricket players over six innings were recorded as follows.
Player A: 31, 34, 42, 28, 30, 41
Player B: 0, 0, 1, 0, 250, 0
Player B was hailed as a hero for his score of 250. Comment on the performance of the two players.

7. A small manufacturing plant employs 80 workers. The table below shows the structure of the plant's workforce.

Position	Salary ($)	Number of employees
Machine operator	18 000	50
Machine mechanic	20 000	15
Floor steward	24 000	10
Manager	62 000	4
Chief Executive Officer	80 000	1

a. Workers are arguing for a pay rise, but the management of the factory claims that workers are well paid because the mean salary of the factory is $22 100. Explain whether this is a sound argument.
b. Suppose that you were representing the factory workers and had to write a short submission in support of the pay rise. How could you explain the management's claim? Provide some other statistics to support your case.

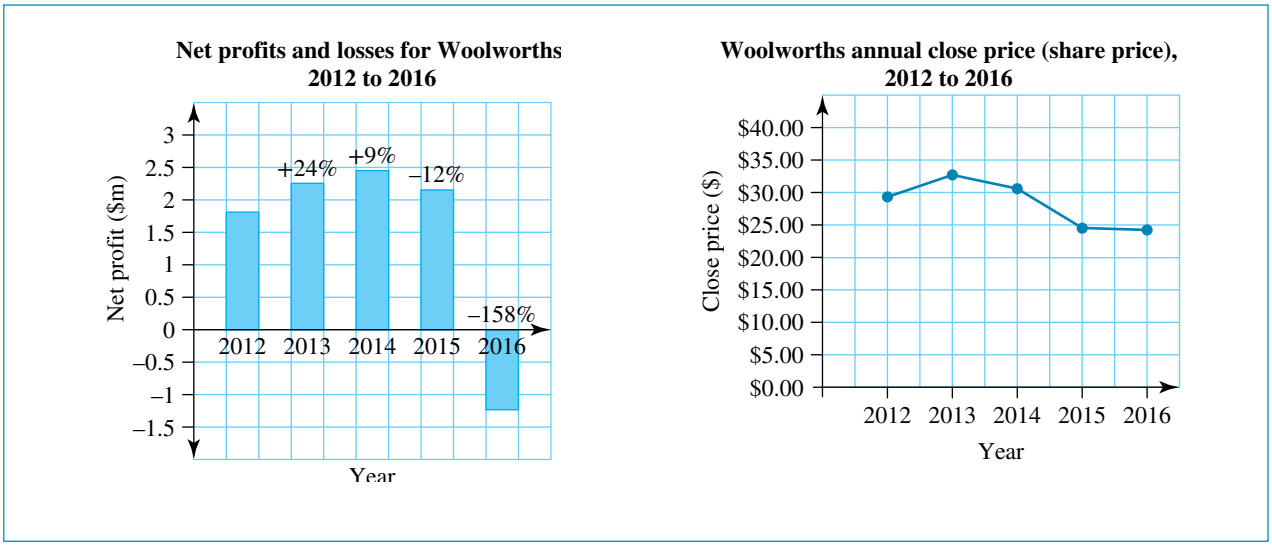
8. WE9, 10 Read the information below and look carefully at the graphs, then answer the following questions.

PROFIT SLIP

On 25 August 2009, Woolworths announced a plan to enter the Australian hardware sector via a joint venture with US-based hardware chain Lowe's. The joint venture was to be called Masters Home Improvement. The Masters chain was operated by Woolworths between September 2011 and December 2016.

On 18 January 2016, Woolworths announced that it was exiting the home improvement business. During 2016, Woolworths incurred costs of $2988.2 million (after tax) relating to its decision to close the Masters chain.

The Woolworths share price at 2:23 pm on 28 April 2017 was $26.88.



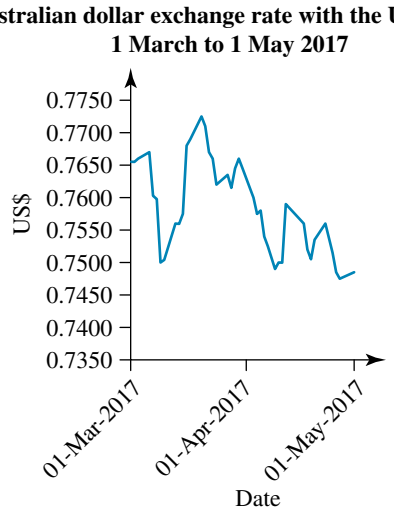
The information provided above paints a picture that could indicate a direct relationship between the profit and share price of Woolworths and the operation of the Woolworths-run home improvement business, Masters.

a. Comment on the above statement and detail any other information that is provided by the graphs.

b. Provide your thoughts as to why the Woolworths share price did not rise sharply after Masters was closed in December 2016.

Communicating, reasoning and problem solving

9. The following graph shows the fluctuation in the Australian dollar in terms of the US dollar during the period 1 March to 1 May 2016. The lower the Australian dollar, the more expensive it is for Australian companies to import goods from overseas, and the more they should be able to charge the Australian public for their goods.



The board of Company XYZ wanted to raise the price of the goods they sold in line with their understanding of the change in exchange rates as shown in the graph. However, the manager of the company produced another graph to support his claim that, because there hadn't been much change in the Australian dollar over that period, there shouldn't be any change in the price he charged for his imported goods to the Australian public. Draw a graph that would support his claim. Explain how you were able to achieve this effect.

10. Two brands of light globes were tested by a consumer organisation. They obtained the following results.

Brand A (Hours lasted)	Brand B (Hours lasted)
385 390 425 426 570	500 555 560 630 720
640 645 730 735 760	735 742 770 820 860

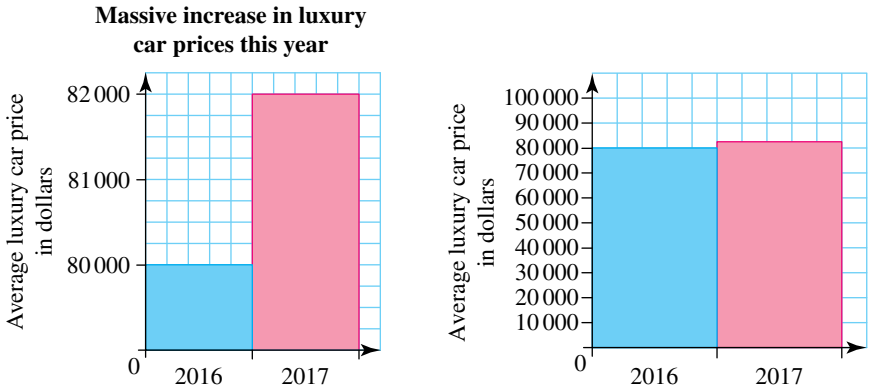
a. Complete a back-to-back stem plot for the data.

b. Which brand had the shortest lifetime?

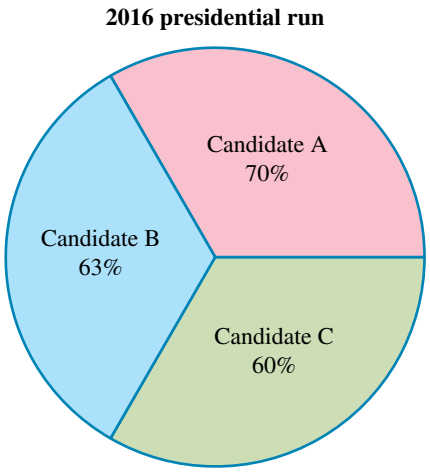
c. Which brand had the longest lifetime?

d. If you wanted to be certain that a globe you bought would last at least 500 hours, which brand would you buy?

11. Look at the following bar charts and discuss why the one on the left is misleading. What characteristics does the graph on the right possess that make it acceptable?



12. a. What is wrong with this pie graph?



b. Why is the following information misleading?

> Did scientists falsify research to support their own theories on global warming?
> 59% somewhat likely
> 35% very likely
> 26% not very likely

c. Discuss the implications of this falsification by statistics.

13. What is the point of drawing a misleading graph in a report?

6.4 Statistical investigations [Stages 5.2 and 5.3]

6.4.1 Using primary data

- This section deals with the steps involved in carrying out a statistical investigation with primary data collection.
- Assume you have been given the task of describing which pizza on the market is the best value for money.
- This is a very broad investigation, and each stage of the investigation must be carefully planned.
  - Collecting the data
  - Organising the data
  - Performing calculations
  - Analysing the data
  - Reporting the results

Collecting the data

- At this initial stage, questions should be posed with regard to the data.
  - What data should be collected?
    - Best value for money involves the price and size of the pizza. Data on both of these need to be collected.
    - Stores have different prices for different sizes.
    - Would size best be measured as area or mass?
    - Not all pizzas are round; some are rectangular.
    - What about the variety of toppings? A standard one should be chosen.
    - Should frozen pizzas be included?
  - How should the data be collected?
    - It is not possible to buy every pizza on the market, so what alternatives are there?
    - A store is probably not willing to allow their pizzas to be weighed, so mass is most likely out of the question.
    - Will the store allow their pizzas to be measured?
  - What problems are likely to be encountered?
    - How many different companies sell pizza?

Organising the data

- The data should be organised into some sort of table format.
  - What format is appropriate for this investigation?
  - A table with column headings 'Price' and 'Measurements' will organise the data.

Price	Measurements	Area	Value for money

- Take time to design the table so figures required for calculation are readily visible.
  - What calculations are required at this stage?
  - Measurements are required to calculate the area of each pizza.
- Think forward and add extra columns for future calculations.
  - What further calculations are needed?
  - 'Area' and 'Value for money' need to be calculated. Provide two extra columns for these.

Performing calculations

- What calculations need to be performed?
- Area and value for money are required in this case. How should these be calculated? Dividing price by area gives results in $\$/cm^2$, whereas dividing area by price gives results in $cm^2/\$$.

Analysing the data

- Are there any anomalies or obvious calculation errors?
- Do the calculated results make sense?
- In this case, if value for money is calculated in units of $\$/cm^2$, the pizza with the smallest of these values is the best value for money. Using units of $cm^2/\$$, the pizza with the highest of these values gives the best value for money.
- Would the inclusion of graphs be appropriate?

Reporting the results

- The results should be reported in a clear, concise manner.
- Justify any conclusions.
- Are there any anomalies or exceptions to mention?

6.4.2 Using secondary data

- The procedure for undertaking a statistical investigation using secondary data is similar to that for primary data, the difference being that you sometimes have to search for data in several areas before you find the appropriate source.
  Suppose you were given this assignment.

> There have been _____ prime ministers of Australia since 1901 until this day.
> There have been _____ elections.
> _____ prime ministers have been defeated at a general election.
> There have been _____ changes of prime minister without an election.
> The average length these prime ministers served in office is _____.
> Undertake a statistical investigation to complete the details.

Collecting the data

- What data should be collected?
- Where can the data be found? The internet is probably a good starting point, but not all sites are reliable.
- If there are multiple sources for the data, are they all in agreement?
- How many of these statements require calculations?

Organising the data

- Design a table to record all the data.
- Consider how many columns are necessary.
- Leave columns for calculations.

Performing calculations

- There is at least one calculation here — to determine the average length of time served in office. Are there any more?

Analysing the data

- Do all the calculated values make sense?
- Would a graph be appropriate?

Reporting the results

- Complete the details.
- Acknowledge the source(s) of the secondary data.

6.4.3 Investigating media reports

- Reports in the media often provide a good starting point for an interesting investigation. Here are a few suggestions.

Media report 1

Here is an article on the number of female MPs in the Turnbull Government. The author is Peta Credlin, a columnist for the Daily Telegraph.

Who's got a woman problem now? It's you, Malcolm

Peta Credlin

THE next time you see footage of the Prime Minister in parliament, take a closer look.

They won't ever admit it but every face you see in the camera shot behind him is deliberately chosen, and they've spent hours doing it.

You might think it's a reward to get one of these prized positions but in fact, it's an indictment; you're either so marginal that it's odds on you will lose your seat at the next election, or you're a woman, and they're desperately trying to make their ranks look healthy when in fact the numbers are in dangerous decline.

With polls consistently backing in Labor, it's well known that the Turnbull Government is in trouble.

Not only is it likely that the Coalition will suffer a generational wipe-out of female MPs at the next election but women voters are leaving the Coalition in droves.

For the man who likes to think he's got the Midas touch when it comes to women, Malcolm Turnbull's result in the Australian Electoral Study is the worst of any Liberal leader in 30 years.

According to the AES which analyses voter data via a significant survey taken immediately after each federal election, Turnbull's 2016 result saw the lowest number of women vote Liberal-National since the survey started in 1987.

In fact, at 35 per cent, his result among women is the only time the Coalition has fallen below 40 per cent over the same period with every other Liberal predecessor at least six percentage points above Turnbull's result.

Throughout his time in office, John Howard collected between 55–48 per cent of the female vote and even his 1987 election loss, at 43 per cent, was eight points better than Turnbull's 'win' last year.

What about Tony Abbott? In 2013, he secured 45 per cent of the female vote, up from 41 per cent in 2010 election survey. Indeed, this analysis busts the myth that between the two men, Abbott was the one with the 'woman problem' because the survey shows his result in 2013 bettered Turnbull's by eight plus percentage points in every age bracket between 18–64 years.

Looking at these AES results, and the Coalition's ranks, it's clear the Turnbull Government has got a problem with women that it must urgently address.

At present, the Government has 76 members in the House of Representatives comprising 63 men, and only 13 women (see now why the seats behind the PM matter?).

If the polling trend continues, it's likely it will head into opposition at the next election with the number of female lower house MPs in single digits.

With more women in the population than men, more women coming through our schools and universities than ever before, this is an unacceptable reality for any political party claiming to represent mainstream modern Australia.

The Coalition has a proud record of female achievement but it has fallen dismally behind when it comes to getting women into parliament.

While significant appointments might be worthy, they don't seem to have impacted how women see the Coalition. Indeed, while the Coalition's had its first female deputy leader for a decade now, things have actually gone backwards.

There's been great work done by the Menzies Research Centre on the reasons why but those in charge of the party organisation have been slow to take them up.

I'm pleased that Abbott's target of 50 per cent female representation by 2025 has been reaffirmed by Turnbull but as I bluntly told the former PM at the time, it means nothing unless there's practical measures to make it happen, and plenty of transparency too.

I'm still not sure if the Coalition is serious when it comes to boosting the number of women in parliament. This week's factional game to undermine and damage Kelly O'Dwyer when there was never a challenger is a case in point.

Maybe the fact that blokes might actually lose their own seats, and government, as female voters desert the Coalition will be enough to shock them into action?

Let's hope so because this isn't just crisis time for women in the Coalition, it's a crisis for the Coalition itself.

Source: *The Daily Telegraph*

- Many articles such as the one above are written and published every day in newspapers, and websites. How could you find evidence of the truth of some of the statements in the article?

Media report 2

How money changes your brain, behaviour and beliefs

I recently went to see Cyndi Lauper and Blondie play at the Bimbadgen Estate in the Hunter Valley.

It was a fabulous afternoon with great music and plenty of good wine shared with friends. But when Lauper sang 'Money Changes Everything' from her 1983 album *She's So Unusual*, I started thinking.

Is it true?

Certainly money – and an older target audience – is likely behind the fact that the expensive seated area was in front of the stage where the dance floor should be. That made me a little sad.

We often hear stories about families tearing themselves apart over an inheritance, or that stress over money problems is one of the highest causes of relationship breakdown.

But can having money change your core personality and values?

Cyndi Lauper sang 'Money Changes Everything'. It turns out she was right.

Social psychologist Paul Piff gave an interesting TEDx Talk in 2013 about how wealth can change someone's behaviour and beliefs about themselves.

He cited one study in the United States in which researchers brought rich and poor members of the community into the lab, and gave each of them the equivalent of $US10. The participants were told they could keep the $10 for themselves, or share a portion of it with an anonymous stranger.

Individuals who made $US25,000 a year gave 44 per cent more of their money to the stranger than individuals making $US150,000 or more a year.

Another study in California suggested that drivers in cheap cars were more likely to stop for pedestrians than those in expensive cars.

But none of this is set in stone. It's not that rich people are born with different attitudes, or that they are immutably set from their upbringing; the evidence suggests they're reacting to their changed circumstances and can shift back again.

The most striking study was one at UC Berkeley where more than 100 strangers played rigged games of Monopoly. One player was given twice as much starting cash and, when they passed Go, they collected twice the salary. They also got to roll both dice instead of one, so they got to move around the board more quickly.

As the game went on, the rich players started to assert their dominance: moving their pieces more loudly, talking about how well they were going, being rude to the other player, and eating more of the pretzels provided for both players.

At the end the researchers asked the players to talk about their experiences during the game. The rich players – who'd been given a huge advantage by the flip of a coin – talked about the strategies they used to buy different properties and how they earned their success in the game.

I've written before about how people tend to undervalue the importance of luck in their own success.

The good news is that people with wealth can remain generous and grateful if they work at it.

In 2010, a series of studies out of UCSF found lower-class participants were better able to read the facial expressions of people in photos, and of strangers in mock interviews, to discern their emotions.

When upper-class participants were told to imagine themselves in the position of lower-class people, it boosted their ability to detect other people's emotions.

This suggests that the effects of wealth on our behaviour are not hardwired, but we can boost our emotional intelligence through deliberate practice.

Piff talked about another study in his lab in which people watched a brief video, just 46 seconds long, about childhood poverty. After that the researchers looked at how willing people were to give up their time to a person in distress.

After watching the video, rich people became just as generous of their own time to help out a stranger as poor people. Again, this suggests that the differences between rich and poor people can be altered or influenced by small changes.

Unfortunately, while poor people may find it easier to understand how others are feeling, they may also find it harder to make good decisions if they're stressed about money.

A 2013 study published in Science suggested that lack of money or time could have negative effects on the brain, possibly because poverty imposes a cognitive load that saps attention and reduces effort.

The researchers looked at shoppers in a New Jersey mall and farmers in Tamil Nadu, India. They found that considering a projected financial decision, such as how to pay for a car repair, affects people's performance on unrelated spatial and reasoning tasks. Both lower-income and higher-income individuals did fine if the cost was low, but only lower-income people performed poorly if the repairs were expensive.

Similarly, the sugarcane farmers from Tamil Nadu performed these tasks better after harvest than before.

It's unfashionable these days to talk about such things as 'character' and 'virtue', unless of course it's framed as 'self improvement' or 'self help'.

If you're taking the advice in the Money section, hopefully you're increasing your personal wealth as a result. There's nothing wrong with that at all.

But if you don't want money to warp your sense of self, I believe these studies are a good reminder to consciously cultivate empathy and respect for others. It turns out being a good person isn't innate; it takes practice.

I've always disliked the attention-seeking nature of politicians and business leaders volunteering in soup kitchens or sleeping rough for the Vinnies CEO Sleepout, but maybe I should rethink my natural opposition.

We'd also do well to think about how this works on a societal level, as the trend to inequality accelerates.

If Australia is losing its egalitarian nature, is that why politics is becoming so polarised?

Source: *The Sydney Morning Herald*

- Do you agree with the contents of the article above? How correct do you believe the author's conclusions are?

Media report 3

Get rich by focusing on process not outcome

Noel Whittaker

Every year has two days when we are reminded to reflect on what has passed, and what we can change in the future. They are January 1, the start of a new calendar year, and July 1, the start of a new financial year. Here we are, just two months from the end of the current financial year.

By now, January is probably a dim memory, and all those New Year's resolutions you made to lose weight and get your finances in order may have gone with it. You're probably berating yourself with statements like, 'I've got no willpower' or 'this happens every year'.

Take heart — the sad truth is that the human body is not wired for long-term planning. Our ancestors were hunters and gatherers who lived by the rule of fight or flight. Their dominant thoughts were purely about survival.

As a result, we instinctively prefer an action with a fast pay-off, over one with a long-term result. The scientific name for it is hyperbolic discounting – it causes people to make choices that can lead to short-term pleasure, but long-term disaster.

Credit card usage is an obvious example. Who cares about paying interest at 20 per cent on their credit card balance, living beyond their means, or getting into financial strife when they can simply swipe their credit card and get a retail fix on the spot?

Research from Johns Hopkins University reveals that only one in 10 coronary bypass patients make the necessary changes to their lifestyle to prevent further attacks. The remaining nine in 10 still opt for the short-term pleasures of unhealthy food and no exercise.

To make it more difficult, long-term progress by its nature is slow and erratic, and is often discouraging. Imagine you got excited about investing $500 a month into a managed fund that matched the All Ordinaries Index. If the market had a great year and produced 12 per cent compound you would have $6341 at year's end. The profit would be just $341. However, if the market had a bad year and went backwards by 5 per cent, your portfolio would be worth $5864. The difference is minimal.

This is the point where most people give up, and move onto something else with the lure of a quick high return. However, if you kept investing that $500 a month for 35 years, and the investment averaged 9 per cent a year, the portfolio would grow to $1.4 million.

It works the same when you are paying off a mortgage. If you owed $300, 000 on your home at 5.5 per cent with monthly repayments of $1703, over the term of 30 years total interest payable would be $314,000.

Suppose you learned about the effect of compound interest and decided to slash your home loan to 20 years by raising your payments to $2064 a month, which would save over $119, 000 in interest. It's a most exciting prospect, but after five long years at the higher payments, you would still owe $253,000, and may well be starting to feel the result isn't worth the effort. But hang in there for another 20 years and it would be paid off. In contrast, leaving the payments at $1703 you would still owe $157,000 in 20 years' time.

This leads us to a fundamental success principle, which is applicable in every aspect of your life – focus on the process and not the outcome. Because success comes slowly, you will almost certainly get discouraged and probably give up if you keep thinking about the outcome. It's like planting a seedling and then digging it up every year to see if it's growing. The secret is to get excited about the process, in the knowledge that the right process, if followed through, will almost always lead to the outcome you are looking for.

Source: *The Sydney Morning Herald*

- The article above has a great deal of sound financial advice. Review the article and provide a summary of the main points made by the author.

ACTIVITY: FACEBOOK USAGE

Below are a few statistics on Facebook users. This data was reported in 2017.

- There are more than 1.86 billion active users.
- In Europe over 307 million people are on Facebook
- More than 300 million photos are uploaded to the site each month.

Write a report summarising the usage of Facebook throughout the world. How will you know that the statistics you are using are correct?

Exercise 6.4 Statistical investigations

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–7, 9	Questions: 1–9	Questions: 1–11

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. a. Write a plan detailing how you would collect primary data to undertake an investigation to determine which pizza on the market is the best value for money.
   b. Undertake your investigation.
   c. Report on the results of your findings.
2. Undertake an investigation on the history of prime ministers in Australia. Report your findings.
3. Find evidence of the facts presented by Peta Credlin in her article entitled 'Who's got a woman problem now? It's you, Malcolm' (Media report 2). Provide your thoughts as to the accuracy of her claims.
4. In Media report 2, the author states: 'Cyndi Lauper sang 'Money Changes Everything'. It turns out she was right.' Comment on this statement.
5. Comment on the items in the summary you prepared after reading the article 'Get rich by focusing on process not outcome' (Media report 3).
6. Search for a media article you would like to investigate. Provide a full report on your findings.

Communicating, reasoning and problem solving

7. An investigation is to be conducted to find the two most popular television programs in Australia. For each of the following samples, explain why they would be biased.
   a. A sample of 100 students from a city secondary school.
   b. A sample of 100 people passing a certain point in a busy city street at lunchtime.
   c. How would you go about selecting a sample of people?
8. There has been a rise in supermarket-owned brands in Australia. These are commonly available in supermarkets such as Woolworths, Coles and Aldi. It has been said that these brands account for almost one-quarter of all grocery sales. It has also been claimed that the quality of supermarket-owned brands is comparable with the equivalent market-leading brands, although the supermarket-owned brands are much cheaper.

   Assume you are planning to undertake a study of a particular grocery line (e.g. baked beans or breakfast cereal). Write a plan of how you would undertake this study.
9. You wish to assess the opinion of a local population on the possible closure of their hospital.
   a. What target population would you use?
   b. What resources would you use?
   c. What are the possible biases in conducting such an experiment?
10. What would you consider to be the most important factor in reporting the results of a statistical investigation?
11. List five positive even integers that have a mean of 8 and a median of 10. How many possible solutions are there?

6.5 Estimating population means and medians [Stages 5.2 and 5.3]

6.5.1 Populations

- The term **population** refers to a complete set of individuals, objects or events belonging to some category.
- When data are collected from a whole population, the process is known as a **census**.
  - It is often not possible, nor cost-effective, to conduct a census.
  - For this reason, **samples** have to be selected carefully from the population. A sample is a subset of its population.

WORKED EXAMPLE 11

List some of the problems you might encounter in trying to collect data on the following populations.
a The life of a mobile phone battery
b The number of possums in a local area
c The number of males in Australia
d The average cost of a loaf of white bread

THINK	WRITE
For each of these scenarios, consider how the data might be collected, and the problems in obtaining these data.	
a The life of a mobile phone battery	**a** The life of a mobile phone battery cannot be measured until it is dead. The battery life also depends on how the phone is used, and how many times it has been recharged.
b The number of possums in a local area	**b** It would be almost impossible to find all the possums in a local area in order to count them. The possums also may stray into other areas.
c The number of males in Australia	**c** The number of males in Australia is constantly changing. There are births and deaths every second.
d The average cost of a loaf of white bread	**d** The price of one particular loaf of white bread varies widely from one location to another. Sometimes the bread is on 'special' and this would affect the calculations.

6.5.2 Why the deviations are squared

- Surveys are conducted using samples. Ideally the sample should reveal generalisations about the population.
- A random sample is generally accepted as being an ideal representation of the population from which it was drawn. However, it must be remembered that different random samples from the same population can produce different results. This means that we must be cautious about making predictions about a population, as results of surveys conducted using random samples may vary.
- A sample size must be sufficiently large. As a general rule, the sample size should be about $\sqrt{N}$, where N is the size of the population.

WORKED EXAMPLE 12

A die was rolled 50 times and the following results were obtained.
6 5 3 1 6 2 3 6 2 5 3 4 1 3 2 6 4 5 5 4 3 1 2 1 6 4 5 2
3 6 1 5 3 3 2 4 1 4 2 3 2 6 3 4 6 2 1 2 4 2.

a Determine the mean of the population (to 1 decimal place).

b A suitable sample size for this population would be 7($\sqrt{50} \approx 7.1$).

i Select a random sample of 7 scores and determine the mean of these scores.

ii Select a second random sample of 7 scores and determine the mean of these.

iii Select a third random sample of 20 scores and determine the mean of these.

c Comment on your answers to parts a and b.

THINK	WRITE
a Calculate the mean by first finding the sum of all the scores, then dividing by the number of scores (50).	**a** Population mean $= \frac{\Sigma x}{n}$ $= \frac{169}{50}$ $= 3.4$
b i Use a calculator to randomly generate 7 scores from 1 to 50. Relate these numbers back to the scores, then calculate the mean.	**b i** The 7 scores randomly selected are numbers 17, 50, 11, 40, 48, 12, 19 in the set of 50 scores. These correspond to the scores 4, 2, 3, 3, 2, 4, 5. The mean of these scores $= \frac{23}{7} = 3.3$.
ii Repeat bi to obtain a second set of 7 randomly selected scores. This second set of random numbers produced the number 1 twice. Try again. Another attempt produced the number 14 twice. Try again. A third attempt produced 7 different numbers. This set of 7 random numbers will then be used to, again, calculate the mean of the scores.	**ii** Ignore the second and third attempts to select 7 random numbers because of repeated numbers. The second set of 7 scores randomly selected is numbers 16, 49, 2, 42, 31, 11, 50 of the set of 50. These correspond to the scores 6, 4, 5, 6, 1, 3, 2. The mean of these scores $= \frac{27}{7} = 3.9$.
iii Repeat for a randomly selected 20 scores.	**iii** The set of 20 randomly selected numbers produced a total of 68. Mean of 20 random scores $= \frac{68}{20} = 3.4$
c Comment on the results.	**c** The population mean is 3.4. The means of the two samples of 7 are 3.3 and 3.9. This shows that, even though the samples are randomly selected, their calculated means may be different. The mean of the sample of 20 scores is 3.4. This indicates that by using a bigger sample the result is more accurate than those obtained with the smaller samples.

ACTIVITY: SAMPLING ERRORS, WAY TO BE RANDOM!

Equipment: pen and paper

1. In groups of 3, think of 3 different ways that you could randomly select 10 people from a group of 50 people (25 males and 25 females ranged in age from 20 to 70) to survey their thoughts on global warming.
2. What factors do you need to consider? What would be considered a random sample? List any possible bias that could come from this group.
3. Using the information in the table below, trial your 3 methods of random sampling and comment on the groups you obtain.

Male	45 68 21 36 70 22 23 37 41 55 59 66 29 33 65 51 24 70 48 26 32 21 57 42 61
Female	33 32 57 52 44 68 70 61 23 25 57 36 41 49 26 47 55 68 70 20 34 49 51 67 42

4. As a class, discuss why it is important that randomly chosen data values have to have an equally likely chance of being selected.

6.5.3 To sample or to conduct a census?

- The particular circumstances determine whether data are collected from a population, or from a sample of the population. For example, suppose you collected data on the height of every Year 10 student in your class. If your class was the only Year 10 class in the school, your class would be the population. If, however, there were several Year 10 classes in your school, your class would be a sample of the Year 10 population.
- Worked example 12 showed that different random samples can produce different results. For this reason, it is important to acknowledge that there could be some uncertainty when using sample results to make predictions about the population.

DISCUSSION

Why are samples collected instead of a census, which gives a complete picture of a data set?

WORKED EXAMPLE 13

For each of the following situations, state whether the information was obtained by census or survey. Justify why that particular method was used.

a A roll call is conducted each morning at school to determine which students are absent.
b TV ratings are collected from a selection of viewers to discover the popular TV shows.
c Every hundredth light bulb off an assembly production line is tested to determine the life of that type of light bulb.
d A teacher records the examination results of her class.

THINK	WRITE
a Every student is recorded as being present or absent at the roll call.	**a** This is a census. If the roll call only applied to a sample of the students, there would not be an accurate record of attendance at school. A census is essential in this case.
b Only a selection of the TV audience contributed to these data.	**b** This is a survey. To collect data from the whole viewer population would be time-consuming and expensive. For this reason, it is appropriate to select a sample to conduct the survey.

c Only 1 bulb in every 100 is tested.

c This is a survey. Light bulbs are tested to destruction (burn-out) to determine their life. If every bulb was tested in this way, there would be none left to sell! A survey on a sample is essential.

d Every student's result is recorded.

d This is a census. It is essential to record the result of every student.

learnON RESOURCES — ONLINE ONLY

Interactivity: Sample sizes (int-6183)
Digital doc: SkillSHEET Determining suitability of questions for a survey (doc-5337)
Digital doc: SkillSHEET Finding proportions (doc-5338)
Digital doc: SkillSHEET Distinguishing between types of data (doc-5339)
Digital doc: SkillSHEET Reading bar graphs (doc-5340)

Exercise 6.5 Estimating population means and medians

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–3, 5, 7, 10	Questions: 1–4, 6, 8, 11, 14	Questions: 1–3, 5, 7, 9–15

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE11 List some of the problems you might encounter in trying to collect data from the following populations.
   a. The life of a laptop computer battery
   b. The number of dogs in your neighbourhood
   c. The number of fish for sale at the fish markets
   d. The average number of pieces of popcorn in a bag of popcorn
2. WE12 A die was rolled 50 times and the following results were obtained.
   6 5 3 1 6 2 3 6 2 5 3 4 1 3 2 6 4 5 5 4 3 1 2 1 6 4 5 2 3 6 1 5 3 3 2
   4 1 4 2 3 2 6 3 4 6 2 1 2 4 2
   The mean of the population is 3.4. Select your own samples for the following questions.
   a. Select a random sample of 7 scores and determine the mean of these scores.
   b. Select a second random sample of 7 scores and determine the mean of these.
   c. Select a third random sample of 20 scores and determine the mean of these.
   d. Comment on your answers to parts **a**, **b** and **c**.
3. WE13 In each of the following scenarios, state whether the information was obtained by census or survey. Justify why that particular method was used.
   a. Seating for all passengers is recorded for each aeroplane flight.
   b. Movie ratings are collected from a selection of viewers to discover the best movies for the week.
   c. Every hundredth soft drink bottle on an assembly production line is measured to determine the volume of its contents.
   d. A car driving instructor records the number of hours each learner driver has spent driving.

4. For each of the following, state whether a census or a survey has been used.

a. Two hundred people in a shopping centre are asked to nominate the supermarket where they do most of their grocery shopping.

b. To find the most popular new car on the road, 500 new car buyers are asked what make and model they purchased.

c. To find the most popular new car on the road, data are obtained from the transport department.

d. Your Year 10 Maths class completed a series of questions on the amount of maths homework for Year 10 students.

5. To conduct a statistical investigation, Gloria needs to obtain information from 630 students.

a. What size sample would be appropriate?

b. Describe a method of generating a set of random numbers for this sample.

6. A local council wants the opinions of its residents regarding its endeavours to establish a new sporting facility for the community. It has specifically requested all residents over 10 years of age to respond to a set of on-line questions.

a. Is this a census or a survey?

b. What problems could you encounter collecting data this way?

7. A poll was conducted at a school a few days before the election for Head Boy and Head Girl. After the election, it was discovered that the polls were completely misleading. Explain how this could have happened.

8. A sampling error is said to occur when results of a sample are different from those of the population from which the sample was drawn. Discuss some factors which could introduce sampling errors.

Communicating, reasoning and problem solving

9. Since 1961, a census has been conducted in Australia every 5 years. Some people object to the census on the basis that their privacy is being invaded. Others say that the expense involved could be directed to a better cause. Others say that a sample could obtain statistics which are just as accurate. What are your views on this? Justify your statements.

10. Australia has a very small population compared with other countries such as China and India. These are the world's most populous nations, so the problems we encounter in conducting a census in Australia would be insignificant compared with those encountered in those countries. What different problems would authorities come across when conducting a census there?

11. The game of Lotto involves picking the same 6 numbers in the range 1 to 45 as have been randomly selected by a machine containing 45 numbered balls. The balls are mixed thoroughly, then 8 balls are selected representing the 6 main numbers, plus 2 extra numbers, called supplementary numbers.

The following two lists show the number of times each number had been drawn over a period of time, and the number of weeks since each particular number has been drawn.

NUMBER OF WEEKS SINCE EACH NUMBER DRAWN

1 **1**	2 **5**	3 **2**	4 **1**	5 **1**	6 **7**	7 **-**	8 **4**
9 **3**	10 **3**	11 **1**	12 **5**	13 **5**	14 **7**	15 **-**	16 **4**
17 **9**	18 **-**	19 **9**	20 **2**	21 **2**	22 **12**	23 **10**	24 **8**
25 **5**	26 **11**	27 **17**	28 **2**	29 **3**	30 **3**	31 **-**	32 **22**
33 **4**	34 **3**	35 **-**	36 **1**	37 **12**	38 **-**	39 **6**	40 **-**
41 **6**	42 **1**	43 **7**	44 **-**	45 **31**			

NUMBER OF TIMES EACH NUMBER DRAWN SINCE DRAW 413

1 **246**	2 **238**	3 **244**	4 **227**	5 **249**	6 **241**	7 **253**	8 **266**
9 **228**	10 **213**	11 **250**	12 **233**	13 **224**	14 **221**	15 **240**	16 **223**
17 **217**	18 **233**	19 **240**	20 **226**	21 **238**	22 **240**	23 **253**	24 **228**
25 **252**	26 **239**	27 **198**	28 **229**	29 **227**	30 **204**	31 **230**	32 **226**
33 **246**	34 **233**	35 **232**	36 **251**	37 **222**	38 **221**	39 **219**	40 **259**
41 **245**	42 **242**	43 **237**	44 **221**	45 **224**			

If these numbers are randomly chosen, explain the differences shown in the tables.

12. A sample of 30 people was selected at random from those attending a local swimming pool. Their ages (in years) were recorded as follows:
19, 7, 58, 41, 17, 23, 62, 55, 40, 37, 32, 29, 21, 18, 16, 10, 40, 36, 33, 59, 65, 68, 15, 9, 20, 29, 38, 24, 10, 30.

a. Find the mean and the median age of the people in this sample.
b. Group the data into class intervals of 10 (0–9 etc.) and complete the frequency distribution table.
c. Use the frequency distribution table to calculate an estimate of the mean age.
d. Calculate the cumulative frequency and, hence, plot the ogive.
e. Estimate the median age from the ogive.
f. Compare the mean and median of the original data in part **a** with the estimates of the mean and the median obtained for the grouped data in parts **c** and **e**.
g. Were the estimates good enough? Explain your answer.

13. The typing speed (words per minute) was recorded for a group of Year 8 and Year 10 students. The results are displayed in this back-to-back stem plot.

Key: 2|6 = 26 wpm

Leaf: Year 8	*Stem*	*Leaf: Year* 10
9 9	0	
9 8 6 5 4 2 0	1	7 9
9 8 8 6 4 2 1 0 0	2	2 3 6 8 9
9 7 7 6 4 1 0	3	0 2 4 5 5 7 8 8
8 6 5 2 0	4	1 2 5 8 8 9 9
	5	0 3 5 7 8
	6	0 0 3

Write a report comparing the typing speeds of the two groups.

14. A well-known saying about statistics is: *Statistics means never having to say you're certain*. What does this saying mean?

15. A fisheries and wildlife officer released 200 tagged trout into a lake. A week later, the officer took a sample of 50 trout and found that 8 of them were tagged. The officer can use this information to estimate the population of trout in the lake. How many trout are now in the lake?

6.6 Review

Investigation | Rich task

Populations and samples

In this task we will see how closely a sample resembles the population. (Digital technology should be used to answer the following questions.)

The following table gives information about literacy rates (as percentages) for the entire population in 100 countries and for males and females in each country. This task will only use the first column of literacy rates for the country. Other conclusions may be drawn from the male and female literacy rates and possibly compared to the overall literacy rates in this investigation.

1. Find the five-figure summary for the 100 countries' literacy rates and draw a boxplot.
2. Find the mean and standard deviation.
3. Take a sample of 20 from the set of 100 countries using a random number generator and make a list of the countries and their literacy rates.
4. Find the five-number summary for your sample.
5. Find the mean and standard deviation for your sample.

6. Draw a boxplot that compares your sample with the whole population.
7. Describe the sample data obtained in terms of the population as a whole. In particular, concentrate on the middle 50 per cent of the data.
8. Take another two random samples of 20 countries and draw boxplots of all three on the same scale. Comment on these three samples.
9. Find the mean and standard deviation for the three samples. How do they compare?
10. From the results you have obtained, how would you describe the reliability of a sample compared with using the whole population?

	Country	Literacy (%)	Male literacy (%)	Female literacy (%)
1	Afghanistan	29	44	14
2	Argentina	98	99	98
3	Armenia	98	100	100
4	Australia	100	100	100
5	Azerbaijan	98	100	100
6	Bahrain	77	55	55
7	Bangladesh	35	47	22
8	Barbados	99	99	99
9	Belarus	99	100	100
10	Bolivia	78	85	71
11	Botswana	72	32	16
12	Brazil	81	82	80
13	Bulgaria	98	99	98
14	Burkina Faso	24	34	14
15	Burundi	50	61	40
16	Cambodia	35	48	22
17	Cameroon	54	66	45
18	Canada	97	97	97
19	Cent. Afri. R	27	33	15
20	Chile	93	94	93
21	China	78	87	68
22	Colombia	87	88	86
23	Costa Rica	93	93	93
24	Croatia	98	99	97
25	Cuba	94	95	93
26	Dominican R.	83	85	82
27	Ecuador	88	90	86
28	Egypt	48	63	34
29	El Salvador	73	76	70
30	Estonia	99	100	100
31	Ethiopia	24	32	16
32	Finland	100	100	100
33	France	99	99	98
34	Gabon	61	74	48
35	Gambia	27	39	16
36	Georgia	99	100	100
37	Germany	99	99	98
38	Greece	93	98	89
39	Guatemala	55	63	47
40	Haiti	53	59	47
41	Honduras	73	76	71
42	Hungary	99	99	98
43	Iceland	100	100	100
44	India	52	64	39
45	Indonesia	77	84	68

	Country	Literacy (%)	Male literacy (%)	Female literacy (%)
46	Iran	54	64	43
47	Iraq	60	70	49
48	Ireland	98	99	97
49	Israel	92	95	89
50	Italy	97	98	96
51	Japan	99	99	99
52	Jordan	80	89	70
53	Kenya	69	80	58
54	Kuwait	73	77	67
55	Latvia	99	100	100
56	Lebanon	80	88	73
57	Liberia	40	50	29
58	Libya	64	75	50
59	Lithuania	99	99	98
60	Malaysia	78	86	70
61	Mexico	87	90	85
62	Morocco	50	61	38
63	N. Korea	99	99	99
64	Netherlands	100	100	100
65	New Zealand	99	99	99
66	Nicaragua	57	57	57
67	Nigeria	51	62	40
68	Norway	99	99	99
69	Oman	71	80	62
70	Pakistan	35	47	21
71	Panama	88	88	88
72	Paraguay	90	92	88
73	Peru	85	92	79
74	Philippines	90	90	90
75	Poland	99	99	98
76	Portugal	85	89	82
77	Russia	99	100	100
78	Rwanda	50	64	37
79	S. Korea	96	99	99
80	Saudi Arabia	62	73	48
81	Senegal	38	52	25
82	Singapore	88	93	84
83	Somalia	24	36	14
84	South Africa	85	86	86
85	Spain	95	97	93
86	Sweden	99	99	99
87	Switzerland	99	99	99
88	Syria	64	78	51
89	Thailand	93	96	90
90	Turkey	81	90	71
91	U. Arab Em.	68	70	63
92	UK	99	99	99
93	USA	97	97	97
94	Uganda	48	62	35
95	Ukraine	97	100	100
96	Uruguay	96	97	96
97	Uzbekistan	97	100	100
98	Venezuela	88	90	87
99	Vietnam	88	93	83
100	Zambia	73	81	65

learnON RESOURCES — ONLINE ONLY

Digital doc: Investigation — Populations and samples (doc-15942)
Digital doc: Code puzzle — Animals that originated in Australia include… (doc-15943)
Digital doc: Summary Topic 6 (doc-22904)
Interactivity: Word search Topic 6 (int-2865)
Interactivity: Crossword Topic 6 (int-2866)
Interactivity: Sudoku Topic 6 (int-3601)

Exercise 6.6 Review questions

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. List some problems you might encounter in trying to collect data from the following populations.
- **a.** The average number of mL in a can of soft drink
- **b.** The number of fish in a dam
- **c.** The number of workers who catch public transport to work each weekday morning

2. a. Calculate the mean of the integers 1 to 100.
- **b. i.** Randomly select 10 numbers in the range 1 to 100.
- **ii.** Calculate the mean of these numbers.
- **c. i.** Randomly select 20 numbers in the range 1 to 100.
- **ii.** Calculate the mean of these numbers.
- **d.** Comment on the similarities/differences between your means calculated in parts **a**, **b** and **c**.

3. For each of the following investigations, state whether a census or a survey has been used.
- **a.** The average price of petrol in Canberra was estimated by averaging the price at 30 petrol stations in the area.
- **b.** The performance of a cricketer is measured by looking at his performance in every match he has played.
- **c.** Public opinion on an issue is sought by a telephone poll of 2000 homes.

4. Traffic lights (red, amber, green) are set so that each colour shows for a set amount of time. Describe how you could use a spinner to simulate the situation so that you could determine (on average) how many sets of lights you must encounter in order to get two green lights in succession.

5. MC John and Bill play squash each week. In any given game they are evenly matched. A device that could not be used to represent the outcomes of the situation is:
- **A.** a die
- **B.** a coin
- **C.** a circular spinner divided into 2 equal sectors
- **D.** a circular spinner divided into 5 equal sectors
- **E.** a circular spinner divided into 4 equal sectors

6. The table at right shows the number of students in each year level from Years 7 to 12.
Draw two separate graphs to illustrate the following.
- **a.** The principal of the school claims a high retention rate in Years 11 and 12 (that is, most of the students from Year 10 continue on to complete Years 11 and 12).
- **b.** The parents claim that the retention rate of students in Years 11 and 12 is low (that is, a large number of students leave at the end of Year 10).

Year	Number of students
7	230
8	200
9	189
10	175
11	133
12	124

7. Records from a school were examined to determine the number of absent days of both boys and girls over the two years of Year 9 and Year 10. The result is shown in this stem-and-leaf plot.

Key: 2|1 = 21 days

Leaf: Boys	*Stem*	*Leaf: Girls*
	0	1 7
7 4 1 0	1	2 4 7 9 9
9 9 7 6 6 5 3 1 1 0	2	1 3 3 4 6 6
8 7 7 5 2	3	4 4 4 8
2	4	3 6
	5	4

a. Calculate the median number of days absent for both boys and girls.
b. Calculate the range for both boys and girls.
c. Comment on the distribution of days absent for each group.

8. Fifteen boys and fifteen girls were randomly chosen from a group of 900 students. Their heights (in metres) were measured as shown below.

Boys: 1.65, 1.71, 1.59, 1.74, 1.66, 1.69, 1.72, 1.66, 1.65, 1.64, 1.68, 1.74, 1.57, 1.59, 1.60

Girls: 1.66, 1.69, 1.58, 1.55, 1.51, 1.56, 1.64, 1.69, 1.70, 1.57, 1.52, 1.58, 1.64, 1.68, 1.67

a. Comment on the size of the sample.
b. Display the data as a back-to-back stem plot.
c. Compare the heights of the boys and girls.

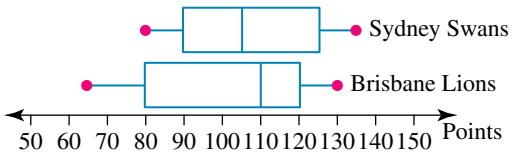
9. The stem plot below is used to display the number of vehicles sold by the Ford and Holden dealerships in a Sydney suburb each week for a three-month period.

Key: 1|5 = 15 vehicles

Leaf: Ford	*Stem*	*Leaf: Holden*
7 4	0	3 9
9 5 2 2 1 0	1	1 1 1 6 6 8
8 5 4 4	2	2 2 7 9
0	3	5

a. State the medians of both distributions.
b. Calculate the ranges of both distributions.
c. Calculate the interquartile ranges of both distributions.
d. Show both distributions on a parallel box plot.

10. The box plots drawn below display statistical data for two AFL teams over a season.



a. Which team had the higher median score?
b. What was the range of scores for each team?
c. For each team calculate the interquartile range.

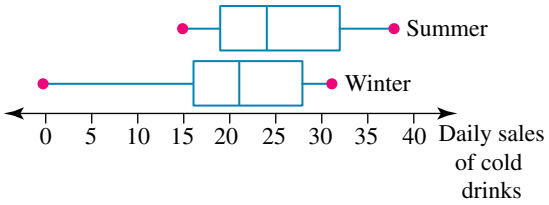
11. Tanya measures the heights (in m) of a group of Year 10 boys and girls and produces the following five-point summaries for each data set.

Boys: 1.45, 1.56, 1.62, 1.70, 1.81

Girls: 1.50, 1.55, 1.62, 1.66, 1.73

a. Draw a box plot for both sets of data and display them on the same scale.
b. What is the median of each distribution?
c. What is the range of each distribution?
d. What is the interquartile range for each distribution?
e. Comment on the spread of the heights among the boys and the girls.

12. The box plots below show the average daily sales of cold drinks at the school canteen in summer and winter.

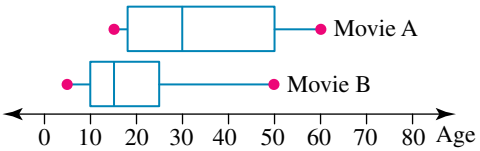


a. Calculate the range of sales in both summer and winter.

b. Calculate the interquartile range of the sales in both summer and winter.

c. Comment on the relationship between the two data sets, both in terms of measures of location and measures of spread.

13. MC A movie theatre has taken a survey of the ages of people at a showing of two of their movies. The results are shown in these box plots.



Which of the following conclusions could be drawn based on the preceding information?

A. Movie A attracts an older audience than Movie B.

B. Movie B attracts an older audience than Movie A.

C. Movie A appeals to a wider age group than Movie B.

D. Movie B appeals to a wider age group than Movie A.

E. More people went to Movie A.

Communicating, reasoning and problem solving

14. The following data show the ages of a group of 30 males and 30 females as they enter hospital for the first time.

Leaf: Male	*Stem*	*Leaf: Female*
9 8	0	5
9 9 8 8 8 6 3 2 1	1	7 7 8 9 9
8 7 7 6 4 3 2 0	2	0 0 1 2 4 5 5 6 7 9
8 6 3 1 0	3	0 1 3 3 5 8
7 5 2	4	2 3 6 8
5 3	5	1 3 4
	6	2
8	7	

a. Construct a pair of parallel boxplots to represent the two sets of data. Show your working for the median and the 1st and 3rd quartiles.

b. Calculate the mean, range and IQR for both sets of data.

c. Determine any outliers, if they exist.

d. Write a short paragraph comparing the data.

15. The times, in seconds, of the duration of 20 TV advertisements shown in the 6–8 pm time slot are recorded below.

16 60 35 23 45 15 25 55 33 20 22 30 28 38 40 18 29 19 35 75

a. From the data, determine the:

i. mode **ii.** median

iii. mean (write your answer correct to 2 decimal places)

iv. range **v.** lower quartile

vi. upper quartile **vii.** interquartile range.

b. Using your results from part **a**, construct a boxplot for the time, in seconds, for the 20 TV advertisements in the 6–8 pm time slot.

c. From your boxplot, determine:

i. the percentage of advertisements that are more than 39 seconds in length

ii. the percentage of advertisements that last between 21 and 39 seconds

iii. the percentage of advertisements that are more than 21 seconds in length

The types of TV advertisements during the 6–8 pm time slot were categorised as fast food, supermarkets, program information, and retail (clothing, sporting goods, furniture). A frequency table for the frequency of these advertisements being shown during this time slot is shown below.

Type	Frequency
Fast food	7
Supermarkets	5
Program information	3
Retail	5

d. What type of data has been collected in the table?

e. What percentage of advertisements are advertisements for fast food outlets?

f. What would be good options for a graphical representation of this type of data?

16. The test scores out of a total score of 50 for two classes, A and B, are shown in the stem plot below.

Leaf: Class A	*Stem*	*Leaf: Class B*
5	0	124
9753	1	145
97754	2	005
886551	3	155
320	4	157789
0	5	00

a. Ms Vinculum teaches both classes. She made the statement that 'Class A's performance on the test showed that the students' ability was more closely matched than the students' ability in Class B.' By finding the median, the first and third quartiles, and the interquartile range for the test scores for each class, explain if Ms Vinculum's statement was correct.

b. Would it be correct to say that Class A performed better on the test than Class B? Justify your answer by comparing the quartiles and median for each class.

17. The speeds, in km/h, of 55 cars travelling along a major road are recorded below.

Speed	Frequency
60–64	1
65–69	1
70–74	10
75–79	13
80–84	9
85–89	8

Speed	Frequency
90–94	6
95–99	3
100–104	2
105–109	1
110–114	1
Total	55

a. By finding the midpoint for each class interval, determine the mean speed, in km/h, of the cars travelling along the road. Write your answer correct to 2 decimal places.

b. The speed limit along the road is 75 km/h. A speed camera is set to photograph the license plates of cars travelling 7% more than the speed limit. A speeding fine is automatically sent to the owners of the cars photographed. Based on the 55 cars recorded, how many speeding fines were issued?

c. Drivers of cars travelling 5 km/h up to 15 km/h over the speed limit are fined $135. Drivers of cars travelling more than 15 km/h and up to 25 km/h over the speed limit are fined $165, and drivers of cars recorded travelling more than 25 km/h and up to 35 km/h are fined $250. Drivers travelling more than 35 km/h pay a $250 fine in addition to having their driver's license suspended. Assume that this data is representative of the speeding habits of drivers along a major road and that there are 30000 cars travelling along this road on any given month.

i. Determine the amount, in dollars, collected in fines throughout the month. Write your answer correct to the nearest cent.

ii. How many drivers would expect to have their licenses suspended throughout the month?

Answers

Topic 6 Statistics in the media

Exercise 6.2 How is data collected?

1. These are simply examples of simulations that could be conducted.

a. Flip a coin (Heads represents 'True', while Tails represents 'False').

b. Flip a coin (Heads represents 'red', while Tails represents 'black').

c. Spinner with 4 equal sectors (each sector representing a different toy)

d. Roll a die (each face represents a particular person).

e. Spinner with 3 equal sectors (each one representing a particular meal)

f. Spinner with 5 equal sectors (each one representing a particular destination)

g. Spinner with 5 sectors, one which will have an angle size of 120°, while the other 4 each have an angle size of 60° (each one representing a particular fast food)

2. Answers will vary. Possible answers include:

Which students have internet access at home?

Do the students need access at night?

What hours would be suitable?

How many would make use of this facility?

3. Answers will vary. Check with your teacher.

4. a. Answers will vary. Possible answers include:

Census, survey, questionnaire, interview, observation, experiment, on-line response…

b. i. Measurement **ii.** Observation

iii. Newspaper recordings **iv.** Survey

5–7. Student's own responses

8. The claim is false. It is not a logical deduction.

9. Student's own response

10. Student's own response

11. Sealy Posturepremier 40% off $\left(\frac{1000}{2499} \times 100\%\right)$,

Sealy Posturepedic 41% off $\left(\frac{1600}{3899} \times 100\%\right)$,

Sleepmaker Casablanca 40% off $\left(\frac{800}{1999} \times 100\%\right)$,

Sleepmaker Umbria 42% off $\left(\frac{1800}{4299} \times 100\%\right)$.

There is at least 40% off these beds.

12. a. 7750 seconds

b. i. Under 20 − (20 − 24): 1000 seconds difference **ii.** Under 20 − (25 − 29): 500 seconds difference

iii. (20 − 24) − (25 − 29): 500 seconds

c. The median time for 20–24-year-olds was the same as for 25–29-year-olds, which was about 250 seconds greater than the median time for te under 20s. The 20–24-year-olds and the under 20s both had a range of 2500 seconds, which was slightly higher than the range for the older group.

13.

Product	Calories	Protein	Carbs	Sugars	Sodium
Rosella Classic Australian Chicken and Vegetables	107	5.8	19	3.0	750
Campbell's Roast Chicken & Winter Vegetable	134	5.1	19.2	4.7	602
Campbell's Country Ladle Butternut Pumpkin	99	3.3	13.0	8.2	673
Campbell's Café Chicken Mushroom	134	7.0	16.8	2.0	675

a. The contents vary, but the breakdowns provided highlight the following.

i. Calories vary from 99 to 134.
ii. Protein varies from 3.3 g to 7.0 g.
iii. Carbs vary from 13.0 g to 19.2 g.
iv. Sugar content varies from 2.0 g to 8.2 g.
v. Sodium content varies from 602 mg to 750 mg.

b. **i.** Campbell's Roast Chicken & Winter Vegetable soup and Campbell's Café Chicken Mushroom soup have the most calories.
ii. Campbell's Café Chicken Mushroom soup has the highest amount of protein.
iii. Rosella Classic Australian Chicken and Vegetable soup has the highest level of sodium.

c. Campbell's Country Ladle Butternut Pumpkin soup has the lowest calories, protein and carbs, but has the highest level of sugars.

14. Hane and Roarne had a higher median and a lower spread, so they appear to have performed better.

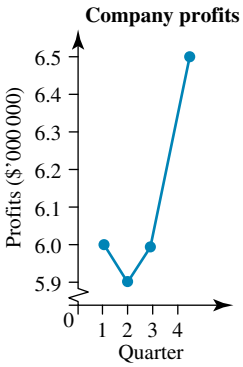
15. Discuss with your teacher.

Exercise 6.3 Evaluating statistical reports

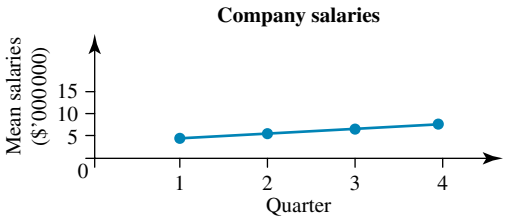
1. a. Primary. There is probably no secondary data available.

b, c Answers will vary. Check with your teacher.

2. Company profits



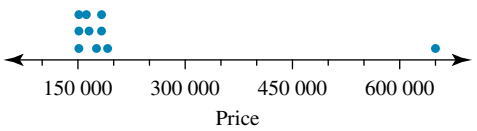
Mean salaries



3. a. Mean = $215 000, median = $170 000, mode = $150 000. The median best represents these land prices. The mean is inflated by one large score, and the mode is the lowest price.

b. Range = $500 000, interquartile range = $30 000. The interquartile range is the better measure of spread.

c.



This dot plot shows how 9 of the scores are grouped close together, while the score of $650 000 is an outlier.

d. The agent is quoting the modal price, which is the lowest price. This is not a true reflection of the average price of these blocks of land.

4. a. False. Mean = 1.82 m, lower quartile = 1.765 m, median = 1.83 m

b. True. This is the definition of interquartile range.

c. Players with heights 1.83 m, 1.83 m, 1.88 m, 1.88 m, 1.83 m

5. a. 7.1 **b.** 7 **c.** 7

d. The mode has the most meaning as this size sells the most.

6. Player B appears to be the better player if the mean result is used. However, Player A is the more consistent player.

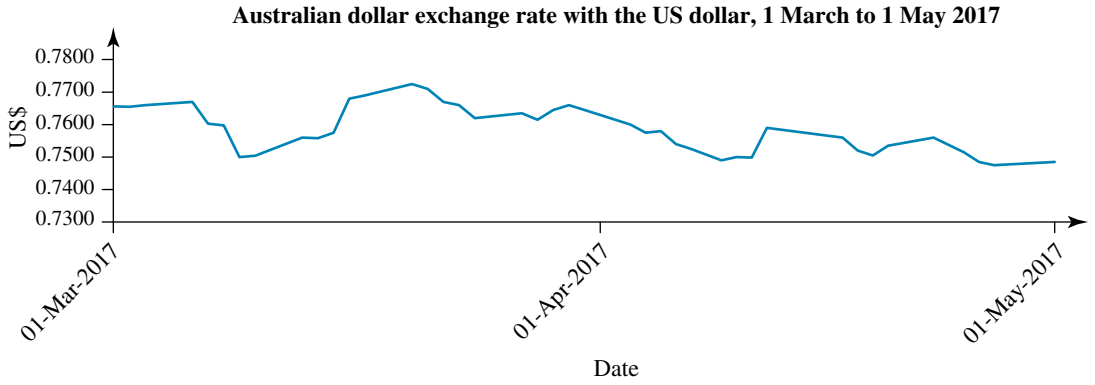
7. a. The statement is true but misleading, as most of the employees earn $18 000.

b. The median and modal salary is $18 000, and only 15 out of 80 (less than 20%) earn more than the mean.

8. a. On face value the statement is true, as the profit and share price from 2013 have both dropped remarkably. The profit rose by only 9% in 2014, then dropped by 12% in 2015 and 158% in 2016. At the same time the share price dropped from $33.85 to $24.10. It was widely known over those years that Masters was continually making a loss. We must be aware, though, that there will be other factors that would have had an impact during the same period. You may wish to visit the Woolworths website to look more closely at their 2016 Annual Report.

b. The final closure of Masters occurred in December 2016, so the sale had not been or had only recently been finalised by April 2017. Investors caused the share price to rise from $24.10 in 2016 to $26.88 as at 28 April 207. It would be interesting to observe the price movement over the next twelve months.

9. Shorten the *y*-axis and expand the *x*-axis.



10. a. Key: 3 | 85 = 385 hours

Leaf: Brand B	*Stem*	*Leaf: Brand A*
	3	85 90
	4	25 26
60 55 00	5	70
30	6	40 45
70 42 35 20	7	30 35
60 20	8	60

Brand A: mean = 570.6, median = 605

Brand B: mean = 689.2, median = 727.5

a. Brand A had the shortest mean lifetime.

b. Brand B had the longest mean lifetime.

c. Brand B

11. The bar chart on the left suggests that prices have tripled in one year; this is because the vertical axis does not start at zero. The bar chart on the right is truly indicative of situation.

12. a and **b** Percentages do not add to 100%.

c. Such representation allows multiple choices to have greater percentages than really exist.

13. To support an idea presented by the creator of the graph

Exercise 6.4 Statistical investigations

The questions in this exercise relate to student investigations, so there will be a variety of answers. Example answers are given below.

1. Answers will vary. Discuss with your teacher.

2. There have been 29 prime ministers of Australia since 1901 until this day.

There have been 45 elections.

11 prime ministers have been defeated at a general election.

There have been 23 changes of prime minister without an election.

3-6. Discuss with your teacher.

7. a. The sample would be biased since respondents would all be from the same area and the same age group, and would not be representative of the total population.

b. The sample would be biased since it is from only one city, so for example the respondents might not be representative of different socio-economic groups and ethnic backgrounds. The survey should include people from the country as well as the city.

8. Student's plan for an investigation — check with your teacher.

9. Answers will vary but could include:

a. All households in the area serviced by the hospital

b. Contacts obtained from the local electoral rolls

c. Bias could be introduced in the wording of the questions; only people who feel passionately about the issue might respond.

10. Discuss with your teacher.

11. 2, 4, 10, 10, 14; 8 solutions

Exercise 6.5 Estimating population means and medians

1. a. When was it first put into the machine? How old was the battery before being purchased? How frequently has the computer been used on battery?

b. Can't always see if a residence has a dog; a census is very time-consuming; perhaps could approach council for dog registrations.

c. This number is never constant with ongoing purchases, and continuously replenishing stock.

d. Would have to sample in this case as a census would involve opening every packet.

2. Answers will vary with the samples chosen.

3. a. Census. The airline must have a record of every passenger on every flight.

b. Survey. It would be impossible to interview everyone.

c. Survey. A census would involve opening every bottle.

d. Census. The instructor must have an accurate record of each learner driver's progress.

4. a. Survey **b.** Survey **c.** Census **d.** Survey

5. a. About 25

b. Answers will vary. Possible answers include drawing numbers from a hat or using the random number generator on a calculator.

6. a. The council is probably hoping it is a census, but it will probably be a survey because not all those over 10 will respond.

b. Residents may not all have internet access. Only those who are highly motivated are likely to respond.

7. The sample could have been biased. The questionnaire may have been unclear.

8. Sample size, randomness of sample

9. Answers will vary. Check with your teacher.

10. Answers will vary. Possible answers include:

The populations of China and India are growing very rapidly.

Many expatriate workers in China have different backgrounds, and forms need to be modified for them (such as people from Hong Kong working in mainland China). There is a large migrant population in New Delhi, and often migrants don't have residency permits (so the truth of their answers is questionable).

Many people live in remote areas that are difficult to access.

Some families in China have more than 1 child and do not disclose this (due to social pressures relating to the Chinese government's former one-child policy).

11. There is quite a variation in the frequency of particular numbers drawn. For example, the number 45 has not been drawn for 31 weeks, while most have been drawn within the last 10 weeks. In the long term, one should find the frequency of drawing each number is roughly the same. It may take a long time for this to happen, as only 8 numbers are drawn each week.

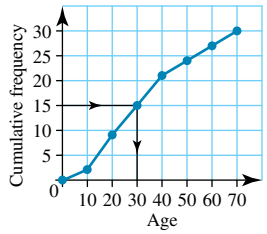
12. a. Mean = 32.03; median = 29

b.

Class interval	Frequency
0–9	2
10–19	7
20–29	6
30–39	6
40–49	3
50–59	3
60–69	3
Total	30

c. Mean = 31.83

d.



e. Median = 30

f. Estimates from parts **c** and **e** were fairly accurate.

g. Yes, they were fairly close to the mean and median of the raw data.

13. Year 8: mean = 26.83, median = 27, range = 39, IQR = 19 Year 10: mean = 40.7, median = 39.5, range = 46, IQR = 20

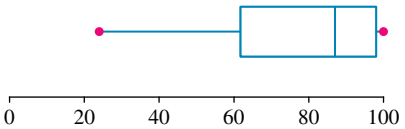
The typing speed of Year 10 students is about 13 to 14 wpm faster than that of Year 8 students. The spread of data in Year 8 is slightly less than the spread in Year 10.

14. Statistics quantifies uncertainty.

15. 1250 trout

Investigation | Rich task

1. Data for the whole population: minimum = 24, Q_1 = 61.5, median = 87, Q_3 = 98, maximum = 100



2. Mean = 77.85, standard deviation = 23

3–10 Answers will vary depending on the sample.

Exercise 6.6 Review questions

1. a. You would need to open every can to determine this.

b. Fish are continuously dying, being born and being caught.

c. Approaching work places and public transport offices

2. a. 50.5 **b, c, d** Answers will vary.

3. a. Survey **b.** Census **c.** Survey

4. Use a spinner with the green and red sectors being an equal size (say 174°) and the amber section being a smaller size (say 12°). Twirl the spinner until a green-green combination has been obtained. This is defined as one experiment. Count the number of trials required for this experiment. Repeat this procedure a number of times and determine an average.

5. D

6. a. Check with your teacher.

b. This graph should look relatively flat, with little decline in the Years 11 and 12 region.

c. This graph should show a sharp decline in the Years 11 and 12 region.

7. a. Boys: median = 26; girls: median = 23.5

b. Boys: range = 32; girls: range = 53

c. Both sets have similar medians, but the girls have a greater range of absenteeism than the boys.

8. a. The sample is an appropriate size as $\sqrt{900} = 30$.

b. Key: 16 | 1 = 1.61

Leaf: Boys	*Stem*	*Leaf: Girls*
9 9 7	15	1 2 5 6 7 8 8
9 8 6 6 5 5 4 0	16	4 4 6 7 8 9 9
4 4 2 1	17	0

c. The boys are generally taller than the girls, with the mean of the boys being 1.66 m and that of the girls being 1.62 m. The five-number summaries are:

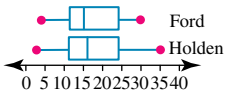
Boys: 1.57 m, 1.60 m, 1.66 m, 1.71 m, 1.74 m

Girls: 1.51 m, 1.56 m, 1.64 m, 1.68 m, 1.70 m

9. a. Ford: median = 15; Holden: median = 16

b. Ford: range = 26; Holden: range = 32

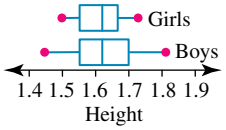
c. Ford: IQR = 14; Holden: IQR = 13.5

d.


10. a. Brisbane Lions

b. Brisbane Lions: range = 65; Sydney Swans: range = 55

c. Brisbane Lions: IQR = 40; Sydney Swans: IQR = 35

11. a.


b. Boys: median = 1.62 m; girls: median = 1.62 m

c. Boys: range = 0.36 m; girls: range = 0.23 m

d. Boys: IQR = 0.14 m; girls: IQR = 0.11 m

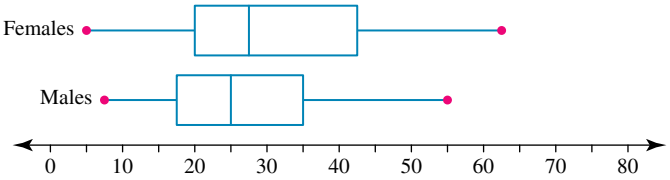
e. Although the boys and girls have the same median height, the spread of heights is greater among boys as shown by the greater range and interquartile range.

12. a. Summer: range = 23; winter: range = 31

b. Summer: IQR = 13; winter: IQR = 12

c. There are generally more cold drinks sold in summer as shown by the higher median. The spread of data is similar as shown by the IQR although the range in winter is greater.

13. A

14. a.


b.

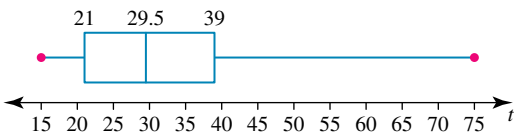
	Males	Females
Mean	28.2	31.1
Range	70	57
IQR	18	22

c. There is one outlier — a male aged 78.

d. Typically males seem to enter hospital for the first time at a younger age than females.

15. a. **i.** 35 s **ii.** 29.5 s **iii.** 33.05 s **iv.** 60 s

v. 21 s **vi.** 39 s **vii.** 18 s

b.


c. **i.** 25% **ii.** 50% **iii.** 75%

d. Categorical

e. 35%

f. Pictogram, pie chart or bar chart

16. a. Class A: $Q_1 = 21.5$, Median = 30, $Q_3 = 38$, IQR = 16.5

Class B: $Q_1 = 14.5$, Median = 33, $Q_3 = 47$, IQR = 32.5

Based on the comparison between Class A's IQR (16.5) and Class B's IQR (32.5), Ms Vinculum was correct in her statement.

b. No, Class B has a higher median and upper quartile score than Class A, whereas Class A has a higher lower quartile. You can't confidently say that either class did better in the test than the other.

17. a. 82.73 km/h

b. 30 cars

c. **i.** $2 607 272.73

ii. About 545

TOPIC 7
Algebraic fractions and equations

[Stages 5.2 and 5.3]

7.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the concepts covered in this topic.

7.1.1 Why learn this?

Do you speak mathematics? Algebra is the language of mathematics; it holds the key to understanding the rules, formulas and relationships that summarise much of our understanding of the universe. Every student of mathematics needs a mathematical tool chest, a set of algebraic skills to manipulate and process mathematical information.

DISCUSSION

Throughout history the Persians made significant contributions to the development of mathematics. How else did the Persians influence our modern world?

LEARNING SEQUENCE

7.1 Overview
7.2 [Stage 5.2] Substitution
7.3 [Stages 5.2 and 5.3] Adding and subtracting algebraic fractions
7.4 [Stages 5.2 and 5.3] Multiplying and dividing algebraic fractions
7.5 [Stage 5.2] Solving simple equations
7.6 [Stage 5.3] Solving complex linear equations
7.7 [Stage 5.3] Rearranging literal equations
7.8 Review

LEARNING OUTCOMES

A student:

- selects appropriate notations and conventions to communicate mathematical ideas and solutions MA5.2-1WM
- interprets mathematical or real-life situations, systematically applying appropriate strategies to solve problems MA5.2-2WM
- constructs arguments to prove and justify results MA5.2-3WM
- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3-1WM
- generalises mathematical ideas and techniques to analyse and solve problems efficiently MA5.3-2WM
- uses deductive reasoning in presenting arguments and formal proofs MA5.3-3WM
- simplifies algebraic fractions. MA5.2-6NA
- solves linear equations MA5.2-8NA
- selects and applies appropriate algebraic techniques to operate with algebraic expressions MA5.3-5NA
- solves complex linear equations and rearranges literal equations MA5.3-7NA

CONTENT DESCRIPTIONS

Students:

Apply the four operations to simple algebraic fractions with numerical denominators (ACMNA232)
Apply the four operations to algebraic fractions with pronumerals in the denominator (NSW)
Add and subtract algebraic fractions with numerical denominators, including those with binomial numerators (NSW)
Solve linear equations (ACMNA215)
Solve linear equations involving simple algebraic fractions (ACMNA240)
Substitute values into formulas to determine an unknown (ACMNA234)
Solve problems involving linear equations, including those derived from formulas (ACMNA235)
Solve complex linear equations involving algebraic fractions (NSW)
Rearrange literal equations (NSW)

Source: NSW Syllabus for the Australian curriculum

learnon RESOURCES — ONLINE ONLY

eLesson: The story of mathematics — The Arabic scholar Al-Khwarizmi (eles-1841)

Note: Your teacher may now set you a pre-test to determine how familiar you are with the content in this topic.

7.2 Substitution [Stage 5.2]

7.2.1 Substitution

- When the numerical values of pronumerals are known, they can be substituted into an algebraic expression and the expression can then be evaluated. It can be useful to place any substituted values in brackets when evaluating an expression.

WORKED EXAMPLE 1

If $a = 4$, $b = 2$ and $c = -7$, evaluate the following expressions.
a $a - b$ **b** $a^3 + 9b - c$

THINK	WRITE
a 1 Write the expression.	**a** $a - b$
2 Substitute $a = 4$ and $b = 2$ into the expression.	$= 4 - 2$
3 Simplify.	$= 2$
b 1 Write the expression.	**b** $a^3 + 9b - c$
2 Substitute $a = 4$, $b = 2$ and $c = -7$ into the expression.	$= (4)^3 + 9(2) - (-7)$
3 Simplify.	$= 64 + 18 + 7$ $= 89$

WORKED EXAMPLE 2

If $c = \sqrt{a^2 + b^2}$, calculate c if $a = 12$ and $b = -5$.

THINK	WRITE
1 Write the expression.	$c = \sqrt{a^2 + b^2}$
2 Substitute $a = 12$ and $b = -5$ into the expression.	$= \sqrt{(12)^2 + (-5)^2}$
3 Simplify.	$= \sqrt{144 + 25}$ $= \sqrt{169}$ $= 13$

7.2.2 Number laws

- Recall from previous studies that when dealing with numbers and pronumerals, particular rules must be obeyed. Before progressing further, let us briefly review the Commutative, Associative, Identity and Inverse Laws.
- Consider any three pronumerals x, y and z, where x, y and z are elements of the set of real numbers.

7.2.3 Commutative Law

1. $x + y = y + x$ (example: $3 + 2 = 5$ and $2 + 3 = 5$)
2. $x - y \neq y - x$ (example: $3 - 2 = 1$ but $2 - 3 = -1$)
3. $x \times y = y \times x$ (example: $3 \times 2 = 6$ and $2 \times 3 = 6$)
4. $x \div y \neq y \div x$ (example: $3 \div 2 = \frac{3}{2}$ but $2 \div 3 = \frac{2}{3}$)

- Therefore, the **Commutative Law** holds true for addition and multiplication, since the order in which two numbers or pronumerals are added or multiplied does not affect the result. However, the Commutative Law does not hold true for subtraction or division.

7.2.4 Associative Law

1. $x + (y + z) = (x + y) + z$ (example: $2 + (3 + 4) = 2 + 7 = 9$ and $(2 + 3) + 4 = 5 + 4 = 9$)
2. $x - (y - z) \neq (x - y) - z$ (example: $2 - (3 - 4) = 2 - -1 = 3$ and $(2 - 3) - 4 = -1 - 4 = -5$)
3. $x \times (y \times z) = (x \times y) \times z$ (example: $2 \times (3 \times 4) = 2 \times 12 = 24$ and $(2 \times 3) \times 4 = 6 \times 4 = 24$)
4. $x \div (y \div z) \neq (x \div y) \div z$ (example: $2 \div (3 \div 4) = 2 \div \frac{3}{4} = 2 \times \frac{4}{3} = \frac{8}{3}$ but $(2 \div 3) \div 4 = \frac{2}{3} \div 4 = \frac{2}{3} \times \frac{1}{4} = \frac{2}{12} = \frac{1}{6}$)

- The **Associative Law** holds true for addition and multiplication, since grouping two or more numbers or pronumerals and calculating them in a different order does not affect the result. However, the Associative Law does not hold true for subtraction or division.

7.2.5 Identity Law

- The **Identity Law** states that in general: $x + 0 = 0 + x = x$
  $$x \times 1 = 1 \times x = x$$
  In both of the examples above, x has not been changed (that is, it has kept its identity) when zero is added to it or it is multiplied by 1.

7.2.6 Inverse Law

- The **Inverse Law** states that in general: $x + -x = -x + x = 0$
  $$x \times \frac{1}{x} = \frac{1}{x} \times x = 1$$
  That is, when the additive inverse of a number or pronumeral is added to itself, it equals 0. When the multiplicative inverse of a number or pronumeral is multiplied by itself, it equals 1.

7.2.7 Closure Law

- A law that you may not yet have encountered is the Closure Law. The **Closure Law** states that, when an operation is performed on an element (or elements) of a set, the result produced must also be an element of that set. For example, addition is closed on natural numbers (that is, positive integers: 1, 2, 3, …), since adding a pair of natural numbers produces a natural number. Subtraction is not closed on natural numbers. For example, 5 and 7 are natural numbers and the result of adding them is 12, a natural number. However, the result of subtracting 7 from 5 is –2, which is not a natural number.

WORKED EXAMPLE 3

Find the value of the following expressions given the integer values $x = 4$ and $y = -12$. Comment on whether the Closure Law for integers holds for each of the expressions when these values are substituted.

a $x + y$ **b** $x - y$ **c** $x \times y$ **d** $x \div y$

THINK	WRITE
a 1 Substitute each pronumeral into the expression.	**a** $x + y = 4 + -12$
2 Evaluate and write the answer.	$= -8$
3 Determine whether the Closure Law holds; that is, is the result an integer?	The Closure Law holds for these substituted values.
b Repeat steps 1–3 of part **a**.	**b** $x - y = 4 - -12$ $= 16$ The Closure Law holds for these substituted values.
c Repeat steps 1–3 of part **a**.	**c** $x \times y = 4 \times -12$ $= -48$ The Closure Law holds for these substituted values.
d Repeat steps 1–3 of part **a**.	**d** $x \div y = 4 \div -12$ $= \frac{4}{-12}$ $= -\frac{1}{3}$ The Closure Law does not hold for these substituted values, since the answer obtained is a fraction, not an integer.

- It is important to note that, although a particular set of numbers may be closed under a given operation, such as multiplication, another set of numbers may not be closed under that same operation. For example, in part **c** of Worked example 3, integers were closed under multiplication. However, the set of irrational numbers is not closed under multiplication; for example, $\sqrt{3} \times \sqrt{3} = \sqrt{9} = 3$. In this example, two irrational numbers produce a rational number under multiplication.

learnON RESOURCES — ONLINE ONLY

Interactivity: Substituting positive and negative numbers (int-3765)
Interactivity: Commutative Law (int-6109)
Interactivity: Associative Law (int-6110)
Interactivity: Identity Law (int-6111)
Interactivity: Inverse Law (int-6112)
eLesson: Substitution (eles-1892)
Digital doc: SkillSHEET Like terms (doc-5183)
Digital doc: SkillSHEET Collecting like terms (doc-5184)
Digital doc: SkillSHEET Finding the highest common factor (doc-5185)
Digital doc: SkillSHEET Order of operations (doc-5189)

Exercise 7.2 Substitution

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–f, 2a–f, 3c–d, 4a–c, 5a–c, 6a–e, 7, 8, 10, 14	Questions: 1c–i, 2a–f, 3c–d, 4, 5a–c, 6d–j, 9, 10, 13, 14, 16	Questions: 1e–l, 2c–i, 3, 5, 6d–j, 9–17

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE1 If $a = 2$, $b = 3$ and $c = 5$, evaluate the following expressions.

a. $a + b$ **b.** $c - b$ **c.** $c - a - b$ **d.** $c - (a - b)$

e. $7a + 8b - 11c$ **f.** $\frac{a}{2} + \frac{b}{3} + \frac{c}{5}$ **g.** abc **h.** $ab(c - b)$

i. $a^2 + b^2 - c^2$ **j.** $c^2 + a$ **k.** $-a \times b \times -c$ **l.** $2.3a - 3.2b$

2. If $d = -6$ and $k = -5$, evaluate the following.

a. $d + k$ **b.** $d - k$ **c.** $k - d$

d. kd **e.** $-d(k + 1)$ **f.** d^2

g. k^3 **h.** $\frac{k - 1}{d}$ **i.** $3k - 5d$

3. If $x = \frac{1}{3}$ and $y = \frac{1}{4}$, evaluate the following.

a. $x + y$ **b.** $y - x$ **c.** xy **d.** $\frac{x}{y}$ **e.** x^2y^3 **f.** $\frac{9x}{y^2}$

4. If $x = 3$, find the value of the following.

a. x^2 **b.** $-x^2$ **c.** $(-x)^2$ **d.** $2x^2$ **e.** $-2x^2$ **f.** $(-2x)^2$

5. If $x = -3$, find the value of the following.

a. x^2 **b.** $-x^2$ **c.** $(-x)^2$ **d.** $2x^2$ **e.** $-2x^2$ **f.** $(-2x)^2$

6. WE2 Calculate the unknown variables in the following real-life mathematical formulas.

a. If $c = \sqrt{a^2 + b^2}$, calculate c if $a = 8$ and $b = 15$.

b. If $A = \frac{1}{2}bh$, determine the value of A if $b = 12$ and $h = 5$.

c. The perimeter, P, of a rectangle is given by $P = 2L + 2W$. Calculate the perimeter, P, of a rectangle, given $L = 1.6$ and $W = 2.4$.

d. If $T = \frac{C}{L}$, determine the value of T if $C = 20.4$ and $L = 5.1$.

e. If $K = \frac{n + 1}{n - 1}$, determine the value of K if $n = 5$.

f. Given $F = \frac{9C}{5} + 32$, calculate F if $C = 20$.

g. If $v = u + at$, evaluate v if $u = 16$, $a = 5$, $t = 6$.

h. The area, A, of a circle is given by the formula $A = \pi r^2$. Calculate the area of a circle, correct to 1 decimal place, if $r = 6$.

i. If $E = \frac{1}{2}mv^2$, calculate m if $E = 40$, $v = 4$.

j. Given $r = \sqrt{\frac{A}{\pi}}$, evaluate A to 1 decimal place if $r = 14.1$.

7. MC **a.** If $p = -5$ and $q = 4$, then pq is equal to:

A. 20 **B.** 1 **C.** -1 **D.** -20 **E.** $-\frac{5}{4}$

b. If $c^2 = a^2 + b^2$, and $a = 6$ and $b = 8$, then c is equal to:

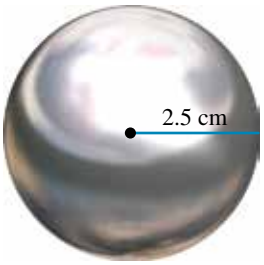
A. 28 **B.** 100 **C.** 10 **D.** 14 **E.** 44

c. Given $h = 6$ and $k = 7$, then kh^2 is equal to:

A. 294 **B.** 252 **C.** 1764 **D.** 5776 **E.** 85

8. Knowing the length of two sides of a right-angled triangle, the third side can be calculated using Pythagoras' theorem. If the two shorter sides have lengths of 1.5 cm and 3.6 cm, calculate the length of the hypotenuse.

9. The volume of a sphere can be calculated using the formula $\frac{4}{3}\pi r^3$. What is the volume of a sphere with a radius of 2.5 cm? Give your answer correct to 2 decimal places.



10. A rectangular park is 200 m by 300 m. If Blake runs along the diagonal of the park, how far will he run? Give your answer to the nearest metre.

Communicating, reasoning and problem solving

11. WE3 Determine the value of the following expressions given the integer values $x = 1$, $y = -2$ and $z = -1$. Comment on whether the Closure Law for integers holds true for each of the expressions when these values are substituted.

a. $x + y$ **b.** $y - z$ **c.** $y \times z$ **d.** $x \div z$ **e.** $z - x$ **f.** $x \div y$

12. Find the value of the following expressions given the natural number values $x = 8$, $y = 2$ and $z = 6$. Comment on whether the Closure Law for natural numbers holds true for each of the expressions.

a. $x + y$ **b.** $y - z$ **c.** $y \times z$ **d.** $x \div z$ **e.** $z - x$ **f.** $x \div y$

13. For each of the following, complete the relationship to illustrate the stated law. Justify your reasoning.

a. $(a + 2b) + 4c =$ _____ Associative Law

b. $(x \times 3y) \times 5c =$ _____ Associative Law

c. $2p \div q \neq$ _____ Commutative Law

d. $5d + q =$ _____ Commutative Law

e. $3z + 0 =$ _____ Identity Law

f. $2x \times$ _____ = _____Inverse Law

g. $(4x \div 3y) \div 5z \neq$ _____ Associative Law

h. $3d - 4y \neq$ _____ Commutative Law

14. Consider the equation $s = ut + \frac{1}{2}at^2$, where t is the time in seconds, s is the displacement in metres, u is the initial velocity and a is the acceleration due to gravity.

a. Calculate s when $u = 16.5$ m/s, $t = 2.5$ seconds and $a = 9.8$ m/s^2.

b. A body has an initial velocity of 14.7 m/s and after t seconds has a displacement of 137.2 metres. Find the value of t if $a = 9.8$ m/s^2.

15. Find the value of m if $n = p\sqrt{1 + \frac{1}{m}}$, when $n = 6$ and $p = 4$.

16. Why is understanding of the Commutative Law useful?

17. The lowest common multiple of four terms is $24a^3bc^2d$. Three of the terms are $12a^2bc$, $8ab$ and $4a^2cd$. The fourth term contains only two pronumerals, and its coefficient is an odd prime number. What is the fourth term?

7.3 Adding and subtracting algebraic fractions [Stages 5.2 and 5.3]

7.3.1 Algebraic fractions

- In an algebraic fraction, the denominator, the numerator or both are algebraic expressions. For example, $\frac{x}{2}$, $\frac{3x+1}{2x-5}$ and $\frac{1}{x^2+5}$ are all **algebraic fractions**.

WORKED EXAMPLE 4

Simplify the following expressions.

a $\frac{2x}{3} - \frac{x}{2}$ **b** $\frac{x+1}{6} + \frac{x+4}{4}$

THINK	WRITE
a 1 Write the expression.	**a** $\frac{2x}{3} - \frac{x}{2}$
2 Rewrite each fraction as an equivalent fraction using the LCD of 3 and 2, which is 6.	$= \frac{2x}{3} \times \frac{2}{2} - \frac{x}{2} \times \frac{3}{3}$ $= \frac{4x}{6} - \frac{3x}{6}$
3 Express as a single fraction.	$= \frac{4x - 3x}{6}$
4 Simplify the numerator.	$= \frac{x}{6}$
b 1 Write the expression.	**b** $\frac{x+1}{6} + \frac{x+4}{4}$
2 Rewrite each fraction as an equivalent fraction using the LCD of 6 and 4, which is 12.	$= \frac{x+1}{6} \times \frac{2}{2} + \frac{x+4}{4} \times \frac{3}{3}$ $= \frac{2(x+1)}{12} + \frac{3(x+4)}{12}$
3 Express as a single fraction.	$= \frac{2(x+1) + 3(x+4)}{12}$
4 Simplify the numerator by expanding brackets and collecting like terms.	$= \frac{2x + 2 + 3x + 12}{12}$ $= \frac{5x + 14}{12}$

7.3.2 Pronumerals in the denominator

- If pronumerals appear in the denominator, the process involved in adding and subtracting the fractions is to find a lowest common denominator as usual.
- When there is an algebraic expression in the denominator of each fraction, we can obtain a common denominator by writing the product of the denominators. For example, if $x + 3$ and $2x - 5$ are in the denominator of each fraction, then a common denominator of the two fractions will be $(x + 3)(2x - 5)$.

WORKED EXAMPLE 5

Simplify $\frac{2}{3x} - \frac{1}{4x}$.

THINK

1 Write the expression.

2 Rewrite each fraction as an equivalent fraction using the LCD of $3x$ and $4x$, which is $12x$.
Note: $12x^2$ is a common denominator but is not the lowest common denominator.

3 Express as a single fraction.

4 Simplify the numerator.

WRITE

$$\frac{2}{3x} - \frac{1}{4x}$$
$$= \frac{2}{3x} \times \frac{4}{4} - \frac{1}{4x} \times \frac{3}{3}$$
$$= \frac{8}{12x} - \frac{3}{12x}$$
$$= \frac{8 - 3}{12x}$$
$$= \frac{5}{12x}$$

WORKED EXAMPLE 6

Simplify $\frac{x+1}{x+3} + \frac{2x-1}{x+2}$ by writing it first as a single fraction.

THINK

1 Write the expression.

2 Rewrite each fraction as an equivalent fraction using the LCD of $x + 3$ and $x + 2$, which is the product $(x + 3)(x + 2)$.

3 Express as a single fraction.

4 Simplify the numerator by expanding brackets and collecting like terms.
Note: The denominator is generally kept in factorised form. That is, it is not expanded.

WRITE

$$\frac{x+1}{x+3} + \frac{2x-1}{x+2}$$
$$= \frac{(x+1)}{(x+3)} \times \frac{(x+2)}{(x+2)} + \frac{(2x-1)}{(x+2)} \times \frac{(x+3)}{(x+3)}$$
$$= \frac{(x+1)(x+2)}{(x+3)(x+2)} + \frac{(2x-1)(x+3)}{(x+3)(x+2)}$$
$$= \frac{(x+1)(x+2) + (2x-1)(x+3)}{(x+3)\,(x+2)}$$
$$= \frac{(x^2 + 2x + x + 2) + (2x^2 + 6x - x - 3)}{(x+3)\,(x+2)}$$
$$= \frac{(x^2 + 3x + 2 + 2x^2 + 5x - 3)}{(x+3)(x+2)}$$
$$= \frac{3x^2 + 8x - 1}{(x+3)(x+2)}$$

WORKED EXAMPLE 7

Simplify $\frac{x+2}{x-3} + \frac{x-1}{(x-3)^2}$ by writing it first as a single fraction.

THINK

1 Write the expression.

WRITE

$$\frac{x+2}{x-3} + \frac{x-1}{(x-3)^2}$$

2 Rewrite each fraction as an equivalent fraction using the LCD of $x - 3$ and $(x - 3)^2$, which is $(x - 3)^2$.

$$= \frac{x+2}{x-3} \times \frac{x-3}{x-3} + \frac{x-1}{(x-3)^2}$$
$$= \frac{(x+2)(x-3)}{(x-3)^2} + \frac{x-1}{(x-3)^2}$$
$$= \frac{x^2 - x - 6}{(x-3)^2} + \frac{x-1}{(x-3)^2}$$

3 Express as a single fraction.

$$= \frac{x^2 - x - 6 + x - 1}{(x-3)^2}$$

4 Simplify the numerator.

$$= \frac{x^2 - 7}{(x-3)^2}$$

learnON RESOURCES — ONLINE ONLY

Interactivity: Adding and subtracting algebraic fractions (int-6113)
Interactivity: Lowest common denominators with pronumerals (int-6114)
Digital doc: SkillSHEET Addition and subtraction of fractions (doc-5186)
Digital doc: SkillSHEET Writing equivalent algebraic fractions with the lowest common denominator (doc-5190)

Exercise 7.3 Adding and subtracting algebraic fractions

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–f, 2a–f, 3a–f, 4–6	Questions: 1d–i, 2a–f, 3a–i, 4–7, 9	Questions: 1g–i, 2e–j, 3d–i, 4–11

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. Simplify each of the following.

a. $\frac{4}{7} + \frac{2}{3}$ **b.** $\frac{1}{8} + \frac{5}{9}$ **c.** $\frac{3}{5} + \frac{6}{15}$
d. $\frac{4}{9} - \frac{3}{11}$ **e.** $\frac{3}{7} - \frac{2}{5}$ **f.** $\frac{1}{5} - \frac{x}{6}$
g. $\frac{5x}{9} - \frac{4}{27}$ **h.** $\frac{3}{8} - \frac{2x}{5}$ **i.** $\frac{5}{x} - \frac{2}{3}$

2. WE4 Simplify the following expressions.

a. $\frac{2y}{3} - \frac{y}{4}$ **b.** $\frac{y}{8} - \frac{y}{5}$ **c.** $\frac{4x}{3} - \frac{x}{4}$ **d.** $\frac{8x}{9} + \frac{2x}{3}$
e. $\frac{2w}{14} - \frac{w}{28}$ **f.** $\frac{y}{20} - \frac{y}{4}$ **g.** $\frac{12y}{5} + \frac{y}{7}$ **h.** $\frac{10x}{5} + \frac{2x}{15}$
i. $\frac{x+1}{5} + \frac{x+3}{2}$ **j.** $\frac{x+2}{4} + \frac{x+6}{3}$ **k.** $\frac{2x-1}{5} - \frac{2x+1}{6}$ **l.** $\frac{3x+1}{2} + \frac{5x+2}{3}$

3. WE5 Simplify the following.

a. $\frac{2}{4x} + \frac{1}{8x}$ **b.** $\frac{3}{4x} - \frac{1}{3x}$ **c.** $\frac{5}{3x} + \frac{1}{7x}$

d. $\frac{12}{5x} + \frac{4}{15x}$ **e.** $\frac{1}{6x} + \frac{1}{8x}$ **f.** $\frac{9}{4x} - \frac{9}{5x}$

g. $\frac{2}{100x} + \frac{7}{20x}$ **h.** $\frac{1}{10x} + \frac{5}{x}$ **i.** $\frac{4}{3x} - \frac{3}{2x}$

4. WE6, 7 Simplify the following by writing as single fractions.

a. $\frac{2}{x+4} + \frac{3x}{x-2}$ **b.** $\frac{2x}{x+5} + \frac{5}{x-1}$ **c.** $\frac{5}{2x+1} + \frac{x}{x-2}$ **d.** $\frac{2x}{x+1} - \frac{3}{2x-7}$

e. $\frac{4x}{x+7} + \frac{3x}{x-5}$ **f.** $\frac{x+2}{x+1} + \frac{x-1}{x+4}$ **g.** $\frac{x+8}{x+1} - \frac{2x+1}{x+2}$ **h.** $\frac{x+5}{x+3} - \frac{x-1}{x-2}$

i. $\frac{x+1}{x+2} - \frac{2x-5}{3x-1}$ **j.** $\frac{2}{x-1} - \frac{3}{1-x}$ **k.** $\frac{4}{(x+1)^2} + \frac{3}{x+1}$ **l.** $\frac{3}{x-1} - \frac{1}{(x-1)^2}$

5. A classmate attempted to complete an algebraic fraction subtraction problem.

$$\begin{aligned}\frac{x}{x-1} - \frac{3}{x-2} &= \frac{x}{x-1} \times \frac{(x-2)}{(x-2)} - \frac{3}{x-2} \times \frac{(x-1)}{(x-1)} \\ &= \frac{x(x-2) - 3(x-1)}{(x-1)(x+2)} \\ &= \frac{x^2 - 2x - 3x - 1}{(x-1)(x+2)} \\ &= \frac{x^2 - 5x - 1}{(x-1)(x+2)}\end{aligned}$$

a. What mistake did she make?

b. What is the correct answer?

Communicating, reasoning and problem solving

Adding and subtracting algebraic fractions can become more complicated if you add a third fraction into the expression.

6. Simplify the following.

a. $\frac{1}{x+2} + \frac{2}{x+1} + \frac{1}{x+3}$ **b.** $\frac{1}{x-1} + \frac{4}{x+2} + \frac{2}{x-4}$

c. $\frac{3}{x+1} + \frac{2}{x+3} - \frac{1}{x+2}$ **d.** $\frac{2}{x-4} - \frac{3}{x-1} + \frac{5}{x+3}$

7. Why is the process that involves finding the lowest common denominator important in question **6**?

8. The reverse process of adding or subtracting algebraic fractions is quite complex. Use trial and error, or technology, to determine the value of a if $\frac{7x-4}{(x-8)(x+5)} = \frac{a}{x-8} + \frac{3}{x+5}$.

9. Simplify $\frac{3}{x^2+7x+12} - \frac{1}{x^2+x-6} + \frac{2}{x^2+2x-8}$.

10. Simplify $\frac{x^2+3x-18}{x^2-x-42} - \frac{x^2-3x+2}{x^2-5x+4}$.

11. Why can't we just add the numerators and the denominators of fractions; for example, $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$?

7.4 Multiplying and dividing algebraic fractions [Stages 5.2 and 5.3]

7.4.1 Simplifying algebraic fractions

- Algebraic fractions can be simplified using the index laws and by cancelling factors common to the numerator and denominator.
- A fraction can only be simplified if:
  - there is a common factor in the numerator and the denominator
  - the numerator and denominator are both written in factorised form, that is, as the *product* of two or more factors.

  For example:

$$\frac{3ab}{12a} = \frac{{}^{1}\cancel{3} \times {}^{1}\cancel{a} \times b}{{}^{4}\cancel{12} \times {}^{1}\cancel{a}} \quad \begin{matrix}\leftarrow \text{product of factors} \\ \leftarrow \text{product of factors}\end{matrix}$$

$$= \frac{b}{4}$$

$$\frac{3a + b}{12a} = \frac{3 \times a + b}{12 \times a} \quad \begin{matrix}\leftarrow \text{not a product of factors} \\ \leftarrow \text{product of factors}\end{matrix}$$

Cannot be simplified

7.4.2 Multiplying algebraic fractions

- Multiplication of algebraic fractions follows the same rules as multiplication of numerical fractions: multiply the numerators, then multiply the denominators.

WORKED EXAMPLE 8

Simplify each of the following.

a $\frac{5y}{3x} \times \frac{6z}{7y}$ **b** $\frac{2x}{(x + 1)(2x - 3)} \times \frac{x + 1}{x}$

THINK	WRITE
a 1 Write the expression.	**a** $\frac{5y}{3x} \times \frac{6z}{7y}$
2 Cancel common factors in the numerator and denominator. The y can be cancelled in the denominator and the numerator. Also, the 3 in the denominator can divide into the 6 in the numerator.	$= \frac{5\cancel{y}^{1}}{{}_{1}\cancel{3}x} \times \frac{\cancel{6}^{2}z}{7\cancel{y}^{1}}$ $= \frac{5}{x} \times \frac{2z}{7}$
3 Multiply the numerators, then multiply the denominators.	$= \frac{10z}{7x}$
b 1 Write the expression.	**b** $\frac{2x}{(x + 1)(2x - 3)} \times \frac{x + 1}{x}$
2 Cancel common factors in the numerator and the denominator. $(x + 1)$ and the x are both common in the numerator and the denominator and can therefore be cancelled.	$= \frac{2\cancel{x}^{1}}{{}^{1}\cancel{(x+1)}(2x - 3)} \times \frac{\cancel{x+1}^{1}}{\cancel{x}^{1}}$ $= \frac{2}{2x - 3} \times \frac{1}{1}$
3 Multiply the numerators, then multiply the denominators.	$= \frac{2}{2x - 3}$

7.4.3 Dividing algebraic fractions

- When dividing algebraic fractions, follow the same rules as for division of numerical fractions: write the division as a multiplication and invert the second fraction. This process is sometimes known as multiplying by the **reciprocal**.

DISCUSSION

What is the result of multiplying an algebraic fraction by its reciprocal? Does this result hold true for all algebraic fractions?

WORKED EXAMPLE 9

Simplify the following expressions.

a $\frac{3xy}{2} \div \frac{4x}{9y}$ **b** $\frac{4}{(x+1)(3x-5)} \div \frac{x-7}{x+1}$

THINK	WRITE
a 1 Write the expression.	**a** $\frac{3xy}{2} \div \frac{4x}{9y}$
2 Change the division sign to a multiplication sign and write the second fraction as its reciprocal.	$= \frac{3xy}{2} \times \frac{9y}{4x}$
3 Cancel common factors in the numerator and denominator and cancel. The pronumeral x is common to both the numerator and denominator and can therefore be cancelled.	$= \frac{3y}{2} \times \frac{9y}{4}$
4 Multiply the numerators, then multiply the denominators.	$= \frac{27y^2}{8}$
b 1 Write the expression.	**b** $\frac{4}{(x+1)(3x-5)} \div \frac{x-7}{x+1}$
2 Change the division sign to a multiplication sign and write the second fraction as its reciprocal.	$= \frac{4}{(x+1)(3x-5)} \times \frac{x+1}{x-7}$
3 Cancel common factors in the numerator and denominator and cancel. $(x+1)$ is common to both the numerator and denominator and can therefore be cancelled.	$= \frac{4}{3x-5} \times \frac{1}{x-7}$
4 Multiply the numerators, then multiply the denominators.	$= \frac{4}{(3x-5)(x-7)}$

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Interactivity: Simplifying algebraic fractions (int-6115)
Interactivity: Multiplying algebraic fractions (int-6116)
Interactivity: Dividing algebraic fractions (int-6117)
Digital doc: SkillSHEET Multiplication of fractions (doc-5187)
Digital doc: SkillSHEET Division of fractions (doc-5188)
Digital doc: SkillSHEET Simplification of algebraic fractions (doc-5191)
Digital doc: WorkSHEET Algebraic fractions (doc-13847)

Exercise 7.4 Multiplying and dividing algebraic fractions

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–f, 2a–f, 3a–i, 4a–b	Questions: 1d–i, 2a–f, 3a–i, 4, 5, 7, 9	Questions: 1g–l, 2e–j, 3d–l, 4–10

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Understanding and fluency

1. WE8a Simplify each of the following.

a. $\frac{x}{5} \times \frac{20}{y}$ **b.** $\frac{x}{4} \times \frac{12}{y}$ **c.** $\frac{y}{4} \times \frac{16}{x}$ **d.** $\frac{x}{2} \times \frac{9}{2y}$

e. $\frac{x}{10} \times \frac{-25}{2y}$ **f.** $\frac{3w}{-14} \times \frac{-7}{x}$ **g.** $\frac{3y}{4x} \times \frac{8z}{7y}$ **h.** $\frac{-y}{3x} \times \frac{6z}{-7y}$

i. $\frac{x}{3z} \times \frac{-9z}{2y}$ **j.** $\frac{5y}{3x} \times \frac{x}{8y}$ **k.** $\frac{-20y}{7x} \times \frac{-21z}{5y}$ **l.** $\frac{y}{-3w} \times \frac{x}{2y}$

2. WE8b Simplify the following expressions.

a. $\frac{2x}{(x-1)(3x-2)} \times \frac{x-1}{x}$ **b.** $\frac{5x}{(x-3)(4x+7)} \times \frac{4x+7}{x}$

c. $\frac{9x}{(5x+1)(x-6)} \times \frac{5x+1}{2x}$ **d.** $\frac{(x+4)}{(x+1)(x+3)} \times \frac{x+1}{x+4}$

e. $\frac{2x}{x+1} \times \frac{x-1}{(x+1)(x-1)}$ **f.** $\frac{2}{x(2x-3)} \times \frac{x(x+1)}{4}$

g. $\frac{2x}{4(a+3)} \times \frac{3a}{15x}$ **h.** $\frac{15c}{12(d-3)} \times \frac{21d}{6c}$

i. $\frac{6x^2}{20(x-2)^2} \times \frac{15(x-2)}{16x^4}$ **j.** $\frac{7x^2(x-3)}{5x(x+1)} \times \frac{3(x-3)(x+1)}{14(x-3)^2(x-1)}$

3. WE9a Simplify the following expressions.

a. $\frac{3}{x} \div \frac{5}{x}$ **b.** $\frac{2}{x} \div \frac{9}{x}$ **c.** $\frac{4}{x} \div \frac{12}{x}$ **d.** $\frac{20}{y} \div \frac{20}{3y}$

e. $\frac{1}{5w} \div \frac{5}{w}$ **f.** $\frac{7}{2x} \div \frac{3}{5x}$ **g.** $\frac{3xy}{7} \div \frac{3x}{4y}$ **h.** $\frac{2xy}{5} \div \frac{5x}{y}$

i. $\frac{6y}{9} \div \frac{3x}{4xy}$ **j.** $\frac{8wx}{5} \div \frac{3w}{4y}$ **k.** $\frac{2xy}{5} \div \frac{3xy}{5}$ **l.** $\frac{10xy}{7} \div \frac{20x}{14y}$

4. WE9b Simplify the following expressions.

a. $\frac{9}{(x-1)(3x-7)} \div \frac{x+3}{x-1}$ **b.** $\frac{1}{(x+2)(2x-5)} \div \frac{x-9}{2x-5}$

c. $\frac{12(x-3)^2}{(x+5)(x-9)} \div \frac{4(x-3)}{7(x-9)}$ **d.** $\frac{13}{6(x-4)^2(x-1)} \div \frac{3(x+1)}{2(x-4)(x-1)}$

Communicating, reasoning and problem solving

5. Is $\frac{3}{x+2}$ the same as $\frac{1}{x+2} + \frac{1}{x+2} + \frac{1}{x+2}$? Explain your reasoning.

6. a. Simplify $\frac{(x-4)(x+3)}{4x-x^2} \times \frac{x^2-x}{(x+3)(x-1)}$.

b. Find and describe the error in the following reasoning.

$$\frac{(x-4)(x+3)}{4x-x^2} \times \frac{x^2-x}{(x+3)(x-1)}$$
$$= \frac{(x-4)(x+3)}{x(4-x)} \times \frac{x(x-1)}{(x+3)(x-1)}$$
$$= 1$$

7. Simplify $\frac{x^2-2x-3}{x^4-1} \times \frac{x^2+4x-5}{x^2-5x+6} \div \frac{x^2+7x+10}{x^4-3x^2-4}$.

8. Simplify $\frac{x+1}{x-\frac{x}{\frac{x}{a}}}$ where $a = \frac{x-1}{x+1}$.

9. How are multiplying and dividing algebraic fractions different to adding and subtracting them?

10. Simplify the expression $\left(1+\frac{\frac{1}{x}}{1+\frac{1}{x}}\right) \times \left(1+\frac{\frac{1}{x}}{1-\frac{1}{x}}\right)$.

7.5 Solving simple equations [Stage 5.2]

7.5.1 Equations

- **Equations** show the equivalence of two expressions.
- Equations can be solved using inverse operations.
- When solving equations, the last operation performed on the pronumeral when building the equation is the first operation undone by applying inverse operations to both sides of the equation. For example, the equation $2x + 3 = 5$ is built from x by multiplying x by 2 and then adding 3 to give the result of 5. To solve the equation, undo the adding 3 by subtracting 3, then undo the multiplying by 2 by dividing by 2.
- Addition (+) and subtraction (–) are inverse operations.
- Multiplication (×) and division (÷) are inverse operations.
- Square (^{2}) and square root ($\sqrt{}$) are inverse operations.

7.5.2 One-step equations

- Equations that require one step to solve are called one-step equations.

WORKED EXAMPLE 10

Solve the following equations.

a $a + 27 = 71$ **b** $\frac{d}{16} = 3\frac{1}{4}$ **c** $\sqrt{e} = 0.87$ **d** $f^2 = \frac{4}{25}$

THINK

a 1 Write the equation.

2 27 has been added to a, resulting in 71. The addition of 27 has to be reversed by subtracting 27 from both sides of the equation to obtain the solution.

WRITE

a $a + 27 = 71$

$a + 27 - 27 = 71 - 27$

$a = 44$

b 1	Write the equation.	**b** $\frac{d}{16} = 3\frac{1}{4}$
2	Express $3\frac{1}{4}$ as an improper fraction.	$\frac{d}{16} = \frac{13}{4}$
3	The pronumeral d has been divided by 16, resulting in $\frac{13}{4}$. Therefore the division has to be reversed by multiplying both sides of the equation by 16 to obtain d.	$\frac{d}{16} \times 16 = \frac{13}{4} \times 16$ $d = 52$
c 1	Write the equation.	**c** $\sqrt{e} = 0.87$
2	The square root of e has been taken to result in 0.87. Therefore, the square root has to be reversed by squaring both sides of the equation to obtain e.	$(\sqrt{e})^2 = 0.87^2$ $e = 0.7569$
d 1	Write the equation.	**d** $f^2 = \frac{4}{25}$
2	The pronumeral f has been squared, resulting in $\frac{4}{25}$. Therefore the squaring has to be reversed by taking the square root of both sides of the equation to obtain f. Note that there are two possible solutions, one positive and one negative, since two negative numbers can also be multiplied together to produce a positive result.	$f = \pm\sqrt{\frac{4}{25}}$ $f = \pm\frac{2}{5}$

7.5.3 Two-step equations

- Two-step equations involve the inverse of two operations in their solutions.

WORKED EXAMPLE 11

Solve the following equations.
a $5y - 6 = 79$ **b** $\frac{4x}{9} = 5$

THINK	WRITE
a 1 Write the equation.	**a** $5y - 6 = 79$
2 Step 1: Add 6 to both sides of the equation.	$5y - 6 + 6 = 79 + 6$ $5y = 85$
3 Step 2: Divide both sides of the equation by 5 to obtain y.	$\frac{5y}{5} = \frac{85}{5}$ $y = 17$
b 1 Write the equation.	**b** $\frac{4x}{9} = 5$
2 Step 1: Multiply both sides of the equation by 9.	$\frac{4x}{9} \times 9 = 5 \times 9$ $4x = 45$
3 Step 2: Divide both sides of the equation by 4 to obtain x.	$\frac{4x}{4} = \frac{45}{4}$ $x = \frac{45}{4}$
4 Express the improper fraction as a mixed number.	$x = 11\frac{1}{4}$

DISCUSSION

What happens when these operations are performed in the opposite order?

7.5.4 Equations where the pronumeral appears on both sides

- In solving equations where the pronumeral appears on both sides, subtract the smaller pronumeral term so that it is eliminated from both sides of the equation.

WORKED EXAMPLE 12

Solve the following equations.

a $5h + 13 = 2h - 2$ **b** $14 - 4d = 27 - d$ **c** $2(x - 3) = 5(2x + 4)$

THINK	WRITE
a 1 Write the equation.	**a** $5h + 13 = 2h - 2$
2 Eliminate the pronumeral from the right-hand side by subtracting $2h$ from both sides of the equation.	$3h + 13 = -2$
3 Subtract 13 from both sides of the equation.	$3h = -15$
4 Divide both sides of the equation by 3 and write your answer.	$h = -5$
b 1 Write the equation.	**b** $14 - 4d = 27 - d$
2 Add $4d$ to both sides of the equation.	$14 = 27 + 3d$
3 Subtract 27 from both sides of the equation.	$-13 = 3d$
4 Divide both sides of the equation by 3.	$-\frac{13}{3} = d$
5 Express the improper fraction as a mixed number.	$-4\frac{1}{3} = d$
6 Write your answer so that d is on the left-hand side.	$d = -4\frac{1}{3}$
c 1 Write the equation.	**c** $2(x - 3) = 5(2x + 4)$
2 Expand the brackets on both sides of the equation.	$2x - 6 = 10x + 20$
3 Subtract $2x$ from both sides of the equation.	$2x - 2x - 6 = 10x - 2x + 20$
4 Subtract 20 from both sides of the equation.	$-6 - 20 = 8x + 20 - 20$ $-26 = 8x$
5 Divide both sides of the equation by 8.	$-\frac{26}{8} = x$
6 Simplify and write your answer with the pronumeral on the left-hand side.	$x = -\frac{13}{4}$

learnon RESOURCES — ONLINE ONLY

Interactivity: Using algebra to solve problems (int-3805)
Interactivity: One-step equations (int-6118)
Interactivity: Two-step equations (int-6119)
eLesson: Solving linear equations (eles-1895)
eLesson: Solving linear equations with the pronumeral on both sides (eles-1901)

Exercise 7.5 Solving simple equations

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–f, 2a–e, 3a–f, 4a–b, 5a–b, 6a–b, 7a–f, 8a–d, 9a–b, 10a–b, 11a–c, 12a–b, 13a–c, 15a–d, 16a–d, 17a–d, 19–21, 25	Questions: 1d–i, 2d–i, 3a–f, 4, 5a–b, 6a–b, 7d–i, 8c–f, 9c–g, 10a–d, 11c–f, 12, 13d–i, 15c–f, 16c–f, 17c–f, 19–21, 23, 25, 26	Questions: 3–6, 7d–i, 8c–f, 9e–i, 10d–f, 11d–f, 12, 14, 15g–i, 16g–i, 17g–i, 18–28

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Understanding and fluency

1. WE10a Solve the following equations.

a. $a + 61 = 85$ **b.** $k - 75 = 46$ **c.** $g + 9.3 = 12.2$

d. $r - 2.3 = 0.7$ **e.** $h + 0.84 = 1.1$ **f.** $i + 5 = 3$

g. $t - 12 = -7$ **h.** $q + \frac{1}{3} = \frac{1}{2}$ **i.** $x - 2 = -2$

2. WE10b Solve the following equations.

a. $\frac{f}{4} = 3$ **b.** $\frac{i}{10} = -6$ **c.** $6z = -42$

d. $9v = 63$ **e.** $6w = -32$ **f.** $\frac{k}{12} = \frac{5}{6}$

g. $4a = 1.7$ **h.** $\frac{m}{19} = \frac{7}{8}$ **i.** $\frac{y}{4} = 5\frac{3}{8}$

3. WE10c, d Solve the following equations.

a. $\sqrt{t} = 10$ **b.** $y^2 = 289$ **c.** $\sqrt{q} = 2.5$

d. $f^2 = 1.44$ **e.** $\sqrt{h} = \frac{4}{7}$ **f.** $p^2 = \frac{9}{64}$

g. $\sqrt{g} = \frac{15}{22}$ **h.** $j^2 = \frac{196}{961}$ **i.** $a^2 = 2\frac{7}{9}$

4. Solve the following equations.

a. $\sqrt{t} - 3 = 2$ **b.** $5x^2 = 180$ **c.** $3\sqrt{m} = 12$

d. $-2t^2 = -18$ **e.** $t^2 + 11 = 111$ **f.** $\sqrt{m} + 5 = 0$

5. Solve the following equations.

a. $\sqrt[3]{x} = 2$ **b.** $x^3 = -27$ **c.** $\sqrt[3]{m} = \frac{1}{2}$

d. $x^3 = \frac{27}{64}$ **e.** $\sqrt[3]{m} = 0.2$ **f.** $w^3 = 15\frac{5}{8}$

6. Solve the following equations.

a. $x^3 + 1 = 0$ **b.** $3x^3 = -24$ **c.** $\sqrt[3]{m} + 5 = 6$

d. $-2 \times \sqrt[3]{w} = 16$ **e.** $\sqrt[3]{t} - 13 = 28$ **f.** $2x^3 - 14 = 2$

7. WE11a Solve the following equations.

a. $5a + 6 = 26$ **b.** $6b + 8 = 44$ **c.** $8i - 9 = 15$

d. $7f - 18 = 45$ **e.** $8q + 17 = 26$ **f.** $10r - 21 = 33$

g. $6s + 46 = 75$ **h.** $5t - 28 = 21$ **i.** $8a + 88 = 28$

8. Solve the following equations.

a. $\frac{f}{4} + 6 = 16$ **b.** $\frac{g}{6} + 4 = 9$ **c.** $\frac{r}{10} + 6 = 5$

d. $\frac{m}{9} - 12 = -10$ **e.** $\frac{n}{8} + 5 = 8.5$ **f.** $\frac{p}{12} - 1.8 = 3.4$

9. Solve the following equations.

a. $6(x + 8) = 56$ **b.** $7(y - 4) = 35$ **c.** $5(m - 3) = 7$

d. $3(2k + 5) = 24$ **e.** $5(3n - 1) = 80$ **f.** $6(2c + 7) = 58$

g. $2(x - 5) + 3(x - 7) = 19$ **h.** $3(x + 5) - 5(x - 1) = 12$ **i.** $3(2x - 7) - (x + 3) = -60$

10. WE11b Solve the following equations.

a. $\frac{3k}{5} = 15$ **b.** $\frac{9m}{8} = 18$ **c.** $\frac{7p}{10} = -8$

d. $\frac{8u}{11} = -3$ **e.** $\frac{11x}{4} = 2$ **f.** $\frac{4v}{15} = 0.8$

11. Solve the following equations.

a. $\frac{x - 5}{3} = 7$ **b.** $\frac{2m + 1}{3} = -3$ **c.** $\frac{3w - 1}{4} = 6$

d. $\frac{t - 5}{2} = 0$ **e.** $\frac{6 - x}{3} = -1$ **f.** $\frac{3n - 5}{4} = -6$

12. MC **a.** The solution to the equation $\frac{p}{5} + 2 = 7$ is:

A. $p = 5$ **B.** $p = 25$ **C.** $p = 45$ **D.** $p = 10$ **E.** $p = 1$

b. If $5h + 8 = 53$, then h is equal to:

A. $\frac{1}{5}$ **B.** 12.2 **C.** 225 **D.** 10 **E.** 9

c. The exact solution to the equation $14x = 75$ is:

A. $x = 5.357\,142\,857$ **B.** $x = 5.357$ (to 3 decimal places)

C. $x = 5\frac{5}{14}$ **D.** $x = 5.4$ **E.** $x = 5.5$

13. Solve the following equations.

a. $-x = 5$ **b.** $2 - d = 3$ **c.** $5 - p = -2$

d. $-7 - x = 4$ **e.** $-5h = 10$ **f.** $-6t = -30$

g. $-\frac{v}{5} = 4$ **h.** $-\frac{r}{12} = \frac{1}{4}$ **i.** $-4g = 3.2$

14. Solve the following equations.

a. $6 - 2x = 8$ **b.** $10 - 3v = 7$ **c.** $9 - 6l = -3$

d. $-3 - 2g = 1$ **e.** $-5 - 4t = -17$ **f.** $-\frac{3e}{5} = 14$

g. $-\frac{8j}{3} = 9$ **h.** $-\frac{k}{4} - 3 = 6$ **i.** $-\frac{4f}{7} + 1 = 8$

15. WE12a Solve the following equations.

a. $6x + 5 = 5x + 7$ **b.** $7b + 9 = 6b + 14$ **c.** $11w + 17 = 6w + 27$

d. $8f - 2 = 7f + 5$ **e.** $10t - 11 = 5t + 4$ **f.** $12r - 16 = 3r + 5$

g. $12g - 19 = 3g - 31$ **h.** $7h + 5 = 2h - 6$ **i.** $5a - 2 = 3a - 2$

16. WE12b Solve the following equations.

a. $5 - 2x = 6 - x$ **b.** $10 - 3c = 8 - 2c$ **c.** $3r + 13 = 9r - 3$

d. $k - 5 = 2k - 6$ **e.** $5y + 8 = 13y + 17$ **f.** $17 - 3g = 3 - g$

g. $14 - 5w = w + 8$ **h.** $4m + 7 = 8 - m$ **i.** $14 - 5p = 9 - 2p$

17. WE12c Solve the following equations.

a. $3(x + 5) = 2x$ **b.** $8(y + 3) = 3y$ **c.** $6(t - 5) = 4(t + 3)$

d. $10(u + 1) = 3(u - 3)$ **e.** $12(f - 10) = 4(f - 5)$ **f.** $2(4r + 3) = 3(2r + 7)$

g. $5(2d + 9) = 3(3d + 13)$ **h.** $5(h - 3) = 3(2h - 1)$ **i.** $2(4x + 1) = 5(3 - x)$

18. MC **a.** The solution to $8 - 4k = -2$ is:

A. $k = 2\frac{1}{2}$ **B.** $k = -2\frac{1}{2}$ **C.** $k = 1\frac{1}{2}$ **D.** $k = -1\frac{1}{2}$ **E.** $k = \frac{2}{5}$

b. The solution to $-\frac{6n}{5} + 3 = -7$ is:

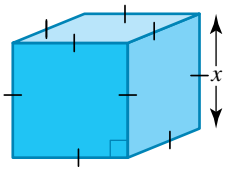
A. $n = 3\frac{1}{3}$ **B.** $n = -3\frac{1}{3}$ **C.** $n = \frac{1}{3}$ **D.** $n = 8\frac{1}{3}$ **E.** $n = -8\frac{1}{3}$

c. The solution to $p - 6 = 8 - 4p$ is:

A. $p = \frac{2}{5}$ **B.** $p = 2\frac{4}{5}$ **C.** $p = 4\frac{2}{3}$ **D.** $p = \frac{2}{3}$ **E.** $p = \frac{4}{5}$

19. If the area of a triangle is 30 square centimetres and the base length is 12 centimetres, find the perpendicular height of the triangle.

20. If the side length of a cube is x cm, then its volume V is given by $V = x^3$. What is the side length (correct to the nearest cm) of a cube that has a volume of:



a. 216 cm^3

b. 2 m^3?

21. The surface area of a cube with side length x cm is given by $A = 6x^2$. Find the side length (correct to the nearest cm) of a cube that has a surface area of:

a. 37.5 cm^2 **b.** 1 m^2.

22. A pebble is dropped down a well. In time t seconds it falls a distance of d metres, given by $d = 5t^2$.

a. How far does the pebble fall in 1s?

b. How many seconds will it take the pebble to fall 40m? (Answer correct to 1 decimal place.)

23. The surface area of a sphere is given by the formula $A = 4\pi r^2$, where r is the radius of the sphere.

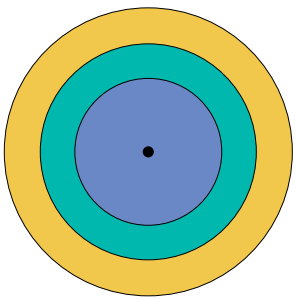
a. Find the surface area of a sphere that has a radius of 5 cm.

b. What is the radius of a sphere that has a surface area equal to 500 cm^2? (Answer correct to the nearest mm.)

Communicating, reasoning and problem solving

24. Find the radius of a circle of area 10 cm^2.

25. The volume of a sphere is given by the formula $V = \frac{4}{3}\pi r^3$, where r is the radius of the sphere. If the sphere can hold 1 litre of water, what is its radius correct to the nearest mm?



26. The width of a room is three-fifths of its length. When the width is increased by 2 metres and the length is decreased by 2 metres, the resultant shape is a square. Find the dimensions of the room.

27. A target board for a dart game has been designed as three concentric circles where each coloured region is the same area. If the radius of the purple circle is r cm and the radius of the outer circle is 10 cm, find the value of r.

28. Describe in one sentence what it means to solve linear equations.

7.6 Solving complex linear equations [Stage 5.3]

7.6.1 Equations with multiple brackets

- Many equations need to be simplified by expanding brackets and collecting like terms before they are solved. Doing this reduces the equation to one of the basic types covered in the previous exercise.

WORKED EXAMPLE 13

Solve each of the following linear equations.
a $6(x + 1) - 4(x - 2) = 0$ **b** $7(5 - x) = 3(x + 1) - 10$

THINK	WRITE
a 1 Write the equation.	**a** $6(x + 1) - 4(x - 2) = 0$
2 Expand all the brackets. (Be careful with the –4.)	$6x + 6 - 4x + 8 = 0$
3 Collect like terms.	$2x + 14 = 0$
4 Subtract 14 from both sides of the equation.	$2x = -14$
5 Divide both sides of the equation by 2 to find the value of x.	$x = -7$
b 1 Write the equation.	**b** $7(5 - x) = 3(x + 1) - 10$
2 Expand all the brackets.	$35 - 7x = 3x + 3 - 10$
3 Collect like terms.	$35 - 7x = 3x - 7$
4 Create a single pronumeral term by adding $7x$ to both sides of the equation.	$35 = 10x - 7$
5 Add 7 to both sides of the equation.	$42 = 10x$
6 Divide both sides of the equation by 10 to solve for x and simplify.	$\frac{42}{10} = x$ $\frac{21}{5} = x$
7 Express the improper fraction as a mixed number fraction.	$4\frac{1}{5} = x$
8 Rewrite the equation so that x is on the left-hand side.	$x = 4\frac{1}{5}$

7.6.2 Equations involving algebraic fractions

- To solve an equation containing algebraic fractions, multiply both sides of the equation by the lowest common multiple (LCM) of the denominators. This gives an equivalent form of the equation without fractions.

WORKED EXAMPLE 14

Solve the equation $\frac{x - 5}{3} = \frac{x + 7}{4}$ and verify the solution.

THINK	WRITE
1 Write the equation.	$\frac{x-5}{3} = \frac{x+7}{4}$
2 The LCM is $3 \times 4 = 12$. Multiply both sides of the equation by 12.	$\frac{{}^{4}\cancel{12}(x-5)}{{}^{1}\cancel{3}} = \frac{{}^{3}\cancel{12}(x+7)}{{}^{1}\cancel{4}}$
3 Simplify the fractions.	$4(x-5) = 3(x+7)$
4 Expand the brackets.	$4x - 20 = 3x + 21$
5 Subtract $3x$ from both sides of the equation.	$x - 20 = 21$
6 Add 20 to both sides of the equation and write the answer.	$x = 41$
7 To verify, check that the answer $x = 41$ is true for both the left-hand side (LHS) and the right-hand side (RHS) of the equation by substitution.	
Substitute $x = 41$ into the LHS.	$\text{LHS} = \frac{41-5}{3}$ $= \frac{36}{3}$ $= 12$
Substitute $x = 41$ into the RHS.	$\text{RHS} = \frac{41+7}{4}$ $= \frac{48}{4}$ $= 12$
8 Write your answer.	Because the LHS = RHS, the solution $x = 41$ is correct.

WORKED EXAMPLE 15

Solve each of the following equations.

a $\frac{5(x+3)}{6} = 4 + \frac{3(x-1)}{5}$ **b** $\frac{4}{3(x-1)} = \frac{1}{x+1}$

THINK	WRITE
a 1 Write the equation.	**a** $\frac{5(x+3)}{6} = 4 + \frac{3(x-1)}{5}$
2 The lowest common denominator of 5 and 6 is 30. Write each term as an equivalent fraction with a denominator of 30.	$\frac{25(x+3)}{30} = \frac{120}{30} + \frac{18(x-1)}{30}$
3 Multiply each term by 30. This effectively removes the denominator.	$25(x+3) = 120 + 18(x-1)$
4 Expand the brackets and collect like terms.	$25x + 75 = 120 + 18x - 18$ $25x + 75 = 102 + 18x$
5 Subtract $18x$ from both sides of the equation.	$7x + 75 = 102$

6	Subtract 75 from both sides of the equation.	$7x = 27$
7	Divide both sides of the equation by 7 to solve for x.	$x = \frac{27}{7}$
8	Express the improper fraction as a mixed number.	$x = 3\frac{6}{7}$

b

1	Write the equation.	$\frac{4}{3(x-1)} = \frac{1}{x+1}$
2	The lowest common denominator of 3, $x + 1$ and $x - 1$ is $3(x - 1)(x + 1)$. Write each term as an equivalent fraction with a common denominator of $3(x - 1)(x + 1)$.	$\frac{4(x+1)}{3(x-1)(x+1)} = \frac{3(x-1)}{3(x-1)(x+1)}$
3	Multiply each term by the common denominator.	$4(x + 1) = 3(x - 1)$
4	Expand the brackets.	$4x + 4 = 3x - 3$
5	Subtract $3x$ from both sides of the equation.	$x + 4 = -3$
6	Subtract 4 from both sides of the equation to solve for x.	$x + 4 - 4 = -3 - 4$ $= -7$

DISCUSSION

When using equations to solve word problems, why is it important to clearly state the meaning of an introduced pronumeral, for example n = number of years?

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Interactivity: Expanding brackets: Distributive law (int-3774)
Interactivity: Solving equations (int-2778)
eLesson: Solving linear equations with algebraic fractions (eles-1934)
Digital doc: WorkSHEET Solving equations with fractions (doc-13848)

Exercise 7.6 Solving complex linear equations

Individual pathways

PRACTISE
Questions:
1, 2a–c, 3a–e, 4a–d, 5, 6, 9

CONSOLIDATE
Questions:
1, 2a–f, 3b–i, 4c–g, 6, 7, 9, 11, 12, 14

MASTER
Questions:
1, 2c–h, 3g–l, 4g–l, 7–14

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE13 Solve each of the following linear equations.

a. $6(4x - 3) + 7(x + 1) = 9$ **b.** $9(3 - 2x) + 2(5x + 1) = 0$ **c.** $8(5 - 3x) - 4(2 + 3x) = 3$
d. $9(1 + x) - 8(x + 2) = 2x$ **e.** $6(4 + 3x) = 7(x - 1) + 1$ **f.** $10(4x + 2) = 3(8 - x) + 6$

2. WE14 Solve each of the following equations and verify the solutions.

a. $\frac{x+1}{2} = \frac{x+3}{3}$ **b.** $\frac{x-7}{5} = \frac{x-8}{4}$

c. $\frac{x-6}{4} = \frac{x-2}{2}$ **d.** $\frac{8x+3}{5} = 2x$

e. $\frac{2x-1}{5} = \frac{x-3}{4}$ **f.** $\frac{4x+1}{3} = \frac{x+2}{4}$

g. $\frac{6-x}{3} = \frac{2x-1}{5}$ **h.** $\frac{8-x}{9} = \frac{2x+1}{3}$

3. Solve each of the following linear equations.

a. $\frac{x}{3} + \frac{4x}{5} = \frac{1}{3}$ **b.** $\frac{x}{4} - \frac{x}{5} = \frac{3}{4}$ **c.** $\frac{x}{4} - \frac{4x}{7} = 2$

d. $\frac{-3x}{5} + \frac{x}{8} = \frac{1}{4}$ **e.** $\frac{2x}{3} - \frac{x}{6} = -\frac{3}{4}$ **f.** $\frac{5x}{8} - 8 = \frac{2x}{3}$

g. $\frac{2}{7} - \frac{x}{8} = \frac{3x}{8}$ **h.** $\frac{4}{x} - \frac{1}{6} = \frac{2}{x}$ **i.** $\frac{15}{x} - 4 = \frac{2}{x}$

j. $\frac{1}{3} + \frac{4}{x} = \frac{5}{x}$ **k.** $\frac{2x-4}{5} + 6 = \frac{x}{2}$ **l.** $\frac{4x-1}{2} - \frac{2x+5}{3} = 0$

4. WE15 Solve each of the following linear equations.

a. $\frac{3(x+1)}{2} + \frac{5(x+1)}{3} = 4$ **b.** $\frac{2(x+1)}{7} + \frac{3(2x-5)}{8} = 0$ **c.** $\frac{2(4x+3)}{5} - \frac{6(x-2)}{2} = \frac{1}{2}$

d. $\frac{8(x+3)}{5} = \frac{3(x+2)}{4}$ **e.** $\frac{5(7-x)}{2} = \frac{2(2x-1)}{7} + 1$ **f.** $\frac{2(6-x)}{3} = \frac{9(x+5)}{6} + \frac{1}{3}$

g. $\frac{-5(x-2)}{3} - \frac{6(2x-1)}{5} = \frac{1}{3}$ **h.** $\frac{9(2x-1)}{7} = \frac{4(x-5)}{3}$ **i.** $\frac{1}{x-1} + \frac{3}{x+1} = \frac{8}{x+1}$

j. $\frac{3}{x+1} + \frac{5}{x-4} = \frac{5}{x+1}$ **k.** $\frac{1}{x-1} - \frac{3}{x} = \frac{-1}{x-1}$ **l.** $\frac{4}{2x-1} - \frac{5}{x} = \frac{-1}{x}$

5. Last week Maya broke into her money box. She spent one-quarter of the money on a birthday present for her brother and one-third of the money on an evening out with her friends, leaving her with $75. How much money was in her money box?

6. At work Keith spends one-fifth of his time in planning and buying merchandise. He spends seven-twelfths of his time in customer service and one-twentieth of his time training the staff. This leaves him ten hours to deal with the accounts. How many hours does he work each week?

7. Last week's school fete was a great success, raising a good deal of money. Three-eighths of the profit came from sales of food and drink, and the market stalls recorded one-fifth of the total. A third of the profit came from the major raffle, and the jumping castle raised $1100. How much money was raised at the fete?

8. Lucy had half as much money as Mel, but since Grandma gave them each $20 she now has three-fifths as much. How much money does Lucy have?

Communicating, reasoning and problem solving

9. a. Which numbers smaller than 100 have exactly 3 factors (including 1 and the number itself)?

b. Which two numbers smaller than 100 have exactly 5 factors?

c. Which number smaller than 100 has exactly 7 factors?

10. To raise money for a charity, a Year 10 class has decided to organise a school lunch. Tickets will cost $6 each. The students have negotiated a special deal for delivery of drinks and pizzas, and they have budgeted $200 for drinks and $250 for pizzas. If they raise $1000 or more, they qualify for a special award.

a. Write an equation to represent the minimum number of tickets required to be sold to qualify for the award.

b. Solve the equation to find the number of tickets they must sell to qualify for the award. Explain your answer.

11. If $\frac{x+7}{(x+2)(x+3)} \equiv \frac{a}{x+2} - \frac{4}{x+3}$, explain why a must be equal to 5. (*Note:* '$\equiv$' means identically equal to.)

12. Solve for x:

$\frac{2}{9}(x-1) - \frac{5}{8}(x-2) = \frac{2}{5}(x-4) - \frac{7}{12}$.

13. If $\frac{2(4x+3)}{(x-3)(x+7)} \equiv \frac{a}{x-3} + \frac{b}{x+7}$, find the values of a and b.

14. Do the rules for the order of operations apply to algebraic fractions? Explain.

7.7 Rearranging literal equations [Stage 5.3]

7.7.1 Literal equations

- A **literal equation** is an equation that is composed of only variables or multiple variables. Solving literal equations involves changing the subject of the equation to a particular pronumeral (letter).
- A **formula** is a literal equation that uses algebra to state an interesting or important real-life relationship.

WORKED EXAMPLE 16

Solve the following literal equations for x.

a $ax^2 + bd = c$

b $ax = cx + b$

THINK	WRITE
a 1 Write the equation.	**a** $ax^2 + bd = c$
2 Subtract bd from both sides of the equation.	$ax^2 = c - bd$
3 Divide both sides by a.	$x^2 = \frac{c - bd}{a}$
4 To solve for x, take the square root of both sides. This gives both a positive and negative result for x.	$x = \pm\sqrt{\frac{c - bd}{a}}$
b 1 Write the equation.	**b** $ax = cx + b$
2 Subtract cx from both sides.	$ax - cx = b$
3 Factorise by taking x as a common factor.	$x(a - c) = b$
4 To solve for x, divide both sides by $a - c$.	$x = \frac{b}{a - c}$

WORKED EXAMPLE 17

Make b the subject of the formula $D = \sqrt{b^2 - 4ac}$.

THINK	WRITE
1 Write the formula.	$D = \sqrt{b^2 - 4ac}$
2 Square both sides.	$D^2 = b^2 - 4ac$
3 Add $4ac$ to both sides of the equation.	$D^2 + 4ac = b^2$
4 Take the square root of both sides.	$\pm\sqrt{D^2 + 4ac} = b$
5 Make b the subject of the formula by solving for b.	$b = \pm\sqrt{D^2 + 4ac}$

7.7.2 Restrictions on variables

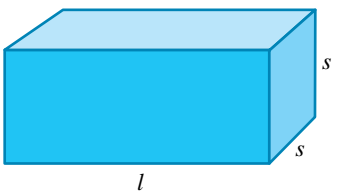
- Some variables may have implicit restrictions on the values that they may be assigned in an equation or formula. For example:
  - if $V = \frac{d}{t}$, then t cannot equal zero, otherwise the value of V would be undefined.
  - if $d = \sqrt{x - 9}$, then:
    - the value of d will be restricted to positive values or 0
    - the value of $x - 9$ must be greater than or equal to zero because the square root of a negative number cannot be found.

    $x - 9 \geq 0$

    $x \geq 9$ (Hence, x must be greater than or equal to 9.)
- Other restrictions may arise once a formula is rearranged. For example, if we look at the formula $V = ls^2$, there are no restrictions on the values that the variables l and s can be assigned. (However, the sign of V must always be the same as the sign of l because s^2 is always positive.) If the formula is transposed to make s the subject, then:

$$V = ls^2$$
$$\frac{V}{l} = s^2$$
$$\text{or } s = \pm\sqrt{\frac{V}{l}}$$

  This shows the restrictions that $l \neq 0$ and $\frac{V}{l} \geq 0$.
- If the formula $V = ls^2$ represents the volume of the rectangular prism shown, additional restrictions become evident: the variables l and s represent a length and must be positive numbers. Hence, when we make s the subject we get $s = \sqrt{\frac{V}{l}}$.



DISCUSSION

Think of some real-life situations which could be modelled by literal linear equations. Are there any restrictions on the variables?

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Interactivity: Restrictions on variables (int-6120)

Exercise 7.7 Rearranging literal equations

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–f, 2a–d, 3a–d, 4a–d, 5, 7	Questions: 1a–g, 2a–f, 3a–f, 4a–f, 5, 7, 8	Questions: 1d–i, 2c–j, 3c–h, 4–9

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE16 Solve the following literal equations for x.

a. $\frac{ax}{bc} = d$ **b.** $\frac{x}{a} - bc = d$ **c.** $\sqrt{x} + n = m$ **d.** $acx^2 = w$ **e.** $\frac{a}{x} = \frac{b}{y}$

f. $\frac{x + m}{n} = w$ **g.** $ab(x + b) = c$ **h.** $\frac{a}{x} = \frac{b}{c} + m$ **i.** $mx = ay - bx$ **j.** $\frac{x}{m} + a = \frac{c}{d}$

2. WE17 Rearrange each of the following literal equations to make the variable in brackets the subject.

a. $V = lbh$ $[I]$ **b.** $P = 2l + 2b$ $[b]$

c. $A = \frac{1}{2}bh$ $[h]$ **d.** $c = \sqrt{a^2 + b^2}$ $[a]$

e. $F = \frac{9C}{5} + 32$ $[C]$ **f.** $A = \pi r^2$ $[r]$

g. $v = u + at$ $[a]$ **h.** $I = \frac{PRN}{100}$ $[N]$

i. $E = \frac{1}{2}mv^2$ $[m]$ **j.** $E = \frac{1}{2}mv^2$ $[v]$

k. $v^2 = u^2 + 2as$ $[a]$ **l.** $v^2 = u^2 + 2as$ $[u]$

m. $\frac{1}{x} = \frac{1}{a} + \frac{1}{b}$ $[a]$ **n.** $x = \frac{nx_1 + mx_2}{m + n}$ $[x_1]$

3. Complete the following.

a. If $c = \sqrt{a^2 + b^2}$, calculate a if $c = 13$ and $b = 5$.

b. If $A = \frac{1}{2}bh$, find the value of h if $A = 56$ and $b = 16$.

c. If $F = \frac{9C}{5} + 32$, find the value of C if $F = 86$.

d. If $v = u + at$, find the value of a if $v = 83.6$, $u = 15$ and $t = 7$.

e. If $V = ls^2$, find the value of s if $V = 2028$ and $l = 12$.

f. If $v^2 = u^2 + 2as$, find the value of u if $v = 16$, $a = 10$ and $s = 6.75$.

g. If $A = \frac{1}{2}h(a + b)$, find the value of a if $A = 360$, $b = 15$ and $h = 18$.

h. If $x = \frac{nx_1 + mx_2}{m + n}$, find the value of x_2 if $x = 10$, $m = 2$, $n = 1$ and $x_1 = 4$.

4. For the following equations:

i. list any restrictions on the variables in the equation

ii. rearrange the equation to make the variable in brackets the subject

iii. list any new restrictions on the variables in the equation formed in part **ii**.

a. $y = x^2 + 4$ $[x]$

b. $y = \dfrac{2}{x - 3}$ $[x]$

c. $v = u + at$ $[t]$

d. $c = \sqrt{a^2 + b^2}$ $[b]$

e. $s = \dfrac{a}{1 - r}$ $[r]$

f. $m = \dfrac{pb + qa}{p + q}$ $[b]$

g. $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $[c]$

h. $m = \dfrac{pb + qa}{p + q}$ $[p]$

Communicating, reasoning and problem solving

5. The volume of a cylinder is given by $V = \pi r^2 h$, where r is the radius and h is the height of the cylinder.

a. State any restrictions on the values of the variables in this formula.

b. Make r the subject of the formula.

c. List any new restrictions on the variables in the formula.

6. T is the period of a pendulum whose length is l and g is the acceleration due to gravity. The formula relating these variables is $T = 2\pi\sqrt{\dfrac{l}{g}}$.

a. What restrictions are applied to the variables T and l?

b. Make l the subject of the equation.

c. Do the restrictions stated in part a still apply?

d. Find the length of a pendulum that has a period of 3 seconds, given that $g = 9.8$ m/s^2. Give your answer correct to 1 decimal place.

7. $F = 32 + \frac{9}{5}C$ is the formula relating degrees Celsius (C) to degrees Fahrenheit (F).

a. Transform the equation to make C the subject.

b. Find the temperature when degrees Celsius is equal to degrees Fahrenheit.

8. Jing Jing and Pieter live on the same main road, but Jing Jing lives a kilometres to the east of Pieter. Both Jing Jing and Pieter set off on their bicycles at exactly the same time and both ride in a westerly direction. Jing Jing rides at j kilometres per hour and Pieter rides at p kilometres per hour. It is known that $j > p$. Find an equation in terms of a, j and p for the distance Jing Jing has ridden in order to catch up with Pieter.

9. Why is it important to consider restrictions on variables when solving literal equations?

7.8 Review

Investigation | Rich task

Checking for data entry errors

When we enter numbers into an electronic device or write numbers down, errors frequently occur. A common type of error is a transposition error, which occurs when two digits are written in the reverse order. Take the number 2869, for example. With this type of error, it could be written as 8269, 2689 or 2896. A common rule for checking these errors is as follows.

If the difference between the correct number and the recorded number is a multiple of 9, a transposition error has occurred.

We can use algebraic expressions to check this rule. Let the digit in the thousands position be represented by a, the digit in the hundreds position by b, the digit in the tens position by c and the digit in the ones position by d. So the real number can be represented as $1000a + 100b + 10c + d$.

1. If the digits in the ones position and the tens position were written in the reverse order, the number would be $1000a + 100b + 10d + c$. The difference between the correct number and the incorrect one would then be: $1000a + 100b + 10c + d - (1000a + 100b + 10d + c)$.
   a. Simplify this expression.
   b. Is the expression a multiple of 9? Explain.

2. If a transposition error had occurred in the tens and hundreds position, the incorrect number would be $1000a + 100c + 10b + d$. Perform the procedure shown in question **1** to determine whether the difference between the correct number and the incorrect one is a multiple of 9.

3. Consider, lastly, a transposition error in the thousands and hundreds positions. Is the difference between the two numbers a multiple of 9?

4. Comment on the checking rule for transposition errors.

learnON RESOURCES — ONLINE ONLY

Digital doc: Investigation — Checking for data entry errors (doc-15918)

Digital doc: Code puzzle — In which country was the first practical ice-making machine and refrigerator produced in 1856? (doc-15919)

Digital doc: Summary Topic 7 (doc-22905)

Interactivity: Word search Topic 7 (int-2829)

Interactivity: Crossword Topic 7 (int-2830)

Interactivity: Sudoku Topic 7 (int-3589)

Exercise 7.8 Review questions

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. MC Given $E = \frac{1}{2}mv^2$ where $m = 0.2$ and $v = 0.5$, the value of E is:

A. 0.000 625 **B.** 0.1 **C.** 0.005 **D.** 0.025 **E.** 0.0025

2. MC The expression $-6d + 3r - 4d - r$ simplifies to:

A. $2d + 2r$ **B.** $-10d + 2r$ **C.** $-10d - 4r$ **D.** $2d + 4r$ **E.** $-8dr$

3. MC The expression $5(2f + 3) + 6(4f - 7)$ simplifies to:

A. $34f + 2$ **B.** $34f - 4$ **C.** $34f - 27$ **D.** $34f + 14$ **E.** $116f - 14$

4. MC The expression $7(b - 1) - (8 - b)$ simplifies to:

A. $8b - 9$ **B.** $8b - 15$ **C.** $6b - 9$ **D.** $6b - 15$ **E.** $8b + 1$

5. MC If $14p - 23 = 6p - 7$, then p equals:

A. -3 **B.** -1 **C.** 1 **D.** 2 **E.** 4

6. Simplify the following by collecting like terms.

a. $3c - 5 + 4c - 8$ **b.** $-3k + 12m - 4k - 9m$

c. $-d + 3c - 8c - 4d$ **d.** $6y^2 + 2y + y^2 - 7y$

7. If $A = \frac{1}{2}bh$, determine the value of A if $b = 10$ and $h = 7$.

8. For each of the following, complete the relationship to illustrate the stated law.

a. $(a + 3b) + 6c =$ _____ Associative Law

b. $12a - 3b \neq$ _____ Commutative Law

c. $7p \times$ _____ = _____ Inverse Law

d. $(x \times 5y) \times 7z =$ _____ Associative Law

e. $12p + 0 =$ _____ Identity Law

f. $(3p \div 5q) \div 7r =$ _____ Associative Law

g. $9d + 11e =$ _____ Commutative Law

h. $4a \div b \neq$ _____ Commutative Law

9. Find the value of the following expressions given the natural number values $x = 12$, $y = 8$ and $z = 4$. Comment on whether the Closure Law holds for each of the expressions when the values are substituted.

a. $x \times y$ **b.** $z \div x$ **c.** $y - x$

10. Simplify the following.

a. $\frac{5y}{3} - \frac{y}{2}$ **b.** $\frac{x + 4}{5} + \frac{x + 2}{2}$ **c.** $\frac{5}{3x} - \frac{1}{5x}$ **d.** $\frac{x - 1}{x + 3} + \frac{2x - 5}{x + 2}$

11. Simplify the following.

a. $\frac{y}{4} \times \frac{32}{x}$ **b.** $\frac{20y}{7x} \times \frac{35z}{16y}$ **c.** $\frac{x+6}{(x+1)(x+3)} \times \frac{5(x+1)}{x+6}$

d. $\frac{25}{x} \div \frac{30}{x}$ **e.** $\frac{xy}{5} \div \frac{10x}{y}$ **f.** $\frac{2x}{(x+8)(x-1)} \div \frac{9x+1}{x+8}$

12. Solve the following equations.

a. $p - 20 = 68$ **b.** $s - 0.56 = 2.45$ **c.** $3b = 48$

d. $\frac{r}{7} = -5$ **e.** $\sqrt{x} = 12$ **f.** $2(x + 5) = -3$

g. $\frac{y}{4} - 3 = 12$ **h.** $a^2 = 36$ **i.** $5 - k = -7$

13. Solve the following.

a. $42 - 7b = 14$ **b.** $12t - 11 = 4t + 5$ **c.** $2(4p - 3) = 2(3p - 5)$

14. Solve each of the following linear equations.

a. $5(x - 2) + 3(x + 2) = 0$ **b.** $7(5 - 2x) - 3(1 - 3x) = 1$

c. $5(x + 1) - 6(2x - 1) = 7(x + 2)$ **d.** $8(3x - 2) + (4x - 5) = 7x$

e. $7(2x - 5) - 4(x + 20) = x - 5$ **f.** $3(x + 1) + 6(x + 5) = 3x + 40$

15. Solve each of the following equations.

a. $\frac{x}{2} + \frac{x}{5} = \frac{3}{5}$ **b.** $\frac{x}{3} - \frac{x}{5} = 3$ **c.** $-\frac{1}{21} = \frac{x}{7} - \frac{x}{6}$

d. $\frac{3}{x} + \frac{2}{5} = \frac{5}{x}$ **e.** $\frac{2x-3}{2} - \frac{3}{5} = \frac{x+3}{5}$ **f.** $\frac{2(x+2)}{3} = \frac{3}{7} + \frac{5(x+1)}{3}$

16. a. Make x the subject of $bx + cx = \frac{d}{2}$.

b. Make r the subject of $V = \frac{4}{3}\pi r^3$.

Communicating, reasoning and problem solving

17. A production is in town and many parents are taking their children. An adult ticket costs $15 and a child's ticket costs $8. Every child must be accompanied by an adult and each adult can have no more than 4 children with them. It costs the company $12 per adult and $3 per child to run the production. There is a seating limit of 300 people and all tickets are sold.

a. Determine how much profit the company makes on each adult ticket and on each child's ticket.

b. To maximise profit, the company should sell as many children's tickets as possible. Of the 300 available seats, determine how many should be allocated to children if there is a maximum of 4 children per adult.

c. Using your answer to part **b**, determine how many adults would make up the remaining seats.

d. Construct an equation to represent the profit that the company can make depending on the number of children and adults attending the production.

e. Substitute your values to calculate the maximum profit the company can make.

18. You are investigating prices for having business cards printed for your new games store. A local printing company charges a flat rate of $250 for the materials used and $40 per hour for labour.

a. If h is the number of hours of labour required to print the cards, construct an equation for the cost of the cards, C.

b. You have budgeted $1000 for the printing job. How many hours of labour can you afford? Give your answer to the nearest minute.

c. The printer estimates that it can print 1000 cards per hour of labour. How many cards will be printed with your current budget?

d. An alternative to printing is photocopying. The company charges 15 cents per side for the first 10000 cards and then 10 cents per side for the remaining cards. Which is the cheaper option for 18750 single-sided cards and by how much?

19. A scientist tried to use a mathematical formula to predict people's moods based on the number of hours of sleep they had the previous night. One formula that he used was what he called the 'grumpy formula', $g = 0.16(h - 8)^2$, which was valid on a 'grumpy scale' from 0 to 10 (least grumpy to most grumpy).

a. Calculate the number of hours needed to not be grumpy.

b. Evaluate the grumpy factor for somebody who has had:

i. 4 hours of sleep **ii.** 6 hours of sleep **iii.** 10 hours of sleep.

c. Calculate the number of hours of sleep required to be most grumpy.

20. Another scientist already had his own grumpy formula and claims that the scientist from question **19** stole his idea and has just simplified it. The second scientist's grumpy formula was

$$g = \frac{0.16(h - 8)}{8 - h} \times \frac{2(8 - h)}{3(h - 8)} \div \frac{2h}{3(h - 8)^2}.$$

a. Write the second scientist's formula in simplified form.

b. Are the second scientist's claims justified? Explain.

Answers

Topic 7 Algebraic fractions and equations

Exercise 7.2 Substitution

1. a. 5 **b.** 2 **c.** 0 **d.** 6 **e.** −17 **f.** 3
g. 30 **h.** 12 **i.** −12 **j.** 27 **k.** 30 **l.** −5

2. a. −11 **b.** −1 **c.** 1 **d.** 30 **e.** −24 **f.** 36
g. −125 **h.** 1 **i.** 15

3. a. $\frac{7}{12}$ **b.** $-\frac{1}{12}$ **c.** $\frac{1}{12}$ **d.** $1\frac{1}{3}$ **e.** $\frac{1}{576}$ **f.** 48

4. a. 9 **b.** −9 **c.** 9 **d.** 18 **e.** −18 **f.** 36

5. a. 9 **b.** −9 **c.** 9 **d.** 18 **e.** −18 **f.** 36

6. a. 17 **b.** 30 **c.** 8 **d.** 4 **e.** 1.5 **f.** 68
g. 46 **h.** 113.1 **i.** 5 **j.** 624.6

7. a. D **b.** C **c.** B

8. 3.9 cm **9.** 65.45 cm^3 **10.** 361 m

11. a. −1 — in this case, addition is closed on integers.
b. −1 — in this case, subtraction is closed on integers.
c. 2 — in this case, multiplication is closed on integers.
d. −1 — in this case, division is closed on integers.
e. −2 — in this case, subtraction is closed on integers.
f. $-\frac{1}{2}$ — in this case, division is not closed on integers.

12. a. 10 — in this case, addition is closed on natural numbers.
b. −4 — in this case, subtraction is not closed on natural numbers.
c. 12— in this case, multiplication is closed on natural numbers.
d. $\frac{4}{3}$ — in this case, division is not closed on natural numbers.
e. −2 — in this case, subtraction is not closed on natural numbers.
f. 4 — in this case, division is closed on natural numbers.

13. a. $(a + 2b) + 4c = a + (2b + 4c)$ **b.** $(x \times 3y) \times 5c = x \times (3y \times 5c)$
c. $2p \div q \neq q \div 2p$ **d.** $5d + q = q + 5d$
e. $3z + 0 = 0 + 3z = 3z$ **f.** $2x \times \frac{1}{2x} = \frac{1}{2x} \times 2x = 1$
g. $(4x \div 3y) \div 5z \neq 4x \div (3y \div 5z)$ **h.** $3d - 4y \neq 4y - 3d$

14. a. $s = 71.875$ metres **b.** $t = 4$ seconds

15. $m = \frac{4}{5}$

16. Check with your teacher.

17. $3a^3c^2$

Exercise 7.3 Adding and subtracting algebraic fractions

1. a. $\frac{26}{21}$ or $1\frac{5}{21}$ **b.** $\frac{49}{72}$ **c.** 1
d. $\frac{17}{99}$ **e.** $\frac{1}{35}$ **f.** $\frac{6 - 5x}{30}$
g. $\frac{15x - 4}{27}$ **h.** $\frac{15 - 16x}{40}$ **i.** $\frac{15 - 2x}{3x}$

2. a. $\frac{5y}{12}$ **b.** $-\frac{3y}{40}$ **c.** $\frac{13x}{12}$ **d.** $\frac{14x}{9}$
e. $\frac{3w}{28}$ **f.** $-\frac{y}{5}$ **g.** $\frac{89y}{35}$ **h.** $\frac{32x}{15}$
i. $\frac{7x + 17}{10}$ **j.** $\frac{7x + 30}{12}$ **k.** $\frac{2x - 11}{30}$ **l.** $\frac{19x + 7}{6}$

3. a. $\frac{5}{8x}$ **b.** $\frac{5}{12x}$ **c.** $\frac{38}{21x}$

d. $\frac{8}{3x}$ **e.** $\frac{7}{24x}$ **f.** $\frac{9}{20x}$

g. $\frac{37}{100x}$ **h.** $\frac{51}{10x}$ **i.** $-\frac{1}{6x}$

4. a. $\frac{3x^2 + 14x - 4}{(x + 4)(x - 2)}$ **b.** $\frac{2x^2 + 3x + 25}{(x + 5)(x - 1)}$ **c.** $\frac{2x^2 + 6x - 10}{(2x + 1)(x - 2)}$ **d.** $\frac{4x^2 - 17x - 3}{(x + 1)(2x - 7)}$

e. $\frac{7x^2 + x}{(x + 7)(x - 5)}$ **f.** $\frac{2x^2 + 6x + 7}{(x + 1)(x + 4)}$ **g.** $\frac{-x^2 + 7x + 15}{(x + 1)(x + 2)}$ **h.** $\frac{x - 7}{(x + 3)(x - 2)}$

i. $\frac{x^2 + 3x + 9}{(x + 2)(3x - 1)}$ **j.** $\frac{5 - 5x}{(x - 1)(1 - x)} = \frac{5}{x - 1}$ **k.** $\frac{3x + 7}{(x + 1)^2}$ **l.** $\frac{3x - 4}{(x - 1)^2}$

5. a. The student transcribed the denominator incorrectly and wrote $(x + 2)$ instead of $(x - 2)$ in line 2.

Also, the student forgot that multiplying a negative number by a negative number gives a positive number.

Line 3 should have $+3$ in the numerator, not -1. They didn't multiply.

b. $\frac{x^2 - 5x + 3}{(x - 1)(x - 2)}$

6. a. $\frac{4x^2 + 17x + 17}{(x + 2)(x + 1)(x + 3)}$ **b.** $\frac{7x^2 - 20x + 4}{(x - 1)(x + 2)(x - 4)}$ **c.** $\frac{4x^2 + 17x + 19}{(x + 1)(x + 3)(x + 2)}$ **d.** $\frac{2(2x^2 - 9x + 25)}{(x - 4)(x - 1)(x + 3)}$

7. The lowest common denominator may not always be the product of the denominators. Each fraction must be multiplied by the correct multiple.

8. $a = 4$

9. $\frac{4(x - 1)}{(x + 3)(x + 4)(x - 2)}$

10. $\frac{2(x - 1)}{(x - 7)(x - 4)}$

11. The lowest common denominator needs to be found before fractions can be added or subtracted.

Exercise 7.4 Multiplying and dividing algebraic fractions

1. a. $\frac{4x}{y}$ **b.** $\frac{3x}{y}$ **c.** $\frac{4y}{x}$ **d.** $\frac{9x}{4y}$ **e.** $\frac{-5x}{4y}$ **f.** $\frac{3w}{2x}$

g. $\frac{6z}{7x}$ **h.** $\frac{2z}{7x}$ **i.** $\frac{-3x}{2y}$ **j.** $\frac{5}{24}$ **k.** $\frac{12z}{x}$ **l.** $\frac{-x}{6w}$

2. a. $\frac{2}{3x - 2}$ **b.** $\frac{5}{x - 3}$ **c.** $\frac{9}{2(x - 6)}$ **d.** $\frac{1}{x + 3}$ **e.** $\frac{2x}{(x + 1)^2}$

f. $\frac{x + 1}{2(2x - 3)}$ **g.** $\frac{a}{10(a + 3)}$ **h.** $\frac{35d}{8(d - 3)}$ **i.** $\frac{9}{32x^2(x - 2)}$ **j.** $\frac{3x}{10(x - 1)}$

3. a. $\frac{3}{5}$ **b.** $\frac{2}{9}$ **c.** $\frac{1}{3}$ **d.** 3 **e.** $\frac{1}{25}$ **f.** $\frac{35}{6}$ or $5\frac{5}{6}$

g. $\frac{4y^2}{7}$ **h.** $\frac{2y^2}{25}$ **i.** $\frac{8y^2}{9}$ **j.** $\frac{32xy}{15}$ **k.** $\frac{2}{3}$ **l.** y^2

4. a. $\frac{9}{(3x - 7)(x + 3)}$ **b.** $\frac{1}{(x + 2)(x - 9)}$ **c.** $\frac{21(x - 3)}{x + 5}$ **d.** $\frac{13}{9(x - 4)(x + 1)}$

5. Yes, because all of the fractions on the right-hand side have the same denominator and therefore can be added together.

6. a. -1

b. $4 - x$ considered to be the same as $x - 4$

7. 1

8. $\frac{(x + 1)^2}{x^2 + 1}$

9. Before we multiply or divide algebraic fractions, they need to be simplified by cancelling out common factors in the numerators and denominators. In contrast, when we add or subtract algebraic fractions, only the lowest common denominator needs to be found.

10. $\frac{x(x + 2)}{(x - 1)(x + 1)}$ or $\frac{x(x + 2)}{x^2 - 1}$

Exercise 7.5 Solving simple equations

1. a. $a = 24$ **b.** $k = 121$ **c.** $g = 2.9$ **d.** $r = 3$ **e.** $h = 0.26$
f. $i = -2$ **g.** $t = 5$ **h.** $q = \frac{1}{6}$ **i.** $x = 0$

2. a. $f = 12$ **b.** $i = -60$ **c.** $z = -7$ **d.** $v = 7$ **e.** $w = -5\frac{1}{3}$
f. $k = 10$ **g.** $a = 0.425$ **h.** $m = 16\frac{5}{8}$ **i.** $y = 21\frac{1}{2}$

3. a. $t = 100$ **b.** $y = \pm 17$ **c.** $q = 6.25$ **d.** $f = \pm 1.2$ **e.** $h = \frac{16}{49}$
f. $p = \pm\frac{3}{8}$ **g.** $g = \frac{225}{484}$ **h.** $j = \pm\frac{14}{31}$ **i.** $a = \pm 1\frac{2}{3}$

4. a. $t = 25$ **b.** $x = \pm 6$ **c.** $m = 16$ **d.** $t = \pm 3$ **e.** $t = \pm 10$ **f.** $m = 25$

5. a. $x = 8$ **b.** $x = -3$ **c.** $m = \frac{1}{8}$ **d.** $x = \frac{3}{4}$ **e.** $m = 0.008$ **f.** $w = 2\frac{1}{2}$

6. a. $x = -1$ **b.** $x = -2$ **c.** $m = 1$ **d.** $w = -512$ **e.** $t = 125$ **f.** $x = 2$

7. a. $a = 4$ **b.** $b = 6$ **c.** $i = 3$ **d.** $f = 9$ **e.** $q = 1\frac{1}{8}$
f. $r = 5\frac{2}{5}$ **g.** $s = 4\frac{5}{6}$ **h.** $t = 9\frac{4}{5}$ **i.** $a = -7\frac{1}{2}$

8. a. $f = 40$ **b.** $g = 30$ **c.** $r = -10$ **d.** $m = 18$ **e.** $n = 28$ **f.** $p = 62.4$

9. a. $x = 1\frac{1}{3}$ **b.** $y = 9$ **c.** $m = 4\frac{2}{5}$ **d.** $k = 1\frac{1}{2}$ **e.** $n = 5\frac{2}{3}$
f. $c = 1\frac{1}{3}$ **g.** $x = 10$ **h.** $x = 4$ **i.** $x = -7\frac{1}{5}$

10. a. $k = 25$ **b.** $m = 16$ **c.** $p = -11\frac{3}{7}$ **d.** $u = -4\frac{1}{8}$ **e.** $x = \frac{8}{11}$ **f.** $v = 3$

11. a. $x = 26$ **b.** $m = -5$ **c.** $w = \frac{25}{3}$ **d.** $t = 5$ **e.** $x = 9$ **f.** $n = -\frac{19}{3}$

12. a. B **b.** E **c.** C

13. a. $x = -5$ **b.** $d = -1$ **c.** $p = 7$ **d.** $x = -11$ **e.** $h = -2$
f. $t = 5$ **g.** $v = -20$ **h.** $r = -3$ **i.** $g = -0.8$

14. a. $x = -1$ **b.** $v = 1$ **c.** $l = 2$ **d.** $g = -2$ **e.** $t = 3$
f. $e = -23\frac{1}{3}$ **g.** $j = -3\frac{3}{8}$ **h.** $k = -36$ **i.** $f = -12\frac{1}{4}$

15. a. $x = 2$ **b.** $b = 5$ **c.** $w = 2$ **d.** $f = 7$ **e.** $t = 3$
f. $r = 2\frac{1}{3}$ **g.** $g = -1\frac{1}{3}$ **h.** $h = -2\frac{1}{5}$ **i.** $a = 0$

16. a. $x = -1$ **b.** $c = 2$ **c.** $r = 2\frac{2}{3}$ **d.** $k = 1$ **e.** $y = -1\frac{1}{8}$
f. $g = 7$ **g.** $w = 1$ **h.** $m = \frac{1}{5}$ **i.** $p = 1\frac{2}{3}$

17. a. $x = -15$ **b.** $y = -4\frac{4}{5}$ **c.** $t = 21$ **d.** $u = -2\frac{5}{7}$ **e.** $f = 12\frac{1}{2}$
f. $r = 7\frac{1}{2}$ **g.** $d = -6$ **h.** $h = -12$ **i.** $x = 1$

18. a. A **b.** D **c.** B

19. 5 cm

20. a. 6 cm **b.** 1.26 m

21. a. 2.5 cm **b.** 41 cm

22. a. 5 m **b.** 2.8 s

23. a. 314 cm^2 **b.** 6.3 cm

24. 1.8 cm

25. 6.2 cm

26. Dimensions are 10 m by 6 m.

27. $\frac{10\sqrt{3}}{3}$ cm

28. Solving a linear equation means finding a value for a pronumeral in the equation that makes the equation true.

Exercise 7.6 Solving complex linear equations

1. a. $x = \frac{20}{31}$ **b.** $x = 3\frac{5}{8}$ **c.** $x = \frac{29}{36}$ **d.** $x = -7$ **e.** $x = -2\frac{8}{11}$ **f.** $x = \frac{10}{43}$

2. a. $x = 3$ **b.** $x = 12$ **c.** $x = -2$ **d.** $x = \frac{3}{2}$
e. $x = -\frac{11}{3}$ or $x = -3\frac{2}{3}$ **f.** $x = \frac{2}{13}$ **g.** $x = 3$ **h.** $x = \frac{5}{7}$

3. a. $x = \frac{5}{17}$ **b.** $x = 15$ **c.** $x = -6\frac{2}{9}$ **d.** $x = -\frac{10}{19}$ **e.** $x = -1\frac{1}{2}$ **f.** $x = -192$
g. $x = \frac{4}{7}$ **h.** $x = 12$ **i.** $x = 3\frac{1}{4}$ **j.** $x = 3$ **k.** $x = 52$ **l.** $x = 1\frac{5}{8}$

4. a. $x = \frac{5}{19}$ **b.** $x = 1\frac{31}{58}$ **c.** $x = 4\frac{11}{14}$ **d.** $x = -3\frac{15}{17}$ **e.** $x = 5\frac{20}{43}$ **f.** $x = -1\frac{10}{13}$

g. $x = 1\frac{2}{61}$ **h.** $x = -4\frac{9}{26}$ **i.** $x = 1.5$ **j.** $x = -4\frac{1}{3}$ **k.** $x = 3$ **l.** $x = 1$

5. $180

6. 60 hours

7. $12 000

8. $60

9. a. 4, 9, 25, 49 **b.** 16, 81 **c.** 64

10. a. $6x - 450 = 1000$

b. $241\frac{1}{3}$ tickets. This means they need to sell 242 tickets to qualify, as the number of tickets must be a whole number.

11. Check with your teacher.

12. 4

13. $a = 3, b = 5$

14. Discuss with your teacher.

Exercise 7.7 Rearranging literal equations

1. a. $x = \frac{bcd}{a}$ **b.** $x = a(d + bc)$ **c.** $x = (m - n)^2$ **d.** $x = \pm\sqrt{\frac{w}{ac}}$ **e.** $x = \frac{ay}{b}$

f. $x = nw - m$ **g.** $x = \frac{c}{ab} - b$ **h.** $x = \frac{ac}{b + mc}$ **i.** $x = \frac{ay}{m + b}$ **j.** $x = \frac{mc - amd}{d}$

2. a. $l = \frac{V}{bh}$ **b.** $b = \frac{P - 2l}{2}$ **c.** $h = \frac{2A}{b}$ **d.** $a = \pm\sqrt{c^2 - b^2}$ **e.** $C = \frac{5}{9}(F - 32)$

f. $r = \pm\sqrt{\frac{A}{\pi}}$ **g.** $a = \frac{v - u}{t}$ **h.** $N = \frac{100I}{PR}$ **i.** $m = \frac{2E}{v^2}$ **j.** $v = \pm\sqrt{\frac{2E}{m}}$

k. $a = \frac{v^2 - u^2}{2s}$ **l.** $u = \pm\sqrt{v^2 - 2as}$ **m.** $a = \frac{xb}{b - x}$ **n.** $x_1 = \frac{x(m + n) - mx_2}{n}$

3. a. $a = \pm 12$ **b.** $h = 7$ **c.** $C = 30$ **d.** $a = 9.8$

e. $s = \pm 13$ **f.** $u = \pm 11$ **g.** $a = 25$ **h.** $x_2 = 13$

4. a. i. No restrictions on x **ii.** $x = \pm\sqrt{y - 4}$ **iii.** $y \geq 4$

b. i. $x \neq 3$ **ii.** $x = \frac{2}{y} + 3$ **iii.** $y \neq 0$

c. i. No restrictions **ii.** $t = \frac{v - u}{a}$ **iii.** $a \neq 0$

d. i. $c \geq 0$ **ii.** $b = \pm\sqrt{c^2 - a^2}$ **iii.** $|c| \geq |a|$

e. i. $r \neq 1$ **ii.** $r = \frac{s - a}{s}$ **iii.** $s \neq 0$

f. i. $p \neq -q$ **ii.** $b = \frac{m(p + q) - qa}{p}$ **iii.** $p \neq 0$

g. i. $a \neq 0,\ b^2 \geq 4ac$

ii. $c = \frac{b^2 - (2ax + b)^2}{4a}$ or $c = -ax^2 - bx$

iii. No new restrictions

h. i. $p \neq -q$ **ii.** $p = \frac{q(a - m)}{m - b}$ **iii.** $m \neq b$

5. a. All values must be positive for a cylinder to exist.

b. $r = \sqrt{\frac{V}{\pi h}}$

c. $h \neq 0$, no new restrictions

6. a. T and l must be greater than zero. **b.** $l = \frac{T^2 g}{4\pi^2}$

c. The restrictions still hold. **d.** 2.2 m

7. a. $C = \frac{5}{9}(F - 32)$ **b.** $-40°$

8. The distance Jing Jing has ridden is $\frac{ja}{j - p}$ kilometres.

9. Discuss with your teacher.

Investigation | Rich task

1. a. $9(c - d)$

b. Yes, this is a multiple of 9 as the number that multiples the brackets is 9.

2. $90(b - c)$; 90 is a multiple of 9 so the difference between the correct and incorrect one is a multiple of 9.

3. $900(a - b)$; again 900 is a multiple of 9.

4. If two adjacent digits are transposed, the difference between the correct number and the transposed number is a multiple of 9.

Exercise 7.8 Review questions

1. D **2.** B **3.** C **4.** B **5.** D

6. a. $7c - 13$ **b.** $-7k + 3m$ **c.** $-5c - 5d$ **d.** $7y^2 - 5y$

7. 35

8. a. $(a + 3b) + 6c = a + (3b + 6c)$ **b.** $12a - 3b \neq 3b - 12a$

c. $7p \times \frac{1}{7p} = \frac{1}{7p} \times 7p = 1$ **d.** $(x \times 5y) \times 7z = x \times (5y \times 7z)$

e. $12p + 0 = 0 + 12p = 12p$ **f.** $(3p \div 5q) \div 7r \neq 3p \div (5q \div 7r)$

g. $9d + 11e = 11e + 9d$ **h.** $4a \div b \neq b \div 4a$

9. a. 96 — in this case, multiplication is closed on natural numbers.

b. $\frac{1}{3}$ — in this case, division is not closed on natural numbers.

c. -4 — in this case, subtraction is not closed on natural numbers.

10. a. $\frac{7y}{6}$ **b.** $\frac{7x + 18}{10}$ **c.** $\frac{22}{15x}$ **d.** $\frac{3x^2 + 2x - 17}{(x + 3)(x + 2)}$

11. a. $\frac{8y}{x}$ **b.** $\frac{25z}{4x}$ **c.** $\frac{5}{x + 3}$

d. $\frac{5}{6}$ **e.** $\frac{y^2}{50}$ **f.** $\frac{2x}{(x - 1)(9x + 1)}$

12. a. $p = 88$ **b.** $s = 3.01$ **c.** $b = 16$ **d.** $r = -35$ **e.** $x = 144$

f. $x = -\frac{13}{2}$ **g.** $y = 60$ **h.** $a = \pm 6$ **i.** $k = 12$

13. a. $b = 4$ **b.** $t = 2$ **c.** $p = -2$

14. a. $x = \frac{1}{2}$ **b.** $x = 6\frac{1}{5}$ **c.** $x = -\frac{3}{14}$

d. $x = 1$ **e.** $x = 12\frac{2}{9}$ **f.** $x = 1\frac{1}{6}$

15. a. $x = \frac{6}{7}$ **b.** $x = 22\frac{1}{2}$ **c.** $x = 2$

d. $x = 5$ **e.** $x = 3\frac{3}{8}$ **f.** $x = -\frac{16}{21}$

16. a. $x = \frac{d}{2(b + c)}$ **b.** $r = \sqrt[3]{\frac{3V}{4\pi}}$

17. a. \$3 per adult ticket; \$5 per child's ticket

b. 240

c. 60

d. $P = 3a + 5c$, where a = number of adults and c = number of children

e. \$1380

18. a. $C = 250 + 40h$

b. 18 hours 45 minutes

c. 18750

d. Printing is the cheaper option by \$1375.

19. a. 8 hours

b. i. 2.56 **ii.** 0.64 **iii.** 0.64

c. 0.094 hours or 15.9 hours

20. a. $g = \frac{0.16(h - 8)^2}{h}$

b. No, the formula is not the same.

TOPIC 8
Linear relationships [Stages 5.2 and 5.3]

8.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the concepts covered in this topic.

8.1.1 Why learn this?

What did you weigh as a baby, and how tall were you? Did you grow at a steady (linear) rate, or were there periods in your life when you grew rapidly? What is the relationship between your height and your weight? We constantly seek to find relationships between variables, and coordinate geometry provides a picture, a visual clue as to what the relationships may be.

DISCUSSION

Descartes and which other philosopher had an influence on the mathematical world?

LEARNING SEQUENCE

8.1 Overview
8.2 [Stages 5.2 and 5.3] The distance between two points
8.3 [Stages 5.2 and 5.3] Sketching linear graphs
8.4 [Stages 5.2 and 5.3] Standard forms of the equation of a straight line
8.5 [Stages 5.2 and 5.3] The midpoint of a line segment
8.6 [Stages 5.2 and 5.3] Parallel and perpendicular lines
8.7 Review

LEARNING OUTCOMES

A student:

- selects appropriate notations and conventions to communicate mathematical ideas and solutions MA5.2-1WM
- constructs arguments to prove and justify results MA5.2-3WM
- uses the gradient–intercept form to interpret and graph linear relationships MA5.2-9NA
- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3-1WM
- generalises mathematical ideas and techniques to analyse and solve problems efficiently MA5.3-2WM
- uses deductive reasoning in presenting arguments and formal proofs MA5.3-3WM
- uses formulas to find midpoint, gradient and distance on the Cartesian plane, and applies standard forms of the equation of a straight line MA5.3-8NA

CONTENT DESCRIPTIONS

Students:

Find the midpoint and gradient of a line segment (interval) on the Cartesian plane (ACMNA294)
Find the distance between two points located on the Cartesian plane (ACMNA214)
Sketch linear graphs using the coordinates of two points (ACMNA215)
Solve problems using various standard forms of the equation of a straight line
Solve problems involving parallel and perpendicular lines (ACMNA238)

Source: NSW Syllabus for the Australian curriculum

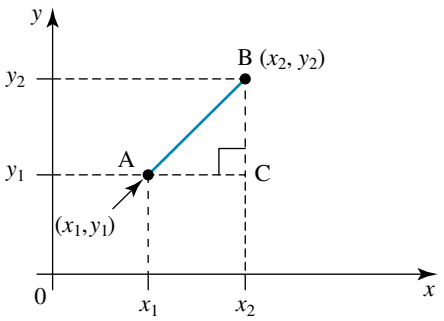
learnON RESOURCES — ONLINE ONLY

eLesson: The story of mathematics — Descartes (eles-1842)

Note: Your teacher may now set you a pre-test to determine how familiar you are with the content in this topic.

8.2 The distance between two points

[Stages 5.2 and 5.3]



- The distance between two points can be calculated using Pythagoras' theorem.
- Consider two points A (x_1, y_1) and B (x_2, y_2) on the Cartesian plane as shown at right.
- If point C is placed as shown, ABC is a right-angled triangle and AB is the hypotenuse.

$$AC = x_2 - x_1$$
$$BC = y_2 - y_1$$

By Pythagoras' theorem:

$$AB^2 = AC^2 + BC^2$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Hence, $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

The distance between two points A (x_1, y_1) and B (x_2, y_2) is:

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

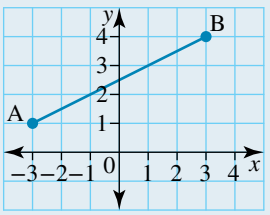
- This distance formula can be used to calculate the distance between any two points on the Cartesian plane.
- The distance formula has many geometric applications.

DISCUSSION

Explain why the formula $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ gives the same value as $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

WORKED EXAMPLE 1

Find the distance between the points A and B in the figure shown. Answer correct to two decimal places.



THINK	WRITE
1 From the graph, locate points A and B.	A (–3, 1) and B (3, 4)
2 Let A have coordinates (x_1, y_1).	Let $(x_1, y_1) = (-3, 1)$.
3 Let B have coordinates (x_2, y_2).	Let $(x_2, y_2) = (3, 4)$.

4 Find the length AB by applying the formula for calculating the distance between two points.

$$\begin{aligned}
AB &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{(3 - (-3))^2 + (4 - 1)^2} \\
&= \sqrt{(6)^2 + (3)^2} \\
&= \sqrt{36 + 9} \\
&= \sqrt{45} \\
&= 3\sqrt{5} \\
&= 6.71 \text{ (correct to 2 decimal places)}
\end{aligned}$$

Note: If the coordinates were named in the reverse order, the formula would still give the same answer. Check this for yourself using $(x_1, y_1) = (3, 4)$ and $(x_2, y_2) = (-3, 1)$.

WORKED EXAMPLE 2

Find the distance between the points P (−1, 5) and Q (3, −2).

THINK

1 Let P have coordinates (x_1, y_1).

2 Let Q have coordinates (x_2, y_2).

3 Find the length PQ by applying the formula for the distance between two points.

WRITE

Let $(x_1, y_1) = (-1, 5)$.

Let $(x_2, y_2) = (3, -2)$

$$\begin{aligned}
PQ &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{(3 - (-1))^2 + (-2 - 5)^2} \\
&= \sqrt{(4)^2 + (-7)^2} \\
&= \sqrt{16 + 49} \\
&= \sqrt{65} \\
&= 8.06 \text{ (correct to 2 decimal places)}
\end{aligned}$$

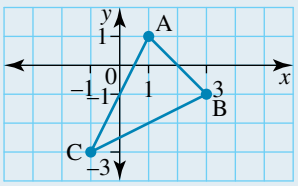
WORKED EXAMPLE 3

Prove that the points A (1, 1), B (3, −1) and C (−1, −3) are the vertices of an isosceles triangle.

THINK

1 Plot the points and draw the triangle.
Note: For triangle ABC to be isosceles, two sides must have the same magnitude.

WRITE/DRAW



2 AC and BC seem to be equal. Find the length AC.
A (1, 1) = (x_2, y_2)
C (−1, −3) = (x_1, y_1)

$$\begin{aligned}
AC &= \sqrt{(1 - (-1))^2 + (1 - (-3))^2} \\
&= \sqrt{(2)^2 + (4)^2} \\
&= \sqrt{20} \\
&= 2\sqrt{5}
\end{aligned}$$

3 Find the length BC.
B (3, −1) = (x_2, y_2)
C (−1, −3) = (x_1, y_1)

$$\begin{aligned}
BC &= \sqrt{(3 - (-1))^2 + (-1 - (-3))^2} \\
&= \sqrt{(4)^2 + (2)^2} \\
&= \sqrt{20} \\
&= 2\sqrt{5}
\end{aligned}$$

4 Find the length AB.
A $(1, 1) = (x_1, y_1)$
B $(3, -1) = (x_2, y_2)$

$$\begin{aligned} AB &= \sqrt{(3-(1))^2 + (-1-(1))^2} \\ &= \sqrt{(2)^2 + (-2)^2} \\ &= \sqrt{4+4} \\ &= \sqrt{8} \\ &= 2\sqrt{2} \end{aligned}$$

5 State your proof.

Since AC = BC ≠ AB, triangle ABC is an isosceles triangle.

learnON RESOURCES — ONLINE ONLY

Interactivity: Distance between two points (int-6051)
Digital doc: SpreadSHEET Distance between two points (doc-5206)

Exercise 8.2 The distance between two points

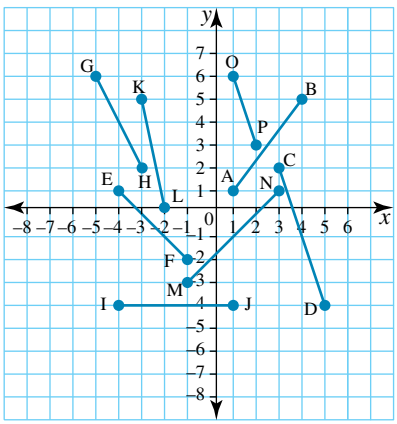
Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1, 2a–d, 5, 8, 9	Questions: 1, 2c–f, 5, 7, 9, 11	Questions: 1, 2e–h, 3–7, 9–13

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE1 Find the distance between each pair of points shown at right.



2. WE2 Find the distances between the following pairs of points.

a. (2, 5), (6, 8)
b. (−1, 2), (4, 14)
c. (−1, 3), (−7, −5)
d. (5, −1), (10, 4)
e. (4, −5), (1, 1)
f. (−3, 1), (5, 13)
g. (5, 0), (−8, 0)
h. (1, 7), (1, −6)
i. (a, b), $(2a, -b)$
j. $(-a, 2b)$, $(2a, -b)$

3. MC If the distance between the points $(3, b)$ and $(-5, 2)$ is 10 units, then the value of b is:

A. -8 **B.** -4 **C.** 4 **D.** 0 **E.** 2

4. MC A rhombus has vertices A$(1, 6)$, B$(6, 6)$, C$(-2, 2)$ and D(x, y). The coordinates of D are:

A. $(2, -3)$ **B.** $(2, 3)$ **C.** $(-2, 3)$ **D.** $(3, 2)$ **E.** $(3, -2)$

5. The vertices of a quadrilateral are A$(1, 4)$, B$(-1, 8)$, C$(1, 9)$ and D$(3, 5)$.

a. Find the lengths of the sides.

b. Find the lengths of the diagonals.

c. What type of quadrilateral is it?

Communicating, reasoning and problem solving

6. WE3 Prove that the points A$(0, -3)$, B$(-2, -1)$ and C$(4, 3)$ are the vertices of an isosceles triangle.

7. The points P$(2, -1)$, Q$(-4, -1)$ and R$(-1, 3\sqrt{3} - 1)$ are joined to form a triangle. Prove that triangle PQR is equilateral.

8. Prove that the triangle with vertices D$(5, 6)$, E$(9, 3)$ and F$(5, 3)$ is a right-angled triangle.

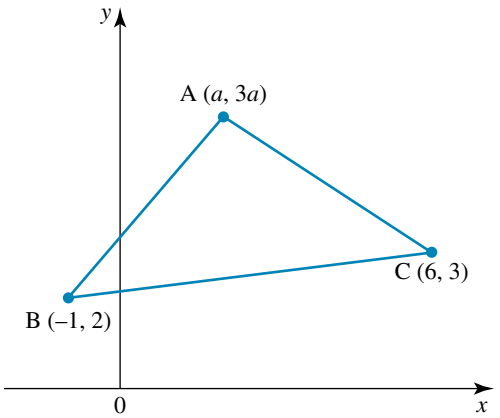
9. A rectangle has vertices A$(1, 5)$, B$(10.6, z)$, C$(7.6, -6.2)$ and D$(-2, 1)$. Find:

a. the length of CD **b.** the length of AD

c. the length of the diagonal AC **d.** the value of z.

10. Show that the triangle ABC with coordinates A(a, a), B$(m, -a)$ and C$(-a, m)$ is isosceles.

11. Triangle ABC is an isosceles triangle where AB = AC, B is the point $(-1, 2)$, C is the point $(6, 3)$ and A is the point $(a, 3a)$. Find the value of the integer constant a.

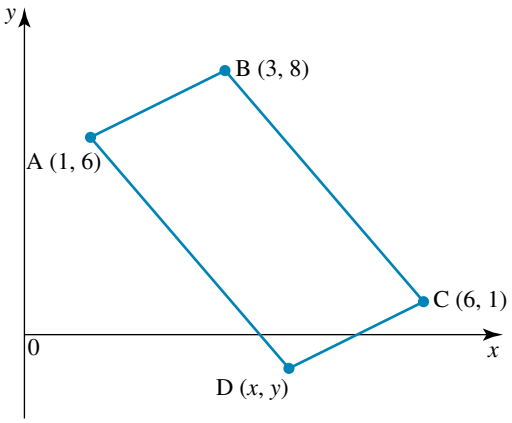


12. ABCD is a parallelogram.

a. Find the gradients of AB and BC.

b. Find the coordinates of the point D(x, y).

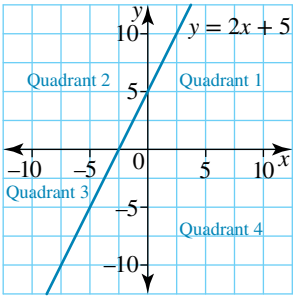
c. Show that the diagonals AC and BD bisect each other.



13. How could you use the distance formula to show that a series of points lay on the circumference of a circle with centre C?

8.3 Sketching linear graphs [Stages 5.2 and 5.3]

8.3.1 Linear graphs



- If a series of points (x, y) is plotted using the rule $y = mx + b$, then the points always lie in a straight line whose gradient equals m and whose y-intercept equals b.
- The rule $y = mx + b$ is called the equation of a straight line written in 'gradient–intercept' form.

8.3.2 Plotting linear graphs

- To plot a **linear graph**, complete a table of values to determine the points.

WORKED EXAMPLE 4

Plot the linear graph defined by the rule $y = 2x - 5$ for the x-values -3, -2, -1, 0, 1, 2 and 3.

THINK

1 Create a table of values using the given x-values.

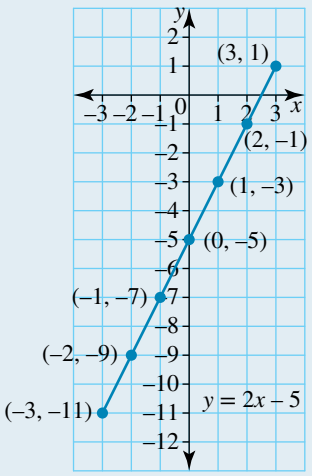
WRITE/DRAW

x	–3	–2	–1	0	1	2	3
y							

2 Find the corresponding y-values by substituting each x-value into the rule.

x	–3	–2	–1	0	1	2	3
y	–11	–9	–7	–5	–3	–1	1

3 Plot the points on a Cartesian plane and rule a straight line through them. Since the x-values have been specified, the line should only be drawn between the x-values of –3 and 3.



4 Label the graph.

8.3.3 Sketching straight lines

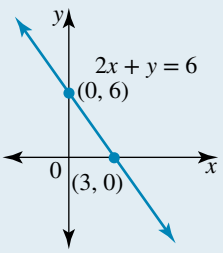
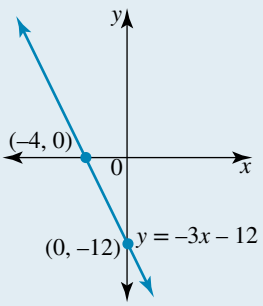
- A minimum of two points are necessary to plot a straight line.
- Two methods can be used to plot a straight line:
  - Method 1: The x- and y-intercept method
  - Method 2: The gradient–intercept method.

8.3.4 Method 1: Sketching a straight line using the x- and y-intercepts

- As the name implies, this method involves plotting the x- and y-intercepts, then joining them to sketch the straight line.
- The line cuts the y-axis where $x = 0$ and the x-axis where $y = 0$.

WORKED EXAMPLE 5

Sketch graphs of the following linear equations by finding the x- and y-intercepts.
a $2x + y = 6$ **b** $y = -3x - 12$

THINK	WRITE/DRAW
a 1 Write the equation.	**a** $2x + y = 6$
2 Find the x-intercept by substituting $y = 0$.	x-intercept: when $y = 0$, $2x + 0 = 6$ $2x = 6$ $x = 3$ The x-intercept is $(3, 0)$.
3 Find the y-intercept by substituting $x = 0$.	y-intercept: when $x = 0$, $2(0) + y = 6$ $y = 6$ The y-intercept is $(0, 6)$.
4 Plot both points and rule the line.	
5 Label the graph.	
b 1 Write the equation.	**b** $y = -3x - 12$
2 Find the x-intercept by substituting $y = 0$. **i** Add 12 to both sides of the equation. **ii** Divide both sides of the equation by -3.	x-intercept: when $y = 0$, $-3x - 12 = 0$ $-3x = 12$ $x = -4$ The x-intercept is $(-4, 0)$
3 Find the y-intercept. The equation is in the form $y = mx + b$, so compare this with our equation to find the y-intercept, b.	$c = -12$ y-intercept is $(0, -12)$
4 Plot both points and rule the line.	
5 Label the graph.	

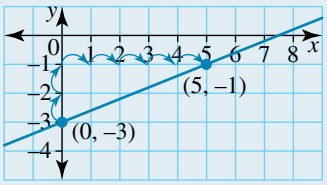
8.3.5 Method 2: Sketching a straight line using the gradient–intercept method

- This method is often used if the equation is in the form $y = mx + b$, where m represents the gradient (slope) of the straight line and b represents the y-intercept.
- The steps below outline how to use the gradient–intercept method to sketch a linear graph.

Step 1: Plot a point at the y-intercept.
Step 2: Write the gradient in the form $m = \frac{\text{rise}}{\text{run}}$. (To write a whole number as a fraction, place it over a denominator of 1.)
Step 3: Starting from the y-intercept, move up the number of units indicated by the rise (move down if the gradient is negative).
Step 4: Move to the right the number of units indicated by the run and plot the second point.
Step 5: Rule a straight line through the two points.

WORKED EXAMPLE 6

Sketch the graph of $y = \frac{2}{5}x - 3$ using the gradient–intercept method.

THINK	WRITE/DRAW
1 Write the equation of the line.	$y = \frac{2}{5}x - 3$
2 Identify the value of b (that is, the y-intercept) and plot this point.	$b = -3$, so the y-intercept is $(0, -3)$.
3 Write the gradient, m, as a fraction.	$m = \frac{2}{5}$
4 $m = \frac{\text{rise}}{\text{run}}$; note the rise and run.	So rise = 2 and run = 5.
5 Starting from the y-intercept at $(0, -3)$, move 2 units up and 5 units to the right to find the second point $(5, -1)$. We have still not found the x-intercept.	

DISCUSSION

Do you prefer to use one of the methods of sketching straight lines over the other? If so, why do you think that is?

8.3.6 Sketching linear graphs of the form $y = mx$

- Graphs of equations in the format $y = mx$ pass through the origin $(0, 0)$, since $b = 0$.
- A second point may be determined using the rule $y = mx$ by substituting a value for x to find y.

WORKED EXAMPLE 7

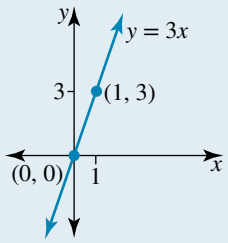
Sketch the graph of $y = 3x$.

THINK	WRITE/DRAW
1 Write the equation.	$y = 3x$
2 Find the x- and y-intercepts. *Note:* By recognising the form of this linear equation, $y = mx$, you can simply state that the graph passes through the origin, $(0, 0)$.	x-intercept: when $y = 0$, $0 = 3x$ $x = 0$ y-intercept: $(0, 0)$ The x- and y-intercepts are both at $(0, 0)$.

3 Find another point to plot by finding the y-value when $x = 1$.

When $x = 1$,
$y = 3 \times 1$
$= 3$
Another point on the line is $(1, 3)$.

4 Plot the two points $(0, 0)$ and $(1, 3)$ and rule a straight line through them.



5 Label the graph.

8.3.7 Sketching linear graphs of the form $y = b$ and $x = a$

- The line $y = b$ is parallel to the x-axis, having a gradient of zero and a y-intercept of b.
- The line $x = a$ is parallel to the y-axis and has an undefined (infinite) gradient.

WORKED EXAMPLE 8

Sketch graphs of the following linear equations.
a $y = -3$ **b** $x = 4$

THINK

a 1 Write the equation.

2 The y-intercept is –3. As x does not appear in the equation, the line is parallel to the x-axis, such that all points on the line have a y-coordinate equal to –3. That is, this line is the set of points $(x, -3)$ where x is an element of the set of real numbers.

3 Sketch a horizontal line through $(0, -3)$.

4 Label the graph.

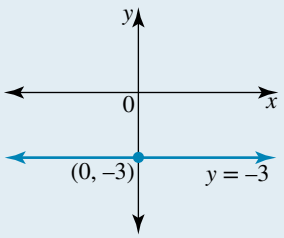
b 1 Write the equation.

2 The x-intercept is 4. As y does not appear in the equation, the line is parallel to the y-axis, such that all points on the line have an x-coordinate equal to 4. That is, this line is the set of points $(4, y)$ where y is an element of the set of real numbers.

WRITE/DRAW

a $y = -3$

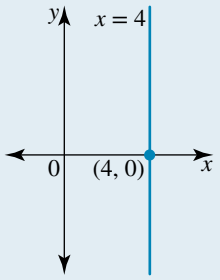
y-intercept = –3: $(0, -3)$



b $x = 4$

x-intercept = 4: $(4, 0)$

3 Sketch a vertical line through (4, 0).



4 Label the graph.

learnON RESOURCES — ONLINE ONLY

Interactivity: Plotting linear graphs (int-3834)
Interactivity: Equations of straight lines (int-6485)
Interactivity: The intercept method (int-3840)
Interactivity: The gradient–intercept method (int-3839)
eLesson: Sketching linear graphs (eles-1919)
eLesson: Sketching linear graphs using the gradient–intercept method (eles-1920)
Digital doc: SkillSHEET Describing the gradient of a line (doc-5197)
Digital doc: SkillSHEET Plotting a line using a table of values (doc-5198)
Digital doc: SkillSHEET Stating the *y*-intercept from a graph (doc-5199)
Digital doc: SkillSHEET Solving linear equations that arise when finding *x*- and *y*-intercepts (doc-5200)
Digital doc: SkillSHEET Using Pythagoras' theorem (doc-5201)
Digital doc: SkillSHEET Substitution into a linear rule (doc-5202)
Digital doc: SkillSHEET Transposing linear equations to standard form (doc-5203)
Digital doc: WorkSHEET Gradient (doc-13849)

Exercise 8.3 Sketching linear graphs

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1, 2, 3a–h, 4a–e, 5a–d, 6a–f, 7a–d, 8a–d, 9, 10, 12	Questions: 1, 2, 3f–m, 4a–e, 5a–d, 6a–f, 7c–f, 8a–f, 9–12	Questions: 1, 2, 3h–o, 4d–i, 5c–f, 6e–i, 7d–h, 8c–h, 9–14

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE4 Generate a table of values and then plot the linear graphs defined by the following rules for the given range of *x*-values.

	Rule	*x*-values
a.	$y = 10x + 25$	$-5, -4, -3, -2, -1, 0, 1$
b.	$y = 5x - 12$	$-1, 0, 1, 2, 3, 4$

Rule	*x*-values
c. $y = -0.5x + 10$	$-6, -4, -2, 0, 2, 4$
d. $y = 100x - 240$	$0, 1, 2, 3, 4, 5$
e. $y = -5x + 3$	$-3, -2, -1, 0, 1, 2$
f. $y = 7 - 4x$	$-3, -2, -1, 0, 1, 2$

2. Plot the linear graphs defined by the following rules for the given range of *x*-values.

a. $y = -3x + 2$

x	–6	–4	–2	0	2	4	6
y							

b. $y = -x + 3$

x	–3	–2	–1	0	1	2	3
y							

c. $y = -2x + 3$

x	–6	–4	–2	0	2	4	6
y							

3. WE5 Sketch graphs of the following linear equations by finding the *x*- and *y*-intercepts.

a. $5x - 3y = 10$ **b.** $5x + 3y = 10$ **c.** $-5x + 3y = 10$
d. $-5x - 3y = 10$ **e.** $2x - 8y = 20$ **f.** $4x + 4y = 40$
g. $5x + 6y = 120$ **h.** $-2x + 8y = -20$ **i.** $10x + 30y = -150$
j. $5x + 30y = -150$ **k.** $-9x + 4y = 36$ **l.** $6x - 4y = -24$
m. $y = 2x - 10$ **n.** $y = -5x + 20$ **o.** $y = -\frac{1}{2}x - 4$

4. WE6 Sketch graphs of the following linear equations using the gradient–intercept method.

a. $y = 4x + 1$ **b.** $y = 3x - 7$ **c.** $y = -2x + 3$
d. $y = -5x - 4$ **e.** $y = \frac{1}{2}x - 2$ **f.** $y = -\frac{2}{7}x + 3$
g. $y = 0.6x + 0.5$ **h.** $y = 8x$ **i.** $y = x - 7$

5. WE7 Sketch the graphs of the following linear equations.

a. $y = 2x$ **b.** $y = 5x$ **c.** $y = -3x$
d. $y = \frac{1}{2}x$ **e.** $y = \frac{2}{3}x$ **f.** $y = -\frac{5}{2}x$

6. WE8 Sketch the graphs of the following linear equations.

a. $y = 10$ **b.** $y = -10$ **c.** $x = 10$
d. $x = -10$ **e.** $y = 100$ **f.** $y = 0$
g. $x = 0$ **h.** $x = -100$ **i.** $y = -12$

7. Transpose each of the equations to standard form (that is, $y = mx + b$). State the *x*- and *y*-intercept for each.

a. $5(y + 2) = 4(x + 3)$ **b.** $5(y - 2) = 4(x - 3)$ **c.** $2(y + 3) = 3(x + 2)$
d. $10(y - 20) = 40(x - 2)$ **e.** $4(y + 2) = -4(x + 2)$ **f.** $2(y - 2) = -(x + 5)$
g. $-5(y + 1) = 4(x - 4)$ **h.** $5(y + 2.5) = 2(x - 3.5)$ **i.** $2.5(y - 2) = -6.5(x - 1)$

8. Find the *x*- and *y*-intercepts of the following lines.

a. $-y = 8 - 4x$ **b.** $6x - y + 3 = 0$ **c.** $2y - 10x = 50$

9. Explain why the gradient of a horizontal line is equal to zero and the gradient of a vertical line is undefined.

Communicating, reasoning and problem solving

10. Determine whether $\frac{x}{3} - \frac{y}{2} = \frac{7}{6}$ is the equation of a straight line by rearranging into an appropriate form and hence sketch the graph, showing all relevant features.

11. Your friend loves to download music. She earns $50 and spends some of it buying music online at $1.75 per song. She saves the remainder. Her saving is given by the function $f(x) = 50 - 1.75x$.
a. What does x represent?
b. Sketch the function.
c. How many songs can she buy and still save $25?

12. A straight line has a general equation defined by $y = mx + b$. This line intersects the lines defined by the rules $y = 7$ and $x = 3$. The lines $y = mx + b$ and $y = 7$ have the same y-intercept while $y = mx + b$ and $x = 3$ have the same x-intercept.
a. On the one set of axes, sketch all three graphs.
b. Determine the y-axis intercept for $y = mx + b$.
c. Determine the gradient for $y = mx + b$.
d. MC The equation of the line defined by $y = mx + b$ is:
A. $x + y - 3 = 0$ **B.** $7x + 3y - 21 = 0$ **C.** $3x + 7y - 21 = 0$
D. $x + y - 7 = 0$ **E.** $7x + 3y - 7 = 0$

13. Water is flowing from a tank at a constant rate. The equation relating the volume of water in the tank, V litres, to the time the water has been flowing from the tank, t minutes, is given by $V = 80 - 4t$, $t \geq 0$.
a. How much water is in the tank initially?
b. Why is it important that $t \geq 0$?
c. At what rate is the water flowing from the tank?
d. How long does it take for the tank to empty?
e. Sketch the graph of V versus t.

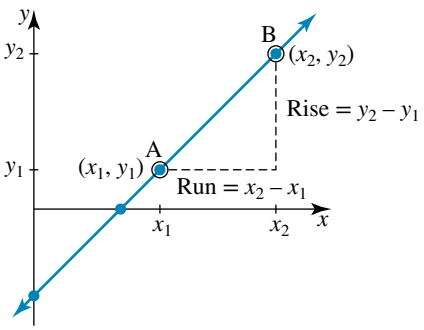
14. What types of straight lines have an x-and y-intercept of the same value?

8.4 Standard forms of the equation of a straight line [Stages 5.2 and 5.3]

8.4.1 Finding a linear equation given two points

- The gradient of a straight line can be calculated from the coordinates of two points (x_1, y_1) and (x_2, y_2) that lie on the line.

$$\text{Gradient} = m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

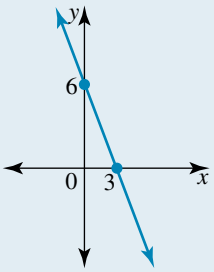


DISCUSSION
Explain why the formula $m = \frac{y_1 - y_2}{x_1 - x_2}$ gives the same value for the gradient as $m = \frac{y_2 - y_1}{x_2 - x_1}$.

- The equation of the straight line can then be found in the form $y = mx + b$, where b is the y-intercept.

WORKED EXAMPLE 9

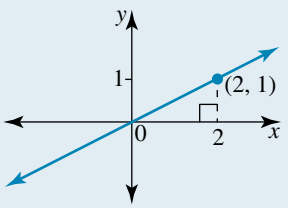
Find the equation of the straight line shown in the graph.



THINK	WRITE
1 There are two points given on the straight line: the x-intercept $(3, 0)$ and the y-intercept $(0, 6)$.	$(3, 0), (0, 6)$
2 Find the gradient of the line by applying the formula $m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$, where $(x_1, y_1) = (3, 0)$ and $(x_2, y_2) = (0, 6)$.	$m = \frac{\text{rise}}{\text{run}}$ $= \frac{y_2 - y_1}{x_2 - x_1}$ $= \frac{6 - 0}{0 - 3}$ $= \frac{6}{-3}$ $= -2$ The gradient $m = -2$.
3 The graph has a y-intercept of 6, so $b = 6$. Substitute $m = -2$ and $b = 6$ into $y = mx + b$ to find the equation.	$y = mx + b$ $y = -2x + 6$

WORKED EXAMPLE 10

Find the equation of the straight line shown in the graph.



THINK	WRITE
1 There are two points given on the straight line: the x- and y-intercept, $(0, 0)$, and another point, $(2, 1)$.	$(0, 0), (2, 1)$
2 Find the gradient of the line by applying the formula $m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$, where $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (2, 1)$.	$m = \frac{\text{rise}}{\text{run}}$ $= \frac{y_2 - y_1}{x_2 - x_1}$ $= \frac{1 - 0}{2 - 0}$ $= \frac{1}{2}$ The gradient $m = \frac{1}{2}$.

3 The y-intercept is 0, so $b = 0$. Substitute $m = \frac{1}{2}$ and $b = 0$ into $y = mx + b$ to determine the equation.	$y = mx + b$ $y = \frac{1}{2}x + 0$ $y = \frac{1}{2}x$

WORKED EXAMPLE 11

Find the equation of the straight line passing through (−2, 5) and (1, −1).

THINK	WRITE
1 Write the general equation of a straight line.	$y = mx + b$
2 Write the formula for calculating the gradient of a line between two points.	$m = \frac{y_2 - y_1}{x_2 - x_1}$
3 Let (x_1, y_1) and (x_2, y_2) be the two points (–2, 5) and (1, –1) respectively. Substitute the values of the pronumerals into the formula to calculate the gradient.	$m = \frac{-1 - 5}{1 - -2}$ $= \frac{-6}{3}$ $= -2$
4 Substitute the value of the gradient into the general rule.	$y = -2x + b$
5 Select either of the two points, say (1, –1), and substitute its coordinates into $y = -2x + b$.	Point $(1, -1)$: $-1 = -2 \times 1 + b$
6 Solve for b; that is, add 2 to both sides of the equation.	$-1 = -2 + b$ $1 = b$
7 State the equation by substituting the value of b into $y = -2x + b$.	The equation of the line is $y = -2x + 1$.

8.4.2 Finding the equation of a straight line using the gradient and one point

- If the gradient of a line is known, only one point is needed to determine the equation of the line.

WORKED EXAMPLE 12

Find the equation of the straight line with gradient of 2 and y-intercept of –5.

THINK	WRITE
1 Write the known information. The other point is the y-intercept, which makes the calculation of b straightforward.	Gradient = 2, y-intercept = –5
2 State the values of m and b.	$m = 2$, $b = -5$
3 Substitute these values into $y = mx + b$ to find the equation.	$y = mx + b$ $y = 2x - 5$

WORKED EXAMPLE 13

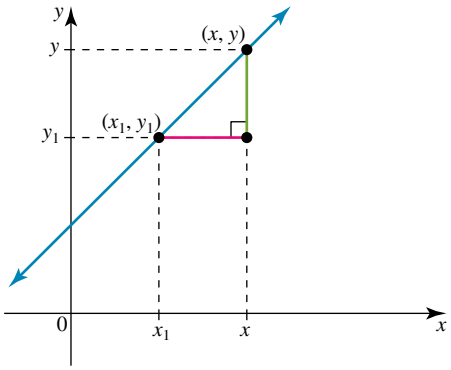
Find the equation of the straight line passing through the point (5, −1) with a gradient of 3.

THINK	WRITE
1 Write the known information.	Gradient = 3, point (5, –1)
2 State the values of m, x and y.	$m = 3$, $(x, y) = (5, -1)$

3 Substitute the values $m = 3, x = 5$ and $y = -1$ into $y = mx + b$ and solve to find b.	$y = mx + b$ $-1 = 3(5) + b$ $-1 = 15 + b$ $-16 = b$
4 Substitute $m = 3$ and $b = -16$ into $y = mx + b$ to determine the equation.	The equation of the line is $y = 3x - 16$.

8.4.3 A simple formula

- The diagram shows a line of gradient m passing through the point (x_1, y_1).



- If (x, y). is any other point on the line, then:

$$m = \frac{\text{rise}}{\text{run}}$$
$$m = \frac{y - y_1}{x - x_1}$$
$$m(x - x_1) = (y - y_1)$$
$$y - y_1 = m(x - x_1)$$

- The formula $y - y_1 = m(x - x_1)$ can be used to write down the equation of a line, given the gradient and the coordinates of one point.

WORKED EXAMPLE 14

Find the equation of the line with a gradient of −2 which passes through the point (3, −4). Write the equation in the form $ax + by + c = 0$, where a, b and c are integers.

THINK	WRITE
1 Use the formula $y - y_1 = m(x - x_1)$. Write the values of (x_1, y_1), and m.	$m = -2, x_1 = 3, y_1 = -4$ $y - y_1 = m(x - x_1)$
2 Substitute for x_1, y_1, and m into the equation.	$y - (-4) = -2(x - 3)$ $y + 4 = -2x + 6$
3 Transpose the equation into the form $ax + by + c = 0$.	$y + 4 + 2x - 6 = 0$ $2x + y - 2 = 0$

learnon RESOURCES — ONLINE ONLY

Interactivity: Linear graphs (int-6484)
eLesson: Finding a linear equation given two points (eles-2313)
Digital doc: SkillSHEET Measuring the rise and the run (doc-5196)
Digital doc: SkillSHEET Finding the gradient given two points (doc-5204)

Exercise 8.4 Standard forms of the equation of a straight line

Individual pathways

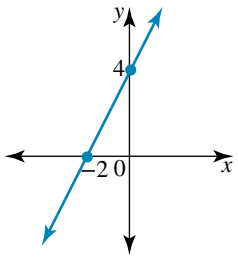
PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–d, 2, 3, 4, 5a–d, 7	Questions: 1a–f, 2, 3, 4, 5c–g, 7, 9, 10, 12	Questions: 1d–h, 2, 3, 4, 5e–j, 6–9, 11, 13

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE9 Determine the equation for each of the straight lines shown.

a.



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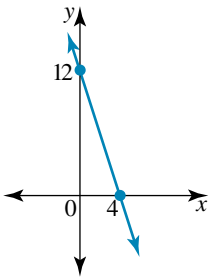
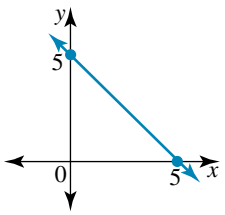
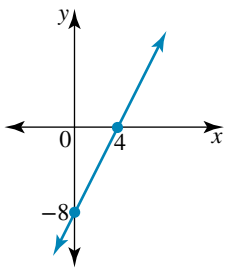


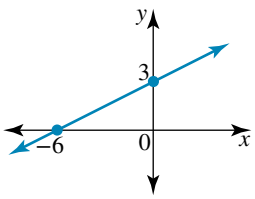
c.



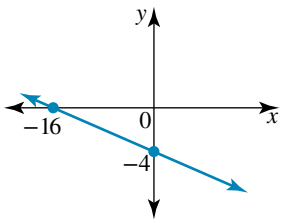
d.



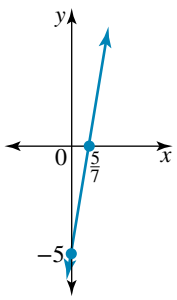
e.



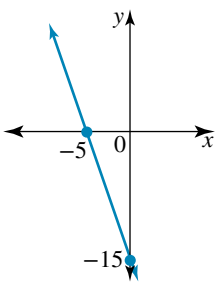
f.



g.

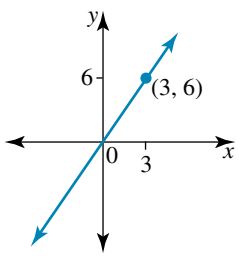


h.

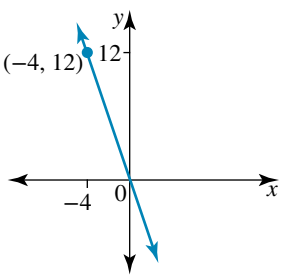


2. WE10 Determine the equation of each of the straight lines shown.

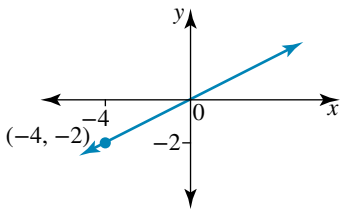
a.



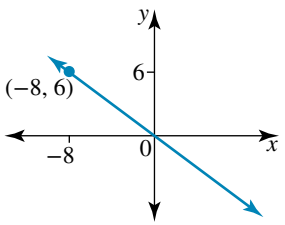
b.



c.



d.



3. WE11 Find the equation of the straight line that passes through each pair of points.

a. (1, 4) and (3, 6)
b. (0, −1) and (3, 5)
c. (−1, 4) and (3, 2)
d. (3, 2) and (−1, 0)
e. (−4, 6) and (2, −6)
f. (−3, −5) and (−1, −7)

4. WE12 Find the linear equation given the information in each case below.

a. Gradient = 3, y-intercept = 3
b. Gradient = −3, y-intercept = 4
c. Gradient = −4, y-intercept = 2
d. Gradient = 4, y-intercept = 2
e. Gradient = −1, y-intercept = –4
f. Gradient = 0.5, y-intercept = –4
g. Gradient = 5, y-intercept = 2.5
h. Gradient = −6, y-intercept = 3
i. Gradient = −2.5, y-intercept = 1.5
j. Gradient = 3.5, y-intercept = 6.5

5. WE13, 14 For each of the following, find the equation of the straight line with the given gradient and passing through the given point. Express your answer in the form $y = mx + b$.

a. Gradient = 5, point = (5, 6)
b. Gradient = –5, point = (5, 6)
c. Gradient = –4, point = (–2, 7)
d. Gradient = 4, point = (8, –2)
e. Gradient = 3, point = (10, –5)
f. Gradient = –3, point = (3, –3)

6. For each of the following, find the equation of the straight line with the given gradient and passing through the given point. Express your answer in the form $ax + by + c = 0$, where a, b and c are integers.

a. Gradient = –2, point = (20, –10)
b. Gradient = 2, point = (2, –0.5)
c. Gradient = 0.5, point = (6, –16)
d. Gradient = –0.5, point = (5, 3)

7. a. Refer to the information given at right. If t represents the time in hours and C represents cost ($), construct a table of values for 0–3 hours for the cost of playing ten-pin bowling at the new alley.

b. Use your table of values to plot a graph of time versus cost. (*Hint:* Ensure your time axis (horizontal axis) extends to 6 hours and your cost axis (vertical axis) extends to $40.)

c. i. What is the y-intercept?

ii. What does the y-intercept represent in terms of the cost?

d. Calculate the gradient and explain what this means in the context of the question.

e. Write a linear equation to describe the relationship between cost and time.

f. Use your linear equation from part **f** to calculate the cost of a 5-hour tournament.

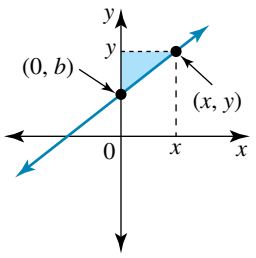
g. Use your graph to check your answer to part **g**.

Save $$$ with Supa-Bowl!!!
NEW Ten-Pin Bowling Alley
Shoe rental just $2 (fixed fee)
Rent a lane for ONLY $6/hour!

Communicating, reasoning and problem solving

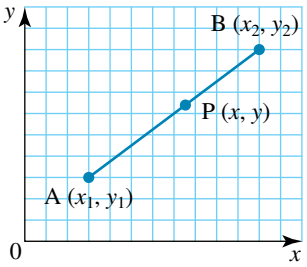
8. When using the gradient to draw a line, does it matter if you rise before you run or run before you rise? Explain your answer.

9. a. Using the graph shown, write a general formula for the gradient m in terms of x, y and b.

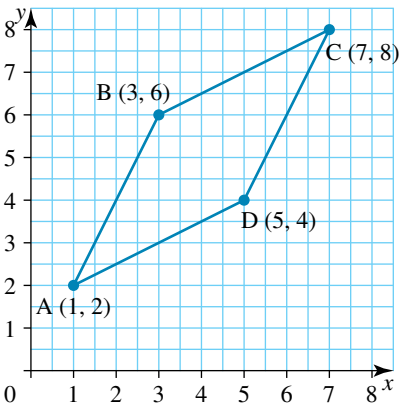


b. Transpose your formula to make y the subject. What do you notice?

10. The points $A(x_1, y_1)$, $B(x_2, y_2)$ and $P(x, y)$ are all on the same straight line. P is a general point that lies anywhere on the line. Show that an equation relating these three points is given by $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$.

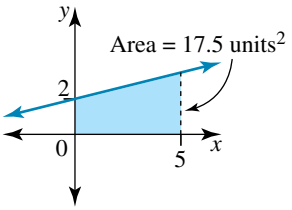


11. Show that the quadrilateral ABCD is a parallelogram.



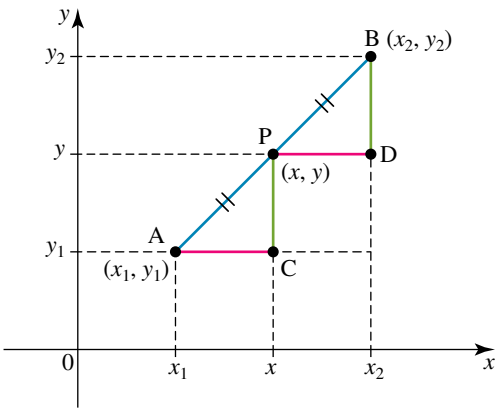
12. What problems might you encounter when calculating the equation of a line whose graph is actually parallel to one of the axes?

13. The graph of the straight line crosses the y-axis at (0, 2). The shaded section represents an area of 17.5 square units. Use this information to determine the equation of the line.



8.5 The midpoint of a line segment [Stages 5.2 and 5.3]

- The **midpoint** of a **line segment** is the halfway point.
- The x- and y-coordinates of the midpoint are halfway between those of the coordinates of the end points.
- The diagram shows the line interval AB joining points $A(x_1, y_1)$ and $B(x_2, y_2)$.
  The midpoint of AB is P, so AP = PB.
  Points $C(x, y_1)$ and $D(x_2, y)$ are used to make the two right-angled triangles ΔAPC and ΔPBD.
  The two triangles are congruent:
  AP = PB (given)
  ∠APC = ∠PBD (corresponding angles)



$\angle CAP = \angle DPB$ (corresponding angles)
So $\Delta APC \equiv \Delta PBD$ (ASA).
This means that AC = PD,
so $x - x_1 = x_2 - x$ (solve for x)

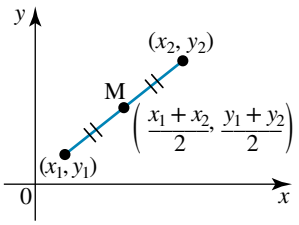
$$2x = x_1 + x_2$$
$$x = \frac{x_1 + x_2}{2}$$

In other words, x is simply the average of x_1 and x_2.

Similarly, $y = \frac{y_1 + y_2}{2}$.

- In general, the coordinates of the midpoint of a line segment joining the points (x_1, y_1) and (x_2, y_2) can be found by averaging the x- and y-coordinates of the end points, respectively.
- The coordinates of the midpoint of the line segment joining (x_1, y_1) and (x_2, y_2) are:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$



WORKED EXAMPLE 15

Find the coordinates of the midpoint of the line segment joining (−2, 5) and (7, 1).

THINK	WRITE
1 Label the given points (x_1, y_1) and (x_2, y_2).	Let $(x_1, y_1) = (-2, 5)$ and $(x_2, y_2) = (7, 1)$.
2 Find the x-coordinate of the midpoint.	$x = \frac{x_1 + x_2}{2}$ $= \frac{-2 + 7}{2}$ $= \frac{5}{2}$ $= 2\frac{1}{2}$
3 Find the y-coordinate of the midpoint.	$y = \frac{y_1 + y_2}{2}$ $= \frac{5 + 1}{2}$ $= \frac{6}{2}$ $= 3$
4 Give the coordinates of the midpoint.	The coordinates of the midpoint are $\left(2\frac{1}{2}, 3\right)$.

WORKED EXAMPLE 16

The coordinates of the midpoint, M, of the line segment AB are (7, 2). If the coordinates of A are (1, −4), find the coordinates of B.

THINK	WRITE/DRAW
1 Let the start of the line segment be (x_1, y_1) and let the midpoint be (x, y).	Let $(x_1, y_1) = (1, -4)$ and $(x, y) = (7, 2)$

2 The average of the x-coordinates is 7. Find the x-coordinate of the end point.

$$x = \frac{x_1 + x_2}{2}$$
$$7 = \frac{1 + x_2}{2}$$
$$14 = 1 + x_2$$
$$x_2 = 13$$

3 The average of the y-coordinates is 2. Find the y-coordinate of the end point.

$$y = \frac{y_1 + y_2}{2}$$
$$2 = \frac{-4 + y_2}{2}$$
$$4 = -4 + y_2$$
$$y_2 = 8$$

learnON RESOURCES — ONLINE ONLY

Interactivity: Midpoints (int-6052)

Digital doc: SpreadSHEET Midpoint of a segment (doc-5207)

Digital doc: WorkSHEET Midpoint of a line segment (doc-13850)

Exercise 8.5 The midpoint of a line segment

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–d, 2, 3a, 4, 9, 11	Questions: 1a–d, 2–6, 9, 11	Questions: 1a–f, 2–13

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE15 Use the formula method to find the coordinates of the midpoint of the line segments joining the following pairs of points.

a. $(-5, 1), (-1, -8)$ **b.** $(4, 2), (11, -2)$

c. $(0, 4), (-2, -2)$ **d.** $(3, 4), (-3, -1)$

e. $(a, 2b), (3a, -b)$ **f.** $(a + 3b, b), (a - b, a - b)$

2. WE16 The coordinates of the midpoint, M, of the line segment AB are $(2, -3)$. If the coordinates of A are $(7, 4)$, find the coordinates of B.

3. A square has vertices A $(0, 0)$, B $(2, 4)$, C $(6, 2)$ and D $(4, -2)$. Find:

a. the coordinates of the centre **b.** the length of a side

c. the length of a diagonal.

4. MC The midpoint of the line segment joining the points $(-2, 1)$ and $(8, -3)$ is:

A. $(6, -2)$ **B.** $(5, 2)$ **C.** $(6, 2)$ **D.** $(3, -1)$ **E.** $(5, -2)$

5. MC If the midpoint of AB is $(-1, 5)$ and the coordinates of B are $(3, 8)$, then A has coordinates:

A. $(1, 6.5)$ **B.** $(2, 13)$ **C.** $(-5, 2)$ **D.** $(4, 3)$ **E.** $(7, 11)$

6. a. The vertices of a triangle are A (2, 5), B (1, –3) and C (–4, 3). Find:
i. the coordinates of P, the midpoint of AC
ii. the coordinates of Q, the midpoint of AB
iii. the length of PQ.
b. Show that BC = 2PQ.

7. a. A quadrilateral has vertices A(6, 2), B (4, –3), C (–4, –3) and D (–2, 2). Find:
i. the midpoint of the diagonal AC
ii. the midpoint of the diagonal BD.
b. What can you infer about the quadrilateral?

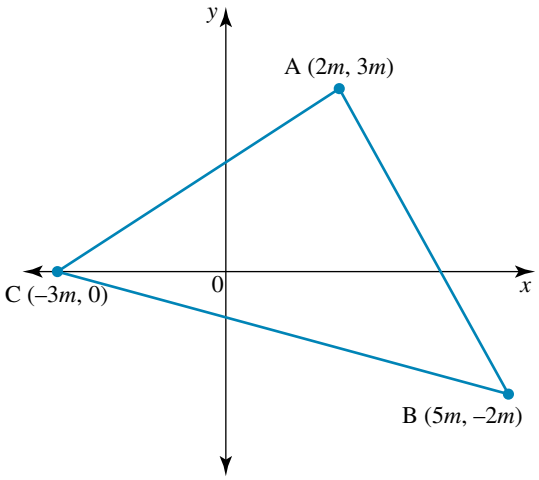
8. a. The points A (–5, 3.5), B (1, 0.5) and C (–6, –6) are the vertices of a triangle. Find:
i. the midpoint, P, of AB
ii. the length of PC
iii. the length of AC
iv. the length of BC.
b. Describe the triangle. What does PC represent?

Communicating, reasoning and problem solving

9. Find the equation of the straight line that passes through the midpoint of A (–2, 5) and B (–2, 3) and has a gradient of –3.

10. Find the equation of the straight line that passes through the midpoint of A (–1, –3) and B (3, –5) and has a gradient of $\frac{2}{3}$.

11. The points A $(2m, 3m)$, B $(5m, -2m)$ and C $(-3m, 0)$ are the vertices of a triangle. Show that this is a right-angled triangle.



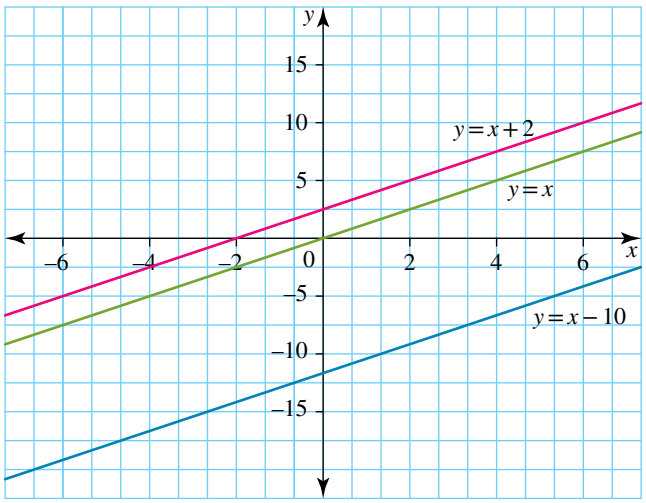
12. Write down the coordinates of the midpoint of the line joining the points $(3k - 1, 4 - 5k)$ and $(5k - 1, 3 - 5k)$. Show that this point lies on the line with equation $5x + 4y = 9$.

13. If the midpoint of a line segment is the origin, what are the possible values of the x- and y-coordinates of the end points?

8.6 Parallel and perpendicular lines [Stages 5.2 and 5.3]

8.6.1 Parallel lines

- Lines that have the same gradient are **parallel** lines. The three lines shown on the graph have a gradient of 1 and are parallel to each other.



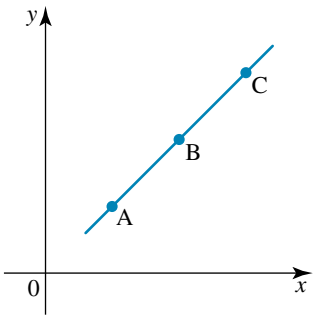
WORKED EXAMPLE 17

Show that AB is parallel to CD given that A has coordinates (−1, −5), B has coordinates (5, 7), C has coordinates (−3, 1) and D has coordinates (4, 15).

THINK	WRITE
1 Find the gradient of AB by applying the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$.	Let A $(-1, -25) = (x_1, y_1)$ and B $(5, 7) = (x_2, y_2)$. $m = \frac{y_2 - y_1}{x_2 - x_1}$ Since $m_{AB} = \frac{7 - (-5)}{5 - (-1)}$ $= \frac{12}{6}$ $= 2$
2 Find the gradient of CD.	Let C $(-3, 1) = (x_1, y_1)$ and D $(4, 15) = (x_2, y_2)$. $m_{CD} = \frac{15 - 1}{4 - (-3)}$ $= \frac{14}{7}$ $= 2$
3 Draw a conclusion. (*Note*: ∥ means 'is parallel to'.)	Since $m_{AB} = m_{CD} = 2$, then AB ∥ CD.

8.6.2 Collinear points

- **Collinear points** are points that all lie on the same straight line.
- If A, B and C are collinear, then $m_{AB} = m_{BC}$.



WORKED EXAMPLE 18

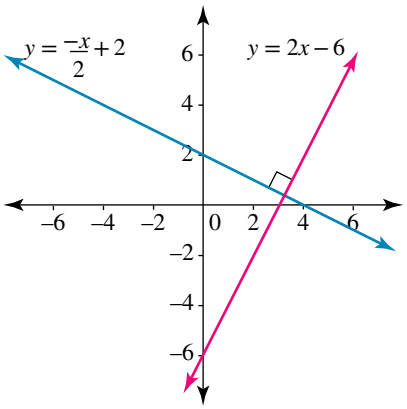
Show that the points A (2, 0), B (4, 1) and C (10, 4) are collinear.

THINK	WRITE
1 Find the gradient of AB.	Let A $(2, 0) = (x_1, y_1)$ and B $(4, 1) = (x_2, y_2)$. Since $m = \frac{y_2 - y_1}{x_2 - x_1}$, $m_{AB} = \frac{1 - 0}{4 - 2}$ $= \frac{1}{2}$
2 Find the gradient of BC.	Let B $(4, 1) = (x_1, y_1)$ and C $(10, 4) = (x_2, y_2)$. $m_{BC} = \frac{4 - 1}{10 - 4}$ $= \frac{3}{6}$ $= \frac{1}{2}$

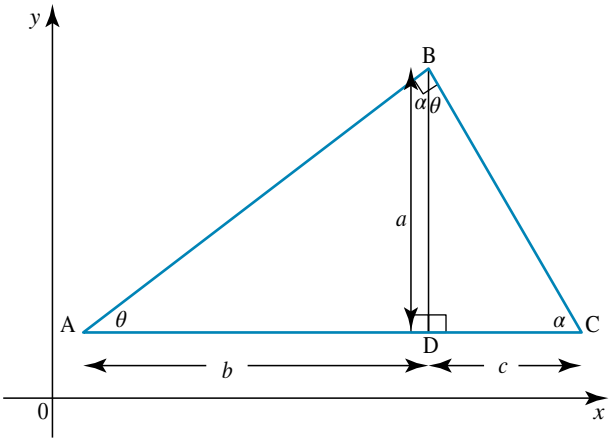
3 Show that A, B and C are collinear.

Since $m_{AB} = m_{BC} = \frac{1}{2}$ and B is common to both line segments, A, B and C are collinear.

8.6.3 Perpendicular lines



- There is a special relationship between the gradients of two **perpendicular** lines. The graph at right shows two perpendicular lines. What do you notice about their gradients?
- Consider the diagram shown below, in which the line segment AB is perpendicular to the line segment BC, AC is parallel to the x-axis, and BD is the perpendicular height of the resulting triangle ABC.



In ΔABD, let $m_{AB} = m_1$

$$= \frac{a}{b}$$
$$= \tan(\theta)$$

In ΔBCD, let $m_{BC} = m_2$

$$= -\frac{a}{c}$$
$$= -\tan(\alpha)$$

In ΔABD, $\tan(\alpha) = \frac{b}{a}$.

So $m_2 = -\frac{b}{a}$

$$= \frac{-1}{m_1}$$

Hence, $m_2 = -\frac{1}{m_1}$

or $m_1 m_2 = -1$.

- Hence, if two lines are perpendicular to each other, then the product of their gradients is –1. Two lines are perpendicular if and only if:

$$m_1 m_2 = -1$$

- If two lines are perpendicular, then their gradients are $\frac{a}{b}$ and $-\frac{b}{a}$ respectively.

DISCUSSION

How can you determine if two lines are parallel or perpendicular from their equations?

WORKED EXAMPLE 19

Show that the lines $y = -5x + 2$ and $5y - x + 15 = 0$ are perpendicular to one another.

THINK	WRITE
1 Find the gradient of the first line.	$y = -5x + 2$ Hence, $m_1 = -5$.
2 Find the gradient of the second line.	$5y - x + 15 = 0$ Rewrite in the form $y = mx + b$: $5y = x - 15$ $y = \frac{x}{5} - 3$ Hence, $m_2 = \frac{1}{5}$. $m_1 m_2 = -5 \times \frac{1}{5}$ $= -1$
3 Test for perpendicularity. (The two lines are perpendicular if the product of their gradients is –1.)	Hence, the two lines are perpendicular.

8.6.4 Determining the equation of a line parallel or perpendicular to another line

- The gradient properties of parallel and perpendicular lines can be used to solve many problems.

WORKED EXAMPLE 20

Find the equation of the line that passes through the point $(3, -1)$ and is parallel to the straight line with equation $y = 2x + 1$.

THINK	WRITE
1 Write the general equation.	$y = mx + b$
2 Find the gradient of the given line. The two lines are parallel, so they have the same gradient.	$y = 2x + 1$ has a gradient of 2. Hence, $m = 2$.
3 Substitute for m in the general equation.	$y = 2x + b$
4 Substitute the given point to find b.	$(x, y) = (3, -1)$ $-1 = 2(3) + b$ $-1 = 6 + b$ $b = -7$
5 Substitute for b in the general equation.	$y = 2x - 7$ or $2x - y - 7 = 0$

WORKED EXAMPLE 21

Find the equation of the line that passes through the point (0, 3) and is perpendicular to a straight line with a gradient of 5.

THINK

1 For perpendicular lines, $m_1 \times m_2 = -1$.
Find the gradient of the perpendicular line.

2 Use the equation $y - y_1 = m\,(x - x_1)$ where $m = -\frac{1}{5}$ and $(x_1, y_1) = (0,\ 3)$.

WRITE

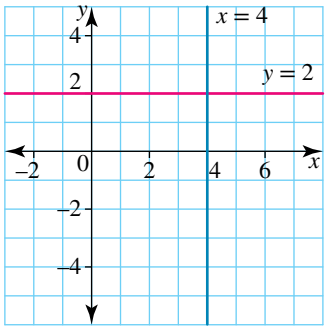
Given: $m_1 = 5$, $m_2 = -\frac{1}{5}$

Since $y - y_1 = m(x - x_1)$
and $(x_1, y_1) = (0,\ 3)$,
then $y - 3 = -\frac{1}{5}(x - 0)$

$$y - 3 = \frac{-x}{5}$$
$$5(y - 3) = -x$$
$$5y - 15 = -x$$
$$x + 5y - 15 = 0$$
$$\text{or } y = -\frac{x}{5} + 3$$

8.6.5 Horizontal and vertical lines

- Horizontal lines are parallel to the x-axis, have a gradient of zero, are expressed in the form $y = b$ and have no x-intercept.
- Vertical lines are parallel to the y-axis, have an undefined (infinite) gradient, are expressed in the form $x = a$ and have no y-intercept.



WORKED EXAMPLE 22

Find the equation of:
a the vertical line that passes through the point (2, −3)
b the horizontal line that passes through the point (−2, 6).

THINK

a The equation of a vertical line is $x = a$.
The x-coordinate of the given point is 2.

b The equation of a horizontal line is $y = b$.
The y-coordinate of the given point is 6.

WRITE

a $x = 2$

b $y = 6$

WORKED EXAMPLE 23

Find the equation of the perpendicular bisector of the line joining the points (0, −4) and (6, 5). (A bisector is a line that crosses another line at right angles and cuts it into two equal lengths.)

THINK

1 Find the gradient of the line joining the given points by applying the formula. $m = \frac{y_2 - y_1}{x_2 - x_1}$.

WRITE

Let $(x_1, y_1) = (0, -4)$.
Let $(x_2, y_2) = (6, 5)$.

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m_1 = \frac{5 - (-4)}{6 - 0}$$
$$= \frac{9}{6}$$
$$= \frac{3}{2}$$

2 Find the gradient of the perpendicular line.
$m_1 \times m_2 = -1$

$$m_2 = \frac{-1}{m_1}$$
$$m_2 = -\frac{2}{3}$$

3 Find the midpoint of the line joining the given points.
$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ where $(x_1, y_1) = (0, -4)$ and $(x_2, y_2) = (6, 5)$.

$$x = \frac{x_1 + x_2}{2} \qquad y = \frac{y_1 + y_2}{2}$$
$$= \frac{0 + 6}{2} \qquad = \frac{-4 + 5}{2}$$
$$= 3 \qquad = \frac{1}{2}$$

Hence, the coordinates of the midpoint are $\left(3, \frac{1}{2}\right)$.

4 Find the equations of the line with gradient $-\frac{2}{3}$ that passes through $\left(3, \frac{1}{2}\right)$.

$$y - y_1 = m(x - x_1),$$
$$y - \frac{1}{2} = -\frac{2}{3}(x - 3).$$

5 Simplify by removing the fractions.
Multiply both sides by 3.

$$3\left(y - \frac{1}{2}\right) = -2(x - 3)$$
$$3y - \frac{3}{2} = -2x + 6$$

Multiply both sides by 2.

$$6y - 3 = -4x + 12$$
$$4x + 6y - 15 = 0$$

learnON RESOURCES — ONLINE ONLY

Interactivity: Parallel lines (int-3841)
Interactivity: Perpendicular lines (int-6124)
Interactivity: Vertical and horizontal lines (int-6049)
Digital doc: SpreadSHEET Perpendicular checker (doc-5209)
Digital doc: SpreadSHEET Equation of a straight line (doc-5210)

Exercise 8.6 Parallel and perpendicular lines

Individual pathways

PRACTISE
Questions:
1a–d, 2, 5, 6a–c, 7, 8, 9a–c, 12, 13, 16a–b, 18, 20a, 21, 23, 26a, 27

CONSOLIDATE
Questions:
1a–d, 2–5, 6c–d, 7, 8, 9a–c, 12, 13, 15, 16a–b, 17a, 18, 20a, 21–23, 26–28, 30, 32

MASTER
Questions:
1c–f, 2, 3, 4, 5, 6e–f, 7–19, 20b, 21, 22, 24–31, 33–39

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Understanding and fluency

1. WE17 Find whether AB is parallel to CD given the following sets of points.
- **a.** A (4, 13), B (2, 9), C (0, –10), D (15, 0)
- **b.** A (2, 4), B (8, 1), C (–6, –2), D (2, –6)
- **c.** A (–3, –10), B (1, 2), C (1, 10), D (8, 16)
- **d.** A (1, –1), B (4, 11), C (2, 10), D (–1, –5)
- **e.** A (1, 0), B (2, 5), C (3, 15), D (7, 35)
- **f.** A (1, –6), B (–5, 0), C (0, 0), D (5, –4)

2. Which pairs of the following straight lines are parallel?

a. $2x + y + 1 = 0$ **b.** $y = 3x - 1$ **c.** $2y - x = 3$ **d.** $y = 4x + 3$

e. $y = \frac{x}{2} - 1$ **f.** $6x - 2y = 0$ **g.** $3y = x + 4$ **h.** $2y = 5 - x$

3. WE18 Show that the points A (0, –2), B (5, 1) and C (–5, –5) are collinear.

4. Show that the line that passes through the points (–4, 9) and (0, 3) also passes through the point (6, –6).

5. WE19 Show that the lines $y = 6x - 3$ and $x + 6y - 6 = 0$ are perpendicular to one another.

6. Determine whether AB is perpendicular to CD, given the following sets of points.
- **a.** A (1, 6), B (3, 8), C (4, –6), D (–3, 1)
- **b.** A (2, 12), B (–1, –9), C (0, 2), D (7, 1)
- **c.** A(1, 3), B (4, 18), C (–5, 4), D (5, 0)
- **d.** A (1, –5), B (0, 0), C (5, 11), D (–10, 8)
- **e.** A (–4, 9), B (2, –6), C (–5, 8), D (10, 14)
- **f.** A (4, 4), B (–8, 5), C (–6, 2), D (3, 11)

7. WE20 Find the equation of the line that passes through the point (4, −1) and is parallel to the line with equation $y = 2x - 5$.

8. WE21 Find the equation of the line that passes through the point (–2, 7) and is perpendicular to a line with a gradient of $\frac{2}{3}$.

9. Find the equations of the following lines.
- **a.** Gradient 3 and passing through the point (1, 5)
- **b.** Gradient –4 and passing through the point (2, 1)
- **c.** Passing through the points (2, –1) and (4, 2)
- **d.** Passing through the points (1, –3) and (6, –5)
- **e.** Passing through the point (5, –2) and parallel to $x + 5y + 15 = 0$
- **f.** Passing through the point (1, 6) and parallel to $x - 3y - 2 = 0$
- **g.** Passing through the point (–1, –5) and perpendicular to $3x + y + 2 = 0$

10. Find the equation of the line that passes through the point (–2, 1) and is:
- **a.** parallel to the line with equation $2x - y - 3 = 0$
- **b.** perpendicular to the line with equation $2x - y - 3 = 0$.

11. Find the equation of the line that contains the point (1, 1) and is:
- **a.** parallel to the line with equation $3x - 5y = 0$
- **b.** perpendicular to the line with equation $3x - 5y = 0$.

12. WE22 Find the equation of:
- **a.** the vertical line that passes through the point (1, –8)
- **b.** the horizontal line that passes through the point (–5, –7).

13. MC **a.** The vertical line passing through the point (3, –4) is given by:

A. $y = -4$ **B.** $x = 3$ **C.** $y = 3x - 4$ **D.** $y = -4x + 3$ **E.** $x = -4$

b. Which of the following points does the horizontal line given by the equation $y = -5$ pass through?

A. (–5, 4) **B.** (4, 5) **C.** (3, –5) **D.** (5, –4) **E.** (5, 5)

c. Which of the following statements is true?
- **A.** Vertical lines have a gradient of zero.
- **B.** The y-coordinates of all points on a vertical line are the same.
- **C.** Horizontal lines have an undefined gradient.
- **D.** The x-coordinates of all points on a vertical line are the same.
- **E.** A horizontal line has the general equation $x = a$.

d. Which of the following statements is false?
- **A.** Horizontal lines have a gradient of zero.
- **B.** The line joining the points (1, –1) and (–7, –1) is vertical.
- **C.** Vertical lines have an undefined gradient.
- **D.** The line joining the points (1, 1) and (–7, 1) is horizontal.
- **E.** A horizontal line has the general equation $y = b$.

14. The triangle ABC has vertices A (9, –2), B (3, 6) and C (1, 4).
a. Find the midpoint, M, of BC.
b. Find the gradient of BC.
c. Show that AM is the perpendicular bisector of BC.
d. Describe triangle ABC.

15. WE23 Find the equation of the perpendicular bisector of the line joining the points (1, 2) and (–5, –4).

16. Find the equation of the perpendicular bisector of the line joining the points (–2, 9) and (4, 0).

17. ABCD is a parallelogram. The coordinates of A, B and C are (4, 1), (1, –2) and (–2, 1) respectively. Find:
a. the equation of AD **b.** the equation of DC **c.** the coordinates of D.

18. For each of the following sets of points, show that ABCD is a parallelogram.
a. A (2, 0), B (4, –3), C (2, –4), D (0, –1)
b. A (2, 2), B (0, –2), C (–2, –3), D (0, 1)
c. A (2.5, 3.5), B (10, –4), C (2.5, –2.5), D (–5, 5)

19. For each of the following sets of points, show that ABCD is a trapezium.
a. A (0, 6), B (2, 2), C (0, –4), D (–5, –9)
b. A (26, 32), B (18, 16), C (1, –1), D (–3, 3)
c. A (2, 7), B (1, –1), C (–0.6, –2.6), D (–2, 3)

20. MC The line that passes through the points (0, –6) and (7, 8) also passes through:
A. (4, 3) **B.** (5, 4) **C.** (–2, 10)
D. (1, –8) **E.** (1, 4)

21. MC The point (–1, 5) lies on a line parallel to $4x + y + 5 = 0$. Another point on the same line as (–1, 5) is:
A. (2, 9) **B.** (4, 2) **C.** (4, 0)
D. (–2, 3) **E.** (3, –11)

22. Find the equation of the straight line given the following conditions.
a. Passes through the point (–1, 3) and parallel to $y = -2x + 5$
b. Passes through the point (4, –3) and parallel to $3y + 2x = -3$

23. Determine which pairs of the following lines are perpendicular.
a. $x + 3y - 5 = 0$ **b.** $y = 4x - 7$
c. $y = x$ **d.** $2y = x + 1$
e. $y = 3x + 2$ **f.** $x + 4y - 9 = 0$
g. $2x + y = 6$ **h.** $x + y = 0$

24. Find the equation of the straight line that cuts the x-axis at 3 and is perpendicular to the line with equation $3y - 6x = 12$.

25. Calculate the value of m for which lines with the following pairs of equations are perpendicular to each other.

a. $2y - 5x = 7$ and $4y + 12 = mx$

b. $5x - 6y = -27$ and $15 + mx = -3y$

26. MC The gradient of the line perpendicular to the line with equation $3x - 6y = 2$ is:

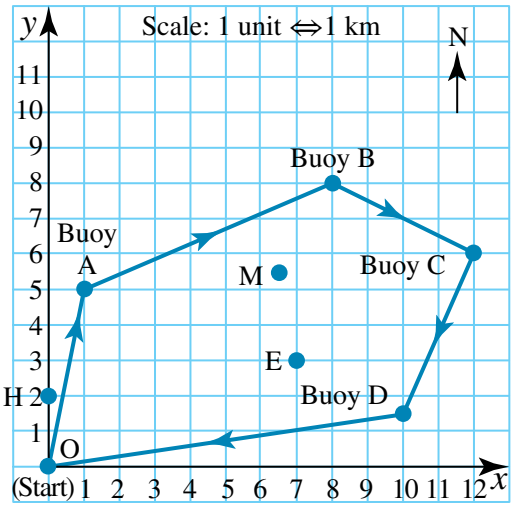
A. 3 **B.** –6 **C.** 2 **D.** $\frac{1}{2}$ **E.** –2

27. MC Triangle ABC has a right angle at B. The vertices are A(–2, 9), B(2, 8) and C(1, z). The value of z is:

A. $8\frac{1}{4}$ **B.** 4 **C.** 12 **D.** $7\frac{3}{4}$ **E.** –4

Communicating, reasoning and problem solving

28. The map shows the proposed course for a yacht race. Buoys have been positioned at A(1, 5), B(8, 8), C(12, 6) and D(10, w).



a. How far is it from the start, O, to buoy A?

b. The race marshal's boat, M, is situated halfway between buoys A and C. What are the coordinates of the boat's position?

c. Stage 4 of the race (from C to D) is perpendicular to stage 3 (from B to C). What is the gradient of CD?

d. Find the linear equation that describes stage 4.

e. Hence determine the exact position of buoy D.

f. An emergency boat is to be placed at point E(7, 3). How far is the emergency boat from the hospital, located at H, 2 km north of the start?

29. Show that the following sets of points form the vertices of a right-angled triangle.

a. A(1, –4), B(2, –3), C(4, –7)

b. A(3, 13), B(1, 3), C(–4, 4)

c. A (0, 5), B(9, 12), C(3, 14)

30. Prove that the quadrilateral ABCD is a rectangle given A(2, 5), B(6, 1), C(3, –2) and D(–1, 2).

31. Prove that the quadrilateral ABCD is a rhombus given A(2, 3), B(3, 5), C(5, 6) and D(4, 4). (*Hint:* A rhombus is a parallelogram with diagonals that intersect at right angles.)

32. **a.** A square has vertices at (0, 0) and (2, 0). Where are the other 2 vertices? (There are 3 sets of answers.)

b. An equilateral triangle has vertices at (0, 0) and (2, 0). Where is the other vertex? (There are 2 answers.)

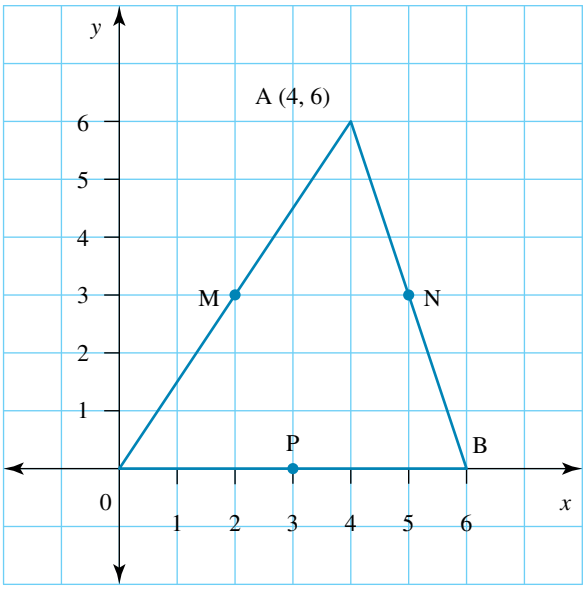
c. A parallelogram has vertices at (0, 0), (2, 0) and (1, 1). Where is the other vertex? (There are 3 sets of answers.)

33. A is the point (0, 0) and B is the point (0, 2).

a. Find the perpendicular bisector of AB.

b. Show that any point on this line is equidistant from A and B.

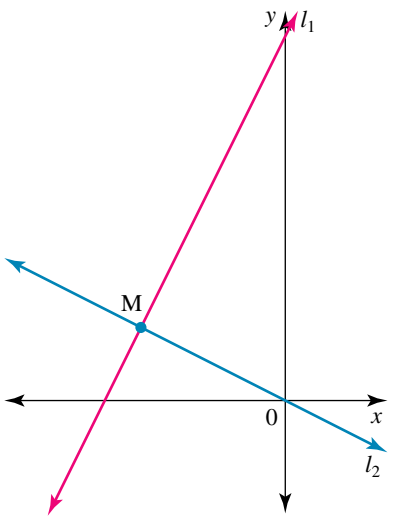
Questions 34 and 35 relate to the diagram shown.
M is the midpoint of OA.
N is the midpoint of AB.
P is the midpoint of OB.



34. A simple investigation:
- **a.** Show that MN is parallel to OB.
- **b.** Is PN parallel to OA?
- **c.** Is PM parallel to AB?

35. A difficult investigation:
- **a.** Find the perpendicular bisectors of OA and OB.
- **b.** Find the point W where the two bisectors intersect.
- **c.** Show that the perpendicular bisector of AB also passes through W.
- **d.** Explain why W is equidistant from O, A and B.
- **e.** W is called the circumcentre of triangle OAB. Using W as the centre, draw a circle through O, A and B.

36. The lines l_1 and l_2 are at right angles to each other. The line l_1 has the equation $px + py + r = 0$.
Show that the distance from M to the origin is given by $\frac{r}{\sqrt{p^2 + p^2}}$.



37. Line A is parallel to the line with equation $2x - y = 7$ and passes through the point (2, 3). Line B is perpendicular to the line with equation $4x - 3y + 3 = 0$ and also passes through the point (2, 3). Line C intersects with line A where it cuts the y-axis and intersects with line B where it cuts the x-axis.
- **a.** Determine the equations for all three lines. Give answers in the form $ax + by + c = 0$.
- **b.** Sketch all three lines on the one set of axes.
- **c.** Determine whether the triangle formed by the three lines is scalene, isosceles or equilateral.

38. How could you use coordinate geometry to design a logo for an organisation?

39. The first six numbers of a particular number pattern are 1, 2, 3, 6, 11 and 20. Given that this pattern continues, what will be the next four numbers? Describe the pattern.

8.7 Review

Investigation | Rich task

What common computer symbol is this?

On computer hardware, and on many different software applications, a broad range of symbols is used. These symbols help us to identify where things need to be plugged into, what buttons we need to push, or what option needs to be selected. The main focus of this task involves constructing a common symbol found on the computer. The instructions are given below.

The construction part of this task requires you to graph nine lines to reveal a common computer symbol. Draw the scale of your graph to accommodate x- and y-values in the following ranges: $-10 \leq x \leq 16$ and $-10 \leq y \leq 16$.

- Line 1 has the equation $y = x - 1$. Graph this line in the range $-7 \leq x \leq -2$.
- Line 2 is perpendicular to line 1 and has a y-intercept of -5. Determine the equation of this line, and then draw the line in the range $-5 \leq x \leq -1$.
- Line 3 is parallel to line 1, with a y-intercept of 3. Determine the equation of the line, and then graph the line in the range $-9 \leq x \leq -4$.
- Line 4 is parallel to line 1, with a y-intercept of -3. Determine the equation of the line, and then graph the line in the range $-1 \leq x \leq 2$.
- Line 5 has the same length as line 4 and is parallel to it. The point $(-2, 3)$ is the starting point of the line, which decreases in both x- and y-values from there.
- Line 6 commences at the same starting point as line 5, and then runs at right angles to line 5. It has an x-intercept of 1 and is the same length as line 2.
- Line 7 commences at the same starting point as both lines 5 and 6. Its equation is $y = 6x + 15$. The point $(-1, 9)$ lies at the midpoint.
- Line 8 has the equation $y = -x + 15$. Its midpoint is the point $(7, 8)$ and its extremities are the points where the line meets line 7 and line 9.
- Line 9 has the equation $6y - x + 8 = 0$. It runs from the intersection of lines 4 and 6 until it meets line 8.

1. What common computer symbol have you drawn?

2. The top section of your figure is a familiar geometric shape. Use the coordinates on your graph together with the distance formula to determine the necessary lengths to calculate the area of this figure.

3. Using any symbol of interest to you, draw your symbol on grid lines and provide instructions for your design. Ensure that your design involves aspects of coordinate geometry that have been used throughout this task.

learnON RESOURCES — ONLINE ONLY

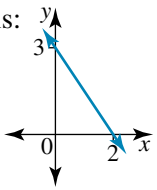
Digital doc: Investigation — What common computer symbol is this? (doc-15920)
Digital doc: Code puzzle — Who won the inaugural 875 km Sydney to Melbourne marathon in 1983? (doc-15921)
Digital doc: Summary Topic 8 (doc-22906)
Interactivity: Word search Topic 8 (int-2832)
Interactivity: Crossword Topic 8 (int-2833)
Interactivity: Sudoku Topic 8 (int-3590)

Exercise 8.7 Review questions

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. MC The equation of the following line is:



A. $3x + 2y = 6$
B. $3x - 2y = 6$
C. $2x + 3y = 6$
D. $2x - 3y = 6$
E. $2x - 3y = -6$

2. MC The equation of a linear graph with gradient –3 and x-intercept of 4 is:
A. $y = -3x - 12$ **B.** $y = -3x + 4$ **C.** $y = -3x - 4$
D. $y = -3x + 12$ **E.** $y = 4x - 3$

3. MC The equation of a linear graph that passes through $(2, -7)$ and $(-2, -2)$ is:
A. $4x - 5y + 18 = 0$ **B.** $5x + 4y + 18 = 0$ **C.** $5x + 4y - 18 = 0$
D. $5x - 4y - 18 = 0$ **E.** $4x + 5y + 18 = 0$

4. MC The distance between the points $(1, 5)$ and $(6, -7)$ is:
A. $\sqrt{53}$ **B.** $\sqrt{29}$ **C.** 13 **D.** $\sqrt{193}$ **E.** 12

5. MC The midpoint of the line segment joining the points $(-4, 3)$ and $(2, 7)$ is:
A. $(-1, 5)$ **B.** $(-2, 10)$ **C.** $(-6, 4)$ **D.** $(-2, 4)$ **E.** $(-1, 2)$

6. MC If the midpoint of the line segment joining the points A $(3, 7)$ and B (x, y) has coordinates $(6, 2)$, then the coordinates of B are:
A. $(15, 3)$ **B.** $(0, -6)$ **C.** $(9, -3)$ **D.** $(4.5, 4.5)$ **E.** $(-9, 3)$

7. MC If the points $(-6, -11)$, $(2, 1)$ and $(x, 4)$ are collinear, then the value of x is:
A. 4 **B.** 3.2 **C.** $\frac{1}{4}$ **D.** $\frac{5}{16}$ **E.** 3

8. MC The gradient of the line perpendicular to $3x - 4y + 7 = 0$ is:
A. $\frac{3}{4}$ **B.** $\frac{4}{3}$ **C.** $-\frac{4}{3}$ **D.** 3 **E.** –4

9. MC The equation of the line perpendicular to $2x + y - 1 = 0$ and passing through the point $(1, 4)$ is:
A. $2x + y - 6 = 0$ **B.** $2x + y - 2 = 0$ **C.** $x - 2y + 7 = 0$
D. $x + 2y + 9 = 0$ **E.** $x - 2y = 0$

10. Produce a table of values, and sketch the graph of the equation $y = -5x + 15$ for values of x between –10 and +10.

11. Sketch the graph of the following linear equations, labelling the x- and y-intercepts.
a. $y = 3x - 2$ **b.** $y = -5x + 15$ **c.** $y = -\frac{2}{3}x + 1$ **d.** $y = \frac{7}{5}x - 3$

12. Find the x- and y-intercepts of the following straight lines.

a. $y = -7x + 6$ **b.** $y = \frac{3}{8}x - 5$ **c.** $y = \frac{4}{7}x - \frac{3}{4}$ **d.** $y = 0.5x + 2.8$

13. Sketch graphs of the following linear equations by finding the x- and y-intercepts.

a. $2x - 3y = 6$ **b.** $3x + y = 0$ **c.** $5x + y = -3$ **d.** $x + y + 3 = 0$

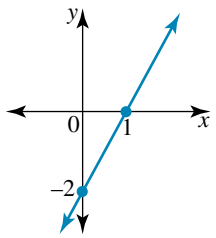
14. Sketch the graph of each of the following.

a. $y = \frac{1}{2}x$ **b.** $y = -4x$ **c.** $x = -2$ **d.** $y = 7$

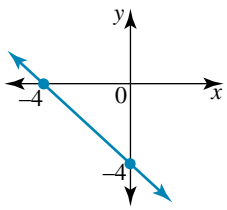
15. Sketch the graph of the equation $3(y - 5) = 6(x + 1)$.

16. Find the equations of the straight lines in the following graphs.

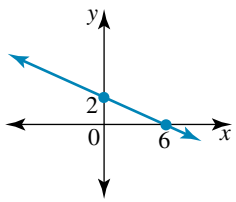
a.



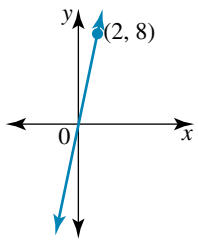
b.



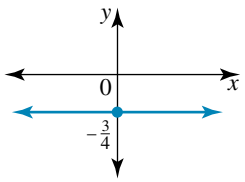
c.



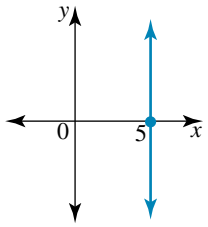
d.



e.



f.



17. Find the linear equation given the information in each case below.

a. Gradient = 3, y-intercept = –4 **b.** Gradient = –2, y-intercept = –5

c. Gradient = $\frac{1}{2}$, y-intercept = 5 **d.** Gradient = 0, y-intercept = 6

18. For each of the following, find the equation of the straight line with the given gradient and passing through the given point.

a. Gradient = 7, point (2, 1) **b.** Gradient = –3, point (1, 1)

c. Gradient = $\frac{1}{2}$, point (–2, 5) **d.** Gradient = $\frac{3}{5}$, point (1, –3)

19. Find the distance between the points (1, 3) and (7, –2) in exact form.

20. Prove that triangle ABC is isosceles given A (3, 1), B (–3, 7) and C (–1, 3).

21. Show that the points A (1, 1), B (2, 3) and C (8, 0) are the vertices of a right-angled triangle.

22. The midpoint of the line segment AB is (6, –4). If B has coordinates (12, 10), find the coordinates of A.

23. Show that the points A (3, 1), B (5, 2) and C (11, 5) are collinear.

24. Show that the lines $y = 2x - 4$ and $x + 2y - 10 = 0$ are perpendicular to one another.

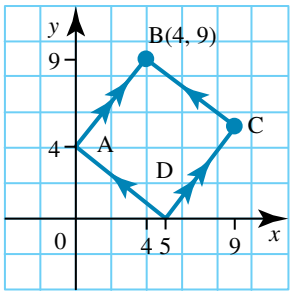
25. Find the equation of the straight line passing through the point (6, –2) and parallel to the line $x + 2y - 1 = 0$.

26. Find the equation of the line perpendicular to $3x - 2y + 6 = 0$ and having the same y-intercept.

27. Find the equation of the perpendicular bisector of the line joining the points (–2, 7) and (4, 11).

28. Find the equation of the straight line joining the point (–2, 5) and the point of intersection of the straight lines with equations $y = 3x - 1$ and $y = 2x + 5$.

29. Using the information given in the diagram:



a. find:

i. the gradient of AD **ii.** the gradient of AB

iii. the equation of BC **iv.** the equation of DC

v. the coordinates of C.

b. describe quadrilateral ABCD.

30. In triangle ABC, A is (1, 5), B is (–2, –3) and C is (8, –2).

a. Find:

i. the gradient of BC **ii.** the midpoint, P, of AB **iii.** the midpoint, Q, of AC.

b. Hence, show that:

i. PQ is parallel to BC **ii.** PQ is half the length of BC.

Communicating, reasoning and problem solving

31. John has a part-time job working as a gardener and is paid $13.50 per hour.

a. Complete the following table of values relating the amount of money received to the number of hours worked.

Number of hours	0	2	4	6	8	10
Pay ($)						

b. Find a linear equation relating the amount of money received to the number of hours worked.

c. Sketch the linear equation on a Cartesian plane over a suitable domain.

d. Using algebra, calculate the pay that John will receive if he works for $6\frac{3}{4}$ hours.

32. A fun park charges a $12.50 entry fee and an additional $2.50 per ride.

a. Complete the following table of values relating the total cost to the number of rides.

Number of rides	0	2	4	6	8	10
Cost ($)						

b. Find a linear equation relating total cost to the number of rides.

c. Sketch the linear equation on a Cartesian plane over a suitable domain.

d. Using algebra, calculate the cost for 7 rides.

33. The cost of hiring a boat is $160 plus $22.50 per hour.

a. Sketch a graph showing the total cost for between 0 and 12 hours.

b. State the equation relating cost to time rented.

c. Predict the cost of hiring a boat for 12 hours and 15 minutes.

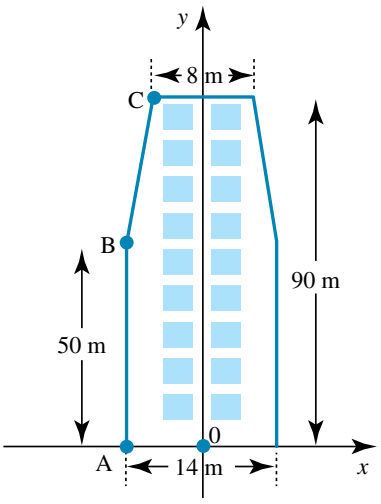
34. ABCD is a quadrilateral with vertices A (4, 9), B (7, 4), C (1, 2) and D (a, 10). Given that the diagonals are perpendicular to each other, find:

a. the equation of the diagonal AC

b. the equation of the diagonal BD

c. the value of a.

35. An architect decides to design a building with a 14-metre-square base such that the external walls are initially vertical to a height of 50 metres, but taper so that their separation is 8 metres at its peak height of 90 metres. A profile of the building is shown with the point (0, 0) marked as a reference at the centre of the base.

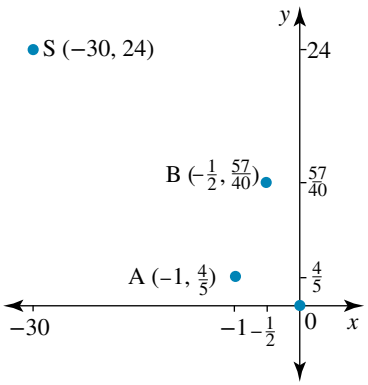


a. Write the equation of the vertical line connecting A and B.

b. Write the coordinates of B and C.

c. Find the length of the tapered section of wall from B to C.

36. In a game of lawn bowls, the object is to bowl a biased ball so that it gets as close as possible to a smaller white ball called a jack. During a game, a player will sometimes bowl a ball quite quickly so that it travels in a straight line in order to displace an opponent's 'guard balls'. In a particular game, player X has 2 guard balls close to the jack. The coordinates of the jack are (0, 0) and the coordinates of the guard balls are A$\left(-1, \frac{4}{5}\right)$ and B $\left(-\frac{1}{2}, \frac{57}{40}\right)$. Player Y bowls a ball so that it travels in a straight line toward the jack. The ball is bowled from position S, which has coordinates (–30, 24).



(Not to scale)

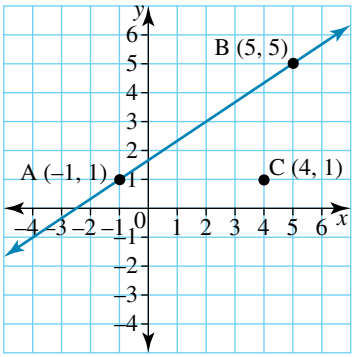
a. Will player Y displace one of the guard balls? If so, which one?

b. Due to bias, the displaced guard ball is knocked so that it begins to travel in a straight line (at right angles to the path found in part **a**). Find the equation of the line of the guard ball.

c. Show that guard ball A is initially heading directly toward guard ball B.

d. Given its initial velocity, guard ball A can travel in a straight line for 1 metre before its bias affects it path. Calculate and explain whether guard ball A will collide with guard ball B.

37. The graph shows the line p passing through the points A (–1, 1) and B (5, 5). Given that C is the point (4, 1), find:



a. the gradient of p

b. the equation of p

c. the area of ΔABC

d. the length of BC, giving your answer correct to 2 decimal places.

38. The temperature of the air (T °C) is related to the height above sea level (h metres) by the formula $T = 18 - 0.005h$.

a. What is the temperature at the heights of:

i. 600 m **ii.** 1000 m **iii.** 3000 m?

b. Draw a graph using the results from part **a**.

c. Use the graph to find the temperature at 1200 m and 2500 m.

d. Predict the height at which the temperature is 9 °C.

39. An old theory on the number of hours of sleep (h) that a child of c years of age should have each night is $h = 8 + \frac{18 - c}{2}$.

a. How many hours should a 10-year-old have?

b. How old is a child that requires 10 hours sleep?

c. For every year, how much less sleep does a child require?

Answers

Topic 8 Linear relationships

Exercise 8.2 The distance between two points

1. AB = 5, CD = $2\sqrt{10}$ or 6.32, EF = $3\sqrt{2}$ or 4.24, GH = $2\sqrt{5}$ or 4.47, IJ = 5, KL = $\sqrt{26}$ or 5.10, MN = $4\sqrt{2}$ or 5.66, OP = $\sqrt{10}$ or 3.16

2. a. 5 **b.** 13 **c.** 10 **d.** 7.07 **e.** 6.71 **f.** 14.42
g. 13 **h.** 13 **i.** $\sqrt{a^2 + 4b^2}$ **j.** $3\sqrt{a^2 + b^2}$

3. B

4. D

5. a. AB = 4.47, BC = 2.24, CD = 4.47, DA = 2.24

b. AC = 5, BD = 5

c. Rectangle

6, 7, 8. Answers will vary.

9. a. 12 **b.** 5 **c.** 13 **d.** −2.2

10. Answers will vary.

11. $a = 2$

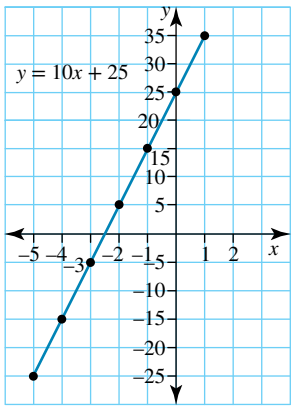
12. a. $m_{AB} = 1$ and $m_{BC} = -\frac{7}{3}$ **b.** D (4, −1) **c.** Check with your teacher.

13. Discuss with your teacher.

Exercise 8.3 Sketching linear graphs

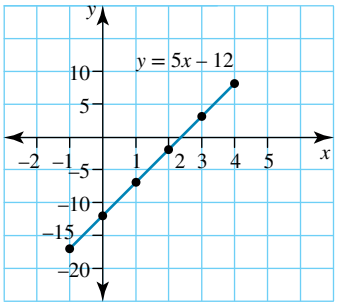
1. a.

x	y
−5	−25
−4	−15
−3	−5
−2	5
−1	15
0	25
1	35



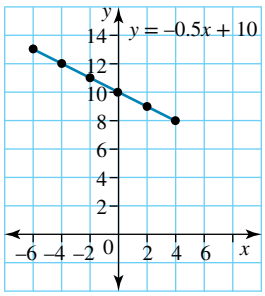
b.

x	y
−1	−17
0	−12
1	−7
2	−2
3	3
4	8



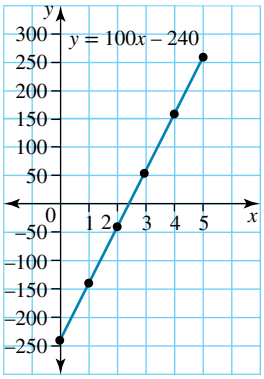
c.

x	y
−6	13
−4	12
−2	11
0	10
2	9
4	8



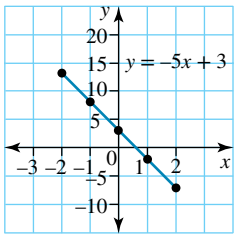
d.

x	y
0	−240
1	−140
2	−40
3	60
4	160
5	260



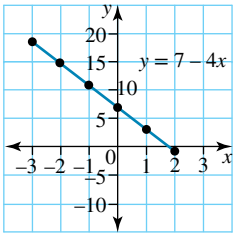
e.

x	y
−3	18
−2	13
−1	8
0	3
1	−2
2	−7



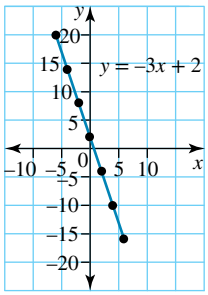
f.

x	y
−3	19
−2	15
−1	11
0	7
1	3
2	−1



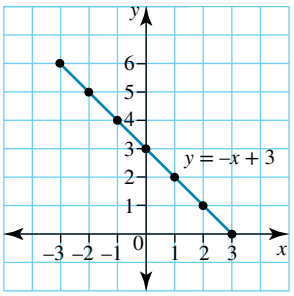
2. a.

x	y
−6	20
−4	14
−2	8
0	2
2	−4
4	−10
6	−16



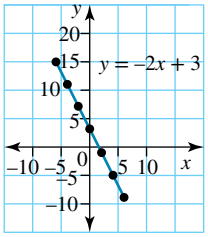
b.

x	y
−3	6
−2	5
−1	4
0	3
1	2
2	1
3	0



c.

x	y
−6	15
−4	11
−2	7
0	3
2	−1
4	−5
6	−9



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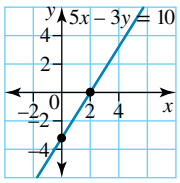

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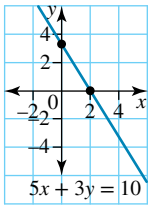

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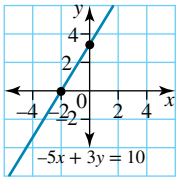

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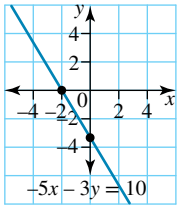

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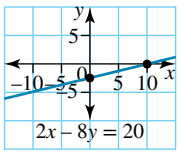

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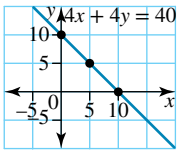

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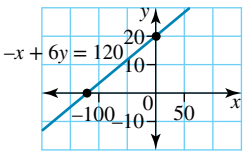

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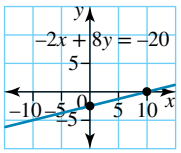

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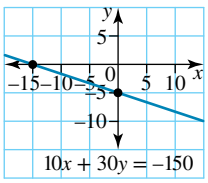

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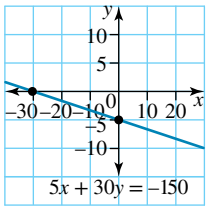

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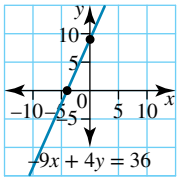

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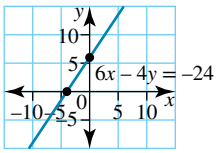

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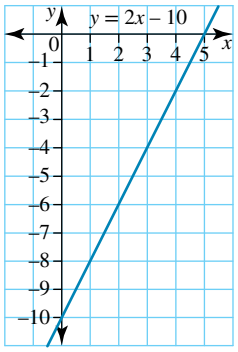

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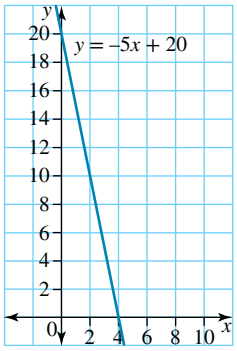

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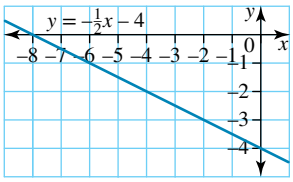

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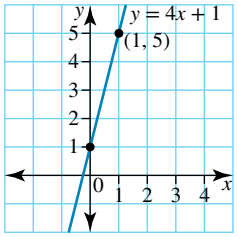

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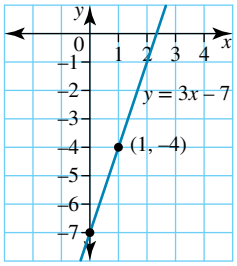

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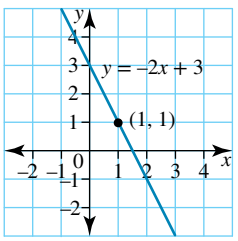

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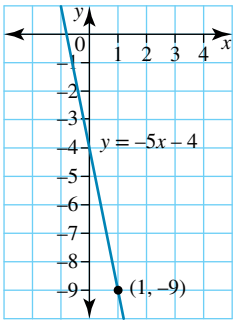

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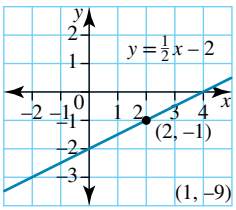

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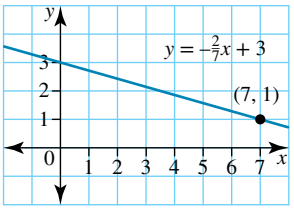

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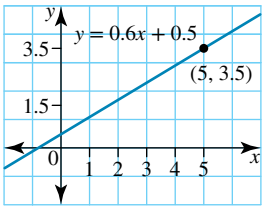

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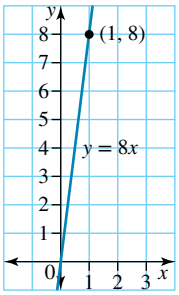

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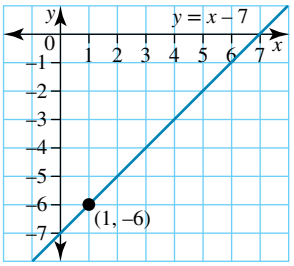

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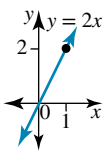

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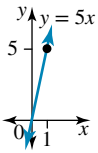

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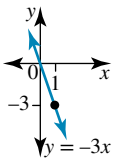

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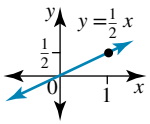

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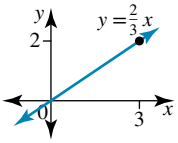

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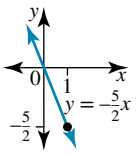

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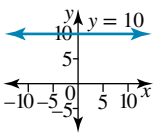

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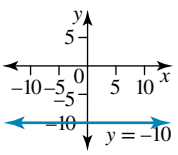

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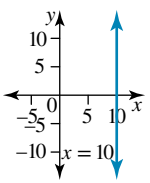

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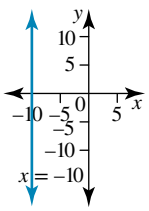

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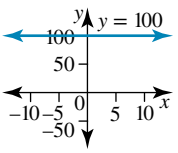
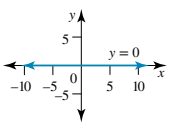


f.



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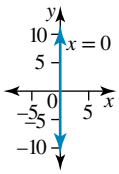
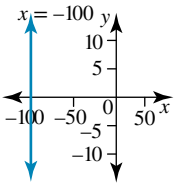
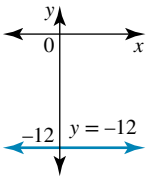


h.



i.



7. a. x-intercept: -0.5; y-intercept: 0.4

b. x-intercept: 0.5; y-intercept: -0.4

c. x-intercept: 0; y-intercept: 0

d. x-intercept: -3; y-intercept: 12

e. x-intercept: -4; y-intercept: -4

f. x-intercept: -1; y-intercept: -0.5

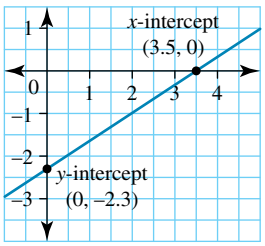
g. x-intercept: 2.75; y-intercept: 2.2 **h.** x-intercept: 9.75; y-intercept: −3.9

i. x-intercept: $\frac{23}{13} \approx 1.77$; y-intercept: 4.6

8. a. $(2, 0), (0, -8)$ **b.** $(-\frac{1}{2}, 0), (0, 3)$ **c.** $(-5, 0), (0, 25)$

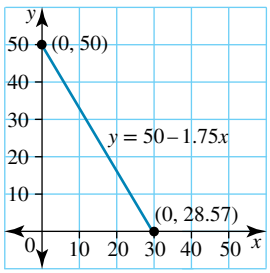
9. Answers will vary.

10. $y = \frac{2}{3}x - \frac{7}{3}$



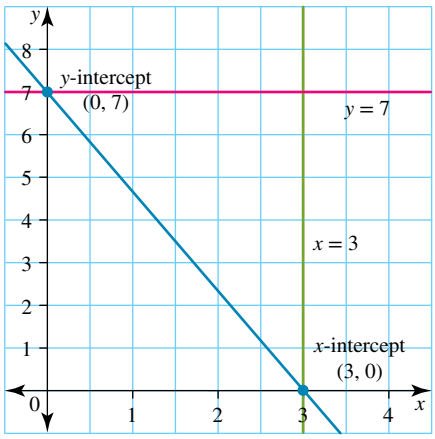
11. a. x represents the number of songs purchased.

b.



c. 14 songs

12. a.



b. 7 **c.** $-\frac{7}{3}$ **d.** B

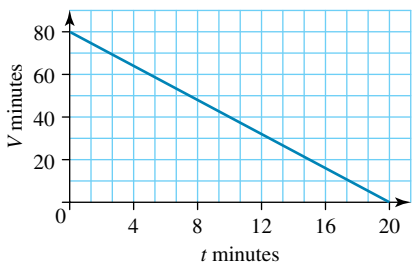
13. a. Initially there are 80 litres of water.

b. Time cannot be negative.

c. 4 litres per minute

d. 20 minutes

e.



14. Lines of the form $x + y = d$, $d \in R$.

Exercise 8.4 Standard forms of the equation of a straight line

1. a. $y = 2x + 4$ **b.** $y = -3x + 12$ **c.** $y = -x + 5$ **d.** $y = 2x - 8$
e. $y = \frac{1}{2}x + 3$ **f.** $y = -\frac{1}{4}x - 4$ **g.** $y = 7x - 5$ **h.** $y = -3x - 15$

2. a. $y = 2x$ **b.** $y = -3x$ **c.** $y = \frac{1}{2}x$ **d.** $y = -\frac{3}{4}x$

3. a. $y = x + 3$ **b.** $y = 2x - 1$ **c.** $y = -\frac{1}{2}x + \frac{7}{2}$ **d.** $y = \frac{1}{2}x + \frac{1}{2}$
e. $y = -2x - 2$ **f.** $y = -x - 8$

4. a. $y = 3x + 3$ **b.** $y = -3x + 4$ **c.** $y = -4x + 2$ **d.** $y = 4x + 2$
e. $y = -x - 4$ **f.** $y = 0.5x - 4$ **g.** $y = 5x + 2.5$ **h.** $y = -6x + 3$
i. $y = -2.5x + 1.5$ **j.** $y = 3.5x + 6.5$

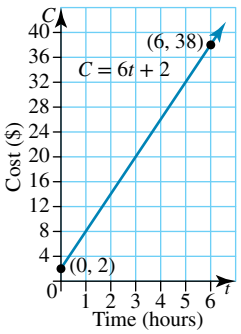
5. a. $y = 5x - 19$ **b.** $y = -5x + 31$ **c.** $y = -4x - 1$ **d.** $y = 4x - 34$
e. $y = 3x - 35$ **f.** $y = -3x + 6$

6. a. $2x + y - 30 = 0$ **b.** $4x - 2y - 9 = 0$ **c.** $x - 2y - 38 = 0$ **d.** $x + 2y - 11 = 0$

7. a.

t	0	1	2	3
C	2	8	14	20

b.



c. i. (0, 2)
ii. The y-intercept represents the initial cost of bowling at the alley, which is the shoe rental.
d. $m = 6$, which represents the cost to hire a lane for an additional hour.
e. $C = 6t + 2$
f. $32
g. Answers will vary.

8. It does not matter if you rise before you run or run before you rise, as long as you take into account whether the rise or run is negative.

9. a. $m = \frac{y - b}{x}$
b. $y = mx + b$

10. Check with your teacher.

11. $m_{AB} = m_{CD} = 2$ and $m_{BC} = m_{AD} = \frac{1}{2}$. As opposite sides have the same gradients, this quadrilateral is a parallelogram.

12. Horizontal lines have any y-value and vertical lines have any x-value.

13. $y = \frac{3}{5}x + 2$

Exercise 8.5 The midpoint of a line segment

1. a. $\left(-3, -3\frac{1}{2}\right)$ **b.** $\left(7\frac{1}{2}, 0\right)$ **c.** $(-1, 1)$ **d.** $\left(0, 1\frac{1}{2}\right)$
e. $\left(2a, \frac{1}{2}b\right)$ **f.** $\left(a + b, \frac{1}{2}a\right)$

2. $(-3, -10)$

3. a. (3, 1) **b.** 4.47 **c.** 6.32

4. D

5. C

6. a. i. $(-1, 4)$ **ii.** $\left(1\frac{1}{2}, 1\right)$ **iii.** 3.91
b. BC = 7.8 = 2 PQ

7. a. i. $(1, -0.5)$ **ii.** $(1, -0.5)$

b. The diagonals bisect each other, so it is a parallelogram.

8. a. i. $(-2, 2)$ **ii.** 8.94 **iii.** 9.55 **iv.** 9.55

b. Isosceles. PC is the perpendicular height of the triangle.

9. $y = -3x - 2$

10. $3y - 2x + 14 = 0$

11. Check with your teacher.

12. $(4k - 1, 3.5 - 5k)$

13. One endpoint must be the negative value of the other end point.

For example:

Endpoint A = $(-17, -7)$

Endpoint B = $(17, 7)$

Exercise 8.6 Parallel and perpendicular lines

1. a. No **b.** Yes **c.** No **d.** No **e.** Yes **f.** No

2. b, **f**; **c**, **e**

3. Answers will vary.

4. Answers will vary.

5. Answers will vary.

6. a. Yes **b.** Yes **c.** No **d.** Yes **e.** Yes **f.** No

7. $y = 2x - 9$

8. $3x + 2y - 8 = 0$

9. a. $y = 3x + 2$ **b.** $y = -4x + 9$ **c.** $3x - 2y - 8 = 0$ **d.** $5y + 2x + 13 = 0$

e. $x + 5y + 5 = 0$ **f.** $x - 3y + 17 = 0$ **g.** $x - 3y - 14 = 0$

10. a. $2x - y + 5 = 0$ **b.** $x + 2y = 0$

11. a. $3x - 5y + 2 = 0$ **b.** $5x + 3y - 8 = 0$

12. a. $x = 1$ **b.** $y = -7$

13. a. B **b.** C **c.** D **d.** B

14. a. $(2,\ 5)$ **b.** 1 **c.** Answers will vary. **d.** Isosceles triangle

15. $y = -x - 3$

16. $4x - 6y + 23 = 0$

17. a. $y = -x + 5$ **b.** $y = x + 3$ **c.** $(1,\ 4)$

18. Answers will vary.

19. Answers will vary.

20. B

21. E

22. a. $y = -2x + 1$ **b.** $3y + 2x + 1 = 0$

23. a, e; b, f; c, h; d, g

24. $y = -\frac{1}{2}x + \frac{3}{2}$ or $x + 2y - 3 = 0$

25. a. $m = 2\frac{8}{5}$ **b.** $m = \frac{18}{5}$

26. E

27. B

28. a. 5.10 km **b.** (6.5, 5.5) **c.** 2 **d.** $y = 2x - 18$ **e.** $(10,\ 2)$ **f.** 7.07 km

29. Answers will vary.

30. Answers will vary.

31. Answers will vary.

32. a. $(0, 2)$, $(2, 2)$ or $(0, -2)$, $(2, -2)$ or $(1, 1)$, $(1, -1)$

b. $(1, \sqrt{3})$ or $(1, -\sqrt{3})$

c. $(3, 1)$, $(-1, 1)$ or $(1, -1)$

33. a. $y = 1$ **b.** Answers will vary.

34. a. Answers will vary. **b.** Yes **c.** Yes

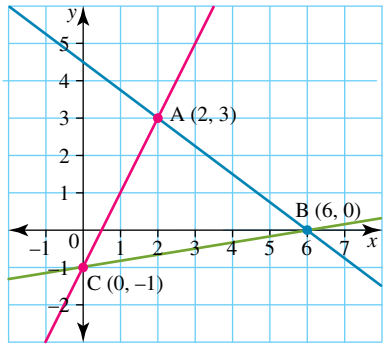
35. a. OA: $2x + 3y - 13 = 0$; OB: $x = 3$ **b.** $\left(3, \frac{7}{3}\right)$

c, d. Answers will vary.

36. Check with your teacher.

37. a. Line A: $2x - y - 1 = 0$, Line B: $3x + 4y - 18 = 0$, Line C: $x - 6y - 6 = 0$

b.

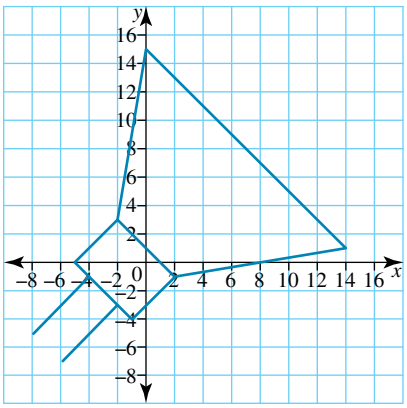


c. Scalene

38. Discuss with your teacher.

39. 37, 68, 125, 230. To find the next number, add the three preceding numbers.

Investigation | Rich task



1. The symbol is the one used to represent a speaker.

2. The shape is a trapezium.

Area $= \frac{1}{2}$(length line 6 + length line 8) $\times$ perpendicular distance between these lines.

$= \frac{1}{2}(4\sqrt{2} + 14\sqrt{2}) \times 7\sqrt{2}$

$= 126$ units^2

3. Check with your teacher.

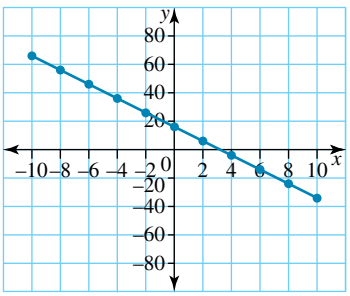
Exercise 8.7 Review questions

1. A **2.** D **3.** B **4.** C **5.** A

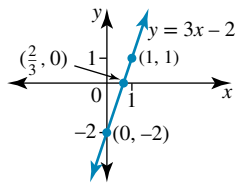
6. C **7.** A **8.** C **9.** C

10.

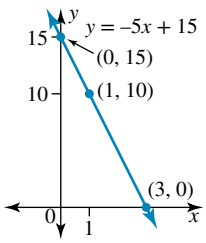
x	−10	−8	−6	−4	−2	0	2	4	6	8	10
y	65	55	45	35	25	15	5	−5	−15	−25	−35



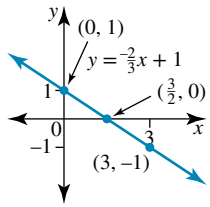
11. a.



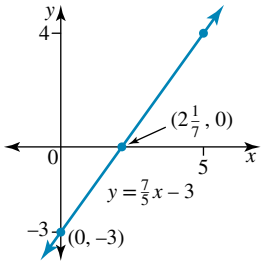
b.



c.



d.



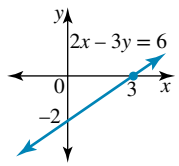
12. a. x-intercept $= \frac{6}{7}$, y-intercept $b = 6$

b. x-intercept $= \frac{40}{3}$, $\left(= 13\frac{1}{3}\right)$, y-intercept $b = -5$

c. x-intercept $= \frac{21}{16}$, $\left(= 1\frac{5}{16}\right)$, y-intercept $b = -\frac{3}{4}$

d. x-intercept $= -5.6$, y-intercept $b = 2.8$

13. a.



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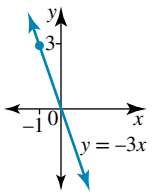

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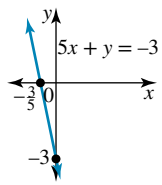

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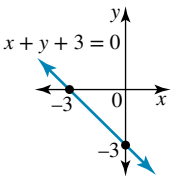
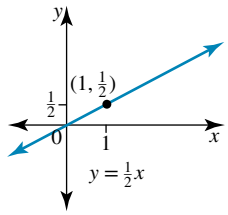
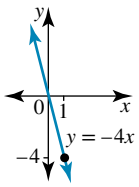


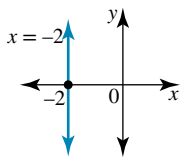
14. a.



b.



c.



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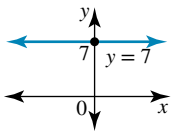
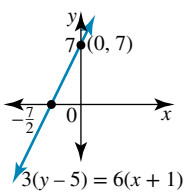


15.



16. a. $y = 2x - 2$

b. $y = -x - 4$

c. $y = -\frac{1}{3}x + 2$

d. $y = 4x$

e. $y = -\frac{3}{4}$

f. $x = 5$

17. a. $y = 3x - 4$ **b.** $y = -2x - 5$ **c.** $y = \frac{1}{2}x + 5$ **d.** $y = 6$

18. a. $y = 7x - 13$ **b.** $y = -3x + 4$ **c.** $y = \frac{1}{2}x + 6$ **d.** $y = \frac{3}{5}x - \frac{18}{15}$

19. $\sqrt{61}$ **20.** Answers will vary. **21.** Answers will vary. **22.** $(0, -18)$

23. Answers will vary. **24.** Answers will vary. **25.** $x + 2y - 2 = 0$ **26.** $2x + 3y - 9 = 0$

27. $3x + 2y - 21 = 0$ **28.** $3x - 2y + 16 = 0$

29. a. i. $-\frac{4}{5}$ **ii.** $\frac{5}{4}$ **iii.** $4x + 5y - 61 = 0$

iv. $5x - 4y - 25 = 0$ **v.** $(9, 5)$

b. Square

30. a. i. $\frac{1}{10}$ **ii.** $\left(-\frac{1}{2}, 1\right)$ **iii.** $\left(4\frac{1}{2}, 1\frac{1}{2}\right)$

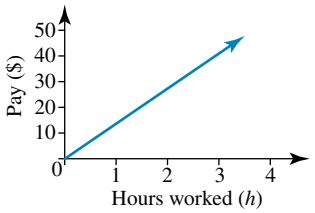
b. Answers will vary.

31. a.

Number of hours	0	2	4	6	8	10
Pay($)	0	27	54	81	108	135

b. Pay = \$13.50 × (number of hours worked)

c.



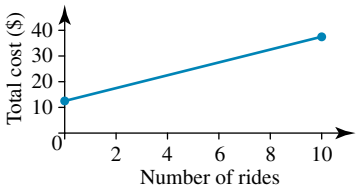
d. \$91.13

32. a.

Number of rides	0	2	4	6	8	10
Cost($)	12.50	17.50	22.50	27.50	32.50	37.50

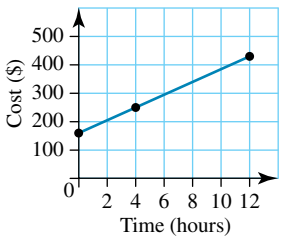
b. Cost = \$2.50 × number of rides + \$12.50

c.



d. \$30

33. a.



b. $C = 22.50h + 160$

c. Approximately \$436

34. a. $7x - 3y - 1 = 0$ **b.** $3x + 7y - 49 = 0$ **c.** –7

35. a. $x = -7$ **b.** B (–7, 50), C (–4, 90) **c.** 40.11 metres

36. a. Since the gradient of SA equals the gradient of SO = –0.8, the points S, A and O are collinear. Player Y will displace guard ball A.

b. $y = \frac{5}{4}x + \frac{41}{20}$ or $25x - 20y + 41 = 0$

c. Since the gradient of the path AB is $\frac{5}{4}$, which is the same as the gradient of the known path of travel from the common point A, the direction of travel is toward B.

d. $d_{AB} = 0.625$. Yes, guard ball A will collide with guard ball B as it will not be deviated from its linear path under 1 metre of travel.

37. a. Gradient $= m = \dfrac{5-1}{5--1} = \dfrac{4}{6} = \dfrac{2}{3}$

b. $y = mx + b$, $y = \frac{2}{3}x + b$

If $x = -1$ and $y = 1$, substitute in the question:

$1 = \frac{2}{3}(-1) + b$

$b = 1\frac{2}{3}$

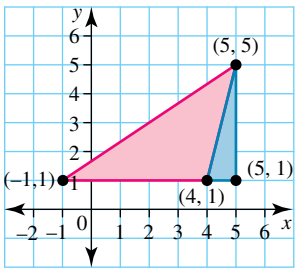
$y = \frac{2}{3}x + 1\frac{2}{3}$

c. Plot the point (5,1).

Area of large $\Delta = \frac{1}{2} \times 6 \times 4 = 12$

Area of small $\Delta = \frac{1}{2} \times 1 \times 4 = 2$

Area of $\Delta ABC = 12 - 2 = 10$ units^2



d. $BC^2 = 4^2 + 1^2$

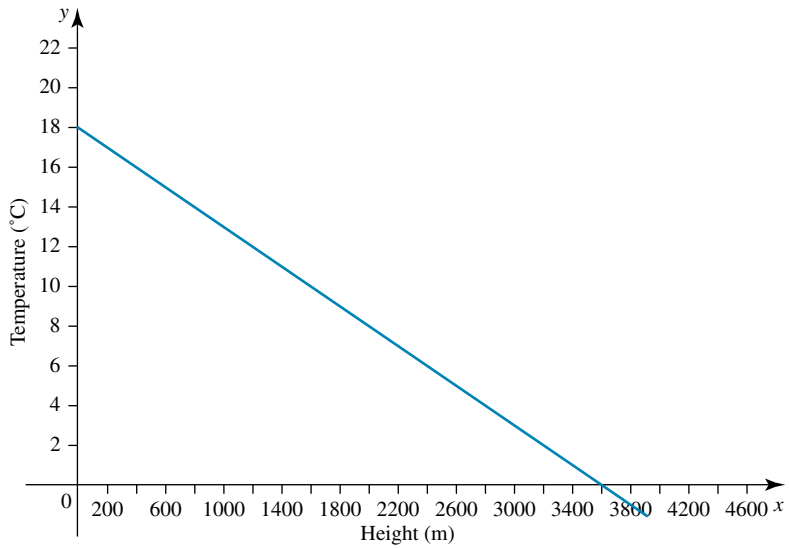
$BC^2 = 16 + 1$

$BC^2 = 17$

$BC = \sqrt{17} \approx 4.12$ units

38. a. **i.** $T = 18 - 0.005(600) = 15$ **ii.** $T = 18 - 0.005(1000) = 13$ **iii.** $T = 18 - 0.005(3000) = 3$

b.



c. 1200 m = 12 °C, 2500 m = 5.5 °C

d. 1800 m

39. a. 12 hours

b. 14 years old

c. $h = 8 + \dfrac{18 - c}{2}$

$2h = 16 + 18 - c$

$2h = -c + 34$

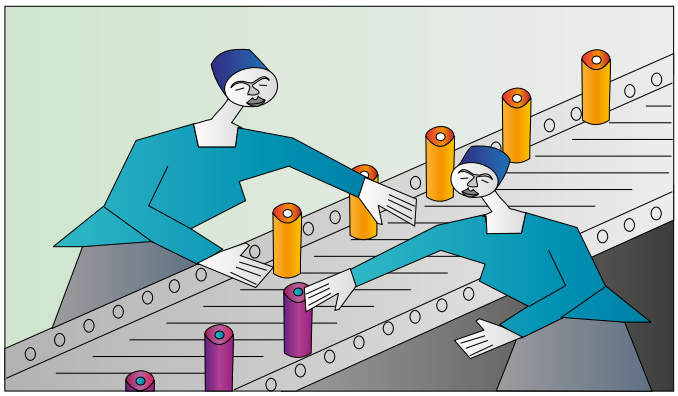
$h = -\frac{1}{2}c + 17$

For every year, the child requires half an hour less sleep.

TOPIC 9
Simultaneous linear equations and inequalities [Stages 5.2 and 5.3]

9.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the concepts covered in this topic.



9.1.1 Why learn this?

Picture this: you own a factory that produces two different products, and you are planning to buy some new machines. The big machines are more expensive than the small ones, take up more floor space and need more staff to operate, but they can produce more. Which machines should you buy?

Solving simultaneous equations will help you determine feasible solutions to questions like this.

DISCUSSION

How is mathematics linked to poetry? Use the internet to help you with your research.

LEARNING SEQUENCE

9.1 Overview
9.2 [Stage 5.2] Graphical solution of simultaneous linear equations
9.3 [Stage 5.2] Solving simultaneous linear equations using substitution
9.4 [Stage 5.2] Solving simultaneous linear equations using elimination
9.5 [Stage 5.2] Applications of simultaneous linear equations
9.6 [Stage 5.3] Solving simultaneous linear and non-linear equations
9.7 [Stage 5.2] Solving linear inequalities
9.8 Review

LEARNING OUTCOMES

A student:

- selects appropriate notations and conventions to communicate mathematical ideas and solutions MA5.2-1WM
- interprets mathematical or real-life situations, systematically applying appropriate strategies to solve problems MA5.2-2WM
- constructs arguments to prove and justify results MA5.2-3WM
- solves linear inequalities and linear simultaneous equations, using analytical and graphical techniques MA5.2-8NA
- solves simultaneous equations MA5.3-7NA

CONTENT DESCRIPTIONS

Students:

Solve linear simultaneous equations, using algebraic and graphical techniques, including with the use of digital technologies (ACMNA237)

Solve simultaneous equations, where one equation is non-linear, using algebraic and graphical techniques, including the use of digital technologies

Solve linear inequalities and graph their solutions on a number line (ACMNA236)

Source: NSW Syllabus for the Australian curriculum

learnon RESOURCES — ONLINE ONLY

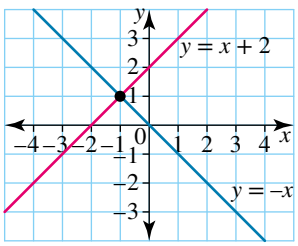
eLesson: The story of mathematics — Khayyam (eles-1843)

Note: Your teacher may now set you a pre-test to determine how familiar you are with the content in this topic.

9.2 Graphical solution of simultaneous linear equations [Stage 5.2]

9.2.1 Simultaneous linear equations

- **Simultaneous** means occurring at the same time.
- When a point belongs to more than one line, the coordinates of the point satisfy all equations. The equations of the lines are called **simultaneous equations**. An example is shown below.



- A **system of equations** is a set of two or more equations with the same variables.
- To solve simultaneous equations is to calculate the values of the variables that satisfy all equations in the system.
- Any two linear graphs will meet at a point, unless they are parallel.
- At this point, the two equations simultaneously share the same x- and y-coordinates, which are referred to as the solution.
- Simultaneous equations can be solved graphically or algebraically.

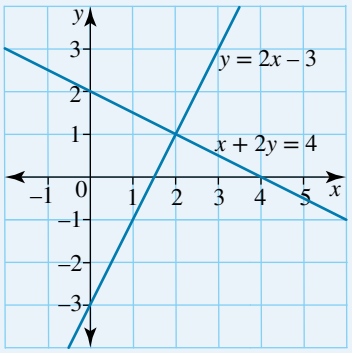
9.2.2 Graphical solution

- The solution to a pair of simultaneous equations can be found by graphing the two equations and identifying the coordinates of the point of intersection.
- An accurate solution depends on drawing an accurate graph.
- Graph paper or graphing software can be used.

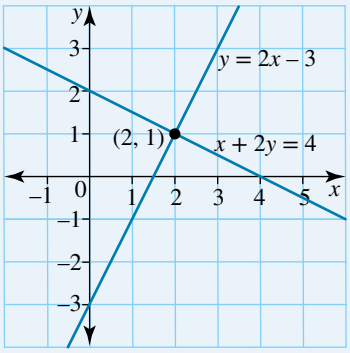
WORKED EXAMPLE 1

Use the graphs of the given simultaneous equations to determine the point of intersection and, hence, the solution of the simultaneous equations.

$x + 2y = 4$
$y = 2x - 3$



THINK	WRITE/DRAW
1 Write the equations and number them.	$x + 2y = 4$ [1] $y = 2x - 3$ [2]
2 Locate the point of intersection of the two lines. This gives the solution.	Point of intersection (2, 1) Solution: $x = 2$ and $y = 1$
3 Check the solution by substituting $x = 2$ and $y = 1$ into the given equations. Comment on the results obtained.	Check equation [1]: LHS $= x + 2y$ RHS $= 4$ $= 2 + 2(1)$ $= 4$ LHS = RHS Check equation [2]: LHS $= y$ RHS $= 2x - 3$ $= 1$ $= 2(2) - 3$ $= 4 - 3$ $= 1$ LHS = RHS In both cases, LHS = RHS. Therefore, the solution set (2, 1) is correct.



WORKED EXAMPLE 2

Check whether the given pair of coordinates, (5, −2), is the solution to the following pair of simultaneous equations.

$3x - 2y = 19$
$4y + x = -3$

THINK	WRITE
1 Write the equations and number them.	$3x - 2y = 19$ [1] $4y + x = -3$ [2]
2 Substitute $x = 5$ and $y = -2$ into equation [1].	Check equation [1]: LHS $= 3x - 2y$ RHS $= 19$ $= 3(5) - 2(-2)$ $= 15 + 4$ $= 19$ LHS = RHS
3 Substitute $x = 5$ and $y = -2$ into equation [2].	Check equation [2]: LHS $= 4y + x$ RHS $= -3$ $= 4(-2) + 5$ $= -8 + 5$ $= -3$ LHS = RHS Therefore, the solution set $(5, -2)$ is a solution to both equations.

WORKED EXAMPLE 3

Solve the following pair of simultaneous equations using a graphical method.

$$x + y = 6$$
$$2x + 4y = 20$$

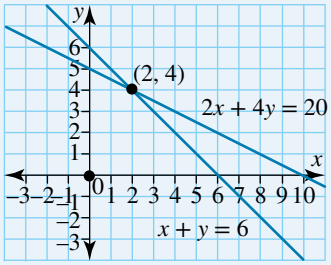
THINK	WRITE/DRAW
1 Write the equations, one under the other, and number them.	$x + y = 6$ [1] $2x + 4y = 20$ [2]
2 Calculate the x- and y-intercepts for equation [1]. For the x-intercept, substitute $y = 0$ into equation [1]. For the y-intercept, substitute $x = 0$ into equation [1].	Equation [1] x-intercept: when $y = 0$, $x + 0 = 6$ $x = 6$ The x-intercept is at $(6, 0)$. y-intercept: when $x = 0$, $0 + y = 6$ $y = 6$ The y-intercept is at $(0, 6)$.
3 Calculate the x- and y-intercepts for equation [2]. For the x-intercept, substitute $y = 0$ into equation [2]. Divide both sides by 2. For the y-intercept, substitute $x = 0$ into equation [2]. Divide both sides by 4.	Equation [2] x-intercept: when $y = 0$, $2x + 0 = 20$ $2x = 20$ $x = 10$ The x-intercept is at $(10, 0)$. y-intercept: when $x = 0$, $0 + 4y = 20$ $4y = 20$ $y = 5$ The y-intercept is at $(0, 5)$.

4 Use graph paper to rule a set of axes and label the x-axis from 0 to 10 and the y-axis from 0 to 6.

5 Plot the x- and y-intercepts for each equation.

6 Produce a graph of each equation by ruling a straight line through its intercepts.

7 Label each graph.



8 Locate the point of intersection of the lines.

The point of intersection is (2, 4).

9 Check the solution by substituting $x = 2$ and $y = 4$ into each equation.

Check [1]: LHS $= x + y$ RHS $= 6$
$= 2 + 4$
$= 6$
LHS = RHS

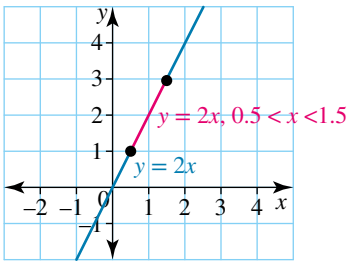
Check [2]: LHS $= 2x + 4y$ RHS $= 20$
$= 2(2) + 4(4)$
$= 4 + 16$
$= 20$
LHS = RHS

10 State the solution.

In both cases, LHS = RHS. Therefore, the solution set (2, 4) is correct.
The solution is $x = 2$, $y = 4$.

9.2.3 Equations with multiple solutions

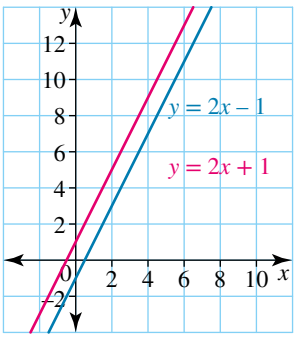
- Two lines are **coincident** if they lie one on top of the other. For example, the line in blue and line segment in red shown in the graph are coincident.
- There are an infinite number of solutions to coincident equations. Every point where the lines coincide satisfies both equations and hence is a solution to the simultaneous equations.
- Coincident equations have the same equation, although the equations may have been transposed so they look different. For example, $y = 2x + 3$ and $2y - 4x = 6$ are coincident equations.



9.2.4 Equations with no solutions

- If two lines do not intersect, there is no simultaneous solution to the equations. For example, the lines shown do not intersect, so there is no point that belongs to both lines.
- Parallel lines have the same gradient but a different y-intercept.
- For straight lines, the only situation in which the lines do not cross is if the lines are parallel and not coincident.
- Writing both equations in the form $y = mx + b$ confirms that the lines are parallel, as the gradients are equal.

$2x - y = 1$ [1]
$-y = 1 - 2x$
$-y = -2x + 1$
$y = 2x - 1$
Gradient $m = 2$



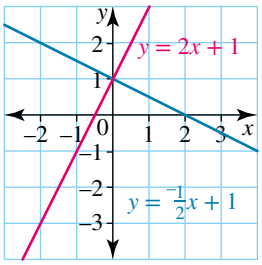
$$
\begin{aligned}
4x - 2y &= -2 \qquad [2] \\
-2y &= -2 - 4x \\
-2y &= -4x - 2 \\
y &= 2x + 1 \\
\text{Gradient } m &= 2
\end{aligned}
$$

DISCUSSION

If the gradients of two lines are the same, how can you tell whether they are coincident or whether they have no solutions?

9.2.5 Perpendicular lines

- Perpendicular lines meet at right angles (90°).
- Perpendicular lines have negative reciprocal gradients: $m_1 = \frac{-1}{m_2}$ or $m_1 m_2 = -1$, where m_1 is the gradient of the first line and m_2 is the gradient of the second line. For example, for the two lines shown, $m_1 = 2$ and $m_2 = -\frac{1}{2}$.



learnON RESOURCES — ONLINE ONLY

Interactivity: Solving simultaneous equations graphically (int-6452)

Interactivity: Parallel lines (int-3841)

Interactivity: Perpendicular lines (int-6124)

Digital doc: SkillSHEET Graphing linear equations using the *x*- and *y*-intercept method (doc-5217)

Exercise 9.2 Graphical solution of simultaneous linear equations

Individual pathways

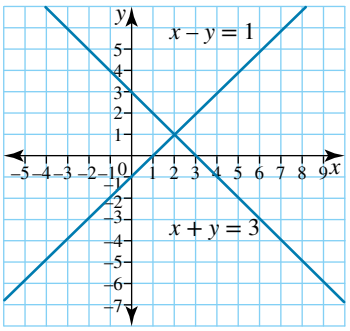
PRACTISE	CONSOLIDATE	MASTER
Questions: 1, 2a–d, 3a–d, 4a–d, 7	Questions: 1, 2c–g, 3a–d, 4a–f, 5, 7, 9, 11	Questions: 1, 2e–j, 3–11

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

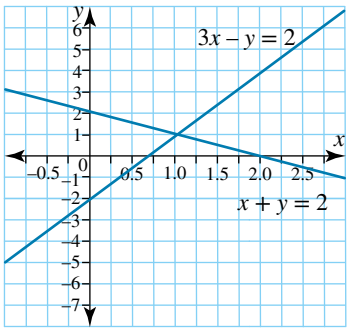
Understanding and fluency

1. **WE1** Use the graphs to find the solutions of the simultaneous equations.

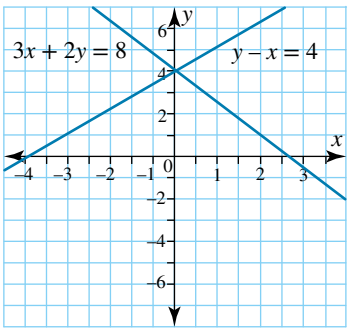
a. $x + y = 3$
$x - y = 1$



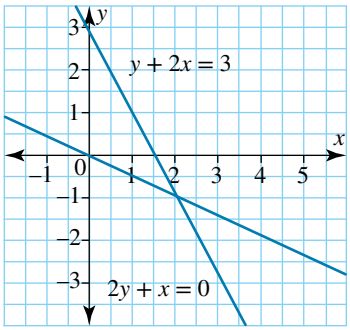
b. $x + y = 2$
$3x - y = 2$



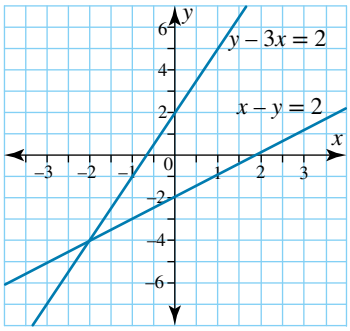
c. $y - 4 = x$
$3x + 2y = 8$



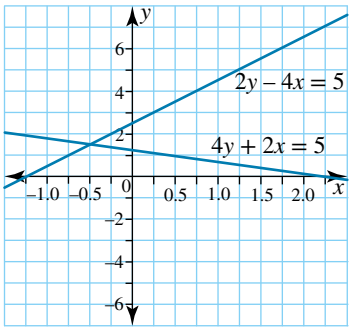
d. $y + 2x = 3$
$2y + x = 0$



e. $y - 3x = 2$
$x - y = 2$



f. $2y - 4x = 5$
$4y + 2x = 5$



2. **WE2** For the following simultaneous equations, use substitution to check if the given pair of coordinates is a solution.

a. (7, 5)
$3x + 2y = 31$
$2x + 3y = 28$

b. (3, 7)
$y - x = 4$
$2y + x = 17$

c. (9, 1)
$x + 3y = 12$
$5x - 2y = 43$

d. (2, 5)
$x - y = 7$
$2x + 3y = 18$

e. (4, −3)
$y = 3x - 15$
$4x + 7y = -5$

f. (6, −2)
$x - 2y = 2$
$3x + y = 16$

g. (4, −2)
$2x + y = 6$
$x - 3y = 8$

h. (5, 1)
$y - 5x = -24$
$3y + 4x = 23$

i. (−2, −5)
$3x - 2y = -4$
$2x - 3y = 11$

j. (−3, −1)
$y - x = 2$
$2y - 3x = 7$

3. **WE3** Solve each of the following pairs of simultaneous equations using a graphical method.

a. $x + y = 5$
$2x + y = 8$

b. $x + 2y = 10$
$3x + y = 15$

c. $2x + 3y = 6$
$2x - y = -10$

d. $x - 3y = -8$
$2x + y = -2$

e. $6x + 5y = 12$
$5x + 3y = 10$

f. $y + 2x = 6$
$2y + 3x = 9$

g. $y = 3x + 10$
$y = 2x + 8$

h. $y = 8$
$3x + y = 17$

i. $4x - 2y = -5$
$x + 3y = 4$

j. $3x + y = 11$
$4x - y = 3$

k. $3x + 4y = 27$
$x + 2y = 11$

l. $3y + 3x = 8$
$3y + 2x = 6$

4. Using technology, determine which of the following pairs of simultaneous equations have no solutions. Confirm by finding the gradient of each line.

a. $y = 2x - 4$
$3y - 6x = 10$

b. $5x - 3y = 13$
$4x - 2y = 10$

c. $x + 2y = 8$
$5x + 10y = 45$

d. $y = 4x + 5$
$2y - 10x = 8$

e. $3y + 2x = 9$
$6x + 4y = 22$

f. $y = 5 - 3x$
$3y = -9x + 18$

g. $4y + 3x = 7$
$12y + 9x = 22$

h. $2y - x = 0$
$14y - 6x = 2$

5. Two straight lines intersect at the point $(3, -4)$. One of the lines has a y-intercept of 8. The second line is a mirror image of the first in the line $x = 3$. Determine the equation of the second line. (*Hint:* Draw a graph of both lines.)

Communicating, reasoning and problem solving

6. At a well-known beach resort it is possible to hire a jet ski by the hour in two different locations. On the northern beach the cost is \$20 plus \$12 per hour, while on the southern beach the cost is \$8 plus \$18 per hour. The jet skis can be rented for up to 5 hours.

a. Write the rules relating cost to the length of rental.

b. On the same set of axes sketch a graph of cost (y-axis) against length of rental (x-axis) for 0–5 hours.

c. For what rental times, if any, is the northern beach rental cheaper than the southern beach rental? Use your graph to justify your answer.

d. For what length of rental time are the two rental schemes identical? Use the graph and your rules to justify your answer.

7. For each of the pairs of simultaneous equations below, determine whether they are the same line, parallel lines, perpendicular lines or intersecting lines. Show your working.

a. $2x - y = -9$
$-4x - 18 = -2y$

b. $x - y = 7$
$x + y = 7$

c. $x + 6 = y$
$2x + y = 6$

d. $x + y = -2$
$x + y = 7$

8. Which of the following problems has one solution, an infinite number of solutions or no solution? Explain your answers.

a. $x - y = 1$
$2x - 3y = 2$

b. $2x - y = 5$
$4x - 2y = -6$

c. $x - 2y = -8$
$4x - 8y = -16$

9. Line A is parallel to the line with equation $y - 3x - 3 = 0$ and passes through the point $(1, 9)$. Line B is perpendicular to the line with equation $2y - x + 6 = 0$ and passes through the point $(2, -3)$.

a. Find the equation of line A.

b. Find the equation of line B.

c. Sketch both lines on the one set of axes to find where they intersect.

10. Solve the system of three simultaneous equations graphically.

$3x - y = 2$
$y + 3x = 4$
$2y - x = 1$

11. What do you think is the major error made when solving simultaneous equations graphically?

9.3 Solving simultaneous linear equations using substitution [Stage 5.2]

9.3.1 Solving simultaneous linear equations

- There are two algebraic methods that are commonly used to solve simultaneous equations.
- They are the **substitution method** and the **elimination method**.

9.3.2 Substitution method

- The substitution method is particularly useful when one (or both) of the equations is in a form where one of the two variables is the subject.
- This variable is then substituted into the other equation, producing a third equation with only one variable.
- This third equation can then be used to determine the value of the variable.

WORKED EXAMPLE 4

Solve the simultaneous equations $y = 2x - 1$ and $3x + 4y = 29$ using the substitution method.

THINK	WRITE
1 Write the equations, one under the other, and number them.	$y = 2x - 1$ [1] $3x + 4y = 29$ [2]
2 y and $2x - 1$ are equal so substitute expression $2x - 1$ for y into equation [2].	Substituting $2x - 1$ into [2]: $3x + 4(2x - 1) = 29$
3 Solve for x. **i** Expand the brackets on the LHS of the equation. **ii** Collect like terms. **iii** Add 4 to both sides of the equation. **iv** Divide both sides by 11.	$3x + 8x - 4 = 29$ $11x - 4 = 29$ $11x = 33$ $x = 3$
4 Substitute $x = 3$ into either of the equations, say [1], to find the value of y.	Substituting $x = 3$ into [1]: $y = 2(3) - 1$ $= 6 - 1$ $= 5$
5 Write your answer.	Solution: $x = 3$, $y = 5$ or $(3, 5)$
6 Check the solution by substituting $(3, 5)$ into equation [2].	Check: Substitute $(3, 5)$ into $3x + 4y = 29$. LHS $= 3(3) + 4(5)$ RHS $= 29$ $= 9 + 20$ $= 29$ As LHS = RHS, the solution is correct.

WORKED EXAMPLE 5

Solve the pair of simultaneous equations $y = 5x - 8$ and $y = -3x + 16$ using the substitution method.

THINK	WRITE
1 Write the equations, one under the other, and number them.	$y = 5x - 8$ [1] $y = -3x + 16$ [2]

2 Both equations are written with y as the subject, so equate them.

$$5x - 8 = -3x + 16$$

3 Solve for x.

i Add $3x$ to both sides of the equation.

ii Add 8 to both sides of the equation.

iii Divide both sides of the equation by 8.

$$\begin{aligned} 8x - 8 &= 16 \\ 8x &= 24 \\ x &= 3 \end{aligned}$$

4 Substitute the value of x into either of the original equations, say [1], and solve for y.

Substituting $x = 3$ into [1]:

$$\begin{aligned} y &= 5(3) - 8 \\ &= 15 - 8 \\ &= 7 \end{aligned}$$

5 Write your answer.

Solution: $x = 3$, $y = 7$ or $(3, 7)$

6 Check the answer by substituting the point of intersection into equation [2].

Check: Substitute into $y = -3x + 16$.

$$\begin{aligned} \text{LHS} &= y \\ &= 7 \\ \text{RHS} &= -3x + 16 \\ &= -3(3) + 16 \\ &= -9 + 16 \\ &= 7 \end{aligned}$$

As LHS = RHS, the solution is correct.

learnON RESOURCES — ONLINE ONLY

Interactivity: Solving simultaneous equations using substitution (int-6453)

eLesson: Solving simultaneous equations using substitution (eles-1932)

Exercise 9.3 Solving simultaneous linear equations using substitution

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–d, 2a–d, 4, 6, 8	Questions: 1a–d, 2c–f, 5–9	Questions: 1–12

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE4 Solve the following pairs of simultaneous equations using the substitution method. Check your solutions using technology.

a. $x = -10 + 4y$
$3x + 5y = 21$

b. $3x + 4y = 2$
$x = 7 + 5y$

c. $3x + y = 7$
$x = -3 - 3y$

d. $3x + 2y = 33$
$y = 41 - 5x$

e. $y = 3x - 3$
$-5x + 3y = 3$

f. $4x + y = 9$
$y = 11 - 5x$

g. $x = -5 - 2y$
$5y + x = -11$

h. $x = -4 - 3y$
$-3x - 4y = 12$

i. $x = 7 + 4y$
$2x + y = -4$

j. $x = 14 + 4y$
$-2x + 3y = -18$

k. $3x + 2y = 12$
$x = 9 - 4y$

l. $y = 2x + 1$
$-5x - 4y = 35$

2. WE5 Solve the following pairs of simultaneous equations using the substitution method. Check your solutions using technology.

a. $y = 2x - 11$ and $y = 4x + 1$
b. $y = 3x + 8$ and $y = 7x - 12$
c. $y = 2x - 10$ and $y = -3x$
d. $y = x - 9$ and $y = -5x$
e. $y = -4x - 3$ and $y = x - 8$
f. $y = -2x - 5$ and $y = 10x + 1$
g. $y = -x - 2$ and $y = x + 1$
h. $y = 6x + 2$ and $y = -4x$
i. $y = 0.5x$ and $y = 0.8x + 0.9$
j. $y = 0.3x$ and $y = 0.2x + 0.1$
k. $y = -x$ and $y = -\frac{2}{7}x + \frac{4}{7}$
l. $y = -x$ and $y = -\frac{3}{4}x - \frac{1}{4}$

3. A small farm has sheep and chickens. There are twice as many chicken as sheep, and there are 104 legs between the sheep and the chickens. How many chickens are there?

4. Use substitution to solve each of the following pairs of simultaneous equations.

a. $5x + 2y = 17$
$y = \dfrac{3x - 7}{2}$
b. $2x + 7y = 17$
$x = \dfrac{1 - 3y}{4}$
c. $2x + 3y = 13$
$y = \dfrac{4x - 15}{5}$
d. $-2x - 3y = -14$
$x = \dfrac{2 + 5y}{3}$
e. $3x + 2y = 6$
$y = 3 - \dfrac{5x}{3}$
f. $-3x - 2y = -12$
$y = \dfrac{5x - 20}{3}$

5. Use substitution to solve each of the following pairs of simultaneous equations for x and y in terms of m and n.

a. $mx + y = n$
$y = mx$
b. $x + ny = m$
$y = nx$
c. $mx - y = n$
$y = nx$
d. $mx - ny = n$
$y = x$
e. $mx - ny = -m$
$x = y - n$
f. $mx + y = m$
$x = \dfrac{y + m}{n}$

6. Determine the values of a and b so that the pair of equations $ax + by = 17$ and $2ax - by = -11$ has a unique solution of $(-2, 3)$.

7. The earliest record of magic squares is from China in about 2200 BC. In magic squares the sums of the numbers of each row, column and diagonal are all equal to a magic number.

Let z be the magic number. By creating a set of equations, solve to find the magic number and the missing values in the magic square.

m	11	7
9		
n	5	10

Communicating, reasoning and problem solving

8. a. For the pair of simultaneous equations below, which of the equations is the logical choice to make x the subject of the equation?

$8x - 7y = 9$
$x + 2y = 4$

b. Use the substitution method to solve the system of equations. Show all your working.

9. A particular Chemistry book costs \$6 less than a particular Physics book. Two such Chemistry books and three such Physics books cost a total of \$123. Construct two simultaneous equations and solve them using the substitution method. Show your working.

10. Use the substitution method to solve the following.

$2x + y - 9 = 0$

$4x + 5y + 3 = 0$

11. Use the substitution method to solve the following.

$\frac{y - x}{2} - \frac{x + y}{3} = \frac{1}{6}$

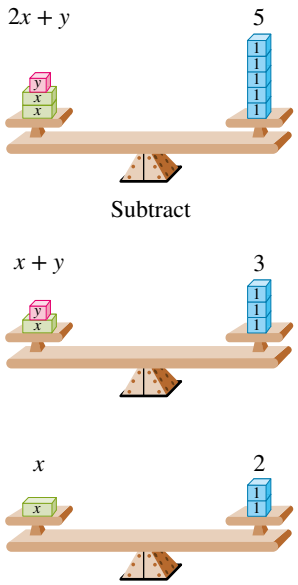
$\frac{x}{5} + \frac{y}{2} = \frac{1}{2}$

12. When would you choose the substitution method in solving simultaneous equations?

9.4 Solving simultaneous linear equations using elimination [Stage 5.2]

9.4.1 The elimination method

- The elimination method is an algebraic method to solve simultaneous equations without graphing.
- If two balanced equations contain the same variables, the equations can be added or subtracted to eliminate one of the variables. For example, the equations $2x + y = 5$ and $x + y = 3$ are shown on balance scales.



If the left-hand side of the second equation is subtracted from the left-hand side of the first equation, and the right-hand side of the second equation is subtracted from the right-hand side of the first equation, the variable y is eliminated, leaving $x = 2$.

Another way to represent this situation is:

$$\begin{array}{r} 2x + y = 5 \\ -(x + y = 3) \\ \hline x \quad\quad = 2 \end{array}$$

In this example, the variable is eliminated by subtraction to reveal the value of x. The value of y can then be calculated by substituting $x = 2$ into either equation.

$$2(2) + y = 5 \Rightarrow y = 1$$

WORKED EXAMPLE 6

Solve the following pair of simultaneous equations using the elimination method.

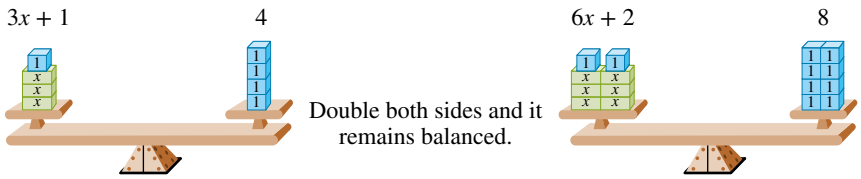
$-2x - 3y = -9$

$2x + y = 7$

THINK	WRITE
1 Write the equations, one under the other, and number them.	$-2x - 3y = -9$ [1] $2x + y = 7$ [2]
2 Look for an addition or subtraction that will eliminate either x or y. *Note:* Adding equations [1] and [2] in order will eliminate x.	[1] + [2]: $-2x - 3y + (2x + y) = -9 + 7$ $-2x - 3y + 2x + y = -2$ $-2y = -2$
3 Solve for y by dividing both sides of the equation by -2.	$y = 1$

4 Substitute the value of y into equation [2]. *Note:* $y = 1$ may be substituted into either equation.	Substituting $y = 1$ into [2]: $2x + 1 = 7$
5 Solve for x. **i** Subtract 1 from both sides of the equation. **ii** Divide both sides of the equation by 2.	$2x = 6$ $x = 3$
6 Write the solution.	Solution: $x = 3$, $y = 1$ or $(3, 1)$
7 Check the solution by substituting $(3, 1)$ into equation [1], since equation [2] was used to find the value of x.	Check: Substitute $x = 3$, $y = 1$ into $-2x - 3y = -9$. LHS $= -2(3) - 3(1)$ $= -6 - 3$ $= -9$ RHS $= -9$ LHS = RHS, so the solution is correct.

- If a variable is not eliminated when the equations are simply added or subtracted, it may be necessary to multiply one or both equations by some number or numbers so that when the equations are added, one of the variables is then eliminated.
- If two equal quantities are multiplied by the same number, the results remain equal.



WORKED EXAMPLE 7

Solve the following pair of simultaneous equations using the elimination method.
$x - 5y = -17$
$2x + 3y = 5$

THINK	WRITE
1 Write the equations, one under the other and number them.	$x - 5y = -17$ [1] $2x + 3y = 5$ [2]
2 Look for a single multiplication that will create the same coefficient of either x or y. Multiply equation [1] by 2 and call the new equation [3].	[1] $\times$ 2: $2x - 10y = -34$ [3]
3 Subtract equation [2] from [3] in order to eliminate x.	[3] $-$ [2]: $2x - 10y - (2x + 3y) = -34 - 5$ $2x - 10y - 2x - 3y = -39$ $-13y = -39$
4 Solve for y by dividing both sides of the equation by 213.	$y = 3$
5 Substitute the value of y into equation [2].	Substituting $y = 3$ into [2]: $2x + 3(3) = 5$ $2x + 9 = 5$

6 Solve for x. **i** Subtract 9 from both sides of the equation. **ii** Divide both sides of the equation by 2.	$2x = -4$ $x = -2$
7 Write the solution.	Solution: $x = -2$, $y = 3$ or $(-2, 3)$
8 Check the solution by substituting into equation [1].	Check: Substitute $x = -2$, $y = 3$ into $x - 5y = -17$. LHS $= (-2) - 5(3)$ $= -2 - 15$ $= -17$ RHS $= -17$ LHS = RHS, so the solution is correct.

Note: In this example, equation [1] could have been multiplied by -2 (instead of by 2), then the two equations added (instead of subtracted) to eliminate x.

WORKED EXAMPLE 8

Solve the following pair of simultaneous equations using the elimination method.
$6x + 5y = 3$
$5x + 4y = 2$

THINK	WRITE
1 Write the equations, one under the other and number them.	$6x + 5y = 3$ [1] $5x + 4y = 2$ [2]
2 Decide which variable to eliminate, say y. Multiply equation [1] by 4 and call the new equation [3]. Multiply equation [2] by 5 and call the new equation [4].	Eliminate y. (1) $\times$ 4: $24x + 20y = 12$ [3] (2) $\times$ 5: $25x + 20y = 10$ [4]
3 Subtract equation [3] from [4] in order to eliminate y.	[4] − [3]: $25x + 20y - (24x + 20y) = 10 - 12$ $25x + 20y - 24x - 20y = -2$ $x = -2$
4 Substitute the value of x into equation [1].	Substituting $x = -2$ into [1]: $6(-2) + 5y = 3$ $-12 + 5y = 3$
5 Solve for y. **i** Add 12 to both sides of the equation. **ii** Divide both sides of the equation by 5.	$5y = 15$ $y = 3$
6 Write your answer.	Solution $x = -2$, $y = 3$ or $(-2, 3)$
7 Check the answer by substituting the solution into equation [2].	Check: Substitute $x = -2$, $y = 3$ into $5x + 4y = 2$. LHS $= 5(-2) + 4(3)$ $= -10 + 12$ $= 2$ RHS $= 2$ LHS = RHS, so the solution is correct.

Note: Equation [1] could have been multiplied by -4 (instead of by 4), then the two equations added (instead of subtracted) to eliminate y.

DISCUSSION

Do you prefer to use one of the methods for solving simultaneous equations over the other? If so, why do you think that is?

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Interactivity: Solving simultaneous equations using elimination (int-6127)

Interactivity: Simultaneous linear equations (int-2780)

eLesson: Solving simultaneous equations using elimination (eles-1931)

Digital doc: WorkSHEET Simultaneous equations I (doc-13851)

Exercise 9.4 Solving simultaneous linear equations using elimination

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1, 2, 3a–c, 4a–c, 5a–c, 6, 7	Questions: 1, 2, 3a–d, 4a–e, 5a–d, 7–10	Questions: 1, 2, 3d–f, 4e–i, 5c–f, 6–12

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE6 Solve the following pairs of simultaneous equations by adding equations to eliminate either x or y.

a. $x + 2y = 5$
$-x + 4y = 1$

b. $5x + 4y = 2$
$5x - 4y = -22$

c. $-2x + y = 10$
$2x + 3y = 14$

2. Solve the following pairs of equations by subtracting equations to eliminate either x or y.

a. $3x + 2y = 13$
$5x + 2y = 23$

b. $2x - 5y = -11$
$2x + y = 7$

c. $-3x - y = 8$
$-3x + 4y = 13$

3. Solve each of the following pairs of simultaneous equations using the elimination method.

a. $6x - 5y = -43$
$6x - y = -23$

b. $x - 4y = 27$
$3x - 4y = 17$

c. $-4x + y = -10$
$4x - 3y = 14$

d. $-5x + 3y = 3$
$-5x + y = -4$

e. $5x - 5y = 1$
$2x - 5y = -5$

f. $4x - 3y - 1 = 0$
$4x + 7y - 11 = 0$

4. WE7 Solve the following pairs of simultaneous equations.

a. $6x + y = 9$
$-3x + 2y = 3$

b. $x + 3y = 14$
$3x + y = 10$

c. $5x + y = 27$
$4x + 3y = 26$

d. $-6x + 5y = -14$
$-2x + y = -6$

e. $2x + 5y = 14$
$3x + y = -5$

f. $-3x + 2y = 6$
$x + 4y = -9$

g. $3x - 5y = 7$
$x + y = -11$

h. $2x + 3y = 9$
$4x + y = -7$

i. $-x + 5y = 7$
$5x + 5y = 19$

5. WE8 Solve the following pairs of simultaneous equations.

a. $-4x + 5y = -9$
$2x + 3y = 21$

b. $2x + 5y = -6$
$3x + 2y = 2$

c. $2x - 2y = -4$
$5x + 4y = 17$

d. $2x - 3y = 6$
$4x - 5y = 9$

e. $\frac{x}{2} + \frac{y}{3} = 2$
$\frac{x}{4} + \frac{y}{3} = 4$

f. $\frac{x}{3} + \frac{y}{2} = \frac{3}{2}$
$\frac{x}{2} + \frac{y}{5} = -\frac{1}{2}$

6. Solve the following simultaneous equations using an appropriate method. Check your answer using technology.

a. $7x + 3y = 16$
$y = 4x - 1$

b. $2x + y = 8$
$4x + 3y = 16$

c. $-3x + 2y = 19$
$4x + 5y = 13$

d. $-3x + 7y = 9$
$4x - 3y = 7$

e. $-4x + 5y = -7$
$x = 23 - 3y$

f. $y = -x$
$y = -\frac{2}{5}x - \frac{1}{5}$

Communicating, reasoning and problem solving

7. Ann, Beth and Celine wanted to weigh themselves, but the scales they had were broken and would only give readings over 100 kg. They decided to weigh themselves in pairs and calculate their weights from the results.

- Ann and Beth weighed 119 kg
- Beth and Celine weighed 112 kg
- Celine and Ann weighed 115 kg

How much did each of the girls weigh? Show your working.

8. a. For the general case $ax + by = e$ [1]
$cx + dy = f$ [2]

y can be found by eliminating x.

i. Multiply equation [1] by c to create equation 3.

ii. Multiply equation [2] by a to create equation 4.

iii. Use the elimination method to find a general solution for y.

b. Use a similar process to that outlined above to find a general solution for x.

c. Use the general solution for x and y to solve each of the following.

i. $2x + 5y = 7$
$7x + 2y = 24$

ii. $3x - 5y = 4$
$x + 3y = 5$

Choose another method to check that your solutions are correct in each part.

d. For y to exist, it is necessary to state that $bc - ad \neq 0$. Why?

e. Is there a necessary condition for x to exist? Explain.

9. Use the method of elimination to solve:

$$\frac{x-4}{3} + y = -2$$
$$\frac{2y-1}{7} + x = 6$$

10. Use an appropriate method to solve:

$2x + 3y + 3z = -1$
$3x - 2y + z = 0$
$z + 2y = 0$

11. How does eliminating one variable help to solve simultaneous equations?

12. If $x + y = 17$, $y + z = 15$ and $x + z = 14$, what is the value of z?

9.5 Applications of simultaneous linear equations [Stage 5.2]

- There are many practical applications of simultaneous equations, some examples of which are shown below.
- When solving practical problems, the following steps can be useful.
  - Define the unknown quantities using appropriate pronumerals.
  - Use the information given in the problem to form two equations in terms of these pronumerals.
  - Solve these equations using an appropriate method.
  - Write the solution in words.
  - Check the solution.

WORKED EXAMPLE 9

Ashley received better results for his Mathematics test than for his English test. If the sum of the two marks is 164 and the difference is 22, calculate the mark he received for each subject.

THINK	WRITE
1 Define the two variables.	Let x = the Mathematics mark. Let y = the English mark.
2 Formulate two equations from the information given and number them. The sum of the two marks is $x + y$. The difference of the two marks is $x - y$.	$x + y = 164$ [1] $x - y = 22$ [2]
3 Use the elimination method by adding equations [1] and [2] to eliminate y.	[1] + [2]: $2x = 186$
4 Solve for x by dividing both sides of the equation by 2.	$x = 93$
5 Substitute the value of x into equation [1].	Substituting $x = 93$ into [1]: $x + y = 164$ $93 + y = 164$
6 Solve for y by subtracting 93 from both sides of the equation.	$y = 71$
7 Write the solution.	Solution: Mathematics mark (x) = 93 English mark (y) = 71
8 Check the solution by substituting $x = 93$ and $y = 71$ into equation [1].	Check: Substitute $x = 93$, $y = 71$ into $x + y = 164$. LHS = 93 + 71 RHS = 164 = 164 As LHS = RHS, the solution is correct.

WORKED EXAMPLE 10

To finish a project, Genevieve buys a total of 25 nuts and bolts from a hardware store. If each nut costs 12 cents, each bolt costs 25 cents and the total purchase price is $4.30, how many nuts and how many bolts does Genevieve buy?

THINK	WRITE
1 Define the two variables.	Let x = the number of nuts. Let y = the number of bolts.
2 Formulate two equations from the information given and number them. *Note:* The total number of nuts and bolts is 25. Each nut cost 12 cents, each bolt cost 25 cents and the total cost is 430 cents ($4.30).	$x + y = 25$ [1] $12x + 25y = 430$ [2]
3 Solve simultaneously using the substitution method, since equation [1] is easy to rearrange.	
4 Rearrange equation [1] to make x the subject by subtracting y from both sides of equation [1].	Rearrange equation [1]: $x + y = 25$ $x = 25 - y$
5 Substitute the expression $(25 - y)$ for x into equation [2].	Substituting $(25 - y)$ into [2]: $12(25 - y) + 25y = 430$
6 Solve for y.	$300 - 12y + 25y = 430$ $300 + 13y = 430$ $13y + 300 = 430$ $13y = 130$ $y = 10$
7 Substitute the value of y into the rearranged equation $x = 25 - y$ from step 4.	Substituting $y = 10$ into $x = 25 - y$: $x = 25 - 10$ $x = 15$
8 Write the solution.	Solution: The number of nuts $(x) = 15$. The number of bolts $(y) = 10$.
9 Check the solution by substituting $x = 15$ and $y = 10$ into equation [1].	Check: Substitute $x = 15$, $y = 10$ into $x + y = 25$. LHS $= 15 + 10$ RHS $= 25$<br$= 25$ As LHS = RHS, the solution is correct.

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Digital doc: WorkSHEET Simultaneous equations II (doc-13852)

Exercise 9.5 Applications of simultaneous linear equations

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–3, 6, 8, 12, 14, 16, 19	Questions: 1, 2, 4, 7, 9, 11, 14, 16, 17, 19	Questions: 1, 2, 5, 7, 9, 10, 13, 15–23

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE9 Rick received better results for his Mathematics test than for his English test. If the sum of his two marks is 163 and the difference is 31, find the mark for each subject.
2. WE10 Rachael buys 30 nuts and bolts to finish a project. If each nut costs 10 cents, each bolt costs 20 cents and the total purchase price is $4.20, how many nuts and how many bolts does she buy?
3. Find two numbers whose difference is 5 and whose sum is 11.
4. The difference between two numbers is 2. If three times the larger number minus twice the smaller number is 13, find the two numbers.
5. One number is 9 less than three times a second number. If the first number plus twice the second number is 16, find the two numbers.
6. A rectangular house has a perimeter of 40 metres and the length is 4 metres more than the width. What are the dimensions of the house?
7. Mike has 5 lemons and 3 oranges in his shopping basket. The cost of the fruit is $3.50. Voula, with 2 lemons and 4 oranges, pays $2.10 for her fruit. How much does each type of fruit cost?

8. A surveyor measuring the dimensions of a block of land finds that the length of the block is three times the width. If the perimeter is 160 metres, what are the dimensions of the block?
9. Julie has $3.10 in change in her pocket. If she has only 50 cent and 20 cent pieces and the total number of coins is 11, how many coins of each type does she have?
10. Mr Yang's son has a total of twenty-one $1 and $2 coins in his moneybox. When he counts his money, he finds that its total value is $30. How many coins of each type does he have?

11. If three Magnums and two Paddlepops cost $8.70 and the difference in price between a Magnum and a Paddlepop is 90 cents, how much does each type of ice-cream cost?

12. If one Redskin and 4 Golden Roughs cost $1.65, whereas 2 Redskins and 3 Golden Roughs cost $1.55, how much does each type of sweet cost?

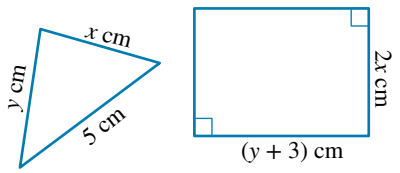
13. A catering firm charges a fixed cost for overheads and a price per person. It is known that a party for 20 people costs $557, whereas a party for 35 people costs $909.50. What is the fixed cost and the cost per person charged by the company?

14. The difference between Sally's PE mark and her Science mark is 12, and the sum of the marks is 154. If the PE mark is the higher mark, what did Sally get for each subject?

15. Mozza's Cheese Supplies sells six Mozzarella cheeses and eight Swiss cheeses to Munga's deli for $83.60, and four Mozzarella cheeses and four Swiss cheeses to Mina's deli for $48. How much does each type of cheese cost?

Communicating, reasoning and problem solving

16. If the perimeter of the triangle in the diagram is 12 cm and the length of the rectangle is 1 cm more than the width, find the value of x and y. Show your working.



17. Mr and Mrs Waugh want to use a caterer for a birthday party for their twin sons. The manager says the cost for a family of four would be $160. However, the sons want to invite 8 friends, making 12 people in all. The cost for this would be $360. If the total cost in each case is made up of the same cost per person and the same fixed cost, find the cost per person and the fixed cost. Show your working.

18. Joel needs to buy some blank DVDs and zip disks to back up a large amount of data that has been generated by an accounting firm. He buys 6 DVDs and 3 zip disks for $96. He later realises these are not sufficient and so buys another 5 DVDs and 4 zip disks for $116. How much did each DVD and each zip disk cost? (Assume the same rate per item was charged for each visit.) Show your working.

19. At the football, hot chips are twice as popular as meat pies and three times as popular as hot dogs. Over the period of half an hour during half time, a fast-food outlet serves 121 people who each bought one item. How many serves of each of the foods were sold during this half-hour period?

20. Three jet skis in a 300-kilometre handicap race leave at two hour intervals. Jet ski 1 leaves first and has an average speed of 25 kilometres per hour for the entire race. Jet ski 2 leaves two hours later and has an average speed of 30 kilometres per hour for the entire race. Jet ski 3 leaves last, two hours after jet ski 2, and has an average speed of 40 kilometres per hour for the entire race.

a. Sketch a graph to show each jet ski's journey on the one set of axes.

b. Determine who wins the race.

c. Check your findings algebraically and describe what happened to each jet ski during the course of the race.

21. How do you decide which method to use when solving problems using simultaneous linear equations?

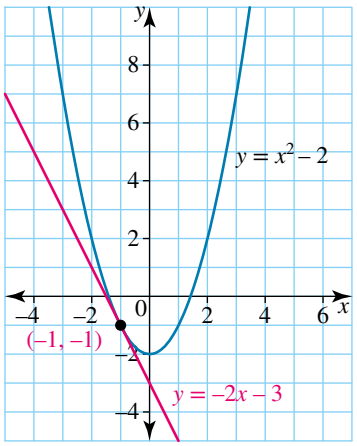
22. At a fun park, the cost of a rollercoaster ride and a Ferris wheel ride is $10. The cost of a Gravitron ride and a Ferris wheel ride is $12. The cost of a rollercoaster ride and a Gravitron ride is $14. What is the cost of each ride?

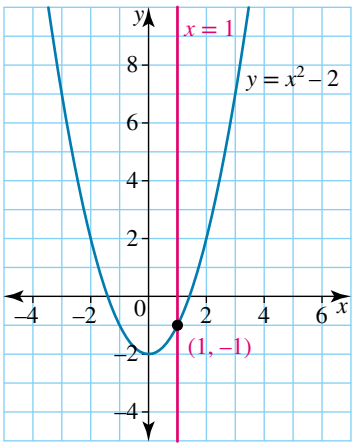
23. A number has five digits. The digit 6 is three places to the right of the digit 9. The digit 4 is somewhere to the left of digit 6 and to the right of digit 2. The digit 8 is three places to the left of the digit 4. What is the number?

9.6 Solving simultaneous linear and non-linear equations [Stage 5.3]

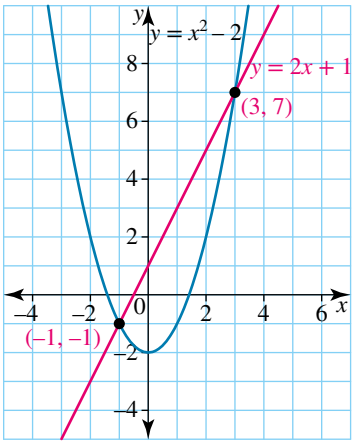
9.6.1 Solving simultaneous linear and quadratic equations

- The graph of a quadratic function is called a parabola.
- A parabola and a straight line may:
  - intersect at only one point, as shown

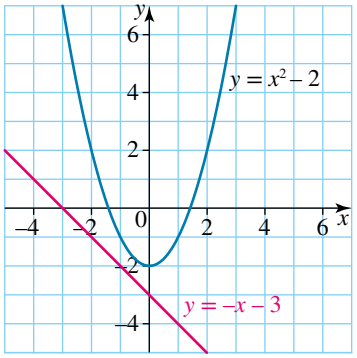




  - intersect at two points, as shown



– never intersect, as shown.



WORKED EXAMPLE 11

Find the points of intersection of $y = x^2 + x - 6$ and $y = 2x - 4$:
a algebraically **b graphically.**

THINK	WRITE/DRAW
a 1 Number the equations. Equate [1] and [2].	**a** $y = x^2 + x - 6$ [1] $y = 2x - 4$ [2] $x^2 + x - 6 = 2x - 4$
2 Collect all the terms on one side and simplify.	$x^2 + x - 6 - 2x + 4 = 2x - 4 - 2x + 4$ $x^2 + x - 6 - 2x + 4 = 0$ $x^2 - x - 2 = 0$
3 Factorise and solve the quadratic equation, using the Null Factor Law.	$(x - 2)(x + 1) = 0$ $x - 2 = 0$ or $x + 1 = 0$ $x = 2$ $x = -1$
4 Identify the y-coordinate for each point of intersection by substituting each x-value into one of the equations.	When $x = 2$, $y = 2(2) - 4$ $= 4 - 4$ $= 0$ Intersection point $(2, 0)$ When $x = -1$, $y = 2(-1) - 4$ $= -2 - 4$ $= -6$
5 Write the solution.	Intersection point $(-1, -6)$
b 1 To sketch the graph of $y = x^2 + x - 6$, find the x- and y-intercepts and the turning point.	**b** x-intercepts: $y = 0$ $0 = x^2 + x - 6$ $0 = (x + 3)(x - 2)$ $x = -3,\ x = 2$ The x-intercepts are $(-3, 0)$ and $(2, 0)$. y-intercept: $x = 0$ $y = -6$ The y-intercept is $(0, -6)$. x-value of the turning point: $\frac{-3 + 2}{2} = -0.5$ $y = (-0.5)^2 + (-0.5) - 6$ $y = -6.25$ The turning point is $(-0.5, -6.25)$.

2 To sketch the graph of $y = 2x - 4$, find the x- and y-intercepts.

x-intercept: $y = 0$

$0 = 2x - 4$

$x = 2$

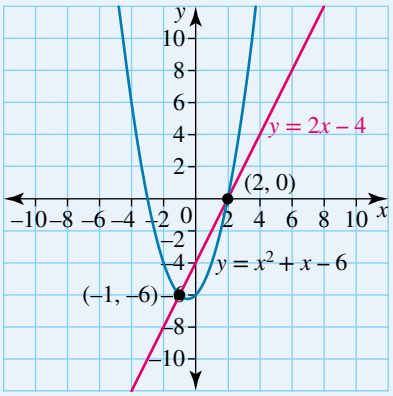
The x-intercept is $(2, 0)$.

y-intercept: $x = 0$

$y = -4$

The y-intercept is $(0, -4)$.

3 On the same set of axes, sketch the graphs of $y = x^2 + x - 6$ and $y = 2x - 4$, labelling both.

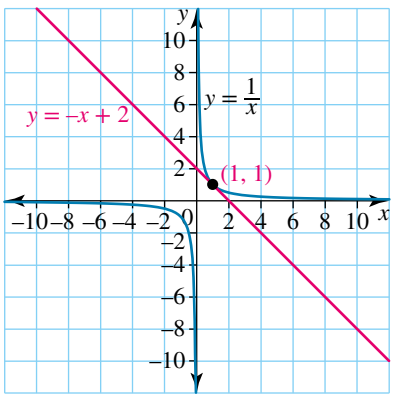


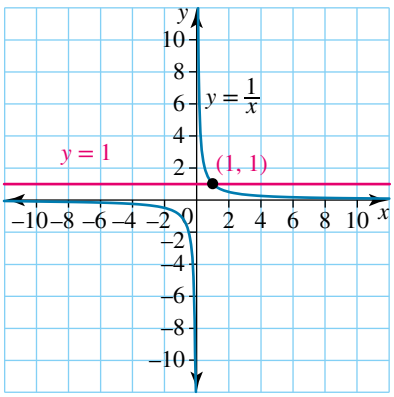
4 On the graph, locate the points of intersection and write the solutions.

The points of intersection are $(2, 0)$ and $(-1, -6)$.

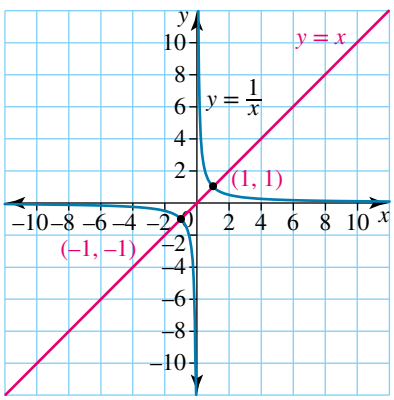
9.6.2 Solving simultaneous linear and hyperbolic equations

- A hyperbola and a straight line may:
  - intersect at only one point, as shown. In the first case, the line is a tangent to the curve.

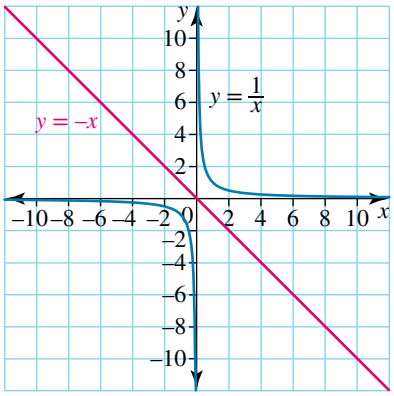




  - intersect at two points, as shown



– never intersect, as shown



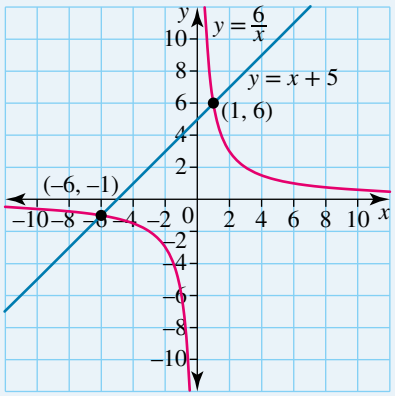
WORKED EXAMPLE 12

Find the point(s) of intersection between $y = x + 5$ and $y = \frac{6}{x}$:

a algebraically **b graphically.**

THINK	WRITE/DRAW
a 1 Number the equations. Equate [1] and [2].	**a** $y = x + 5$ [1] $y = \frac{6}{x}$ [2] $x + 5 = \frac{6}{x}$ $x(x + 5) = 6$
2 Collect all the terms on one side, factorise and simplify.	$x^2 + 5x - 6 = 0$ $(x + 6)(x - 1) = 0$ $x = -6,\ x = 1$
3 To find the y-coordinates of the points of intersection, substitute the values of x into [1].	$x = -6$ $y = -6 + 5$ $y = -1$ $x = 1$ $y = 1 + 5$ $y = 6$
4 Write the solutions.	The points of intersection are $(-6, -1)$ and $(1, 6)$.
b 1 To sketch the graph of $y = \frac{6}{x}$, draw a table of values.	**b** x: -6, -5, -4, -3, -2, -1, 0, 1, 2 y: -1, $-1\frac{1}{5}$, $-1\frac{1}{2}$, -2, -3, -6, Undef., 6, 3
2 To sketch the graph of $y = x + 5$, find the x- and y-intercepts.	x-intercept: $y = 0$ $0 = x + 5$ $x = -5$ The x-intercept is $(-5, 0)$. y-intercept: $x = 0$ $y = 5$ The iy-intercept is $(0, 5)$.

3 On the same set of axes, sketch the graphs of $y = x + 5$ and $y = \frac{6}{x}$, labelling both.

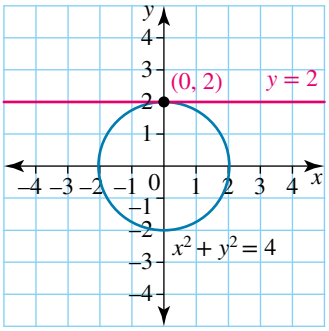


4 On the graph, locate the points of intersection and write the solutions.

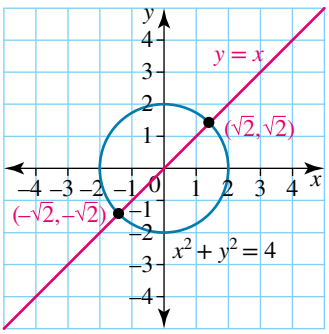
The points of intersection are (1, 6) and (−6, −1).

9.6.3 Solving simultaneous linear equations and circles

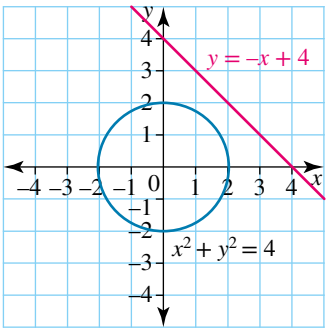
- A circle and a straight line may:
  - intersect at only one point, as shown. In this case, the line is a tangent to the curve.



  - intersect at two points, as shown



  - never intersect, as shown



learnON RESOURCES — ONLINE ONLY

Interactivity: Solving simultaneous linear and non-linear equations (int-6128)

Exercise 9.6 Solving simultaneous linear and non-linear equations

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1, 2a–c, 4, 5, 7a–b, 8a–b, 9, 12	Questions: 1, 2, 3c–e, 4–6, 7b–d, 8b–d, 9, 12	Questions: 1, 2, 3d–f, 4–6, 7c–d, 8c–d, 9–12

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. Describe how a parabola and straight line may intersect. Use diagrams to illustrate your explanation.

2. WE11 Find the points of intersection of the following:

i. using algebra

ii. algebraically using a calculator

iii. graphically using a calculator.

a. $y = x^2 + 5x + 4$ and $y = -x - 1$

b. $y = -x^2 + 2x + 3$ and $y = -2x + 7$

c. $y = -x^2 + 2x + 3$ and $y = -6$

3. Find the points of intersection of the following.

a. $y = -x^2 + 2x + 3$ and $y = 3x - 8$

b. $y = -(x - 1)^2 + 2$ and $y = x - 1$

c. $y = x^2 + 3x - 7$ and $y = 4x + 2$

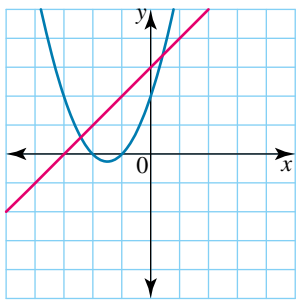
d. $y = 6 - x^2$ and $y = 4$

e. $y = 4 + x - x^2$ and $y = \dfrac{3 - x}{2}$

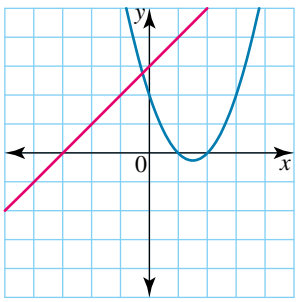
f. $x = 3$ and $y = 2x^2 + 7x - 2$

4. MC Which of the following graphs shows the parabola $y = x^2 + 3x + 2$, $x \in R$, and the straight line $y = x + 3$?

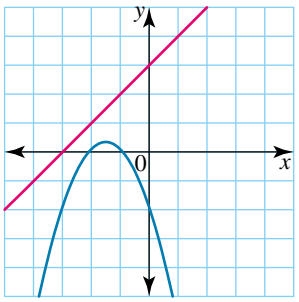
A.



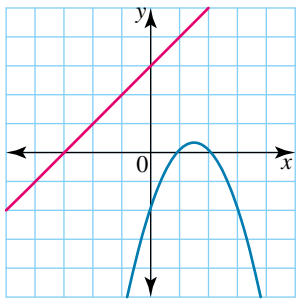
B.



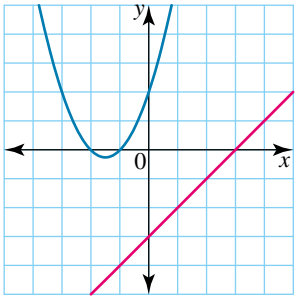
C.



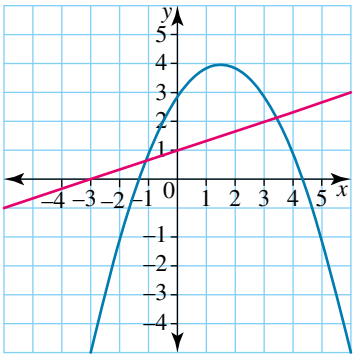
D.



E.



5. MC The diagram below could show which of the following?



A. $y = 0.5(x + 1.5)^2 + 4$ and $y = -\frac{1}{3}x + 1$

B. $y = -0.5(x + 1.5)^2 - 4$ and $y = -\frac{1}{3}x + 1$

C. $y = -0.5(x - 1.5)^2 + 4$ and $y = \frac{1}{3}x + 1$

D. $y = 0.5(x - 1.5)^2 + 4$ and $y = -\frac{1}{3}x + 1$

E. $y = 0.5(x - 1.5)^2 + 4$ and $y = -\frac{1}{3}x + 1$

6. Determine whether the following graphs intersect.

a. $y = -x^2 + 3x + 4$ and $y = x - 4$

b. $y = -x^2 + 3x + 4$ and $y = 2x + 5$

c. $y = -(x + 1)^2 + 3$ and $y = -4x - 1$

d. $y = (x - 1)^2 + 5$ and $y = -4x - 1$

7. WE12 Find the point(s) of intersection between the following.

a. $y = x$
$y = \frac{1}{x}$

b. $y = x - 2$
$y = \frac{1}{x}$

c. $y = 3x$
$y = \frac{5}{x}$

d. $y = \frac{6}{x}$
$y = \frac{x}{2} + 2$

8. Find the point(s) of intersection between the following.

a. $y = 3x$
$x^2 + y^2 = 10$

b. $x^2 + y^2 = 25$
$3x + 4y = 0$

c. $x^2 + y^2 = 50$
$y = 5 - 2x$

d. $x^2 + y^2 = 9$
$y = 2 - x$

Communicating, reasoning and problem solving

9. Show that there is at least one point of intersection between the parabola $y = -2(x + 1)^2 - 5$, where $y = f(x)$, and the straight line $y = mx - 7$, where $y = f(x)$.

10. a. Find the point(s) of intersection between the circle $x^2 + y^2 = 50$ and the linear equation $y = 2x - 5$.

b. Confirm your solution to part **a** by plotting the equation of the circle and the linear equation on the same graph.

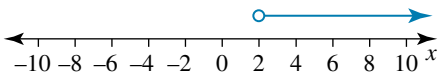
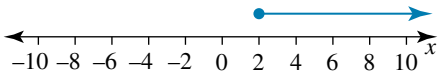
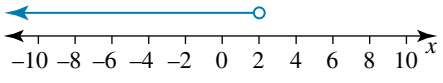
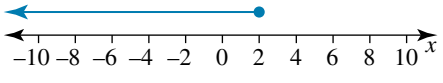
11. The sum of two positive numbers is 21. Twice the square of the larger number minus three times the square of the smaller number is 45. Find the value of the two numbers.

12. What does it mean if a straight line touches a curve only once?

9.7 Solving linear inequalities [Stage 5.2]

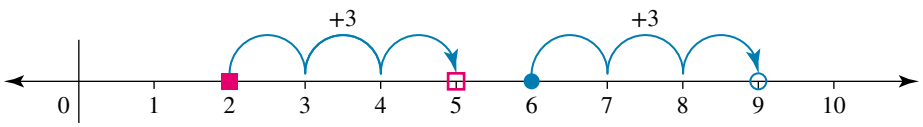
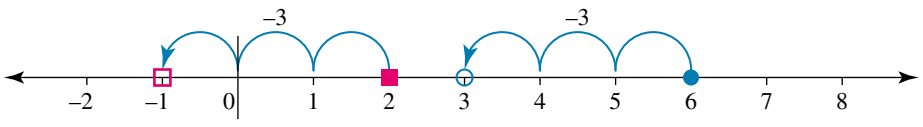
9.7.1 Inequalities between two expressions

- An equation is a statement of *equality* such as $x = 2$; an **inequation** is a statement of **inequality** between two expressions such as $x < 2$ (x is less than 2).
- The solution to a linear equation is a single point on a number line, but the solution to an inequation is a portion of the number line. That is, there are an infinite number of solutions to an inequality.
- The following table shows examples of four types of simple inequalities and their corresponding representation on a number line.
- Note that an open circle placed over the 2 indicates that 2 is not included; that is, 2 does not satisfy the inequality. A closed or solid circle indicates that 2 is included; that is, it does satisfy the inequality.

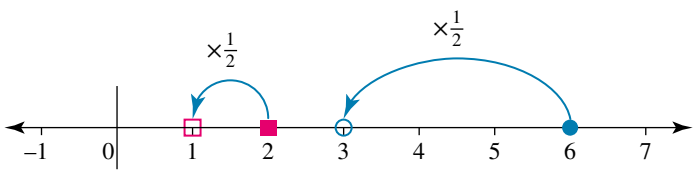
Mathematical statement	English statement	Number line diagram
$x > 2$	x is greater than 2	
$x \geq 2$	x is greater than or equal to 2	
$x < 2$	x is less than 2	
$x \leq 2$	x is less than or equal to 2	

9.7.2 Solving inequalities

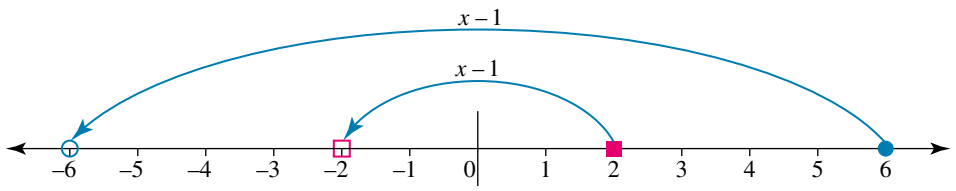
- The following things may be done to both sides of an inequality without affecting its truth.
  – A number can be added or subtracted from both sides of the inequality.

Adding or subtracting a number:		
e.g. $6 > 2$	Add 3 to both sides:	
$9 > 5$	(True)	
e.g. $6 \geq 2$	Subtract 3 from both sides:	
$3 \geq -1$	(True)	
Adding or subtracting moves both numbers the same distance along the number line.		

– A number can be multiplied or divided by a positive number.

Multiplying or dividing by a positive number:		
e.g. $6 > 2$	Multiply both sides by $\frac{1}{2}$:	
$3 > 1$	(True)	
The distance between the numbers has changed, but their relative position has not.		

– Care must be taken when multiplying or dividing by a negative number.

Multiplying or dividing by a negative number:		
e.g. $6 > 2$	Multiply both sides by -1:	
$-6 < -2$	(False)	
Multiplying or dividing by a negative number reflects numbers about $x = 0$. Their relative positions are reversed.		

- **When solving inequalities, if both sides are multiplied or divided by a negative number, then the inequality sign must be reversed.**
  For example, $6 > 2$ implies that $-6 < -2$.

DISCUSSION

Why must the inequality sign be reversed when multiplying or dividing by a negative number? Explain your reasoning to a partner using some numerical examples.

WORKED EXAMPLE 13

Solve each of the following linear inequalities and show the solution on a number line.
a $4x - 1 < -2$ **b** $6x - 7 \geq 3x + 5$

THINK

a 1 Write the inequality.

2 Add 1 to both sides of the inequality.

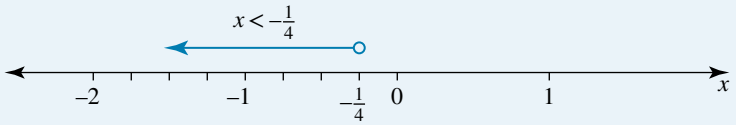
3 Obtain x by dividing both sides of the inequality by 4.

WRITE/DRAW

a $4x - 1 < -2$

$4x - 1 + 1 < -2 + 1$
$4x < -1$

$\frac{4x}{4} < -\frac{1}{4}$
$x < -\frac{1}{4}$



b 1 Write the inequality.

2 Subtract $3x$ from both sides of the inequality.

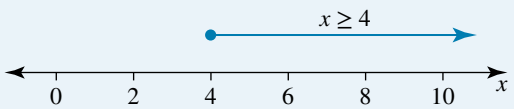
3 Add 7 to both sides of the inequality.

4 Obtain x by dividing both sides of the inequality by 3.

b $6x - 7 \geq 3x + 5$

$6x - 7 - 3x \geq 3x + 5 - 3x$
$3x - 7 \geq 5$

$3x - 7 + 7 \geq 5 + 7$
$3x \geq 12$
$\frac{3x}{3} \geq \frac{12}{3}$
$x \geq 4$



WORKED EXAMPLE 14

Solve each of the following linear inequalities.
a $-3m + 5 < -7$ **b** $5(x - 2) \geq 7(x + 3)$

THINK	WRITE
a 1 Write the inequality.	**a** $-3m + 5 < -7$
2 Subtract 5 from both sides of the inequality. (No change to the inequality sign is required.)	$-3m + 5 - 5 < -7 - 5$ $-3m < -12$
3 Obtain m by dividing both sides of the inequation by -3. Reverse the inequality sign, since you are dividing by a negative number.	$\frac{-3m}{-3} > \frac{-12}{-3}$ $m > 4$
b 1 Write the inequality.	**b** $5(x - 2) \geq 7(x + 3)$
2 Expand both brackets.	$5x - 10 \geq 7x + 21$
3 Subtract $7x$ from both sides of the inequality.	$5x - 10 - 7x \geq 7x + 21 - 7x$ $-2x - 10 \geq 21$
4 Add 10 to both sides of the inequation.	$-2x - 10 + 10 \geq 21 + 10$ $-2x \geq 31$
5 Divide both sides of the inequality by -2. Since we need to divide by a negative number, reverse the direction of the inequality sign.	$\frac{-2x}{-2} \leq \frac{31}{-2}$ $x \leq \frac{-31}{2}$ $x \leq -15\frac{1}{2}$

learnON RESOURCES — ONLINE ONLY

Interactivity: Inequality on the number line (int-6129)
Digital doc: SkillSHEET Checking whether a given point makes the inequation a true statement (doc-5218)
Digital doc: SkillSHEET Writing equations from worded statements (doc-5219)

Exercise 9.7 Solving linear inequalities

Individual pathways

PRACTISE
Questions:
1a–f, 2, 3a–f, 4a–f, 5a–c, 6a–g, 7, 8a–c, 9a–f, 10, 13

CONSOLIDATE
Questions:
1e–l, 2, 3d–i, 4d–i, 5a–c, 6a–l, 7, 8a–c, 9a–i, 10, 12, 13

MASTER
Questions:
1j–l, 2g–l, 3g–l, 4d–l, 5d–f, 6f–o, 8d–f, 9–15

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE13a Solve each of the following inequalities and show the solution on a number line.

a. $x + 1 > 3$ **b.** $a + 2 > 1$ **c.** $y - 3 \geq 4$ **d.** $m - 1 \geq 3$

e. $p + 4 < 5$ **f.** $x + 2 < 9$ **g.** $m - 5 \leq 4$ **h.** $a - 2 \leq 5$

i. $x - 4 > -1$ **j.** $5 + m \geq 7$ **k.** $6 + q \geq 2$ **l.** $5 + a > -3$

2. Solve each of the following inequalities. Check your solutions by substitution.

a. $3m > 9$ **b.** $5p \leq 10$ **c.** $2a < 8$ **d.** $4x \geq 20$

e. $5p > -25$ **f.** $3x \leq -21$ **g.** $2m \geq -1$ **h.** $4b > -2$

i. $\frac{m}{3} > 6$ **j.** $\frac{x}{2} < 4$ **k.** $\frac{a}{7} \leq -2$ **l.** $\frac{m}{5} \geq 5$

3. WE13a Solve each of the following inequalities.

a. $2m + 3 < 12$ **b.** $3x + 4 \geq 13$ **c.** $5p - 9 > 11$ **d.** $4n - 1 \leq 7$

e. $2b - 6 < 4$ **f.** $8y - 2 > 14$ **g.** $10m + 4 \leq -6$ **h.** $2a + 5 \geq -5$

i. $3b + 2 < -11$ **j.** $6c + 7 \leq 1$ **k.** $4p - 2 > -10$ **l.** $3a - 7 \geq -28$

4. WE13b Solve each of the following inequalities.

a. $2m + 1 > m + 4$ **b.** $2a - 3 \geq a - 1$ **c.** $5a - 3 < a - 7$

d. $3a + 4 \leq a - 2$ **e.** $5x - 2 > 40 - 2x$ **f.** $7x - 5 \leq 11 - x$

g. $7b + 5 < 2b + 25$ **h.** $2(a + 4) > a + 13$ **i.** $3(m - 1) < m + 1$

j. $5(2m - 3) \leq 3m + 6$ **k.** $3(5b + 2) \leq -10 + 4b$ **l.** $5(3m + 1) \geq 2(m + 9)$

5. Solve each of the following inequalities.

a. $\frac{x + 1}{2} \leq 4$ **b.** $\frac{x - 2}{5} \geq -4$ **c.** $\frac{x + 7}{3} < -1$

d. $\frac{2x + 3}{4} > 6$ **e.** $\frac{3x - 1}{7} \geq 2$ **f.** $\frac{5x + 9}{6} < 0$

6. WE14 Solve each of the following inequalities.

a. $-2m > 4$ **b.** $-5p \leq 15$ **c.** $-2a \geq -10$ **d.** $-p - 3 \leq 2$

e. $10 - y \geq 13$ **f.** $14 - x < 7$ **g.** $1 - 6p > 1$ **h.** $2 - 10a \leq 0$

i. $2(3 - x) < 12$ **j.** $-4(a + 9) \geq 8$ **k.** $-15 \leq -3(2 + b)$ **l.** $2x - 3 > 5x + 6$

m. $k + 5 < 2k - 3$ **n.** $3(x - 4) < 5(x + 5)$ **o.** $7(a + 4) \geq 4(2a - 3)$

7. MC When solving the inequality $-2x > -7$, we need to:

A. change the sign to $\geq$ **B.** change the sign to $<$ **C.** change the sign to $=$

D. change the sign to $\leq$ **E.** keep the sign unchanged

8. Solve each of the following inequalities.

a. $\frac{2 - x}{3} > 1$ **b.** $\frac{5 - m}{4} \geq 2$ **c.** $\frac{-3 - x}{5} < -4$

d. $\frac{3 - 8a}{2} < -1$ **e.** $\frac{4 - 3m}{2} \leq 0$ **f.** $\frac{-2m + 6}{10} \leq 3$

9. Solve each of the following inequalities.

a. $3k > 6$ **b.** $-a - 7 < -2$ **c.** $5 - 3m \geq 0$

d. $x + 4 > 9$ **e.** $10 - y \leq 3$ **f.** $5 + 3d < -1$

g. $\frac{7p}{3} \geq -2$ **h.** $\frac{1 - x}{3} \leq 2$ **i.** $\frac{-4 - 2m}{5} > 0$

j. $5a - 2 < 4a + 7$ **k.** $6p + 2 \leq 7p - 1$ **l.** $2(3x + 1) > 2x - 16$

10. Write linear inequalities for the following statements, using x to represent the unknown. (Do not attempt to solve the equations.)

a. The product of 5 and a certain number is greater than 10.

b. When three is subtracted from a certain number the result is less than or equal to 5.

c. The sum of 7 and three times a certain number is less than 42.

Communicating, reasoning and problem solving

11. Given the positive numbers a, b, c and d and the variable x, there is the following relationship: $-c < ax + b < -d$.

a. Find the possible range of values of x if $a = 2$, $b = 3$, $c = 10$ and $d = 1$.

b. Rewrite the original relationship in terms of x only (x by itself between the $<$ signs), using a, b, c and d.

12. Two speed boats are racing along a section of Lake Quikalong. The speed limit along this section of the lake is 50 km/h. Ross is travelling 6 km/h faster than Steven, and the sum of the speeds at which they are travelling is greater than 100 km/h.

a. Write an inequation and solve it to describe all of the possible speeds that Steven could be travelling at.

b. At Steven's lowest possible speed, is he over the speed limit?

c. The water police issue a warning to Ross for exceeding the speed limit on the lake. Show that the police were justified in issuing a warning to Ross.

13. Mick the painter has fixed costs (e.g. insurance, equipment, etc) of $3400 per year. His running cost to travel to jobs is based on $0.75 per kilometre. Last year Mick had costs that were less than $16 000.

a. Write an inequality to show this information and solve it to find how many kilometres Mick travelled for the year.

b. Explain the information you have found.

14. I have $40 000 to invest. Part of this I intend to invest in a stable 5% simple interest account. The remainder will be invested in my friend's IT business. She has said that she will pay me 7.5% interest on any money I give to her. I am saving for a European trip, so I want the best return for my money. What is the least amount of money I should invest with my friend so that I receive at least $2500 interest per year from my investments?

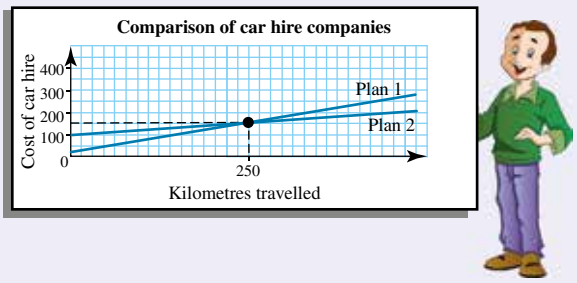
15. What is similar and different when solving linear inequations to linear equations?

9.8 Review

Investigation | Rich task

Documenting business expenses

In business, expenses can be represented graphically, so that relevant features are clearly visible. The graph shown compares the costs of hiring cars from two different car rental companies. Plan 1 will be cheaper if you are travelling distances less than 250 kilometres, but Plan 2 will be cheaper if you are travelling more than 250 kilometres. Both plans will cost the same if you are travelling exactly 250 kilometres.



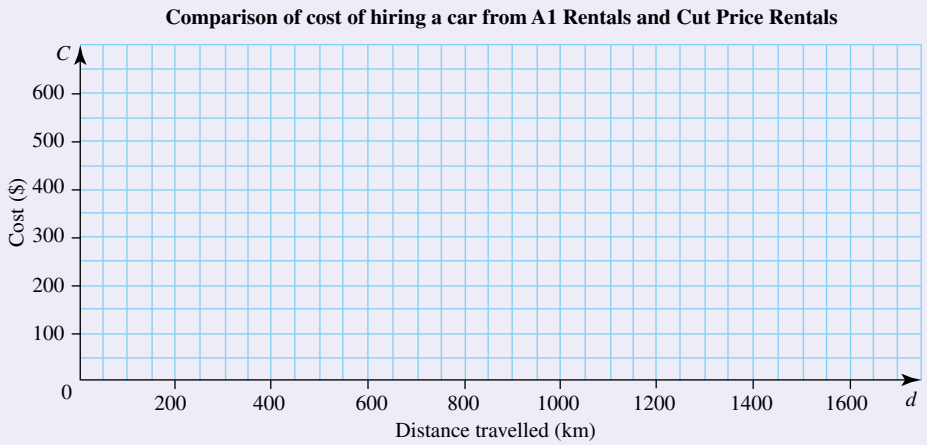
Jim works as a travelling sales representative. He needs to plan his next business trip to Port Hedland, which he anticipates will take him away from the office for 3 or 4 days. Due to other work commitments, he is not sure whether he can make the trip by the end of this month or early next month. He plans to fly to Port Hedland and use a hire car to travel when he arrives. Jim's boss has asked him to supply documentation detailing the anticipated costs for the hire car, based on the following quotes received.

A1 Rentals	\$35 per day plus 28c per kilometre of travel
Cut Price Rentals	\$28 per day plus 30c per kilometre of travel

Jim is aware that, although the Cut Price Rentals deal looks cheaper, it could work out more expensive in the long run, because of the higher cost per kilometre of travel. (Jim intends to travel a considerable distance by car.) Jim is advised by both rental companies that their daily hire charges are due to rise by \$2 per day from the first day of next month.

Assuming that Jim is able to travel this month and his trip will last 3 days, use the information given to answer questions 1 to 3.

1. Write equations to represent the costs of hiring a car from A1 Rentals and Cut Price Rentals. Use C to represent the cost (in dollars) and d to represent the distance travelled (in kilometres).
2. Copy the set of axes shown. Plot the two equations from question **1** on the axes to show how the costs compare over 1500 km.



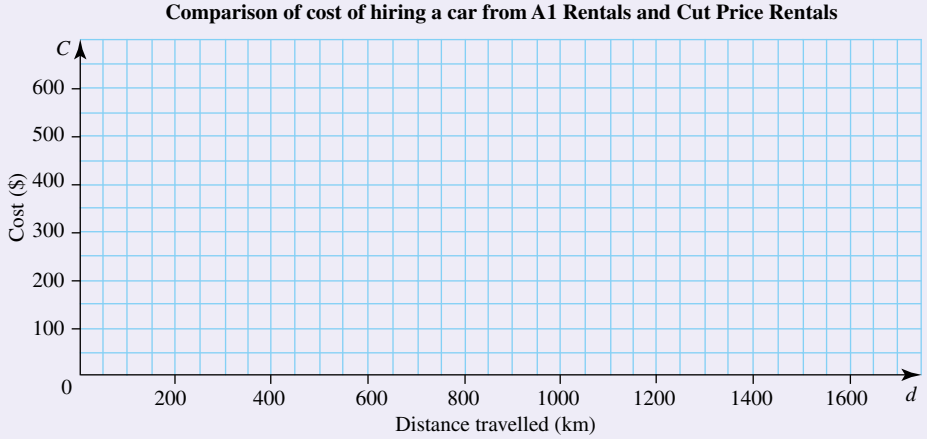
3. Use the graph to determine how many kilometres Jim would have to travel to make the hire costs the same for both rental companies.

4. Assume Jim's trip is extended to four days. Use an appropriate method to show how this changes the answer found in question **3**.

For questions 5 to 7, assume that Jim has delayed his trip until next month when the hire charges have increased.

5. Write equations to show the cost of hiring a car from both car rental companies for a trip lasting:
a. 3 days **b.** 4 days.

6. Copy the set of axes shown. Plot the two equations from question **5** on the axes to show how the costs compare over 1500 km.



7. Comment on the results displayed in your graph.

8. Jim needs to provide his boss with documentation of the hire car costs, catering for all options. On a separate sheet of paper, prepare a document for Jim to hand to his boss.

learnon RESOURCES — ONLINE ONLY

Digital doc: Investigation — Documenting business expenses (doc-15922)
Digital doc: Code puzzle — This world-first medical event took place in 1986 (doc-15923)
Digital doc: Summary Topic 9 (doc-22907)
Interactivity: Word search Topic 9 (int-2835)
Interactivity: Crossword Topic 9 (int-2836)
Interactivity: Sudoku Topic 9 (int-3591)

Exercise 9.8 Review questions

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. MC The equation of a linear graph that passes through the origin with gradient –3 is:

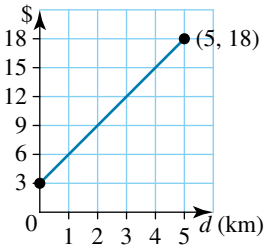
A. $y = -3$ **B.** $x = -3$ **C.** $y = -3x$ **D.** $y = 3 - 3x$ **E.** $y = 3x - 3$

2. MC An online music shop charges a flat rate of $5 postage for 2 CDs and $11 for 5 CDs. The equation that best represents this, if C is the cost and n is the number of CDs, is:

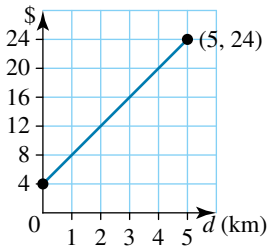
A. $C = 5n + 11$ **B.** $C = 6n + 5$ **C.** $C = n + 2$ **D.** $C = 5n + 1$ **E.** $C = 2n + 1$

3. MC During a walk-a-thon, Sarah receives $4 plus $3 per kilometre. The graph that best represents Sarah walking up to 5 kilometres is:

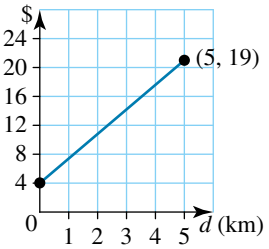
A.



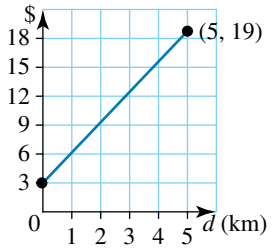
B.



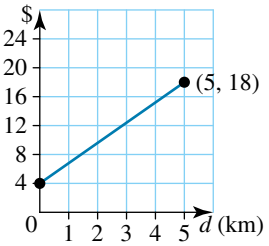
C.



D.



E.



4. MC Which of the following pairs of coordinates is the solution to the given simultaneous equations?

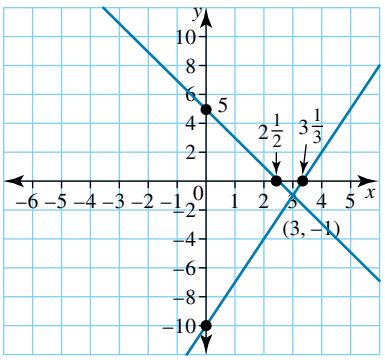
$2x + 3y = 18$
$5x - y = 11$

A. (6, 2) **B.** (3, –4) **C.** (3, 9) **D.** (3, 4) **E.** (5, 11)

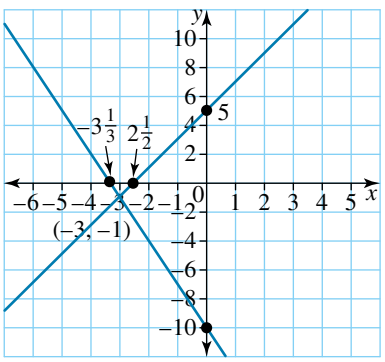
5. MC The graphical solution to the following pair of simultaneous equations is:

$y = 5 - 2x$
$y = 3x - 10$

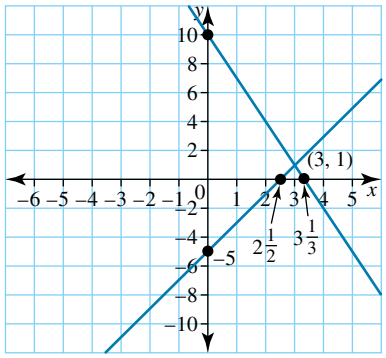
A.



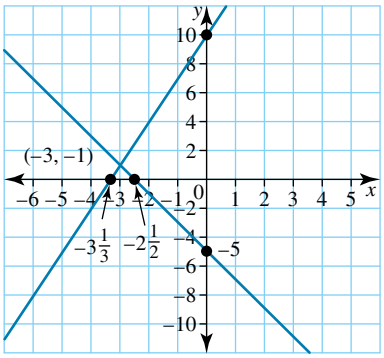
B.



C.



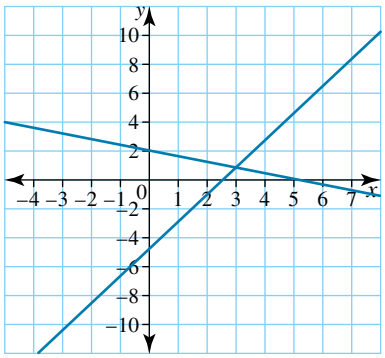
D.



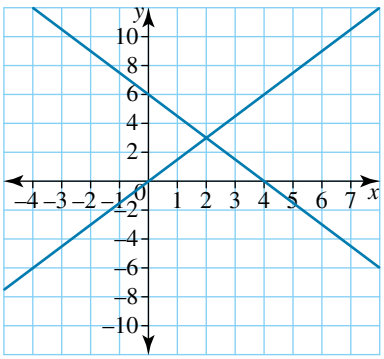
E. none of the above

6. The graphs below show the given simultaneous equations. Use each graph to write the point of intersection and, hence, the solution of each pair of simultaneous equations.

a. $x + 3y = 6$
$y = 2x - 5$



b. $3x + 2y = 12$
$2y = 3x$



7. Use substitution to check if the given pairs of coordinates are solutions to the given simultaneous equations.

a. (7, 1) $x - 2y = 5$
$5y + 2x = 18$

b. (4, 3) $y = 7 - x$
$5y - 2x = 7$

8. Solve each of the following pairs of simultaneous equations using a graphical method.

a. $4y - 2x = 8$
$x + 2y = 0$

b. $y = 2x - 2$
$x - 4y = 8$

c. $2x + 5y = 20$
$y = 7$

9. Solve the following pairs of simultaneous equations using the substitution method.

a. $y = 3x + 1$
$x + 2y = 16$

b. $y = 2x + 7$
$3y - 4x = 11$

c. $2x + 5y = 6$
$y = \frac{3}{2}x + 5$

d. $y = -x$
$y = 8x + 21$

e. $y = 3x - 11$
$y = 5x + 17$

f. $y = 4x - 17$
$y = 6x - 22$

10. Solve the following pairs of simultaneous equations using the elimination method.

a. $3x + y = 17$
$7x - y = 33$

b. $4x + 3y = 1$
$-4x + y = 11$

c. $3x - 7y = -2$
$-2x - 7y = 13$

d. $4y - 3x = 9$
$y + 3x = 6$

e. $5x + 2y = 6$
$4x + 3y = 2$

f. $x - 4y = -4$
$4x - 2y = 12$

11. Solve the following pairs of simultaneous equations using an appropriate method.

a. $3x + 2y = 6$
$3y + 5x = 9$

b. $6x - 4y = -6$
$7x + 3y = -30$

c. $6x + 2y = 14$
$x = -3 + 5y$

12. Find the point(s) of intersection for each of the following pairs of lines.

a. $y = x^2 - 6$
$y = 5x - 3$

b. $y = \frac{2}{x}$
$y = 5x - 3$

c. $x^2 + y^2 = 2$
$y = 5x - 3$

Communicating, reasoning and problem solving

13. Write each of the following as a pair of simultaneous equations and solve the equations.

a. Find two numbers whose difference is 5 and whose sum is 23.

b. A rectangular house has a total perimeter of 34 metres and the width is 5 metres less than the length. What are the dimensions of the house?

c. If two Chupa Chups and three Wizz Fizzes cost $2.55, but five Chupa Chups and seven Wizz Fizzes cost $6.10, find the price of each type of lolly.

14. Laurie buys milk and bread for his family on the way home from school each day, paying with a $10 note. If he buys three cartons of milk and two loaves of bread, he receives 5 cents in change. If he buys two cartons of milk and one loaf of bread, he receives $4.15 in change. How much does each item cost?

15. A paddock contains some cockatoos (which have 2 legs) and kangaroos (which have 4 legs). The total number of animals is 21 and they have 68 legs in total. Using simultaneous equations, determine how many cockatoos and kangaroos there are in the paddock.

16. Warwick was solving a pair of simultaneous equations using the elimination method and reached the result that $0 = -5$. Suggest a solution to the problem, giving a reason for your answer.

17. There are two sections to a concert hall. Seats in the dress circle are arranged in rows of 40 and cost $140 each. Seats in the stalls are arranged in rows of 70 and cost $60 each. There are 10 more rows in the dress circle than in the stalls, and the capacity of the hall is 7000.

a. Let d represent the number of rows in the dress circle and let b represent the number of rows in the stalls. Write an equation in terms of these two variables based on the fact that there are 10 more rows in the dress circle than in the stalls.

b. Write an equation in terms of these two variables based on the fact that the capacity of the hall is 7000 seats.

c. Solve the two equations from **a** and **b** simultaneously to find the number of rows in each section, using the method of your choice.

d. Now that you have the number of rows in each section, calculate the number of seats in each section.

e. Hence, calculate the total receipts for a concert where all tickets are sold.

18. John is comparing two car rental companies: Golden Ace Rental Company and Silver Diamond Rental Company. Golden Ace Rental Company charges a flat rate of $38 per day and $0.20 per kilometre. Silver Diamond Rental Company charges a flat rate of $30 per day plus $0.32 per kilometre.

a. Write an algebraic equation for the cost of renting a car for three days from Golden Ace Rental Company in terms of the number of kilometres travelled, k.

b. Write an algebraic equation for the cost of renting a car for three days from Silver Diamond Rental Company in terms of the number of kilometres travelled, k.

c. How many kilometres would John have to travel so that the cost of hiring from each company for three days is the same?

d. Write an inequation that, when solved, will tell you the number of kilometres for which it is cheaper to use Golden Ace Rental Company when renting for three days.

e. For what number of kilometres will it be cheaper to use Silver Diamond Rental Company for three days' hire?

19. Frederika has $24 000 saved for a holiday and a new stereo. Her travel expenses are $5400 and her daily expenses are $260.

a. Write down an equation for the cost of her holiday if she stays for d days. Upon her return from holidays Frederika wants to purchase a new stereo system that will cost her $2500.

b. How many days can she spend on her holiday if she wishes to purchase a new stereo upon her return?

Answers

Topic 9 Simultaneous linear equations and inequalities

Exercise 9.2 Graphical solution of simultaneous linear equations

1. a. (2, 1) **b.** (1, 1) **c.** (0, 4)
d. (2, −1) **e.** (−2, −4) **f.** (−0.5, 1.5)

2. a. No **b.** Yes **c.** Yes **d.** No
e. Yes **f.** No **g.** No **h.** Yes
i. No **j.** Yes

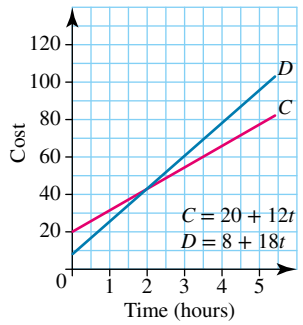
3. a. (3, 2) **b.** (4, 3) **c.** (−3, 4) **d.** (−2, 2)
e. (2, 0) **f.** (3, 0) **g.** (−2, 4) **h.** (3, 8)
i. $(-\frac{1}{2}, 1\frac{1}{2})$ **j.** (2, 5) **k.** (5, 3) **l.** $(2, \frac{2}{3})$

4. a. No solution **b.** (2, −1) **c.** No solution **d.** (1, 9)
e. (3, 1) **f.** No solution **g.** No solution **h.** (2, 1)

5. $y = 4x - 16$

6. a. Northern beach: $C = 20 + 12t$, $0 \leq t \leq 5$
Southern beach: $D = 8 + 18t$, $0 \leq t \leq 5$

b. Northern beach in red, southern beach in blue



c. Time > 2 hours

d. Time = 2 hours, cost = $44

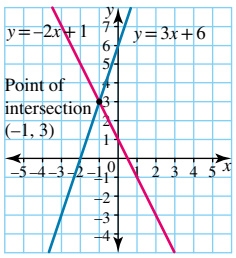
7. a. Same line **b.** Perpendicular **c.** Intersecting **d.** Parallel

8. a. 1 solution **b.** No solution (parallel lines) **c.** No solution (parallel lines)

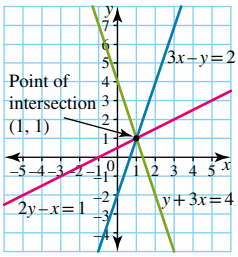
9. a. $y = 3x + 6$

b. $y = -2x + 1$

c.



10.



11. Discuss with your teacher.

Exercise 9.3 Solving simultaneous linear equations using substitution

1. a. $(2, 3)$ **b.** $(2, -1)$ **c.** $(3, -2)$ **d.** $(7, 6)$
e. $(3, 6)$ **f.** $(2, 1)$ **g.** $(-1, -2)$ **h.** $(-4, 0)$
i. $(-1, -2)$ **j.** $(6, -2)$ **k.** $\left(3, 1\frac{1}{2}\right)$ **l.** $(-3, -5)$

2. a. $(-6, -23)$ **b.** $(5, 23)$ **c.** $(2, -6)$ **d.** $\left(\frac{3}{2}, -\frac{15}{2}\right)$
e. $(1, -7)$ **f.** $\left(-\frac{1}{2}, -4\right)$ **g.** $\left(-\frac{3}{2}, -\frac{1}{2}\right)$ **h.** $\left(-\frac{1}{5}, \frac{4}{5}\right)$
i. $(-3, -1.5)$ **j.** $(1, 0.3)$ **k.** $\left(-\frac{4}{5}, \frac{4}{5}\right)$ **l.** $(1, -1)$

3. 26 chickens

4. a. $(3, 1)$ **b.** $(-2, 3)$ **c.** $(5, 1)$
d. $(4, 2)$ **e.** $(0, 3)$ **f.** $(4, 0)$

5. a. $x = \dfrac{n}{2m}, y = \dfrac{n}{2}$
b. $x = \dfrac{m}{n^2 + 1}, y = \dfrac{mn}{n^2 + 1}$
c. $x = \dfrac{n}{m - n}, y = \dfrac{n^2}{m - n}$
d. $x = \dfrac{n}{m - n}, y = \dfrac{n}{m - n}$
e. $x = \dfrac{-(m - n^2)}{m - n}, y = \dfrac{m(n - 1)}{m - n}$
f. $x = \dfrac{2m}{m + n}, y = \dfrac{m(n - m)}{m + n}$

6. $a = -1, b = 5$

7. $z = 24, m = 6, n = 9$

m	11	7
9	8	7
n	5	10

8. a. $x + 2y = 4$ **b.** $x = 2, y = 1$

9. Chemistry \$21, Physics \$27

10. $x = 8, y = -7$

11. $x = 0, y = 1$

12. The substitution method is suitable when one or both of the equations is in a form where one of the two variables is the subject.

Exercise 9.4 Solving simultaneous linear equations using elimination

1. a. $(3, 1)$ **b.** $(-2, 3)$ **c.** $(-2, 6)$

2. a. $(5, -1)$ **b.** $(2, 3)$ **c.** $(-3, 1)$

3. a. $(-3, 5)$ **b.** $(-5, -8)$ **c.** $(2, -2)$
d. $\left(1\frac{1}{2}, 3\frac{1}{2}\right)$ **e.** $\left(2, 1\frac{4}{5}\right)$ **f.** $(1, 1)$

4. a. $(1, 3)$ **b.** $(2, 4)$ **c.** $(5, 2)$
d. $(4, 2)$ **e.** $(-3, 4)$ **f.** $\left(-3, -1\frac{1}{2}\right)$
g. $(-6, -5)$ **h.** $(-3, 5)$ **i.** $(2, 1.8)$

5. a. $(6, 3)$ **b.** $(2, -2)$ **c.** $(1, 3)$
d. $(-1.5, -3)$ **e.** $(-8, 18)$ **f.** $(-3, 5)$

6. a. $(1, 3)$ **b.** $(4, 0)$ **c.** $(-3, 5)$
d. $(4, 3)$ **e.** $(8, 5)$ **f.** $\left(\frac{1}{3}, -\frac{1}{3}\right)$

7. Ann 61 kg, Beth 58 kg, Celine 54 kg

8. a. **i.** $acx + bcy = ce$ (3)
ii. $acx + ady = af$ (4)
iii. $y = \dfrac{ce - af}{bc - ad}$

b. $x = \dfrac{de - bf}{ad - bc}$

c. i. $\left(\frac{106}{31}, \frac{1}{31}\right)$ **ii.** $\left(\frac{37}{14}, \frac{11}{14}\right)$

d. Because you cannot divide by 0

e. $ad - bc \neq 0$

9. $x = 7, y = -3$

10. $x = 4, y = 3, z = -6$

11. It simplifies the equation and allows you to evaluate the value of the other variable.

12. $z = 6$

Exercise 9.5 Applications of simultaneous linear equations

1. Maths mark = 97, English mark = 66

2. 18 nuts, 12 bolts

3. 8 and 3

4. 9 and 7

5. 6 and 5

6. Length 12 m, width 8 m

7. Lemons cost 55 cents and oranges cost 25 cents.

8. Length 60 m, width 20 m

9. Eight 20-cent coins and three 50-cent coins

10. Twelve $1 coins and nine $2 coins

11. Paddlepops cost $1.20 and a Magnum costs $2.10.

12. Cost of the Golden Rough = 35 cents; cost of the Redskin = 25 cents

13. Fixed costs = $87; cost per person = $23.50

14. Sally's PE mark is 83 and her Science mark is 71.

15. Mozzarella costs $6.20 and Swiss cheese costs $5.80.

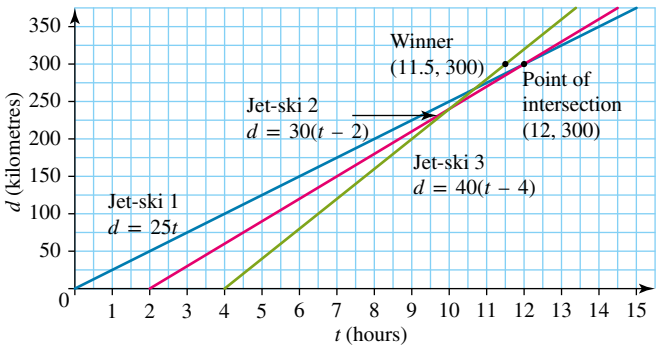
16. $x = 3, y = 4$

17. Fixed costs = $60; cost per person = $25

18. $4 each for DVDs, $24 each for zip disks

19. 66 cups of hot chips, 33 meat pies and 22 hot dogs were sold during the half-hour period.

20. a.



b. Jet ski 3 wins the race.

c. Jet ski 1 and 2 reach the destination at the same time although jet ski 2 started two hours after jet ski 1. Jet ski 3 overtakes jet ski 1 6 hours and 40 minutes after its race begins or 10 hours and 40 minutes after jet ski 1 starts the race.

Jet ski 3 overtakes jet ski 2 6 hours after it starts the race or 8 hours after jet ski 2 started the race.

21. Discuss with your teacher.

22. Rollercoaster = $6, Ferris wheel = $4, Gravitron = $8

23. 89 246

Exercise 9.6 Solving simultaneous linear and non-linear equations

1. A parabola may intersect with a straight line twice, once or not at all.

2. a. $(-5, 4)$ and $(-1, 0)$

b. $(2, 3)$

c. $(1 - \sqrt{10}, -6)$ and $(1 + \sqrt{10}, -6)$

3. a. $\left(\frac{-1}{2} - \frac{3\sqrt{5}}{2}, \frac{-19}{2} - \frac{9\sqrt{5}}{2}\right)$ and $\left(\frac{-1}{2} + \frac{3\sqrt{5}}{2}, \frac{-19}{2} + \frac{9\sqrt{5}}{2}\right)$

b. $(-1, -2)$ and $(2, 1)$

c. $(-2.54, -8.17)$ and $(3.54, 16.17)$

d. $(-1.41, 4)$ and $(1.41, 4)$

e. $(-1, 2)$ and $\left(\frac{5}{2}, \frac{1}{4}\right)$

f. $(3, 37)$

4. B

5. C

6. a. Yes **b.** No **c.** Yes **d.** No

7. a. $(1, 1), (-1, -1)$

b. $(1 + \sqrt{2}, -1 + \sqrt{2}), (1 - \sqrt{2}, -1 - \sqrt{2})$ or $(2.41, 0.41), (-0.41, -2.41)$

c. $\left(\frac{-\sqrt{15}}{3}, -\sqrt{15}\right)\right), \left(\frac{\sqrt{15}}{3}, \sqrt{15}\right)$ or $(-1.29, -3.87), (1,29, 3.87)$

d. $(-6, -1), (2, 3)$

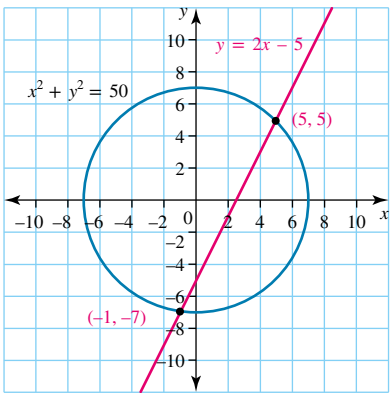
8. a. $(-1, -3), (1, 3)$ **b.** $(-4, 3), (4, -3)$ **c.** $(-1, 7), (5, -5)$

d. $\left(\frac{-(\sqrt{14} - 2)}{2}, \frac{\sqrt{14} + 2}{2}\right), \left(\frac{\sqrt{14} + 2}{2}, \frac{-(\sqrt{14} - 2)}{2}\right)$ or $(-0.87, 2.87), (2.87, -0.87)$

9. The straight line crosses the parabola at $(0, -7)$, so no matter what value m takes, there will be at least one intersection point.

10. a. $(5, 5), (-1, -7)$

b.



11. 9, 12

12. The line is a tangent to the curve.

Exercise 9.7 Solving linear inequalities

1. a. $x > 2$ **b.** $a > -1$ **c.** $y \geq 7$ **d.** $m \geq 4$

e. $p < 1$ **f.** $x < 7$ **g.** $m \leq 9$ **h.** $a \leq 7$

i. $x > 3$ **j.** $m \geq 2$ **k.** $q \geq -4$ **l.** $a > -8$

2. a. $m > 3$ **b.** $p \leq 2$ **c.** $a < 4$ **d.** $x \geq 5$

e. $p > -5$ **f.** $x \leq -7$ **g.** $m \geq -0.5$ **h.** $b > -0.5$

i. $m > 18$ **j.** $x < 8$ **k.** $a \leq -14$ **l.** $m \geq 25$

3. a. $m < 4.5$ **b.** $x \geq 3$ **c.** $p > 4$ **d.** $n \leq 2$

e. $b < 5$ **f.** $y > 2$ **g.** $m \leq -1$ **h.** $a \geq -5$

i. $b < -4\frac{1}{3}$ **j.** $c \leq -1$ **k.** $p > -2$ **l.** $a \geq -7$

4. a. $m > 3$ **b.** $a \geq 2$ **c.** $a < -1$ **d.** $a \leq -3$

e. $x > 6$ **f.** $x \leq 2$ **g.** $b < 4$ **h.** $a > 5$

i. $m < 2$ **j.** $m \leq 3$ **k.** $b \leq -\frac{16}{11}$ **l.** $m \geq 1$

5. a. $x \leq 7$ **b.** $x \geq -18$ **c.** $x < -10$
d. $x > 10\frac{1}{2}$ **e.** $x \geq 5$ **f.** $x < -1\frac{4}{5}$

6. a. $m < -2$ **b.** $p \geq -3$ **c.** $a \leq 5$ **d.** $p \geq -5$
e. $y \leq -3$ **f.** $x > 7$ **g.** $p < 0$ **h.** $a \geq \frac{1}{5}$
i. $x > -3$ **j.** $a \leq -11$ **k.** $b \leq 3$ **l.** $x < -3$
m. $k > 8$ **n.** $x > -18\frac{1}{2}$ **o.** $a \leq 40$

7. B

8. a. $x < -1$ **b.** $m \leq -3$ **c.** $x > 17$
d. $a > \frac{5}{8}$ **e.** $m \geq 1\frac{1}{3}$ **f.** $m \geq -12$

9. a. $k > 2$ **b.** $a > -5$ **c.** $m \leq 1\frac{2}{3}$ **d.** $x > 5$
e. $y \geq 7$ **f.** $d < -2$ **g.** $p \geq \frac{-6}{7}$ **h.** $x \geq -5$
i. $m < -2$ **j.** $a < 9$ **k.** $p \geq 3$ **l.** $x > -4\frac{1}{2}$

10. a. $5x > 10$ **b.** $x - 3 \leq 5$ **c.** $7 + 3x < 42$

11. a. $-6.5 < x < -2$ **b.** $\frac{-c - b}{a} < x < \frac{-d - b}{a}$

12. a. $S > 47$ **b.** No **c.** Answers will vary.

13. a. $n < 16\,800$ km

b. Mick travelled less than 16 800 km for the year and his costs stayed below \$16 000.

14. \$20 000

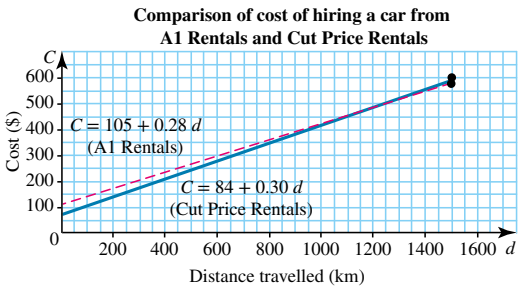
15. Discuss with your teacher.

Investigation | Rich task

1. A1 Rentals: $C = \$35 \times 3 + 0.28d$

Cut Price Rentals: $C = \$28 \times 3 + 0.3d$

2.



3. 1050 km

4. 1400 km

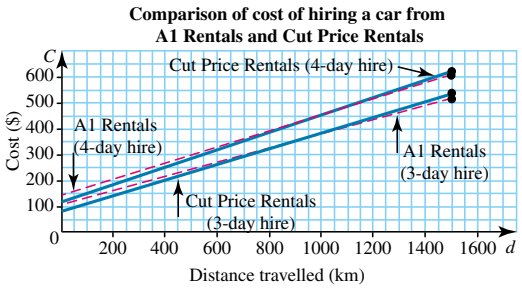
5. a. A1 Rentals: $C = \$37 \times 3 + 0.28d$

Cut Price Rentals: $C = \$30 \times 3 + 0.3d$

b. A1 Rentals: $C = \$37 \times 4 + 0.28d$

Cut Price Rentals: $C = \$30 \times 4 + 0.3d$

6.



7. The extra cost of \$2 per day for both rental companies has not affected the charges they make for the distances travelled. However, the overall costs have increased.

8. Presentation of the answers will vary. Answers will include:

Travelling 3 days this month:

- If Jim travels 1050 km, the cost will be the same for both rental companies; that is, \$399.

- If he travels less than 1050 km, Cut Price Rentals is cheaper.
- If he travels more than 1050 km, A1 Rentals is cheaper.

Travelling 4 days this month:

- If Jim travels 1400 km, the cost will be the same for both rental companies; that is, $532.
- If he travels less than 1400 km, Cut Price Rentals is cheaper.
- If he travels more than 1400 km, A1 Rentals is cheaper.

Travelling 3 days next month:

- If Jim travels 1050 km, the cost will be the same for both rental companies; that is, $405.
- If he travels less than 1050 km, Cut Price Rentals is cheaper.
- If he travels more than 1050 km, A1 Rentals is cheaper.

Travelling 4 days next month:

- If Jim travels 1400 km, the cost will be the same for both rental companies; that is, $540.
- If he travels less than 1400 km, Cut Price Rentals is cheaper.
- If he travels more than 1400 km, A1 Rentals is cheaper.

Exercise 9.8 Review questions

1. C

2. E

3. C

4. D

5. A

6. a. (3, 1) **b.** (2, 3)

7. a. No **b.** Yes

8. a. (−2, 1) **b.** (0, −2) **c.** (–7.5, 7)

9. a. (2, 7) **b.** (−5, −3) **c.** (−2, 2)
d. $\left(-\frac{7}{3}, \frac{7}{3}\right)$ **e.** (−14, −53) **f.** $\left(\frac{5}{2}, -7\right)$

10. a. (5, 2) **b.** (−2, 3) **c.** (−3, −1)
d. (1, 3) **e.** (2, −2) **f.** (4, 2)

11. a. (0, 3) **b.** (−3, −3) **c.** (2, 1)

12. a. $\left(\frac{5 - \sqrt{37}}{2}, \frac{19 - 5\sqrt{37}}{2}\right), \left(\frac{5 + \sqrt{37}}{2}, \frac{19 + 5\sqrt{37}}{2}\right)$
b. $\left(-\frac{2}{5}, -5\right)$, (1, 2)
c. $\left(\frac{15 + \sqrt{43}}{26}, \frac{-3 + 5\sqrt{43}}{26}\right), \left(\frac{15 - \sqrt{43}}{26}, \frac{-3 - 5\sqrt{43}}{26}\right)$

13. a. The numbers are 9 and 14.
b. Length 11 metres, width 6 metres
c. Chupa Chups cost 45 cents and Whizz Fizzes cost 55 cents.

14. Milk $1.75, bread $2.35

15. 13 kangaroos and 8 cockatoos

16. Any false statement that occurs during the solving of simultaneous equations indicates the lines are parallel, and have no points of intersection.

17. a. $d = b + 10$ **b.** $7000 = 70b + 40d$ **c.** $b = 60$ and $d = 70$
d. The number of seats in the stalls is 4200, and the number of seats in the dress circle is 2800.
e. $644 000

18. a. $C_G = 114 + 0.2k$ **b.** $C_S = 90 + 0.32k$ **c.** 200 km
d. $114 + 0.2k < 90 + 0.32k$ **e.** $k < 200$

19. a. $5400 + 260d = C$ **b.** 61 days

TOPIC 10
Surface area and volume

[Stages 5.1, 5.2 and 5.3]

10.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the concepts covered in this topic.

10.1.1 Why learn this?

Humans must measure! How much paint or carpet will you need to redecorate your bedroom? How many litres of water will it take to fill the new pool? How far is it to the end of the universe? These are just a few examples of where measurement skills are needed. Measuring tools have advanced significantly in their capacity to measure extremely small and extremely large amounts, leading to many breakthroughs in medicine, engineering, science, architecture and astronomy.

DISCUSSION

Investigate extinct megafauna from other countries. Try to find out why these animals became extinct.

LEARNING SEQUENCE

10.1 Overview
10.2 [Stage 5.1] Area
10.3 [Stages 5.2 and 5.3] Total surface area
10.4 [Stages 5.2 and 5.3] Volume
10.5 Review

LEARNING OUTCOMES

A student:

- calculates the areas of composite shapes, and the surface areas of rectangular and triangular prisms MA5.1-8MG
- selects appropriate notations and conventions to communicate mathematical ideas and solutions MA5.2-1WM
- interprets mathematical or real-life situations, systematically applying appropriate strategies to solve problems MA5.2-2WM
- calculates the surface areas of right prisms, cylinders and related composite solids MA5.2-11MG
- applies formulas to calculate the volumes of composite solids composed of right prisms and cylinders MA5.2-12MG
- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3-1WM
- generalises mathematical ideas and techniques to analyse and solve problems efficiently MA5.3-2WM
- applies formulas to find the surface areas of right pyramids, right cones, spheres and related composite solids MA5.3-13MG
- applies formulas to find the volumes of right pyramids, right cones, spheres and related composite solids MA5.3-14MG

CONTENT DESCRIPTIONS

Students:
Calculate the areas of composite shapes (ACMMG216)
Calculate the surface areas of cylinders and solve related problems (ACMMG217)
Solve problems involving the surface areas and volumes of right prisms (ACMMG218)
Solve problems involving surface area and volume for a range of prisms, cylinders and composite solids (ACMMG242)
Solve problems involving the surface areas and volume of right pyramids, right cones, spheres and related composite solids (ACMMG271)

Source: NSW Syllabus for the Australian curriculum

learnon RESOURCES — ONLINE ONLY

eLesson: The story of mathematics — Australian megafauna (eles-1845)

Note: Your teacher may now set you a pre-test to determine how familiar you are with the content in this topic.

10.2 Area [Stage 5.1]

10.2.1 Definition of area

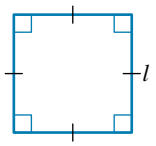
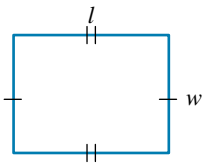
- The **area** of a figure is the amount of surface covered by the figure.
- The units used for area are mm^2, cm^2, m^2, km^2 or ha (hectares), depending upon the size of the figure.

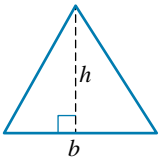
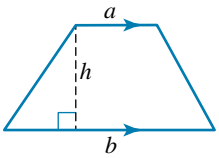
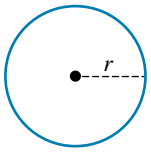
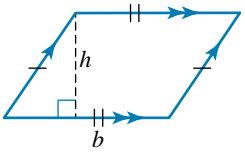
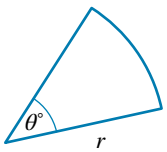
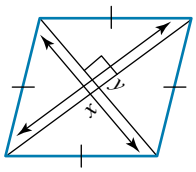
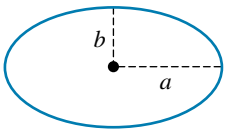
$$1\,\text{ha} = 10\,000\,(\text{or}\,10^4)\,\text{m}^2$$

- There are many real-life situations that require an understanding of the area concept. Some are 'the area to be painted', 'the floor area of a room or house', 'how much land one has' and 'how many tiles are needed for a wall'.
- It is important that you are familiar with converting units of area.

10.2.2 Using area formulas

- The areas of many plane figures can be found by using a formula. The table below shows the formulas for the area of some common shapes.

Shape	Formula
Square 	$A = l^2$
Rectangle 	$A = lw$

Triangle		$A = \frac{1}{2}bh$
Trapezium		$A = \frac{1}{2}(a + b)h$
Circle		$A = \pi r^2$
Parallelogram		$A = bh$
Sector		$A = \frac{\theta^\circ}{360^\circ} \times \pi r^2$
Kite (including rhombus)		$A = \frac{1}{2}xy$, where x and y are diagonals.
Ellipse		$A = \pi ab$, where a and b are the lengths of the semi-major and semi-minor axes respectively.

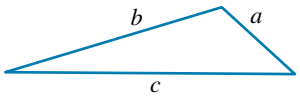
Note: A calculator uses a stored value for π of approximately 3.141 592 654. Before calculators were in common usage, π was often taken to be approximately $\frac{22}{7}$ or 3.14. You are advised to use the π button on your calculator rather than $\frac{22}{7}$ or 3.14.

DISCUSSION

Can you think of a way to easily remember some or all of these area formulas? Share your best memory techniques with your classmates.

10.2.3 Heron's formula

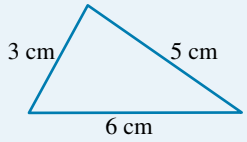
- If the lengths of all three sides of a triangle are known, its area, A, can be found by using **Heron's formula**: $A = \sqrt{s(s-a)(s-b)(s-c)}$, where a, b and c are the lengths of the three sides and s is the semi-perimeter, $s = \frac{a+b+c}{2}$.



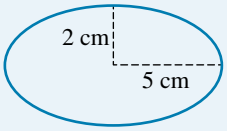
WORKED EXAMPLE 1

Find the areas of the following plane figures, correct to 2 decimal places.

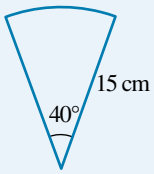
a



b



c



THINK	WRITE
a 1 Three side lengths are known, but not the height. In this case apply Heron's formula.	**a** $A = \sqrt{s(s-a)(s-b)(s-c)}$
2 Identify the values of a, b and c.	$a = 3, b = 5, c = 6$
3 Calculate the value of s, the semi-perimeter of the triangle.	$s = \frac{a+b+c}{2}$ $= \frac{3+5+6}{2}$ $= \frac{14}{2}$ $= 7$
4 Substitute the values of a, b, c and s into Heron's formula and evaluate, correct to 2 decimal places.	$A = \sqrt{7(7-3)(7-5)(7-6)}$ $= \sqrt{7 \times 4 \times 2 \times 1}$ $= \sqrt{56}$ $= 7.48 \text{ cm}^2$
b 1 The shape shown is an ellipse. Write the appropriate area formula.	**b** $A = \pi ab$
2 Identify the values of a and b (the semi-major and semi-minor axes).	$a = 5, b = 2$
3 Substitute the values of a and b into the formula and evaluate, correct to 2 decimal places.	$A = \pi \times 5 \times 2$ $= 31.42 \text{ cm}^2$
c 1 The shape shown is a sector. Write the formula for finding the area of a sector.	**c** $A = \frac{\theta}{360°} \times \pi r^2$
2 Write the value of θ and r.	$\theta = 40°, r = 15$
3 Substitute and evaluate the expression, correct to 2 decimal places.	$A = \frac{40°}{360°} \times \pi \times 15^2$ $= 78.54 \text{ cm}^2$

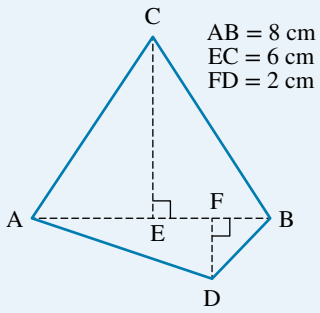
10.2.4 Areas of composite figures

- A **composite figure** is a figure made up of a combination of simple figures.
- The area of a composite figure can be calculated by:
  - calculating the sum of the areas of the simple figures that make up the composite figure
  - calculating the area of a larger shape and then subtracting the extra area involved.

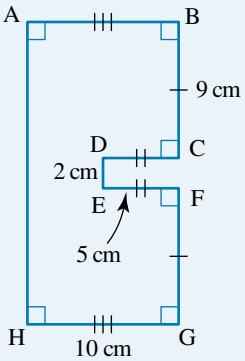
WORKED EXAMPLE 2

Find the area of each of the following composite shapes.

a



b



THINK

a 1 ACBD is a quadrilateral that can be split into two triangles: ΔABC and ΔABD.

2 Write the formula for the area of a triangle containing base and height.

3 Identify the values of b and h for ΔABC.

4 Substitute the values of the pronumerals into the formula and, hence, calculate the area of ΔABC.

5 Identify the values of b and h for ΔABD.

6 Calculate the area of ΔABD.

7 Add the areas of the two triangles together to find the area of the quadrilateral ACBD.

b 1 One way to find the area of the shape shown is to find the total area of the rectangle ABGH and then subtract the area of the smaller rectangle DEFC.

2 Write the formula for the area of a rectangle.

3 Identify the values of the pronumerals for the rectangle ABGH.

4 Substitute the values of the pronumerals into the formula to find the area of the rectangle ABGH.

5 Identify the values of the pronumerals for the rectangle DEFC.

WRITE

a Area ACBD = Area ΔABC + Area ΔABD

$A_{\text{triangle}} = \frac{1}{2}bh$

ΔABC: $b = \text{AB} = 8$, $h = \text{EC} = 6$

$$\begin{aligned}\text{Area of } \Delta\text{ABC} &= \tfrac{1}{2} \times \text{AB} \times \text{EC} \\ &= \tfrac{1}{2} \times 8 \times 6 \\ &= 24 \text{ cm}^2\end{aligned}$$

ΔABD: $b = \text{AB} = 8$, $h = \text{FD} = 2$

$$\begin{aligned}\text{Area of } \Delta\text{ABD} &= \tfrac{1}{2}\text{AB} \times \text{FD} \\ &= \tfrac{1}{2} \times 8 \times 2 \\ &= 8 \text{ cm}^2\end{aligned}$$

$$\begin{aligned}\text{Area of ACBD} &= 24 \text{ cm}^2 + 8 \text{ cm}^2 \\ &= 32 \text{ cm}^2\end{aligned}$$

b Area = Area ABGH − Area DEFC

$A_{\text{rectangle}} = l \times w$

$$\begin{aligned}\text{Rectangle ABGH: } l &= 9 + 2 + 9 \\ &= 20 \\ w &= 10\end{aligned}$$

$$\begin{aligned}\text{Area of ABGH} &= 20 \times 10 \\ &= 200 \text{ cm}^2\end{aligned}$$

Rectangle DEFC: $l = 5$, $w = 2$

6 Substitute the values of the pronumerals into the formula to find the area of the rectangle DEFC.

$$\text{Area of DEFC} = 5 \times 2 = 10 \text{ cm}^2$$

7 Subtract the area of the rectangle DEFC from the area of the rectangle ABGH to find the area of the given shape.

$$\text{Area} = 200 - 10 = 190 \text{ cm}^2$$

learnON RESOURCES — ONLINE ONLY

Interactivity: Conversion chart for area (int-3783)
Interactivity: Area of rectangles (int-3784)
Interactivity: Area of trapeziums 1 (int-3789)
Interactivity: Area of trapeziums 2 (int-3790)
Interactivity: Area of circles (int-3788)
Interactivity: Area of parallelograms (int-3786)
Interactivity: Area of a sector (int-6076)
Interactivity: Area of a kite (int-6136)
Interactivity: Area of an ellipse (int-6137)
Interactivity: Using Heron's formula to find the area of a triangle (int-6475)
eLesson: Heron's formula (eles-0177)
eLesson: Composite area (eles-1886)
Digital doc: SkillSHEET Conversion of area units (doc-5236)
Digital doc: SkillSHEET Using a formula to find the area of a common shape (doc-5237)
Digital doc: WorkSHEET Area (doc-5241)

Exercise 10.2 Area

Individual pathways

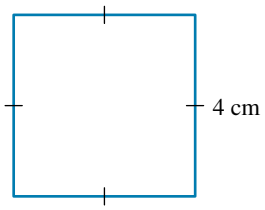
PRACTISE	CONSOLIDATE	MASTER
Questions: 1, 3–5, 8, 9, 11, 12, 14	Questions: 1–6, 8–10, 12, 14, 16, 18, 19	Questions: 1–9, 12–20

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

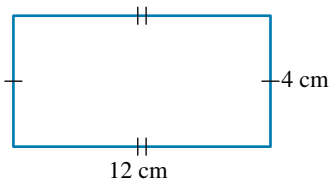
Understanding and fluency

1. Find the areas of the following shapes.

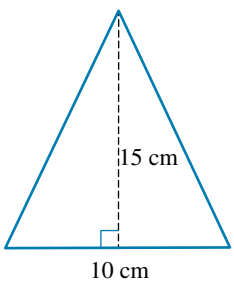
a.



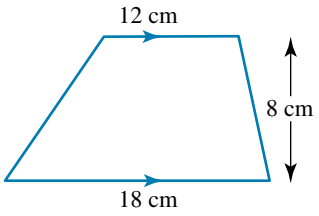
b.



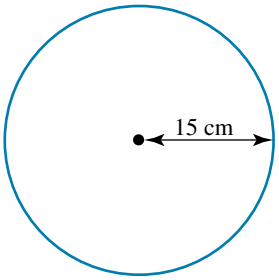
c.



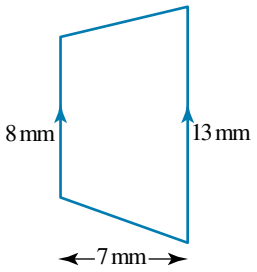
d.



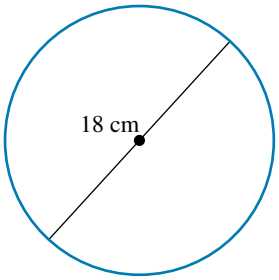
e.



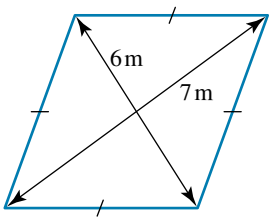
f.



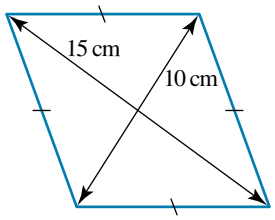
g.



h.



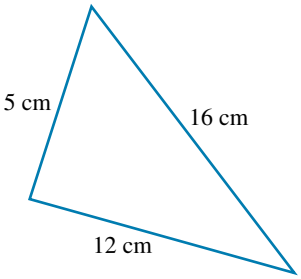
i.



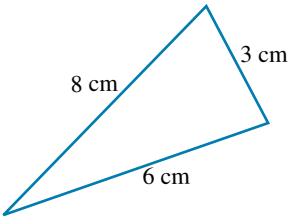
2. Express the areas in questions **1e** and **1g** in terms of π.

3. WE1a Use Heron's formula to find the areas of the following triangles correct to 2 decimal places.

a.

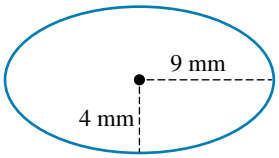


b.

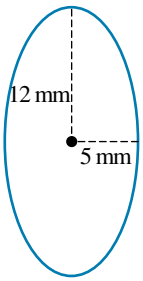


4. WE1b Find the areas of the following ellipses. Answer correct to 1 decimal place.

a.



b.

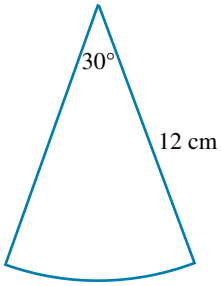


5. WE1c Find the areas of the following shapes:

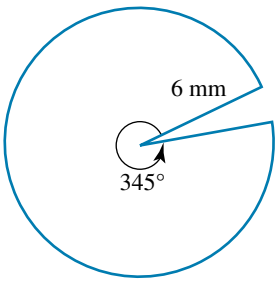
i. stating the answer exactly; that is, in terms of π

ii. correct to 2 decimal places.

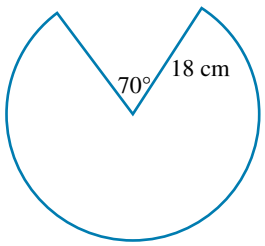
a.



b.



c.



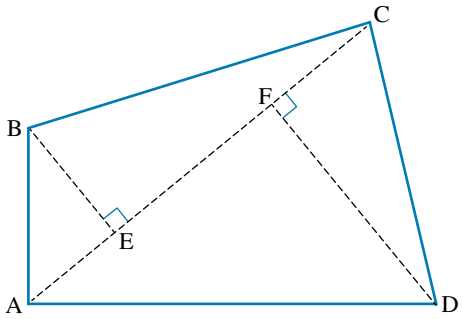
6. MC A figure has an area of about 64 cm^2. Which of the following *cannot* possibly represent the figure?

A. A triangle with base length 16 cm and height 8 cm

B. A circle with radius 4.51 cm

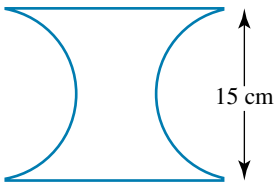
C. A rectangle with dimensions 16 cm and 4 cm
D. A square with side length 8 cm
E. A rhombus with diagonals 16 cm and 4 cm

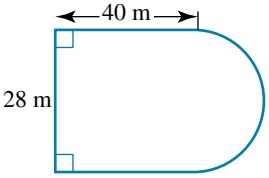
7. MC The area of the quadrilateral shown at right is to be calculated. Which of the following lists all the lengths required to calculate the area?

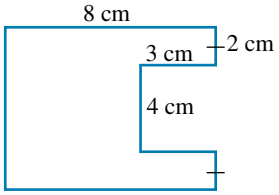


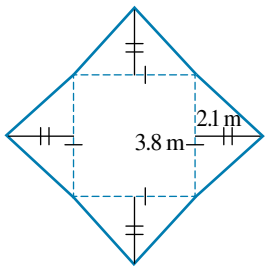
A. AB, BC, CD and AD
B. AB, BE, AC and CD
C. BC, BE, AD and CD
D. AC, BE and FD
E. AC, CD and AB

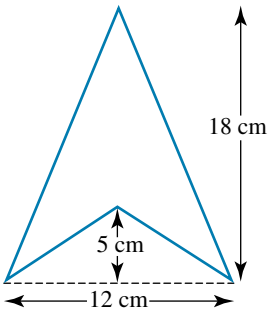
8. WE2 Find the areas of the following composite shapes.

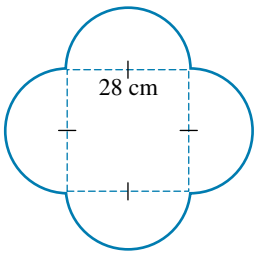
a.


b.


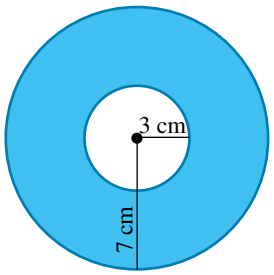
c.


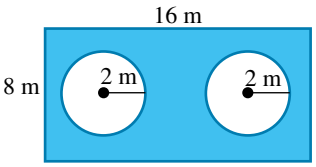
d.


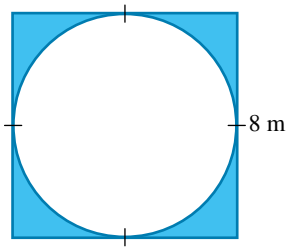
e.


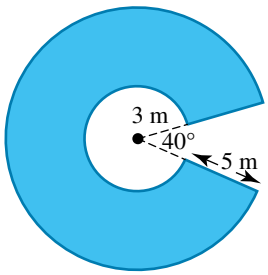
f.


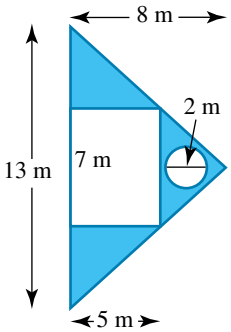
9. Find the shaded area in each of the following figures.

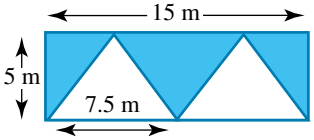
a.


b.


c.


d.


e.


f.


10. A sheet of cardboard is 1.6 m by 0.8 m. The following shapes are cut from the cardboard:
- a circular piece with radius 12 cm
- a rectangular piece 20 cm by 15 cm

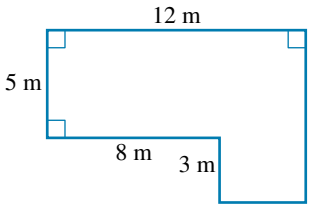
- 2 triangular pieces with base 30 cm and height 10 cm
- a triangular piece with side length 12 cm, 10 cm and 8 cm.

What is the area of the remaining piece of cardboard?

11. A rectangular block of land, 12 m by 8 m, is surrounded by a concrete path 0.5 m wide. Find the area of the path.

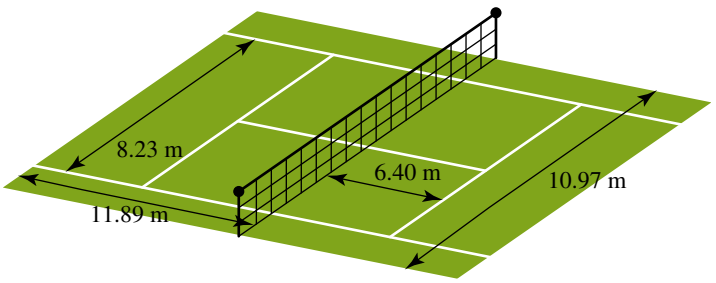
12. Concrete slabs 1 m by 0.5 m are used to cover a footpath 20 m by 1.5 m. How many slabs are needed?

13. A city council builds a 0.5-m wide concrete path around the garden as shown below.



Find the cost of the job if the workman charges \$40.00 per m^2.

14. A tennis court used for doubles is 10.97 m wide, but a singles court is only 8.23 m wide, as shown in the diagram.



a. What is the area of the doubles tennis court?

b. What is the area of the singles court?

c. What percentage of the doubles court is used for singles? Give your answer to the nearest whole number.

15. Ron the excavator operator has 100 metres of barricade mesh and needs to enclose an area to work in safely. He chooses to make a rectangular region with dimensions x and y.

a. Write an equation that connects x, y and the perimeter.

b. Write y in terms of x.

c. Write an equation for the area of the region in terms of x.

d. Fill in the table for different values of x.

x	0	5	10	15	20	25	30	35	40	45	50
Area (m^2)											

e. Can x have a value more than 50? Why?

f. Sketch a graph of area against x.

g. Determine the value of x that makes the area a maximum.

h. What is the value of y for maximum area?

i. What shape encloses the maximum area?

j. Calculate the maximum area.

Ron decides to make a circular area with the barricade mesh.

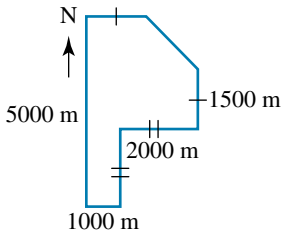
k. What is the radius of this circular region?

l. What area is enclosed in this circular region?

m. How much extra area does Ron now have compared to his rectangular region?

Communicating, reasoning and problem solving

16. Dan has purchased a country property with layout and dimensions as shown in the diagram.



a. Show that the property has a total area of 987.5 ha.

b. Dan wants to split the property in half (in terms of area) by building a straight fence running either north–south or east–west through the property. Assuming the cost of the fencing is a fixed amount per linear metre, justify where the fence should be built (that is, how many metres from the top left-hand corner and in which direction) to minimise the cost.

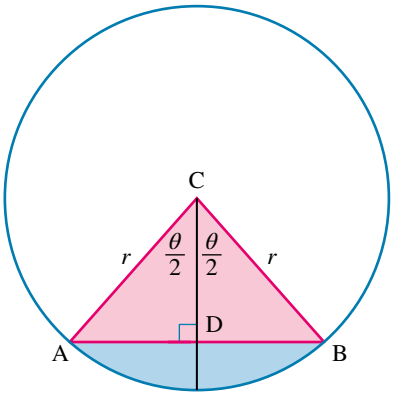
17. In question **15**, Ron the excavator operator could choose to enclose a rectangular or circular area with 100 m of barricade mesh. In this case, the circular region resulted in a larger safe work area.

a. Show that for 150 m of barricade mesh, a circular region again results in a larger safe work area as opposed to a rectangular region.

b. Show that for n metres of barricade mesh, a circular region will result in a larger safe work area as opposed to a rectangular region.

18. Proving the segment formula

Follow the steps below to prove the formula for the area of a segment using the fact that the area of the segment $=$ area of sector ABC $-$ 2 $\times$ area of triangle ACD.



a. Using trigonometry, show that $\frac{\text{AD}}{r} = \sin\left(\frac{\theta}{2}\right)$.

b. Show that $\frac{\text{CD}}{r} = \cos\left(\frac{\theta}{2}\right)$.

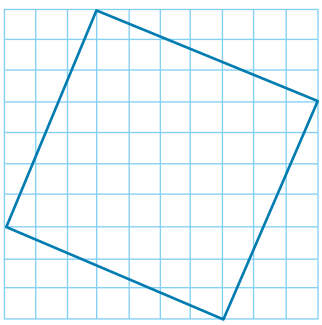
c. Show that the area of triangle ACD is $\frac{r^2 \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2}$.

Note that this formula is the same if θ is in degrees or radians.

d. Finally, show that the area of the segment (in purple) is $r^2\left(\pi \times \frac{\theta}{360^\circ} - \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right)$ if θ is in degrees.

19. How are perimeter and area different but fundamentally related?

20. The diagram shows one smaller square drawn inside a larger square on grid paper. Represent the area of the smaller square as a fraction of the larger square.

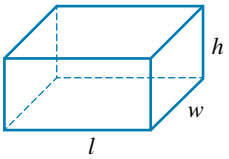


10.3 Total surface area [Stages 5.2 and 5.3]

10.3.1 TSA of rectangular prisms and cubes

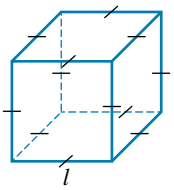
- The **total surface area (TSA)** of a solid is the sum of the areas of all the faces of that solid.
- The formula for finding the TSA of a **rectangular prism (cuboid)** is:

$$\text{TSA} = 2(lh + lw + wh)$$



- A special case of the rectangular prism is the **cube**, where all sides are equal ($l = w = h$).

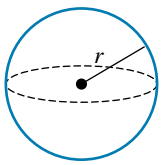
$$\text{TSA} = 6l^2$$



10.3.2 TSA of spheres and cylinders

Sphere

- $\text{TSA} = 4\pi r^2$



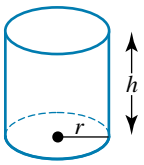
Note: The mathematics required to prove the formula for the total surface area of a sphere is beyond the scope of Year 10.

DISCUSSION

How does the surface area of a sphere compare to the area of a circle? How about the surface area of a hemisphere?

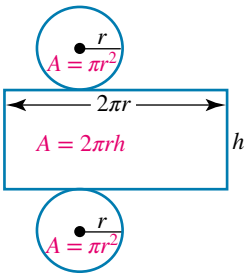
Cylinder

- $\text{TSA} = 2\pi r(r + h)$ or $2\pi r^2 + 2\pi rh$



- The formula for the TSA of a cylinder is found from the area of the net as shown.

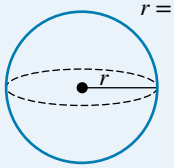
$$\begin{aligned}\text{TSA} &= \pi r^2 + \pi r^2 + 2\pi rh \\ &= 2\pi r^2 + 2\pi rh \\ &= 2\pi r(r + h)\end{aligned}$$



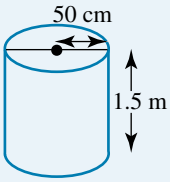
WORKED EXAMPLE 3

Find the total surface areas of the solids shown, correct to the nearest cm^2.

a



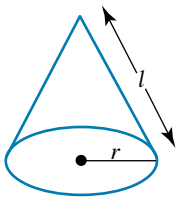
b

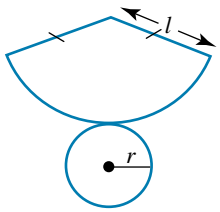


THINK	WRITE
a 1 Write the formula for the TSA of a sphere.	**a** TSA $= 4\pi r^2$
2 Identify the value for r.	$r = 7$
3 Substitute and evaluate.	TSA $= 4 \times \pi \times 7^2$ ≈ 615.8 cm^2 ≈ 616 cm^2
b 1 Write the formula for the TSA of a cylinder.	**b** TSA $= 2\pi r(r + h)$
2 Identify the values for r and h. Note that the units will need to be the same.	$r = 50$ cm, $h = 1.5$ m $= 150$ cm
3 Substitute and evaluate.	TSA $= 2 \times \pi \times 50 \times (50 + 150)$ $\approx 62\,831.9$ cm^2 $\approx 62\,832$ cm^2

10.3.3 TSA of cones

- The total surface area of a **cone** can be found by considering its net, which is comprised of a small circle and a sector of a larger circle.





r = radius of the cone
l = slant height of the cone

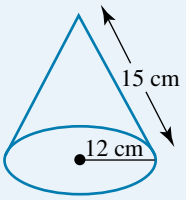
- The sector is a fraction of the full circle of radius l with circumference $2\pi l$.
- The sector has an arc length equivalent to the circumference of the base of the cone, $2\pi r$.
- The fraction of the full circle represented by the sector can be found by writing the arc length as a fraction of the circumference of the full circle, $\frac{2\pi r}{2\pi l} = \frac{r}{l}$.

  Area of a sector = fraction of the circle $\times \pi l^2$

  $$= \frac{r}{l} \times \pi l^2$$
  $$= \pi r l$$
- Thus, SA $= A_{\text{circular base}} + A_{\text{curved surface}}$
  $= \pi r^2 + \pi r l$
  $= \pi r(r + l)$
- Cone: TSA $= \pi r(r + l)$ or $\pi r^2 + \pi r l$

WORKED EXAMPLE 4

Find the total surface area of the cone shown correct to 1 decimal place.



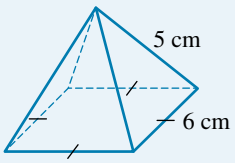
THINK	WRITE
1 Write the formula for the TSA of a cone.	TSA $= \pi r(r + l)$
2 State the values of r and l.	$r = 12$, $l = 15$
3 Substitute and evaluate.	TSA $= \pi \times 12 \times (12 + 15)$ $= 1017.9\ \text{cm}^2$

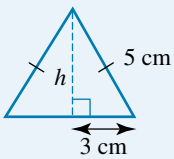
10.3.4 TSA of other solids

- The total surface area of a solid can be found by summing the areas of each face.
- The area of each face may need to be calculated separately.
- Check the total number of faces to ensure that none are left out.

WORKED EXAMPLE 5

Find the total surface area of the square-based pyramid shown.



THINK	WRITE/DRAW
1 There are five faces: the square base and four identical triangles.	TSA = Area of square base + area of four triangular faces
2 Find the area of the square base.	Area of base $= l^2$, where $l = 6$ Area of base $= 6^2$ $= 36\ \text{cm}^2$
3 Draw and label one triangular face and write the formula for finding its area.	 Area of a triangular face $= \frac{1}{2}bh$; $b = 6$
4 Find the height of the triangle, h, using Pythagoras' theorem.	$a^2 = c^2 - b^2$, where $a = h$, $b = 3$, $c = 5$ $= 5^2 - 3^2$ $h^2 = 25 - 9$ $h^2 = 16$ $h = 4\,\text{cm}$
5 Calculate the area of the triangular face by substituting $b = 6$ and $h = 4$.	Area of triangular face $= \frac{1}{2} \times 6 \times 4$ $= 12\ \text{cm}^2$

6 Calculate the TSA by adding the area of the square base and the area of four identical triangular faces together.

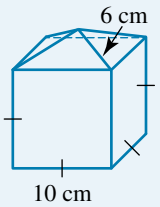
$$\begin{aligned}\text{TSA} &= 36 + 4 \times 12 \\ &= 36 + 48 \\ &= 84 \text{ cm}^2\end{aligned}$$

10.3.5 TSA of composite solids

- Composite solids are formed when two or more simple solids are joined together.
- The total surface area of a composite solid is calculated by summing the areas of the solid's external faces.

WORKED EXAMPLE 6

Find the total surface area of the solid shown correct to 1 decimal place.



THINK

1 The solid shown has 9 faces: five identical squares and four identical triangles.

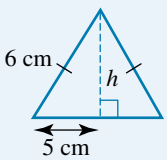
WRITE/DRAW

$\text{TSA} = 5 \times \text{area of a square} + 4 \times \text{area of a triangle}$

2 Find the area of one square face with the side length 10 cm.

$$\begin{aligned}A_{\text{square}} &= l^2, \text{ where } l = 10 \\ A &= 10^2 \\ A &= 100 \text{ cm}^2\end{aligned}$$

3 Draw a triangular face and work out its height using Pythagoras' theorem.



$$\begin{aligned}a^2 &= c^2 - b^2, \text{ where } a = h, b = 5, c = 6 \\ h^2 &= 6^2 - 5^2 \\ h^2 &= 36 - 25 \\ h^2 &= 11 \\ h &= 3.316\,62 \ldots \text{ cm (or with rounding, } h = 3.3)\end{aligned}$$

4 Find the area of one triangular face.

$$\begin{aligned}A_{\text{triangle}} &= \tfrac{1}{2}bh, \text{ where } b = 10, h = 3.316\,62 \\ &= \tfrac{1}{2} \times 10 \times 3.316\,62... \\ &= 16.5831... \text{ cm}^2 \text{ (or, with rounding, } A_{\text{triangle}} = 16.6 \text{ cm}^2)\end{aligned}$$

5 Find the TSA of the solid by adding the area of 5 squares and 4 triangles together.

$$\begin{aligned}\text{TSA} &= 5 \times 100 + 4 \times 16.5831... \\ &= 500 + 66.3324... \\ &\approx 566.3 \text{ cm}^2 \text{ (or } = 566 \text{ using the previously rounded value)}\end{aligned}$$

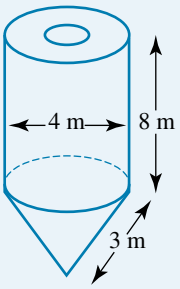
Note: Rounding is not done until the final step. If h had been rounded to 3.3 in step 3 and this value used in steps 4 and 5, the decimal place value of the TSA would have been lost. It is important to realise that rounding too early can affect the accuracy of results.

WORKED EXAMPLE 7

The silo shown is to be built from metal. The top portion of the silo is a cylinder of diameter 4 m and height 8 m. The bottom part of the silo is a cone of slant height 3 m. The silo has a circular opening of radius 30 cm on the top.

a What area of metal (to the nearest m^2) is required to build the silo?

b If it costs \$12.50 per m^2 to cover the surface with an anti-rust material, how much will it cost to cover the silo completely?



THINK

a 1 The surface area of the silo consists of an annulus, the curved part of the cylinder and the curved section of the cone.

2 To find the area of the annulus, subtract the area of the small circle from the area of the larger circle. Let R = radius of small circle.

3 The middle part of the silo is the curved part of a cylinder. Find its area. (Note that in the formula $TSA_{cylinder} = 2\pi r^2 + 2\pi rh$, the curved part is represented by $2\pi rh$.)

4 The bottom part of the silo is the curved section of a cone. Find its area. (Note that in the formula $TSA_{cone} = \pi r^2 + \pi rl$, the curved part is given by πrl.)

5 Find the total surface area of the silo by finding the sum of the surface areas calculated above.

6 Write the answer in words.

b To find the total cost, multiply the total surface area of the silo by the cost of the anti-rust material per m^2(\$12.50).

WRITE

a TSA = area of annulus
+ area of curved section of a cylinder
+ area of curved section of a cone

Area of annulus $= A_{\text{large scale}} - A_{\text{small circle}}$
$= \pi r^2 - \pi R^2$

where $r = \frac{4}{2} = 2$ m and $R = 30$ cm $= 0.3$ m.

Area of annulus $= \pi \times 2^2 - \pi \times 0.3^2$
$= 12.28$ m

Area of curved section of cylinder $= 2\pi rh$

where $r = 2$ and $h = 8$.

Area of curved section of cylinder
$= 2 \times \pi \times 2 \times 8$
$= 100.53$ m^2

Area of curved section of cone $= \pi rl$

where $r = 2$ and $l = 3$.

Area of curved section of cone $= \pi \times 2 \times 3$
$= 18.85$ m^2

TSA $= 12.28 + 100.53 + 18.85$
$= 131.66$ m^2

The area of metal required is 132 m^2, correct to the nearest square metre.

b Cost $= 132 \times \$12.50$
$= \$1650.00$

learnON RESOURCES — ONLINE ONLY

Interactivity: Surface of a prism (int-6079)
Interactivity: Surface area of a cylinder (int-6080)
Interactivity: Surface area (int-6477)
Interactivity: TSA — sphere (int-2782)
eLesson: Total surface area of prisms (eles-1909)
Digital doc: WorkSHEET Surface area (doc-5242)

Exercise 10.3 Total surface area

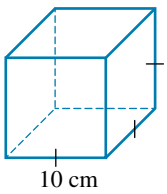
Individual pathways

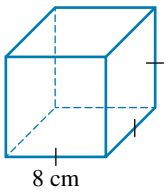
PRACTISE	CONSOLIDATE	MASTER
Questions: 1–4, 6a–e, 7, 10, 12	Questions: 1–4, 6, 7, 9–12, 15, 18	Questions: 1–8, 10–18

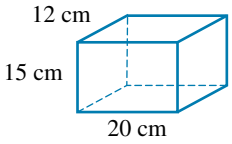
To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

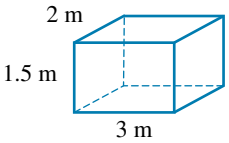
Understanding and fluency

1. Find the total surface areas of the solids shown.

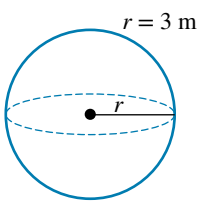
a.


b.


c.


d.


2. WE3 Find the total surface areas of the solids shown. Give your answers correct to 1 decimal place.

a.


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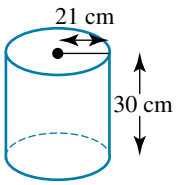
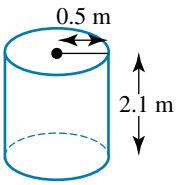
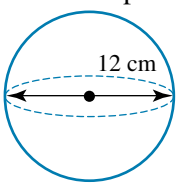
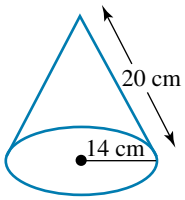
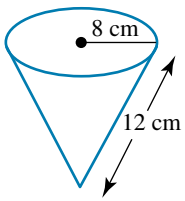


c.


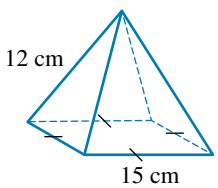
d.


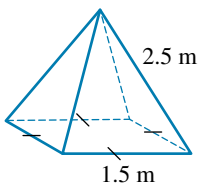
3. WE4 Find the total surface areas of the cones shown. Give your answers correct to 1 decimal place.

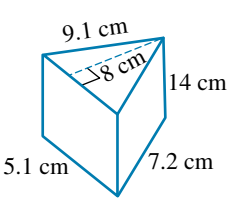
a.


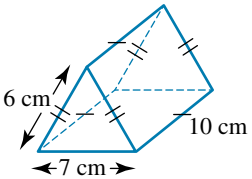
b.


4. WE5 Find the total surface areas of the solids shown. Give your answers correct to 1 decimal place.

a.


b.


c.


d.


5. Find the surface areas of the following.

a. A cube of side length 1.5 m

b. A rectangular prism 6 m × 4 m × 2.1 m

c. A cylinder of radius 30 cm and height 45 cm, open at one end

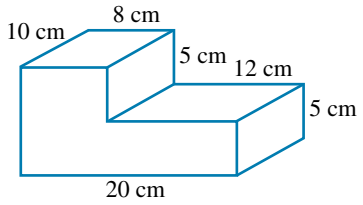
d. A sphere of radius 28 mm

e. An open cone of radius 4 cm and slant height 10 cm

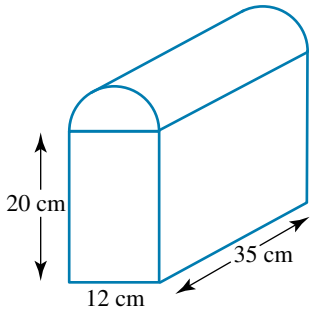
f. A square pyramid of base length 20 cm and slant edge 30 cm

6. WE6 Find the total surface areas of the objects shown.

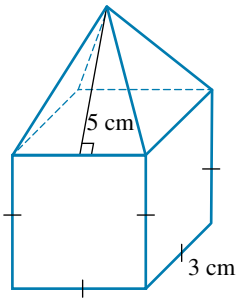
a.



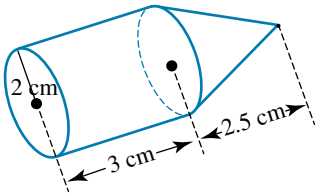
b.



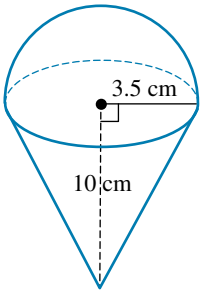
c.



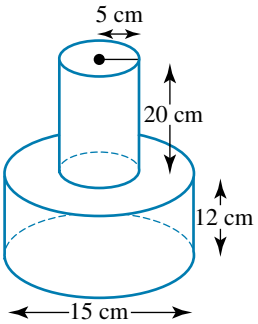
d.



e.



f.

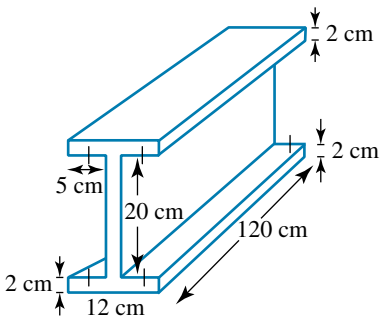


7. MC A cube has a total surface area of 384 cm^2. The length of the edge of the cube is:

A. 9 cm **B.** 8 cm **C.** 7 cm **D.** 6 cm **E.** 5 cm

8. Open cones are made from nets cut from a large sheet of paper 1.2 m × 1.0 m. If a cone has a radius of 6 cm and a slant height of 10 cm, how many cones can be made from the sheet? (Assume there is 5% wastage of paper.)

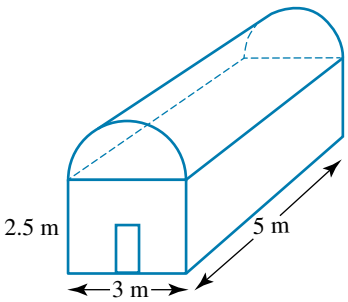
9. A steel girder is to be painted. Calculate the area of the surface to be painted.



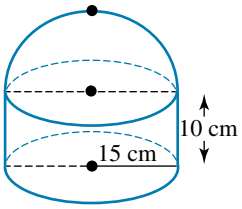
10. WE7 The greenhouse shown below is to be built using shade cloth. It has a wooden door of dimensions 1.2 m × 0.5 m.

a. Find the total area of shade cloth needed to complete the greenhouse.

b. Find the cost of the shade cloth at $6.50 per m^2.



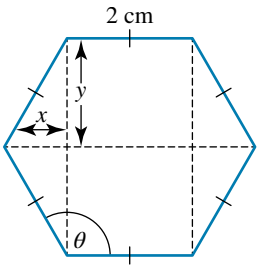
11. A cylinder is joined to a hemisphere to make a cake holder, as shown below. The surface of the cake holder is to be chromed at 5.5 cents per cm^2.



a. Find the total surface area to be chromed.

b. Find the cost of chroming the cake holder.

12. A soccer ball is made up of a number of hexagons sewn together on its surface. Each hexagon can be considered to have dimensions as shown in the diagram.



a. Calculate $\theta°$.

b. Calculate the values of x and y exactly.

c. Calculate the area of the trapezium in the diagram.

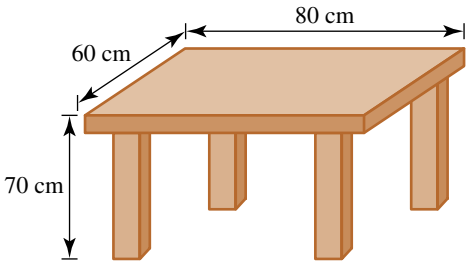
d. Hence, determine the area of the hexagon.

e. If the total surface area of the soccer ball is $192\sqrt{3}\,cm^2$, how many hexagons are on the surface of the soccer ball?

Communicating, reasoning and problem solving

Complete the following question without the aid of a calculator.

13. The table shown below is to be varnished (including the base of each leg). The table top has a thickness of 180 mm, and the cross-sectional dimensions of the legs are 50 mm by 50 mm.



A friend completes the calculation as shown. Assume there are no simple calculating errors. Analyse the working presented and justify if the TSA calculated is correct.

Table top (inc. leg bases)	0.96	$2 \times (0.8 \times 0.6)$
Legs	0.416	$16 \times (0.52 \times 0.05)$
Table top edging	0.504	$0.18 \times (2(0.8 + 0.6))$
TSA	1.88 m^2	

14. A shower recess with dimensions 1500 mm (back wall) by 900 mm (side wall) needs to have the back and two side walls tiled to a height of 2 m.

a. Calculate the area to be tiled in m^2.

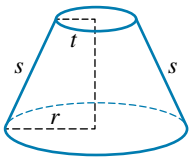
b. Justify that 180 tiles (including those that need to be cut) of dimension 20 cm by 20 cm will be required. Disregard the grout and assume that once a tile is cut, only one piece of the tile can be used.

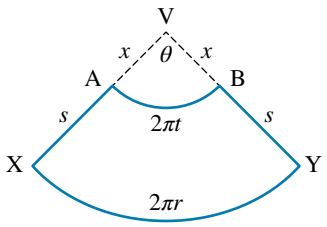
c. Evaluate the cheapest option of tiling out of the following options: \$1.50/tile; \$39.50/box, where a box covers $1\,m^2$; or tiles of dimension 30 cm by 30 cm costing \$3.50/tile.

15. If the surface area of a sphere to that of a cylinder is in the ratio 4 : 3 and the sphere has a radius of $3a$, show that if the radius of the cylinder is equal to its height, then the radius of the cylinder is $\frac{3\sqrt{3}a}{2}$.

16. Frustum of a cone

A frustum is a cone with the top sliced off (see the diagram). When the curved side is 'opened up', it creates a shape, ABYX, as shown in the diagram.





a. Write an expression for the arc length XY in terms of the angle θ. Write another expression for the arc length AB in terms of the same angle θ. Show that, in radians, $\theta = \frac{2\pi(r-t)}{s}$.

i. Using the above formula for θ, show that $x = \frac{st}{(r-t)}$.

ii. Use similar triangles to confirm this formula.

b. Determine the area of sectors AVB and XVY and hence determine the area of ABYX. Add the areas of the 2 circles to the area of ABYX to determine the TSA of a frustum.

17. Tina is re-covering the cushion of a footstool in the shape of a cylinder with diameter 50 cm and height 30 cm. She also intends to cover the base (that is, the underside) of the cushion. She has 1 m^2 of fabric to cover the cushion. When calculating the area of fabric required, allow an extra 20% of the total surface area to cater for seams and pattern placings. Explain whether Tina has enough material to cover the cushion.

18. Why is calculating the total surface area of a composite solid more difficult than for a simple solid such as a rectangular prism or cylinder?

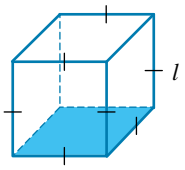
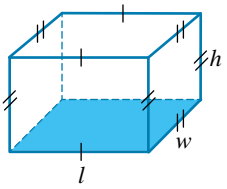
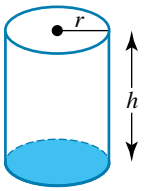
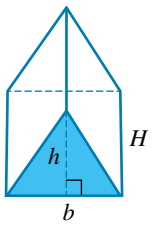
10.4 Volume [Stages 5.2 and 5.3]

10.4.1 Definition of volume

- The **volume** of a 3-dimensional object is the amount of space it takes up.
- Volume is commonly measured in units of mm^3, cm^3 and m^3.

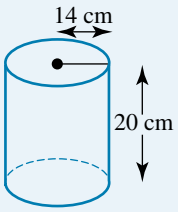
10.4.2 Volume of prisms and cylinders

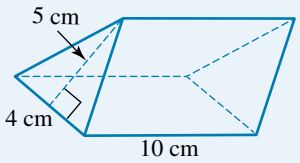
- The volume of any solid with a uniform cross-sectional area is given by the formula $V = AH$, where A is the cross-sectional (or base) area and H is the height of the solid.

Shape	Formula
Cube 	Volume $= AH$ $=$ area of a square $\times$ height $= l^2 \times l$ $= l^3$
Rectangular prism 	Volume $= AH$ $=$ area of a rectangle $\times$ height $= lwh$
Cylinder 	Volume $= AH$ $=$ area of a circle $\times$ height $= \pi r^2 h$
Triangular prism 	Volume $= AH$ $=$ area of a triangle $\times$ height $= \frac{1}{2}bh \times H$

WORKED EXAMPLE 8

Find the volumes of the following shapes.

a. 

b. 

THINK	WRITE
a 1 Write the formula for the volume of the cylinder (prism).	**a** $V = AH$ $= \pi r^2 h$
2 Identify the value of the pronumerals.	$r = 14, h = 20$
3 Substitute and evaluate.	$V = \pi \times 14^2 \times 20$ $\approx 12315.04 \text{ cm}^3$
b 1 Write the formula for the volume of a triangular prism.	**b** $V = AH = \frac{1}{2}bh \times H$
2 Identify the value of the pronumerals. (Note that h is the height of the triangle and H is the depth of the prism.)	$b = 4, h = 5, H = 10$
3 Substitute and evaluate.	$V = \frac{1}{2} \times 4 \times 5 \times 10$ $= 100 \text{ cm}^3$

WORKED EXAMPLE 9

a What effect will doubling each of the side lengths of a cube have on its volume?
b What effect will halving the radius and doubling the height of a cylinder have on its volume?

THINK	WRITE
a 1 Write the formula for the volume of the cube.	**a** $V = l^3$
2 Identify the value of the pronumeral. *Note:* Doubling is the same as multiplying by 2.	$l_{\text{new}} = 2l$
3 Substitute and evaluate.	$V_{\text{new}} = (2l)^3$
4 Compare the answer obtained in step 3 with the volume of the original shape.	$= 8l^3$
5 Write your answer.	Doubling each side length of a cube increases the volume by a factor of 8; that is, the new volume will be 8 times as large as the original volume.
b 1 Write the formula for the volume of the cylinder.	**b** $V = \pi r^2 h$
2 Identify the value of the pronumerals. *Note:* Halving is the same as dividing by 2.	$r_{\text{new}} = \frac{r}{2}$, $h_{\text{new}} = 2h$
3 Substitute and evaluate.	$V_{\text{new}} = \pi\left(\frac{r}{2}\right)^2 2h$
	$= \pi \times \frac{r^2}{2^{\cancel{4}}} \times \cancel{2}h$
	$= \frac{\pi r^2 h}{2}$
4 Compare the answer obtained in step 3 with the volume of the original shape.	$= \frac{1}{2}\pi r^2 h$
5 Write your answer.	Halving the radius and doubling the height of a cylinder decreases the volume by a factor of 2; that is, the new volume will be half the original volume.

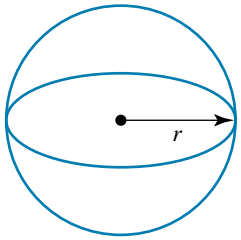
ACTIVITY: THE BEST SOFT-DRINK CAN

An aluminium soft-drink can has a capacity of 375 mL. The manufacturer of the can wishes to use the least amount of aluminium. What are the dimensions of a can that would achieve this end?

1. In pairs, use the formula for the volume of a cylinder to show that the can's height, h, can be calculated by $h = \frac{375}{\pi r^2}$.
2. Substitute this expression into the formula for the volume of a cylinder.
3. Using trial and error, or another method, determine the value of r that produces the smallest surface area. (This would use the least amount of aluminium.)
4. Using the formula for h from step 1 and the value for r from step 3, determine the value of h.
5. Discuss reasons why actual drink cans have different dimensions from your results.

10.4.3 Volume of spheres

- The volume of a sphere of radius r can be calculated using the formula $V = \frac{4}{3}\pi r^3$.



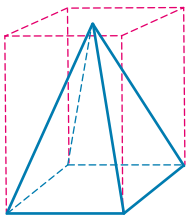
WORKED EXAMPLE 10

Find the volume of a sphere of radius 9 cm. Answer correct to 1 decimal place.

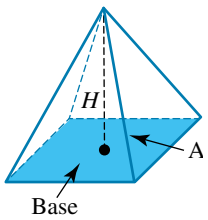
THINK	WRITE
1 Write the formula for the volume of a sphere.	$V = \frac{4}{3}\pi r^3$
2 Identify the value of r.	$r = 9$
3 Substitute and evaluate.	$V = \frac{4}{3} \times \pi \times 9^3$ $= 3053.6\,\text{cm}^3$

10.4.4 Volume of pyramids

- Pyramids are not prisms, as the cross-section of a pyramid changes from the base upwards.

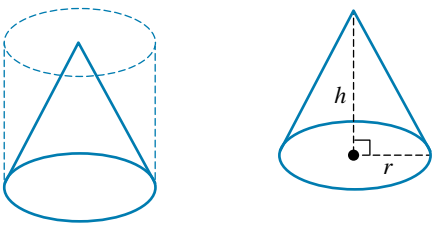


- The volume of a pyramid is one-third the volume of an equivalent prism with the same base area and height.



Volume of a pyramid $= \frac{1}{3}AH$

- Since a cone is a pyramid with a circular cross-section, the volume of a cone is one-third the volume of a cylinder with the same base area and height.



Volume of a cone $= \frac{1}{3}AH$

$= \frac{1}{3}\pi r^2 h$

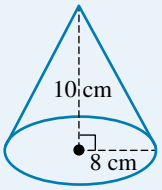
DISCUSSION

If a triangular prism and a rectangular prism have the same volume, what can be said about their surface areas?

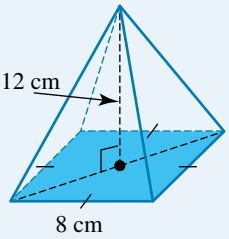
WORKED EXAMPLE 11

Find the volume of each of the following solids.

a



b



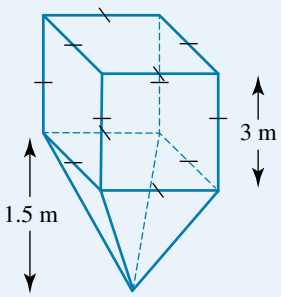
THINK	WRITE
a 1 Write the formula for the volume of a cone.	**a** $V = \frac{1}{3}\pi r^2 h$
2 Identify the values of r and h.	$r = 8, h = 10$
3 Substitute and evaluate.	$V = \frac{1}{3} \times \pi \times 8^2 \times 10$ $= 670.21 \text{ cm}^3$
b 1 Write the formula for the volume of a pyramid.	**b** $V = \frac{1}{3}AH$
2 Find the area of the square base.	$A = l^2$ where $l = 8$ $A = 8^2$ $= 64 \text{ cm}$
3 Identify the value of H.	$H = 12$
4 Substitute and evaluate.	$V = \frac{1}{3} \times 64 \times 12$ $= 256 \text{ cm}^3$

10.4.5 Volume of composite solids

- A composite solid is a combination of a number of solids.
- The volume of each smaller solid component can be calculated separately.
- The volume of a composite solid is calculated by summing the volumes of each of the smaller solid components.

WORKED EXAMPLE 12

Calculate the volume of the composite solid shown.



THINK	WRITE
1 The given solid is a composite figure, made up of a cube and a square-based pyramid.	V = volume of cube + volume of pyramid
2 Find the volume of the cube.	$V_{\text{cube}} = l^3$ where $l = 3$ $V_{\text{cube}} = 3^3$ $= 27 \text{ m}^3$
3 Write the formula for finding the volume of a square-based pyramid.	$V_{\text{square-based pyramid}} = \frac{1}{3}AH$
4 Find the area of the square base.	$A = l^2$ $= 3^2$ $= 9 \text{ m}^2$
5 Identify the value of H.	$H = 1.5$
6 Substitute and evaluate the volume of the pyramid.	$V_{\text{square-based pyramid}} = \frac{1}{3} \times 9 \times 1.5$ $= 4.5 \text{ m}^3$
7 Find the total volume by adding the volume of the cube and pyramid.	$V = 27 + 4.5$ $= 31.5 \text{ m}^3$

10.4.6 Capacity

- Some 3-dimensional objects are hollow and can be filled with liquid or some other substance.
- The amount of substance which a container can hold is called its **capacity**.
- Capacity is essentially the same as volume but is usually measured in mL, L and kL, where 1 mL = 1 cm^3
  1 L = 1000 cm^3
  1 kL = 1 m^3.

WORKED EXAMPLE 13

Find the capacity (in litres) of a cuboidal aquarium, which is 50 cm long, 30 cm wide and 40 cm high.

THINK	WRITE
1 Write the formula for the volume of a rectangular prism.	$V = lwh$
2 Identify the values of the pronumerals.	$l = 50$, $w = 30$, $h = 40$
3 Substitute and evaluate.	$V = 50 \times 30 \times 40$ $= 60\,000 \text{ cm}^3$

4 State the capacity of the container in millilitres, using 1 cm^3 = 1 mL. = 60 000 mL

5 Since 1 L = 1000 mL, to convert millilitres to litres, divide by 1000. = 60 L

6 Give a worded answer. The capacity of the fish tank is 60 L.

learnON RESOURCES — ONLINE ONLY

Interactivity: Conversion chart for volume (int-3791)
Interactivity: Volume of solids (int-3794)
Interactivity: Volume (int-6476)
Interactivity: Maximising the volume of a cuboid (int-1150)
Digital doc: SkillSHEET Conversion of volume units (doc-5239)
Digital doc: SkillSHEET Volume of cubes and rectangular prisms (doc-5240)
Digital doc: WorkSHEET Volume (doc-6733)

Exercise 10.4 Volume

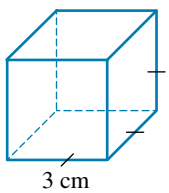
Individual Pathways

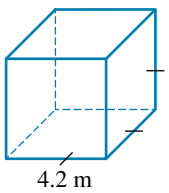
PRACTISE	CONSOLIDATE	MASTER
Questions: 1–4, 6–8, 9a, 10, 13, 14, 20	Questions: 1–8, 10–12, 14, 16, 19, 20, 22, 25	Questions: 1–18, 20–26, 28

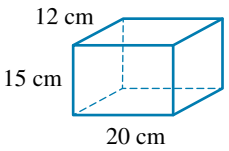
To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

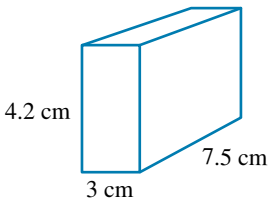
Understanding and fluency

1. Find the volumes of the following prisms.

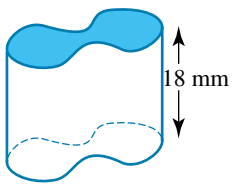
a.


b.


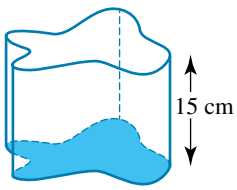
c.


d.


2. Calculate the volumes of the following solids.

a.


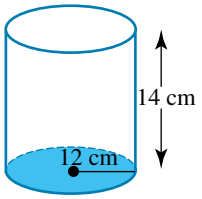
[Base area: 25 mm^2]

b.


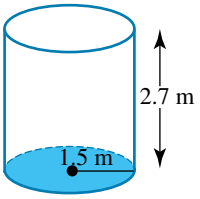
[Base area: 24 cm^2]

3. WE8 Find the volumes of the following solids. Give each answer correct to 1 decimal place where appropriate.

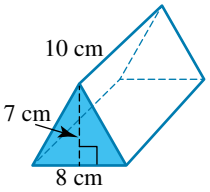
a.



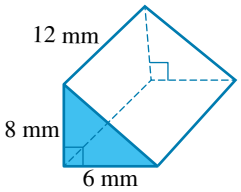
b.



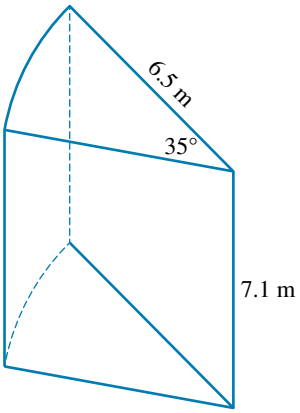
c.



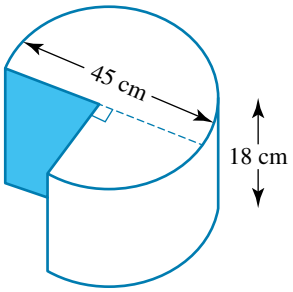
d.



e.



f.

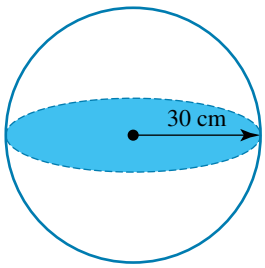


4. WE10 Find the volume of a sphere (correct to 1 decimal place) with a radius of:

a. 1.2 m **b.** 15 cm **c.** 7 mm **d.** 50 cm.

5. Find the volume of each of these figures, correct to 2 decimal places.

a.



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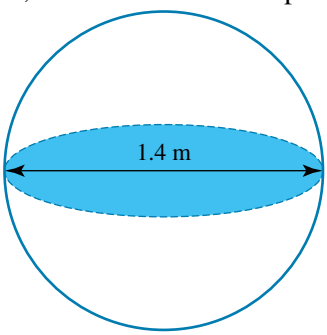

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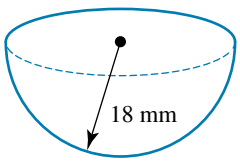
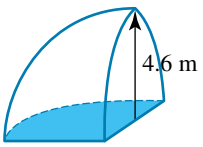
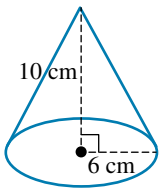


d.

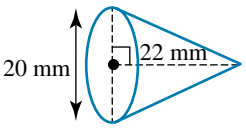


6. WE11a Find the volume of each of the following cones, correct to 1 decimal place.

a.

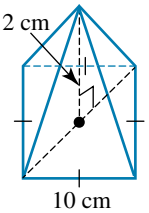


b.

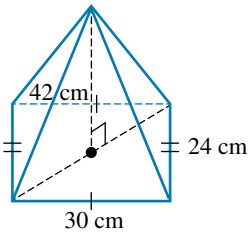


7. WE11b Find the volume of each of the following pyramids.

a.

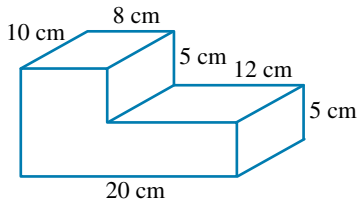


b.

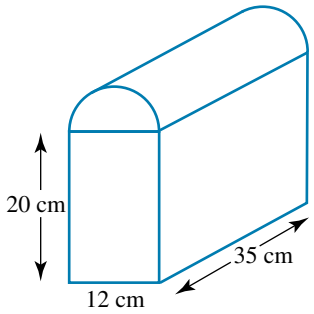


8. WE12 Calculate the volume of each of the following composite solids, correct to 2 decimal places where appropriate.

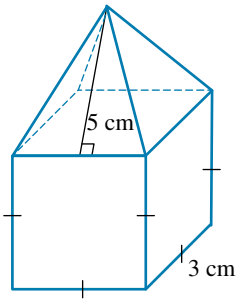
a.



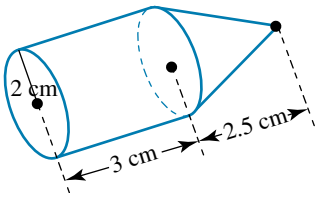
b.



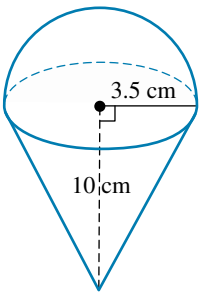
c.



d.



e.



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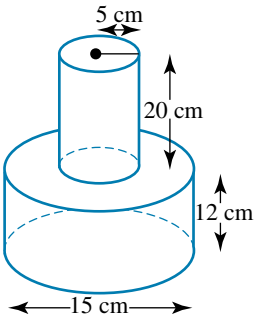


9. WE9 **a.** What effect will tripling each of the side lengths of a cube have on its volume?

b. What effect will halving each of the side lengths of a cube have on its volume?

c. What effect will doubling the radius and halving the height of a cylinder have on its volume?

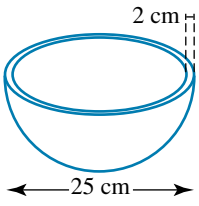
d. What effect will doubling the radius and dividing the height of a cylinder by 4 have on its volume?

e. What effect will doubling the length, halving the width and tripling the height of a rectangular prism have on its volume?

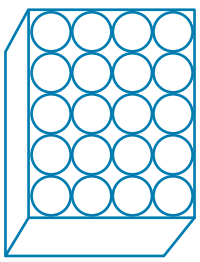
10. MC A hemispherical bowl has a thickness of 2 cm and an outer diameter of 25 cm. If the bowl is filled with water, the capacity of the water will be closest to:

A. 1.526 L **B.** 1.308 33 L **C.** 3.052 08 L

D. 2.616 66 L **E.** 2.424 52 L



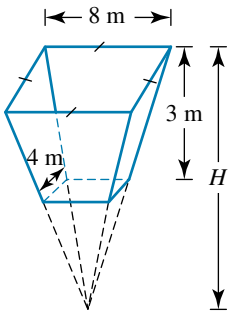
11. Tennis balls of diameter 8 cm are packed in a box 40 cm × 32 cm × 10 cm, as shown. How much space is left unfilled?



12. WE13 A cylindrical water tank has a diameter of 1.5 m and a height of 2.5 m. What is the capacity (in litres) of the tank?

13. A monument in the shape of a rectangular pyramid (base length of 10 cm, base width of 6 cm, height of 8 cm), a spherical glass ball (diameter of 17 cm) and conical glassware (radius of 14 cm, height of 10 cm) are packed in a rectangular prism of dimensions 30 cm by 25 cm by 20 cm. The extra space in the box is filled up by a packing material. What volume of packing material is required?

14. A swimming pool is being constructed so that it is the upper part of an inverted square-based pyramid.



a. Calculate H.
b. Calculate the volume of the pool.
c. How many 6-m^3 bins will be required to take the dirt away?
d. How many litres of water are required to fill this pool?
e. How deep is the pool when it is half-filled?

15. A soft drink manufacturer is looking to repackage cans of soft drink to minimise the cost of packaging while keeping the volume constant. Consider a can of soft drink with a capacity of 400 mL.

a. If the soft drink was packaged in a spherical can:
i. find the radius of the sphere
ii. find the total surface area of this can.
b. If the soft drink was packaged in a cylindrical can with a radius of 3 cm:
i. find the height of the cylinder
ii. find the total surface area of this can.
c. If the soft drink was packaged in a square-based pyramid with a base side length of 6 cm:
i. find the height of the pyramid
ii. find the total surface area of this can.
d. Which can would you recommend the soft drink manufacturer use for its repackaging? Why?

16. The volume of a cylinder is given by the formula $V = \pi r^2 h$.

a. Transpose the formula to make h the subject.

b. A given cylinder has a volume of 1600 cm^3. Find its height if it has a radius of:

i. 4 cm **ii.** 8 cm.

c. Transpose the formula to make r the subject.

d. What restrictions must be placed on r? Why?

e. A given cylinder has a volume of 1800 cm^3. Find its radius if it has a height of:

i. 10 cm **ii.** 15 cm.

17. A toy maker has enough rubber to make one super-ball of radius 30 cm. How many balls of radius 3 cm can he make from this rubber?

18. A manufacturer plans to make a cylindrical water tank to hold 2000 L of water.

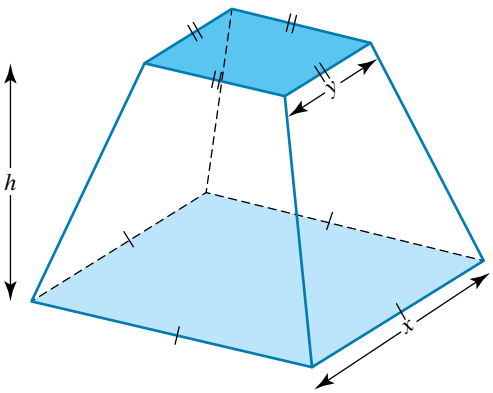
a. What must the height be if he uses a radius of 500 cm?

b. What must the radius be if he uses a height of 500 cm?

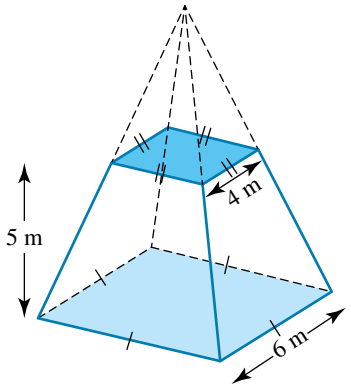
c. What will be the surface area of each of the two tanks? Assume the tank is a closed cylinder and give your answer in square metres.

19. The ancient Egyptians knew that the volume of the frustum of a square-based pyramid was given by the formula $V = \frac{1}{3}h(x^2 + xy + y^2)$, although how they discovered this is unclear. (A frustum is the part of a cone or pyramid that is left when the top is cut off.)

a. Find the volume of the frustum shown below.



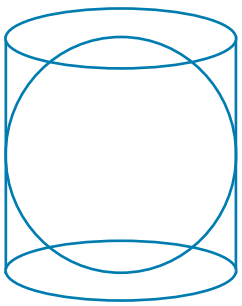
b. What would be the volume of the missing portion of the square-based pyramid shown?



Communicating, reasoning and problem solving

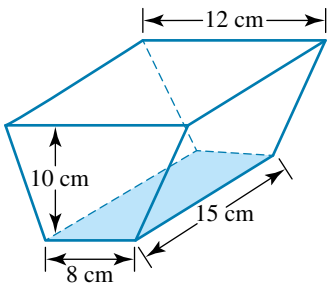
20. Archimedes is considered to be one of the three greatest mathematicians of all time (along with Newton and Gauss). He discovered several of the formulas used in this chapter. Inscribed on his tombstone was a diagram of his proudest discovery. It shows a sphere inscribed (fitting exactly) into a cylinder.

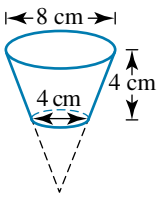
$$\text{Show that } \frac{\text{volume of the cylinder}}{\text{volume of the sphere}} = \frac{\text{surface area of the cylinder}}{\text{surface area of the sphere}}.$$



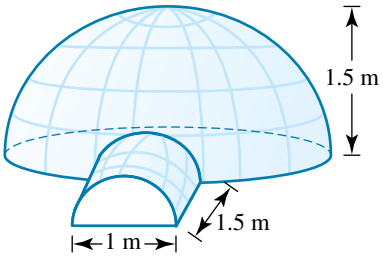
21. Marion has mixed together ingredients for a cake. The recipe requires a baking tin that is cylindrical in shape with a diameter of 20 cm and a height of 5 cm. Marion only has a tin as shown below left and a muffin tray consisting of 24 muffin cups. Each of the muffin cups in the tray is a portion of a cone as shown in the diagram below right.

Should Marion use the tin or the muffin tray? Explain.





22. Nathaniel and Andrew are going to the snow for survival camp. They plan to construct an igloo, consisting of an entrance and hemispherical structure, as shown. Nathaniel and Andrew are asked to redraw their plans and increase the size of the liveable region (hemispherical structure) so that the total volume (including the entrance) is doubled. How can this be achieved?



23. Sam is having his 16th birthday party and wants to make an ice trough to keep drinks cold. He has found a square piece of sheet metal with a side length of 2 metres. He cuts squares of side length x metres from each corner, then bends the sides of the remaining sheet. When four squares of the appropriate side length are cut from the corners, the capacity of the trough can be maximised at 588 litres. Explain how Sam should proceed to maximise the capacity of the trough.

24. The Hastings family house has a rectangular roof with dimensions 17 m × 10 m providing water to three cylindrical water tanks, each with a radius of 1.25 m and a height of 2.1 m. Show that approximately 182 millimetres of rain must fall on the roof to fill the tanks.

25. Six tennis balls are just contained in a cylinder as the balls touch the sides and the end sections of the cylinder. Each tennis ball has a radius of R cm.

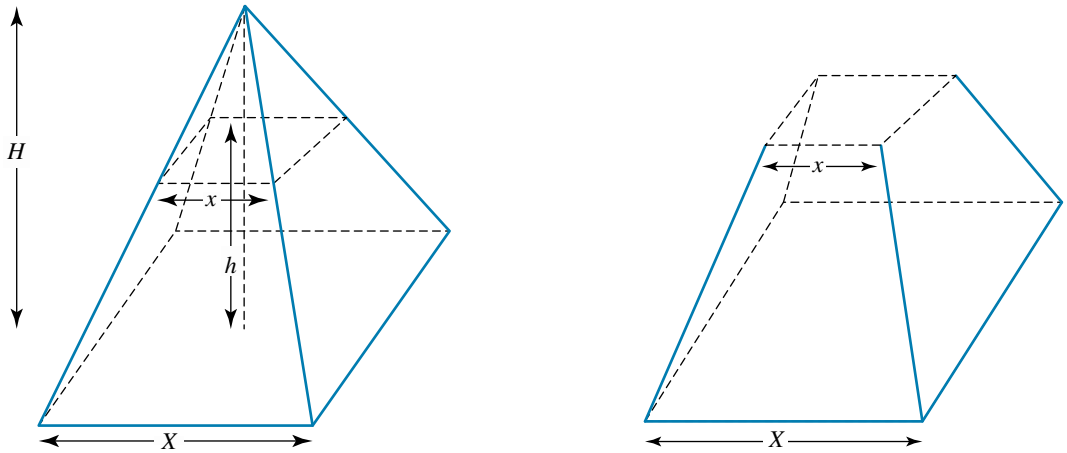
a. Express the height of the cylinder in terms of R.

b. Find the total volume of the tennis balls.

c. Find the volume of the cylinder in terms of R.

d. Show that the ratio of the volume of the tennis balls to the volume of the cylinder is 2 : 3.

26. A frustum of a square-based pyramid is a square pyramid with the top sliced off. H is the height of the full pyramid and h is the height of the frustum.



a. Find the volume of the large pyramid, which has a square base side of X cm.

b. Find the volume of the small pyramid, which has a square base side of x cm.

c. Show that the relationship between H and h is given by $H = \frac{Xh}{X - x}$.

d. Show that the volume of the frustum is given by $\frac{1}{3}h(X^2 + x^2 + Xx)$.

27. Volume is measured in cubic units. How is this reflected in the volume formula?

28. A large container is five-eighths full of ice-cream. After removing 27 identical scoops it is one-quarter full. How many scoops of ice-cream are left in the container?

10.5 Review

Investigation | Rich task

So close!

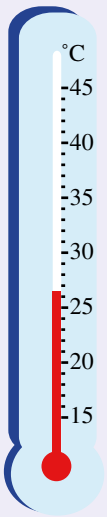
Humans must measure! Imagine what a chaotic world it would be if we didn't measure anything. Some of the things we measure are time, length, weight and temperature. We also use other measures derived from these, such as area, volume, speed.

Accurate measurement is important. The accuracy of a measurement depends on the instrument being used to measure and the interpretation of the measurement. There is no such thing as a perfectly accurate measurement. The best we can do is learn how to make meaningful use of the numbers we read off our devices. It is also important to use appropriate units of measurement.

Measurement errors

When we measure a quantity by using a scale, the accuracy of our measurement depends on the markings on the scale. For example, the ruler shown can measure both in centimetres and millimetres.

Measurements made with this ruler would have $\pm$ 0.5 mm added to the measurement. The quantity $\pm$ 0.5 is called the tolerance of measurement or measurement error.



$$\text{Tolerance of measurement} = \frac{1}{2} \times \text{size of smallest marked unit}$$

For a measurement of 5.6 ± 0.5 mm, the largest possible value is 5.6 cm + 0.5 mm = 5.65 cm, and the smallest value is 5.6 cm − 0.5 mm = 5.55 cm.

1. For the thermometer scale shown:
- **a.** determine the temperature
- **b.** state the measurement with its tolerance
- **c.** determine the largest and smallest possible values.

2. Calculate the largest and smallest values for:
- **a.** $(56.2 \pm 0.1) - (19.07 \pm 0.05)$
- **b.** $(78.4 \pm 0.25) \times (34 \pm 0.1)$.

Significant figures in measurement

A significant figure is any non zero-digit, any zero appearing between two non-zero digits, any trailing zero in a number containing a decimal point, or any digit in the decimal places. For example, the number 345.6054 has 7 significant figures, whereas 300 has 1 significant figure.

The number of significant figures is an expression of the accuracy of a measurement. The greater the number of significant figures, the more accurate the measurement. For example, a fast food chain claims it has sold 6000000000 hamburgers, not 6453456102. The first measurement has only 1 significant figure and is a very rough approximation of the actual number sold, which has 10 significant figures.

Reducing the number of significant figures is a process that is similar to rounding.

Rounding and measurement error in calculations

When you perform calculations, it is important to keep as many significant digits as practical, and to perform any rounding as the *final* step. For example, calculating 5.34×341 by rounding to 2 significant figures *before* multiplying gives $5.30 \times 340 = 1802$, compared with 1820 if the rounding is carried out after the multiplication.

Calculations that involve numbers from measurements containing errors can result in answers with even larger errors. The smaller the tolerances, the more accurate the answers will be.

3. a. Calculate $45943.4503 \times 86.765303$ by:

i. first rounding each number to 2 significant figures

ii. rounding only the answer to 2 significant figures.

b. Compare the two results.

Error in area and volume resulting from an error in a length measurement

The side length of a cube is measured and incorrectly recorded as 5 cm. The actual size is 6 cm. The effect of the length measurement error used on calculations of the surface area is shown below. Complete the calculations for volume.

Error used in length measurement $= 1$ cm

Surface area calculated with incorrectly recorded value $= 5^2 \times 6 = 150\ \text{cm}^2$

Surface area calculated with actual value $= 6^2 \times 6 = 216\ \text{cm}^2$

Percentage error $= \dfrac{216 - 150}{6} \times 100\% \approx 30.5\%$

4. a. Complete a similar calculation for the volume of the cube using the incorrectly recorded length. What conclusion can you make regarding errors when the number of dimensions increase?

b. Give three examples of a practical situation where an error in measuring or recording would have a potentially disastrous impact.

learnON RESOURCES — ONLINE ONLY

Digital doc: Investigation — So close! (doc-15926)

Digital doc: Code puzzle — Australian inventions! (doc-15927)

Digital doc: Summary Topic 10 (doc-22908)

Interactivity: Word search Topic 10 (int-2841)

Interactivity: Crossword Topic 10 (int-2842)

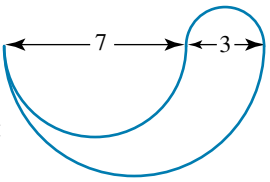
Interactivity: Sudoku Topic 10 (int-3593)

Exercise 10.5 Review questions

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

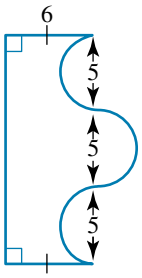
1. MC If all measurements are in cm, the area of the figure at right is:



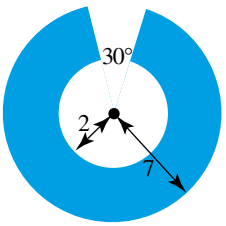
A. 16.49 cm^2 **B.** 39.25 cm^2 **C.** 9.81 cm^2
D. 23.56 cm^2 **E.** 30 cm^2

2. MC If all measurements are in centimetres, the area of the figure below is:

A. 50.73 cm^2 **B.** 99.82 cm^2 **C.** 80.18 cm^2
D. 90 cm^2 **E.** 119.45 cm^2

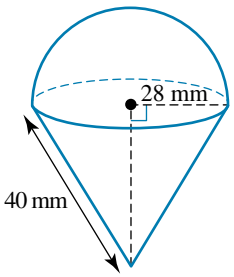


3. MC If all measurements are in centimetres, the shaded area of the figure below is:



A. 3.93 cm^2 **B.** 11.52 cm^2 **C.** 388.77 cm^2
D. 141.11 cm^2 **E.** 129.59 cm^2

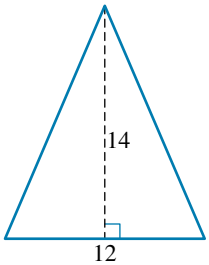
4. MC The total surface area of the solid below is:



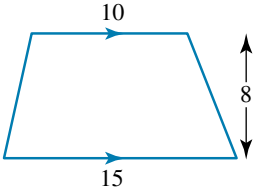
A. 8444.6 mm^2 **B.** 9221 mm^2 **C.** 14146.5 mm^2
D. 50271.1 mm^2 **E.** 16609.5 mm^2

5. Find the areas of the following plane figures. All measurements are in cm.

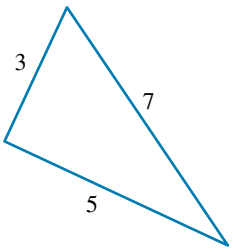
a.



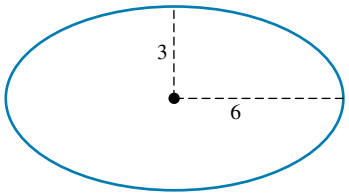
b.



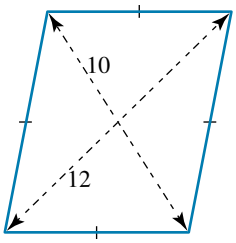
c.



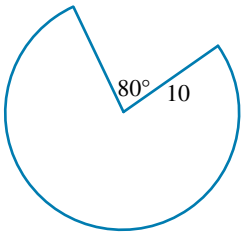
d.



e.

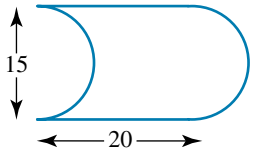


f.

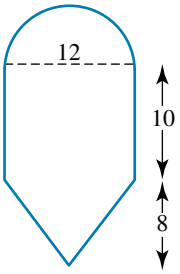


6. Find the areas of the following figures. All measurements are in cm.

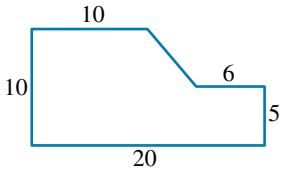
a.



b.

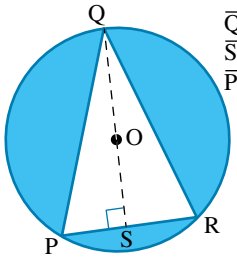


c.



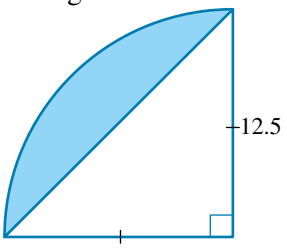
7. Find the shaded area in each of the following. All measurements are in cm.

a.

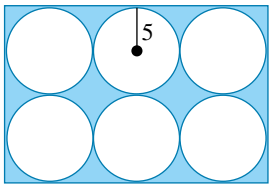


$\overline{QO} = 15$ cm
$\overline{SO} = 8$ cm
$\overline{PR} = 18$ cm

b.

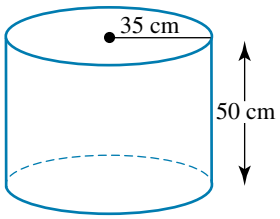


c.

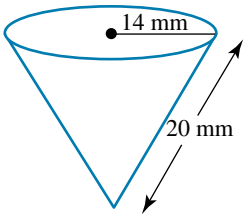


8. Find the total surface area of each of the following solids.

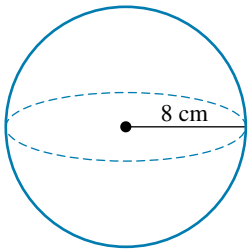
a.



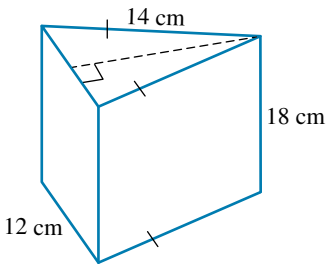
b.



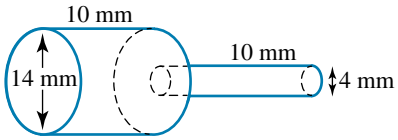
c.



d.

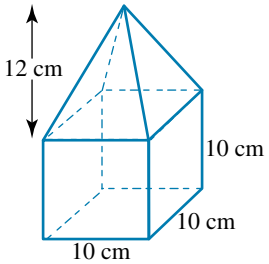


e.



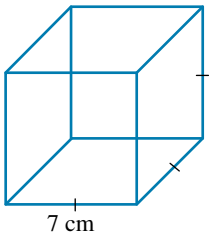
[closed at both ends]

f.

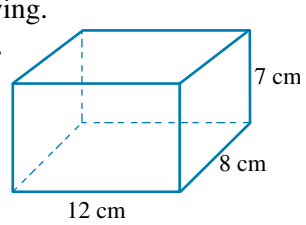


9. Find the volume of each of the following.

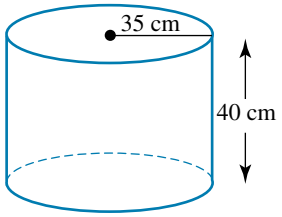
a.



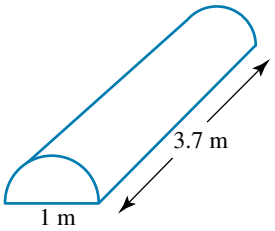
b.



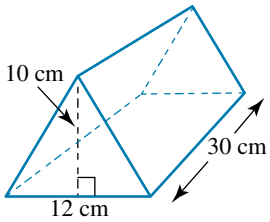
c.



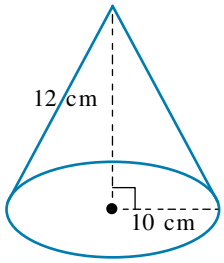
d.



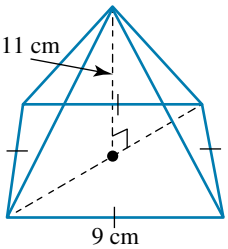
e.



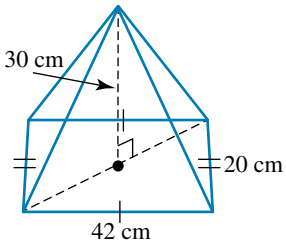
f.



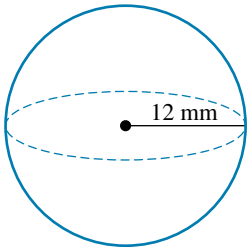
g.



h.



i.



Communicating, reasoning and problem solving

10. A rectangular block of land 4 m × 25 m is surrounded by a concrete path 1 m wide.
- **a.** Calculate the area of the path.
- **b.** Calculate the cost of concreting at $45 per square metre.

11. What effect will tripling the radius and dividing the height of a cylinder by 6 have on its volume (in comparison with the original shape)?

12. What effect will halving the length, tripling the width and doubling the height of a rectangular prism have on its volume (in comparison with the original shape)?

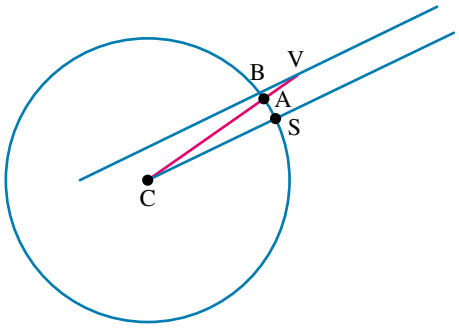
13. A cylinder of radius 14 cm and height 20 cm is joined to a hemisphere of radius 14 cm to form a bread holder.
- **a.** Find the total surface area.
- **b.** Find the cost of chroming the bread holder on the outside at $0.05 per cm^2.
- **c.** What is the storage volume of the bread holder?
- **d.** How much more space is in this new bread holder than the one it is replacing, which had a quarter circle end with a radius of 18 cm and a length of 35 cm?

14. Bella Silos has two rows of silos for storing wheat. Each row has 16 silos and all the silos are identical, with a cylindrical base (height of 5 m, diameter of 1.5 m) and conical top (diameter of 1.5 m, height of 1.1 m).
- **a.** What is the slant height of the conical tops?
- **b.** What is the total surface area of all the silos?
- **c.** What will it cost to paint the silos if one litre of paint covers 40 m^2 at a bulk order price of $28.95 per litre?
- **d.** How much wheat can be stored altogether in these silos?

e. Wheat is pumped from these silos into cartage trucks with rectangular containers 2.4 m wide, 5 m long and 2.5 m high. How many truckloads are necessary to empty all the silos?

f. If wheat is pumped out of the silos at 2.5 m^3/min, how long will it take to fill one truck?

15. The Greek mathematician Eratosthenes developed an accurate method for calculating the circumference of the Earth 2200 years ago! The figure below illustrates how he did this. In this figure, A is the town of Alexandria and S is the town of Syene, exactly 787 km due south of Alexandria. When the sun's rays (blue lines) were vertical at Syene, they formed an angle of 7.2° at Alexandria ($\angle BVA = 7.2°$), obtained by placing a stick at A and measuring the angle formed by the sun's shadow with the stick.



a. Assuming that the sun's rays are parallel, what is the angle $\angle SCA$?

b. Given that the arc AS = 787 km, determine the radius of the Earth, SC.

c. Given that the true radius is 6380 km, determine Eratosthenes' percentage error.

Answers

Topic 10 Surface area and volume

Exercise 10.2 Area

1. a. 16 cm^2 **b.** 48 cm^2 **c.** 75 cm^2
d. 120 cm^2 **e.** 706.86 cm^2 **f.** 73.5 mm^2
g. 254.47 cm^2 **h.** 21 m^2 **i.** 75 cm^2

2. Part **e** = 225π cm^2; part **g** = 81π cm^2

3. a. 20.66 cm^2 **b.** 7.64 cm^2

4. a. 113.1 mm^2 **b.** 188.5 mm^2

5. a. i. 12π cm^2 **ii.** 37.70 cm^2
b. i. $\frac{69\pi}{2}$ mm^2 **ii.** 108.38 mm^2
c. i. 261π cm^2 **ii.** 819.96 cm^2

6. E

7. D

8. a. 123.29 cm^2 **b.** 1427.88 m^2 **c.** 52 cm^2
d. 30.4 m^2 **e.** 78 cm^2 **f.** 2015.5 cm^2

9. a. 125.66 cm^2 **b.** 102.87 m^2 **c.** 13.73 m^2
d. 153.59 m^2 **e.** 13.86 m^2 **f.** 37.5 m^2

10. 11 707.92 cm^2

11. 21 m^2

12. 60

13. $840

14. a. 260.87 m^2 **b.** 195.71 m^2 **c.** 75%

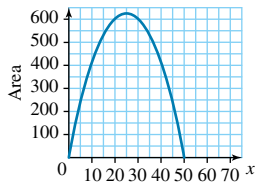
15. a. $50 = x + y$ **b.** $y = 50 - x$ **c.** Area $= 50x - x^2$

d.

x	0	5	10	15	20	25	30	35	40	45	50
Area (m^2)	0	225	400	525	600	625	600	525	400	225	0

e. No, it is impossible to make a rectangle if x is greater than 50.

f.



g. $x = 25$ **h.** $y = 25$ **i.** Square **j.** 625 m^2
k. r = 15.92 m **l.** 795.77 m^2 **m.** 170.77 m^2

16. a. Check with your teacher. **b.** 2020.83 m; horizontal

17. a. Circular area, 1790.49 m^2; rectangular area, 1406.25 m^2
b. Circular area, $\left(\frac{1}{4\pi}n^2\right)$ m^2; rectangular (square) area, $\left(\frac{1}{16}n^2\right)$ m^2. Circular area is always $\frac{4}{\pi}$ or 1.27 times larger.

18. Check with your teacher.

19. The calculation of both area and perimeter rely on the dimensions of the shape.

20. $\frac{29}{50}$

Exercise 10.3 Total surface area

1. a. 600 cm^2 **b.** 384 cm^2 **c.** 1440 cm^2 **d.** 27 m^2

2. a. 113.1 m^2 **b.** 6729.3 cm^2 **c.** 8.2 m^2 **d.** 452.4 cm^2

3. a. 1495.4 cm^2 **b.** 502.7 cm^2

4. a. 506.0 cm^2 **b.** 9.4 m^2 **c.** 340.4 cm^2 **d.** 224.1 cm^2

5. a. 13.5 m^2 **b.** 90 m^2 **c.** 11 309.7 cm^2
d. 9852.0 mm^2 **e.** 125.7 cm^2 **f.** 1531.4 cm^2

6. a. 880 cm^2 **b.** 3072.8 cm^2 **c.** 75 cm^2
d. 70.4 cm^2 **e.** 193.5 cm^2 **f.** 1547.2 cm^2

7. B

8. 60

9. 11 216 cm^2

10. a. 70.0 m^2 **b.** $455

11. a. 3063.1 cm^2 **b.** $168.47

12. a. $\theta = 120°$ **b.** $x = 1;\ y = \sqrt{3}$ **c.** $3\sqrt{3}$ cm^2
d. $6\sqrt{3}$ cm^2 **e.** 32

13. The calculation is correct.

14. a. 6.6 m^2

b. Back wall = 80 tiles

c. Side wall = 50 tiles

d. 80 + 50 + 50 = 180 tiles

e. Cheapest: 30 cm by 30 cm, $269.50; 20 cm by 20 cm (individually) $270; 20 cm by 20 cm (boxed) $276.50

15. $r = \dfrac{3\sqrt{3}a}{2}$

16. a. Arc length XY = $(x + s)\ \theta$

Arc length AB = $x\ \theta$

i. $x = \dfrac{2\pi t}{\theta} = \dfrac{st}{r - t}$ **ii.** $\dfrac{x}{x + s} = \dfrac{t}{r}$

b. Area of sector AVB = $\dfrac{x^2\theta}{2}$

Area of sector XVY = $\dfrac{(s + x)^2\theta}{2}$

Area of ABYX = $\dfrac{s\theta\ (s + 2x)}{2}$

TSA of frustum = $\pi(t^2 + r^2) + \dfrac{s\theta\ (s + 2x)}{2}$

17. The area of material required is 1.04 m^2. If Tina is careful in placing the pattern pieces, she may be able to cover the footstool.

18. Calculating total surface area of a composite solid requires determining the different shapes that make up the composite shape and then calculating the surface area for each one. The surface area of a standard shape can usually be calculated using a formula.

Exercise 10.4 Volume

1. a. 27 cm^3 **b.** 74.088 m^3 **c.** 3600 cm^3 **d.** 94.5 cm^3

2. a. 450 mm^3 **b.** 360 cm^3

3. a. 6333.5 cm^3 **b.** 19.1 m^3 **c.** 280 cm^3
d. 288 mm^3 **e.** 91.6 m^3 **f.** 21470.8 cm^3

4. a. 7.2 m^3 **b.** 14137.2 cm^3 **c.** 1436.8 mm^3 **d.** 523598.8 cm^3

5. a. 113 097.34 cm^3 **b.** 1.44 m^3 **c.** 12214.51 mm^3 **d.** 101.93 m^3

6. a. 377.0 cm^3 **b.** 2303.8 mm^3

7. a. 400 cm^3 **b.** 10080 cm^3

8. a. 1400 cm^3 **b.** 10379.20 cm^3 **c.** 41.31 cm^3
d. 48.17 cm^3 **e.** 218.08 cm^3 **f.** 3691.37 cm^3

9. a. $V_{new} = 27l^3$, the volume will be 27 times as large as the original volume.

b. $V_{new} = \frac{1}{8}l^2$, the volume will be $\frac{1}{8}$ of the original volume.

c. $V_{new} = 2\pi r^2 h$, the volume will be twice as large as the original volume.

d. $V_{new} = \pi r^2 h$, the volume will remain the same.

e. $V_{new} = 3lwh$, the volume will be 3 times as large as the original value.

10. E

11. 7438.35 cm^3

12. 4417.9 L

13. 10215.05 cm^3

14. a. $H = 6$ m **b.** 112 m^3 **c.** 19 bins

d. 112000 L **e.** 1.95 m from floor

15. a. i. 4.57 cm **ii.** 262.5 cm^2

b. i. 14.15 cm **ii.** 323.27 cm^2

c. i. 33.33 cm **ii.** 437.62 cm^2

d. Sphere. Costs less for a smaller surface area.

16. a. $h = \frac{V}{\pi r^2}$

b. i. 31.8 cm **ii.** 8.0 cm

c. $\sqrt{\frac{V}{\pi h}}$

d. $r \geq 0$, since r is a length

e. i. 7.6 cm **ii.** 6.2 cm

17. 1000

18. a. 2.55 cm **b.** 35.68 cm

c. $A_a = 157.88$ m^2, $A_b = 12.01$ m^2

19. a. 126.67 m^3 **b.** 53.33 m^3

20. Answers will vary.

21. Required volume = 1570.80 cm^3; tin volume = 1500 cm^3; muffin tray volume = 2814.72 cm^3. Marion could fill the tin and have a small amount of mixture left over, or she could almost fill 14 of the muffin cups and leave the remaining cups empty.

22. Increase radius of hemispherical section to 1.92 m.

23. Cut squares of side length, $s = 0.3$ m or 0.368 m from the corners.

24. Volume of water needed; 30.9 m^3.

25. a. $H = 12R$ **b.** $8\pi R^3$ **c.** $12\pi R^3$ **d.** 8 : 12 = 2 : 3

26. a. $\frac{1}{3}X^2H$ **b.** $\frac{1}{3}x^2(H - h)$ **c, d.** Check with your teacher.

27. Volume needs the measurement of three dimensions to be calculated, this is reflected in the units of volume measurement.

28. 18 scoops

Investigation | Rich task

1. a. The temperature reading is 26.5° C.

b. The smallest unit mark is 1° C, so the tolerance is 0.5.

c. Largest possible value = 27° C, smallest possible value = 26° C

2. a. Largest value = 37.28, smallest value = 36.98

b. Largest value = 2681.965, smallest value = 2649.285

3. a. i. 4002000 **ii.** 4000000

b. The result for **i** has 4 significant figures, whereas **ii** has only 1 significant figure after rounding. However, **ii** is closer to the actual value (3986297.386144940 9).

4. a. Volume using the incorrectly recorded value = 125 cm^3

Volume using the actual value = 216 cm^3

The percentage error is 42.1%, which shows that the error compounds as the number of dimensions increases.

b. Check with your teacher.

Exercise 10.5 Review questions

1. D

2. C

3. E

4. A

5. a. 84 cm^2 **b.** 100 cm^2 **c.** 6.50 cm^2

d. 56.52 cm^2 **e.** 60 cm^2 **f.** 244.35 cm^2

6. a. 300 cm^2 **b.** 224.55 cm^2 **c.** 160 cm^2

7. a. 499.86 cm^2 **b.** 44.59 cm^2 **c.** 128.76 cm^2

8. a. 18 692.48 cm^2 **b.** 1495.40 cm^2 **c.** 804.25 cm^2
d. 871.79 cm^2 **e.** 873.36 mm^2 **f.** 760 cm^2

9. a. 343 cm^3 **b.** 672 cm^3 **c.** 153 938.04 cm^3
d. 1.45 m^3 **e.** 1800 cm^3 **f.** 1256.64 cm^3
g. 297 cm^3 **h.** 8400 cm^3 **i.** 7238.23 mm^3

10. a. 62 m^2 **b.** $2790

11. $V = \frac{3}{2}\pi r^2 h$; the volume will be 1.5 times as large as the original volume.

12. $V = 3lwh$; the volume will be 3 times as large as (or triple) the original volume.

13. a. 3606.55 cm^2 **b.** $180.33 **c.** 18 062.06 cm^3 **d.** 9155.65 cm^3

14. a. 1.33 m **b.** 910.91 m^2 **c.** $618.35
d. 303.48 m^3 **e.** 11 trucks **f.** 12 minutes

15. a. 7.2° **b.** 6263 km **c.** 1.8% error

TOPIC 11
Quadratic expressions [Stages 5.2 and 5.3]

11.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the concepts covered in this topic.

11.1.1 Why learn this?

How is your algebraic tool kit? Is there some room to expand your skills? As expressions become more complex, more power will be needed to manipulate them and to carry out basic skills such as adding, multiplying and factorising. Dealing with quadratic expressions is the first step to higher-level skills.

DISCUSSION

What are the advantages and disadvantages of having more than one language in which we can communicate? Do you think the human race will move towards one common language?

LEARNING SEQUENCE

11.1 Overview
11.2 [Stages 5.2 and 5.3] Expanding algebraic expressions
11.3 [Stages 5.2 and 5.3] Factorising expressions with three terms
11.4 [Stage 5.3] Factorising expressions with two or four terms
11.5 [Stage 5.3] Factorising by completing the square
11.6 [Stage 5.3] Mixed factorisation
11.7 Review

LEARNING OUTCOMES

A student:

- selects appropriate notations and conventions to communicate mathematical ideas and solutions MA5.2-1WM
- constructs arguments to prove and justify results MA5.2 3WM
- expands and factorises quadratic expressions MA5.2-6NA
- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3-1WM
- selects and applies appropriate algebraic techniques to operate with algebraic expressions MA5.3-5NA

CONTENT DESCRIPTIONS

Students:
Apply the distributive law to the expansion of algebraic expressions, including binomials, and collect like terms where appropriate (ACMNA213)
Expand binomial products and factorise monic quadratic expressions using a variety of strategies (ACMNA233)
Factorise monic and non-monic quadratic expressions (ACMNA269)

Source: NSW Syllabus for the Australian curriculum

learnON RESOURCES — ONLINE ONLY

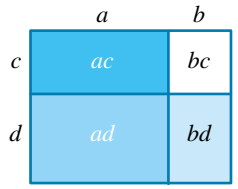
eLesson: The story of mathematics — Adelard of Bath (eles-1846)

Note: Your teacher may now set you a pre-test to determine how familiar you are with the content in this topic.

11.2 Expanding algebraic expressions [Stages 5.2 and 5.3]

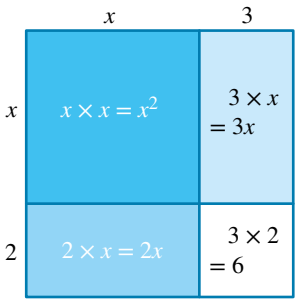
11.2.1 Binomial expansion

- Consider the rectangle of length $a + b$ and width $c + d$ shown below. Its area is equal to $(a + b)(c + d)$.



The diagram shows that $\underbrace{(a + b)(c + d)}_{\text{factorised form}} = \underbrace{ac + ad + bc + bd}_{\text{expanded form}}$.

- Expansion of the binomial expression $(x + 3)(x + 2)$ can be shown by this area model.



- Expressed mathematically this is:

$$\underbrace{(x + 3)(x + 2)}_{\text{expanded form}} = x^2 + 2x + 3x + 6 = \underbrace{x^2 + 5x + 6}_{\text{factorised form}}$$

- There are several methods that can be used to expand binomial factors.

DISCUSSION

Can you think of mathematical situations where you would want an expression to be in factorised or expanded form?

11.2.2 The FOIL method

- The word **FOIL** provides us with an acronym for the expansion of a binomial product.
  First: multiply the first terms in each bracket

  Outer: multiply the two outer terms

F
$(x + a)(x - b)$
O
$(x + a)(x - b)$

Inner: multiply the two inner terms

Last: multiply the last terms in each bracket

$$(x + a)(x - b) \quad \text{(I: } a \times -b\text{)}$$

$$(x + a)(x - b) \quad \text{(L: } x \times -b\text{)}$$

WORKED EXAMPLE 1

Expand each of the following.
a $(x + 3)(x + 2)$ **b** $(x - 7)(6 - x)$

THINK	WRITE
a 1 Write the expression.	**a** $(x + 3)(x + 2)$
2 Multiply the first terms in each bracket, then the outer terms, the inner terms and finally the last terms.	$= x \times x, x \times 2, x \times 3, 3 \times 2$ $= x^2 + 2x + 3x + 6$
3 Collect like terms.	$= x^2 + 5x + 6$
b 1 Write the expression.	**b** $(x - 7)(6 - x)$
2 Multiply the first terms in each bracket, then the outer terms, the inner terms and finally the last terms.	$= x \times 6, x \times -x, -7 \times 6, -7 \times -x$
3 Remove the brackets by multiplying each term in the brackets by the term outside the bracket. Remember to change the sign when the term outside the bracket is negative.	$= 6x - x^2 - 42 + 7x$
4 Collect like terms.	$= -x^2 + 13x - 42$

- If there is a term outside the pair of brackets, expand the brackets and then multiply each term of the expansion by that term.

WORKED EXAMPLE 2

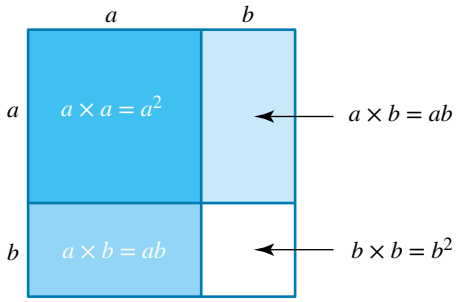
Expand $3(x + 8)(x + 2)$.

THINK	WRITE
1 Write the expression.	$3(x + 8)(x + 2)$
2 Use FOIL to expand the pair of brackets.	$= 3(x^2 + 2x + 8x + 16)$
3 Collect like terms within the brackets.	$= 3(x^2 + 10x + 16)$
4 Multiply each of the terms inside the brackets by the term outside the brackets.	$= 3x^2 + 30x + 48$

- If an expression contains three brackets, expand any two of the brackets and then multiply the expanded expression by the third bracket.

11.2.3 The square of a binomial

- The expansion of $(a + b)^2$ can be represented by this area model.



$$(a + b)^2 = a^2 + ab + ab + b^2$$
$$= a^2 + 2ab + b^2$$

- Similarly, $(a - b)^2 = a^2 - 2ab + b^2$.
- This expansion is often memorised. To find the square of a binomial:
  - square the first term
  - multiply the two terms together and then double them
  - square the last term.

WORKED EXAMPLE 3

Expand and simplify each of the following.
a $(2x - 5)^2$ **b** $-3(2x + 7)^2$

THINK	WRITE
a 1 Write the expression.	**a** $(2x - 5)^2$
2 Expand using the rule $(a - b)^2 = a^2 - 2ab + b^2$.	$= (2x)^2 - 2 \times 2x \times 5 + (5)^2$ $= 4x^2 - 20x + 25$
b 1 Write the expression.	**b** $-3(2x + 7)^2$
2 Expand the brackets using the rule $(a + b)^2 = a^2 + 2ab + b^2$.	$= -3[(2x)^2 + 2 \times 2x \times 7 + (7)^2]$ $= -3(4x^2 + 28x + 49)$
3 Multiply every term inside the brackets by the term outside the brackets.	$= -12x^2 - 84x - 147$

11.2.4 The difference of two squares

- When $a + b$ is multiplied by $a - b$ (or vice-versa),
  $(a + b)(a - b) = a^2 - ab + ab - b^2$
  $= a^2 - b^2$.
  The expression is called the difference of two squares and is often referred to as DOTS. This result can be memorised as a short cut.

WORKED EXAMPLE 4

Expand and simplify each of the following.
a $(3x + 1)(3x - 1)$ **b** $4(2x - 7)(2x + 7)$

THINK	WRITE
a 1 Write the expression.	**a** $(3x + 1)(3x - 1)$

2 Expand using the rule $(a + b)(a - b) = a^2 - b^2$.

$= (3x^2) - (1)^2$
$= 9x^2 - 1$

b 1 Write the expression.

b $4(2x - 7)(2x + 7)$

2 Expand using the difference of two squares rule.

$= 4[(2x)^2 - (7)^2]$
$= 4(4x^2 - 49)$

3 Multiply by 4.

$= 16x^2 - 196$

learnON RESOURCES — ONLINE ONLY

Interactivity: Expanding binomial factors (int-6033)
Interactivity: Difference of two squares (int-6036)
eLesson: Expansion of binomial expressions (eles-1908)
Digital doc: SkillSHEET Expanding brackets (doc-5244)
Digital doc: SkillSHEET Expanding a pair of brackets (doc-5245)

Exercise 11.2 Expanding algebraic expressions

Individual pathways

PRACTISE
Questions:
1a–f, 2a–h, 3a–d, 5a–d, 6a, 8a–d, 9a–d, 10a–f, 11–14

CONSOLIDATE
Questions:
1d–l, 2d–j, 3c–f, 4a–c, 5c–f, 6, 7, 8c–f, 9c–f, 10–17, 19

MASTER
Questions:
1d–l, 2f–l, 3e–i, 4, 5e–h, 6, 7, 8g–l, 9e–i, 10–22

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. Expand each of the following.

a. $2(x + 3)$ **b.** $4(x - 5)$ **c.** $3(7 - x)$ **d.** $-(x + 3)$
e. $x(x + 2)$ **f.** $2x(x - 4)$ **g.** $3x(5x - 2)$ **h.** $5x(2 - 3x)$
i. $2x(4x + 1)$ **j.** $2x^2(2x - 3)$ **k.** $3x^2(2x - 1)$ **l.** $5x^2(3x + 4)$

2. WE1 Expand each of the following.

a. $(x + 3)(x - 4)$ **b.** $(x + 1)(x - 3)$ **c.** $(x - 7)(x + 2)$ **d.** $(x - 1)(x - 5)$
e. $(2 - x)(x + 3)$ **f.** $(x - 4)(x - 2)$ **g.** $(2x - 3)(x - 7)$ **h.** $(x - 1)(3x + 2)$
i. $(3x - 1)(2x - 5)$ **j.** $(3 - 2x)(7 - x)$ **k.** $(5 - 2x)(3 + 4x)$ **l.** $(11 - 3x)(10 + 7x)$

3. WE2 Expand each of the following.

a. $2(x + 1)(x - 3)$ **b.** $4(2x + 1)(x - 4)$ **c.** $-2(x + 1)(x - 7)$
d. $2x(x - 1)(x + 1)$ **e.** $3x(x - 5)(x + 5)$ **f.** $6x(x - 3)(x + 3)$
g. $-2x(3 - x)(x - 3)$ **h.** $-5x(2 - x)(x - 4)$ **i.** $6x(x + 5)(4 - x)$

4. Expand each of the following.

a. $(x - 1)(x + 1)(x + 2)$ **b.** $(x - 3)(x - 1)(x + 2)$ **c.** $(x - 5)(x + 1)(x - 1)$
d. $(x - 1)(x - 2)(x - 3)$ **e.** $(2x - 1)(x + 1)(x - 4)$ **f.** $(3x + 1)(2x - 1)(x - 1)$

5. Expand each of the following and simplify.

a. $(x + 2)(x - 1) - 2x$ **b.** $3x - (2x - 5)(x + 2)$
c. $(2x - 3)(x + 1) + (3x + 1)(x - 2)$ **d.** $(3 - 2x)(2x - 1) + (4x - 5)(x + 4)$

e. $(x + 1)(x - 7) - (x + 2)(x - 3)$ **f.** $(x - 2)(x - 5) - (x - 1)(x - 4)$

g. $(x - 3)(x + 1) + \sqrt{3}x$ **h.** $(\sqrt{2} - 3x)(\sqrt{3} + 2x) - \sqrt{5}x$

6. MC **a.** $(3x - 1)(2x + 4)$ expands to:

A. $6x^2 + 10x - 4$ **B.** $5x^2 - 24x + 3$ **C.** $3x^2 + 2x - 4$

D. $6x^2 - 10x - 4$ **E.** $6x^2 - 4$

b. $-2x(x - 1)(x + 3)$ expands to:

A. $x^2 + 2x - 3$ **B.** $-2x^2 - 4x + 6$ **C.** $-2x^3 - 4x^2 + 6x$

D. $-2x^3 + 4x^2 - 6x$ **E.** $-2x^3 - 3$

7. MC The expression $(x - 1)(x - 3)(x + 2)$ is *not* the same as:

A. $(x - 3)(x - 1)(x + 2)$ **B.** $(x + 3)(x - 1)(x - 2)$ **C.** $(x - 1)(x + 2)(x - 3)$

D. $(x + 2)(x - 1)(x - 3)$ **E.** $(x - 3)(x + 2)(x - 1)$

8. WE3a Expand and simplify each of the following.

a. $(x - 1)^2$ **b.** $(x + 2)^2$ **c.** $(x + 5)^2$ **d.** $(4 + x)^2$

e. $(7 - x)^2$ **f.** $(12 - x)^2$ **g.** $(3x - 1)^2$ **h.** $(12x - 3)^2$

i. $(5x + 2)^2$ **j.** $(2 - 3x)^2$ **k.** $(5 - 4x)^2$ **l.** $(1 - 5x)^2$

9. WE3b Expand and simplify each of the following.

a. $2(x - 3)^2$ **b.** $4(x - 7)^2$ **c.** $3(x + 1)^2$

d. $-(2x + 3)^2$ **e.** $-(7x - 1)^2$ **f.** $2(2x - 3)^2$

g. $-3(2 - 9x)^2$ **h.** $-5(3 - 11x)^2$ **i.** $-4(2x + 1)^2$

10. WE4 Expand and simplify each of the following.

a. $(x + 7)(x - 7)$ **b.** $(x + 9)(x - 9)$ **c.** $(x - 5)(x + 5)$

d. $(x - 1)(x + 1)$ **e.** $(2x - 3)(2x + 3)$ **f.** $(3x - 1)(3x + 1)$

g. $(7 - x)(7 + x)$ **h.** $(8 + x)(8 - x)$ **i.** $(3 - 2x)(3 + 2x)$

11. The length of the side of a rectangle is $(x + 1)$ cm and the width is $(x - 3)$ cm.

a. Find an expression for the area of the rectangle.

b. Simplify the expression by expanding.

c. If $x = 5$ cm, find the dimensions of the rectangle and, hence, its area.

12. Chickens are kept in a square enclosure with sides measuring x m. The number of chickens is increasing, so the size of the enclosure is to have 1 metre added to one side and 2 metres to the adjacent side.

a. Draw a diagram of the original enclosure.

b. Add to the first diagram or draw another one to show the new enclosure. Mark the lengths on each side on your diagram.

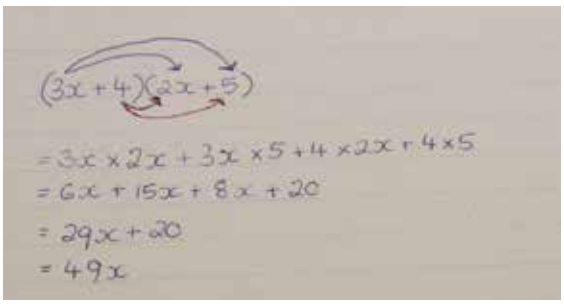
c. Write an expression for the area of the new enclosure in factorised form.

d. Expand and simplify the expression by removing the brackets.

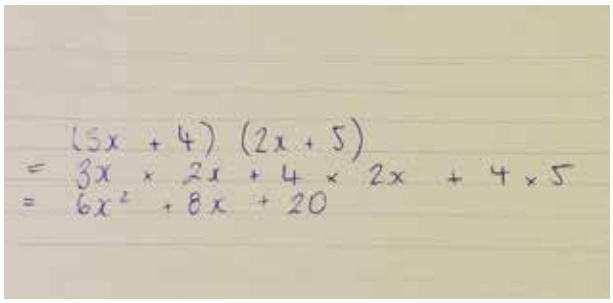
e. If the original enclosure had sides of 2 metres, find the area of the original square and then the area of the new enclosure.

13. Shown below are three students' attempts at expanding $(3x + 4)(2x + 5)$.

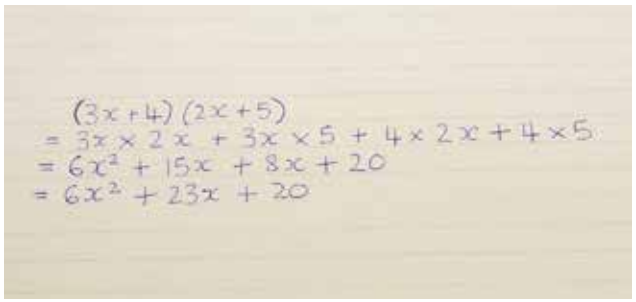
STUDENT A



STUDENT B



STUDENT C



a. Which student's work was correct?

b. Copy each of the incorrect answers into your workbook and correct the mistakes in each as though you were the teacher of these students.

14. If $a = 5$ and $b = 3$, show that $(a - b)(a + b) = a^2 - b^2$ by evaluating both expressions.

15. If $a = 5$ and $b = 3$, show that $(a + b)^2 = a^2 + 2ab + b^2$ by evaluating both expressions.

16. Write an expression in factorised and expanded form that is:

a. a quadratic trinomial

b. the square of a binomial

c. the difference of two squares

d. both **a** and **b**.

Communicating, reasoning and problem solving

17. Explain the difference between 'the square of a binomial' and 'the difference between two squares'.

18. Show that $(a + b)(c + d) = (c + d)(a + b)$.

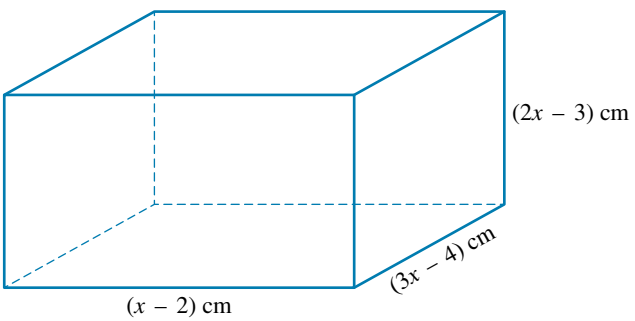
19. Expand:

a. $(2x + 3y - 5z)^2$

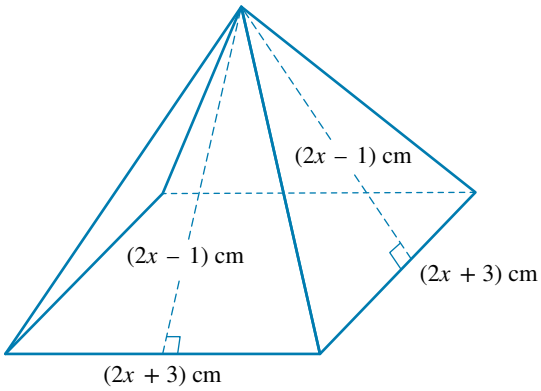
b. $\left(\left(1 + \frac{1}{2x}\right) - 2x\right)^2$

20. Find an expanded expression for:

a. the volume of the cuboid



b. the total surface area of the square-based pyramid.



21. Why does the difference of two squares rule have that name?

22. Find all the positive integers, a, which make the expression $(a - 10)(a - 14)$ a perfect square. Consider 0 to be the first perfect square.

11.3 Factorising expressions with three terms [Stages 5.2 and 5.3]

11.3.1 Factorising monic quadratic trinomials

- A monic quadratic expression is an expression in the form $ax^2 + bx + c$ where $a = 1$.
- The area model of binomial expansion can be used to find a pattern for factorising a general quadratic expression. For example,

$$(x + f)(x + h) = x^2 + fx + hx + fh$$
$$= x^2 + (f + h)x + fh$$

$$(x + 4)(x + 3) = x^2 + 4x + 3x + 12$$
$$= x^2 + 7x + 12$$

	x	f
x	x^2	fx
h	hx	fh

	x	4
x	x^2	$4x$
3	$3x$	12

- To factorise a general quadratic, look for factors of c that add to b.

$x^2 + bx + c = (x + f)(x + h)$

Factors of c that add to b

$3 + 4 = 7$

For example, $x^2 + 7x + 12 = (x + 3)(x + 4)$.

$3 \times 4 = 12$

WORKED EXAMPLE 5

Factorise the following quadratic expressions.

a $x^2 + 5x + 6$ **b** $x^2 + 10x + 24$

THINK

a 1 Write the expression and:

- **i** check for a common factor. There is no common factor.
- **ii** check for a DOTS pattern. The expression is not in the form $a^2 - b^2$.
- **iii** check for a perfect squares pattern. 6 is not a perfect square.
- **iv** This must be a general quadratic expression.

2 i The general quadratic expression has the pattern $x^2 + 5x + 6 = (x + f)(x + h)$. f and h are a factor pair of 6 that add to 5.

ii Calculate the sums of factor pairs of 6. The factors of 6 that add to 5 are 2 and 3, as shown in blue.

3 Substitute the values of f and h into the expression in its factorised form.

b 1 Check for patterns of common factors, DOTS and perfect squares patterns. None of these apply, so the expression is a general quadratic.

2 i The general quadratic expression has the pattern $x^2 + 10x + 24 = (x + f)(x + h)$, where f and h are a factor pair of 24 that add to 10.

ii Calculate the sums of factor pairs of 24. The factors of 24 that add to 10 are 4 and 6, as shown in blue.

3 Substitute the values of f and h into the expression in its factorised form.

WRITE

a $x^2 + 5x + 6$

Factors of 6	Sum of factors
1 and 6	7
2 and 3	5

$x^2 + 5x + 6 = (x + 2)(x + 3)$

b $x^2 + 10x + 24$

Factors of 24	Sum of factors
1 and 24	25
2 and 12	14
3 and 8	11
4 and 16	10

$x^2 + 10x + 24 = (x + 4)(x + 6)$

11.3.2 Factorising non-monic quadratic trinomials

- A non-monic quadratic expression is $ax^2 + bx + c$ where $a \neq 1$.
- When a quadratic trinomial in the form $ax^2 + bx + c$ is written as $ax^2 + mx + nx + c$, where $m + n = b$, the four terms can be factorised by grouping.

$$\begin{aligned} 2x^2 + 11x + 12 &= 2x^2 + 8x + 3x + 12 \\ &= 2x^2 + 8x + 3x + 12 \\ &= 2x(x + 4) + 3(x + 4) \\ &= (x + 4)(2x + 3) \end{aligned}$$

- There are many combinations of numbers that satisfy $m + n = b$; however, only one particular combination can be grouped and factorised. For example,

$$2x^2 + 11x + 12 = 2x^2 + 7x + 4x + 12$$
$$= 2x^2 + 7x + 4x + 12$$
$$= (2x + 7) + 4(x + 3)$$
cannot be factorised further

or

$$2x^2 + 11x + 12 = 2x^2 + 8x + 3x + 12$$
$$= 2x^2 + 8x + 3x + 12$$
$$= 2x(x + 4) + 3(x + 4)$$
$$= (x + 4)(2x + 3)$$

- In examining the general binomial expansion, a pattern emerges that can be used to help identify which combination to use for $m + n = b$.

$$(dx + e)(fx + g) = dfx^2 + efx + eg$$
$$= dfx^2 + (dg + ef)x + eg$$
$$m + n = dg + ef \text{ and } m \times n = dg \times ef$$
$$= b \qquad = dgef$$
$$= dfeg$$
$$= ac$$

Therefore, m and n are factors of ac that sum to b.

- To factorise a general quadratic where $a \neq 1$, look for factors of ac that sum to b. Then rewrite the quadratic trinomial with four terms that can then be grouped and factorised.

$$ax^2 + bx + c = ax^2 + mx + nx + c$$

Factors of ac that sum to b

WORKED EXAMPLE 6

Factorise $6x^2 - 11x - 10$.

THINK

WRITE

1 Write the expression and look for common factors and special patterns. The expression is a general quadratic with $a = 6$, $b = -11$ and $c = -10$.

$6x^2 - 11x - 10 = 6x^2 + \ - 11x + \ - 10$

2 Since $a \neq 1$, rewrite $ax^2 + bx + c$ as $ax^2 + mx + nx + c$, where m and n are factors of ac (6×-10) that sum to b (-11).
Calculate the sums of factor pairs of -60.
As shown in blue, 4 and -15 are factors of -60 that add to -11.

Factors of -60 (6×-10)	Sum of factors
$-60, 1$	-59
$-20, 3$	-17
$-30, 2$	-28
$15, -4$	11
$-15, 4$	-11

3 Rewrite the quadratic expression: $ax^2 + bx + c = ax^2 + mx + nx + c$ with $m = 4$ and $n = -15$.

$6x^2 - 11x - 10 = 6x^2 + 4x + -15x - 10$

4 Factorise using the grouping method:
$6x^2 + 4x = 2x(3x + 2)$ and
$-15x - 10 = -5(3x + 12)$.
Write the answer.

$6x^2 - 11x - 10 = 2x(3x + 2) + -5(3x + 2)$
$= (3x + 2)(2x - 5)$

learnON RESOURCES — ONLINE ONLY

Interactivity: Factorising monic quadratic trinomials (int-6143)
Interactivity: Factorising trinomials by grouping (int-6144)
eLesson: Factorisation of trinomials (eles-1921)
Digital doc: SkillSHEET Finding a factor pair that adds to a given number (doc-5250)
Digital doc: WorkSHEET Factorising and expanding (doc-5251)

Exercise 11.3 Factorising expressions with three terms

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–e, 2a–e, 3a–e, 4–6, 7a–e, 8, 11, 15	Questions: 1f–j, 2f–i, 3f–i, 4–6, 7d–h, 8, 9a–d, 11, 12, 15, 16	Questions: 1k–o, 2k–l, 3j–l, 4–6, 7i–l, 8, 9, 10, 13–18

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE5 Factorise each of the following.

a. $x^2 + 3x + 2$	**b.** $x^2 + 4x + 3$	**c.** $x^2 + 10x + 16$	**d.** $x^2 + 8x + 16$
e. $x^2 - 2x - 3$	**f.** $x^2 - 3x - 4$	**g.** $x^2 - 11x - 12$	**h.** $x^2 - 4x - 12$
i. $x^2 + 3x - 4$	**j.** $x^2 + 4x - 5$	**k.** $x^2 + 6x - 7$	**l.** $x^2 + 3x - 10$
m. $x^2 - 4x + 3$	**n.** $x^2 - 9x + 20$	**o.** $x^2 + 9x - 70$	

2. WE6 Factorise each of the following.

a. $-2x^2 - 20x - 18$	**b.** $-3x^2 - 9x - 6$	**c.** $-x^2 - 3x - 2$	**d.** $-x^2 - 11x - 10$
e. $-x^2 - 7x - 10$	**f.** $-x^2 - 13x - 12$	**g.** $-x^2 - 7x - 12$	**h.** $-x^2 - 8x - 12$
i. $2x^2 + 14x + 20$	**j.** $3x^2 + 33x + 30$	**k.** $5x^2 + 105x + 100$	**l.** $5x^2 + 45x + 100$

3. Factorise each of the following.

a. $a^2 - 6a - 7$	**b.** $t^2 - 6t + 8$	**c.** $b^2 + 5b + 4$
d. $m^2 + 2m - 15$	**e.** $p^2 - 13p - 48$	**f.** $c^2 + 13c - 48$
g. $k^2 + 22k + 57$	**h.** $s^2 - 16s - 57$	**i.** $g^2 - g - 72$
j. $v^2 - 28v + 75$	**k.** $x^2 + 14x - 32$	**l.** $x^2 - 19x + 60$

4. MC **a.** To factorise $-14x^2 - 49x + 21$, the first step is to:

A. find factors of 14 and 21 that will add to –49
B. take out 14 as a common factor
C. take out –7 as a common factor
D. find factors of 14 and –49 that will add to make 21
E. take out –14 as a common factor

b. The expression $42x^2 - 9x - 6$ can be completely factorised to:

A. $(6x - 3)(7x + 2)$ **B.** $3(2x - 1)(7x + 2)$ **C.** $(2x - 1)(21x + 6)$
D. $3(2x + 1)(7x - 2)$ **E.** $42(x - 3)(x + 2)$

5. MC When factorised, $(x + 2)^2 - (y + 3)^2$ equals:

A. $(x + y - 2)(x + y + 2)$ **B.** $(x - y - 1)(x + y - 1)$ **C.** $(x - y - 1)(x + y + 5)$
D. $(x - y + 1)(x + y + 5)$ **E.** $(x + y - 1)(x + y + 2)$

6. Factorise each of the following using an appropriate method.

a. $2x^2 + 5x + 2$	**b.** $2x^2 - 3x + 1$	**c.** $4x^2 - 17x - 15$
d. $4x^2 + 4x - 3$	**e.** $2x^2 - 9x - 35$	**f.** $3x^2 + 10x + 3$
g. $6x^2 - 17x + 7$	**h.** $12x^2 - 13x - 14$	**i.** $10x^2 - 9x - 9$
j. $20x^2 + 3x - 2$	**k.** $12x^2 + 5x - 2$	**l.** $15x^2 + x - 2$

7. Factorise each of the following, remembering to look for a common factor first.

a. $4x^2 + 2x - 6$	**b.** $9x^2 - 60x - 21$	**c.** $72x^2 + 12x - 12$
d. $-18x^2 + 3x + 3$	**e.** $-60x^2 + 150x + 90$	**f.** $24ax^2 + 18ax - 105a$
g. $-8x^2 + 22x - 12$	**h.** $-10x^2 + 31x + 14$	**i.** $-24x^2 + 35x - 4$
j. $-12x^2 - 2xy + 2y^2$	**k.** $-30x^2 + 85xy + 70y^2$	**l.** $-600x^2 - 780xy - 252y^2$

8. Consider the expression $(x - 1)^2 + 5(x - 1) - 6$.

a. Substitute $w = x - 1$ in this expression.

b. Factorise the resulting quadratic.

c. Replace w with $x - 1$ and simplify each factor. This is the factorised form of the original expression.

9. Use the method outlined in question **8** to factorise each of the following expressions.

a. $(x + 1)^2 + 3(x + 1) - 4$ **b.** $(x + 2)^2 + (x + 2) - 6$ **c.** $(x - 3)^2 + 4(x - 3) + 4$

d. $(x + 3)^2 + 8(x + 3) + 12$ **e.** $(x - 7)^2 - 7(x - 7) - 8$ **f.** $(x - 5)^2 - 3(x - 5) - 10$

10. Factorise $x^2 + x - 0.75$.

11. Students decide to make Valentine's Day cards. The total area of each card is equal to $(x^2 - 4x - 5)\,\text{cm}^2$.

a. Factorise the expression to find the dimensions of the cards in terms of x.

b. Write down the length of the shorter side in terms of x.

c. If the shorter sides of a card are 10 cm in length and the longer sides are 16 cm in length, find the value of x.

d. Find the area of the card proposed in part **c**.

e. If the students want to make 3000 Valentine's Day cards, how much cardboard will be required? Give your answer in terms of x.

12. The area of a rectangular playground is given by the general expression $(6x^2 + 11x + 3)\,\text{m}^2$ where x is a positive whole number.

a. Find the length and width of the playground in terms of x.

b. Write an expression for the perimeter of the playground.

c. If the perimeter of a particular playground is 88 metres, find x.

Communicating, reasoning and problem solving

13. Cameron wants to build an in-ground 'endless' pool. Basic models have a depth of 2 metres and a length triple the width. A spa will also be attached to the end of the pool.

a. The pool needs to be tiled. Write an expression for the surface area of the empty pool (that is, the floor and walls only).

b. The spa needs an additional 16 m^2 of tiles. Write an expression for the total area of tiles needed for both the pool and the spa.

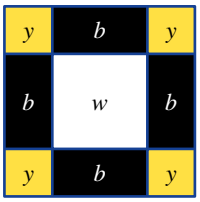
c. Factorise this expression.

d. Cameron decides to use tiles that are selling at a discount price, but there are only 280 m^2 of the tile available. Find the maximum dimensions of the pool he can build if the width is in whole metres. Assume the spa is to be included in the tiling.

e. What area of tiles is actually needed to construct the spa and pool?

f. What volume of water can the pool hold?

14. Fabric pieces comprising yellow squares, white squares and black rectangles are sewn together to make larger squares (patches) as shown in the diagram. The length of each black rectangle is twice its width. These patches are then sewn together to make a patchwork quilt. A finished square quilt, made from 100 patches, has an area of 1.44 m^2.



a. Determine the size of each yellow, black and white section in one fabric piece. Show your working.

b. How much (in m^2) of each of the coloured fabrics would be needed to construct the quilt? (Ignore seam allowances.)

c. Sketch a section of the finished product.

15. Each factorisation below contains an error. Identify the error in each statement.

a. $x^2 - 7x + 12 = (x + 3)(x - 4)$ **b.** $x^2 - x - 12 = (x - 3)(x + 4)$

c. $x^2 - x - 2 = (x - 1)(x + 2)$ **d.** $x^2 - 4x - 21 = (x - 3)(x - 7)$

e. $x^2 + 4x - 21 = (x + 3)(x - 7)$ **f.** $x^2 - x - 30 = (x - 5)(x + 6)$

g. $x^2 + 7x - 8 = (x + 1)(x - 8)$ **h.** $x^2 - 11x + 30 = (x - 5)(x + 6)$

16. Factorise:

a. $6(3a - 1)^2 - 13(3a - 1) - 5$ **b.** $3m^4 - 19m^2 - 14$ **c.** $2\sin^2(x) - 3\sin(x) + 1$.

17. Factorise:

a. $2x^2 + 3\sqrt{3}x - 6$ **b.** $(z + 1)^3 + (z - 1)^3$.

18. In your own words, describe how you would factorise a quadratic trinomial.

11.4 Factorising expressions with two or four terms [Stage 5.3]

11.4.1 Factorising expressions with two terms

- If the terms in an expanded expression have a common factor, the highest common factor is written at the front of the brackets and the remaining factor for each term in the expression is written in the brackets. For example, $4x^2 - 36 = 4(x^2 - 9)$.
- A Difference of Two Squares (DOTS) expression in expanded form has two squared terms separated by a subtraction symbol.

$$a^2 - b^2 = (a - b)(a + b)$$

Expanded form Factorised form

DISCUSSION

Can you explain to a friend how to easily determine if an expression is in DOTS form, and how to expand it?

WORKED EXAMPLE 7

Factorise the following.

a $12k^2 + 18$ **b** $16a^2 - 25b^4$

THINK

a 1 Write the expression and look for common factors. The terms have a highest common factor of 6. Write the 6 in front of a set of brackets, then determine what must go inside the brackets. $12k^2 = 6 \times 2k^2$, $18 = 6 \times 3$

WRITE

a $12k^2 + 18 = 6(2k^2 + 3)$

2 Look for patterns in the expression inside the brackets to factorise further. The expression inside the brackets cannot be factorised further.

b 1 Write the expression and look for common factors. The expression has no common factor.

2 Look for the DOTS pattern in the expression. Write the equation showing squares.

3 Use the pattern for DOTS to write the factors.
$a^2 - b^2 = (a + b)(a - b)$

b $16a^2 - 25b^4$

$= 4^2a^2 - 5^2(b^2)^2$
$= (4a)^2 - (5b^2)^2$

$= (4a + 5b^2)(4a - 5b^2)$

11.4.2 Factorising expressions with four terms

- If there are four terms to be factorised, look for a common factor first. Then group the terms in pairs and look for a common factor in each pair. It may be that a new common factor emerges as a bracket (common binomial factor).
- If an expression has four terms, it may require grouping to factorise it.
- In the process known as grouping 'two and two', the terms of the expression are grouped into two pairs, then a common factor is removed from each pair.
- When selecting terms to place as pairs, each pair after factorising should result in a common binomial factor. For example:

$$2a - 6b + 3ac - 9bc = 2(a - 3b) + 3c(a - 3b)$$

WORKED EXAMPLE 8

Factorise each of the following.
a $x - 4y + mx - 4my$ **b** $x^2 + 3x - y^2 + 3y$

THINK

a 1 Write the expression and look for a common factor. (There isn't one.)

2 Group the terms so that those with common factors are next to each other.

3 Take out a common factor from each group (it may be 1).

4 Factorise by taking out a common binomial factor. The factor $(x - 4y)$ is common to both groups.

b 1 Write the expression and look for a common factor.

2 Group the terms so that those with common factors are next to each other.

3 Factorise each group.

4 Factorise by taking out a common binomial factor. The factor $(x + y)$ is common to both groups.

WRITE

a $x - 4y + mx - 4my$

$= (x - 4y) + (mx - 4my)$

$= 1(x - 4y) + m(x - 4y)$

$= (x - 4y)(1 + m)$

b $x^2 + 3x - y^2 + 3y$

$= (x^2 - y^2) + (3x + 3y)$

$= (x + y)(x - y) + 3(x + y)$

$= (x + y)(x - y + 3)$

- Now we will look at grouping a different combination, known as grouping 'three and one'.

WORKED EXAMPLE 9

Factorise the following expression: $x^2 + 12x + 36 - y^2$.

THINK	WRITE
1 Write the expression and look for a common factor.	$x^2 + 12x + 36 - y^2$
2 Group the terms so that those that can be factorised are next to each other.	$= (x^2 + 12x + 36) - y^2$
3 Factorise the quadratic trinomial. This is the form of a perfect square.	$= (x + 6)(x + 6) - y^2$ $= (x + 6)^2 - y^2$
4 Factorise the expression using $a^2 - b^2 = (a + b)(a - b)$.	$= (x + 6 + y)(x + 6 - y)$

learnON RESOURCES — ONLINE ONLY

Interactivity: Factorising expressions with four terms (int-6145)
Digital doc: SkillSHEET Factorising by taking out the highest common factor (doc-5246)
Digital doc: SkillSHEET Factorising by taking out a common binomial factor (doc-5247)

Exercise 11.4 Factorising expressions with two or four terms

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–4, 8a–h, 9a–d, 10a–c, 14, 15	Questions: 1–4, 5a, 7a–c, 8a–h, 9a–d, 10a–d, 11–13, 15, 16, 18	Questions: 1, 2, 5, 7, 9–11, 19, 20

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. Factorise each of the following by taking out a common factor.

a. $x^2 + 3x$	**b.** $x^2 - 4x$	**c.** $3x^2 - 6x$
d. $4x^2 + 16x$	**e.** $9x^2 - 3x$	**f.** $8x - 8x^2$
g. $12x - 3x^2$	**h.** $8x - 12x^2$	**i.** $8x^2 - 11x$

2. Factorise each of the following by taking out a common binomial factor.

a. $3x(x - 2) + 2(x - 2)$	**b.** $5(x + 3) - 2x(x + 3)$	**c.** $(x - 1)^2 + 6(x - 1)$
d. $(x + 1)^2 - 2(x + 1)$	**e.** $(x + 4)(x - 4) + 2(x + 4)$	**f.** $7(x - 3) - (x + 3)(x - 3)$

3. WE7 Factorise each of the following.

a. $x^2 - 1$	**b.** $x^2 - 9$	**c.** $x^2 - 25$
d. $x^2 - 100$	**e.** $y^2 - k^2$	**f.** $4x^2 - 9y^2$
g. $16a^2 - 49$	**h.** $25p^2 - 36q^2$	**i.** $1 - 100d^2$

4. Factorise each of the following.

a. $4x^2 - 4$ **b.** $5x^2 - 80$ **c.** $ax^2 - 9a$

d. $2b^2 - 8d^2$ **e.** $100x^2 - 1600$ **f.** $3ax^2 - 147a$

g. $4px^2 - 256p$ **h.** $36x^2 - 16$ **i.** $108 - 3x^2$

5. MC **a.** If the factorised expression is $(x + 7)(x - 7)$, then the expanded expression must have been:

A. $x^2 - 7$ **B.** $x^2 + 7$ **C.** $x^2 - 49$

D. $x^2 + 49$ **E.** $x^2 - 14x + 49$

b. If the factorised expression is $\left(\frac{x}{4} - \frac{3}{5}\right)\left(\frac{x}{4} + \frac{3}{5}\right)$, then the original expression must have been:

A. $\frac{x^2}{4} - \frac{3}{5}$ **B.** $\frac{x^2}{16} - \frac{9}{25}$ **C.** $\frac{x^2}{4} - \frac{(\sqrt{3})^2}{(\sqrt{5})^2}$

D. $\frac{x^2}{4} - \frac{9}{25}$ **E.** $\frac{x^2}{16} - \frac{(\sqrt{3})^2}{(\sqrt{5})^2}$

c. The factorised form of $64x^2 - 9y^2$ is:

A. $(64x + 9y)(64x - 9y)$ **B.** $(8x + 3y)(8x - 3y)$ **C.** $(8x - 3y)(8x - 3y)$

D. $(8x + 3y)(8x + 3y)$ **E.** $(16x + 3y)(16x - 3y)$

6. MC Which of the following expressions would be factorised by grouping 'two and two'?

A. $x^2 - a^2 + 12a - 36$ **B.** $x^2 - 7x - 10$ **C.** $2x^2 - 6x - xy + 3y$

D. $(s - 5)^2 - 25(s + 3)^2$ **E.** $(r + 5) - (r + 3)(r + 5)$

7. Factorise each of the following over the set of real numbers.

a. $x^2 - 11$ **b.** $x^2 - 7$ **c.** $x^2 - 15$

d. $4x^2 - 13$ **e.** $9x^2 - 19$ **f.** $3x^2 - 66$

g. $5x^2 - 15$ **h.** $2x^2 - 4$ **i.** $12x^2 - 36$

8. Factorise each of the following expressions.

a. $(x - 1)^2 - 4$ **b.** $(x + 1)^2 - 25$ **c.** $(x - 2)^2 - 9$

d. $(x + 3)^2 - 16$ **e.** $49 - (x + 1)^2$ **f.** $36 - (x - 4)^2$

g. $(x - 1)^2 - (x - 5)^2$ **h.** $4(x + 2)^2 - 9(x - 1)^2$ **i.** $25(x - 2)^2 - 16(x + 3)^2$

9. WE8a Factorise each of the following.

a. $x - 2y + ax - 2ay$ **b.** $2x + ax + 2y + ay$ **c.** $ax - ay + bx - by$

d. $4x + 4y + xz + yz$ **e.** $ef - 2e + 3f - 6$ **f.** $mn - 7m + n - 7$

g. $6rt - 3st + 6ru - 3su$ **h.** $7mn - 21n + 35m - 105$ **i.** $64 - 8j + 16k - 2jk$

j. $3a^2 - a^2b + 3ac - abc$ **k.** $5x^2 + 10x + x^2y + 2xy$ **l.** $2m^2 - m^2n + 2mn - mn^2$

10. Factorise each of the following.

a. $xy + 7x - 2y - 14$ **b.** $mn + 2n - 3m - 6$ **c.** $pq + 5p - 3q - 15$

d. $s^2 + 3s - 4st - 12t$ **e.** $a^2b - cd - bc + a^2d$ **f.** $xy - z - 5z^2 + 5xyz$

11. WE8b Factorise each of the following.

a. $a^2 - b^2 + 4a - 4b$ **b.** $p^2 - q^2 - 3p + 3q$ **c.** $m^2 - n^2 + lm + ln$

d. $7x + 7y + x^2 - y^2$ **e.** $5p - 10pq + 1 - 4q^2$ **f.** $49g^2 - 36h^2 - 28g - 24h$

12. WE9 Factorise each of the following.

a. $x^2 + 14x + 49 - y^2$ **b.** $x^2 + 20x + 100 - y^2$ **c.** $a^2 - 22a + 121 - b^2$

d. $9a^2 + 12a + 4 - b^2$ **e.** $25p^2 - 40p + 16 - 9t^2$ **f.** $36t^2 - 12t + 1 - 5v^2$

13. MC **a.** In the expression $3(x - 2) + 4y(x - 2)$, the common binomial factor is:

A. $3 + 4y$ **B.** $3 - 4y$ **C.** x

D. $-x + 2$ **E.** $x - 2$

b. Which of the following terms is a perfect square?

A. 9 **B.** $(x + 1)(x - 1)$ **C.** $3x^2$

D. $5(a + b)^2$ **E.** $25x$

c. Which of the following expressions can be factorised using grouping?

A. $x^2 - y^2$ **B.** $1 + 4y - 2xy + 4x^2$ **C.** $3a^2 + 8a + 4$

D. $x^2 + x + y - y^2$ **E.** $2a + 4b - 6ab + 18$

14. MC When factorised, $6(a + b) - x(a + b)$ equals:

A. $6 - x(a + b)$ **B.** $(6 - x)(a + b)$ **C.** $6(a + b - x)$

D. $(6 + x)(a - b)$ **E.** $(6 + x)(a + b)$

15. The area of a rectangle is $(x^2 - 25)$ cm^2.

a. Factorise the expression.

b. Find the width of the rectangle if the length is $x + 5$ cm.

c. If $x = 7$ cm, find the dimensions of the rectangle.

d. Hence, find the area of the rectangle.

e. If $x = 13$ cm, how much bigger would the area of this rectangle be?

Communicating, reasoning and problem solving

16. A circular garden of diameter $2r$ m is to have a gravel path laid around it. The path is to be 1 m wide.

a. Find the area of the garden in terms of r.

b. Find the area of the garden and path together in terms of r, using the formula for the area of a circle.

c. Write an expression for the area of the path in fully factorised form.

d. If the radius of the garden is 5 m, then find the area of the path, correct to 2 decimal places. Show your working.

17. A roll of material is $(x + 2)$ metres wide. Annie buys $(x + 3)$ metres of the material and Bronwyn buys 5 metres of the material.

a. Write an expression, in terms of x, for the area of each piece of material purchased.

b. If Annie has bought more material than Bronwyn, write an expression for how much more she has than Bronwyn.

c. Factorise and simplify this expression.

d. Find the width of the material if Annie has 5 m^2 more than Bronwyn.

e. How much material does each person have? Explain your answer.

Questions 18 and 19 use the following information.

A polynomial in the form $a^3 - b^3$ is known as the difference of two cubes.

The difference of two cubes can be factorised as:

$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Use either the difference of two squares or the difference of two cubes to answer these questions.

18. Factorise:

a. $x^2 - 4xy + 4y^2 - a^2 + 6ab - 9b^2$ **b.** $x^3 - 8$.

19. Factorise:

a. $27x^3 - 1$ **b.** $12x^2 - 75y^2 - 9(4x - 3)$.

20. What do you always check for first when factorising?

11.5 Factorising by completing the square [Stage 5.3]

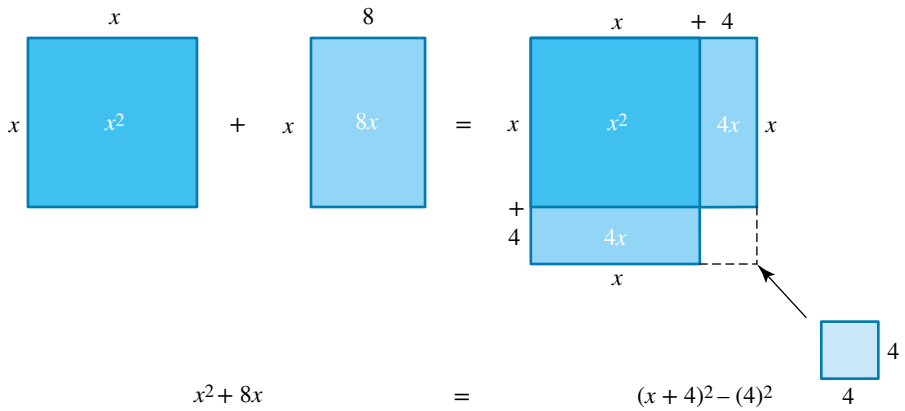
11.5.1 Completing the square

- **Completing the square** is the process of writing a general quadratic expression in turning point form.

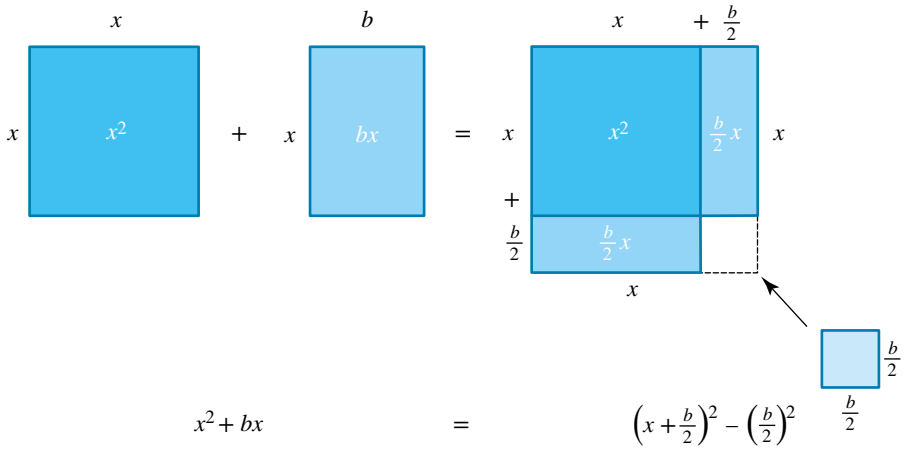
Complete the square

$$\underbrace{ax^2 + bx + c}_{\text{General form}} = \underbrace{a(x - h)^2 + k}_{\text{Turning point form}}$$

- The expression $x^2 + 8x$ can be modelled as a square with a smaller square missing from the corner, as shown below.



- In 'completing the square', the general equation is written as the area of the large square minus the area of the small square.
- In general, to complete the square for $x^2 + bx$, the small square has a side length equal to half of the coefficient of x; that is, the area of the small square is $\left(\frac{b}{2}\right)^2$.



WORKED EXAMPLE 10

Write the following in turning point form by completing the square.

a $x^2 + 4x$ **b** $x^2 + 7x + 1$

THINK

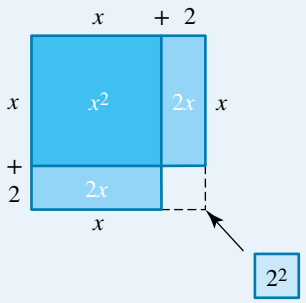
a
- The square will consist of a square that has an area of x^2 and two identical rectangles with a total area of $4x$.
- The length of the large square is $(x + 2)$ so its area is $(x + 2)^2$.
- The area of the smaller square is $(2)^2$.
- Write $x^2 + 4x$ in turning point form.

b **1**
- Complete the square with the terms containing x.
- The square will consist of a square that has an area of x^2 and two identical rectangles with a total area of $7x$.
- The length of the large square is $\left(x + \frac{7}{2}\right)$ so its area is $\left(x + \frac{7}{2}\right)^2$.
- The area of the smaller square is $\left(\frac{7}{2}\right)^2$.
- Write $x^2 + 7x + 1$ in turning point form.

2 Simplify the last two terms.

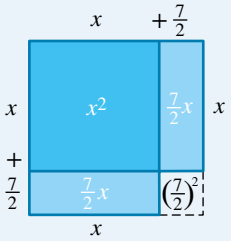
WRITE

a



$$x^2 + 4x = (x + 2)^2 - (2)^2$$
$$= (x + 2)^2 - 4$$

b



$$x^2 + 7x + 1 = \left(x + \frac{7}{2}\right)^2 - \left(\frac{7}{2}\right)^2 + 1$$
$$= \left(x + \frac{7}{2}\right)^2 - \frac{49}{4} + \frac{4}{4}$$
$$= \left(x + \frac{7}{2}\right)^2 - \frac{45}{4}$$

- The process of completing the square is sometimes described as the process of adding the square of half of the coefficient of x then subtracting it, as shown in green below. The result of this process is a perfect square that is then factorised, as shown in blue.

$$x^2 + bx = x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$
$$= x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$
$$= \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

- For example, factorise $x^2 + 8x + 2$ by completing the square.

$$x^2 + 8x + \left(\frac{8}{2}\right)^2 - \left(\frac{8}{2}\right)^2 + 2$$
$$= x^2 + 8x + (4)^2 - (4)^2 + 2$$
$$= x^2 + 8x + 16 - 16 + 2$$
$$= (x + 4)^2 - 14$$

11.5.2 Factorising by completing the square

- When an equation is written in turning point form, it can be factorised as a difference of two squares.

WORKED EXAMPLE 11

Factorise the following by completing the square.
a $x^2 + 4x + 2$ **b** $x^2 - 9x + 1$

THINK	WRITE
a 1 To complete the square, add the square of half of the coefficient of x and then subtract it.	**a** $x^2 + 4x + 2 = x^2 + 4x + \left(\frac{4}{2}\right)^2 - \left(\frac{4}{2}\right)^2 + 2$
2 Write the perfect square created in its factorised form.	$= x^2 + 4x + (2)^2 - (2)^2 + 2$ $= (x + 2)^2 - (2)^2 + 2$
3 Write the expression as a difference of two squares by: • simplifying the numerical terms • writing the numerical term as a square $(2 = (\sqrt{2})^2)$.	$= (x + 2)^2 - 4 + 2$ $= (x + 2)^2 - 2$ $= (x + 2)^2 - (\sqrt{2})^2$
4 Use the pattern for DOTS, $a^2 - b^2 = (a - b)(a + b)$, where $a = (x + 2)$ and $b = \sqrt{2}$.	$= (x + 2 + \sqrt{2})(x + 2 - \sqrt{2})$
b 1 To complete the square, add the square of half of the coefficient of x, then subtract it.	**b** $x^2 - 9x + 1 = x^2 - 9x + \left(\frac{9}{2}\right)^2 - \left(\frac{9}{2}\right)^2 + 1$
2 Write the perfect square created in its factorised form.	$= x^2 - 9x + \left(\frac{9}{2}\right)^2 - \left(\frac{9}{2}\right)^2 + 1$ $= \left(x - \frac{9}{2}\right)^2 - \left(\frac{9}{2}\right)^2 + 1$
3 Write the expression as a difference of two squares by: • simplifying the numerical terms • writing the numerical term as a square. $\frac{77}{4} = \left(\sqrt{\frac{77}{4}}\right)^2 = \left(\frac{\sqrt{77}}{2}\right)^2$	$= \left(x - \frac{9}{2}\right)^2 - \frac{81}{4} + 1$ $= \left(x - \frac{9}{2}\right)^2 - \frac{\sqrt{77}}{2}$ $= \left(x - \frac{9}{2}\right)^2 - \left(\frac{\sqrt{77}}{2}\right)^2$
4 Use the pattern for DOTS: $a^2 - b^2 = (a - b)(a + b)$, where $a = \left(x + \frac{9}{2}\right)$ and $b = \frac{\sqrt{77}}{2}$.	$= \left(x - \frac{9}{2} + \frac{\sqrt{77}}{2}\right)\left(x - \frac{9}{2} - \frac{\sqrt{77}}{2}\right)$

- Remember that you can expand the brackets to check your answer.
- If the coefficient of $x^2 \neq 1$, factorise the expression before completing the square.

learnon RESOURCES — ONLINE ONLY

Interactivity: Completing the square (int-2559)
Interactivity: Completing the square (int-2783)
eLesson: Factorisation by completing the square (eles-1939)

Exercise 11.5 Factorising by completing the square

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–d, 2a–d, 3a–d, 4a–d, 5–7, 9	Questions: 1e–i, 2e–h, 3e–h, 4e–h, 5–8, 10	Questions: 1g–i, 2g–i, 3g–i, 4g–i, 5–12

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE10 Complete the square for each of the following expressions.

a. $x^2 + 10x$ **b.** $x^2 + 6x$ **c.** $x^2 - 4x$ **d.** $x^2 + 16x$
e. $x^2 - 20x$ **f.** $x^2 + 8x$ **g.** $x^2 - 14x$ **h.** $x^2 + 50x$
i. $x^2 + 7x$ **j.** $x^2 - x$

2. WE11a Factorise each of the following by first completing the square.

a. $x^2 - 4x - 7$ **b.** $x^2 + 2x - 2$ **c.** $x^2 - 10x + 12$ **d.** $x^2 + 6x - 10$
e. $x^2 + 16x - 1$ **f.** $x^2 - 14x + 43$ **g.** $x^2 + 8x + 9$ **h.** $x^2 - 4x - 13$
i. $x^2 - 12x + 25$

3. WE11b Factorise each of the following by first completing the square.

a. $x^2 - x - 1$ **b.** $x^2 - 3x - 3$ **c.** $x^2 + x - 5$ **d.** $x^2 + 3x - 1$
e. $x^2 + 5x + 2$ **f.** $x^2 + 5x - 2$ **g.** $x^2 - 7x - 1$ **h.** $x^2 - 9x + 13$
i. $x^2 - x - 3$

4. Factorise each of the following by first looking for a common factor and then completing the square.

a. $2x^2 + 4x - 4$ **b.** $4x^2 - 8x - 20$ **c.** $5x^2 + 30x + 5$ **d.** $3x^2 - 12x - 39$
e. $5x^2 - 30x + 10$ **f.** $6x^2 + 24x - 6$ **g.** $3x^2 + 30x + 39$ **h.** $2x^2 - 8x - 14$
i. $6x^2 + 36x - 30$

5. Which method of factorising is the most appropriate for each of the following expressions?

a. Factorising using common factors
b. Factorising using the difference of two squares rule
c. Factorising by grouping
d. Factorising quadratic trinomials
e. Completing the square

i. $3x^2 - 8x - 3$ **ii.** $49m^2 - 16n^2$
iii. $x^2 + 8x + 4 - y^2$ **iv.** $7x^2 - 28x$
v. $6a - 6b + a^2 - b^2$ **vi.** $x^2 + x - 5$
vii. $(x - 3)^2 + 3(x - 3) - 10$ **viii.** $x^2 - 7x - 1$

6. MC **a.** To complete the square, the term which should be added to $x^2 + 4x$ is:

A. 16 **B.** 4 **C.** $4x$ **D.** 2 **E.** $2x$

b. To factorise the expression $x^2 - 3x + 1$, the term that must be both added and subtracted is:

A. 9 **B.** 3 **C.** $3x$ **D.** $\frac{3}{2}$ **E.** $\frac{9}{4}$

7. MC The factorised form of $x^2 - 6x + 2$ is:

A. $(x + 3 - \sqrt{7})(x + 3 + \sqrt{7})$ **B.** $(x + 3 - \sqrt{7})(x - 3 + \sqrt{7})$
C. $(x - 3 - \sqrt{7})(x - 3 - \sqrt{7})$ **D.** $(x - 3 - \sqrt{7})(x + 3 + \sqrt{7})$
E. $(x - 3 + \sqrt{7})(x - 3 - \sqrt{7})$

Communicating, reasoning and problem solving

8. A square measuring x cm in side length has a cm added to its length and b cm added to its width. The resulting rectangle has an area of $(x^2 + 6x + 3)$ cm^2. Find the values of a and b, correct to 2 decimal places.

9. Show that $x^2 + 4x + 6$ cannot be factorised by completing the square.

10. For each of the following, complete the square to factorise the expression.

a. $2x^2 + 8x + 1$ **b.** $3x^2 - 7x + 5$

11. Use the technique of completion of the square to factorise $x^2 + 2(1 - p)x + p(p - 2)$.

12. Why is this method called completing the square?

11.6 Mixed factorisation [Stage 5.3]

- Apply what has been covered in this chapter to the following exercise. Factorise monic and non-monic trinomials using:
  - factorising by grouping
  - difference of two squares
  - completing the square.

DISCUSSION

Explain how you can determine, by looking at an expression, how best to factorise it. Can all expressions be factorised?

learnON RESOURCES — ONLINE ONLY

Digital doc: SkillSHEET Simplifying algebraic fractions (doc-5248)

Digital doc: SkillSHEET Simplifying surds (doc-5249)

Digital doc: SkillSHEET Factorising by grouping three and one (doc-5252)

Digital doc: WorkSHEET Mixed factorisation (doc-5254)

Exercise 11.6 Mixed factorisation

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–6, 8–10, 12–15, 18, 20, 21, 25, 26, 31–35, 46, 48	Questions: 1, 2, 4–9, 11, 13–16, 18–20, 22, 26–29, 30, 34, 36, 38, 40, 42, 44, 46, 47a–g, 48, 49, 52	Questions: 1–9, 13–15, 17, 18, 20, 23, 24, 26, 34–37, 39, 41, 43, 45, 47h–j, 48, 50–55

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

Factorise each of the expressions in questions **1–45**.

1. $3x + 9$
2. $x^2 + 4x + 4 - 9y^2$
3. $x^2 - 36$
4. $x^2 - 49$
5. $5x^2 - 9x - 2$
6. $15x - 20y$
7. $5c + de + dc + 5e$
8. $5x^2 - 80$
9. $-x^2 - 6x - 5$
10. $x^2 + x - 12$
11. $mn + 1 + m + n$
12. $x^2 - 7$
13. $16x^2 - 4x$
14. $5x^2 + 60x + 100$
15. $18 + 9x - 6y - 3xy$
16. $x^2 - 8x + 16 - y^2$
17. $4x^2 + 8$
18. $fg + 2h + 2g + fh$
19. $x^2 - 5$
20. $10mn - 5n + 10m - 5$
21. $x^2 + 6x + 5$
22. $x^2 - 10x - 11$
23. $x^2 - 4$
24. $-5a + bc + ac - 5b$
25. $xy - 1 + x - y$
26. $3x^2 + 5x + 2$
27. $7x^2 - 28$
28. $-4x^2 - 28x - 24$
29. $2p - rs + pr - 2s$
30. $3x^2 - 27$
31. $-3u + tv + ut - 3v$
32. $x^2 - 11$
33. $12x^2 - 7x + 1$
34. $(x - 1)^2 - 4$
35. $(x + 2)^2 - 16$
36. $(2x + 3)^2 - 25$
37. $3(x + 5)^2 - 27$
38. $25 - (x - 2)^2$
39. $4(3 - x)^2 - 16y^2$
40. $(x + 2y)^2 - (2x + y)^2$
41. $(x + 3)^2 - (x + 1)^2$
42. $(2x - 3y)^2 - (x - y)^2$
43. $(x + 3)^2 + 5(x + 3) + 4$
44. $(x - 3)^2 + 3(x - 3) - 10$
45. $2(x + 1)^2 + 5(x + 1) + 2$

46. Consider the following product of algebraic fractions.

$$\frac{x^2 + 3x - 10}{x^2 - 4} \times \frac{x^2 + 4x + 4}{x^2 - 2x - 8}$$

a. Factorise the expression in each numerator and denominator.
b. Cancel factors common to both the numerator and the denominator.
c. Simplify the expression as a single fraction.

47. Use the procedure in question **46** to factorise and simplify each of the following.

a. $\frac{x^2 - 4x + 3}{x^2 - 4x - 12} \times \frac{x^2 + 5x + 6}{x^2 - 9}$
b. $\frac{3x^2 - 17x + 10}{6x^2 + 5x - 6} \times \frac{x^2 - 1}{x^2 - 6x + 5}$
c. $\frac{6x - 12}{x^2 - 4} \times \frac{3x + 6}{x(x - 5)}$
d. $\frac{6x^2 - x - 2}{2x^2 + 3x + 1} \times \frac{2x^2 + x - 1}{3x^2 + 10x - 8}$
e. $\frac{x^2 + 4x - 5}{x^2 + x - 2} \div \frac{x^2 + 10x + 25}{x^2 + 4x + 4}$
f. $\frac{x^2 - 7x + 6}{x^2 + x - 2} \div \frac{x^2 - x - 12}{x^2 - 2x - 8}$
g. $\frac{4ab + 8a}{(c - 3)} \div \frac{5ac + 5a}{c^2 - 2c - 3}$
h. $\frac{p^2 - 7p}{p^2 - 49} \div \frac{p^2 + p - 6}{p^2 + 14p + 49}$
i. $\frac{m^2 + 4m + 4 - n^2}{4m^2 - 4m - 15} \div \frac{2m^2 + 4m - 2mn}{10m^2 + 15m}$
j. $\frac{d^2 - 6d + 9 - 25e^2}{4d^2 - 5d - 6} \div \frac{4d - 12 - 20e}{15d - 10}$

48. Find the original expression if the factorised expression is $\left(\frac{x}{4} + \frac{3}{5}\right)\left(\frac{x}{4} - \frac{3}{5}\right)$.

49. Factorise the following using grouping 'three and one' and DOTS.

a. $x^2 - 18x + 81 - y^2$
b. $4x^2 + 12x - 16y^2 + 9$

Communicating, reasoning and problem solving

50. Expand the binomial factors $(x + a)(x - a)$ and $(-x + a)(-x - a)$. What do you notice? Use your findings to factorise each of the following, giving two possible answers.

a. $x^2 - 169$ **b.** $36b^2 - 144c^2$ **c.** $225x^4y^2 - 169x^2y^6$

51. Use grouping 'two and two' and DOTS to factorise the following. Show your working.

a. $x^2 + 3x - y^2 + 3y$ **b.** $7x + 7y + x^2 - y^2$ **c.** $5p - 10pq + 1 - 4q^2$

52. Simplify

$$\frac{2a^2 - 7a + 6}{a^3 + 8} \times \frac{5a^2 + 11a + 2}{a^3 - 8} \div \frac{10a^2 - 13a - 3}{a^2 - 2a + 4}.$$

Note: Use the difference of two cubes (see page 447).

53. Factorise $x^2 + 12x + 40 - 4x^2y^2 - 4$.

54. When an expression is fully factorised, what should it look like?

55. The expansion of perfect squares

The rules $(a + b)^2 = a^2 + 2ab + b^2$ and $(a - b)^2 = a^2 + 2ab + b^2$ can be used to simplify some arithmetic calculations. For example:

$$\begin{aligned} 97^2 &= (100 - 3)^2 \\ &= 100^2 - 2 \times 100 \times 3 + 3^2 \\ &= 9409 \end{aligned}$$

Use this method to calculate the following.

a. 103^2

b. 62^2

c. 997^2

d. 1012^2

e. 53^2

f. 98^2

11.7 Review

Investigation | Rich task

Celebrity squares and doubles

In small groups or as a class, use the process of elimination to find your 'square and double pair' by playing 'Celebrity squares and doubles' as outlined below.

Equipment: roll of calculator paper, scissors, sticky tape, marker pen

1. Set-up

- Make a class set of headbands. Each headband will be part of a matching pair made by a number being squared and that same original number being doubled (16 and 8 would be a pair, because $4^2 = 16$ and $4 \times 2 = 8$). Your teacher will direct the class as to what number should be written on each headband.
- Place the headbands randomly on a table.

2. Beginning the game

- There is to be no communication between players at this time.
- Your teacher will randomly allocate a headband to each player by placing a headband on their head without the player seeing the number on their headband.

3. Playing the game

The object of the game

- The object of the game is to use the process of elimination for you to find your pair. A possible train of thought is shown in the illustration.

Starting the game

- Once all headbands have been allocated, stand in a circle or walk around freely.
- Without speaking, determine who is a match; then, by a process of elimination, determine who might be your match.

Making a match

- When you think you have found your match, approach that person and say 'I think I am your match.'
- The other player should now check to see if you have a match elsewhere and can reply by saying one of two things: 'Yes, I think I am your match,' or 'I know your match is still out there.'
- If a match is agreed upon, the players should sit out for the remainder of the game. If a match is not agreed upon, players should continue looking.

Ending the game

- The class should continue until everyone is in a pair, at which time the class can check their results.
- The class should now discuss the different trains of thought they used to find their pair and how this relates to factorising quadratic trinomials.

learnON RESOURCES — ONLINE ONLY

Digital doc: Investigation — Celebrity squares and doubles (doc-15928)
Digital doc: Code puzzle — Isaac Newton demonstrated this in 1665 (doc-15929)
Digital doc: Summary Topic 11 (doc-22909)
Interactivity: Word search Topic 11 (int-2844)
Interactivity: Crossword Topic 11 (int-2845)
Interactivity: Sudoku Topic 11 (int-3594)

Exercise 11.7 Review questions

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. MC When expanded, $-3x(x + 4)(5 - x)$ becomes:
A. $-3x^3 - 3x^2 - 27x$ **B.** $-3x^3 + 3x - 27x$ **C.** $3x^3 + 3x^2 - 60x$
D. $-3x^3 + 3x^2 - 60x$ **E.** $3x^3 - 3x^2 - 60x$

2. MC When expanded, $(3x + 7)^2$ becomes:
A. $9x^2 + 49$ **B.** $3x^2 + 49$ **C.** $3x^2 + 21x + 49$
D. $9x^2 + 42x + 49$ **E.** $9x^2 + 21x + 49$

3. MC The factorised form of $-3d^2 - 9d + 30$ is:
A. $-3(d - 5)(d - 2)$ **B.** $-3(d + 5)(d - 6)$ **C.** $-(3d + 5)(d - 2)$
D. $-(3d + 5)(d - 6)$ **E.** $-3(d + 5)(d - 2)$

4. MC If the factorised expression is $(2x - 5)(2x + 5)$, then the original expression must have been:
A. $2x^2 - 5$ **B.** $4x^2 - 5$ **C.** $4x^2 - 25$ **D.** $4x^2 - 20x + 25$ **E.** $2x^2 + 25$

5. MC To factorise $-5x^2 - 45x + 100$, the first step is to:
A. find factors of 5 and 100 that sum to -45
B. take out 5 as a common factor
C. take out -5 as a common factor
D. find factors of 5 and -45 that will add to make 100
E. take out $-5x$ as a common factor

6. MC To complete the square, the term that should be added to $x^2 - 12x$ is:
A. 36 **B.** -12 **C.** $-12x$ **D.** -6 **E.** $-6x$

7. MC Which of the following is equivalent to $5x^2 - 20x - 5$?
A. $5(x - 2)^2$ **B.** $5(x - 2)^2 - 3$ **C.** $5(x - 2)^2 - 15$
D. $5(x - 2)^2 - 20$ **E.** $5(x - 2)^2 - 25$

8. MC In the expanded form of $(x - 3)(x + 5)$, which of the following is incorrect?
A. The value of the constant is –15.
B. The coefficient of the x term is 2.
C. The coefficient of the x term is –8.
D. The coefficient of the x^2 term is 1.
E. The expansion shows this to be a trinomial expression.

9. Expand each of the following and simplify where necessary.
a. $3x(x - 4)$ **b.** $-7x(3x + 1)$ **c.** $(x - 7)(x + 1)$
d. $(2x - 5)(x - 3)$ **e.** $(4x - 1)(3x - 5)$ **f.** $3(x - 4)(2x + 7)$
g. $(2x - 5)(x + 3)(x + 7)$
h. $(x + 5)(x + 7) + (2x - 5)(x - 6)$
i. $(x + 3)(5x - 1) - 2x$

10. Expand and simplify each of the following.
a. $(x - 7)^2$ **b.** $(2 - x)^2$ **c.** $(3x + 1)^2$
d. $-2(3x - 2)^2$ **e.** $-7(2x + 5)^2$ **f.** $-10(4x - 5)^2$
g. $(x + 9)(x - 9)$ **h.** $(3x - 1)(3x + 1)$ **i.** $(5 + 2x)(5 - 2x)$

11. Factorise each of the following.
a. $2x^2 - 8x$ **b.** $-4x^2 + 12x$ **c.** $3ax - 2ax^2$
d. $(x + 1)^2 + (x + 1)$ **e.** $3(2x - 5) - (2x - 5)^2$ **f.** $(x - 4)(x + 2) - (x - 4)$

12. Factorise each of the following.
a. $x^2 - 16$ **b.** $x^2 - 25$ **c.** $2x^2 - 72$
d. $3x^2 - 27y^2$ **e.** $4ax^2 - 16ay^2$ **f.** $(x - 4)^2 - 9$

13. Factorise each of the following by grouping.

a. $ax - ay + bx - by$ **b.** $7x + ay + ax + 7y$ **c.** $xy + 2y + 5x + 10$

d. $mn - q - 2q^2 + 2mnq$ **e.** $pq - 5r^2 - r + 5pqr$ **f.** $uv - u + 9v - 9$

g. $a^2 - b^2 + 5a - 5b$ **h.** $d^2 - 4c^2 - 3d + 6c$ **i.** $2 + 2m + 1 - m^2$

14. Factorise each of the following by grouping.

a. $4x^2 + 12x + 9 - y^2$ **b.** $49a^2 - 28a + 4 - 4b^2$ **c.** $64s^2 - 16s + 1 - 3t$

15. Factorise each of the following.

a. $x^2 + 10x + 9$ **b.** $x^2 - 11x + 18$ **c.** $x^2 - 4x - 21$ **d.** $x^2 + 3x - 28$

e. $-x^2 + 6x - 9$ **f.** $3x^2 + 33x - 78$ **g.** $-2x^2 + 8x + 10$ **h.** $-3x^2 + 24x - 36$

i. $8x^2 + 2x - 1$ **j.** $6x^2 + x - 1$ **k.** $8x^2 + 4x - 12$ **l.** $105x^2 - 10x - 15$

m. $-12x^2 + 62x - 70$ **n.** $-45x^2 - 3x + 6$ **o.** $-60x^2 - 270x - 270$

16. Factorise each of the following by completing the square.

a. $x^2 + 6x + 1$ **b.** $x^2 - 10x - 3$ **c.** $x^2 + 4x - 2$

d. $x^2 - 5x + 2$ **e.** $x^2 + 7x - 1$ **f.** $2x^2 + 18x - 2$

17. Factorise each of the following using the most appropriate method.

a. $3x^2 - 12x$ **b.** $x^2 + 6x + 2$ **c.** $4x^2 - 25$

d. $2x^2 + 9x + 10$ **e.** $2ax + 4x + 3a + 6$ **f.** $-3x^2 - 3x + 18$

18. First factorise then simplify each of the following.

a. $\dfrac{x+4}{5x-30} \times \dfrac{2x-12}{x+1}$ **b.** $\dfrac{3x+6}{4x-24} \times \dfrac{7x-42}{6x+12}$ **c.** $\dfrac{x^2-4}{x^2+5x} \times \dfrac{x^2+4x-5}{x^2-2x-8}$

Communicating, reasoning and problem solving

19. A large storage box has a square base with sides measuring $(x + 2)$ cm and is 32 cm high.

a. Write an expression for the area of the base of the box.

b. Write an expression for the volume of the box (V = area of base × height).

c. Simplify the expression by expanding the brackets.

d. If $x = 30$ cm, find the volume of the box in cm^3.

20. A section of garden is to have a circular pond of radius $2r$ with a 2 m path around its edge.

a. State the diameter of the pond.

b. State the radius of the pond and path.

c. State the area of the pond.

d. State the area of the pond and path.

e. Write an expression to find the area of the path only and write it in factorised form.

f. If the radius of the pond is 3 metres, find the area of the path.

21. In order to make the most of the space available for headlines and stories, the front page of a newspaper is given an area of $x^2 - 5x - 14\,\text{cm}^2$.

a. If the length is $x + 2$ cm, what is the width?

b. Write the length of the shorter side in terms of x.

c. If the shorter side of the front page is 28 cm, find the value of x.

d. Find the area of this particular paper.

22. Here is a well-known puzzle. Let $a = b = 1$.

Step 1: Write $a = b$.	$a = b$
Step 2: Multiply both sides by a.	$a^2 = ab$
Step 3: Subtract b^2 from both sides.	$a^2 - b^2 = ab - b^2$
Step 4: Factorise.	$(a + b)(a - b) = b(a - b)$
Step 5: Simplify by dividing by $(a - b)$.	$(a + b) = b$
Step 6: Substitute $a = b = 1$.	$1 + 1 = 1$

Where is the error? Show your thinking.

Answers

Topic 11 Quadratic expressions

Exercise 11.2 Expanding algebraic expressions

1. a. $2x + 6$ **b.** $4x - 20$ **c.** $21 - 3x$
d. $-x - 3$ **e.** $x^2 + 2x$ **f.** $2x^2 - 8x$
g. $15x^2 - 6x$ **h.** $10x - 15x^2$ **i.** $8x^2 + 2x$
j. $4x^3 - 6x^2$ **k.** $6x^3 - 3x^2$ **l.** $15x^3 + 20x^2$

2. a. $x^2 - x - 12$ **b.** $x^2 - 2x - 3$ **c.** $x^2 - 5x - 14$
d. $x^2 - 6x + 5$ **e.** $-x^2 - x + 6$ **f.** $x^2 - 6x + 8$
g. $2x^2 - 17x + 21$ **h.** $3x^2 - x - 2$ **i.** $6x^2 - 17x + 5$
j. $21 - 17x + 2x^2$ **k.** $15 + 14x - 8x^2$ **l.** $110 + 47x - 21x^2$

3. a. $2x^2 - 4x - 6$ **b.** $8x^2 - 28x - 16$ **c.** $-2x^2 + 12x + 14$
d. $2x^3 - 2x$ **e.** $3x^3 - 75x$ **f.** $6x^3 - 54x$
g. $2x^3 - 12x^2 + 18x$ **h.** $5x^3 - 30x^2 + 40x$ **i.** $-6x^3 - 6x^2 + 120x$

4. a. $x^3 + 2x^2 - x - 2$ **b.** $x^3 - 2x^2 - 5x + 6$ **c.** $x^3 - 5x^2 - x + 5$
d. $x^3 - 6x^2 + 11x - 6$ **e.** $2x^3 - 7x^2 - 5x + 4$ **f.** $6x^3 - 7x^2 + 1$

5. a. $x^2 - x - 2$ **b.** $-2x^2 + 4x + 10$ **c.** $5x^2 - 6x - 5$ **d.** $19x - 23$
e. $-5x - 1$ **f.** $-2x + 6$ **g.** $x^2 - 2x - 3 + \sqrt{3}x$
h. $\sqrt{6} + 2\sqrt{2}x - 3\sqrt{3}x - 6x^2 - \sqrt{5}x$

6. a. A **b.** C

7. B

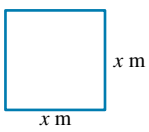
8. a. $x^2 - 2x + 1$ **b.** $x^2 + 4x + 4$ **c.** $x^2 + 10x + 25$ **d.** $16 + 8x + x^2$
e. $49 - 14x + x^2$ **f.** $144 - 24x + x^2$ **g.** $9x^2 - 6x + 1$ **h.** $144x^2 - 72x + 9$
i. $25x^2 + 20x + 4$ **j.** $4 - 12x + 9x^2$ **k.** $25 - 40x + 16x^2$ **l.** $1 - 10x + 25x^2$

9. a. $2x^2 - 12x + 18$ **b.** $4x^2 - 56x + 196$ **c.** $3x^2 + 6x + 3$
d. $-4x^2 - 12x - 9$ **e.** $-49x^2 + 14x - 1$ **f.** $8x^2 - 24x + 18$
g. $-12 + 108x - 243x^2$ **h.** $-45 + 330x - 605x^2$ **i.** $-16x^2 - 16x - 4$

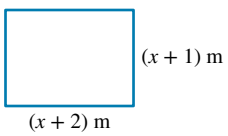
10. a. $x^2 - 49$ **b.** $x^2 - 81$ **c.** $x^2 - 25$
d. $x^2 - 1$ **e.** $4x^2 - 9$ **f.** $9x^2 - 1$
g. $49 - x^2$ **h.** $64 - x^2$ **i.** $9 - 4x^2$

11. a. $(x + 1)(x - 3)$ **b.** $x^2 - 2x - 3$ **c.** 6 cm, 2 cm, 12 cm^2

12. a.



b.



c. $(x + 1)(x + 2)$
d. $x^2 + 3x + 2$ **e.** 4 m^2, 12 m^2

13. a. Student C

b. Student B:

$(3x + 4)(2x + 5)$
$= 3x \times 2x + 3x \times 5 + 4 \times 2x + 4 \times 5$
$= 6x^2 + 23x + 20$

Student A:

$(3x + 4)(2x + 5)$
$= 3x \times 2x + 3x \times 5 + 4 \times 2x + 4 \times 5$
$= 6x^2 + 15x + 8x + 20$
$= 6x^2 + 23x + 20$

14. $(a - b)(a + b) = a^2 - b^2$

LHS:
$(5 - 3)(5 + 3)$
$= 2 \times 8$
$= 16$
RHS:
$5^2 - 3^2$
$= 25 - 9$
$= 16$
LHS = RHS ⇒ True

15. $(a + b)^2 = a^2 + 2ab + b^2$

LHS:
$(5 + 3)^2$
$= 8^2$
$= 64$
RHS:
$5^2 + 2 \times 5 \times 3 + 3^2$
$= 25 + 30 + 9$
$= 64$
LHS = RHS ⇒ True

16. Answers will vary; examples are shown.

a. $(x + 4)(x + 3) = x^2 + 7x + 12$

b. $(x + 4)^2 = x^2 + 8x + 16$

c. $(x + 4)(x - 4) = x^2 - 16$

d. $(x + 4)^2 = x^2 + 8x + 16$

17. The square of a binomial is a trinomial; the difference of two squares has two terms.

18. $(a + b)(c + d) = (c + d)(a + b)$

LHS:
$(a + b)(c + d) = ac + ad + bc + bd$
RHS:
$(c + d)(a + b) = ca + cb + da + db$
LHS = RHS ⇒ True

19. a. $4x^2 + 12xy - 20xz + 9y^2 - 30yz + 25z^2$

b. $\frac{1}{x} + \frac{1}{4x^2} - 4x + 4x^2 - 1$

20. a. $V = 6x^3 - 29x^2 + 46x - 24$

b. $TSA = 12x^2 + 20x + 3$

21. Discuss with your teacher.

22. 10, 11, 13, 18, 35

Exercise 11.3 Factorising expressions with three terms

1. a. $(x + 2)(x + 1)$ **b.** $(x + 3)(x + 1)$ **c.** $(x + 8)(x + 2)$ **d.** $(x + 4)^2$
e. $(x - 3)(x + 1)$ **f.** $(x - 4)(x + 1)$ **g.** $(x - 12)(x + 1)$ **h.** $(x - 6)(x + 2)$
i. $(x + 4)(x - 1)$ **j.** $(x + 5)(x - 1)$ **k.** $(x + 7)(x - 1)$ **l.** $(x + 5)(x - 2)$
m. $(x - 3)(x - 1)$ **n.** $(x - 4)(x - 5)$ **o.** $(x + 14)(x - 5)$

2. a. $-2(x + 9)(x + 1)$ **b.** $-3(x + 2)(x + 1)$ **c.** $-(x + 2)(x + 1)$
d. $-(x + 10)(x + 1)$ **e.** $-(x + 2)(x + 5)$ **f.** $-(x + 12)(x + 1)$
g. $-(x + 3)(x + 4)$ **h.** $-(x + 2)(x + 6)$ **i.** $2(x + 2)(x + 5)$
j. $3(x + 1)(x + 10)$ **k.** $5(x + 20)(x + 1)$ **l.** $5(x + 4)(x + 5)$

3. a. $(a - 7)(a + 1)$ **b.** $(t - 4)(t - 2)$ **c.** $(b + 4)(b + 1)$
d. $(m + 5)(m - 3)$ **e.** $(p - 16)(p + 3)$ **f.** $(c + 16)(c - 3)$
g. $(k + 19)(k + 3)$ **h.** $(s - 19)(s + 3)$ **i.** $(g + 8)(g - 9)$
j. $(v - 25)(v - 3)$ **k.** $(x + 16)(x - 2)$ **l.** $(x - 15)(x - 4)$

4. a. C **b.** B

5. C

6. a. $(2x + 1)(x + 2)$ **b.** $(2x - 1)(x - 1)$ **c.** $(4x + 3)(x - 5)$ **d.** $(2x - 1)(2x + 3)$
e. $(x - 7)(2x + 5)$ **f.** $(3x + 1)(x + 3)$ **g.** $(3x - 7)(2x - 1)$ **h.** $(4x - 7)(3x + 2)$
i. $(5x + 3)(2x - 3)$ **j.** $(4x - 1)(5x + 2)$ **k.** $(3x + 2)(4x - 1)$ **l.** $(3x - 1)(5x + 2)$

7. a. $2(x - 1)(2x + 3)$ **b.** $3(3x + 1)(x - 7)$ **c.** $12(2x + 1)(3x - 1)$ **d.** $-3(3x + 1)(2x - 1)$
e. $-30(2x + 1)(x - 3)$ **f.** $3a(4x - 7)(2x + 5)$ **g.** $-2(4x - 3)(x - 2)$ **h.** $-(2x - 7)(5x + 2)$
i. $-(8x - 1)(3x - 4)$ **j.** $-2(3x - y)(2x + y)$ **k.** $-5(2x - 7y)(3x + 2y)$ **l.** $-12(5x + 3y)(10x + 7y)$

8. a. $w^2 + 5w - 6$ **b.** $(w + 6)(w - 1)$ **c.** $(x + 5)(x - 2)$

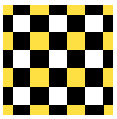
9. a. $x(x + 5)$ **b.** $x(x + 5)$ **c.** $(x - 1)^2$
d. $(x + 9)(x + 5)$ **e.** $(x - 15)(x - 6)$ **f.** $(x - 10)(x - 3)$

10. $(x - 0.5)(x + 1.5)$

11. a. $(x - 5)(x + 1)$ **b.** $(x - 5)$ cm **c.** $x = 15$ cm **d.** 160 cm^2
e. $3000(x - 5)(x + 1)$ cm^2 or $(3000x^2 - 12000x - 15000)$ cm^2

12. a. $(2x + 3)(3x + 1)$ **b.** $P = 10x + 8$ **c.** $x = 8$ metres

13. a. SA $= 3x^2 + 16x$ **b.** Total area $= 3x^2 + 16x + 16$ **c.** $(3x + 4)(x + 4)$
d. $l = 21$ m; $w = 7$ m; $d = 2$ m **e.** 275 m^2 **f.** 294 m^3

14. a. Yellow = 3 cm × 3 cm
Black = 3 cm × 6 cm
White = 6 cm × 6 cm
b. Yellow = 0.36 m^2
Black = 0.72 m^2
White = 0.36 m^2
c. 

15. a. $x^2 - 7x + 12 = (x - 3)(x - 4)$ **b.** $x^2 - x - 12 = (x + 3)(x - 4)$
c. $x^2 - x - 2 = (x - 2)(x + 1)$ **d.** $x^2 - 4x - 21 = (x + 3)(x - 7)$
e. $x^2 + 4x - 21 = (x - 3)(x + 7)$ **f.** $x^2 - x - 30 = (x + 5)(x - 6)$
g. $x^2 + 7x - 8 = (x - 1)(x + 8)$ **h.** $x^2 - 11x + 30 = (x - 5)(x - 6)$

16. a. $(9a - 2)(6a - 7)$ **b.** $(3m^2 + 2)(m - \sqrt{7})(m + \sqrt{7})$ **c.** $(2\sin(x) - 1)(\sin(x) - 1)$

17. a. $(x + 2\sqrt{3})(2x - \sqrt{3})$ **b.** $2z(z^2 + 3)$

18. Discuss with your teacher.

Exercise 11.4 Factorising expressions with two or four terms

1. a. $x(x + 3)$ **b.** $x(x - 4)$ **c.** $3x(x - 2)$
d. $4x(x + 4)$ **e.** $3x(3x - 1)$ **f.** $8x(1 - x)$
g. $3x(4 - x)$ **h.** $4x(2 - 3x)$ **i.** $x(8x - 11)$

2. a. $(x - 2)(3x + 2)$ **b.** $(x + 3)(5 - 2x)$ **c.** $(x - 1)(x + 5)$
d. $(x + 1)(x - 1)$ **e.** $(x + 4)(x - 2)$ **f.** $(x - 3)(4 - x)$

3. a. $(x + 1)(x - 1)$ **b.** $(x + 3)(x - 3)$ **c.** $(x + 5)(x - 5)$
d. $(x + 10)(x - 10)$ **e.** $(y + k)(y - k)$ **f.** $(2x + 3y)(2x - 3y)$
g. $(4a + 7)(4a - 7)$ **h.** $(5p + 6q)(5p - 6q)$ **i.** $(1 + 10d)(1 - 10d)$

4. a. $4(x + 1)(x - 1)$ **b.** $5(x + 4)(x - 4)$ **c.** $a(x + 3)(x - 3)$
d. $2(b + 2d)(b - 2d)$ **e.** $100(x + 4)(x - 4)$ **f.** $3a(x + 7)(x - 7)$
g. $4p(x + 8)(x - 8)$ **h.** $4(3x + 2)(3x - 2)$ **i.** $3(6 + x)(6 - x)$

5. a. C **b.** B **c.** B

6. C

7. a. $(x + \sqrt{11})(x - \sqrt{11})$ **b.** $(x + \sqrt{7})(x - \sqrt{7})$ **c.** $(x + \sqrt{15})(x - \sqrt{15})$
d. $(2x + \sqrt{13})(2x - \sqrt{13})$ **e.** $(3x + \sqrt{19})(3x - \sqrt{19})$ **f.** $3(x + \sqrt{22})(x - \sqrt{22})$
g. $5(x + \sqrt{3})(x - \sqrt{3})$ **h.** $2(x + \sqrt{2})(x - \sqrt{2})$ **i.** $12(x + \sqrt{3})(x - \sqrt{3})$

8. a. $(x - 3)(x + 1)$ **b.** $(x - 4)(x + 6)$ **c.** $(x - 5)(x + 1)$
d. $(x - 1)(x + 7)$ **e.** $(6 - x)(x + 8)$ **f.** $(10 - x)(x + 2)$
g. $8(x - 3)$ **h.** $(7 - x)(5x + 1)$ **i.** $(x - 22)(9x + 2)$

9. a. $(x - 2y)(1 + a)$ **b.** $(x + y)(2 + a)$ **c.** $(x - y)(a + b)$ **d.** $(x + y)(4 + z)$
e. $(f - 2)(e + 3)$ **f.** $(n - 7)(m + 1)$ **g.** $3(2r - s)(t + u)$ **h.** $7(m - 3)(n + 5)$
i. $2(8 - j)(4 + k)$ **j.** $a(3 - b)(a + c)$ **k.** $x(5 + y)(x + 2)$ **l.** $m(m + n)(2 - n)$

10. a. $(y + 7)(x - 2)$ **b.** $(m + 2)(n - 3)$ **c.** $(q + 5)(p - 3)$
d. $(s + 3)(s - 4t)$ **e.** $(b + d)(a^2 - c)$ **f.** $(1 + 5z)(xy - z)$
11. a. $(a - b)(a + b + 4)$ **b.** $(p - q)(p + q - 3)$ **c.** $(m + n)(m - n + l)$
d. $(x + y)(7 + x - y)$ **e.** $(1 - 2q)(5p + 1 + 2q)$ **f.** $(7g + 6h)(7g - 6h - 4)$
12. a. $(x + 7 + y)(x + 7 - y)$ **b.** $(x + 10 + y)(x + 10 - y)$ **c.** $(a - 11 + b)(a - 11 - b)$
d. $(3a + 2 + b)(3a + 2 - b)$ **e.** $(5p - 4 + 3t)(5p - 4 - 3t)$ **f.** $(6t - 1 + \sqrt{5}v)(6t - 1 - \sqrt{5}v)$
13. a. E **b.** A **c.** D
14. B
15. a. $(x - 5)(x + 5)$ **b.** $(x - 5)$ cm **c.** 2 cm, 12 cm
d. 24 cm^2 **e.** 120 cm^2 or 6 times bigger
16. a. $A_1 = \pi r^2$ m^2 **b.** $A_2 = \pi(r + 1)^2$ m^2
c. $A = \pi(r + 1)^2 - \pi r^2 = \pi(2r + 1)$ m^2 **d.** 34.56 m^2
17. a. Annie $= (x + 3)(x + 2)$ m^2
Bronwyn $= 5(x + 2)$ m^2
b. $(x + 3)(x + 2) - 5(x + 2)$
c. $(x + 2)(x - 2) = x^2 - 4$
d. Width $= 5$ m
e. Annie has 30 m^2 and Bronwyn has 25 m^2.
18. a. $(x - 2y - a + 3b)(x - 2y + a - 3b)$ **b.** $(x - 2)(x^2 + 2x + 4)$
19. a. $(3x - 1)(9x^2 + 3x + 1)$ **b.** $3(2x - 5y - 3)(2x + 5y - 3)$
20. Look for the common factor to factorise something out.

Exercise 11.5 Factorising by completing the square

1. a. 25 **b.** 9 **c.** 4 **d.** 64 **e.** 100
f. 16 **g.** 49 **h.** 625 **i.** $\frac{49}{4}$ **j.** $\frac{1}{4}$
2. a. $(x - 2 + \sqrt{11})(x - 2 - \sqrt{11})$ **b.** $(x + 1 + \sqrt{3})(x + 1 - \sqrt{3})$ **c.** $(x - 5 + \sqrt{13})(x - 5 - \sqrt{13})$
d. $(x + 3 + \sqrt{19})(x + 3 - \sqrt{19})$ **e.** $(x + 8 + \sqrt{65})(x + 8 - \sqrt{65})$ **f.** $(x - 7 + \sqrt{6})(x - 7 - \sqrt{6})$
g. $(x + 4 + \sqrt{7})(x + 4 - \sqrt{7})$ **h.** $(x - 2 + \sqrt{17})(x - 2 - \sqrt{17})$ **i.** $(x - 6 + \sqrt{11})(x - 6 - \sqrt{11})$
3. a. $\left(x - \frac{1}{2} + \frac{\sqrt{5}}{2}\right)\left(x - \frac{1}{2} - \frac{\sqrt{5}}{2}\right)$ **b.** $\left(x - \frac{3}{2} + \frac{\sqrt{21}}{2}\right)\left(x - \frac{3}{2} - \frac{\sqrt{21}}{2}\right)$ **c.** $\left(x + \frac{1}{2} + \frac{\sqrt{21}}{2}\right)\left(x + \frac{1}{2} - \frac{\sqrt{21}}{2}\right)$
d. $\left(x + \frac{3}{2} + \frac{\sqrt{13}}{2}\right)\left(x + \frac{3}{2} - \frac{\sqrt{13}}{2}\right)$ **e.** $\left(x + \frac{5}{2} + \frac{\sqrt{17}}{2}\right)\left(x + \frac{5}{2} - \frac{\sqrt{17}}{2}\right)$ **f.** $\left(x + \frac{5}{2} + \frac{\sqrt{33}}{2}\right)\left(x + \frac{5}{2} - \frac{\sqrt{33}}{2}\right)$
g. $\left(x - \frac{7}{2} + \frac{\sqrt{53}}{2}\right)\left(x - \frac{7}{2} - \frac{\sqrt{53}}{2}\right)$ **h.** $\left(x - \frac{9}{2} + \frac{\sqrt{29}}{2}\right)\left(x - \frac{9}{2} - \frac{\sqrt{29}}{2}\right)$ **i.** $\left(x - \frac{1}{2} + \frac{\sqrt{13}}{2}\right)\left(x - \frac{1}{2} - \frac{\sqrt{13}}{2}\right)$
4. a. $2(x + 1 + \sqrt{3})(x + 1 - \sqrt{3})$ **b.** $4(x - 1 + \sqrt{6})(x - 1 - \sqrt{6})$ **c.** $5(x + 3 + 2\sqrt{2})(x + 3 - 2\sqrt{2})$
d. $3(x - 2 + \sqrt{17})(x - 2 - \sqrt{17})$ **e.** $5(x - 3 + \sqrt{7})(x - 3 - \sqrt{7})$ **f.** $6(x + 2 + \sqrt{5})(x + 2 - \sqrt{5})$
g. $3(x + 5 + 2\sqrt{3})(x + 5 - 2\sqrt{3})$ **h.** $2(x - 2 + \sqrt{11})(x - 2 - \sqrt{11})$ **i.** $6(x + 3 + \sqrt{14})(x + 3 - \sqrt{14})$
5. a. iv **b. ii** **c. iii, v** **d. i, vi, vii** **e. viii**
6. a. B **b.** E
7. E
8. $a = 0.55$; $b = 5.45$
9. Check with your teacher.
10. a. $2\left(x + 2 - \frac{\sqrt{14}}{2}\right)\left(x + 2 + \frac{\sqrt{14}}{2}\right)$
b. This expression cannot be factorised as there is no difference of two squares.
11. $(x - p)(x - p + 2)$
12. This method is called 'completing the square' because it uses the pattern found in the expansion and factorising of perfect squares.

Exercise 11.6 Mixed factorisation

1. $3(x + 3)$ **2.** $(x + 2 + 3y)(x + 2 - 3y)$ **3.** $(x + 6)(x - 6)$ **4.** $(x + 7)(x - 7)$
5. $(5x + 1)(x - 2)$ **6.** $5(3x - 4y)$ **7.** $(c + e)(5 + d)$ **8.** $5(x + 4)(x - 4)$
9. $-(x + 5)(x + 1)$ **10.** $(x + 4)(x - 3)$ **11.** $(m + 1)(n + 1)$ **12.** $(x + \sqrt{7})(x - \sqrt{7})$
13. $4x(4x - 1)$ **14.** $5(x + 10)(x + 2)$ **15.** $3(3 - y)(x + 2)$ **16.** $(x - 4 + y)(x - 4 - y)$

17. $4(x^2 + 2)$ **18.** $(g + h)(f + 2)$ **19.** $(x + \sqrt{5})(x - \sqrt{5})$ **20.** $5(n + 1)(2m - 1)$

21. $(x + 5)(x + 1)$ **22.** $(x + 1)(x - 11)$ **23.** $(x + 2)(x - 2)$ **24.** $(a + b)(c - 5)$

25. $(y + 1)(x - 1)$ **26.** $(3x + 2)(x + 1)$ **27.** $7(x + 2)(x - 2)$ **28.** $-4(x + 6)(x + 1)$

29. $(2 + r)(p - s)$ **30.** $3(x + 3)(x - 3)$ **31.** $(u + v)(t - 3)$ **32.** $(x + \sqrt{11})(x - \sqrt{11})$

33. $(4x - 1)(3x - 1)$ **34.** $(x + 1)(x - 3)$ **35.** $(x + 6)(x - 2)$ **36.** $4(x - 1)(x + 4)$

37. $3(x + 2)(x + 8)$ **38.** $(3 + x)(7 - x)$ **39.** $4(3 - x + 2y)(3 - x - 2y)$ **40.** $3(y + x)(y - x)$

41. $4(x + 2)$ **42.** $(3x - 4y)(x - 2y)$ **43.** $(x + 7)(x + 4)$ **44.** $(x + 2)(x - 5)$

45. $(2x + 3)(x + 3)$

46. a. $\dfrac{(x + 5)(x - 2)}{(x + 2)(x - 2)} \times \dfrac{(x + 2)(x + 2)}{(x - 4)(x + 2)}$ **b.** $\dfrac{(x + 5)\cancel{(x - 2)}}{\cancel{(x + 2)}\cancel{(x - 2)}} \times \dfrac{\cancel{(x + 2)}\cancel{(x + 2)}}{(x - 4)\cancel{(x + 2)}}$ **c.** $\dfrac{x + 5}{x - 4}$

47. a. $\dfrac{x - 1}{x - 6}$ **b.** $\dfrac{x + 1}{2x + 3}$ **c.** $\dfrac{18}{x(x - 5)}$

d. $\dfrac{2x - 1}{x + 4}$ **e.** $\dfrac{x + 2}{x + 5}$ **f.** $\dfrac{x - 6}{x + 3}$

g. $\dfrac{4(b + 2)}{5}$ **h.** $\dfrac{p(p + 7)}{(p + 3)(p - 2)}$ **i.** $\dfrac{5(m + 2 + n)}{2(2m - 5)}$

j. $\dfrac{5(3d - 2)(d - 3 + 5e)}{4(d - 2)(4d + 3)}$

48. $\dfrac{x^2}{16} - \dfrac{9}{25}$

49. a. $(x - 9 - y)(x - 9 + y)$ **b.** $(2x + 3 - 4y)(2x + 3 + 4y)$

50. $x^2 - a^2$; they are the same.

a. $(x - 13)(x + 13)$ or $(-x - 13)(-x + 13)$

b. $36(b - 2c)(b + 2c)$ or $36(-b - 2c)(-b + 2c)$

c. $x^2y^2(15x - 13y^2)(15x + 13y^2)$ or $x^2y^2(-15x - 13y^2)(-15x + 13y^2)$

51. a. $(x + y)(x - y + 3)$ **b.** $(x + y)(7 + x - y)$ **c.** $(1 - 2q)(5p + 1 + 2q)$

52. $\dfrac{1}{a^2 + 2a + 4}$

53. $(x + 6 - 2xy)(x + 6 + 2xy)$

54. A fully factorised expression should be formed from products of terms and expressions.

55. a. 10609 **b.** 3844 **c.** 99409 **d.** 1024144 **e.** 2809 **f.** 9604

Investigation | Rich task

Check with your teacher.

Exercise 11.7 Review questions

1. E **2.** D **3.** E **4.** C

5. C **6.** A **7.** E **8.** C

9. a. $3x^2 - 12x$ **b.** $-21x^2 - 7x$ **c.** $x^2 - 6x - 7$

d. $2x^2 - 11x + 15$ **e.** $12x^2 - 23x + 5$ **f.** $6x^2 - 3x - 84$

g. $2x^3 + 15x^2 - 8x - 105$ **h.** $3x^2 - 5x + 65$ **i.** $5x^2 + 12x - 3$

10. a. $x^2 - 14x + 49$ **b.** $4 - 4x + x^2$ **c.** $9x^2 + 6x + 1$

d. $-18x^2 + 24x - 8$ **e.** $-28x^2 - 140x - 175$ **f.** $-160x^2 + 400x - 250$

g. $x^2 - 81$ **h.** $9x^2 - 1$ **i.** $25 - 4x^2$

11. a. $2x(x - 4)$ **b.** $-4x(x - 3)$ **c.** $ax(3 - 2x)$

d. $(x + 1)(x + 2)$ **e.** $2(2x - 5)(4 - x)$ **f.** $(x - 4)(x + 1)$

12. a. $(x + 4)(x - 4)$ **b.** $(x + 5)(x - 5)$ **c.** $2(x + 6)(x - 6)$

d. $3(x + 3y)(x - 3y)$ **e.** $4a(x + 2y)(x - 2y)$ **f.** $(x - 1)(x - 7)$

13. a. $(x - y)(a + b)$ **b.** $(x + y)(7 + a)$ **c.** $(x + 2)(y + 5)$

d. $(1 + 2q)(mn - q)$ **e.** $(5r + 1)(pq - r)$ **f.** $(v - 1)(u + 9)$

g. $(a - b)(a + b + 5)$ **h.** $(d - 2c)(d + 2c - 3)$ **i.** $(1 + m)(3 - m)$

14. a. $(2x + 3 + y)(2x + 3 - y)$ **b.** $(7a - 2 + 2b)(7a - 2 - 2b)$ **c.** $(8s - 1 + \sqrt{3}t)(8s - 1 - \sqrt{3}t)$

15. a. $(x + 9)(x + 1)$ **b.** $(x - 9)(x - 2)$ **c.** $(x - 7)(x + 3)$

d. $(x + 7)(x - 4)$ **e.** $-(x - 3)^2$ **f.** $3(x + 13)(x - 2)$

g. $-2(x - 5)(x + 1)$ **h.** $-3(x - 6)(x - 2)$ **i.** $(4x - 1)(2x + 1)$

j. $(3x - 1)(2x + 1)$ **k.** $4(2x + 3)(x - 1)$ **l.** $5(7x - 3)(3x + 1)$

m. $-2(3x - 5)(2x - 7)$ **n.** $-3(3x - 1)(5x + 2)$ **o.** $-30(2x + 3)(x + 3)$

16. a. $(x + 3 + 2\sqrt{2})(x + 3 - 2\sqrt{2})$ **b.** $(x - 5 + 2\sqrt{7})(x - 5 - 2\sqrt{7})$ **c.** $(x + 2 + \sqrt{6})(x + 2 - \sqrt{6})$

d. $\left(x - \frac{5}{2} + \frac{\sqrt{17}}{2}\right)\left(x - \frac{5}{2} - \frac{\sqrt{17}}{2}\right)$ **e.** $\left(x + \frac{7}{2} + \frac{\sqrt{53}}{2}\right)\left(x + \frac{7}{2} - \frac{\sqrt{53}}{2}\right)$ **f.** $2\left(x + \frac{9}{2} + \frac{\sqrt{85}}{2}\right)\left(x + \frac{9}{2} - \frac{\sqrt{85}}{2}\right)$

17. a. $3x(x - 4)$ **b.** $(x + 3 + \sqrt{7})(x + 3 - \sqrt{7})$ **c.** $(2x + 5)(2x - 5)$

d. $(2x + 5)(x + 2)$ **e.** $(a + 2)(2x + 3)$ **f.** $-3(x - 2)(x + 3)$

18. a. $\dfrac{2(x + 4)}{5(x + 1)}$ **b.** $\dfrac{7}{8}$ **c.** $\dfrac{(x - 2)(x - 1)}{x(x - 4)}$

19. a. $(x + 2)^2$ **b.** $32(x + 2)^2$ **c.** $32x^2 + 128x + 128$ **d.** $32\,768\ \text{cm}^3$

20. a. $4r$ **b.** $2r + 2$ **c.** $4\pi r^2$

d. $(4r^2 + 8r + 4)\pi$ **e.** $4\pi(2r + 1)$ **f.** $28\pi\ \text{m}^2$

21. a. $(x - 7)$ **b.** $x - 7$ cm **c.** 35 **d.** $1036\ \text{cm}^2$

22. Division by zero in Step 5

TOPIC 12
Quadratic equations [Stage 5.3]

12.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the concepts covered in this topic.

12.1.1 Why learn this?

The Guggenheim Museum in Bilbao (Spain) is covered with thin metal plates like the scales of a fish, each one designed and shaped by a computer. This project required the solving of thousands of non-linear equations. Parabolic shapes are widely used by engineers and architects.

DISCUSSION

In Chinese culture, certain numbers have different meaning. For example, some numbers are lucky, and other numbers are unlucky. Research the meaning of numbers in Chinese culture and present your research to your class.

LEARNING SEQUENCE

12.1 Overview
12.2 [Stage 5.3] Solving quadratic equations algebraically
12.3 [Stage 5.3] The quadratic formula
12.4 [Stage 5.3] Solving quadratic equations graphically
12.5 [Stage 5.3] The discriminant
12.6 Review

LEARNING OUTCOMES

A student:

- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3-1WM
- generalises mathematical ideas and techniques to analyse and solve problems efficiently MA5.3-2WM
- uses deductive reasoning in presenting arguments and formal proofs MA5.3-3WM
- solves quadratic equations MA5.3-7NA

CONTENT DESCRIPTIONS

Students:

Solve a wide range of quadratic equations derived from a variety of contexts (ACMNA269)

Source: NSW Syllabus for the Australian curriculum

learnON RESOURCES — ONLINE ONLY

eLesson: The story of mathematics — The Chinese Golden Age of Mathematics (eles-1847)

Note: Your teacher may now set you a pre-test to determine how familiar you are with the content in this topic.

12.2 Solving quadratic equations algebraically [Stage 5.3]

12.2.1 Quadratic equations

- The general form of a quadratic equation is $ax^2 + bx + c = 0$.
- To solve an equation means to find the value of the pronumeral(s) or variables that, when substituted, will make the equation a true statement.

12.2.2 The Null Factor Law

- The **Null Factor Law** states that if the product of two numbers is zero then one or both of the numbers must equal zero.
- In other words, there are two solutions to the equation $pq = 0$; they are $p = 0$ and $q = 0$.
- When solving quadratic equations by applying the Null Factor Law, it is best to write the equations equal to zero.

DISCUSSION

Explain the Null Factor Law to a friend. When would you use it in a mathematical context?

WORKED EXAMPLE 1

Solve the equation $(x - 7)(x + 11) = 0$.

THINK	WRITE
1 Write the equation and check that the right-hand side equals zero. (The product of the two numbers is zero.)	$(x - 7)(x + 11) = 0$
2 The left-hand side is factorised, so apply the Null Factor Law.	$x - 7 = 0$ or $x + 11 = 0$
3 Solve for x.	$x = 7$ $x = -11$

WORKED EXAMPLE 2

Solve each of the following equations.

a $x^2 - 3x = 0$ **b** $3x^2 - 27 = 0$ **c** $x^2 - 13x + 42 = 0$ **d** $36x^2 - 21x = 2$

THINK	WRITE
a 1 Write the equation. Check that the right-hand side equals zero.	**a** $x^2 - 3x = 0$
2 Factorise by taking out the common factor of x^2 and $3x$, which is x.	$x(x - 3) = 0$
3 Apply the Null Factor Law.	$x = 0$ or $x - 3 = 0$
4 Solve for x.	$x = 0$ $x = 3$

b 1 Write the equation. Check that the right-hand side equals zero.

b $3x^2 - 27 = 0$

2 Factorise by taking out the common factor of $3x^2$ and 27, which is 3.

$3(x^2 - 9) = 0$

3 Factorise using the difference of two squares rule.

$3(x^2 - 3^2) = 0$
$3(x + 3)(x - 3) = 0$

4 Apply the Null Factor Law.

$x + 3 = 0$ or $x - 3 = 0$

5 Solve for x.

$x = -3$ $x = 3$
(Alternatively, $x = \pm 3$.)

c 1 Write the equation. Check that the right-hand side equals zero.

c $x^2 - 13x + 42 = 0$

2 Factorise by finding a factor pair of 42 that adds to –13.

Factors of 42	Sum of factors
−6 and −7	−13

$(x - 6)(x - 7) = 0$

3 Use the Null Factor Law to write two linear equations.

$x - 6 = 0$ or $x - 7 = 0$

4 Solve for x.

$x = 6$ $x = 7$

d 1 Write the equation. Check that the right-hand side equals zero. (It does not.)

d $36x^2 - 21x = 2$

2 Rearrange the equation so the right-hand side of the equation equals zero.

$36x^2 - 21x - 2 = 0$

3 Recognise that the expression to factorise is a quadratic trinomial.

Factors of −72	Sum of factors
3 and −24	−21

$36x^2 - 24x + 3x - 2 = 0$

4 Factorise the expression.

$12x(3x - 2) + (3x - 2) = 0$
$(3x - 2)(12x + 1) = 0$

5 Use the Null Factor Law to write two linear equations.

$3x - 2 = 0$ or $12x + 1 = 0$
$3x = 2$ $12x = -1$

6 Solve for x.

$x = \frac{2}{3}$ $x = -\frac{1}{12}$

12.2.3 Solving quadratic equations by completing the square

- Sometimes it is necessary to complete the square in order to factorise a quadratic trinomial.
- This is often necessary if the solutions are not rational numbers.

WORKED EXAMPLE 3

Find the solutions to the equation $x^2 + 2x - 4 = 0$. Give exact answers.

THINK

1 Write the equation.

WRITE

$x^2 + 2x - 4 = 0$

2 Identify the coefficient of x, halve it and square the result.	$\left(\frac{1}{2} \times 2\right)^2$
3 Add the result of step 2 to the equation, placing it after the x-term. To balance the equation, we need to subtract the same amount as we have added.	$x^2 + 2x + \left(\frac{1}{2} \times 2\right)^2 - 4 - \left(\frac{1}{2} \times 2\right)^2 = 0$ $x^2 + 2x + (1)^2 - 4 - (1)^2 = 0$ $x^2 + 2x + 1 - 4 - 1 = 0$
4 Insert brackets around the first three terms to group them and then simplify the remaining terms.	$(x^2 + 2x + 1) - 5 = 0$
5 Factorise the first three terms to produce a perfect square.	$(x + 1)^2 - 5 = 0$
6 Express as the difference of two squares and then factorise.	$(x + 1)^2 - (\sqrt{5})^2 = 0$ $(x + 1 + \sqrt{5})(x + 1 - \sqrt{5}) = 0$
7 Apply the Null Factor Law to find linear equations.	$x + 1 + \sqrt{5} = 0$ or $x + 1 - \sqrt{5} = 0$
8 Solve for x. Keep the answer in surd form to provide an exact answer.	$x = -1 - \sqrt{5}$ or $x = -1 + \sqrt{5}$ (Alternatively, $x = -1 \pm \sqrt{5}$.)

12.2.4 Solving problems

- There are many problems that can be modelled by a quadratic equation. You should first form the quadratic equation that represents the situation before attempting to solve such problems.
- Recall that worded problems should always be answered with a sentence.

WORKED EXAMPLE 4

When two consecutive numbers are multiplied together, the result is 20. Determine the numbers.

THINK	WRITE
1 Define the unknowns. Let the first number = x and the second number = $x + 1$.	Let the two numbers be x and $(x + 1)$.
2 Write an equation using the information given in the question.	$x(x + 1) = 20$
3 Transpose the equation so that the right-hand side equals zero.	$x(x + 1) - 20 = 0$
4 Expand to remove the brackets.	$x^2 + x - 20 = 0$
5 Factorise.	$(x + 5)(x - 4) = 0$
6 Apply the Null Factor Law to solve for x.	$x + 5 = 0$ or $x - 4 = 0$ $x = -5$ $x = 4$
7 Use the answer to determine the second number.	If $x = -5$, $x + 1 = -4$. If $x = 4$, $x + 1 = 5$.
8 Check the solutions.	Check: $4 \times 5 = 20$ $-5 \times -4 = 20$
9 Answer the question in a sentence.	The numbers are 4 and 5 or -5 and -4.

WORKED EXAMPLE 5

The height of a football after being kicked is determined by the formula $h = -0.1d^2 + 3d$, where d is the horizontal distance from the player.

a How far away is the ball from the player when it hits the ground?

b What horizontal distance has the ball travelled when it first reaches a height of 20 m?

THINK	WRITE
a 1 Write the formula.	**a** $h = -0.1d^2 + 3d$
2 The ball hits the ground when $h = 0$. Substitute $h = 0$ into the formula.	$-0.1d^2 + 3d = 0$
3 Factorise.	$-0.1d^2 + 3d = 0$ $d(-0.1d + 3) = 0$
4 Apply the Null Factor Law and simplify.	$d = 0$ or $-0.1d + 3 = 0$ $-0.1d = -3$ $d = \frac{-3}{-0.1}$ $d = 30$
5 Interpret the solutions.	$d = 0$ is the origin of the kick. $d = 30$ is the distance from the origin that the ball has travelled when it lands.
6 Answer the question in a sentence.	The ball is 30 m from the player when it hits the ground.
b 1 The height of the ball is 20 m, so, substitute $h = 20$ into the formula.	**b** $h = -0.1d^2 + 3d$ $20 = -0.1d^2 + 3d$
2 Transpose the equation so that zero is on the right-hand side.	$0.1d^2 - 3d + 20 = 0$
3 Multiply both sides of the equation by 10 to remove the decimal from the coefficient.	$d^2 - 30d + 200 = 0$
4 Factorise.	$(d - 20)(d - 10) = 0$
5 Apply the Null Factor Law.	$d - 20 = 0$ or $d - 10 = 0$
6 Solve.	$d = 20$ $d = 10$

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Interactivity: The Null Factor Law (int-6095)
eLesson: The Null Factor Law (eles-2312)
Digital doc: SkillSHEET Factorising by taking out the highest common factor (doc-5256)
Digital doc: SkillSHEET Finding a factor pair that adds to a given number (doc-5257)
Digital doc: SkillSHEET Simplifying surds (doc-5258)
Digital doc: SkillSHEET Substituting into quadratic equations (doc-5259)
Digital doc: SkillSHEET Equation of a vertical line (doc-5260)
Digital doc: WorkSHEET Solving quadratic equations (doc-5261)

Exercise 12.2 Solving quadratic equations algebraically

Individual pathways

PRACTISE
Questions:
1a–f, 2a–c, 3a–d, 4a–c, 5a–h, 6–8, 10, 14, 16, 19, 20, 22

CONSOLIDATE
Questions:
1b–g, 2a–d, 3a–f, 4a–f, 5d–l, 6, 7, 8a–d, 9, 11, 14–16, 18, 20–22, 26

MASTER
Questions:
1, 2, 3f–i, 4f–l, 5g–l, 6, 7, 8g–l, 9f–i, 10d–i, 11d–i, 12–29

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE1 Solve each of the following equations.

a. $(x + 7)(x - 9) = 0$ **b.** $(x - 3)(x + 2) = 0$ **c.** $(x - 2)(x - 3) = 0$
d. $x(x - 3) = 0$ **e.** $x(x - 1) = 0$ **f.** $x(x + 5) = 0$
g. $2x(x - 3) = 0$ **h.** $9x(x + 2) = 0$ **i.** $\left(x - \frac{1}{2}\right)\left(x + \frac{1}{2}\right) = 0$
j. $-(x + 1.2)(x + 0.5) = 0$ **k.** $2(x - 0.1)(2x - 1.5) = 0$ **l.** $(x + \sqrt{2})(x - \sqrt{3}) = 0$

2. Solve each of the following equations

a. $(2x - 1)(x - 1) = 0$ **b.** $(3x + 2)(x + 2) = 0$ **c.** $(4x - 1)(x - 7) = 0$
d. $(7x + 6)(2x - 3) = 0$ **e.** $(5x - 3)(3x - 2) = 0$ **f.** $(8x + 5)(3x - 2) = 0$
g. $x(x - 3)(2x - 1) = 0$ **h.** $x(2x - 1)(5x + 2) = 0$ **i.** $x(x + 3)(5x - 2) = 0$

3. WE2a Solve each of the following equations.

a. $x^2 - 2x = 0$ **b.** $x^2 + 5x = 0$ **c.** $x^2 = 7x$
d. $3x^2 = -2x$ **e.** $4x^2 - 6x = 0$ **f.** $6x^2 - 2x = 0$
g. $4x^2 - 2\sqrt{7}x = 0$ **h.** $3x^2 + \sqrt{3}x = 0$ **i.** $15x - 12x^2 = 0$

4. WE2b Solve each of the following equations.

a. $x^2 - 4 = 0$ **b.** $x^2 - 25 = 0$ **c.** $3x^2 - 12 = 0$
d. $4x^2 - 196 = 0$ **e.** $9x^2 - 16 = 0$ **f.** $4x^2 - 25 = 0$
g. $9x^2 = 4$ **h.** $36x^2 = 9$ **i.** $x^2 - \frac{1}{25} = 0$
j. $\frac{1}{36}x^2 - \frac{4}{9} = 0$ **k.** $x^2 - 5 = 0$ **l.** $9x^2 - 11 = 0$

5. WE2c Solve each of the following equations. Verify the solutions by substituting them into the original quadratic equations.

a. $x^2 - x - 6 = 0$ **b.** $x^2 + 6x + 8 = 0$ **c.** $x^2 - 6x - 7 = 0$
d. $x^2 - 8x + 15 = 0$ **e.** $x^2 - 2x + 1 = 0$ **f.** $x^2 - 3x - 4 = 0$
g. $x^2 - 10x + 25 = 0$ **h.** $x^2 - 3x - 10 = 0$ **i.** $x^2 - 8x + 12 = 0$

j. $x^2 - 4x - 21 = 0$ **k.** $x^2 - x - 30 = 0$ **l.** $x^2 - 7x + 12 = 0$
m. $x^2 - 8x + 16 = 0$ **n.** $x^2 + 10x + 25 = 0$ **o.** $x^2 - 20x + 100 = 0$

6. MC The solutions to the equation $x^2 + 9x - 10 = 0$ are:
A. $x = 1$ and $x = 10$ **B.** $x = 1$ and $x = -10$ **C.** $x = -1$ and $x = 10$
D. $x = -1$ and $x = -10$ **E.** $x = 1$ and $x = 9$

7. MC The solutions to the equation $x^2 - 100 = 0$ are:
A. $x = 0$ and $x = 10$ **B.** $x = 0$ and $x = -10$ **C.** $x = -10$ and $x = 10$
D. $x = 0$ and $x = 100$ **E.** $x = -100$ and $x = 100$

8. WE2d Solve each of the following equations.
a. $2x^2 - 5x = 3$ **b.** $3x^2 + x - 2 = 0$ **c.** $5x^2 + 9x = 2$
d. $6x^2 - 11x + 3 = 0$ **e.** $14x^2 - 11x = 3$ **f.** $12x^2 - 7x + 1 = 0$
g. $6x^2 - 7x = 20$ **h.** $12x^2 + 37x + 28 = 0$ **i.** $10x^2 - x = 2$
j. $6x^2 - 25x + 24 = 0$ **k.** $30x^2 + 7x - 2 = 0$ **l.** $3x^2 - 21x = -36$

9. WE3 Find the solutions for each of the following equations. Give exact answers.
a. $x^2 - 4x + 2 = 0$ **b.** $x^2 + 2x - 2 = 0$ **c.** $x^2 + 6x - 1 = 0$
d. $x^2 - 8x + 4 = 0$ **e.** $x^2 - 10x + 1 = 0$ **f.** $x^2 - 2x - 2 = 0$
g. $x^2 + 2x - 5 = 0$ **h.** $x^2 + 4x - 6 = 0$ **i.** $x^2 + 4x - 11 = 0$

10. Find the solutions for each of the following equations. Give exact answers.
a. $x^2 - 3x + 1 = 0$ **b.** $x^2 + 5x - 1 = 0$ **c.** $x^2 - 7x + 4 = 0$
d. $x^2 - 5 = x$ **e.** $x^2 - 11x + 1 = 0$ **f.** $x^2 + x = 1$
g. $x^2 + 3x - 7 = 0$ **h.** $x^2 - 3 = 5x$ **i.** $x^2 - 9x + 4 = 0$

11. Solve each of the following equations, rounding answers to 2 decimal places.
a. $2x^2 + 4x - 6 = 0$ **b.** $3x^2 + 12x - 3 = 0$ **c.** $5x^2 - 10x - 15 = 0$
d. $4x^2 - 8x - 8 = 0$ **e.** $2x^2 - 6x + 2 = 0$ **f.** $3x^2 - 9x - 3 = 0$
g. $5x^2 - 15x - 25 = 0$ **h.** $7x^2 + 7x - 21 = 0$ **i.** $4x^2 + 8x - 2 = 0$

12. WE4 When two consecutive numbers are multiplied, the result is 72. Find the numbers.

13. When two consecutive even numbers are multiplied, the result is 48. Find the numbers.

14. When a number is added to its square, the result is 90. Find the number.

15. Twice a number is added to three times its square. If the result is 16, find the number.

16. Five times a number is added to two times its square. If the result is 168, find the number.

17. WE5 A soccer ball is kicked. The height, h, in metres, of the soccer ball t seconds after it is kicked can be represented by the equation $h = -t(t - 6)$. Find how long it takes for the soccer ball to hit the ground again.

18. The length of an Australian flag is twice its width and the diagonal length is 45 cm.
a. If x cm is the width of the flag, find the length in terms of x.
b. Draw a diagram of the flag marking in the diagonal. Mark the length and the width in terms of x.
c. Use Pythagoras' theorem to write an equation relating the lengths of the sides to the length of the diagonal.
d. Solve the equation to find the dimensions of the Australian flag. Round your answer to the nearest cm.

19. If the length of a paddock is 2 m more than its width and the area is 48 m^2, find the length and width of the paddock.

20. Solve the following for x.
a. $x + 5 = \frac{6}{x}$ **b.** $x = \frac{24}{x - 5}$ **c.** $x = \frac{1}{x}$

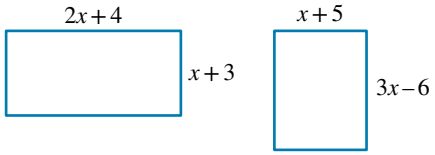
Communicating, reasoning and problem solving

21. The sum of the first n numbers $1, 2, 3, 4 \ldots n$ is given by the formula $S = \dfrac{n(n+1)}{2}$.

a. Use the formula to find the sum of the first 6 counting numbers.

b. How many numbers are added to give a sum of 153?

22. If these two rectangles have the same area, what is the value of x?



23. Henrietta is a pet rabbit who lives in an enclosure that is 2 m wide and 4 m long. Her human family has decided to purchase some more rabbits to keep her company, so the size of the enclosure must be increased.

a. Draw a diagram of Henrietta's enclosure, clearly marking the lengths of the sides.

b. If the length and width of the enclosure are increased by x m, find the new dimensions.

c. If the new area is to be 24 m^2, write an equation relating the sides and the area of the enclosure (area = length × width).

d. Use the equation to find the value of x and, hence, the length of the sides of the new enclosure. Justify your answer.

24. The cost per hour, C, in thousands of dollars of running two cruise ships, *Annabel* and *Betty*, travelling at a speed of s knots is given by the following relationships.

$C_{Annabel} = 0.3s^2 + 4.2s + 12$ and

$C_{Betty} = 0.4s^2 + 3.6s + 8$

a. Determine the cost per hour for each ship if they are both travelling at 28 knots.

b. Find the speed in knots at which both ships must travel for them to have the same cost.

c. Explain why only one of the solutions obtained in your working for part **b** is valid.

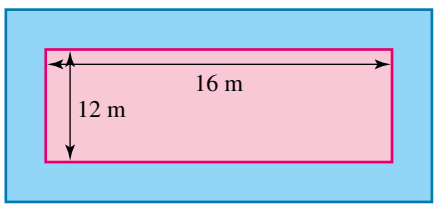
25. Explain why the equation $x^2 + 4x + 10 = 0$ has no real solutions.

26. Solve $(x^2 - x)^2 - 32(x^2 - x) + 240 = 0$ for x.

27. Solve $\dfrac{3z^2 - 35}{16} - z = 0$ for z.

28. What does the Null Factor Law mean?

29. A garden measuring 12 metres by 16 metres is to have a pedestrian pathway installed all around it, increasing the total area to 285 square metres. What will be the width of the pathway?



12.3 The quadratic formula [Stage 5.3]

- The method of solving quadratic equations by completing the square can be generalised to produce what is called the **quadratic formula**.
- The general equation $ax^2 + bx + c = 0$ can be solved by completing the square. We will first follow the steps involved in completing the square.

1. Divide both sides of the equation by a. $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$
2. Complete the square. $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$
3. Factorise the first three terms as a perfect square. $\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$
4. Add the final two terms. $\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} = 0$
5. Write as the difference of two squares. $\left(x + \frac{b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2 = 0$
6. Factorise using the difference of two squares rule. $\left(x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a}\right)\left(x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}\right) = 0$
7. Solve the two linear factors. $x + \frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} = 0$ or $x + \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} = 0$

$$x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \text{ or } x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

- The solution can be summarised as $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ where a is the coefficient of x^2, b is the coefficient of x and c is the constant or the term without an x.
- The quadratic formula can be used to solve any quadratic equation.
- If the value inside the square root sign is negative, then there are no solutions to the equation.

DISCUSSION

Is it possible to rearrange the quadratic formula to solve for *a*, *b* or *c*? Use the internet to help you find the answer.

WORKED EXAMPLE 6

Use the quadratic formula to solve each of the following equations.
a $3x^2 + 4x + 1 = 0$ **(exact answer)**
b $-3x^2 - 6x - 1 = 0$ **(round to 2 decimal places)**

THINK	WRITE
a 1 Write the equation.	**a** $3x^2 + 4x + 1 = 0$
2 Write the quadratic formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

3 State the values for a, b and c.

where $a = 3, b = 4, c = 1$

4 Substitute the values into the formula.

$$x = \frac{-4 \pm \sqrt{(4)^2 - (4 \times 3 \times 1)}}{2 \times 3}$$

5 Simplify and solve for x.

$$= \frac{-4 \pm \sqrt{4}}{6}$$

$$= \frac{-4 \pm 2}{6}$$

$$x = \frac{-4 \pm 2}{6} \text{ or } x = \frac{-4 - 2}{6}$$

6 Write the two solutions.

$$x = -\frac{1}{3} \qquad x = -1$$

b 1 Write the equation.

b $-3x^2 - 6x - 1 = 0$

2 Write the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3 State the values for a, b and c.

where $a = -3, b = -6, c = -1$

4 Substitute the values into the formula.

$$x = \frac{-(-6) \pm \sqrt{36 - 4 \times -3 \times -1}}{2 \times -3}$$

5 Simplify the fraction.

$$= \frac{6 \pm \sqrt{24}}{-6}$$

$$= \frac{6 \pm 2\sqrt{6}}{-6}$$

$$= \frac{3 \pm \sqrt{6}}{-3}$$

$$x = \frac{3 \pm \sqrt{6}}{-3} \text{ or } \frac{3 - \sqrt{6}}{-3}$$

6 Write the two solutions correct to 2 decimal places.

$x \approx -1.82$ or $x \approx -0.18$

learnON RESOURCES — ONLINE ONLY

Interactivity: The quadratic formula (int-2561)

eLesson: The quadratic formula (eles-2314)

Digital doc: SkillSHEET Substituting into the quadratic formula (doc-5262)

Exercise 12.3 The quadratic formula

Individual pathways

PRACTISE
Questions:
1a–d, 2a–f, 3a–f, 4–7, 8a–g, 9

CONSOLIDATE
Questions:
1d–g, 2d–h, 3d–h, 4–7, 8d–i, 10, 12, 13

MASTER
Questions:
1e–h, 2g–l, 3g–n, 4–7, 8i–o, 9–16

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. State the values for a, b and c in each of the following equations of the form $ax^2 + bx + c = 0$.

a. $3x^2 - 4x + 1 = 0$ **b.** $7x^2 - 12x + 2 = 0$

c. $8x^2 - x - 3 = 0$ **d.** $x^2 - 5x + 7 = 0$

e. $5x^2 - 5x - 1 = 0$ **f.** $4x^2 - 9x - 3 = 0$

g. $12x^2 - 29x + 103 = 0$ **h.** $43x^2 - 81x - 24 = 0$

2. WE6a Use the quadratic formula to solve each of the following equations. Give exact answers.

a. $x^2 + 5x + 1 = 0$ **b.** $x^2 + 3x - 1 = 0$ **c.** $x^2 - 5x + 2 = 0$

d. $x^2 - 4x - 9 = 0$ **e.** $x^2 + 2x - 11 = 0$ **f.** $x^2 - 7x + 1 = 0$

g. $x^2 - 9x + 2 = 0$ **h.** $x^2 - 6x - 3 = 0$ **i.** $x^2 + 8x - 15 = 0$

j. $-x^2 + x + 5 = 0$ **k.** $-x^2 + 5x + 2 = 0$ **l.** $-x^2 - 2x + 7 = 0$

3. WE6b Use the quadratic formula to solve each of the following equations. Give approximate answers rounded to 2 decimal places.

a. $3x^2 - 4x - 3 = 0$ **b.** $4x^2 - x - 7 = 0$ **c.** $2x^2 + 7x - 5 = 0$

d. $7x^2 + x - 2 = 0$ **e.** $5x^2 - 8x + 1 = 0$ **f.** $2x^2 - 13x + 2 = 0$

g. $-3x^2 + 2x + 7 = 0$ **h.** $-7x^2 + x + 8 = 0$ **i.** $-12x^2 + x + 9 = 0$

j. $-6x^2 + 4x + 5 = 0$ **k.** $-11x^2 - x + 1 = 0$ **l.** $-4x^2 - x + 7 = 0$

m. $-2x^2 + 12x - 1 = 0$ **n.** $-5x^2 + x + 3 = 0$

4. MC The solutions of the equation $3x^2 - 7x - 2 = 0$ are:

A. 1, 2 **B.** 1, −2 **C.** −0.257, 2.59

D. −0.772, 7.772 **E.** −1.544, 15.544

5. MC In the expansion of $(6x - 5)(3x + 4)$, the coefficient of x is:

A. 18 **B.** −15 **C.** 9 **D.** 6 **E.** −2

6. MC In the expanded form of $(x - 2)(x + 4)$, which of the following is incorrect?

A. The value of the constant is −8. **B.** The coefficient of the x term is −6.

C. The coefficient of the x term is 2. **D.** The coefficient of the x^2 term is 1.

E. The expansion shows this to be a trinomial expression.

7. MC An exact solution to the equation $x^2 + 2x - 5 = 0$ is:

A. −3.449 **B.** $-1 + \sqrt{24}$ **C.** $-1 + \sqrt{6}$ **D.** $\frac{2 + \sqrt{-16}}{2}$ **E.** $\frac{2 + \sqrt{24}}{2}$

8. Solve each of the following equations using any suitable method. Round to 3 decimal places where appropriate.

a. $2x^2 - 7x + 3 = 0$ **b.** $x^2 - 5x = 0$ **c.** $x^2 - 2x - 3 = 0$

d. $x^2 - 3x + 1 = 0$ **e.** $x^2 - 7x + 2 = 0$ **f.** $x^2 - 6x + 8 = 0$

g. $x^2 - 5x + 8 = 0$ **h.** $x^2 - 7x - 8 = 0$ **i.** $x^2 + 2x - 9 = 0$

j. $3x^2 + 3x - 6 = 0$ **k.** $2x^2 + 11x - 21 = 0$ **l.** $7x^2 - 2x + 1 = 0$

m. $-x^2 + 9x - 14 = 0$ **n.** $-6x^2 - x + 1 = 0$ **o.** $-6x^2 + x - 5 = 0$

9. The surface area of a closed cylinder is given by the formula $SA = 2\pi r(r + h)$, where r cm is the radius of the can and h cm is the height. The height of a can of wood finish is 7 cm and its surface area is 231 cm^2.

a. Substitute values into the formula to form a quadratic equation using the pronumeral r.

b. Use the quadratic formula to solve the equation and, hence, find the radius of the can correct to 1 decimal place.

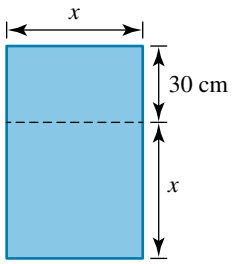
c. Calculate the area of the curved surface of the can, correct to the nearest square centimetre.

10. To satisfy lighting requirements, a window must have an area of 1500 cm^2.

a. Find an expression for the area of the window in terms of x.

b. Write an equation so that the window satisfies the lighting requirements.

c. Use the quadratic formula to solve the equation and find x to the nearest mm.



Communicating, reasoning and problem solving

11. Two competitive neighbours build rectangular pools that cover the same area but are different shapes. Pool A has a width of $(x + 3)$ m and a length that is 3 m longer than its width. Pool B has a length that is double the width of pool A. The width of pool B is 4 m shorter than its length.

a. Find the exact dimensions of each pool if their areas are the same.

b. Verify that the areas are the same.

c. Explain why one of the solutions to the quadratic equation generated to solve this problem may not have been a possible solution to the problem.

12. A block of land is in the shape of a right-angled triangle with a perimeter of 150 m and a hypotenuse of 65 m. Determine the lengths of the other two sides. Show your working.

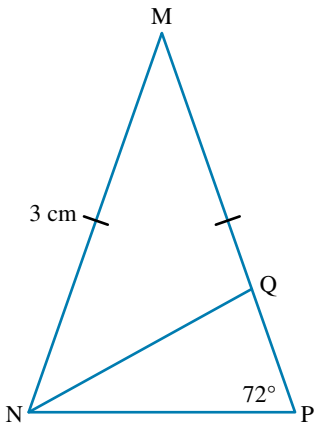
13. Solve $\left(x + \frac{1}{x}\right)^2 - 14\left(x + \frac{1}{x}\right) = 72$ for x.

14. Triangle MNP is an isosceles triangle with sides MN = MP = 3 cm. Angle MPN is equal to 72°. The line NQ bisects the angle MNP.

a. Prove that triangles MNP and NPQ are similar.

b. If NP = m cm and PQ = $3 - m$, show that $m^2 + 3m - 9 = 0$.

c. Solve the equation $m^2 + 3m - 9 = 0$ and find the side length of NP, giving your answer correct to 2 decimal places.



15. What kind of answer will you get if the value inside the square root sign in the quadratic formula is zero?

16. The equation $ax^4 + bx^2 + c = 0$ can be solved by applying substitution and the rules used to solve quadratics. For example, $x^4 - 5x^2 + 4 = 0$ is solved for x as follows.

Notice that $x^4 - 5x^2 + 4 = (x^2)^2 - 5(x)^2 + 4$. Now let $x^2 = u$ and substitute.

$(x^2)^2 - 5(x)^2 + 4 = u^2 - 5u + 4$.

Solve for u. That is,

$u^2 - 5u + 4 = 0$

$(u - 4)(u - 1) = 0$

$u - 4 = 0$ or $u - 1 = 0$

$u = 4$ or $u = 1$

Since $x^2 = u$, that implies that

$x^2 = 4$ or $x^2 = 1$

$x = \pm 2$ or $x = \pm 1$

Using this or another method, solve the following for x.

a. $x^4 - 13x^2 + 36 = 0$

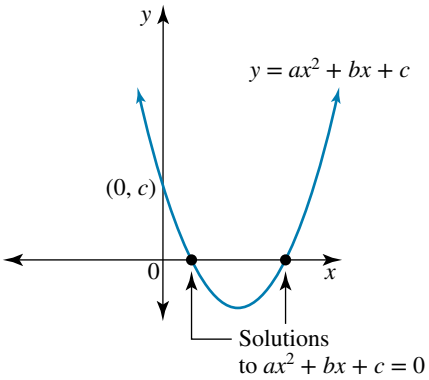
b. $4x^4 - 17x^2 = -4$

12.4 Solving quadratic equations graphically

[Stage 5.3]

12.4.1 The parabola

- The graph of $y = ax^2 + bx + c$ is in the shape of a parabola.
- The graph can be used to locate the solutions to quadratic equations such as $ax^2 + bx + c = 0$.



DISCUSSION

Explain the meaning of finding the solutions of a quadratic equation with reference to the graph.

WORKED EXAMPLE 7

Determine the solutions of each of the following quadratic equations by inspecting their corresponding graphs. Give answers to 1 decimal place where appropriate.

a $x^2 + x - 2 = 0$ **b** $2x^2 + 4x - 5 = 0$

THINK

a 1 Examine the graph of $y = x^2 + x - 2$ and locate the points where $y = 0$, that is, where the graph intersects the x-axis.

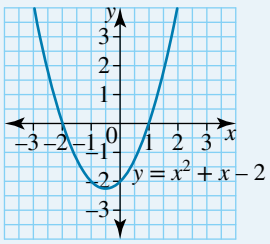
2 The graph cuts the x-axis ($y = 0$) at $x = 1$ and $x = -2$. Write the solutions.

b 1 The graph of $y = 2x^2 - 4x - 5$ is equal to zero when $y = 0$.
Look at the graph to see where $y = 0$, that is, where it intersects the x-axis. By sight, we can only give estimates of the solutions.

2 The graph cuts the x-axis at approximately $x = -0.9$ and approximately $x = 2.9$. Write the solutions.

WRITE/DRAW

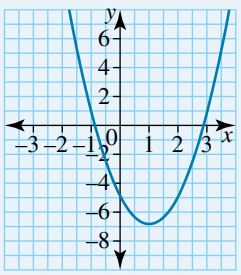
a



$x^2 + x - 2 = 0$

From the graph, the solutions are $x = 1$ and $x = -2$.

b

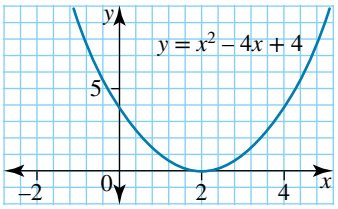


$2x^2 - 4x - 5 = 0$

From the graph, the solutions are $x \approx -0.9$ and $x \approx 2.9$.

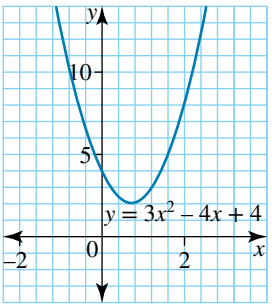
12.4.2 Quadratic equations with only one solution

- Some quadratic equations have only one solution. For example, the graph of $y = x^2 - 4x + 4$ has only one solution at $x = 2$. That is, the graph of $y = x^2 - 4x + 4$ touches the x-axis at one point only at $x = 2$.



12.4.3 Quadratic equations with no solutions

- There are also quadratic equations that have no real solutions. For example, the graph of $y = 3x^2 - 4x + 4$ does not intersect the x-axis and so $3x^2 - 4x + 4 = 0$ has no real solutions (that is, no solutions that are real numbers).



12.4.4 Confirming solutions

- It is possible to confirm the solutions obtained by sight. This is achieved by substituting the solution or solutions into the original quadratic equation, and determining whether they make a true statement.

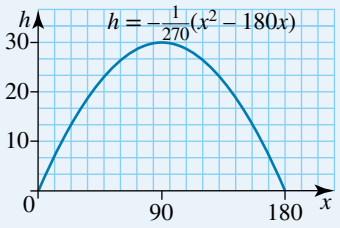
WORKED EXAMPLE 8

Confirm, by substitution, the solutions obtained in Worked example 7.
$x^2 + x - 2 = 0$; solutions: $x = 1$ and $x = -2$

THINK	WRITE
1 Write the left-hand side of the equation and substitute $x = 1$ into the expression.	When $x = 1$, LHS: $x^2 + x - 2 = 1^2 + 1 - 2$ $= 0$
2 Write the right-hand side.	RHS: $= 0$
3 Confirm the solution.	LHS = RHS $\Rightarrow$ Solution is confirmed.
4 Write the left-hand side and substitute $x = -2$.	When $x = -2$, LHS: $x^2 + x - 2 = (-2)^2 + -2 - 2$
5 Write the right-hand side.	$= 4 - 2 - 2$ $= 0$ RHS: $= 0$
6 Confirm.	LHS = RHS $\Rightarrow$ Solution is confirmed.

WORKED EXAMPLE 9

A golf ball hit along a fairway follows the path shown in the following graph. The height, h metres after it has travelled x metres horizontally, follows the rule $h = -\frac{1}{270}(x^2 - 180x)$. Use the graph to find how far the ball landed from the golfer.



THINK	WRITE
On the graph, the ground is represented by the x-axis since this is where $h = 0$. The golf ball lands when the graph intersects the x-axis.	The golf ball lands 180 m from the golfer.

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Interactivity: Solving quadratic equations graphically (int-6148)

Exercise 12.4 Solving quadratic equations graphically

Individual pathways

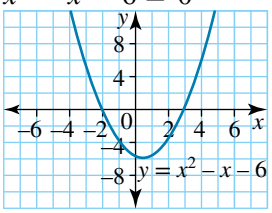
PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–d, 2–4, 7	Questions: 1c–h, 2–5, 7	Questions: 1e–j, 2–9

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

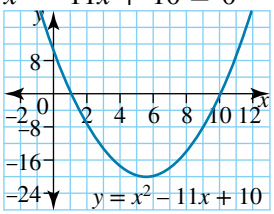
Understanding and fluency

1. WE7 Determine the solutions of each of the following quadratic equations by inspecting the corresponding graphs. Give answers correct to 1 decimal place where appropriate.

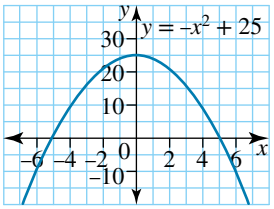
a. $x^2 - x - 6 = 0$



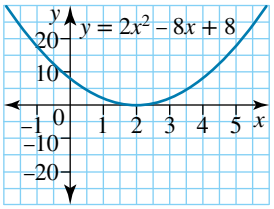
b. $x^2 - 11x + 10 = 0$



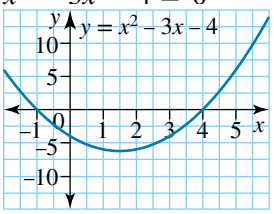
c. $-x^2 + 25 = 0$



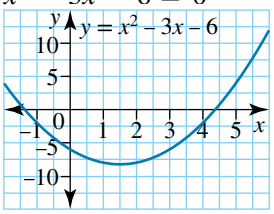
d. $2x^2 - 8x + 8 = 0$



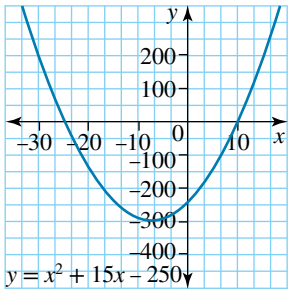
e. $x^2 - 3x - 4 = 0$



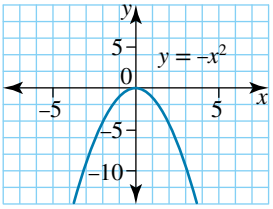
f. $x^2 - 3x - 6 = 0$



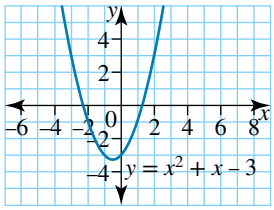
g. $x^2 + 15x - 250 = 0$



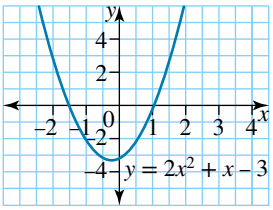
h. $-x^2 = 0$



i. $x^2 + x - 3 = 0$

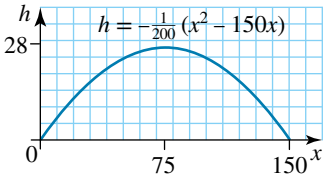


j. $2x^2 + x - 3 = 0$

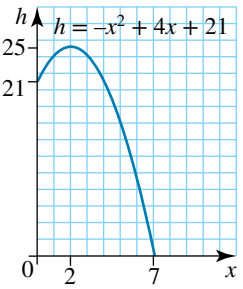


2. **WE8** Confirm, by substitution, the solutions obtained in question **1**.

3. **WE9** A golf ball hit along a fairway follows the path shown in the graph. The height, h metres after it has travelled x metres horizontally, follows the rule $h = -\frac{1}{200}(x^2 - 150x)$. Use the graph to find how far the ball lands from the golfer.



4. A ball is thrown upwards from a building and follows the path shown in the graph until it lands on the ground.
The ball is h metres above the ground when it is a horizontal distance of x metres from the building. The path of the ball follows the rule $h = -x^2 + 4x + 21$. Use the graph to find how far from the building the ball lands.



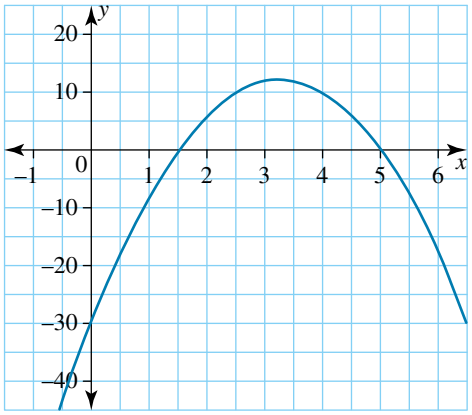
Communicating, reasoning and problem solving

5. **a.** The x-intercepts of a particular equation are $x = 2$ and $x = 5$. Suggest a possible equation.
b. If the y-intercept in part **a** is $(0, 4)$, give the exact equation.

6. **a.** The x-intercepts of a particular equation are $x = p$, q. Suggest a possible equation.
b. If the y-intercept in part **a** is $(0, r)$, give the exact equation.

7. A platform diver follows a path determined by the equation $h = -0.5d^2 + 2d + 6$, where h represents the height of the diver above the water and d represents the distance from the diving board. Both pronumerals are measured in metres.
Use the graph to determine:
a. how far the diver landed from the edge of the diving board
b. how high the diving board is above the water.

8. Find the equation of the given parabola. Give your answer in the form $y = ax^2 + bx + c$.



9. What does 'the solution of a graph' mean?

12.5 The discriminant [Stage 5.3]

12.5.1 Definition of the discriminant

- Where $ax^2 + bx + c = 0$, the expression $\Delta = b^2 - 4ac$ is known as the **discriminant**.

Discriminant
$\Delta = b^2 - 4ac$

- The symbol used for the discriminant, Δ, is the Greek capital letter delta.
- The discriminant is found in the quadratic formula, as shown below.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{\Delta}}{2a}$$

- The discriminant is the value that determines the number of solutions to the quadratic equation.
  - If $\Delta < 0$, there are no real solutions to the quadratic equation.
  - If $\Delta = 0$, there is only one solution to the quadratic equation.
  - If $\Delta > 0$, there are two solutions to the quadratic equation.

WORKED EXAMPLE 10

Calculate the value of the discriminant for each of the following and use it to determine how many solutions the equation will have.

a $2x^2 + 9x - 5 = 0$ **b** $x^2 + 10 = 0$

THINK	WRITE
a 1 Write the expression and determine the values of a, b and c given $ax^2 + bx + c = 0$.	**a** $2x^2 + 9x - 5 = 0$ $2x^2 + 9x + -5 = 0$ $a = 2, b = 9, c = -5$
2 Write the formula for the discriminant and substitute values of a, b and c.	$\Delta = b^2 - 4ac$ $= 9^2 - 4 \times 2 \times -5$
3 Simplify the equation and solve.	$= 81 - -40$ $= 121$

4 State the number of solutions. In this case $\Delta > 0$, which means there are two solutions.

$\Delta > 0$
There will be two solutions to the equation $2x^2 + 9x - 5 = 0$.

b 1 Write the expression and determine the values of a, b and c given $ax^2 + bx + c = 0$.

b $x^2 + 10 = 0$
$1x^2 + 0x + 10 = 0$
$a = 1, b = 0, c = 10$

2 Write the formula for the discriminant and substitute the values of a, b and c.

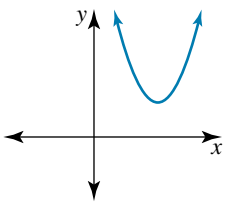
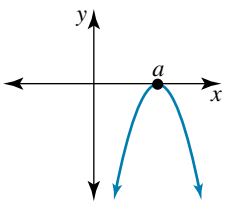
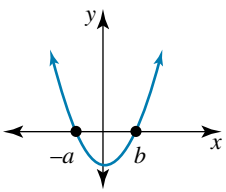
$\Delta = b^2 - 4ac$
$= (0)^2 - 4 \times 1 \times 10$
$= 0 - 40$
$= -40$

3 State the number of solutions. In this case $\Delta < 0$, which means there are no solutions.

$\Delta < 0$, so there will be no solutions to the equation $x^2 + 10 = 0$.

12.5.2 Types of solutions

- The table below summarises the types of solutions indicated by the discriminant.

	$\Delta < 0$ (negative)	$\Delta = 0$ (zero)	$\Delta > 0$ (positive)	
			Perfect square	Not a perfect square
Number of solutions	No real solutions	1 rational solution	2 rational solutions	2 irrational (surd) solutions
Description	Graph does not cross or touch the x-axis	Graph touches the x-axis	Graph intersects the x-axis twice	
Graph				

WORKED EXAMPLE 11

By using the discriminant, determine whether the following equations have:
i two rational solutions
ii two irrational solutions
iii one rational solution (two equal solutions)
iv no real solutions.

a $x^2 - 9x - 10 = 0$ **b** $x^2 - 2x - 14 = 0$
c $x^2 - 2x + 14 = 0$ **d** $x^2 - 14x = -49$

THINK

a 1 Write the equation.

2 Identify the coefficients a, b and c.

WRITE

a $x^2 - 9x - 10 = 0$

$a = 1, b = -9, c = -10$

3 Find the discriminant.	$\Delta = b^2 - 4ac$ $= (-9)^2 - (4 \times 1 \times -10)$ $= 121$
4 Identify the number and type of solutions when $\Delta > 0$ and is a perfect square.	The equation has two rational solutions.
b 1 Write the equation.	**b** $x^2 - 2x - 14 = 0$
2 Identify the coefficients a, b and c.	$a = 1, b = -2, c = -14$
3 Find the discriminant.	$\Delta = b^2 - 4ac$ $= (-2)^2 - 4 \times 1 \times -14$ $= 60$
4 Identify the number and type of solutions when $\Delta > 0$ but not a perfect square.	The equation has two irrational solutions.
c 1 Write the equation.	**c** $x^2 - 2x + 14 = 0$
2 Identify the coefficients a, b and c.	$a = 1, b = -2, c = 14$
3 Find the discriminant.	$\Delta = b^2 - 4ac$ $= (-2)^2 - (4 \times 1 \times 14)$ $= -52$
4 Identify the number and type of solutions when $\Delta < 0$.	The equation has no real solutions.
d 1 Write the equation, then rewrite it so the right side equals zero.	**d** $x^2 + 14x = -49$ $x^2 + 14x + 49 = 0$
2 Identify the coefficients a, b and c.	$a = 1, b = 14, c = 49$
3 Find the discriminant.	$\Delta = b^2 - 4ac$ $= 14^2 - (4 \times 1 \times 49)$ $= 0$
4 Identify the number and types of solutions when $\Delta = 0$.	The equation has 1 rational solution.

12.5.3 Using the discriminant to determine if graphs intersect

- The discriminant can be used to determine the number of points of intersection between graphs.

WORKED EXAMPLE 12

Determine whether the parabola $y = x^2 - 2$ and the line $y = x - 3$ intersect.

THINK	WRITE
1 If the parabola and the line intersect, there will be at least one solution to the simultaneous equations. Let $y_1 = y_2$.	$y_1 = x^2 - 2$ $y_2 = x - 3$ $y_1 = y_2$ $x^2 - 2 = x - 3$
2 Collect all terms on one side and simplify.	$x^2 - 2 - x + 3 = x - 3 - x + 3$ $x^2 - 2 - x + 3 = 0$ $x^2 - x + 1 = 0$

3 Use the discriminant to check if any solutions exist. If $\Delta < 0$, then no solutions exist.

$\Delta = b^2 - 4ac$
$a = 1, b = -1$
$\Delta = (-1)^2 - 4(1)(1)$
$= 1 - 4$
$= -3$
$\Delta < 0 \therefore$ no solutions exist

4 Answer the question.

The parabola and the line do not intersect.

learnON RESOURCES — ONLINE ONLY

Interactivity: The discriminant (int-2560)
Interactivity: Simultaneous quadratic equations (int-2784)
eLesson: The discriminant (eles-1946)
Digital doc: WorkSHEET Using the discriminant (doc-13854)

Exercise 12.5 The discriminant

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–f, 2a–f, 3, 4, 6–8, 10, 13, 15	Questions: 1e–j, 2d–i, 3, 5–9, 11, 12, 14, 15, 17	Questions: 1h–n, 2f–l, 3, 5–19

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE10 Calculate the value of the discriminant for each of the following and use it to determine how many solutions the equation will have.

a. $6x^2 + 13x - 5 = 0$ **b.** $x^2 + 9x - 90 = 0$ **c.** $x^2 + 4x - 2 = 0$
d. $36x^2 - 1 = 0$ **e.** $x^2 + 2x + 8 = 0$ **f.** $x^2 - 5x - 14 = 0$
g. $36x^2 + 24x + 4 = 0$ **h.** $x^2 - 19x + 88 = 0$ **i.** $x^2 - 10x + 17 = 0$
j. $30x^2 + 17x - 21 = 0$ **k.** $x^2 + 16x + 62 = 0$ **l.** $9x^2 - 36x + 36 = 0$
m. $2x^2 - 16x = 0$ **n.** $x^2 - 64 = 0$

2. WE11 By using the discriminant, determine whether the equations below have:

i. two rational solutions **ii.** two irrational solutions
iii. one rational solution (two equal solutions) **iv.** no real solutions.

a. $x^2 - 3x + 5$ **b.** $4x^2 - 20x + 25 = 0$ **c.** $x^2 + 9x - 22 = 0$
d. $9x^2 + 12x + 4$ **e.** $x^2 + 3x - 7 = 0$ **f.** $25x^2 - 10x + 1 = 0$
g. $3x^2 - 2x - 4 = 0$ **h.** $2x^2 - 5x + 4 = 0$ **i.** $x^2 - 10x + 26 = 0$
j. $3x^2 + 5x - 7 = 0$ **k.** $2x^2 + 7x - 10 = 0$ **l.** $x^2 - 11x + 30 = 0$

3. WE12 Determine whether the following graphs intersect.

a. $y = -x^2 + 3x + 4$ and $y = x - 4$ **b.** $y = -x^2 + 3x + 4$ and $y = 2x + 5$
c. $y = -(x + 1)^2 + 3$ and $y = -4x - 1$ **d.** $y = (x - 1)^2 + 5$ and $y = -4x - 1$

4. Consider the equation $3x^2 + 2x + 7 = 0$.

a. What are the values of a, b and c?

b. What is the value of $b^2 - 4ac$?

c. How many real solutions, and hence x-intercepts, are there for this equation?

5. Consider the equation $-6x^2 + x + 3 = 0$.

a. What are the values of a, b and c?

b. What is the value of $b^2 - 4ac$?

c. How many real solutions, and hence x-intercepts, are there for this equation?

d. With the information gained from the discriminant, use the most efficient method to solve the equation. Give an exact answer.

6. MC The discriminant of the equation $x^2 - 4x - 5 = 0$ is:

A. 36 **B.** 11 **C.** 4 **D.** 0 **E.** -4

7. MC Which of the following quadratic equations has two irrational solutions?

A. $x^2 - 8x + 16 = 0$ **B.** $2x^2 - 7x = 0$ **C.** $x^2 + 8x + 9 = 0$

D. $x^2 - 4 = 0$ **E.** $x^2 - 6x + 15 = 0$

8. MC The equation $x^2 = 2x - 3$ has:

A. two rational solutions **B.** exactly one solution

C. no solutions **D.** two irrational solutions

E. one rational and one irrational solution

9. Find the value of k if $x^2 - 2x - k = 0$ has one solution.

10. Find the value of m for which $mx^2 - 6x + 5 = 0$ has one solution.

11. Find the values of n when $x^2 - 3x - n = 0$ has two solutions.

12. Show that $3x^2 + px - 2 = 0$ will have real solutions for all values of p.

13. The path of a dolphin as it leaps out of the water can be modelled by the equation $h = -0.4d^2 + d$, where h is the dolphin's height above water and d is the horizontal distance from its starting point. Both h and d are in metres.

a. How high above the water is the dolphin when it has travelled 2 m horizontally from its starting point?

b. What horizontal distance has the dolphin covered when it first reaches a height of 25 cm?

c. What horizontal distance has the dolphin covered when it next reaches a height of 25 cm? Explain your answer.

d. What horizontal distance does the dolphin cover in one leap? (*Hint:* What is the value of h when the dolphin has completed its leap?)

e. During a leap, can this dolphin reach a height of:
  i. 0.5 m
  ii. 1 m?
  How can you determine this without actually solving the equation?
f. Find the greatest height the dolphin reaches during a leap.

14. The parabolas $y = x^2 - 4$ and $y = 4 - x^2$ intersect in two places. Find the coordinates of their points of intersection.

Communicating, reasoning and problem solving

15. a. For what values of a will the straight line $y = ax + 1$ have one intersection with the parabola $y = -x^2 - x - 8$?
b. For what values of b will the straight line $y = 2x + b$ *not* intersect the parabola $y = x^2 + 3x - 5$?

16. a. Find how many points of intersection exist between the parabola $y = -2(x + 1)^2 - 5$, where $y = f(x)$, $x \in R$, and the straight line $y = mx - 7$, where $y = f(x)$, $x \in R$.
b. Find $m\,(m < 0)$ such that $y = mx - 7$ has one intersection point with $y = -m(x + 1)^2 - 5$.

17. The parabola with the general equation $y = ax^2 + bx + 9$ where $0 < a < 10$ and $0 < b < 20$ touches the x-axis at one point only. The graph passes through the point $(1, 25)$. Find the values of a and b.

18. The line with equation $kx + y = 3$ is a tangent to the curve with equation $y = kx^2 + kx - 1$. Find the value of k.

19. What does the discriminant tell us?

12.6 Review

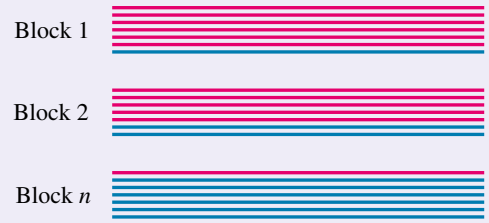
Investigation | Rich task

Weaving

Many articles of clothing are sewn from materials that show designs and patterns made by weaving together threads of different colours. Intricate and complex designs can result. Let's investigate some very simple repetitive-type patterns. Knowledge of quadratic equations and the quadratic formula is helpful in creating these designs.

We need to understand the process of weaving. Weaving machines have parts called *warps*. Each warp is divided into a number of *blocks*. Consider a pattern that is made up of a series of blocks, where the first block is all one colour except for the last thread, which is a different colour. Let's say our pattern is red and blue. The first block contains all red threads, except for the last one, which is blue. The next block has all red threads, except for the last two threads, which are blue. The pattern continues in this manner. The last block has the first thread as red and the remainder as blue. The warp consists of a particular number of threads; let's say 42 threads. How many blocks and how many threads per block would be necessary to create a pattern of this type?

To produce this pattern, we need to divide the warp into equally sized blocks, if possible. What size block and how many threads per block would give us the 42-thread warp? We will need to look for a mathematical pattern. Look at the table, where we consider the smallest block consisting of 2 threads through to a block consisting of 7 threads.



Pattern	Number of threads per block	Number of blocks	Total threads in warp
RB	2	1	2
RRB RBB	3	2	6
RRRB RRBB RBBB	4		
	5		
	6		
	7		

1. Complete the entries in the table.

2. Consider a block consisting of n threads.

a. How many blocks would be needed?

b. What would be the total number of threads in the warp?

The 42-thread warp was chosen as a simple example to show the procedure involved in determining the number of blocks required and the number of threads per block. In this particular case, 6 blocks of 7 threads per block would give us our design for a 42-thread warp. In practice, you would not approach the problem by drawing up a table to determine the number of blocks and the size of each block.

3. Take your expression from question **2b** and let it equal 42. This should form a quadratic equation. Solve this equation to verify that you would need 6 blocks with 7 threads per block to fulfil the size of a 42-thread warp.

4. In reality, the size of each block is not always clearly defined. Also, the thread warp sizes are generally much larger, say, about 250. Let's determine the number of threads per block and the number of blocks required for a 250-thread warp.

a. Form your quadratic equation with the thread warp size equal to 250.

b. A solution to this equation can be found using the quadratic formula. Use the quadratic formula to determine a solution.

c. The number of threads per block is represented by n, and this obviously must be a whole number. Round your solution down to the nearest whole number.

d. How many whole blocks are needed?

e. Use your solutions to **c** and **d** to determine the total number of threads used for the pattern.

f. How many more threads do you need to make the warp size equal to 250 threads?

g. Distribute these threads by including them at the beginning of the first block and the end of the last block. Describe your overall pattern.

5. Investigate the number of blocks required and threads per block required for a 400-thread warp.

6. Investigate changing the pattern. Let the first block be all red. In the next block change the colour of the first and last threads to blue. With each progressive block, change the colour of an extra thread at the top and bottom to blue until the last block is all blue. On a separate sheet of paper, draw a table to determine the thread warp size for a block size of n threads. Draw the pattern and describe the result for a particular warp size.

learnON RESOURCES — ONLINE ONLY

Digital doc: Investigation — Weaving (doc-15930)
Digital doc: Code puzzle — Aussie fact (doc-15931)
Digital doc: Summary Topic 12 (doc-22910)
Interactivity: Word search Topic 12 (int-2847)
Interactivity: Crossword Topic 12 (int-2848)
Interactivity: Sudoku Topic 12 (int-3595)

Exercise 12.6 Review questions

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. MC The solutions to the equation $x^2 + 10x - 11 = 0$ are:
A. $x = 1$ and $x = 11$
B. $x = 1$ and $x = -11$
C. $x = -1$ and $x = 11$
D. $x = -1$ and $x = -11$
E. $x = 1$ and $x = 10$

2. MC The solutions to the equation $-5x^2 + x + 3 = 0$ are:
A. $x = 1$ and $x = \frac{3}{5}$
B. $x = -0.68$ and $x = 0.88$
C. $x = 3$ and $x = -5$
D. $x = 0.68$ and $x = -0.88$
E. $x = 1$ and $x = -\frac{3}{5}$

3. MC The discriminant of the equation $x^2 - 11x + 30 = 0$ is:
A. 1 **B.** 241 **C.** 91 **D.** 19 **E.** -11

4. MC Which of the following quadratic equations has two irrational solutions?
A. $x^2 - 6x + 9 = 0$
B. $4x^2 - 11x = 0$
C. $x^2 - 25 = 0$
D. $x^2 + 8x + 2 = 0$
E. $x^2 - 4x + 10 = 0$

5. The area of a pool is $(6x^2 + 11x + 4)\,m^2$. Find the length of the rectangular pool if its width is $(2x + 1)$ m.

6. Solve each of the following quadratic equations by first factorising the left-hand side of the equation.
a. $x^2 + 8x + 15 = 0$ **b.** $x^2 + 7x + 6 = 0$ **c.** $x^2 + 11x + 24 = 0$
d. $x^2 + 4x - 12 = 0$ **e.** $x^2 - 3x - 10 = 0$ **f.** $x^2 + 3x - 28 = 0$
g. $x^2 - 4x + 3 = 0$ **h.** $x^2 - 11x + 30 = 0$ **i.** $x^2 - 2x - 35 = 0$

7. Solve each of the following quadratic equations.
a. $2x^2 + 16x + 24 = 0$ **b.** $3x^2 + 9x + 6 = 0$ **c.** $4x^2 + 10x - 6 = 0$
d. $5x^2 + 25x - 70 = 0$ **e.** $2x^2 - 7x - 4 = 0$ **f.** $6x^2 - 8x - 8 = 0$
g. $2x^2 - 6x + 4 = 0$ **h.** $6x^2 - 25x + 25 = 0$ **i.** $2x^2 + 13x - 7 = 0$

8. Solve each of the following by completing the square. Give an exact answer for each one.
a. $x^2 + 8x - 1 = 0$ **b.** $3x^2 + 6x - 15 = 0$ **c.** $-4x^2 - 3x + 1 = 0$

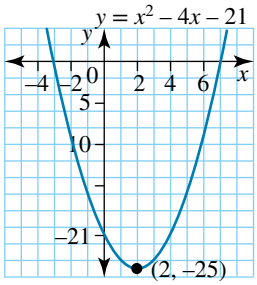
9. Ten times an integer is added to seven times its square. If the result is 152, what was the original number?

10. Solve each of the following by using the quadratic formula, rounding your answers to 3 decimal places.
a. $4x^2 - 2x - 3 = 0$ **b.** $7x^2 + 4x - 1 = 0$ **c.** $-8x^2 - x + 2 = 0$

11. Solve each of the following equations, rounding your answers to 3 decimal places.

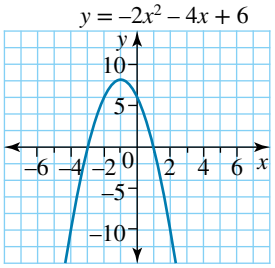
a. $18x^2 - 2x - 7 = 0$ **b.** $29x^2 - 105x - 24 = 0$ **c.** $-5x^2 + 2 = 0$

12. The graph of $y = x^2 - 4x - 21$ is shown.



Use the graph to find the solutions to the quadratic equation $x^2 - 4x - 21 = 0$.

13. Determine the roots of the quadratic graph shown.



14. Identify whether each of the equations below has no real solutions, one solution or two solutions. State whether the solutions are rational or irrational.

a. $x^2 + 11x + 9 = 0$ **b.** $3x^2 + 2x - 5 = 0$ **c.** $x^2 - 3x + 4 = 0$

15. Solve the following pairs of simultaneous equations.

a. $y = x^2 + 4x - 10$, $y = 6 - 2x$ **b.** $y = x^2 - 7x + 20$, $y = 3x - 5$ **c.** $y = x^2 + 7x + 11$, $y = x$

16. For each of the following pairs of equations:

i. solve the pair simultaneously to find the points of intersection

ii. illustrate the solution (or lack of solution) using a sketch graph.

a. $y = x^2 + 6x + 5$ and $y = 11x - 1$ **b.** $y = x^2 + 5x - 6$ and $y = 8x - 8$

c. $y = x^2 + 9x + 14$ and $y = 3x + 5$ **d.** $y = x^2 - 7x + 10$ and $y = -11x + 6$

e. $y = -x^2 + 14x - 48$ and $y = 13x - 54$ **f.** $y = -x^2 + 4x + 12$ and $y = 9x + 16$

Communicating, reasoning and problem solving

17. When a number is added to its square, the result is 56. Determine the number.

18. Leroy measures his bedroom and finds that its length is 3 metres more than its width. If the area of the bedroom is 18 m^2, calculate the length and width of the room.

19. The surface area of a cylinder is given by the formula $SA = 2\pi r(r + h)$, where r cm is the radius of the cylinder and h cm is the height. The height of a can of soft drink is 10 cm and its surface area is 245 cm^2.

a. Substitute values into the formula to form a quadratic equation using the pronumeral r.

b. Use the quadratic formula to solve the equation and, hence, find the radius of the can. Round your answer to 1 decimal place.

c. Calculate the area of the label on the can. The label covers the entire curved surface. Round the answer to the nearest square centimetre.

20. Find the value of d when $2x^2 - 5x - d = 0$ has one solution.

21. For what values of k does $(k - 1)x^2 - (k - 1)x + 2 = 0$ have two distinct solutions?

22. Let m and n be the solutions to the quadratic equation $x^2 - 2\sqrt{5}x - 2 = 0$. Determine the value of $m^2 + n^2$.

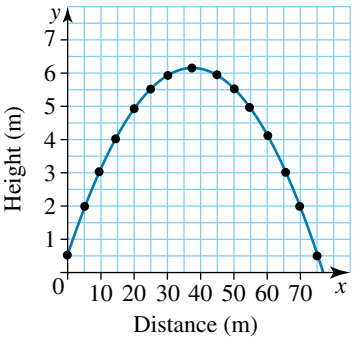
23. A minimum of 2 points are required to determine the graph of a line; however, a minimum of 3 points are required to determine the shape of a parabola. The general equation of a parabola is $y = ax^2 + bx + c$, where a, b and c are the constants to be determined.

a. Determine the equation of the parabola that has a y-intercept of $(0, -2)$ and passes though the points $(1, -5)$ and $(-2, 16)$.

b. Determine the equation of a parabola that goes through the points $(0, 0)$, $(2, 2)$ and $(5, 5)$. Show full working to justify your answer.

24. When the radius of a circle increases by 6 cm, its area increases by 25%. Use the quadratic formula to find the exact radius of the original circle.

25. A football player received a hand pass and ran directly towards goal. Right on the 50-metre line, he kicked the ball and scored a goal. The graph at right represents the path of the ball. Using the graph, answer the following questions.



a. At what height from the ground was the ball when it was kicked?

b. What was the greatest height the ball reached?

c. How long was the kick?

d. If there were defenders in the goal square, would it have been possible for one of them to mark the ball right on the goal line to prevent a goal? Explain your answer. (*Hint:* What was the height of the ball when it crossed the goal line?)

e. As the footballer kicked the ball, a defender rushed at him to try to smother the kick. If the defender can reach a height of 3 m when he jumps, how close to the player kicking the ball must he be to just touch the football as it passes over his outstretched hands?

26. The quadratic formula is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

An alternative form of the quadratic formula is $x = \dfrac{2c}{-b \pm \sqrt{b^2 - 4ac}}$.

Choose a quadratic equation and show that the two formulas give the same answers.

Answers

Topic 12 Quadratic equations

Exercise 12.2 Solving quadratic equations algebraically

1. **a.** $-7, 9$ **b.** $-2, 3$ **c.** $2, 3$ **d.** $0, 3$ **e.** $0, 1$
f. $-5, 0$ **g.** $0, 3$ **h.** $-2, 0$ **i.** $-\frac{1}{2}, \frac{1}{2}$ **j.** $-1.2, -0.5$
k. $0.1, 0.75$ **l.** $-\sqrt{2}, \sqrt{3}$

2. **a.** $\frac{1}{2}, 1$ **b.** $-2, -\frac{2}{3}$ **c.** $\frac{1}{4}, 7$ **d.** $-\frac{6}{7}, 1\frac{1}{2}$ **e.** $\frac{3}{5}, \frac{2}{3}$
f. $-\frac{5}{8}, \frac{2}{3}$ **g.** $0, \frac{1}{2}, 3$ **h.** $0, \frac{1}{2}, -\frac{2}{5}$ **i.** $0, -3, \frac{2}{5}$

3. **a.** $0, 2$ **b.** $-5, 0$ **c.** $0, 7$ **d.** $-\frac{2}{3}, 0$ **e.** $0, 1\frac{1}{2}$
f. $0, \frac{1}{3}$ **g.** $0, \frac{\sqrt{7}}{2}$ **h.** $-\frac{\sqrt{3}}{3}, 0$ **i.** $0, 1\frac{1}{4}$

4. **a.** $-2, 2$ **b.** $-5, 5$ **c.** $-2, 2$ **d.** $-7, 7$ **e.** $-1\frac{1}{3}, 1\frac{1}{3}$
f. $-2\frac{1}{2}, 2\frac{1}{2}$ **g.** $-\frac{2}{3}, \frac{2}{3}$ **h.** $-\frac{1}{2}, \frac{1}{2}$ **i.** $-\frac{1}{5}, \frac{1}{5}$ **j.** $-4, 4$
k. $-\sqrt{5}, \sqrt{5}$ **l.** $-\frac{\sqrt{11}}{3}, \frac{\sqrt{11}}{3}$

5. **a.** $-2, 3$ **b.** $-4, -2$ **c.** $-1, 7$ **d.** $3, 5$ **e.** 1
f. $-1, 4$ **g.** 5 **h.** $-2, 5$ **i.** $2, 6$ **j.** $-3, 7$
k. $-5, 6$ **l.** $3, 4$ **m.** 4 **n.** -5 **o.** 10

6. B

7. C

8. **a.** $-\frac{1}{2}, 3$ **b.** $\frac{2}{3}, -1$ **c.** $-2, \frac{1}{5}$ **d.** $\frac{1}{3}, 1\frac{1}{2}$ **e.** $-\frac{3}{14}, 1$
f. $\frac{1}{4}, \frac{1}{3}$ **g.** $-1\frac{1}{3}, 2\frac{1}{2}$ **h.** $-1\frac{3}{4}, -1\frac{1}{3}$ **i.** $-\frac{2}{5}, \frac{1}{2}$ **j.** $1\frac{1}{2}, 2\frac{2}{3}$
k. $-\frac{2}{5}, \frac{1}{6}$ **l.** $3, 4$

9. **a.** $2 + \sqrt{2}, 2 - \sqrt{2}$ **b.** $-1 + \sqrt{3}, -1 - \sqrt{3}$ **c.** $-3 + \sqrt{10}, -3 - \sqrt{10}$
d. $4 + 2\sqrt{3}, 4 - 2\sqrt{3}$ **e.** $5 + 2\sqrt{6}, 5 - 2\sqrt{6}$ **f.** $1 + \sqrt{3}, 1 - \sqrt{3}$
g. $-1 + \sqrt{6}, -1 - \sqrt{6}$ **h.** $-2 + \sqrt{10}, -2 - \sqrt{10}$ **i.** $-2 + \sqrt{15}, -2 - \sqrt{15}$

10. **a.** $\frac{3}{2} + \frac{\sqrt{5}}{2}, \frac{3}{2} - \frac{\sqrt{5}}{2}$ **b.** $-\frac{5}{2} + \frac{\sqrt{29}}{2}, -\frac{5}{2} - \frac{\sqrt{29}}{2}$ **c.** $\frac{7}{2} + \frac{\sqrt{33}}{2}, \frac{7}{2} - \frac{\sqrt{33}}{2}$
d. $\frac{1}{2} + \frac{\sqrt{21}}{2}, \frac{1}{2} - \frac{\sqrt{21}}{2}$ **e.** $\frac{11}{2} + \frac{\sqrt{117}}{2}, \frac{11}{2} - \frac{\sqrt{117}}{2}$ **f.** $-\frac{1}{2} + \frac{\sqrt{5}}{2}, -\frac{1}{2} - \frac{\sqrt{5}}{2}$
g. $-\frac{3}{2} + \frac{\sqrt{37}}{2}, -\frac{3}{2} - \frac{\sqrt{37}}{2}$ **h.** $\frac{5}{2} + \frac{\sqrt{37}}{2}, \frac{5}{2} - \frac{\sqrt{37}}{2}$ **i.** $\frac{9}{2} + \frac{\sqrt{65}}{2}, \frac{9}{2} - \frac{\sqrt{65}}{2}$

11. **a.** $-3, 1$ **b.** $-4.24, 0.24$ **c.** $-1, 3$ **d.** $-0.73, 2.73$ **e.** $0.38, 2.62$
f. $-0.30, 3.30$ **g.** $-1.19, 4.19$ **h.** $-2.30, 1.30$ **i.** $-2.22, 0.22$

12. 8 and 9 or -8 and -9

13. 6 and 8, -6 and -8

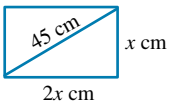
14. 9 or -10

15. 2 or $-2\frac{2}{3}$

16. 8 or $-10\frac{1}{2}$

17. 6 seconds

18. **a.** $l = 2x$

b.


c. $x^2 + (2x)^2 = 45^2,\ 5x^2 = 2025$

d. Length 40 cm, width 20 cm

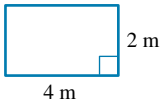
19. 8 m, 6 m

20. **a.** $-6, 1$ **b.** $8, -3$ **c.** $x = \pm 1$

21. **a.** 21 **b.** 17

22. 7

23. a.



b. $(2 + x)$ m, $(4 + x)$ m

c. $(2 + x)(4 + x) = 24$

d. $x = 2$, 4 m wide, 6 m long

24. a. $C_{Annabel}(28) = \$364\,800$, $C_{Betty}(28) = \$422\,400$

b. 10 knots

c. Speed can only be a positive quantity, so the negative solution is not valid.

25. No real solutions — when we complete the square we get the sum of two squares, not the difference of two squares and we cannot factorise the expression.

26. $x = 5, -4, 4, -3$

27. $z = -\frac{5}{3}, 7$

28. The Null Factor Law states that if the product of two numbers is equal to zero, then one or both numbers must equal zero.

29. The width of the pathway is 1.5 m.

Exercise 12.3 The quadratic formula

1. a. $a = 3, b = -4, c = 1$ **b.** $a = 7, b = -12, c = 2$ **c.** $a = 8, b = -1, c = -3$

d. $a = 1, b = -5, c = 7$ **e.** $a = 5, b = -5, c = -1$ **f.** $a = 4, b = -9, c = -3$

g. $a = 12, b = -29, c = 103$ **h.** $a = 43, b = -81, c = -24$

2. a. $\frac{-5 \pm \sqrt{21}}{2}$ **b.** $\frac{-3 \pm \sqrt{13}}{2}$ **c.** $\frac{5 \pm \sqrt{17}}{2}$ **d.** $2 \pm \sqrt{13}$ **e.** $-1 \pm 2\sqrt{3}$

f. $\frac{7 \pm 3\sqrt{5}}{2}$ **g.** $\frac{9 \pm \sqrt{73}}{2}$ **h.** $3 \pm 2\sqrt{3}$ **i.** $-4 \pm \sqrt{31}$ **j.** $\frac{1 \pm \sqrt{21}}{2}$

k. $\frac{5 \pm \sqrt{33}}{2}$ **l.** $-1 \pm 2\sqrt{2}$

3. a. $-0.54, 1.87$ **b.** $-1.20, 1.45$ **c.** $-4.11, 0.61$ **d.** $-0.61, 0.47$ **e.** $0.14, 1.46$

f. $0.16, 6.34$ **g.** $-1.23, 1.90$ **h.** $-1.00, 1.14$ **i.** $-0.83, 0.91$ **j.** $-0.64, 1.31$

k. $-0.35, 0.26$ **l.** $-1.45, 1.20$ **m.** $0.08, 5.92$ **n.** $-0.68, 0.88$

4. C

5. C

6. B

7. C

8. a. 0.5, 3 **b.** 0, 5 **c.** –1, 3 **d.** 0.382, 2.618 **e.** 0.298, 6.702

f. 2, 4 **g.** No real solution **h.** –1, 8 **i.** –4.162, 2.162 **j.** –2, 1

k. –7, 1.5 **l.** No real solution **m.** 2, 7 **n.** $-\frac{1}{2}, \frac{1}{3}$ **o.** No real solution

9. a. $2\pi r^2 + 14\pi r - 231 = 0$ **b.** 3.5 cm **c.** 154 cm^2

10. a. $x(x + 30)$ **b.** $x(x + 30) = 1500$ **c.** 265 mm

11. a. Pool A: $3\frac{2}{3}$ m by $6\frac{2}{3}$ m; pool B: $3\frac{1}{3}$ m by $7\frac{1}{3}$ m

b. The area of each is $24\frac{4}{9}$ m^2.

c. A negative solution is not possible when finding the dimensions of a swimming pool.

12. 25 m, 60 m

13. $-2 \pm \sqrt{3}, 9 \pm 4\sqrt{59}$

14. a, b. Check with your teacher.

c. $m = 1.85$, so NP is 1.85 cm.

15. There is only one solution to the quadratic equation.

16. a. $x = \pm 2$ or $x = \pm 3$ **b.** $x = \pm\frac{1}{2}$ or $x = \pm 2$

Exercise 12.4 Solving quadratic equations graphically

1. a. $x = -2, x = 3$ **b.** $x = 1, x = 10$ **c.** $x = -5, x = 5$ **d.** $x = 2$

e. $x = -1, x = 4$ **f.** $x \approx -1.4, x \approx 4.4$ **g.** $x = -25, x = 10$ **h.** $x = 0$

i. $x \approx -2.3, x \approx 1.3$ **j.** $x \approx -1.5, x = 1$

2. a–j. Confirm by substitution of above values into quadratic equations.

3. 150 m

4. 7 m

5. a. $y = a(x - 2)(x - 5)$ **b.** $y = \frac{2}{5}(x - 2)(x - 5)$

6. a. $y = a(x - p)(x - q)$ **b.** $y = \frac{r}{pq}(x - p)(x - q)$

7. a. 6 m **b.** 6 m

8. $y = -4x^2 + 26x - 30$

9. The solutions of a graph generally refer to the x-intercepts.

Exercise 12.5 The discriminant

1. a. $\Delta = 289$, 2 solutions **b.** $\Delta = 441$, 2 solutions **c.** $\Delta = 24$, 2 solutions

d. $\Delta = 144$, 2 solutions **e.** $\Delta = -28$, 0 solutions **f.** $\Delta = 81$, 2 solutions

g. $\Delta = 0$, 1 solution **h.** $\Delta = 9$, 2 solutions **i.** $\Delta = 32$, 2 solutions

j. $\Delta = 2809$, 2 solutions **k.** $\Delta = 8$, 2 solutions **l.** $\Delta = 0$, 1 solution

m. $\Delta = 256$, 2 solutions **n.** $\Delta = 256$, 2 solutions

2. a. No real solutions **b.** 1 rational solution **c.** 2 rational solutions

d. 1 rational solution **e.** 2 irrational solutions **f.** 1 rational solution

g. 2 irrational solutions **h.** No real solutions **i.** No real solutions

j. 2 irrational solutions **k.** 2 irrational solutions **l.** 2 rational solutions

3. a. Yes **b.** No **c.** Yes **d.** No

4. a. $a = 3, b = 2, c = 7$ **b.** -80 **c.** No real solutions

5. a. $a = -6, b = 1, c = 3$ **b.** 73 **c.** 2 real solutions **d.** $\frac{1 \pm \sqrt{73}}{12}$

6. A

7. C

8. C

9. $k = -1$

10. $m = 1.8$

11. $n > -\frac{9}{4}$

12. p^2 can only give a positive number, which, when added to 24, is always a positive solution.

13. a. 0.4 m **b.** 0.28 m **c.** 2.22 m **d.** 2.5 m

e. i. Yes

ii. No

Find the halfway point between the beginning and the end of the leap, and substitute this value into the equation to find the maximum height.

f. 0.625 m

14. (–2, 0), (2, 0)

15. a. a = –7 or 5 will give one intersection point.

b. For values of $b < -\frac{21}{4}$, there will be no intersection points.

16. a. The straight line crosses the parabola at (0, –7), so no matter what value m takes, there will be at least one intersection point and a maximum of two.

b. $m = -\frac{8}{5}$

17. $a = 4, b = 12$

18. $k = -4$

19. The discriminant tells us the number of real solutions of a quadratic equation.

Investigation | Rich task

1.

Pattern	Number of threads per block	Number of blocks	Total threads in warp
RB	2	1	2
RRB RRB	3	2	6
RRRB RRBB RBBB	4	3	12
RRRRB RRRBB RRBBB RBBBB	5	4	20
RRRRRB RRRRBB RRRBBB RRBBBB RBBBBB	6	5	30
RRRRRRB RRRRRBB RRRRBBB RRRBBBB RRBBBBB RBBBBBB	7	6	42

2. a. $n - 1$ **b.** $n^2 - n$

3. Check with your teacher.

4. a. $n^2 - n = 250$ **b.** $n = \dfrac{\sqrt{1001} + 1}{2}$ **c.** $n = 16$ **d.** 15 **e.** 240 **f.** 10

g. Answers will vary.

5. Answers will vary.

6. Answers will vary.

Exercise 12.6 Review questions

1. B

2. B

3. A

4. D

5. $(3x + 4)$ m

6. a. –5, –3 **b.** –6, –1 **c.** –8, –3 **d.** 2, –6 **e.** 5, –2 **f.** 4, –7

g. 3, 1 **h.** 5, 6 **i.** 7, –5

7. a. –2, –6 **b.** –2, –1 **c.** $\frac{1}{2}$, –3 **d.** 2, –7 **e.** $-\frac{1}{2}$, 4 **f.** $-\frac{2}{3}$, 2

g. 2, 1 **h.** $\frac{5}{3}$, $\frac{5}{2}$ **i.** -7, $\frac{1}{2}$

8. a. $-4 \pm \sqrt{17}$ **b.** $-1 \pm \sqrt{6}$ **c.** -1, $\frac{1}{4}$

9. 4

10. a. –0.651, 1.151 **b.** –0.760, 0.188 **c.** 0.441, –0.566

11. a. –0.571, 0.682 **b.** –0.216, 3.836 **c.** –0.632, 0.632

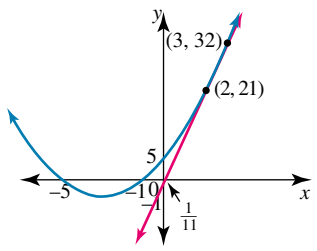
12. –3, 7

13. –3, 1

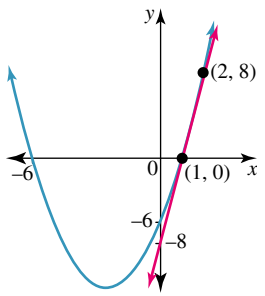
14. a. 2 irrational solutions **b.** 2 rational solutions **c.** No real solutions

15. a. (–8, 22) and (2, 2) **b.** (5, 10) **c.** No solution

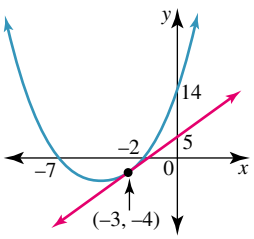
16. a.



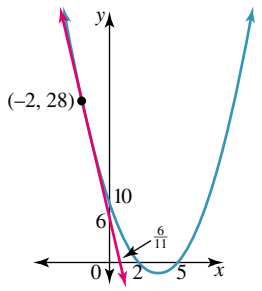
b.



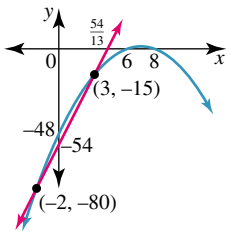
c.



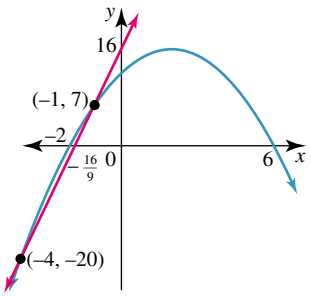
d.



e.



f.



17. –8 and 7

18. Length = 6 m, width = 3 m

19. a. $2\pi r(r + 10) = 245$ **b.** 3.0 cm **c.** 188 cm^2

20. $-\frac{25}{8}$

21. $k > 9$ and $k < 1$

22. 24

23. a. $y = 2x^2 - 5x - 2$

b. No parabola is possible. The points are on the same straight line.

24. $12(\sqrt{5} + 2)$ cm

25. a. 0.5 m

b. 6.1 m

c. 76.5 m

d. No, the ball is 5.5 m off the ground and nobody can reach it.

e. 9.5 m away

26. Check with your teacher.

TOPIC 13
Non-linear relationships [Stage 5.3]

13.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the concepts covered in this topic.

13.1.1 Why learn this?

The world around us, both natural and artificial, is full of elegant curves and shapes that are not straight lines. Achieving the brilliance of modern and ancient architecture, or understanding the motion of planets and tennis balls requires an understanding of non-linear relationships.

DISCUSSION

Describe two examples of how the Golden Ratio has been used in different cultures throughout history.

LEARNING SEQUENCE

13.1 Overview
13.2 [Stage 5.3] Plotting parabolas
13.3 [Stage 5.3] Sketching parabolas
13.4 [Stage 5.3] Sketching parabolas of the form $y = a(x - h)^2 + k$
13.5 [Stage 5.3] Sketching parabolas of the form $y = ax^2 + bx + c$
13.6 [Stage 5.3] Exponential functions and graphs
13.7 [Stage 5.3] The hyperbola
13.8 [Stage 5.3] The circle
13.9 Review

LEARNING OUTCOMES

A student:

- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3-1WM
- uses deductive reasoning in presenting arguments and formal proofs MA5.3-3WM
- sketches and interprets a variety of non-linear relationships MA5.3-9NA

CONTENT DESCRIPTIONS

Students:

Describe, interpret and sketch parabolas, hyperbolas, circles and exponential functions and their transformations (ACMNA267)

Source: NSW Syllabus for the Australian curriculum

learnON RESOURCES — ONLINE ONLY

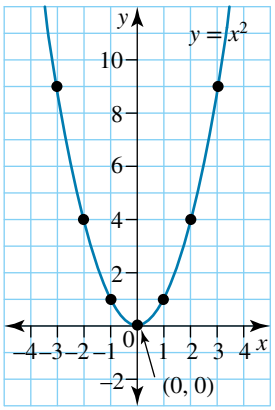
eLesson: The story of mathematics — Fibonacci (eles-1848)

Note: Your teacher may now set you a pre-test to determine how familiar you are with the content in this topic.

13.2 Plotting parabolas [Stage 5.3]

13.2.1 Parabolas

- The graphs of all quadratic relationships are called **parabolas**.
- If the equation of a parabola is given, a table of values can be produced by substituting x-values into the equation to obtain the corresponding y-values. These x- and y-values provide the coordinates for points that can be plotted and joined to form the shape of the graph. When plotting graphs, use grid or graph paper for accuracy.
- The graph of $y = x^2$ shown has been produced by generating a table of values.



x	−3	−2	−1	0	1	2	3
y	9	4	1	0	1	4	9

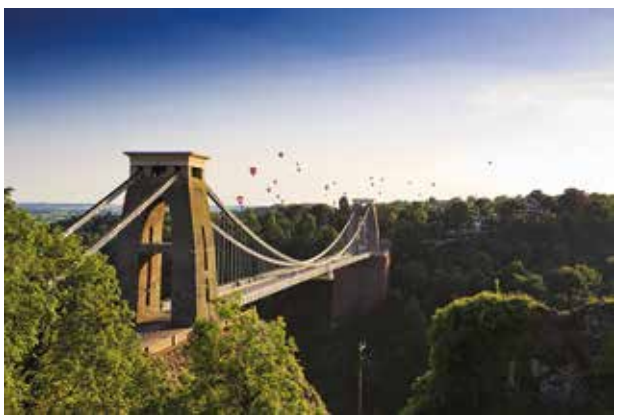
- Parabolas are symmetrical. In other words, they have an axis of symmetry. In the parabola above, the axis of symmetry is the y-axis, also called the line $x = 0$.
- A parabola has a vertex or turning point. In this case the vertex is at the origin and is called a 'minimum vertex' or 'minimum turning point'.
- Parabolas with the shape ∪ are said to be 'concave up' or upright, and have a minimum vertex. Parabolas with the shape ∩ are said to be 'concave down' or inverted, and have a maximum vertex.

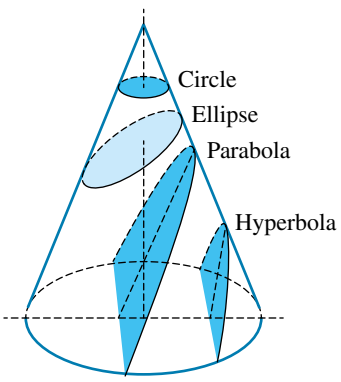
13.2.2 Parabolas in the world around us

- Parabolas abound in the world around us. Here are some examples.

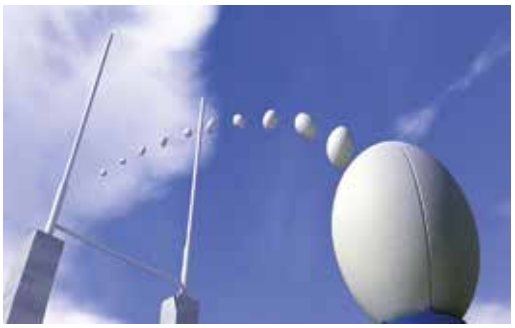
Satellite dishes

Water droplets from a hose

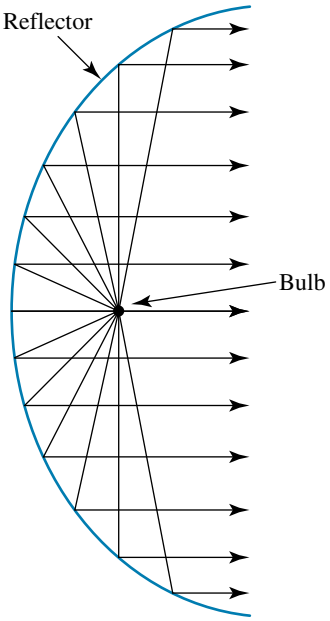

The cables from a suspension bridge



A cone when sliced parallel to its edge reveals a parabola.



The trajectory of a football when it is thrown or kicked



The reflectors in a car headlight

DISCUSSION

Where else do parabolas appear in the natural world?

WORKED EXAMPLE 1

Plot the graph of each of the following equations. In each case, use the values of x shown as the values in your table. State the equation of the axis of symmetry and the coordinates of the vertex. You may also use digital technologies to sketch these graphs.

a $y = 2x^2$ **for** $-3 \leq x \leq 3$ **b** $y = \frac{1}{2}x^2$ **for** $-3 \leq x \leq 3$

THINK

a 1 Write the equation.

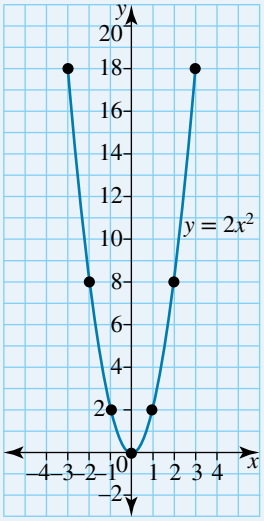
2 Produce a table of values using x-values from –3 to 3.

WRITE/DRAW

a $y = 2x^2$

x	–3	–2	–1	0	1	2	3
y	18	8	2	0	2	8	18

3 Draw a set of clearly labelled axes, plot the points and join them with a smooth curve. An appropriate scale would be from 20 to −2 on the y-axis and −4 to 4 on the x-axis.



4 Label the graph.

5 Write the equation of the axis of symmetry that divides the parabola exactly in half.

The equation of the axis of symmetry is $x = 0$.

6 Write the coordinates of the vertex.

The vertex is $(0, 0)$.

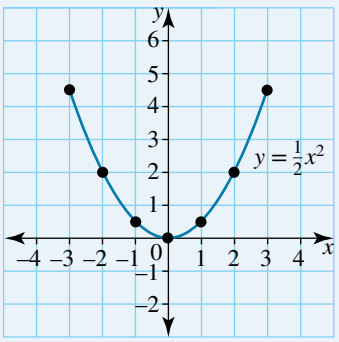
b 1 Write the equation.

b $y = \frac{1}{2}x^2$

2 Produce a table of values using x-values from −3 to 3.

x	−3	−2	−1	0	1	2	3
y	4.5	2	0.5	0	0.5	2	4.5

3 Draw a set of clearly labelled axes, plot the points and join them with a smooth curve. An appropriate scale would be from 6 to −2 on the y-axis and −4 to 4 on the x-axis.



4 Label the graph.

5 Write the equation of the line that divides the parabola exactly in half.

The equation of the axis of symmetry is $x = 0$.

6 Write the coordinates of the vertex.

The vertex is $(0, 0)$.

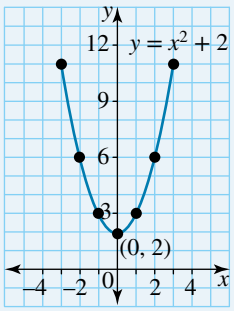
WORKED EXAMPLE 2

Plot the graph of each of the following equations. In each case, use the values of x shown as the values in your table. State the equation of the axis of symmetry, the coordinates of the vertex and the y-intercept for each one.

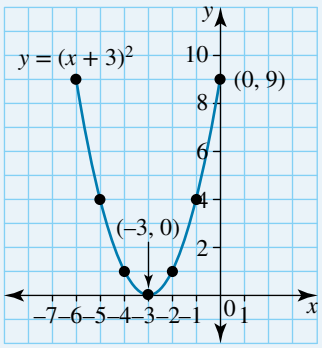
a $y = x^2 + 2$ **for** $-3 \leq x \leq 3$ **b** $y = (x + 3)^2$ **for** $-6 \leq x \leq 0$ **c** $y = -x^2$ **for** $-3 \leq x \leq 3$

THINK	WRITE/DRAW
a 1 Write the equation.	**a** $y = x^2 + 2$
2 Produce a table of values.	

x	−3	−2	−1	0	1	2	3
y	11	6	3	2	3	6	11

3 Draw a set of clearly labelled axes, plot the points and join them with a smooth curve. An appropriate scale on the y-axis would be from 0 to 12 and −4 to 4 on the x-axis.	
4 Label the graph.	
5 Write the equation of the line that divides the parabola exactly in half.	The equation of the axis of symmetry is $x = 0$.
6 Write the coordinates of the vertex.	The vertex is $(0, 2)$.
7 Find the y-coordinate of the point where the graph crosses the y-axis.	The y-intercept is 2.
b 1 Write the equation.	**b** $y = (x + 3)^2$
2 Produce a table of values.	

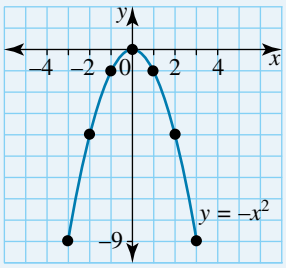
x	−6	−5	−4	−3	−2	−1	0
y	9	4	1	0	1	4	9

3 Draw a set of clearly labelled axes, plot the points and join them with a smooth curve. An appropriate scale on the y-axis would be from 0 to 10 and −7 to 11 on the x-axis.	
4 Label the graph.	
5 Write the equation of the line that divides the parabola exactly in half.	The equation of the axis of symmetry is $x = -3$.
6 Write the coordinates of the vertex.	The vertex is $(-3, 0)$.
7 Find the y-coordinate of the point where the graph crosses the y-axis.	The y-intercept is 9.
c 1 Write the equation.	**c** $y = -x^2$
2 Produce a table of values.	

x	−3	−2	−1	0	1	2	3
y	−9	−4	−1	0	−1	−4	−9

3 Draw a set of clearly labelled axes, plot the points and join them with a smooth curve. An appropriate scale on the y-axis would be from -10 to 1 and from -4 to 4 on the x-axis.

4 Label the graph.



5 Write the equation of the line that divides the parabola exactly in half.

The equation of the axis of symmetry is $x = 0$.

6 Write the coordinates of the vertex.

The vertex is $(0, 0)$.

7 Find the y-coordinate of the point where the graph crosses the y-axis.

The y-intercept is 0.

learnON RESOURCES — ONLINE ONLY

Interactivity: Plotting quadratic graphs (int-6150)

Digital doc: SkillSHEET Substitution into quadratic equations (doc-5266)

Digital doc: SkillSHEET Equation of a vertical line (doc-5267)

Exercise 13.2 Plotting parabolas

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–8, 9a–d, 10–15, 17	Questions: 1–8, 9a–d, g, 10–14, 16, 17, 18	Questions: 1–20

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

You may wish to use graphing software for this exercise.

Understanding and fluency

1. WE1 Plot the graph of each of the following equations. In each case, use the values of x shown as the values in your table. State the equation of the axis of symmetry and the coordinates of the vertex.

a. $y = 3x^2$ for $-3 \leq x \leq 3$ **b.** $y = \frac{1}{4}x^2$ for $-3 \leq x \leq 3$

2. Compare the graphs you have drawn for question **1** with that of $y = x^2$. Explain how placing a number in front of x^2 affects the graph obtained.

3. WE2a Plot the graph of each of the following for values of x between -3 and 3. State the equation of the axis of symmetry, the coordinates of the vertex and the y-intercept for each one.

a. $y = x^2 + 1$ **b.** $y = x^2 + 3$ **c.** $y = x^2 - 3$ **d.** $y = x^2 - 1$

4. Compare the graphs you have drawn for question **3** with the graph of $y = x^2$. Explain how adding to or subtracting from x^2 affects the graph obtained.

5. WE2b Plot the graph of each of the following equations. In each case, use the values of x shown as the values in your table. State the equation of the axis of symmetry, the coordinates of the vertex and the y-intercept for each one.

a. $y = (x + 1)^2, -5 \leq x \leq 3$ **b.** $y = (x + 2)^2, -6 \leq x \leq 2$

c. $y = (x - 2)^2, -1 \leq x \leq 5$ **d.** $y = (x - 1)^2, -2 \leq x \leq 4$

6. Compare the graphs you have drawn for question **5** with that for $y = x^2$. Explain how adding to or subtracting from x before squaring affects the graph obtained.

7. WE2c Plot the graph of each of the following equations. In each case, use the values of x shown as the values in your table. State the equation of the axis of symmetry, the coordinates of the vertex and the y-intercept for each one.

a. $y = -x^2 + 1, -3 \leq x \leq 3$ **b.** $y = -(x - 1)^2, -2 \leq x \leq 4$

c. $y = -(x + 2)^2, -5 \leq x \leq 1$ **d.** $y = -x^2 - 3, -3 \leq x \leq 3$

8. Compare the graphs you have drawn for question **7** with that of $y = x^2$. Explain how a negative sign in front of x^2 affects the graph obtained. Also compare the graphs obtained in question **7** with those in questions **3** and **5**. Which graphs have the same vertex? How are they different?

9. Plot the graph of each of the following, and state:

i. the equation of the axis of symmetry

ii. the coordinates of the vertex and whether it is a maximum or a minimum

iii. the y-intercept.

a. $y = (x - 5)^2 + 1, 0 \leq x \leq 6$ **b.** $y = 2(x + 2)^2 - 3, -5 \leq x \leq 1$

c. $y = 2(x - 3)^2 + 4, 0 \leq x \leq 6$ **d.** $y = -3(x - 1)^2 + 2, -2 \leq x \leq 4$

e. $y = x^2 + 4x - 5, -6 \leq x \leq 2$ **f.** $y = -x^2 - 2x + 15, -6 \leq x \leq 4$

g. $y = 23x^2 - 6x + 24, -5 \leq x \leq 3$ **h.** $y = (x - 2)^2 + 1, -2 \leq x \leq 4$

10. Use the equation $y = a(x - b)^2 + c$ to answer the following.

a. Explain how you can determine whether a parabola has a minimum or maximum vertex by looking only at its equation.

b. Explain how you can determine the coordinates of the vertex of a parabola by looking only at the equation.

c. Explain how you can obtain the equation of the axis of symmetry by looking only at the equation of the parabola.

11. MC For the graph of $y = (x - 2)^2 + 5$, the vertex is:

A. $(5, 2)$ **B.** $(2, -5)$ **C.** $(2, 5)$ **D.** $(-2, -5)$ **E.** $(-2, 5)$

12. MC For the graph of $y = 3(x - 1)^2 + 12$, the vertex is:

A. $(3, 12)$ **B.** $(1, 12)$ **C.** $(-1, 12)$ **D.** $(-3, 12)$ **E.** $(-1, -12)$

13. MC For the graph of $y = (x + 2)^2 - 7$, the y-intercept is:

A. -2 **B.** -7 **C.** -3 **D.** -11 **E.** 7

14. MC Which of the following is true for the graph of $y = 2(x - 3)^2 + 4$?

A. Vertex $(3, 4)$, y-intercept -5 **B.** Vertex $(3, 4)$, y-intercept 5

C. Vertex $(-3, 4)$, y-intercept -5 **D.** Vertex $(-3, 4)$, y-intercept 5

E. Vertex $(3, -4)$, y-intercept 13

Communicating, reasoning and problem solving

15. A ball is thrown into the air. The height, h metres, of the ball at any time, t seconds, can be found by using the equation $h = -(t - 4)^2 + 16$.

a. Plot the graph for values of t between 0 and 8.

b. Use the graph to find:

i. the maximum height of the ball

ii. how long it takes for the ball to fall back to the ground from the moment it is thrown.

16. From a crouching position in a ditch, an archer wants to fire an arrow over a horizontal tree branch, which is 15 metres above the ground. The height, in metres (h), of the arrow t seconds after it has been fired is given by the equation $h = -8t(t - 3)$.

a. Plot the graph for $t = 0, 1, 1.5, 2, 3$.

b. From the graph, find:

i. the maximum height the arrow reaches

ii. whether the arrow clears the branch and the distance by which it clears or falls short of the branch

iii. the time it takes to reach maximum height

iv. how long it takes for the arrow to hit the ground after it has been fired.

17. There are 0, 1, 2 and infinite possible points of intersection for two parabolas.

a. Illustrate these on separate graphs.

b. Explain why infinite points of intersection are possible. Give an example.

c. How many points of intersection are possible for a parabola and a straight line? Illustrate these.

18. The path taken by a netball thrown by a rising Australian player is given by the quadratic equation $y = -x^2 + 3.2x + 1.8$, where y is the height of the ball and x is the horizontal distance from the player's upstretched hand.

a. Complete a table of values for $-1 \leq x \leq 4$.

b. Plot the graph.

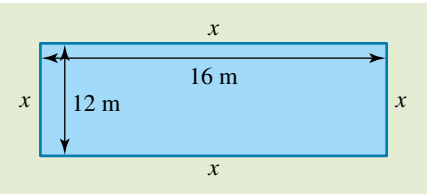
c. What values of x are 'not reasonable' and why?

d. What is the maximum height reached by the netball?

e. Assuming that nothing hits the netball, how far away from the player will the netball strike the ground?

19. The values of a, b and c in the equation $y = ax^2 + bx + c$ can be calculated using three points that lie on the parabola. This requires solving triple simultaneous equations by algebra. This can also be done using graphing software. If the points $(0, 1)$, $(1, 0)$ and $(2, 3)$ all lie on one parabola, find the equation of the parabola.

20. What x-values can a parabola have? What y-values can a parabola have?



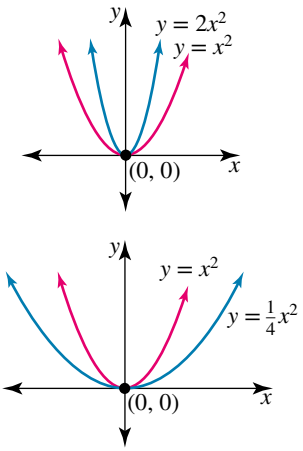
13.3 Sketching parabolas [Stage 5.3]

13.3.1 Sketching parabolas

- A sketch graph of a parabola does not show a series of plotted points, but it does accurately locate important features such as x- and y-intercepts and the vertex.
- The basic quadratic graph has the equation $y = x^2$. Transformations or changes in the features of the graph can be observed when the equation changes. These transformations include:
  - dilation
  - translation
  - reflection.

13.3.2 Dilation

- Compare the graph of $y = 2x^2$ with that of $y = x^2$. This graph is thinner or closer to the y-axis and has a dilation factor of 2. As the coefficient of x^2 increases, the graph becomes narrower and closer to the y-axis.
- The vertex has not changed under the transformation and is still $(0, 0)$.
- Compare the graph of $y = \frac{1}{4}x^2$ with that of $y = x^2$.
  The graph is wider or closer to the x-axis and has a dilation factor of $\frac{1}{4}$.
  The vertex has not changed and is still $(0, 0)$.
  As the coefficient of x^2 decreases (but remains positive), the graph becomes wider or closer to the x-axis.



WORKED EXAMPLE 3

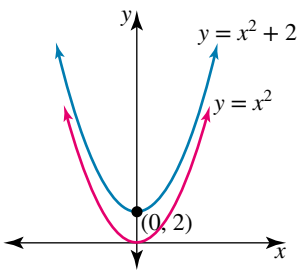
State whether each of the following graphs is wider or narrower than the graph of $y = x^2$ and state the coefficients of the vertex of each one.

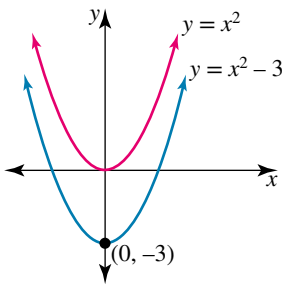
a $y = \frac{1}{5}x^2$ **b** $y = 4x^2$

THINK	WRITE
a 1 Write the equation.	**a** $y = \frac{1}{5}x^2$
2 Look at the coefficient of x^2 and decide whether it is greater than or less than 1.	$\frac{1}{5} < 1$, so the graph is wider than that of $y = x^2$.
3 The dilation doesn't change the vertex.	The vertex is $(0, 0)$.
b 1 Write the equation.	**b** $y = 4x^2$
2 Look at the coefficient of x^2 and decide whether it is greater than or less than 1.	$4 > 1$, so the graph is narrower than that of $y = x^2$.
3 The dilation doesn't change the vertex.	The vertex is $(0, 0)$.

13.3.3 Vertical translation

- Compare the graph of $y = x^2 + 2$ with that of $y = x^2$.
  The whole graph has been moved or translated 2 units upwards.
  The vertex has become $(0, 2)$.
- Compare the graph of $y = x^2 - 3$ with that of $y = x^2$.
  The whole graph has been moved or translated 3 units downwards.
  The vertex has become $(0, -3)$.





WORKED EXAMPLE 4

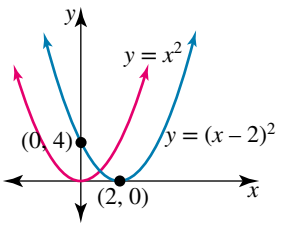
State the vertical translation and the coordinates of the vertex for the graphs of the following equations when compared to the graph of $y = x^2$.

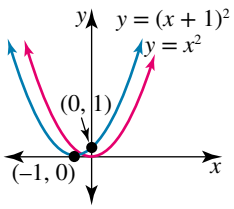
a $y = x^2 + 5$ **b** $y = x^2 - 4$

THINK	WRITE
a 1 Write the equation.	**a** $y = x^2 + 5$
2 +5 means the graph is translated upwards 5 units.	Vertical translation of 5 units up
3 Translate the vertex of $y = x^2$, which is (0, 0). The x-coordinate of the vertex remains 0, and the y-coordinate has 5 added to it.	The vertex becomes (0, 5).
b 1 Write the equation.	**b** $y = x^2 - 4$
2 −4 means the graph is translated downwards 4 units.	Vertical translation of 4 units down
3 Translate the vertex of $y = x^2$, which is (0, 0). The x-coordinate of the vertex remains 0, and the y-coordinate has 4 subtracted from it.	The vertex becomes (0, –4).

13.3.4 Horizontal translation

- Compare the graph of $y = (x - 2)^2$ with that of $y = x^2$.
- The whole graph has been moved or translated 2 units to the right. The vertex has become (2, 0).
- Compare the graph of $y = (x + 1)^2$ with that of $y = x^2$.
- The whole graph has been moved or translated 1 unit left. The vertex has become (–1, 0).





WORKED EXAMPLE 5

State the horizontal translation and the coordinates of the vertex for the graphs of the following equations when compared to the graph of $y = x^2$.

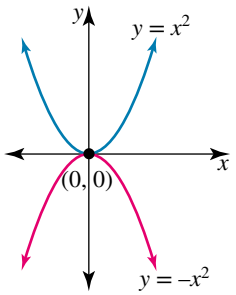
a $y = (x - 3)^2$ **b** $y = (x + 2)^2$

THINK	WRITE
a 1 Write the equation.	**a** $y = (x - 3)^2$
2 –3 means the graph is translated to the right 3 units.	Horizontal translation of 3 units to the right
3 Translate the vertex of $y = x^2$, which is (0, 0). The y-coordinate of the vertex remains 0, and the x-coordinate has 3 added to it.	The vertex becomes (3, 0).

b 1 Write the equation.	**b** $y = (x + 2)^2$
2 +2 means the graph is translated to the left 2 units.	Horizontal translation of 2 units to the left
3 Translate the vertex of $y = x^2$, which is (0, 0). The y-coordinate of the vertex remains 0, and the x-coordinate has 2 subtracted from it.	The vertex becomes (−2, 0).

13.3.5 Reflection

- Compare the graph of $y = -x^2$ with that of $y = x^2$.



In each case the axis of symmetry is the line $x = 0$ and the vertex is (0, 0). The only difference between the equations is the negative sign in $y = -x^2$, and the difference between the graphs is that $y = x^2$ 'sits' on the x-axis and $y = -x^2$ 'hangs' from the x-axis. (One is a reflection or mirror image of the other.) The graph of $y = x^2$ has a minimum vertex, and the graph of $y = -x^2$ has a maximum vertex.

- Any quadratic graph where x^2 is positive has a ∪ shape and is said to be concave up. Conversely, if x^2 is negative, the graph has a ∩ shape and is said to be concave down.

WORKED EXAMPLE 6

For each of the following graphs, give the coordinates of the vertex and state whether it is a maximum or a minimum.

a $y = -(x - 7)^2$ **b** $y = 5 - x^2$

THINK	WRITE
a 1 Write the equation.	**a** $y = -(x - 7)^2$
2 It is a horizontal translation of 7 units to the right, so 7 units is added to the x-coordinate of (0, 0).	The vertex is (7, 0).
3 The sign in front of the x^2 term is negative, so it is concave down.	Maximum vertex — concave down
b 1 Write the equation.	**b** $y = 5 - x^2$
2 Rewrite the equation so that the x^2 term is first.	$y = -x^2 + 5$
3 The vertical translation is 5 units up, so 5 units is added to the y-coordinate of (0, 0).	The vertex is (0, 5).
4 The sign in front of the x^2 term is negative, so the graph is concave down.	Maximum vertex — concave down

WORKED EXAMPLE 7

For each of the following quadratic equations:

i state the appropriate dilation, reflection and translation of the graph of $y = x^2$ needed to obtain the graph

ii state the coordinates of the vertex

iii hence, sketch the graph.

a $y = (x + 3)^2$ **b** $y = -2x^2$

THINK

a 1 Write the quadratic equation.

2 Identify the transformation needed — horizontal translation only, no dilation or reflection.

3 State the vertex.

4 Sketch the graph of $y = (x + 3)^2$.
You may find it helpful to lightly sketch the graph of $y = x^2$ on the same set of axes first.

b 1 Write the quadratic equation.

2 Identify the transformations needed — dilation (2 in front of x^2) and reflection (negative in front of x^2 term), no translation.

3 The vertex remains the same as there is no translation.

4 Sketch the graph of $y = -2x^2$. You may find it helpful to lightly sketch the graph of $y = x^2$ on the same set of axes first.

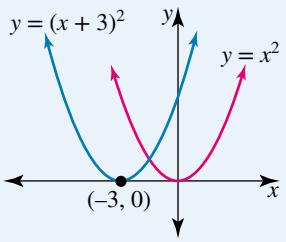
WRITE

a $y = (x + 3)^2$

i Horizontal translation of 3 units to the left

ii The vertex is $(-3, 0)$.

iii

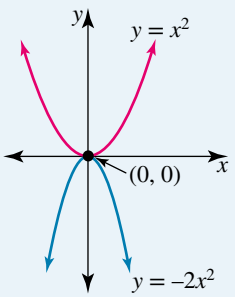


b $y = -2x^2$

i This is a reflection, so the graph is inverted. As $2 > 1$, the graph is narrower than that of $y = x^2$.

ii The vertex is $(0, 0)$.

iii



learnon RESOURCES — ONLINE ONLY

Interactivity: Dilation of parabolas (int-6096)
Interactivity: Vertical translations of parabolas (int-6097)
Interactivity: Horizontal translations of parabolas (int-6098)
Interactivity: Reflection of parabolas (int-6151)
Interactivity: Dilation of $y = x^2$ (int-1148)
Interactivity: Vertical translation of $y = x^2 + c$ (int-1192)
Interactivity: Horizontal translation of $y = (x - h)^2$ (int-1193)
Digital doc: WorkSHEET Quadratic graphs (doc-5272)

Exercise 13.3 Sketching parabolas

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–d, 2a–d, 3a–d, 4a–d, 5a–d, 6a–d, 7	Questions: 1c–f, 2c–f, 3c–f, 4c–f, 5c–f, 6c–f, 7–9, 11	Questions: 1c–h, 2c–h, 3c–h, 4e–h, 5e–h, 6i–p, 7–10, 12

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE3 State whether each of the following graphs is wider or narrower than the graph of $y = x^2$ and state the coordinates of the vertex of each one.

a. $y = 5x^2$ **b.** $y = \frac{1}{3}x^2$ **c.** $y = 7x^2$ **d.** $y = 10x^2$

e. $y = \frac{2}{5}x^2$ **f.** $y = 0.25x^2$ **g.** $y = 1.3x^2$ **h.** $y = \sqrt{3}x^2$

2. WE4 State the vertical translation and the coordinates of the vertex for the graphs of each of the following equations when compared to the graph of $y = x^2$.

a. $y = x^2 + 3$ **b.** $y = x^2 - 1$ **c.** $y = x^2 - 7$ **d.** $y = x^2 + \frac{1}{4}$

e. $y = x^2 - \frac{1}{2}$ **f.** $y = x^2 - 0.14$ **g.** $y = x^2 + 2.37$ **h.** $y = x^2 + \sqrt{3}$

3. WE5 State the horizontal translation and the coordinates of the vertex for the graphs of the following equations when compared to the graph of $y = x^2$.

a. $y = (x - 1)^2$ **b.** $y = (x - 2)^2$ **c.** $y = (x + 10)^2$ **d.** $y = (x + 4)^2$

e. $y = (x - \frac{1}{2})^2$ **f.** $y = (x + \frac{1}{5})^2$ **g.** $y = (x + 0.25)^2$ **h.** $y = (x + \sqrt{3})^2$

4. WE6 For each of the following graphs, give the coordinates of the vertex and state whether it is a maximum or a minimum.

a. $y = -x^2 + 1$ **b.** $y = x^2 - 3$ **c.** $y = -(x + 2)^2$ **d.** $y = 3x^2$

e. $y = 4 - x^2$ **f.** $y = -2x^2$ **g.** $y = (x - 5)^2$ **h.** $y = 1 + x^2$

5. In each of the following state whether the graph is wider or narrower than $y = x^2$ and whether it has a maximum or a minimum vertex.

a. $y = 3x^2$ **b.** $y = -3x^2$ **c.** $y = \frac{1}{2}x^2$ **d.** $y = -\frac{1}{5}x^2$

e. $y = -\frac{4}{3}x^2$ **f.** $y = 0.25x^2$ **g.** $y = \sqrt{3}x^2$ **h.** $y = -0.16x^2$

6. WE7 For each of the following quadratic equations:

i. state the appropriate dilation, reflection and translation of the graph of $y = x^2$ needed to obtain the graph

ii. state the coordinates of the vertex

iii. hence, sketch the graph.

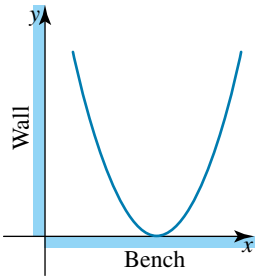
a. $y = (x + 1)^2$ **b.** $y = -3x^2$ **c.** $y = x^2 + 1$ **d.** $y = \frac{1}{3}x^2$

e. $y = x^2 - 3$ **f.** $y = (x - 4)^2$ **g.** $y = -\frac{2}{5}x^2$ **h.** $y = 5x^2$

i. $y = -x^2 + 2$ **j.** $y = -(x - 6)^2$ **k.** $y = -x^2 - 4$ **l.** $y = -(x + 1)^2$

m. $y = 2(x + 1)^2 - 4$ **n.** $y = \frac{1}{2}(x - 3)^2 + 2$ **o.** $y = -\frac{1}{3}(x + 2)^2 + \frac{1}{4}$ **p.** $y = -\frac{7}{4}(x - 1)^2 - \frac{3}{2}$

7. A vase 25 cm tall is positioned on a bench near a wall as shown. The shape of the vase follows the curve $y = (x - 10)^2$, where y cm is the height of the vase and x cm is the distance of the vase from the wall.



a. How far is the base of the vase from the wall?
b. What is the shortest distance from the top of the vase to the wall?
c. If the vase is moved so that the top just touches the wall, find the new distance from the wall to the base.
d. Find the new equation that follows the shape of the vase.

8. A ball is thrown vertically upwards. Its height in metres after t seconds is given by $h = 7t - t^2$.

a. Sketch the path of the ball.
b. What is the highest point reached by the ball?
A second ball is thrown vertically upwards. Its total time in flight lasts 3 seconds longer than the first ball.
c. State the equation for the flight of the second ball.
d. On the same set of axes used for part **a**, sketch the path of the second ball.
e. State the difference in the highest point reached by the two balls.

Communicating, reasoning and problem solving

9. Consider the quadratic equation $y = x^2 - 4x + 7$.
a. Determine the equivalent inverted equation of the quadratic that just touches the one above at the vertex.
b. Confirm your result graphically.

10. A parabola has the equation $y = -\frac{1}{2}(x - 3)^2 + 4$. A second parabola has an equation defined by $Y = 2(y - 1) - 3$.
a. Find the equation relating Y to x.
b. State the appropriate dilation, reflection and translation of the graph of $Y = x^2$ required to obtain the graph of $Y = 2(y - 1) - 3$.
c. State the coordinates of the vertex.
d. Sketch the graph of $Y = 2(y - 1) - 3$.

11. What are the vertices of the graphs $y = x^2 + k$ and $y = (x - h)^2$?

12. A ball blasted upwards followed a parabolic path. It reached a maximum height of 200 m when its horizontal distance from its starting point was 10 m. When the ball's horizontal distance from the starting point was 1 m, the ball had reached a height of 38 m. Suggest an equation to model the ball's flight, clearly defining your chosen pronumerals.

13.4 Sketching parabolas of the form $y = a(x - h)^2 + k$ [Stage 5.3]

13.4.1 Vertex form

- When a quadratic equation is expressed in the form $y = a(x - h)^2 + k$:
  - the vertex is the point (h, k)
  - the axis of symmetry is $x = h$
  - the x-intercepts are calculated by solving $a(x - h)^2 + k = 0$.
- Changing the values of a, h and k in the equation transforms the shape and position of the parabola when compared with the parabola $y = x^2$.

$y = a(x - h)^2 + k$

Reflects and dilates	Translates left and right	Translates up and down a

WORKED EXAMPLE 8

For each of the following equations, state the coordinates of the vertex of the graph and whether it is a maximum or a minimum.

a $y = (x - 6)^2 - 4$ **b** $y = -(x + 3)^2 + 2$

THINK	WRITE
a 1 Write the equation.	**a** $y = (x - 6)^2 - 4$
2 Identify the transformations — a horizontal translation of 6 units to the right and a vertical translation of 4 units down. State the vertex.	The vertex is $(6, -4)$.
3 As a is positive ($a = 1$), the graph is upright with a minimum vertex.	Minimum vertex
b 1 Write the equation	**b** $y = -(x + 3)^2 + 2$
2 Identify the transformations — a horizontal translation of 3 units to the left and a vertical translation of 2 units up. State the vertex.	The vertex is $(-3, 2)$.
3 As a is negative ($a = -1$), the graph is inverted with a maximum vertex.	Maximum vertex

13.4.2 *x*- and *y*-intercepts of quadratic graphs

- Other key features such as the x- and y-intercepts can also be determined from the equation of a parabola.
- The point(s) where the graph cuts or touches the x-axis are called the x-intercept(s). At these points, $y = 0$.
- The point where the graph cuts the y-axis is called the y-intercept. At this point, $x = 0$.

WORKED EXAMPLE 9

Determine the *y*-intercept and the *x*-intercepts (where they exist) for the parabolas with equations:

a $y = (x + 3)^2 - 4$ **b** $y = 2(x - 1)^2$ **c** $y = -(x + 2)^2 - 1$.

THINK	WRITE
a 1 Write the equation.	**a** $y = (x + 3)^2 - 4$
2 Find the y-intercept by substituting $x = 0$ into the equation.	y-intercept: when $x = 0$, $y = (0 + 3)^2 - 4$ $= 9 - 4$ $= 5$ The y-intercept is 5.
3 Find the x-intercepts by substituting $y = 0$ into the equation and solving for x. Add 4 to both sides of the equation. Take the square root of both sides of the equation. Subtract 3 from both sides of the equation. Solve for x.	x-intercepts: when $y = 0$, $(x + 3)^2 - 4 = 0$ $(x + 3)^2 = 4$ $(x + 3) = +2$ or -2 $x = 2 - 3$ or $x = -2 - 3$ $x = -1$ $x = -5$ The x-intercepts are –5 and –1.

b 1 Write the equation.	**b** $y = 2(x - 1)^2$
2 Find the y-intercept by substituting $x = 0$ into the equation.	y-intercept: when $x = 0$, $y = 2(0 - 1)^2$ $= 2 \times 1$ $= 2$ The y-intercept is 2.
3 Find the x-intercepts by substituting $y = 0$ into the equation and solving for x. Note that there is only one solution for x and so there is only one x-intercept. (The graph touches the x-axis.)	x-intercepts: when $y = 0$, $2(x - 1)^2 = 0$ $(x - 1)^2 = 0$ $x - 1 = 0$ $x = 0 + 1$ $x = 1$ The x-intercept is 1.
c 1 Write the equation.	**c** $y = -(x + 2)^2 - 1$
2 Find the y-intercept by substituting $x = 0$ into the equation.	y-intercept: when $x = 0$, $y = -(0 + 2)^2 - 1$ $= -4 - 1$ $= -5$ The y-intercept is -5.
3 Find the x-intercepts by substituting $y = 0$ into the equation and solving for x. We cannot take the square root of 21 to obtain real solutions; therefore, there are no x-intercepts.	x-intercepts: when $y = 0$, $-(x + 2)^2 - 1 = 0$ $(x + 2)^2 = -1$ There are no real solutions, so there are no x-intercepts.

WORKED EXAMPLE 10

For each of the following:

i write the coordinates of the vertex
ii state whether the graph has a maximum or a minimum vertex
iii state whether the graph is wider, narrower or the same width as the graph of $y = x^2$
iv find the y-intercept
v find the x-intercepts
vi sketch the graph.

a $y = (x - 2)^2 + 3$ **b** $y = -2(x + 1)^2 + 6$

THINK	WRITE
a 1 Write the equation.	**a** $y = (x - 2)^2 + 3$
2 State the coordinates of the vertex from the equation. Use (h, k) as the equation is in the vertex form of $y = a(x - h)^2 + k$ where $a = 1$, $h = 2$ and $k = 3$.	**i** The vertex is $(2, 3)$.
3 State the nature of the vertex by considering the sign of a.	**ii** The graph has a minimum vertex as the sign of a is positive.

4 Specify the width of the graph by considering the magnitude of a.

iii The graph has the same width as $y = x^2$ since $a = 1$.

5 Find the y-intercept by substituting $x = 0$ into the equation.

iv y-intercept: when $x = 0$,

$y = (0 - 2)^2 + 3$

$= 4 + 3$

$= 7$

y-intercept is 7.

6 Find the x-intercepts by substituting $y = 0$ into the equation and solving for x.
As we have to take the square root of a negative number, we cannot solve for x.

v x-intercepts: when $y = 0$,

$(x - 2)^2 + 3 = 0$

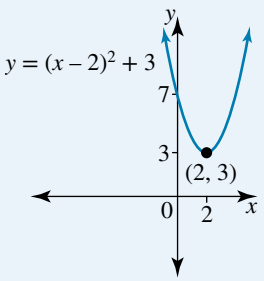
$(x - 2)^2 = -3$

There are no real solutions, and hence no x-intercepts.

7 Sketch the graph, clearly showing the vertex and the y-intercept.

8 Label the graph.

vi



b 1 Write the equation.

b $y = -2(x + 1)^2 + 6$

2 State the coordinates of the vertex from the equation. Use (h, k) as the equation is in the vertex form of $y = a(x - h)^2 + k$ where $a = -2$, $h = -1$ and $k = 6$.

i The vertex is $(-1, 6)$.

3 State the nature of the vertex by considering the sign of a.

ii The graph has a maximum vertex as the sign of a is negative.

4 Specify the width of the graph by considering the magnitude of a.

iii The graph is narrower than $y = x^2$ since $|a| > 1$.

5 Find the y-intercept by substituting $x = 0$ into the equation.

iv y-intercept: when $x = 0$,

$y = -2(0 + 1)^2 + 6$

$= -2 \times 1 + 6$

$= 4$

The y-intercept is 4.

6 Find the x-intercepts by substituting $y = 0$ into the equation and solving for x.

v x-intercepts: when $y = 0$,

$-2(x + 1)^2 + 6 = 0$

$2(x + 1)^2 = 6$

$(x + 1)^2 = 3$

$x + 1 = \sqrt{3}$ or $x + 1 = -\sqrt{3}$

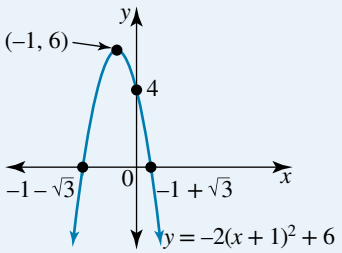
$x = -1\sqrt{3}$ $x = -1 - \sqrt{3}$

The x-intercepts are $-1 - \sqrt{3}$ and $-1 + \sqrt{3}$ (or approximately –2.73 and 0.73).

7 Sketch the graph, clearly showing the vertex and the x- and y-intercepts.

8 Label the graph.

vi



Note: Unless otherwise stated, exact values for the intercepts should be shown on sketch graphs.

learnON RESOURCES — ONLINE ONLY

Interactivity: Quadratic functions (int-2562)

eLesson: Sketching quadratics in turning point form (eles-1926)

eLesson: Solving quadratic equations in turning point form (eles-1941)

Exercise 13.4 Sketching parabolas of the form $y = a(x - h)^2 + k$

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–d, 2a–c, 3a–c, 4, 5, 6a–d, 7, 10, 12	Questions: 1c–f, 2c–e, 3c–e, 4, 5, 6c–f, 7, 11, 12, 14	Questions: 1c–i, 2c–f, 3c–f, 4, 5, 6e–i, 7–16

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE8 For each of the following equations, state the coordinates of the vertex of the graph and whether it is a maximum or a minimum.

a. $y = (x - 1)^2 + 2$ **b.** $y = (x + 2)^2 - 1$ **c.** $y = (x + 1)^2 + 1$

d. $y = -(x - 2)^2 + 3$ **e.** $y = -(x - 5)^2 + 3$ **f.** $y = (x + 2)^2 - 6$

g. $y = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$ **h.** $y = \left(x - \frac{1}{3}\right)^2 + \frac{2}{3}$ **i.** $y = (x + 0.3)^2 - 0.4$

2. For each of the following, state:

i. the coordinates of the vertex

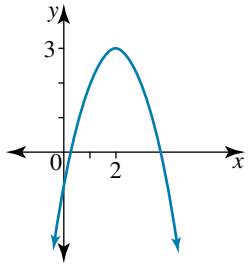
ii. whether the graph has a maximum or a minimum vertex

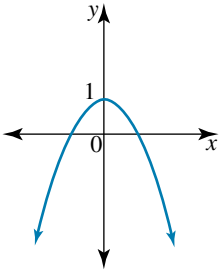
iii. whether the graph is wider, narrower or the same width as that of $y = x^2$.

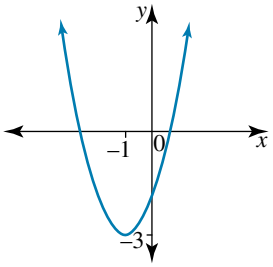
a. $y = 2(x + 3)^2 - 5$ **b.** $y = -(x - 1)^2 + 1$ **c.** $y = -5(x + 2)^2 - 4$

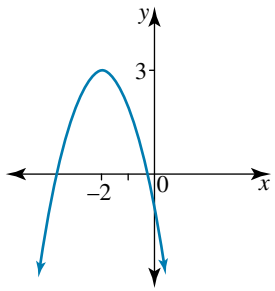
d. $y = \frac{1}{4}(x - 3)^2 + 2$ **e.** $y = -\frac{1}{2}(x + 1)^2 + 7$ **f.** $y = 0.2\left(x - \frac{1}{5}\right)^2 - \frac{1}{2}$

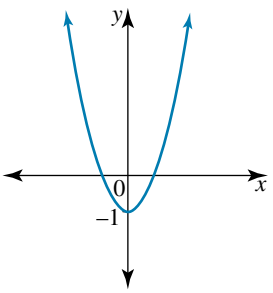
3. Select the equation that best suits each of the following graphs.

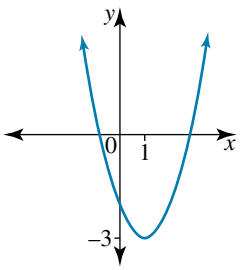
a.


b.


c.


d.


e.


f.


i. $y = (x - 1)^2 - 3$
ii. $y = -(x - 2)^2 + 3$
iii. $y = x^2 - 1$
iv. $y = -(x + 2)^2 + 3$
v. $y = -x^2 + 1$
vi. $y = (x + 1)^2 - 3$

4. MC **a.** The translations required to change $y = x^2$ into $y = \left(x - \frac{1}{2}\right)^2 + \frac{1}{3}$ are:

A. right $\frac{1}{2}$, up $\frac{1}{3}$
B. left $\frac{1}{2}$, down $\frac{1}{3}$
C. right $\frac{1}{2}$, down $\frac{1}{3}$
D. left $\frac{1}{2}$, up $\frac{1}{3}$
E. right $\frac{1}{3}$, up $\frac{1}{2}$

b. For the graph $\frac{1}{4}\left(x - \frac{1}{2}\right)^2 + \frac{1}{3}$, the effect of the $\frac{1}{4}$ on the graph is:

A. no effect
B. to make the graph narrower
C. to make the graph wider
D. to invert the graph
E. to translate the graph up $\frac{1}{4}$ of a unit

c. Compared to the graph of $y = x^2$, $y = -2(x + 1)^2 - 4$ is:

A. inverted and wider
B. inverted and narrower
C. upright and wider
D. upright and narrower
E. inverted and the same width

d. A graph that has a minimum vertex (1, 5) and that is narrower than the graph of $y = x^2$ is:

A. $y = (x - 1)^2 + 5$
B. $y = \frac{1}{2}(x + 1)^2 + 5$
C. $y = 2(x - 1)^2 + 5$
D. $y = 2(x + 1)^2 + 5$
E. $y = \frac{1}{2}(x - 1)^2 + 5$

e. Compared to the graph of $y = x^2$, the graph of $y = -3(x - 1)^2 - 2$ has the following features.

A. Maximum vertex at (–1, –2), narrower
B. Maximum vertex at, narrower
C. Maximum vertex at (1, 2), wider
D. Minimum vertex at, narrower
E. Minimum vertex at, wider

5. WE9 Determine **i** the y-intercept and **ii** the x-intercepts (where they exist) for the parabolas with the following equations.

a. $y = (x + 1)^2 - 4$
b. $y = 3(x - 2)^2$
c. $y = -(x + 4)^2 - 2$
d. $y = (x - 2)^2 - 9$
e. $y = 2x^2 + 4$
f. $y = (x + 3)^2 - 5$

6. WE10 For each of the following:

i. write the coordinates of the vertex
ii. state whether the graph has a maximum or a minimum vertex
iii. state whether the graph is wider, narrower or the same width as the graph of $y = x^2$

iv. find the y-intercept
v. find the x-intercepts
vi. sketch the graph.

a. $y = (x - 4)^2 + 2$ **b.** $y = (x - 3)^2 - 4$ **c.** $y = (x + 1)^2 + 2$
d. $y = (x + 5)^2 - 3$ **e.** $y = -(x - 1)^2 + 2$ **f.** $y = -(x + 2)^2 - 3$
g. $y = -(x + 3)^2 - 2$ **h.** $y = 2(x - 1)^2 + 3$ **i.** $y = -3(x + 2)^2 + 1$

7. Consider the equation $2x^2 - 3x - 8 = 0$.
a. Complete the square.
b. Use the result to determine the exact solutions to the original equation.
c. Determine the vertex of $y = 2x^2 - 3x - 8$ and indicate its type.

8. a. Find the equation of a quadratic that has a vertex of (–4, 6) and has an x-intercept at (–1, 0).
b. State the other x-intercept (if any).

9. Write the new equation for the parabola $y = x^2$ that has been:
a. reflected in the x-axis
b. dilated by a factor of 7 away from the x-axis
c. translated 3 units in the negative direction of the x-axis
d. translated 6 units in the positive direction of the y-axis
e. dilated by a factor of $\frac{1}{4}$ from the x-axis, reflected in the x-axis, and translated 5 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis.

Communicating, reasoning and problem solving

10. The price of shares in fledgling company 'Lollies'r'us' plunged dramatically one afternoon, following the breakout of a small fire on the premises. However, Ms Sarah Sayva of Lollies Anonymous agreed to back the company, and share prices began to rise.

Sarah noted at the close of trade that afternoon that the company's share price followed the curve $P = 0.1(t - 3)^2 + 1$, where $\$P$ is the price of shares t hours after noon.

a. Sketch a graph of the relationship between time and share price to represent the situation.
b. What was the initial share price?
c. What was the lowest price of shares that afternoon?
d. At what time was the price at its lowest?
e. What was the final price of 'Lollies'r'us' shares as trade closed at 5 pm?

11. Rocky is practising for a football kicking competition. After being kicked, the path that the ball follows can be modelled by the quadratic relationship:

$$h = -\frac{1}{30}(d - 15)^2 + 8$$

where h is the vertical distance the ball reaches (in metres) and d is the horizontal distance (in metres).

a. Determine the initial vertical height of the ball.

b. Determine the exact maximum horizontal distance the ball travels.

c. Write down both the maximum height and the horizontal distance when the maximum height is reached.

12. a. If the vertex of a particular parabola is $(2, 6)$, suggest a possible equation for the parabola.

b. If the y-intercept in part **a** is $(0, 4)$, give the exact equation for the parabola.

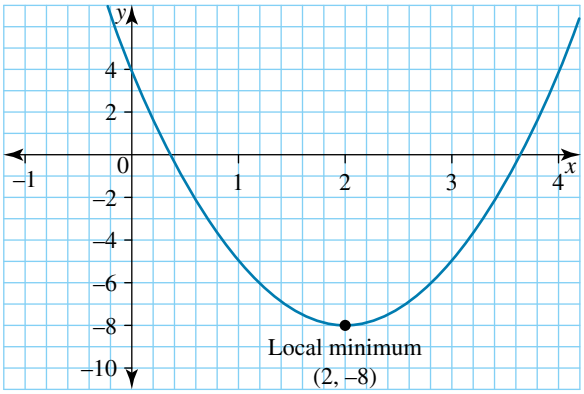
13. a. If the vertex of a particular parabola is (p, q), suggest a possible equation for the parabola.

b. If the y-intercept in part **a** is $(0, r)$, give the exact equation for the parabola.

14. Use the completing the square method to write each of the following in vertex form and sketch the parabola for each part.

a. $y = x^2 - 8x + 1$ **b.** $y = x^2 + 4x - 5$ **c.** $y = x^2 + 3x + 2$

15. a. Find the equation of the parabola shown.



b. State the dilation and translation transformations that have been applied to $y = x^2$ to achieve this parabola.

c. This graph is reflected in the x-axis. State the equation of the reflected graph.

d. Sketch the graph of the reflected parabola.

16. Does a in the equation $y = a(x - h)^2 + k$ have any impact on the vertex?

13.5 Sketching parabolas of the form $y = ax^2 + bx + c$ [Stage 5.3]

13.5.1 Parabolas of the form $y = ax^2 + bx$

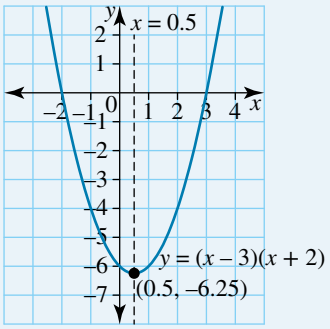
- The general form of a quadratic equation is $y = ax^2 + bx + c$ where a, b and c are constants.
- A sketch of a parabola usually shows x- and y-intercepts and the vertex.
- The x-coordinate of the vertex lies midway between the x-intercepts.
- The x-coordinate of the vertex can also be found using the formula $x = \frac{-b}{2a}$ from the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. For example, for the equation $y = x^2 + 4x - 6$, the x-coordinate of the vertex will be at $x = \frac{-4}{2 \times 1} = -2$.

  The y-coordinate of the vertex can be found by substitution. Continuing the example, given $x = -2$, $y = (-2)^2 + 4(-2) - 6 = -10$. The vertex is $(-2, -10)$.

- $x = \frac{-b}{2a}$ is also the equation of the axis of symmetry.
- If the equation is not written in vertex form, the coordinates of the vertex may be found by:
  - finding the midpoint between the x-intercepts
  - using the formula $x = \frac{-b}{2a}$
  - writing the equation in vertex form by completing the square.

WORKED EXAMPLE 11

Sketch the graph of $y = (x - 3)(x + 2)$.

THINK	WRITE/DRAW
1 The equation is in factorised form. To find the x-intercepts, let $y = 0$ and use the Null Factor Law.	$y = (x - 3)(x + 2)$ $0 = (x - 3)(x + 2)$ $x - 3 = 0$ or $x + 2 = 0$ (Null Factor Law) $x = 3$ or $x = -2$ x-intercepts: $(3, 0), (-2, 0)$
2 The x-coordinate of the vertex is midway between the x-intercepts. Find the average of the two x-intercepts to find the midpoint between them.	$x_{\text{vertex}} = \frac{3 + (-2)}{2}$ $= 0.5$
3 State the equation of the axis of symmetry.	$x = 0.5$
4 To find the y-coordinate of the vertex, substitute x_{vertex} into the equation.	$y = (x - 3)(x + 2)$ $y_{\text{vertex}} = (0.5 - 3)(0.5 + 2)$ $= -6.25$
5 State the vertex.	Vertex: $(0.5, -6.25)$
6 To find the y-intercept, let $x = 0$ and substitute.	$y = (0 - 3)(0 + 2)$ $= -6$
7 State the y-intercept.	y-intercept: $(0, -6)$
8 Sketch the graph, showing all the important features.	
9 Label the graph.	

WORKED EXAMPLE 12

Sketch the graph of $y = 2x^2 - 6x - 6$.

THINK	WRITE/DRAW
1 The equation is not in factorised form, but there is a common factor of 2. Take out the common factor of 2.	$y = 2x^2 - 6x - 6$ $= 2(x^2 - 3x - 3)$

2 The equation cannot be factorised (no factors of –3 add to –3), so use completing the square to write the equation in vertex form.
- Halve and then square the coefficient of x.
- Add this and then subtract it from the equation.
- Collect the terms for and write the perfect square.
- Simplify the brackets to write the equation in vertex form.
- Identify the coordinates of the vertex (h, k).

$$y = 2\left(x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 - 3\right)$$
$$= 2\left(\left[x - \frac{3}{2}\right]^2 - \left(\frac{3}{2}\right)^2 - 3\right)$$
$$= 2\left(\left[x - \frac{3}{2}\right]^2 - \frac{9}{4} - 3\right)$$
$$= 2\left(\left[x - \frac{3}{2}\right]^2 - \frac{21}{4}\right)$$
$$= 2\left(x - \frac{3}{2}\right)^2 - 2 \times \frac{21}{2}$$
$$= 2\left(x - \frac{3}{2}\right)^2 - \frac{21}{2}$$

Vertex: $\left(\frac{3}{2}, \frac{-21}{2}\right)$

3 To find the x-intercepts, let $y = 0$.
No factors of –3 add to –3, so use the quadratic formula to find the x-intercepts.

x-intercepts: let $y = 0$.

$$0 = 2x^2 - 6x - 6$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $a = 2$, $b = -6$, $c = -6$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(2)(-6)}}{2(2)}$$
$$= \frac{6 \pm \sqrt{36 + 48}}{4}$$
$$= \frac{6 \pm \sqrt{84}}{4}$$
$$= \frac{6 \pm 2\sqrt{21}}{4}$$

4 State the x-intercepts.

The x-intercepts are:

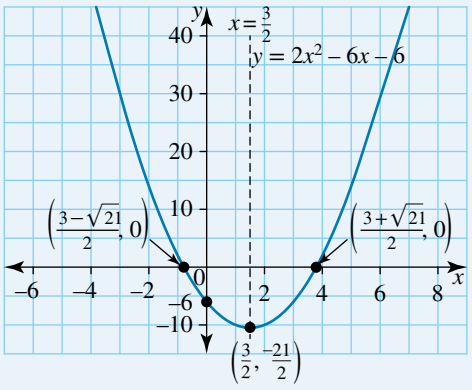
$$x = \frac{3 + \sqrt{21}}{2} \text{ and } x = \frac{3 - \sqrt{21}}{2}$$
$$x \approx 3.79 \qquad x \approx -0.79$$

5 To find the y-intercepts, let $x = 0$ and substitute. State the y-intercept.

$$y = 2x^2 - 6x - 6$$
$$y = 2(0)^2 - 6(0) - 6$$
$$= -6$$

y-intercept: $(0, -6)$

6 Sketch the graph, showing all the important features.



7 Label the graph, showing the exact values of the x-intercepts.

learnON RESOURCES — ONLINE ONLY

Interactivity: Sketching parabolas (int-2785)
eLesson: Sketching parabolas using the quadratic formula (eles-1945)
eLesson: Sketching quadratics in factorised form (eles-1927)
Digital doc: SkillSHEET Completing the square (doc-5268)
Digital doc: SkillSHEET Solving quadratic equations using the quadratic formula (doc-5269)
Digital doc: SkillSHEET Solving quadratic equations of the type $y = ax^2 + bx + c = 0$ where $a = 1$ (doc-5270)
Digital doc: SkillSHEET Solving quadratic equations of the type $y = ax^2 + bx + c = 0$ where $a \neq 1$ (doc-5271)
Digital doc: WorkSHEET $y = ax^2 + bx + c$ (doc-5273)

Exercise 13.5 Sketching parabolas of the form $y = ax^2 + bx + c$

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1, 2a–c, 3a–c, 4, 5, 7	Questions: 1, 2b–d, 3c–d, 4, 6–8, 10	Questions: 1, 2c–f, 3c–g, 5, 7–9, 11, 12

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. What information is necessary to be able to sketch a parabola?
2. WE11 Sketch the graph of each of the following.
   a. $y = (x - 5)(x - 2)$
   b. $y = (x + 4)(x - 7)$
   c. $y = (x + 3)(x + 5)$
   d. $y = (2x + 3)(x + 5)$
   e. $y = (4 - x)(x + 2)$
   f. $y = \left(\frac{x}{2} + 3\right)(5 - x)$
3. WE12 Sketch the graph of each of the following.
   a. $y = x^2 + 4x + 2$
   b. $y = x^2 - 4x - 5$
   c. $y = 2x^2 - 4x - 3$
   d. $y = -2x^2 + 11x + 5$
   e. $y = -2x^2 + 12x$
   f. $y = 3x^2 + 6x + 1$
   g. $y = -3x^2 - 5x + 2$
4. The path of a soccer ball kicked by the goal keeper can be modelled by the equation $y = -\frac{1}{144}(x^2 - 24x)$ where y is the height of the soccer ball and x is the horizontal distance from the goalie, both in metres.
   a. Sketch the graph.
   b. How far away from the player does the ball first bounce?
   c. What is the maximum height of the ball?

5. The monthly profit or loss, p, (in thousands of dollars) for a new brand of chicken loaf is given by $p = 3x^2 - 15x - 18$, where x is the number of months after its introduction (when $x = 0$).
   a. Sketch the graph.
   b. During which month was a profit first made?
   c. In which month is the profit $54 000?

6. The height, h metres, of a model rocket above the ground t seconds after launch is given by the equation $h = 4t(50 - t)$ for $0 \leq t \leq 50$.
   a. Sketch the graph of the rocket's flight.
   b. Find the height of the rocket above the ground when it is launched.
   c. What is the greatest height reached by the rocket?
   d. How long does the rocket take to reach its greatest height?
   e. For how long is the rocket in the air?

Communicating, reasoning and problem solving

7. The equation $y = x^2 + bx + 7500$ has x-intercepts of $(-150, 0)$ and $(-50, 0)$. What is the value of b in the equation? Justify your answer.

8. The equation $y = x^2 + bx + c$ has x-intercepts of m and n. What is the value of b in the equation? Justify your answer.

9. **a.** What path does a spaceship take to get to the Moon?
   b. What is a transfer orbit?
   c. Is any part of the flight path parabolic?

10. A ball is thrown upwards from a building and follows the path given by the formula $h = -x^2 + 4x + 21$. The ball is h metres above the ground when it is a horizontal distance of x metres from the building.
   a. Sketch the graph of the path of the ball.
   b. What is the maximum height the ball reaches?
   c. How far is the ball from the wall when it reaches the maximum height?
   d. How far from the building does the ball land?

11. During an 8-hour period, an experiment is done in which the temperature of a room follows the relationship $T = h^2 - 8h + 21$, where T is the temperature in degrees Celsius h hours after starting the experiment.
   a. Sketch the graph of this quadratic.
   b. What is the initial temperature?
   c. After 3 hours, is the temperature increasing or decreasing?
   d. After 5 hours, is the temperature increasing or decreasing?
   e. State the minimum temperature and when it occurred.
   f. What is the temperature after 8 hours?

12. What strategy can you use to remember all of the information necessary to sketch a parabola?

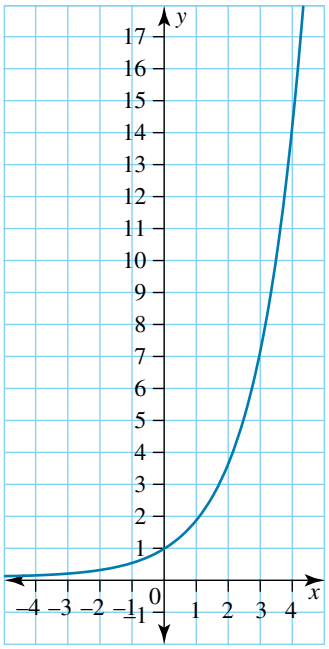
13.6 Exponential functions and graphs [Stage 5.3]

13.6.1 Exponential functions

- Relationships of the form $y = a^x$ are called **exponential functions** with base a, where a is a real number not equal to 1, and x is the index power or exponent.
- The term 'exponential' is used because x is an exponent (or index).

- For example, the graph of the exponential function $y = 2^x$ can be plotted by completing a table of values.
  Remember that $2^{-3} = \frac{1}{2^3}$
  $= \frac{1}{8}$, and so on.

x	−4	−3	−2	−1	0	1	2	3	4
y	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16



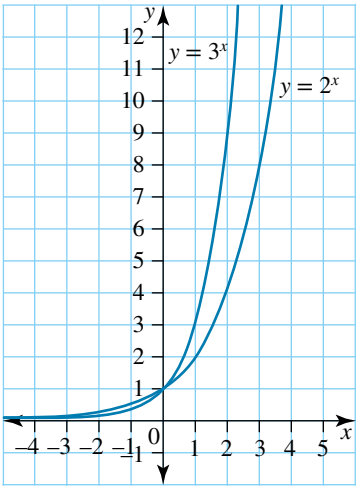
- The graph has many significant features.
  - The y-intercept is 1.
  - The value of y is always greater than zero.
  - As x decreases, y gets closer to but never reaches zero. So the graph gets closer to but never reaches the x-axis. The x-axis (or the line $y = 0$) is called an **asymptote**.
  - As x increases, y becomes very large.

DISCUSSION

Where do exponential functions appear in real life? Use the internet to help you research some real-life examples.

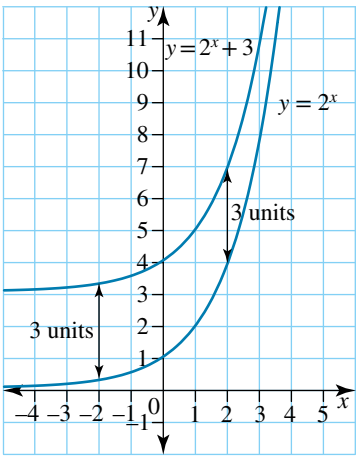
13.6.2 Comparing exponential graphs

- The diagram shows the graphs of $y = 2^x$ and $y = 3^x$.
- The graphs both pass through the point (0, 1).
- The graph of $y = 3^x$ climbs more steeply than the graph of $y = 2^x$.
- $y = 0$ is an asymptote for both graphs.



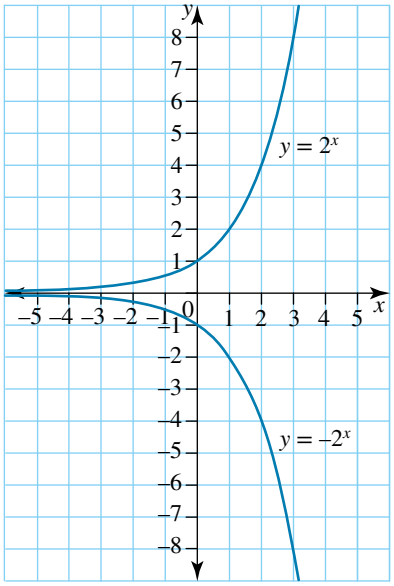
13.6.3 Vertical translation

- The diagram shows the graphs of $y = 2^x$ and $y = 2^x + 3$.
- The graphs have identical shape.
- Although they appear to get closer to each other, the graphs are constantly 3 units apart.
- As x becomes very small, the graph of $y = 2^x + 3$ approaches but never reaches the line $y = 3$, so $y = 3$ is the horizontal asymptote.
- When the graph of $y = 2^x$ is translated 3 units upward, it becomes the graph of $y = 2^x + 3$.



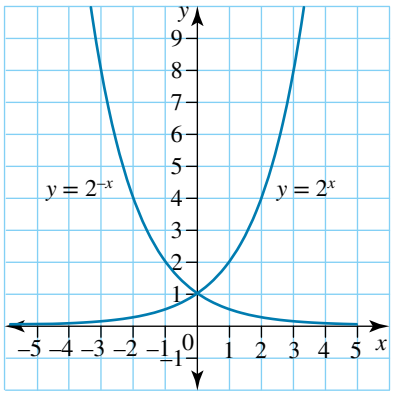
13.6.4 Reflection about the *x*-axis

- The diagram at right shows the graphs of $y = 2^x$ and $y = -2^x$.
- The graphs have identical shape.
- The graph of $y = -2^x$ is a reflection about the *x*-axis of the graph of $y = 2^x$.
- The *x*-axis ($y = 0$) is an asymptote for both graphs.
- In general, the graph of $y = -a^x$ is a reflection about the *x*-axis of the graph of $y = a^x$.



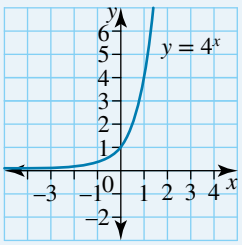
13.6.5 Reflection about the *y*-axis

- The diagram shows the graphs of $y = 2^x$ and $y = 2^{-x}$.
- The graphs have identical shape.
- The graph of $y = 2^{-x}$ is a reflection about the *y*-axis of the graph of $y = 2^x$.
- Both graphs pass through the point (0, 1).
- The *x*-axis ($y = 0$) is an asymptote for both graphs.
- In general, the graph of $y = a^{-x}$ is a reflection about the *y*-axis of the graph of $y = a^x$.



WORKED EXAMPLE 13

Given the graph of $y = 4^x$, sketch on the same axes the graphs of:
a $y = 4^x - 2$
b $y = -4^x$
c $y = 4^{-x}$.

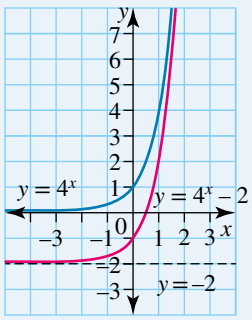


THINK

a The graph of $y = 4^x$ has already been drawn. It has a *y*-intercept of 1 and a horizontal asymptote at $y = 0$. The graph of $y = 4^x - 2$ has the same shape as $y = 4^x$ but is translated 2 units vertically down. It has a *y*-intercept of –1 and a horizontal asymptote at $y = -2$.

DRAW

a

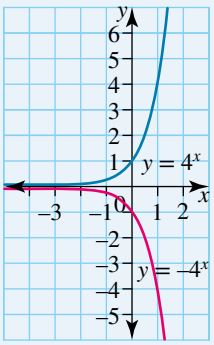


THINK

b $y = -4^x$ has the same shape as $y = 4^x$ but is reflected about the x-axis. It has a y-intercept of -1 and a horizontal asymptote at $y = 0$.

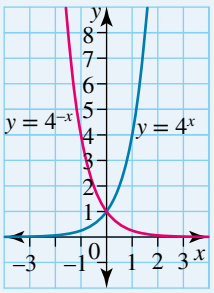
DRAW

b



c $y = 4^{-x}$ has the same shape as $y = 4^x$ but is reflected about the y-axis. The graphs have the same y-intercept and the same horizontal asymptote ($y = 0$).

c



13.6.6 Combining transformations

- It is possible to combine translations, dilations and reflections in one graph.

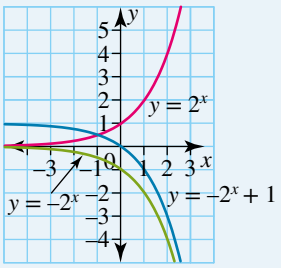
WORKED EXAMPLE 14

By considering transformations to the graph of $y = 2^x$, sketch the graph of $y = -2^x + 1$.

THINK

Start by sketching $y = 2^x$.
It has a y-intercept of 1 and a horizontal asymptote at $y = 0$.
Sketch $y = -2^x$ by reflecting $y = 2^x$ about the x-axis.
It has a y-intercept of -1 and a horizontal asymptote at $y = 0$.
Sketch $y = -2^x + 1$ by translating $y = -2^x$ upwards by 1 unit.
The graph has a y-intercept of 0 and a horizontal asymptote at $y = 1$.

DRAW



learnon RESOURCES — ONLINE ONLY

Interactivity: Exponential functions (int-5959)

Interactivity: Exponential graphs (int-1149)

Exercise 13.6 Exponential functions and graphs

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–16	Questions: 1–17	Questions: 1–18

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. Complete the table below and use the table to plot the graph of $y = 3^x$ for $-3 \leq x \leq +3$.

x	−3	−2	−1	0	1	2	3
y							

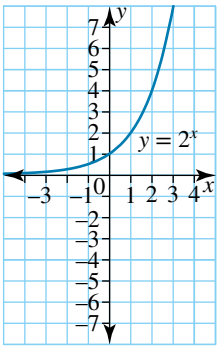
2. If $x = 1$, find the value of y when:
a. $y = 2^x$ **b.** $y = 3^x$ **c.** $y = 4^x$ **d.** $y = 10^x$ **e.** $y = a^x$.

3. Using a calculator or graphing program, sketch the graphs of $y = 2^x$, $y = 3^x$ and $y = 4^x$ on the same set of axes.
a. What do the graphs have in common?
b. How does the value of the base (2, 3, 4) affect the graph?
c. Predict where the graph $y = 8^x$ would lie and sketch it in.

4. Using graphing technology, sketch the following graphs on one set of axes.
$y = 3^x$, $y = 3^x + 2$, $y = 3^x + 5$, $y = 3^x - 3$
a. What remains the same in all of these graphs?
b. What is changed?
c. For the graph of $y = 3^x + 10$, write down:
i. the y-intercept
ii. the equation of the horizontal asymptote.

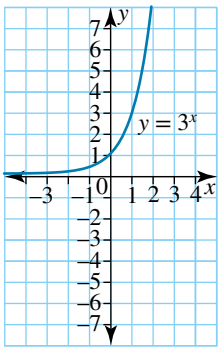
5. a. Using graphing technology, sketch the graphs of:
i. $y = 2^x$ and $y = -2^x$ **ii.** $y = 3^x$ and $y = -3^x$
iii. $y = 6^x$ and $y = -6^x$.
b. What is the relationship between these pairs of graphs?

6. a. Using graphing technology, sketch the graphs of:
i. $y = 2^x$ and $y = 2^{-x}$ **ii.** $y = 3^x$ and $y = 3^{-x}$
iii. $y = 6^x$ and $y = 6^{-x}$.
b. What is the relationship between these pairs of graphs?



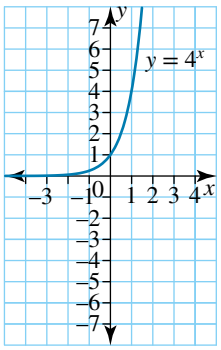
7. WE13 Given the graph of $y = 2^x$, sketch on the same axes the graphs of:
a. $y = 2^x + 6$ **b.** $y = -2^x$ **c.** $y = 2^{-x}$.

8. Given the graph of $y = 3^x$, sketch on the same axes the graphs of:
a. $y = 3^x + 2$
b. $y = -3^x$.



9. Given the graph of $y = 4^x$, sketch on the same axes the graphs of:

a. $y = 4^x - 3$ **b.** $y = 4^{-x}$.



10. **WE14** By considering transformations of the graph of $y = 2^x$, sketch the following graphs on the same set of axes.

a. $y = 2^{-x} + 2$ **b.** $y = -2^x + 3$

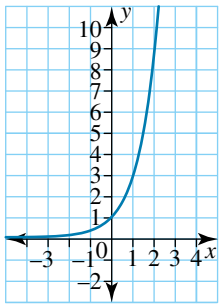
11. By considering transformations of the graph of $y = 5^x$, sketch the following graphs on the same set of axes.

a. $y = -5^x + 10$ **b.** $y = 5^{-x} + 10$

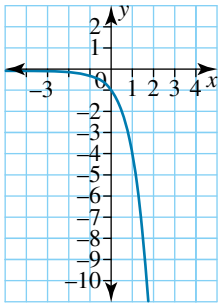
12. Match each graph with its correct label.

A. $y = 2^x$ **B.** $y = 3^x$ **C.** $y = -4^x$ **D.** $y = 5^{-x}$

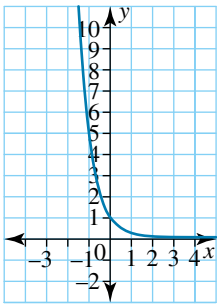
a.



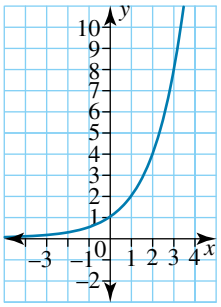
b.



c.



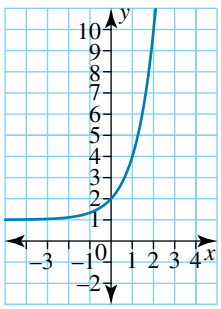
d.



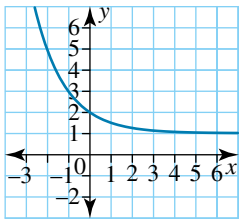
13. Match each graph with its correct label. Explain your answer.

A. $y = 2^x + 1$ **B.** $y = 3^x + 1$ **C.** $y = -2^x + 1$ **D.** $y = 2^{-x} + 1$

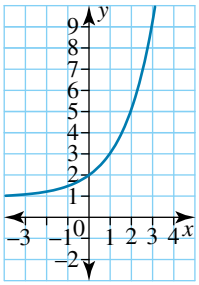
a.



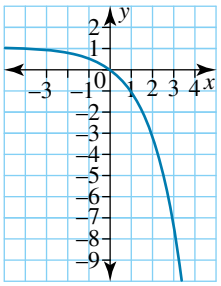
b.



c.



d.



Communicating, reasoning and problem solving

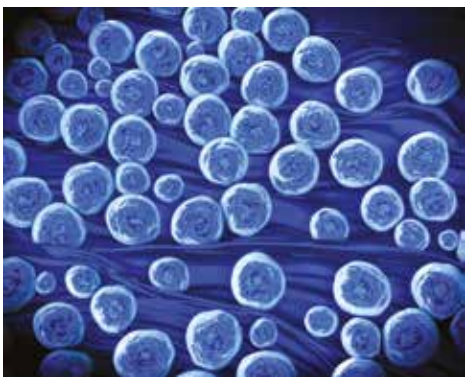
14. By considering transformations of the graph of $y = 3^x$, sketch the graph of $y = -3^{-x} - 3$.

15. The graph of $f(x) = 16^x$ can be used to solve for x in the exponential equation $16^x = 32$. Draw a graph of $f(x) = 16^x$ and use it to solve $16^x = 32$.

16. The number of bacteria, N, in a certain culture is reduced by a third every hour, so:

$$N = N_0 \times \left(\frac{1}{3}\right)^t$$

where t is the time in hours after 12 noon on a particular day. Initially there are 10000 bacteria present.



a. Find the value of N_0.

b. Find the number of bacteria, correct to the nearest whole number, in the culture when:

i. $t = 2$ **ii.** $t = 5$ **iii.** $t = 10$.

17. a. The table below shows the population of a city between 1850 and 1930. Is the population growth exponential?

Year	1850	1860	1870	1880	1890	1900	1910	1920	1930
Population (million)	1.0	1.3	1.69	2.197	2.856	3.713	4.827	6.275	8.157

b. What is the common ratio in part **a**?
c. What is the annual percentage increase?
d. Estimate the population in 1895.
e. Estimate the population in 1980.

18. Will the graph of an exponential function always have a horizontal asymptote? Why?

13.7 The hyperbola [Stage 5.3]

- A **hyperbola** is a function of the form $xy = k$ or $y = \frac{k}{x}$.

WORKED EXAMPLE 15

Complete the table of values below and use it to plot the graph of $y = \frac{1}{x}$.

x	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3
y									

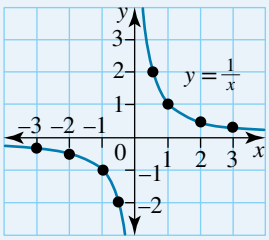
THINK

1 Substitute each x-value into the function $y = \frac{1}{x}$ to obtain the corresponding y-value.

2 Draw a set of axes and plot the points from the table. Join them with a smooth curve.

WRITE/DRAW

x	-3	-2	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3
y	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	Undefined	2	1	$\frac{1}{2}$	$\frac{1}{3}$



- The graph in Worked example 15 has several important features.
  1. There is no function value (y-value) when $x = 0$. At this point the hyperbola is undefined. When this occurs, the line that the graph approaches ($x = 0$) is called a vertical asymptote.
  2. As x becomes larger and larger, the graph gets very close to but will never touch the x-axis. The same is true as x becomes smaller and smaller. The hyperbola also has a horizontal asymptote at $y = 0$.
  3. The hyperbola has two separate branches. It cannot be drawn without lifting your pen from the page and is an example of a discontinuous graph.
- Graphs of the form $y = \frac{k}{x}$ are the same basic shape as $y = \frac{1}{x}$ with y-values dilated by a factor of k.

WORKED EXAMPLE 16

a Plot the graph of $y = \frac{4}{x}$ for $-2 \leq x \leq 2$.

b Write down the equation of each asymptote.

THINK

a 1 Prepare a table of values taking x-values from –2 to 2. Fill in the table by substituting each x-value into the given equation to find the corresponding y-value.

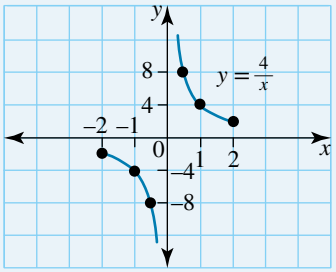
2 Draw a set of axes and plot the points from the table. Join them with a smooth curve.

b Consider any lines that the curve approaches but does not cross.

WRITE/DRAW

a

x	–2	–1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
y	–2	–4	–8	Undefined	8	4	2



b The vertical asymptote is $x = 0$.
The horizontal asymptote is $y = 0$.

WORKED EXAMPLE 17

Plot the graph of $y = \frac{-3}{x}$ for $-3 \leq x \leq 3$.

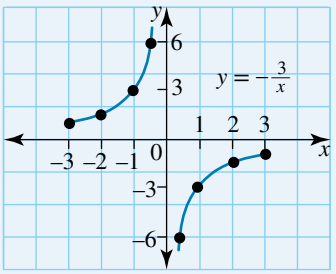
THINK

1 Draw a table of values and substitute each x-value into the given equation to find the corresponding y-value.

2 Draw a set of axes and plot the points from the table. Join them with a smooth curve.

WRITE/DRAW

x	–3	–2	–1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2	3
y	1	1.5	3	6	Undefined	–6	–3	–1.5	–1



DISCUSSION

Why is it useful to consider both very small and very large numbers when constructing a table of values for a hyperbola?

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Interactivity: Hyperbolas (int-6155)

Exercise 13.7 The hyperbola

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–12	Questions: 1–14	Questions: 1–16

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE15 Complete the table of values below and use it to plot the graph of $y = \frac{10}{x}$.

x	−5	−4	−3	−2	−1	0	1	2	3	4	5
y											

2. WE16 **a.** Plot the graph of each of the following hyperbolas.

i. $y = \frac{5}{x}$ **ii.** $y = \frac{20}{x}$ **iii.** $y = \frac{100}{x}$

b. Write down the equation of each asymptote for the graphs in part **a**.

3. On the same set of axes, draw the graphs of $y = \frac{2}{x}$, $y = \frac{3}{x}$ and $y = \frac{4}{x}$.

4. Use your answer to question **3** to describe the effect of increasing the value of k on the graph of $y = \frac{k}{x}$.

5. WE17 Plot the graph of $y = \frac{-10}{x}$ for $-5 \leq x \leq 5$.

6. On the same set of axes, draw the graphs of $y = \frac{6}{x}$ and $y = \frac{-6}{x}$.

7. Use your answer to question **6** to describe the effect of the negative in $y = \frac{-k}{x}$.

8. Complete the table of values below and use the points to plot $y = \frac{1}{x-1}$. State the equation of the vertical asymptote.

x	−3	−2	−1	0	1	2	3	4
y								

9. Plot the graph of each hyperbola and label the vertical asymptote.

a. $y = \frac{1}{x-2}$ **b.** $y = \frac{1}{x-3}$ **c.** $y = \frac{1}{x+1}$

10. Use your answers to question **9** to describe the effect of a in $y = \frac{1}{x-a}$.

11. Sketch each of the following, showing the position of the vertical asymptote.

a. $y = \frac{-4}{x+1}$ **b.** $y = \frac{2}{x-1}$ **c.** $y = \frac{5}{x+2}$

Communicating, reasoning and problem solving

12. Give an example of the equation of a hyperbola that has a vertical asymptote of:

a. $x = 3$ **b.** $x = -10$.

13. The graph of $y = \frac{1}{x}$ is reflected in the x-axis, dilated by a factor of 2 parallel to the y-axis or from the x-axis and translated 3 units to the left and down 1 unit. Find the equation of the resultant hyperbola and give the equations of any asymptotes.

14. a. Complete the following table in order to graph the hyperbola defined by $y = \frac{1}{x^2}$.

x	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2
y						

This hyperbola is also known as a truncus. Give the equations of any asymptotes.

b. Find the equation of the truncus that results when $y = \frac{1}{x^2}$ is reflected in the x-axis.

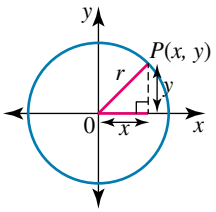
c. Find the equation of the truncus that results when $y = \frac{1}{x^2}$ is reflected in the y-axis.

15. Consider again the truncus defined by $y = \frac{1}{x^2}$. This hyperbola is reflected in the x-axis, dilated by a factor of 3 parallel to the y-axis or from the x-axis, and translated 1 unit to the left and up 2 units. Find the equation of the resulting hyperbola and give the equations of any asymptotes.

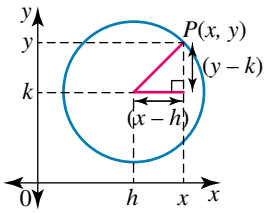
16. How could you summarise the effect of the transformations dealt with in this exercise on the shape of the basic hyperbola $y = \frac{1}{x}$?

13.8 The circle [Stage 5.3]

- A circle is the path traced out by a point at a constant distance (the radius) from a fixed point (the centre).
- Consider the circles shown below. The first circle has its centre at the origin and radius r.
  Let $P(x, y)$ be a point on the circle.
  By Pythagoras, $x^2 + y^2 = r^2$.
  This relationship is true for all points, P, on the circle.



- The equation of a circle with centre $(0, 0)$ and radius r is:
$$x^2 + y^2 = r^2$$
- If the circle is translated h units to the right, parallel to the x-axis, and k units upwards, parallel to the y-axis, then:



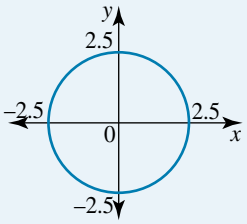
- The equation of a circle with centre (h, k) and radius r is:
$$(x - h)^2 + (y - k)^2 = r^2$$

DISCUSSION

How can you determine whether a particular point is inside, on or outside a given circle?

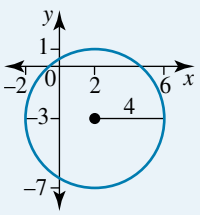
WORKED EXAMPLE 18

Sketch the graph of $4x^2 + 4y^2 = 25$, stating the centre and radius.

THINK	WRITE/DRAW
1 Express the equation in general form by dividing both sides by 4.	$x^2 + y^2 = r^2$ $4x^2 + 4y^2 = 25$ $x^2 + y^2 = \frac{25}{4}$
2 State the coordinates of the centre.	Centre $(0, 0)$
3 Find the length of the radius by taking the square root of both sides. (Ignore the negative results.)	$r^2 = \frac{25}{4}$ $r = \frac{5}{2}$ Radius = 2.5 units
4 Sketch the graph.	

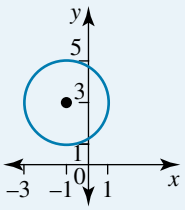
WORKED EXAMPLE 19

Sketch the graph of $(x - 2)^2 + (y + 3)^2 = 16$, clearly showing the centre and radius.

THINK	WRITE/DRAW
1 Express the equation in general form.	$(x - h)^2 + (y - k)^2 = r^2$ $(x - 2)^2 + (y + 3)^2 = 16$
2 State the coordinates of the centre.	Centre $(2, -3)$
3 State the length of the radius.	$r^2 = 16$ $r = 4$ Radius = 4 units
4 Sketch the graph.	

WORKED EXAMPLE 20

Sketch the graph of the circle $x^2 + 2x + y^2 - 6y + 6 = 0$.

THINK	WRITE/DRAW
1 Express the equation in general form by completing the square on the x terms and again on the y terms.	$(x - h)^2 + (y - k)^2 = r^2$ $x^2 + 2x + y^2 - 6y + 6 = 0$ $(x^2 + 2x + 1) - 1 + (y^2 - 6y + 9) - 9 + 6 = 0$ $(x + 1)^2 + (y - 3)^2 - 4 = 0$ $(x + 1)^2 + (y - 3)^2 = 4$
2 State the coordinates of the centre.	Centre $(-1, 3)$
3 State the length of the radius.	$r^2 = 4$ $r = 2$ Radius $= 2$ units
4 Sketch the graph.	

learnON RESOURCES — ONLINE ONLY

Interactivity: Graphs of circles (int-6156)

Interactivity: Compare and contrast types of graphs (int-3920)

Exercise 13.8 The circle

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–c, 2a–c, 3a–c, 4, 5, 7	Questions: 1c–e, 2c–e, 3c–e, 4–7, 10, 12	Questions: 1d–f, 2d–f, 3d–f, 4–11, 13

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE18 Sketch the graphs of the following circles, stating the centre and radius of each.

a. $x^2 + y^2 = 49$ **b.** $x^2 + y^2 = 4^2$ **c.** $x^2 + y^2 = 36$

d. $x^2 + y^2 = 81$ **e.** $2x^2 + 2y^2 = 50$ **f.** $9x^2 + 9y^2 = 100$

2. WE19 Sketch the graphs of the following circles, clearly showing the centre and the radius.

a. $(x - 1)^2 + (y - 2)^2 = 5^2$ **b.** $(x + 2)^2 + (y + 3)^2 = 6^2$ **c.** $(x + 3)^2 + (y - 1)^2 = 49$

d. $(x - 4)^2 + (y + 5)^2 = 64$ **e.** $x^2 + (y + 3)^2 = 4$ **f.** $(x - 5)^2 + y^2 = 100$

3. WE20 Sketch the graphs of the following circles.

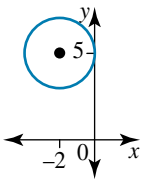
a. $x^2 + 4x + y^2 + 8y + 16 = 0$ **b.** $x^2 - 10x + y^2 - 2y + 10 = 0$

c. $x^2 - 14x + y^2 + 6y + 9 = 0$ **d.** $x^2 + 8x + y^2 - 12y - 12 = 0$

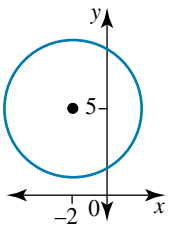
e. $x^2 + y^2 - 18y - 19 = 0$ **f.** $2x^2 - 4x + 2y^2 + 8y - 8 = 0$

4. MC The graph of $(x - 2)^2 + (y + 5)^2 = 4$ is:

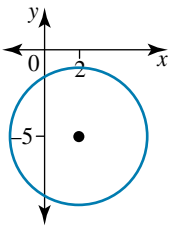
A.



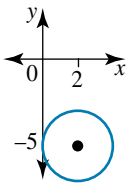
B.



C.



D.



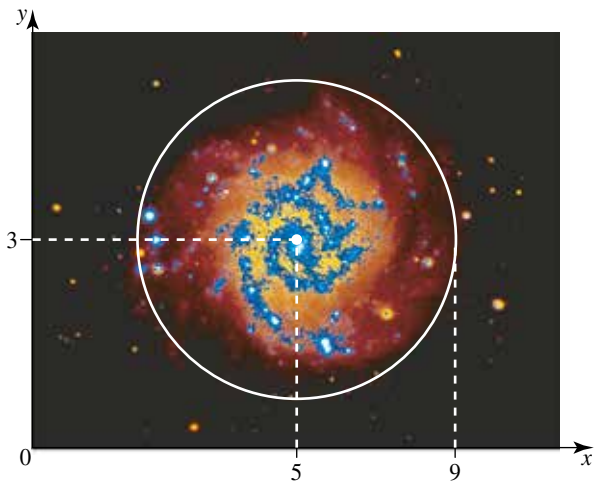
5. MC The centre and radius of the circle $(x + 1)^2 + (y - 3)^2 = 4$ is:

A. $(1, -3), 4$ **B.** $(-1, 3), 2$ **C.** $(3, -1), 4$

D. $(1, -3), 2$ **E.** $(1, 3), 2$

Communicating, reasoning and problem solving

6. Find the equation representing the outer edge of the galaxy as shown in the photo at below, using the astronomical units provided.



7. Circular ripples are formed when a water drop hits the surface of a pond.

If one ripple is represented by the equation $x^2 + y^2 = 4$ and then 3 seconds later by $x^2 + y^2 = 190$, where the length of measurements are in centimetres:

a. find the radius (in cm) of the ripple in each case

b. calculate how fast the ripple is moving outwards. (State your answers to 1 decimal place.)

8. Two circles with equations $x^2 + y^2 = 4$ and $(x - 1)^2 + y^2 = 9$ intersect. Determine the point(s) of intersection. Show your working.

9. a. Graph the line $y = x$, the parabola $y = x^2$ and the circle $x^2 + y^2 = 1$ on the one set of axes.

b. Find algebraically the points of intersection of:

i. the line and the circle

ii. the line and the parabola

iii. the parabola and the circle.

10. Find the point(s) of intersection of the circles $x^2 + y^2 - 2x - 2y - 2 = 0$ and $x^2 + y^2 - 8x - 2y + 16 = 0$ both algebraically and graphically.

11. The general equation of a circle is given by $x^2 + y^2 + ax + by + c = 0$. Find the equation of the circle which passes through the points $(4, 5)$, $(2, 3)$ and $(0, 5)$. State the centre of the circle and its radius.

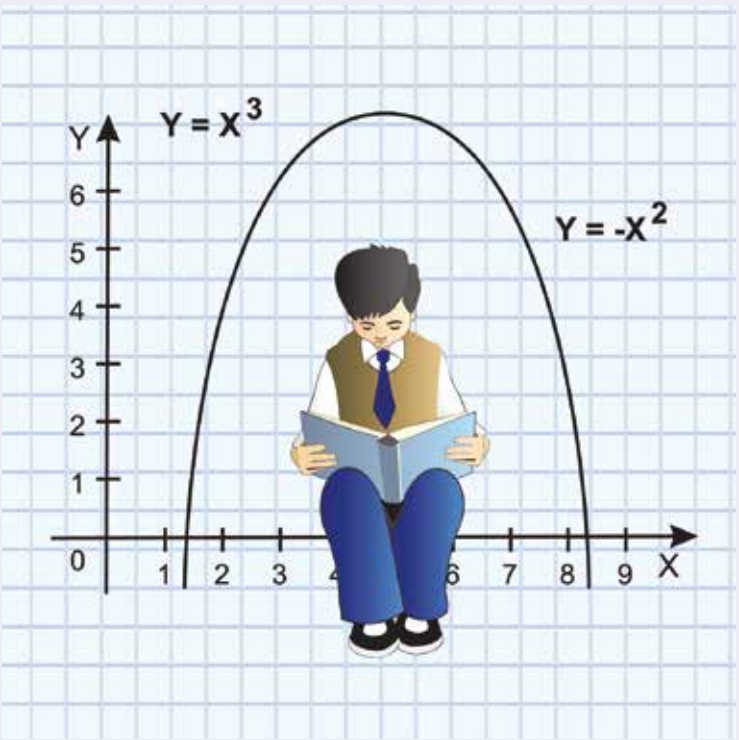
12. How could you write equations representing a set of concentric circles (circles with the same centre but different radii)?

13. Does the point $(1, 1.5)$ lie on, inside or outside the circle with equation $x^2 + 2x + y^2 - 6y + 6 = 0$?
Hint: It is important to know the length of the radius and the location of the centre of the circle.

13.9 Review

Investigation | Rich task

Parametric equations



You are familiar with the quadratic equation $y = x^2$ and its resulting graph. Let us consider an application of this equation by forming a relationship between x and y through a third variable, say, t.

$$x = t \text{ and } y = t^2$$

It is obvious that these two equations are equivalent to the equation $y = x^2$. This third variable t is known as a parameter, and the two equations are now called parametric equations. We cannot automatically assume that the resulting graph of these two parametric equations is the same as that of $y = x^2$ for all real values of x. It is dependent on the range of values of t.

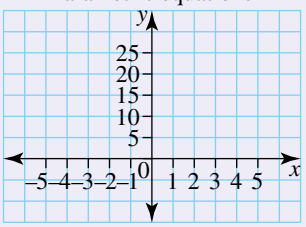
For questions 1 to 3, consider the parametric equations $x = t$ and $y = t^2$ for values of the parameter $t \geq 0$.

1. Complete the following table by calculating x- and y-values from the corresponding t-value.

t	x	y
0		
1		
2		
3		
4		
5		

2. Copy this Cartesian plane in your workbook and graph the x-values and corresponding y-values. Join the points with a smooth curve and place an arrow on the curve to indicate the direction of increasing t-values.

Parametric equations

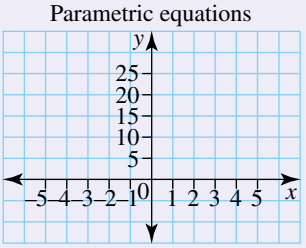


3. Is there any difference between this graph and that of $y = x^2$? Explain your answer.

4. Consider now the parametric equations $x = 1 - t$ and $y = (1 - t)^2$. These are clearly also equivalent to the equation $y = x^2$. Complete the table and draw the graph of these two equations for values of the parameter $t \geq 0$. Draw an arrow on the curve in the direction of increasing t-values.

t	x	y
0		
1		
2		
3		
4		
5		

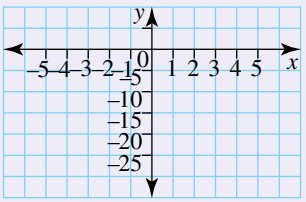
Parametric equations



Describe the shape of your resulting graph. What values of the parameter t would produce the same curve as that obtained in question **2**?

5. The graph of $y = -x^2$ is a reflection of $y = x^2$ in the x-axis. Construct a table and draw the graph of the parametric equations $x = t$ and $y = -t^2$ for parameter values $t \geq 0$. Remember to place an arrow on the curve in the direction of increasing t-values.

Parametric equations
$x = t$ and $y = -t^2$



6. Without constructing a table, predict the shape of the graph of the parametric equations $x = 1 - t$ and $y = -(1 - t)^2$ for parameter values $t \geq 0$. Draw a sketch of the shape.

7. This task requires you to produce the shape of the parabola $y = x^2$ in the range $-2 \leq x \leq 2$ by considering two different parametric equations to those already considered.
a. State your two equations and the range of the parameter values.
b. Construct a table showing calculated values.
c. Draw a sketch of the graph.

learnON RESOURCES — ONLINE ONLY

Digital doc: Investigation — Parametric equations (doc-15932)
Digital doc: Code puzzle — Olympic games facts (doc-15933)
Digital doc: Summary Topic 13 (doc-22911)
Interactivity: Word search Topic 13 (int-2850)
Interactivity: Crossword Topic 13 (int-2851)
Interactivity: Sudoku Topic 13 (int-3596)

Exercise 13.9 Review questions

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. MC The vertex for the graph $y = 3x^2 - 4x + 9$ is:

A. $\left(\frac{1}{3}, 1\frac{2}{3}\right)$ **B.** $\left(\frac{1}{3}, \frac{2}{3}\right)$ **C.** $\left(\frac{1}{6}, 1\frac{1}{6}\right)$ **D.** $\left(\frac{2}{3}, 7\frac{2}{3}\right)$ **E.** $\left(\frac{2}{3}, 6\frac{2}{3}\right)$

2. MC Which graph of the following equations has the x-intercepts closest together?

A. $y = x^2 + 3x + 2$ **B.** $y = x^2 + x - 2$ **C.** $y = 2x^2 + x - 15$
D. $y = 4x^2 + 27x - 7$ **E.** $y = x^2 - 2x - 8$

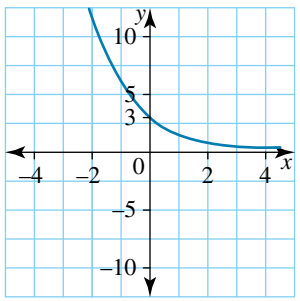
3. MC Which graph of the equations below has the largest y-intercept?

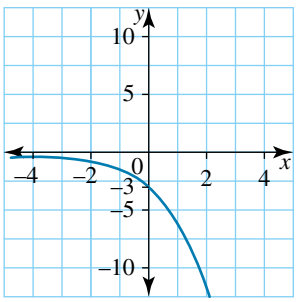
A. $y = 3(x - 2)^2 + 9$ **B.** $y = 5(x - 1)^2 + 8$ **C.** $y = 2(x - 1)^2 + 19$
D. $y = 2(x - 5)^2 + 4$ **E.** $y = 12(x - 1)^2 + 10$

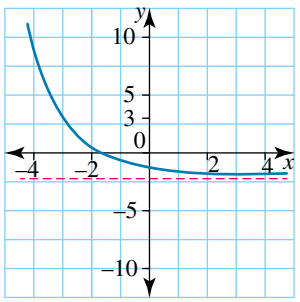
4. MC The translation required to change $y = x^2$ into $y = (x - 3)^2 + \frac{1}{4}$ is:

A. right 3, up $\frac{1}{4}$ **B.** right 3, down $\frac{1}{4}$ **C.** left 3, down $\frac{1}{4}$
D. left 3, up $\frac{1}{4}$ **E.** right $\frac{1}{4}$, up 3

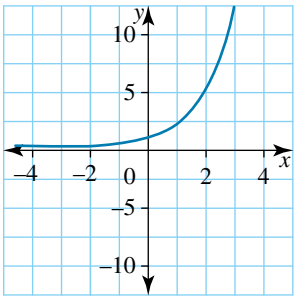
5. MC The graph of $y = -3 \times 2^x$ is best represented by:

A.


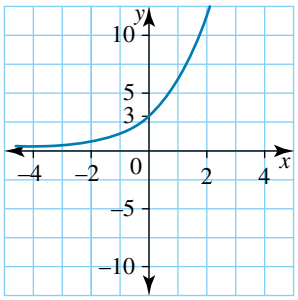
B.


C.


D.



E.



6. Use the completing the square method to find the vertex for each of the following graphs.
a. $y = x^2 - 8x + 1$ **b.** $y = x^2 + 4x - 5$

7. For the graph of the equation $y = x^2 + 8x + 7$, produce a table of values for the x-values between –9 and 1, and then plot the graph. Show the y-intercept and vertex. From your graph, state the x-intercepts.

8. For each of the following, find the coordinates of the vertex and the x- and y-intercepts and sketch the graph.
a. $y = (x - 3)^2 + 1$ **b.** $y = 2(x + 1)^2 - 5$

9. For the equation $y = -x^2 - 2x + 15$, sketch the graph and determine the x- and y-intercepts and the coordinates of the vertex.

10. For the exponential function $y = 5^x$:
a. complete the table of values below

x	y
–3	
–2	
–1	
0	
1	
2	
3	

b. plot the graph.

11. Draw the graph of $y = 10 \times 3^x$ for $-4 \leq x \leq 4$.

12. Draw the graph of $y = 10^{-x}$ for $-4 \leq x \leq 4$.

13. a. On the same axes draw the graphs of $y = (1.2)^x$ and $y = (1.5)^x$.
b. Use your answer to part **a** to explain the effect of changing the value of a in the equation of $y = a^x$.

14. a. On the one set of axes draw the graphs of $y = 2 \times 3^x$, $y = 5 \times 3^x$ and $y = \frac{1}{2} \times 3^x$.
b. Use your answer to part **a** to explain the effect of changing the value of k in the equation of $y = ka^x$.

15. a. On the same set of axes sketch the graphs of $y = (2.5)^x$ and $y = (2.5)^{-x}$.
b. Use your answer to part **a** to explain the effect of a negative index on the equation $y = a^x$.

16. Sketch each of the following.
a. $y = \frac{4}{x}$ **b.** $y = -\frac{2}{x}$

17. Sketch $y = \frac{-3}{x - 2}$.

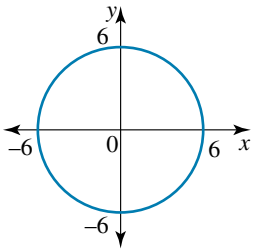
18. Give an example of an equation of a hyperbola that has a vertical asymptote at $x = -3$.

19. Sketch each of these circles. Clearly show the centre and the radius.
a. $x^2 + y^2 = 16$ **b.** $(x - 5)^2 + (y + 3)^2 = 64$

20. Sketch the following circles. Remember to first complete the square.

a. $x^2 + 4x + y^2 - 2y = 4$ **b.** $x^2 + 8x + y^2 + 8y = 32$

21. Find the equation of this circle.



Communicating, reasoning and problem solving

22. The height, h, in metres of a golf ball t seconds after it is hit is given by the formula $h = 4t - t^2$.

a. Sketch the graph of the path of the ball.

b. What is the maximum height the golf ball reaches?

c. How long does it take for the ball to reach the maximum height?

d. How long is it before the ball lands on the ground after it has been hit?

23. A soccer ball is kicked upwards in the air. The height, h, in metres, t seconds after the kick is modelled by the quadratic equation $h = -5t^2 + 20t$.

a. Sketch the graph of this relationship.

b. For how many seconds is the ball in the air?

c. For how many seconds is the ball above a height of 15 m? That is, solve the quadratic inequation $-5t^2 + 20t > 15$.

d. For how many seconds is the ball above a height of 20 m?

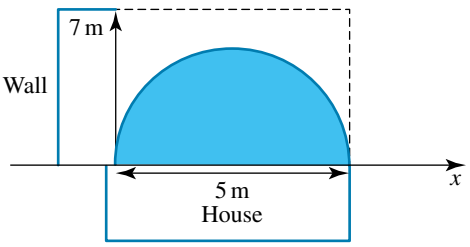
24. The height of the water level in a cave is determined by the tides. At any time, t, in hours after 9 am, the height, $h(t)$, in metres, can be modelled by the function $h(t) = t^2 - 12t + 32$, $0 \leq t \leq 12$.

a. What values of t is the model valid for? Write your answer in interval notation.

b. Determine the initial height of the water.

c. Bertha has dropped her keys onto a ledge that is 7 metres from the bottom of the cave. Determine the times in which she would be able to climb down to retrieve her keys. Write your answers correct to the nearest minute.

25. A grassed area is planted in a courtyard that has a width of 5 metres and length of 7 metres. The shape of the grassed area is described by the function $P = -x^2 + 5x$, where P is the distance, in metres, from the house and x is the distance, in metres from the side wall. The diagram represents this information on a Cartesian plane.



a. In terms of P, write down an inequality that describes the region where the grass has been planted.

b. Determine the maximum distance the grass area is planted from the house.

c. The owners of the house have decided that they would prefer all of the grass to be within a maximum distance of 3.5 metres from the house. The shape of the lawn following this design can be described by the equation

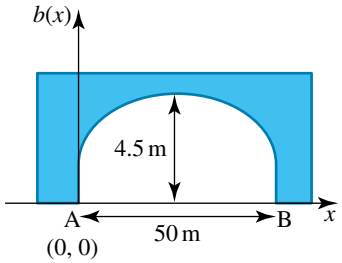
$$N(x) = ax^2 + bx + c.$$

i. Using algebra, show that this new design can be described by the function $N(x) = -0.56x(x - 5)$.

ii. Describe the transformation that maps $P(x)$ to $N(x)$.

d. If the owners decide on the first design, $P(x)$, the percentage of area within the courtyard without grass is 40.5%. By using any method, find the approximate percentage of area of courtyard without lawn with the new design, $N(x)$.

26. A stone arch bridge has a span of 50 metres. The shape of the curve AB can be modelled using a quadratic equation.



a. Taking A as the origin, $(0, 0)$, and given that the maximum height of the arch above the water level is 4.5 metres, show using algebra, that the shape of the arch can be modelled using the equation $b(x) = -0.0072x^2 + 0.36x$, where $b(x)$ is the vertical height of the bridge, in metres, and x is the horizontal distance, in metres.

b. A floating platform p metres high is towed under the bridge. Given that the platform needs to have a clearance of at least 30 centimetres on each side, explain why the maximum value of p is 10.7 centimetres.

27. When a drop of water hits the flat surface of a pool, circular ripples are made. One ripple is represented by the equation $x^2 + y^2 = 9$ and 5 seconds later, the ripple is represented by the equation $x^2 + y^2 = 225$, where the lengths of the radii are in cm.

a. State the radius of each of the ripples.

b. Sketch these graphs.

c. How fast is the ripple moving outwards?

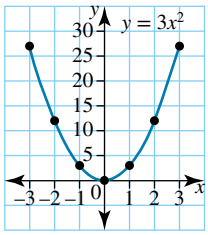
d. If the ripple continues to move at the same rate, when will it hit the edge of the pool which is 2 m from its centre?

Answers

Topic 13 Non-linear relationships

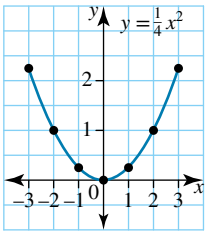
Exercise 13.2 Plotting parabolas

1. a.



$x = 0, (0,\ 0)$

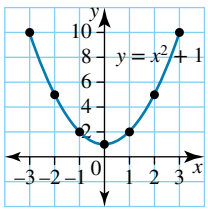
b.



$x = 0, (0,\ 0)$

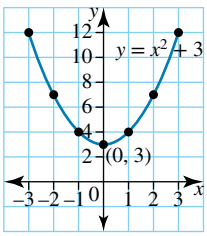
2. Placing a number greater than 1 in front of x^2 makes the graph thinner. Placing a number greater than 0 but less than 1 in front of x^2 makes the graph wider.

3. a.



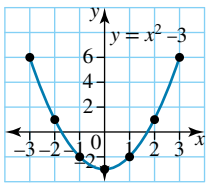
$x = 0, (0, 1), 1$

b.



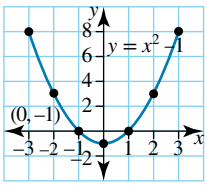
$x = 0, (0, 3), 3$

c.



$x = 0, (0,\ -3), -3$

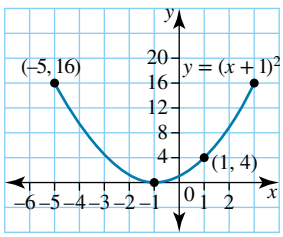
d.



$x = 0, (0, -1), -1$

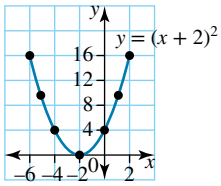
4. Adding a number raises the graph of $y = x^2$ vertically that number of units. Subtracting a number lowers the graph of $y = x^2$ vertically that number of units.

5. a.



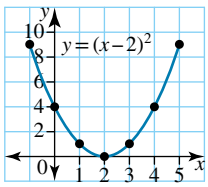
$x = -1, (-1, 0), 1$

b.



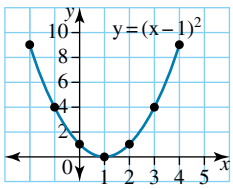
$x = -2,\ (-2, 0), 4$

c.



$x = 2, (2, 0), 4$

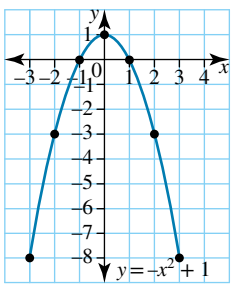
d.



$x = 1, (1, 0), 1$

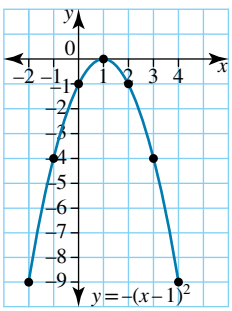
6. Adding a number to x moves the graph of $y = x^2$ horizontally to the left by that number of units. Subtracting a number from x moves the graph of $y = x^2$ horizontally to the right by that number of units.

7. a.



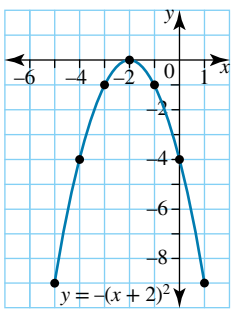
$x = 0, (0, 1), 1$

b.



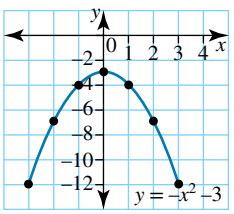
$x = 1, (1, 0), -1$

c.



$x = -2, (-2, 0), -4$

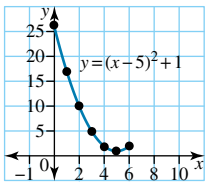
d.



$x = 0,\ (0, -3), -3$

8. The negative sign inverts the graph of $y = x^2$. The graphs with the same vertices are: $y = x^2 + 1$ and $y = 2x^2 + 1$; $y = (x - 1)^2$ and $y = -(x - 1)^2$; $y = (x + 2)$ and $y = -(x + 2)^2$; $y = x^2 - 3$ and $y = -x^2 - 3$.

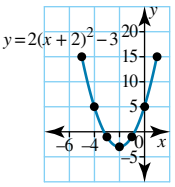
They differ in that the first graph is upright while the second graph is inverted.

9. a. 

i. $x = 5$

ii. (5, 1), minimum

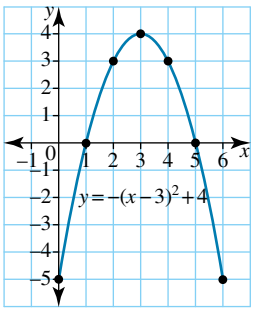
iii. 26

b. 

i. $x = -2$

ii. $(-2, -3)$, minimum

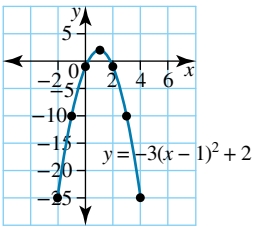
iii. 5

c. 

i. $x = 3$

ii. (3, 4), maximum

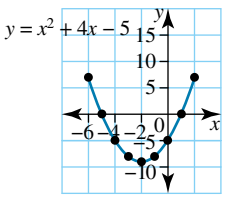
iii. -5

d. 

i. $x = 1$

ii. (1, 2), maximum

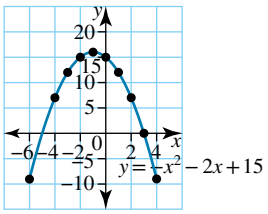
iii. -1

e. 

i. $x = -2$

ii. $(-2, -9)$, minimum

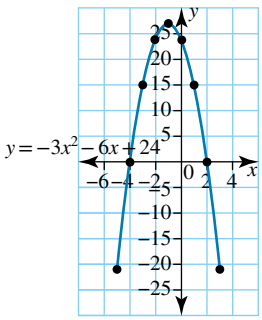
iii. -5

f. 

i. $x = -1$

ii. $(-1, 16)$, maximum

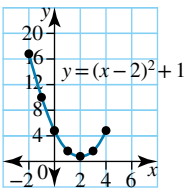
iii. 15

g. 

i. $x = 1$

ii. $(-1, 27)$, maximum

iii. 24

h. 

i. $x = 2$

ii. (2, 1), minimum

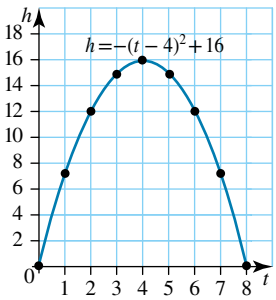
iii. 5

10. a. If the x^2 term is positive, the parabola has a minimum vertex. If the x^2 term is negative, the parabola has a maximum vertex.

b. If the equation is of the form $y = a(x - b)^2 + c$, the vertex has coordinates (b, c).

c. The equation of the axis of symmetry can be found from the x-coordinate of the vertex. That is, $x = b$.

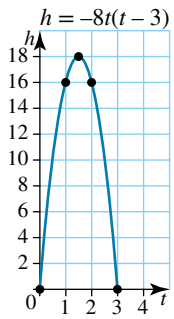
11. C **12.** B **13.** C **14.** A

15. a. 

b. i. 16 m

ii. 8 s

16. a.



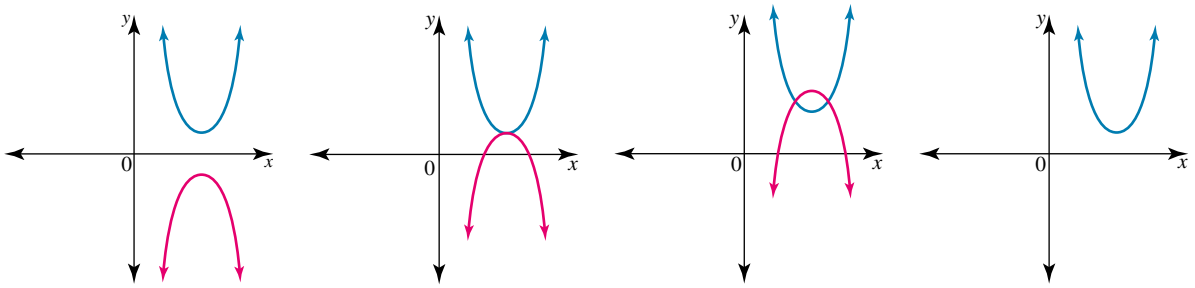
b. **i.** 18 m

ii. Yes, by 3 m

iii. 1.5 s

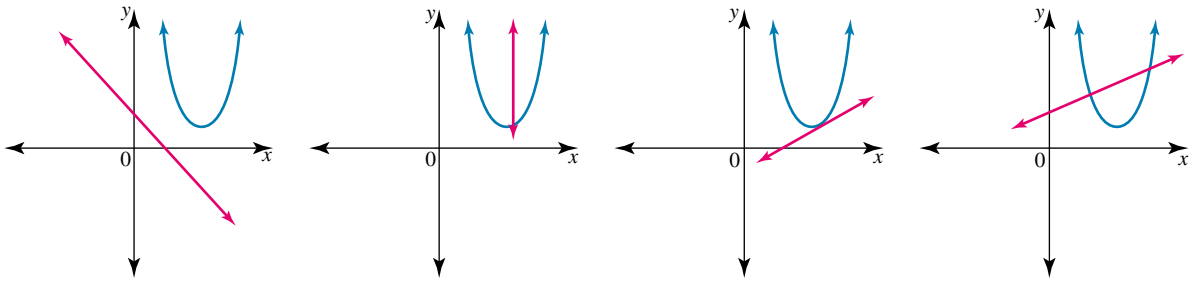
iv. 3 s

17. a.



b. An infinite number of points of intersection occur when the two equations represent the same parabola, with the effect that the two parabolas superimpose. For example $y = x^2 + 4x + 3$ and $2y = 2x^2 + 8x + 6$.

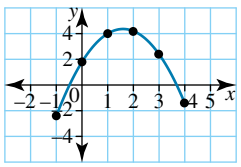
c. It is possible to have 0, 1 or 2 points of intersection.



18. a.

x	−1	0	1	2	3	4
y	−2.4	1.8	4	4.2	2.4	−1.4

b.



c. x cannot equal −1 as this would put the ball behind her; at $x = 4$, the ball is under ground level.

d. The maximum height reached is 4.36 m.

e. The ball will hit the ground 3.688 m from the player.

19. $y = 2x^2 – 3x + 1$

20. A parabola is defined for all x values, unless the x values are restricted for a particular parabola. The y values will be dependent on the x values.

Exercise 13.3 Sketching parabolas

1. a. Narrower, vertex (0, 0)
b. Wider, (0, 0)
c. Narrower, vertex (0, 0)
d. Narrower, vertex (0, 0)
e. Wider, vertex (0, 0)
f. Wider, vertex (0, 0)
g. Narrower, vertex (0, 0)
h. Narrower, vertex (0, 0)

2. a. Vertical 3 up, vertex (0, 3)
b. Vertical 1 down, vertex (0, −1)
c. Vertical 7 down, vertex (0, −7)
d. Vertical up, vertex $\left(0, \frac{1}{4}\right)$
e. Vertical down, vertex $\left(0, -\frac{1}{2}\right)$
f. Vertical 0.14 down, vertex (0, −0.14)
g. Vertical 2.37 up, vertex (0, 2.37)
h. Vertical $\sqrt{3}$ up, vertex $(0, \sqrt{3})$

3. a. Horizontal 1 right, (1, 0)
b. Horizontal 2 right, (2, 0)
c. Horizontal 10 left, (−10, 0)
d. Horizontal 4 left, (−4, 0)
e. Horizontal $\frac{1}{2}$ right, $\left(\frac{1}{2}, 0\right)$
f. Horizontal $\frac{1}{5}$ left, $\left(-\frac{1}{5}, 0\right)$
g. Horizontal 0.25 left, (−0.25, 0)
h. Horizontal $\sqrt{3}$ left, $(-\sqrt{3}, 0)$

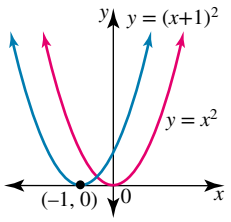
4. a. (0, 1), maximum **b.** (0, –3), minimum **c.** (–2, 0), maximum **d.** (0, 0), minimum
e. (0, 4), maximum **f.** (0, 0), maximum **g.** (5, 0), minimum **h.** (0, 1), minimum

5. a. Narrower, minimum **b.** Narrower, maximum **c.** Wider, minimum **d.** Wider, maximum
e. Narrower, maximum **f.** Wider, minimum **g.** Narrower, minimum **h.** Wider, maximum

6. a. i. Horizontal translation 1 left

ii. (–1, 0)

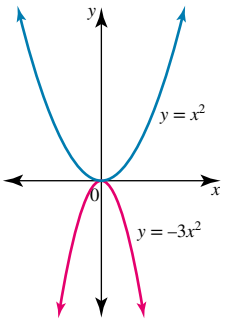
iii.



b. i. Reflected, narrower (dilation)

ii. (0, 0)

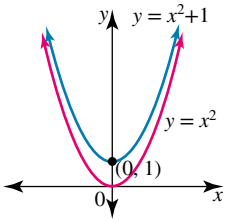
iii.



c. i. Vertical translation 1 up

ii. (0, 1)

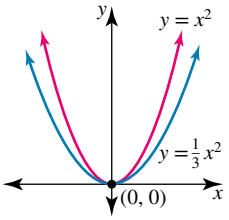
iii.



d. i. Wider (dilation)

ii. (0, 0)

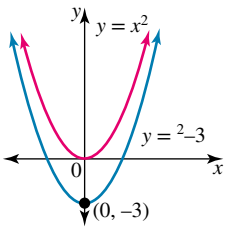
iii.



e. i. Vertical translation 3 down

ii. (0, –3)

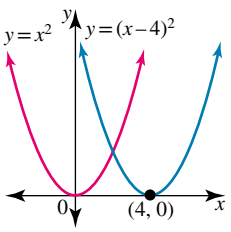
iii.



f. i. Horizontal translation 4 right

ii. (4, 0)

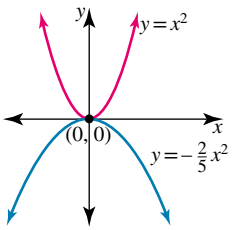
iii.



g. i. Reflected, wider (dilation)

ii. (0, 0)

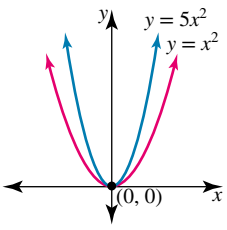
iii.



h. i. Narrower (dilation)

ii. (0, 0)

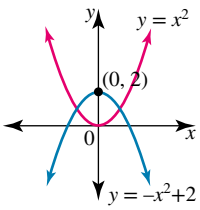
iii.



i. i. Reflected, vertical translation 2 up

ii. (0, 2)

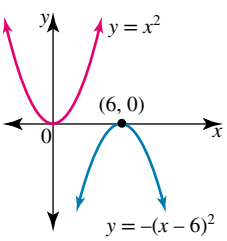
iii.



j. i. Reflected, horizontal translation 6 right

ii. (6, 0)

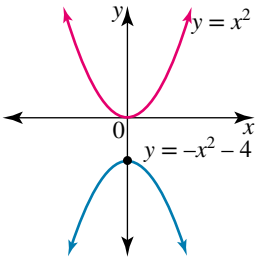
iii.



k. **i.** Reflected, vertical translation 4 down

ii. (0, −4)

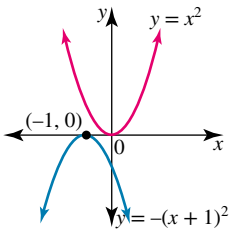
iii.



l. **i.** Reflected, horizontal translation 1 left

ii. (−1, 0)

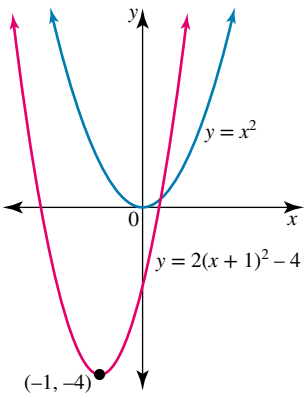
iii.



m. **i.** Narrower (dilation), horizontal translation 1 left, vertical translation 4 down

ii. (−1, −4)

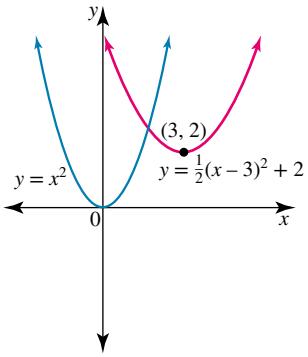
iii.



n. **i.** Wider (dilation), horizontal translation 3 right, vertical translation 2 up

ii. (3, 2)

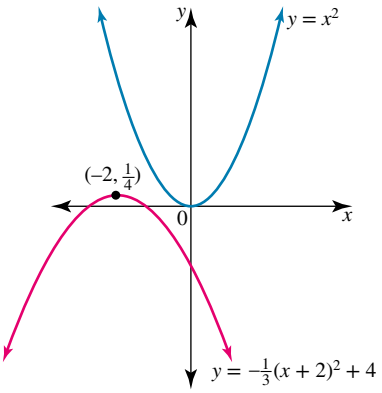
iii.



o. **i.** Wider (dilation), reflected, horizontal translation 2 left, vertical translation $\frac{1}{4}$ up

ii. $\left(-2, \frac{1}{4}\right)$

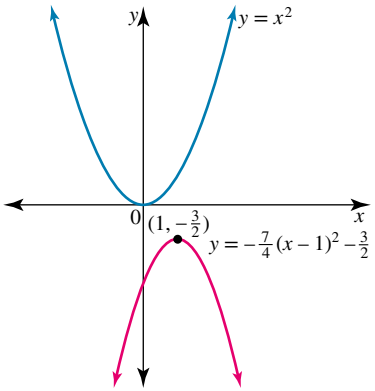
iii.



p. **i.** Narrower (dilation), reflected, horizontal translation 1 right, vertical translation $\frac{3}{2}$ down

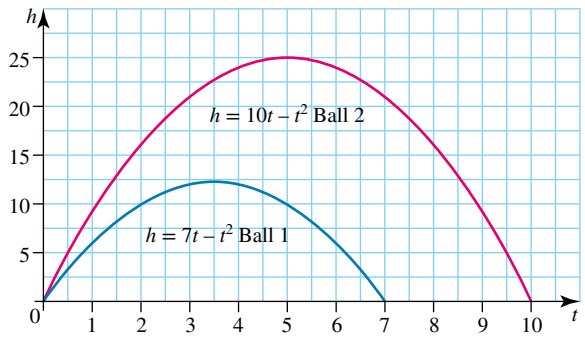
ii. $\left(1, -\frac{3}{2}\right)$

iii.



7. **a.** 10 cm **b.** 5 cm **c.** 5 cm **d.** $y = (x - 5)^2$

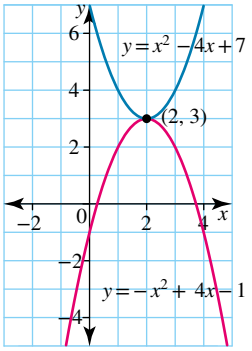
8. **a.** and **d.**



b. 12.25 m **c.** $h = t(10 - t)$ **d.** 12.75 m

9. a. $y = -(x - 2)^2 + 3 = -x^2 + 4x - 1$

b.

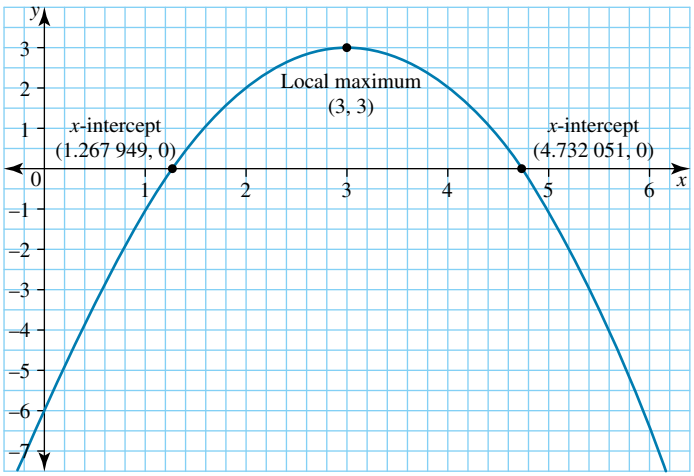


10. a. $Y = -(x - 3)^2 + 3$

b. Reflected in x-axis, translated 3 units to the right and up 3 units. No dilation.

c. (3, 3)

d.



11. $(0, k)$ and $(-h, 0)$

12. $v = -2h^2 + 40h$, where v is the vertical distance and h is the horizontal distance.

Exercise 13.4 Sketching parabolas of the form $y = a(x - h)^2 + k$

1. a. (1, 2), minimum **b.** (−2, −1), minimum **c.** (−1, 1), minimum

d. (2, 3), maximum **e.** (5, 3), maximum **f.** (−2, −6), minimum

g. $\left(\frac{1}{2}, \frac{3}{4}\right)$, minimum **h.** $\left(\frac{1}{3}, \frac{2}{3}\right)$, minimum **i.** (−0.3, −0.4), minimum

2. a. i. (–3, –5) **ii.** Minimum **iii.** Narrower

b. i. (1, 1) **ii.** Maximum **iii.** Same

c. i. (–2, –4) **ii.** Maximum **iii.** Narrower

d. i. (3, 2) **ii.** Minimum **iii.** Wider

e. i. (–1, 7) **ii.** Maximum **iii.** Wider

f. i. $\left(\frac{1}{5}, -\frac{1}{2}\right)$ **ii.** Minimum **iii.** Wider

3. a. ii, $y = -(x - 2)^2 + 3$ **b.** v, $y = -x^2 + 1$ **c.** vi, $y = (x + 1)^2 - 3$

d. iv, $y = -(x + 2)^2 + 3$ **e.** iii, $y = x^2 - 1$ **f.** i, $y = (x - 1)^2 - 3$

4. a. A **b.** C **c.** B **d.** C **e.** B

5. a. i. −3 **ii.** −3, 1

b. i. 12 **ii.** 2 **c. i.** −18 **ii.** No x-intercepts

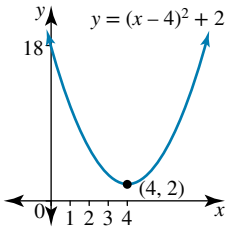
d. i. −5 **ii.** −1, 5

e. i. 4 **ii.** No x-intercepts

f. i. 4 **ii.** $-3 - \sqrt{5}$, $-3 + \sqrt{5}$ (approx. −5.24, −0.76)

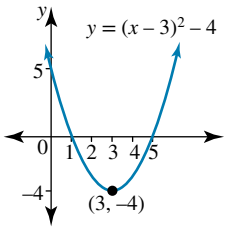
6. a. **i.** (4, 2) **ii.** Minimum **iii.** Same width

iv. 18 **v.** No x-intercepts **vi.**



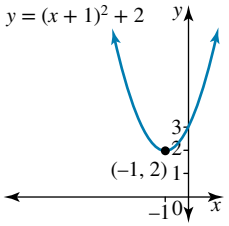
b. **i.** (3, –4) **ii.** Minimum **iii.** Same width

iv. 5 **v.** 1, 5 **vi.**



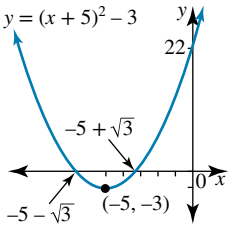
c. **i.** (–1, 2) **ii.** Minimum **iii.** Same width

iv. 3 **v.** No x-intercepts **vi.**



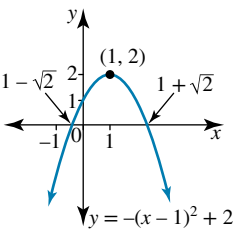
d. **i.** (–5, –3) **ii.** Minimum **iii.** Same width

iv. 22 **v.** $-5 - \sqrt{3}, -5 + \sqrt{3}$ (approx. −6.73, −3.27) **vi.**



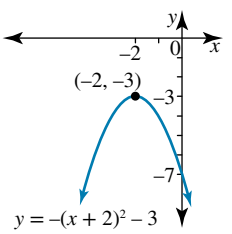
e. **i.** (1, 2) **ii.** Maximum **iii.** Same width

iv. 1 **v.** $1 - \sqrt{2}, 1 + \sqrt{2}$ (approx. −0.41, 2.41) **vi.**



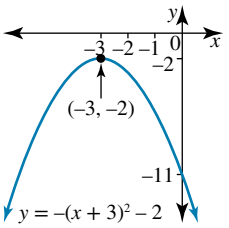
f. **i.** (–2, –3) **ii.** Maximum **iii.** Same width

iv. –7 **v.** No x-intercepts **vi.**



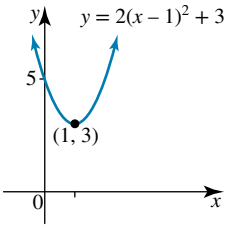
g. **i.** $(-3, -2)$ **ii.** Maximum **iii.** Same width

iv. -11 **v.** No x-intercepts **vi.**



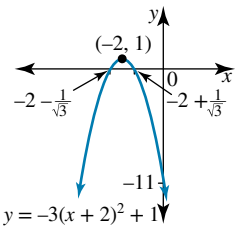
h. **i.** $(1, 3)$ **ii.** Minimum **iii.** Narrower

iv. 5 **v.** No x-intercepts **vi.**



i. **i.** $(-2, 1)$ **ii.** Maximum **iii.** Narrower

iv. -11 **v.** $-2 - \frac{1}{\sqrt{3}}, -2 + \frac{1}{\sqrt{3}}$ (approx. $-2.58, -1.42$) **vi.**



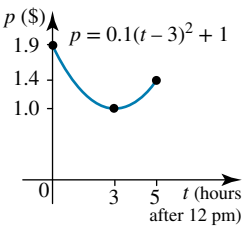
7. **a.** $2\left(x - \frac{3}{4}\right)^2 - \frac{73}{8} = 0$ **b.** $x = \frac{3}{4} \pm \frac{\sqrt{73}}{4}$ **c.** $\left(\frac{3}{4}, -\frac{73}{8}\right)$, minimum

8. **a.** $y = -\frac{2}{3}(x + 4)^2 + 6$ **b.** $(-7, 0)$

9. **a.** $y = -x^2$ **b.** $y = 7x^2$ **c.** $y = (x + 3)^2$

d. $y = x^2 + 3$ **e.** $y = -\frac{1}{4}(x - 5)^2 - 3$

10. **a.**



b. \$1.90 **c.** \$1

d. 3 pm **e.** \$1.40

11. **a.** 0.5 m

b. $(15 + 4\sqrt{15})$ m

c. The maximum height is 8 metres when horizontal distance is 15 metres.

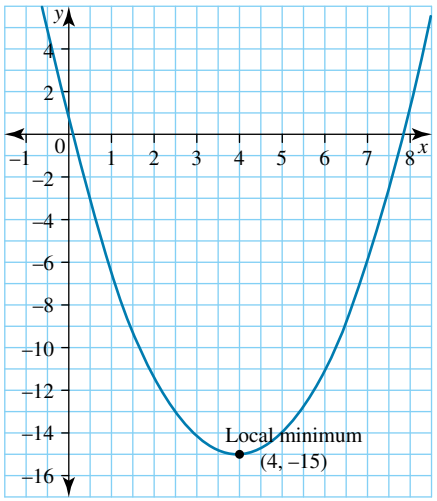
12. **a.** Answers will vary. An example is $y = (x - 2)^2 + 6$.

b. $y = -\frac{1}{2}(x - 2)^2 + 6$

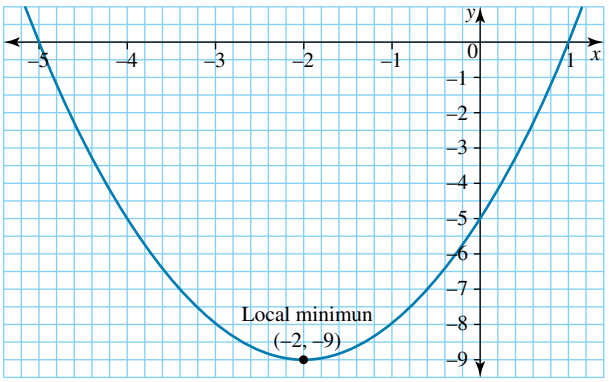
13. **a.** Answers will vary. An example is $y = (x - p)^2 + q$.

b. $y = \left(\frac{r - q}{p^2}\right)(x - p)^2 + q$

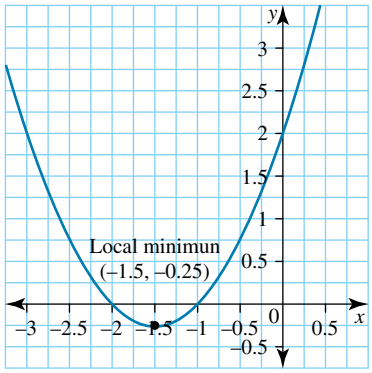
14. a. $y = (x - 4)^2 - 15$



b. $y = (x + 2)^2 - 9$



c. $y = \left(x + \frac{3}{2}\right)^2 - \frac{1}{4}$

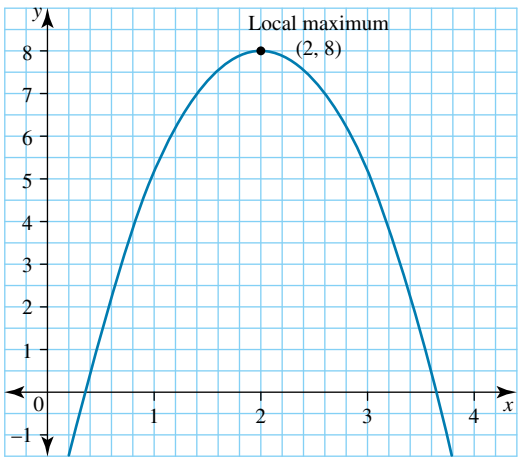


15. a. $y = 3(x - 2)^2 - 8$

b. The parabola has been dilated by a factor of 3 parallel to the y-axis or from the x-axis as well as being translated 2 units to the right and down 8 units.

c. $y = -3(x - 2)^2 + 8$

d.

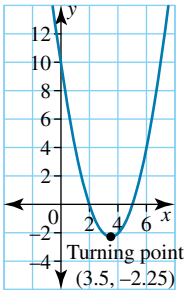


16. No, it does not.

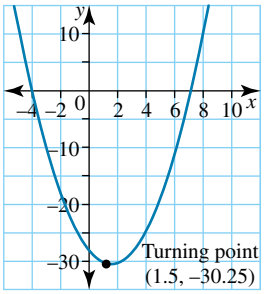
Exercise 13.5 Sketching parabolas of the form $y = ax^2 + bx + c$

1. You need the x-intercepts, the y-intercept and the vertex to sketch a parabola.

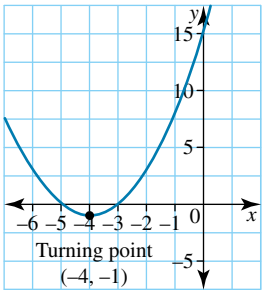
2. a. $y = (x - 5)(x - 2)$



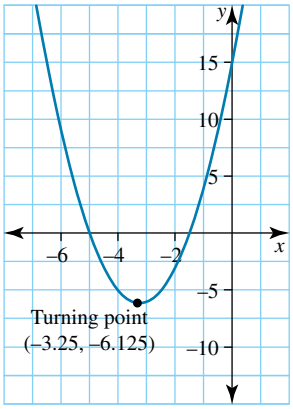
b. $y = (x + 4)(x - 7)$



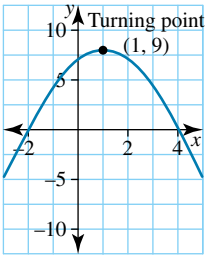
c. $y = (x + 3)(x + 5)$



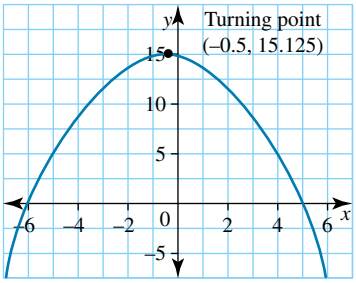
d. $y = (2x + 3)(x + 5)$



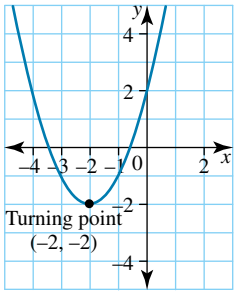
e. $y = (4 - x)(x + 2)$



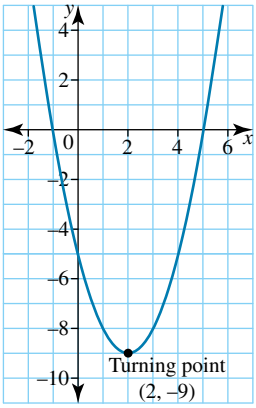
f. $y = \left(\frac{x}{2} + 3\right)(5 - x)$



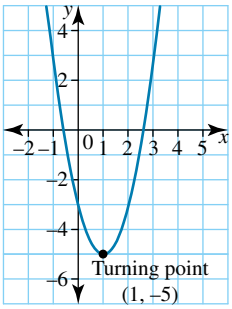
3. a. $y = x^2 + 4x + 2$



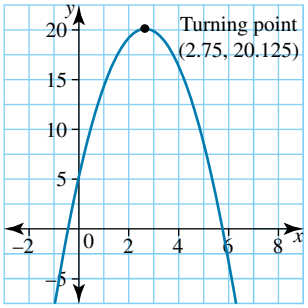
b. $y = x^2 - 4x - 5$



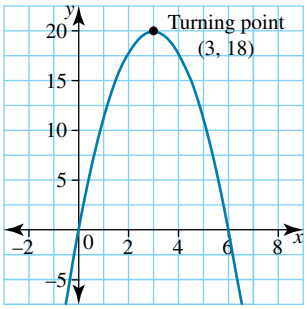
c. $y = 2x^2 - 4x - 3$



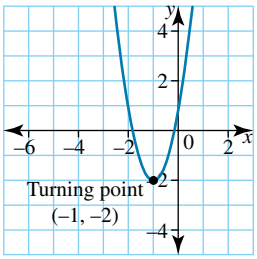
d. $y = -2x^2 - 11x + 5$



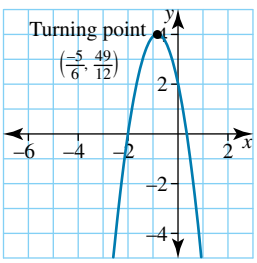
e. $y = -2x^2 + 12x$



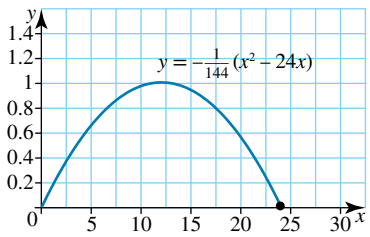
f. $y = 3x^2 + 6x + 1$



g. $y = -3x^2 - 5x + 2$

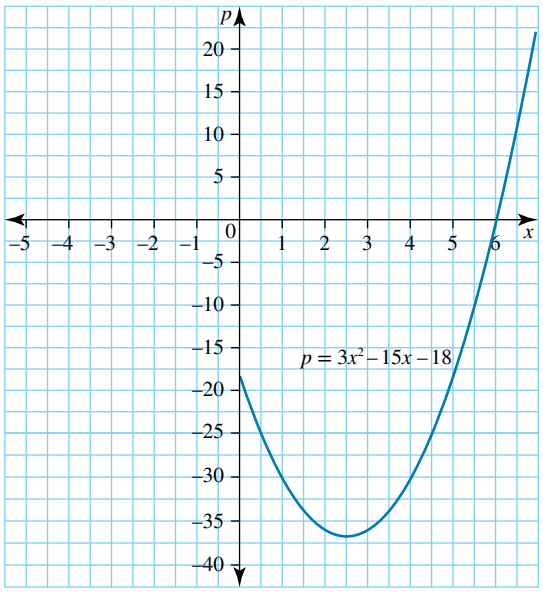


4. a.



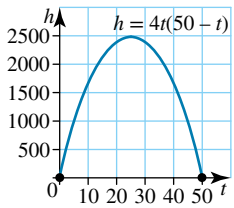
b. 24 m **c.** 1 m

5. a.



b. 7th month **c.** 8th month

6. a.

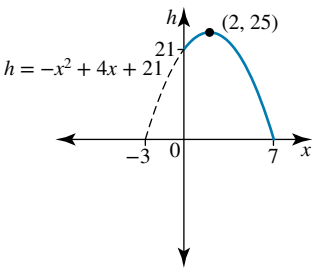


b. 0 **c.** 2500 m

d. 25 seconds **e.** 50 seconds

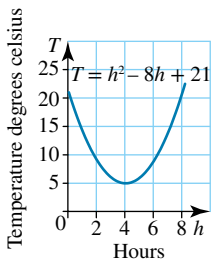
7. 200 **8.** $-(m + n)$ **9.** Answers will vary.

10. a.



b. 25 m **c.** 2 m **d.** 7 m

11. a.



b. 21°C **c.** Decreasing

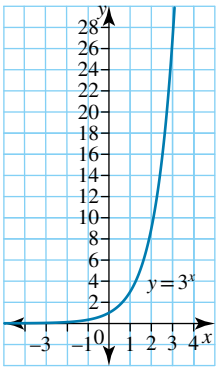
d. Increasing **e.** 5 °C after 4 hours **f.** 21 °C

12. One way to remember the information necessary to sketch a parabola is to construct a table with the different forms of a parabolic equation and the information that can be obtained from each form.

Exercise 13.6 Exponential functions and graphs

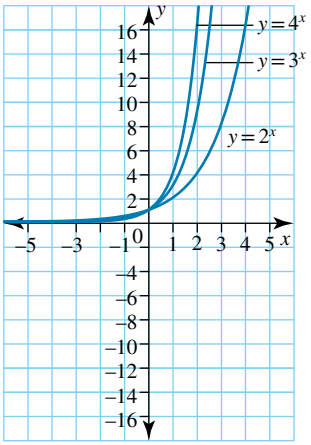
1.

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27



2. a. 2 **e.** 3 **f.** 4 **g.** 10 **h.** a

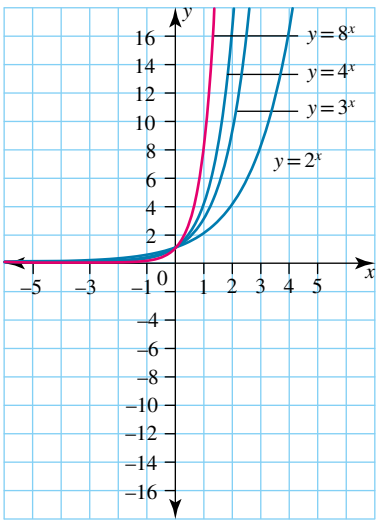
3.



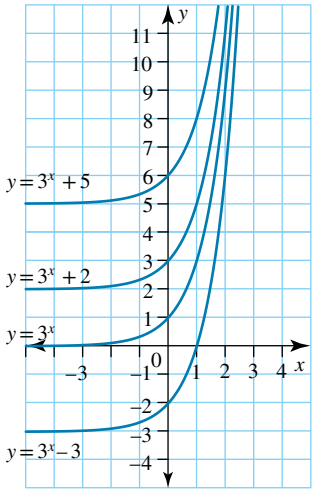
a. The graphs all pass through $(0, 1)$. The graphs have the same horizontal asymptote ($y = 0$). The graphs are all very steep.

b. As the base grows larger, the graphs become steeper.

c.



4.

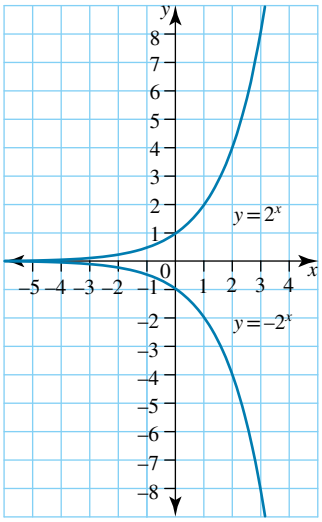


a. The shape of each graph is the same.

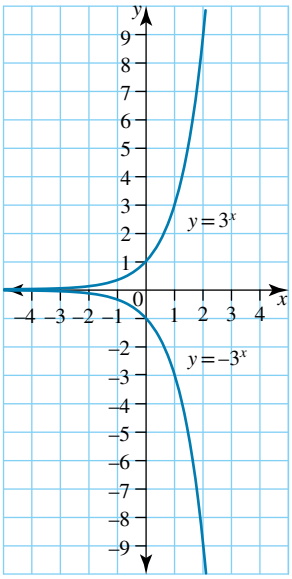
b. Each graph has a different y-intercept and a different horizontal asymptote.

c. i. (0, 11) **ii.** $y = 10$

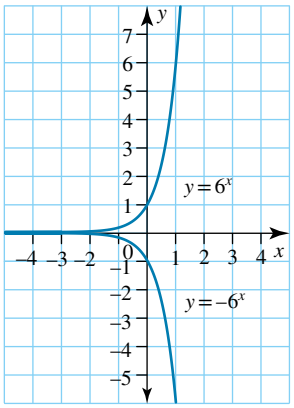
5. a. i.



ii.

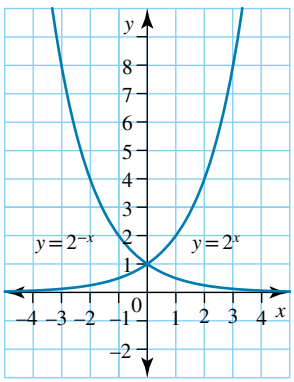


iii.

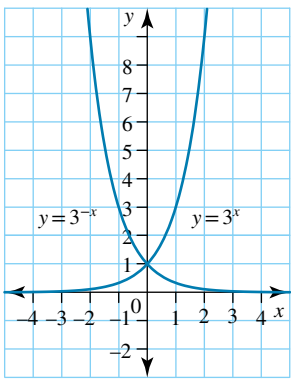


b. In each case the graphs are symmetric about the x-axis.

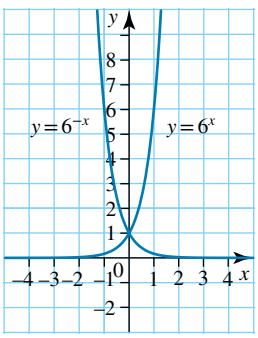
6. a. i.



ii.

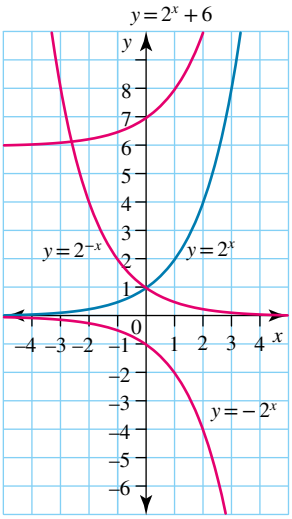


iii.

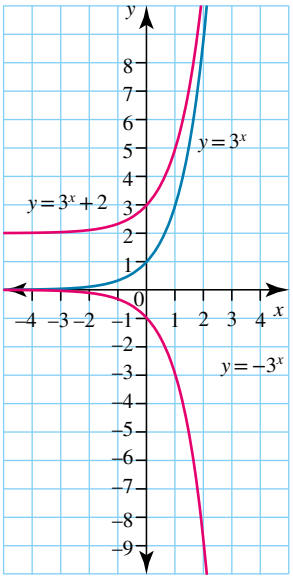


b. In each case the graphs are symmetric about the y-axis.

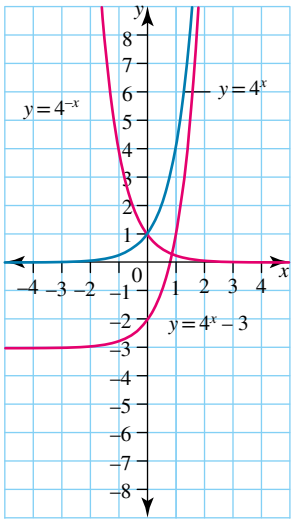
7. a–c



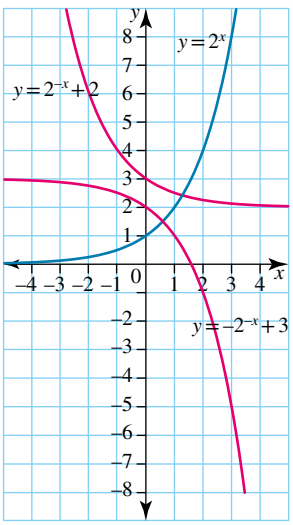
8. a, b



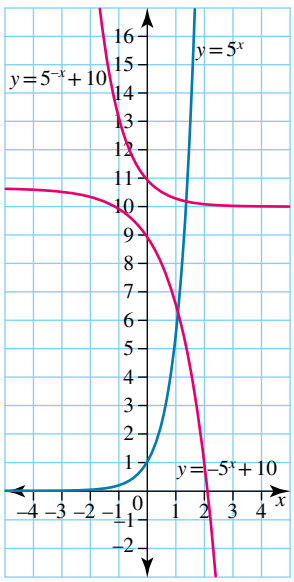
9. a, b



10. a, b



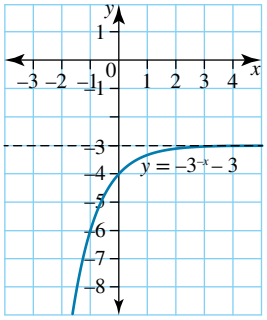
11. a, b



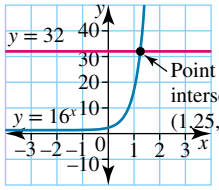
12. a. B **b.** C **c.** D **d.** A

13. a. B **b.** D **c.** A **d.** C

14.



15.



$x = 1.25$

16. a. 10000 **b. i.** 1111 **ii.** 41 **iii.** 0

17. a. Yes **b.** There is a constant ratio of 1.3. **c.** 30%

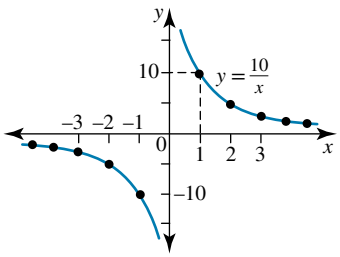
d. 3.26 million **e.** 30 million

18. Yes, $y = a^x$ will always have a horizontal asymptote as $a^x \neq 0$.

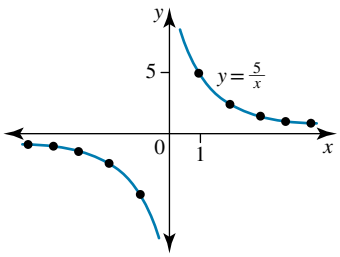
Exercise 13.7 The hyperbola

1.

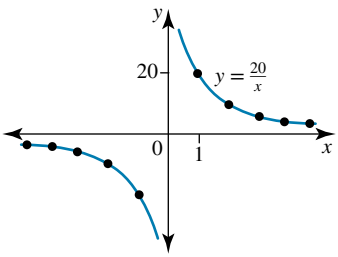
x	–5	–4	–3	–2	–1	0	1	2	3	4	5
y	2	–2.5	–3.3	–5	–10	Undefined	10	5	3.3	2.5	2



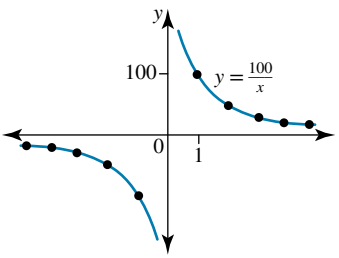
2. a. i.



ii.

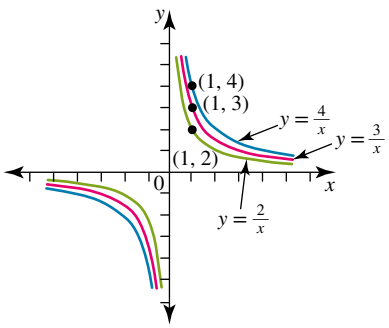


iii.



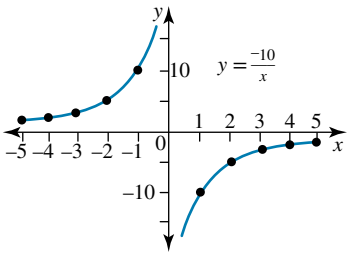
b. i. $x = 0,\ y = 0$ **ii.** $x = 0,\ y = 0$ **iii.** $x = 0,\ y = 0$

3.

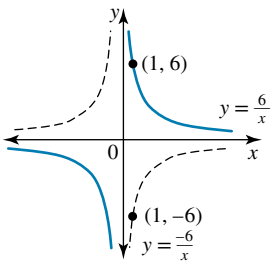


4. It increases the y-values by a factor of k and hence dilates the curve from the x-axis by a factor of k.

5.



6.

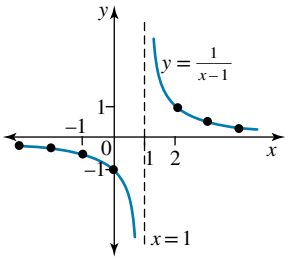


7. The negative reflects the curve $y = \frac{k}{x}$ in the x-axis.

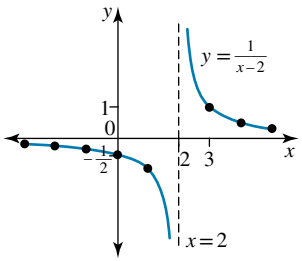
8.

x	−3	−2	−1	0	1	2	4	3
y	−0.2	−0.33	−0.5	−1	Undefined	1	0.5	0.33

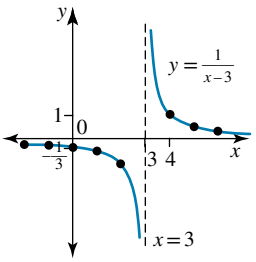
The equation of the vertical asymptote is $x = 1$.



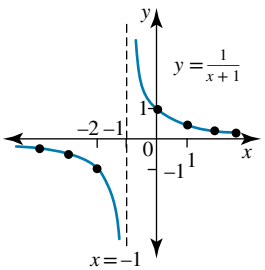
9. a.



b.

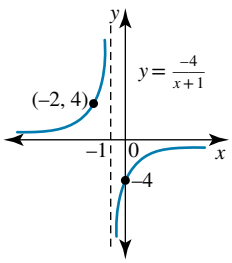


c.

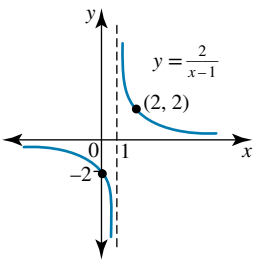


10. The a translates the graph left or right, and $x = a$ becomes the vertical asymptote.

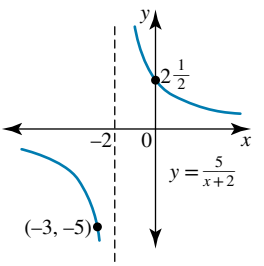
11. a.



b.



c.



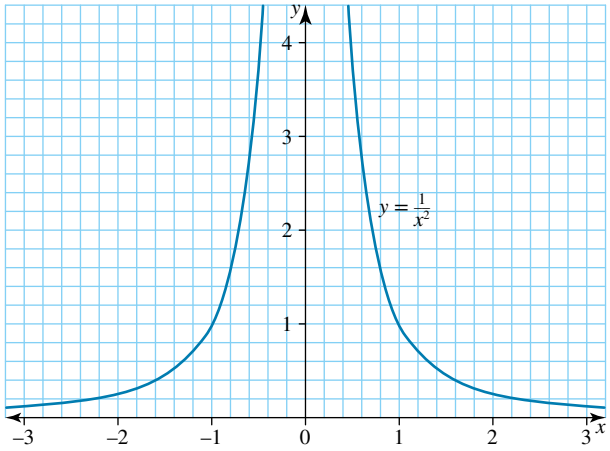
12. Check with your teacher. Possible answers:

a. $y = \dfrac{1}{x - 3}$ **b.** $y = \dfrac{1}{x + 10}$

13. $y = \dfrac{-2}{x + 3} - 1$, $x = -3$, $y = -1$

14. a.

x	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2
y	$\frac{1}{4}$	1	4	4	1	$\frac{1}{4}$



$x = 0$, $y = 0$

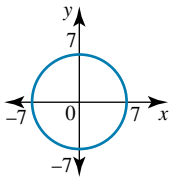
b. $y = -\dfrac{1}{x^2}$ **c.** $y = \dfrac{1}{x^2}$

15. $y = -\dfrac{3}{(x + 1)^2} + 2$, $x = -1$, $y = 2$

16. Construct a table to summarise the information regarding transformations of hyperbolas.

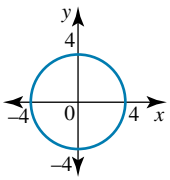
Exercise 13.8 The circle

1. a. $x^2 + y^2 = 49$



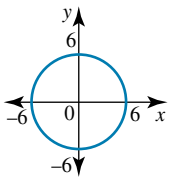
Centre $(0, 0)$, radius 7

b. $x^2 + y^2 = 16$



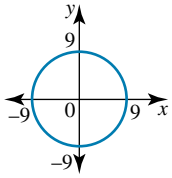
Centre $(0, 0)$, radius 4

c. $x^2 + y^2 = 36$



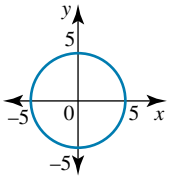
Centre $(0, 0)$, radius 6

d. $x^2 + y^2 = 81$



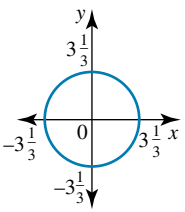
Centre $(0, 0)$, radius 9

e. $2x^2 + 2y^2 = 50$



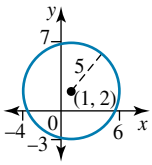
Centre $(0, 0)$, radius 5

f. $9x^2 + 9y^2 = 100$

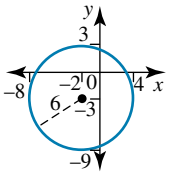


Centre $(0, 0)$, radius $\frac{10}{3}$

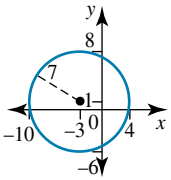
2. a. $(x - 1)^2 + (y - 2)^2 = 25$



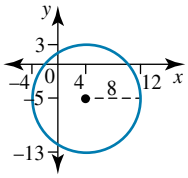
b. $(x + 2)^2 + (y + 3)^2 = 36$



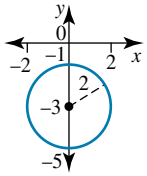
c. $(x + 3)^2 + (y - 1)^2 = 49$



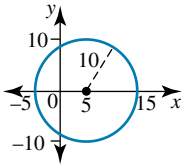
d. $(x - 4)^2 + (y + 5)^2 = 64$



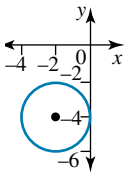
e. $x^2 + (y + 3)^2 = 4$



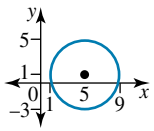
f. $(x - 5)^2 + y^2 = 100$



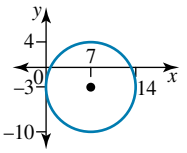
3. a. $(x + 2)^2 + (y + 4)^2 = 2^2$



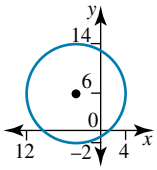
b. $(x - 5)^2 + (y - 1)^2 = 4^2$



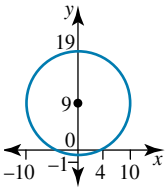
c. $(x - 7)^2 + (y + 3)^2 = 7^2$



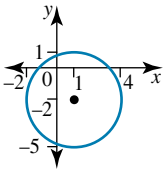
d. $(x + 4)^2 + (y - 6)^2 = 8^2$



e. $x^2 + (y - 9)^2 = 10^2$



f. $(x - 1)^2 + (y + 2)^2 = 3^2$



4. D

5. B

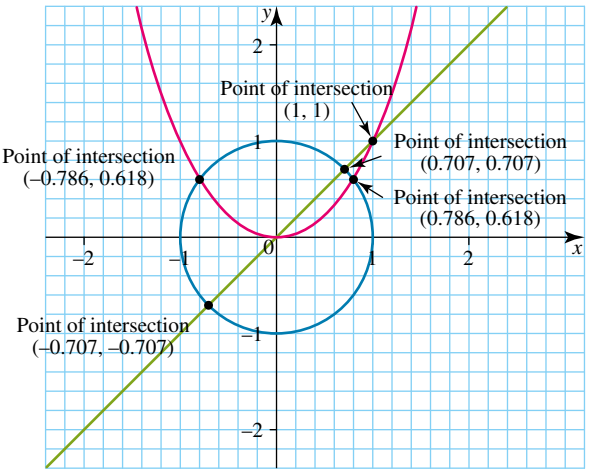
6. $(x - 5)^2 + (y - 3)^2 = 16$

7. a. 2 cm, 13.8 cm

b. 3.9 cm/s

8. (–2, 0)

9. a.

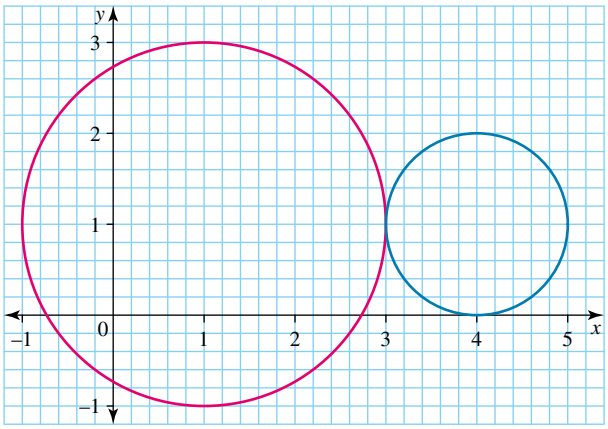


b. **i.** (0.707, 0.707) and (−0.707, − 0.707)

ii. (0, 0) and (1, 1)

iii. (0.786, 0.618) and (−0.786, 0.618)

10. $(x-1)^2 + (y-1)^2 = 4$: centre at $(1, 1)$ and radius of 2 units
$(x-4)^2 + (y-1)^2 = 1$: centre at $(4, 1)$ and radius of 1 unit
The circles intersect (touch) at $(3, 1)$.



11. $(x-2)^2 + (y-5)^2 = 4$: centre at $(2, 5)$ and radius of 2 units.

12. Increase or decrease the radius.

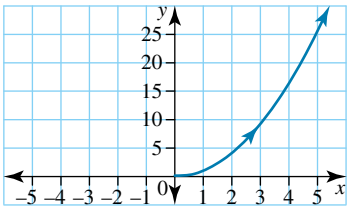
13. The distance between the centre and the point $(1, 1.5)$ is 2.5, which is greater than the length of the radius. Therefore, the point $(1, 1.5)$ lies outside the circle with equation $x^2 + 2x + y^2 - 6y + 6 = 0$.

Investigation | Rich task

1.

t	x	y
0	0	0
1	1	1
2	2	4
3	3	9
4	4	16
5	5	25

2. Parametric equations
$x = t$ and $y = t^2$

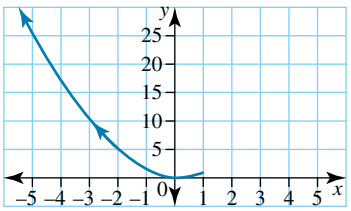


3. Answers will vary.

4.

t	x	y
0	1	1
1	0	0
2	−1	1
3	−2	4
4	−3	9
5	−4	16

Parametric equations
$x = 1 - t$ and $y = (1-t)^2$

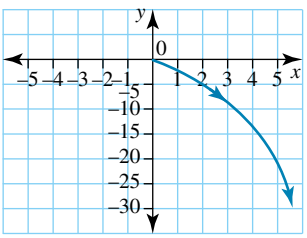


Answers will vary;
$t = 1, 0, -1, -2, -3, -4$.

5.

t	x	y
0	0	0
1	1	−1
2	2	−4
3	3	−9
4	4	−16
5	5	−25

Parametric equations
$x = t$ and $y = -t^2$



6. Answers will vary.

7. Answers will vary.

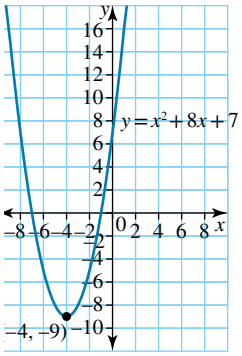
Exercise 13.9 Review questions

1. D **2.** A **3.** D **4.** A **5.** B

6. a. $(4, -15)$ **b.** $(-2, -9)$

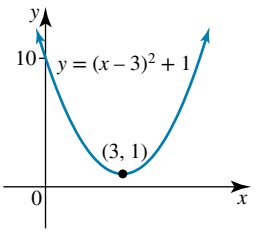
7.

x	−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1
y	16	7	0	−5	−8	−9	−8	−5	0	7	16

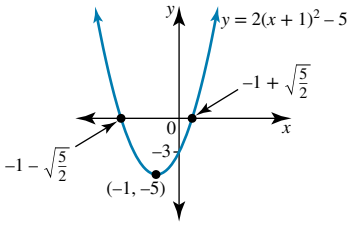


Vertex $(-4, -9)$; x-intercepts: -7 and -1

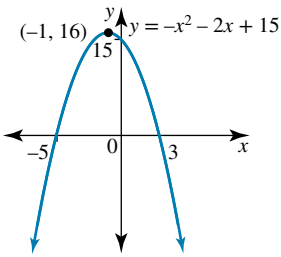
8. a. Vertex $(3, 1)$; no x-intercepts; y-intercept: $(0, 10)$



b. Vertex $(-1, -5)$; x-intercepts: $-1 - \sqrt{\frac{5}{2}}$, $-1 + \sqrt{\frac{5}{2}}$; y-intercept: $(0, -3)$



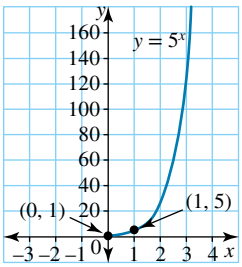
9. Vertex $(-1, 16)$; x-intercepts: -5 and 3; y-intercept: $(0, 15)$



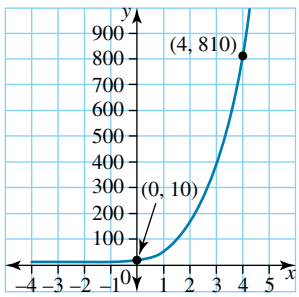
10. a.

x	−3	−2	−1	0	1	2	3
y	0.008	0.04	0.2	1	5	25	125

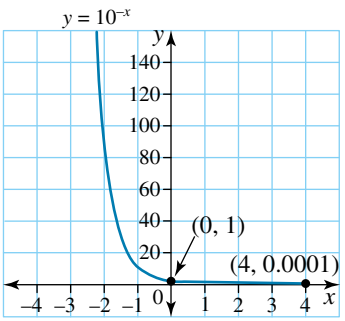
b.



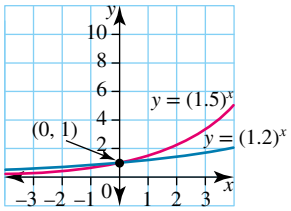
11.



12.

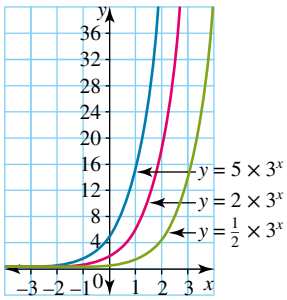


13. a.



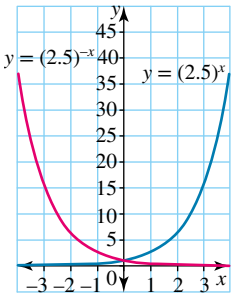
b. Increasing the value of a makes the graph steeper for positive x-values and flatter for negative x-values.

14. a.



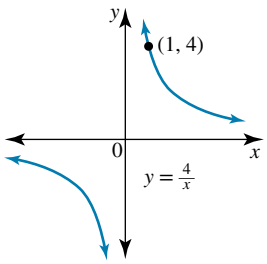
b. Increasing the value of k makes the graph steeper.

15. a.

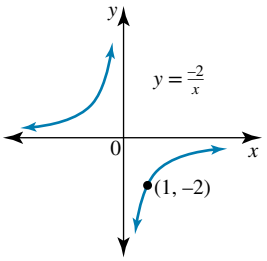


b. Changing the sign of the index reflects the graph in the y-axis.

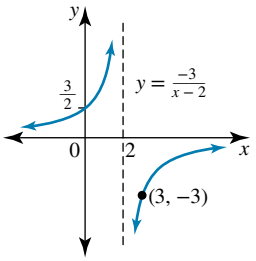
16. a.



b.

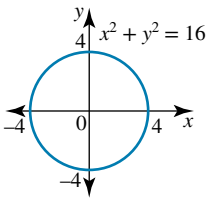


17.

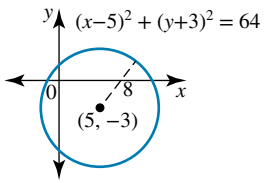


18. Check with your teacher. A possible answer is $y = \frac{1}{x + 3}$.

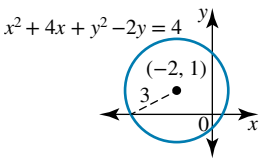
19. a.



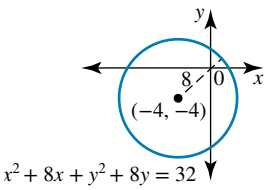
b.



20. a.

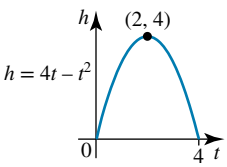


b.



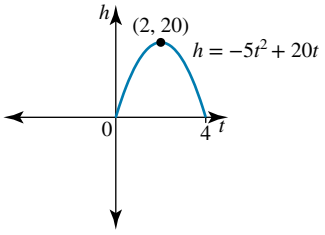
21. $x^2 + y^2 = 36$

22. a.



b. 4 m **c.** 2 s **d.** 4 s

23. a.



b. 4 s

c. 2 s ($1 < t < 3$)

d. The ball is never above a height of 20 m.

24. a. [0, 12]

b. 32 m

c. 11:41 am to 6:19 pm

25. a. $P \leq -x^2 + 5x$, $0 \leq x \leq 5$

b. 6.25 m

c. i. Check with your teacher.

ii. Dilation by a factor of 0.56 parallel to the y-axis

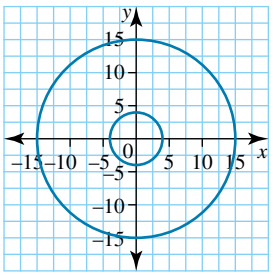
d. 66.7%

26. a. Check with your teacher.

b. When $x = 0.3$, $b = 10.7$. Therefore if p is greater than 10.7 cm, the platform will hit the bridge.

27. a. First ripple's radius is 3 cm, second ripple's radius is 15 cm.

b.



c. 2.4 cm/s

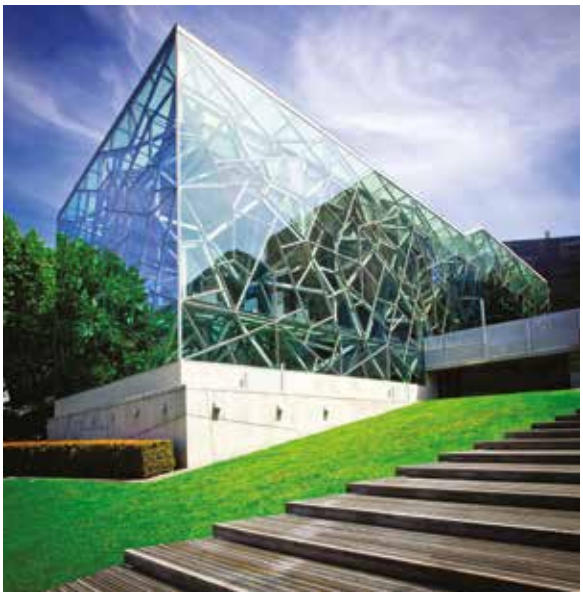
d. 1 minute 22.1 seconds after it is dropped

TOPIC 14
Properties of geometrical figures

[Stages 5.2 and 5.3]

14.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the concepts covered in this topic.



14.1.1 Why learn this?

Learning about geometry includes being able to reason deductively and to prove logically that certain mathematical statements are true. It is important to be able to prove theories rigorously and step by step in order to show that the conclusions reached are soundly based. Mathematicians spend most of their time trying to prove new theories, and they rely heavily on all the proofs that have gone before. Reasoning skills, and hence the ability to prove theories, can be developed and learned through practice and application.

DISCUSSION

Euclid is known as 'the father of geometry'. Who are the fathers of some of the other branches of mathematics and the sciences?

LEARNING SEQUENCE

14.1 Overview
14.2 [Stages 5.2 and 5.3] Angles, triangles and congruence
14.3 [Stages 5.2 and 5.3] Similar triangles
14.4 [Stages 5.2 and 5.3] Quadrilaterals
14.5 [Stages 5.2 and 5.3] Polygons
14.6 Review

LEARNING OUTCOMES

A student:

- selects appropriate notations and conventions to communicate mathematical ideas and solutions MA5.2-1WM
- interprets mathematical or real-life situations, systematically applying appropriate strategies to solve problems MA5.2-2WM
- constructs arguments to prove and justify results MA5.2-3WM
- calculates the angle sum of any polygon and uses minimum conditions to prove triangles are congruent or similar MA5.2-14MG
- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3-1WM
- generalises mathematical ideas and techniques to analyse and solve problems efficiently MA5.3-2WM
- uses deductive reasoning in presenting arguments and formal proofs MA5.3-3WM
- proves triangles are similar, and uses formal geometric reasoning to establish properties of triangles and quadrilaterals MA5.3-16MG

CONTENT DESCRIPTIONS

Students:

Formulate proofs involving congruent triangles and angle properties (ACMMG243)
Use the enlargement transformations to explain similarity and to develop the conditions for triangles to be similar (ACMMG220)
Apply logical reasoning, including the use of congruence and similarity, to proofs and numerical exercises involving plane shapes (ACMMG244)

Source: NSW Syllabus for the Australian curriculum

learnon RESOURCES — ONLINE ONLY

eLesson: The story of mathematics — Euclid (eles-1849)

Note: Your teacher may now set you a pre-test to determine how familiar you are with the content in this topic.

14.2 Angles, triangles and congruence [Stages 5.2 and 5.3]

14.2.1 Proofs and theorems

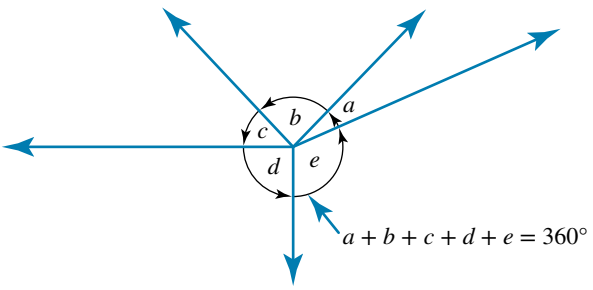
- Euclid (c. 300 BC) was the mathematician who developed a systematic approach to geometry, now referred to as Euclidean geometry, that relied on mathematical proofs.
- A **proof** is an argument that shows why a statement is true.
- A **theorem** is a statement that can be demonstrated to be true. To demonstrate that a statement is proved, formal language needs to be used. It is conventional to use the following structure when setting out a theorem.
  - **Given:** a summary of the information given
  - **To prove:** a statement that needs to be proved
  - **Construction:** a description of any additions to the diagram given
  - **Proof:** a sequence of steps that can be justified and form part of a formal mathematical proof.

DISCUSSION

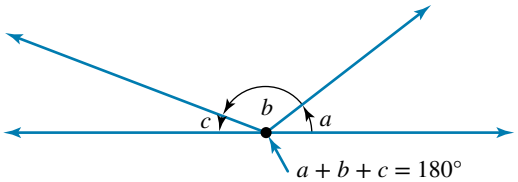
Can you explain the difference between a proof and a theorem to a friend?

14.2.2 Angles at a point

- The sum of the angles at a point is 360°.
  $a + b + c + d + e = 360°$



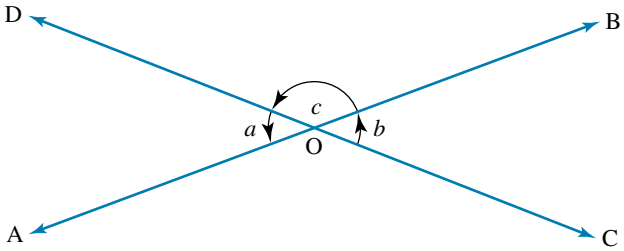
14.2.3 Supplementary angles



- The sum of the angles on a straight line is 180°.
- Angles a, b and c are **supplementary angles**.
  $a + b + c = 180°$
- The sum of the angles at a point is 360°.

14.2.4 Vertically opposite angles

- **Theorem 1:** **Vertically opposite angles** are equal.



Given: Straight lines AB and CD intersect at O.

To prove: $\angle AOD = \angle BOC$ and $\angle BOD = \angle AOC$

Construction: Label $\angle AOD$ as a, $\angle BOC$ as b and $\angle BOD$ as c.

Proof: Let $\angle AOD = a°$, $\angle BOC = b°$ and $\angle BOD = c°$.

$a + c = 180°$ (supplementary angles)

$b + c = 180°$ (supplementary angles)

$\therefore a + c = b + c$

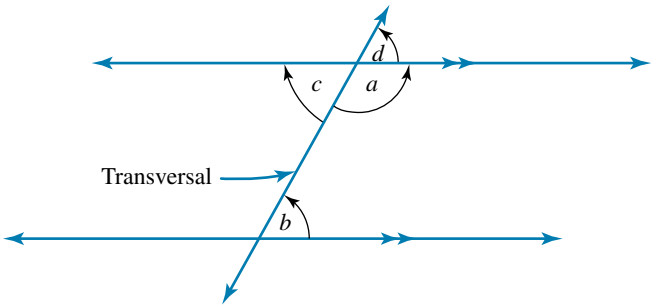
$\therefore a = b$

So, $\angle AOD = \angle BOC$.

Similarly, $\angle BOD = \angle AOC$.

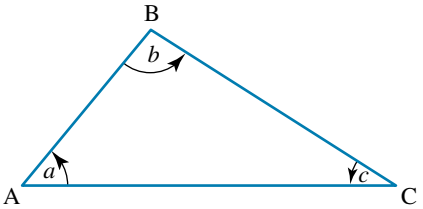
14.2.5 Parallel lines

- If two lines are parallel and cut by a **transversal**, then:
  - co-interior angles are supplementary. For example, $a + b = 180°$.
  - corresponding angles are equal. For example, $b = d$.
  - alternate angles are equal. For example, $b = c$.
  - vertically opposite angles are equal. For example, $c = d$.



14.2.6 Angle properties of triangles

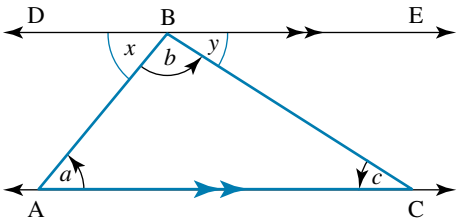
- **Theorem 2:** The sum of the interior angles of a triangle is 180°.



Given: ΔABC with interior angles a, b and c

To prove: $a + b + c = 180°$

Construction: Draw a line parallel to AC, passing through B, and label it DE as shown. Label ∠ABD as x and ∠CBE as y.



Proof:

$a = x$ (alternate angles)

$c = y$ (alternate angles)

$x + b + y = 180°$ (supplementary angles)

$\therefore a + b + c = 180°$

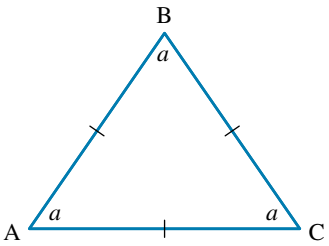
14.2.7 Equilateral triangles

- It follows from Theorem 2 that each interior angle of an **equilateral triangle** is 60°, and, conversely, if the three angles of a triangle are equal, then the triangle is equiangular.

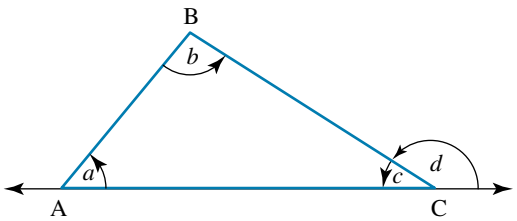
$a + a + a = 180°$ (sum of interior angles in a triangle is 180°)

$3a = 180°$

$a = 60°$



- **Theorem 3:** The exterior angle of a triangle is equal to the sum of the opposite interior angles.



Given: ΔABC with the exterior angle labelled d

To prove: $d = a + b$

Proof:

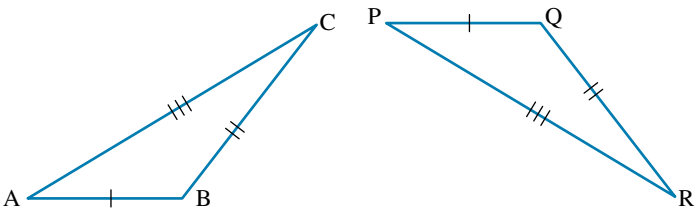
$c + d = 180°$ (supplementary angles)

$a + b + c = 180°$ (sum of interior angles in a triangle is 180°)

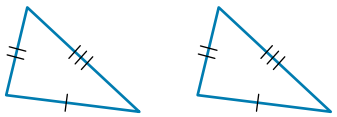
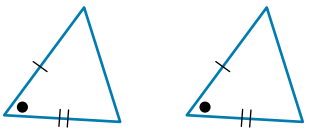
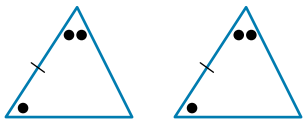
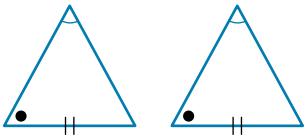
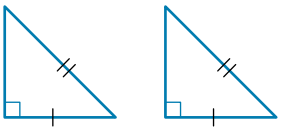
$\therefore d = a + b$

14.2.8 Congruent triangles

- **Congruent triangles** have the same size and the same shape; that is, they are identical in all respects.
- The symbol used for congruency is ≡.
- For example, ΔABC in the diagram below is congruent to ΔPQR. This is written as ΔABC ≡ ΔPQR.



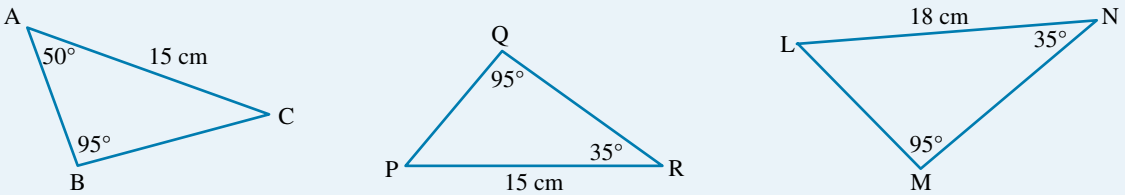
- Note that the vertices of the two triangles are written in corresponding order.
- There are five tests designed to check whether triangles are congruent. The tests are summarised in the table below.

Test	Diagram	Abbreviation
All three sides in one triangle are equal in length to the corresponding sides in the other triangle.		**SSS**
Two corresponding sides and the included angle are the same in both triangles.		**SAS**
Two corresponding angles and a pair of corresponding sides are the same in both triangles.		**ASA**
A pair of corresponding angles and a non-included side are equal in both triangles.		**AAS**
The hypotenuse and one pair of the other corresponding sides in two right-angled triangles are the same in two right-angled triangles.		**RHS**

- In each of the tests we need to show three equal measurements about a pair of triangles in order to show they are congruent.
- When completing formal assessment, use the full names of the tests for congruence, not their abbreviations.

WORKED EXAMPLE 1

Select a pair of congruent triangles from the diagrams below, giving a reason for your answer.



THINK

1 In each triangle the length of the side opposite the 95° angle is given. If triangles are to be congruent, the sides opposite the angles of equal size must be equal in length. Draw your conclusion.

2 To test whether ΔABC is congruent to ΔPQR, first find the angle C.

WRITE

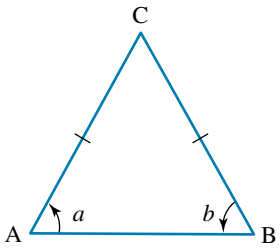
All three triangles have equal angles, but the sides opposite the angle 95° are not equal.
AC = PR = 15 and LN = 18 cm

ΔABC: ∠A = 50°, ∠B = 95°,
∠C = 180° − 50° − 95°
= 35°

3 Apply a test for congruence. Triangles ABC and PQR have a pair of corresponding sides equal in length and 2 pairs of angles the same, so draw your conclusion.

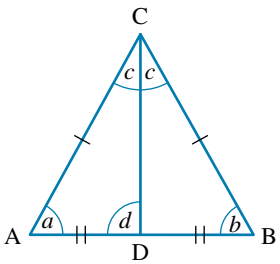
A pair of corresponding angles $\angle B = \angle Q$ and $\angle C = \angle R$ and a non-included side (AP = PR) are equal.
$\Delta ABC \equiv \Delta PQR$ (AAS)

14.2.9 Isosceles triangles



- A triangle is isosceles if the lengths of two sides are equal but the third side is not equal.
- **Theorem 4:** The angles at the base of an **isosceles triangle** are equal.

Given: AC = CB
To prove: $\angle BAC = \angle CBA$
Construction: Draw a line from the vertex C to the midpoint of the base AB and label the midpoint D. CD is the bisector of $\angle ACB$.



Proof:

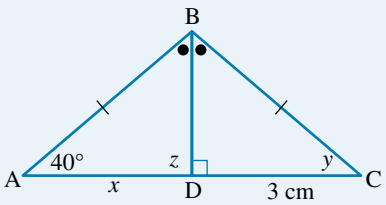
In ΔACD and ΔBCD,
CD = CD (common side)
AD = DB (construction, D is the midpoint of AB)
AC = CB (given)
$\Rightarrow \Delta ACD \equiv \Delta BCD$ (SSS)
$\therefore \angle BAC = \angle CBA$

- Conversely, if two angles of a triangle are equal, then the sides opposite those angles are equal.
- It also follows that $\angle ADC = \angle BDC = d$

and that
$2d = 180°$
$\Rightarrow d = 90°$ (supplementary)

WORKED EXAMPLE 2

Given that $\Delta ABD \equiv \Delta CBD$, find the values of the pronumerals in the figure below.



THINK

1 In congruent triangles corresponding sides are equal in length. Side AD (marked x) corresponds to side DC, so state the value of x.

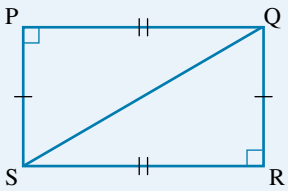
WRITE

$\Delta ABD \equiv \Delta CBD$
AD = CD, AD = x, CD = 3
So x = 3 cm.

2 Since the triangles are congruent, corresponding angles are equal. State the angles corresponding to y and z and hence find the values of these pronumerals.	$\angle BAD = \angle BCD$ $\angle BAD = 40°$, $\angle BCD = y$ So $y = 40°$. $\angle BDA = \angle BDC$ $\angle BDA = z$, $\angle BDC = 90°$ So $z = 90°$.

WORKED EXAMPLE 3

Prove that ΔPQS is congruent to ΔRSQ.



THINK	WRITE
1 Write the information given.	Given: Rectangle PQRS with diagonal QS
2 Write what needs to be proved.	To prove: that ΔPQS is congruent to ΔRSQ QP = SR (given) $\angle SPQ = \angle SRQ = 90°$ (given) QS is common.
3 Select the appropriate congruency test for proof. (In this case it is RHS because the triangles have an equal side, a right angle and a common hypotenuse.)	So ΔPQS ≡ ΔRSQ (RHS).

learnon RESOURCES — ONLINE ONLY

Interactivity: Angles at a point (int-6157)
Interactivity: Supplementary angles (int-6158)
Interactivity: Vertically opposite and adjacent angles (int-3968)
Interactivity: Co-interior angles (int-3970)
Interactivity: Corresponding angles (int-3969)
Interactivity: Alternate angles (int-3971)
Interactivity: Angles in a triangle (int-3965)
Interactivity: Interior and exterior angles of a triangle (int-3966)
Interactivity: Congruent triangles (int-3754)
Interactivity: Congruency tests (int-3755)
Interactivity: Angles in an isosceles triangle (int-6159)
Digital doc: SkillSHEET Naming angles, lines and figures (doc-5276)
Digital doc: SkillSHEET Corresponding sides and angles of congruent figures (doc-5277)
Digital doc: SkillSHEET Angles and parallel lines (doc-5280)

Exercise 14.2 Angles, triangles and congruence

Individual pathways

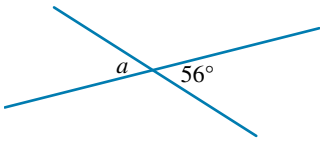
PRACTISE	CONSOLIDATE	MASTER
Questions: 1–5, 7, 9, 11	Questions: 1–5, 6, 8–10, 12, 13	Questions: 1–15

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

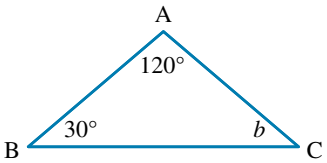
Understanding and fluency

1. Determine the values of the unknown in each of the following.

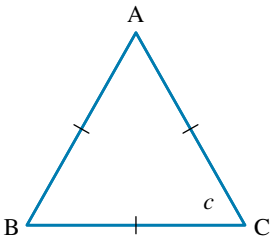
a.



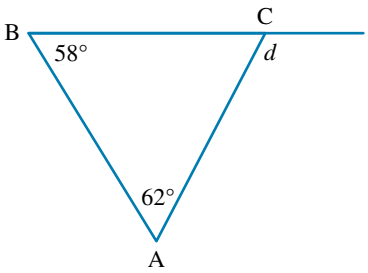
b.



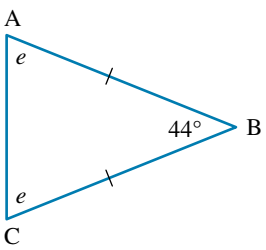
c.



d.

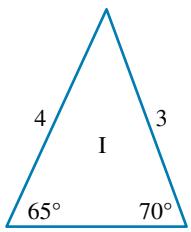


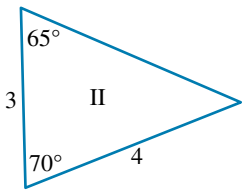
e.

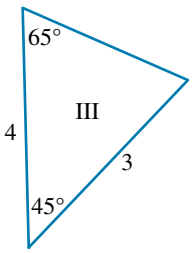


2. WE1 Select a pair of congruent triangles in each of the following, giving a reason for your answer. All side lengths are in cm.

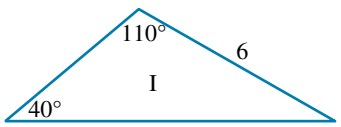
a.

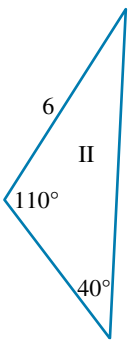


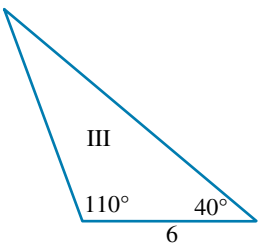




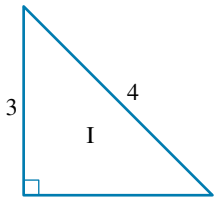
b.

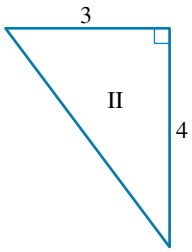


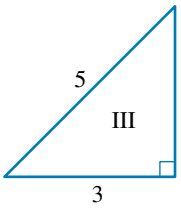




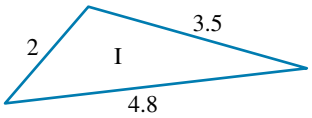
c.

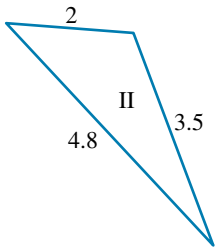


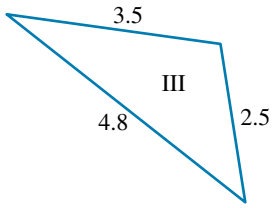




d.

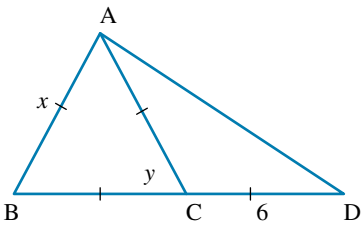




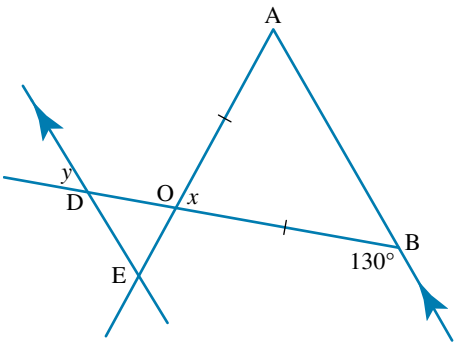


3. Find the missing values of x and y in each of the following diagrams. Give reasons for your answers.

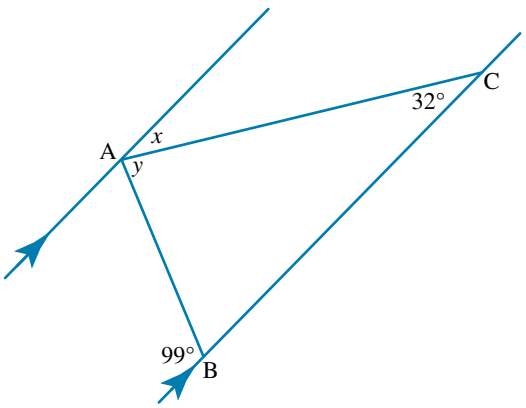
a.



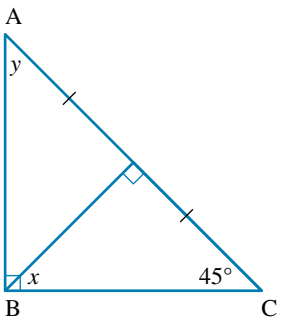
b.



c.

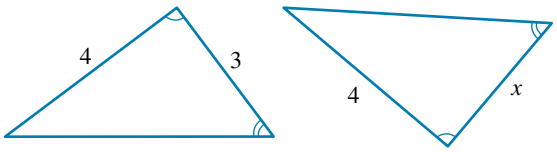


d.

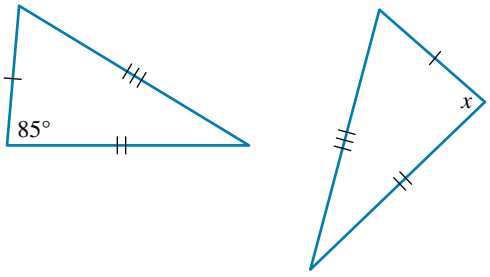


4. WE2 Find the value of the pronumeral in each of the following pairs of congruent triangles. All side lengths are in cm.

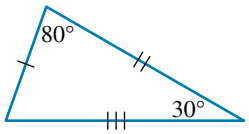
a.

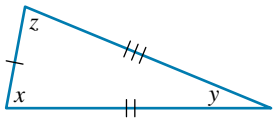


b.

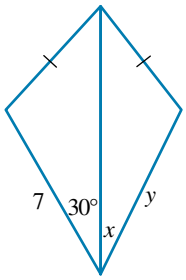


c.

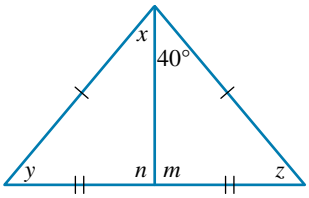




d.



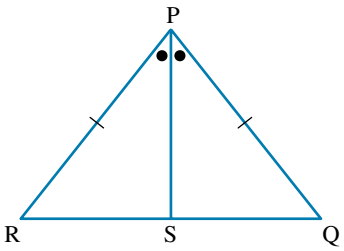
e.



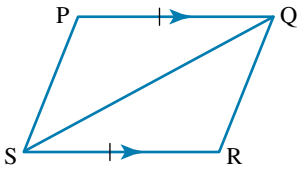
Communicating, reasoning and problem solving

5. WE3 Prove that each of the following pairs of triangles are congruent.

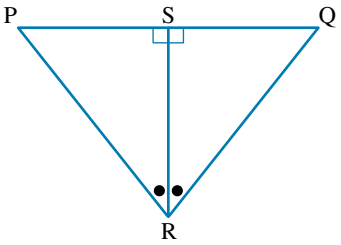
a.



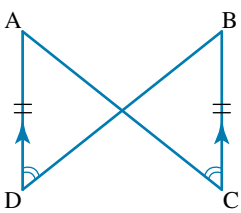
b.



c.



d.



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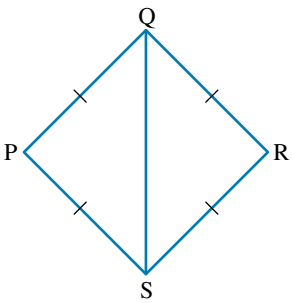
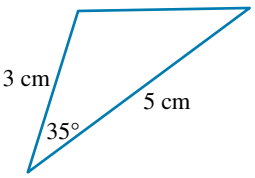
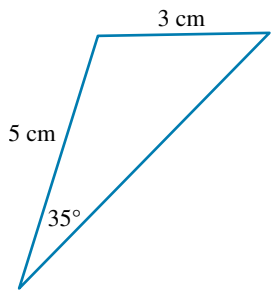


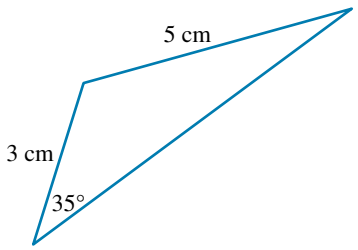
6. MC Which of the following is congruent to the triangle shown?
Note: There may be more than one correct answer.



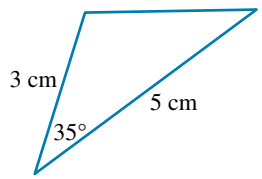
A.



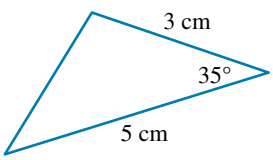
B.



C.

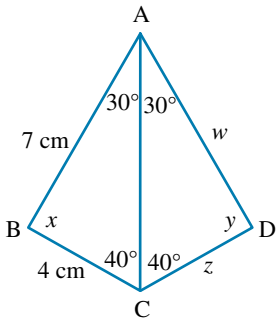


D.

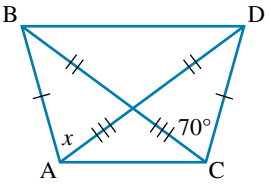


7. Prove that $\Delta ABC \equiv \Delta ADC$ and hence find the values of the pronumerals in each of the following.

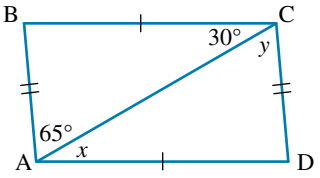
a.



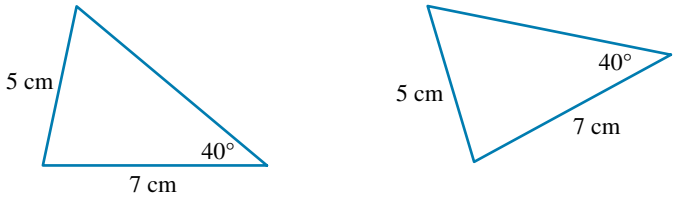
b.



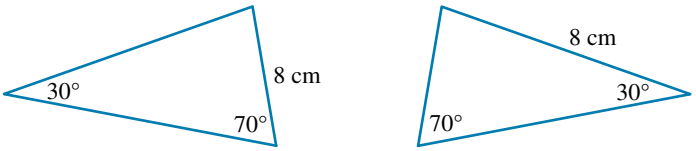
c.



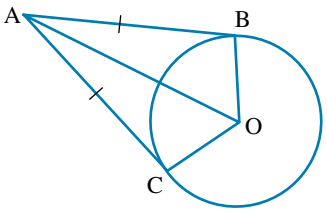
8. Explain why the triangles shown are not necessarily congruent.



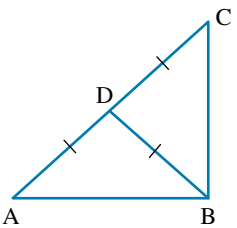
9. Explain why the triangles shown are not congruent.



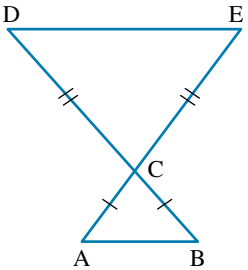
10. Show that $\Delta ABO \equiv \Delta ACO$, if O is the centre of the circle.



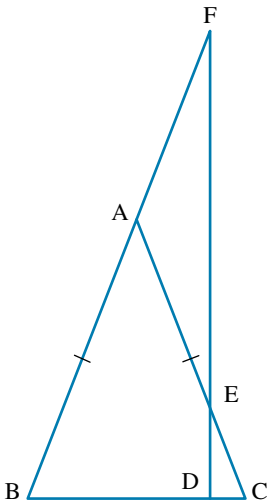
11. If DA = DB = DC, prove that $\angle ABC$ is a right angle.



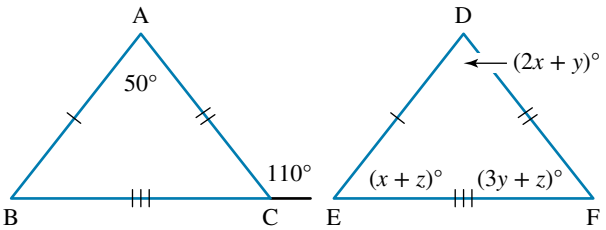
12. If AC = CB and DC = CE in the diagram shown, prove that AB || DE.



13. ABC is an isosceles triangle in which AB and AC are equal in length. BDF is a right-angled triangle. Show that triangle AEF is an isosceles triangle.



14. Triangles ABC and DEF are congruent. Find the values of x, y and z.



15. Use congruent triangle results to prove the following.

a. If two sides of a triangle are equal in length, prove that the angels opposite the equal sides are equal.

b. If two angles of a triangle are equal, prove that the sides opposite those angles are equal.

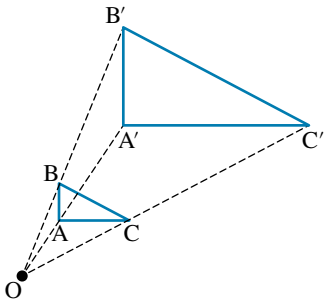
c. If three sides of a triangle are equal, prove that each interior angle is 60°.

14.3 Similar triangles [Stages 5.2 and 5.3]

14.3.1 Similar figures

- Two geometric shapes are **similar** when one is an **enlargement** or reduction of the other shape.

– An enlargement increases the length of each side of a figure in all directions by the same factor. For example, in the diagram shown, triangle A′B′C′ is an enlargement of triangle ABC by a factor of 3 from its **centre of enlargement** at O.



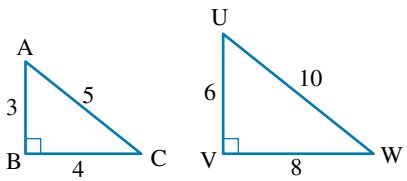
- The symbol for similarity is ||| and is read as 'is similar to'.
- The **image** of the original object is the enlarged or reduced shape.
- To create a similar shape, use a **scale factor** to enlarge or reduce the original shape.
- The scale factor can be found using the formula below and the lengths of a pair of corresponding sides.

$$\text{Scale factor} = \frac{\text{image side length}}{\text{object side length}}$$

- If the scale factor is less than 1, the image is a reduced version of the original shape. If the scale factor is greater than 1, the image is an enlarged version of the original shape.

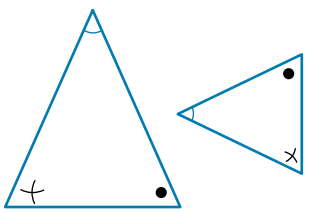
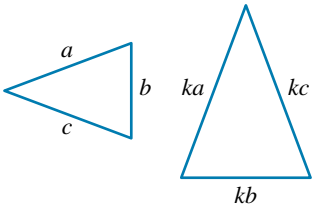
14.3.2 Similar triangles

- Two triangles are similar if:
  - the angles are equal, or
  - the corresponding sides are proportional.
- Consider the pair of **similar triangles** shown.
- The following statements are true for these triangles.
  - Triangle UVW is similar to triangle ABC or, using symbols, $\Delta UVW \mid\mid\mid \Delta ABC$.
  - The corresponding angles of the two triangles are equal in size: $\angle CAB = \angle WUV$, $\angle ABC = \angle UVW$ and $\angle ACB = \angle UWV$.
  - The corresponding sides of the two triangles are in the same ratio. $\frac{UV}{AB} = \frac{VW}{BC} = \frac{UW}{AC} = 2$; that is, ΔUVW has each of its sides twice as long as the corresponding sides in ΔABC.
  - The scale factor is 2.

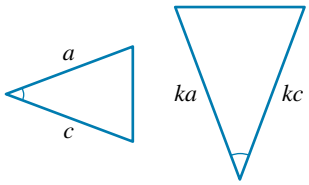
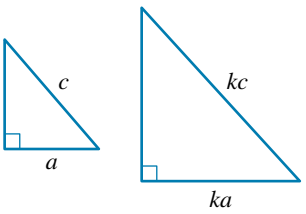


14.3.3 Testing triangles for similarity

- Triangles can be checked for similarity using one of the tests described in the table below.

Test	Diagram	Abbreviation
Two angles of a triangle are equal to two angles of another triangle. This implies that the third angles are equal, as the sum of angles in a triangle is 180°.		**AAA**
The three sides of a triangle are proportional to the three sides of another triangle.		**SSS**

(*continued*)

Test	Diagram	Abbreviation
Two sides of a triangle are proportional to two sides of another triangle, and the included angles are equal.		SAS
The hypotenuse and a second side of a right-angled triangle are proportional to the hypotenuse and a second side of another right-angled triangle.		RHS

- *Note:* When using the equiangular test, only two corresponding angles have to be checked. Since the sum of the interior angles in any triangle is a constant number (180°), the third pair of corresponding angles will automatically be equal, provided that the first two pairs match exactly.
- When completing formal assessment, use the full names of the tests for similarity, not their abbreviations.

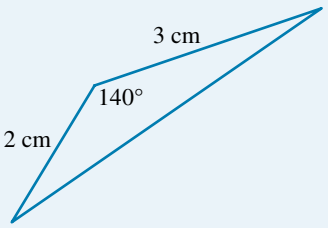
DISCUSSION

Explain the differences between the tests for similarity and congruency for triangles.

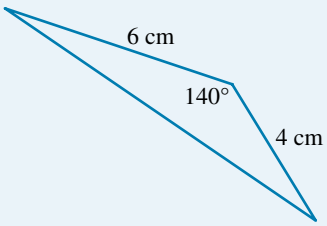
WORKED EXAMPLE 4

Find a pair of similar triangles among those shown. Give a reason for your answer.

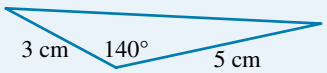
a



b



c



THINK

1 In each triangle the lengths of two sides and the included angle are known, so the SAS test can be applied. Since all included angles are equal (140°), we need to the find ratios of corresponding sides, taking two triangles at a time.

2 Only triangles **a** and **b** have corresponding sides in the same ratio (and included angles are equal). State your conclusion, specifying the similarity test that has been used.

WRITE

For triangles **a** and **b**: $\frac{6}{3} = \frac{4}{2} = 2$

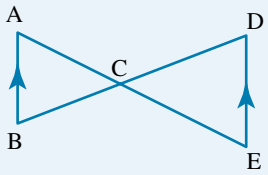
For triangles **a** and **c**: $\frac{5}{3} = 1.6$, $\frac{3}{2} = 1.5$

For triangles **b** and **c**: $\frac{5}{6} = 0.83$, $\frac{3}{4} = 0.75$

Triangle **a** ||| triangle **b** (SAS)

WORKED EXAMPLE 5

Prove that ΔABC is similar to ΔEDC.



THINK	WRITE
1 Write the information given. AB is parallel to DE. Transversal BD forms two alternate angles: ∠ABC and ∠EDC.	**1** Given: ΔABC and ΔDCE AB ∥ DE C is common.
2 Write what is to be proved.	**2** To prove: ΔABC ||| ΔEDC
3 Write the proof.	**3** Proof: ∠ABC = ∠EDC (alternate angles) ∠BAC = ∠DEC (alternate angles) ∠BCA = ∠DCE (vertically opposite angles) ∴ ΔABC ||| ΔEDC (equiangular, AAA)

learnON RESOURCES — ONLINE ONLY

Interactivity: Scale factors (int-6041)
Interactivity: Angle-angle-angle condition of similarity (AAA) (int-6042)
Interactivity: Side-side-side condition of similarity (SSS) (int-6448)
Interactivity: Side-angle-side condition of similarity (SAS) (int-6447)
eLesson: Similar triangles (eles-1925)
Digital doc: SkillSHEET Writing similarity statements (doc-5278)
Digital doc: SkillSHEET Calculating unknown side lengths in a pair of similar triangles (doc-5281)
Digital doc: WorkSHEET Deductive geometry I (doc-5282)

Exercise 14.3 Similar triangles

Individual pathways

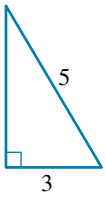
PRACTISE	CONSOLIDATE	MASTER
Questions: 1–10	Questions: 1–11, 15	Questions: 1–18

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE4 Find a pair of similar triangles among those shown in each part. Give a reason for your answer.

a. i.



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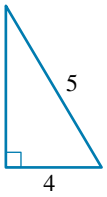

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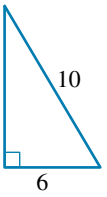
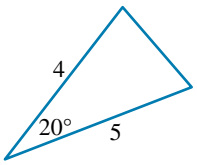
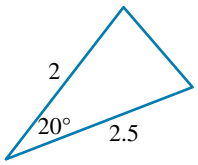


b. i.



ii.



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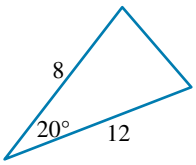
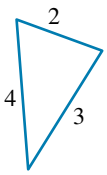
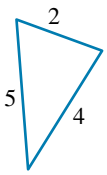


c. i.



ii.



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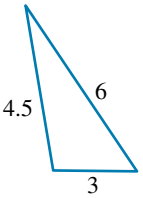
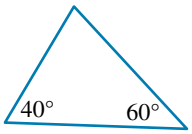
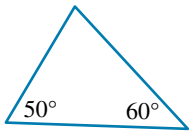


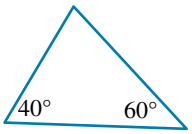
d. i.



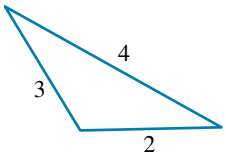
ii.



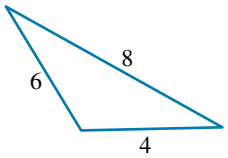
iii.



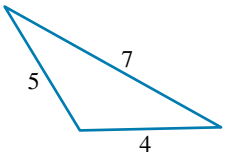
e. i.



ii.

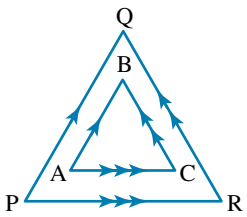


iii.

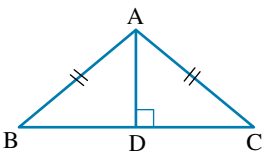


2. Name two similar triangles in each of the following figures.

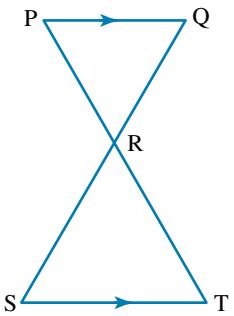
a.



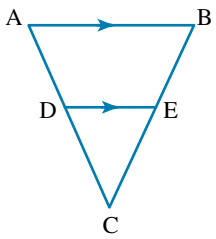
b.



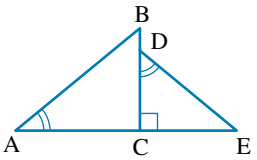
c.



d.

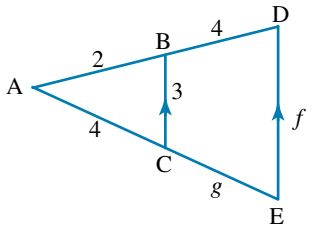


e.

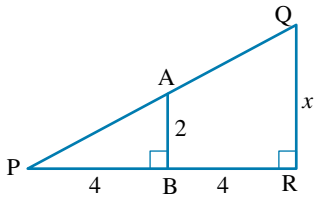


3. a. Complete this statement: $\frac{AB}{AD} = \frac{BC}{\quad} = \frac{\quad}{AE}$.

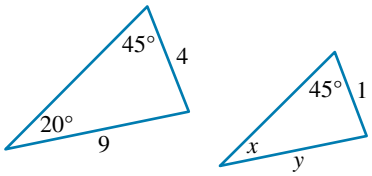
b. Find the value of the pronumerals.



4. Find the value of the pronumeral in the diagram shown.

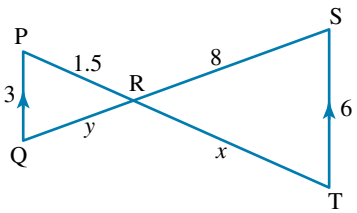


5. The triangles shown are similar. Find the value of x and y.

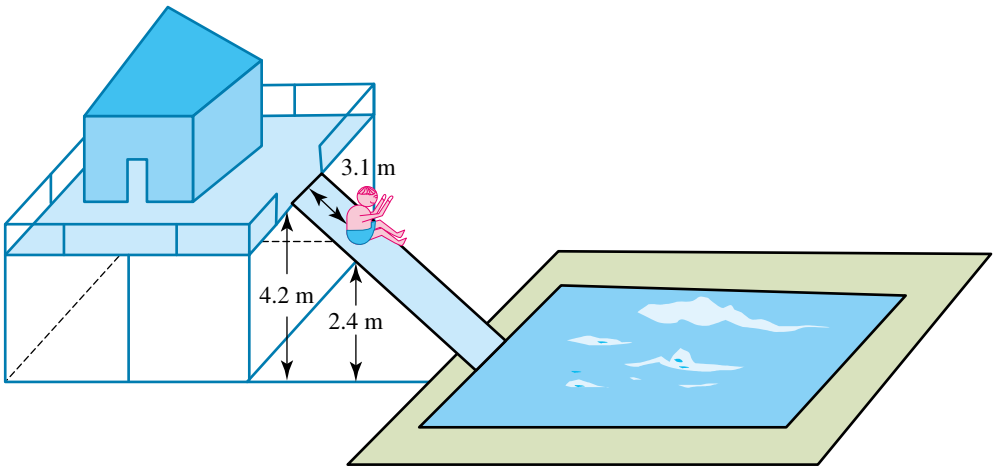


6. a. State why these two triangles shown are similar.

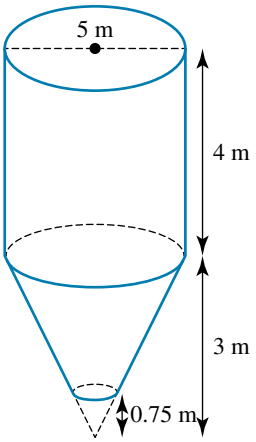
b. Find the values of x and y in the diagram.



7. A waterslide is 4.2 m high and has a support 2.4 m tall. If a student reaches this support when she is 3.1 m down the slide, how long is the slide?

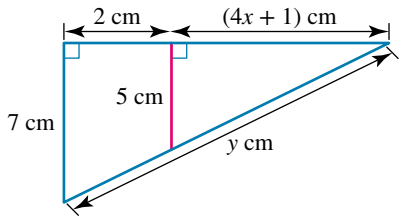


8. A storage tank as shown in the diagram is made of a 4-m-tall cylinder joined by a 3-m-tall cone. If the diameter of the cylinder is 5 m, what is the radius of the end of the cone if 0.75 m has been cut off the tip?

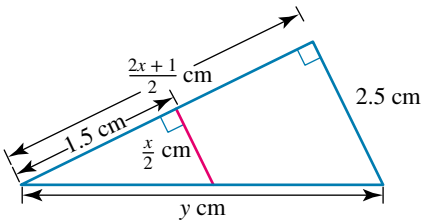


9. Calculate the values of the pronumerals in the following diagrams.

a.



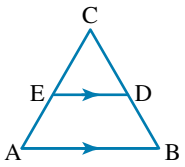
b.



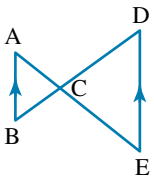
Communicating, reasoning and problem solving

10. WE5 Prove that ΔABC is similar to ΔEDC in each of the following.

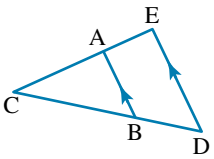
a.



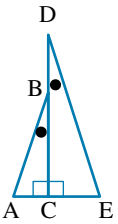
b.



c.



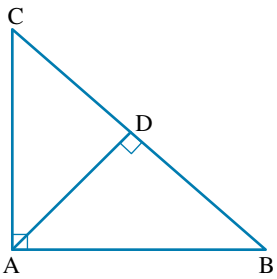
d.



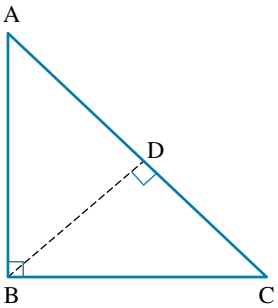
11. ΔABC is a right-angled triangle. A line is drawn from A to D as shown so that AD ⊥ BC. Prove that:

a. ΔABD ||| ΔACB

b. ΔACD ||| ΔACB.



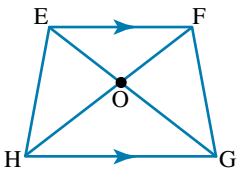
12. Explain why the AAA test cannot be used to prove congruence but can be used to prove similarity.



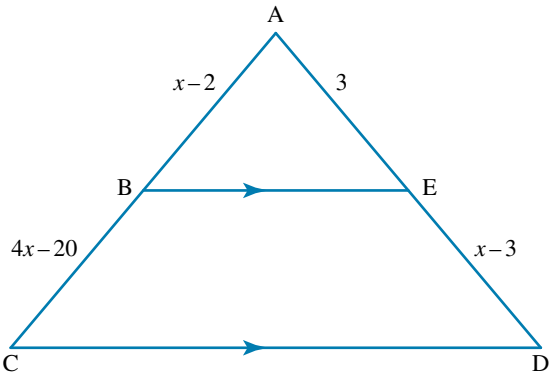
13. a. Prove Pythagoras' theorem, $AC^2 = AB^2 + BC^2$, using similar triangles.

b. Show that the converse of Pythagoras' theorem holds true; that is, if the square on one side of a triangle equals the sum of the squares on the other two sides, then the angle between these other two sides is a right angle.

14. Prove that $\Delta EFO \parallel\!\parallel\!\parallel \Delta GHO$.



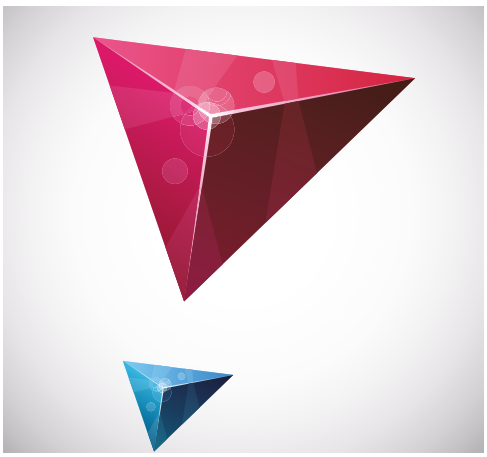
15. Solve for x.



16. How can you be certain that two figures are similar?

17. A tetrahedron (regular triangular-based pyramid) has an edge length of 2 cm. A similar tetrahedron has a total surface area of $36\sqrt{3}$ cm^2. What is the scale factor relationship between the side lengths of the two tetrahedra?

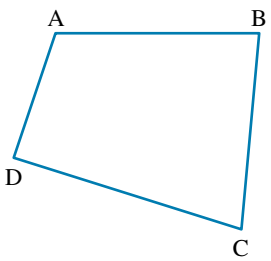
18. Explain why the remaining (third) angles must also be equal if two angles of a triangle are equal to two angles of another triangle.



14.4 Quadrilaterals [Stages 5.2 and 5.3]

14.4.1 Definition of a quadrilateral

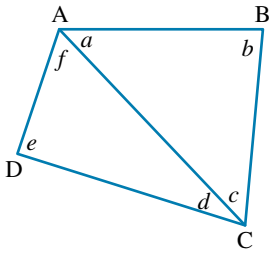
- A quadrilateral is a four-sided plane shape whose interior angles sum to 360°.
- **Theorem 5:** The sum of the interior angles in a quadrilateral is 360°.



Given: A quadrilateral ABCD

To prove: $\angle ABC + \angle BCD + \angle ADC + \angle BAD = 360°$

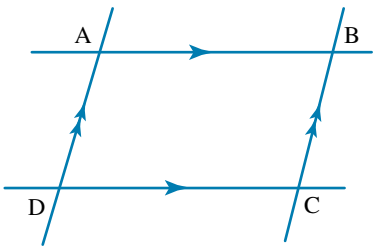
Construction: Draw a line joining vertex A to vertex C. Label the interior angles of the triangles formed.



Proof:
$a + b + c = 180°$ (sum of interior angles in a triangle is 180°)
$d + e + f = 180°$ (sum of interior angles in a triangle is 180°)
$\Rightarrow a + b + c + d + e + f = 360°$
$\therefore \angle ABC + \angle BCD + \angle ADC + \angle BAD = 360°$

14.4.2 Parallelograms

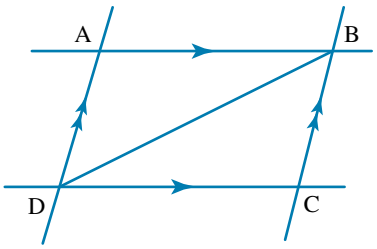
- A **parallelogram** is a quadrilateral with two pairs of parallel sides.
- **Theorem 6:** Opposite angles of a parallelogram are equal.



Given: AB || DC and AD || BC

To prove: $\angle ABC = \angle ADC$

Construction: Draw a diagonal from B to D.

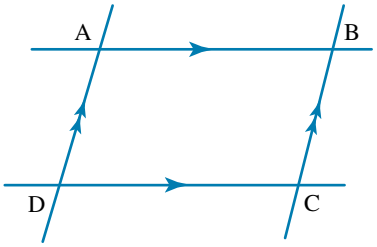


Proof:
$\angle ABD = \angle BDC$ (alternate angles)
$\angle ADB = \angle CBD$ (alternate angles)
$\angle ABC = \angle ABD + \angle CBD$ (by construction)
$\angle ADC = \angle BDC + \angle ADB$ (by construction)
$\therefore \angle ABC = \angle ADC$

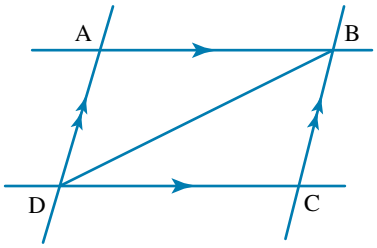
- Conversely, if each pair of opposite angles of a quadrilateral is equal, then it is a parallelogram.

14.4.3 Theorem 7

- **Theorem 7:** Opposite sides of a parallelogram are equal.



Given: AB || DC and AD || BC
To prove: AB = DC
Construction: Draw a diagonal from B to D.

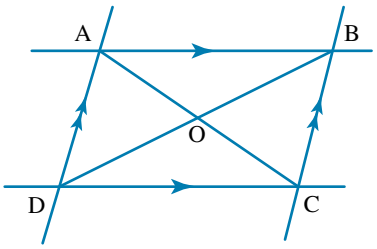


Proof:
∠ABD = ∠BDC (alternate angles)
∠ADB = ∠CBD (alternate angles)
BD is common to ΔABD and ΔBCD.
⇒ ΔABD ≡ ΔBCD (ASA)
∴ AB = DC

- Conversely, if each pair of opposite sides of a quadrilateral is equal, then it is a parallelogram.

14.4.4 Theorem 8

- **Theorem 8:** The diagonals of a parallelogram bisect each other.



Given: AB || DC and AD || BC with diagonals AC and BD
To prove: AO = OC and BO = OD
Proof: In ΔAOB and ΔCOD,
∠OAB = ∠OCD (alternate angles)
∠OBA = ∠ODC (alternate angles)
AB = CD (opposite sides of a parallelogram)
⇒ ΔAOB ≡ ΔCOD (ASA)
⇒ AO = OC (corresponding sides in congruent triangles)
and BO = OD (corresponding sides in congruent triangles)

14.4.5 Rectangles

- A rectangle is a parallelogram with four right angles.
- **Theorem 9:** A parallelogram with a right angle is a rectangle.

Given: Parallelogram ABCD with $\angle BAD = 90°$

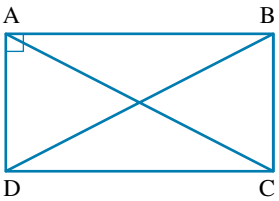
To prove: $\angle BAD = \angle ABC = \angle BCD = \angle ADC = 90°$

Proof:

AB ∥ CD	(properties of a parallelogram)
$\Rightarrow \angle BAD + \angle ADC = 180°$	(co-interior angles)
But $\angle BAD = 90°$	(given)
$\Rightarrow \angle ADC = 90°$	
Similarly, $\angle BCD = \angle ADC = 90°$	
$\therefore \angle BAD = \angle ABC = \angle BCD = \angle ADC = 90°$	

14.4.6 Theorem 10

- **Theorem 10:** The diagonals of a rectangle are equal.



Given: Rectangle ABCD with diagonals AC and BD

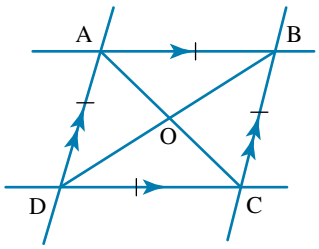
To prove: AC = BD

Proof:

In ΔADC and ΔBCD,	
AD = BC	(opposite sides equal in a rectangle)
DC = CD	(common)
$\angle ADC = \angle BCD = 90°$	(right angles in a rectangle)
$\Rightarrow \Delta ADC \equiv \Delta BCD$	(SAS)
$\therefore AC = BD$	

14.4.7 Rhombuses

- A **rhombus** is a parallelogram with four equal sides.
- **Theorem 11:** The diagonals of a rhombus are perpendicular.



Given: Rhombus ABCD with diagonals AC and BD

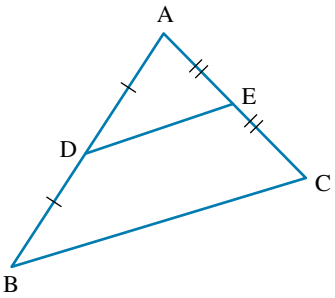
To prove: AC ⊥ BD

Proof:

In ΔAOB and ΔBOC,	
AO = OC	(property of parallelogram)
AB = BC	(property of rhombus)
BO = OB	(common)
$\Rightarrow \Delta AOB \equiv \Delta BOC$	(SSS)
$\Rightarrow \angle AOB = \angle BOC$	
But $\angle AOB + \angle BOC = 180°$	(supplementary angles)
$\Rightarrow \angle AOB = \angle BOC = 90°$	
Similarly, $\angle AOD = \angle DOC = 90°$.	
Hence, AC ⊥ BD.	

14.4.8 The midpoint theorem

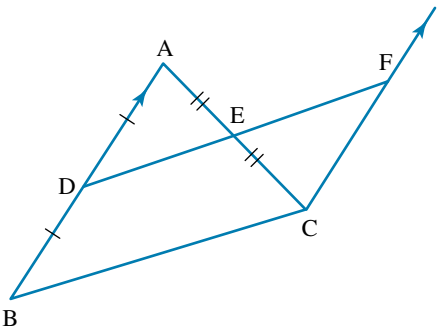
- Now that the properties of quadrilaterals have been explored, the midpoint theorem can be tackled.
- **Theorem 12:** The interval joining the midpoints of two sides of a triangle is parallel to the third side and half its length.



Given: ΔABC in which AD = DB and AE = EC

To prove: $DE \parallel BC$ and $DE = \frac{1}{2}BC$

Construction: Draw a line through C parallel to AB. Extend DE to F on the parallel line.

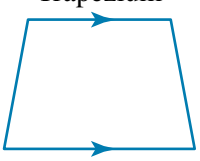


Proof: In ΔADE and ΔCEF,

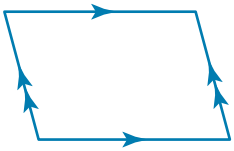
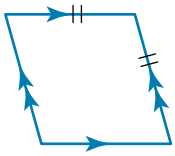
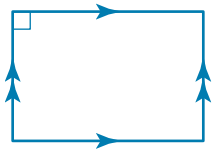
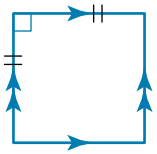
AE = EC	(E is the midpoint of AC, given)
$\angle AED = \angle CEF$	(vertically opposite angles)
$\angle EAD = \angle ECF$	(alternate angles)
$\Rightarrow \Delta ADE \equiv \Delta CEF$	(ASA)
$\therefore$ AD = CF and DE = EF	(corresponding sides in congruent triangles)
So, AD = DB = CF.	
We have $AB \parallel CF$	(by construction)
So BDFC is a parallelogram.	
$\Rightarrow DE \parallel BC$	
Also, BC = DF	(opposite sides in parallelogram)
But DE = DF	(sides in congruent triangles)
$\Rightarrow DE = \frac{1}{2}BC$	

Therefore, $DE \parallel BC$ and $DE = \frac{1}{2}BC$.

- Conversely, if a line interval is drawn parallel to a side of a triangle and half the length of that side, then the line interval bisects each of the other two sides of the triangle.
- A summary of the definitions and properties of quadrilaterals is shown in the table.

Shape	Definition	Properties
Trapezium 	A trapezium is a quadrilateral with one pair of opposite sides parallel.	• One pair of opposite sides is parallel but not equal in length.

(*continued*)

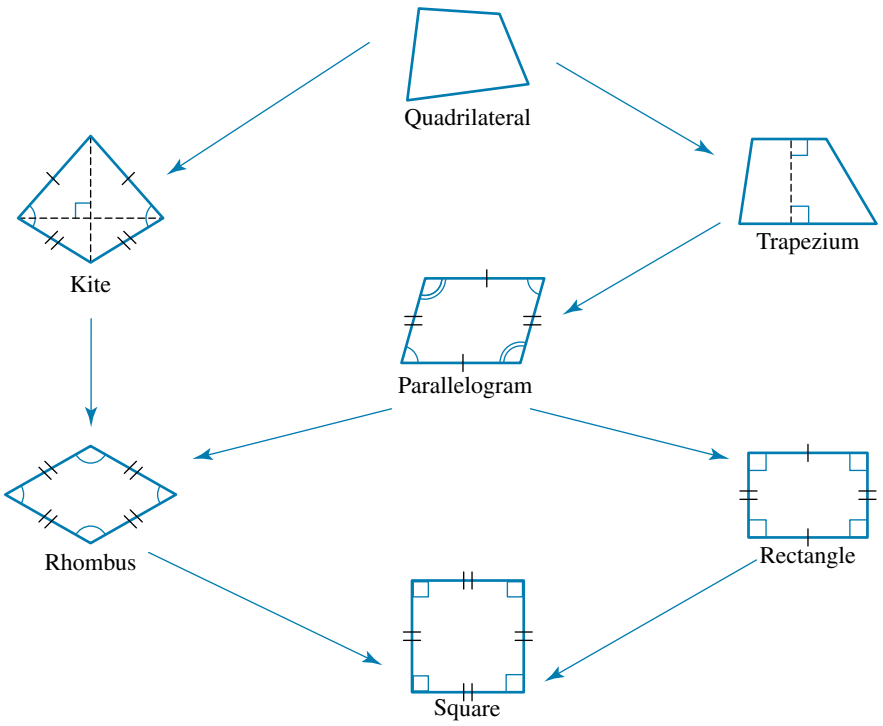
Shape	Definition	Properties
Parallelogram 	A parallelogram is a quadrilateral with both pairs of opposite sides parallel.	• Opposite angles are equal. • Opposite sides are equal. • Diagonals bisect each other.
Rhombus 	A rhombus is a parallelogram with four equal sides.	• Diagonals bisect each other at right angles. • Diagonals bisect the angles at the vertex through which they pass.
Rectangle 	A rectangle is a parallelogram whose interior angles are right angles.	• Diagonals are equal in length and bisect each other.
Square 	A square is a parallelogram whose interior angles are right angles with four equal sides.	• All angles are right angles. • All side lengths are equal. • Diagonals are equal in length and bisect each other at right angles. • Diagonals bisect the vertex through which they pass (45°).

DISCUSSION

What is the minimum information needed to identify a figure as a rhombus or a trapezium?

14.4.9 Relationships between quadrilaterals

- The flowchart below shows the relationships between quadrilaterals.



learnON RESOURCES — ONLINE ONLY

Interactivity: Angles in a quadrilateral (int-3967)
Interactivity: Opposite angles of a parallelogram (int-6160)
Interactivity: Opposite sides of a parallelogram (int-6161)
Interactivity: Diagonals of a parallelogram (int-6162)
Interactivity: Diagonals of a rectangle (int-6163)
Interactivity: Diagonals of a rhombus (int-6164)
Interactivity: The midpoint theorem (int-6165)
Interactivity: Quadrilateral definitions (int-2786)
Digital doc: SkillSHEET Identifying quadrilaterals (doc-5279)
Digital doc: WorkSHEET Deductive geometry II (doc-5283)

Exercise 14.4 Quadrilaterals

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–10	Questions: 1–14, 16	Questions: 1–21

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

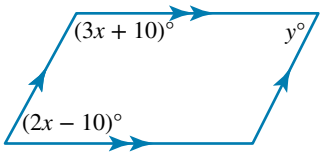
Understanding and fluency

1. Use the definitions of the five special quadrilaterals to decide if the following statements are true or false.

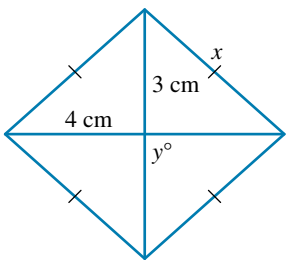
a. A square is a rectangle.
b. A rhombus is a parallelogram.
c. A square is a rhombus.
d. A rhombus is a square.
e. A square is a trapezium.
f. A parallelogram is a rectangle.
g. A trapezium is a rhombus.
h. A rectangle is a square.

2. Determine the values of x and y in each of the following figures.

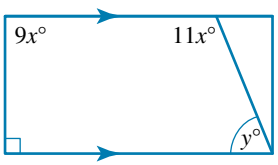
a.



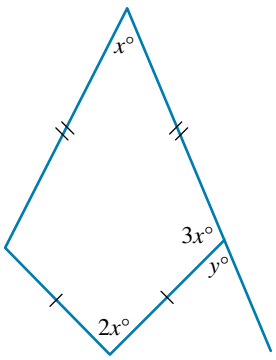
b.



c.



d.



3. Draw three different trapeziums. Using your ruler, compass and protractor, decide which of the following properties are true in a trapezium.

a. Opposite sides are equal.
b. All sides are equal.
c. Opposite angles are equal.
d. All angles are equal.
e. Diagonals are equal in length.
f. Diagonals bisect each other.
g. Diagonals are perpendicular.
h. Diagonals bisect the angles they pass through.

4. Draw three different parallelograms. Using your ruler and protractor to measure, decide which of the following properties are true in a parallelogram.

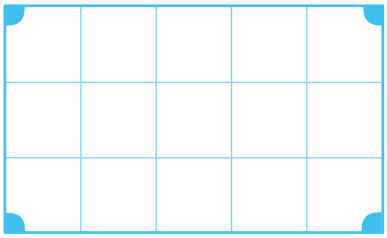
a. Opposite sides are equal.
b. All sides are equal.
c. Opposite angles are equal.
d. All angles are equal.
e. Diagonals are equal in length.
f. Diagonals bisect each other.
g. Diagonals are perpendicular.
h. Diagonals bisect the angles they pass through.

5. Name four quadrilaterals that have at least one pair of opposite sides that are parallel and equal.

6. Name a quadrilateral that has equal diagonals that bisect each other and bisect the angles they pass through.

7. Pool is played on a rectangular table. Balls are hit with a cue and bounce off the sides of the table until they land in one of the holes or pockets.

a. Draw a rectangular pool table measuring 5 cm by 3 cm on graph paper. Mark the four holes, one in each corner.

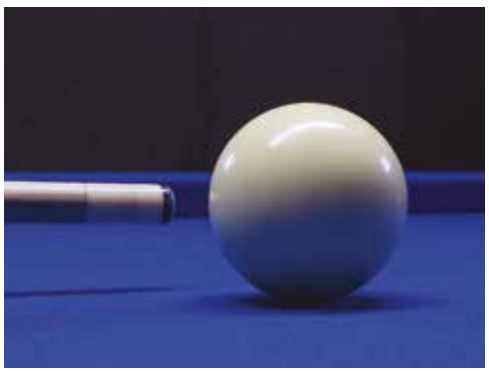


b. A ball starts at A. It is hit so that it travels at a 45° diagonal across the grid. When it hits the side of the table, it bounces off at a 45° diagonal as well. How many sides does the ball bounce off before it goes in a hole?

c. A different table is sized 7 cm by 2 cm. How many sides does a ball bounce off before it goes in a hole when hit from A?

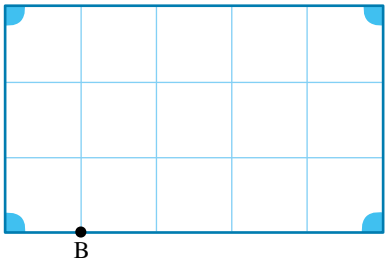
d. Complete the following table.

Table size	Number of sides hit
5 cm × 3 cm	
7 cm × 2 cm	
4 cm × 3 cm	
4 cm × 2 cm	
6 cm × 3 cm	
9 cm × 3 cm	
12 cm × 4 cm	

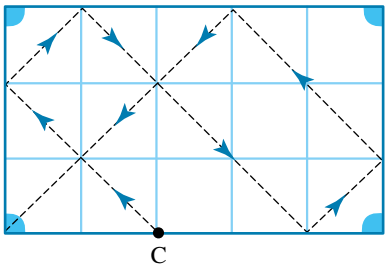


e. Can you see a pattern? How many sides would a ball bounce off before going in a hole when hit from A on an $m \times n$ table?

f. The ball is now hit from B on a 5 cm × 3 cm pool table. How many *different* paths can a ball take when hit along 45° diagonals? Do these paths all hit the same number of sides before going in a hole? Does the ball end up in the same hole each time? Justify your answer.



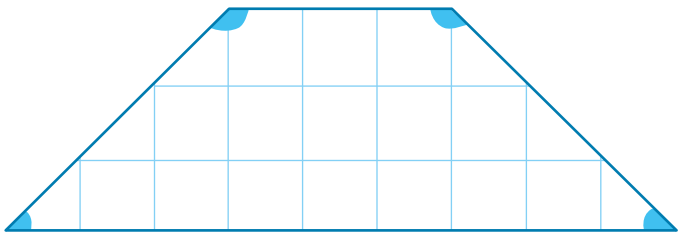
g. The ball is now hit from C along the path shown. What type of triangles and quadrilaterals are formed by the path of the ball with itself and the sides of the table? Are any of the triangles congruent?



h. A ball is hit from C on a 6 cm by 3 cm table. What shapes are formed by the path of the ball with itself and the sides of the table? Is there only one path possible?

i. A ball is hit from A along 45° diagonals. The table is $m \times n$. Can you find a formula to predict which hole the ball will go in?

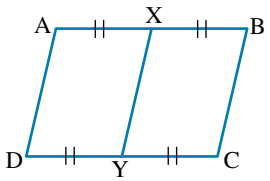
j. What would happen if the game was played on a trapezoidal table?



Communicating, reasoning and problem solving

8. Prove that the diagonals of a rhombus bisect each other.

9. ABCD is a parallelogram. X is the midpoint of AB and Y is the midpoint of DC. Prove that AXYD is also a parallelogram.

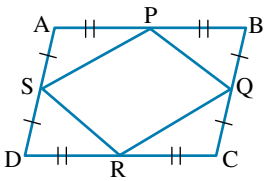


10. ABCD is a parallelogram. P, Q, R and S are all midpoints of their respective sides of ABCD.

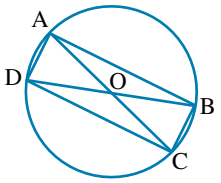
a. Prove ΔPAS ≡ ΔRCQ.

b. Prove ΔSDR ≡ ΔPBQ.

c. Hence, prove that PQRS is also a parallelogram.

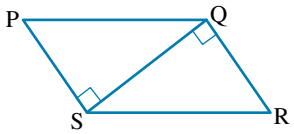


11. AC and BD are diameters of a circle with centre O. Prove that ABCD is a rectangle.

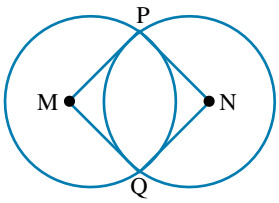


12. The diagonals of a parallelogram meet at right angles. Prove that the parallelogram is a rhombus.

13. Two congruent right-angled triangles are arranged as shown. Show that PQRS is a parallelogram.

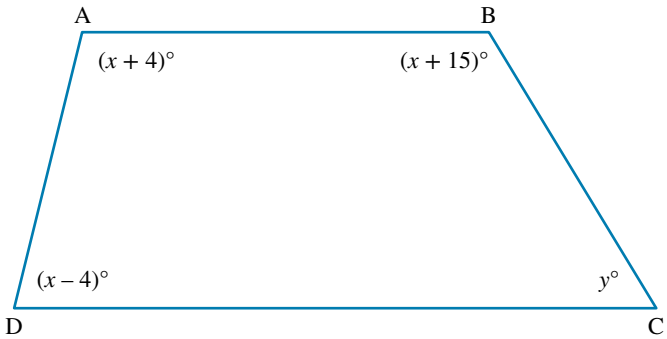


14. Two circles, centred at M and N, have equal radii and intersect at P and Q. Prove that PNQM is a rhombus.



15. Give reasons why a square is a rhombus but a rhombus is not necessarily a square.

16. ABCD is a trapezium.



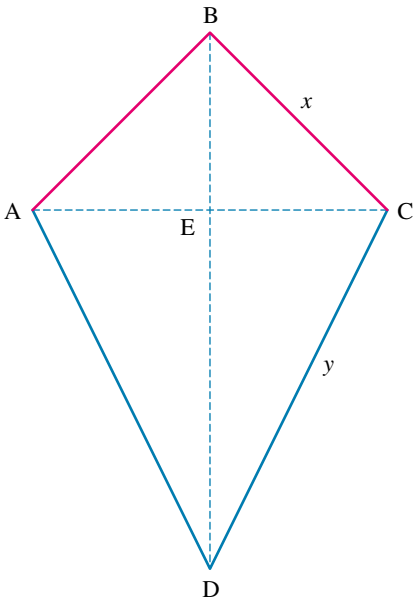
a. What fact do you know about a trapezium?

b. Find the values of x and y.

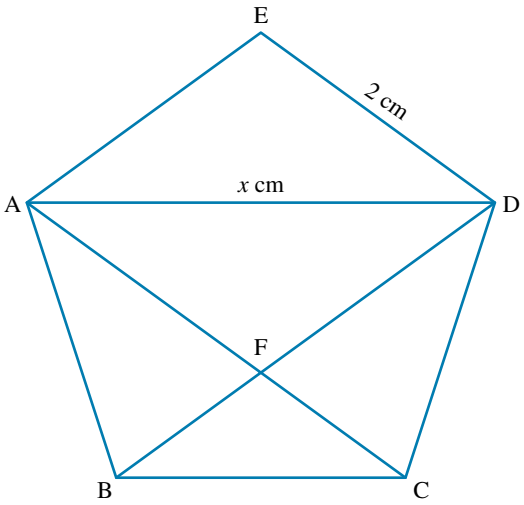
17. ABCD is a kite where AC = 8 cm, BE = 5 cm and ED = 9 cm. Find the exact values of:

a. i. x **ii.** y

b. Find angle BAD and hence angle BCD.

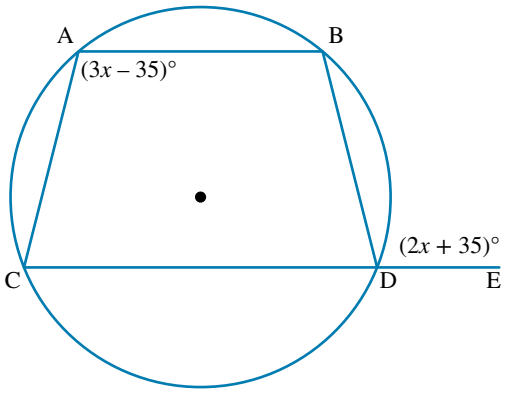


18. ABCDE is a regular pentagon whose side lengths are 2 cm. Each diagonal is x cm long.



a. What kind of shape is AEDF and what is the length of FD?
b. What kind of shape is ABCD?
c. If ∠EDA is 40°, find the value of ∠ACB, giving reasons for your findings.
d. Which triangle is similar to AED?
e. Explain why FB = $(x - 2)$ cm.
f. Show that $x^2 - 2x - 4 = 0$.
g. Solve the equation $x^2 - 2x - 4 = 0$, giving your answer as an exact value.

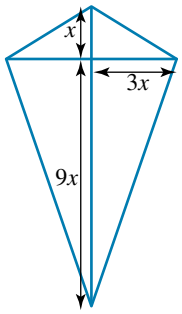
19. ABCD is called a cyclic quadrilateral because it is inscribed inside a circle.

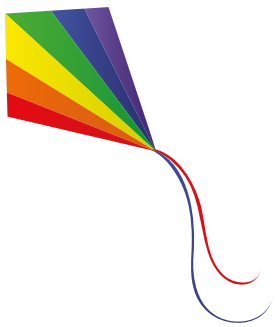


A characteristic of a cyclic quadrilateral is that the opposite angles are supplementary. Determine the value of x.

20. How do you know if a quadrilateral is a rhombus?

21. The perimeter of this kite is 80 cm. Determine the exact value of x.

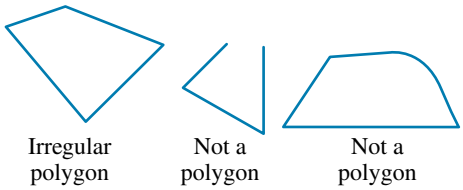




14.5 Polygons [Stages 5.2 and 5.3]

14.5.1 Definition of a polygon

- A **polygon** is a closed shape that has three or more straight sides.



- **Regular polygons** are polygons with sides of the same length and interior angles of the same size, like the pentagon shown in the centre of the photo above.
- **Convex polygons** are polygons with no interior reflex angles.
- **Concave polygons** are polygons with at least one reflex interior angle. For example, the pentagon shown above is a concave polygon as well as a regular polygon.

14.5.2 Interior angles of a polygon

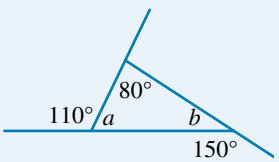
- The interior angles of a polygon are the angles inside the polygon at each vertex.
- The sum of the interior angles of a polygon is given by the formula:

$$\text{Angle sum} = 180° \times (n - 2)$$

where n = the number of sides of the polygon

WORKED EXAMPLE 6

Calculate the values of the pronumerals in the figure below.



THINK

1 Angles a and 110° form a straight line, so they are supplementary (add to 180°).

WRITE

$a + 110° = 180°$
$a + 110° - 110° = 180° - 110°$
$a = 70°$

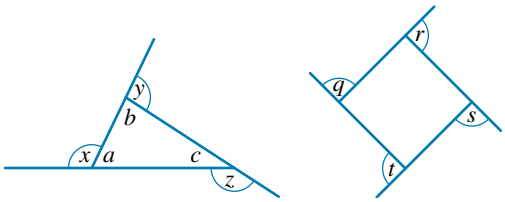
2 The interior angles of a triangle sum to 180°.

3 Substitute 70° for a and solve for b.

$$b + a + 80° = 180°$$
$$b + 70° + 80° = 180°$$
$$b + 150° = 180$$
$$b = 30°$$

14.5.3 Exterior angles of a polygon

- The exterior angles of a polygon are formed by the side of the polygon and an extension of its adjacent side. For example, x, y and z are exterior angles for the polygon (triangle) below.



- The exterior angle and interior angle at that vertex are supplementary (add to 180°). For example, $x + a = 180°$.
- Exterior angles of polygons can be measured in a clockwise or anticlockwise direction.
- In a regular polygon, the size of the exterior angle can be found by dividing 360° by the number of sides.

$$\text{Exterior angle} = \frac{360°}{n}$$

- The sum of the exterior angles of a polygon equals 360°.
- The exterior angle of a triangle is equal to the sum of the opposite interior angles.

learnON RESOURCES — ONLINE ONLY

Interactivity: Interior angles of a polygon (int-6166)
Interactivity: Exterior angles of a polygon (int-6167)
Interactivity: Angle sum of a polygon (int-0818)
Interactivity: Exterior angles of a polygon (int-0819)
Digital doc: WorkSHEET Deductive geometry III (doc-5284)

Exercise 14.5 Polygons

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–7, 9	Questions: 1–9, 12	Questions: 1–13

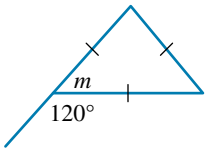
To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

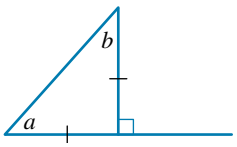
1. How are the interior and exterior angles of a polygon related to the number of sides in a polygon?

2. WE6 Calculate the values of the pronumerals in the diagrams below.

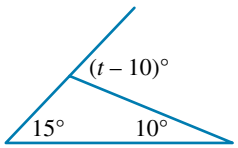
a.



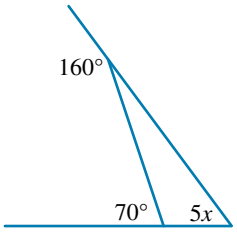
b.



c.

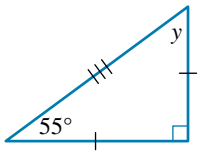


d.

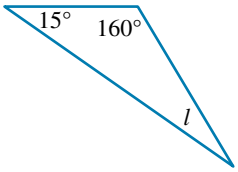


3. For the five triangles below, evaluate the pronumerals and determine the size of the interior angles.

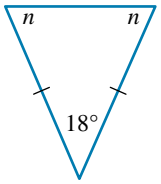
a.



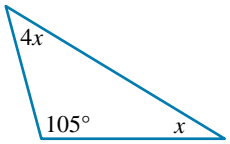
b.



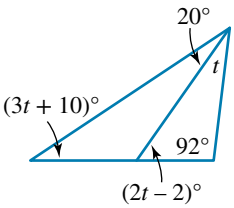
c.



d.



e.

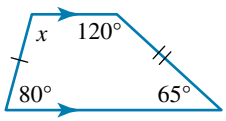


4. For the five quadrilaterals below:

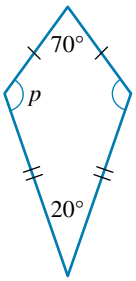
i. label the quadrilaterals as regular or irregular

ii. determine the value of the pronumeral for each shape.

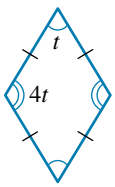
a.



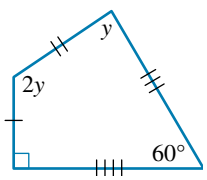
b.



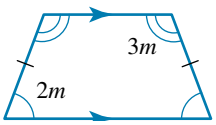
c.



d.



e.

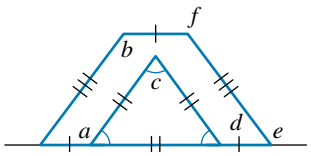


5. The photograph below shows a house built on the side of a hill. Use your knowledge of angles to calculate the values of the pronumerals.

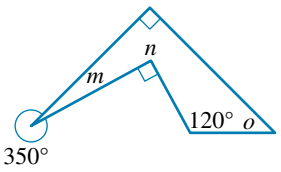
6. Calculate the values of the four interior angles of the front face of the building in the photograph below.

7. Calculate the values of the pronumerals for the irregular polygons below.

a.



b.



8. Calculate the size of the exterior angle of a regular hexagon (6 sides).

Communicating, reasoning and problem solving

9. A diagonal of a polygon joins two vertices.

a. Calculate the number of diagonals in a regular polygon with:

i. 4 sides **ii.** 5 sides **iii.** 6 sides **iv.** 7 sides.

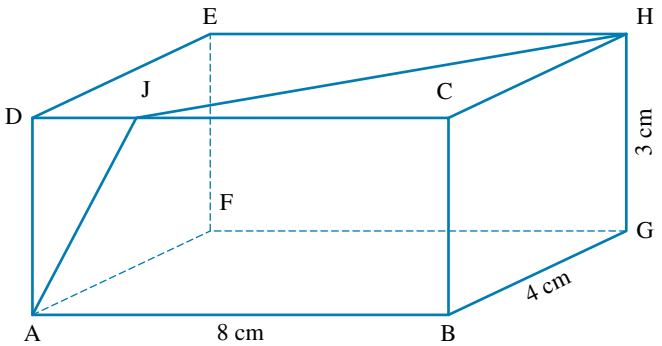
b. Write a formula that relates the number of diagonals for an n-sided polygon.

10. The exterior angle of a polygon can be calculated using the formula:

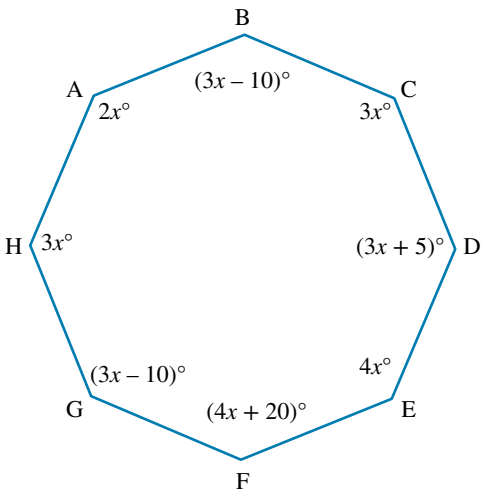
$$\text{exterior angle} = \frac{360°}{n}$$

Use the relationship between interior and exterior angles of a polygon to write a formula for the interior angle of a regular polygon.

11. A piece of string is fixed at A and H as shown. The string is tight and fixed to the surface of the cuboid. Locate the exact position of J on the edge CD.



12. ABCDEFGH is an octagon.



a. What is the sum of the interior angles of an octagon?

b. Find the value of x.

13. How are the angles associated with polygons related to each other and the polygon?

14.6 Review

Investigation | Rich task

Enlargement activity

Enlargement is the construction of a bigger picture from a small one. The picture is identical to the other except that it is bigger. The new picture is often called the image. This can also be called creating a similar figure.

The geometrical properties shared by a shape and its image under enlargement can be listed as:

- lines are enlarged as lines
- sides are enlarged to corresponding sides by the same factor
- matching angles on the two shapes are equal.

In this activity, we will start with a small cartoon character, and then 'blow it up' to almost life-size.

Equipment: ruler, pencil, cartoon print, butcher's paper or some other large piece of paper.

1. Do some research on the internet and select a cartoon character or any character of your choice.

2. Draw a grid of 2-cm squares over the small cartoon character.

Example: The small Casper is 9 squares wide and 7 squares tall.

3. Label the grids with letters across the top row and numbers down the first column, as shown in the example.

4. Get a large piece of paper and draw the same number of squares. You will have to work out the ratio of similitude (e.g. 2 : 8).
5. If your small cartoon character stretches from one side of the 'small' paper (the paper the image is printed on) to the other, your 'large' Casper must stretch from one side of the 'big' paper to the other. Your large grid squares may have to be 8 cm by 8 cm or larger, depending on the paper size.
6. Draw this enlarged grid on your large paper. Use a metre ruler or some other long straight-edged tool. Be sure to keep all of your squares the same size.
   - At this point, you are ready to DRAW. Remember, you do NOT have to be an artist to produce an impressive enlargement.
   - All you do is draw EXACTLY what you see in each small cell into its corresponding large cell.
   - For example, in cell B3 of the Casper enlargement, you see the tip of his finger, so draw this in the big grid.
   - If you take your time and are very careful, you will produce an extremely impressive enlargement.
   - What you have used is called a RATIO OF SIMILITUDE. This ratio controls how large the new picture will be.

A 2 : 5 ratio will give you a smaller enlargement than a 2 : 7 ratio, because for every two units on the original you are generating only 5 units of enlargement instead of 7.

If Casper's ratio is 1 : 4, it produces a figure that has a linear measure that is four times bigger.

Big Casper's overall area, however, will be 16 times larger than small Casper's. This is because area is found by taking length times width.

The length is 4 times longer and the width is 4 times longer.

Thus the area is $4 \times 4 = 16$ times larger than the original Casper.

His overall volume will be $4 \times 4 \times 4$ or 64 times larger! This means that big Casper will weigh 64 times more than small Casper.

learnon RESOURCES — ONLINE ONLY

Digital doc: Investigation — Enlargement activity (doc-15934)
Digital doc: Code puzzle — Why was the archaeologist upset? (doc-15935)
Digital doc: Summary Topic 14 (doc-22912)
Interactivity: Word search Topic 14 (int-2853)
Interactivity: Crossword Topic 14 (int-2854)
Interactivity: Sudoku Topic 14 (int-3597)

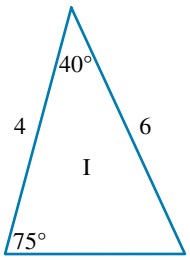
Exercise 14.6 Review questions

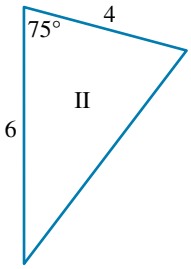
To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

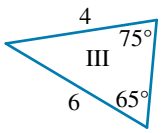
Understanding and fluency

1. Select a pair of congruent triangles in each of the following sets of triangles, giving a reason for your answer. All angles are in degrees and side lengths in cm. (The figures are not drawn to scale.)

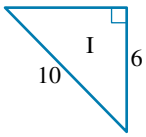
a.

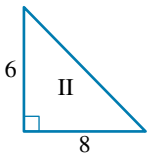


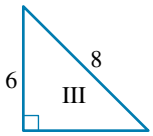




b.

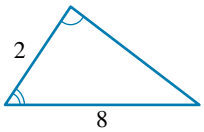


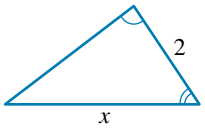




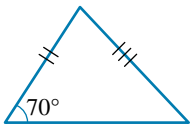
2. Find the value of the pronumeral in each pair of congruent triangles. All angles are given in degrees and side lengths in cm.

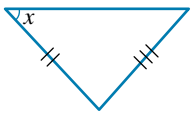
a.



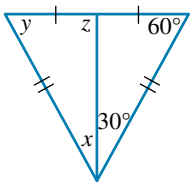


b.

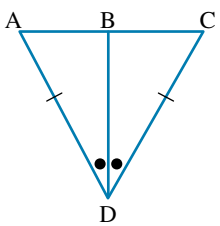




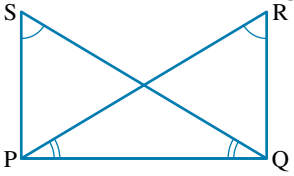
c.



3. a. Prove that the two triangles shown in the diagram below are congruent.

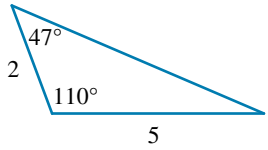


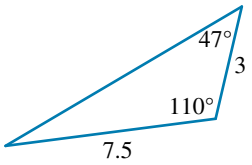
b. Prove that ΔPQR is congruent to ΔQPS.



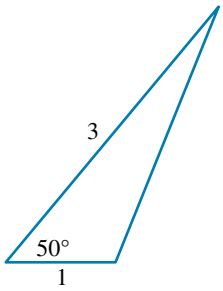
4. Test whether the following pairs of triangles are similar. For similar triangles find the scale factor. All angles are in degrees and side lengths in cm.

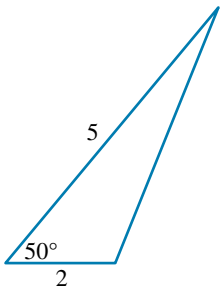
a.



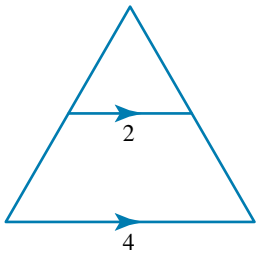


b.



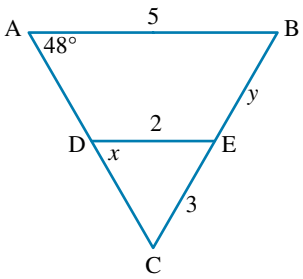


c.

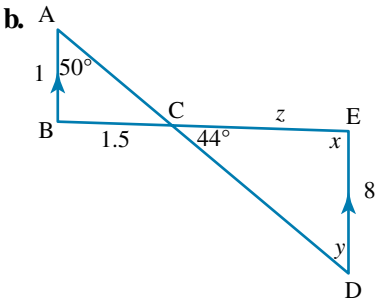


5. Find the value of the pronumeral in each pair of similar triangles. All angles are given in degrees and side lengths in cm.

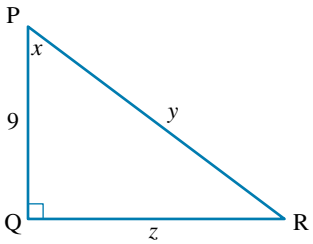
a.

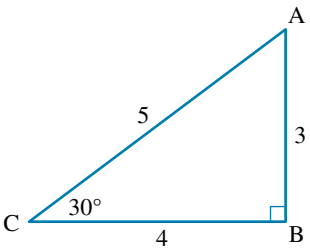


b.

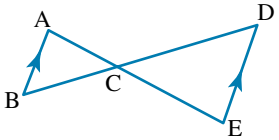


c.

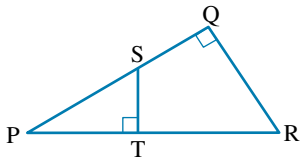




6. Prove that $\Delta ABC \;|||\; \Delta EDC$.



7. Prove that $\Delta PST \,|||\, \Delta PRQ$.



8. Prove that the angles opposite the equal sides in an isosceles triangle are equal.

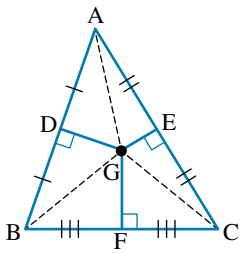
9. MC Two corresponding sides in a pair of similar octagons have lengths of 4 cm and 60 mm. The respective scale factor in length is:

A. 4 : 60 **B.** 6 : 40 **C.** 40 : 60 **D.** 60 : 40 **E.** 40 : 6

10. A regular nonagon has side length x cm. Use a scale factor of $\frac{x+1}{x}$ to find the side length of a similar nonagon.

Communicating, reasoning and problem solving

11. ABC is a triangle. D is the midpoint of AB, E is the midpoint of AC and F is the midpoint of BC. DG ⊥ AB, EG ⊥ AC and FG ⊥ BC.



a. Prove that $\Delta GDA \equiv \Delta GDB$.

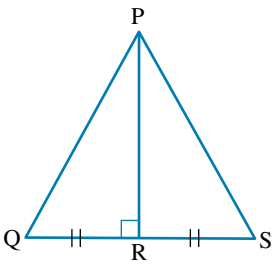
b. Prove that $\Delta GAE \equiv \Delta GCE$.

c. Prove that $\Delta GBF \equiv \Delta GCF$.

d. What does this mean about AG, BG and CG?

e. A circle centred at G is drawn through A. What other points must it pass through?

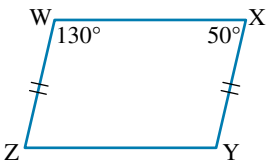
12. PR is the perpendicular bisector of QS. Prove that ΔPQS is isosceles.



13. Name any quadrilaterals that have diagonals that bisect the angles they pass through.

14. State three tests that can be used to show that a quadrilateral is a rhombus.

15. Prove that WXYZ is a parallelogram.



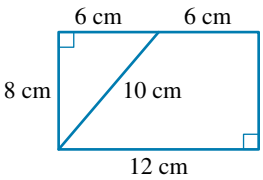
16. Prove that the diagonals in a rhombus bisect the angles they pass through.

17. Explain why the triangles shown are not congruent.

18. State the definition of a rhombus.

19. Name any quadrilaterals that have equal diagonals.

20. This 8 cm by 12 cm rectangle is cut into two sections as shown.



a. Draw labelled diagrams to show how the two sections can be rearranged to form:

i. a parallelogram

ii. a right-angled triangle

iii. a trapezium.

b. Comment on the perimeters of the figures.

Answers

Topic 14 Properties of geometrical figures

Exercise 14.2 Angles, triangles and congruence

1. a. $a = 56°$ **b.** $b = 30°$ **c.** $c = 60°$ **d.** $d = 120°$ **e.** $e = 68°$

2. a. I and III, SAS **b.** I and II, AAS **c.** II and III, RHS **d.** I and II, SSS

3. a. $x = 6$, $y = 60°$ **b.** $x = 80°$, $y = 50°$ **c.** $x = 32°$, $y = 67°$ **d.** $x = 45°$, $y = 45°$

4. a. $x = 3$ cm

b. $x = 85°$

c. $x = 80°$, $y = 30°$, $z = 70°$

d. $x = 30°$, $y = 7$ cm

e. $x = 40°$, $y = 50°$, $z = 50°$, $m = 90°$, $n = 90°$

5. a. Use SAS **b.** Use SAS. **c.** Use ASA. **d.** Use ASA. **e.** Use SSS.

6. C, D

7. a. $x = 110°$, $y = 110°$, $z = 4$ cm, $w = 7$ cm

b. $x = 70°$

c. $x = 30°$, $y = 65°$

8. The third sides are not necessarily equal.

9. Corresponding sides are not the same.

10. Use SSS.

11, 12, 13. Check with your teacher.

14. $x = 20°$, $y = 10°$, $z = 40°$

15. Check with your teacher.

Exercise 14.3 Similar triangles

1. a. i and **iii**, RHS **b. i** and **ii**, SAS **c. i** and **iii**, SSS **d. i** and **iii**, AAA **e. i** and **ii**, SSS

2. a. Triangles PQR and ABC **b.** Triangles ADB and ADC **c.** Triangles PQR and TSR

d. Triangles ABC and DEC **e.** Triangles ABC and DEC

3. a. $\frac{AB}{AD} = \frac{BC}{DE} = \frac{AC}{AE}$ **b.** $f = 9$, $g = 8$

4. $x = 4$

5. $x = 20°$, $y = 2\frac{1}{4}$

6. a. AAA **b.** $x = 3$, $y = 4$

7. The slide is 7.23 m long.

8. Radius = 0.625 m

9. a. $x = 1$, $y = 7\sqrt{2}$ **b.** $x = 2.5$, $y = 3.91$

10. Check with your teacher. One method would be to use Pythagorean triples.

11. a. $\angle ABD = \angle ABC$ (common angle)

$\angle ADB = \angle BAC = 90°$

$\Delta ABD \,|||\, \Delta ACB$ (AAA)

b. $\angle ACD = \angle BCA$ (common angle)

$\angle ADC = \angle CAB = 90°$

$\Delta ACD \,|||\, \Delta ACB$ (AAA)

12. Congruent triangles must be identical; that is, the angles must be equal and the side lengths must be equal. Therefore, it is not enough just to prove that the angles are equal.

13. Check with your teacher.

14. $\angle FEO = \angle OGH$ (alternate angles equal as EF || HG)

$\angle EFO = \angle OHG$ (alternate angles equal as EF || HG)

$\angle EOF = \angle HOG$ (vertically opposite angles equal)

$\therefore \Delta EFO \,|||\, \Delta GHO$ (equiangular)

15. $x = 6$ or 11

16. Similar triangles have equal angles or corresponding sides are proportional.

17. 3

18. Check with your teacher.

Exercise 14.4 Quadrilaterals

1. a. True **b.** True **c.** True **d.** False

e. False **f.** False **g.** False **h.** False

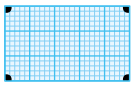
2. a. $x = 36°$, $y = 62°$ **b.** $x = 5$ cm, $y = 90°$ **c.** $x = 10°$, $y = 70°$ **d.** $x = 40°$, $y = 60°$

3. None are true, unless the trapezium is a regular trapezium, then e is true.

4. **a**, **c**, **f**

5. Parallelogram, rhombus, rectangle, square

6. Square

7. a. 

b. 6 sides

c. 7 sides

d.

Table size	Number of sides hit
5 cm × 3 cm	6
7 cm × 2 cm	7
4 cm × 3 cm	5
4 cm × 2 cm	1
6 cm × 3 cm	1
9 cm × 3 cm	2
12 cm × 4 cm	2

e. If the ratio of the sides is written in simplest form, then the pattern is $m + n - 2$.

f. There are two routes for the ball when hit from B. Either 2 or 3 sides are hit. The ball does not end up in the same hole each time.

A suitable justification would be a diagram — student to draw.

g. Isosceles triangles and parallelograms. The triangles are congruent.

h. The shapes formed are parallelograms. There is only one possible path although the ball could be hit in either of two directions initially.

i. Given: $m : n$ is the ratio length to width in simplest form.

When m is even and n, is odd the destination pocket will be the upper left.

When m and n are both odd, the destination pocket will be the upper right.

When m is odd and n is even, the destination pocket will be the lower right.

j. Students to investigate.

8. Check with your teacher.

9. AX ∥ DY because ABCD is a parallelogram.

AX = DY (given)

∴ AXYD is a parallelogram since opposite sides are equal and parallel.

10. a. Use SAS. **b.** Use SAS. **c.** Opposite sides are equal.

11. AC = DB (diameters of the same circle are equal)

AO = OC and OD = OB (radii of the same circle are equal)

∴ ABCD is a rectangle (diagonals are equal and bisect each other).

12. Check with your teacher.

13. PS = QR (corresponding sides in congruent triangles are equal)

PS∥QR (alternate angles are equal)

∴ PQRS is a parallelogram since one pair of opposite sides are parallel and equal.

14. MP = MQ (radii of same circle)
PN = QN (radii of same circle) and circles have equal radii.
∴ All sides are equal.
∴ PNQM is a rhombus.

15. Check with your teacher.

16. a. One pair of opposite sides is parallel.

b. $x = 90°$, $y = 75°$

17. a. i. $x = \sqrt{41}$ **ii.** $y = \sqrt{97}$

b. $\angle BAD = \angle BCD = 117°23'$

18. a. Rhombus, 2 cm **b.** Trapezium **c.** 40° **d.** Triangle BFC

e. Check with your teacher. **f.** Check with your teacher. **g.** $x = (1 + \sqrt{5})$ cm

19. 70°

20. Rhombuses have four equal sides.

21. $x = \sqrt{10}$ cm

Exercise 14.5 Polygons

1. The sum of the interior angles is based on the number of sides of the polygon.
The size of the exterior angle can be found by dividing 360° by the number of sides.

2. a. $m = 60°$ **b.** $a = 45°$, $b = 45°$ **c.** $t = 35°$ **d.** $x = 10°$

3. a. $y = 35°$ **b.** $t = 5°$ **c.** $n = 81°$ **d.** $x = 15°$ **e.** $t = 30°$

4. a. i. Irregular **ii.** $x = 95°$

b. i. Irregular **ii.** $p = 135°$

c. i. Irregular **ii.** $t = 36°$

d. i. Irregular **ii.** $y = 70°$

e. i. Irregular **ii.** $p = 36°$

5. $w = 75°$, $x = 105°$, $y = 94°$, $z = 133°$

6. 82.5°, 82.5°, 97.5°, 97.5°

7. a. $a = 120°$, $b = 120°$, $c = 60°$, $d = 60°$, $e = 120°$, $f = 240°$

b. $m = 10°$, $n = 270°$, $o = 50°$

8. 60°

9. a. i. 2 **ii.** 5 **iii.** 9 **iv.** 14

b. Number of diagonals $= \frac{1}{2}n(n - 3)$

10. Interior angle $= 180° - \frac{360°}{n}$

11. J is $\frac{24}{7}$ cm from D.

12. a. 1080° **b.** 43°

13. Discuss with teacher

Investigation | Rich task

Check with your teacher.

Exercise 14.6 Review questions

1. a. I and III, ASA or SAS **b.** I and II, RHS

2. a. $x = 8$ cm **b.** $x = 70°$ **c.** $x = 30°$, $y = 60°$, $z = 90°$

3. a. Use SAS. **b.** Use ASA.

4. a. Similar, scale factor = 1.5

b. Not similar

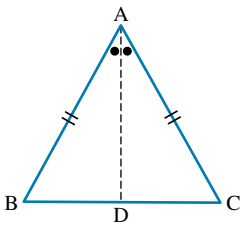
c. Similar, scale factor = 2

5. a. $x = 48°$, $y = 4.5$ cm **b.** $x = 86°$, $y = 50°$, $z = 12$ cm **c.** $x = 60°$, $y = 15$ cm, $z = 12$ cm

6. Use the equiangular test.

7. Use the equiangular test.

8.



Bisect ∠BAC.

AB = AC (given)

∠BAD = ∠DAC

AD is common.

∴ ΔABD ≡ ΔACD (SAS)

∴ ∠ABD = ∠ACD (corresponding sides in congruent triangles are equal)

9. C

10. $x + 1$

11. a. Use SAS. **b.** Use SAS. **c.** Use SAS.

d. They are all the same length.

e. B and C

12. Use SAS.

PQ = PS (corresponding sides in congruent triangles are equal)

13. Rhombus, square

14. A quadrilateral is a rhombus if:

a. all sides are equal

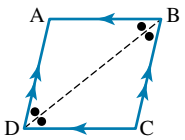
b. the diagonals bisect each other at right angles

c. the diagonals bisect the angles they pass through.

15. WZ ∥ XY (co-interior angles are supplementary) and WZ = XY (given)

∴ WXYZ is a parallelogram since one pair of sides is parallel and equal.

16.



∠ABD = ∠ADB (angles opposite the equal sides in an isosceles triangle are equal)

∠ABD = ∠BDC (alternate angles equal as AB ∥ DC)

∴ ∠ADB = ∠BDC

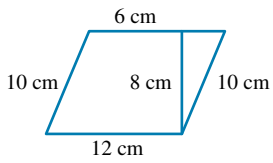
∴ Diagonals bisect the angles they pass through.

17. Corresponding sides are not the same.

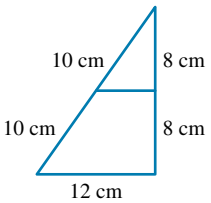
18. A rhombus is a parallelogram with two adjacent sides equal in length.

19. Rectangle, square

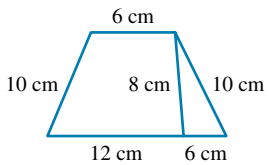
20. a. i.



ii.



iii.



b. Perimeter of rectangle = 40 cm

Perimeter of parallelogram = 44 cm

Perimeter of triangle = 48 cm

Perimeter of trapezium = 44 cm

The triangle has the largest perimeter, and the rectangle has the smallest.

TOPIC 15
Bivariate data analysis [Stages 5.2 and 5.3]

15.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the concepts covered in this topic.

15.1.1 Why learn this?

Observations of two or more variables are often recorded, for example the heights and weights of individuals. Studying the data allows us to investigate whether there is any relationship between the variables, how strong the relationship is, and whether one variable can be effectively predicted from information about another variable. Statistics can be applied to medical research, sport, agriculture, sustainability, weather forecasting and fashion trends, to name but a few fields. The capacity to analyse data and draw conclusions is an essential skill in a world where information is readily available and often manipulated.

DISCUSSION

Choose an area of interest from the following list: medical research, sport, sustainability and weather forecasting. Think of two variables that could be investigated to determine if a relationship exists between the variables.

LEARNING SEQUENCE

15.1 Overview
15.2 [Stage 5.2] Bivariate data
15.3 [Stage 5.3] Lines of best fit
15.4 [Stages 5.2 and 5.3] Time as the independent variable
15.5 Review

LEARNING OUTCOMES

A student:

- selects appropriate notations and conventions to communicate mathematical ideas and solutions MA5.2-1WM
- constructs arguments to prove and justify results MA5.2-3WM
- investigates relationships between two statistical variables, including their relationship over time MA5.2-16SP
- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3-1WM
- generalises mathematical ideas and techniques to analyse and solve problems efficiently MA5.3-2WM
- investigates the relationship between numerical variables using lines of best fit MA5.3-19SP

CONTENT DESCRIPTIONS

Students:
Investigate and describe bivariate numerical data where the independent variable is time (ACMSP252)
Use scatter plots to investigate and comment on relationships between two numerical variables (ACMSP251)
Use information technologies to investigate bivariate numerical data sets; where appropriate, students use a straight line to describe the relationship, allowing for variation (ACMSP279)

Source: NSW Syllabus for the Australian curriculum

learnon RESOURCES — ONLINE ONLY

eLesson: The story of mathematics — Florence Nightingale (eles-1853)

Note: Your teacher may now set you a pre-test to determine how familiar you are with the content in this topic.

15.2 Bivariate data [Stage 5.2]

15.2.1 Definition of bivariate data

- **Bivariate data** are data with two variables.
- A list of bivariate data can be considered as numerical pairs of the type:
$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$
- Bivariate data are usually represented graphically on scatter plots.

DISCUSSION

Using some examples, describe the difference between bivariate data and single variable data.

15.2.2 Scatter plots

- A **scatter plot** is a graph that shows whether there is a relationship between two variables.
- Each data value on a scatter plot is shown by a point on a Cartesian plane.

15.2.3 Dependent and independent variables

- One variable is generally the **dependent variable**, and the other is the **independent variable**.
- The dependent variable, as the name suggests, is the one whose value depends on the other variable. The independent variable takes on values that do not depend on the value of the other variable.
- When data are expressed in the form of a table, generally the independent variable is written in the first row or the first column.
- The independent variable is placed on the x-axis and the dependent variable on the y-axis.

DISCUSSION

In your own words, explain the relationship between independent and dependent variables. How can you identify which variable is the independent variable and which is the dependent variable?

WORKED EXAMPLE 1

The table shows the total revenue from selling tickets for a number of different chamber music concerts. Represent these data on a scatter plot.

Number of tickets sold	400	200	450	350	250	300	500	400	350	250
Total revenue ($)	8000	3600	8500	7700	5800	6000	11 000	7500	6600	5600

THINK

1 Determine which is the dependent variable and which is the independent variable.

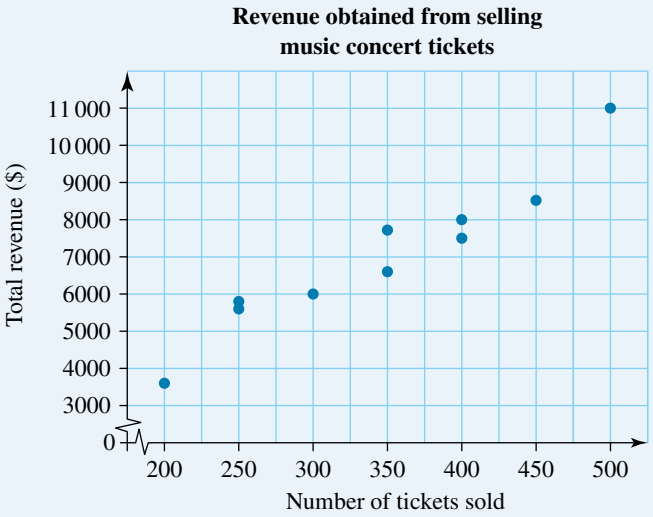
2 Rule up a set of axes. Label the title of the graph. Label the horizontal axis 'Number of tickets sold' and the vertical axis 'Total revenue ($)'.

3 Use an appropriate scale on the horizontal and vertical axes.

4 Plot the points on the scatter plot.

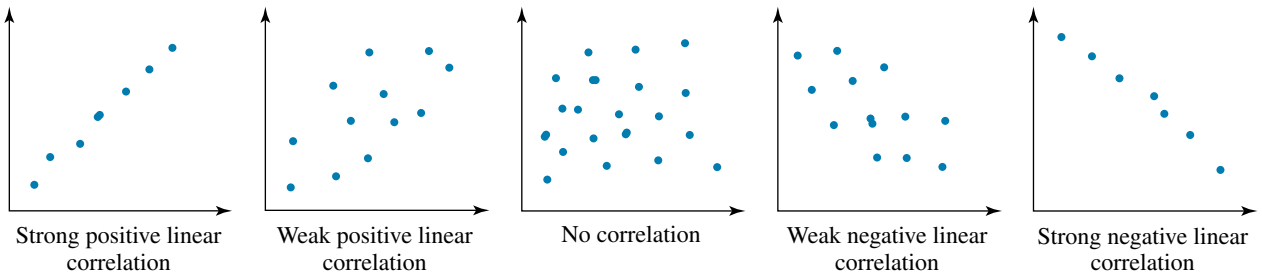
WRITE/DRAW

The total revenue depends on the number of tickets being sold, so the number of tickets is the independent variable and the total revenue is the dependent variable.



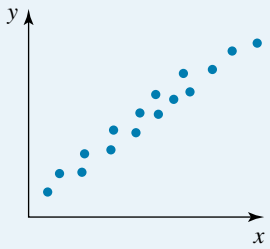
15.2.4 Describing the relationship between two variables

- The **strength**, the **direction** and the **form** of the relationship between two variables may be displayed in a scatter plot.
- The closer the points are to a straight line form, the stronger the relationship between the two variables. The strength is described as **weak**, **moderate** or **strong**.
- If the points on a scatter plot have a generally positive slope, the relationship has a positive direction. If the slope is negative, the direction is negative.



WORKED EXAMPLE 2

State the type of relationship between the variables *x* and *y* shown on the scatter plot.



THINK	WRITE
Carefully analyse the scatter plot and comment on its form, direction and strength.	The points on the scatter plot are close together and constantly increasing therefore the relationship is linear. The path is directed from the bottom left corner to the top right corner, and the value of *y* increases as *x* increases. Therefore the relationship is positive. The points are close together so the relationship can be classified as strong. There is a strong, positive, linear relationship between *x* and *y*.

15.2.5 A relationship does not always mean causation

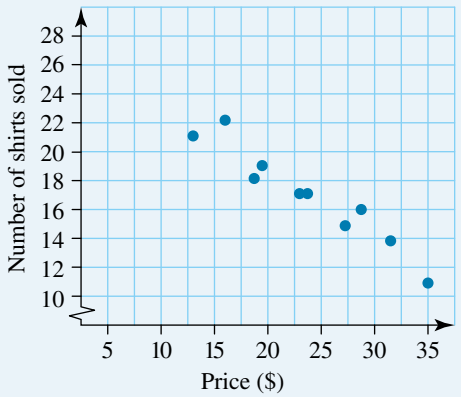
- Even a strong relationship does not necessarily mean that the increase or decrease in the level of one variable *causes* an increase or decrease in the level of the other. It is best to avoid statements such as: 'An increase in rainfall *causes* an increase in the wheat growth.'
- The following guidelines should be closely followed in order to draw a conclusion about the relationship between the two variables based on the scatter plot.
  - If the relationship between *x* and *y* is *weak*, we can conclude that *there is little evidence to show* that the larger *x* is, the larger (positive relationship) or smaller (negative relationship) *y* is.
  - If the relationship between *x* and *y* is *moderate*, we can conclude that *there is evidence to show* that the larger *x* is, the larger (positive relationship) or smaller (negative relationship) *y* is.
  - If the relationship between *x* and *y* is *strong*, we can conclude that the larger *x* is, the larger (positive relationship) or smaller (negative relationship) *y* is.

WORKED EXAMPLE 3

Mary sells business shirts in a department store. She always records the number of different styles of shirt sold during the day. The table below shows her sales over one week.

Price ($)	14	18	20	21	24	25	28	30	32	35
Number of shirts sold	21	22	18	19	17	17	15	16	14	11

a Construct a scatter plot of the data.
b State the type of relationship between the two variables and draw a corresponding conclusion.

THINK	WRITE/DRAW
a Draw the scatter plot showing 'Price ($)' (independent variable) on the horizontal axis and 'Number of shirts sold' (dependent variable) on the vertical axis.	**a** 
b 1 Carefully analyse the scatter plot and comment on its form, direction and strength.	**b** The points on the plot form a path that resembles a straight, narrow band, directed from the top left corner to the bottom right corner. The points are close to forming a straight line. There is a strong, negative, linear relationship between the two variables.
2 Draw a conclusion corresponding to the analysis of the scatter plot.	The price of the shirt appears to affect the number sold; that is, the more expensive the shirt, the fewer are sold.

learnON RESOURCES — ONLINE ONLY

Digital doc: SkillSHEET Solving linear equations that arise when finding *x*- and *y*-intercepts (doc-5406)
Digital doc: SkillSHEET Transposing linear equations to standard form (doc-5407)
Digital doc: SkillSHEET Measuring the rise and the run (doc-5408)
Digital doc: SkillSHEET Graphing linear equations using the *x*- and *y*- intercept method (doc-5410)
Digital doc: SkillSHEET Determining independent and dependent variables (doc-5411)
Digital doc: SkillSHEET Determining the type of relationship (doc-5413)

Exercise 15.2 Bivariate data

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–d, 2–6, 8, 12	Questions: 1d–g, 2–5, 7, 9, 12	Questions: 1e–h, 2–4, 7–13

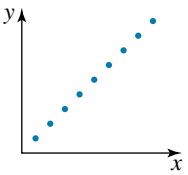
To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

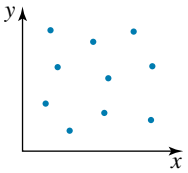
Understanding and fluency

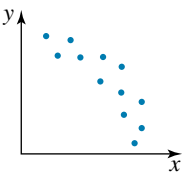
1. For each of the following pairs, decide which of the variables is independent and which is dependent.
   - **a.** Number of hours spent studying for a Mathematics test and the score on that test
   - **b.** Daily amount of rainfall (in mm) and daily attendance at the Botanical Gardens
   - **c.** Number of hours per week spent in a gym and the annual number of visits to the doctor
   - **d.** The amount of computer memory taken by an essay and the length of the essay (in words)
   - **e.** The cost of care in a childcare centre and attendance at the childcare centre
   - **f.** The cost of the property (real estate) and the age of the property
   - **g.** The entry requirements for a certain tertiary course and the number of applications for that course
   - **h.** The heart rate of a runner and the running speed
2. **WE1** The following table shows the cost of a wedding reception at 10 different venues. Represent the data on a scatter plot.

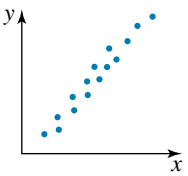
No of guests	30	40	50	60	70	80	90	100	110	120
Total cost (× $1000)	1.5	1.8	2.4	2.3	2.9	4	4.3	4.5	4.6	4.6

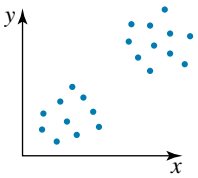
3. **WE2** State the type of relationship between x and y for each of the following scatter plots.

a.


b.


c.


d.


e.


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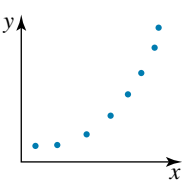
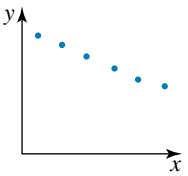
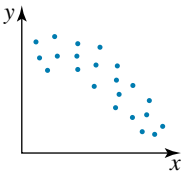
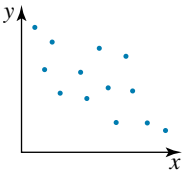
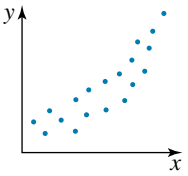
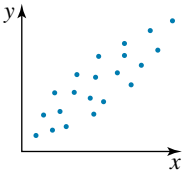


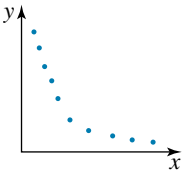
g. 

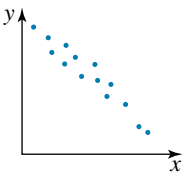
h. 

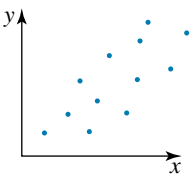
i. 

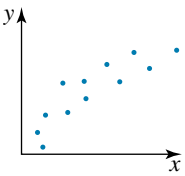
j. 

k. 

l. 

m. 

n. 

o. 

4. WE3 Eugene is selling leather bags at the local market. During the day he keeps records of his sales. The table below shows the number of bags sold over one weekend and their corresponding prices (to the nearest dollar).

Price ($) of a bag	30	35	40	45	50	55	60	65	70	75	80
Number of bags sold	10	12	8	6	4	3	4	2	2	1	1

a. Construct a scatter plot of the data.

b. State the type of relationship between the two variables and draw a corresponding conclusion.

5. The table below shows the number of bedrooms and the price of each of 30 houses.

Number of bedrooms	**Price (× $1000)**	**Number of bedrooms**	**Price (× $1000)**	**Number of bedrooms**	**Price (× $1000)**
2	180	3	279	3	243
2	160	2	195	3	198
3	240	6	408	3	237
2	200	4	362	2	226
2	155	2	205	4	359
4	306	7	420	4	316
3	297	5	369	2	200
5	383	1	195	2	158
2	212	3	265	1	149
4	349	2	174	3	286

a. Construct a scatter plot of the data.
b. State the type of relationship between the number of bedrooms and the price of the house and, hence, draw a corresponding conclusion.
c. Suggest other factors that could contribute to the price of the house.

6. The table below shows the number of questions solved by each student on a test, and the corresponding total score on that test.

Number of questions	2	4	7	10	5	2	6	3	9	4	8	3	6
Total score (%)	22	39	69	100	56	18	60	36	87	45	84	32	63

a. Construct a scatter plot of the data.
b. What type of relationship does the scatter plot suggest?
c. Give a possible explanation as to why the scatter plot is not perfectly linear.

7. A sample of 25 drivers who had obtained a full licence within the last month was asked to recall the approximate number of driving lessons they had taken (to the nearest 5) and the number of accidents they had had while being on P plates. The results are summarised in the table that follows.

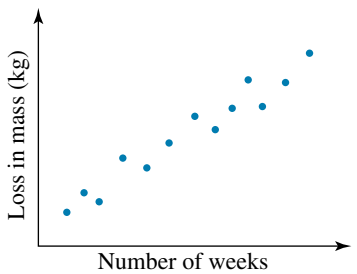
Number of lessons	**Number of accidents**
5	6
20	2
15	3
25	3
10	4
35	0
5	5
15	1
10	3
20	1
40	2
25	2
10	5

Number of lessons	**Number of accidents**
5	5
20	3
40	0
25	4
30	1
15	4
35	1
5	4
30	0
15	2
20	3
10	4

a. Represent these data on a scatter plot.
b. Specify the relationship suggested by the scatter plot.
c. Suggest some reasons why this scatter plot is not perfectly linear.

Communicating, reasoning and problem solving

8. Each point on the scatter plot below shows the time (in weeks) spent by a person on a healthy diet and the corresponding mass lost (in kg).



Study the scatter plot and state whether each of the following statements is true or false.

a. The number of weeks that the person stays on a diet is the independent variable.

b. The y-coordinates of the points represent the time spent by a person on a diet.

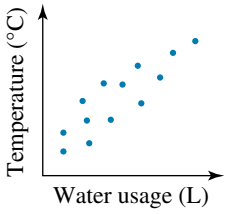
c. There is evidence to suggest that the longer the person stays on a diet, the greater the loss in mass.

d. The time spent on a diet is the only factor that contributes to the loss in mass.

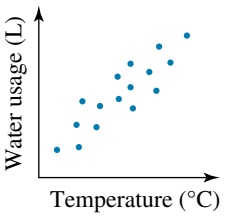
e. The relationship between the number of weeks on a diet and the number of kilograms lost is positive.

9. MC The scatter plot that best represents the relationship between the amount of water consumed daily by a certain household for a number of days in summer and the daily temperature is:

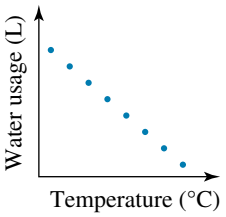
A.



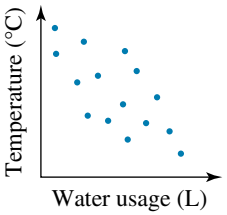
B.



C.

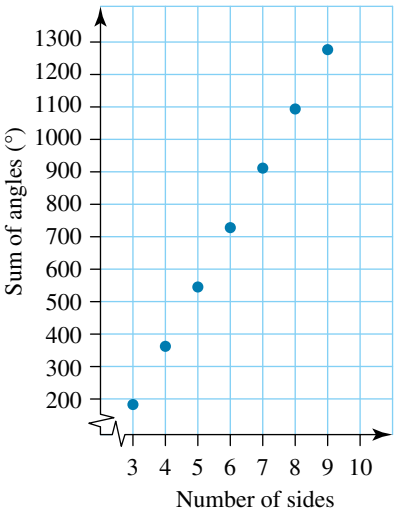


D.



E. None of the above

10. MC The scatter plot at right shows the number of sides and the sum of interior angles for a number of polygons. Which of the following statements is not true?



A. The relationship between the number of sides and the angle sum of the polygon is perfectly linear.

B. The increase in the number of sides causes the increase in the size of the angle sum.

C. The number of sides depends on the sum of the angles.

D. The relationship between the two variables is positive.

E. The relationship between the two variables is strong.

11. MC After studying a scatter plot, it was concluded that there was evidence that the greater the level of one variable, the smaller the level of the other variable. The scatter plot must have shown a:

A. strong, positive relationship

B. strong, negative relationship

C. moderate, positive relationship

D. moderate, negative relationship

E. weak, positive relationship

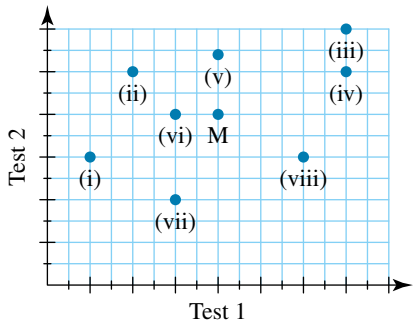
12. The table below gives the number of kicks and handballs obtained by the top 8 players in an AFL game.

Player	A	B	C	D	E	F	G	H
Number of kicks	20	27	21	19	17	18	21	22
Number of handballs	11	3	11	6	5	1	9	7

a. Represent this information on a scatter plot by using the x-axis as the number of kicks and the y-axis as the number of handballs.

b. Does the scatter plot support the claim that the more kicks a player obtains, the more handballs he gives?

13. The scatter plot shown gives the marks obtained by students in two mathematics tests. Mardi's score in the tests is represented by M. Which point represents each of the following students?



- **a.** Mandy, who got the highest mark in both tests
- **b.** William, who got the top mark in test 1 but not in test 2
- **c.** Charlotte, who did better on test 1 than Mardi but not as well in test 2
- **d.** Dario, who did not do as well as Charlotte in both tests
- **e.** Edward, who got the same mark as Mardi in test 2 but did not do so well in test 1
- **f.** Cindy, who got the same mark as Mardi for test 1 but did better than her for test 2
- **g.** Georgina, who was the lowest in test 1
- **h.** Harrison, who had the greatest discrepancy between his two marks

14. How could you determine whether the change in one variable *causes* the change in another variable?

15.3 Lines of best fit [Stage 5.3]

15.3.1 Definition of lines of best fit

- If the points on a scatter plot appear to lie fairly closely distributed in a linear pattern, a straight line can be drawn through the data. The line can then be used to make predictions about the data.
- A **line of best fit** is a line on a scatter plot that is positioned so that it is as *close as possible to all the data points*.
- A line of best fit is used to generalise the relationship between two variables.

15.3.2 Lines of best fit by eye

- A line of best fit can be drawn on a scatter plot **by eye**. This means that a line is positioned so that there is an equal number of points above and below the line.
- Once a line of best fit has been placed on the scatter plot, an equation for this line can be established, using the coordinates of any two points on the line. These two points do not necessarily have to be actual data points, but if any data points do lie on the line, these are chosen as their values are known accurately.
- The equation for the line passing through the two selected points can then be calculated.
  The equation through the two points (x_1, y_1) and (x_2, y_2) is given by:

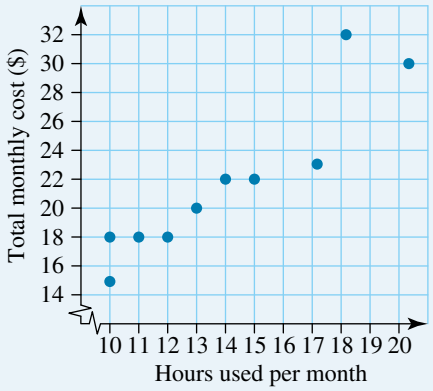
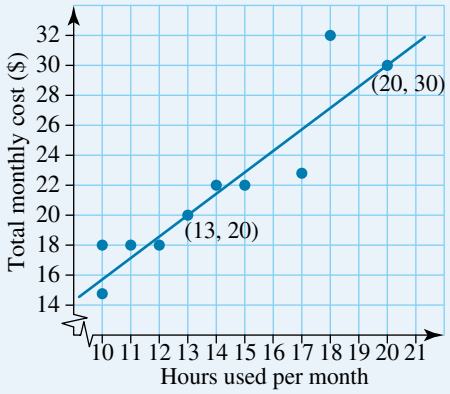
$$y = mx + b \text{ where } m = \frac{y_2 - y_1}{x_2 - x_1}.$$

WORKED EXAMPLE 4

The data in the table show the cost of using the internet at a number of different internet cafes based on hours used per month.

Hours used per month	10	12	20	18	10	13	15	17	14	11
Total monthly cost ($)	15	18	30	32	18	20	22	23	22	18

a Construct a scatter plot of the data.
b Draw in the line of best fit by eye.
c Find the equation of the line of best fit in terms of the variables *n* (number of hours) and *C* (monthly cost).

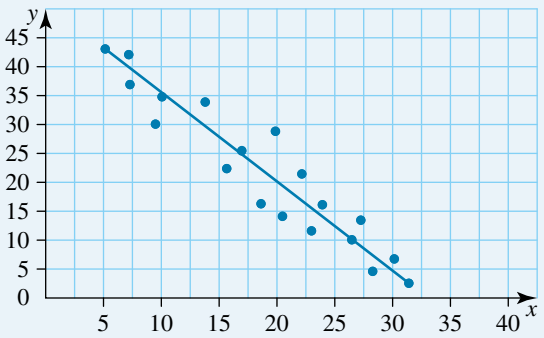
THINK	WRITE/DRAW
a Draw the scatter plot placing the independent variable (hours used per month) on the horizontal axis and the dependent variable (total monthly cost) on the vertical axis. Label the axes.	**a** 
b 1 Carefully analyse the scatter plot. **2** Position the line of best fit so there is approximately an equal number of data points on either side of the line and so that all points are close to the line. *Note:* With the line of best fit, there is no single definite solution.	**b** 
c 1 Select two points on the line that are not too close to each other.	**c** Let $(x_1, y_1) = (13, 20)$ and $(x_2, y_2) = (20, 30)$.
2 Calculate the gradient of the line.	$m = \frac{y_2 - y_1}{x_2 - x_1}$ $m = \frac{30 - 20}{20 - 13}$ $= \frac{10}{7}$
3 Write the rule for the equation of a straight line.	$y = mx + b$
4 Substitute the known values into the equation.	$y = \frac{10}{7}x + b$
5 Substitute one pair of coordinates (say, (13, 20)) into the equation to calculate b.	$20 = \frac{10}{7}(13) + b$ $b = 20 - \frac{130}{7}$ $= \frac{140 - 130}{7}$ $= \frac{10}{7}$
6 Write the equation.	$y = \frac{10}{7}x + \frac{10}{7}$
7 Replace x with n (number of hours used) and y with C (the total monthly cost) as required.	$C = \frac{10}{7}n + \frac{10}{7}$

15.3.3 Making predictions

- The line of best fit can be used to predict the value of one variable from that of another. Because of the subjective nature of the line, it should be noted that predictions are not accurate values, but rough estimates. Although this is the case, predictions using this method are considered valuable when no other methods are available.
- If the equation of the line of best fit is known, or can be derived, predictions can be made by substituting known values into the equation of the line of best fit.

WORKED EXAMPLE 5

Use the given scatter plot and line of best fit to predict:
a the value of y when $x = 10$
b the value of x when $y = 10$.



THINK

a 1 Locate 10 on the x-axis and draw a vertical line until it meets with the line of best fit. From that point, draw a horizontal line to the y-axis. Read the value of y indicated by the horizontal line.

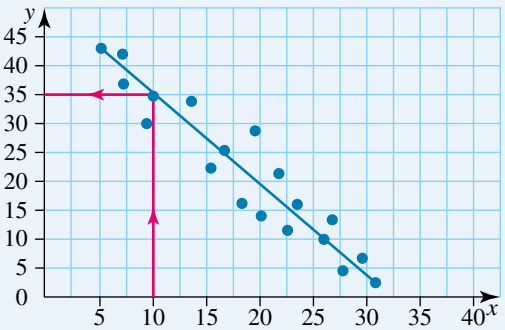
2 Write your answer.

b 1 Locate 10 on the y-axis and draw a horizontal line until it meets with the line of best fit. From that point draw a vertical line to the x-axis. Read the value of x indicated by the vertical line.

2 Write your answer.

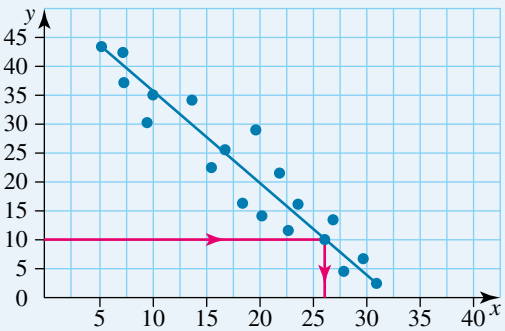
WRITE/DRAW

a



When $x = 10$, y is predicted to be 35.

b



When $y = 10$, x is predicted to be 27.

WORKED EXAMPLE 6

The table below shows the number of boxes of tissues purchased by hayfever sufferers and the number of days affected by hayfever during the blooming season in spring.

Number of days affected by hayfever (d)	3	12	14	7	9	5	6	4	10	8
Total number of boxes of tissues purchased (T)	1	4	5	2	3	2	2	2	3	3

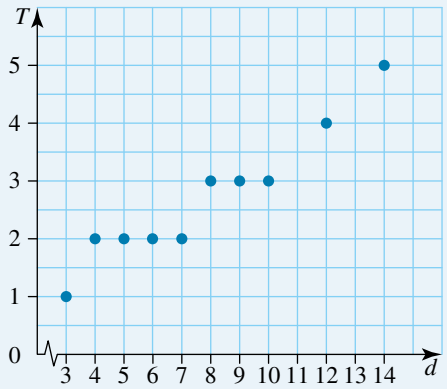
a Construct a scatter plot of the data and draw a line of best fit.
b Determine the equation of the line of best fit.
c Interpret the meaning of the gradient.
d Use the equation of the line of best fit to predict the number of boxes of tissues purchased by people suffering from hayfever over a period of:
i 11 days **ii 15 days.**

THINK

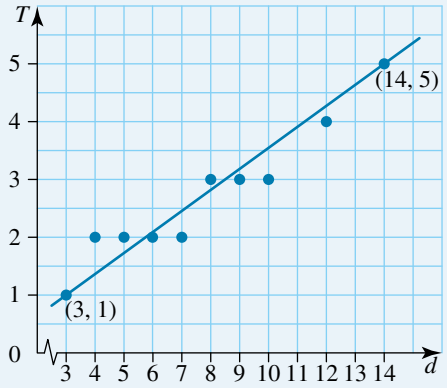
a 1 Draw the scatter plot showing the independent variable (number of days affected by hayfever) on the horizontal axis and the dependent variable (total number of boxes of tissues purchased) on the vertical axis.

WRITE/DRAW

a



2 Position the line of best fit on the scatter plot so there is approximately an equal number of data points on either side of the line.



b 1 Select two points on the line that are not too close to each other.

b Let $(x_1, y_1) = (3, 1)$ and $(x_2, y_2) = (14, 5)$.

2 Calculate the gradient of the line.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$m = \frac{5 - 1}{14 - 3} = \frac{4}{11}$$

3 Write the rule for the equation of a straight line.

$$y = mx + b$$

4 Substitute the known values into the equation, say, 3, 1, into the equation to calculate b.

$$y = \frac{4}{11}x + b$$
$$1 = \frac{4}{11}(3) + b$$
$$b = 1 - \frac{12}{11}$$
$$= -\frac{1}{11}$$
$$y = \frac{4}{11}x - \frac{1}{11}$$

5 Replace x with d (number of days with hay fever) and y with T (total number of boxes of tissues used) as required.

$$T = \frac{4}{11}d - \frac{1}{11}$$

c Interpret the meaning of the gradient of the line of best fit.

c The gradient indicates an increase in sales of tissues as the number of days affected by hayfever increases. A hayfever sufferer is using on average $\frac{4}{11}$ (or about 0.36) of a box of tissues per day.

d i 1 Substitute the value $d = 11$ into the equation and evaluate.

d i When $d = 11$,

$$T = \frac{4}{11} \times 11 - \frac{1}{11}$$
$$= 4 - \frac{1}{11}$$
$$= 3\frac{10}{11}$$

2 Interpret and write your answer.

In 11 days the hayfever sufferer will need about 4 boxes of tissues.

ii 1 Substitute the value $d = 15$ into the equation of the line of best fit and evaluate.

ii When $d = 15$,

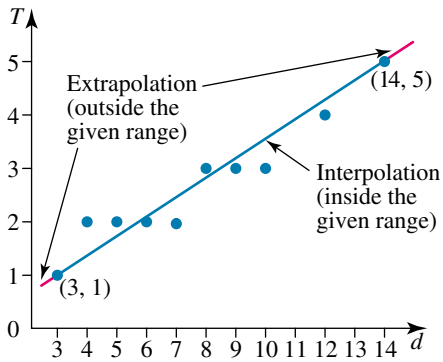
$$T = \frac{4}{11} \times 15 - \frac{1}{11}$$
$$= \frac{60}{11} - \frac{1}{11}$$
$$= 5\frac{4}{11}$$

2 Interpret and write your answer.

In 15 days the hayfever sufferer will need 6 boxes of tissues.

15.3.4 Interpolation and extrapolation

- **Interpolation** is the term used for predicting a value of a variable from within the range of the given data.
- **Extrapolation** occurs when the value of the variable being predicted is outside the range of the given data.



- In Worked example 6, the number of days ranged from 3 days to 14 days. Making a prediction for 11 days is an example of interpolation, whereas making a prediction for 15 days is an example of extrapolation.
- Predictions involving interpolation are considered to be quite reliable. Those involving extrapolation should be treated with caution, as they rely on the **trend** of the line remaining unchanged beyond the range of the data.

15.3.5 Reliability of predictions

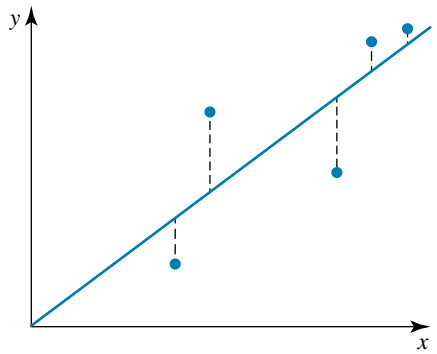
- When predictions of any type are made, it is useful to know whether they are reliable or not.
- If the line of best fit is used to make predictions, they can be considered to be reliable if each of the following is observed.
  - The number of data values is large.
  - The scatter plot indicates reasonably strong relationship between the variables.
  - The predictions are made using interpolation.

DISCUSSION

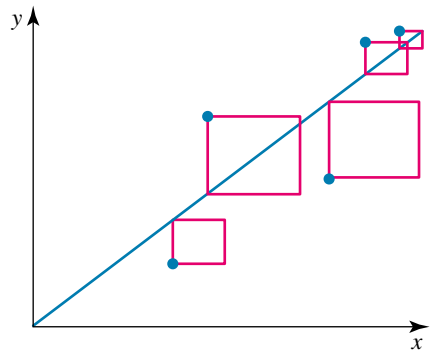
Why are predictions made using extrapolation less reliable than predictions made using interpolation? Does it matter how far outside the range of the given data the extrapolation is taking place?

15.3.6 Least squares regression

- Least squares regression involves an exact mathematical approach to fitting a line of best fit to bivariate data that show a strong linear relationship.
- Consider the regression line shown. The vertical lines give an indication of how well the line best 'fits' the data. The line of best fit is placed so that these 'error' lines are minimised, by balancing the errors above and below the line.



- Least squares regression takes these error lines, forms squares, and minimises the sum of the areas of the squares.



- The actual calculation of the equation of a least squares regression line is complicated. However, the use of digital technologies, such as a spreadsheet, makes constructing the line of best fit very easy.

WORKED EXAMPLE 7

The height (H) and arm span (A) in cm of 20 students is shown below.

a Using an Excel spreadsheet, draw a scatter plot of the data and use excel to construct a line of best fit (Linear Trendline).

b Write the equation of the line of best fit.

c Using the equation from part b, calculate the height of a student with an arm span of 162 cm and comment on the accuracy of the interpolation.

H	180	156	168	147	131	159	154	155	178	175
A	181	154	168	148	128	160	142	154	181	175

H	135	180	178	161	139	154	156	153	170	156
A	130	175	186	158	139	148	156	154	175	152

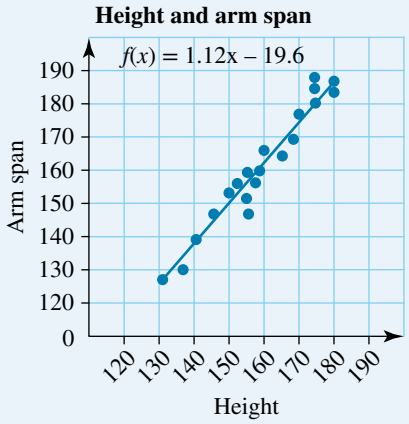
Source: Data from the Australian Bureau of Statistics.

THINK

a Use Excel to draw the scatter plot and the line of best fit.

WRITE

a



b Find the equation from the spreadsheet.

c Substitute Arm span = 162 cm and calculate the height.

b Arm span = 1.1165 × Height − 19.597

$$\begin{aligned}\text{Arm span} &= 1.1165 \times \text{Height} - 19.597\\ 162 &= 1.1165 \times \text{Height} - 19.597\\ \text{Height} &= \frac{162 + 19.597}{1.1165}\\ &= 162.65\,\text{cm}\end{aligned}$$

From an inspection of the scatter plot, the data points appear very close to the line of best fit, so the value for the height of 16.65 should be a close prediction for an arm span of 162 cm.

learnON RESOURCES — ONLINE ONLY

Interactivity: Lines of best fit (int-6180)
Interactivity: Interpolation and extrapolation (int-6181)
Interactivity: Applying lines of best fit (int-2798)
Interactivity: Extrapolation (int-1154)
Digital doc: WorkSHEET Lines of best fit (doc-14599)

Exercise 15.3 Lines of best fit

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–7, 11	Questions: 1–8, 10, 11	Questions: 1–13

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE4 The data in the table below show the distances travelled by 10 cars and the amount of petrol used for their journeys (to the nearest litre).

Distance travelled, *d* (km)	52	36	83	12	44	67	74	23	56	95
Petrol used, *P* (L)	7	5	9	2	7	9	12	3	8	14

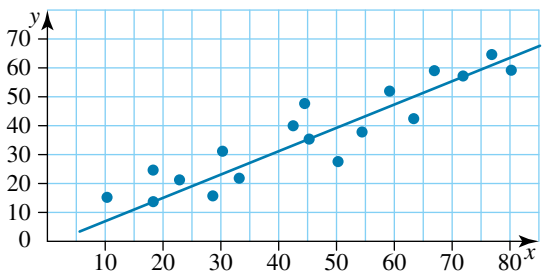
a. Construct a scatter plot of the data.
b. Draw in the line of best fit.
c. Determine the equation of the line of best fit in terms of the variables d (distance travelled) and P (petrol used).

2. A random sample of ten Year 10 students who have part-time jobs was selected. Each student was asked to state the average number of hours they work per week and their average weekly earnings (to the nearest dollar). The results are summarised in the table below.

Hours worked, *h*	4	8	15	18	10	5	12	16	14	6
Weekly earnings ($), *E*	23	47	93	122	56	33	74	110	78	35

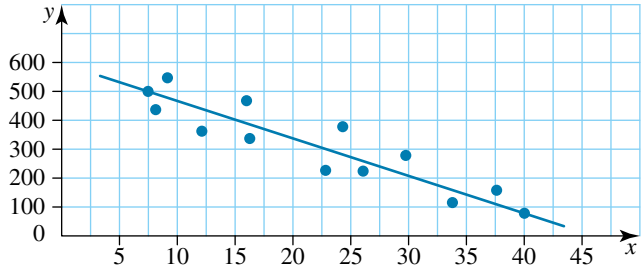
a. Construct a scatter plot of the data using technology.
b. Draw in the line of best fit using technology.
c. Write the equation of the line of best fit, in terms of variables h (hours worked) and E (weekly earnings).
d. Interpret the meaning of the gradient.

3. WE5 Use the given scatter plot and line of best fit to predict:



a. the value of y when $x = 45$
b. the value of x when $y = 15$.

4. Analyse the following graph.



a. Use the line of best fit to predict the value of y when the value of x is:
i. 7 **ii.** 22 **iii.** 36.

b. Use the line of best fit to predict the value of x when the value of y is:
i. 120 **ii.** 260 **iii.** 480.

c. Determine the equation of the line of best fit if it is known that the line passes through the points (5, 490) and (40, 80).

d. Use the equation of the line to verify the values obtained from the graph in parts **a** and **b**.

5. WE6 The following table shows the average weekly expenditure on food for households of various sizes.

Number of people in a household	1	2	4	7	5	4	3	5
Cost of food ($ per week)	70	100	150	165	150	140	120	155

Number of people in a household	2	4	6	5	3	1	4
Cost of food ($ per week)	90	160	160	160	125	75	135

a. Construct a scatter plot of the data and draw in the line of best fit.

b. Determine the equation of the line of best fit. Write it in terms of variables n (for the number of people in a household) and C (weekly cost of food).

c. Interpret the meaning of the gradient.

d. Use the equation of the line of best fit to predict the weekly food expenditure for a family of:
i. 8
ii. 9
iii. 10.

6. The following table shows the gestation time and the birth weight of 10 babies.

Gestation time (weeks)	31	32	33	34	35	36	37	38	39	40
Birth weight (kg)	1.080	1.470	1.820	2.060	2.230	2.540	2.750	3.110	3.080	3.370

Use technology to answer the following questions.

a. Construct a scatter plot of the data. What type of relationship does the scatter plot suggest?

b. Draw in the line of best fit and determine its equation. Write it in terms of the variables t (gestation time) and M (birth weight).

c. What does the value of the gradient represent?

d. Although the full term of gestation is considered to be 40 weeks, some pregnancies last longer. Use the equation obtained in part **b** to predict the birth weight of babies born after 41 and 42 weeks of gestation.

e. Many babies are born prematurely. Using the equation obtained in part **b**, predict the birth weight of a baby whose gestation time was 30 weeks.

f. If the birth weight of a baby was 2.390 kg, what was the baby's gestation time (to the nearest week)?

Communicating, reasoning and problem solving

7. As a part of her project, Rachel is growing a crystal. Every day she measures the crystal's mass using special laboratory scales and records it. The table below shows the results of her experiment.

Day number	1	2	3	4	5	8	9	10	11	12	15	16
Mass (g)	2.5	3.7	4.2	5.0	6.1	8.4	9.9	11.2	11.6	12.8	16.1	17.3

Measurements on days 6, 7, 13 and 14 are missing because these were 2 consecutive weekends, and as Rachel's crystal is kept in the school laboratory, she was not able to measure it on those days.
Note: An Excel spreadsheet or other digital technology may be used to answer this question.

a. Construct the scatter plot of the data and draw in the line of best fit.

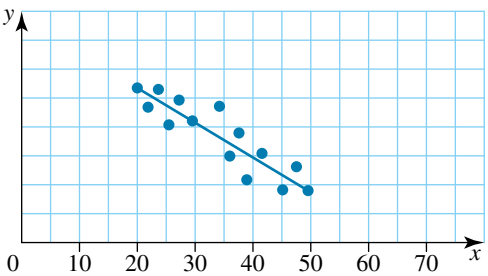
b. Determine the equation of the line of best fit. Write the equation using variables d (day of the experiment) and M (mass of the crystal).

c. Interpret the meaning of the gradient.

d. For her report, Rachel would like to fill in the missing measurements (that is, the mass of the crystal on days 6, 7, 13 and 14). Use the equation of the line of best fit to help Rachel find these measurements. Is this an example of interpolation or extrapolation? Explain your answer.

e. Rachel needed to continue her experiment for 2 more days, but she fell ill and had to miss school. Help Rachel to predict the mass of the crystal on those two days (that is, days 17 and 18), using the equation of the line of best fit. Are these predictions reliable? Explain your answer.

8. MC Consider the figure shown below.



The line of best fit on the scatter plot is used to predict the values of y when $x = 15$, $x = 40$ and $x = 60$.

a. Interpolation would be used to predict the value of y when the value of x is:

A. 15 and 40 **B.** 15 and 60 **C.** 15 only **D.** 40 only **E.** 60 only

b. The prediction of the y-value(s) can be considered reliable when:

A. $x = 15$ and $x = 40$

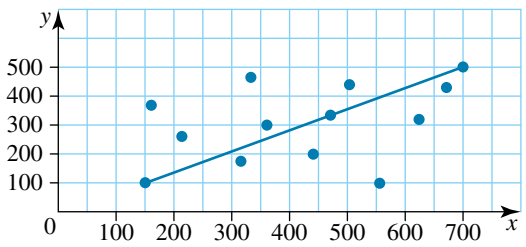
B. $x = 15$, $x = 40$ and $x = 60$

C. $x = 40$

D. $x = 40$ and $x = 60$

E. $x = 60$

9. MC The scatter plot below is used to predict the value of y when $x = 300$. This prediction is:



A. reliable, because it is obtained using interpolation
B. not reliable, because it is obtained using extrapolation
C. not reliable, because only x-values can be predicted with confidence
D. reliable because the scatter plot contains a large number of points
E. not reliable, because there is no relationship between x and y

10. For his birthday, Ari was given a small white rabbit. To monitor the rabbit's growth, Ari decided to measure it once a week. The table below shows the length of the rabbit for various weeks.
(*Note:* An Excel spreadsheet or other digital technologies may be used to answer this question.)

Week number, n	1	2	3	4	6	8	10	13	14	17	20
Length (cm), l	20	21	23	24	25	30	32	35	36	37	39

a. Construct a scatter plot of the data.
b. Draw a line of best fit and determine its equation.
c. As can be seen from the table, Ari did not measure his rabbit on weeks 5, 7, 9, 11, 12, 15, 16, 18 and 19. Use the equation of the line of best fit to predict the length of the rabbit for those weeks.
d. Were the predictions made in part **c** an example of interpolation or extrapolation? Explain.
e. Predict the length of the rabbit in the next three weeks (that is, weeks 21–23), using the line of best fit from part **b**.
f. Are the predictions that have been made in part **e** reliable? Explain.

11. Laurie is training for the long jump, hoping to make the Australian Olympic team. His best jump each year is shown in the table below.
(*Note:* An Excel spreadsheet or other digital technologies may be used to answer this question.)

Age (a)	**Best jump (B) (metres)**
8	4.31
9	4.85
10	5.29
11	5.74
12	6.05
13	6.21
14	—
15	6.88
16	7.24
17	7.35
18	7.57

a. Plot the points generated by the table on a scatter plot.
b. Join the points generated with straight line segments.
c. Draw a line of best fit and determine its equation.
d. The next Olympic Games will occur when Laurie is 20 years old. Use the equation of the line of best fit to estimate Laurie's best jump that year and whether it will pass the qualifying mark of 8.1 metres.
e. Is a line of best fit a good way to predict future improvement in this situation? What problems are there with using a line of best fit?
f. Olympic Games will also be held when Laurie is 24 years old and 28 years old. Using extrapolation, what length would you predict Laurie could jump at these two ages? Is this realistic?

g. When Laurie was 14, he twisted a knee in training and did not compete for the whole season. In that year, a national junior championship was held. The winner of that championship jumped 6.5 metres. Use your line of best fit to predict whether Laurie would have won that championship.

12. Why is extrapolation considered to be not reliable?

13. Sam has a mean score of 88 per cent for his first nine tests of the semester. In order to receive an A^+ his score must be 90 per cent or higher. There is one test remaining for the semester. Is it possible for him to receive an A^+? Why or why not?

15.4 Time as the independent variable

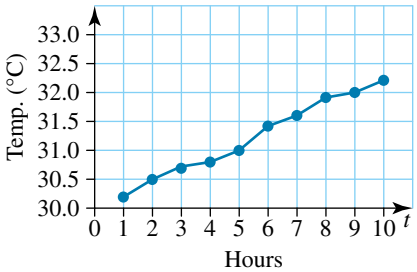
[Stages 5.2 and 5.3]

15.4.1 Time series

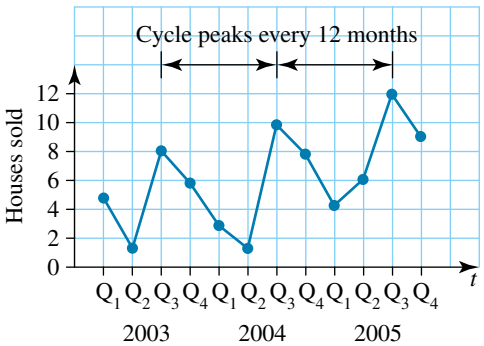
- **Time series** data are any data that have time as the independent variable.
- The data are graphed and the graphs are used to determine if a trend is present in the data.
- Identifying a trend can help when making predictions about the future.

15.4.2 Types of trends

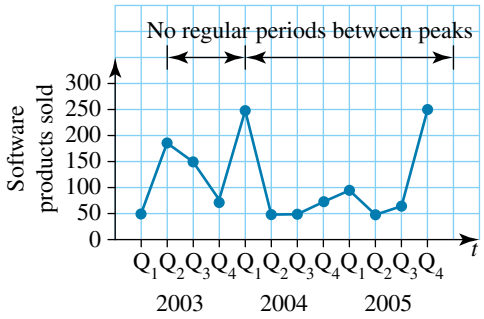
- A **general upward or downward trend** is a graph that overall goes up or down as illustrated in the graph below.



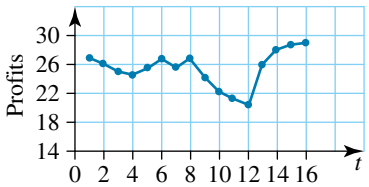
- A **seasonal pattern** displays fluctuations that repeat at the same time each week, month or quarter and usually last less than one year. The graph below illustrates that the peak selling time for houses is in the spring.



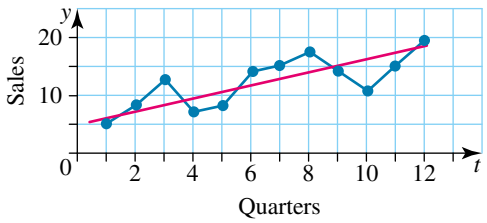
- A **cyclical pattern** displays fluctuations that repeat but will usually take longer than a year to repeat. An example of this is shown in the graph below, which depicts software products sold.



- **Random patterns** do not show any regular fluctuation. They are usually caused by unpredictable events such as the economic recession illustrated in the graph below.



- Trends can work in combinations; for example, you can have a seasonal pattern with an upward trend.

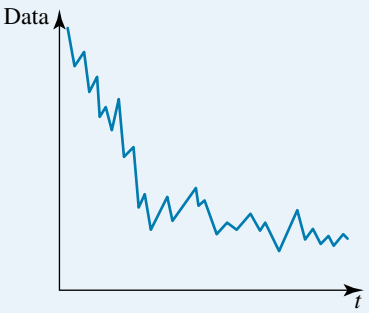


DISCUSSION

Can you think of some data sets that might fit each of the different types of trends?

WORKED EXAMPLE 8

Classify the trend suggested by the time series graph shown as being linear or non-linear, and upward, downward or no trend.



THINK

Carefully analyse the given graph and comment on whether the graph resembles a straight line or not and whether the values of y increase or decrease over time.

WRITE

The time series graph does not resemble a straight line and overall the level of the variable, y, decreases over time. The time series graph suggests a non-linear downward trend.

WORKED EXAMPLE 9

The data below show the average daily mass of a person (to the nearest 100 g), recorded over a period of 4 weeks.
63.6, 63.8, 63.5, 63.7, 63.2, 63.0, 62.8, 63.3, 63.1, 62.7, 62.6, 62.5,
62.9, 63.0, 63.1, 62.9, 62.6, 62.8, 63.0, 62.6, 62.5, 62.1, 61.8, 62.2, 62.0, 61.7, 61.5, 61.2

a Plot these masses as a time series graph.
b Comment on the trend.

THINK

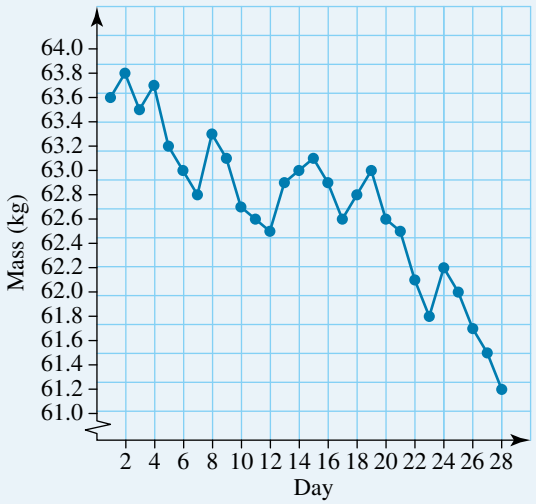
a 1 Draw the points on a scatter plot with day on the horizontal axis and mass on the vertical axis.

2 Join the points with straight line segments to create a time series plot.

b Carefully analyse the given graph and comment on whether the graph resembles a straight line or not and whether the values of y (in this case, mass) increase or decrease over time.

WRITE/DRAW

a



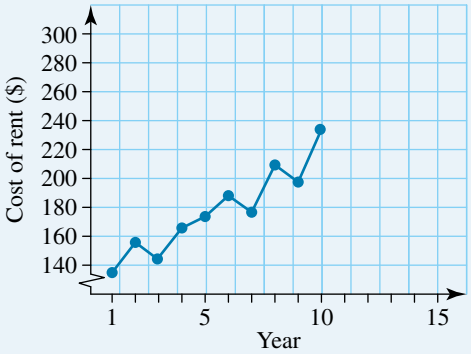
b The graph resembles a straight line that slopes downwards from left to right (that is, mass decreases with increase in time). Although a person's mass fluctuates daily, the time series graph suggests a downward trend. That is, overall, the person's mass has decreased over the 28-day period.

15.4.3 Trend lines

- A **trend line** is a type of line of best fit. Trend lines indicate the general trend of the data.
- Trend lines are useful in forecasting, or making predictions about the future, by extrapolation. Extrapolation can have limited reliability, as predictions are based on the assumption that the current trend will continue into the future.

WORKED EXAMPLE 10

The graph shows the average cost of renting a one-bedroom flat, as recorded over a 10-year period.



a If appropriate, draw in a line of best fit and comment on the type of trend.
b Assuming that the current trend will continue, use the line of best fit to predict the cost of rent in 5 years' time.

THINK

a 1 Analyse the graph and observe what occurs over a period of time. Draw a line of best fit.

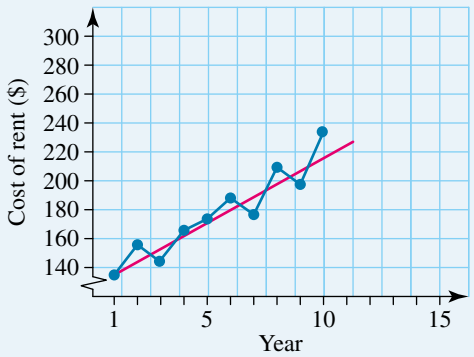
2 Comment on the type of trend observed.

b 1 Extend the line of best fit drawn in part **a**. The last entry corresponds to the 10th year and we need to predict the cost of rent in 5 years' time, that is, in the 15th year.

2 Locate the 15th year on the time axis and draw a vertical line until it meets with the line of best fit. From the trend line (line of best fit) draw a horizontal line to the cost axis.

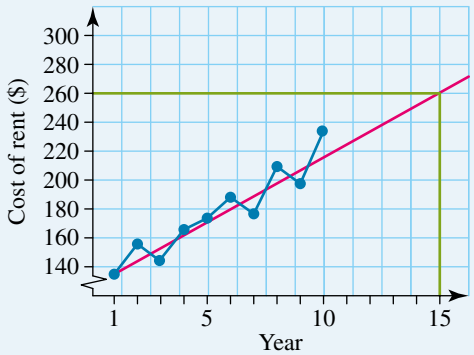
WRITE/DRAW

a



The graph illustrates that the cost of rent increases steadily over the years. The time series graph indicates an upward linear trend.

b



3 Read the cost from the vertical axis.

Cost of rent = $260

4 Write your answer.

Assuming that the cost of rent will continue to increase at the present rate, in 5 years we can expect the cost of rent to reach $260 per week.

15.4.4 Using spreadsheets to determine the relationship

- It is possible to draw scatter plots and time series graphs using a variety of digital technologies, including spreadsheets.

WORKED EXAMPLE 11

Data were recorded about the number of families who moved from Sydney to Newcastle over the past 10 years.

Year	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013
Number moved	97	118	125	106	144	155	162	140	158	170

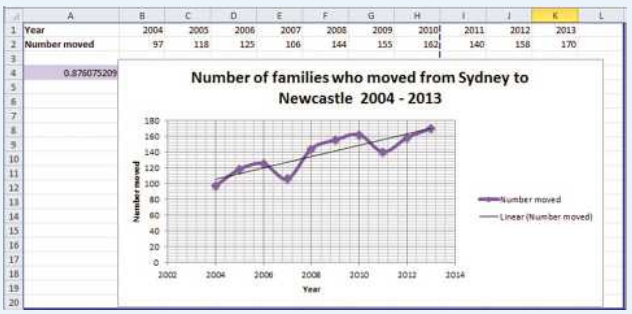
a Use technology to construct a time series graph, with a line of best fit, that represents the data.
b Describe the trend.
c Measure the relationship.
d Comment on the results.

THINK

a Enter the data into a spreadsheet.
Select the type of graph — a scatter plot with the points joined and a trend line fitted.

WRITE

a



b Describe the trend.

b There appears to be an upward trend over the 10 years.

c To measure the relationship, place the cursor in a cell — see the purple cell. From the menu ribbon, select 'More formulas', then 'Statistics', then 'CORREL'. Complete the values for the relevant cells and press Enter.
The relationship value will be shown.

c The relationship is 0.8761.

d Interpret the results.

d Over the last 10 years, an increasing number of families have decided to make the move from Sydney to Newcastle. The relationship is strong and positive (0.8761), making it possible to predict that this trend is likely to continue.

learnON RESOURCES — ONLINE ONLY

eLesson: Fluctuations and cycles (eles-0181)

Digital doc: WorkSHEET Time series (doc-14600)

Exercise 15.4 Time as the independent variable

Individual pathways

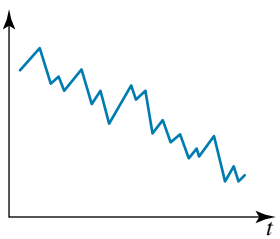
PRACTISE	CONSOLIDATE	MASTER
Questions: 1–3, 5–7, 9	Questions: 1–7, 9, 10	Questions: 1–13

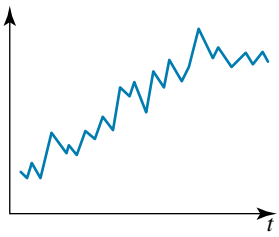
To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

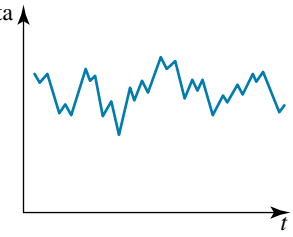
Understanding and fluency

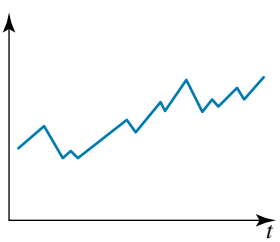
Where appropriate, digital technology should be used to answer the following questions.

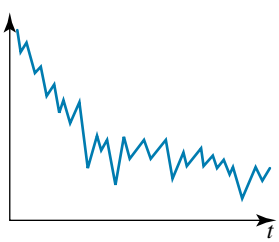
1. WE8 Classify the trend suggested by each time series graph as being linear or non-linear, and upward, downward or stationary in the mean (no trend).

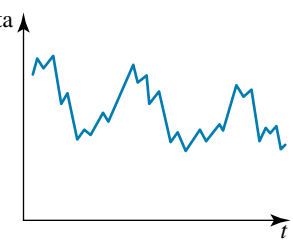
a.


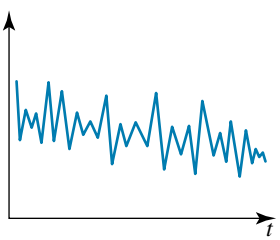
b.


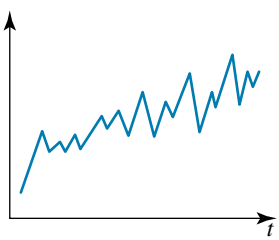
c.


d.


e.


f.


g.


h.


2. WE9 The data below show the average daily temperatures recorded in June.

17.6, 17.4, 18.0, 17.2, 17.5, 16.9, 16.3, 17.1, 16.9, 16.2, 16.0, 16.6, 16.1, 15.4, 15.1, 15.5, 16.0, 16.0, 15.4, 15.2, 15.0, 15.5, 15.1, 14.8, 15.3, 14.9, 14.6, 14.4, 15.0, 14.2

a. Plot these temperatures as a time series graph.

b. Comment on the trend.

3. The data below show the quarterly sales (in thousands of dollars) recorded by the owner of a sheepskin product store over a period of 4 years.

Quarter	2006	2007	2008	2009
1	57	59	50	52
2	100	102	98	100
3	125	127	120	124
4	74	70	72	73

a. Plot the time series.

b. The time series plot displays seasonal fluctuations of period 4 (since there are four quarters). Explain in your own words what this means. Also write one or two possible reasons for the occurrence of these fluctuations.

c. Overall, does the time series plot indicate upward, downward or no trend?

4. The table below shows the total monthly revenue (in thousands of dollars) obtained by the owners of a large reception hall. The revenue comes from rent and catering for various functions over a period of 3 years.

	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
2007	60	65	40	45	40	50	45	50	55	50	55	70
2008	70	65	60	65	55	60	60	65	70	75	80	85
2009	80	70	65	70	60	65	70	75	80	85	90	100

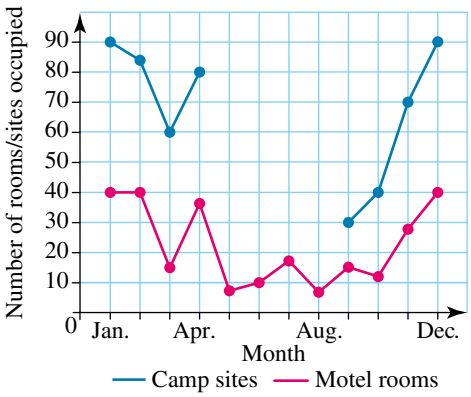
a. Construct a time series plot for these data.

b. Describe the graph (peaks and troughs, long-term trend, any other patterns).

c. Try to give possible reasons for monthly fluctuations.

d. Does the graph show seasonal fluctuations of period 12? Are there any patterns that repeat from year to year?

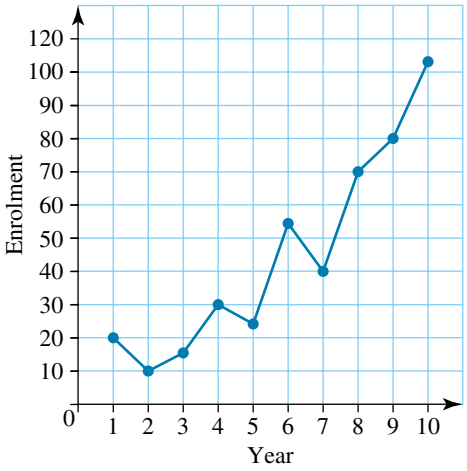
5. The owner of a motel and caravan park in a small town keeps records of the total number of rooms and total number of camp sites occupied per month. The time series plots based on his records are shown below.



a. Describe each graph, discussing general trend, peaks and troughs and so on. Explain particular features of the graphs and give possible reasons.

b. Compare the two graphs and write a short paragraph commenting on any similarities and differences between them.

6. WE10 The graph below shows enrolments in the Health and Nutrition course at a local college over a 10-year period.



a. If appropriate, draw in a line of best fit and comment on the type of the trend.
b. Assuming that the trend will continue, use the line of best fit to predict the enrolment for the course in 5 years' time, that is, in the 15th year.

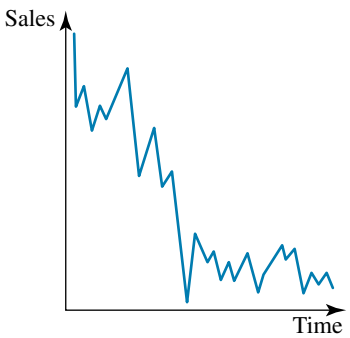
Communicating, reasoning and problem solving

7. WE11 In June a new childcare centre was opened. The number of children attending full time (according to the enrolment at the beginning of each month) during the first year of operation is shown in the table below.

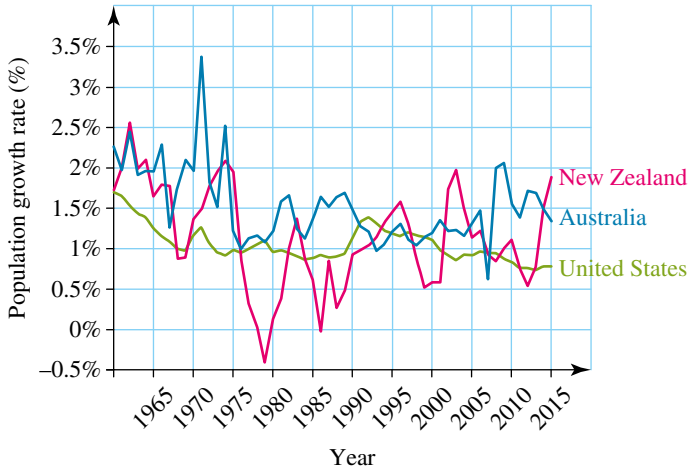
June	July	Aug.	Sept.	Oct.	Nov.	Dec.	Jan.	Feb.	Mar.	Apr.	May
6	8	7	9	10	9	12	10	11	13	12	14

a. Plot this time series using a digital technology. (*Hint:* Let June = 1, July = 2 etc.)
b. Is the child care business going well? Justify your answer.
c. Draw a line of best fit and find its equation, using coordinates of any two points on the line.
d. Use your equation of the line of best fit to predict the enrolment in the centre during the second year of operation at the beginning of:
i. August **ii.** January.
What assumptions have you made?

8. The graph below shows the monthly sales of a certain book since its publication. Explain in your own words why linear trend forecasting of the future sales of this book is not appropriate.



9. Consider the graph below, which shows the changing percentage population growth of Australia, New Zealand and the United States between 1960 and 2015.



a. Describe the percentage population growth for Australia, New Zealand and the United States over the period between 1960 and 2015.

b. Suggest reasons for the percentage population growth for Australia in 2007 and 2008 and 2009.

10. Running a small business is difficult when your business is affected by seasonal patterns. It can make it difficult to do a budget when you have a fluctuating income. For example, if you owned a cafe at Mt Buller, you would have a lot of business during the ski season but very little business over summer. If you wished to remain open over the summer, you would need to find a way to remove the impact of the seasons from your sales data to see if remaining open is viable. Investigate a way of removing the effects of the seasons from business data.

11. The table below gives the quarterly sales figures for a second-hand car dealer over a three-year period.

Year	Q1	Q2	Q3	Q4
2012	75	65	92	99
2013	91	79	115	114
2014	93	85	136	118

a. Represent this data on a time series plot.

b. Briefly describe how the car sales have altered over the time period.

c. Does it appear that the car dealer can sell more cars in a particular period each year?

12. Why are predictions in the future appropriate for time series even though they involve extrapolation?

13. Mr MacDonald recorded the test marks for his Year 10 class of 25 students. He calculated the average mark to be 72. Sandra's mark of 86 was incorrectly marked as 36. What was the correct average mark for the test?

15.5 Review

Investigation | Rich task

Collecting, recording and analysing data over time

A time series is a sequence of measurements taken at regular intervals (daily, weekly, monthly and so on) over a certain period of time. Time series are best represented using time-series plots, which are line graphs with the time plotted on the horizontal axis. Examples of time series include daily temperature, monthly unemployment rates and daily share prices.

When data are recorded on a regular basis, the value of the variable may go up and down in what seems to be an erratic pattern. These changes are called fluctuations. However, over a long period of time, the time series usually suggests a certain trend. These trends can be classified as being linear or non-linear, and upward, downward or stationary (no trend).

Time series are often used for forecasting, that is, making predictions about the future. The predictions made with the help of time series are always based on the assumption that the observed trend will continue in the future.

1. Choose a subject that is of interest to you and that can be observed and measured during one day or over the period of a week or more. (Suitable subjects are shown in the list on the next page.)

2. Prepare a table for recording your results. Select appropriate regular time intervals. An example is shown below.

Time	8 am	9 am	10 am	11 am	12 pm	1 pm	2 pm	3 pm	4 pm	5 pm
Pulse rate										

3. Take your measurements at the selected time intervals and record them in the table.

4. Use your data to plot the time series. You can use software such as Excel or draw the scatter plot by hand.

5. Describe the graph and comment on its trend.

6. If appropriate, draw a line of best fit and predict the next few data values.

7. Take the actual measurements during the hours you have made predictions for. Compare the predictions with the actual measurements. Were your predictions good? Give reasons.

Here are some suitable subjects for data observation and recording:

- minimum and maximum temperatures each day for 2 weeks (use the TV news or newspaper as resources)
- the value of a stock on the share market (e.g. Telstra, Wesfarmers and Rio Tinto)
- your pulse over 12 hours (ask your teacher how to do this or check on the internet)

- the value of sales each day at the school canteen
- the number of students absent each day
- the position of a song in the Top 40 over a number of weeks
- petrol prices each day for 2 weeks
- other measurements (check with your teacher)
- world population statistics over time.

learnON RESOURCES — ONLINE ONLY

Digital doc: Investigation — Collecting, recording and analysing data over time (doc-15940)
Digital doc: Code puzzle — What did the chewing gum say to the shoe? (doc-15941)
Digital doc: Summary Topic 15 (doc-22913)
Interactivity: Word search Topic 15 (int-2886)
Interactivity: Crossword Topic 15 (int-2887)
Interactivity: Sudoku Topic 15 (int-3600)

Exercise 15.5 Review questions

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

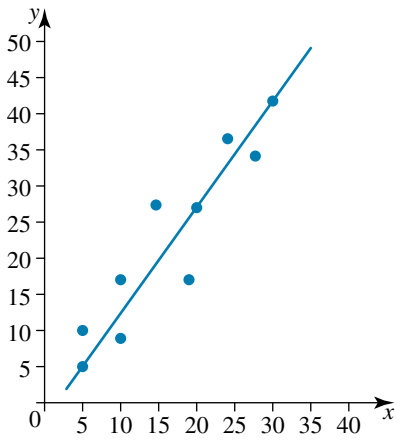
1. As preparation for a Mathematics test, a group of 20 students was given a revision sheet containing 60 questions. The table below shows the number of questions from the revision sheet successfully completed by each student and the mark, out of 100, of that student on the test.

Number of questions	9	12	37	60	55	40	10	25	50	48	60
Test result	18	21	52	95	100	67	15	50	97	85	89
Number of questions	50	48	35	29	19	44	49	20	16	58	52
Test result	97	85	62	54	30	70	82	37	28	99	80

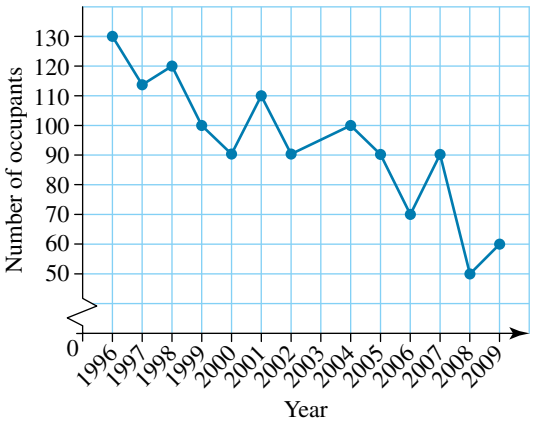
a. State which of the variables is dependent and which is independent.
b. Construct a scatter plot of the data.
c. State the type of relationship between the two variables suggested by the scatter plot and draw a corresponding conclusion.
d. Suggest why the relationship is not perfectly linear.

2. a. Use the line of best fit shown on the graph at the top of the next page to predict the value of y, when the value of x is:
i. 10 **ii.** 35.
b. Use the line of best fit to predict the value of x, when the value of y is:
i. 15 **ii.** 30.
c. Find the equation of a line of best fit if it is known that it passes through the points (5, 5) and (20, 27).

d. Use the equation of the line to algebraically verify the values obtained from the graph in parts **a** and **b**.



3. The graph shows the number of occupants of a large nursing home over the last 14 years.



a. Comment on the type of trend displayed.
b. Explain why it is appropriate to draw in a line of best fit.
c. Draw a line of best fit and use it to predict the number of occupants in the nursing home in 3 years' time.
d. What assumption has been made when predicting figures for part **c**?

4. The table below shows the advertised sale price ('000s dollars) and the land size (m^2) for ten vacant blocks of land.

Land size (m^2)	Sale price (×$1000)
632	36
1560	58
800	40
1190	44
770	41
1250	52
1090	43
1780	75
1740	72
920	43

a. Construct a scatter plot using digital technology and determine the equation of the line of best fit.
b. What does the gradient represent?

c. Using the line of best fit, predict the approximate sale price, to the nearest thousand dollars for a block of land with an area of 1600 m^2.

d. Using the line of best fit, predict the approximate land size, to the nearest 10 square metres, you could purchase with $50 000.

5. The table below shows, for 15 students, the amount of pocket money they receive and spend at the school canteen in an average week.

Pocket money ($)	Canteen spending ($)
30	16
40	17
15	12
25	14
40	16
15	14
30	16
30	17
25	15
15	13
50	19
20	14
35	17
20	15
10	13

a. Construct a scatter plot using digital technology and determine the equation of the line of best fit.

b. What does the gradient represent?

c. Using your line of best fit, predict the amount of money spent at the canteen for a student receiving $45 pocket money a week.

d. Using your line of best fit, predict the amount of money spent at the canteen by a student who receives $100 pocket money each week? Does this seem reasonable? Explain.

6. The table below shows, for 10 ballet students, the number of hours a week spent training and the number of pirouettes in a row they can complete.

Training (h)	11	11	2	8	4	16	11	16	5	3
Number of pirouettes	15	13	3	12	7	17	13	16	8	5

a. Construct a scatter plot using digital technology and determine the equation of the line of best fit.

b. What does the gradient represent?

c. Using your line of best fit, predict the number of pirouettes that could be complete if a student undertakes 14 hours of training.

d. Professional ballet dancers may undertake up to 30 hours of training a week. Using your line of best fit, predict the number of pirouettes they should be able to do in a row. Comment on your findings.

7. a. Use the data given below to draw a scatter plot and determine the equation of the line of best fit.

Age in years	7	11	8	16	9	8	14	19	17	10	20	15
Hours of television watched in a week	20	19	25	55	46	50	53	67	59	25	70	58

b. Use the line of best fit to predict the value of the number of hours of television watched by a person aged 15.

c. Determine the age when the number of hours of television watched is 60.

Communicating, reasoning and problem solving

8. The following time series data show the mean monthly daily hours of sunshine in Melbourne from January to December. Using a scatter plot, describe the trends present in the data.

Month	1	2	3	4	5	6	7	8	9	10	11	12
Daily hours of sunshine	8.7	8.0	7.5	6.4	4.8	4.0	4.5	5.5	6.3	7.3	7.5	8.3

9. The existence of the following situations is often considered an obstacle to making estimates from data.

a. Outlier

b. Extrapolation

c. Small range of data

d. Small number of data points

Explain why each of these situations is considered an obstacle to making estimates of data and how each might be overcome.

10. The table below shows the heights of 10 students and the distances along the ground between their feet as they attempt to do the splits. (*Note:* Digital technologies can be used to assist you in answering this question.)

Height (cm)	**Distance stretched (cm)**
134.5	150
156	160
133.5	147
145	160
160	162
135	149
163	163
138	149
152	158
159	160

Using the data, estimate the distance a person 1.8 m tall can achieve when attempting the splits. Write a detailed analysis of your result. Include:

- an explanation of the method(s) used
- any plots or formula generated
- comments on validity of the estimate
- any ways the validity of the estimate could be improved.

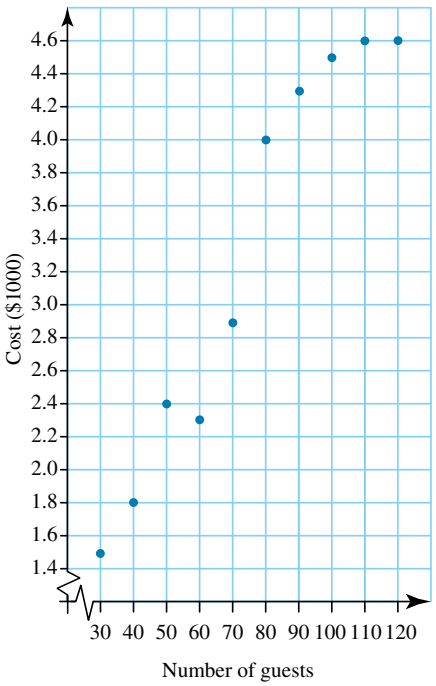
Answers

Topic 15 Bivariate data analysis

Exercise 15.2 Bivariate data

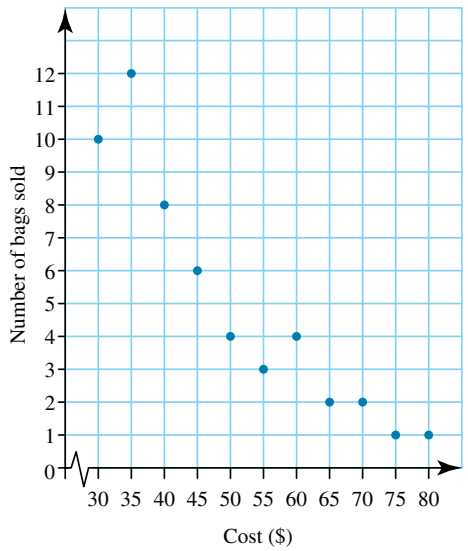
1.	Independent	Dependent
a.	Number of hours	Test results
b.	Rainfall	Attendance
c.	Hours in gym	Visits to the doctor
d.	Lengths of essay	Memory taken
e.	Cost of care	Attendance
f.	Age of property	Cost of property
g.	Number of applicants	Cut-off OP score
h.	Running speed	Heart rate

2.



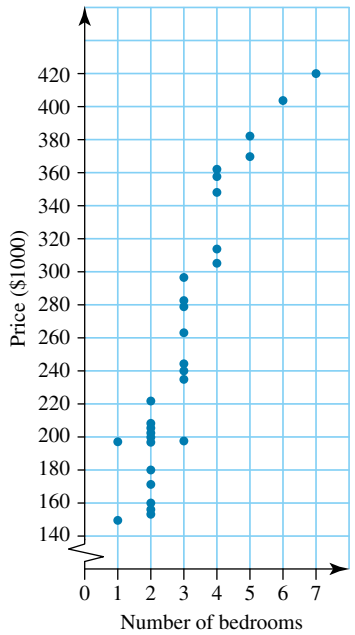
3. a. Perfectly linear, positive
b. No relationship
c. Non-linear, negative, moderate
d. Strong, positive, linear
e. No relationship
f. Non-linear, positive, strong
g. Strong, negative, negative
h. Non-linear, moderate, negative
i. Weak, negative, linear
j. Non-linear, moderate, positive
k. Positive, moderate, linear
l. Non-linear, strong, negative
m. Strong, negative, linear
n. Weak, positive, linear
o. Non-linear, moderate, positive

4. a.



b. Negative, linear, moderate. The price of the bag appeared to affect the numbers sold; that is, the more expensive the bag, the fewer sold.

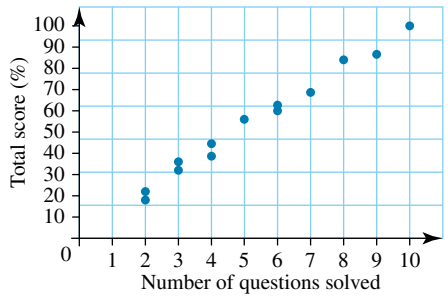
5. a.



b. Moderate positive linear relationship. There is evidence to show that the larger the number of bedrooms, the higher the price of the house.

c. Various answers; location, age, number of people interested in the house, and so on.

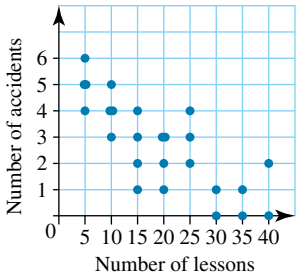
6. a.



b. Strong, positive, linear relationship

c. Various answers — some students are of different ability levels and they may have attempted the questions but had incorrect answers.

7. a.



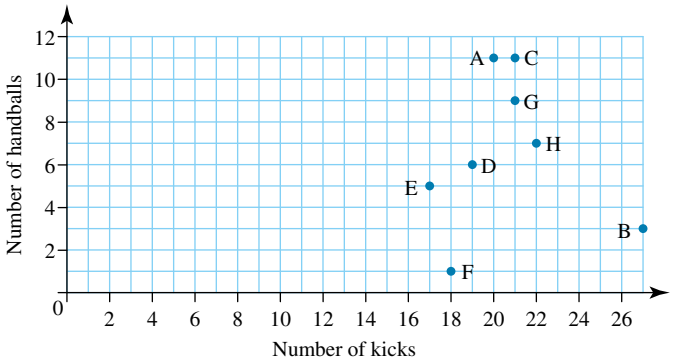
b. Weak, negative, linear relation

c. Various answers, such as some drivers are better than others, live in lower traffic areas, traffic conditions etc.

8. a. T **b.** F **c.** T **d.** F **e.** T

9. B **10.** C **11.** D

12. a.



b. This scatter plot does not support the claim.

13. a. Mandy (iii) **b.** William (iv) **c.** Charlotte (viii) **d.** Dario (vii)

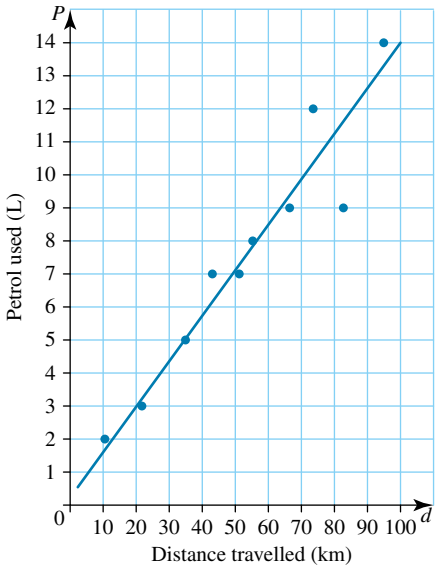
e. Edward (vi) **f.** Cindy (v) **g.** Georgina (i) **h.** Harrison (ii)

14. Consider all the reasons there may have been a change in each individual variable.

Exercise 15.3 Lines of best fit

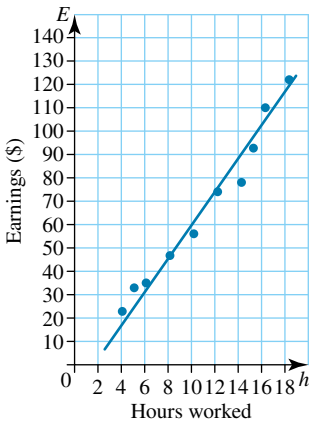
Note: Answers may vary depending on the line of best fit drawn.

1. a, b



c. Using (23, 3) and (56, 8), the equation is $P = \frac{5}{33}d - \frac{16}{33}$.

2. a, b



a. Using (8, 47) and (12, 74), the equation is $E = 6.75h - 7$.

b. On average, students were paid $6.75 per hour.

3. a. 38 **b.** 18

4. a. i. 505 **ii.** 310 **iii.** 100

b. i. 39 **ii.** 24 **iii.** 6

c. $y = -11.71x + 548.57$

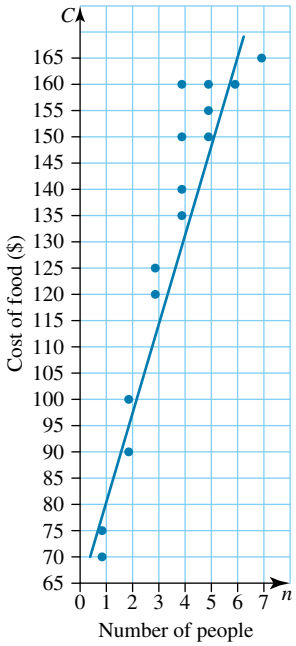
d. y-values:

i. 466.60 **ii.** 290.95 **iii.** 127.01

x-values:

i. 36.60 **ii.** 24.64 **iii.** 5.86

5. a.

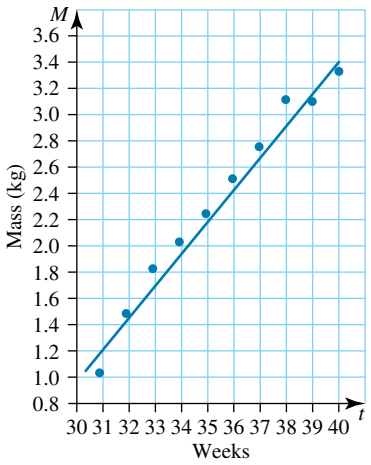


b. Using (1, 75) and (5, 150), the equation is $C = 18.75n + 56.25$.

c. On average, weekly cost of food increases by $18.75 for every extra person.

d. i. $206.25 **ii.** $225.00 **iii.** $243.75

6. a.



Positive, strong, linear relationship

b. Using (32, 1.470) and (35, 2.230), $M = 0.25t - 6.5$.

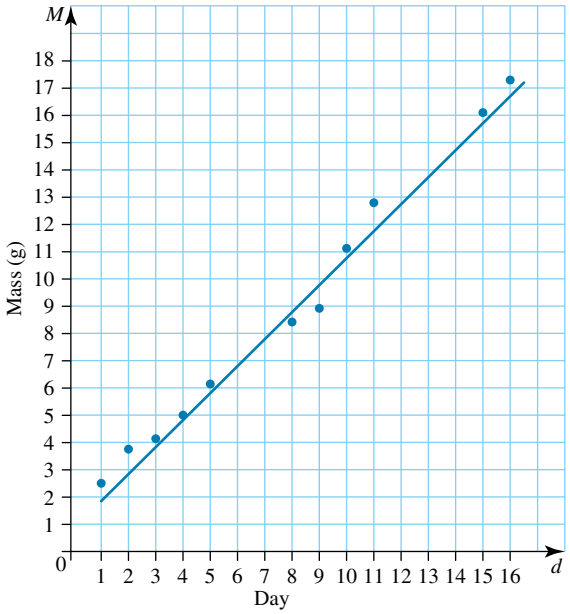
c. With every week of gestation the weight of the baby increases by approximately 250 g.

d. 3.75 kg; 4 kg

e. Approximately 1 kg

f. Between 35 and 36 weeks

7. a.



b. Using (2, 3.7) and (10, 11.2), $M = 0.973d + 1.285$.

c. Each day Rachel's crystal gains 0.973 g in mass.

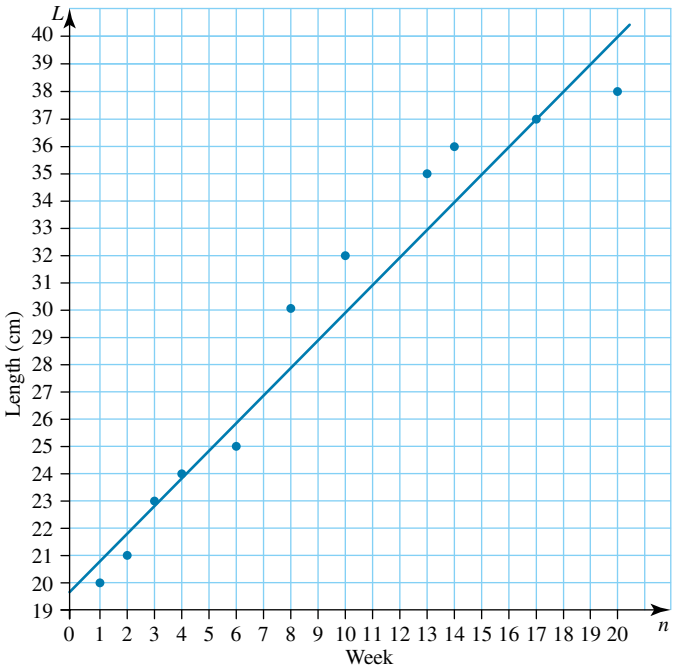
d. 7.123 g; 8.096 g; 13.934 g; 14.907 g; interpolation (within the given range of 1–16)

e. 17.826 g; 18.799 g; predictions are not reliable, since they were obtained using extrapolation.

8. a. D **b.** C

9. E

10. a.



b. Using (2, 21) and (17, 37), $L = 1.07n + 18.9$.

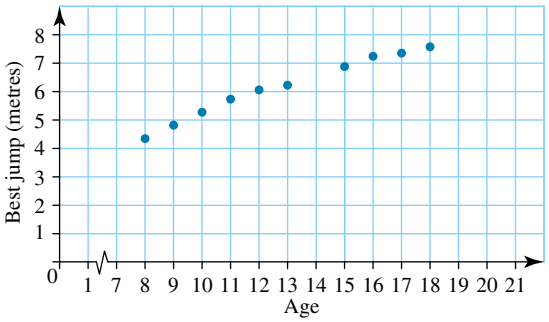
c. 24.25 cm; 26.39 cm; 28.53 cm; 30.67 cm; 31.74 cm; 34.95 cm; 36.02 cm; 38.16 cm; 39.23 cm

d. Interpolation (within the given range of 1–20)

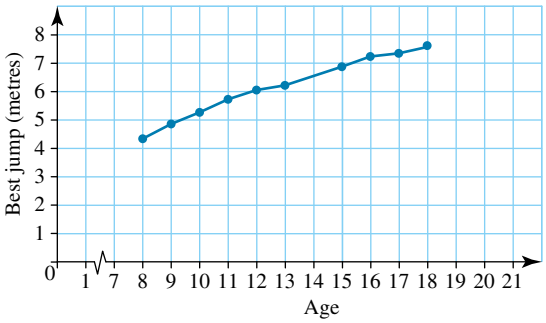
e. 41.37 cm; 42.44 cm; 43.51 cm

f. Not reliable, because extrapolation has been used.

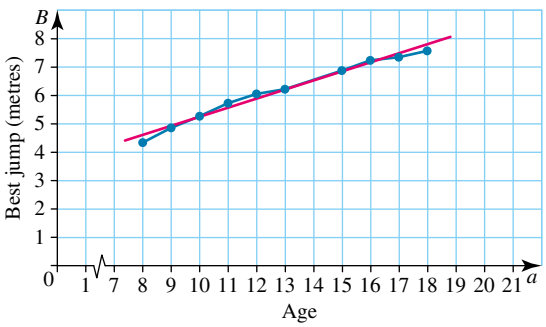
11. a.



b.



c.



Using technology, $B = 0.34a + 1.8$.

d. Estimated best jump = 8.6 m
e. No, trends work well over the short term but long term are affected by other variables.
f. 24 years old: 9.97 m; 28 years old: 11.33 m. It is unrealistic to expect his jumping distance to increase indefinitely.
g. Equal first

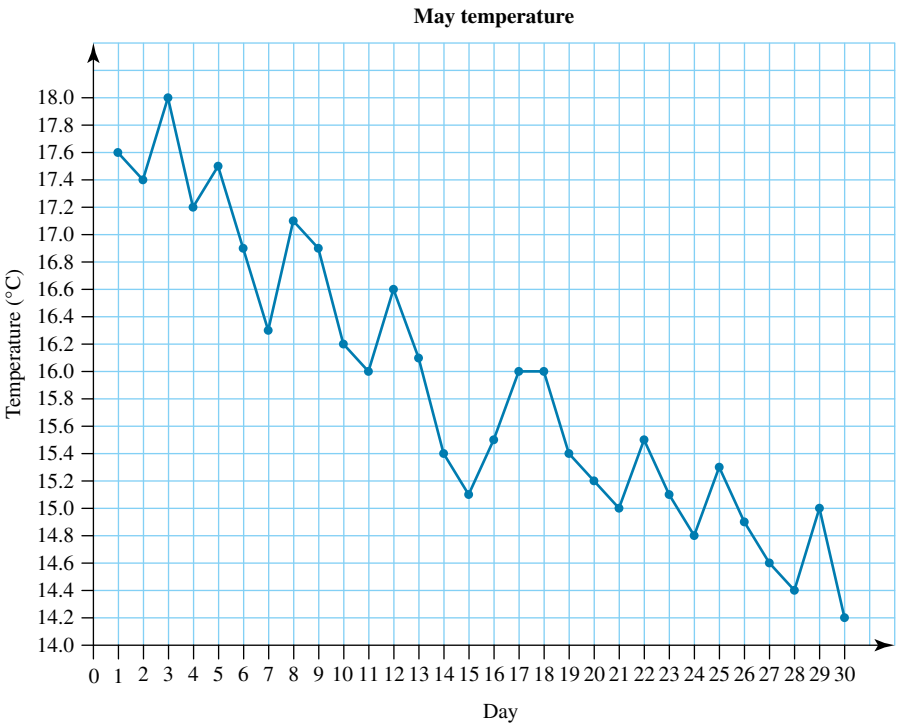
12. Extrapolation is outside the range of data given.

13. No. He would have to get 108%, which would be impossible on a test.

Exercise 15.4 Time as the independent variable

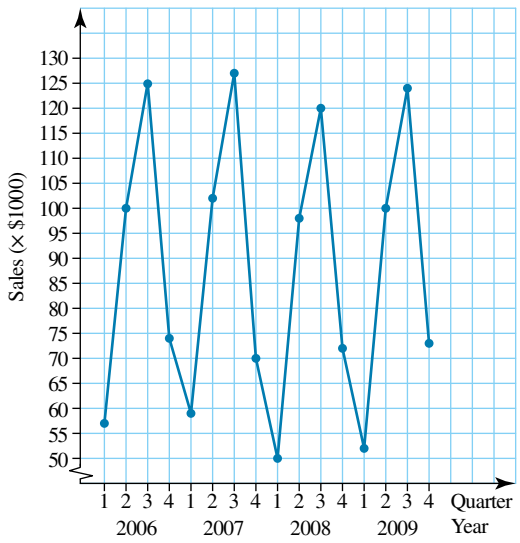
1. a. Linear, downward
b. Non-linear, upward
c. Non-linear, stationary in the mean
d. Linear, upward
e. Non-linear, downward
f. Non-linear, stationary in the mean
g. Non-linear, stationary in the mean
h. Linear, upward

2. a.



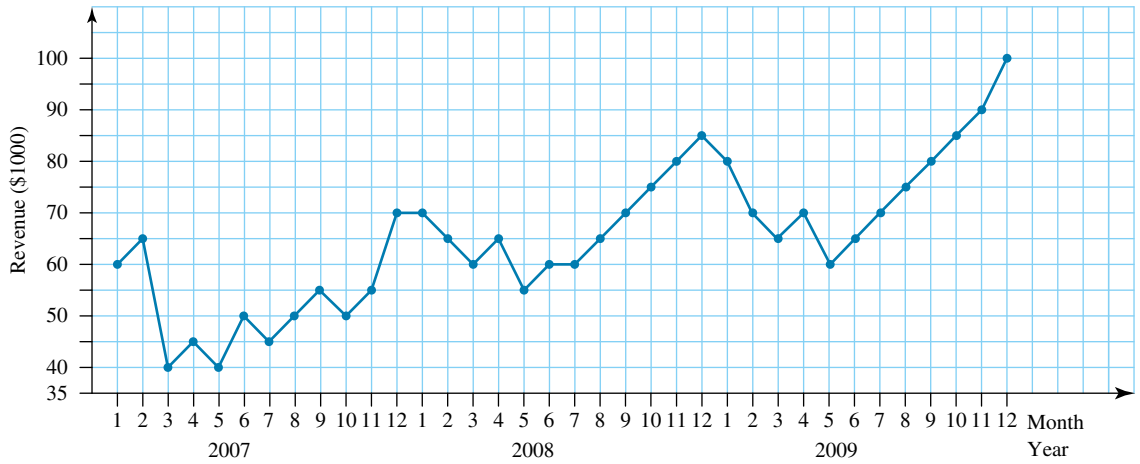
b. Linear downward trend

3. a.



b. Sheepskin products more popular in the third quarter (presumably winter) — discount sales, increase in sales, and so on.
c. No trend

4. a.



b. General upward trend with peaks around December and troughs around April.

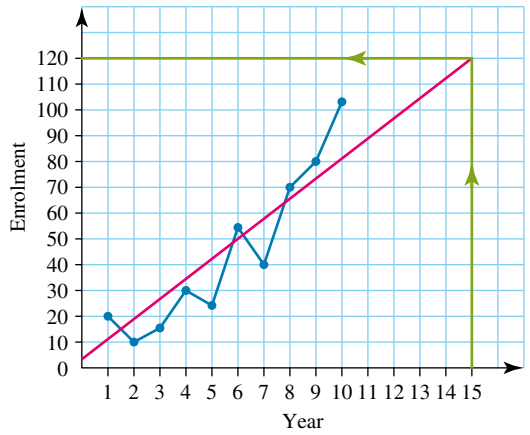
c. Peaks around Christmas where people have lots of parties, troughs around April where weather gets colder and people less inclined to go out.

d. Yes. Peaks in December, troughs in April.

5. a. Peaks around Christmas holidays and a minor peak at Easter. No camping in colder months.

b. Check with your teacher.

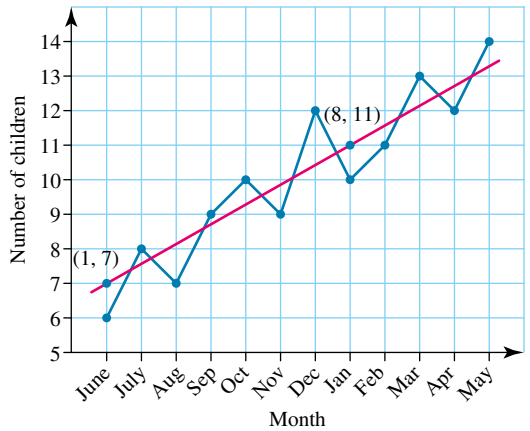
6. a.



Upward linear

b. In the 15th year, the expected amount is 122.

7. a.



b. Yes, the graph shows an upward trend.

c. $y = \frac{4}{7}x + \frac{45}{7}$

d. i. 15

ii. 18 (The assumption made was that business will continue on a linear upward trend.)

8. The trend is non-linear, therefore unable to forecast future sales.

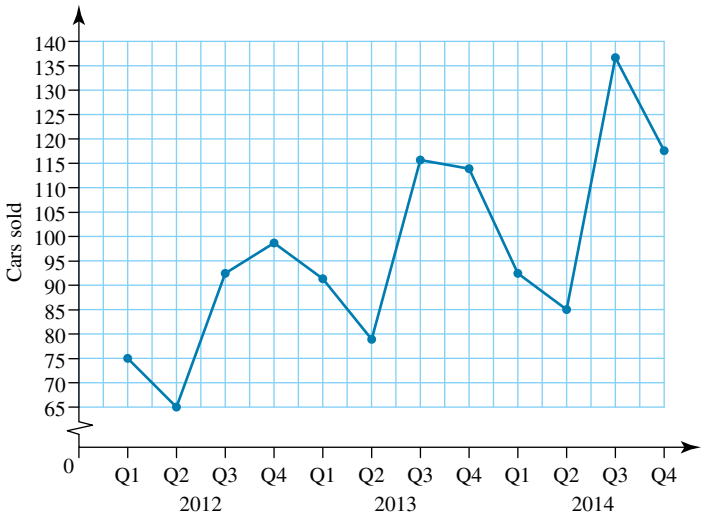
9. a. The percentage population growth in Australia, New Zealand and the United States in 1960 was 2.25%, 1.75% and 1.75% respectively. The percentage population growth in Australia, New Zealand and the United States in 2015 was 1.3%, 1.8% and 0.75% respectively. In this period, the USA has not had big fluctuations in population growth, and has stayed between a range of 1.75% in 1970 to 0.75% in 2015. New Zealand, on the other hand, has had large fluctuations over that time, from a high growth of 2.6% in 1962 to a low of −0.5% in 1979.

Australia has maintained a population growth of between 1% and 2.5% except for a spike in 1971 of 3.4% and a dip in 2007 to 0.6%.

b. In 2007, the world was not yet aware of the imminent Global Financial Crisis (GFC); hence, employment and living standards in the developed world were still high. This may have had an impact on Australia's net population growth. After the GFC, Australia experienced a higher level of interest in immigration as the Australian economy was not significantly affected by the GFC.

10. Check with your teacher. Some methods could involve 'smoothing' and 'seasonal adjustments'.

11. a.



b. Second-hand car sales per quarter have shown a general upward trend but with some major fluctuations.

c. More cars are sold in the third and fourth quarters compared to the first and second quarters.

12. Predictions into the future using time series data is appropriate if there are large amounts of data already provided and hence fluctuations and trends are visible based on knowledge of events, seasons and cycles.

13. 74

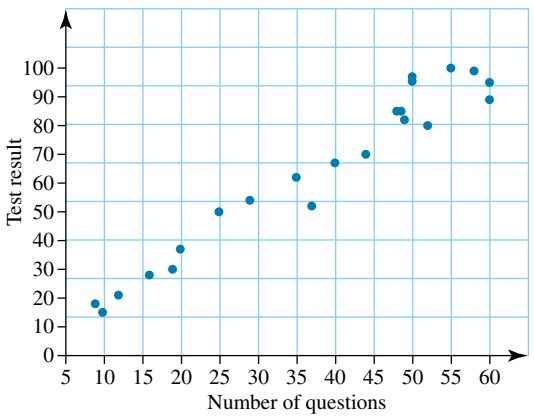
Investigation | Rich task

Answers will vary; check with your teacher.

Exercise 15.5 Review questions

1. a. Number of questions — independent; test result — dependent

b.



c. Strong, positive, linear relationship; the larger the number of completed revision questions, the higher the mark on the test.

d. Different abilities of the students

2. a. i. 12.5 **ii.** 49

b. i. 12 **ii.** 22.5

c. $y = \frac{22}{15}x - \frac{7}{3}$

d. For part **a**: **i.** 12.33 **ii.** 49

For part **b**: **i.** 11.82 **ii.** 22.05

3. a. Linear downwards

b. The trend is linear.

c. About 65 occupants

d. Assumes that the current trend will continue.

4. a. $P = 31.82a + 13\,070.4$, where P is the sale price and a is the land area.

b. The price of land is approximately \$31.82 per square metre.

c. \$64 000

d. 1160 m^2

5. a. $C = 0.15p + 11.09$, where C is the money spent at the canteen and p represents the pocket money received.

b. Students spend 15 cents at the canteen per dollar received for pocket money.

c. \$18

d. \$26. This involves extrapolation, which is considered unreliable. It does not seem reasonable that, if a student receives more money, they will eat more or have to purchase more than any other student.

6. a. $P = 0.91t + 2.95$, where P is the number of pirouettes and t is the number of hours of training.

b. Ballet students can do approximately 0.91 pirouettes for each hour of training.

c. Approximately 15 pirouettes

d. Approximately 30 pirouettes. This estimate is based on extrapolation, which is considered unreliable. To model this data linearly as the number of hours of training becomes large is unrealistic.

7. a. Hours of TV watched = 3.31 × age + 3.05

b. Approximately 53 hours

c. Approximately 17 years old

8. Overall the data appears to be following a seasonal trend, with peaks at either end of the year and a trough in the middle.

9. a. Outliers can unfairly skew data and as such dramatically alter the line of best fit. Identify and remove any outliers from the data before determining the line of best fit.

b. Extrapolation involves making estimates outside the data range and this is considered unreliable. When extrapolation is required, consider the data and the likelihood that the data would remain linear if extended. When giving results, make comment on the validity of the estimation.

c. A small range may not give a fair indication if a data set shows a strong linear relationship. Try to increase the range of the data set by taking more measurements or undertaking more research.

d. A small number of data points may not be able to establish with confidence the existence of a strong linear relationship. Try to increase the number of data points by taking more measurements or undertaking more research.

10. About 170 cm; data was first plotted as a scatter plot. (145, 160) was identified as an outlier and removed from the data set. A line of best fit was then fitted to the remaining data and its equation determined as $d = 0.5h + 80$, where d is the distance stretched and h is the height. Substitution was used to obtain the estimate.

The estimation requires extrapolation and cannot be considered reliable. The presence of the outlier may indicate variation in flexibility rather than a strong linear relationship between the data. The estimate is based on a small set of data. More data should be collected in order to determine the suitability of least squares regression.

TOPIC 16
Surds, indices and logarithms

[Stage 5.3]

16.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the concepts covered in this topic.

16.1.1 Why learn this?

A knowledge of number is crucial if we are to understand the world around us. Over time, you have been building your knowledge of the concept of number, starting with the counting numbers, also known as natural numbers. Moving on, you needed to include zero. You then had to learn about integers and fractions, which are also called rational numbers. But even the rational numbers do not include all of the numbers on the number line, as they do not include numbers that cannot be written as fractions. That brings us to the concept of real numbers, the set of numbers that includes both rational and irrational numbers.

DISCUSSION

Imagine a world without numbers. How would it be different? What would it look like?

LEARNING SEQUENCE

16.1 Overview
16.2 [Stage 5.3] Rational and irrational numbers
16.3 [Stage 5.3] Surds
16.4 [Stage 5.3] Operations with surds
16.5 [Stage 5.3] Fractional indices
16.6 [Stage 5.3] Negative indices
16.7 [Stage 5.3] Logarithms
16.8 [Stage 5.3] Logarithm laws
16.9 [Stage 5.3] Solving equations
16.10 Review

LEARNING OUTCOMES

A student:

- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3-1WM
- generalises mathematical ideas and techniques to analyse and solve problems efficiently MA5.3-2WM
- uses deductive reasoning in presenting arguments and formal proofs MA5.3-3WM
- performs operations with surds and indices MA5.3-6NA
- uses the definition of a logarithm to establish and apply the laws of logarithms MA5.3-11NA

CONTENT DESCRIPTIONS

Students:

Define rational and irrational numbers and perform operations with surds and fractional indices (ACMNA264)
Use the definition of a logarithm to establish and apply the laws of logarithms (ACMNA265)
Solve simple exponential equations (ACMNA270)

Source: NSW Syllabus for the Australian curriculum

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eLesson: The story of mathematics — Real numbers (eles-2019)

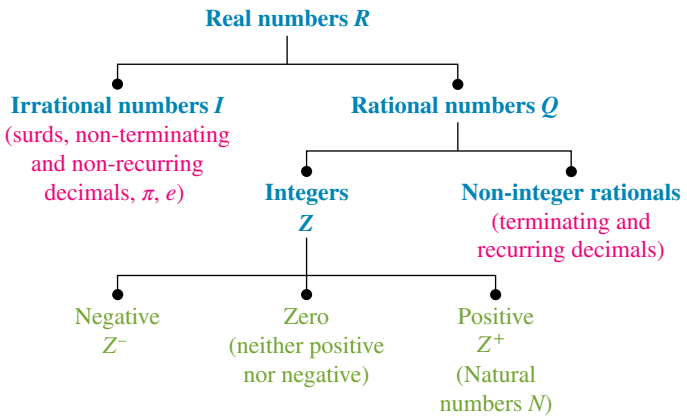
Note: Your teacher may now set you a pre-test to determine how familiar you are with the content in this topic.

16.2 Rational and irrational numbers [Stage 5.3]

16.2.1 The number system

- The number systems used today evolved from a basic and practical need of primitive people to count and measure magnitudes and quantities such as livestock, people, possessions, time and so on.
- As societies grew and architecture and engineering developed, number systems became more sophisticated. Number use developed from solely whole numbers to fractions, decimals and irrational numbers.

- The real number system contains the set of rational and irrational numbers. It is denoted by the symbol R. The set of real numbers contains a number of subsets, which can be classified as shown in the chart below.



16.2.2 Rational numbers (Q)

- A **rational number** is a number that can be expressed as a ratio of two whole numbers in the form $\frac{a}{b}$, where $b \neq 0$.
- Rational numbers are given the symbol Q. Examples are:

$$\tfrac{1}{5}, \tfrac{2}{7}, \tfrac{3}{10}, \tfrac{9}{4}, 7, -6, 0.35, 1.\dot{4}$$

16.2.3 Integers (Z)

- Rational numbers may be expressed as **integers**. Examples are:

$$\frac{5}{1} = 5, -\frac{4}{1} = -4, \frac{27}{1} = 27, -\frac{15}{1} = -15.$$

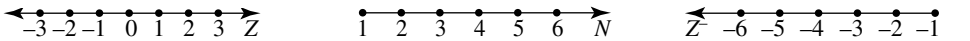
- The set of integers consists of positive and negative whole numbers and 0 (which is neither positive nor negative). They are denoted by the letter Z and can be further divided into subsets. That is:

$$Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$
$$Z^+ = \{1, 2, 3, 4, 5, 6, \ldots\}$$
$$Z^- = \{\ldots, -5, -4, -3, -2, -1\}$$

- Positive integers are also known as **natural numbers** (or counting numbers) and are denoted by the letter N. That is:

$$N = \{1, 2, 3, 4, 5, 6, \ldots\}$$

- Integers may be represented on the number line as illustrated below.



The set of integers | The set of positive integers or natural numbers | The set of negative integers

Note: Integers on the number line are marked with a solid dot to indicate that they are the only points in which we are interested.

16.2.4 Non-integer rational numbers

- Rational numbers may be expressed as **terminating decimals**. Examples are:

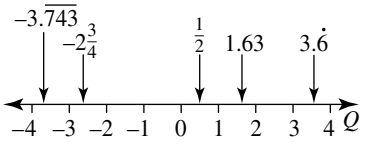
$$\tfrac{7}{10} = 0.7, \tfrac{1}{4} = 0.25, \tfrac{5}{8} = 0.625, \tfrac{9}{5} = 1.8.$$

These decimal numbers terminate after a specific number of digits.

- Rational numbers may be expressed as **recurring decimals** (non-terminating or periodic decimals). For example:

$$\tfrac{1}{3} = 0.333\,333\ldots \text{ or } 0.\dot{3}$$
$$\tfrac{9}{11} = 0.818\,181\ldots \text{ or } 0.\dot{8}\dot{1} \text{ (or } 0.\overline{81})$$
$$\tfrac{5}{6} = 0.833\,333\ldots \text{ or } 0.8\dot{3}$$
$$\tfrac{3}{13} = 0.230\,769\,230\,769\ldots \text{ or } 0.\dot{2}30\,769\dot{0} \text{ (or } 0.\overline{230\,769})$$

- These decimal numbers do not terminate, and the specific digit (or number of digits) is repeated in a pattern. Recurring decimals are represented by placing a dot or line above the repeating digit or pattern.



- Rational numbers are defined in set notation as:
  $Q =$ rational numbers
  $Q = \left\{\frac{a}{b}, a, b \in Z, b \neq 0\right\}$ where $\in$ means 'an element of'.

ACTIVITY: RECURRING DECIMAL PATTERNS

1. In pairs, express $0.\dot{1}$, $0.\dot{2}$, $0.\dot{4}$ and $0.\dot{5}$ as fractions.
2. What do you notice about the fractions before you simplify?
3. Express $0.\dot{1}\dot{2}$, $0.\dot{2}\dot{7}$, $0.\dot{4}\dot{4}$ and $0.\dot{5}\dot{8}$ as fractions.
4. What do you notice about the fractions before you simplify?
5. Make a prediction about the fraction that will result in expressing recurring decimals with three repeating digits, then confirm your prediction by completing at least **two** conversions.
6. (a) Discuss any patterns found.
   (b) Write a rule that will predict the fraction for any recurring decimal regardless of the number of repeating digits.

16.2.5 Irrational numbers (*I*)

- An **irrational number** is a number that cannot be expressed as a ratio of two whole numbers in the form $\frac{a}{b}$, where $b \neq 0$.
- Irrational numbers are given the symbol I. Examples are:

$$\sqrt{7},\ \sqrt{13},\ 5\sqrt{21},\ \frac{\sqrt{7}}{9},\ \pi,\ e.$$

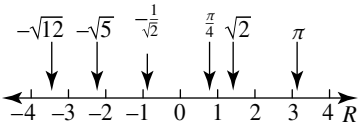
- Irrational numbers may be expressed as decimals. For example:

$$\sqrt{5} = 2.236\,067\,977\,5\ldots$$
$$\sqrt{18} = 4.242\,640\,687\,12\ldots$$
$$\pi = 3.141\,592\,653\,59\ldots$$
$$\sqrt{0.03} = 0.173\,205\,080\,757\ldots$$
$$2\sqrt{7} = 5.291\,502\,622\,13\ldots$$
$$e = 2.718\,281\,828\,46\ldots$$

- These decimal numbers do not terminate, and the digits do not repeat themselves in any particular pattern or order (that is, they are non-terminating and non-recurring).

16.2.6 Real numbers

- Rational and irrational numbers belong to the set of **real numbers** (denoted by the symbol ***R***). They can be positive, negative or 0. The real numbers may be represented on a number line as shown at right (irrational numbers above the line; rational numbers below it).



- To classify a number as either rational or irrational:
  1. Determine whether it can be expressed as a whole number, a fraction or a terminating or recurring decimal.
  2. If the answer is yes, the number is rational; if the answer is no, the number is irrational.

16.2.7 π (pi)

- The symbol **π (pi)** is used for a particular number: the circumference of a circle whose diameter length is 1 unit.
- It can be approximated as a decimal that is non-terminating and non-recurring. Therefore, π is classified as an irrational number. (It is also called a **transcendental number** and cannot be expressed as a surd.)
- In decimal form, $\pi = 3.141\,592\,653\,589\,793\,23\ldots$ It has been calculated to 29 000 000 (29 million) decimal places with the aid of a computer.

WORKED EXAMPLE 1

Specify whether the following numbers are rational or irrational.

a $\frac{1}{5}$ **b** $\sqrt{25}$ **c** $\sqrt{13}$ **d** 3π **e** 0.54 **f** $\sqrt[3]{64}$ **g** $\sqrt[3]{32}$ **h** $\sqrt[3]{\frac{1}{27}}$

THINK	WRITE
a $\frac{1}{5}$ is already a rational number.	**a** $\frac{1}{5}$ is rational.
b 1 Evaluate $\sqrt{25}$.	**b** $\sqrt{25} = 5$
2 The answer is an integer, so classify $\sqrt{25}$.	$\sqrt{25}$ is rational.
c 1 Evaluate $\sqrt{13}$.	**c** $\sqrt{13} = 3.605\,551\,275\,46\ldots$
2 The answer is a non-terminating and non-recurring decimal; classify $\sqrt{13}$.	$\sqrt{13}$ is irrational.
d 1 Use your calculator to find the value of 3π.	**d** $3\pi = 9.424\,777\,960\,77\ldots$
2 The answer is a non-terminating and non-recurring decimal; classify 3π.	3π is irrational.
e 0.54 is a terminating decimal; classify it accordingly.	**e** 0.54 is rational.
f 1 Evaluate $\sqrt[3]{64}$.	**f** $\sqrt[3]{64} = 4$
2 The answer is a whole number, so classify $\sqrt[3]{64}$.	$\sqrt[3]{64}$ is rational.
g 1 Evaluate $\sqrt[3]{32}$.	**g** $\sqrt[3]{32} = 3.174\,802\,103\,94\ldots$
2 The result is a non-terminating and non-recurring decimal; classify $\sqrt[3]{32}$.	$\sqrt[3]{32}$ is irrational.
h 1 Evaluate $\sqrt[3]{\frac{1}{27}}$.	**h** $\sqrt[3]{\frac{1}{27}} = \frac{1}{3}$
2 The result is a number in a rational form.	$\sqrt[3]{\frac{1}{27}}$ is rational.

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Interactivity: The number system (int-6027)
Interactivity: Recurring decimals (int-6189)
Interactivity: Classifying numbers (int-2792)

Exercise 16.2 Rational and irrational numbers

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–6, 8, 10	Questions: 1–8, 10, 12, 13	Questions: 1–15

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE1 Specify whether the following numbers are rational (Q) or irrational (I).

a. $\sqrt{4}$	**b.** $\frac{4}{5}$	**c.** $\frac{7}{9}$	**d.** $\sqrt{2}$	**e.** $\sqrt{7}$
f. $\sqrt{0.04}$	**g.** $2\frac{1}{2}$	**h.** $\sqrt{5}$	**i.** $\frac{9}{4}$	**j.** 0.15
k. -2.4	**l.** $\sqrt{100}$	**m.** $\sqrt{14.4}$	**n.** $\sqrt{14.4}$	**o.** π
p. $\sqrt{\frac{25}{9}}$	**q.** 7.32	**r.** $-\sqrt{21}$	**s.** $\sqrt{1000}$	**t.** 7.216 349 157 …
u. $-\sqrt{81}$	**v.** 3π	**w.** $\sqrt[3]{62}$	**x.** $\sqrt{\frac{1}{16}}$	**y.** $\sqrt[3]{0.0001}$

2. Specify whether the following numbers are rational (Q), irrational (I) or neither.

a. $\frac{1}{8}$	**b.** $\sqrt{625}$	**c.** $\frac{11}{4}$	**d.** $\frac{0}{8}$	**e.** $-6\frac{1}{7}$
f. $\sqrt[3]{81}$	**g.** $-\sqrt{11}$	**h.** $\sqrt{\frac{1.44}{4}}$	**i.** $\sqrt{\pi}$	**j.** $\frac{8}{0}$
k. $\sqrt[3]{21}$	**l.** $\frac{\pi}{7}$	**m.** $\sqrt[3]{(-5)^2}$	**n.** $-\frac{3}{11}$	**o.** $\sqrt{\frac{1}{100}}$
p. $\frac{64}{16}$	**q.** $\sqrt{\frac{2}{25}}$	**r.** $\frac{\sqrt{6}}{2}$	**s.** $\sqrt[3]{27}$	**t.** $\frac{1}{\sqrt{4}}$
u. $\frac{22\pi}{7}$	**v.** $\sqrt[3]{-1.728}$	**w.** $6\sqrt{4}$	**x.** $4\sqrt{6}$	**y.** $(\sqrt{2})^4$

3. MC Which of the following best represents a rational number?

A. π **B.** $\sqrt{\frac{4}{9}}$ **C.** $\sqrt{\frac{9}{12}}$ **D.** $\sqrt[3]{3}$ **E.** $\sqrt{5}$

4. MC Which of the following best represents an irrational number?

A. $-\sqrt{81}$ **B.** $\frac{6}{5}$ **C.** $\sqrt[3]{343}$ **D.** $\sqrt{22}$ **E.** $\sqrt{144}$

5. MC Which of the following statements regarding the numbers -0.69, $\sqrt{7}$, $\frac{\pi}{3}$, $\sqrt{49}$ is correct?

A. $\frac{\pi}{3}$ is the only rational number.

B. $\sqrt{7}$ and $\sqrt{49}$ are both irrational numbers.

C. -0.69 and $\sqrt{49}$ are the only rational numbers.

D. -0.69 is the only rational number.

E. $\sqrt{7}$ is the only rational number.

6. MC Which of the following statements regarding the numbers $2\frac{1}{2}$, $-\frac{11}{3}$, $\sqrt{624}$, $\sqrt[3]{99}$ is correct?

A. $-\frac{11}{3}$ and $\sqrt{624}$ are both irrational numbers.

B. $\sqrt{624}$ is an irrational number and $\sqrt[3]{99}$ is a rational number.

C. $\sqrt{624}$ and $\sqrt[3]{99}$ are both irrational numbers.

D. $2\frac{1}{2}$ is a rational number and $-\frac{11}{3}$ is an irrational number.

E. $\sqrt[3]{99}$ is the only rational number.

7. Simplify $\sqrt{\frac{a^2}{b^2}}$.

8. MC If $p < 0$, then $\sqrt{p}$ is:

A. positive **B.** negative **C.** rational
D. irrational **E.** none of the above

9. MC If $p < 0$, then $\sqrt{p^2}$ must be:

A. positive **B.** negative **C.** rational
D. irrational **E.** any of the above

Communicating, reasoning and problem solving

10. Simplify $(\sqrt{p} - \sqrt{q}) \times (\sqrt{p} + \sqrt{q})$.

11. Prove that if $c^2 = a^2 + b^2$, it does not follow that $a = b + c$.

12. Find the value of m and n if $\frac{36}{11}$ is written as:

a. $3 + \frac{1}{\frac{m}{n}}$ **b.** $3 + \frac{1}{3 + \frac{m}{n}}$ **c.** $3 + \frac{1}{3 + \frac{1}{\frac{m}{n}}}$ **d.** $3 + \frac{1}{3 + \frac{1}{1 + \frac{m}{n}}}$.

13. If x^{-1} means $\frac{1}{x}$, what is the value of $\frac{3^{-1} - 4^{-1}}{3^{-1} + 4^{-1}}$?

14. Follow the example below to convert the recurring decimal 0.090909… to a fraction.

Step 1: Let $x = 0.090909\ldots$

Step 2: Count the number of recurring digits and hence move the decimal point on both sides of the equation an equivalent number of decimal places.

In this example there are two recurring digits, so the decimal point needs to be moved two places to the right and hence x needs to be multiplied by 100.

$x = 0.090909\ldots$ [1]

$100x = 9.0909\ldots$ [2]

Step 3: Subtract equation 1 from equation 2.

[2]−[1]:

$99x = 9$

$x = \frac{1}{11}$

Using the process outlined above or another method, write the following recurring decimals as fractions

a. 0.1111…

b. 0.166666…

c. 0.714285714285 71…

15. Why is it important to understand the real number system?

16.3 Surds [Stage 5.3]

16.3.1 Definition of a surd

- A **surd** is an irrational number that is represented by a root sign or a radical sign, for example: $\sqrt{}$, $\sqrt[3]{}$, $\sqrt[4]{}$.
- Examples of surds include: $\sqrt{7}$, $\sqrt{5}$, $\sqrt[3]{11}$, $\sqrt[4]{15}$.
- Examples that are not surds include: $\sqrt{9}$, $\sqrt{16}$, $\sqrt[3]{125}$, $\sqrt[4]{81}$.
- Numbers that are not surds can be simplified to rational numbers, that is: $\sqrt{9} = 3$, $\sqrt{16} = 4$, $\sqrt[3]{125} = 5$, $\sqrt[4]{81} = 3$.
- A decimal approximation for a surd can be found, but if an exact value is required, the decimal approximation cannot be used.

WORKED EXAMPLE 2

Which of the following numbers are surds?

a $\sqrt{16}$ **b** $\sqrt{13}$ **c** $\sqrt{\frac{1}{16}}$ **d** $\sqrt[3]{17}$ **e** $\sqrt[4]{63}$ **f** $\sqrt[3]{1728}$

THINK	WRITE
a 1 Evaluate $\sqrt{16}$.	**a** $\sqrt{16} = 4$
2 The answer is rational (since it is a whole number), so state your conclusion.	$\sqrt{16}$ is not a surd.
b 1 Evaluate $\sqrt{13}$.	**b** $\sqrt{13} = 3.605\,551\,275\,46\ldots$
2 The answer is irrational (since it is a non-recurring and non-terminating decimal), so state your conclusion.	$\sqrt{13}$ is a surd.
c 1 Evaluate $\sqrt{\frac{1}{16}}$.	**c** $\sqrt{\frac{1}{16}} = \frac{1}{4}$
2 The answer is rational (a fraction); state your conclusion.	$\sqrt{\frac{1}{16}}$ is not a surd.
d 1 Evaluate $\sqrt[3]{17}$.	**d** $\sqrt[3]{17} = 2.571\,281\,590\,66\ldots$
2 The answer is irrational (a non-terminating and non-recurring decimal), so state your conclusion.	$\sqrt[3]{17}$ is a surd.
e 1 Evaluate $\sqrt[4]{63}$.	**e** $\sqrt[4]{63} = 2.817\,313\,247\,26\ldots$
2 The answer is irrational, so classify $\sqrt[4]{63}$ accordingly.	$\sqrt[4]{63}$ is a surd.
f 1 Evaluate $\sqrt[3]{1728}$.	**f** $\sqrt[3]{1728} = 12$
2 The answer is rational; state your conclusion.	$\sqrt[3]{1728}$ is not a surd. So **b**, **d** and **e** are surds.

ACTIVITY: SURD MEMORY

Equipment: paper, scissors

1. Write down five surds and their simplified versions on a piece of paper, then cut them out into cards.
2. Swap your cards with a partner, then match up each surd with its simplified version. Compare your results and check that every simplification works.
3. Swap all the cards with another pair, then take the simplified surds and arrange them in ascending order.
4. Lay all the cards face down on the table in front of you and play Memory with the cards.
5. What strategies can you use to help you identify what factors to use when simplifying surds?

16.3.2 Proof that a number is irrational

- In Mathematics you are required to study a variety of types of proofs. One such method is called proof by contradiction.
- This proof is so named because the logical argument of the proof is based on an assumption that leads to contradiction within the proof. Therefore, the original assumption must be false.
- An irrational number is one that cannot be expressed in the form $\frac{a}{b}$ (where a and b are integers). The next worked example sets out to prove that $\sqrt{2}$ is irrational.

WORKED EXAMPLE 3

Prove that $\sqrt{2}$ is irrational.

THINK	WRITE
1 Assume that $\sqrt{2}$ is rational; that is, it can be written as $\frac{a}{b}$ in simplest form. We need to show that a and b have no common factors.	Let $\sqrt{2} = \frac{a}{b}$, where $b \neq 0$.
2 Square both sides of the equation.	$2 = \frac{a^2}{b^2}$
3 Rearrange the equation to make a^2 the subject of the formula.	$a^2 = 2b^2$ [1]
4 $2b^2$ is an even number and $2b^2 = a^2$.	$\therefore a^2$ is an even number and a must also be even; that is, a has a factor of 2.
5 Since a is even it can be written as $a = 2r$.	$\therefore a = 2r$
6 Square both sides.	$a^2 = 4r^2$ [2] But $a^2 = 2b^2$ from [1].
7 Equate [1] and [2].	$2b^2 = 4r^2$ $b^2 = \frac{4r^2}{2}$ $b^2 = 2r^2$ $\therefore b^2$ is an even number and b must also be even; that is, b has a factor of 2.
8 Repeat the steps for b as previously done for a.	Both a and b have a common factor of 2. This contradicts the original assumption that $\sqrt{2} = \frac{a}{b}$, where a and b have no common factor. $\therefore \sqrt{2}$ is not rational. $\therefore$ It must be irrational.

- *Note:* An irrational number written in surd form gives an exact value of the number, whereas the same number written in decimal form (for example, to 4 decimal places) gives an approximate value.

DISCUSSION

Can you use proof by contradiction to prove that all irrational surds are actually irrational?

learnon RESOURCES — ONLINE ONLY

Interactivity: Surds on the number line (int-6029)

Digital doc: SkillSHEET Identifying surds (doc-5354)

Exercise 16.3 Surds

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–8, 10	Questions: 1–11	Questions: 1–13

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE2 Which of the numbers below are surds?

a. $\sqrt{81}$	**b.** $\sqrt{48}$	**c.** $\sqrt{16}$	**d.** 1.6
e. $\sqrt{0.16}$	**f.** $\sqrt{11}$	**g.** $\sqrt{\frac{3}{4}}$	**h.** $\sqrt[3]{\frac{3}{37}}$
i. $\sqrt{1000}$	**j.** $\sqrt{1.44}$	**k.** $4\sqrt{100}$	**l.** $2 + \sqrt{10}$
m. $\sqrt[3]{32}$	**n.** $\sqrt{361}$	**o.** $\sqrt[3]{100}$	**p.** $\sqrt[3]{125}$
q. $\sqrt{6} + \sqrt{6}$	**r.** 2π	**s.** $\sqrt[3]{169}$	**t.** $\sqrt{\frac{7}{8}}$
u. $\sqrt[4]{16}$	**v.** $(\sqrt{7})^2$	**w.** $\sqrt[3]{33}$	**x.** $\sqrt{0.0001}$
y. $\sqrt[5]{32}$	**z.** $\sqrt{80}$		

2. MC The correct statement regarding the set of numbers $\left\{\frac{\sqrt{6}}{9}, \sqrt{20}\sqrt{54}\sqrt[3]{27}, \sqrt{9}\right\}$ is:

A. $\sqrt[3]{27}$ and $\sqrt{9}$ are the only rational numbers of the set.

B. $\sqrt{\frac{6}{9}}$ is the only surd of the set.

C. $\sqrt{\frac{6}{9}}$ and $\sqrt{20}$ are the only surds of the set.

D. $\sqrt{20}$ and $\sqrt{54}$ are the only surds of the set.

E. $\sqrt{9}$ and $\sqrt{20}$ are the only surds of the set.

3. MC Which of the numbers of the set $\left\{\sqrt{\frac{1}{4}}, \sqrt[3]{\frac{1}{27}}, \sqrt{\frac{1}{8}}, \sqrt{21}, \sqrt[3]{8}\right\}$ are surds?

A. $\sqrt{21}$ only **B.** $\sqrt{\frac{1}{8}}$ only **C.** $\sqrt{\frac{1}{8}}$ and $\sqrt[3]{8}$

D. $\sqrt{\frac{1}{8}}$ and $\sqrt{21}$ only **E.** $\sqrt{\frac{1}{4}}$ and $\sqrt{21}$ only

4. MC Which statement regarding the set of numbers $\left\{\pi, \sqrt{\frac{1}{49}}, \sqrt{12}, \sqrt{16}, \sqrt{3} + 1\right\}$ is *not* true?

A. $\sqrt{12}$ is a surd. **B.** $\sqrt{12}$ and $\sqrt{16}$ are surds.

C. π is irrational but not a surd. **D.** $\sqrt{12}$ and $\sqrt{3} + 1$ are not rational.

E. π is not a surd.

5. MC Which statement regarding the set of numbers $\left\{6\sqrt{7}, \sqrt{\frac{144}{16}}, 7\sqrt{6}, 9\sqrt{2}, \sqrt{18}, \sqrt{25}\right\}$ is *not* true?

A. $\sqrt{\frac{144}{16}}$ when simplified is an integer. **B.** $\sqrt{\frac{144}{16}}$ and $\sqrt{25}$ are not surds.

C. $7\sqrt{6}$ is smaller than $9\sqrt{2}$. **D.** $9\sqrt{2}$ is smaller than $6\sqrt{7}$.

E. $\sqrt{18}$ is a surd.

6. Complete the following statement by selecting appropriate words from the suggestions in brackets:
$\sqrt[6]{a}$ is definitely not a surd, if a is … (any multiple of 4; a perfect square and cube).

7. Find the smallest value of m, where m is a positive integer, so that $\sqrt[3]{16m}$ is not a surd.

Communicating, reasoning and problem solving

8. WE3 Prove that the following numbers are irrational, using a proof by contradiction:
a. $\sqrt{3}$ **b.** $\sqrt{5}$ **c.** $\sqrt{7}$

9. π is an irrational number and so is $\sqrt{3}$. Therefore, determine whether $(\pi - \sqrt{3})(\pi + \sqrt{3})$ is an irrational number.

10. Many composite numbers have a variety of factor pairs. For example, factor pairs of 24 are 1 and 24, 2 and 12, 3 and 8, 4 and 6.
a. Use each pair of possible factors to simplify the following surds.
i. $\sqrt{48}$ **ii.** $\sqrt{72}$
b. Does the factor pair chosen when simplifying a surd affect the way the surd is written in simplified form?
c. Does the factor pair chosen when simplifying a surd affect the value of the surd when it is written in simplified form? Explain.

11. Solve $\sqrt{3}x - \sqrt{12} = \sqrt{3}$ and indicate whether the result is rational or irrational and integral or not integral.

12. An equilateral triangle has a side length of 10 cm. Calculate the perpendicular height of the triangle, leaving your answer as an exact value.

13. How can you be certain that $\sqrt{a}$ is a surd?

16.4 Operations with surds [Stage 5.3]

16.4.1 Simplifying surds

- To simplify a surd means to make a number (or an expression) under the radical sign ($\sqrt{\ }$) as small as possible.
- To simplify a surd (if it is possible), it should be rewritten as a product of two factors, one of which is a perfect square, that is, 4, 9, 16, 25, 36, 49, 64, 81, 100 and so on.
- We must always aim to obtain the largest perfect square when simplifying surds so that there are fewer steps involved in obtaining the answer. For example, $\sqrt{32}$ can be written as $\sqrt{4 \times 8} = 2\sqrt{8}$; however, $\sqrt{8}$ can be further simplified to $2\sqrt{2}$, so $\sqrt{32} = 2 \times 2\sqrt{2}$; that is $\sqrt{32} = 4\sqrt{2}$. If, however, the largest perfect square is selected and $\sqrt{32}$ is written as $\sqrt{16 \times 2} = \sqrt{16} \times \sqrt{2} = 4\sqrt{2}$, the same answer is obtained in fewer steps.

WORKED EXAMPLE 4

Simplify the following surds. Assume that x and y are positive real numbers.
a $\sqrt{384}$ **b** $\sqrt[3]{405}$ **c** $-\frac{1}{8}\sqrt{175}$ **d** $5\sqrt{180x^3y^5}$

THINK

a 1 Express 384 as a product of two factors where one factor is the largest possible perfect square.

2 Express $\sqrt{64 \times 6}$ as the product of two surds.

3 Simplify the square root from the perfect square (that is, $\sqrt{64} = 8$).

b 1 Express 405 as a product of two factors, one of which is the largest possible perfect square.

2 Express $\sqrt{81 \times 5}$ as a product of two surds.

WRITE

a $\sqrt{384} = \sqrt{64 \times 6}$
$= \sqrt{64} \times \sqrt{6}$
$= 8\sqrt{6}$

b $3\sqrt{405} = 3\sqrt{81 \times 5}$
$= 3\sqrt{81} \times \sqrt{5}$

3 Simplify $\sqrt{81}$.

4 Multiply together the whole numbers outside the square root sign (3 and 9).

$= 3 \times 9\sqrt{5}$

$= 27\sqrt{5}$

c 1 Express 175 as a product of two factors in which one factor is the largest possible perfect square.

2 Express $\sqrt{25 \times 7}$ as a product of 2 surds.

3 Simplify $\sqrt{25}$.

4 Multiply together the numbers outside the square root sign.

c $-\frac{1}{8}\sqrt{175} = -\frac{1}{8}\sqrt{25 \times 7}$

$= -\frac{1}{8} \times \sqrt{25} \times \sqrt{7}$

$= -\frac{1}{8} \times 5\sqrt{7}$

$= -\frac{5}{8}\sqrt{7}$

d 1 Express each of 180, x^3 and y^5 as a product of two factors where one factor is the largest possible perfect square.

2 Separate all perfect squares into one surd and all other factors into the other surd.

3 Simplify $\sqrt{36x^2y^4}$.

4 Multiply together the numbers and the pronumerals outside the square root sign.

d $5\sqrt{180x^3y^5} = 5\sqrt{36 \times 5 \times x^2 \times x \times y^4 \times y}$

$= 5 \times \sqrt{36x^2y^4} \times \sqrt{5xy}$

$= 5 \times 6 \times x \times y^2 \times \sqrt{5xy}$

$= 30xy^2\sqrt{5xy}$

16.4.2 Addition and subtraction of surds

- Surds may be added or subtracted only if they are *alike*.
- Examples of *like* surds include $\sqrt{7}$, $3\sqrt{7}$ and $-5\sqrt{7}$. Examples of *unlike* surds include $\sqrt{11}$, $\sqrt{5}$, $2\sqrt{13}$ and $-2\sqrt{3}$.
- In some cases surds will need to be simplified before you decide whether they are like or unlike, and then addition and subtraction can take place. The concept of adding and subtracting surds is similar to adding and subtracting like terms in algebra.

WORKED EXAMPLE 5

Simplify each of the following expressions containing surds. Assume that a and b are positive real numbers.

a $3\sqrt{6} + 17\sqrt{6} - 2\sqrt{6}$ **b** $5\sqrt{3} + 2\sqrt{12} - 5\sqrt{2} + 3\sqrt{8}$

c $\frac{1}{2}\sqrt{100a^3b^2} + ab\sqrt{36a} - 5\sqrt{4a^2b}$

THINK	WRITE
a All 3 terms are alike because they contain the same surd ($\sqrt{6}$). Simplify.	**a** $3\sqrt{6} + 17\sqrt{6} - 2\sqrt{6} = (3 + 17 - 2)\sqrt{6}$ $= 18\sqrt{6}$
b 1 Simplify surds where possible.	**b** $5\sqrt{3} + 2\sqrt{12} - 5\sqrt{2} + 3\sqrt{8}$ $= 5\sqrt{3} + 2\sqrt{4 \times 3} - 5\sqrt{2} + 3\sqrt{4 \times 2}$ $= 5\sqrt{3} + 2 \times 2\sqrt{3} - 5\sqrt{2} + 3 \times 2\sqrt{2}$ $= 5\sqrt{3} + 4\sqrt{3} - 5\sqrt{2} + 6\sqrt{2}$
2 Add like terms to obtain the simplified answer.	$= 9\sqrt{3} + \sqrt{2}$

c 1 Simplify surds where possible.

c $\frac{1}{2}\sqrt{100a^3b^2} + ab\sqrt{36a} - 5\sqrt{4a^2b}$

$= \frac{1}{2} \times 10\sqrt{a^2 \times a \times b^2} + ab \times 6\sqrt{a} - 5 \times 2 \times a\sqrt{b}$

$= \frac{1}{2} \times 10 \times a \times b\sqrt{a} + ab \times 6\sqrt{a} - 5 \times 2 \times a\sqrt{b}$

$= 5ab\sqrt{a} + 6ab\sqrt{a} - 10a\sqrt{b}$

$= 11ab\sqrt{a} - 10a\sqrt{b}$

2 Add like terms to obtain the simplified answer.

16.4.3 Multiplying surds

- To multiply surds, multiply together the expressions under the radical signs. For example, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$, where a and b are positive real numbers.
- When multiplying surds it is best to first simplify them (if possible). Once this has been done and a mixed surd has been obtained, the coefficients are multiplied with each other and then the surds are multiplied together. For example,

$$m\sqrt{a} \times n\sqrt{b} = mn\sqrt{ab}.$$

WORKED EXAMPLE 6

Multiply the following surds, expressing answers in the simplest form. Assume that x and y are positive real numbers.

a $\sqrt{11} \times \sqrt{7}$ **b** $5\sqrt{3} \times 8\sqrt{5}$

c $6\sqrt{12} \times 2\sqrt{6}$ **d** $\sqrt{15x^5y^2} \times \sqrt{12x^2y}$

THINK	WRITE
a Multiply the surds together, using $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ (that is, multiply expressions under the square root sign). *Note:* This expression cannot be simplified any further.	**a** $\sqrt{11} \times \sqrt{7} = \sqrt{11 \times 7}$ $= \sqrt{77}$
b Multiply the coefficients together and then multiply the surds together.	**b** $5\sqrt{3} \times 8\sqrt{5} = 5 \times 8 \times \sqrt{3} \times \sqrt{5}$ $= 40 \times \sqrt{3 \times 5}$ $= 40\sqrt{15}$
c 1 Simplify $\sqrt{12}$. **2** Multiply the coefficients together and multiply the surds together. **3** Simplify the surd.	**c** $6\sqrt{12} \times 2\sqrt{6} = 6\sqrt{4 \times 3} \times 2\sqrt{6}$ $= 6 \times 2\sqrt{3} \times 2\sqrt{6}$ $= 12\sqrt{3} \times 2\sqrt{6}$ $= 24\sqrt{18}$ $= 24\sqrt{9 \times 2}$ $= 24 \times 3\sqrt{2}$ $= 72\sqrt{2}$

d 1 Simplify each of the surds.

2 Multiply the coefficients together and the surds together.

3 Simplify the surd.

d $\sqrt{15x^5y^2} \times \sqrt{12x^2y}$

$= \sqrt{15 \times x^4 \times x \times y^2} \times \sqrt{4 \times 3 \times x^2 \times y}$

$= x^2 \times y \times \sqrt{15 \times x} \times 2 \times x \times \sqrt{3 \times y}$

$= x^2y\sqrt{15x} \times 2x\sqrt{3y}$

$= x^2y \times 2x\sqrt{15x \times 3y}$

$= 2x^3y\sqrt{45xy}$

$= 2x^3y\sqrt{9 \times 5xy}$

$= 2x^3y \times 3\sqrt{5xy}$

$= 6x^3y\sqrt{5xy}$

- When working with surds, it is sometimes necessary to multiply surds by themselves; that is, square them. Consider the following examples:

$$(\sqrt{2})^2 = \sqrt{2} \times \sqrt{2} = \sqrt{4} = 2$$
$$(\sqrt{5})^2 = \sqrt{5} \times \sqrt{5} = \sqrt{25} = 5$$

- Observe that squaring a surd produces the number under the radical sign. This is not surprising, because squaring and taking the square root are *inverse operations* and, when applied together, leave the original unchanged.
- When a surd is squared, the result is the number (or expression) under the radical sign; that is, $(\sqrt{a})^2 = a$, where a is a positive real number.

WORKED EXAMPLE 7

Simplify each of the following.

a $(\sqrt{6})^2$ **b** $(3\sqrt{5})^2$

THINK

a Use $(\sqrt{a})^2 = a$, where $a = 6$.

b 1 Square 3 and apply $(\sqrt{a})^2 = a$ to square $\sqrt{5}$.

2 Simplify.

WRITE

a $(\sqrt{6})^2 = 6$

b $3(\sqrt{5})^2 = 3^2 \times (\sqrt{5})^2$

$= 9 \times 5$

$= 45$

16.4.4 Dividing surds

- To divide surds, divide the expressions under the radical signs; that is, $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$, where a and b are whole numbers.
- When dividing surds it is best to simplify them (if possible) first. Once this has been done, the coefficients are divided next and then the surds are divided.

WORKED EXAMPLE 8

Divide the following surds, expressing answers in the simplest form. Assume that x and y are positive real numbers.

a $\frac{\sqrt{55}}{\sqrt{5}}$ **b** $\frac{\sqrt{48}}{\sqrt{3}}$ **c** $\frac{9\sqrt{88}}{6\sqrt{99}}$ **d** $\frac{\sqrt{36xy}}{\sqrt{25x^9y^{11}}}$

THINK	WRITE
a 1 Rewrite the fraction, using $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$.	**a** $\frac{\sqrt{55}}{\sqrt{5}} = \sqrt{\frac{55}{5}}$
2 Divide the numerator by the denominator (that is, 55 by 5).	$= \sqrt{11}$
3 Check if the surd can be simplified any further.	
b 1 Rewrite the fraction, using $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$.	**b** $\frac{\sqrt{48}}{\sqrt{3}} = \sqrt{\frac{48}{3}}$
2 Divide 48 by 3.	$= \sqrt{16}$
3 Evaluate $=\sqrt{16}$.	$= 4$
c 1 Rewrite the fraction, using $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$.	**c** $\frac{9\sqrt{88}}{6\sqrt{99}} = \frac{9}{6}\sqrt{\frac{88}{99}}$
2 Simplify the fraction under the radical by dividing both numerator and denominator by 11.	$= \frac{9}{6}\sqrt{\frac{8}{9}}$
3 Simplify the surds.	$= \frac{9 \times 2\sqrt{2}}{6 \times 3}$
4 Multiply the whole numbers in the numerator together and those in the denominator together.	$= \frac{18\sqrt{2}}{18}$
5 Cancel the common factor of 18.	$= \sqrt{2}$
d 1 Simplify each surd.	**d** $\frac{\sqrt{36xy}}{\sqrt{25x^9y^{11}}} = \frac{6\sqrt{xy}}{5\sqrt{x^8 \times x \times y^{10} \times y}}$
	$= \frac{6\sqrt{xy}}{5x^4y^5\sqrt{xy}}$
2 Cancel any common factors — in this case, $\sqrt{xy}$.	$= \frac{6}{5x^4y^5}$

16.4.5 Rationalising denominators

- If the **denominator** of a fraction is a surd, it can be changed into a rational number through multiplication. In other words, it can be rationalised.
- As discussed earlier in this chapter, squaring a simple surd (that is, multiplying it by itself) results in a rational number. This fact can be used to rationalise denominators as follows.

$$\frac{\sqrt{a}}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{\sqrt{ab}}{b}, \text{ where } \frac{\sqrt{b}}{\sqrt{b}} = 1$$

- If both the numerator and the denominator of a fraction are multiplied by the surd contained in the denominator, the denominator becomes a rational number. The fraction takes on a different appearance, but its numerical value is unchanged, because multiplying the numerator and denominator by the same number is equivalent to multiplying by 1.

WORKED EXAMPLE 9

Express the following in their simplest form with a rational denominator.

a $\dfrac{\sqrt{6}}{\sqrt{13}}$ **b** $\dfrac{2\sqrt{12}}{3\sqrt{54}}$ **c** $\dfrac{\sqrt{17}-3\sqrt{14}}{\sqrt{7}}$

THINK

a 1 Write the fraction.

2 Multiply both the numerator and denominator by the surd contained in the denominator (in this case $\sqrt{13}$). This has the same effect as multiplying the fraction by 1, because $\dfrac{\sqrt{13}}{\sqrt{13}} = 1$.

b 1 Write the fraction.

2 Simplify the surds. (This avoids dealing with large numbers.)

3 Multiply both the numerator and denominator by $\sqrt{6}$. (This has the same effect as multiplying the fraction by 1, because $\dfrac{\sqrt{6}}{\sqrt{6}} = 1$.)

Note: We need to multiply only by the surd part of the denominator (that is, by $\sqrt{6}$ rather than by $9\sqrt{6}$).

4 Simplify $\sqrt{18}$.

5 Divide both the numerator and denominator by 6 (cancel down).

c 1 Write the fraction.

2 Multiply both the numerator and denominator by $\sqrt{7}$. Use grouping symbols (brackets) to make it clear that the whole numerator must be multiplied by $\sqrt{7}$.

WRITE

a $\dfrac{\sqrt{6}}{\sqrt{13}}$

$= \dfrac{\sqrt{6}}{\sqrt{13}} \times \dfrac{\sqrt{13}}{\sqrt{13}}$

$= \dfrac{\sqrt{78}}{13}$

b $\dfrac{2\sqrt{12}}{3\sqrt{54}}$

$\dfrac{2\sqrt{12}}{3\sqrt{54}} = \dfrac{2\sqrt{4 \times 3}}{3\sqrt{9 \times 6}}$

$= \dfrac{2 \times 2\sqrt{3}}{3 \times 3\sqrt{6}}$

$= \dfrac{4\sqrt{3}}{9\sqrt{6}}$

$= \dfrac{4\sqrt{3}}{9\sqrt{6}} \times \dfrac{\sqrt{6}}{\sqrt{6}}$

$= \dfrac{4\sqrt{18}}{9 \times 6}$

$= \dfrac{4\sqrt{9 \times 2}}{9 \times 6}$

$= \dfrac{4 \times 3\sqrt{2}}{54}$

$= \dfrac{12\sqrt{2}}{54}$

$= \dfrac{2\sqrt{2}}{9}$

c $\dfrac{\sqrt{17}-3\sqrt{14}}{\sqrt{7}}$

$= \dfrac{(\sqrt{17}-3\sqrt{14})}{\sqrt{7}} \times \dfrac{\sqrt{7}}{\sqrt{7}}$

3 Apply the Distributive Law in the numerator: $a(b + c) = ab + ac$.

4 Simplify $\sqrt{98}$.

$$= \frac{\sqrt{17} \times \sqrt{7} - 3\sqrt{14} \times \sqrt{7}}{\sqrt{7} \times \sqrt{7}}$$
$$= \frac{\sqrt{119} - 3\sqrt{98}}{7}$$
$$= \frac{\sqrt{119} - 3\sqrt{49 \times 2}}{7}$$
$$= \frac{\sqrt{119} - 3 \times 7\sqrt{2}}{7}$$
$$= \frac{\sqrt{119} - 21\sqrt{2}}{7}$$

16.4.6 Rationalising denominators using conjugate surds

- The product of pairs of **conjugate surds** results in a rational number. Examples of pairs of conjugate surds include $\sqrt{6} + 11$ and $\sqrt{6} - 11$; $\sqrt{a} + b$ and $\sqrt{a} - b$; and $2\sqrt{5} - \sqrt{7}$ and $2\sqrt{5} + \sqrt{7}$.
  This fact is used to rationalise denominators containing a sum or a difference of surds.
  To rationalise a denominator that contains a sum or a difference of surds, multiply both the numerator and the denominator by the conjugate of the denominator. Two examples are given below.
  1. To rationalise the denominator of the fraction $\frac{1}{\sqrt{a} + \sqrt{b}}$, multiply it by $\frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} - \sqrt{b}}$.
  2. To rationalise the denominator of the fraction $\frac{1}{\sqrt{a} - \sqrt{b}}$, multiply it by $\frac{\sqrt{a} + \sqrt{b}}{\sqrt{a} + \sqrt{b}}$.
- A quick way to simplify the denominator is to use the difference of two squares identity:
$$(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2$$
$$= a - b$$

WORKED EXAMPLE 10

Rationalise the denominator and simplify the following.

a $\frac{1}{4 - \sqrt{3}}$ **b** $\frac{\sqrt{6} + 3\sqrt{2}}{3 + \sqrt{3}}$

THINK

a 1 Write the fraction.

2 Multiply the numerator and denominator by the conjugate of the denominator.
(Note that $\frac{(4 + \sqrt{3})}{(4 + \sqrt{3})} = 1$.)

3 Apply the Distributive Law in the numerator and the difference of two squares identity in the denominator.

WRITE

a $\frac{1}{4 - \sqrt{3}}$
$= \frac{1}{(4 - \sqrt{3})} \times \frac{(4 + \sqrt{3})}{(4 + \sqrt{3})}$
$= \frac{4 + \sqrt{3}}{(4)^2 - (\sqrt{3})^2}$

4 Simplify.

$$= \frac{4 + \sqrt{3}}{16 - 3}$$
$$= \frac{4 + \sqrt{3}}{13}$$

b 1 Write the fraction.

b $\dfrac{\sqrt{6} + 3\sqrt{2}}{3 + \sqrt{3}}$

2 Multiply the numerator and denominator by the conjugate of the denominator.
(Note that $\dfrac{(3 - \sqrt{3})}{(3 - \sqrt{3})} = 1$.)

$$= \frac{(\sqrt{6} + 3\sqrt{2})}{(3 + \sqrt{3})} \times \frac{(3 - \sqrt{3})}{(3 - \sqrt{3})}$$

3 Multiply the expressions in grouping symbols in the numerator, and apply the difference of two squares identity in the denominator.

$$= \frac{\sqrt{6} \times 3 + \sqrt{6} \times -\sqrt{3} + 3\sqrt{2} \times 3 + 3\sqrt{2} \times -\sqrt{3}}{(3)^2 - (\sqrt{3})^2}$$

4 Simplify.

$$= \frac{3\sqrt{6} - \sqrt{18} + 9\sqrt{2} - 3\sqrt{6}}{9 - 3}$$
$$= \frac{-\sqrt{18} + 9\sqrt{2}}{6}$$
$$= \frac{-\sqrt{9 \times 2} + 9\sqrt{2}}{6}$$
$$= \frac{-3\sqrt{2} + 9\sqrt{2}}{6}$$
$$= \frac{6\sqrt{2}}{6}$$
$$= \sqrt{2}$$

learnON RESOURCES — ONLINE ONLY

Interactivity: Simplifying surds (int-6028)
Interactivity: Addition and subtraction of surds (int-6190)
Interactivity: Multiplying surds (int-6191)
Interactivity: Dividing surds (int-6192)
Interactivity: Conjugate surds (int-6193)
eLesson: Surds (eles-1906)
eLesson: Rationalisation of surds (eles-1948)
Digital doc: SkillSHEET Simplifying surds (doc-5355)
Digital doc: SkillSHEET Adding and subtracting surds (doc-5356)
Digital doc: SkillSHEET Multiplying surds (doc-5357)
Digital doc: SkillSHEET Rationalising denominators (doc-5360)
Digital doc: SkillSHEET Conjugate pairs (doc-5361)
Digital doc: SkillSHEET Applying the difference of two squares rule to surds (doc-5362)
Digital doc: WorkSHEET Real numbers I (doc-14612)

Exercise 16.4 Operations with surds

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–h, 2a–h, 3a–h, 4a–d, 5a–h, 6a–h, 7a–h, 8a–d, 9a–d, 10a–h, 11a–f, 12a–c, 13, 15	Questions: 1e–j, 2e–j, 3e–k, 4c–f, 5c–i, 6e–j, 7g–l, 8d–f, 9g–k, 10f–j, 11e–h, 12d–f, 13–15	Questions: 1g–l, 2g–l, 3g–l, 4e–h, 5g–l, 6g–l, 7j–r, 8e–h, 9i–n, 10k–o, 11i–l, 12g–i, 13–18

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

Where appropriate, digital technology should be used to answer the following questions.

1. WE4a Simplify the following surds.

a. $\sqrt{12}$ **b.** $\sqrt{24}$ **c.** $\sqrt{27}$ **d.** $\sqrt{125}$
e. $\sqrt{54}$ **f.** $\sqrt{112}$ **g.** $\sqrt{68}$ **h.** $\sqrt{180}$
i. $\sqrt{88}$ **j.** $\sqrt{162}$ **k.** $\sqrt{245}$ **l.** $\sqrt{448}$

2. WE4b, c Simplify the following surds.

a. $2\sqrt{8}$ **b.** $8\sqrt{90}$ **c.** $9\sqrt{80}$ **d.** $7\sqrt{54}$
e. $-6\sqrt{75}$ **f.** $-7\sqrt{80}$ **g.** $16\sqrt{48}$ **h.** $\frac{1}{7}\sqrt{392}$
i. $\frac{1}{9}\sqrt{162}$ **j.** $\frac{1}{4}\sqrt{192}$ **k.** $\frac{1}{9}\sqrt{135}$ **l.** $\frac{3}{10}\sqrt{175}$

3. WE4d Simplify the following surds. Assume that a, b, c, d, e, f, x and y are positive real numbers.

a. $\sqrt{16a^2}$ **b.** $\sqrt{72a^2}$ **c.** $\sqrt{90a^2b}$ **d.** $\sqrt{338a^4}$
e. $\sqrt{338a^3b^3}$ **f.** $\sqrt{68a^3b^5}$ **g.** $\sqrt{125x^6y^4}$ **h.** $5\sqrt{80x^3y^2}$
i. $6\sqrt{162c^7d^5}$ **j.** $2\sqrt{405c^7d^9}$ **k.** $\frac{1}{2}\sqrt{88ef}$ **l.** $\frac{1}{2}\sqrt{392e^{11}f^{11}}$

4. WE5a Simplify the following expressions containing surds. Assume that x and y are positive real numbers.

a. $3\sqrt{5} + 4\sqrt{5}$ **b.** $2\sqrt{3} + 5\sqrt{3} + \sqrt{3}$
c. $8\sqrt{5} + 3\sqrt{3} + 7\sqrt{5} + 2\sqrt{3}$ **d.** $6\sqrt{11} - 2\sqrt{11}$
e. $7\sqrt{2} + 9\sqrt{2} - 3\sqrt{2}$ **f.** $9\sqrt{6} + 12\sqrt{6} - 17\sqrt{6} - 7\sqrt{6}$
g. $12\sqrt{3} - 8\sqrt{7} + 5\sqrt{3} - 10\sqrt{7}$ **h.** $2\sqrt{x} + 5\sqrt{y} + 6\sqrt{x} - 2\sqrt{y}$

5. WE5b Simplify the following expressions containing surds. Assume that a and b are positive real numbers.

a. $\sqrt{200} - \sqrt{300}$ **b.** $\sqrt{125} - \sqrt{150} + \sqrt{600}$
c. $\sqrt{27} - \sqrt{3} + \sqrt{75}$ **d.** $2\sqrt{20} - 3\sqrt{5} + \sqrt{45}$
e. $6\sqrt{12} + 3\sqrt{27} - 7\sqrt{3} + \sqrt{18}$ **f.** $\sqrt{150} + \sqrt{24} - \sqrt{96} + \sqrt{108}$
g. $3\sqrt{90} - 5\sqrt{60} + 3\sqrt{40} + \sqrt{100}$ **h.** $5\sqrt{11} + 7\sqrt{44} - 9\sqrt{99} + 2\sqrt{121}$
i. $2\sqrt{30} + 5\sqrt{120} + \sqrt{60} - 6\sqrt{135}$ **j.** $6\sqrt{ab} - \sqrt{12ab} + 2\sqrt{9ab} + 3\sqrt{27ab}$
k. $\frac{1}{2}\sqrt{98} + \frac{1}{3}\sqrt{48} + \frac{1}{3}\sqrt{12}$ **l.** $\frac{1}{8}\sqrt{32} - \frac{7}{6}\sqrt{18} + 3\sqrt{72}$

6. WE5c Simplify the following expressions containing surds. Assume that a and b are positive real numbers.

a. $7\sqrt{a} - \sqrt{8a} + 8\sqrt{9a} - \sqrt{32a}$ **b.** $10\sqrt{a} - 15\sqrt{27a} + 8\sqrt{12a} + 14\sqrt{9a}$
c. $\sqrt{150ab} + \sqrt{96ab} - \sqrt{54ab}$ **d.** $16\sqrt{4a^2} - \sqrt{24a} + 4\sqrt{8a^2} + \sqrt{96a}$

e. $\sqrt{8a^3} + \sqrt{72a^3} - \sqrt{98a^3}$ **f.** $\frac{1}{2}\sqrt{36a} + \frac{1}{4}\sqrt{128a} - \frac{1}{6}\sqrt{144a}$

g. $\sqrt{9a^3} + \sqrt{3a^5}$ **h.** $6\sqrt{a^5b} + \sqrt{a^3b} - 5\sqrt{a^5b}$

i. $ab\sqrt{ab} + 3ab\sqrt{a^2b} + \sqrt{9a^3b^3}$ **j.** $\sqrt{a^3b} + 5\sqrt{ab} - 2\sqrt{ab} + 5\sqrt{a^3b}$

k. $\sqrt{32a^3b^2} - 5ab\sqrt{8a} + \sqrt{48a^5b^6}$ **l.** $\sqrt{4a^2b} + 5\sqrt{a^2b} - 3\sqrt{9a^2b}$

7. **WE6** Multiply the following surds, expressing answers in the simplest form. Assume that a, b, x and y are positive real numbers.

a. $\sqrt{2} \times \sqrt{7}$ **b.** $\sqrt{6} \times \sqrt{7}$ **c.** $\sqrt{8} \times \sqrt{6}$

d. $\sqrt{10} \times \sqrt{10}$ **e.** $\sqrt{21} \times \sqrt{3}$ **f.** $\sqrt{27} \times 3\sqrt{3}$

g. $5\sqrt{3} \times 2\sqrt{11}$ **h.** $10\sqrt{15} \times 6\sqrt{3}$ **i.** $4\sqrt{20} \times 3\sqrt{5}$

j. $10\sqrt{6} \times 3\sqrt{8}$ **k.** $\frac{1}{4}\sqrt{48} \times 2\sqrt{2}$ **l.** $\frac{1}{9}\sqrt{48} \times 2\sqrt{3}$

m. $\frac{1}{10}\sqrt{60} \times \frac{1}{5}\sqrt{40}$ **n.** $\sqrt{xy} \times \sqrt{x^3y^2}$ **o.** $\sqrt{3a^4b^2} \times \sqrt{6a^5b^3}$

p. $\sqrt{12a^7b} \times \sqrt{6a^3b^4}$ **q.** $\sqrt{15x^3y^2} \times \sqrt{6x^2y^3}$ **r.** $\frac{1}{2}\sqrt{15a^3b^3} \times 3\sqrt{3a^2b^6}$

8. **WE7** Simplify each of the following.

a. $(\sqrt{2})^2$ **b.** $(\sqrt{5})^2$ **c.** $(\sqrt{12})^2$ **d.** $(\sqrt{15})^2$

e. $(3\sqrt{2})^2$ **f.** $(4\sqrt{5})^2$ **g.** $(2\sqrt{7})^2$ **h.** $(5\sqrt{8})^2$

9. **WE8** Simplify the following surds, expressing answers in the simplest form. Assume that a, b, x and y are positive real numbers.

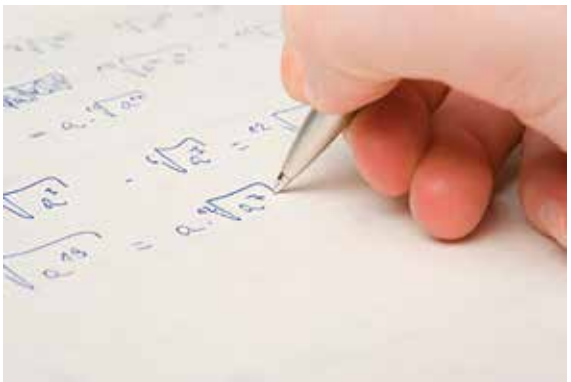
a. $\frac{\sqrt{15}}{\sqrt{3}}$ **b.** $\frac{\sqrt{8}}{\sqrt{2}}$ **c.** $\frac{\sqrt{60}}{\sqrt{10}}$

d. $\frac{\sqrt{128}}{\sqrt{8}}$ **e.** $\frac{\sqrt{18}}{4\sqrt{6}}$ **f.** $\frac{\sqrt{65}}{2\sqrt{13}}$

g. $\frac{\sqrt{96}}{\sqrt{8}}$ **h.** $\frac{7\sqrt{44}}{14\sqrt{11}}$ **i.** $\frac{9\sqrt{63}}{15\sqrt{7}}$

j. $\frac{\sqrt{2040}}{\sqrt{30}}$ **k.** $\frac{\sqrt{x^4y^3}}{\sqrt{x^2y^5}}$ **l.** $\frac{\sqrt{16xy}}{\sqrt{8x^7y^9}}$

m. $\frac{\sqrt{xy}}{\sqrt{x^5y^7}} \times \frac{\sqrt{12x^8y^{12}}}{\sqrt{x^2y^3}}$ **n.** $\frac{2\sqrt{2a^2b^4}}{\sqrt{5a^3b^6}} \times \frac{\sqrt{10a^9b^3}}{3\sqrt{a^7b}}$



10. **WE9a, b** Express the following in their simplest form with a rational denominator.

a. $\frac{5}{\sqrt{2}}$ **b.** $\frac{7}{\sqrt{3}}$ **c.** $\frac{4}{\sqrt{11}}$ **d.** $\frac{8}{\sqrt{6}}$ **e.** $\frac{\sqrt{12}}{\sqrt{7}}$

f. $\frac{\sqrt{15}}{\sqrt{6}}$ **g.** $\frac{2\sqrt{3}}{\sqrt{5}}$ **h.** $\frac{3\sqrt{7}}{\sqrt{5}}$ **i.** $\frac{5\sqrt{2}}{2\sqrt{3}}$ **j.** $\frac{4\sqrt{3}}{3\sqrt{5}}$

k. $\frac{5\sqrt{14}}{7\sqrt{8}}$ **l.** $\frac{16\sqrt{3}}{6\sqrt{5}}$ **m.** $\frac{8\sqrt{3}}{7\sqrt{7}}$ **n.** $\frac{8\sqrt{60}}{\sqrt{28}}$ **o.** $\frac{2\sqrt{35}}{3\sqrt{14}}$

11. **WE9c** Express the following in their simplest form with a rational denominator.

a. $\frac{\sqrt{6} + \sqrt{12}}{\sqrt{3}}$ **b.** $\frac{\sqrt{15} - \sqrt{22}}{\sqrt{6}}$ **c.** $\frac{6\sqrt{2} - \sqrt{15}}{\sqrt{10}}$ **d.** $\frac{2\sqrt{18} + 3\sqrt{2}}{\sqrt{5}}$

e. $\frac{3\sqrt{5} + 6\sqrt{7}}{\sqrt{8}}$ **f.** $\frac{4\sqrt{2} + 3\sqrt{8}}{2\sqrt{3}}$ **g.** $\frac{3\sqrt{11} - 4\sqrt{5}}{\sqrt{18}}$ **h.** $\frac{2\sqrt{7} - 2\sqrt{5}}{\sqrt{12}}$

i. $\frac{7\sqrt{12} - 5\sqrt{6}}{6\sqrt{3}}$ **j.** $\frac{6\sqrt{2} - \sqrt{5}}{4\sqrt{8}}$ **k.** $\frac{6\sqrt{3} - 5\sqrt{5}}{7\sqrt{20}}$ **l.** $\frac{3\sqrt{5} + 7\sqrt{3}}{5\sqrt{24}}$

12. WE10 Rationalise the denominator and simplify.

a. $\frac{1}{\sqrt{5}+2}$ **b.** $\frac{1}{\sqrt{8}-\sqrt{5}}$ **c.** $\frac{4}{2\sqrt{11}-\sqrt{13}}$

d. $\frac{5\sqrt{3}}{3\sqrt{5}+4\sqrt{2}}$ **e.** $\frac{\sqrt{8}-3}{\sqrt{8}+3}$ **f.** $\frac{\sqrt{12}-\sqrt{7}}{\sqrt{12}+\sqrt{7}}$

g. $\frac{\sqrt{3}-1}{\sqrt{5}+1}$ **h.** $\frac{3\sqrt{6}-\sqrt{15}}{\sqrt{6}+2\sqrt{3}}$ **i.** $\frac{\sqrt{5}-\sqrt{3}}{4\sqrt{2}-\sqrt{3}}$

Communicating, reasoning and problem solving

13. Express the average of $\frac{1}{2\sqrt{x}}$ and $\frac{1}{3-2\sqrt{x}}$, writing your answer with a rational denominator.

14. a. Show that $(\sqrt{a}+\sqrt{b})^2 = a+b+2\sqrt{ab}$.

b. Use this result to find:

i. $\sqrt{8+2\sqrt{15}}$ **ii.** $\sqrt{8-2\sqrt{15}}$ **iii.** $\sqrt{7+4\sqrt{3}}$.

15. Simplify $\frac{\sqrt{5}+\sqrt{3}}{\sqrt{3}+\sqrt{3}+\sqrt{5}} - \frac{\sqrt{5}-\sqrt{3}}{\sqrt{3}+\sqrt{3}-\sqrt{5}}$.

16. Solve the following for x.

a. $\sqrt{9+x}-\sqrt{x} = \frac{5}{\sqrt{9+x}}$ **b.** $\frac{9\sqrt{x}-7}{3\sqrt{x}} = \frac{3\sqrt{x}+1}{\sqrt{x}+5}$

17. Explain why $\sqrt{2}+\sqrt{3} \neq \sqrt{5}$.

18. Under what circumstance might you need to rationalise the denominator of a fraction?

16.5 Fractional indices [Stage 5.3]

16.5.1 Fractional indices and roots

- Consider the expression $a^{\frac{1}{2}}$. Now consider what happens if we square that expression.
$$\left(a^{\frac{1}{2}}\right)^2 = a \text{ (using the Fourth Index Law, } (a^m)^n = a^{m\times n})$$
- Now, from our work on surds we know that $\left(\sqrt{a}\right)^2 = a$.
- From this we can conclude that $\left(a^{\frac{1}{2}}\right)^2 = \left(\sqrt{a}\right)^2$ and further conclude that $a^{\frac{1}{2}} = \sqrt{a}$.
- We can similarly show that $a^{\frac{1}{3}} = \sqrt[3]{a}$.
- This pattern can be continued and generalised to produce $a^{\frac{1}{n}} = \sqrt[n]{a}$.

WORKED EXAMPLE 11

Evaluate each of the following without using a calculator.

a $9^{\frac{1}{2}}$ **b** $64^{\frac{1}{3}}$

THINK	WRITE
a 1 Write $9^{\frac{1}{2}}$ as $\sqrt{9}$.	**a** $9^{\frac{1}{2}} = \sqrt{9}$
2 Evaluate.	$= 3$
b 1 Write $64^{\frac{1}{3}}$ as $\sqrt[3]{64}$.	**b** $64^{\frac{1}{3}} = \sqrt[3]{64}$
2 Evaluate.	$= 4$

WORKED EXAMPLE 12

Use a calculator to find the value of the following, correct to 1 decimal place.

a $10^{\frac{1}{4}}$ **b** $200^{\frac{1}{5}}$

THINK	WRITE
a Use a calculator to produce the answer.	**a** $10^{\frac{1}{4}} = 1.77827941$ ≈ 1.8
b Use a calculator to produce the answer.	**b** $200^{\frac{1}{5}} = 2.885399812$ ≈ 2.9

- Consider the expression $\left(a^m\right)^{\frac{1}{n}}$. From earlier, we know that $\left(a^m\right)^{\frac{1}{n}} = \sqrt[n]{a^m}$.

  We also know $\left(a^m\right)^{\frac{1}{n}} = a^{\frac{m}{n}}$ using the index laws.

  We can therefore conclude that $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.
- Such expressions can be evaluated on a calculator either by using the index function (which is usually either ^ or x^y) and entering the fractional index, or by separating the two functions for power and root.

WORKED EXAMPLE 13

Evaluate $3^{\frac{2}{7}}$, correct to 1 decimal place.

THINK	WRITE
Use a calculator to evaluate $3^{\frac{2}{7}}$.	$3^{\frac{2}{7}} \approx 1.4$

- The index law $a^{\frac{1}{2}} = \sqrt{a}$ can be applied to convert between expressions that involve fractional indices and surds.

WORKED EXAMPLE 14

Write each of the following expressions in simplest surd form.

a $10^{\frac{1}{2}}$ **b** $5^{\frac{3}{2}}$

THINK	WRITE
a Since an index of $\frac{1}{2}$ is equivalent to taking the square root, this term can be written as the square root of 10.	**a** $10^{\frac{1}{2}} = \sqrt{10}$
b 1 A power of $\frac{3}{2}$ means the square root of the number cubed.	**b** $5^{\frac{3}{2}} = \sqrt{5^3}$
2 Evaluate 5^3.	$= \sqrt{125}$
3 Simplify $\sqrt{125}$.	$= 5\sqrt{5}$

WORKED EXAMPLE 15

Simplify each of the following.

a $m^{\frac{1}{5}} \times m^{\frac{2}{5}}$ **b** $\left(a^2b^3\right)^{\frac{1}{6}}$ **c** $\left(\dfrac{x^{\frac{2}{3}}}{y^{\frac{3}{4}}}\right)^{\frac{1}{2}}$

THINK	WRITE
a 1 Write the expression.	**a** $m^{\frac{1}{5}} \times m^{\frac{2}{5}}$
2 Multiply numbers with the same base by adding the indices.	$= m^{\frac{3}{5}}$
b 1 Write the expression.	**b** $\left(a^2b^3\right)^{\frac{1}{6}}$
2 Multiply each index inside the grouping symbols (brackets) by the index on the outside.	$= a^{\frac{2}{6}}b^{\frac{3}{6}}$
3 Simplify the fractions.	$= a^{\frac{1}{3}}b^{\frac{1}{2}}$
c 1 Write the expression.	**c** $\left(\dfrac{x^{\frac{2}{3}}}{y^{\frac{3}{4}}}\right)^{\frac{1}{2}}$
2 Multiply the index in both the numerator and denominator by the index outside the grouping symbols.	$= \dfrac{x^{\frac{1}{3}}}{y^{\frac{3}{8}}}$

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Interactivity: Fractional indices (int-6107)

eLesson: Fractional indices (eles-1950)

Digital doc: WorkSHEET Real numbers II (doc-14613)

Exercise 16.5 Fractional indices

Individual pathways

PRACTISE
Questions:
1–5, 6a–f, 7a–c, 8a–f, 9a–d, 10a–d, 11a–d, 12–14, 16

CONSOLIDATE
Questions:
1–5, 6d–g, 7b–d, 8d–f, 9b–d, 10c–f, 11c–f, 12–16

MASTER
Questions:
1–5, 6g–i, 7d–f, 8f–i, 9c–f, 10e–i, 11e–i, 12–18

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE11 Evaluate each of the following without using a calculator.

a. $16^{\frac{1}{2}}$ **b.** $25^{\frac{1}{2}}$ **c.** $81^{\frac{1}{2}}$

d. $8^{\frac{1}{3}}$ **e.** $27^{\frac{1}{3}}$ **f.** $125^{\frac{1}{3}}$

2. WE12 Use a calculator to evaluate each of the following, correct to 1 decimal place.

a. $81^{\frac{1}{4}}$ **b.** $16^{\frac{1}{4}}$ **c.** $3^{\frac{1}{3}}$

d. $5^{\frac{1}{2}}$ **e.** $7^{\frac{1}{5}}$ **f.** $8^{\frac{1}{9}}$

3. WE13 Use a calculator to find the value of each of the following, correct to 1 decimal place.

a. $12^{\frac{3}{8}}$ **b.** $100^{\frac{5}{9}}$ **c.** $50^{\frac{2}{3}}$

d. $(0.6)^{\frac{4}{5}}$ **e.** $\left(\frac{3}{4}\right)^{\frac{3}{4}}$ **f.** $\left(\frac{4}{5}\right)^{\frac{2}{3}}$

4. WE14 Write each of the following expressions in simplest surd form.

a. $7^{\frac{1}{2}}$ **b.** $12^{\frac{1}{2}}$ **c.** $72^{\frac{1}{2}}$

d. $2^{\frac{5}{2}}$ **e.** $3^{\frac{3}{2}}$ **f.** $10^{\frac{5}{2}}$

5. Write each of the following expressions with a fractional index.

a. $\sqrt{5}$ **b.** $\sqrt{10}$ **c.** $\sqrt{x}$

d. $\sqrt{m^3}$ **e.** $2\sqrt{t}$ **f.** $\sqrt[3]{6}$

6. WE15a Simplify each of the following. Leave your answer in index form.

a. $4^{\frac{3}{5}} \times 4^{\frac{1}{5}}$ **b.** $2^{\frac{1}{8}} \times 2^{\frac{3}{8}}$ **c.** $a^{\frac{1}{2}} \times a^{\frac{1}{3}}$

d. $x^{\frac{3}{4}} \times x^{\frac{2}{5}}$ **e.** $5m^{\frac{1}{3}} \times 2m^{\frac{1}{5}}$ **f.** $\frac{1}{2}b^{\frac{3}{7}} \times 4b^{\frac{2}{7}}$

g. $-4y^2 \times y^{\frac{2}{9}}$ **h.** $\frac{2}{5}a^{\frac{3}{8}} \times 0.05a^{\frac{3}{4}}$ **i.** $5x^3 \times x^{\frac{1}{2}}$

7. Simplify each of the following.

a. $a^{\frac{2}{3}}b^{\frac{3}{4}} \times a^{\frac{1}{3}}b^{\frac{3}{4}}$ **b.** $x^{\frac{3}{5}}y^{\frac{2}{9}} \times x^{\frac{1}{5}}y^{\frac{1}{3}}$ **c.** $2ab^{\frac{1}{3}} \times 3a^{\frac{3}{5}}b^{\frac{4}{5}}$

d. $6m^{\frac{3}{7}} \times \frac{1}{3}m^{\frac{1}{4}}n^{\frac{2}{5}}$ **e.** $x^3y^{\frac{1}{2}}z^{\frac{1}{3}} \times x^{\frac{1}{6}}y^{\frac{1}{3}}z^{\frac{1}{2}}$ **f.** $2a^{\frac{2}{5}}b^{\frac{3}{8}}c^{\frac{1}{4}} \times 4b^{\frac{3}{4}}c^{\frac{3}{4}}$

8. Simplify each of the following.

a. $3^{\frac{1}{2}} \div 3^{\frac{1}{3}}$ **b.** $5^{\frac{2}{3}} \div 5^{\frac{1}{4}}$ **c.** $12^2 \div 12^{\frac{3}{2}}$

d. $a^{\frac{6}{7}} \div a^{\frac{3}{7}}$ **e.** $x^{\frac{3}{2}} \div x^{\frac{1}{4}}$ **f.** $\dfrac{m^{\frac{4}{5}}}{m^{\frac{5}{9}}}$

g. $\dfrac{2x^{\frac{3}{4}}}{4x^{\frac{3}{5}}}$ **h.** $\dfrac{7n^2}{21n^{\frac{4}{3}}}$ **i.** $\dfrac{25b^{\frac{3}{5}}}{20b^{\frac{1}{4}}}$

9. Simplify each of the following.

a. $x^3y^2 \div x^{\frac{4}{3}}y^{\frac{3}{5}}$ **b.** $a^{\frac{5}{9}}b^{\frac{2}{3}} \div a^{\frac{2}{5}}b^{\frac{2}{5}}$ **c.** $m^{\frac{3}{8}}n^{\frac{4}{7}} \div 3n^{\frac{3}{8}}$

d. $10x^{\frac{4}{5}}y \div 5x^{\frac{2}{3}}y^{\frac{1}{4}}$ **e.** $\dfrac{5a^{\frac{3}{4}}b^{\frac{3}{5}}}{20a^{\frac{1}{5}}b^{\frac{1}{4}}}$ **f.** $\dfrac{p^{\frac{7}{8}}q^{\frac{1}{4}}}{7p^{\frac{2}{3}}q^{\frac{1}{6}}}$

10. Simplify each of the following.

a. $\left(2^{\frac{3}{4}}\right)^{\frac{3}{5}}$ **b.** $\left(5^{\frac{2}{3}}\right)^{\frac{1}{4}}$ **c.** $\left(7^{\frac{1}{5}}\right)^{6}$

d. $\left(a^{3}\right)^{\frac{1}{10}}$ **e.** $\left(m^{\frac{4}{9}}\right)^{\frac{3}{8}}$ **f.** $\left(2b^{\frac{1}{2}}\right)^{\frac{1}{3}}$

g. $4\left(p^{\frac{3}{7}}\right)^{\frac{14}{15}}$ **h.** $\left(x^{\frac{m}{n}}\right)^{\frac{n}{p}}$ **i.** $\left(3m^{\frac{a}{b}}\right)^{\frac{b}{c}}$

11. WE15b, c Simplify each of the following.

a. $\left(a^{\frac{1}{2}}b^{\frac{1}{3}}\right)^{\frac{1}{2}}$ **b.** $(a^{4}b)^{\frac{3}{4}}$ **c.** $\left(x^{\frac{3}{5}}y^{\frac{7}{8}}\right)^{2}$

d. $\left(3a^{\frac{1}{3}}b^{\frac{3}{5}}c^{\frac{3}{4}}\right)^{\frac{1}{3}}$ **e.** $\left(x^{\frac{1}{2}}y^{\frac{2}{3}}z^{\frac{2}{5}}\right)^{\frac{1}{2}}$ **f.** $\left(\frac{a^{\frac{3}{4}}}{b}\right)^{\frac{2}{3}}$

g. $\left(\frac{m^{\frac{4}{5}}}{n^{\frac{7}{8}}}\right)^{2}$ **h.** $\left(\frac{b^{\frac{3}{5}}}{c^{\frac{4}{9}}}\right)^{\frac{2}{3}}$ **i.** $\left(\frac{4x^{7}}{2y^{\frac{3}{4}}}\right)^{\frac{1}{2}}$

12. MC *Note:* There may be more than one correct answer.

If $\left(a^{\frac{3}{4}}\right)^{\frac{m}{n}}$ is equal to $a^{\frac{1}{4}}$, then m and n could not be:

A. 1 and 3 **B.** 2 and 6 **C.** 3 and 8

D. 4 and 9 **E.** 5 and 15

13. Simplify each of the following.

a. $\sqrt{a^{8}}$ **b.** $\sqrt[3]{b^{9}}$ **c.** $\sqrt[4]{m^{16}}$

d. $\sqrt{16x^{4}}$ **e.** $\sqrt[3]{8y^{9}}$ **f.** $\sqrt[4]{16x^{8}y^{12}}$

g. $\sqrt[3]{27m^{9}n^{15}}$ **h.** $\sqrt[5]{32p^{5}q^{10}}$ **i.** $\sqrt[3]{216a^{6}b^{18}}$

Communicating, reasoning and problem solving

14. Manning's formula is used to calculate the flow of water in a river during a flood situation. The formula is $v = \frac{R^{\frac{2}{3}}S^{\frac{1}{2}}}{n}$, where R is the hydraulic radius, S is the slope of the river and n is the roughness coefficient. This formula is used by meteorologists and civil engineers to analyse potential flood situations.

a. Find the flow of water in metres per second in the river if $R = 8$, $S = 0.0025$ and $n = 0.625$.

b. To find the volume of water flowing through the river, we multiply the flow rate by the average cross-sectional area of the river. If the average cross-sectional area is 52 m^2, find the volume of water (in L) flowing through the river each second. (Remember 1 m^3 = 1000 L.)

c. If water continues to flow at this rate, what will be the total amount of water to flow through in one hour? Justify your answer.

d. Use the internet to find the meaning of the terms 'hydraulic radius' and 'roughness coefficient'.

15. Find x if $m^x = \dfrac{\sqrt{m^{10}}}{\left(\sqrt{m^4}\right)^2}$.

16. Simplify:

a. $\dfrac{x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y - z}{\left(x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}}\right)}$ **b.** $\sqrt[5]{\dfrac{t^2}{\sqrt{t^3}}}$.

17. Expand $\left(m^{\frac{3}{4}} + m^{\frac{1}{2}}n^{\frac{1}{2}} + m^{\frac{1}{4}}n + n^{\frac{3}{2}}\right)\left(m^{\frac{1}{4}} - n^{\frac{1}{2}}\right)$.

18. How will you remember the rule for fractional indices?

16.6 Negative indices [Stage 5.3]

- Consider the following division:
  $\dfrac{2^3}{2^4} = 2^{-1}$ (using the Second Index Law).
  Alternatively, $\dfrac{2^3}{2^4} = \dfrac{8}{16} = \dfrac{1}{2}$.
- We can conclude that $2^{-1} = \dfrac{1}{2}$.
- In general form:

$$a^{-1} = \frac{1}{a} \text{ and } a^{-n} = \frac{1}{a^n}.$$

WORKED EXAMPLE 16

Evaluate each of the following using a calculator.
a 4^{-1} **b** 2^{-4}

THINK	WRITE
a Use a calculator to evaluate 4^{-1}.	**a** $4^{-1} = 0.25$
b Use a calculator to evaluate 2^{-4}.	**b** $2^{-4} = 0.0625$

- Consider the index law $a^{-1} = \dfrac{1}{a}$. Now consider the case in which a is fractional.

  Consider the expression $\left(\dfrac{a}{b}\right)^{-1}$.

$$\left(\frac{a}{b}\right)^{-1} = \frac{1}{\frac{a}{b}}$$
$$= 1 \times \frac{b}{a}$$
$$= \frac{b}{a}$$

We can therefore consider an index of –1 to be a reciprocal function.

WORKED EXAMPLE 17

Write the value of each of the following without the use of a calculator.

a $\left(\frac{2}{3}\right)^{-1}$ **b** $\left(\frac{1}{5}\right)^{-1}$ **c** $\left(1\frac{1}{4}\right)^{-1}$

THINK	WRITE
a To evaluate $\left(\frac{2}{3}\right)^{-1}$, take the reciprocal of $\frac{2}{3}$.	**a** $\left(\frac{2}{3}\right)^{-1} = \frac{3}{2}$
b 1 To evaluate $\left(\frac{1}{5}\right)^{-1}$, take the reciprocal of $\frac{1}{5}$.	**b** $\left(\frac{1}{5}\right)^{-1} = \frac{5}{1}$
2 Write $\frac{5}{1}$ as a whole number.	$= 5$
c 1 Write $1\frac{1}{4}$ as an improper fraction.	**c** $\left(1\frac{1}{4}\right)^{-1} = \left(\frac{5}{4}\right)^{-1}$
2 Take the reciprocal of $\frac{5}{4}$.	$= \frac{4}{5}$

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Interactivity: Negative indices (int-6064)
eLesson: Negative indices (eles-1910)
Digital doc: WorkSHEET Real numbers III (doc-14614)

Exercise 16.6 Negative indices

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–e, 2a–e, 3a–e, 4a–e, 5a–e, 6a–d, 7a–d, 8–12	Questions: 1d–f, 2d–f, 3d–f, 4d–f, 5e–h, 6c–f, 7c–f, 8–12	Questions: 1e–h, 2e–h, 3e–h, 4e–h, 5g–l, 6e–h, 7e–h, 8–14

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE16 Evaluate each of the following using a calculator.

a. 5^{-1}	**b.** 3^{-1}	**c.** 8^{-1}	**d.** 10^{-1}
e. 2^{-3}	**f.** 3^{-2}	**g.** 5^{-2}	**h.** 10^{-4}

2. Find the value of each of the following, correct to 3 significant figures.

a. 6^{-1}	**b.** 7^{-1}	**c.** 6^{-2}	**d.** 9^{-3}
e. 6^{-3}	**f.** 15^{-2}	**g.** 16^{-2}	**h.** 5^{-4}

3. Find the value of each of the following, correct to 2 significant figures.

a. $(2.5)^{-1}$	**b.** $(0.4)^{-1}$	**c.** $(1.5)^{-2}$	**d.** $(0.5)^{-2}$
e. $(2.1)^{-3}$	**f.** $(10.6)^{-4}$	**g.** $(0.45)^{-3}$	**h.** $(0.125)^{-4}$

4. Find the value of each of the following, correct to 2 significant figures.

a. $(-3)^{-1}$	**b.** $(-5)^{-1}$	**c.** $(-2)^{-2}$	**d.** $(-4)^{-2}$
e. $(-1.5)^{-1}$	**f.** $(-2.2)^{-1}$	**g.** $(-0.6)^{-1}$	**h.** $(-0.85)^{-2}$

5. WE17 Write the value of each of the following without the use of a calculator.

a. $\left(\frac{4}{5}\right)^{-1}$ **b.** $\left(\frac{3}{10}\right)^{-1}$ **c.** $\left(\frac{7}{8}\right)^{-1}$ **d.** $\left(\frac{13}{20}\right)^{-1}$

e. $\left(\frac{1}{2}\right)^{-1}$ **f.** $\left(\frac{1}{4}\right)^{-1}$ **g.** $\left(\frac{1}{8}\right)^{-1}$ **h.** $\left(\frac{1}{10}\right)^{-1}$

i. $\left(1\frac{1}{2}\right)^{-1}$ **j.** $\left(2\frac{1}{4}\right)^{-1}$ **k.** $\left(1\frac{1}{10}\right)^{-1}$ **l.** $\left(5\frac{1}{2}\right)^{-1}$

6. Find the value of each of the following, leaving your answer in cf1tion form if necessary.

a. $\left(\frac{1}{2}\right)^{-2}$ **b.** $\left(\frac{2}{5}\right)^{-2}$ **c.** $\left(\frac{2}{3}\right)^{-3}$ **d.** $\left(\frac{1}{4}\right)^{-2}$

e. $\left(1\frac{1}{2}\right)^{-2}$ **f.** $\left(2\frac{1}{4}\right)^{-2}$ **g.** $\left(1\frac{1}{3}\right)^{-3}$ **h.** $\left(2\frac{1}{5}\right)^{-3}$

7. Find the value of each of the following.

a. $\left(-\frac{2}{3}\right)^{-1}$ **b.** $\left(-\frac{3}{5}\right)^{-1}$ **c.** $\left(-\frac{1}{4}\right)^{-1}$ **d.** $\left(-\frac{1}{10}\right)^{-1}$

e. $\left(-\frac{2}{3}\right)^{-2}$ **f.** $\left(-\frac{1}{5}\right)^{-2}$ **g.** $\left(-1\frac{1}{2}\right)^{-1}$ **h.** $\left(-2\frac{3}{4}\right)^{-2}$

8. Without using a calculator, evaluate $\dfrac{\left(\frac{2^{-1}}{3}\right)^{-1}}{\left(\frac{4}{5^{-1}}\right)}$.

9. Simplify $\left(\sqrt{\frac{a^2}{b^2}}\right)^{-1}$.

Communicating, reasoning and problem solving

10. Consider the equation $y = \frac{6}{x}$. Clearly $x \neq 0$, as $\frac{6}{x}$ would be undefined. What happens to the value of y as x gets closer to zero coming from:

a. the positive direction

b. the negative direction?

11. Consider the expression 2^{-n}. Explain what happens to the value of this expression as n increases.

12. Solve the following pair of simultaneous equations.

$3^{y+1} = \frac{1}{9}$ and $\frac{5^y}{125^x} = 125$

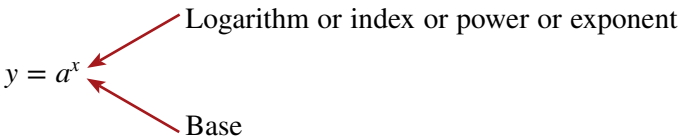
13. Simplify $\dfrac{x^{n+2} + x^{n-2}}{x^{n-4} + x^n}$.

14. How can division be used to explain negative indices?

16.7 Logarithms [Stage 5.3]

16.7.1 Definition of a logarithm

- The index, power or exponent in the statement $y = a^x$ is also known as a **logarithm** (or log for short).



- The statement $y = a^x$ can be written in an alternative form as $\log_a y = x$, which is read as 'the logarithm of y to the base a is equal to x'. These two statements are equivalent.

$a^x = y \Leftrightarrow$	$\log_a y = x$
Index form	Logarithmic form

- For example, $3^2 = 9$ can be written as $\log_3 9 = 2$. The log form would be read as 'the logarithm of 9, to the base of 3, is 2'. In both forms, the base is 3 and the logarithm is 2.

WORKED EXAMPLE 18

Write the following in logarithmic form.
a $10^4 = 10\,000$ **b** $6^x = 216$

THINK	WRITE
a 1 Write the given statement.	**a** $10^4 = 10000$
2 Identify the base (10) and the logarithm (4), and write the equivalent statement in logarithmic form. (Use $a^x = y \Leftrightarrow \log_a y = x$, where the base is a and the log is x.)	$\log_{10} 10000 = 4$
b 1 Write the given statement.	**b** $6^x = 216$
2 Identify the base (6) and the logarithm (x), and write the equivalent statement in logarithmic form.	$\log_6 216 = x$

WORKED EXAMPLE 19

Write the following in index form.
a $\log_2 8 = 3$ **b** $\log_{25} 5 = \frac{1}{2}$

THINK	WRITE
a 1 Write the statement.	**a** $\log_2 8 = 3$
2 Identify the base (2) and the log (3), and write the equivalent statement in index form. Remember that the log is the same as the index.	$2^3 = 8$
b 1 Write the statement.	**b** $\log_{25} 5 = \frac{1}{2}$
2 Identify the base (25) and the log $\left(\frac{1}{2}\right)$, and write the equivalent statement in index form.	$25^{\frac{1}{2}} = 5$

- In the previous examples, we found that:
$$\log_2 8 \Leftrightarrow 2^3 = 8 \text{ and } \log_{10} 10000 = 4 \Leftrightarrow 10^4 = 10000$$
- We could also write $\log_2 8 = 3$ as $\log_2 2^3 = 3$ and $\log_{10} 10000 = 4$ as $\log_{10} 10^4 = 4$.
- Can this pattern be used to work out the value of $\log_3 81$? We need to find the power when the base of 3 is raised to that power to give 81.

WORKED EXAMPLE 20

Evaluate $\log_3 81$.

THINK	WRITE
1 Write the log expression.	$\log_3 81$
2 Express 81 in index form with a base of 3.	$= \log_3 3^4$
3 Write the value of the logarithm.	$= 4$

16.7.2 Using logarithmic scales in measurement

- Logarithms can also be used to display data sets that cover a range of values which vary greatly in size. For example, when measuring the amplitude of earthquake waves, some earthquakes will have

amplitudes of 10000, whereas other earthquakes may have amplitudes of 10000000 (1000 times greater). Rather than trying to display this data on a linear scale, we can take the logarithm of the amplitude, which gives us the magnitude of each earthquake. The Richter scale uses the magnitudes of earthquakes to display the difference in their power.

WORKED EXAMPLE 21

Convert the following amplitudes of earthquakes into values on the Richter scale, correct to 1 decimal place.

a 1989 Newcastle earthquake: amplitude 398000

b 2010 Canterbury earthquake: amplitude 12600000

c 2010 Chile earthquake: amplitude 631000000

THINK

a Use a calculator to calculate the logarithmic value of the amplitude. Round the answer to 1 decimal place.
Write the answer in words.

b Use a calculator to calculate the logarithmic value of the amplitude. Round the answer to 1 decimal place.
Write the answer in words.

c Use a calculator to calculate the logarithmic value of the amplitude. Round the answer to 1 decimal place.
Write the answer in words.

WRITE

a $\log 398000 = 5.599\ldots$
$= 5.6$
The 1989 Newcastle earthquake rated 5.6 on the Richter scale.

b $\log 12600000 = 7.100\ldots$
$= 7.1$
The 2010 Canterbury earthquake rated 7.1 on the Richter scale.

c $\log 631000000 = 8.800\ldots$
$= 8.8$
The 2010 Chile earthquake rated 8.8 on the Richter scale.

learnON RESOURCES

Interactivity: Logarithms (int-6194)

Exercise 16.7 Logarithms

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–e, 2, 3a–e, 4, 5a–e, 6–8, 10	Questions: 1e–k, 2, 3d–i, 4, 5e–h, 6–10	Questions: 1i–p, 2, 3g–l, 4, 5g–l, 6–13

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Understanding and fluency

1. WE18 Write the following in logarithmic form.

a. $4^2 = 16$ **b.** $2^5 = 32$ **c.** $3^4 = 81$ **d.** $6^2 = 36$

e. $1000 = 10^3$ **f.** $25 = 5^2$ **g.** $4^3 = x$ **h.** $5^x = 125$

i. $7^x = 49$ **j.** $p^4 = 16$ **k.** $9^{\frac{1}{2}} = 3$ **l.** $0.1 = 10^{-1}$

m. $2 = 8^{\frac{1}{3}}$ **n.** $2^{-1} = \frac{1}{2}$ **o.** $a^0 = 1$ **p.** $4^{\frac{3}{2}} = 8$

2. MC The statement $w = h^t$ is equivalent to:

A. $w = \log_t h$ **B.** $h = \log_t w$ **C.** $t = \log_w h$ **D.** $t = \log_h w$ **E.** $w = \log_h t$

3. WE19 Write the following in index form.

a. $\log_2 16 = 4$ **b.** $\log_3 27 = 3$ **c.** $\log_{10} 100000 = 6$ **d.** $\log_5 125 = 3$

e. $\log_{16} 4 = \frac{1}{2}$ **f.** $\log_4 64 = x$ **g.** $\frac{1}{2} = \log_{49} 7$ **h.** $\log_3 x = 5$

i. $\log_{81} 9 = \frac{1}{2}$ **j.** $\log_{10} 0.01 = -2$ **k.** $\log_8 8 = 1$ **l.** $\log_{64} 4 = \frac{1}{3}$

4. MC The statement $q = \log_r p$ is equivalent to:

A. $q = r^p$ **B.** $p = r^q$ **C.** $r = p^q$ **D.** $r = q^p$ **E.** $q = p^r$

5. WE20 Evaluate the following logarithms.

a. $\log_2 16$ **b.** $\log_4 16$ **c.** $\log_{11} 121$ **d.** $\log_{10} 100000$

e. $\log_3 243$ **f.** $\log_2 128$ **g.** $\log_5 1$ **h.** $\log_9 3$

i. $\log_3\left(\frac{1}{3}\right)$ **j.** $\log_6 6$ **k.** $\log_{10}\left(\frac{1}{100}\right)$ **l.** $\log_{125} 5$

6. Write the value of each of the following.

a. $\log_{10} 1$ **b.** $\log_{10} 10$ **c.** $\log_{10} 100$ **d.** $\log_{10} 1000$

e. $\log_{10} 10000$ **f.** $\log_{10} 100000$

7. Use your results for question **6** to answer the following.

a. Between which two whole numbers would $\log_{10} 7$ lie?

b. Between which two whole numbers would $\log_{10} 4600$ lie?

c. Between which two whole numbers would $\log_{10} 85$ lie?

d. Between which two whole numbers would $\log_{10} 12750$ lie?

e. Between which two whole numbers would $\log_{10} 110$ lie?

f. Between which two whole numbers would $\log_{10} 81000$ lie?

8. WE21 Convert the following amplitudes of earthquakes into values on the Richter Scale, correct to 1 decimal place.

a. 2016 Northern Territory earthquake: amplitude 1 260 000

b. 2011 Christchurch earthquake: amplitude 1 200 000

c. 1979 Tumaco earthquake: amplitude 1 580 000 000

Communicating, reasoning and problem solving

9. a. If $\log_{10} g = k$, find the value of $\log_{10} g^2$. Justify your answer.

b. If $\log_x y = 2$, find the value of $\log_y x$. Justify your answer.

c. By referring to the equivalent index statement, explain why x must be a positive number given $\log_4 x = y$, for all values of y.

10. Calculate each of the following logarithms.

a. $\log_2 (64)$ **b.** $\log_3\left(\frac{1}{81}\right)$ **c.** $\log_{10} (0.00001)$

11. Find the value of x in each of the following.

a. $\log_x\left(\frac{1}{243}\right) = -5$ **b.** $\log_x (343) = 3$ **c.** $\log_{64} (x) = -\frac{1}{2}$

12. Simplify $10^{\log_{10}(x)}$.

13. How are indices and logarithms related?

16.8 Logarithm laws [Stage 5.3]

16.8.1 The index laws and logarithm laws

- The index laws are:
  1. $a^m \times a^n = a^{m+n}$
  2. $\frac{a^m}{a^n} = a^{m-n}$
  3. $(a^m)^n = a^{mn}$
  4. $a^0 = 1$
  5. $a^1 = a$
  6. $a^{-1} = \frac{1}{a}$
- The index laws can be used to produce equivalent logarithm laws.

16.8.2 Logarithm law 1

- If $x = a^m$ and $y = a^n$, then $\log_a x = m$ and $\log_a y = n$ (equivalent log form).

Now	$xy = a^m \times a^n$	
or	$xy = a^{m+n}$	(First Index Law).
So	$\log_a (xy) = m + n$	(equivalent log form)
or	$\log_a (xy) = \log_a x + \log_a y$	(substituting for m and n).

$$\log_a x + \log_a y = \log_a (xy)$$

- This means that the sum of two logarithms with the same base is equal to the logarithm of the product of the numbers.

WORKED EXAMPLE 22

Evaluate $\log_{10} 20 + \log_{10} 5$.

THINK

1 Since the same base of 10 is used in each log term, use $\log_a x + \log_a y = \log_a (xy)$ and simplify.

2 Evaluate. (Remember that $100 = 10^2$.)

WRITE

$\log_{10} 20 + \log_{10} 5 = \log_{10} (20 \times 5)$

$= \log_{10} 100$

$= 2$

16.8.3 Logarithm law 2

- If $x = a^m$ and $y = a^n$, then $\log_a x = m$ and $\log_a y = n$ (equivalent log form).

Now	$\frac{x}{y} = \frac{a^m}{a^n}$	
or	$\frac{x}{y} = a^{m-n}$	(Second Index Law).
So	$\log_a \left(\frac{x}{y}\right) = m - n$	(equivalent log form)
or	$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$	(substituting for m and n).

$$\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$$

- This means that the difference of two logarithms with the same base is equal to the logarithm of the quotient of the numbers.

WORKED EXAMPLE 23

Evaluate $\log_4 20 - \log_4 5$.

THINK

1 Since the same base of 4 is used in each log term, use $\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$ and simplify.

2 Evaluate. (Remember that $4 = 4^1$.)

WRITE

$$\log_4 20 - \log 45 = \log_4\left(\frac{20}{5}\right)$$
$$= \log_4 4$$
$$= 1$$

WORKED EXAMPLE 24

Evaluate $\log_5 35 + \log_5 15 - \log_5 21$.

THINK

1 Since the first two log terms are being added, use $\log_a x + \log_a y = \log_a (xy)$ and simplify.

2 To find the difference between the two remaining log terms, use $\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$ and simplify.

3 Evaluate. (Remember that $25 = 5^2$.)

WRITE

$$\log_5 35 + \log_5 15 - \log_5 21$$
$$= \log_5 (35 \times 15) - \log_5 21$$
$$= \log_5 525 - \log_5 21$$
$$= \log_5\left(\frac{525}{21}\right)$$
$$= \log_5 25$$
$$= 2$$

- Once you have gained confidence in using the first two laws, you can reduce the number of steps of working by combining the application of the laws. In Worked example 24, we could write:

$$\log_5 35 + \log_5 15 - \log_5 21 = \log_5\left(\frac{35 \times 15}{21}\right)$$
$$= \log_5 25$$
$$= 2$$

16.8.4 Logarithm law 3

- If $x = a^m$, then $\log_a x = m$ (equivalent log form).

Now $x^n = (a^m)^n$

or $x^n = a^{mn}$ (Third Index Law).

So $\log_a x^n = mn$ (equivalent log form)

or $\log_a x^n = (\log_a x) \times n$ (substituting for m)

or $\log_a x^n = n \log_a x$

$$\boxed{\log_a x^n = n \log_a x}$$

- This means that the logarithm of a number raised to a power is equal to the product of the power and the logarithm of the number.

WORKED EXAMPLE 25

Evaluate $2\log_6 3 + \log_6 4$.

THINK	WRITE
1 The first log term is not in the required form to use the log law relating to sums. Use $\log_a x^n = n\log_a x$ to rewrite the first term in preparation for applying the first log law.	$2\log_6 3 + \log_6 4 = \log_6 3^2 + \log_6 4$ $= \log_6 9 + \log_6 4$
2 Use $\log_a x + \log_a y = \log_a (xy)$ to simplify the two log terms to one.	$= \log_6 (9 \times 4)$ $= \log_6 36$
3 Evaluate. (Remember that $36 = 6^2$.)	$= 2$

16.8.5 Logarithm law 4

- As $a^0 = 1$ (Fourth Index Law)
  $\log_a 1 = 0$ (equivalent log form).

$$\boxed{\log_a 1 = 0}$$

- This means that the logarithm of 1 with any base is equal to 0.

16.8.6 Logarithm law 5

- As $a^1 = a$ (Fifth Index Law),
  $\log_a a = 1$ (equivalent log form).

$$\boxed{\log_a a = 1}$$

- This means that the logarithm of any number a with base a is equal to 1.

16.8.7 Logarithm law 6

- Now $\log_a \left(\frac{1}{x}\right) = \log_a x^{-1}$ (Sixth Index Law)
  or $\log_a \left(\frac{1}{x}\right) = -1 \times \log_a x$ (using the fourth log law)
  or $\log_a \left(\frac{1}{x}\right) = -\log_a x$.

$$\boxed{\log_a \left(\frac{1}{x}\right) = -\log_a x}$$

16.8.8 Logarithm law 7

- Now $\log_a a^x = x\log_a a$ (using the third log law)
  or $\log_a a^x = x \times 1$ (using the fifth log law)
  or $\log_a a^x = x$.

$$\boxed{\log_a a^x = x}$$

learnON RESOURCES — ONLINE ONLY

Interactivity: The first law of logarithms (int-6195)
Interactivity: The second law of logarithms (int-6196)
Interactivity: The third law of logarithms (int-6197)
Interactivity: The fourth law of logarithms (int-6198)
Interactivity: The fifth law of logarithms (int-6199)
Interactivity: The sixth law of logarithms (int-6200)
Interactivity: The seventh law of logarithms (int-6201)

Exercise 16.8 Logarithm laws

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–7, 8a–f, 9a–f, 10, 11a–g, 12, 13, 15	Questions: 1–7, 8d–i, 9e–j, 10, 11e–i, 12–15	Questions: 1–7, 8g–l, 9g–l, 10, 11g–l, 12–18

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Understanding and fluency

1. Use a calculator to evaluate the following, correct to 5 decimal places.
a. $\log_{10} 50$ **b.** $\log_{10} 25$ **c.** $\log_{10} 5$ **d.** $\log_{10} 2$

2. Use your answers to question **1** to show that each of the following statements is true.
a. $\log_{10} 25 + \log_{10} 2 = \log_{10} 50$
b. $\log_{10} 50 - \log_{10} 2 = \log_{10} 25$
c. $\log_{10} 25 = 2\log_{10} 5$
d. $\log_{10} 50 - \log_{10} 25 - \log_{10} 2 = \log_{10} 1$

3. WE22 Evaluate the following.
a. $\log_6 3 + \log_6 2$ **b.** $\log_4 8 + \log_4 8$ **c.** $\log_{10} 25 + \log_{10} 4$
d. $\log_8 32 + \log_8 16$ **e.** $\log_6 108 + \log_6 12$ **f.** $\log_{14} 2 + \log_{14} 7$

4. WE23 Evaluate the following.
a. $\log_2 20 - \log_2 5$ **b.** $\log_3 54 - \log_3 2$ **c.** $\log_4 24 - \log_4 6$
d. $\log_{10} 30000 - \log_{10} 3$ **e.** $\log_6 648 - \log_6 3$ **f.** $\log_2 224 - \log_2 7$

5. WE24 Evaluate the following.
a. $\log_3 27 + \log_3 2 - \log_3 6$
b. $\log_4 24 - \log_4 2 - \log_4 6$
c. $\log_6 78 - \log_6 13 + \log_6 1$
d. $\log_2 120 - \log_2 3 - \log_2 5$

6. Evaluate $2\log_4 8$.

7. WE25 Evaluate the following.
a. $2\log_{10} 5 + \log_{10} 4$
b. $\log_3 648 - 3\log_3 2$
c. $4\log_5 10 - \log_5 80$
d. $\log_2 50 + \frac{1}{2}\log_2 16 - 2\log_2 5$

8. Evaluate the following.
a. $\log_8 8$ **b.** $\log_5 1$ **c.** $\log_2\left(\frac{1}{2}\right)$ **d.** $\log_4 4^5$
e. $\log_6 6^{-2}$ **f.** $\log_{20} 20$ **g.** $\log_2 1$ **h.** $\log_3\left(\frac{1}{9}\right)$
i. $\log_4\left(\frac{1}{2}\right)$ **j.** $\log_4 \sqrt{4}$ **k.** $\log_3\left(\frac{1}{\sqrt{3}}\right)$ **l.** $\log_2 8\sqrt{2}$

9. Use the logarithm laws to simplify each of the following.

a. $\log_a 5 + \log_a 8$
b. $\log_a 12 + \log_a 3 - \log_a 2$
c. $4\log_x 2 + \log_x 3$
d. $\log_x 100 - 2\log_x 5$
e. $3\log_a x - \log_a x^2$
f. $5\log_a a - \log_a a^4$
g. $\log_x 6 - \log_x 6x$
h. $\log_a a^7 + \log_a 1$
i. $\log_p \sqrt{p}$
j. $\log_k k\sqrt{k}$
k. $6\log_a\left(\frac{1}{a}\right)$
l. $\log_a\left(\frac{1}{\sqrt[3]{a}}\right)$

10. MC *Note:* There may be more than one correct answer.

a. The equation $y = 10^x$ is equivalent to:
A. $x = 10^y$ **B.** $x = \log_{10} y$ **C.** $y = \log_x 10$ **D.** $x = \log_y 10$ **E.** $y = \log_{10} x$

b. The equation $y = 10^{4x}$ is equivalent to:
A. $x = \log_{10}\sqrt{4y}$ **B.** $x = \log_{10}\sqrt[4]{y}$ **C.** $x = 10^{\frac{1}{4y}}$ **D.** $x = \frac{1}{4}\log_{10} y$ **E.** $x = 4\log_{10} y$

c. The equation $y = 10^{3x}$ is equivalent to:
A. $x = \frac{1}{3}\log_{10} y$ **B.** $x = \log_{10} y^{\frac{1}{3}}$ **C.** $x = \log_{10} y - 3$ **D.** $x = 10^{y-3}$ **E.** $x = 3\log_{10} y$

d. The equation $y = ma^{nx}$ is equivalent to:
A. $x = \frac{1}{n}a^{my}$ **B.** $x = \log a\left(\frac{m}{y}\right)^n$ **C.** $x = \frac{1}{n}(\log_a y - \log_a m)$
D. $x = \frac{n}{m}\log_a y$ **E.** $x = \frac{1}{n}\log_a\left(\frac{y}{m}\right)$

11. Simplify, and evaluate where possible, each of the following without a calculator.

a. $\log_2 8 + \log_2 10$ **b.** $\log_3 7 + \log_3 15$ **c.** $\log_{10} 20 + \log_{10} 5$
d. $\log_6 8 + \log_6 7$ **e.** $\log_2 20 - \log_2 5$ **f.** $\log_3 36 - \log_3 12$
g. $\log_5 100 - \log_5 8$ **h.** $\log_2 \frac{1}{3} + \log_2 9$ **i.** $\log_4 25 + \log_4 \frac{1}{5}$
j. $\log_{10} 5 - \log_{10} 20$ **k.** $\log_3 \frac{4}{5} - \log_3 \frac{1}{5}$ **l.** $\log_2 9 + \log_2 4 - \log_2 12$
m. $\log_3 8 - \log_3 2 + \log_3 5$ **n.** $\log_4 24 - \log_4 2 - \log_4 6$

12. MC **a.** The expression $\log_{10} xy$ is equal to:
A. $\log_{10} x \times \log_{10} y$ **B.** $\log_{10} x - \log_{10} y$ **C.** $\log_{10} x + \log_{10} y$
D. $y\log_{10} x$ **E.** $\log_{10}(x + y)$

b. The expression $\log_{10} x^y$ is equal to:
A. $x\log_{10} y$ **B.** $y\log_{10} x$ **C.** $10\log_x y$
D. $\log_{10} x + \log_{10} y$ **E.** $\log_{10} x \times \log_x y$

c. The expression $\frac{1}{3}\log_2 64 + \log_2 10$ is equal to:
A. $\log_2 40$ **B.** $\log_2 80$ **C.** $\log_2 \frac{64}{10}$
D. 1 **E.** 2

Communicating, reasoning and problem solving

13. For each of the following, write the possible strategy you intend to use.

a. Evaluate $(\log_3 81)(\log_3 27)$.

b. Evaluate $\dfrac{\log_a 81}{\log_a 3}$.

c. Evaluate $5^{\log_5 7}$.

In each case, explain how you obtained your final answer.

14. Simplify $\log_2\left(\frac{8}{125}\right) - 3\log_2\left(\frac{3}{5}\right) - 4\log_2\left(\frac{1}{2}\right)$.

15. Simplify $\log_a(a^5 + a^3) - \log_a(a^4 + a^2)$.

16. If $2\log_a(x) = 1 + \log_a(8x - 15a)$, find x in terms of a where a is a positive constant and x is positive.

17. What technique will you use to remember the log laws?

18. Evaluate $\dfrac{\log_2 8 \times \log_2 16}{4^{\log_4 8}}$.

16.9 Solving equations [Stage 5.3]

- The equation $\log_a y = x$ is an example of a general logarithmic equation. Laws of logarithms and indices are used to solve these equations.

WORKED EXAMPLE 26

Solve for x in the following equations.

a $\log_2 x = 3$ **b** $\log_6 x = -2$ **c** $\log_3 x^4 = -16$ **d** $\log_5(x - 1) = 2$

THINK	WRITE
a 1 Write the equation.	**a** $\log_2 x = 3$
2 Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	$2^3 = x$
3 Rearrange and simplify.	$x = 8$
b 1 Write the equation.	**b** $\log_6 x = -2$
2 Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	$6^{-2} = x$
3 Rearrange and simplify.	$x = \dfrac{1}{6^2}$ $= \frac{1}{36}$
c 1 Write the equation.	**c** $\log_3 x^4 = -16$
2 Rewrite using $\log_a x^n = n\log_a x$.	$4\log_3 x = -16$
3 Divide both sides by 4.	$\log_3 x = -4$
4 Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	$3^{-4} = x$
5 Rearrange and simplify.	$x = \dfrac{1}{3^4}$ $= \frac{1}{81}$
d 1 Write the equation.	**d** $\log_5(x - 1) = 2$
2 Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	$5^2 = x - 1$
3 Solve for x.	$x - 1 = 25$ $x = 26$

WORKED EXAMPLE 27

Solve for x in $\log_x 25 = 2$, given that $x > 0$.

THINK	WRITE
1 Write the equation.	$\log_x 25 = 2$
2 Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	$x^2 = 25$
3 Solve for x. *Note:* $x = -5$ is rejected as a solution because $x > 0$.	$x = 5$ (because $x > 0$)

WORKED EXAMPLE 28

Solve for x in the following.

a $\log_2 16 = x$ **b** $\log_3\left(\frac{1}{3}\right) = x$ **c** $\log_9 3 = x$

THINK	WRITE
a 1 Write the equation.	**a** $\log_2 16 = x$
2 Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	$2^x = 16$
3 Write 16 with base 2.	$= 2^4$
4 Equate the indices.	$x = 4$
b 1 Write the equation.	**b** $\log_3\left(\frac{1}{3}\right) = x$
2 Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	$3^x = \frac{1}{3}$ $= \frac{1}{3^1}$
3 Write $\frac{1}{3}$ with base 3.	$3^x = 3^{-1}$
4 Equate the indices.	$x = -1$
c 1 Write the equation.	**c** $\log_9 3 = x$
2 Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	$9^x = 3$
3 Write 9 with base 3.	$(3^2)^x = 3$
4 Remove the grouping symbols.	$(3^2)^x = 3^1$
5 Equate the indices.	$2x = 1$
6 Solve for x.	$x = \frac{1}{2}$

WORKED EXAMPLE 29

Solve for x in the equation $\log_2 4 + \log_2 x - \log_2 8 = 3$.

THINK	WRITE
1 Write the equation.	$\log_2 4 + \log_2 x - \log_2 8 = 3$
2 Simplify the left-hand side. Use $\log_a x + \log_a y = \log_a (xy)$ and $\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$.	$\log_2\left(\frac{4 \times x}{8}\right) = 3$

3 Simplify.	$\log_2\left(\frac{x}{2}\right) = 3$
4 Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	$2^3 = \frac{x}{2}$
5 Solve for x.	$x = 2 \times 2^3$ $= 2 \times 8$ $= 16$

- When solving an equation such as $\log_2 8 = x$, it could be rewritten in index form as $2^x = 8$. This can be written with the same base of 2 to produce $2^x = 2^3$. Equating the indices gives us a solution of $x = 3$.
- Can we do this to solve the equation $2^x = 7$? Consider the method shown in the next worked example. It involves the use of logarithms and the $\log_{10}$ function on a calculator.

WORKED EXAMPLE 30

Solve for x, correct to 3 decimal places, if:
a $2^x = 7$ **b** $3^{-x} = 0.4$.

THINK	WRITE
a 1 Write the equation.	**a** $2^x = 7$
2 Take $\log_{10}$ of both sides.	$\log_{10} 2^x = \log_{10} 7$
3 Use the logarithm-of-a-power law to bring the power, x, to the front of the logarithmic equation.	$x \log_{10} 2 = \log_{10} 7$
4 Divide both sides by $\log_{10} 2$ to get x by itself.	Therefore, $x = \frac{\log_{10} 7}{\log_{10} 2}$
5 Use a calculator to evaluate the logarithms and write the answer correct to 3 decimal places.	$= 2.807$
b 1 Write the equation.	**b** $3^{-x} = 0.4$
2 Take $\log_{10}$ of both sides.	$\log_{10} 3^{-x} = \log_{10} 0.4$
3 Use the logarithm of a power law to bring the power, x, to the front of the logarithmic equation.	$-x \log_{10} 3 = \log_{10} 0.4$
4 Divide both sides by $\log_{10} 3$ to get $-x$ by itself.	$-x = \frac{\log_{10} 0.4}{\log_{10} 3}$
5 Use a calculator to evaluate the logarithms and write the answer correct to 3 decimal places.	$-x = -0.834$
6 Divide both sides by -1 to get x by itself.	$x = 0.834$

- Therefore, we can state the following rule:

$$\textbf{If } a^x = b \textbf{, then } x = \frac{\log_{10} b}{\log_{10} a}.$$

- This rule applies to any base, but since your calculator has base 10, this is the most commonly used base for this solution technique.

DISCUSSION

Why do you think your calculator has 10 as a standard base for logarithms? Do you find it easier to work in base 10 than in other bases?

learnON RESOURCES — ONLINE ONLY

Interactivity: Solving logarithmic equations (int-6202)

Digital doc: WorkSHEET Real numbers IV (doc-14615)

Exercise 16.9 Solving equations

Individual pathways

PRACTISE
Questions:
1a–h, 2a–e, 3a–f, 4a–h, 5, 6a–h, 7a–f, 8, 9, 11

CONSOLIDATE
Questions:
1d–k, 2d–f, 3c–f, 4e–j, 5, 6e–l, 7d–i, 8–12

MASTER
Questions:
1g–l, 2d–h, 3e–j, 4i–n, 5, 6i–o, 7g–l, 8–14

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE26 Solve for x in the following.

a. $\log_5 x = 2$ **b.** $\log_3 x = 4$ **c.** $\log_2 x = -3$

d. $\log_4 x = -2$ **e.** $\log_{10} x^2 = 4$ **f.** $\log_2 x^3 = 12$

g. $\log_3 (x + 1) = 3$ **h.** $\log_5 (x - 2) = 3$ **i.** $\log_4 (2x - 3) = 0$

j. $\log_{10} (2x + 1) = 0$ **k.** $\log_2 (-x) = -5$ **l.** $\log_3 (-x) = -2$

m. $\log_5 (1 - x) = 4$ **n.** $\log_{10} (5 - 2x) = 1$

2. WE27 Solve for x in the following, given that $x > 0$.

a. $\log_x 9 = 2$ **b.** $\log_x 16 = 4$ **c.** $\log_x 25 = \frac{2}{3}$

d. $\log_x 125 = \frac{3}{4}$ **e.** $\log_x \left(\frac{1}{8}\right) = -3$ **f.** $\log_x \left(\frac{1}{64}\right) = -2$

g. $\log_x 6^2 = 2$ **h.** $\log_x 4^3 = 3$

3. WE28 Solve for x in the following.

a. $\log_2 8 = x$ **b.** $\log_3 9 = x$ **c.** $\log_5 \left(\frac{1}{5}\right) = x$

d. $\log_4 \left(\frac{1}{16}\right) = x$ **e.** $\log_4 2 = x$ **f.** $\log_8 2 = x$

g. $\log_6 1 = x$ **h.** $\log_8 1 = x$ **i.** $\log_{\frac{1}{2}} 2 = x$

j. $\log_{\frac{1}{3}} 9 = x$

4. WE29 Solve for x in the following.

a. $\log_2 x + \log_2 4 = \log_2 20$ **b.** $\log_5 3 + \log_5 x = \log_5 18$

c. $\log_3 x - \log_3 2 = \log_3 5$ **d.** $\log_{10} x - \log_{10} 4 = \log_{10} 2$

e. $\log_4 8 - \log_4 x = \log_4 2$ **f.** $\log_3 10 - \log_3 x = \log_3 5$

g. $\log_6 4 + \log_6 x = 2$ **h.** $\log_2 x + \log_2 5 = 1$

i. $3 - \log_{10} x = \log_{10} 2$ **j.** $5 - \log_4 8 = \log_4 x$

k. $\log_2 x + \log_2 6 - \log_2 3 = \log_2 10$ **l.** $\log_2 x + \log_2 5 - \log_2 10 = \log_2 3$

m. $\log_3 5 - \log_3 x + \log_3 2 = \log_3 10$ **n.** $\log_5 4 - \log_5 x + \log_5 3 = \log_5 6$

5. MC **a.** The solution to the equation $\log_7 343 = x$ is:

A. $x = 2$ **B.** $x = 3$ **C.** $x = 1$

D. $x = 0$ **E.** $x = 7$

b. If $\log_8 x = 4$, then x is equal to:

A. 4096 **B.** 512 **C.** 64

D. 2 **E.** $\frac{1}{3}$

c. Given that $\log_x 3 = \frac{1}{2}$, x must be equal to:

A. 3 **B.** 6 **C.** 81

D. 9 **E.** 2

d. If $\log_a x = 0.7$, then $\log_a x^2$ is equal to:

A. 0.49 **B.** 1.4 **C.** 0.35

D. 0.837 **E.** 4.9

6. Solve for x in the following equations.

a. $2^x = 128$ **b.** $3^x = 9$ **c.** $7^x = \frac{1}{49}$

d. $9^x = 1$ **e.** $5^x = 625$ **f.** $64^x = 8$

g. $6^x = \sqrt{6}$ **h.** $2^x = 2\sqrt{2}$ **i.** $3^x = \dfrac{1}{\sqrt{3}}$

j. $4^x = 8$ **k.** $9^x = 3\sqrt{3}$ **l.** $2^x = \dfrac{1}{4\sqrt{2}}$

m. $3^{x+1} = 27\sqrt{3}$ **n.** $2^{x-1} = \dfrac{1}{32\sqrt{2}}$ **o.** $4^{x+1} = \dfrac{1}{8\sqrt{2}}$

7. WE30 Solve the following equations, correct to 3 decimal places.

a. $2^x = 11$ **b.** $2^x = 0.6$ **c.** $3^x = 20$

d. $3^x = 1.7$ **e.** $5^x = 8$ **f.** $0.7^x = 3$

g. $0.4^x = 5$ **h.** $3^{x+2} = 12$ **i.** $7^{-x} = 0.2$

j. $8^{-x} = 0.3$ **k.** $10^{-2x} = 7$ **l.** $8^{2-x} = 0.75$

8. The decibel (dB) scale for measuring loudness, d, is given by the formula $d = 10 \log_{10}(I \times 10^{12})$, where I is the intensity of sound in watts per square metre.

a. Find the number of decibels of sound if the intensity is 1.
b. Find the number of decibels of sound produced by a jet engine at a distance of 50 metres if the intensity is 10 watts per square metre.
c. Find the intensity of sound if the sound level of a pneumatic drill 10 metres away is 90 decibels.
d. Find how the value of d changes if the intensity is doubled. Give your answer to the nearest decibel.
e. Find how the value of d changes if the intensity is 10 times as great.
f. By what factor does the intensity of sound have to be multiplied in order to add 20 decibels to the sound level?

Communicating, reasoning and problem solving

9. The Richter scale is used to describe the energy of earthquakes. A formula for the Richter scale is $R = \frac{2}{3}\log_{10} K - 0.9$, where R is the Richter scale value for an earthquake that releases K kilojoules (kJ) of energy.
a. Find the Richter scale value for an earthquake that releases the following amounts of energy.
i. 1000 kJ **ii.** 2000 kJ **iii.** 3000 kJ
iv. 10 000 kJ **v.** 100 000 kJ **vi.** 1 000 000 kJ
b. Does doubling the energy released double the Richter scale value? Justify your answer.
c. Find the energy released by an earthquake of:
i. magnitude 4 on the Richter scale
ii. magnitude 5 on the Richter scale
iii. magnitude 6 on the Richter scale.
d. What is the effect (on the amount of energy released) of increasing the Richter scale value by 1?
e. Why is an earthquake measuring 8 on the Richter scale so much more devastating than one that measures 5?

10. Solve the following for x.
a. $3^{x+1} = 7$ **b.** $3^{x+1} = 7^x$
11. Solve the following for x.
$(27 \times 3^x)^3 = 81^x \times 3^2$
12. Solve $\{x : (3^x)^2 = 30 \times 3^x - 81\}$.
13. Tables of logarithms were used in classrooms before calculators were used. Would using logarithms have any effect on the accuracy of calculations?
14. Consider solving the logarithmic equation $\log_{10}(x + 1) + \log_{10} x - \log_{10} 6 = 0$.

The first step in the solution could be $\log_{10}\left(\frac{x(x+1)}{6}\right) = 0$.

Continue the solution by converting the logarithmic equation into a quadratic equation, then solving for x.

16.10 Review

Investigation | Rich task

Other number systems

Throughout history, different systems have been used to aid with counting. Ancient tribes are known to have used stones, bones and knots in rope to help keep count. The counting system that is used around the world today is called the Hindu–Arabic system. This system had its origin in India in the 3rd century BCE. The Arabs brought this method of counting to Europe in the Middle Ages.

The Hindu–Arabic method is known as the decimal or base 10 system, as it is based on counting in lots of ten. This system uses the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9. Notice that the largest digit is one less than the base number; that is, the largest digit in base 10 is 9. To make larger numbers, digits are grouped together. The position of the digit tells us about its value. We call this *place value*. For example, in the number 325, the 3 has a value of 'three lots of a hundred', the 2 has a value of 'two lots of ten' and the 5 has a value of 'five lots of units'. Another way to write this is:

$$3 \times 100 + 2 \times 10 + 5 \times 1 \textit{ or } 3 \times 10^2 + 2 \times 10^1 + 5 \times 10^0$$

In a decimal system, every place value is based on the number 10 raised to a power. The smallest place value (units) is described by 10^0, the tens place value by 10^1, the hundreds place value by 10^2, the thousands by 10^3 and so on.

Computers do not use a decimal system. The system for computer languages is based on the number 2 and is known as the binary system. The only digits needed in the binary system are the digits 0 and 1. Can you see why?

Decimal number	0	1	2	3	4	5	6	7	8	9	10	11	12	13
Binary number	0	1	10	11	100	101	110	111	1000	1001	1010	1011	1100	1101

Consider the decimal number 7. From the table above, you can see that its binary equivalent is 111. How can you be sure this is correct?

$$111 = 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 4 + 2 + 1 = 7$$

Notice that this time each place value is based on the number 2 raised to a power. You can use this technique to change any binary number into a decimal number. (The same pattern applies to other bases, for example, in base 6 the place values are based on the number 6 raised to a power.)

Binary operations

When adding in the decimal system, each time the addition is greater than 9, we need to 'carry over' into the next place value. In the example below, the units column adds to more than 9, so we need to carry over into the next place value.

$$\begin{array}{r} {}^{1}17 \\ +\ 13 \\ \hline 30 \end{array}$$

The same is true when adding in binary, except we need to 'carry over' every time the addition is greater than 1.

$$\begin{array}{r} {}^{1}01 \\ +\ 01 \\ \hline 10 \end{array}$$

1. Perform the following binary additions.

a. $11_2 + 01_2$

b. $111_2 + 110_2$

c. $1011_2 + 101_2$

2. Perform the following binary subtractions. Remember that if you need to borrow a number from a column on the left-hand side, you will actually be borrowing a 2 (not a 10).

a. $11_2 - 01_2$

b. $111_2 - 110_2$

c. $1011_2 - 101_2$

3. Try some multiplication. Remember to carry over lots of 2.

a. $11_2 \times 01_2$

b. $111_2 \times 110_2$

c. $1011_2 \times 101_2$

4. What if our number system had an 8 as its basis (that is, we counted in lots of 8)? The only digits available for use would be 0, 1, 2, 3, 4, 5, 6 and 7. (Remember the maximum digit is 1 less than the base value.) Give examples to show how numbers would be added, subtracted and multiplied using this base system. Remember that you would 'carry over' or 'borrow' lots of 8.

5. The hexadecimal system has 16 as its basis. Investigate this system. Explain how it would be possible to have 15, for example, in a single place position. Give examples to show how the system would add, subtract and multiply.

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Digital doc: Investigation — Other number systems (doc-15946)
Digital doc: Code puzzle — Who is Jørn Utzon? (doc-15947)
Digital doc: Summary Topic 16 (doc-22914)
Interactivity: Word search Topic 16 (int-2871)
Interactivity: Crossword Topic 16 (int-2872)
Interactivity: Sudoku Topic 16 (int-3891)

Exercise 16.10 Review questions

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. MC Which of the given numbers are rational?

$$\sqrt{\frac{6}{12}},\ \sqrt{0.81},\ 5,\ -3.26,\ 0.5,\ \frac{\pi}{5},\ \sqrt{\frac{3}{12}}$$

A. $\sqrt{0.81}$, 5, –3.26, 0.5 and $\sqrt{\frac{3}{12}}$

B. $\sqrt{\frac{6}{12}}$ and $\frac{\pi}{5}$

C. $\sqrt{\frac{6}{12}}$, $\sqrt{0.81}$ and $\sqrt{\frac{3}{12}}$

D. 5, –3.26 and $\sqrt{\frac{6}{12}}$

E. 5, –3.26 and 0.5

2. For each of the following, state whether the number is rational or irrational and give the reason for your answer:

a. $\sqrt{12}$

b. $\sqrt{121}$

c. $\frac{2}{9}$

d. $0.\dot{6}$

e. $\sqrt[3]{0.08}$

3. MC Which of the numbers of the given set are surds?

$$\left\{3\sqrt{2}, 5\sqrt{7}, 9\sqrt{4}, 6\sqrt{10}, 7\sqrt{12}, 12\sqrt{64}\right\}$$

A. $9\sqrt{4}$ and $12\sqrt{64}$ only **B.** $3\sqrt{2}$ and $7\sqrt{12}$ only

C. $3\sqrt{2}$, $5\sqrt{7}$ and $6\sqrt{10}$ only **D.** $3\sqrt{2}$, $5\sqrt{7}$, $6\sqrt{10}$ and $7\sqrt{12}$

E. $9\sqrt{4}$, $6\sqrt{10}$, $7\sqrt{12}$ and $12\sqrt{64}$

4. Which of $\sqrt{2m}$, $\sqrt{25m}$, $\sqrt{\frac{m}{16}}$, $\sqrt{\frac{20}{m}}$, $\sqrt[3]{m}$ and $\sqrt[3]{8m}$ are surds:

a. if $m = 4$? **b.** if $m = 8$?

5. Simplify each of the following.

a. $\sqrt{50}$ **b.** $\sqrt{180}$ **c.** $2\sqrt{32}$ **d.** $5\sqrt{80}$

6. MC The expression $\sqrt{392x^8y^7}$ may be simplified to:

A. $196x^4y^3\sqrt{2y}$ **B.** $2x^4y^3\sqrt{14y}$ **C.** $14x^4y^3\sqrt{2y}$ **D.** $14x^4y^3\sqrt{2}$ **E.** $14x^8y^7\sqrt{2}$

7. Simplify the following surds. Give the answers in the simplest form.

a. $4\sqrt{648x^7y^9}$ **b.** $-\frac{2}{5}\sqrt{\frac{25}{64}x^5y^{11}}$

8. Simplify the following, giving answers in the simplest form.

a. $7\sqrt{12} + 8\sqrt{147} - 15\sqrt{27}$ **b.** $\frac{1}{2}\sqrt{64a^3b^3} - \frac{3}{4}ab\sqrt{16ab} + \frac{1}{5ab}\sqrt{100a^5b^5}$

9. Simplify each of the following.

a. $\sqrt{3} \times \sqrt{5}$ **b.** $2\sqrt{6} \times 3\sqrt{7}$ **c.** $3\sqrt{10} \times 5\sqrt{6}$ **d.** $\left(\sqrt{5}\right)^2$

10. Simplify the following, giving answers in the simplest form.

a. $\frac{1}{5}\sqrt{675} \times \sqrt{27}$ **b.** $10\sqrt{24} \times 6\sqrt{12}$

11. Simplify the following.

a. $\frac{\sqrt{30}}{\sqrt{10}}$ **b.** $\frac{6\sqrt{45}}{3\sqrt{5}}$ **c.** $\frac{3\sqrt{20}}{12\sqrt{6}}$ **d.** $\frac{\left(\sqrt{7}\right)^2}{14}$

12. Rationalise the denominator of each of the following.

a. $\frac{2}{\sqrt{6}}$ **b.** $\frac{\sqrt{3}}{2\sqrt{6}}$ **c.** $\frac{2}{\sqrt{5} - 2}$ **d.** $\frac{\sqrt{3} - 1}{\sqrt{3} + 1}$

13. Evaluate each of the following, correct to 1 decimal place if necessary.

a. $64^{\frac{1}{3}}$ **b.** $20^{\frac{1}{2}}$ **c.** $10^{\frac{1}{3}}$ **d.** $50^{\frac{1}{2}}$

14. Evaluate each of the following, correct to 1 decimal place.

a. $20^{\frac{2}{3}}$ **b.** $2^{\frac{3}{4}}$ **c.** $(0.7)^{\frac{3}{5}}$ **d.** $\left(\frac{2}{3}\right)^{\frac{2}{3}}$

15. Write each of the following in simplest surd form.

a. $2^{\frac{1}{2}}$ **b.** $18^{\frac{1}{2}}$ **c.** $5^{\frac{3}{2}}$ **d.** $8^{\frac{4}{3}}$

16. Evaluate each of the following, without using a calculator. Show all working.

a. $\frac{16^{\frac{3}{4}} \times 81^{\frac{1}{4}}}{6 \times 16^{\frac{1}{2}}}$ **b.** $\left(125^{\frac{2}{3}} - 27^{\frac{2}{3}}\right)^{\frac{1}{2}}$

17. Evaluate each of the following, giving your answer as a fraction.

a. 4^{-1} **b.** 9^{-1} **c.** 4^{-2} **d.** 10^{-3}

18. Find the value of each of the following, correct to 3 significant figures.

a. 12^{-1} **b.** 7^{-2} **c.** $(1.25)^{-1}$ **d.** $(0.2)^{-4}$

19. Write the value of each of the following.

a. $\left(\frac{2}{3}\right)^{-1}$ **b.** $\left(\frac{7}{10}\right)^{-1}$ **c.** $\left(\frac{1}{5}\right)^{-1}$ **d.** $\left(3\frac{1}{4}\right)^{-1}$

20. MC The expression $\sqrt{250}$ may be simplified to:

A. $25\sqrt{10}$ **B.** $5\sqrt{10}$ **C.** $10\sqrt{5}$ **D.** $5\sqrt{50}$ **E.** $10\sqrt{5}$

21. MC When expressed in its simplest form, $2\sqrt{98} - 3\sqrt{72}$ is equal to:

A. $-4\sqrt{2}$ **B.** -4 **C.** $-2\sqrt{4}$ **D.** $4\sqrt{2}$ **E.** -8

22. MC When expressed in its simplest form, $\sqrt{\frac{8x^3}{32}}$ is equal to:

A. $\frac{x\sqrt{x}}{2}$ **B.** $\frac{\sqrt{x^3}}{4}$ **C.** $\frac{\sqrt{x^3}}{2}$ **D.** $\frac{x\sqrt{x}}{4}$ **E.** $2x\sqrt{x}$

23. Find the value of the following, giving your answer in fraction form.

a. $\left(\frac{2}{5}\right)^{-1}$ **b.** $\left(\frac{2}{3}\right)^{-2}$

24. Find the value of each of the following, leaving your answer in fraction form.

a. 2^{-1} **b.** 3^{-2} **c.** 4^{-3} **d.** $\left(\frac{1}{2}\right)^{-1}$

25. Evaluate the following.

a. $\log_{12} 18 + \log_{12} 8$ **b.** $\log_4 60 - \log_4 15$

c. $\log_9 9^8$ **d.** $2\log_3 6 - \log_3 4$

26. Use the logarithm laws to simplify each of the following.

a. $\log_a 16 + \log_a 3 - \log_a 2$ **b.** $\log_x x\sqrt{x}$

c. $4\log_a x - \log_a x^2$ **d.** $5\log_x\left(\frac{1}{x}\right)$

27. Solve for x in the following equations, given that $x > 0$.

a. $\log_2 x = 9$ **b.** $\log_5 x = -2$ **c.** $\log_x 25 = 2$

d. $\log_x 2^6 = 6$ **e.** $\log_3 729 = x$ **f.** $\log_7 1 = x$

28. Solve for x in the following equations.

a. $\log_5 4 + \log_5 x = \log_5 24$ **b.** $\log_3 x - \log_3 5 = \log_3 7$

29. Solve for x in the following equations.

a. $6^x = \frac{1}{36}$ **b.** $7^x = \frac{1}{\sqrt{7}}$ **c.** $2^{x+1} = 8\sqrt{2}$

30. Solve for x in the following equations, correct to 3 decimal places.

a. $2^x = 25$ **b.** $0.6^x = 7$ **c.** $9^{-x} = 0.84$

Communicating, reasoning and problem solving

31. Answer the following. Explain how you reached your answer.

a. What is the hundreds digit in 3^{3^3}?

b. What is the ones digit in 6^{704}?

c. What is the thousands digit in 9^{1000}?

32. a. Plot a graph of $y = 4^x$ by first producing a table of values. Label the y-intercept and the equation of any asymptotes.

b. Draw the line $y = x$ on the same set of axes.

c. Reflect the graph of $y = 4^x$ in the line $y = x$. This is the graph of $y = \log_4 x$. It is called the inverse graph of $y = 4^x$. Label any intercepts and the equation of any asymptotes.

d. Use graphing software to check your graphs.

33. Solve for x: $\left(\frac{6}{x}\right)^{-1} + \frac{1}{6} = x^{-1}$.

34. Simplify $\left(\left(\frac{(a^2)^{-1}}{b^{\frac{1}{2}}}\right)^{-1}\right)^{-1}$.

Answers

Topic 16 Surds, indices and logarithms

Exercise 16.2 Rational and irrational numbers

1. **a.** Q **b.** Q **c.** Q **d.** I
e. I **f.** Q **g.** Q **h.** I
i. Q **j.** Q **k.** Q **l.** Q
m. I **n.** Q **o.** I **p.** Q
q. Q **r.** I **s.** I **t.** I
u. Q **v.** I **w.** I **x.** Q
y. I

2. **a.** Q **b.** Q **c.** Q **d.** Q
e. Q **f.** I **g.** I **h.** Q
i. I **j.** Undefined **k.** I **l.** I
m. I **n.** Q **o.** Q **p.** Q
q. I **r.** I **s.** Q **t.** Q
u. I **v.** Q **w.** Q **x.** I
y. Q

3. B

4. D

5. C

6. C

7. $\frac{a}{b}$

8. E

9. A

10. $p - q$

11. Check with your teacher.

12. **a.** $m = 11, n = 3$ **b.** $m = 2, n = 3$ **c.** $m = 3, n = 2$ **d.** $m = 1, n = 2$

13. $\frac{1}{7}$ or 7^{-1}

14. **a.** $\frac{1}{9}$ **b.** $\frac{1}{6}$ **c.** $\frac{1}{14}$

15. Discuss with teacher

Exercise 16.3 Surds

1. b d f g h i l m o q s t w z

2. A

3. D

4. B

5. C

6. Any perfect square

7. $m = 4$

8. Check with your teacher.

9. Irrational

10. **a.** **i.** $4\sqrt{3}$ **ii.** $6\sqrt{2}$
b. Yes. If you don't choose the largest perfect square, then you will need to simplify again.
c. No

11. $x = 3$; integral and rational

12. $5\sqrt{3}$ cm

13. It is a surd if it cannot be simplified to a rational number

Exercise 16.4 Operations with surds

1. a. $2\sqrt{3}$ **b.** $2\sqrt{6}$ **c.** $3\sqrt{3}$ **d.** $5\sqrt{5}$
e. $3\sqrt{6}$ **f.** $4\sqrt{7}$ **g.** $2\sqrt{17}$ **h.** $6\sqrt{5}$
i. $2\sqrt{22}$ **j.** $9\sqrt{2}$ **k.** $7\sqrt{5}$ **l.** $8\sqrt{7}$

2. a. $4\sqrt{2}$ **b.** $24\sqrt{10}$ **c.** $36\sqrt{5}$ **d.** $21\sqrt{6}$
e. $-30\sqrt{3}$ **f.** $-28\sqrt{5}$ **g.** $64\sqrt{3}$ **h.** $2\sqrt{2}$
i. $\sqrt{2}$ **j.** $2\sqrt{3}$ **k.** $\frac{1}{3}\sqrt{15}$ **l.** $\frac{3}{2}\sqrt{7}$

3. a. $4a$ **b.** $6a\sqrt{2}$ **c.** $3a\sqrt{10b}$ **d.** $13a^2\sqrt{2}$
e. $13ab\sqrt{2ab}$ **f.** $2ab^2\sqrt{17ab}$ **g.** $5x^3y^2\sqrt{5}$ **h.** $20xy\sqrt{5x}$
i. $54c^3d^2\sqrt{2cd}$ **j.** $18c^3d^4\sqrt{5cd}$ **k.** $\sqrt{22ef}$ **l.** $7e^5f^5\sqrt{2ef}$

4. a. $7\sqrt{5}$ **b.** $8\sqrt{3}$ **c.** $15\sqrt{5} + 5\sqrt{3}$ **d.** $4\sqrt{11}$
e. $13\sqrt{2}$ **f.** $-3\sqrt{6}$ **g.** $17\sqrt{3} - 18\sqrt{7}$ **h.** $8\sqrt{x} + 3\sqrt{y}$

5. a. $10(\sqrt{2} - \sqrt{3})$ **b.** $5(\sqrt{5} + \sqrt{6})$ **c.** $7\sqrt{3}$ **d.** $4\sqrt{5}$
e. $14\sqrt{3} + 3\sqrt{2}$ **f.** $3\sqrt{6} + 6\sqrt{3}$ **g.** $15\sqrt{10} - 10\sqrt{15} + 10$ **h.** $-8\sqrt{11} + 22$
i. $12\sqrt{30} - 16\sqrt{15}$ **j.** $12\sqrt{ab} + 7\sqrt{3ab}$ **k.** $\frac{7}{2}\sqrt{2} + 2\sqrt{3}$ **l.** $15\sqrt{2}$

6. a. $31\sqrt{a} - 6\sqrt{2a}$ **b.** $52\sqrt{a} - 29\sqrt{3a}$ **c.** $6\sqrt{6ab}$ **d.** $32a + 2\sqrt{6a} + 8a\sqrt{2}$
e. $a\sqrt{2a}$ **f.** $\sqrt{a} + 2\sqrt{2a}$ **g.** $3a\sqrt{a} + a^2\sqrt{3a}$ **h.** $(a^2 + a)\sqrt{ab}$
i. $4ab\sqrt{ab} + 3a^2b\sqrt{b}$ **j.** $3\sqrt{ab}(2a + 1)$ **k.** $-6ab\sqrt{2a} + 4a^2b^3\sqrt{3a}$ **l.** $-2a\sqrt{b}$

7. a. $\sqrt{14}$ **b.** $\sqrt{42}$ **c.** $4\sqrt{3}$ **d.** 10
e. $3\sqrt{7}$ **f.** 27 **g.** $10\sqrt{33}$ **h.** $180\sqrt{5}$
i. 120 **j.** $120\sqrt{3}$ **k.** $2\sqrt{6}$ **l.** $2\frac{2}{3}$
m. $\frac{2}{5}\sqrt{6}$ **n.** $x^2y\sqrt{y}$ **o.** $3a^4b^2\sqrt{2ab}$ **p.** $6a^5b^2\sqrt{2b}$
q. $3x^2y^2\sqrt{10xy}$ **r.** $\frac{9}{2}a^2b^4\sqrt{5ab}$

8. a. 2 **b.** 5 **c.** 12 **d.** 15
e. 18 **f.** 80 **g.** 28 **h.** 200

9. a. $\sqrt{5}$ **b.** 2 **c.** $\sqrt{6}$ **d.** 4
e. $\frac{\sqrt{3}}{4}$ **f.** $\frac{\sqrt{5}}{2}$ **g.** $2\sqrt{3}$ **h.** 1
i. $1\frac{4}{5}$ **j.** $2\sqrt{17}$ **k.** $\frac{x}{y}$ **l.** $\frac{\sqrt{2}}{x^3y^4}$
m. $2xy\sqrt{3y}$ **n.** $\frac{4\sqrt{a}}{3}$

10. a. $\frac{5\sqrt{2}}{2}$ **b.** $\frac{7\sqrt{3}}{3}$ **c.** $\frac{4\sqrt{11}}{11}$ **d.** $\frac{4\sqrt{6}}{3}$
e. $\frac{2\sqrt{21}}{7}$ **f.** $\frac{\sqrt{10}}{2}$ **g.** $\frac{2\sqrt{15}}{5}$ **h.** $\frac{3\sqrt{35}}{5}$
i. $\frac{5\sqrt{6}}{6}$ **j.** $\frac{4\sqrt{15}}{15}$ **k.** $\frac{5\sqrt{7}}{14}$ **l.** $\frac{8\sqrt{15}}{15}$
m. $\frac{8\sqrt{21}}{49}$ **n.** $\frac{8\sqrt{105}}{7}$ **o.** $\frac{\sqrt{10}}{3}$

11. a. $\sqrt{2} + 2$ **b.** $\frac{3\sqrt{10} - 2\sqrt{33}}{6}$ **c.** $\frac{12\sqrt{5} - 5\sqrt{6}}{10}$ **d.** $\frac{9\sqrt{10}}{5}$
e. $\frac{3\sqrt{10} + 6\sqrt{14}}{4}$ **f.** $\frac{5\sqrt{6}}{3}$ **g.** $\frac{3\sqrt{22} - 4\sqrt{10}}{6}$ **h.** $\frac{\sqrt{21} - \sqrt{15}}{3}$
i. $\frac{14 - 5\sqrt{2}}{6}$ **j.** $\frac{12 - \sqrt{10}}{16}$ **k.** $\frac{6\sqrt{15} - 25}{70}$ **l.** $\frac{\sqrt{30} + 7\sqrt{2}}{20}$

12. a. $\sqrt{5} - 2$ **b.** $\frac{2\sqrt{2} + \sqrt{5}}{3}$ **c.** $\frac{8\sqrt{11} + 4\sqrt{13}}{31}$ **d.** $\frac{15\sqrt{15} - 20\sqrt{6}}{13}$
e. $12\sqrt{2} - 17$ **f.** $\frac{19 - 4\sqrt{21}}{5}$ **g.** $\frac{\sqrt{15} - \sqrt{3} - \sqrt{5} + 1}{4}$ **h.** $\frac{-6 + 6\sqrt{2} + \sqrt{10} - 2\sqrt{5}}{2}$
i. $\frac{4\sqrt{10} + \sqrt{15} - 4\sqrt{6} - 3}{29}$

13. $\frac{9\sqrt{x} + 6x}{36x - 16x^2}$

14. a. Check with your teacher.

b. i. $\sqrt{5} + \sqrt{3}$ **ii.** $\sqrt{5} - \sqrt{3}$ **iii.** $\sqrt{3} + 2$

15. $\frac{2}{7}$

16. a. $x = 16$ **b.** $x = 1$

17. Check with your teacher.

18. One reason to rationalise the denominator would be if you wanted to add or subtract two surds with irrational denominators.

Exercise 16.5 Fractional indices

1. a. 4 **b.** 5 **c.** 9

d. 2 **e.** 3 **f.** 5

2. a. 3 **b.** 2 **c.** 1.4

d. 2.2 **e.** 1.5 **f.** 1.3

3. a. 2.5 **b.** 12.9 **c.** 13.6

d. 0.7 **e.** 0.8 **f.** 0.9

4. a. $\sqrt{7}$ **b.** $2\sqrt{3}$ **c.** $6\sqrt{2}$

d. $4\sqrt{2}$ **e.** $3\sqrt{3}$ **f.** $100\sqrt{10}$

5. a. $5^{\frac{1}{2}}$ **b.** $10^{\frac{1}{2}}$ **c.** $x^{\frac{1}{2}}$

d. $m^{\frac{3}{2}}$ **e.** $2t^{\frac{1}{2}}$ **f.** $6^{\frac{1}{3}}$

6. a. $4^{\frac{4}{5}}$ **b.** $2^{\frac{1}{2}}$ **c.** $a^{\frac{5}{6}}$

d. $x^{\frac{23}{20}}$ **e.** $10m^{\frac{8}{15}}$ **f.** $2b^{\frac{5}{7}}$

g. $-4y^{\frac{20}{9}}$ **h.** $0.02a^{\frac{9}{8}}$ **i.** $5x^{\frac{7}{2}}$

7. a. $ab^{\frac{3}{2}}$ **b.** $x^{\frac{4}{5}}y^{\frac{5}{9}}$ **c.** $6a^{\frac{8}{5}}b^{\frac{17}{15}}$

d. $2m^{\frac{19}{28}}n^{\frac{2}{5}}$ **e.** $x^{\frac{19}{6}}y^{\frac{5}{6}}z^{\frac{5}{6}}$ **f.** $8a^{\frac{2}{5}}b^{\frac{9}{8}}c$

8. a. $3^{\frac{1}{6}}$ **b.** $5^{\frac{5}{12}}$ **c.** $12^{\frac{1}{2}}$

d. $a^{\frac{3}{7}}$ **e.** $x^{\frac{5}{4}}$ **f.** $m^{\frac{11}{45}}$

g. $\frac{1}{2}x^{\frac{3}{20}}$ **h.** $\frac{1}{3}n^{\frac{2}{3}}$ **i.** $\frac{5}{4}b^{\frac{7}{20}}$

9. a. $x^{\frac{5}{3}}y^{\frac{7}{5}}$ **b.** $a^{\frac{7}{45}}b^{\frac{4}{15}}$ **c.** $\frac{1}{3}m^{\frac{3}{8}}n^{\frac{11}{56}}$

d. $2x^{\frac{2}{15}}y^{\frac{3}{4}}$ **e.** $\frac{1}{4}a^{\frac{11}{20}}b^{\frac{7}{20}}$ **f.** $\frac{1}{7}p^{\frac{5}{24}}q^{\frac{1}{12}}$

10. a. $2^{\frac{9}{20}}$ **b.** $5^{\frac{1}{6}}$ **c.** $7^{\frac{6}{5}}$

d. $a^{\frac{3}{10}}$ **e.** $m^{\frac{1}{6}}$ **f.** $2^{\frac{1}{3}}b^{\frac{1}{6}}$

g. $4p^{\frac{2}{5}}$ **h.** $x^{\frac{m}{p}}$ **i.** $3^{\frac{b}{c}}m^{\frac{a}{c}}$

11. a. $a^{\frac{1}{4}}b^{\frac{1}{6}}$ **b.** $a^3b^{\frac{3}{4}}$ **c.** $x^{\frac{6}{5}}y^{\frac{7}{4}}$

d. $3^{\frac{1}{3}}a^{\frac{1}{9}}b^{\frac{1}{5}}c^{\frac{1}{4}}$ **e.** $x^{\frac{1}{4}}y^{\frac{1}{3}}z^{\frac{1}{5}}$ **f.** $\frac{a^{\frac{1}{2}}}{b^{\frac{2}{3}}}$

g. $\frac{m^{\frac{8}{5}}}{n^{\frac{7}{4}}}$ **h.** $\frac{b^{\frac{2}{5}}}{c^{\frac{8}{27}}}$ **i.** $\frac{2^{\frac{1}{2}}x^{\frac{7}{2}}}{y^{\frac{3}{8}}}$

12. C, D

13. a. a^4 **b.** b^3 **c.** m^4

d. $4x^2$ **e.** $2y^3$ **f.** $2x^2y^3$

g. $3m^3n^5$ **h.** $2pq^2$ **i.** $6a^2b^6$

14. a. 0.32 m/s **b.** 16640 L/s

c. 59904000 L/hr (16640 × 60 × 60)

d. The hydraulic radius is the measure of a channel flow efficiency. The roughness coefficient is the resistance of the bed of a channel to the flow of water in it.

15. $x = 1$

16. a. $x^{\frac{1}{2}} + y^{\frac{1}{2}} - z^{\frac{1}{2}}$ **b.** $t^{\frac{1}{10}}$

17. $m - n^2$

18. Discuss with your teacher.

Exercise 16.6 Negative indices

1. a. $\frac{1}{5} = 0.2$ **b.** $\frac{1}{3} = 0.\dot{3}$ **c.** $\frac{1}{8} = 0.125$ **d.** $\frac{1}{10} = 0.1$

e. $\frac{1}{8} = 0.125$ **f.** $\frac{1}{9} = 0.\dot{1}$ **g.** $\frac{1}{25} = 0.04$ **h.** $\frac{1}{10000} = 0.0001$

2. a. 0.167 **b.** 0.143 **c.** 0.0278 **d.** 0.00137

e. 0.00463 **f.** 0.00444 **g.** 0.003 91 **h.** 0.00160

3. a. 0.40 **b.** 2.5 **c.** 0.44 **d.** 4.0

e. 0.11 **f.** 0.000079 **g.** 11 **h.** 4100

4. a. –0.33 **b.** –0.20 **c.** 0.25 **d.** 0.063

e. –0.67 **f.** –0.45 **g.** –1.7 **h.** 1.4

5. a. $\frac{5}{4}$ or $1\frac{1}{4}$ **b.** $\frac{10}{3}$ or $3\frac{1}{3}$ **c.** $\frac{8}{7}$ or $1\frac{1}{7}$ **d.** $\frac{20}{13}$ or $1\frac{7}{13}$

e. 2 **f.** 4 **g.** 8 **h.** 10

i. $\frac{2}{3}$ **j.** $\frac{4}{9}$ **k.** $\frac{10}{11}$ **l.** $\frac{2}{11}$

6. a. 4 **b.** $6\frac{1}{4}$ **c.** $3\frac{3}{8}$ **d.** 16

e. $\frac{4}{9}$ **f.** $\frac{16}{81}$ **g.** $\frac{27}{64}$ **h.** $\frac{125}{1331}$

7. a. $-\frac{3}{2}$ **b.** $-\frac{5}{3}$ **c.** –4 **d.** –10

e. $\frac{9}{4}$ **f.** 25 **g.** $-\frac{2}{3}$ **h.** $\frac{16}{121}$

8. $\frac{3}{10}$ **9.** $\frac{b}{a}$

10. a. $y \to \infty$ **b.** $y \to -\infty$

11. As the value of n increases, the value of 2^{-n} gets closer to 0.

12. $x = -2$, $y = -3$

13. x^2

14. Division can be used in the following way to explain negative indices.

$$\frac{2^3}{2^5} = \frac{\cancel{2} \times \cancel{2} \times \cancel{2}}{\cancel{2} \times \cancel{2} \times \cancel{2} \times 2 \times 2} = \frac{1}{2 \times 2} = \frac{1}{2^2}$$

But $\frac{2^3}{2^5} = 2^{-2}$ using the second law of indices.

Therefore, $\frac{1}{2^2} = 2^{-2}$.

Exercise 16.7 Logarithms

1. a. $\log_4 16 = 2$ **b.** $\log_2 32 = 5$ **c.** $\log_3 81 = 4$ **d.** $\log_6 36 = 2$

e. $\log_{10} 1000 = 3$ **f.** $\log_5 25 = 2$ **g.** $\log_4 x = 3$ **h.** $\log_5 125 = x$

i. $\log_7 49 = x$ **j.** $\log_p 16 = 4$ **k.** $\log_9 3 = \frac{1}{2}$ **l.** $\log_{10} 0.1 = -1$

m. $\log_8 2 = \frac{1}{3}$ **n.** $\log_2 \frac{1}{2} = -1$ **o.** $\log_a 1 = 0$ **p.** $\log_4 8 = \frac{3}{2}$

2. D

3. a. $2^4 = 16$ **b.** $3^3 = 27$ **c.** $10^6 = 1000000$ **d.** $5^3 = 125$

e. $16^{\frac{1}{2}} = 4$ **f.** $4^x = 64$ **g.** $49^{\frac{1}{2}} = 7$ **h.** $3^5 = x$

i. $81^{\frac{1}{2}} = 9$ **j.** $10^{22} = 0.01$ **k.** $8^1 = 8$ **l.** $64^{\frac{1}{3}} = 4$

4. B

5. a. 4 **b.** 2 **c.** 2 **d.** 5

e. 5 **f.** 7 **g.** 0 **h.** $\frac{1}{2}$

i. –1 **j.** 1 **k.** –2 **l.** $\frac{1}{3}$

6. a. 0 **b.** 1 **c.** 2

d. 3 **e.** 4 **f.** 5

7. a. 0 and 1 **b.** 3 and 4 **c.** 1 and 2

d. 4 and 5 **e.** 2 and 3 **f.** 4 and 5

8. a. 6.1 **b.** 6.3 **c.** 8.2

9. a. $\log_{10} g = k$ implies that $g = 10^k$, so $g^2 = (10^k)^2 = 10^{2k}$. Therefore, $\log_{10} g^2 = 2k$.

b. $\log_x y = 2$ implies that $y = x^2$, so $x = y^{\frac{1}{2}}$; therefore, $\log_y x = \frac{1}{2}$.

c. The equivalent exponential statement is $x = 4^y$, and we know that 4^y is greater than zero for all values of y. Therefore, x is a positive number.

10. a. 6 **b.** −4 **c.** −5

11. a. 3 **b.** 7 **c.** $\frac{1}{8}$

12. x

13. An index is a power, an exponent and a logarithm.

Exercise 16.8 Logarithm laws

1. a. 1.69897 **b.** 1.39794 **c.** 0.69897 **d.** 0.30103

2. Teacher to check.

3. a. 1 **b.** 3 **c.** 2

d. 3 **e.** 4 **f.** 1

4. a. 2 **b.** 3 **c.** 1

d. 4 **e.** 3 **f.** 5

5. a. 2 **b.** $\frac{1}{2}$ **c.** 1 **d.** 3

6. 3

7. a. 2 **b.** 4 **c.** 3 **d.** 3

8. a. 1 **b.** 0 **c.** –1 **d.** 5

e. –2 **f.** 1 **g.** 0 **h.** –2

i. $-\frac{1}{2}$ **j.** $\frac{1}{2}$ **k.** $-\frac{1}{2}$ **l.** $\frac{7}{2}$

9. a. $\log_a 40$ **b.** $\log_a 18$ **c.** $\log_x 48$ **d.** $\log_x 4$

e. $\log_a x$ **f.** 1 **g.** –1 **h.** 7

i. $\frac{1}{2}$ **j.** $\frac{3}{2}$ **k.** –6 **l.** $-\frac{1}{3}$

10. a. B **b.** B, D **c.** A, B **d.** C, E

11. a. $\log_2 80$ **b.** $\log_3 105$ **c.** $\log_{10} 100 = 2$ **d.** $\log_6 56$

e. $\log_2 4 = 2$ **f.** $\log_3 3 = 1$ **g.** $\log_5 12.5$ **h.** $\log_2 3$

i. $\log_4 5$ **j.** $\log_{10} \frac{1}{4}$ **k.** $\log_3 4$ **l.** $\log_2 3$

m. $\log_3 20$ **n.** $\log_4 2 = \frac{1}{2}$

12. a. C **b.** B **c.** A

13. a. 12 (Evaluate each logarithm separately and then find the product.)

b. 4 (First simplify the numerator by expressing 81 as a power of 3.)

c. 7 (Let $y = 5^{\log 5^7}$ and write an equivalent statement in logarithmic form.)

14. $7 - 3\log_2 3$ **15.** 1 **16.** $x = 3a, 5a$

17. If you write out the index laws that you can remember, then try to remember which index law is linked to which log law.

18. $\frac{3}{2}$

Exercise 16.9 Solving equations

1. a. 25 **b.** 81 **c.** $\frac{1}{8}$ **d.** $\frac{1}{16}$
e. 100, –100 **f.** 16 **g.** 26 **h.** 127
i. 2 **j.** 0 **k.** $-\frac{1}{32}$ **l.** $-\frac{1}{9}$
m. –624 **n.** –2.5

2. a. 3 **b.** 2 **c.** 125 **d.** 625
e. 2 **f.** 8 **g.** 6 **h.** 4

3. a. 3 **b.** 2 **c.** –1 **d.** –2
e. $\frac{1}{2}$ **f.** $\frac{1}{3}$ **g.** 0 **h.** 0
i. –1 **j.** –2

4. a. 5 **b.** 6 **c.** 10 **d.** 8
e. 4 **f.** 2 **g.** 9 **h.** $\frac{2}{5}$
i. 500 **j.** 128 **k.** 5 **l.** 6
m. 1 **n.** 2

5. a. B **b.** A **c.** D **d.** B

6. a. 7 **b.** 2 **c.** –2 **d.** 0
e. 4 **f.** $\frac{1}{2}$ **g.** $\frac{1}{2}$ **h.** $\frac{3}{2}$
i. $-\frac{1}{2}$ **j.** $\frac{3}{2}$ **k.** $\frac{3}{4}$ **l.** $-\frac{5}{2}$
m. $\frac{5}{2}$ **n.** $-\frac{9}{2}$ **o.** $-\frac{11}{4}$

7. a. 3.459 **b.** –0.737 **c.** 2.727 **d.** 0.483
e. 1.292 **f.** –3.080 **g.** –1.756 **h.** 0.262
i. 0.827 **j.** 0.579 **k.** –0.423 **l.** 2.138

8. a. 120 **b.** 130 **c.** 0.001 **d.** 3 dB are added.
e. 10 dB are added. **f.** 100

9. a. i. 1.1 **ii.** 1.3 **iii.** 1.418 **iv.** 1.77
v. 2.43 **vi.** 3.1
b. No; see answers to **9a i** and **ii** above.
c. i. 22 387 211 kJ **ii.** 707 945 784 kJ **iii.** 22 387 211 385 kJ.
d. The energy is increased by a factor of 31.62.
e. It releases 31.62^3 times more energy.

10. a. $x = 0.7712$ **b.** $x = 1.2966$

11. $x = 7$ **12.** $x = 1, 3$

13. Using log tables to round answers did mean that the calculations were only accurate to the number of decimal places in the tables.

14. The remaining steps of the solution are $\frac{x(x+1)}{6} = 1$, $x^2 + x - 6 = 0$, $x = -3$ or $x = 2$.

Investigation | Rich task

1. a. 100_2 **b.** 1101_2 **c.** 10000_2

2. a. 10_2 **b.** 101_2 **c.** 110_2

3. a. 11_2 **b.** 1001_2 **c.** 10101_2

4. Answers will vary; check with your teacher.

5. Answers will vary; check with your teacher. The numbers 10, 11, 12, 13, 14 and 15 are allocated the letters A, B, C, D, E and F respectively.

Exercise 16.10 Review questions

1. A

2. a. Irrational, since equal to non-recurring and non-terminating decimal
b. Rational, since can be expressed as a whole number
c. Rational, since given in a rational form
d. Rational, since it is a recurring decimal
e. Irrational, since equal to non-recurring and non-terminating decimal

3. D

4. a. $\sqrt{2m}, \sqrt{\frac{20}{m}}, \sqrt[3]{m}, \sqrt[3]{8m}$ **b.** $\sqrt{25m}, \sqrt{\frac{m}{16}}, \sqrt{\frac{20}{m}}$

5. a. $5\sqrt{2}$ **b.** $6\sqrt{5}$ **c.** $8\sqrt{2}$ **d.** $20\sqrt{5}$

6. C

7. a. $72x^3y^4\sqrt{2xy}$ **b.** $-\frac{1}{4}x^2y^5\sqrt{xy}$

8. a. $25\sqrt{3}$ **b.** $3ab\sqrt{ab}$

9. a. $\sqrt{15}$ **b.** $6\sqrt{42}$ **c.** $30\sqrt{15}$ **d.** 5

10. a. 27 **b.** $720\sqrt{2}$

11. a. $\sqrt{3}$ **b.** 6 **c.** $\frac{\sqrt{10}}{4\sqrt{3}}$ or $\frac{\sqrt{30}}{12}$ **d.** $\frac{1}{2}$

12. a. $\frac{\sqrt{6}}{3}$ **b.** $\frac{\sqrt{2}}{4}$ **c.** $2\sqrt{5} + 4$ **d.** $2 - \sqrt{3}$

13. a. 4 **b.** 4.5 **c.** 2.2 **d.** 2.7

14. a. 7.4 **b.** 1.7 **c.** 0.8 **d.** 0.8

15. a. $\sqrt{2}$ **b.** $3\sqrt{2}$ **c.** $5\sqrt{5}$ **d.** 16

16. a. 1 **b.** 4

17. a. $\frac{1}{4}$ **b.** $\frac{1}{9}$ **c.** $\frac{1}{16}$ **d.** $\frac{1}{1000}$

18. a. 0.0833 **b.** 0.0204 **c.** 0.800 **d.** 625

19. a. $1\frac{1}{2}$ **b.** $1\frac{3}{7}$ **c.** 5 **d.** $\frac{4}{13}$

20. B **21.** A **22.** A

23. a. $2\frac{1}{2}$ **b.** $2\frac{1}{4}$

24. a. $\frac{1}{2}$ **b.** $\frac{1}{9}$ **c.** $\frac{1}{64}$ **d.** $\frac{2}{1}$

25. a. 2 **b.** 1 **c.** 8 **d.** 2

26. a. $\log_a 24$ **b.** $\frac{3}{2}$ **c.** $\log_a x^2$ or $2\log_a x$ **d.** –5

27. a. 512 **b.** $\frac{1}{25}$ **c.** 5 **d.** 2

e. 6 **f.** 0

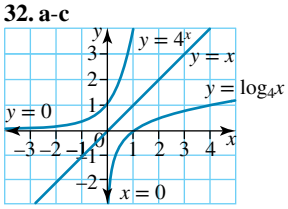
28. a. 6 **b.** 35

29. a. –2 **b.** $-\frac{1}{2}$ **c.** $\frac{5}{2}$

30. a. 4.644 **b.** –3.809 **c.** 0.079

31. a. 9 **b.** 6 **c.** 1

32. a-c



33. $x = 2, -3$

34. $\frac{1}{a^2 b^{\frac{1}{2}}}$

TOPIC 17
Polynomials

[Stage 5.3]

17.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the concepts covered in this topic.

17.1.1 Why learn this?

Just as number is learned in stages, so too are graphs. You have been building your knowledge of graphs and functions over time. First you encountered linear functions, then quadratic and hyperbolic functions. Polynomials are higher-order functions represented by smooth and continuous curves. They can be used to model situations in many fields, such as business, science, architecture, design and engineering.

DISCUSSION

Mary Somerville grew up in a time when it was not thought proper to fully educate women. Why do you think it took so long for this idea to be challenged? Is education for males and females equal in different countries around the world?

LEARNING SEQUENCE

17.1 Overview
17.2 [Stage 5.3] Polynomials
17.3 [Stage 5.3] Adding, subtracting and multiplying polynomials
17.4 [Stage 5.3] Long division of polynomials
17.5 [Stage 5.3] Polynomial values
17.6 [Stage 5.3] The remainder and factor theorems
17.7 [Stage 5.3] Factorising polynomials
17.8 [Stage 5.3] Solving polynomial equations
17.9 Review

LEARNING OUTCOMES

A student:

- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3-1WM
- generalises mathematical ideas and techniques to analyse and solve problems efficiently MA5.3-2WM
- uses deductive reasoning in presenting arguments and formal proofs MA5.3-3WM
- recognises, describes and sketches polynomials, and applies the factor and remainder theorems to solve problems MA5.3-10NA

CONTENT DESCRIPTIONS

Students:

Investigate the concept of a polynomial and apply the factor and remainder theorems to solve problems (ACMNA266)

Apply an understanding of polynomials to sketch a range of curves and describe the features of these curves from their equations (ACMNA268)

Source: NSW Syllabus for the Australian curriculum

learnon RESOURCES — ONLINE ONLY

eLesson: The story of mathematics — Mary Somerville (eles-2020)

Note: Your teacher may now set you a pre-test to determine how familiar you are with the content in this topic.

17.2 Polynomials [Stage 5.3]

17.2.1 Definition of a polynomial

- A **polynomial** *in* x, sometimes denoted $P(x)$, is an expression containing only non-negative integer powers of x.
- The **degree** of a polynomial in x is the highest power of x in the expression. For example:

$3x + 1$	is a polynomial of degree 1, or linear polynomial.
$x^2 + 4x - 7$	is a polynomial of degree 2, or quadratic polynomial.
$-5x^3 + \frac{x}{2}$	is a polynomial of degree 3, or cubic polynomial.
10	is a polynomial of degree 0 (think of 10 as $10x^0$).

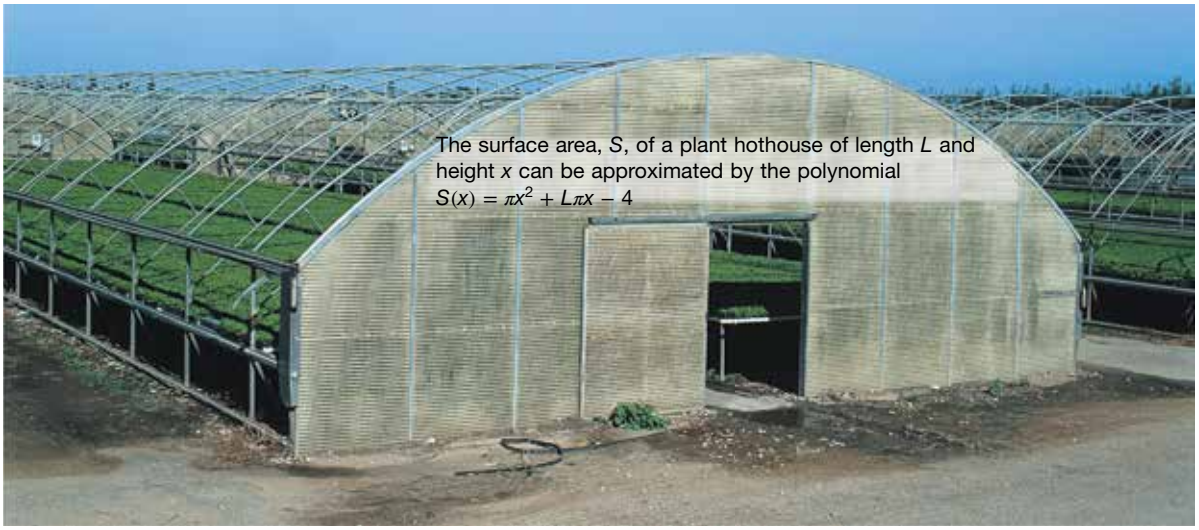
- Expressions containing a term similar to any of the following terms are not polynomials:

$$\frac{1}{x}, \quad x^{-2}, \quad \sqrt{x}, \quad 2^x, \quad \sin x.$$

For example, the following are not polynomials.

$$3x^2 - 4x + \frac{2}{x} \qquad -5x^4 + x^3 - 2\sqrt{x} \qquad x^2 + \sin x + 1$$

- In the expression $6x^3 + 13x^2 - x + 1$:
  x is the *variable*.
  6 is the *coefficient* of x^3.
  13 is the *coefficient* of x^2.
  -1 is the *coefficient* of x.
  $6x^3$, $13x^2$, $-x$ and $+1$ are all *terms*.
  The *constant* term is $+1$.
  The *degree* of the polynomial is 3.
- The leading term is $6x^3$ because it is the term that contains the highest power of x.
- The leading coefficient is 6.
- Any polynomial with a leading coefficient of 1 is called *monic*.
- An example of where polynomials are useful is shown below.



The surface area, S, of a plant hothouse of length L and height x can be approximated by the polynomial $S(x) = \pi x^2 + L\pi x - 4$

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Interactivity: Degrees of polynomials (int-6203)

Exercise 17.2 Polynomials

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a, b, f, 2–5, 7, 8, 11	Questions: 1c–e, g, i, 2–4, 6, 8, 10, 11	Questions: 1a, c, f–i, 2–4, 6–13

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. State the degree of each of the following polynomials.

a. $x^3 - 9x^2 + 19x + 7$ **b.** $65 + 2x^7$ **c.** $3x^2 - 8 + 2x$

d. $x^6 - 3x^5 + 2x^4 + 6x + 1$ **e.** $y^8 + 7y^3 - 5$ **f.** $\frac{1}{2}u^5 - \frac{u^4}{3} + 2u - 6$

g. $18 - \frac{e^5}{6}$ **h.** $2g - 3$ **i.** $1.5f^6 - 800f$

2. State the variable for each polynomial in question **1**.

3. Which polynomials in question **1** are:

a. linear **b.** quadratic **c.** cubic **d.** monic?

4. State whether each of the following is a polynomial (P) or not (N).

a. $7x + 6x^2 + \frac{5}{x}$ **b.** $33 - 4p$ **c.** $\frac{x^2}{9} + x$

d. $3x^4 - 2x^3 - 3\sqrt{x} - 4$ **e.** $k^{-2} + k - 3k^3 + 7$ **f.** $5r - r^9 + \frac{1}{3}$

g. $\frac{4c^6 - 3c^3 + 1}{2}$ **h.** $2^x - 8x + 1$ **i.** $\sin\ x + x^2$

5. Consider the polynomial $P(x) = -2x^3 + 4x^2 + 3x + 5$.

a. What is the degree of the polynomial?

b. What is the variable?

c. What is the coefficient of x^2?
d. What is the value of the constant term?
e. Which term has a coefficient of 3?
f. Which is the leading term?

6. Consider the polynomial $P(w) = 6w^7 + 7w^6 - 9$.
a. What is the degree of the polynomial?
b. What is the variable?
c. What is the coefficient of w^6?
d. What is the coefficient of w?
e. What is the value of the constant term?
f. Which term has a coefficient of 6?

7. Consider the polynomial $f(x) = 4 - x^2 + x^4$.
a. What is the degree of the polynomial?
b. What is the coefficient of x^4?
c. What is the leading term?
d. What is the leading coefficient?

8. A sports scientist determines the following equation for the velocity of a breaststroke swimmer during one complete stroke:

$$v(t) = 63.876t^6 - 247.65t^5 + 360.39t^4 - 219.41t^3 + 53.816t^2 + 0.4746t$$

a. What is the degree of the polynomial?
b. What is the variable?
c. How many terms are there?
d. Use a graphics calculator or graphing software to draw the graph of this polynomial.
e. Match what happens during one complete stroke with points on the graph.

Communicating, reasoning and problem solving

9. The distance travelled by a body after t seconds is given by $d(t) = t^3 + 2t^2 - 4t + 5$. Using a graphing calculator or suitable computer software, draw a graph of the above motion for $0 \leq t \leq 3$. Use the graph to help you answer the following:
a. What information does the constant term give?
b. What is the position of the body after 1 second?
c. Describe in words the motion in the first 2 seconds.

10. Write the following polynomials as simply as possible, arranging terms in descending powers of x.
a. $P(x) = 7x + 2x^2 - 8x + 15 + 4x^3 - 9x + 3$
b. $P(x) = x^2 - 8x^3 + 3x^4 - 2x^2 + 7x + 5x^3 - 7$
c. $P(x) = x^3 - 5x^2 - 11x - 1 + 4x^3 - 2x + x^2 - 5$

11. If $x^2 + 2x - 1 \equiv (x - 1)^2 + a(x + 1) + b$, find the values of a and b.

12. If $x^3 + 9x^2 + 12x + 7 \equiv x^3 + (ax + b)^2 + 3$, find the values of a and b.

13. How can you tell what the degree of a polynomial is?

17.3 Adding, subtracting and multiplying polynomials [Stage 5.3]

- To add or subtract polynomials, we simply add or subtract any like terms in the expressions.

WORKED EXAMPLE 1

Simplify each of the following.

a $(5x^3 + 3x^2 - 2x - 1) + (x^4 + 5x^2 - 4)$

b $(5x^3 + 3x^2 - 2x - 1) - (x^4 + 5x^2 - 4)$

THINK	WRITE
a 1 Write the expression.	**a** $(5x^3 + 3x^2 - 2x - 1) + (x^4 + 5x^2 - 4)$
2 Remove any grouping symbols, watching any signs.	$= 5x^3 + 3x^2 - 2x - 1 + x^4 + 5x^2 - 4$
3 Identify any like terms and change the order.	$= x^4 + 5x^3 + 3x^2 + 5x^2 - 2x - 1 - 4$
4 Simplify by collecting like terms.	$= x^4 + 5x^3 + 8x^2 - 2x - 5$
b 1 Write the expression.	**b** $(5x^3 + 3x^2 - 2x - 1) - (x^4 + 5x^2 - 4)$
2 Remove any grouping symbols, watching any signs.	$= 5x^3 + 3x^2 - 2x - 1 - x^4 - 5x^2 + 4$
3 Identify any like terms and change the order.	$= -x^4 + 5x^3 + 3x^2 - 5x^2 - 2x - 1 + 4$
4 Simplify by collecting like terms.	$= -x^4 + 5x^3 - 2x^2 - 2x + 3$

- If we expand linear factors, for example $(x + 1)(x + 2)(x - 7)$, we may also get a polynomial, as the following worked example shows.

WORKED EXAMPLE 2

Expand and simplify:

a $x(x + 2)(x - 3)$

b $(x - 1)(x + 5)(x + 2)$.

THINK	WRITE
a 1 Write the expression.	**a** $x(x + 2)(x - 3)$
2 Expand the last two linear factors.	$= x(x^2 - 3x + 2x - 6)$ $= x(x^2 - x - 6)$
3 Multiply the expression in the grouping symbols by x.	$= x^3 - x^2 - 6x$
b 1 Write the expression.	**b** $(x - 1)(x + 5)(x + 2)$
2 Expand the last two linear factors.	$= (x - 1)(x^2 + 2x + 5x + 10)$
3 Multiply the expression in the second pair of grouping symbols by x and then by -1.	$= (x - 1)(x^2 + 7x + 10)$ $= x^3 + 7x^2 + 10x - x^2 - 7x - 10$
4 Collect like terms.	$= x^3 + 6x^2 + 3x - 10$

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Interactivity: Adding and subtracting polynomials (int-6204)

Digital doc: SkillSHEET Expanding the product of two linear factors (doc-5366)

Exercise 17.3 Adding, subtracting and multiplying polynomials

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–c, 2a–c, 3a–c, 4, 5a–c, 6, 7, 9, 12	Questions: 1b–d, 2b–d, 3b–d, 4, 5b, d, f, h, i, 6, 8, 10, 12	Questions: 1c–e, 2c–e, 3–5, 6b, d, f, 7–14

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE1a Simplify each of the following.

a. $(x^4 + x^3 - x^2 + 4) + (x^3 - 14)$

b. $(x^6 + x^4 - 3x^3 + 6x^2) + (x^4 + 3x^2 + 5)$

c. $(x^3 + x^2 + 2x - 4) + (4x^3 - 6x^2 + 5x - 9)$

d. $(2x^4 - 3x^3 + 7x^2 + 9) + (6x^3 + 5x^2 - 4x + 5)$

e. $(15x^4 - 3x^2 + 4x - 7) + (x^5 - 2x^4 + 3x^2 - 4x - 3)$

2. WE1b Simplify each of the following.

a. $(x^4 + x^3 + 4x^2 + 5x + 5) - (x^3 + 2x^2 + 3x + 1)$

b. $(x^6 + x^3 + 1) - (x^5 - x^2 - 1)$

c. $(5x^7 + 6x^5 - 4x^3 + 8x^2 + 5x - 3) - (6x^5 + 8x^2 - 3)$

d. $(10x^4 - 5x^2 + 16x + 11) - (2x^2 - 4x + 6)$

e. $(6x^3 + 5x^2 - 7x + 12) - (4x^3 - x^2 + 3x - 3)$

3. WE2a Expand and simplify each of the following.

a. $x(x + 6)(x + 1)$

b. $x(x - 9)(x + 2)$

c. $x(x - 3)(x + 11)$

d. $2x(x + 2)(x + 3)$

e. $-3x(x - 4)(x + 4)$

f. $5x(x + 8)(x + 2)$

g. $x^2(x + 4)$

h. $-2x^2(7 - x)$

i. $(5x)(-6x)(x + 9)$

j. $-7x(x + 4)^2$

4. WE2b Expand and simplify each of the following.

a. $(x + 7)(x + 2)(x + 3)$

b. $(x - 2)(x + 4)(x - 5)$

c. $(x - 1)(x - 4)(x + 8)$

d. $(x - 1)(x - 2)(x - 3)$

e. $(x + 6)(x - 1)(x + 1)$

f. $(x - 7)(x + 7)(x + 5)$

g. $(x + 11)(x + 5)(x - 12)$

h. $(x + 5)(x - 1)^2$

i. $(x + 2)(x - 7)^2$

j. $(x + 1)(x - 1)(x + 1)$

5. Expand and simplify each of the following.

a. $(x - 2)(x + 7)(x + 8)$

b. $(x + 5)(3x - 1)(x + 4)$

c. $(4x - 1)(x + 3)(x - 3)(x + 1)$

d. $(5x + 3)(2x - 3)(x - 4)$

e. $(1 - 6x)(x + 7)(x + 5)$

f. $3x(7x - 4)(x - 4)(x + 2)$

g. $-9x(1 - 2x)(3x + 8)$

h. $(6x + 5)(2x - 7)^2$

i. $(3 - 4x)(2 - x)(5x + 9)(x - 1)$

j. $2(7 + 2x)(x + 3)(x + 4)$

6. Expand and simplify each of the following.

a. $(x + 2)^3$ **b.** $(x + 5)^3$ **c.** $(x - 1)^3$

d. $(x - 3)^4$ **e.** $(2x - 6)^3$ **f.** $(3x + 4)^4$

7. Simplify $2(ax + b) - 5(c - bx)$.

8. Expand and simplify $(x + a)(x - b)(x^2 - 3bx + 2a)$.

Communicating, reasoning and problem solving

9. If $(x - 3)^4 = ax^4 + bx^3 + cx^2 + dx + e$, find a, b, c, d and e.

10. Simplify $(2x - 3)^3 - (4 - 3x)^2$.

11. Find the difference in volume between a cube of side $\frac{3(x - 1)}{2}$ and a cuboid whose sides are x, $(x + 1)$ and $(2x + 1)$.

12. Find the constants a, b and c if

$$\frac{5x - 7}{(x - 1)(x + 1)(x - 2)} \equiv \frac{a}{(x - 1)} + \frac{b}{(x + 1)} + \frac{c}{(x - 2)}.$$

13. Write $\frac{3x - 5}{(x^2 + 1)(x - 1)}$ in the form $\frac{ax + b}{(x^2 + 1)} + \frac{c}{(x - 1)}$ and hence find the values of a, b and c.

14. How do you add or subtract polynomials?

17.4 Long division of polynomials [Stage 5.3]

- The reverse of expanding is factorising (expressing a polynomial as a product of its linear factors).
- Before learning how to factorise, you must be familiar with long division of polynomials. You will remember in earlier year levels doing long division questions.
- The same process can be used to divide polynomials by polynomial factors.

Consider $(x^3 + 2x^2 - 13x + 10) \div (x - 3)$.

$$x - 3\overline{)x^3 + 2x^2 - 13x + 10}$$

x into x^3 goes x^2 times (consider only the leading terms). Write x^2 at the top.

$$\begin{array}{r} x^2 \\ x - 3\overline{)x^3 + 2x^2 - 13x + 10} \end{array}$$

$x^2 \times (x - 3) = x^3 - 3x^2$
Write the $x^3 - 3x^2$.

$$\begin{array}{r} x^2 \\ x - 3\overline{)x^3 + 2x^2 - 13x + 10} \\ x^3 - 3x^2 \end{array}$$

Subtract.
$(x^3 - x^3 = 0,\ 2x^2 - -3x^2 = 5x^2)$

$$\begin{array}{r} x^2 \\ x - 3\overline{)x^3 + 2x^2 - 13x + 10} \\ \underline{x^3 - 3x^2} \\ 5x^2 \end{array}$$

Note: Subtracting a negative is the same as changing the sign and adding.

Bring down the $-13x$.

$$\begin{array}{r} x^2 \\ x - 3\overline{)x^3 + 2x^2 - 13x + 10} \\ \underline{x^3 - 3x^2} \\ 5x^2 - 13x \end{array}$$

x into $5x^2$ goes $5x$. Write $+5x$ at the top.

$$\begin{array}{r} x^2 + 5x \\ x - 3\overline{)x^3 + 2x^2 - 13x + 10} \\ \underline{x^3 - 3x^2} \\ 5x^2 - 13x \end{array}$$

$5x \times (x - 3) = 5x^2 - 15x$

$$x - 3\overline{)x^3 + 2x^2 - 13x + 10}$$

Write the $5x^2 - 15x$.

$$\begin{array}{r} x^2 + 5x \\ x - 3\overline{)x^3 + 2x^2 - 13x + 10} \\ \underline{x^3 - 3x^2} \\ 5x^2 - 13x \\ 5x^2 - 15x \end{array}$$

Subtract.
Note: $5x^2 - 5x^2 = 0$, $-13x - -15x = +2x$.

$$\begin{array}{r} x^2 + 5x \\ x - 3\overline{)x^3 + 2x^2 - 13x + 10} \\ \underline{x^3 - 3x^2} \\ 5x^2 - 13x \\ \underline{5x^2 - 15x} \\ 2x \end{array}$$

Bring down the 10.

$$\begin{array}{r} x^2 + 5x \\ x - 3\overline{)x^3 + 2x^2 - 13x + 10} \\ \underline{x^3 - 3x^2} \\ 5x^2 - 13x \\ \underline{5x^2 - 15x} \\ 2x + 10 \end{array}$$

x into $2x$ goes 2. Write $+2$ at the top.

$$\begin{array}{r} x^2 + 5x + 2 \\ x - 3\overline{)x^3 + 2x^2 - 13x + 10} \\ \underline{x^3 - 3x^2} \\ 5x^2 - 13x \\ \underline{5x^2 - 15x} \\ 2x + 10 \end{array}$$

$2 \times (x - 3) = 2x - 6$

Write the $2x - 6$.

$$\begin{array}{r} x^2 + 5x + 2 \\ x - 3\overline{)x^3 + 2x^2 - 13x + 10} \\ \underline{x^3 - 3x^2} \\ 5x^2 - 13x \\ \underline{5x^2 - 15x} \\ 2x + 10 \\ \underline{2x - 6} \end{array}$$

Subtract to get 16.

$$\begin{array}{r} x^2 + 5x + 2 \\ x - 3\overline{)x^3 + 2x^2 - 13x + 10} \\ \underline{x^3 - 3x^2} \\ 5x^2 - 13x \\ \underline{5x^2 - 15x} \\ 2x + 10 \\ \underline{2x - 6} \\ 16 \end{array}$$

$x^2 + 5x + 2$ ← Quotient

16 ← Remainder

Answer: $x^2 + 5x + 2$ remainder 16

DISCUSSION

How does long division of polynomials compare with long division of integers? What are the similarities and differences?

WORKED EXAMPLE 3

Perform the following long divisions and state the quotient and remainder.

a $(x^3 + 3x^2 + x + 9) \div (x + 2)$ **b** $(x^3 - 4x^2 - 7x - 5) \div (x - 1)$

c $(2x^3 + 6x^2 - 3x + 2) \div (x - 6)$

THINK	WRITE
a 1 Write the question in long division format.	**a** $x + 2\overline{)x^3 + 3x^2 + x + 9}$
2 Perform the long division process.	$x^2 + x - 1 \leftarrow Q$ $x^3 + 2x^2$ $x^2 + x$ $x^2 + 2x$ $-x + 9$ $-x - 2$ $11 \leftarrow R$
3 Write the quotient and remainder.	The quotient is $x^2 + x - 1$; the remainder is 11.
b 1 Write the question in long division format.	**b** $x - 1\overline{)x^3 - 4x^2 - 7x - 5}$
2 Perform the long division process.	$x^2 - 3x - 10 \leftarrow Q$ $x^3 - x^2$ $-3x^2 - 7x$ $-3x^2 + 3x$ $-10x - 5$ $-10x + 10$ $-15 \leftarrow R$
3 Write the quotient and remainder.	The quotient is $x^2 - 3x - 10$; the remainder is -15.
c 1 Write the question in long division format.	**c** $x - 6\overline{)2x^3 + 6x^2 - 3x + 2}$
2 Perform the long division process.	$2x^2 + 18x + 105 \leftarrow Q$ $2x^3 - 12x^2$ $18x^2 - 3x$ $18x^2 - 108x$ $105x + 2$ $105x - 630$ $632 \leftarrow R$
3 Write the quotient and remainder.	The quotient is $2x^2 + 18x + 105$; the remainder is 632.

WORKED EXAMPLE 4

State the quotient and remainder for $(x^3 - 7x + 1) \div (x + 5)$.

THINK	WRITE
1 Write the question in long division format. Note that there is no x^2 term in this equation. Include $0x^2$ as a 'placeholder'.	$x + 5\overline{)x^3 + 0x^2 - 7x + 1}$
2 Perform the long division process.	$x^2 - 5x + 18 \leftarrow Q$ $x^3 + 5x^2$ $-5x^2 - 7x$ $-5x^2 - 25x$ $18x + 1$ $18x + 90$ $-89 \leftarrow R$
3 Write the quotient and remainder.	The quotient is $x^2 - 5x + 18$; the remainder is -89.

WORKED EXAMPLE 5

Divide the polynomial $P(x) = x^4 - 3x^3 + 2x^2 - 8$ by the linear expression $x + 2$ and express $P(x)$ in the form $P(x) = (x - a)Q(x) + c$, where $x - a$ is $x + 2$, $Q(x)$ is the quotient and c is the remainder.

THINK	WRITE
1 Set out the long division with the polynomial in descending powers of x. If one of the powers of x is missing, include it with 0 as the coefficient.	$x + 2\overline{)x^4 - 3x^3 + 2x^2 + 0x - 8}$
2 Divide x into x^4 and write the result above.	$\begin{array}{r} x^3 \\ x + 2\overline{)x^4 - 3x^3 + 2x^2 + 0x - 8} \end{array}$
3 Multiply the result x^3 by $x + 2$ and write the result underneath.	$\begin{array}{r} x^3 \\ x + 2\overline{)x^4 - 3x^3 + 2x^2 + 0x - 8} \\ x^4 + 2x^3 \end{array}$
4 Subtract and bring down the remaining terms to complete the expression.	$\begin{array}{r} x^3 - 5x^2 \\ x + 2\overline{)x^4 - 3x^3 + 2x^2 + 0x - 8} \\ \underline{-(x^4 + 2x^3)} \\ -5x^3 + 2x^2 + 0x - 8 \end{array}$
5 Divide x into $-5x^3$ and write the result above.	$\begin{array}{r} x^3 - 5x^2 + 12x - 24 \\ x + 2\overline{)x^4 - 3x^3 + 2x^2 + 0x - 8} \\ \underline{-(x^4 + 2x^3)} \\ -5x^3 + 2x^2 + 0x - 8 \end{array}$
6 Continue this process to complete the long division.	$\begin{array}{r} x^3 - 5x^2 + 12x - 24 \\ x + 2\overline{)x^4 - 3x^3 + 2x^2 + 0x - 8} \\ \underline{-(x^4 + 2x^3)} \\ -5x^3 + 2x^2 + 0x - 8 \\ \underline{-(-5x^3 - 10x^2)} \\ 12x^2 + 0x - 8 \\ \underline{-(12x^2 + 24x)} \\ -24x - 8 \\ \underline{-(-24x - 48)} \\ 40 \end{array}$
7 The polynomial $x^3 - 5x^2 + 12x - 24$, at the top, is the quotient.	The quotient is $x^3 - 5x^2 + 12x - 24$.
8 The result of the final subtraction, 40, is the remainder.	The remainder is 40.
9 Write in the form $P(x) = (x - a)Q(x) + c$.	$x^4 - 3x^3 2x^2 - 8$ $= (x + 2)(x^3 - 5x^2 + 12x - 24) + 40$

learnON RESOURCES —ONLINE ONLY

Interactivity: Long division of polynomials (int-2564)

Interactivity: Long division of polynomials (int-2793)

Exercise 17.4 Long division of polynomials

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–d, 2a–d, 3, 4a, b, 5, 6a–c, 7a–c, 8, 10	Questions: 1e–h, 2e–h, 3a, c, e, 4c, d, 5, 6d–f, 7d–f, 8, 10	Questions: 1g–j, 2e–h, 3b, d, f, 4e, f, 5, 6e–h, 7d–f, 8–12

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE3a Perform the following long divisions and state the quotient and remainder for each.

a. $(x^3 + 4x^2 + 4x + 9) \div (x + 2)$
b. $(x^3 + 2x^2 + 4x + 1) \div (x + 1)$
c. $(x^3 + 6x^2 + 3x + 1) \div (x + 3)$
d. $(x^3 + 3x^2 + x + 3) \div (x + 4)$
e. $(x^3 + 4x^2 + 3x + 4) \div (x + 2)$
f. $(x^3 + 6x^2 + 2x + 2) \div (x + 2)$
g. $(x^3 + x^2 + x + 3) \div (x + 1)$
h. $(x^3 + 8x^2 + 5x + 4) \div (x + 8)$
i. $(x^3 + x^2 + 4x + 1) \div (x + 2)$
j. $(x^3 + 9x^2 + 3x + 2) \div (x + 5)$

2. WE3b State the quotient and remainder for each of the following.

a. $(x^3 + 2x^2 - 5x - 9) \div (x - 2)$
b. $(x^3 + x^2 + x + 9) \div (x - 3)$
c. $(x^3 + x^2 - 9x - 5) \div (x - 2)$
d. $(x^3 - 4x^2 + 10x - 2) \div (x - 1)$
e. $(x^3 - 5x^2 + 3x - 8) \div (x - 3)$
f. $(x^3 - 7x^2 + 9x - 7) \div (x - 1)$
g. $(x^3 + 9x^2 + 2x - 1) \div (x - 5)$
h. $(x^3 + 4x^2 - 5x - 4) \div (x - 4)$

3. WE3c For each of the following, divide the first polynomial by the second and state the quotient and remainder.

a. $3x^3 - x^2 + 6x + 5,\ x + 2$
b. $4x^3 - 4x^2 + 10x - 4,\ x + 1$
c. $2x^3 - 7x^2 + 9x + 1,\ x - 2$
d. $2x^3 + 8x^2 - 9x - 1,\ x + 4$
e. $4x^3 - 10x^2 - 9x + 8,\ x - 3$
f. $3x^3 + 16x^2 + 4x - 7,\ x + 5$

4. For each of the following, divide the first polynomial by the second and state the quotient and remainder.

a. $6x^3 - 7x^2 + 4x + 4,\ 2x - 1$
b. $6x^3 + 23x^2 + 2x - 31,\ 3x + 4$
c. $8x^3 + 6x^2 - 39x - 13,\ 2x + 5$
d. $2x^3 - 15x^2 + 34x - 13,\ 2x - 7$
e. $3x^3 + 5x^2 - 16x - 23,\ 3x + 2$
f. $9x^3 - 6x^2 - 5x + 9,\ 3x - 4$

5. State the quotient and remainder for each of the following.

a. $\dfrac{-x^3 - 6x^2 - 7x - 16}{x + 1}$
b. $\dfrac{-3x^3 + 7x^2 + 10x - 15}{x - 3}$
c. $\dfrac{-2x^3 + 9x^2 + 17x + 15}{2x + 1}$
d. $\dfrac{4x^3 - 20x^2 + 23x - 2}{-2x + 3}$

6. WE4 State the quotient and remainder for each of the following.

a. $(x^3 - 3x + 1) \div (x + 1)$
b. $(x^3 + 2x^2 - 7) \div (x + 2)$
c. $(x^3 - 5x^2 + 2x) \div (x - 4)$
d. $(-x^3 - 7x + 8) \div (x - 1)$
e. $(5x^2 + 13x + 1) \div (x + 3)$
f. $(2x^3 + 8x^2 - 4) \div (x + 5)$
g. $(-2x^3 - x + 2) \div (x - 2)$
h. $(-4x^3 + 6x^2 + 2x) \div (2x + 1)$

7. WE5 Divide each polynomial by the linear expression given and express $P(x)$ in the form $P(x) = (x - a)Q(x) + c$, where $x - a$ is $x + 2$, $Q(x)$ is the quotient and c is the remainder.

a. $P(x) = x^4 + x^3 + 3x^2 - 7x,\ x - 1$
b. $P(x) = x^4 - 13x^2 + 36,\ x - 2$
c. $P(x) = x^5 - 3x^3 + 4x + 3,\ x + 3$
d. $P(x) = 2x^6 - x^4 + x^3 + 6x^2 - 5x,\ x + 2$
e. $P(x) = 6x^4 - x^3 + 2x^2 - 4x,\ x - 3$
f. $P(x) = 3x^4 - 6x^3 + 12x,\ 3x + 1$

Communicating, reasoning and problem solving

8. Find the quotient and remainder when $ax^2 + bx + c$ is divided by $(x - d)$.

9. A birthday cake in the shape of a cube had side length $(x + p)$ cm. The cake was divided between $(x - p)$ guests. The left-over cake was used for lunch the next day. There were q^3 guests for lunch the next day and each received c^3 cm^3 of cake, which was then all finished. Find q in terms of p and c.

10. When $x^3 - 2x^2 + 4x + a$ is divided by x–1, the remainder is zero. Use long division to determine the value of a.

11. When $2x^2 + ax + b$ is divided by $x - 1$, the remainder is zero, but when $2x^2 + ax + b$ is divided by $x - 2$, the remainder is 9. Use long division to determine the value of a and b.

12. Can you think of an alternative way to divide polynomials?

17.5 Polynomial values [Stage 5.3]

- Consider the polynomial $P(x) = x^3 - 5x^2 + x + 1$.
- The value of the polynomial when $x = 3$ is denoted by $P(3)$ and is found by substituting $x = 3$ into the equation in place of x. That is:

$$P(3) = (3)^3 - 5(3)^2 + (3) + 1$$
$$P(3) = 27 - 5(9) + 3 + 1$$
$$P(3) = 27 - 45 + 4$$
$$P(3) = -14$$

WORKED EXAMPLE 6

If $P(x) = 2x^3 + x^2 - 3x - 4$, find:
a $P(1)$ **b** $P(-2)$ **c** $P(a)$ **d** $P(2b)$ **e** $P(x + 1)$.

THINK	WRITE
a 1 Write the expression.	**a** $P(x) = 2x^3 + x^2 - 3x - 4$
2 Replace each occurrence of x with 1.	$P(1) = 2(1)^3 + (1)^2 - 3(1) - 4$
3 Simplify.	$= 2 + 1 - 3 - 4$ $= -4$
b 1 Write the expression.	**b** $P(x) = 2x^3 + x^2 - 3x - 4$
2 Replace each occurrence of x with –2.	$P(-2) = 2(-2)^3 + (-2)^2 - 3(-2) - 4$
3 Simplify.	$= 2(-8) + (4) + 6 - 4$ $= -16 + 4 + 6 - 4$ $= -10$

c 1 Write the expression.

2 Replace each occurrence of x with a.

3 No further simplification is possible, so stop here.

c $P(x) = 2x^3 + x^2 - 3x - 4$

$P(a) = 2a^3 + a^2 - 3a - 4$

d 1 Write the expression.

2 Replace each occurrence of x with $2b$.

3 Simplify.

d $P(x) = 2x^3 + x^2 - 3x - 4$

$P(2b) = 2(2b)^3 + (2b)^2 - 3(2b) - 4$

$= 2(8b^3) + 4b^2 - 6b - 4$

$= 16b^3 + 4b^2 - 6b - 4$

e 1 Write the expression.

2 Replace each occurrence of x with $(x + 1)$.

3 Expand the right-hand side and collect like terms.

e $P(x) = 2x^3 + x^2 - 3x - 4$

$P(x + 1) = 2(x + 1)^3 + (x + 1)^2 - 3(x + 1) - 4$

$= 2(x + 1)(x + 1)(x + 1) + (x + 1)(x + 1) - 3(x + 1) - 4$

$= 2(x + 1)(x^2 + 2x + 1) + x^2 + 2x + 1 - 3x - 3 - 4$

$= 2(x^3 + 2x^2 + x + x^2 + 2x + 1) + x^2 - x - 6$

$= 2(x^3 + 3x^2 + 3x + 1) + x^2 - x - 6$

$= 2x^3 + 6x^2 + 6x + 2 + x^2 - x - 6$

$= 2x^3 + 7x^2 + 5x - 4$

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Digital doc: SkillSHEET Substitution into quadratic equations (doc-5367)

Digital doc: WorkSHEET Polynomials I (doc-14618)

Exercise 17.5 Polynomial values

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–d, 2–8, 11	Questions: 1a, e–h, 2–7, 9, 11	Questions: 1a, i–l, 2–13

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE6 If $P(x) = 2x^3 - 3x^2 + 2x + 10$, find the following.

a. $P(0)$ **b.** $P(1)$ **c.** $P(2)$ **d.** $P(3)$

e. $P(-1)$ **f.** $P(-2)$ **g.** $P(-3)$ **h.** $P(a)$

i. $P(2b)$ **j.** $P(x + 2)$ **k.** $P(x - 3)$ **l.** $P(-4y)$

2. Copy the following table.

Column 1	Column 2	Column 3	Column 4	Column 5	Column 6	Column 7	Column 8	Column 9
$P(x)$	$P(1)$	$P(2)$	$P(-1)$	$P(-2)$	Rem when divided by $(x - 1)$	Rem when divided by $(x - 2)$	Rem when divided by $(x + 1)$	Rem when divided by $(x + 2)$
a								
b								
c								
d								

Complete columns 2 to 5 of the table for each of the following polynomials.

a. $P(x) = x^3 + x^2 + x + 1$ **b.** $P(x) = x^3 + 2x^2 + 5x + 2$

c. $P(x) = x^3 - x^2 + 4x - 1$ **d.** $P(x) = x^3 - 4x^2 - 7x + 3$

3. Find the remainder when each polynomial in question **2** is divided by $(x - 1)$ and complete column 6 of the table.

4. Find the remainder when each polynomial in question **2** is divided by $(x - 2)$ and complete column 7 of the table.

5. Find the remainder when each polynomial in question **2** is divided by $(x + 1)$ and complete column 8 of the table.

6. Find the remainder when each polynomial in question **2** is divided by $(x + 2)$ and complete column 9 of the table.

7. Copy and complete the following.

a. A quick way of finding the remainder when $P(x)$ is divided by $(x + 8)$ is to calculate _____.

b. A quick way of finding the remainder when $P(x)$ is divided by $(x - 7)$ is to calculate _____.

c. A quick way of finding the remainder when $P(x)$ is divided by $(x - a)$ is to calculate _____.

Communicating, reasoning and problem solving

8. If $P(x) = 2(x - 3)^5 + 1$, find:

a. $P(2)$ **b.** $P(-2)$ **c.** $P(a)$ **d.** $P(-2a)$.

9. When $x^2 + bx + 2$ is divided by $(x - 1)$, the remainder is $b^2 - 4b + 7$. Find the possible values of b.

10. If $P(x) = -2x^3 - 3x^2 + x + 3$, find:

a. $P(a) + 1$ **b.** $P(a + 1)$.

11. If $P(x) = 3x^3 - 2x^2 - x + c$ and $P(2) = 8P(1)$, find the value of c.

12. If $P(x) = 5x^2 + bx + c$ and $P(-1) = 12$ while $P(2) = 21$, find the values of b and c.

13. Is there a quick way to find a remainder when dividing polynomials?

17.6 The remainder and factor theorems [Stage 5.3]

17.6.1 The remainder theorem

- In the previous exercise, you may have noticed that:

  the remainder when $P(x)$ is divided by $(x - a)$ is equal to $P(a)$.

  That is, $R = P(a)$.

  This is called the **remainder theorem**.

- If $P(x) = x^3 + x^2 + x + 1$ is divided by $(x - 2)$, the quotient is $x^2 + 3x + 7$ and the remainder is $P(2)$, which equals 15. That is:

$$(x^3 + x^2 + x + 1) \div (x - 2) = x^2 + 3x + 7 + \frac{15}{x - 2} \text{ and}$$
$$(x^3 + x^2 + x + 1) = (x^2 + 3x + 7)(x - 2) + 15.$$

- In general, if $P(x)$ is divided by $(x - a)$, the quotient is $Q(x)$ and the remainder is R, we can write:

$$P(x) \div (x - a) = Q(x) + \frac{R}{(x - a)} \text{ where } R = P(a)$$
$$\Rightarrow P(x) = (x - a)\,Q(x) + R$$

17.6.2 The factor theorem

- The remainder when 12 is divided by 4 is zero, since 4 is a factor of 12.
- Similarly, if the remainder (R) when $P(x)$ is divided by $(x - a)$ is zero, then $(x - a)$ is a factor of $P(x)$.
- Since $R = P(a)$, find a value of a that makes $P(a) = 0$; then $(x - a)$ is a factor.

  If $P(a) = 0$, then $(x - a)$ is a factor of $P(x)$.
- This is called the **factor theorem**.
- Imagine $P(x)$ could be factorised as follows:

  $P(x) = (x - a)\,Q(x)$, where $Q(x)$ is 'the other' factor of $P(x)$.
- If $P(a) = 0$, $(x - a)$ is a factor.

DISCUSSION

What do the terms roots, zeros and solutions mean for a polynomial of degree n?

WORKED EXAMPLE 7

Without actually dividing, find the remainder when $x^3 - 7x^2 - 2x + 4$ is divided by:
a $x - 3$ **b $x + 6$.**

THINK	WRITE
a 1 Name the polynomial.	**a** Let $P(x) = x^3 - 7x^2 - 2x + 4$.
2 The remainder when $P(x)$ is divided by $(x - 3)$ is equal to $P(3)$.	$R = P(3)$ $= 3^3 - 7(3)^2 - 2(3) + 4$ $= 27 - 7(9) - 6 + 4$ $= 27 - 63 - 6 + 4$ $= -38$
b The remainder when $P(x)$ is divided by $(x + 6)$ is equal to $P(-6)$.	**b** $R = P(-6)$ $= (-6)^3 - 7(-6)^2 - 2(-6) + 4$ $= -216 - 7(36) + 12 + 4$ $= -216 - 252 + 12 + 4$ $= -452$

WORKED EXAMPLE 8

$(x - 2)$ is a factor of $x^3 + kx^2 + x - 2$. Find the value of k.

THINK	WRITE
1 Name the polynomial.	Let $P(x) = x^3 + kx^2 + x - 2$.
2 The remainder when $P(x)$ is divided by $(x - 2)$ is equal to $P(2) = 0$.	$0 = P(2)$ $= 2^3 + k(2)^2 + 2 - 2$ $0 = 8 + 4k$
3 Solve for k.	$4k = -8$ $k = -2$

learnON RESOURCES — ONLINE ONLY

Interactivity: The remainder and factor theorems (int-2565)

Exercise 17.6 The remainder and factor theorems

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1, 2a–d, 3a–d, 4, 7a–d	Questions: 1, 2e–h, 3e–h, 4–6, 7d–g, 8	Questions: 1, 2g–j, 3f–h, 4–11

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Understanding and fluency

1. WE7 Without actually dividing, find the remainder when $x^3 + 3x^2 - 10x - 24$ is divided by:

a. $x - 1$ **b.** $x + 2$ **c.** $x - 3$ **d.** $x + 5$
e. $x - 0$ **f.** $x - k$ **g.** $x + n$ **h.** $x + 3c$.

2. For each of the following, find the remainder when the first polynomial is divided by the second without performing long division.

a. $x^3 + 2x^2 + 3x + 4$, $x - 3$
b. $x^3 - 4x^2 + 2x - 1$, $x + 1$
c. $x^3 + 3x^2 - 3x + 1$, $x + 2$
d. $x^3 - x^2 - 4x - 5$, $x - 1$
e. $2x^3 + 3x^2 + 6x + 3$, $x + 5$
f. $-3x^3 - 2x^2 + x + 6$, $x + 1$
g. $x^3 + x^2 + 8$, $x - 5$
h. $x^3 - 3x^2 - 2$, $x - 2$
i. $-x^3 + 8$, $x + 3$
j. $x^3 + 2x^2$, $x - 7$

3. WE8 **a.** The remainder when $x^3 + kx + 1$ is divided by $(x + 2)$ is –19. Find the value of k.
b. The remainder when $x^3 + 2x^2 + mx + 5$ is divided by $(x - 2)$ is 27. Find the value of m.
c. The remainder when $x^3 - 3x^2 + 2x + n$ is divided by $(x - 1)$ is 1. Find the value of n.
d. The remainder when $ax^3 + 4x^2 - 2x + 1$ is divided by $(x - 3)$ is –23. Find the value of a.
e. The remainder when $x^3 - bx^2 - 2x + 1$ is divided by $(x + 1)$ is 0. Find the value of b.
f. The remainder when $-4x^2 + 2x + 7$ is divided by $(x - c)$ is –5. Find a possible whole number value of c.

g. The remainder when $x^2 - 3x + 1$ is divided by $(x + d)$ is 11. Find the possible values of d.

h. The remainder when $x^3 + ax^2 + bx + 1$ is divided by $(x - 5)$ is –14. When the cubic polynomial is divided by $(x + 1)$, the remainder is –2. Find a and b.

4. MC *Note:* There may be more than one correct answer.

a. When $x^3 + 2x^2 - 5x - 5$ is divided by $(x + 2)$, the remainder is:

A. –5 **B.** –2 **C.** 2 **D.** 5 **E.** 1

b. Which of the following is a factor of $2x^3 + 15x^2 + 22x - 15$?

A. $(x - 1)$ **B.** $(x - 2)$ **C.** $(x + 3)$ **D.** $(x + 5)$ **E.** $(x - 5)$

c. When $x^3 - 13x^2 + 48x - 36$ is divided by $(x - 1)$, the remainder is:

A. –3 **B.** –2 **C.** –1 **D.** 0 **E.** 1

d. Which of the following is a factor of $x^3 - 5x^2 - 22x + 56$?

A. $(x - 2)$ **B.** $(x + 2)$ **C.** $(x - 7)$ **D.** $(x + 4)$ **E.** $(x + 7)$

5. Find one factor of each of the following cubic polynomials.

a. $x^3 - 3x^2 + 3x - 1$ **b.** $x^3 - 7x^2 + 16x - 12$

c. $x^3 + x^2 - 8x - 12$ **d.** $x^3 + 3x^2 - 34x - 120$

Communicating, reasoning and problem solving

6. Prove that each of the following is a linear factor of $x^3 + 4x^2 - 11x - 30$ by substituting values into the cubic function: $(x + 2)$, $(x - 3)$, $(x + 5)$.

7. Without actually dividing, show that the first polynomial is exactly divisible by the second (that is, the second polynomial is a factor of the first).

a. $x^3 + 5x^2 + 2x - 8$, $x - 1$

b. $x^3 - 7x^2 - x + 7$, $x - 7$

c. $x^3 - 7x^2 + 4x + 12$, $x - 2$

d. $x^3 + 2x^2 - 9x - 18$, $x + 2$

e. $x^3 + 3x^2 - 9x - 27$, $x + 3$

f. $-x^3 + x^2 + 9x - 9$, $x - 1$

g. $-2x^3 + 9x^2 - x - 12$, $x - 4$

h. $3x^3 + 22x^2 + 37x + 10$, $x + 5$

8. When $x^4 + ax^3 - 4x^2 + b$ and $x^3 - ax^2 - 7x + b$ are each divided by $(x - 2)$, the remainders are 26 and 8 respectively. Find the values of a and b.

9. Both $(x - 1)$ and $(x - 2)$ are factors of $P(x) = x^4 + ax^3 - 7x^2 + bx - 30$. Find the values of a and b and the remaining two linear factors.

10. How are the remainder and factor theorems related?

11. The remainder when $2x - 1$ is divided into $6x^3 - x^2 + 3x + k$ is the same as when it is divided into $4x^3 - 8x^2 - 5x + 2$. What is the value of k?

17.7 Factorising polynomials [Stage 5.3]

17.7.1 Using long division

- Once one factor of a polynomial has been found (using the factor theorem as in the previous section), long division may be used to find other factors. In the case of a cubic polynomial, one — possibly two — other factors may be found.

WORKED EXAMPLE 9

Use long division to factorise the following.
a $x^3 - 5x^2 - 2x + 24$ **b** $x^3 - 19x + 30$ **c** $-2x^3 - 8x^2 + 6x + 4$

THINK	WRITE
a 1 Name the polynomial.	**a** $P(x) = x^3 - 5x^2 - 2x + 24$
2 Look for a value of x such that $P(x) = 0$. For cubics containing a single x^3, try a factor of the constant term (24 in this case). Try $P(1)$.	$P(1) = 1^3 - 5 \times 1^2 - 2 \times 1 + 24$ $= 1 - 5 - 2 + 24$ $= 18$ $\neq 0$
$P(1) \neq 0$, so $(x - 1)$ is not a factor. Try $P(2)$.	$P(2) = 2^3 - 5 \times 2^2 - 2 \times 2 + 24$ $= 8 - 20 - 4 + 24$ $\neq 0$
$P(2) \neq 0$, so $(x - 2)$ is not a factor. Try $P(-2)$.	$P(-2) = (-2)^3 - 5 \times (-2)^2 - 2 \times (-2) + 24$ $= -8 - 20 + 4 + 24$ $= -28 + 28$ $= 0$
$P(-2)$ does equal 0, so $(x + 2)$ is a factor.	$(x + 2)$ is a factor.
3 Divide $(x + 2)$ into $P(x)$ using long division to find a quadratic factor.	$x^2 - 7x + 12$ $x + 2 \overline{)x^3 - 5x^2 - 2x + 24}$ $x^3 + 2x^2$ $-7x^2 - 2x$ $-7x^2 - 14x$ $12x + 14$ $12x + 14$ 0
4 Write $P(x)$ as a product of the two factors found so far.	$P(x) = (x + 2)(x^2 - 7x + 12)$
5 Factorise the quadratic factor if possible.	$P(x) = (x + 2)(x - 3)(x - 4)$
b 1 Name the polynomial. *Note:* There is no x^2 term, so include $0x^2$.	**b** $P(x) = x^3 - 19x + 30$ $P(x) = x^3 + 0x^2 - 19x + 30$
2 Look at the last term in $P(x)$, which is 30. This suggests it is worth trying $P(5)$ or $P(-5)$. Try $P(-5)$. $P(-5) = 0$ so $(x + 5)$ is a factor.	$P(-5) = (-5)^3 - 19 \times (-5) + 30$ $= -125 + 95 + 30$ $= 0$ So $(x + 5)$ is a factor.

3 Divide $(x + 5)$ into $P(x)$ using long division to find a quadratic factor.

$$\begin{array}{r} x^2 - 5x + 6 \\ x + 5\overline{)x^3 + 0x^2 - 19x + 30} \\ \underline{x^3 + 5x^2} \\ -5x^2 - 19x \\ \underline{-5x^2 - 25x} \\ 6x + 30 \\ \underline{6x + 30} \\ 0 \end{array}$$

4 Write $P(x)$ as a product of the two factors found so far.

$P(x) = (x + 5)(x^2 - 5x + 6)$

5 Factorise the quadratic factor if possible.

$P(x) = (x + 5)(x - 2)(x - 3)$

c 1 Write the given polynomial.

c Let $P(x) = -2x^3 - 8x^2 + 6x + 4$.

2 Take out a common factor of –2. (We could take out + 2 as the common factor, but taking out –2 results in a positive leading term in the part still to be factorised.)

$= -2(x^3 + 4x^2 - 3x - 2)$

3 Let $Q(x) = (x^3 + 4x^2 - 3x - 2)$. (We have already used P earlier.)

Let $Q(x) = (x^3 + 4x^2 - 3x - 2)$.

4 Evaluate $Q(1)$. $Q(1) = 0$, so $(x - 1)$ is a factor.

$Q(1) = 1 + 4 - 3 - 2$
$= 0$
So $(x - 1)$ is a factor.

5 Divide $(x - 1)$ into $Q(x)$ using long division to find a quadratic factor.

$$\begin{array}{r} x^2 + 5x + 2 \\ x - 1\overline{)x^3 + 4x^2 - 3x - 2} \\ \underline{x^3 - x^2} \\ 5x^2 - 3x \\ \underline{5x^2 - 5x} \\ 2x - 2 \\ \underline{2x - 2} \\ 0 \end{array}$$

6 Write the original polynomial $P(x)$ as a product of the factors found so far.

$P(x) = -2(x - 1)(x^2 + 5x + 2)$

7 In this case, it is not possible to further factorise $P(x)$.

- *Note:* In these examples, $P(x)$ may have been factorised without long division by finding all three values of x that make $P(x) = 0$ (and hence three factors) and then checking that the three factors multiply to give $P(x)$.

17.7.2 Using short division

- The process of long division can be quite time (and space) consuming. An alternative is short division, which may take a little longer to understand, but is quicker once mastered.
- Consider $P(x) = x^3 + 2x^2 - 13x + 10$. Using the factor theorem, we can find that $(x - 1)$ is a factor of $P(x)$. So, $P(x) = (x - 1)(?)$.

Actually, we know more than this: as $P(x)$ begins with x^3 and ends with $+10$, we could write:

$$P(x) = (x - 1)(x^2 + ? - 10).$$

The x^2 in the second pair of grouping symbols produces the desired x^3 (the leading term in $P(x)$) when the expressions are multiplied. The -10 in the second pair of grouping symbols produces $+10$ (the last term in $P(x)$) when the expressions are multiplied.

- Imagine expanding this version of $P(x)$. Multiplying x in the first pair of grouping symbols by x^2 in the second would produce x^3, which is what we want, but multiplying -1 in the first pair of grouping symbols by x^2 in the second gives $-1x^2$.

Since $P(x) = x^3 + 2x^2 - 13x + 10$, we really need $+ 2x^2$, not $-1x^2$. That is, we need $+ 3x^2$ more. To get this, the ? must be $3x$, because when x in the first pair of grouping symbols is multiplied by $3x$ in the second pair, $+3x^2$ results. That is, we have deduced $P(x) = (x - 1)(x^2 + 3x - 10)$.

Factorising the expression in the second pair of grouping symbols gives

$$P(x) = (x - 1)(x + 5)(x - 2).$$

- This procedure, which we will call *short division*, can be confusing at first, but with persistence it can be a quick and easy method for factorising polynomials.
- The following worked example is a repeat of a previous one, but explains the use of short, rather than long, division.

WORKED EXAMPLE 10

Use short division to factorise $x^3 - 5x^2 - 2x + 24$.

THINK	WRITE
1 Name the polynomial.	Let $P(x) = x^3 - 5x^2 - 2x + 24$.
2 Look for a value of x such that $P(x) = 0$. Try $P(-2)$. $P(-2)$ does equal 0, so $(x + 2)$ is a factor.	$P(-2) = (-2)^3 - 5 \times (-2)^2 - 2 \times (-2) + 24$ $= -8 - 20 + 4 + 24$ $= -28 + 28$ $= 0$ So $(x + 2)$ is a factor.
3 Look again at the original: $P(x) = x^3 - 5x^2 - 2x + 24$. The first term in the grouping symbols must be x^2, and the last term must be 12.	$P(x) = (x + 2)(x^2 + ax + 12)$
4 Imagine the expansion of the expression in step 3. We have x^3 and $2x^2$, but require $-5x^2$. We need an extra $-7x^2$. We get this by inserting a $-7x$ term in the second pair of grouping symbols.	$P(x) = (x + 2)(x^2 - 7x + 12)$
5 Factorise the expression in the second pair of grouping symbols if possible.	$P(x) = (x + 2)(x - 3)(x - 4)$

DISCUSSION

Do you find the short division method easier or harder to use than the long division method?

learnON RESOURCES — ONLINE ONLY

Digital doc: SkillSHEET Factorising quadratic trinomials (doc-5368)

Exercise 17.7 Factorising polynomials

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–c, 2a–d, 3a–d, 4a–c, 5, 6	Questions: 1d–f, 2e–h, 3e–h, 4d–g, 5, 6, 8	Questions: 1d–f, 2i–n, 3g–j, 4h–k, 5–11

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE9 Use long division to factorise each dividend.

a. $x + 1\overline{)x^3 + 10x^2 + 27x + 18}$

b. $x + 2\overline{)x^3 + 8x^2 + 17x + 10}$

c. $x + 9\overline{)x^3 + 12x^2 + 29x + 18}$

d. $x + 1\overline{)x^3 + 8x^2 + 19x + 12}$

e. $x + 3\overline{)x^3 + 14x^2 + 61x + 84}$

f. $x + 7\overline{)x^3 + 12x^2 + 41x + 42}$

g. $x + 2\overline{)x^3 + 4x^2 + 5x + 2}$

h. $x + 3\overline{)x^3 + 7x^2 + 16x + 12}$

i. $x + 5\overline{)x^3 + 14x^2 + 65x + 100}$

j. $x\overline{)x^3 + 13x^2 + 40x}$

k. $x\overline{)x^3 + 7x^2 + 12x}$

l. $x + 5\overline{)x^3 + 10x^2 + 25x}$

m. $x + 1\overline{)x^3 + 6x^2 + 5x}$

n. $x + 6\overline{)x^3 + 6x^2}$

2. WE9, 10 Factorise the following as fully as possible.

a. $x^3 + x^2 - x - 1$

b. $x^3 - 2x^2 - x + 2$

c. $x^3 + 7x^2 + 11x + 5$

d. $x^3 + x^2 - 8x - 12$

e. $x^3 + 9x^2 + 24x + 16$

f. $x^3 - 5x^2 - 4x + 20$

g. $x^3 + 2x^2 - x - 2$

h. $x^3 - 7x - 6$

i. $x^3 + 3x^2 - 4$

j. $x^3 + x^2 + x + 6$

k. $x^3 + 8x^2 + 17x + 10$

l. $x^3 + x^2 - 9x - 9$

m. $x^3 - x^2 - 8x + 12$

n. $x^3 + 9x^2 - 12x - 160$

3. Factorise the following as fully as possible.

a. $2x^3 + 5x^2 - x - 6$

b. $3x^3 + 14x^2 + 7x - 4$

c. $3x^3 + 2x^2 - 12x - 8$

d. $4x^3 + 35x^2 + 84x + 45$

e. $5x^3 + 9x^2 + 3x - 1$

f. $x^3 + x^2 + x + 1$

g. $4x^3 + 16x^2 + 21x + 9$

h. $6x^3 - 23x^2 + 26x - 8$

i. $10x^3 + 19x^2 - 94x - 40$

j. $7x^3 + 12x^2 - 60x + 16$

4. Factorise the following as fully as possible.

a. $3x^3 - x^2 - 10x$

b. $4x^3 + 2x^2 - 2x$

c. $3x^3 - 6x^2 - 24x$

d. $-2x^3 - 12x^2 - 18x$

e. $6x^3 - 6x^2$

f. $-x^3 - 7x^2 - 12x$

g. $-x^3 - 3x^2 + x + 3$

h. $-2x^3 + 10x^2 - 12x$

i. $-6x^3 - 5x^2 + 12x - 4$

j. $-5x^3 + 24x^2 - 36x + 16$

k. $-x^5 - x^4 + 21x^3 + 49x^2 - 8x - 60$

Communicating, reasoning and problem solving

5. Factorise $x^4 - 9x^2 - 4x + 12$.

6. Factorise $-x^5 + 6x^4 + 11x^3 - 84x^2 - 28x + 240$.

7. Two of the factors of $x^3 + px^2 + qx + r$ are $(x + a)$ and $(x + b)$. Find the third factor.

8. $(x - 1)$ and $(x - 2)$ are known to be factors of $x^5 + ax^4 - 2x^3 + bx^2 + x - 2$. Find the values of a and b and hence fully factorise this fifth-degree polynomial.

9. Factorise $x^5 - 5x^4 + 5x^3 + 5x^2 - 6x$.

10. List the steps in factorising polynomials.

11. The polynomial $x^4 - 6x^3 + 13x^2 - 12x - 32$ has three factors, one of which is $x^2 - 3x + 8$. What are the other two factors?

17.8 Solving polynomial equations [Stage 5.3]

- A polynomial equation of the form $P(x) = 0$ may be solved by factorising $P(x)$ and applying the Null Factor Law. The solutions are also called zeros. They are the x-intercepts on the graph of $P(x)$.
- The Null Factor Law applies to polynomial equations just as it does for quadratics.
- If $P(x) = (x - a)(x - b)(x - c) = 0$, then the solutions can be found as follows.
  Let each factor = 0:
  $$x - a = 0 \quad x - b = 0 \quad x - c = 0$$
- Solving each of these equations produces the solutions (roots)
  $$x = a \quad x = b \quad x = c$$
- If $P(x) = k(lx - a)(mx - b)(nx - c) = 0$, then the solutions can be found as follows. Let each factor = 0:
  $$lx - a = 0 \quad mx - b = 0 \quad nx - c = 0$$
  Solving each of these equations produces the solutions
  $$x = \frac{a}{l} \quad x = \frac{b}{m} \quad x = \frac{c}{n}$$
- *Note:* The coefficient k used in this example does not produce a solution because $k \neq 0$.

WORKED EXAMPLE 11

Solve:
a $x^3 = 9x$ **b** $-2x^3 + 4x^2 + 70x = 0$ **c** $2x^3 - 11x^2 + 18x - 9 = 0$.

THINK	WRITE
a 1 Write the equation.	**a** $x^3 = 9x$
2 Rearrange so all terms are on the left.	$x^3 - 9x = 0$
3 Take out a common factor of x.	$x(x^2 - 9) = 0$
4 Factorise the expression in the grouping symbols using the difference of squares rule.	$x(x + 3)(x - 3) = 0$
5 Use the Null Factor Law to solve.	$x = 0$, $x + 3 = 0$ or $x - 3 = 0$ $x = 0$, $x = -3$ or $x = 3$
b 1 Write the equation.	**b** $-2x^3 + 4x^2 + 70x = 0$
2 Take out a common factor of $-2x$.	$-2x(x^2 - 2x - 35) = 0$
3 Factorise the expression in the grouping symbols.	$-2x(x - 7)(x + 5) = 0$

4 Use the Null Factor Law to solve.

$-2x = 0$, $x - 7 = 0$ or $x + 5 = 0$
$x = 0$, $x = 7$ or $x = -5$

c 1 Name the polynomial.

c Let $P(x) = 2x^3 - 11x^2 + 18x - 9$.

2 Use the factor theorem to find a factor (search for a value a such that $P(a) = 0$). Consider factors of the constant term (that is, factors of 9 such as 1, 3). The simplest value to try is 1.

$P(1) = 2 - 11 + 18 - 9$
$= 0$
So $(x - 1)$ is a factor.

3 Use long or short division to find another factor of $P(x)$.

$$\begin{array}{r} 2x^2 - 9x + 9 \\ x - 1 \overline{)\,2x^3 - 11x^2 + 18x - 9} \\ \underline{2x^3 - 2x^2} \\ -9x^2 + 18x \\ \underline{-9x^2 + 9x} \\ 9x - 9 \\ \underline{9x - 9} \\ 0 \end{array}$$

$P(x) = (x - 1)(2x^2 - 9x + 9)$

4 Factorise the quadratic factor.

$P(x) = (x - 1)(2x - 3)(x - 3)$

5 Consider the factorised equation to solve.

For $(x - 1)(2x - 3)(x - 3) = 0$

6 Use the Null Factor Law to solve.

$x - 1 = 0$, $2x - 3 = 0$ or $x - 3 = 0$
$x = 1$, $x = \frac{3}{2}$ or $x = 3$

learnON RESOURCES — ONLINE ONLY

Digital doc: SkillSHEET Factorising difference of two squares expressions (doc-5369)
Digital doc: SkillSHEET Solving quadratic equations (doc-5370)
Digital doc: WorkSHEET Polynomials II (doc-14619)

Exercise 17.8 Solving polynomial equations

Individual pathways

PRACTISE
Questions:
1a–d, 2a–d, 3, 4, 5a–d, 6a–c, 7, 10

CONSOLIDATE
Questions:
1e–h, 2e–h, 3, 4, 5e–h, 6d–f, 8, 10

MASTER
Questions:
1i–n, 2i–n, 3, 4, 5e–h, 6d–f, 7–12

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE11a, b Solve the following.

a. $x^3 - 4x = 0$
b. $x^3 - 16x = 0$
c. $2x^3 - 50x = 0$
d. $-3x^3 + 81 = 0$
e. $x^3 + 5x^2 = 0$
f. $x^3 - 2x^2 = 0$
g. $-4x^3 + 8x = 0$
h. $12x^3 + 3x^2 = 0$
i. $4x^2 - 20x^3 = 0$
j. $x^3 - 5x^2 + 6x = 0$
k. $x^3 - 8x^2 + 16x = 0$
l. $x^3 + 6x^2 = 7x$
m. $9x^2 = 20x + x^3$
n. $x^3 + 6x = 4x^2$

2. WE11c Use the factor theorem to solve the following.

a. $x^3 - x^2 - 16x + 16 = 0$ **b.** $x^3 - 6x^2 - x + 30 = 0$

c. $x^3 - x^2 - 25x + 25 = 0$ **d.** $x^3 + 4x^2 - 4x - 16 = 0$

e. $x^3 - 4x^2 + x + 6 = 0$ **f.** $x^3 - 4x^2 - 7x + 10 = 0$

g. $x^3 + 6x^2 + 11x + 6 = 0$ **h.** $x^3 - 6x^2 - 15x + 100 = 0$

i. $x^3 - 3x^2 - 6x + 8 = 0$ **j.** $x^3 + 2x^2 - 29x + 42 = 0$

k. $2x^3 + 15x^2 + 19x + 6 = 0$ **l.** $-4x^3 + 16x^2 - 9x - 9 = 0$

m. $-2x^3 - 9x^2 - 7x + 6 = 0$ **n.** $2x^3 + 4x^2 - 2x - 4 = 0$

3. MC Which of the following is a solution to $x^3 - 7x^2 + 2x + 40 = 0$? *Note:* There may be more than one correct answer.

A. 5 **B.** -4 **C.** -2 **D.** 1 **E.** 0

4. MC A solution of $x^3 - 9x^2 + 15x + 25 = 0$ is $x = 5$. How many other (distinct) solutions are there?

A. 0 **B.** 1 **C.** 2 **D.** 3 **E.** 4

5. Solve $P(x) = 0$ for each of the following.

a. $P(x) = x^3 + 4x^2 - 3x - 18$ **b.** $P(x) = 3x^3 - 13x^2 - 32x + 12$

c. $P(x) = -x^3 + 12x - 16$ **d.** $P(x) = 8x^3 - 4x^2 - 32x - 20$

e. $P(x) = x^4 + 2x^3 - 13x^2 - 14x + 24$ **f.** $P(x) = -72 - 42x + 19x^2 + 7x^3 - 2x^4$

g. $P(x) = x^4 + 2x^3 - 7x^2 - 8x + 12$ **h.** $P(x) = 4x^4 + 12x^3 - 24x^2 - 32x$

6. Solve each of the following equations.

a. $x^3 - 3x^2 - 6x + 8 = 0$ **b.** $x^3 + x^2 - 9x - 9 = 0$

c. $3x^3 + 3x^2 - 18x = 0$ **d.** $2x^4 + 10x^3 - 4x^2 - 48x = 0$

e. $2x^4 + x^3 - 14x^2 - 4x + 24 = 0$ **f.** $x^4 - 2x^2 + 1 = 0$

Communicating, reasoning and problem solving

7. Solve for a if $x = 2$ is a solution of $ax^3 - 6x^2 + 3x - 4 = 0$.

8. Solve for p if $x = \frac{p}{2}$ is a solution of $x^3 - 5x^2 + 2x + 8 = 0$.

9. Show that it is possible for a cuboid of side lengths x cm, $(x - 1)$ cm and $(x + 2)$ cm to have a volume that is 4 cm^3 less than twice the volume of a cube of side length x cm. Comment on the shape of such a cuboid.

10. Solve the following for x.

$x^3 + 8 = x(5x - 2)$

11. Solve the following for z.

$z(z - 1)^3 = -2(z^3 - 5z^2 + z + 3)$

12. Can you predict the number of solutions a polynomial might have?

17.9 Review

Investigation | Rich task

Investigating polynomials

A polynomial is a function involving the sum of integer powers of a variable (for example, $y = -4x^3 + 3x^2 - 4$). The highest power of the variable determines the degree of the polynomial. In the case of the given example, the degree is 3. A polynomial of the first degree is a linear function (for example, $y = 3x - 8$), and a second-degree function is a quadratic (for example, $y = 5x^2 - 6x + 7$). Let us investigate how the degree of a polynomial affects the shape of its graph.

Use digital technology to help you with answering the following questions.

1. Consider the following polynomials.

a. $y_1 = (x + 1)$

b. $y_2 = (x + 1)(x - 2)$

c. $y_3 = (x + 1)(x - 2)(x + 3)$

d. $y_4 = (x + 1)(x - 2)(x + 3)(x - 4)$

e. $y_5 = (x + 1)(x - 2)(x + 3)(x - 4)(x + 5)$

f. $y_6 = (x + 1)(x - 2)(x + 3)(x - 4)(x + 5)(x - 6)$

For each of the functions:

i. give the degree of the polynomial

ii. sketch the graph, marking in the x-intercepts

iii. describe how the degree of the polynomial affects the shape of the graph.

2. Let us now look at the effect that the exponent of each factor has on the shape of the graph of the polynomial. Consider the following functions.

a. $y_1 = (x + 1)(x - 2)(x + 3)$

b. $y_2 = (x + 1)^2(x - 2)(x + 3)$

c. $y_3 = (x + 1)^2(x - 2)^2(x + 3)$

d. $y_4 = (x + 1)^2(x - 2)(x + 3)^3$

e. $y_5 = (x + 1)^3(x - 2)(x + 3)^4$

f. $y_6 = (x + 1)^5(x - 2)^3(x + 3)^2$

i. On a separate sheet of paper, draw a sketch of each of the polynomials, marking in the x-intercepts.

ii. Explain how the power of the factor affects the behaviour of the graph at the x-intercept.

3. Create and draw a sketch of polynomials with the following given characteristics. Complete your graphs on a separate sheet of paper.

a. A first-degree polynomial that:

i. crosses the x-axis

ii. does not cross the x-axis

b. A second-degree polynomial that:

i. crosses the x-axis twice

ii. touches the x-axis at one and only one point

c. A third-degree polynomial that crosses the x-axis:

i. three times

ii. twice

iii. once

d. A fourth-degree polynomial that crosses the x-axis:

i. four times

ii. three times

iii. twice

iv. once

4. Considering the powers of factors of polynomials, write a general statement outlining the conditions under which the graph of a polynomial will pass through the x-axis or just touch the x-axis.

learnon RESOURCES — ONLINE ONLY

Digital doc: Investigation — Investigating polynomials (doc-15948)

Digital doc: Code puzzle — Who were Australia's first three Prime Ministers? (doc-15949)

Digital doc: Summary Topic 17 (doc-22915)

Interactivity: Word search Topic 17 (int-2874)

Interactivity: Crossword Topic 17 (int-2875)

Interactivity: Sudoku Topic 17 (int-3892)

Exercise 17.9 Review questions

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. MC Which of the following is *not* a polynomial?

A. $x^3 - \frac{x^2}{3} + 7x - 1$ **B.** $a^4 + 4a^3 + 2a + 2$ **C.** $\sqrt{x^2 + 3x + 2}$

D. 5 **E.** x^2

2. Consider the polynomial $f(x) = -\frac{1}{7}x^4 + x^5 + 3$.

a. What is the degree of $f(x)$? **b.** What is the coefficient of x^4?

c. What is the constant term? **d.** What is the leading term?

3. MC The expansion of $(x + 5)(x + 1)(x - 6)$ is:

A. $x^3 - 30$ **B.** $x^3 + 12x^2 - 31x + 30$ **C.** $x^3 - 31x - 30$

D. $x^3 + 5x^2 - 36x - 30$ **E.** $x^3 + 12x^2 + 41x + 30$

4. MC $x^3 + 5x^2 + 3x - 9$ is the expansion of:

A. $(x + 3)^3$ **B.** $x(x + 3)(x - 3)$ **C.** $(x - 1)(x + 3)^2$

D. $(x - 1)(x + 1)(x + 3)$ **E.** $(x - 3)(x + 1)(x + 3)$

5. Expand each of the following.

a. $(x - 2)^2(x + 10)$ **b.** $(x + 6)(x - 1)(x + 5)$

c. $(x - 7)^3$ **d.** $(5 - 2x)(1 + x)(x + 2)$

6. MC Consider the following long division.

$$
\begin{array}{r}
x^2 + x + 2 \\
x - 4 \overline{)\, x^3 + 5x^2 + 6x - 1} \\
\underline{x^3 + 4x^2} \\
x^2 + 6x \\
\underline{x^2 + 4x} \\
2x - 1 \\
\underline{2x + 8} \\
-9
\end{array}
$$

a. The quotient is:

A. –9 **B.** 4 **C.** $x + 4$ **D.** $x^2 + x + 2$ **E.** $x - 4$

b. The remainder is:

A. –9 **B.** 2 **C.** 4 **D.** $2x - 1$ **E.** $x - 4$

7. Divide $P(x)$ by the linear expression and give the answer in the form $P(x) = (x - a)Q(x) + c$.

a. $P(x) = x^3 + 2x^2 - 16x - 3, x + 2$

b. $P(x) = x^3 + 3x^2 - 13x - 7, x - 3$

c. $P(x) = -x^3 + x^2 + 4x - 7, x + 1$

8. MC If $P(x) = x^3 - 3x^2 + 7x + 1$, then $P(-2)$ equals:

A. –34 **B.** –33 **C.** –9 **D.** 7 **E.** 11

9. If $P(x) = -3x^3 + 2x^2 + x - 4$, find:

a. $P(1)$ **b.** $P(-4)$ **c.** $P(2a)$

10. Without dividing, find the remainder when $x^3 + 3x^2 - 16x + 5$ is divided by $x - 1$.

11. Show that $x + 3$ is a factor of $x^3 - 2x^2 - 29x - 42$.

12. Factorise $x^3 + 4x^2 - 100x - 400$.

13. Solve:

a. $(2x + 1)(x - 3)^2 = 0$

b. $x^3 - 9x^2 + 26x - 24 = 0$

c. $x^4 - 4x^3 - x^2 + 16x - 12 = 0$

Communicating, reasoning and problem solving

14. Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial where the coefficients are integers. Also let $P(w) = 0$ where w is an integer. Show that w is a factor of a_0.

15. Find the area of a square whose sides are $(2x - 3)$ cm. Expand and simplify your answer. If the area is 16 cm^2, find x.

16. A window is in the shape of a semicircle above a rectangle. The height of the window is $(6x + 1)$ cm and its width is $(2x + 2)$ cm.

a. Find the total area of the window.

b. Expand and simplify your answer.

c. What is the perimeter of the window?

17. a. Find the volume of a cube of side $(x + 4)$ cm.

b. Find the surface area of the cube.

c. Find the value of x for which the volume and surface are numerically equal.

d. Find x if the numerical value of the volume is 5 less than the numerical value of the surface area.

18. Find the quotient and remainder when $mx^2 + nx + q$ is divided by $(x - p)$.

19. When $P(x)$ is divided by $(x - n)$, the quotient is $x^2 - 2x + n$ and the remainder is $(n + 1)$. Find $P(x)$.

Answers

Topic 17 Polynomials

Exercise 17.2 Polynomials

1. a. 3 **b.** 7 **c.** 2 **d.** 6 **e.** 8
f. 5 **g.** 5 **h.** 1 **i.** 6

2. a. x **b.** x **c.** x **d.** x **e.** y
f. u **g.** e **h.** g **i.** f

3. a. Polynomial 1h **b.** Polynomial 1c
c. Polynomial 1a **d.** Polynomials 1a, 1d and 1e

4. a. N **b.** P **c.** P **d.** N **e.** N
f. P **g.** P **h.** N **i.** N

5. a. 3 **b.** x **c.** 4 **d.** 5 **e.** $3x$ **f.** $-2x^3$

6. a. 7 **b.** w **c.** 7 **d.** 0 **e.** -9 **f.** $6w^7$

7. a. 4 **b.** 1 **c.** x^4 **d.** 1

8. a. 6 **b.** t **c.** 6 **d, e.** Check with your teacher.

9. a. Start at 5 units to the right of the origin. **b.** 4 units to the right of the origin
c. The body moves towards the origin, then away.

10. a. $P(x) = 4x^3 + 2x^2 - 10x + 18$ **b.** $P(x) = 3x^4 - 3x^3 - x^2 + 7x - 7$ **c.** $P(x) = 5x^3 - 4x^2 - 13x - 6$

11. $a = 4$, $b = -6$ **12.** $a = \pm 3$, $b = \pm 2$

13. The degree of a polynomial is the highest power of x.

Exercise 17.3 Adding, subtracting and multiplying polynomials

1. a. $x^4 + 2x^3 - x^2 - 10$ **b.** $x^6 + 2x^4 - 3x^3 + 9x^2 + 5$ **c.** $5x^3 - 5x^2 + 7x - 13$
d. $2x^4 + 3x^3 + 12x^2 - 4x + 14$ **e.** $x^5 + 13x^4 - 10$

2. a. $x^4 + 2x^2 + 2x + 4$ **b.** $x^6 - x^5 + x^3 + x^2 + 2$ **c.** $5x^7 - 4x^3 + 5x$
d. $10x^4 - 7x^2 + 20x + 5$ **e.** $2x^3 + 6x^2 - 10x + 15$

3. a. $x^3 + 7x^2 + 6x$ **b.** $x^3 - 7x^2 - 18x$ **c.** $x^3 + 8x^2 - 33x$
d. $2x^3 + 10x^2 + 12x$ **e.** $48x - 3x^3$ **f.** $5x^3 + 50x^2 + 80x$
g. $x^3 + 4x^2$ **h.** $2x^3 - 14x^2$ **i.** $-30x^3 - 270x^2$
j. $-7x^3 - 56x^2 - 112x$

4. a. $x^3 + 12x^2 + 41x + 42$ **b.** $x^3 - 3x^2 - 18x + 40$ **c.** $x^3 + 3x^2 - 36x + 32$
d. $x^3 - 6x^2 + 11x - 6$ **e.** $x^3 + 6x^2 - x - 6$ **f.** $x^3 + 5x^2 - 49x - 245$
g. $x^3 + 4x^2 - 137x - 660$ **h.** $x^3 + 3x^2 - 9x + 5$ **i.** $x^3 - 12x^2 + 21x + 98$
j. $x^3 + x^2 - x - 1$

5. a. $x^3 + 13x^2 + 26x - 112$ **b.** $3x^3 + 26x^2 + 51x - 20$ **c.** $4x^4 + 3x^3 - 37x^2 - 27x + 9$
d. $10x^3 - 49x^2 + 27x + 36$ **e.** $-6x^3 - 71x^2 - 198x + 35$ **f.** $21x^4 - 54x^3 - 144x^2 + 96x$
g. $54x^3 + 117x^2 - 72x$ **h.** $24x^3 - 148x^2 + 154x + 245$ **i.** $20x^4 - 39x^3 - 50x^2 + 123x - 54$
j. $4x^3 + 42x^2 + 146x + 168$

6. a. $x^3 + 6x^2 + 12x + 8$ **b.** $x^3 + 15x^2 + 75x + 125$ **c.** $x^3 - 3x^2 + 3x - 1$
d. $x^4 - 12x^3 + 54x^2 - 108x + 81$ **e.** $8x^3 - 72x^2 + 216x - 216$ **f.** $81x^4 + 432x^3 + 864x^2 + 768x + 256$

7. $(2a + 5b)x + (2b - 5c)$

8. $x^4 + (a - 4b)x^3 + (2a - 4ab + 3b^2)x^2 + (2a^2 - 2ab + 3ab^2)x - 2a^2b$

9. $a = 1$, $b = -12$, $c = 54$, $d = -108$, $e = 81$

10. $8x^3 - 45x^2 + 78x - 43$ **11.** $\frac{1}{8}(11x^3 - 105x^2 + 73x - 27)$

12. $a = 1$, $b = -2$ and $c = 1$ **13.** $a = 1$, $b = 4$ and $c = -1$

14. Add or subtract polynomials by adding or subtracting like terms in the expression.

Exercise 17.4 Long division of polynomials

1. a. $x^2 + 2x, 9$ **b.** $x^2 + x + 3, -2$ **c.** $x^2 + 3x - 6, 19$ **d.** $x^2 - x + 5, -17$
e. $x^2 + 2x - 1, 6$ **f.** $x^2 + 4x - 6, 14$ **g.** $x^2 + 1, 2$ **h.** $x^2 + 5, -36$
i. $x^2 - x + 6, -11$ **j.** $x^2 + 4x - 17, 87$

2. a. $x^2 + 4x + 3, -3$ **b.** $x^2 + 4x + 13, 48$ **c.** $x^2 + 3x - 3, -11$ **d.** $x^2 - 3x + 7, 5$
e. $x^2 - 2x - 3, -17$ **f.** $x^2 - 6x + 3, -4$ **g.** $x^2 + 14x + 72, 359$ **h.** $x^2 + 8x + 27, 104$

3. a. $3x^2 - 7x + 20, -35$ **b.** $4x^2 - 8x + 18, -22$ **c.** $2x^2 - 3x + 3, 7$ **d.** $2x^2 - 9, 35$
e. $4x^2 + 2x - 3, -1$ **f.** $3x^2 + x - 1, -2$

4. a. $3x^2 - 2x + 1, 5$ **b.** $2x^2 + 5x - 6, -7$ **c.** $4x^2 - 7x - 2, -3$ **d.** $x^2 - 4x + 3, 8$
e. $x^2 + x - 6, -11$ **f.** $3x^2 + 2x + 1, 13$

5. a. $-x^2 - 5x - 2, -14$ **b.** $-3x^2 - 2x + 4, -3$ **c.** $-x^2 + 5x + 6, 9$ **d.** $-2x^2 + 7x - 1, 1$

6. a. $x^2 - x - 2, 3$ **b.** $x^2, -7$ **c.** $x^2 - x - 2, -8$ **d.** $-x^2 - x - 8,\ 0$
e. $5x - 2, 7$ **f.** $2x^2 - 2x + 10, -54$ **g.** $-2x^2 - 4x - 9, -16$ **h.** $-2x^2 + 4x - 1, 1$

7. a. $x^4 + x^3 + 3x^2 - 7x = (x - 1)(x^3 + 2x^2 + 5x - 2) - 2$
b. $x^4 - 13x^2 + 36 = (x - 2)(x^3 + 2x^2 - 9x - 18) + 0$
c. $x^5 - 3x^3 + 4x + 3 = (x + 3)(x^4 - 3x^3 + 6x^2 - 18x + 58)\ - 171$
d. $2x^6 - x^4 + x^3 + 6x^2 - 5x = (x + 2)(2x^5 - 4x^4 + 7x^3 - 13x^2 + 32x - 69) +\ 138$
e. $6x^4 - x^3 + 2x^2 - 4x = (x - 3)(6x^3 + 17x^2 + 53x + 155) + 465$
f. $3x^4 - 6x^3 + 12x = (3x + 1)\left(x^3 - \frac{7}{3}x^2 + \frac{7}{9}x + 3\frac{20}{27}\right) - 3\frac{20}{27}$

8. Quotient $= ax + (b + ad)$
Remainder $= c + d(b + ad)$

9. $q = \dfrac{2p}{c}$ **10.** $a = -3$ **11.** $a = 3, b = -5$

12. An alternative way to divide a polynomial is to use equations. For example, a cubic divided by a linear expression should result in a quadratic quotient and a remainder.
$(x^3 - 3x + 1) = (x + 1)(ax^2 + bx + c) + R$
Expand the right-hand side and equate the coefficients.

Exercise 17.5 Polynomial values

1. a. 10 **b.** 11 **c.** 18 **d.** 43
e. 3 **f.** –22 **g.** –77 **h.** $2a^3 - 3a^2 + 2a + 10$
i. $16b^3 - 12b^2 + 4b + 10$ **j.** $2x^3 + 9x^2 + 14x + 18$ **k.** $2x^3 - 21x^2 + 74x - 77$ **l.** $-128y^3 - 48y^2 - 8y + 10$

2 to 6.

Column 1	Column 2	Column 3	Column 4	Column 5	Column 6	Column 7	Column 8	Column 9
$P(x)$	$P(1)$	$P(2)$	$P(-1)$	$P(-2)$	Rem when divided by $(x - 1)$	Rem when divided by $(x - 2)$	Rem when divided by $(x + 1)$	Rem when divided by $(x + 2)$
a	4	15	0	−5	4	15	0	−5
b	10	28	−2	−8	10	28	−2	−8
c	3	11	−7	−21	3	11	−7	−21
d	−7	−19	5	−7	−7	−19	5	−7

7. a. $P(-8)$ **b.** $P(7)$ **c.** $P(a)$

8. a. –1 **b.** –6249 **c.** $2(a - 5)^5 + 1$ **d.** $-2(2a + 5)^5 + 1$

9. $b = 1, 4$

10. a. $-2a^3 - 3a^2 + a + 4$ **b.** $-2a^3 - 9a^2 - 11a - 1$

11. $c = 2$

12. $b = -2, c = 5$

13. When $P(x)$ is divided by $(x - a)$, the remainder is $P(a)$.

Exercise 17.6 The remainder and factor theorems

1. a. -30 **b.** 0 **c.** 0 **d.** -24
e. -24 **f.** $k^3 + 3k^2 - 10k - 24$ **g.** $-n^3 + 3n^2 + 10n - 24$ **h.** $-27c^3 + 27c^2 + 30c - 24$

2. a. 58 **b.** -8 **c.** 11 **d.** -9
e. -202 **f.** 6 **g.** 158 **h.** -6
i. 35 **j.** 441

3. a. 6 **b.** 3 **c.** 1 **d.** -2
e. 2 **f.** 2 **g.** $-5, 2$ **h.** $a = -5, b = -3$

4. a. D **b.** C, D **c.** D **d.** A, C, D

5. a. $(x - 1)$ **b.** $(x - 3)$ or $(x - 2)$
c. $(x - 3)$ or $(x + 2)$ **d.** $(x - 6)$ or $(x + 4)$ or $(x + 5)$

6. Show $P(-2) = 0$, $P(3) = 0$ and $P(-5) = 0$.

7. a. Show $P(1) = 0$. **b.** Show $P(7) = 0$. **c.** Show $P(2) = 0$. **d.** Show $P(-2) = 0$.
e. Show $P(-3) = 0$. **f.** Show $P(1) = 0$. **g.** Show $P(4) = 0$. **h.** Show $P(-5) = 0$.

8. $a = 3, b = 2$ **9.** $a = -5, b = 41$, $(x + 3)$ and $(x - 5)$

10. If $P(a) = 0$, then $(x - a)$ is a factor of $P(x)$. **11.** $k = -4$

Exercise 17.7 Factorising polynomials

1. a. $(x + 1)(x + 3)(x + 6)$ **b.** $(x + 1)(x + 2)(x + 5)$ **c.** $(x + 1)(x + 2)(x + 9)$ **d.** $(x + 1)(x + 3)(x + 4)$
e. $(x + 3)(x + 4)(x + 7)$ **f.** $(x + 2)(x + 3)(x + 7)$ **g.** $(x + 1)^2(x + 2)$ **h.** $(x + 2)^2(x + 3)$
i. $(x + 4)(x + 5)^2$ **j.** $x(x + 5)(x + 8)$ **k.** $x(x + 3)(x + 4)$ **l.** $x(x + 5)^2$
m. $x(x + 1)(x + 5)$ **n.** $x^2(x + 6)$

2. a. $(x - 1)(x + 1)^2$ **b.** $(x - 2)(x - 1)(x + 1)$ **c.** $(x + 1)^2(x + 5)$ **d.** $(x - 3)(x + 2)^2$
e. $(x + 1)(x + 4)^2$ **f.** $(x - 5)(x - 2)(x + 2)$ **g.** $(x - 1)(x + 1)(x + 2)$ **h.** $(x - 3)(x + 1)(x + 2)$
i. $(x - 1)(x + 2)^2$ **j.** $(x + 2)(x^2 - x + 3)$ **k.** $(x + 1)(x + 2)(x + 5)$ **l.** $(x - 3)(x + 1)(x + 3)$
m. $(x - 2)^2(x + 3)$ **n.** $(x - 4)(x + 5)(x + 8)$

3. a. $(2x + 3)(x - 1)(x + 2)$ **b.** $(3x - 1)(x + 1)(x + 4)$ **c.** $(3x + 2)(x - 2)(x + 2)$ **d.** $(4x + 3)(x + 3)(x + 5)$
e. $(5x - 1)(x + 1)^2$ **f.** $(x + 1)(x^2 + 1)$ **g.** $(x + 1)(2x + 3)^2$ **h.** $(x - 2)(2x - 1)(3x - 4)$
i. $(x + 4)(2x - 5)(5x + 2)$ **j.** $(7x - 2)(x - 2)(x + 4)$

4. a. $x(x - 2)(3x + 5)$ **b.** $2x(x + 1)(2x - 1)$ **c.** $3x(x - 4)(x + 2)$ **d.** $-2x(x + 3)^2$
e. $6x^2(x - 1)$ **f.** $-x(x + 4)(x + 3)$ **g.** $-(x - 1)(x + 1)(x + 3)$ **h.** $-2x(x - 3)(x - 2)$
i. $-(x + 2)(2x - 1)(3x - 2)$ **j.** $-(x - 2)^2(5x - 4)$ **k.** $-(x - 1)(x + 3)(x - 5)(x + 2)^2$

5. $(x - 1)(x + 2)(x + 2)(x - 3)$ **6.** $-(x - 2)(x + 2)(x + 3)(x - 4)(x - 5)$

7. $(x + p - (a + b))$ **8.** $a = -2, b = 4$, $(x - 1)^2(x + 1)^2(x - 2)$

9. $x(x - 1)(x + 1)(x - 2)(x - 3)$

10. Step 1: Use the factor theorem to find a linear factor of the polynomial.
Step 2: Divide the polynomial by the linear factor
Step 3: Repeat step 1 and step 2 until the quotient is a quadratic then factorise the quadratic.

11. The other two factors are $(x - 4)$ and $(x + 1)$.

Exercise 17.8 Solving polynomial equations

1. a. $-2, 0, 2$ **b.** $-4, 0, 4$ **c.** $-5, 0, 5$ **d.** 3
e. $-5, 0$ **f.** $0, 2$ **g.** $-\sqrt{2},\ 0,\ \sqrt{2}$ **h.** $-\frac{1}{4}, 0$
i. $0, \frac{1}{5}$ **j.** $0, 2, 3$ **k.** $0, 4$ **l.** $-7, 0, 1$
m. $0, 4, 5$ **n.** 0

2. a. $-4, 1, 4$ **b.** $-2, 3, 5$ **c.** $-5, 1, 5$ **d.** $-4, -2, 2$
e. $-1, 2, 3$ **f.** $-2, 1, 5$ **g.** $-3, -2, -1$ **h.** $-4, 5$
i. $-2, 1, 4$ **j.** $-7, 2, 3$ **k.** $-6, -\frac{1}{2}, -1$ **l.** $-\frac{1}{2}, \frac{3}{2}, 3$
m. $-3, -2, \frac{1}{2}$ **n.** $-2, -1, 1$

3. A, C

4. B

5. a. –3, 2 **b.** $-2, \frac{1}{3}, 6$ **c.** –4, 2 **d.** $-1, \frac{5}{2}$

e. –4, –2, 1, 3 **f.** $-2, -\frac{3}{2}, 3, 4$ **g.** –3, –2, 1, 2 **h.** –4, –1, 0, 2

6. a. –2, 1, 4 **b.** –3, –1, 3 **c.** –3, 0, 2 **d.** –4, –3, 0, 2

e. $-2, \frac{3}{2}, 2$ **f.** –1, 1

7. 2.75

8. –2, 4, 8

9. Proof — check with your teacher.

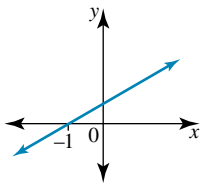
10. $x = -1$, 4 and 2

11. $z = -1$, 1, -2 and 3

12. From previous studies of quadratics, we know there may be 0, 1 or 2 real solutions to a quadratic equation. If we take into consideration complex roots (to be studied in higher levels) the highest degree of a polynomial is also the number of solutions.

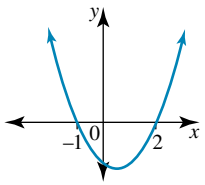
Investigation | Rich task

1. a. i. 1 **ii.**



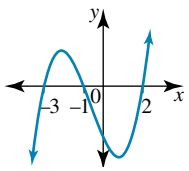
iii. The graph is linear and crosses the x-axis once (at $x = -1$).

b. i. 2 **ii.**



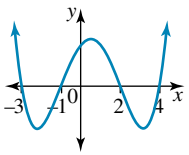
iii. The graph is quadratic and crosses the x-axis twice (at $x = -1$ and $x = 2$).

c. i. 3 **ii.**



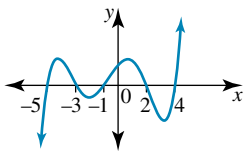
iii. The graph is a curve and crosses the x-axis 3 times (at $x = -1$, $x = 2$ and $x = -3$).

d. i. 4 **ii.**



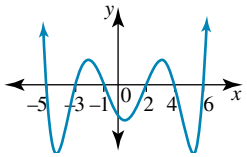
iii. The graph is a curve and crosses the x-axis 4 times (at $x = -1$, $x = 2$, $x = -3$ and $x = 4$).

e. i. 5 **ii.**



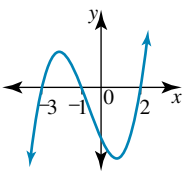
iii. The graph is a curve and crosses the x-axis 5 times (at $x = -1$, $x = 2$, $x = -3$, $x = 4$ and $x = -5$).

f. i. 6 **ii.**



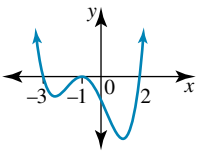
iii. The graph is a curve and crosses the x-axis 6 times (at $x = -1$, $x = 2$, $x = -3$, $x = 4$, $x = -5$ and $x = 6$).

2. a. i.



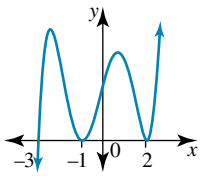
ii. Each factor is raised to the power 1. The polynomial is of degree 3 and the graph crosses the x-axis in 3 places (–3, –1 and 2).

b. i.



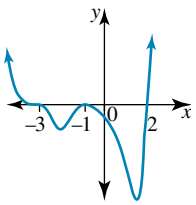
ii. The factor $(x + 1)$ is raised to the power 2 and the other two factors are raised to the power 1. The power 2 causes the curve not to cross the x-axis at $x = -1$ but to be curved back on itself.

c. i.



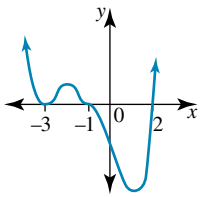
ii. The power 2 on the two factors $(x + 1)$ and $(x - 2)$ causes the curve to be directed back on itself and not to cross the x-axis at those two points ($x = -1$ and $x = 2$).

d. i.



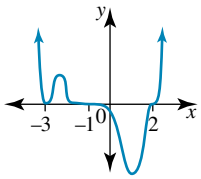
ii. The power 3 on the factor $(x + 3)$ causes the curve to run along the axis at that point, then cross the axis (at $x = -3$).

e. i.



ii. The power 3 on the factor $(x + 1)$ causes the curve to run along the axis at $x = -1$, then cross the axis. The power 4 on the factor $(x + 3)$ causes the curve to be directed back on itself without crossing the axis at $x = -3$.

f. i.



ii. The power 5 on the factor $(x + 1)$ causes the curve to run along the axis at $x = -1$, then cross the axis.

3. Answers will vary; check with your teacher. Possible answers could be as follows.

a. i. $y = 3x + 2$ **ii.** $y = 4$

b. i. $y = (x + 1)(x + 2)$ **ii.** $y = (x + 1)^2$

c. i. $y = (x + 1)(x + 2)(x + 3)$ **ii.** Not possible **iii.** $y = (x + 1)^2(x + 2)$

d. i. $y = (x + 1)(x + 2)(x + 3)(x + 4)$ **ii.** Not possible

iii. $y = (x + 1)^2(x + 2)(x + 3)$, $y = (x + 1)^3(x + 2)$ **iv.** Not possible

4. If the power of the factor of a polynomial is an odd integer, the curve will pass through the x-axis. If the power is 1, the curve passes straight through. If the power is 3, 5 …, the curve will run along the x-axis before passing through it. On the other hand, an even power of a factor causes the curve to just touch the x-axis then move back on the same side of the x-axis.

Exercise 17.9 Review questions

1. C

2. a. 5 **b.** $-\frac{1}{7}$ **c.** 3 **d.** x^5

3. C **4.** C

5. a. $x^3 + 6x^2 - 36x + 40$ **b.** $x^3 + 10x^2 + 19x - 30$ **c.** $x^3 - 21x^2 + 147x - 343$ **d.** $-2x^3 - x^2 + 11x + 10$

6. a. D **b.** A

7. a. $x^3 + 2x^2 - 16x - 3 = (x + 2)(x^2 - 16) + 29$

b. $x^3 + 3x^2 - 13x - 7 = (x - 3)(x^2 + 6x + 5) + 8$

c. $-x^3 + x^2 + 4x - 7 = (x + 1)(-x^2 + 2x + 2) - 9$

8. B

9. a. –4 **b.** 216 **c.** $-24a^3 + 8a^2 + 2a - 4$

10. –7 **11.** Show $P(-3) = 0$. **12.** $(x - 10)(x + 4)(x + 10)$

13. a. $-\frac{1}{2}$, 3 **b.** 2, 3, 4 **c.** –2, 1, 2, 3

14. Teacher to check. For example, given $P(x) = x^3 - x^2 - 34x - 56$ and $P(7) = 0$, $(x - 7)$ is a factor and 7 is a factor of 56.

15. $4x^2 - 12x + 9$; $x = -\frac{1}{2}, \frac{7}{2}$

16. a, b. Area $= \left(\frac{1}{2}\pi + 10\right)x^2 + (\pi + 10)x + \frac{\pi}{2}$ **c.** Perimeter $= (12 + \pi)x + (2 + \pi)$

17. a. $(x + 4)^3$ **b.** $6(x + 4)^2$

c. $x = 2$ **d.** $-3, \frac{-3 + 3\sqrt{5}}{2}, \frac{-3 - 3\sqrt{5}}{2}$

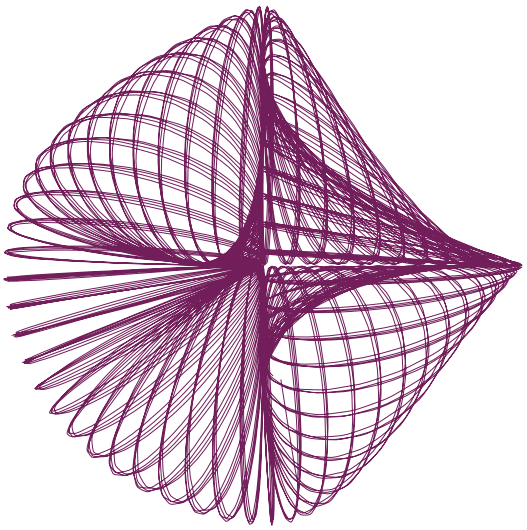
18. $mx + (n + mp)$; $q + p(n + mp)$

19. $x^3 - (2 + n)x^2 + 3nx - (n^2 - n - 1)$

TOPIC 18
Functions and relations [Stage 5.3]

18.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the concepts covered in this topic.



18.1.1 Why learn this?

A relation is a set of ordered pairs; functions are special types of relations. It is important to understand relationships between variables in order to be able to model the relationships. Many different functions can be used to model events in the world. You have already studied lines of best fit, where straight lines are fitted to data in order to make predictions. Exponential functions are used to model growth and decay. There are many examples of functions used in engineering, science, finance, architecture and medicine.

DISCUSSION

Persistence, exemplified by Amalie Noether, is seen as one of the most important factors in identifying how successful we become in adult life. Why do you think this is? How would you rate your persistence?

LEARNING SEQUENCE

18.1 Overview
18.2 [Stage 5.3] Functions and relations
18.3 [Stage 5.3] Exponential functions
18.4 [Stage 5.3] Cubic functions
18.5 [Stage 5.3] Quartic functions
18.6 [Stage 5.3] Transformations
18.7 Review

LEARNING OUTCOMES

A student:

- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3-1WM
- generalises mathematical ideas and techniques to analyse and solve problems efficiently MA5.3-2WM
- uses deductive reasoning in presenting arguments and formal proofs MA5.3-3WM
- sketches and interprets a variety of non-linear relationships MA5.3-9NA
- recognises, describes and sketches polynomials MA5.3-10NA
- uses function notation to describe and sketch functions MA5.3-12NA

CONTENT DESCRIPTIONS

Students:

Describe, interpret and sketch functions (NSW)
Describe, interpret and sketch cubics, other curves and their transformations (NSW)
Apply an understanding of polynomials to sketch a range of curves and describe the features of these curves from their equation (ACMNA268)

Source: NSW Syllabus for the Australian curriculum

learnon RESOURCES — ONLINE ONLY

eLesson: The story of mathematics — Amalie Noether (eles-2021)

Note: Your teacher may now set you a pre-test to determine how familiar you are with the content in this topic.

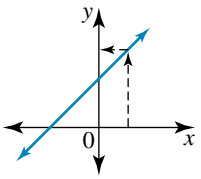
18.2 Functions and relations [Stage 5.3]

18.2.1 Relations

- A **relation** is a set of ordered pairs of values such as all the points on the circle $x^2 + y^2 = 4$ or all the points on the exponential $y = 2^x$. Relations can be grouped into the following four categories.

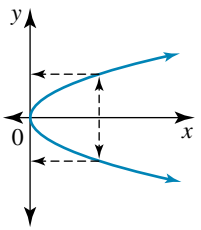
One-to-one relations

- A **one-to-one relation** exists if for any x-value there is only one corresponding y-value and vice versa. For example:



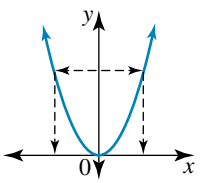
One-to-many relations

- A **one-to-many relation** exists if for any x-value there is more than one y-value, but for any y-value there is only one x-value. For example:



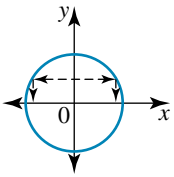
Many-to-one relations

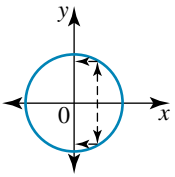
- A **many-to-one relation** exists if there is more than one x-value for any y-value but for any x-value there is only one y-value. For example:



Many-to-many relations

- A **many-to-many relation** exists if there is more than one x-value for any y-value and vice versa. For example:

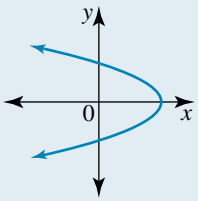




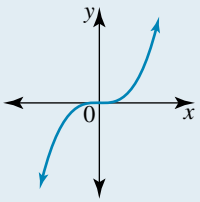
WORKED EXAMPLE 1

What type of relation does each graph represent?

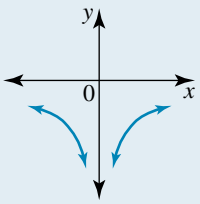
a



b

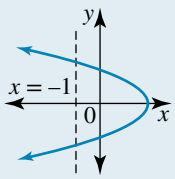


c

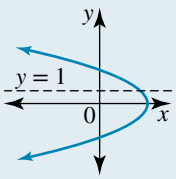


THINK

a 1 For some x-values there is more than one y-value. A line through some x-values shows that 2 y-values are available.



2 For any y-value there is only one x-value. A line through any y-value shows that only one x-value is available.



WRITE

a One-to-many relation

b 1 For any x-value there is only one y-value.

2 For any y-value there is only one x-value.

b One-to-one relation

c 1 For any x-value there is only one y-value.

2 For some y-values there is more than one x-value.

c Many-to-one relation

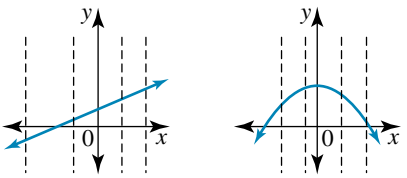
18.2.2 Functions

- Relations that are one-to-one or many-to-one are called **functions**. That is, a function is a relation where for any x-value there is at most one y-value.

Vertical line test

- To determine if a graph is a function, a vertical line is drawn anywhere on the graph. If it does not intersect with the curve more than once, then the graph is a function.

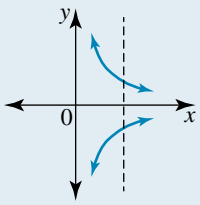
- For example, in each of the two graphs below, each vertical line intersects the graph only once.



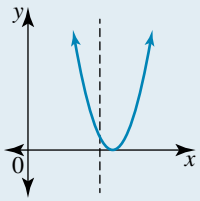
WORKED EXAMPLE 2

State whether or not each of the following relations are functions.

a



b



THINK	WRITE
a It is possible for a vertical line to intersect with the curve more than once.	**a** Not a function
b It is not possible for any vertical line to intersect with the curve more than once.	**b** Function

DISCUSSION

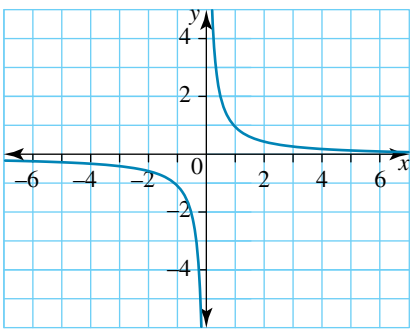
Do straight line graphs always, sometimes or never represent a function?

18.2.3 Function notation

- Consider the relation $y = 2x$, which is a function.
  The y-values are determined from the x-values, so we say 'y is a function of x', which is abbreviated to $y = f(x)$.
  Thus, the rule $y = 2x$ can also be written as $f(x) = 2x$.
  If $x = 1$, then $y = f(1)$
  $= 2 \times 1$
  $= 2$
  If $x = 2$, then $y = f(2)$
  $= 2 \times 2$
  $= 4$, and so on

Domain and range

- The **domain** of a function is the set of all allowable values of x. It is sometimes referred to as the **maximal domain**.
- The **range** of a function is the set of y-values produced by the function.
- For example, the domain of the function $f(x) = 2x + 3$ is the set of all real numbers ($x \in R$), and the range is the set of all real numbers ($y \in R$).
- The domain of the function $f(x) = \frac{1}{x}$ is the set of all real numbers apart from 0 ($x \in R\backslash 0$), and the range is the set of all real numbers apart from 0 ($y \in R\backslash 0$).



Evaluating functions

- For a given function $y = f(x)$, the value of y when $x = 1$ is written as $f(1)$, the value of y when $x = 5$ is written as $f(5)$, the value of y when $x = a$ as $f(a)$, and so on.

WORKED EXAMPLE 3

If $f(x) = x^2 - 3$, find:
a $f(1)$ **b** $f(a)$ **c** $3f(2a)$ **d** $f(a) + f(b)$ **e** $f(a + b)$

THINK	WRITE
a 1 Write the rule.	**a** $f(x) = x^2 - 3$
2 Substitute $x = 1$ into the rule.	$f(1) = 1^2 - 3$
3 Simplify.	$= 1 - 3$ $= -2$
b 1 Write the rule.	**b** $f(x) = x^2 - 3$
2 Substitute $x = a$ into the rule.	$f(a) = a^2 - 3$
c 1 Write the rule.	**c** $f(x) = x^2 - 3$
2 Substitute $x = 2a$ into the rule and simplify.	$f(2a) = (2a)^2 - 3$ $= 2^2a^2 - 3$ $= 4a^2 - 3$
3 Multiply the answer by 3 and simplify.	$3f(2a) = 3(4a^2 - 3)$ $= 12a^2 - 9$
d 1 Write the rule.	**d** $f(x) = x^2 - 3$
2 Evaluate $f(a)$.	$f(a) = a^2 - 3$
3 Evaluate $f(b)$.	$f(b) = b^2 - 3$
4 Evaluate $f(a) + f(b)$.	$f(a) + f(b) = a^2 - 3 + b^2 - 3$ $= a^2 + b^2 - 6$
e 1 Write the rule.	**e** $f(x) = x^2 - 3$
2 Evaluate $f(a + b)$.	$f(a + b) = (a + b)^2 - 3$ $= (a + b)(a + b) - 3$ $= a^2 + 2ab + b^2 - 3$

18.2.4 Identifying features of functions

- We can identify features of certain functions by observing what happens to the function value (y-value) when x approaches a very small value such as 0 ($x \to 0$) or a very large value such as ∞ ($x \to \infty$).

WORKED EXAMPLE 4

Describe what happens to these functions as the value of x increases, that is, as $x \to \infty$.
a $f(x) = x^2$ **b** $f(x) = 2^{-x}$ **c** $f(x) = \frac{1}{x} + 1$

THINK	WRITE
a 1 Write the function.	**a** $f(x) = x^2$
2 Substitute large x-values into the function, such as $x = 10000$ and $x = 1000000$.	$f(10000) = 100000000$ $f(1000000) = 1 \times 10^{12}$
3 Write a conclusion.	As $x \to \infty$, $f(x)$ also increases; that is, $f(x) \to \infty$.
b 1 Write the function.	**b** $f(x) = 2^{-x}$
2 Substitute large x-values into the function, such as $x = 10000$ and $x = 1000000$.	$f(10000) \approx 0$ $f(1000000) \approx 0$
3 Write a conclusion.	As $x \to \infty$, $f(x) \to 0$.
c 1 Write the function.	**c** $f(x) = \frac{1}{x} + 1$
2 Substitute large x-values into the function, such as $x = 10000$ and $x = 1000000$.	$f(10000) = 1.0001$ $f(1000000) = 1.000001$
3 Write a conclusion.	As $x \to \infty$, $f(x) \to 1$.

Points of intersection

- If two functions are drawn on the one set of axes, there may be a point or points where the curves intersect. The function equations can be solved simultaneously to find the coordinates of these points of intersection.

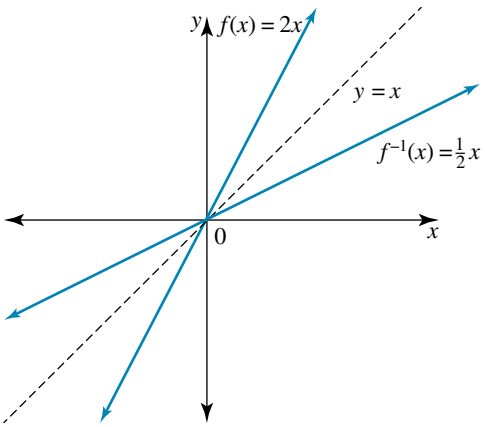
WORKED EXAMPLE 5

Find any points of intersection between $f(x) = 2x + 1$ and $g(x) = \frac{1}{x}$.

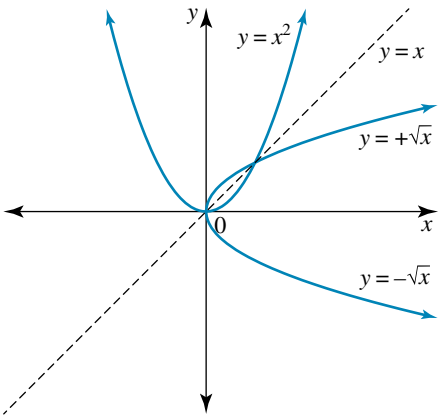
THINK	WRITE
1 Write the two equations.	$f(x) = 2x + 1$ $g(x) = \frac{1}{x}$
2 Points of intersection are common values between the two curves. To solve the equations simultaneously, equate both functions.	For points of intersection: $2x + 1 = \frac{1}{x}$
3 Rearrange the resulting equation and solve for x.	$2x^2 + x = 1$ $2x^2 + x - 1 = 0$ $(2x - 1)(x + 1) = 0$ $x = \frac{1}{2}$ or -1
4 Substitute the x-values into either function to find the y-values.	$f\left(\frac{1}{2}\right) = 2 \times \frac{1}{2} + 1 = 2$ $f(-1) = 2 \times -1 + 1 = -1$
5 Write the coordinates of the two points of intersection.	The points of intersection are $\left(\frac{1}{2}, 2\right)$ and $(-1, -1)$.

18.2.5 Inverse functions

- An inverse graph is created when a graph is reflected in the line $y = x$, the 45° line. Algebraically the resulting equation is created by interchanging x and y in the original equation.
- For example, take the function $f(x) = 2x$.
  Let $y = 2x$.
  Interchange x and y: $x = 2y$
  Make y the subject: $y = \frac{1}{2}x$
  Write the inverse function using function notation.
  The inverse of the function $f(x) = 2x$ is $f^{-1}(x) = \frac{1}{2}x$.



- Not all functions have inverses that are functions. For example, the inverse of the function $y = x^2$ is $x = y^2$ or $y = \pm\sqrt{x}$.

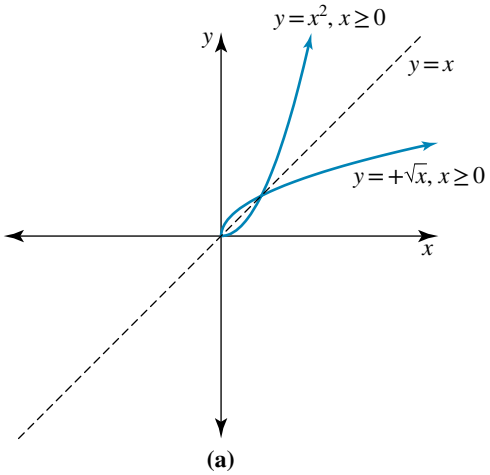


  – Notice that $y = \pm\sqrt{x}$ is not a function, as it is a one-to-many relation.
  – For this inverse to be a function, a restriction must be placed on the domain of the original quadratic function. Normally, either the part of the graph to the left of the vertex (turning point) or the part to the right is chosen. The original function is restated as two functions:

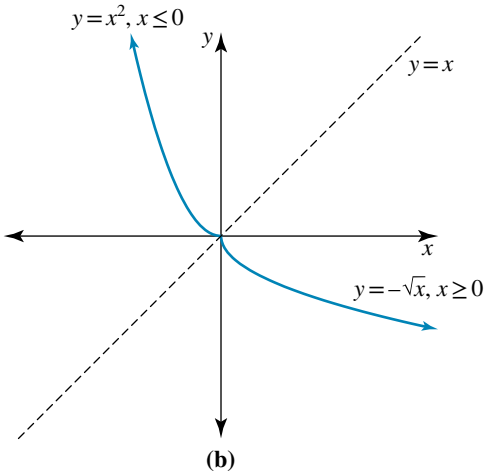
$$y = x^2,\ x \geq 0 \text{ and } y = x^2,\ x \leq 0.$$

  Thus the inverse functions are:

$$y = +\sqrt{x},\ x \geq 0 \text{ and } y = -\sqrt{x},\ x \geq 0.$$



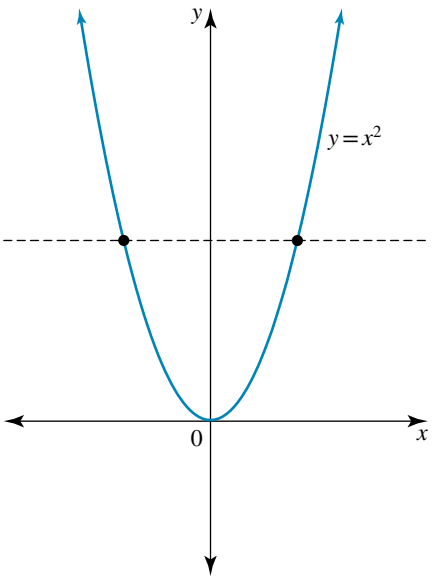
(a)

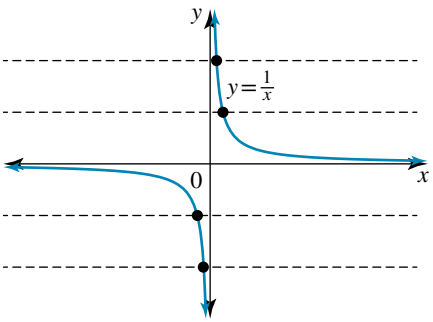


(b)

The horizontal line test

- Not all functions have inverses that are functions. For a function to have an inverse function, it must be a one-to-one function. It was shown above that $f(x) = x^2$ does not have an inverse function unless restrictions are placed.
- A function has an inverse function when a horizontal line cannot be drawn that cuts through the graph more than once.
- The inverse of the function $f(x) = x^2$ is not a function, as its graph does not satisfy the horizontal line test.
- The inverse of the function $f(x) = \frac{1}{x}$ is a function, as its graph satisfies the horizontal line test.





WORKED EXAMPLE 6

a. i Show that the function $f(x) = x(x - 5)$ will have not have an inverse function.
ii Suggest a restriction that would result in an inverse function.
b. i Show that the function $f(x) = x^2 + 4$, $x \geq 0$ will have an inverse function.
ii Determine the equation of the inverse function.

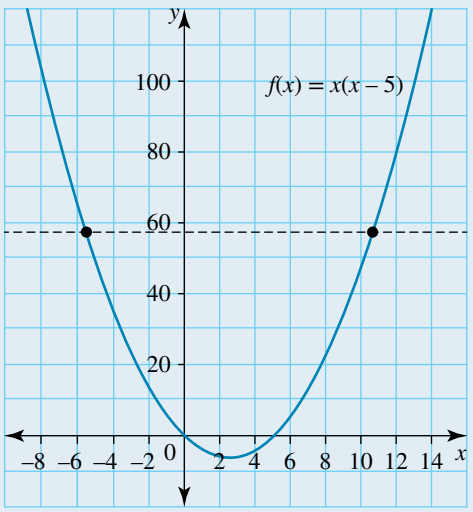
THINK

a i 1 Sketch the graph of $f(x) = x(x - 5)$.

2 Draw a dotted horizontal line(s) through the graph.

WRITE/DRAW

a i



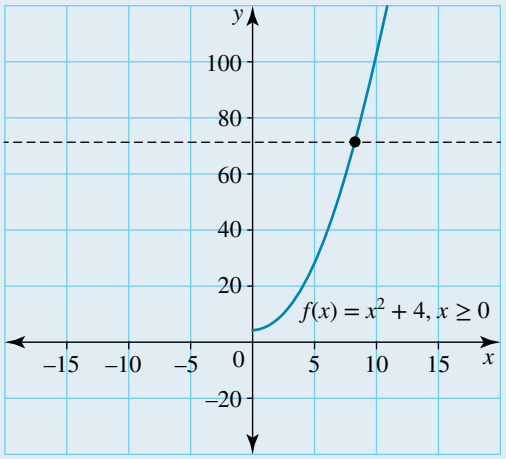
The graph does not satisfy the horizontal line test, so the function $f(x) = x(x - 5)$ will not have an inverse function.

ii Apply a restriction to the function so that it will have an inverse.

ii An inverse function will exist if $f(x) = x(x - 5), x \leq 2.5$ or $f(x) = x(x - 5), x \geq 2.5$.

b i **1** Sketch the graph of $f(x) = x^2 + 4, x > 0$.

b i



2 Draw a dotted horizontal line through the graph.

The graph satisfies the horizontal line test, so the function $f(x) = x^2 + 4, x \geq 0$ has an inverse function.

ii 1 Determine the equation of the inverse function by interchanging x and y and simplifying.

ii Let $y = x^2 + 4, x \geq 0$.
Interchange x and y.
$x = y^2 + 4$
Make y the subject.
$x = y^2 + 4$
$x - 4 = y^2$
$\sqrt{x - 4} = y$
$y = \sqrt{x - 4}$

2 Write the answer in correct form, noting the domain.

The inverse of $f(x) = x^2 + 4$ is $f^{-1}(x) = \sqrt{x - 4}, x \geq -4$.

learnon RESOURCES — ONLINE ONLY

Interactivity: Relations (int-6208)
Interactivity: Evaluating functions (int-6209)
Digital doc: SkillSHEET Finding the gradient and y-intercept (doc-5378)
Digital doc: SkillSHEET Sketching straight lines (doc-5379)
Digital doc: SkillSHEET Sketching parabolas (doc-5380)
Digital doc: SkillSHEET Completing the square (doc-5381)
Digital doc: SkillSHEET Identifying equations of straight lines and parabolas (doc-5382)
Digital doc: SkillSHEET Finding points of intersection (doc-5383)
Digital doc: SkillSHEET Substitution into index expressions (doc-5384)
Digital doc: WorkSHEET Functions and relations (doc-14622)

Exercise 18.2 Functions and relations

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–9, 11, 12	Questions: 1–3, 6, 8, 10, 11, 12, 14	Questions: 1–15

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE1 What type of relation does each graph represent?

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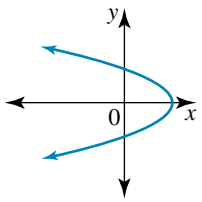

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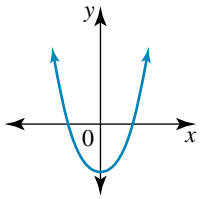

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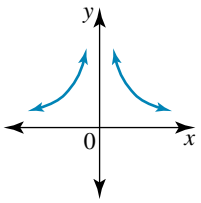

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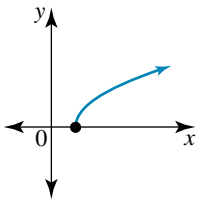

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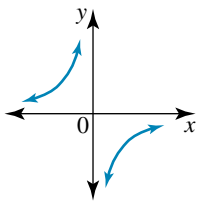

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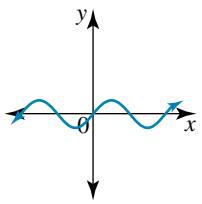

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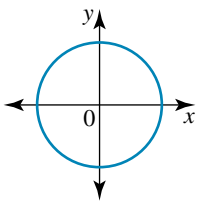

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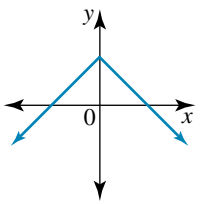

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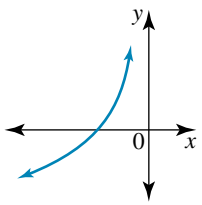

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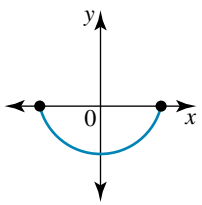

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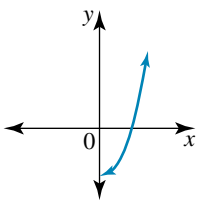

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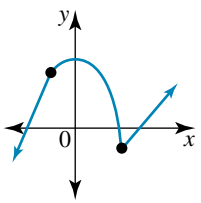


2. WE2 **a.** Use the vertical line test to determine which of the relations in question **1** are functions.

b. Which of these functions have inverses that are also functions?

3. WE3 **a.** If $f(x) = 3x + 1$, find:

i. $f(0)$ **ii.** $f(2)$ **iii.** $f(-2)$ **iv.** $f(5)$.

b. If $g(x) = \sqrt{x + 4}$, find:

i. $g(0)$ **ii.** $g(-3)$ **iii.** $g(5)$ **iv.** $g(-4)$.

c. If $g(x) = 4 - \frac{1}{x}$, find:

i. $g(1)$ **ii.** $g\left(\frac{1}{2}\right)$ **iii.** $g\left(-\frac{1}{2}\right)$ **iv.** $g\left(-\frac{1}{5}\right)$.

d. If $f(x) = (x + 3)^2$, find:

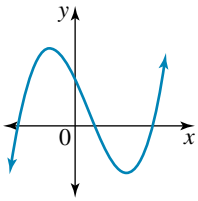
i. $f(0)$ **ii.** $f(-2)$ **iii.** $f(1)$ **iv.** $f(a)$.

e. If $h(x) = \frac{24}{x}$, find:

i. $h(2)$ **ii.** $h(4)$ **iii.** $h(-6)$ **iv.** $h(12)$.

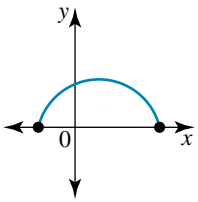
4. MC Which of the following relations is a function? *Note:* There may be more than one correct answer.

A.



B. $x^2 + y^2 = 9$ **C.** $y = 8x - 3$

D.



E. $y = \pm 2\sqrt{x}$

5. Which of the following relations are functions?

a. $y = 2x + 1$ **b.** $y = x^2 + 2$ **c.** $y = 2^x$ **d.** $x^2 + y^2 = 25$

e. $x^2 + 4x + y^2 + 6y = 14$ **f.** $y = -4x$

6. Given that $f(x) = \frac{10}{x} - x$, find:

a. $f(2)$ **b.** $f(-5)$ **c.** $f(2x)$

d. $f(x^2)$ **e.** $f(x + 3)$ **f.** $f(x - 1)$.

7. Find the value (or values) of x for which each function has the value given.

a. $f(x) = 3x - 4, f(x) = 5$ **b.** $g(x) = x^2 - 2, g(x) = 7$

c. $f(x) = \frac{1}{x}, f(x) = 3$ **d.** $h(x) = x^2 - 5x + 6, h(x) = 0$

e. $g(x) = x^2 + 3x, g(x) = 4$ **f.** $f(x) = \sqrt{8 - x}, f(x) = 3$

Communicating, reasoning and problem solving

8. WE4 Describe what happens to:

a. $f(x) = x^2 + 3$ as $x \to \infty$ **b.** $f(x) = 2^x$ as $x \to -\infty$

c. $f(x) = \frac{1}{x}$ as $x \to \infty$ **d.** $f(x) = x^3$ as $x \to -\infty$

e. $f(x) = -5^x$ as $x \to -\infty$.

9. WE5 Find any points of intersection between the following curves.

a. $f(x) = 2x - 4$ and $g(x) = x^2 - 4$

b. $f(x) = -3x + 1$ and $g(x) = -\frac{2}{x}$

c. $f(x) = x^2 - 4$ and $g(x) = 4 - x^2$

d. $f(x) = \frac{3}{4}x - 6\frac{1}{4}$ and $x^2 + y^2 = 25$

10. Find the equation of the inverse function of each of the following, placing restrictions on the original x-values as required.

a. $f(x) = 2x - 1$ **b.** $f(x) = x^2 - 3$ **c.** $f(x) = (x - 2)^2 + 4$

11. WE6 **a i.** Show that the function $f(x) = x(x + 2)$ will not have an inverse function.

ii. Suggest a restriction that would result in an inverse function.

b. i. Show that the function $f(x) = -x^2 + 4, x \leq 0$ will have an inverse function.

ii. Determine the equation of the inverse function.

12. Find the value(s) of x for which:

a. $f(x) = x^2 + 7$ and $f(x) = 16$

b. $g(x) = \frac{1}{x - 2}$ and $g(x) = 3$

c. $h(x) = \sqrt{8 + x}$ and $h(x) = 6$.

13. Consider the function defined by the rule

$$f: R \to R,\ f(x) = (x - 1)^2 + 2.$$

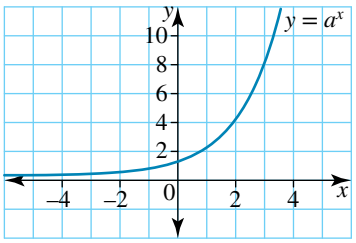
- **a.** State the range of the function.
- **b.** Determine the type of mapping for the function.
- **c.** Sketch the graph of the function stating where it cuts the y-axis and its vertex.
- **d.** Select the maximal domain where x is positive such that f is a one-to-one function.
- **e.** Determine the inverse function. Give the domain and range of the inverse function.
- **f.** Sketch the graph of the inverse function on the same set of axes used for part **c**.
- **g.** Find where f and the function $g(x) = x + 3$ intersect each other.

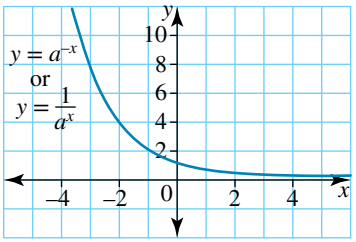
14. How do you determine the difference between functions and relations?

15. Draw and compare the graphs of the inverse functions $y = a^x$ and $y = \log_a x$, choosing various values for a. Explain why these graphs are inverses.

18.3 Exponential functions [Stage 5.3]

- **Exponential functions** can be used to model many real situations involving natural growth and decay.
- **Exponential growth** is when a quantity grows by a constant percentage in each fixed period of time. Examples of exponential growth include growth of investment at a certain rate of compound interest and growth in the number of cells in a bacterial colony.
- **Exponential decay** is when a quantity decreases by a constant percentage in each fixed period of time. Examples of exponential decay include yearly loss of value of an item (called depreciation) and radioactive decay.
- Both exponential growth and decay can be modelled by exponential functions of the type $y = ka^x$ ($y = k \times a^x$). The difference is in the value of the base a. When $a > 1$, there is exponential growth, and when $0 < a < 1$, there is exponential decay.
- The value of k corresponds to the initial quantity that is growing or decaying.





WORKED EXAMPLE 7

The number of bacteria, N, in a Petri dish after x hours is given by the equation $N = 50 \times 2^x$.

a Determine the initial number of bacteria in the Petri dish.

b Determine the number of bacteria in the Petri dish after 3 hours.

c Draw the graph of the function of N against x.

d Use the graph to estimate the length of time it will take for the initial number of bacteria to treble.

THINK

a 1 Write the equation.

2 Substitute $x = 0$ into the given formula and evaluate. (Notice that this is the value of k for equations of the form $y = k \times a^x$.)

WRITE/DRAW

a $N = 50 \times 2^x$

When $x = 0$,

$N = 50 \times 2^0$

$= 50 \times 1$

$= 50$

3 Write the answer in a sentence.

The initial number of bacteria in the Petri dish is 50.

b 1 Substitute $x = 3$ into the formula and evaluate.

b When $x = 3$,

$$N = 50 \times 2^3$$
$$= 50 \times 8$$
$$= 400$$

2 Write the answer in a sentence.

After 3 hours there are 400 bacteria in the Petri dish.

c 1 Draw a set of axes, labelling the horizontal axis as x and the vertical axis as N.

c

2 Plot the points generated by the answers to parts **a** and **b**.

3 Calculate the value of N when $x = 1$ and $x = 2$ and plot the points generated.

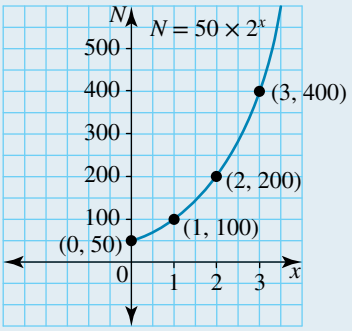
At $x = 1$,

$$N = 50 \times 2^1$$
$$= 50 \times 2$$
$$= 100$$

At $x = 2$,

$$N = 50 \times 2^2$$
$$= 50 \times 4$$
$$= 200$$

4 Join the points plotted with a smooth curve.

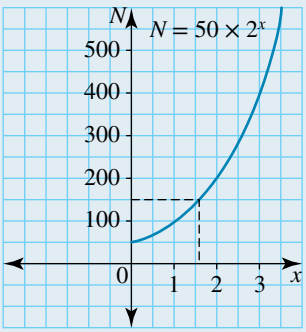


5 Label the graph.

d 1 Determine the number of bacteria required.

d Number of bacteria $= 3 \times 50$
$= 150$

2 Draw a horizontal line from $N = 150$ to the curve and from this point draw a vertical line to the x-axis.



3 The point on the x-axis will be the estimate of the time taken for the number of bacteria to treble.

4 Write the answer in a sentence.

The time taken will be approximately 1.6 hours.

WORKED EXAMPLE 8

A new computer costs $3000. It is estimated that each year it will be losing 12% of the previous year's value.

a Determine the value, $\$V$, of the computer after the first year.

b Determine the value of the computer after the second year.

c Determine the equation that relates the value of the computer to the number of years, n, it has been used.

d Use your equation to determine the value of the computer in 10 years' time.

THINK	WRITE
a 1 State the original value of the computer.	**a** $V_0 = 3000$
2 Since 12% of the value is being lost each year, the value of the computer will be 88% or (100 − 12)% of the previous year's value. Therefore, the value after the first year (V_1) is 88% of the original cost.	$V_1 = 88\%$ of 3000 $= 0.88 \times 3000$ $= 2640$
3 Write the answer in a sentence.	The value of the computer after 1 year is $2640.
b 1 The value of the computer after the second year, V_2, is 88% of the value after the first year.	**b** $V_2 = 88\%$ of 2640 $= 0.88 \times 2640$ $= 2323.2$
2 Write the answer in a sentence.	The value of the computer after the second year is $2323.20.
c 1 The original value is V_0.	**c** $V_0 = 3000$
2 The value after the first year, V_1, is obtained by multiplying the original value by 0.88.	$V_1 = 3000 \times 0.88$
3 The value after the second year, V_2, is obtained by multiplying V_1 by 0.88, or by multiplying the original value, V_0, by $(0.88)^2$.	$V_2 = (3000 \times 0.88) \times 0.88$ $= 3000 \times (0.88)^2$
4 The value after the third year, V_3, is obtained by multiplying V_2 by 0.88, or V_0 by $(0.88)^3$.	$V_3 = 3000 \times (0.88)^2 \times 0.88$ $= 3000 \times (0.88)^3$
5 By observing the pattern we can generalise as follows: the value after the nth year, V_n, can be obtained by multiplying the original value, V_0, by 0.88 n times; that is, by $(0.88)^n$.	$V_n = 3000 \times (0.88)^n$
d 1 Substitute $n = 10$ into the equation obtained in part **c** to find the value of the computer after 10 years.	**d** When $n = 10$, $V_{10} = 3000 \times (0.88)^{10}$ $= 835.50$
2 Write the answer in a sentence.	The value of the computer after 10 years is $835.50.

- Sometimes the relationship between the two variables closely resembles an exponential pattern, but cannot be described exactly by an exponential function. In such cases, part of the data are used to model the relationship with exponential growth or the decay function.

WORKED EXAMPLE 9

The population of a certain city is shown in the table below.

Year	1985	1990	1995	2000	2005	2010
Population (×1000)	128	170	232	316	412	549

Assume that the relationship between the population, P, and the year, x, can be modelled by the function $P = ka^x$, where x is the number of years after 1985. The value of P must be multiplied by 1000 in order to find the actual population.

a State the value of k, which is the population, in thousands, at the start of the period.

b Use a middle point in the data set to find the value of a, correct to 2 decimal places. Hence, write the formula, connecting the population, P, with the number of years, x, since 1985.

c For the years given, find the size of the population using the formula obtained in part b. Compare it with the actual size of the population in those years.

d Predict the population of the city in the years 2015 and 2020.

THINK	WRITE
a From the given table, state the value of k that corresponds to the population of the city in the year 1985.	**a** $k = 128$
b 1 Write the given formula for the population of the city.	**b** $P = ka^x$
2 Replace the value of k with the value found in a.	$P = 128 \times a^x$
3 Using a middle point of the data, replace x with the number of years since 1985 and P with the corresponding value.	The middle point is (1995, 232). When $x = 10$, $P = 232$, so $232 = 128 \times a^{10}$.
4 Solve the equation for a.	$a^{10} = \frac{232}{128}$ $a^{10} = 1.8125$ $a = \sqrt[10]{1.8125}$ $a = 1.0613...$
5 Round the answer to 2 decimal places.	$a \approx 1.06$
6 Rewrite the formula with this value of a.	So $P = 128 \times (1.06)^x$.
c 1 Draw a table of values and enter the given years, the number of years since 1985, x, and the population for each year, P. Round values of P to the nearest whole number.	**c** **Year** 1985 1990 1995 2000 2005 2010 **x** 0 5 10 15 20 25 **P** 128 171 229 307 411 549
2 Comment on the closeness of the fit.	The values for the population obtained using the formula closely resemble the actual data.
d 1 Find the value of x, the number of years after 1985.	**d** For the year 2015, $x = 30$.

2 Substitute this value of x into the formula and evaluate.	$P = 128 \times (1.06)^{30}$ $= 735.166\,87\ldots$
3 Round to the nearest whole number.	$P \approx 735$
4 Answer the question in a sentence.	The predicted population for 2015 is 735 000.
5 Repeat for the year 2020.	For the year 2020, $x = 35$. $P = 128 \times (1.06)^{35}$ $= 983.819\ldots$ $P \approx 984$ The predicted population for 2020 is 984 000.

DISCUSSION

Can you think of more situations that could potentially be represented by exponential functions?

learnON RESOURCES — ONLINE ONLY

Interactivity: Exponential growth and decay (int-6211)
eLesson: Exponential growth (eles-0176)
Digital doc: SkillSHEET Converting a percentage to a decimal (doc-5386)
Digital doc: SkillSHEET Decreasing a quantity by a percentage (doc-5387)
Digital doc: WorkSHEET Exponential growth and decay (doc-14623)

Exercise 18.3 Exponential functions

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1, 3, 5, 6, 8–10, 13, 15, 16	Questions: 1, 2, 4, 6, 7, 9, 11, 12, 14–16, 18	Questions: 1, 4–6, 9–18

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE7 The number of micro-organisms, N, in a culture dish after x hours is given by the equation $N = 2000 \times 3^x$.
- **a.** Determine the initial number of micro-organisms in the dish.
- **b.** Determine the number of micro-organisms in a dish after 5 hours.
- **c.** Draw the graph of N against x.
- **d.** Use the graph to estimate the number of hours needed for the initial number of micro-organisms to quadruple.

2. The value of an investment (in dollars) after n years is given by $A = 5000 \times (1.075)^n$.

a. Determine the size of the initial investment.

b. Determine the value of the investment (to the nearest dollar) after 6 years.

c. Draw the graph of A against n.

d. Use the graph to estimate the number of years needed for the initial investment to double.

3. MC **a.** The function $P = 300 \times (0.89)^n$ represents an:

A. exponential growth with the initial amount of 300

B. exponential growth with the initial amount of 0.89

C. exponential decay with the initial amount of 300

D. exponential decay with the initial amount of 0.89

E. exponential decay with the initial amount of 300×0.89

b. The relationship between two variables, A and t, is described by the function $A = 45 \times (1.095)^t$, where t is the time, in months, and A is the amount, in dollars. This function indicates:

A. a monthly growth of \$45

B. a monthly growth of 9.5 cents

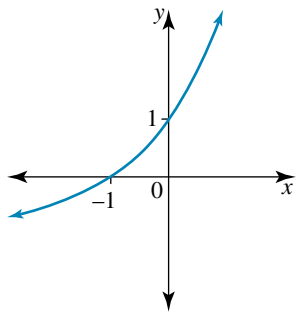
C. a monthly growth of 1.095%

D. a monthly growth of 9.5%

E. a yearly growth of 9.5%

4. MC The graph of $y = 2^{x+1} - 1$ is best represented by:

A.



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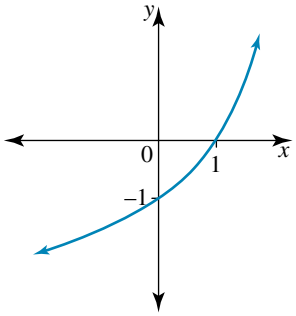
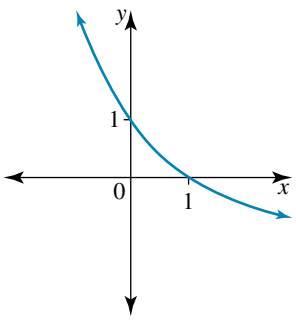
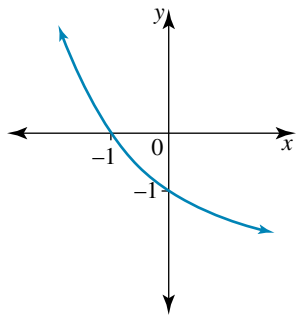


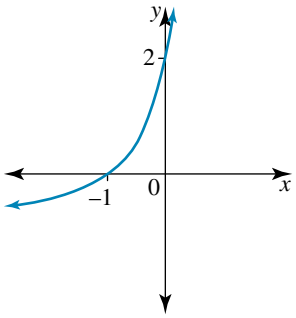
C.



D.



E.



5. MC The graph of $y = 3^{x-2} + 2$ has an asymptote and y-intercept respectively at:

A. $y = 0, 2\frac{1}{9}$ **B.** $y = 2, 2\frac{1}{9}$ **C.** $y = 2, 2$ **D.** $y = 2, 1\frac{8}{9}$ **E.** $y = 0, 2$

6. WE8 A new washing machine costs \$950. It is estimated that each year it will be losing 7% of the previous year's value.

a. Calculate the value of the machine after the first year.

b. Calculate the value of the machine after the second year.

c. Determine the equation that relates the value of the machine, $\$V$, to the number of years, n, that it has been used.

d. Use your equation to find the value of the machine in 12 years' time.

7. A certain radioactive element decays in such a way that every 50 years the amount present decreases by 15%. In 1900, 120 mg of the element was present.
- **a.** Calculate the amount present in 1950.
- **b.** Calculate the amount present in the year 2000.
- **c.** Determine the rule that connects the amount of the element present, A, with the number of 50-year intervals, t, since 1900.
- **d.** Calculate the amount present in the year 2010. Round your answer to 3 decimal places.
- **e.** Graph the function of A against t.
- **f.** Use the graph to estimate the half-life of this element (that is, the number of years needed for half the initial amount to decay).

8. When a shirt made of a certain fabric is washed, it loses 2% of its colour.

- **a.** Determine the percentage of colour that remains after:
  - **i.** two washes
  - **ii.** five washes.
- **b.** Write a function for the percentage of colour, C, remaining after w washings.
- **c.** Draw the graph of C against w.
- **d.** Use the graph to estimate the number of washes after which there is only 85% of the original colour left.

9. WE9 The population of a certain country is shown in the table below.

Year	Population (in millions)
1990	118
1995	130
2000	144
2005	160
2010	178

Assume that the relationship between the population, P, and the year, n, can be modelled by the formula $P = ka^n$, where n is the number of years since 1990.
- **a.** State the value of k.
- **b.** Use the middle point of the data set to find the value of a rounded to 2 decimal places. Hence, write the formula that connects the two variables, P and n.
- **c.** For the years given in the table, find the size of the population, using your formula. Compare the numbers obtained with the actual size of the population.
- **d.** Predict the population of the country in the year 2035.

10. The temperature in a room (in degrees Celsius), recorded at 10-minute intervals after the air conditioner was turned on, is shown in the table below.

Time (min)	0	10	20	30	40
Temperature (°C)	32	26	21	18	17

Assume that the relationship between the temperature, T, and the time, t, can be modelled by the formula $T = ca^t$, where t is the time, in minutes, since the air conditioner was turned on.
- **a.** State the value of c.
- **b.** Use the middle point in the data set to find the value of a to 2 decimal places.
- **c.** Write the rule connecting T and t.
- **d.** Using the rule, find the temperature in the room 10, 20, 30 and 40 minutes after the air conditioner was turned on and compare your numbers with the recorded temperature. Comment on your findings. (Give answers correct to 1 decimal place.)

11. The population of a species of dogs (D) increases exponentially and is described by the equation $D = 60(1 - 0.6^t) + 3$, where t represents the time in years.

a. Calculate the initial number of dogs.

b. Calculate the number of dogs after 1 year.

c. Determine the time taken for the population to reach 50 dogs.

12. Carbon-14 decomposes in such a way that the amount present can be calculated using the equation, $Q = Q_0(1 - 0.038)^t$, where Q is measured in milligrams and t in centuries.

a. If there is 40 mg present initially, how much is present in:

i. 10 years **ii.** 2000 years?

b. How many years will it take for there to be less than 10 mg?

Communicating, reasoning and problem solving

13. Fiona is investing $20000 in a fixed term deposit earning 6% p.a. interest. When Fiona has $30000 she intends to put a deposit on a house.

a. Determine an exponential function that will model the growth of Fiona's investment.

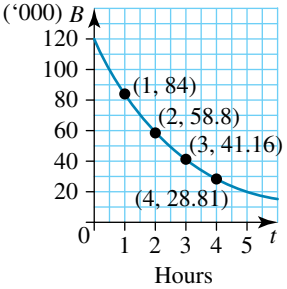
b. Graph this function.

c. Determine the length of time (correct to the nearest year) that it will take for Fiona's investment to grow to $30000.

d. Suppose Fiona had been able to invest at 8% p.a. How much quicker would Fiona's investment have grown to the $30000 she needs?

e. Alvin has $15000 to invest. Find the interest rate at which Alvin must invest his money, if his investment is to grow to $30000 in less than 8 years.

14. A Petri dish containing a bacteria colony was exposed to an antiseptic. The number of bacteria within the colony, B, over time, t, in hours is shown in the graph.



a. Using the graph, predict the number of bacteria in the Petri dish after 5 hours.

b. Using the points from the graph, show that if B (in thousands can be modelled by the function $B = ab^t$, then $a = 120$ and $b = 0.7$.

c. After 8 hours, another type of antiseptic was added to the Petri dish. Within three hours, the number of bacteria in the Petri dish had decreased to 50. If the number of bacteria decreased at a constant rate, show that the total number of bacteria that had decreased within two hours was approximately 6700.

15. One hundred people were watching a fireworks display at a local park. As the fireworks were set off, more people started to arrive to see the show. The number of people, P, at time t minutes, after the start of the fireworks display, can be modelled by the function $P = ab^t$.

a. If after 5 minutes there were approximately 249 people, show that the number of people arriving at the park to watch the fireworks increased by 20% each minute.

b. The fireworks display lasted for 40 minutes. After 40 minutes, people started to leave the park. The number of people leaving the park could be modelled by an exponential function. Fifteen minutes after the fireworks ceased, there were only 700 people in the park.

Derive an exponential function that can determine the number of people, N, remaining in the park after the fireworks had finished at any time, m, in minutes.

16. A hot plate used as a camping stove is cooling down. The formula that describes this cooling pattern is $T = 500 \times 0.5^t$ where T is the temperature in degrees Celsius and t is the time in hours.
a. What is the initial temperature of the stove?
b. What is the temperature of the stove after 2 hours?
c. Decide when the stove will be cool enough to touch and give reasons.

17. The temperature in a greenhouse is monitored when the door is left open. The following measurements are taken.

Time (min)	0	5	10	15	20
Temperature (°C)	45	35	27	21	16

a. State the initial temperature of the greenhouse.
b. Determine an exponential equation to fit the collected data.
c. What will the temperature be after 30 minutes?
It is discovered that one of the temperature readings is incorrect.
d. Recalculate all the temperatures using the exponential rule found in part **b**.
e. If the original incorrect temperature was omitted from the data, does this change the rule?
f. Will the temperature ever reach 0°C? Explain.

18. What are the main differences between a graph modelling exponential growth and one modelling exponential decay?

18.4 Cubic functions [Stage 5.3]

18.4.1 Definition of cubic functions

- **Cubic functions** are polynomials where the highest power of x is 3. These include functions such as $y = x^3$ or $y = (x + 1)(x - 2)(x + 3)$.

WORKED EXAMPLE 10

Plot the graph of $y = x^3 - 1$ by completing a table of values.

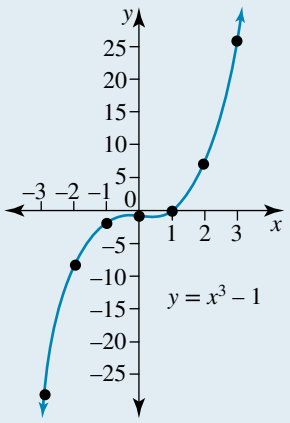
THINK

1 Prepare a table of values, taking x-values from -3 to 3. Fill in the table by substituting each x-value into the given equation to find the corresponding y-value.

WRITE/DRAW

x	-3	-2	-1	0	1	2	3
y	-28	-9	-2	-1	0	7	26

2 Draw a set of axes and plot the points from the table. Join them with a smooth curve.



WORKED EXAMPLE 11

Plot the curve of $y = x(x - 2)(x + 2)$ by completing a table of values.

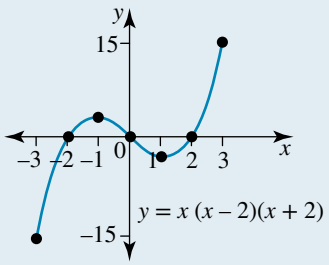
THINK

1 Prepare a table of values, taking x-values from −3 to 3. Fill in the table by substituting each x-value into the given equation.

2 Draw a set of axes and plot the points from the table. Join them with a smooth curve.

WRITE/DRAW

x	−3	−2	−1	0	1	2	3
y	−15	0	3	0	−3	0	15



ACTIVITY: CUBICS AND THEIR TRANSFORMATIONS

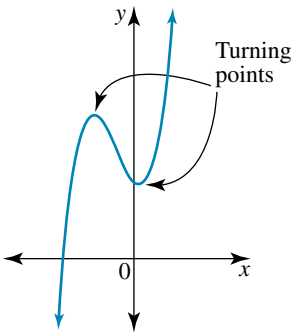
(Digital technologies may be used to help sketch the following graphs.)

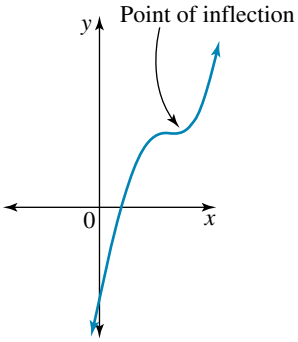
1. (a) Sketch the graphs of:
   (i) $y = x^3$ (ii) $y = -2x^3$ (iii) $y = 2x^3$
   (b) For the graph of $y = ax^3$, describe the effect on the graph of changing the value of a.
2. (a) Sketch the graphs of:
   (i) $y = -3x^3 + 1$ (ii) $y = 3x^3 - 1$
   (b) For the graph of $y = ax^3 + d$, describe the effect on the graph of changing the value of d.
3. (a) Sketch the graphs of:
   (i) $y = -2(x - 1)(x + 2)(x + 3)$ (ii) $y = (x - 2)(x + 2)(x - 1)$
   (b) For the graph of $y = a(x - r)(x - s)(x - t)$, describe the effect on the graph of changing the value of a, r, s and t.

- A good sketch of a cubic function shows:
  1. x- and y-intercepts
  2. the behaviour of the function at extreme values of x, that is, as x approaches infinity ($x \to +\infty$) and as x approaches negative infinity ($x \to -\infty$)
  3. the general location of turning points.

Note that for cubic functions, 'humps' are not symmetrical as they are for parabolas, but are skewed to one side.

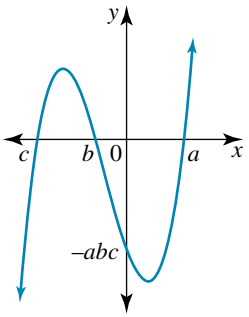
- The graphs below show the two main types of cubic graph.





- Consider the general factorised cubic $y = (x - a)(x - b)(x - c)$.
  The x-intercepts occur when $y = 0$, that is, when $x = a$ or $x = b$ or $x = c$.
  The y-intercept occurs when $x = 0$, that is, the y-intercept is

$$y = (0 - a)(0 - b)(0 - c)$$
$$= -abc$$



DISCUSSION

Is there an easy way to determine whether a cubic graph has turning points or a point of inflection?

18.4.2 Single, double and triple roots of a polynomial equation

- The solutions to the polynomial equation $P(x) = 0$ are called the roots of the polynomial or the zeroes of the polynomial.
- A quadratic equation of the form $y = (x - h)^2$ is said to have a double root at $x = h$ and at that point, the graph touches the x-axis but does not cross it.
- A cubic equation of the form $y = (x - h)^3$ is said to have a triple root at $x = h$ and at that point, the graph crosses the x-axis and it is a point of inflection.
- A cubic equation of the form $y = (x - a)(x - b)^2$ has a single root at $x = a$, that is an x-intercept, and a double root at $x = b$, where the graph touches the x-axis but does not cross it.

WORKED EXAMPLE 12

Sketch the following, showing all intercepts.

a $y = (x - 2)(x - 3)(x + 5)$ **b** $y = (x - 6)^2(4 - x)$ **c** $y = (x - 2)^3$

THINK

a 1 Write the equation.

2 The y-intercept occurs where $x = 0$. Substitute $x = 0$ into the equation.

WRITE/DRAW

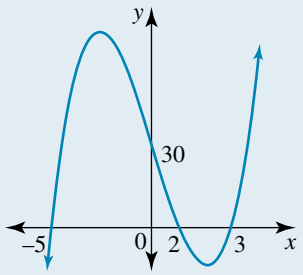
a $y = (x - 2)(x - 3)(x + 5)$

y-intercept: if $x = 0$,
$y = (-2)(-3)(5)$
$= 30$
Point: (0, 30)

3 Solve $y = 0$ to find the x-intercepts.
This is a graph with three single roots, so there are three x-intercepts.

x-intercepts: if $y = 0$,
$x - 2 = 0$, $x - 3 = 0$ or $x + 5 = 0$
$x = 2$, $x = 3$ or $x = -5$
Points: $(2, 0)$, $(3, 0)$, $(-5, 0)$

4 Combine the above steps to sketch the graph.



b 1 Write the equation.

b $y = (x - 6)^2(4 - x)$

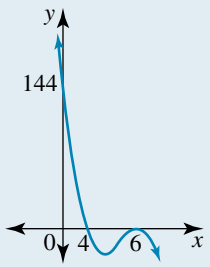
2 Substitute $x = 0$ to find the y-intercept.

y-intercept: if $x = 0$,
$y = (-6)^2(4)$
$= 144$
Point: $(0, 144)$

3 Solve $y = 0$ to find the x-intercepts.
$x = 6$ is a double root, so the graph will touch the x-axis at $x = 6$.

x-intercepts: if $y = 0$,
$x - 6 = 0$ or $4 - x = 0$
$x = 6$ or $x = 4$
Points: $(6, 0)$, $(4, 0)$

4 Combine all information and sketch the graph.



c 1 Write the equation.

c $y = (x - 2)^3$

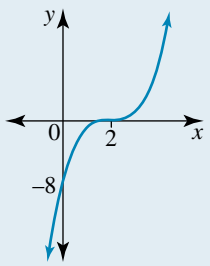
2 Substitute $x = 0$ to find the y-intercept.

y-intercept: if $x = 0$,
$y = (-2)^3$
$= -8$

3 Solve $y = 0$ to find the x-intercepts.
$x = 2$ is a triple root, so the graph will cross the x-axis and is appoint of inflection at $x = 2$.

x-intercept: if $y = 0$,
$x - 2 = 0$
$x = 2$

4 Combine all information and sketch the graph.



learnON RESOURCES — ONLINE ONLY

Interactivity: Cubic polynomials (int-2566)

Exercise 18.4 Cubic functions

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–f, 2a–f, 3–8	Questions: 1e–h, 2e–h, 3–8, 10	Questions: 1i–l, 2i–l, 3–11

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

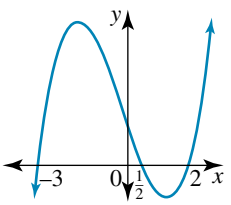
1. WE10, 11, 12 Sketch the following, showing all intercepts.

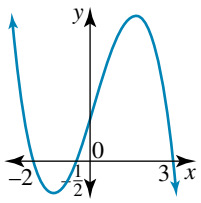
a. $y = (x - 1)(x - 2)(x - 3)$
b. $y = (x - 3)(x - 5)(x + 2)$
c. $y = (x + 6)(x + 1)(x - 7)$
d. $y = (x + 4)(x + 9)(x + 3)$
e. $y = (x + 8)(x - 11)(x + 1)$
f. $y = (2x - 6)(x - 2)(x + 1)$
g. $y = (2x - 5)(x + 4)(x - 3)$
h. $y = (3x + 7)(x - 5)(x + 6)$
i. $y = (4x - 3)(2x + 1)(x - 4)$
j. $y = (2x + 1)(2x - 1)(x + 2)$
k. $y = (x - 3)^2(x - 6)$
l. $y = (x + 2)(x + 5)^2$

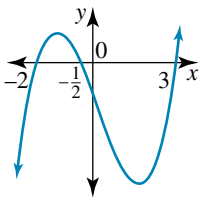
2. Sketch the following (a mixture of positive and negative cubics). You may need to use techniques for factorising polynomials from Topic 17.

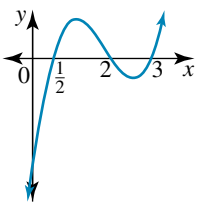
a. $y = (2 - x)(x + 5)(x + 3)$
b. $y = (1 - x)(x + 7)(x - 2)$
c. $y = (x + 8)(x - 8)(2x + 3)$
d. $y = (x - 2)(2 - x)(x + 6)$
e. $y = x^3 - x^2 - 2x$
f. $y = -2x^3 - 8x^2 - 2x + 12$
g. $y = 3(x + 1)(x + 10)(x + 5)$
h. $y = -3x^3 + 24x^2 - 48x$
i. $y = 4x^2(x + 8)$
j. $y = (5 - 3x)(x - 1)(2x + 9)$
k. $y = (6x - 1)^2(x + 7)$
l. $y = -2x^2(7x + 3)$

3. MC Which of the following is a reasonable sketch of $y = (x + 2)(x - 3)(2x + 1)$?

A.


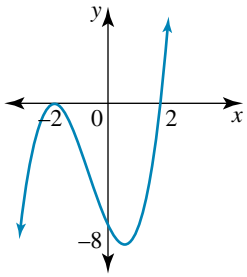
B.


C.


D.


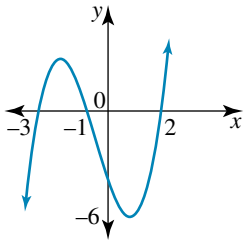
E. None of the above

4. MC The graph shown could be that of:



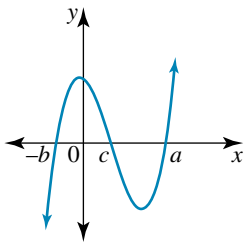
A. $y = x^2(x + 2)$ **B.** $y = (x + 2)^3$
C. $y = (x - 2)(x + 2)^2$ **D.** $y = (x - 2)^2(x + 2)$
E. $y = (x - 2)^3$

5. MC The graph shown has the equation:



A. $y = (x + 1)(x + 2)(x + 3)$ **B.** $y = (x + 1)(x - 2)(x + 3)$
C. $y = (x - 1)(x + 2)(x + 3)$ **D.** $y = (x - 1)(x + 2)(x - 3)$
E. $y = (x - 1)(x - 2)(x - 3)$

6. MC If a, b and c are positive numbers, the equation of the graph shown could be:



A. $y = (x - a)(x + b)(x + c)$ **B.** $y = (x + a)(x - b)(x + c)$
C. $y = (x + a)(x + b)(x - c)$ **D.** $y = (x - a)(x + b)(x - c)$
E. $y = (x - a)(x - b)(x - c)$

7. Sketch the graph of each of the following.

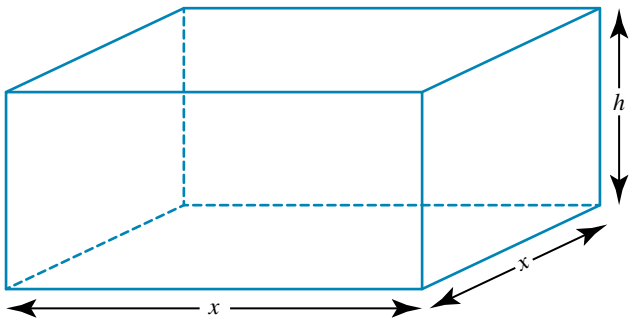
a. $y = x(x - 1)^2$ **b.** $y = -(x + 1)^2(x - 1)$
c. $y = (2 - x)(x^2 - 9)$ **d.** $y = -x(1 - x^2)$

Communicating, reasoning and problem solving

8. The function $f(x) = x^3 + ax^2 + bx + 4$ has x-intercepts at $(1, 0)$ and $(-4, 0)$. Find the values of a and b.

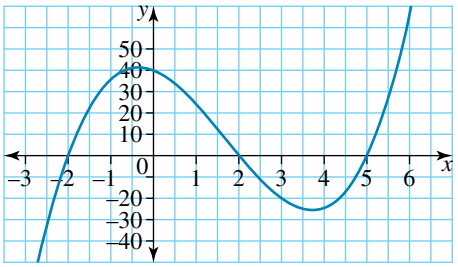
9. The graphs of the functions $f(x) = x^3 + (a + b)x^2 + 3x - 4$ and $g(x) = (x - 3)^3 + 1$ touch. Express a in terms of b.

10. A girl uses 140 cm of wire to make a frame of a cuboid with a square base as shown.
The base length of the cuboid is x cm and the height is h cm.



a. Explain why the volume in cm^3 is given by $V = 35x^2 - 2x^3$.
b. What possible values can x assume?
c. Find the volume of the cuboid when the base area is $81\,cm^2$.
d. Sketch the graph of V versus x.
e. Use technology to determine the coordinates of the maximum turning point. Explain what these coordinates mean.

11. Find the rule for the cubic function shown.

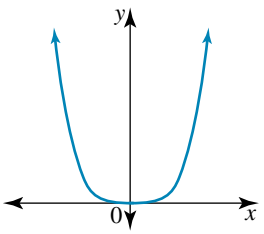


18.5 Quartic functions [Stage 5.3]

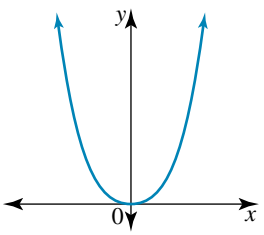
18.5.1 Basic shapes of quartic graphs

- **Quartic functions** are polynomials where the highest power of x is 4. These include functions such as $y = x^4$ or $y = (x + 1)(x - 2)(x + 3)(x - 5)$.
- When sketching the graphs of quartic functions, it is necessary to find all the intercepts on both the x- and y-axes.
- The direction of a quartic graph is determined by the coefficient of the x^4 term. This is similar to the effect the coefficient of x^2 has on the shape of a parabola. Consider the coefficient of x^4 to be a.
  - **When a is positive ($a > 0$)**

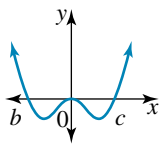
1. $y = ax^4$



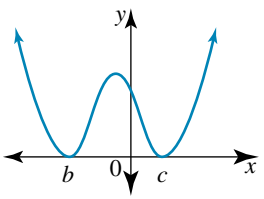
2. $y = ax^4 + cx^2$, $c \geq 0$



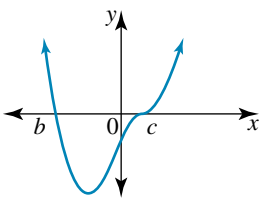
3. $y = ax^2(x - b)(x - c)$



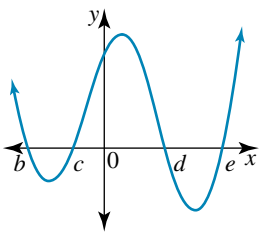
4. $y = a(x - b)^2(x - c)^2$



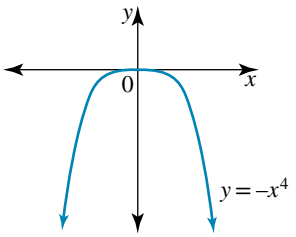
5. $y = a(x - b)(x - c)^3$



6. $y = a(x - b)(x - c)(x - d)(x - e)$

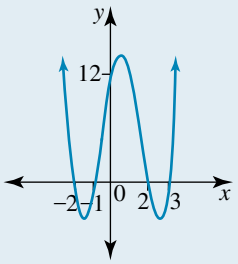


– **When a is negative ($a < 0$)**
If a was negative in each of the previous graphs, they would be reflected in the x-axis.



WORKED EXAMPLE 13

Sketch the graph of $y = x^4 - 2x^3 - 7x^2 + 8x + 12$, showing all intercepts.

THINK	WRITE/DRAW
1 Find the y-intercept.	When $x = 0$, $y = 12$. The y-intercept is 12.
2 Let $P(x) = y$.	Let $P(x) = x^4 - 2x^3 - 7x^2 + 8x + 12$.
3 Find two linear factors of the quartic expressions, if possible, using the factor theorem.	$P(1) = (1)^4 - 2(1)^3 - 7(1)^2 + 8(1) + 12$ $= 12$ $\neq 0$ $P(-1) = (-1)^4 - 2(-1)^3 - 7(-1)^2 + 8(-1) + 12$ $= 0$ $(x + 1)$ is a factor. $P(2) = (2)^4 - 2(2)^3 - 7(2)^2 + 8(2) + 12$ $= 0$ $(x - 2)$ is a factor.
4 Find the product of the two linear factors.	$(x + 1)(x - 2) = x^2 - x - 2$
5 Use long division to divide the quartic by the quadratic factor $x^2 - x - 2$.	$x^2 - x - 6$ $x^2 - x - 2 \overline{)x^4 - 2x^3 - 7x^2 + 8x + 12}$ $x^4 - x^3 - 2x^2$ $-x^3 - 5x^2 + 8x$ $-x^3 + x^2 + 2x$ $-6x^2 + 6x + 12$ $-6x^2 + 6x + 12$ 0
6 Express the quartic in factorised form.	$y = (x + 1)(x - 2)(x^2 - x - 6)$ $= (x + 1)(x - 2)(x - 3)(x + 2)$
7 To find the x-intercepts, solve $y = 0$.	If $0 = (x + 1)(x - 2)(x - 3)(x + 2)$ $x = -1, 2, 3, -2$
8 State the x-intercepts.	The x-intercepts are $-2, -1, 2, 3$.
9 Sketch the graph of the quartic.	

ACTIVITY: QUARTICS AND BEYOND

Investigate graphs of functions of the form $f(x) = ax^n$ for integer values of n from 4 to 9. Use digital technology to assist you in sketching these graphs, and use the graphs to answer the following questions.

1. What do graphs of functions for which n is even have in common?
2. What does an odd value of n do to the graph?
3. What effect do different integer values of a have on the graph of the function?

Write a report on your findings.

learnON RESOURCES — ONLINE ONLY

Interactivity: Quartic functions (int-6213)

Digital doc: WorkSHEET Cubic and quartic functions (doc-14624)

Exercise 18.5 Quartic functions

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–d, 2a–d, 3–6, 9	Questions: 1c–f, 2c–f, 3, 4, 6, 7, 9, 11	Questions: 1d–h, 2d–h, 3–12

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE13 Sketch the graph of each of the following, showing all intercepts. You may like to verify the shape of each graph using digital technology.

a. $y = (x - 2)(x + 3)(x - 4)(x + 1)$
b. $y = (x^2 - 1)(x + 2)(x - 5)$
c. $y = 2x^4 + 6x^3 - 16x^2 - 24x + 32$
d. $y = x^4 + 4x^3 - 11x^2 - 30x$
e. $y = x^4 + 4x^3 - 12x - 9$
f. $y = x^4 - 4x^2 + 4$
g. $y = 30x - 37x^2 + 15x^3 - 2x^4$
h. $y = 6x^4 + 11x^3 - 37x^2 - 36x + 36$

2. Sketch each of the following.

a. $y = x^4 - 2x^3 + x^2$
b. $y = -x^4 + 6x^3 - x^2 - 24x - 16$
c. $y = -x(x - 3)^3$
d. $y = -x^4 + 6x^3 - 7x^2 - 6x + 8$
e. $y = (x - a)(b - x)(x + c)(x + d)$, $a, b, c, d > 0$

3. MC A quartic touches the x-axis at $x = -3$ and $x = 2$. It crosses the y-axis at $y = -9$. A possible equation is:

A. $y = \frac{1}{4}(x + 3)^2(x - 2)^2$
B. $y = -\frac{1}{6}(x + 3)^3(x - 2)$
C. $y = -\frac{3}{8}(x + 3)(x - 2)^3$
D. $y = -\frac{1}{4}(x + 3)^2(x - 2)^2$
E. $y = -\frac{1}{4}(x - 3)^2(x + 2)^2$

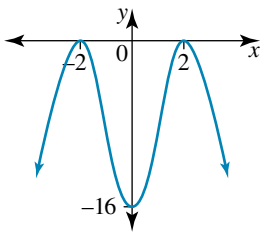
4. MC Consider the function $f(x) = x^4 - 8x^2 - 16$.

a. When factorised, $f(x)$ is equal to:

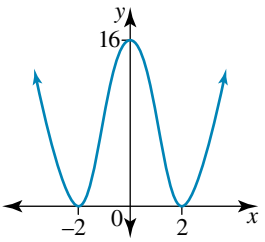
A. $(x + 2)(x - 2)(x - 1)(x + 4)$
B. $(x + 3)(x - 2)(x - 1)(x + 1)$
C. $(x - 2)^3(x + 2)$
D. $(x - 2)^2(x + 2)^2$
E. $(x - 2)^4$

b. The graph of $f(x)$ is best represented by:

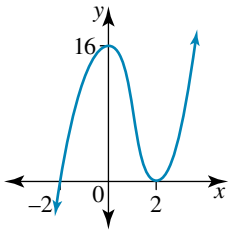
A.



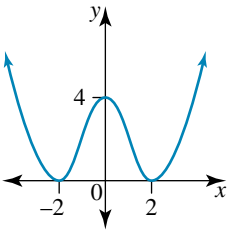
B.



C.



D.



E. None of the above

Communicating, reasoning and problem solving

5. Sketch the graph of each of the following functions. Verify your answers using a graphics calculator.

a. $y = x^4 - 3x^3 + 3x^2 - x$

b. $y = (2 - x)(x^2 - 4)(x + 3)$

c. $y = x^4 - x^2$

d. $y = 9x^4 - 30x^3 + 13x^2 + 20x + 4$

e. $y = -x^4 + 2x^3 + 3x^2 - 4x - 4$

f. $y = x^4 - 6x^2 - 27$

g. $y = (x + 2)^3(x - 3)$

h. $y = 4x^2 - x^4$

6. The function $f(x) = x^4 + ax^3 - 4x^2 + bx + 6$ has x-intercepts $(2, 0)$ and $(-3, 0)$. Find the values of a and b.

7. The functions $y = (a - 2b)x^4 - 3x - 2$ and $y = x^4 - x^3 + (a + 5b)x^2 - 5x + 7$ both have an x-intercept of 1. Find the value of a and b.

8. Patterns emerge when we graph polynomials with repeated factors, that is, polynomials of the form $P(x) = (x - a)^n$, $n > 1$. Describe what happens if:

a. n is even

b. n is odd.

9. A carnival ride has a piece of the track modelled by the rule

$$h = -\frac{1}{300}x(x - 12)^2(x - 20) + 15,\ 0 \leq x \leq 20$$

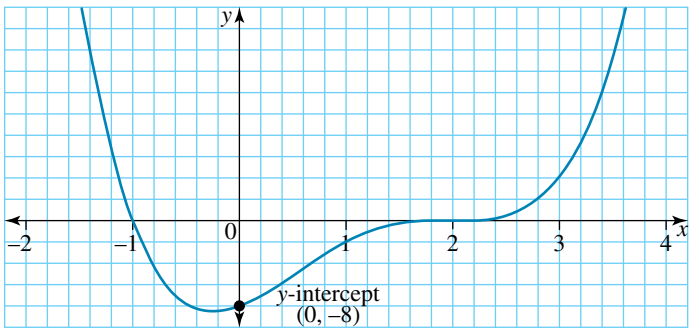
where x metres is the horizontal displacement from the origin and h metres is the vertical displacement of the track above the horizontal ground.

a. How high above the ground level is the track at the origin?

b. Use technology to sketch the function. Give the coordinates of any stationary points (that is, turning points or points of inflection).

c. How high above ground level is the track when $x = 3$?

10. Find the rule for the quartic function shown.



11. What are the basic differences between cubic and quartic functions?

12. Two functions $f(x)$ and $g(x)$ are inverses of each other if $f(g(x)) = x$ and $g(f(x)) = x$. The domain of $f(g(x))$ must be the domain of $g(x)$, and the domain of $g(f(x))$ must be the domain of $f(x)$. Using this information, show that $f(x) = x^2$, $x \geq 0$ and $g(x) = \sqrt{x}$, $x \geq 0$ are inverses of each other.

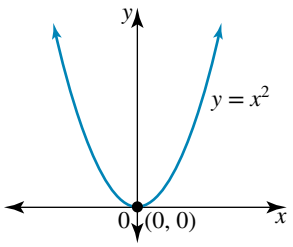
18.6 Transformations [Stage 5.3]

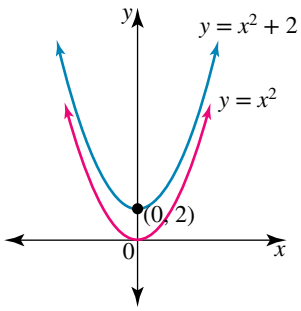
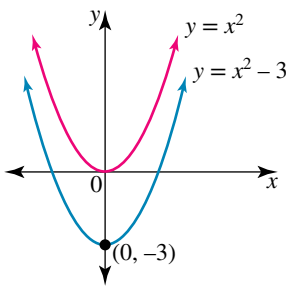
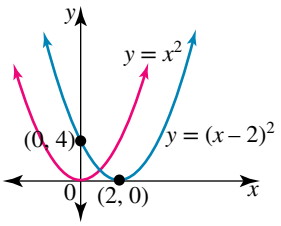
18.6.1 Transformations of functions and relations

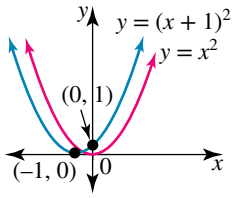
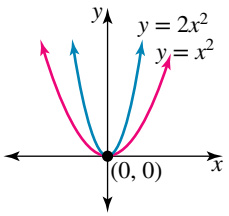
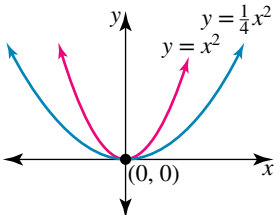
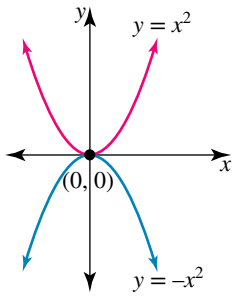
- Once the basic shape of the graph of a particular function or relation is known, it is not difficult to predict the shape of a related function, which is a transformation of the basic function or relation. **Transformations** of parabolas have been dealt with previously, but for the sake of comparison with other functions, they are included in this chapter. Other functions and relations considered are circles, hyperbolas, exponential functions, cubic and quartic functions. Below is a summary of transformations of functions discussed previously.

18.6.2 Quadratic functions

- The basic quadratic function is $y = x^2$. The shape of its graph is:

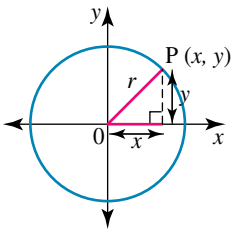


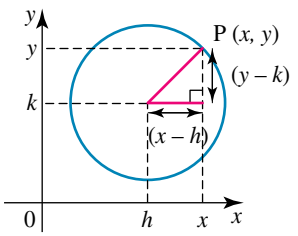
Transformation	Description	Graph
Vertical translation	Adding or subtracting a constant to $y = x^2$ moves the curve up or down the y-axis.	 
Horizontal translation	If the graph of $y = x^2$ is translated b units horizontally, the equation becomes $y = (x - b)^2$.	

		
Dilation	If the graph of $y = x^2$ is dilated by a factor of a, the graph becomes narrower if $a > 1$ and wider if $0 < a < 1$.	 
Reflection	If the x^2 term is positive, the graph is concave up, while if there is a negative sign in front of the x^2 term, the graph is concave down.	

18.6.3 Circles

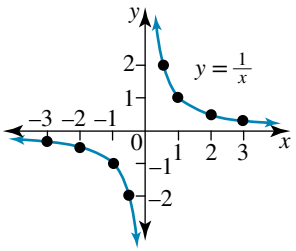
- The equation of a circle (relation) with centre (0, 0) and radius r is $x^2 + y^2 = r^2$.

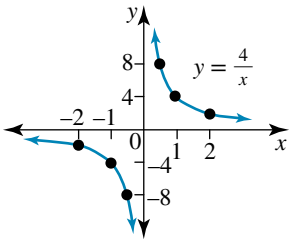
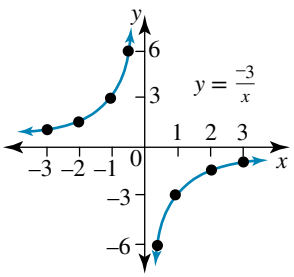


Transformation	Description	Graph
Translation	If the circle is translated h units to the right, parallel to the x-axis, and k units upwards, parallel to the y-axis, the equation of the circle with centre (h, k) becomes $(x - h)^2 + (y - k)^2 = r^2$.	

18.6.4 Hyperbolas

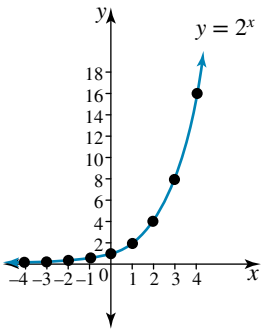
- The hyperbola is a function of the form $xy = k$ or $y = \frac{k}{x}$.
- The graph of $y = \frac{1}{x}$ has the shape shown.

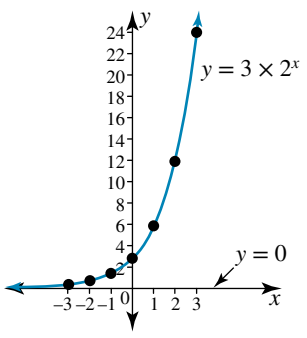


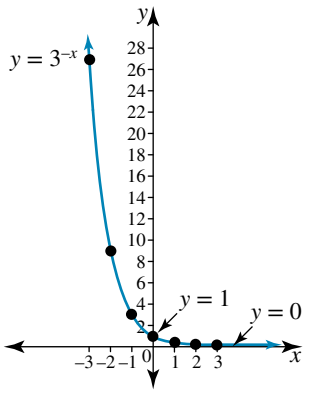
Transformation	Description	Graph
Dilation	Graphs of the form $y = \frac{k}{x}$ are the same basic shape as $y = \frac{1}{x}$, with y-values dilated by a factor of k.	
Negative values of k	Negative values of k cause the graph to be reflected across the y-axis.	

18.6.5 Exponential functions

- These functions are of the form $y = a^x$, where $a \neq 1$. The basic shape has a y-intercept of 1.

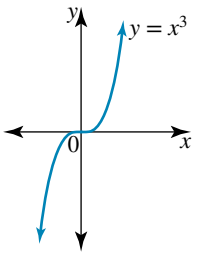


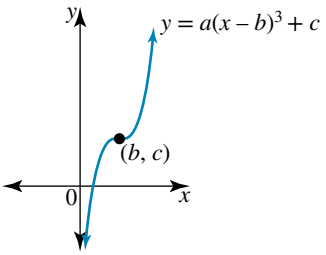
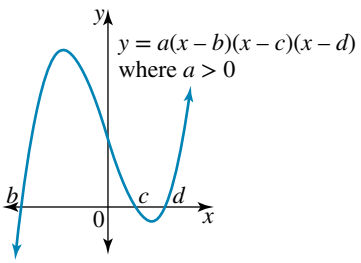
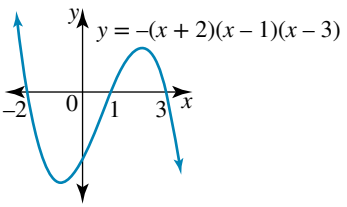
Transformation	Description	Graph
Functions for the form $y = k \times ax$	Multiplying by a factor of k causes the y-intercept to move to the point $(0, k)$.	

Functions with a negative exponent	This causes the graph to be reflected in the y-axis.	

18.6.6 Cubic functions

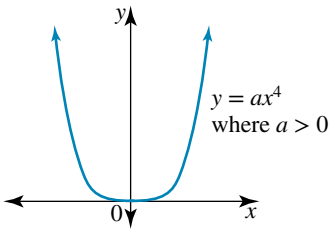
- The basic form of a cubic function is $y = x^3$. This can also be expressed in the form $y = a(x - b)^3 + c$, where $a = 1$, $b = 0$ and $c = 0$.

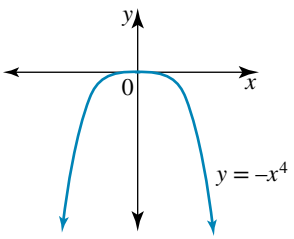


Transformation	Description	Graph
Translation	If $a \neq 1$, $b \neq 0$ and $c \neq 0$, the graph is translated $+b$ units in the x direction, $+c$ units in the y direction, and dilated by a factor of a in the y direction.	
Reflection	The cubic function can be expressed in factor form as $y = a(x - b)(x - c)(x - d)$, where b, c and d are the x-intercepts. If the value of a is negative, this causes the curve to be reflected in the x-axis.	 

18.6.7 Quartic functions

- The basic form of the quartic function $y = ax^4$, when a is positive, has the following shape.



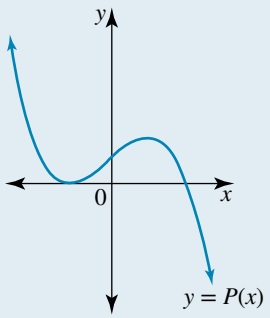
Transformation	Description	Graph
Reflection	Negative values of a cause the graph to be reflected in the x-axis.	

18.6.8 Transformation in general polynomials

- With knowledge of the transformations that occur in the functions just discussed, it is possible to generate many other graphs without knowing the equation of the original function. Consider a basic polynomial $y = P(x)$ and what happens to the shape of the curve as the function is changed.

WORKED EXAMPLE 14

Use the sketch of $y = P(x)$ shown to sketch:
a $y = P(x) + 1$ **b** $y = P(x) - 1$ **c** $y = -P(x)$.

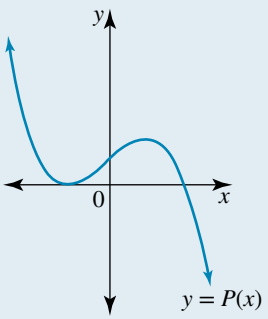


THINK

a 1 Sketch the original $y = P(x)$.

WRITE/DRAW

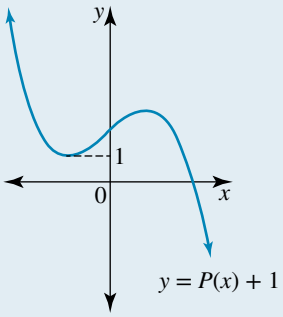
a



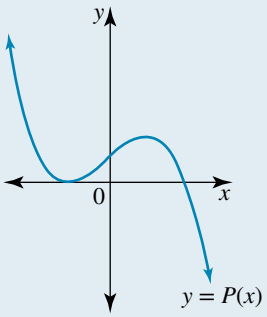
2 Consider the x-values. They remain unchanged — there is no horizontal translation.

3 Consider the y-values. They are increased by 1 — the curve is shifted up 1 unit.

4 Sketch the graph of $y = P(x) + 1$ using a similar scale to the original.



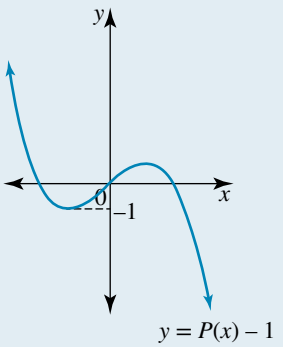
b 1 Sketch the original $y = P(x)$.



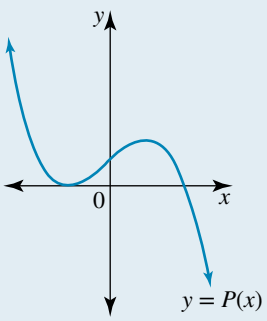
2 Consider the x-values. They remain unchanged — there is no horizontal translation.

3 Consider the y-values. They are decreased by 1 — the curve is shifted down 1 unit.

4 Sketch the graph of $y = P(x) - 1$ using a similar scale to the original.



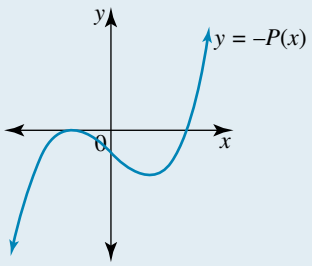
c 1 Sketch the original $y = P(x)$.



2 Consider the x-values. They remain unchanged — there is no horizontal translation.

3 Consider the y-values. They will all change sign — the curve will be reflected in the x-axis. That is, negative becomes positive and positive becomes negative.

4 Sketch the graph of $y = -P(x)$ using a similar scale to the original.



DISCUSSION

What are the similarities and differences between all of the different transformations covered in this section?

learnON RESOURCES — ONLINE ONLY

Interactivity: Vertical translations of parabolas (int-6097)
Interactivity: Horizontal translations of parabolas (int-6098)
Interactivity: Dilation of parabolas (int-6096)
Interactivity: Reflection of parabolas (int-6151)
Interactivity: Translations of circles (int-6214)
Interactivity: Hyperbolas (int-6155)
Interactivity: The rectangular hyperbola (int-2573)
Interactivity: Exponential functions (int-5959)
Interactivity: Transformations of exponentials (int-6216)
Interactivity: Transformations of cubics (int-6217)
Interactivity: Polynomial transformations (int-2794)

Exercise 18.6 Transformations

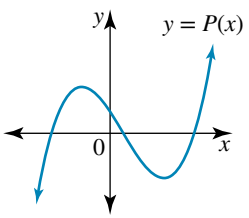
Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–3, 5, 7	Questions: 1–4, 6, 7, 9	Questions: 1–12

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

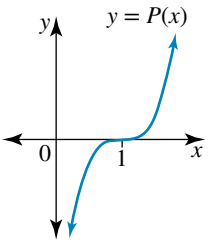
Understanding and fluency

1. WE14 Use the sketch of $y = P(x)$ shown to sketch:



a. $y = P(x) + 1$
b. $y = P(x) - 2$
c. $y = -P(x)$
d. $y = 2P(x)$.

2. Consider the sketch of $y = P(x)$ shown at right. Sketch:



a. $y = P(x) + 1$

b. $y = -P(x)$

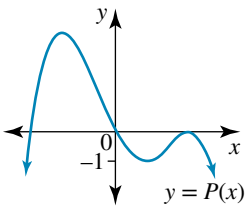
c. $y = P(x + 2)$.

3. Draw any polynomial $y = P(x)$. Discuss the similarities and differences between the graphs of $y = P(x)$ and $y = -P(x)$.

4. Draw any polynomial $y = P(x)$. Discuss the similarities and differences between the graphs of $y = P(x)$ and $y = 2P(x)$.

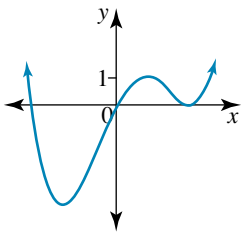
5. Draw any polynomial $y = P(x)$. Discuss the similarities and differences between the graphs of $y = P(x)$ and $y = P(x) - 2$.

6. Consider the sketch of $y = P(x)$ shown at right.

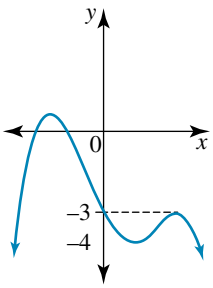


Give a possible equation for each of the following in terms of $P(x)$.

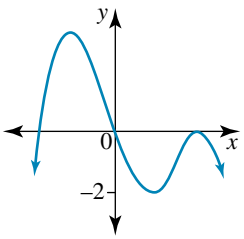
a.



b.



c.



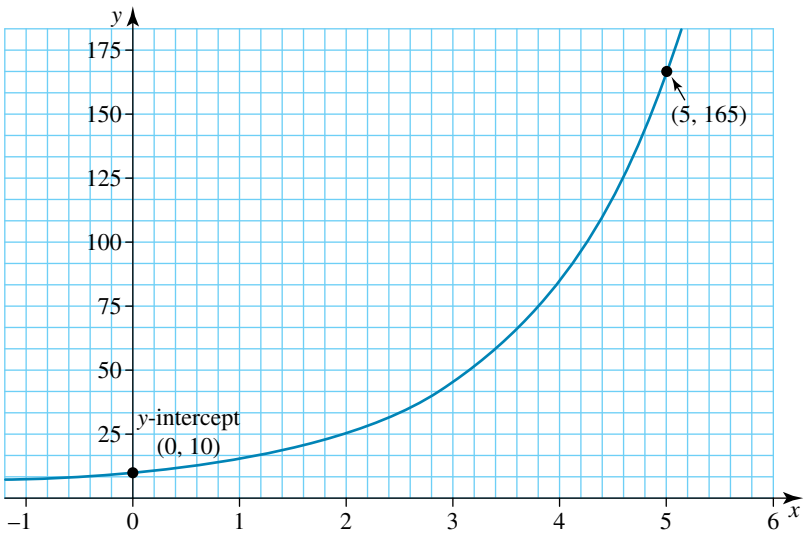
Communicating, reasoning and problem solving

7. The equations $y = x(x - 2)(x - 3)$ and $y = -2x(x - 2)(x - 3)$ are graphed on the same set of axes. Describe the relationship between the two graphs using the language of transformations.

8. If $y = -hr^{-q(x + p)} - r$, what translations take place from the original graph, $y = r^x$?

9. The graph of $y = \frac{1}{x}$ is reflected in the x-axis, dilated by a factor of 2 parallel to the y-axis, translated 2 units to the left and up 1 unit. Find the equation of the resultant curve. Give the equations of any asymptotes.

10. The graph of an exponential function is shown.



Its general rule is given by $y = a(2^x) + b$.

a. Find the values of a and b.

b. Describe any transformations that had to be applied to the graph of $y = 2^x$ to achieve this graph.

11. Why is it important to understand how transformations can affect the shape of a graph?

12. Using digital technology or by hand, sketch the graph of $y_1 = x^3 + x^2 + x$. Without sketching $y_2 = x^3 + x^2 + x + 1$, describe the similarities and differences between y_1 and y_2.

18.7 Review

Investigation | Rich task

Shaping up!

Many beautiful patterns are created by starting with a single function or relation and transforming and repeating it over and over.

In this task you will apply what you have learned about functions, relations and transformations (dilations, reflections and translations) to explore mathematical patterns.

Exploring patterns using transformations

1. a. On the same set of axes, draw the graphs of:

i. $y = x^2 - 4x + 1$ **ii.** $y = x^2 - 3x + 1$ **iii.** $y = x^2 - 2x + 1$

iv. $y = x^2 + 2x + 1$ **v.** $y = x^2 + 3x + 1$ **vi.** $y = x^2 + 4x + 1$.

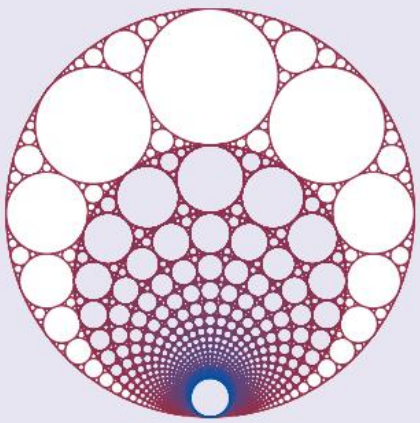
b. Describe the pattern formed by your graphs. Use mathematical terms such as intercepts, vertices, shape and transformations.

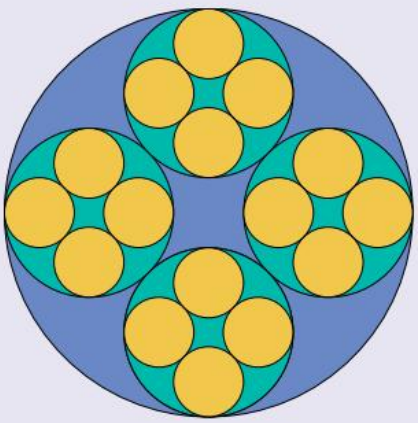
What you have drawn is referred to as a **family of curves** — curves in which the shape of the curve changes if the values of a, b and c in the general equation $y = ax^2 + bx + c$ change.

c. Explore the family of parabolas formed by changing the values of a and c. Comment on your findings.

d. Explore exponential functions belonging to the family of curves with equation $y = ka^x$, families of cubic functions with equations $y = ax^3$ or $y = ax^3 + bx^2 + cx + d$, and families of quartic functions with equations $y = ax^4$ or $y = ax^4 + bx^3 + cx^2 + dx + e$. Comment on your findings.

e. Choose one of the designs shown below and recreate it (or a simplified version of it). Record the mathematical equations used to complete the design.



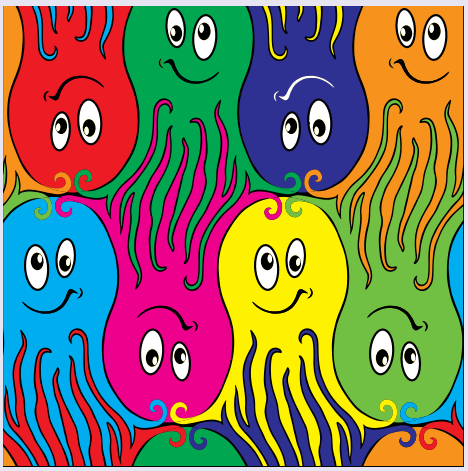


Coming up with your design

2. Use what you know about transformations to functions and relations to create your own design from a basic graph. You could begin with a circle, add some line segments and then repeat the pattern with some change.

Record all the equations and restrictions you use. It may be helpful to apply your knowledge of inverse functions too. Digital technology will be very useful for this task.

Create a poster of your design to share with the class.



learnON RESOURCES — ONLINE ONLY

Digital doc: Investigation — Shaping up! (doc-15950)
Digital doc: Code puzzle — 2 important events of 1973 (doc-15951)
Digital doc: Summary Topic 18 (doc-22916)
Interactivity: Word search Topic 18 (int-2877)
Interactivity: Crossword Topic 18 (int-2878)
Interactivity: Sudoku Topic 18 (int-3893)

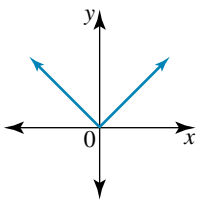
Exercise 18.7 Review questions

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

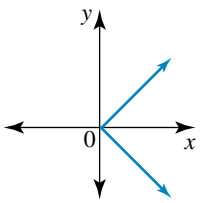
Understanding and fluency

1. Which of the following are functions?

a.



b.



2. Which of the following are functions? For each equation you identified as a function, state the equation of the inverse function, if it exists.

a. $y = 2x - 7$ **b.** $x^2 + y^2 = \sqrt{30}$ **c.** $y = 2^x$ **d.** $y = \dfrac{1}{x+1}$

3. Consider $f(x) = \sqrt{4 - x^2}$.

a. Find:

i. $f(0)$ **ii.** $f(1)$ **iii.** $f(2)$.

b. Is $f^{-1}(x)$ a function? If so, what is it equation?

4. Sketch each of the following curves, showing all intercepts.

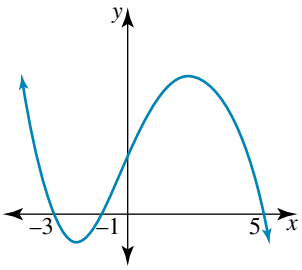
a. $y = (x - 1)(x + 2)(x - 3)$ **b.** $y = (2x + 1)(x + 5)^2$

5. Give an example of the equation of a cubic that would just touch the x-axis and cross it at another point.

6. Match each equation with its type of curve.

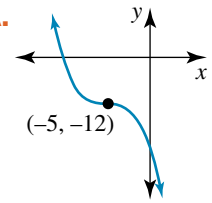
a $y = x^2 + 2$	A circle
b $x^2 + y^2 = 9$	B cubic
c $f(x) = \dfrac{2}{x + 2}$	C exponential
d $g(x) = 6^{-x}$	D parabola
e $h(x) = (x + 1)(x - 3)(x + 5)$	E hyperbola

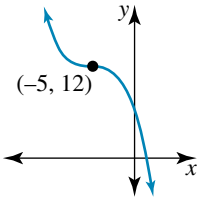
7. MC The equation for this graph could be:

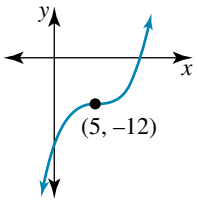


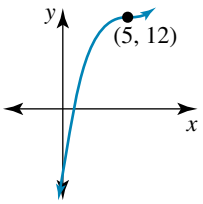
A. $y = (x - 5)(x + 1)(x + 3)$
B. $y = (x - 3)(x - 1)(x + 5)$
C. $y = (x - 3)(x + 1)(x + 5)$
D. $y = (5 - x)(1 + x)(3 + x)$
E. None of the above

8. MC Which of the following shows the graph of $y = -2(x + 5)^3 - 12$?

A.


B.


C.


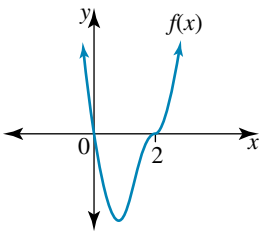
D.


E. None of the above

9. Sketch:

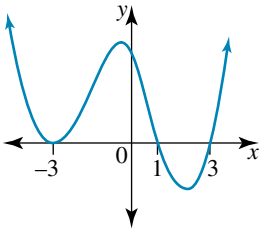
a. $y = x(x - 2)(x + 11)$ **b.** $y = x^3 + 6x^2 - 15x + 8$ **c.** $y = -2x^3 + x^2$.

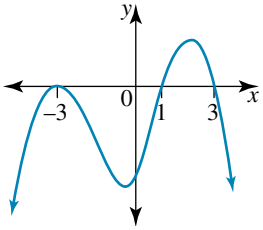
10. MC The rule for the graph shown below could be:

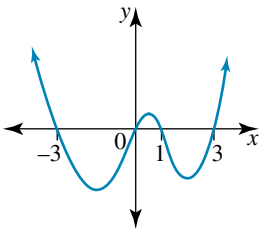


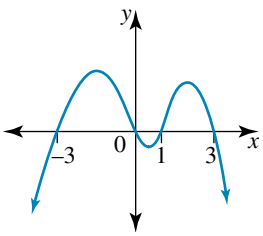
A. $f(x) = x(x + 2)^3$
B. $f(x) = -x(x - 2)^2$
C. $f(x) = x^2(x - 2)^2$
D. $f(x) = x(x - 2)^3$
E. $f(x) = x^2(x + 2)^2$

11. MC The graph of $y = (x + 3)^2 (x - 1)(x - 3)$ is best represented by:

A.


B.


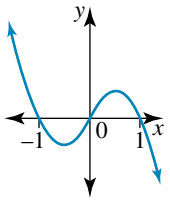
C.


D.


E. None of the above

12. Sketch the graph of $y = x^4 - 7x^3 + 12x^2 + 4x - 16$, showing all intercepts.

13. Consider the sketch of $y = P(x)$ shown below. Sketch $y = -P(x)$.



14. Draw any polynomial $y = P(x)$. Discuss the similarities and differences between the graphs of $y = P(x)$ and $y = P(x) + 3$.

Communicating, reasoning and problem solving

15. Describe what happens to $f(x) = -2^x$ as $x \to \infty$ and $x \to -\infty$.

16. Find any points of intersection between $f(x) = x^2 - 4$ and $g(x) = x^3 + x^2 - 12$.

17. The concentration of alcohol (mg/L) in a bottle of champagne is modelled by $C = C_0 \times 0.33^{kt}$, where t represents the time in days after the bottle is opened. If the initial concentration is 80 mg/L and the concentration after 1 day is 70 mg/L, find the concentration remaining after:

a. 3 days **b.** 1 week **c.** 18 hours.

18. The number of hyenas, H, in the zoo is given by $H = 20(10^{0.1t})$, where t is the number of years since counting started. At the same time, the number of dingoes, D, is given by $D = 25(10^{0.05t})$.

a. Calculate the number of:

i. hyenas **ii.** dingoes

when counting began.

b. Calculate the numbers of each after:

i. 1 year **ii.** 18 months.

c. Which of the animals is the first to reach a population of 40 and by how long?

d. After how many months are the populations equal and what is this population?

19. a. Consider the equation $f(x) = a(x - h)^2 + k$. By restricting the x-values, find the equation of the inverse function.

b. Show that the function $f(x) = \frac{a}{x} + b$ and its inverse function intersect on the line $y = x$.

20. A shend is a type of tropical pumpkin grown by the people of Outer Thrashia. The diameter (Dm) of a shend increases over a number of months (m) according to the rule $D = 0.25 \times (10)^{0.01m}$.

a. Determine the diameter of the shend after 4 months.

b. If the shend is not harvested, it will explode when it reaches a critical diameter of 0.5 metres. Show that it takes approximately 30 months for an unharvested shend to explode.

21. The surface area of a lake is evaporating at a rate of 5% per year due to climate change. To model this situation, the surface area of the lake (S km^2) over time is given by $S = 20000 \times 0.95^x$, where x is the time in years.

a. Explain whether this is an exponential relationship.

b. What is the surface area of the lake initially?

c. What will the surface area be in 10 years' time?

d. Plot a graph for this relationship.

e. What will the surface area be in 100 years' time?

f. Explain whether this is a realistic model.

Answers

Topic 18 Functions and relations

Exercise 18.2 Functions and relations

1. a. One-to-many **b.** Many-to-one **c.** Many-to-one
d. One-to-one **e.** One-to-one **f.** Many-to-one
g. Many-to-many **h.** Many-to-one **i.** One-to-one
j. Many-to-one **k.** One-to-one **l.** Many-to-one

2. a. b, c, d, e, f, h, i, j, k, l **b.** d, e, i, k

3. a. i. 1 **ii.** 7 **iii.** -5 **iv.** 16
b. i. 2 **ii.** 1 **iii.** 3 **iv.** 0
c. i. 3 **ii.** 2 **iii.** 6 **iv.** 9
d. i. 9 **ii.** 1 **iii.** 16 **iv.** $a^2 + 6a + 9$
e. i. 12 **ii.** 6 **iii.** –4 **iv.** 2

4. A, C, D

5. a, **b**, **c**, **f**

6. a. 3 **b.** 3 **c.** $\frac{5}{x} - 2x$ **d.** $\frac{10}{x^2} - x^2$ **e.** $\frac{10}{x+3} - x - 3$ **f.** $\frac{10}{x-1} - x + 1$

7. a. 3 **b.** -3 or 3 **c.** $\frac{1}{3}$ **d.** 2 or 3 **e.** -4 or 1 **f.** -1

8. a. $f(x) \to \infty$ **b.** $f(x) \to 0$ **c.** $f(x) \to 0$ **d.** $f(x) \to -\infty$ **e.** $f(x) \to 0$

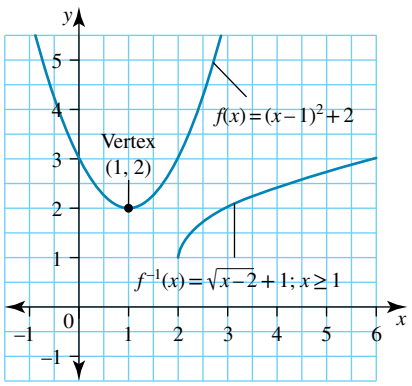
9. a. (0, –4), (2, 0) **b.** (1, –2), $(-\frac{2}{3}, 3)$ **c.** (2, 0), (–2, 0) **d.** (3, –4)

10. a. $f^{-1}x = \frac{x+1}{2}$
b. $f^{-1}(x) = \sqrt{x+3}$ or $f^{-1}(x) = -\sqrt{x+3}$
c. $f(x)^{-1} = \sqrt{x-4} + 2$ or $f(x)^{-1} = -\sqrt{x-4} + 2$

11. a. i. The horizontal line test fails.
ii. An inverse function will exist for $f(x) = x(x-2)$, $x < 1$ or $f(x) = x(x-2)$, $x > 1$.
b. i. The horizontal line test is upheld.
ii. $f^{-1}(x) = -\sqrt{4-x}$, $x \leq 4$.

12. a. $x = \pm 3$ **b.** $x = 2\frac{1}{3}$ **c.** $x = 28$

13. a. Range = $[2, \infty)$
b. Many-to-one
c. and f



d. Domain = $[1, \infty)$
e. $f^{-1}(x) = \sqrt{x-2} + 1$, Domain = $[2, \infty)$, range = $[1, \infty)$
f. (0, 3) and (3, 6)

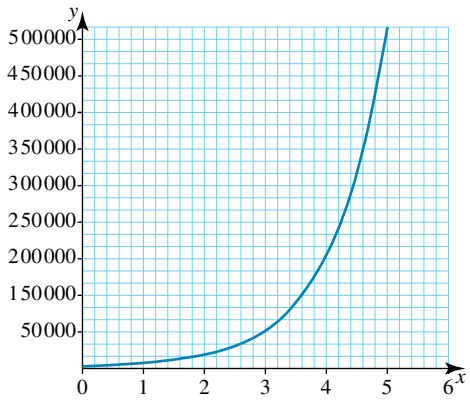
14. Use the vertical line test. If the graph is a function, a vertical line will only cross the graph once.

15. These graphs are inverse because they are the mirror images of each other through the line $y = x$.

Exercise 18.3 Exponential functions

1. a. 2000 **b.** 486 000

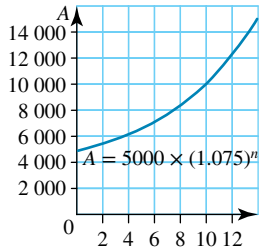
c.



d. 1.26 h

2. a. $5000 **b.** $7717

c.



d. 10 years

3. a. C **b.** D

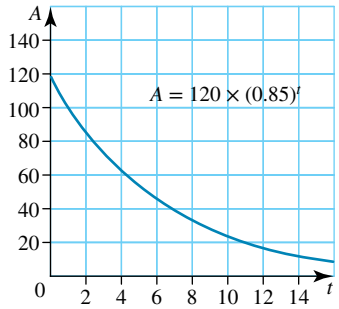
4. A

5. B

6. a. $883.50 **b.** $821.66 **c.** $V = 950 \times (0.93)^n$ **d.** $397.67

7. a. 102 mg **b.** 86.7 mg **c.** $A = 120 \times (0.85)^t$ **d.** 83.927 mg

e.

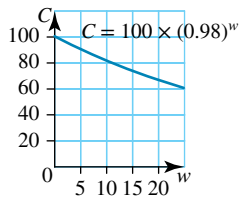


f. Approximately 210 years

8. a. i. 96.04% **ii.** 90.39%

b. $C = 100(0.98)^w$

c.



d. 8 washings

9. a. 118 (million) **b.** $a = 1.02$; $P = 118 \times (1.02)^n$

c.

Year	1990	1995	2000	2005	2010
Population	118	130	144	159	175

The calculated population is less accurate after 10 years.

d. 288 (million)

10. a. 32 **b.** 0.98 **c.** $T = 32 \times (0.98)^t$

d. 26.1, 21.4, 17.5, 14.3; values are close except for $t = 40$.

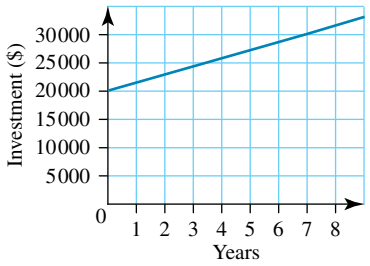
11. a. 3 dogs **b.** 27 dogs **c.** 3 years

12. a. i. 39.85 mg **ii.** 18.43 mg

b. More than 35.78 centuries

13. a. $A = 20000 \times 1.06^x$

b.



c. 7 years **d.** 6 years — 1 year quicker **e.** 9.05% p.a.

14. a. Approximately 20000 **b, c.** Check with your teacher.

15. a. $a = 100$, $b = 1.20$, increase = 20%/min **b.** $N = 146\,977 \times 0.70^m$

16. a. 500°C **b.** 125°C

c. Between 5 and 6 hours once it has cooled to below 15°C

17. a. 45°C **b.** $T = 45 \times 0.95^t$ **c.** 10°C

d.

Time (min)	0	5	10	15	20
Temperature (°C)	45	35	27	21	16

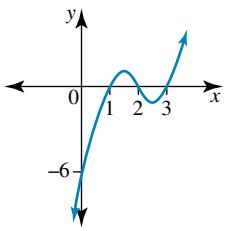
e. No

f. No. The line $T = 0$ is an asymptote.

18. A graph modelling exponential growth will have large values for y when the x-values are large; a graph modelling exponential decay will have very small values for y when x-values are large.

Exercise 18.4 Cubic functions

1. a.



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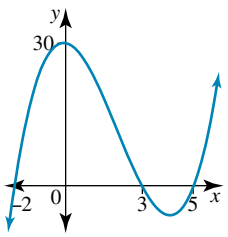

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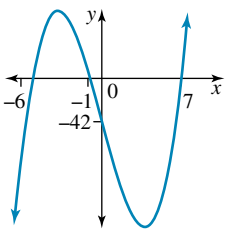

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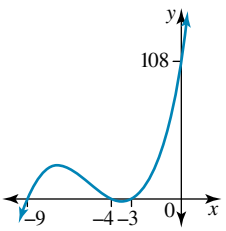

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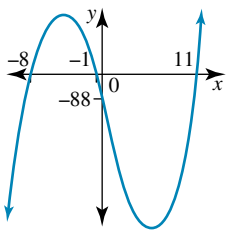

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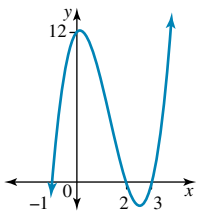

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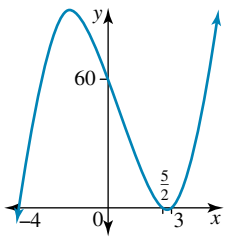

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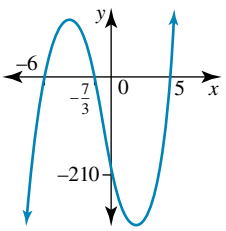

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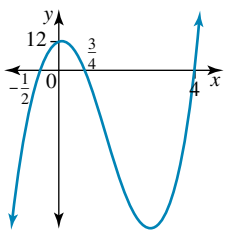

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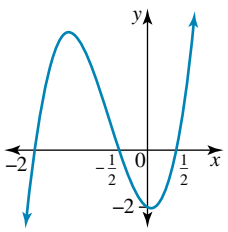

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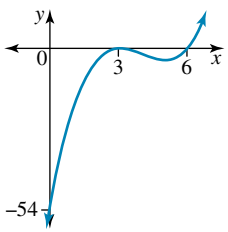

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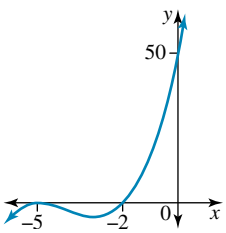

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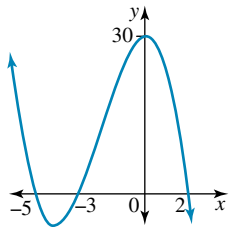

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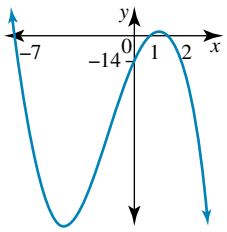

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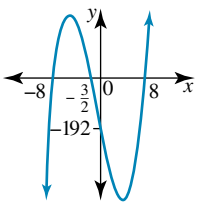

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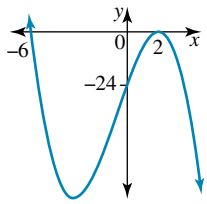

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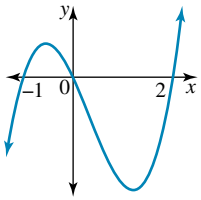

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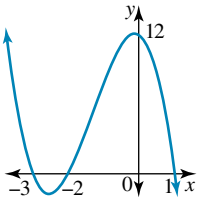

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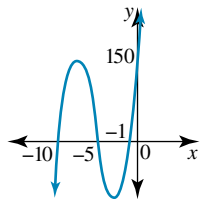

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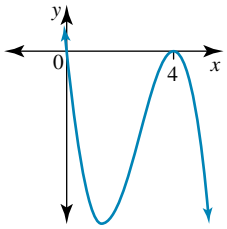
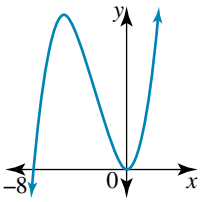
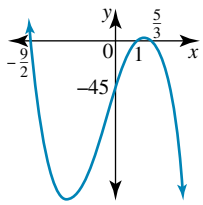


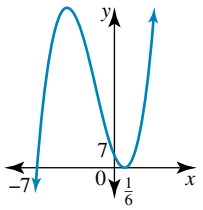
i.



j.



k.



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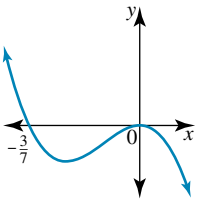


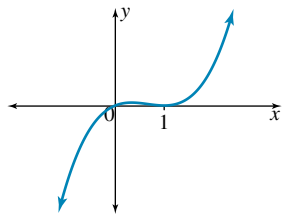
3. C

4. C

5. B

6. D

7. a.



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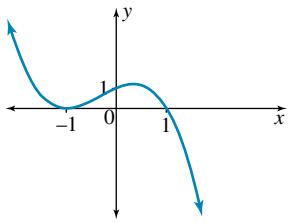

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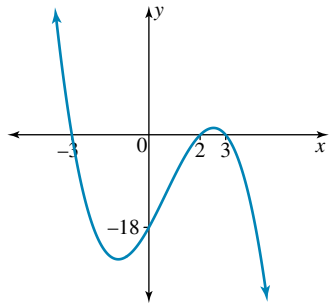

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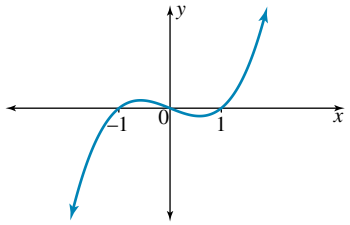
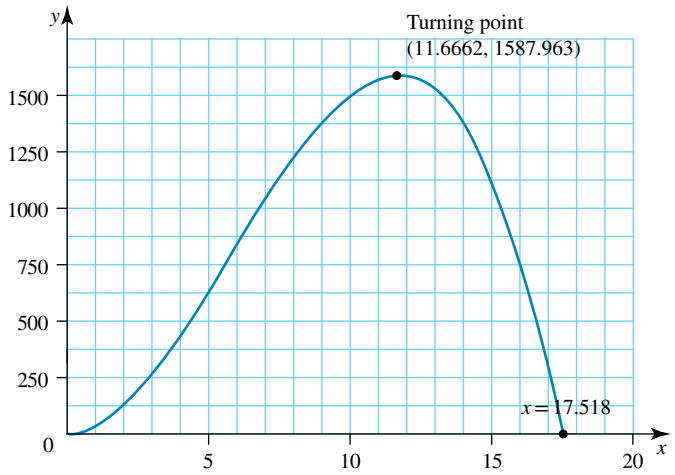


8. $a = 2, b = -7$

9. $a = \dfrac{-(27 + 11b)}{11}$

10. a. Check with your teacher. **b.** $0 < x < 17.5$ **c.** $1377\,\text{cm}^3$

d.

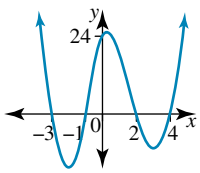


e. (11.6662, 1587.963); this is the value of x that creates the maximum volume.

11. $y = 2(x + 2)(x - 2)(x - 5)$

Exercise 18.5 Quartic functions

1. a.



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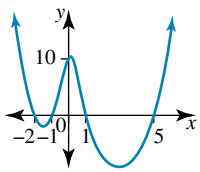

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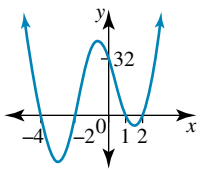

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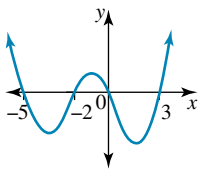

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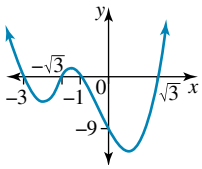

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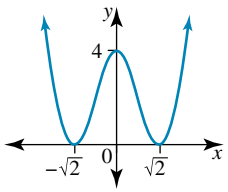

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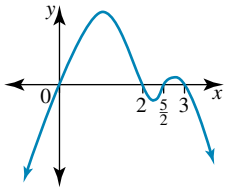

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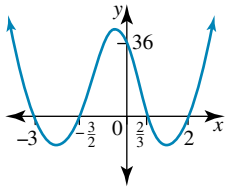

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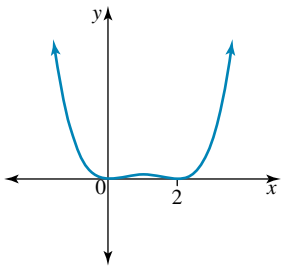

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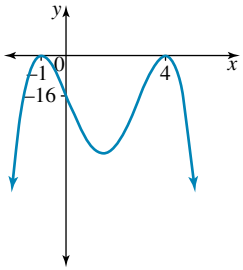

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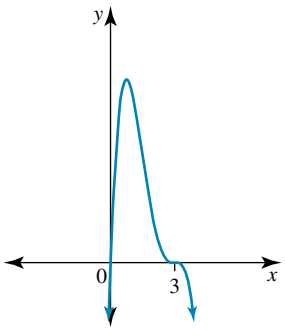

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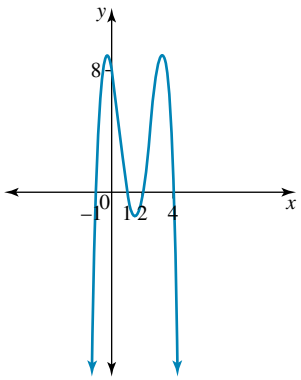
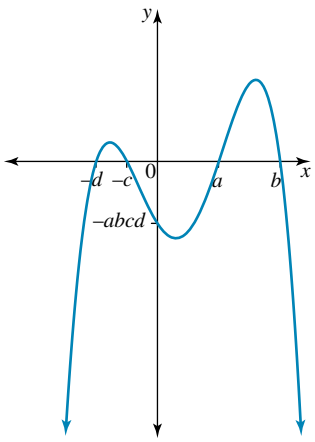


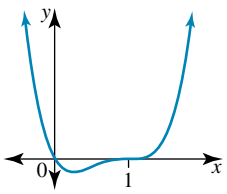
e.



3. D

4. a. D **b.** B

5. a.



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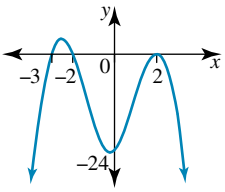

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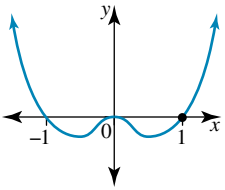

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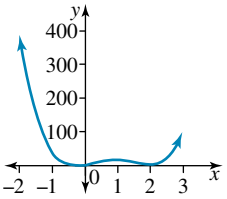

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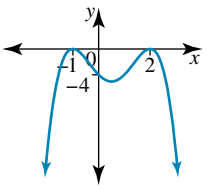
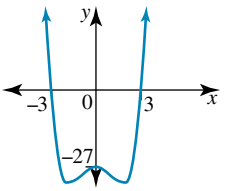
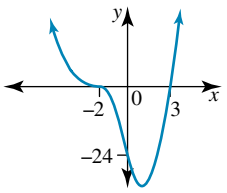


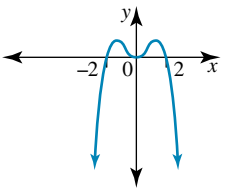
f.



g.



h.



6. $a = 4, b = -19$

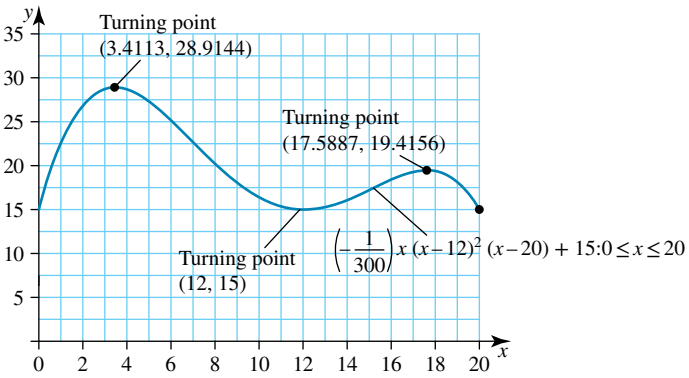
7. $a = 3, b = -1$

8. a. If n is even, the graph touches the x-axis.

b. If n is odd, the graph cuts the x-axis.

9. a. 15 m

b.



c. 28.77 m

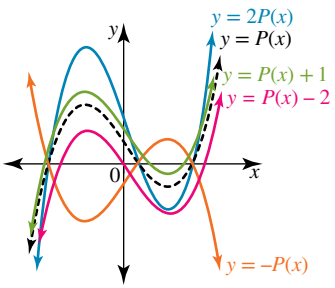
10. $y = (x + 1)(x - 2)^3$

11. A cubic function may have one point of inflection or three x-intercepts or one intercept and a point of contact with the x-axis. A quartic may have one point of contact with x-axis or 4 x-intercepts or one point of inflection and an x-intercept or two points of contact with the x-axis.

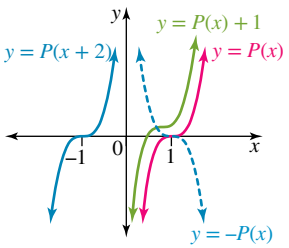
12. Check with your teacher.

Exercise 18.6 Transformations

1.



2.



3. They have the same x-intercepts, but $y = -P(x)$ is a reflection of $y = P(x)$ in the x-axis.

4. They have the same x-intercepts, but the y-values in $y = 2P(x)$ are all twice as large.

5. The entire graph is moved down 2 units. The shape is identical.

6. a. $y = -P(x)$ **b.** $y = P(x) - 3$ **c.** $y = 2P(x)$

7. The original graph has been reflected in the x-axis and dilated by a factor of 2 in the y direction. The location of the intercepts remains unchanged.

8. Dilation by a factor of h from the x-axis, reflection in the x-axis, dilation by a factor of $\frac{1}{q}$ from the y-axis, reflection in the y -axis, translation of p units left, translation of r units down.

9. $y = -\frac{2}{(x + 2)} + 1$, $x = -2$, $y = 1$

10. a. $y = 5(2^x) + 5$

b. Dilation by a factor of 5 parallel to the y-axis and translation of 5 units up. The graph asymptotes to $y = 5$.

11. By understanding transformations, it is much easier to draw the many different graphs.

12. The graphs of y_1 and y_2 will be the same shape. The graph of y_2 will be the graph of y_1 translated 1 unit in the vertical direction.

Investigation | Rich task

Check with your teacher.

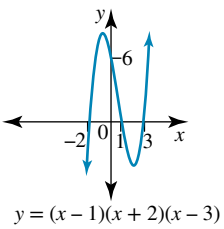
Exercise 18.7 Review questions

1. **a**

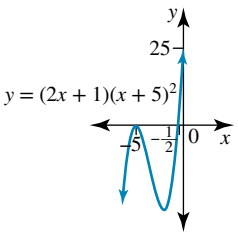
2. **a**, **c**, **d**; $\frac{x + 7}{2}$, $\log_2 x$, $\frac{1}{x} - 1$

3. a. **i.** 2 **ii.** $\sqrt{3}$ **iii.** 0 **iv.** No.

4. a.



b.



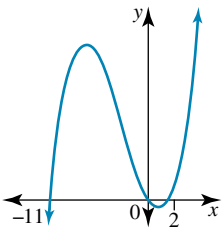
5. Check with your teacher. One possible answer is $y = (x - 1)(x - 2)^2$.

6. a. D **b.** A **c.** E **d.** C **e.** B

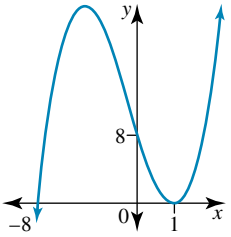
7. D

8. A

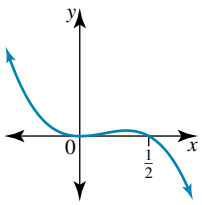
9. a.



b.



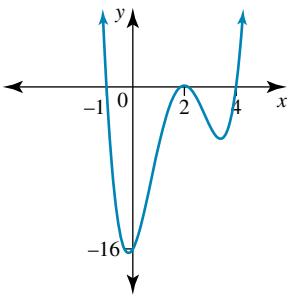
c.



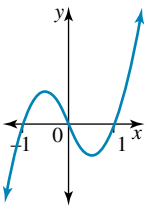
10. D

11. A

12.



13.



14. The entire graph is moved up 3 units. The shape is identical.

15. As $x \to \infty$, $f(x) \to -\infty$.

As $x \to -\infty$, $f(x) \to 0$.

16. (2, 0)

17. a. 53.59 mg/L **b.** 31.42 mg/L **c.** 72.38 mg/L

18. a. i. 20 **ii.** 25

b. i. $H = 25$; $D = 28$ **ii.** $H = 28$; $D = 30$

c. Hyenas after 3 years; dingoes after 4 years

d. After about 23 months; 31 animals

19. a. $f^{-1}(x) = \sqrt{\dfrac{x-k}{a}} + h,\ x \geq \dfrac{k}{a}$

b. Check with your teacher.

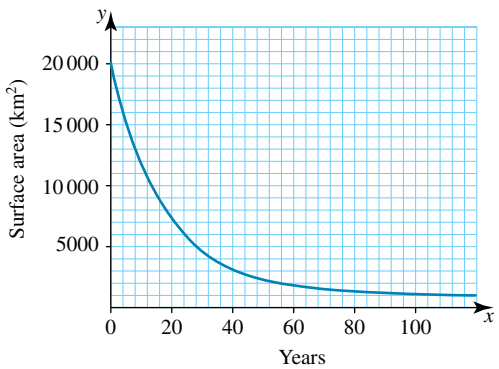
20. a. 0.27 m **b.** Check with your teacher.

21. a. Yes, because the relationship involves a variable as an exponent

b. 20 000 km^2

c. 11 975 km^2

d.



e. 118 km^2

f. No, this is not a realistic model as is it does not take into account changes to climate, rain, runoff from mountains, glaciers etc.

TOPIC 19
Circle geometry

[Stage 5.3]

19.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the concepts covered in this topic.

19.1.1 Why learn this?

For thousands of years humans have been fascinated by circles. Since they first looked upwards towards the sun and moon, which, from a distance at least, looked circular, humans have created circular monuments to nature. The most famous circular invention, one that has been credited as the most important invention of all, is the wheel. Scholars as early as Socrates and Plato have been fascinated with the sheer beauty of the properties of circles, and many scholars made a life's work out of studying them. Euclid was probably the most famous of these. It is in circle geometry that the concepts of congruence and similarity, studied earlier, have a powerful context.

DISCUSSION

Throughout history there has always been a strong link between philosophy and mathematics. Why do you think this is?

LEARNING SEQUENCE

19.1 Overview
19.2 [Stage 5.3] Angles in a circle
19.3 [Stage 5.3] Intersecting chords, secants and tangents
19.4 [Stage 5.3] Cyclic quadrilaterals
19.5 [Stage 5.3] Tangents, secants and chords
19.6 Review

LEARNING OUTCOMES

A student:

- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3-1WM
- generalises mathematical ideas and techniques to analyse and solve problems efficiently MA5.3-2WM
- uses deductive reasoning in presenting arguments and formal proofs MA5.3-3WM
- applies deductive reasoning to prove circle theorems and to solve related problems MA5.3-17MG

CONTENT DESCRIPTIONS

Students:
Prove and apply angle and chord properties of circles (ACMMG272)
Prove and apply tangent and secant properties of circles (NSW)

Source: NSW Syllabus for the Australian curriculum

learnON RESOURCES — ONLINE ONLY

eLesson: The story of mathematics — Piscopia (eles-2022)

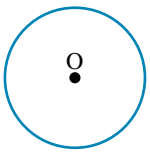
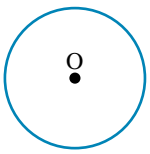
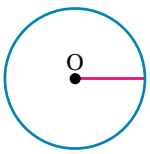
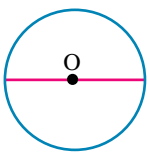
Note: Your teacher may now set you a pre-test to determine how familiar you are with the content in this topic.

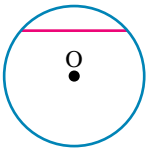
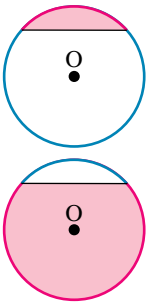
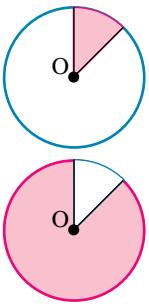
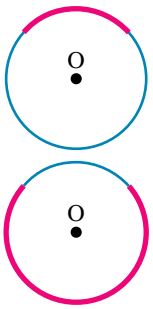
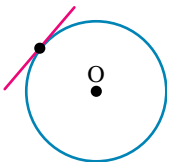
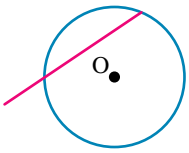
19.2 Angles in a circle [Stage 5.3]

19.2.1 Circles

- A **circle** is a set of points that lie a fixed distance (the **radius**) from a fixed point (the **centre**).
- In circle geometry, there are many theorems that can be used to solve problems. It is important that we are also able to prove these theorems.
- To prove a theorem:
  1. state the aim of the proof
  2. use given information and previously established theorems to establish the result
  3. give a reason for each step of the proof
  4. state a clear conclusion.

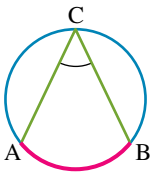
19.2.2 Parts of a circle

Part (name)	Description	Diagram
Centre	The middle point, equidistant from all points on the circumference. It is usually shown by a dot and labelled O.	
Circumference	The outside length or the boundary forming the circle. It is the circle's perimeter.	
Radius	A straight line from the centre to any point on the circumference	
Diameter	A straight line from one point on the circumference to another, passing through the centre	

Part (name)	Description	Diagram
Chord	A straight line from one point on the circumference to another	
Segment	The area of the circle between a chord and the circumference. The smaller segment is called the minor segment and the larger segment is the major segment.	
Sector	An area of a circle enclosed by 2 radii and the circumference	
Arc	A portion of the circumference	
Tangent	A straight line that touches the circumference at one point only	
Secant	A chord extended beyond the circumference on one side	

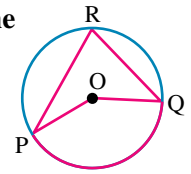
19.2.3 Angles in a circle

- In the diagram, chords AC and BC form the angle ACB. Arc AB has *subtended* angle ACB.

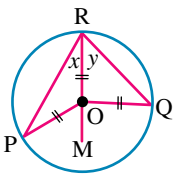


- **Theorem 1** **Code** (A)

The angle subtended at the centre of a circle is twice the angle subtended at the circumference, standing on the same arc.



Proof:



Let $\angle PRO = x$ and $\angle QRO = y$.

RO = PO = QO (radii of the same circle are equal)

$\angle RPO = x$

and $\angle RQO = y$

$\angle POM = 2x$ (exterior angle of triangle)

and $\angle QOM = 2y$ (exterior angle of triangle)

$\angle POQ = 2x + 2y$

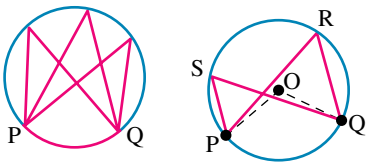
$= 2(x + y)$

which is twice the size of $\angle PRQ = x + y$.

The angle subtended at the centre of a circle is twice the angle subtended at the circumference, standing on the same arc.

- **Theorem 2** **Code** (X)

All angles that have their vertex on the circumference and are subtended by the same arc are equal.



Proof:

Join P and Q to O, the centre of the circle.

Let $\angle PSQ = x$.

$\angle POQ = 2x$ (angle at the centre is twice the angle at the circumference)

$\angle PRQ = x$ (angle at the circumference is half the angle of the centre)

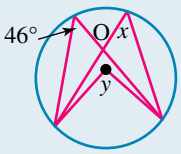
$\angle PSQ = \angle PRQ$.

Angles at the circumference subtended by the same arc are equal.

The application of the first two circle geometry theorems can be seen in the following worked example.

WORKED EXAMPLE 1

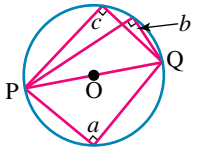
Find the values of the pronumerals in the diagram, giving reasons for your answers.



THINK	WRITE
1 Angles x and 46° are angles subtended by the same arc and both have their vertex on the circumference.	$x = 46°$ (X)
2 Angles y and 46° stand on the same arc. The 46° angle has its vertex on the circumference and y has its vertex at the centre. The angle at the centre is twice the angle at the circumference.	$y = 2 \times 46°$ (A) $= 92°$

- **Theorem 3** **Code**

Angles subtended by the diameter, that is, angles in a semicircle, are right angles.



In the diagram, PQ is the diameter. Angles a, b and c are right angles. This theorem is in fact a special case of Theorem 1.

Proof:

$\angle POQ = 180°$ (straight line)

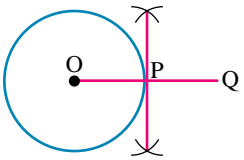
Let S refer to the angle at the circumference subtended by the diameter. In the figure, S could be at the points where a, b and c are represented on the diagram.

$\angle PSQ = 90°$ (angle at the circumference is half the angle at the centre)

Angles subtended by a diameter are right angles.

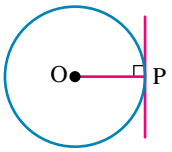
19.2.4 Constructing a tangent

- There are a number of ways to construct a tangent to a circle, as explained using the following steps.
  1. Draw a circle of radius 5 cm and centre O.
  2. Draw a radius.
  3. Call the point of intersection of the radius and the circumference, P.
  4. Extend this radius through P to the point Q, 5 cm outside the circle.
  5. Using O and Q as centres, draw intersecting arcs above and below the line OQ.
  6. Draw a straight line joining the points of intersection. This line is the tangent.
  7. What do you notice about the angle between OQ and the tangent?
  8. Investigate another technique for constructing a tangent to a circle.
  9. Write a set of instructions for this method of constructing a tangent.



- **Theorem 4** **Code**

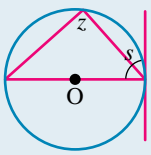
If a radius is drawn to any point on the circumference and a tangent is drawn at the same point, then the radius will be perpendicular to the tangent.



In the diagram, the radius is drawn to a point, P, on the circumference. The tangent to the circle is also drawn at P. The radius and the tangent meet at right angles, that is, the angle at P equals 90°.

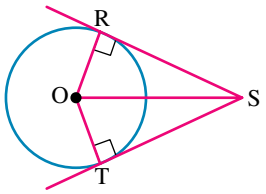
WORKED EXAMPLE 2

Find the values of the pronumerals in the diagram, giving a reason for your answer.



THINK	WRITE
1 Angle z is subtended by the diameter. Use an appropriate theorem to state the value of z.	$z = 90°$
2 Angle s is formed by a tangent and a radius, drawn to the point of contact. Apply the corresponding theorem to find the value of s.	$s = 90°$

- **Theorem 5 Code**
  The angle formed by two tangents meeting at an external point is bisected by a straight line joining the centre of the circle to that external point.



Proof:

Consider ΔSOR and ΔSOT.

OR = OT (radii of the same circle are equal)

OS is common.

∠ORS = ∠OTS = 90° (angle between a tangent and radii is 90°)

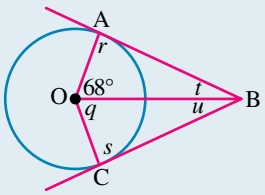
∴ ΔSOR ≡ ΔSOT(RHS)

So ∠ROS = ∠TOS and ∠OSR = ∠OST (corresponding angles in congruent triangles are equal).

The angle formed by two tangents meeting at an external point is bisected by a straight line joining the centre of the circle to the external point.

WORKED EXAMPLE 3

Given that BA and BC are tangents to the circle, find the values of the pronumerals in the diagram. Give reasons for your answers.



THINK	WRITE
1 Angles r and s are angles formed by the tangent and the radius, drawn to the same point on the circle. State their size.	$s = r = 90°$
2 In the triangle ABO, two angles are already known and so angle t can be found using our knowledge of the sum of the angles in a triangle.	ΔABO: $t + 90° + 68° = 180°$ $t + 158° = 180°$ $t = 22°$
3 ∠ABC is formed by the two tangents, so the line BO, joining the vertex B with the centre of the circle, bisects this angle. This means that angles t and u are equal.	∠ABO = ∠CBO ∠ABO = $t = 22°$, ∠CBO = u $u = 22°$
4 ΔAOB and ΔCOB are similar triangles.	In ΔAOB and ΔCOB $r + t + 68° = 180°$ $s + u + q = 180°$ $r = s = 90°$ $t = u = 22°$ (proved previously) $\therefore q = 68°$ (proved previously)

- Dynamic geometry software can be used to investigate the angle properties of circles.

learnON RESOURCES — ONLINE ONLY

Interactivity: Circle theorem 1 (int-6218)
Interactivity: Circle theorem 2 (int-6219)
Interactivity: Circle theorem 3 (int-6220)
Interactivity: Circle theorem 4 (int-6221)
Interactivity: Circle theorem 5 (int-6222)
Interactivity: Angles in a circle (int-2795)
Digital doc: SkillSHEET Using tests to prove congruent triangles (doc-5390)
Digital doc: SkillSHEET Corresponding sides and angles of congruent figures (doc-5391)
Digital doc: SkillSHEET Using tests to prove similar triangles (doc-5392)
Digital doc: SkillSHEET Angles in a triangle (doc-5393)
Digital doc: SkillSHEET More angle relations (doc-5394)
Digital doc: WorkSHEET Circle geometry I (doc-14627)

Exercise 19.2 Angles in a circle

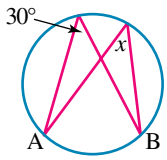
Individual pathways

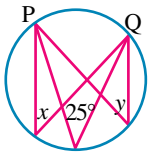
PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–e, 2a–c, 3a–d, 4–7	Questions: 1d–i, 2b–f, 3c–f, 4–10	Questions: 1–12

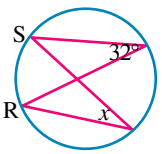
To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

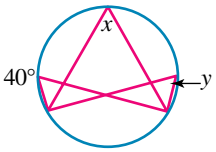
Understanding and fluency

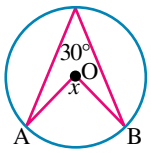
1. WE1 Find the values of the pronumerals in each of the following figures, giving reasons for your answers.

a.


b.


c.


d.


e.


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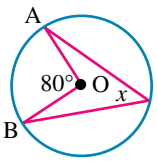
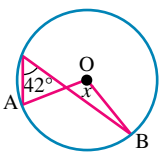
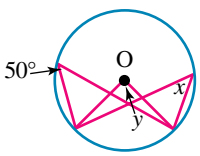
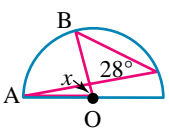
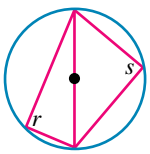


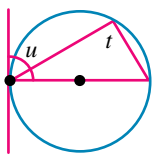
g.


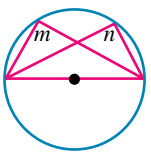
h.


i.


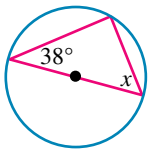
2. WE2 Find the values of the pronumerals in each of the following figures, giving reasons for your answers.

a.


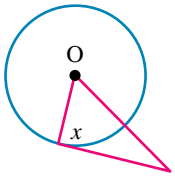
b.


c.


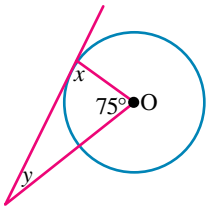
d.



e.

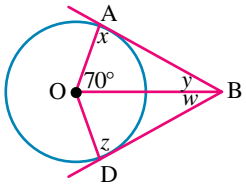


f.

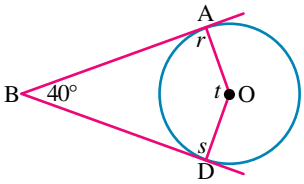


3. WE3 Given that AB and DB are tangents, find the values of the pronumerals in each of the following figures, giving reasons for your answers.

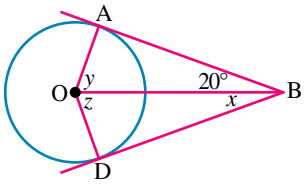
a.



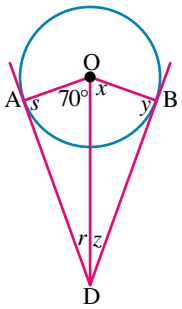
b.



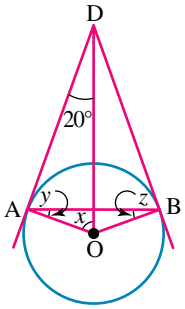
c.



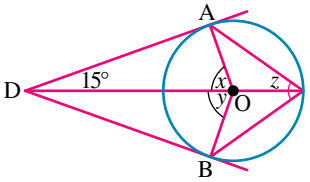
d.



e.



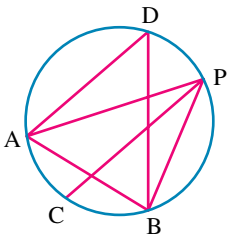
f.



4. MC In the diagram at right, which angle is subtended by the same arc as ∠APB? *Note:* There may be more than one correct answer.

A. ∠APC **B.** ∠BPC **C.** ∠ABP

D. ∠ADB **E.** None of the above

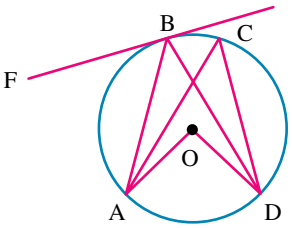


5. MC Referring to the diagram at right, which of the statements is true? *Note:* There may be more than one correct answer.

A. $2\angle AOD = \angle ABD$ **B.** $\angle AOD = 2\angle ACD$

C. $\angle ABF = \angle ABD$ **D.** $\angle ABD = \angle ACD$

E. None of the above

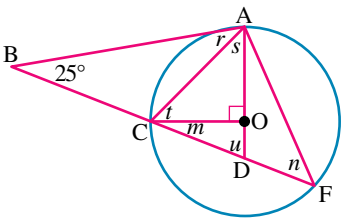


Communicating, reasoning and problem solving

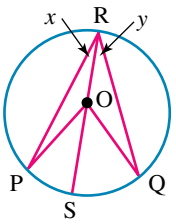
6. Values are suggested for the pronumerals in the diagram at right. AB is a tangent to a circle and O is the centre. In each case give reasons to justify suggested values.

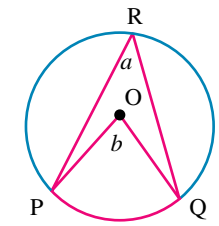
a. $s = t = 45°$ **b.** $r = 45°$ **c.** $u = 65°$

d. $m = 25°$ **e.** $n = 45°$



7. Set out below is the proof of this result: The angle at the centre of a circle is twice the angle at the circumference standing on the same arc. Copy and complete the following to show that $\angle POQ = 2 \times \angle PRQ$. Construct a diameter through R. Let the opposite end of the diameter be S.

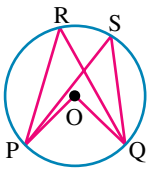




Let ∠ORP = x and ∠ORQ = y.
OR = OP (_____)
∠OPR = x (_____)
∠SOP = $2x$ (exterior angle equals _____)
OR = OQ (_____)
∠OQR = _____ (_____)
∠SOQ = _____ (_____)
Now ∠PRQ = _____ and ∠POQ = _____.
Therefore, ∠POQ = 2 × ∠PRQ.

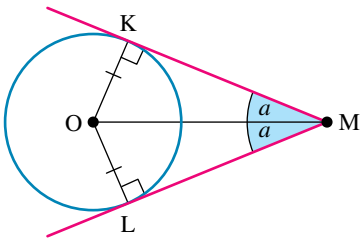
8. Prove that the segments formed by drawing tangents from an external point to a circle are equal in length.

9. Use the figure shown to prove that angles subtended by the same arc are equal.

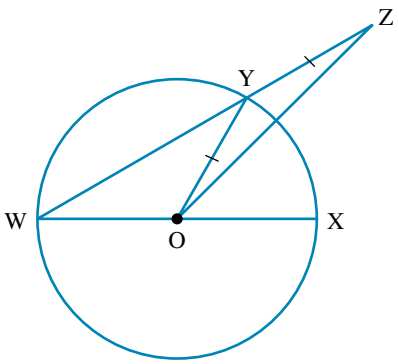


10. Use your knowledge of types of triangles, angles in triangles and the fact that the radius of a circle meets the tangent to the circle at right angles *to prove* the following theorem:

The angle formed between two tangents meeting at an external point is bisected by a line from the centre of the circle to the external point.



11. WX is the diameter of a circle with centre at O. Y is a point on the circle and WY is extended to Z so that OY = YZ. Prove that angle ZOX is 3 times angle YOZ.

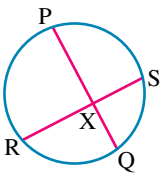


12. What are the common steps in proving a theorem?

19.3 Intersecting chords, secants and tangents [Stage 5.3]

19.3.1 Intersecting chords

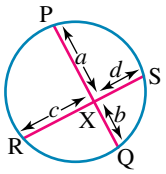
- In the diagram, chords PQ and RS intersect at X.



- **Theorem 6** **Code**

If the two chords intersect inside a circle, then the point of intersection divides each chord into two segments so that the product of the lengths of the segments for both chords is the same.

$$\mathbf{PX \times QX = RX \times SX \text{ or } } \boldsymbol{a \times b = c \times d}$$



Proof:

Join PR and SQ.

Consider ΔPRX and ΔSQX.

$\angle PXR = \angle SXQ$ (vertically opposite angles are equal)

$\angle RSQ = \angle RPQ$ (angles at the circumference standing on the same arc are equal)

$\angle PRS = \angle PQS$ (angles at the circumference standing on the same arc are equal)

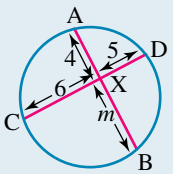
$\Delta PRX \equiv \Delta SQX$ (equiangular)

$\frac{PX}{SX} = \frac{RX}{QX}$ (ratio of sides in similar triangles is equal)

or $PX \times QX = RX \times SX$

WORKED EXAMPLE 4

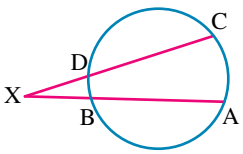
Find the value of the pronumeral.



THINK	WRITE
1 Chords AB and CD intersect at X. Point X divides each chord into two parts so that the products of the lengths of these parts are equal. Write this as a mathematical statement.	$AX \times BX = CX \times DX$
2 Identify the lengths of the line segments.	$AX = 4, BX = m, CX = 6, DX = 5$
3 Substitute the given lengths into the formula and solve for m.	$4m = 6 \times 5$ $m = \frac{30}{4}$ $= 7.5$

19.3.2 Intersecting secants

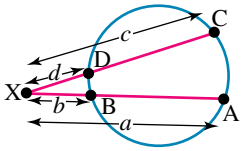
- In the diagram below, chords CD and AB are extended to form secants CX and AX respectively. They intersect at X.



- **Theorem 7 Code**

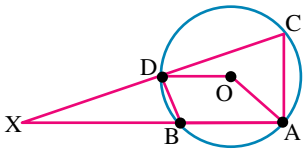
If two secants intersect outside the circle as shown, then the following relationship is always true:

$$\mathbf{AX \times XB = XC \times DX \text{ or } } a \times b = c \times d.$$



Proof:

Join D and A to O, the centre of the circle.



Let $\angle DCA = x$.

$\angle DOA = 2x$	(angle at the centre is twice the angle at the circumference standing on the same arc)
Reflex $\angle DOA = 360° - 2x$	(angles in a revolution add 360°)
$\angle DBA = 180° - x$	(angle at the centre is twice the angle at the circumference standing on the same arc)
$\angle DBX = x$	(angle sum of a straight line is 180°)
$\angle DCA = \angle DBX$	

Consider ΔBXD and ΔCXA.

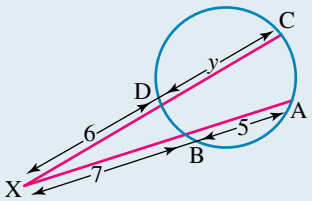
$\angle BXD$ is common.	
$\angle DCA = \angle DBX$	(shown previously)
$\angle XAC = \angle XDB$	(angle sum of a triangle is 180°)
$\Delta AXC \,\|\|\| \Delta DXB$	(equiangular)

$$\frac{AX}{DX} = \frac{XC}{XB}$$

or $AX \times XB = XC \times DX$

WORKED EXAMPLE 5

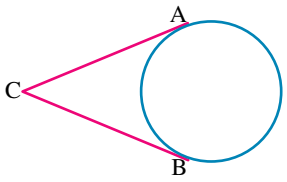
Find the value of the pronumeral.



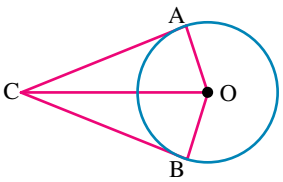
THINK	WRITE
1 Secants XC and AX intersect outside the circle at X. Write the rule connecting the lengths of XC, DX, AX and XB.	$XC \times DX = AX \times XB$
2 State the length of the required line segments.	$XC = y + 6$ $\quad DX = 6$ $AX = 7 + 5$ $\quad XB = 7$ $= 12$
3 Substitute the length of the line segments and solve the equation for y.	$(y + 6) \times 6 = 12 \times 7$ $6y + 36 = 84$ $6y = 48$ $y = 8$

19.3.3 Intersecting tangents

- In the following diagram, tangents AC and BC intersect at C and AC = BC.



- **Theorem 8** **Code**
  If two tangents meet outside a circle, then the lengths from the external point to where they meet the circle are equal.
  Proof:
  Join A and B to O, the centre of the circle.
  Consider ΔOCA and ΔOCB.
  OC is common.

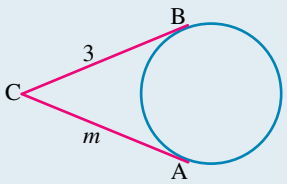


OA = OB	(radii of the same circle are equal)
∠OAC = ∠OBC	(radius is perpendicular to tangent through the point of contact)
ΔOCA ≡ ΔOCB	(RHS)
AC = BC	(corresponding sides of congruent triangles are equal)

If two tangents meet outside a circle, the lengths from the external point to the point of contact are equal.

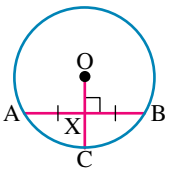
WORKED EXAMPLE 6

Find the value of the pronumeral.

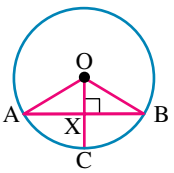


THINK	WRITE
BC and AC are tangents intersecting at C. State the rule that connects the lengths BC and AC.	AC = BC
State the lengths of BC and AC.	AC = m, BC = 3
Substitute the required lengths into the equation to find the value of m.	m = 3

19.3.4 Chords and radii



- In the diagram at right, the chord AB and the radius OC intersect at X at 90°; that is, ∠OXB = 90°. OC bisects the chord AB; that is, AX = XB.
- **Theorem 9** **Code**
  If a radius and a chord intersect at right angles, then the radius bisects the chord.
  Proof:
  Join OA and OB.
  Consider ΔOAX and ΔOBX.

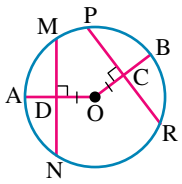


OA = OB	(radii of the same circle are equal)
∠OXB = ∠OXA	(given)
OX is common.	

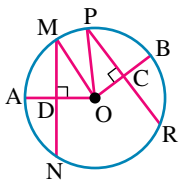
$\Delta OAX \cong \Delta OBX$ (RHS)
$AX = BX$ (corresponding sides in congruent triangles are equal)
If a radius and a chord intersect at right angles, then the radius bisects the chord.

- **The converse is also true:**
  If a radius bisects a chord, the radius and the chord meet at right angles.
- **Theorem 10 Code**

Chords equal in length are equidistant from the centre.



This theorem states that if the chords MN and PR are of equal length, then OD = OC.
Proof:
Construct OA ⊥ MN and OB ⊥ PR.
Then OA bisects MN and OB bisects PR (Theorem 9).
Because MN = PR, MD = DN = PC = CR.
Construct OM and OP, and consider ΔODM and ΔOCP.

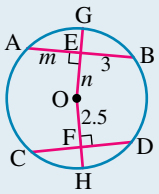


MD = PC	(shown above)
OM = OP	(radii of the same circle are equal)
$\angle ODM = \angle OCP = 90°$	(by construction)
$\Delta ODM \equiv \Delta OCP$	(RHS)
So OD = OC	(corresponding sides in congruent triangles are equal)

Chords equal in length are equidistant from the centre.

WORKED EXAMPLE 7

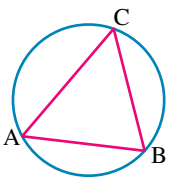
Find the values of the pronumerals, given that AB = CD.



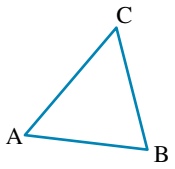
THINK	WRITE
1 Since the radius OG is perpendicular to the chord AB, the radius bisects the chord.	AE = EB
2 State the lengths of AE and EB.	AE = m, EB = 3
3 Substitute the lengths into the equation to find the value of m.	$m = 3$
4 AB and CD are chords of equal length and OE and OF are perpendicular to these chords. This implies that OE and OF are equal in length.	OE = OF
5 State the lengths of OE and OF.	OE = n, OF = 2.5
6 Substitute the lengths into the equation to find the value of n.	$n = 2.5$

19.3.5 The circumcentre of a triangle

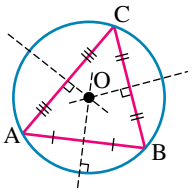
- In the diagram, a circle passes through the vertices of the triangle ABC.



- The circle is called the **circumcircle** of triangle ABC, and the centre of the circle is called the **circumcentre**.
- The circumcentre is located as follows.
  Draw any triangle ABC. Label the vertices.



  Construct perpendicular bisectors of AB, AC and BC, and let the bisectors intersect at O. This means that OA = OB = OC, so a circle can be drawn through A, B and C with a centre at O.



DISCUSSION

Can a circle be drawn so that it touches all three points of any triangle?

- Dynamic geometry software can be used to investigate the chord, secant and tangent properties of circles.

learnON RESOURCES — ONLINE ONLY

Interactivity: Circle theorem 6 (int-6223)
Interactivity: Circle theorem 7 (int-6224)
Interactivity: Circle theorem 8 (int-6225)
Interactivity: Circle theorem 9 (int-6226)
Interactivity: Circle theorem 10 (int-6227)

Exercise 19.3 Intersecting chords, secants and tangents

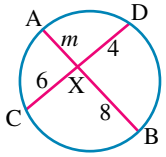
Individual pathways

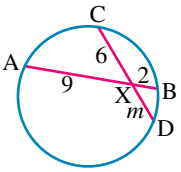
PRACTISE	CONSOLIDATE	MASTER
Questions: 1–7, 10	Questions: 1–8, 10, 11	Questions: 1–13

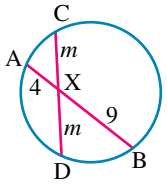
To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

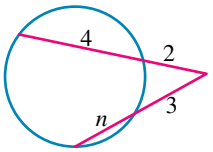
1. WE4 Find the value of the pronumeral in each of the following.

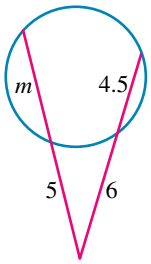
a.


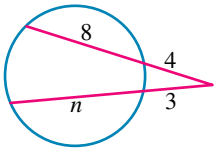
b.


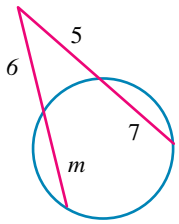
c.


2. WE5 Find the value of the pronumeral in each of the following.

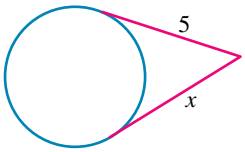
a.


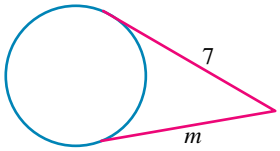
b.


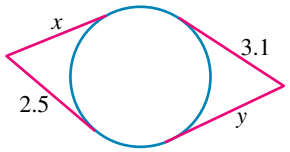
c.


d.


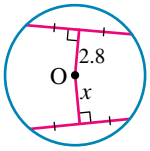
3. WE6 Find the values of the pronumerals in each of the following.

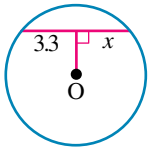
a.


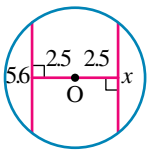
b.


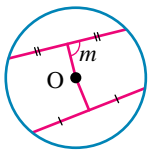
c.


4. WE7 Find the value of the pronumeral in each of the following.

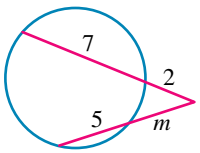
a.


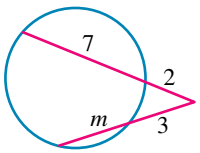
b.


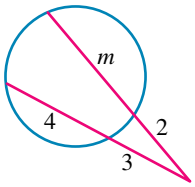
c.


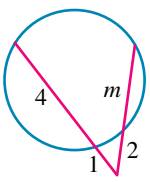
d.


5. MC In which of the following figures is it possible to find the value of m through solving a linear equation? *Note:* There may be more than one correct answer.

A.


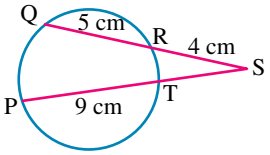
B.


C.


D.


E. None of the above

6. Find the length of ST in the diagram below.



Communicating, reasoning and problem solving

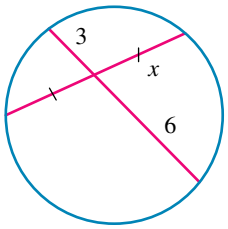
7. Prove the result: If a radius bisects a chord, then the radius meets the chord at right angles. Remember to provide reasons for your statements.

8. Prove the result: Chords that are an equal distance from the centre are equal in length. Provide reasons for your statements.

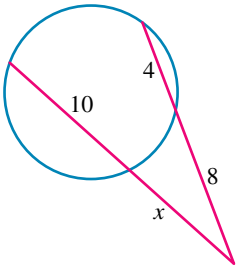
9. Prove that the line joining the centres of two intersecting circles bisects their common chord at right angles. Provide reasons for your statements.

10. Calculate the values of the pronumerals in each of the following diagrams.

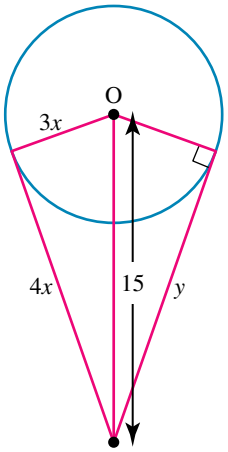
a.



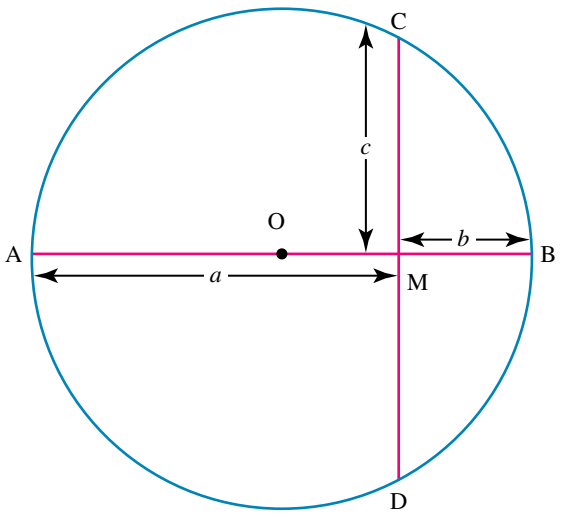
b.



c.



11. AOB is the diameter of the circle. CD is a chord perpendicular to AB and meeting AB at M.



a. Why is M the midpoint of CD?

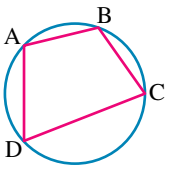
b. If $CM = c$, $AM = a$ and $MB = b$, prove that $c^2 = ab$.

c. Explain why the radius of the circle is equal to $\frac{a + b}{2}$.

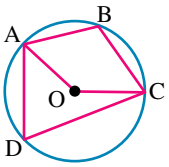
12. What techniques will you use to prove circle theorems?

13. An astroid is the curve traced by a point on the circumference of a small circle as it rolls around the inside circumference of a circle that is four times larger than it. Draw the shape of an astroid.

19.4 Cyclic quadrilaterals [Stage 5.3]



- A **cyclic quadrilateral** has all four vertices on the circumference of a circle; that is, the quadrilateral is inscribed in the circle.
- In the diagram, points A, B, C and D lie on the circumference; hence, ABCD is a cyclic quadrilateral.
- It can also be said that points A, B, C and D are **concyclic**; that is, the circle passes through all the points.
- **Theorem 11** **Code**
  The opposite angles of a cyclic quadrilateral are supplementary (add to 180°).
  Proof:
  Join A and C to O, the centre of the circle.
  Let $\angle ABC = x$.



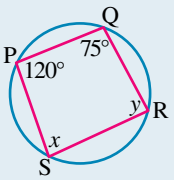
  Reflex $\angle AOC = 2x$ (angle at the centre is twice the angle at the circumference standing on the same arc)
  Reflex $\angle AOC = 360° - 2x$ (angles in a revolution add to 360°)
  $\angle ADC = 180° - x$ (angle at the centre is twice the angle at the circumference standing on the same arc)
  $\angle ABC + \angle ADC = 180°$
  Similarly, $\angle DAB + \angle DCB = 180°$.
  Opposite angles in a cyclic quadrilateral are supplementary.
- **The converse is also true:**
  If opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

DISCUSSION

Does the proof of Theorem 11 also prove the stated converse?

WORKED EXAMPLE 8

Find the values of the pronumerals in the diagram below. Give reasons for your answers.



THINK

1 PQRS is a cyclic quadrilateral, so its opposite angles are supplementary. First find the value of x by considering a pair of opposite angles $\angle PQR$ and $\angle RSP$ and forming an equation to solve.

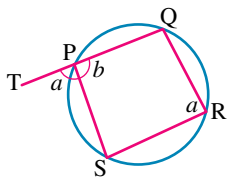
WRITE

$\angle PQR + \angle RSP = 180°$ (The opposite angles of a cyclic quadrilateral are supplementary.)
$\angle PQR = 75°$, $\angle RSP = x$
$x + 75° = 180°$
$x = 105°$

2 Find the value of y by considering the other pair of opposite angles ($\angle SPQ$ and $\angle QRS$).

$\angle SPQ + \angle QRS = 180°$
$\angle SPQ = 120°$, $\angle QRS = y$
$y + 120° = 180°$
$y = 60°$

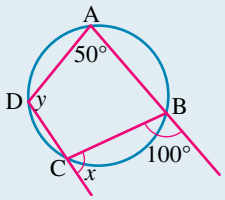
- **Theorem 12** Code
  The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.



Proof:
$\angle QPS + \angle QRS = 180°$ (opposite angles of a cyclic quadrilateral)
$\angle QPS + \angle SPT = 180°$ (adjacent angles on a straight line)
Therefore, $\angle SPT = \angle QRS$.
The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

WORKED EXAMPLE 9

Find the value of the pronumerals in the diagram.



THINK

1 ABCD is a cyclic quadrilateral. The exterior angle, x, is equal to its interior opposite angle, $\angle DAB$.

2 The exterior angle, 100°, is equal to its interior opposite angle, $\angle ADC$.

WRITE

$x = \angle DAB$, $\angle DAB = 50°$
So $x = 50°$.

$\angle ADC = 100°$, $\angle ADC = y$
So $y = 100°$.

- Dynamic geometry software can be used to investigate the properties of cyclic quadrilaterals.

learnon RESOURCES — ONLINE ONLY

Interactivity: Circle theorem 11 (int-6228)
Interactivity: Circle theorem 12 (int-6229)
Digital doc: SkillSHEET Angles in a quadrilateral (doc-5396)
Digital doc: WorkSHEET Circle geometry II (doc-14628)

Exercise 19.4 Cyclic quadrilaterals

Individual pathways

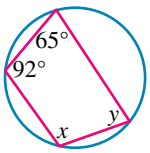
PRACTISE	CONSOLIDATE	MASTER
Questions: 1–6, 8	Questions: 1–8	Questions: 1–10

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

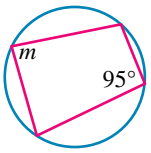
Understanding and fluency

1. WE8 Find the values of the pronumerals in each of the following figures.

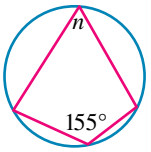
a.



b.



c.



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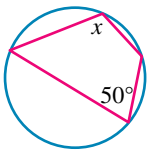

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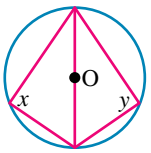

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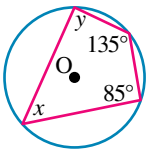
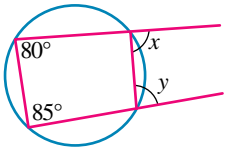
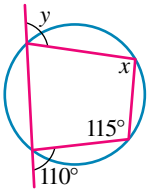


2. WE9 Find the values of the pronumerals in each of the following figures.

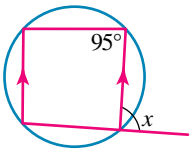
a.



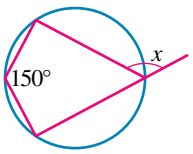
b.



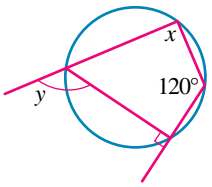
c.



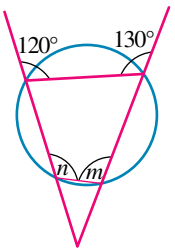
d.



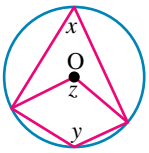
e.



f.

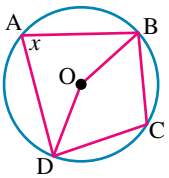


3. MC Which of the following correctly states the relationship between x, y and z in the diagram shown? *Note:* There may be more than one correct answer.



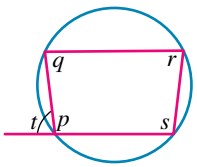
A. $x = y$ and $x = 2z$
B. $x = 2y$ and $y + z = 180°$
C. $z = 2x$ and $y = 2z$
D. $x + y = 180°$ and $z = 2x$
E. None of the above

4. The steps below show you how to set out the proof that the opposite angles of a cyclic quadrilateral are equal.



a. Find the size of ∠DOB.
b. Find the size of the reflex angle DOB.
c. Find the size of ∠BCD.
d. Find ∠DAB + ∠BCD.

5. MC *Note:* There may be more than one correct answer.



a. Which of the following statements is always true for the diagram shown?

A. $r = t$ **B.** $r = p$ **C.** $r = q$

D. $r = s$ **E.** None of the above

b. Which of the following statements is correct for the diagram shown?

A. $r + p = 180°$ **B.** $q + s = 180°$ **C.** $t + p = 180°$

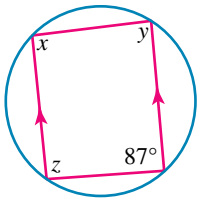
D. $t = r$ **E.** None of the above

Communicating, reasoning and problem solving

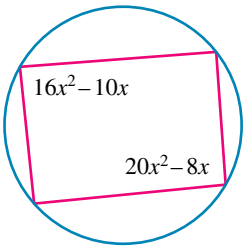
6. Prove that the exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

7. Calculate the values of the pronumerals in each of these diagrams.

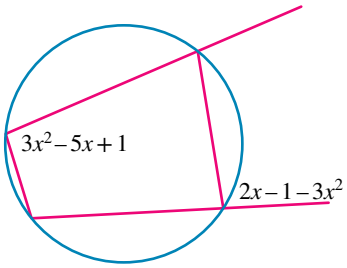
a.



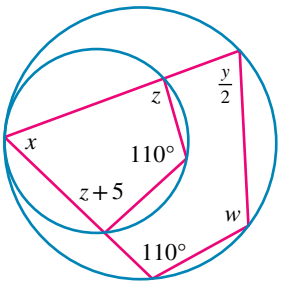
b.



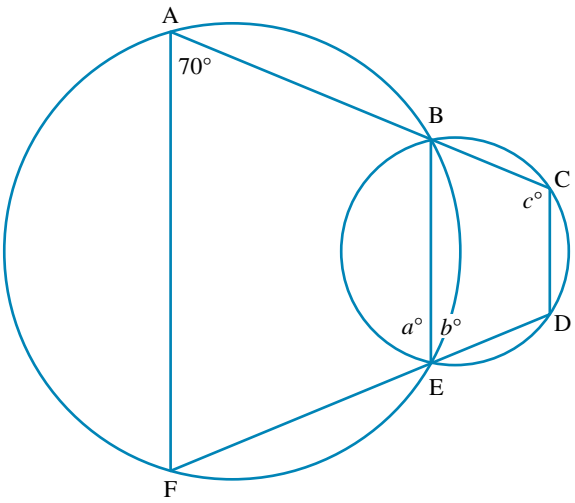
c.



8. Calculate the value of each pronumeral in the diagram.



9. $\angle FAB = 70°$, $\angle BEF = a°$, $\angle BED = b°$ and $\angle BCD = c°$.



a. Find the values of a, b and c.

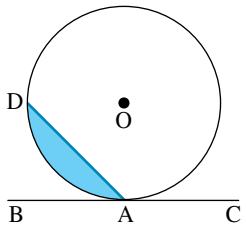
b. Prove that CD is parallel to AF.

10. What is a cyclic quadrilateral?

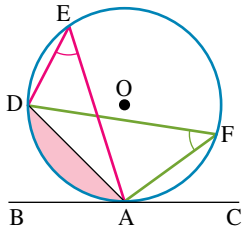
19.5 Tangents, secants and chords [Stage 5.3]

19.5.1 The alternate segment theorem

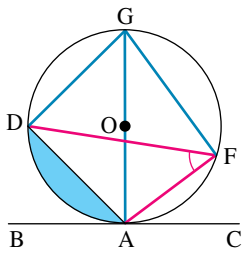
- Consider the figure shown. Line BC is a tangent to the circle at the point A.



- A line is drawn from A to anywhere on the circumference, point D.
  The angle ∠BAD defines a segment (the shaded area).
  The unshaded part of the circle is called the alternate segment to ∠BAD.
  Now consider angles subtended by the chord AD in the alternate segment, such as the angles marked in red and blue.



- The alternate segment theorem states that these are equal to the angle that made the segment, namely:
  $\angle BAD = \angle AED$ and $\angle BAD = \angle AFD$.
- **Theorem 13** Code
  The angle between a tangent and a chord is equal to the angle in the alternate segment.
  Proof:
  We are required to prove that ∠BAD = ∠AFD.
  Construct the diameter from A through O, meeting the circle at G.
  Join G to the points D and F.



$\angle BAG = \angle CAG = 90°$ (radii $\perp$ tangent at point of contact)
$\angle GFA = 90°$ (angle in a semicircle is 90°)
$\angle GDA = 90°$ (angle in a semicircle is 90°)
Consider ΔGDA. We know that ∠GDA = 90°.

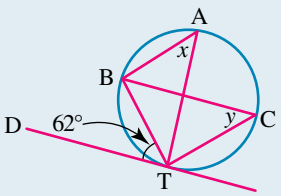
$$\angle GDA + \angle DAG + \angle AGD = 180°$$
$$90° + \angle DAG + \angle AGD = 180°$$
$$\angle DAG + \angle AGD = 90°$$

∠BAG is also a right angle.
$\angle BAG = \angle BAD + \angle DAG = 90°$

Equate the two results.
$\angle DAG + \angle AGD = \angle BAD + \angle DAG$
Cancel the equal angles on both sides.
$\angle AGD = \angle BAD$
Now consider the fact that both triangles DAG and DAF are subtended from the same chord (DA).
$\angle AGD = \angle AFD$ (Angles in the same segment standing on the same arc are equal).
Equate the two equations.
$\angle AFD = \angle BAD$

WORKED EXAMPLE 10

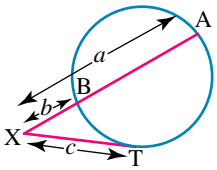
Find the value of x and y, giving reasons.



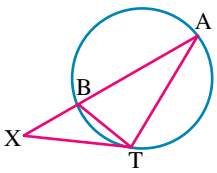
THINK	WRITE
1 Use the alternate segment theorem to find x.	$x = 62°$ (angle between a tangent and a chord is equal to the angle in the alternate segment)
2 The value of y is the same as x because x and y are subtended by the same chord BT.	$y = 62°$ (angles in the same segment standing on the same arc are equal)

19.5.2 Tangents and secants

- **Theorem 14 Code**
  If a tangent and a secant intersect as shown, the following relationship is always true:
  $$\mathbf{XA \times XB = (XT)^2 \text{ or } a \times b = c^2.}$$



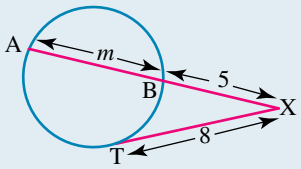
Proof:
Join BT and AT.



Consider ΔTXB and ΔAXT.
$\angle TXB$ is common.
$\angle XTB = \angle XAT$ (angle between a tangent and a chord is equal to the angle in the alternate segment)
$\angle XBT = \angle XTA$ (angle sum of a triangle is 180°)
$\Delta TXB \text{ ||| } \Delta AXT$ (equiangular)
So $\frac{XB}{XT} = \frac{XT}{XA}$
or $XA \times XB = (XT)^2$.

WORKED EXAMPLE 11

Find the value of the pronumeral.



THINK	WRITE
1 Secant XA and tangent XT intersect at X. Write the rule connecting the lengths of XA, XB and XT.	$XA \times XB = (XT)^2$
2 State the values of XA, XB and XT.	$XA = m + 5, XB = 5, XT = 8$
3 Substitute the values of XA, XB and XT into the equation and solve for m.	$(m + 5) \times 5 = 8^2$ $5m + 25 = 64$ $5m = 39$ $m = 7.8$

DISCUSSION

Do you have a method to remember some or all of these circle theorems?

- Dynamic geometry software can be used to investigate the chord, secant and tangent properties of circles.

learnON RESOURCES — ONLINE ONLY

Interactivity: Circle theorem 13 (int-6230)
Interactivity: Circle theorem 14 (int-6231)

Exercise 19.5 Tangents, secants and chords

Individual pathways

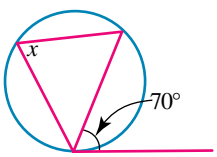
PRACTISE	CONSOLIDATE	MASTER
Questions: 1, 2, 4–6, 8, 10, 13–15, 17, 20, 22	Questions: 1–3, 5, 7, 9, 11, 13, 14, 16, 19, 21–23	Questions: 1–25

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

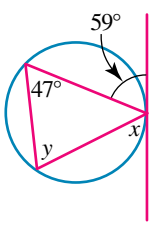
Understanding and fluency

1. WE10 Find the values of the pronumerals in the following figures.

a.

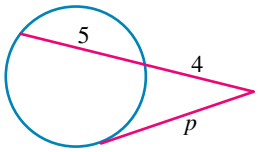


b.

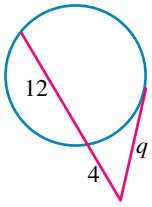


2. WE11 Find the values of the pronumerals in the following figures.

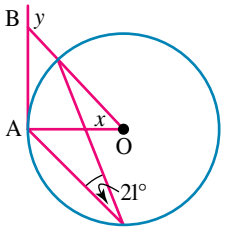
a.



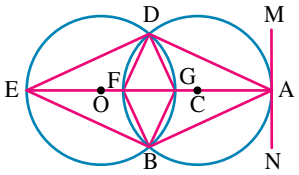
b.



3. Line AB is a tangent to the circle as shown in the figure. Find the values of the angles labelled x and y.



Questions 4 to 6 refer to the figure below. The line MN is a tangent to the circle, and EA is a straight line. The circles have the same radius.



4. Find 6 different right angles.

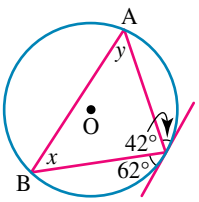
5. MC If $\angle DAC = 20°$, then $\angle CFD$ and $\angle FDG$ are respectively:

A. 70° and 50° **B.** 70° and 40° **C.** 40° and 70°
D. 70° and 70° **E.** 40° and 30°

6. MC A triangle similar to FDA is:

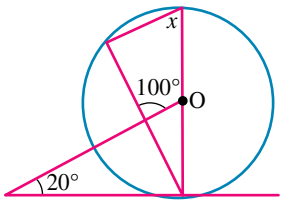
A. FDG **B.** FGB **C.** EDA
D. GDE **E.** FDE

7. Find the values of the angles x and y in the figure.

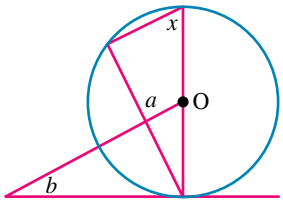


8. Show that if the sum of the two given angles in question **7** is 90°, then the line AB must be a diameter.

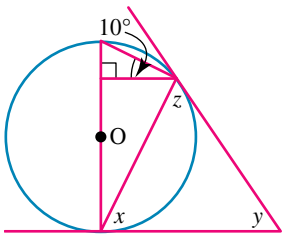
9. Find the value of x in the figure, given that the line underneath the circle is a tangent.



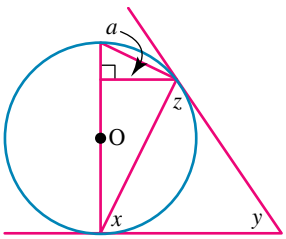
10. In the figure, express x in terms of a and b. This is the same drawing as in question **9**.



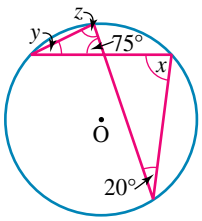
11. Two tangent lines to a circle meet at an angle y, as shown in the figure. Find the values of the angles x, y and z.



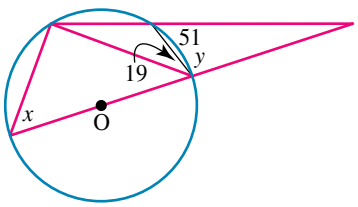
12. Solve question **11** in the general case (see the figure below) and show that $y = 2a$. This result is important for space navigation (imagine the circle to be the Earth) in that an object at y can be seen by people at x and z at the same time.



13. In the figure, find the values of the angles x, y and z.

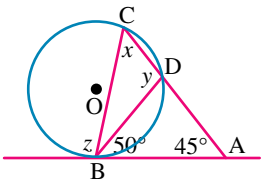


14. MC Examine the figure. The angles x and y (in degrees) are respectively:



A. 51 and 99 **B.** 51 and 129 **C.** 39 and 122
D. 51 and 122 **E.** 39 and 141

Questions 15 to 17 refer to the figure below. The line BA is a tangent to the circle at point B. Line AC is a chord that meets the tangent at A.



15. Find the values of the angles x and y.

16. MC The triangle which is similar to triangle BAD is:

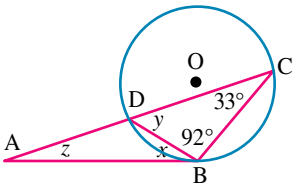
A. CAB **B.** BCD **C.** BDC
D. AOB **E.** None of the above

17. MC The value of the angle z is:

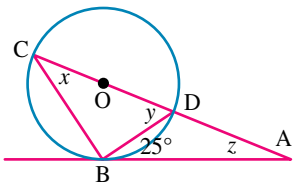
A. 50° **B.** 85° **C.** 95°
D. 100° **E.** 130°

Communicating, reasoning and problem solving

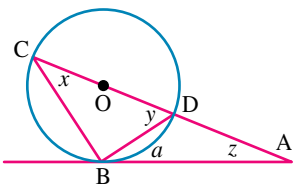
18. Find the values of the angles x, y and z in the figure. The line AB is tangent to the circle at B.



19. Find the values of the angles x, y and z in the figure. The line AB is tangent to the circle at B. The line CD is a diameter.



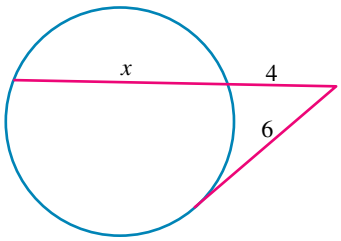
20. Solve question **19** in the general case; that is, express angles x, y and z in terms of a (see the figure below).



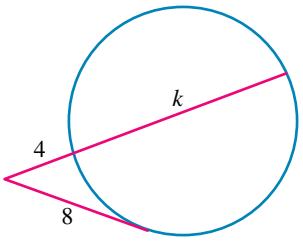
21. Prove that, when two circles touch, their centres and the point of contact are collinear.

22. Find the values of the pronumerals in the following figures.

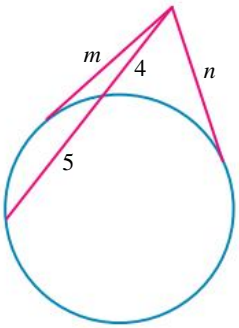
a.



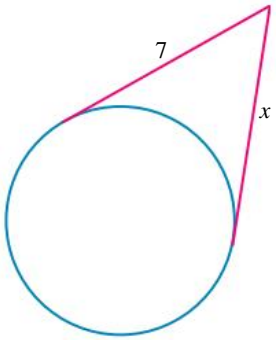
b.



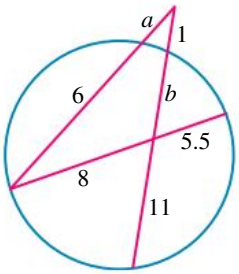
c.



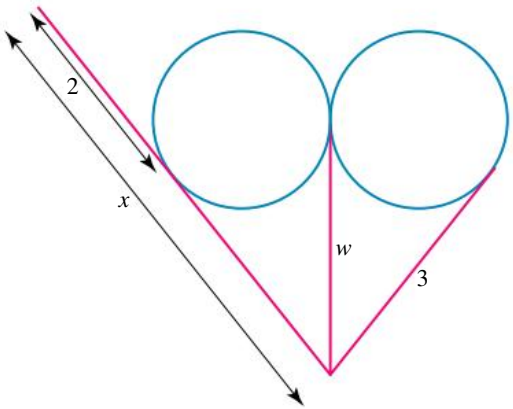
d.



e.

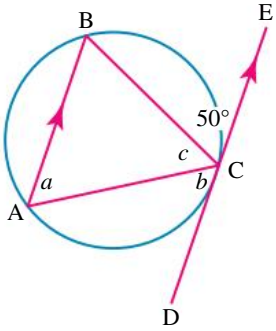


f.

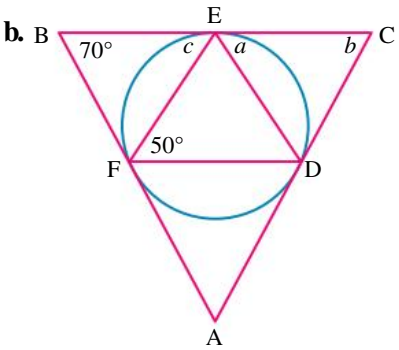


23. Find the values of a, b and c in each case.

a. $\angle BCE = 50°$ and $\angle ACE = c$



b.



24. Describe the alternate segment of a circle.

25. How can an annulus be cut into seven pieces with three straight lines?

19.6 Review

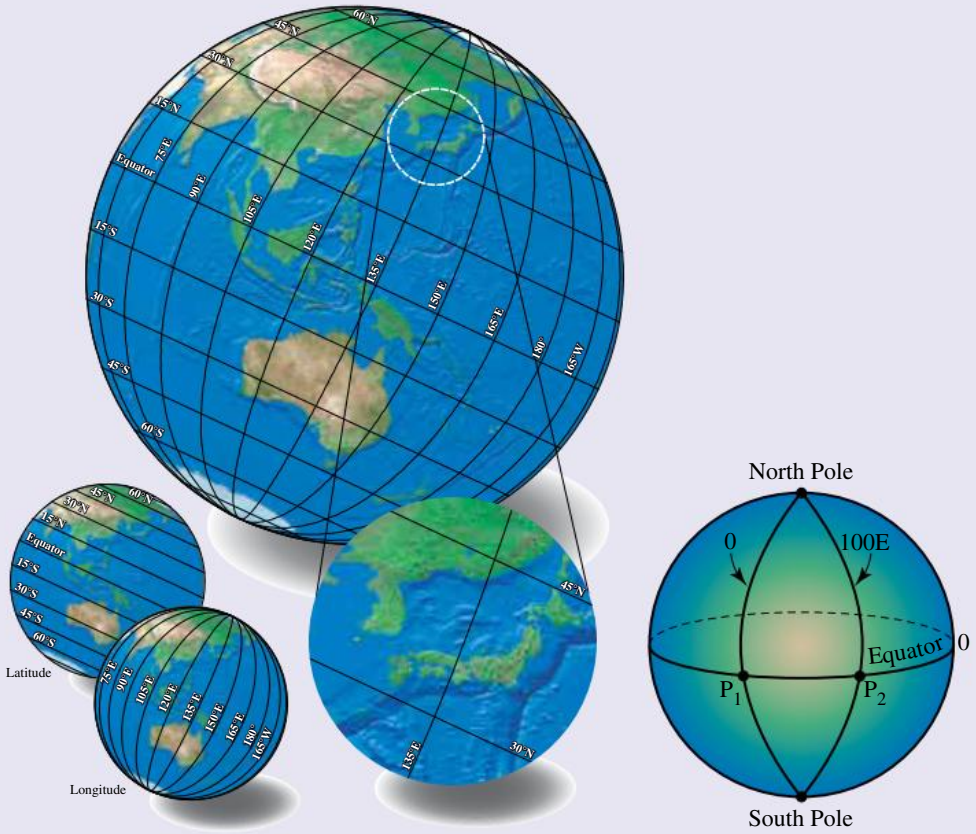
Investigation | Rich task

Variation of distance

The Earth approximates the shape of a sphere. Lines of longitude travel between the North and South poles, and lines of latitude travel east–west, parallel to the equator. Although the lines of longitude are all approximately the same length, this is not the case with lines of latitude. The line of latitude at the equator is the maximum length and these lines decrease in length on approaching both the North and South poles.

This investigation looks at how the distance between points on two given lines of longitude and the same line of latitude changes as we move from the equator to the pole.

Consider two lines of longitude, 0° and 100°E. Take two points, P_1 and P_2, lying on the equator on lines of longitude 0° and 100°E respectively.



The distance (in km) between two points on the same line of latitude is given by the formula:

Distance = angle sector between the two points × 111 × cos (degree of latitude).

1. The size of the angle sector between P_1 and P_2 is 100°, and these two points lie on 0° latitude. The distance between the points would be calculated as $100 \times 111 \times \cos 0°$. Determine this distance.

2. Move the two points to the 10° line of latitude. Calculate the distance between P_1 and P_2 in this position. Round your answer to the nearest kilometre.

3. Complete the following table showing the distance (rounded to the nearest kilometre) between the points P_1 and P_2 as they move from the equator towards the pole.

Latitude	Distance between P_1 and P_2 (km)
0°	
10°	
20°	
30°	
40°	
50°	
60°	
70°	
80°	
90°	

4. Describe what happens to the distance between P_1 and P_2 as we move from the equator to the pole. Is there a constant change? Explain.

5. You would perhaps assume that, at a latitude of 45°, the distance between P_1 and P_2 is half the distance between the points at the equator. This is not the case. At what latitude does this occur?

6. Sketch a graph displaying the change in distance between the points in moving from the equator to the pole.

7. Consider the points P_1 and P_2 on lines of longitude separated by 1°. On what line of latitude (to the nearest degree) would the points be 100 km apart?

8. Keeping the points P_1 and P_2 on the same line of latitude, and varying their lines of longitude, investigate the rate that the distance between them changes from the equator to the pole. Is it more or less rapid in comparison to what you found earlier?

learnON RESOURCES — ONLINE ONLY

Digital doc: Investigation — Variation of distance (doc-15952)

Digital doc: Code puzzle — Where is 2 litres of hydrochloric acid produced each day? (doc-15953)

Digital doc: Summary Topic 19 (doc-22917)

Interactivity: Word search Topic 19 (int-2880)

Interactivity: Crossword Topic 19 (int-2881)

Interactivity: Sudoku Topic 19 (int-3894)

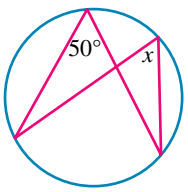
Exercise 19.6 Review questions

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

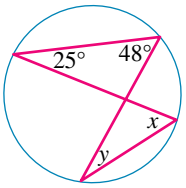
Understanding and fluency

1. Determine the values of the pronumerals in each of the following figures.

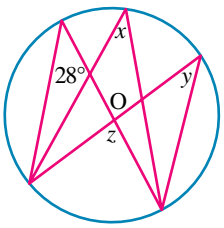
a.



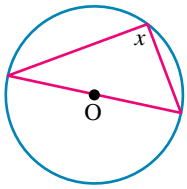
b.



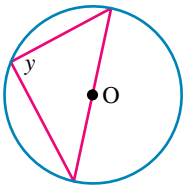
c.



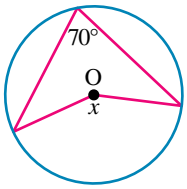
d.



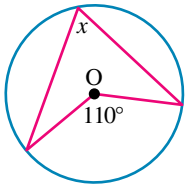
e.



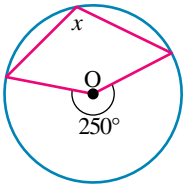
f.



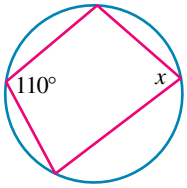
g.



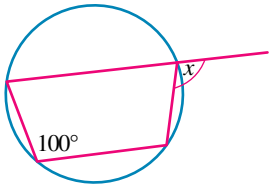
h.



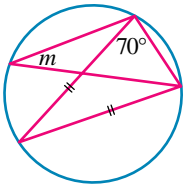
i.



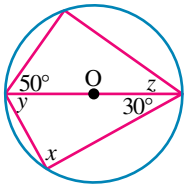
j.



k.

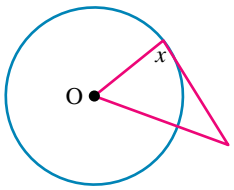


l.

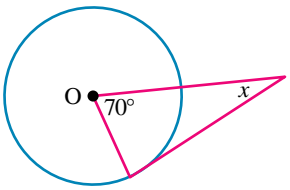


2. Find the value of the pronumeral in each case.

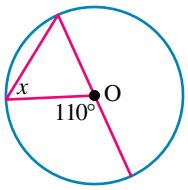
a.



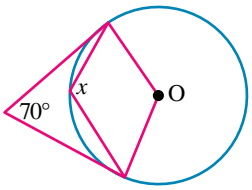
b.



c.

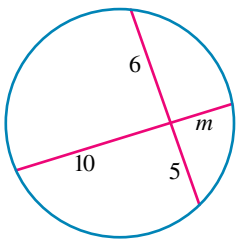


d.

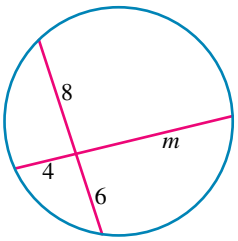


3. Find the value of m in each of the following figures.

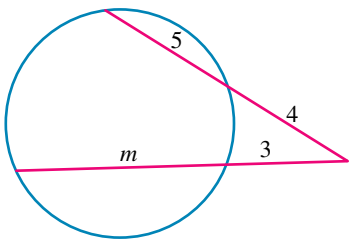
a.



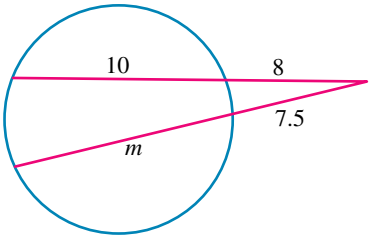
b.



c.

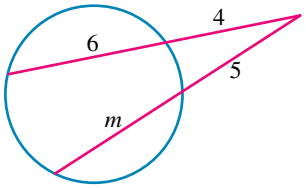


d.

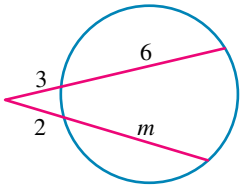


4. MC In which of the following figures is it possible to get a reasonable value for the pronumeral?
Note: There may be more than one correct answer.

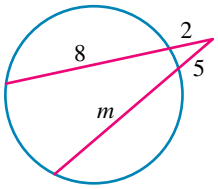
A.



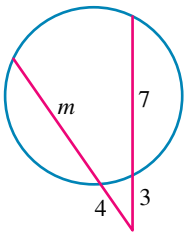
B.



C.

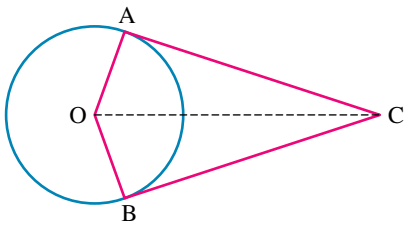


D.



E. None of the above

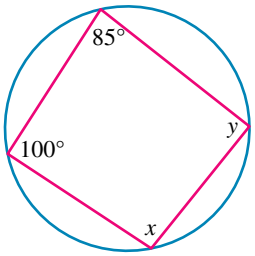
5. MC Which of the following statements is true for the diagram shown? *Note:* There may be more than one correct answer.



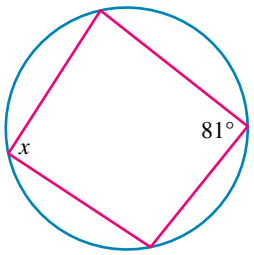
A. AO = BO **B.** AC = BC **C.** ∠OAC = ∠OBC
D. ∠AOC = 90° **E.** None of the above

6. Find the values of the pronumerals in the following figures.

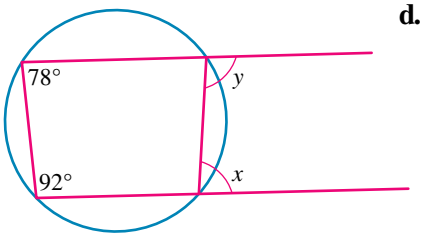
a.



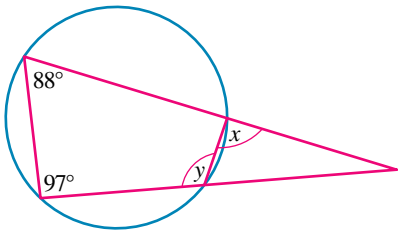
b.



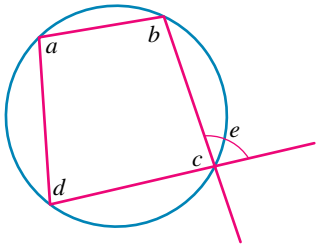
c.



d.



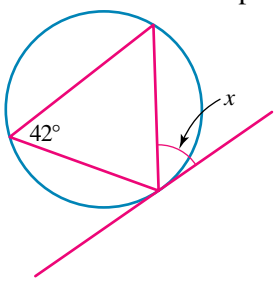
7. MC Which of the following statements is *not* always true for the diagram below?
A. ∠*a* + ∠*c* = 180° **B.** ∠*b* + ∠*d* = 180° **C.** ∠*e* + ∠*c* = 180°
D. ∠*a* + ∠*e* = 180° **E.** ∠*a* + ∠*b* + ∠*c* + ∠*d* = 360°



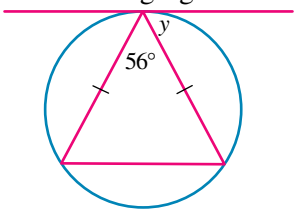
Communicating, reasoning and problem solving

8. Find the values of the pronumerals in the following figures.

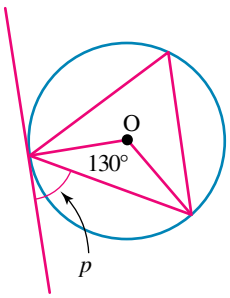
a.



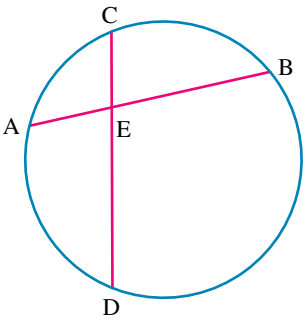
b.



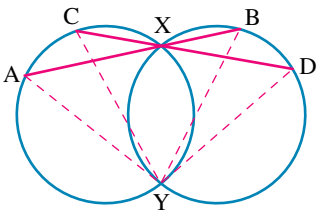
c.



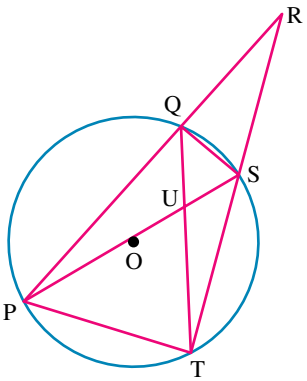
9. Two chords, AB and CD, intersect at E as shown. If AE = CE, prove that EB = ED.



10. Two circles intersect at X and Y. Two lines, AXB and CXD, intersect one circle at A and C, and the other at B and D, as shown. Prove that $\angle AYC = \angle BYD$.



11. Name at least five pairs of equal angles in the following diagram.



12. Prove that the perpendicular bisector of a chord passes through the centre of a circle.

Answers

Topic 19 Circle geometry

Exercise 19.2 Angles in a circle

1. a. $x = 30°$ (theorem 2)

b. $x = 25°$, $y = 25°$ (theorem 2 for both angles)

c. $x = 32°$ (theorem 2)

d. $x = 40°$, $y = 40°$ (theorem 2 for both angles)

e. $x = 60°$ (theorem 1)

f. $x = 40°$ (theorem 1)

g. $x = 84°$ (theorem 1)

h. $x = 50°$ (theorem 2); $y = 100°$ (theorem 1)

i. $x = 56°$ (theorem 1)

2. a. $s = 90°$, $r = 90°$ (theorem 3 for both angles)

b. $u = 90°$ (theorem 4); $t = 90°$ (theorem 3)

c. $m = 90°$, $n = 90°$ (theorem 3 for both angles)

d. $x = 52°$ (theorem 3 and angle sum in a triangle = 180°)

e. $x = 90°$ (theorem 4)

f. $x = 90°$ (theorem 4); $y = 15°$ (angle sum in a triangle = 180°)

3. a. $x = z = 90°$ (theorem 4); $y = w = 20°$ (theorem 5 and angle sum in a triangle = 180°)

b. $s = r = 90°$ (theorem 4); $t = 140°$ (angle sum in a quadrilateral = 360°)

c. $x = 20°$ (theorem 5); $y = z = 70°$ (theorem 4 and angle sum in a triangle = 180°)

d. $s = y = 90°$ (theorem 4); $x = 70°$ (theorem 5); $r = z = 20°$ (angle sum in a triangle = 180°)

e. $x = 70°$ (theorem 4 and angle sum in a triangle = 180°); $y = z = 20°$ (angle sum in a triangle = 180°)

f. $x = y = 75°$ (theorem 4 and angle sum in a triangle = 180°); $z = 75°$ (theorem 1)

4. D

5. B, D

6. a. Base angles of a right-angled isosceles triangle

b. $r + s = 90°$, $s = 45° \Rightarrow r = 45°$

c. u is the third angle in ΔABD, which is right-angled.

d. m is the third angle in ΔOCD, which is right-angled.

e. $\angle AOC$ and $\angle AFC$ stand on the same arc with $\angle AOC$ at the centre and $\angle AFC$ at the circumference.

7. OR = OP (radii of the circle)

$\angle OPR = x$ (equal angles lie opposite equal sides)

$\angle SOP = 2x$ (exterior angle equals the sum of the two interior opposite angles)

OR = OQ (radii of the circle)

$\angle OQR = y$ (equal angles lie opposite equal sides)

$\angle SOQ = 2y$ (exterior angle equals the sum of the two interior opposite angles)

Now $\angle PRQ = x + y$ and $\angle POQ = 2x + 2y = 2(x + y)$.

Therefore $\angle POQ = 2 \times \angle PRQ$.

8–11. Check with your teacher.

12. 1. State the aim

2. Use given information and established theorems to establish the result.

3. Give a reason for each step of the proof.

4. State a clear conclusion.

Exercise 19.3 Intersecting chords, secants and tangents

1. a. $m = 3$ **b.** $m = 3$ **c.** $m = 6$

2. a. $n = 1$ **b.** $m = 7.6$ **c.** $n = 13$ **d.** $m = 4$

3. a. $x = 5$ **b.** $m = 7$ **c.** $x = 2.5$, $y = 3.1$

4. a. $x = 2.8$ **b.** $x = 3.3$ **c.** $x = 5.6$ **d.** $m = 90°$

5. B, C, D

6. ST = 3 cm

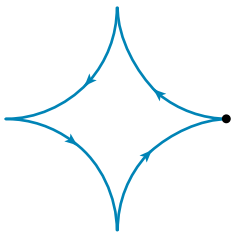
7–9. Check with your teacher.

10. a. $x = 3\sqrt{2}$ **b.** $x = 6$ **c.** $x = 3$, $y = 12$

11. Check with your teacher.

12. Consider the proof you are being asked to prove and all related theorems and knowledge.

13.



Exercise 19.4 Cyclic quadrilaterals

1. a. $x = 115°$, $y = 88°$ **b.** $m = 85°$ **c.** $n = 25°$

d. $x = 130°$ **e.** $x = y = 90°$ **f.** $x = 45°$, $y = 95°$

2. a. $x = 85°$, $y = 80°$ **b.** $x = 110°$, $y = 115°$ **c.** $x = 85°$

d. $x = 150°$ **e.** $x = 90°$, $y = 120°$ **f.** $m = 120°$, $n = 130°$

3. D

4. a. $2x$ **b.** $360° - 2x$ **c.** $180° - x$ **d.** 180°

5. a. A **b.** A, B, C, D

6. Check with your teacher.

7. a. $x = 93°$, $y = 87°$, $z = 93°$

b. $x = -2°$ or $\frac{5}{2}^\circ$

c. $x = \frac{2}{3}^\circ$ or $\frac{1}{2}^\circ$

8. $w = 110°$, $x = 70°$, $y = 140°$, $z = 87.5°$

9. a. $a = 110°$, $b = 70°$, $c = 110°$

b. Check with your teacher.

10. A cyclic quadrilateral has all four vertices on the circumference of the same circle.

Exercise 19.5 Tangents, secants and chords

1. a. $x = 70°$ **b.** $x = 47°$, $y = 59°$

2. a. $p = 6$ **b.** $q = 8$

3. $x = 42°$, $y = 132°$

4. MAC, NAC, FDA, FBA, EDG, EBG

5. B

6. D

7. $x = 42°$, $y = 62°$

8. Answers will vary.

9. 60°

10. $x = 180° - a - b$

11. $x = 80°$, $y = 20°$, $z = 80°$

12. Answers will vary.

13. $x = 85°$, $y = 20°$, $z = 85°$

14. D

15. $x = 50°$, $y = 95°$

16. A

17. C

18. $x = 33°$, $y = 55°$, $z = 22°$

19. $x = 25°$, $y = 65°$, $z = 40°$

20. $x = a$, $y = 90° - a$, $z = 90° - 2a$

21. Check with your teacher.

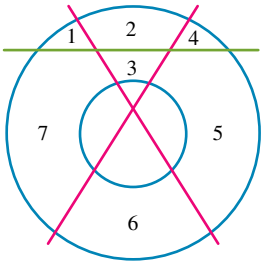
22. a. $x = 5$ **b.** $k = 12$ **c.** $m = 6$, $n = 6$

d. $x = 7$ **e.** $b = 4$, $a = 2$ **f.** $w = 3$, $x = 5$

23. a. $a = 50°$, $b = 50°$, $c = 80°$ **b.** $a = 50°$, $b = 70°$, $c = 70°$

24. A tangent BC is drawn at point A on the circumference of a circle and a chord is then drawn from point A to another point D on the circumference. The alternate segment is not the segment that contains angle BAD.

25.



Investigation | Rich task

1. 11 100 km

2. 10 931 km

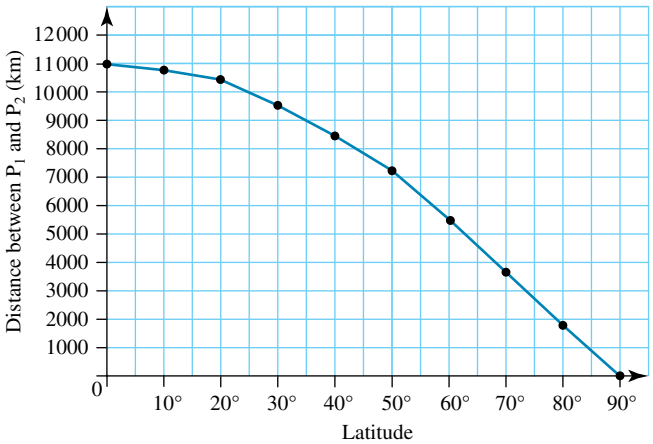
3.

Latitude	Distance between P_1 and P_2 (km)
0°	11 100
10°	10 931
20°	10 431
30°	9613
40°	8503
50°	7135
60°	5550
70°	3796
80°	1927
90°	0

4. The distance between P_1 and P_2 decreases from 11 100 km at the equator to 0 km at the pole. The change is not constant. The distance between the points decreases more rapidly on moving towards the pole.

5. Latitude 60°

6.



7. Latitude 26°

8. Answers will vary. Check with your teacher.

Exercise 19.6 Review questions

1. a. $x = 50°$ **b.** $x = 48°$, $y = 25°$ **c.** $x = y = 28°$, $z = 56°$ **d.** $x = 90°$

e. $y = 90°$ **f.** $x = 140°$ **g.** $x = 55°$ **h.** $x = 125°$

i. $x = 70°$ **j.** $x = 100°$ **k.** $m = 40°$ **l.** $x = 90°$, $y = 60°$, $z = 40°$

2. a. $x = 90°$ **b.** $x = 20°$ **c.** $x = 55°$ **d.** $x = 125°$

3. a. $m = 3$ **b.** $m = 12$ **c.** $m = 9$ **d.** $m = 11.7$

4. A, B, D

5. A, B, C

6. a. $x = 95°$, $y = 80°$ **b.** $x = 99°$ **c.** $x = 78°$, $y = 92°$ **d.** $x = 97°$, $y = 92°$

7. D

8. a. $x = 42°$ **b.** $y = 62°$ **c.** $p = 65°$

9. CE × ED = AE × EB

AE = CE (given)

∴ ED = EB

10. ∠AYC = ∠AXC

∠BXD = ∠BYD

But ∠AXC = ∠BXD

⇒ ∠AYC = ∠BYD

11. ∠PQT and ∠PST, ∠PTS and ∠RQS, ∠TPQ and ∠QSR, ∠QPS and ∠QTS, ∠TPS and ∠TQS, ∠PQS and PTS, ∠PUT and ∠QUS, ∠PUQ and TUS

12. Check with your teacher.

TOPIC 20 Trigonometry

[Stage 5.3]

20.1 Overview

Numerous **videos** and **interactivities** are embedded just where you need them, at the point of learning, in your learnON title at www.jacplus.com.au. They will help you to learn the concepts covered in this topic.

20.1.1 Why learn this?

The ability to calculate distances using angles has long been critical. Early explorers, using rudimentary calculations, were able to navigate their way around the planet and were even able to map coastlines along the way. It is essential to be able to calculate distances that can't be physically measured. Our explorations have now turned towards the skies. By applying trigonometry we can approximate the distances to other planets and beyond.

DISCUSSION

Maria Agnesi was fluent in 7 languages by her 11th birthday and went on to devote her life to the study of mathematics. Do you think being fluent in foreign languages could help you develop your mathematical skills? Why or why not?

LEARNING SEQUENCE

20.1 Overview
20.2 [Stage 5.3] The sine rule
20.3 [Stage 5.3] The cosine rule
20.4 [Stage 5.3] Area of triangles
20.5 [Stage 5.3] The unit circle
20.6 [Stage 5.3] Trigonometric functions
20.7 [Stage 5.3] Solving trigonometric equations
20.8 Review

LEARNING OUTCOMES

A student:

- uses and interprets formal definitions and generalisations when explaining solutions and/or conjectures MA5.3-1WM
- generalises mathematical ideas and techniques to analyse and solve problems efficiently MA5.3-2WM
- uses deductive reasoning in presenting arguments and formal proofs MA5.3-3WM
- applies trigonometric relationships, the sine rule, the cosine rule and the area rule to solve problems, including problems involving three dimensions MA5.3-15MG

CONTENT DESCRIPTIONS

Students:

Establish the sine, cosine and area rules for any triangle and solve related problems (ACMMG273)
Use the unit circle to define trigonometric functions, and graph them, with and without the use of digital technologies (ACMMG274)
Solve simple trigonometric equations (ACMMG275)

Source: NSW Syllabus for the Australian curriculum

learnon RESOURCES — ONLINE ONLY

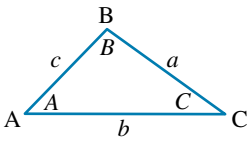
eLesson: The story of mathematics — Agnesi (eles-2023)

Note: Your teacher may now set you a pre-test to determine how familiar you are with the content in this topic.

20.2 The sine rule [Stage 5.3]

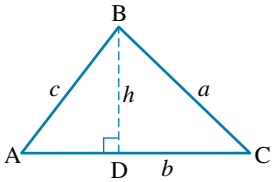
20.2.1 The sine rule

- In non–right-angled triangles, it is usual to label the angles A, B and C, and the sides a, b and c, so that side a is the side opposite angle A, side b is the side opposite angle B and side c is the side opposite angle C.



- In a non–right-angled triangle, a perpendicular line, h, can be drawn from the angle B to side b.

  This divides the triangle into two right-angled triangles, ABD and CBD.



- Using triangle ABD and the sine trigonometric ratio for right-angled triangles, we obtain $\sin A = \frac{h}{c}$. Using triangle CBD and the sine trigonometric ratio for right-angled triangles, we obtain $\sin C = \frac{h}{a}$.

$\frac{h}{c} = \sin A$ and $\frac{h}{a} = \sin C$

- Transposing each equation to make h the subject, we obtain: $h = c \sin A$ and $h = a \sin C$. Since h is common to both triangles, the two equations may be equated to get $c \sin A = a \sin C$.
- Dividing both sides of the equation by $\sin A$ gives:

$$c = \frac{a \sin C}{\sin A}$$

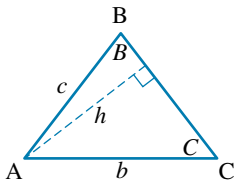
  Dividing both sides of the equation by $\sin C$ gives:

$$\frac{c}{\sin C} = \frac{a}{\sin A}$$

- In a similar way, if a perpendicular line is drawn from angle A to side a, the two right-angle triangles would give $h = c \sin B$ and $h = b \sin C$.

  This would give: $\frac{b}{\sin B} = \frac{c}{\sin C}$

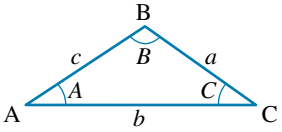
  From this, the **sine rule** can be derived.



$h = c \sin B$ and $h = b \sin C$

- In any triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

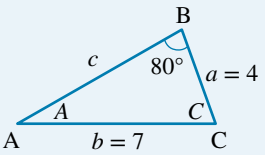


Notes

1. When using this rule, depending on the values given, any combination of the two equalities may be used to solve a particular triangle.
2. To solve a triangle means to find all unknown side lengths and angles.
   The sine rule can be used to solve non–right-angled triangles if we are given:
   - two angles and one side length
   - two side lengths and an angle opposite one of these side lengths.

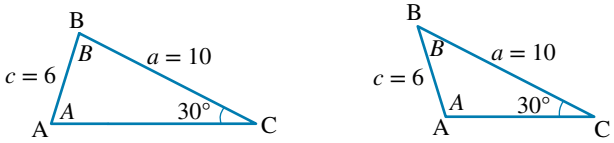
WORKED EXAMPLE 1

In the triangle ABC, $a = 4$ m, $b = 7$ m and $B = 80°$. Find A, C and c.

THINK	WRITE/DRAW
1 Draw a labelled diagram of the triangle ABC and fill in the given information.	
2 Check that one of the criteria for the sine rule has been satisfied.	The sine rule can be used since two side lengths and an angle opposite one of these side lengths have been given.
3 Write the sine rule to find A.	To find angle A: $\frac{a}{\sin A} = \frac{b}{\sin B}$
4 Substitute the known values into the rule.	$\frac{4}{\sin A} = \frac{7}{\sin 80°}$
5 Transpose the equation to make $\sin A$ the subject.	$4 \sin 80° = 7 \sin A$ $\sin A = \frac{4 \sin 80°}{7}$
6 Evaluate.	$A = \sin^{-1}\left(\frac{4 \sin 80°}{7}\right)$ $\approx 34.246004\,71°$
7 Round the answer to degrees and minutes.	$\approx 34°15'$
8 Determine the value of angle C using the fact that the angle sum of any triangle is 180°.	$C \approx 180° - (80° + 34°15')$ $= 65°45'$
9 Write the sine rule to find c.	To find side length c: $\frac{c}{\sin C} = \frac{b}{\sin B}$
10 Substitute the known values into the rule.	$\frac{c}{\sin 65°45'} = \frac{b}{\sin 80°}$
11 Transpose the equation to make c the subject.	$c = \frac{7 \sin 65°45'}{\sin 80°}$
12 Evaluate. Round the answer to 2 decimal places and include the appropriate unit.	≈ 6.48 m

20.2.2 The ambiguous case

- If two side lengths and an angle opposite one of these side lengths are given, then two different triangles may be drawn. For example, if $a = 10$, $c = 6$ and $C = 30°$, two possible triangles could be created.

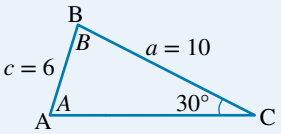


- In the first case (above left), angle A is an acute angle, and in the second case (above right), angle A is an obtuse angle.
- When using the sine rule to find an angle, the inverse sine function is used. If we are finding an angle, given the sine value, it is important to remember that an angle between 0° and 90° has the same sine value as its supplement. For example, $\sin 40° = 0.6427$, and $\sin 140° = 0.6427$.

WORKED EXAMPLE 2

In the triangle ABC, $a = 10$ m, $c = 6$ m and $C = 30°$. Find two possible values of A, and hence two possible values of B and b.

Case 1

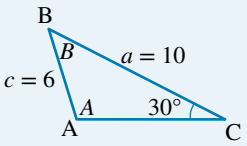
THINK	WRITE/DRAW
1 Draw a labelled diagram of the triangle ABC and fill in the given information.	
2 Check that one of the criteria for the sine rule has been satisfied.	The sine rule can be used since two side lengths and an angle opposite one of these side lengths have been given.
3 Write the sine rule to find A.	To find angle A: $\frac{a}{\sin A} = \frac{c}{\sin C}$
4 Substitute the known values into the rule.	$\frac{10}{\sin A} = \frac{6}{\sin 30°}$ $10 \sin 30° = 6 \sin A$
5 Transpose the equation to make $\sin A$ the subject.	$\sin A = \frac{10 \sin 30°}{6}$
6 Evaluate angle A.	$A = \sin^{-1}\left(\frac{10 \sin 30°}{6}\right)$ $\approx 56.442\,690\,24°$
7 Round the answer to degrees and minutes.	$A = 56°27'$
8 Determine the value of angle B, using the fact that the angle sum of any triangle is 180°.	$B \approx 180° - (30° + 56°27')$ $= 93°33'$

9 Write the sine rule to find b.	To find side length b: $\frac{b}{\sin B} = \frac{c}{\sin C}$
10 Substitute the known values into the rule.	$\frac{b}{\sin 93°33'} = \frac{6}{\sin 30°}$
11 Transpose the equation to make b the subject.	$b = \frac{6 \sin 93°33'}{\sin 30°}$
12 Evaluate. Round the answer to 2 decimal places and include the appropriate unit.	$\approx 11.98\,\text{m}$

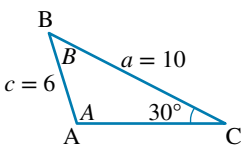
The values we have just obtained are only one set of possible answers for the given dimensions of the triangle ABC.

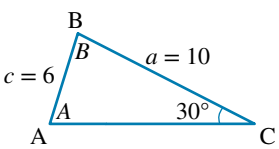
We are told that $a = 10$ m, $c = 6$ m and $C = 30°$. Since side a is larger than side c, it follows that angle A will be larger than angle C. Angle A must be larger than 30°; therefore it may be an acute angle or an obtuse angle.

Case 2

THINK	WRITE/DRAW
1 Draw a labelled diagram of the triangle ABC and fill in the given information.	
2 Write the alternative value for angle A. Simply subtract the value obtained for A in Case 1 from 180°.	To find the alternative angle A: If $\sin A = 0.8333$, then A could also be: $A \approx 180° - 56°27'$ $= 123°33'$
3 Determine the alternative value of angle B, using the fact that the angle sum of any triangle is 180°.	$B \approx 180° - (30° + 123°33')$ $= 26°27'$
4 Write the sine rule to find the alternative b.	To find side length b: $\frac{b}{\sin B} = \frac{c}{\sin C}$
5 Substitute the known values into the rule.	$\frac{b}{\sin 26°27'} = \frac{6}{\sin 30°}$
6 Transpose the equation to make b the subject.	$b = \frac{6 \sin 26°27'}{\sin 30°}$
7 Evaluate. Round the answer to 2 decimal places and include the appropriate unit.	$\approx 5.34\,\text{m}$

- For this example, there were two possible solutions, as shown by the diagrams below.





- The ambiguous case does not apply to every question. Consider Worked example 1, in which we were required to solve the triangle ABC given $a = 4\,\text{m}$, $b = 7\,\text{m}$ and $B = 80°$. For angle A, we obtained $A = 34°15'$. However, angle A could also have been $A = 145°45'$ (since there are two possible values of A between 0° and 180° whose sine is the same; that is, $\sin 34°15' = 0.5628$ and $\sin 145°45' = 0.5628$).
  We will now see whether or not $A = 145°45'$ is a possible solution.
  To obtain C, subtract angles A and B from 180°.
$$\begin{aligned} C &= 180° - (80° + 145°45') \\ &= 180° - 225°45' \\ &= -45°45' \text{ (not possible)} \end{aligned}$$
  Hence, for Worked example 1 only one possible solution exists.
- The ambiguous case exists if C is an acute angle and $a > c > a\sin C$, or any equivalent statement; for example, if B is an acute angle and $a > b > a\sin B$, and so on.
  - In Worked example 2, where $a = 10\,\text{m}$, $c = 6\,\text{m}$ and $C = 30°$, there were two possible solutions because C was an acute angle and $a > c > a\sin C$, since $10 > 6 > 10 \times 0.5$.
  - In Worked example 1, where $a = 4\,\text{m}$, $b = 7\,\text{m}$ and $B = 80°$, there was only one possible solution because even though B was an acute angle, the condition $a > b > a\sin B$ could not be satisfied.

DISCUSSION

Can you explain the ambiguous case of the sine rule? Why is it important to understand the ambiguous case?

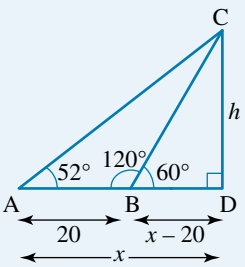
WORKED EXAMPLE 3

To calculate the height of a building, Kevin measures the angle of elevation to the top as 52°. He then walks 20 m closer to the building and measures the angle of elevation as 60°. How high is the building?

THINK

1 Draw a labelled diagram of the situation and fill in the given information.

WRITE/DRAW



2 Check that one of the criteria for the sine rule has been satisfied for triangle ABC.

The sine rule can be used for triangle ABC since two angles and one side length have been given.

3 Determine the value of angle ACB, using the fact that the angle sum of any triangle is 180°.

$$\begin{aligned} \angle ACB &= 180° - (52° + 120°) \\ &= 8° \end{aligned}$$

4 Write the sine rule to find b (or AC).

To find side length b of triangle ABC:
$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

5 Substitute the known values into the rule.

$$\frac{b}{\sin 120°} = \frac{20}{\sin 8°}$$

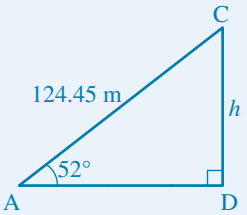
6 Transpose the equation to make b the subject.

$$b = \frac{20 \times \sin 120°}{\sin 8°}$$

7 Evaluate. Round the answer to 2 decimal places and include the appropriate unit.

≈ 124.45 m

8 Draw a diagram of the situation, that is, triangle ADC, labelling the required information.
Note: There is no need to solve the rest of the triangle in this case, as the values will not assist in finding the height of the building.



9 Write what is given for the triangle.

Have: angle and hypotenuse

10 Write what is needed for the triangle.

Need: opposite side

11 Determine which of the trigonometric ratios is required (SOH–CAH–TOA).

$$\sin\theta = \frac{\text{O}}{\text{H}}$$

12 Substitute the given values into the appropriate ratio.

$$\sin 52^\circ = \frac{h}{124.45}$$

13 Transpose the equation and solve for h.

$$124.45 \sin 52^\circ = h$$
$$h = 124.45 \sin 52^\circ$$

14 Round the answer to 2 decimal places.

$$\approx 98.07$$

15 Answer the question.

The height of the building is 98.07 m.

learnON RESOURCES — ONLINE ONLY

Interactivity: The sine rule (int-6275)
Interactivity: The ambiguous case (int-4818)
Digital doc: SkillSHEET Labelling right-angled triangles (doc-5398)
Digital doc: SkillSHEET Calculating sin, cos or tan of an angle (doc-5399)
Digital doc: SkillSHEET Finding side lengths in right-angled triangles (doc-5400)
Digital doc: SkillSHEET Calculating the angle from a sin, cos or tan ratio (doc-5401)
Digital doc: SkillSHEET Finding angles in right-angled triangles (doc-5402)
Digital doc: WorkSHEET The sine rule (doc-14631)

Exercise 20.2 The sine rule

Individual pathways

PRACTISE
Questions:
1, 2, 4, 5, 7, 10, 11, 14, 16, 18, 20, 22–24

CONSOLIDATE
Questions:
1, 3, 5, 6, 8, 10–12, 14, 16, 18, 20, 22, 24

MASTER
Questions:
1, 5–17, 19, 21, 23–26

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE1 In the triangle ABC, $a = 10$, $b = 12$ and $B = 58^\circ$. Find A, C and c.
2. In the triangle ABC, $c = 17.35$, $a = 26.82$ and $A = 101^\circ 47'$. Find C, B and b.

3. In the triangle ABC, $a = 5$, $A = 30°$ and $B = 80°$. Find C, b and c.
4. In the triangle ABC, $c = 27$, $C = 42°$ and $A = 105°$. Find B, a and b.
5. In the triangle ABC, $a = 7$, $c = 5$ and $A = 68°$. Find the perimeter of the triangle.
6. Find all unknown sides and angles for the triangle ABC, given $A = 57°$, $B = 72°$ and $a = 48.2$.
7. Find all unknown sides and angles for the triangle ABC, given $a = 105$, $B = 105°$ and $C = 15°$.
8. Find all unknown sides and angles for the triangle ABC, given $a = 32$, $b = 51$ and $A = 28°$.
9. Find the perimeter of the triangle ABC if $a = 7.8$, $b = 6.2$ and $A = 50°$.
10. MC *Note:* There may be more than one correct answer.
In a triangle ABC, $B = 40°$, $b = 2.6$ and $c = 3$. The value of C is approximately:
A. 47° **B.** 48° **C.** 132° **D.** 133°
11. WE2 In the triangle ABC, $a = 10$, $c = 8$ and $C = 50°$. Find two possible values of A, and hence two possible values of b.
12. In the triangle ABC, $a = 20$, $b = 12$ and $B = 35°$. Find two possible values for the perimeter of the triangle.
13. Find all unknown sides and angles for the triangle ABC, given $A = 27°$, $B = 43°$ and $c = 6.4$.
14. Find all unknown sides and angles for the triangle ABC, given $A = 100°$, $b = 2.1$ and $C = 42°$.
15. Find all unknown sides and angles for the triangle ABC, given $A = 25°$, $b = 17$ and $a = 13$.
16. WE3 To calculate the height of a building, Kevin measures the angle of elevation to the top as 48°. He then walks 18 m closer to the building and measures the angle of elevation as 64°. How high is the building?

Communicating, reasoning and problem solving

17. A river has parallel banks that run directly east–west. From the south bank, Kylie takes a bearing to a tree on the north side. The bearing is 047° T. She then walks 10 m due east, and takes a second bearing to the tree. This is 305° T. Find:
a. her distance from the second measuring point to the tree
b. the width of the river, to the nearest metre.
18. A ship sails on a bearing of S20°W for 14 km; then it changes direction and sails for 20 km and drops anchor. Its bearing from the starting point is now N65°W.
a. How far is it from the starting point?
b. On what bearing did it sail the 20 km leg?

19. A cross-country runner runs at 8 km/h on a bearing of 150° T for 45 minutes; then she changes direction to a bearing of 053° T and runs for 80 minutes at a different speed until she is due east of the starting point.

a. How far was the second part of the run?
b. What was her speed for this section?
c. How far does she need to run to get back to the starting point?

20. From a fire tower, A, a fire is spotted on a bearing of N42°E. From a second tower, B, the fire is on a bearing of N12°W. The two fire towers are 23 km apart, and A is N63°W of B. How far is the fire from each tower?

21. MC A boat sails on a bearing of N15°E for 10 km and then on a bearing of S85°E until it is due east of the starting point. The distance from the starting point to the nearest kilometre is:

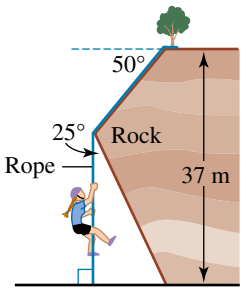
A. 10 km **B.** 38 km **C.** 113 km **D.** 114 km

22. MC A hill slopes at an angle of 30° to the horizontal. A tree that is 8 m tall and leaning downhill is growing at an angle of 10° to the vertical and is part-way up the slope. The vertical height of the top of the tree above the slope is:

A. 7.37 m **B.** 8.68 m **C.** 10.84 m **D.** 15.04 m

23. A cliff is 37 m high. The rock slopes outward at an angle of 50° to the horizontal and then cuts back at an angle of 25° to the vertical, meeting the ground directly below the top of the cliff.

Carol wishes to abseil from the top of the cliff to the ground as shown in the diagram. Her climbing rope is 45 m long, and she needs 2 m to secure it to a tree at the top of the cliff. Will the rope be long enough to allow her to reach the ground?

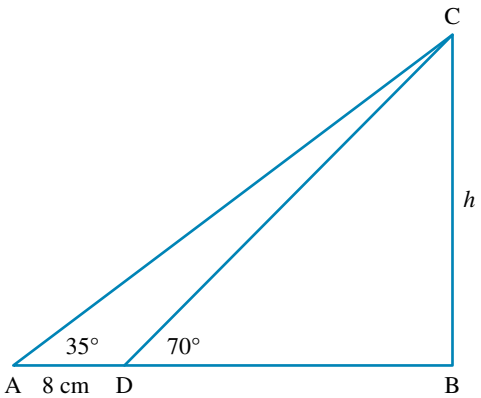


24. A yacht sets sail from a marina and sails on a bearing of 065°T for 3.5 km. It then turns and sails on a bearing of 127°T for another 5 km.

a. How far is the yacht from the marina?

b. On what bearing to the nearest minute should the yacht travel if it was to sail directly back to the marina?

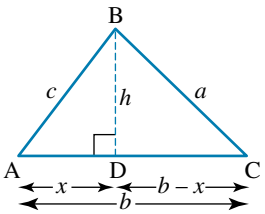
25. Find the value of h, correct to 1 decimal place.



26. In what situations can the sine rule be used?

20.3 The cosine rule [Stage 5.3]

- In any non–right-angled triangle ABC, a perpendicular line can be drawn from angle B to side b. Let D be the point where the perpendicular line meets side b, and let the length of the perpendicular line be h. Let the length AD = x units. The perpendicular line creates two right-angled triangles, ADB and CDB.



- Using triangle ADB and Pythagoras' theorem, we obtain:

$$c^2 = h^2 + x^2 \quad [1]$$

Using triangle CDB and Pythagoras' theorem, we obtain:

$$a^2 = h^2 + (b - x)^2 \quad [2]$$

Expanding the brackets in equation [2]:

$$a^2 = h^2 + b^2 - 2bx + x^2$$

Rearranging equation [2] and using $c^2 = h^2 + x^2$ from equation [1]:

$$\begin{aligned} a^2 &= h^2 + x^2 + b^2 - 2bx \\ &= c^2 + b^2 - 2bx \\ &= b^2 + c^2 - 2bx \end{aligned}$$

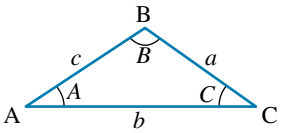
From triangle ABD, $x = c \cos A$; therefore $a^2 = b^2 + c^2 - 2bx$ becomes

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

- This is called the **cosine rule** and is a generalisation of Pythagoras' theorem.
- In a similar way, if the perpendicular line was drawn from angle A to side a or from angle C to side c, the two right-angled triangles would give $c^2 = a^2 + b^2 - 2ab\cos C$ and $b^2 = a^2 + c^2 - 2ac\cos B$ respectively. From this, the cosine rule can be stated:

  In any triangle ABC:

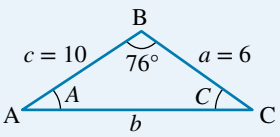
$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$



- The cosine rule can be used to solve non–right-angled triangles if we are given:
  1. three sides of the triangle
  2. two sides of the triangle and the included angle (the angle between the given sides).

WORKED EXAMPLE 4

Find the third side of triangle ABC given $a = 6$, $c = 10$ and $B = 76°$.

THINK	WRITE/DRAW
1 Draw a labelled diagram of the triangle ABC and fill in the given information.	
2 Check that one of the criteria for the cosine rule has been satisfied.	Yes, the cosine rule can be used since two side lengths and the included angle have been given.
3 Write the appropriate cosine rule to find side b.	To find side b $b^2 = a^2 + c^2 - 2ac\cos B$

4 Substitute the given values into the rule.	$= 6^2 + 10^2 - 2 \times 6 \times 10 \times \cos 76°$
5 Evaluate.	≈ 106.9693725
6 Round the answer to 2 decimal places.	$b \approx \sqrt{106.9693725}$ ≈ 10.34

- *Note:* Once the third side has been found, the sine rule could be used to find other angles if necessary.
- If three sides of a triangle are known, an angle could be found by transposing the cosine rule to make $\cos A$, $\cos B$ or $\cos C$ the subject.

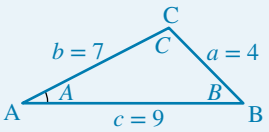
$$a^2 = b^2 + c^2 - 2bc\cos A \Rightarrow \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$b^2 = a^2 + c^2 - 2ac\cos B \Rightarrow \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$c^2 = a^2 + b^2 - 2ab\cos C \Rightarrow \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

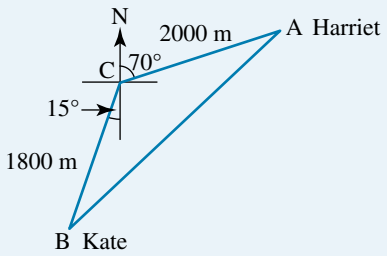
WORKED EXAMPLE 5

Find the smallest angle in the triangle with sides 4 cm, 7 cm and 9 cm.

THINK	WRITE/DRAW
1 Draw a labelled diagram of the triangle, call it ABC and fill in the given information. *Note:* The smallest angle will be opposite the smallest side.	 Let $a = 4$ $b = 7$ $c = 9$
2 Check that one of the criteria for the cosine rule has been satisfied.	The cosine rule can be used since three side lengths have been given.
3 Write the appropriate cosine rule to find angle A.	$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$
4 Substitute the given values into the rearranged rule.	$= \frac{7^2 + 9^2 - 4^2}{2 \times 7 \times 9}$
5 Evaluate.	$= \frac{114}{126}$
6 Transpose the equation to make A the subject by taking the inverse cos of both sides.	$A = \cos^{-1}\left(\frac{114}{126}\right)$ $\approx 25.2087653°$
7 Round the answer to degrees and minutes.	$\approx 25°13'$

WORKED EXAMPLE 6

Two rowers, Harriet and Kate, set out from the same point. Harriet rows N70°E for 2000 m and Kate rows S15°W for 1800 m. How far apart are the two rowers?

THINK	WRITE/DRAW
1 Draw a labelled diagram of the triangle, call it ABC and fill in the given information.	
2 Check that one of the criteria for the cosine rule has been satisfied.	The cosine rule can be used since two side lengths and the included angle have been given.
3 Write the appropriate cosine rule to find side C.	To find side c: $c^2 = a^2 + b^2 - 2ab\cos C$
4 Substitute the given values into the rule.	$= 2000^2 + 1800^2 - 2 \times 2000 \times 1800\cos 125°$
5 Evaluate.	$\approx 11\,369\,750.342$ $c \approx \sqrt{11\,369\,750.342}$
6 Round the answer to 2 decimal places.	≈ 3371.91
7 Answer the question.	The rowers are 3371.91 m apart.

learnON RESOURCES — ONLINE ONLY

Interactivity: The cosine rule (int-6276)

Exercise 20.3 The cosine rule

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1, 3–5, 7, 9, 11, 13, 15	Questions: 1, 2, 4, 6–9, 11–13, 15, 16, 18	Questions: 1, 3, 4, 6, 7, 9–14, 16–20

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE4 Find the third side of triangle ABC given $a = 3.4$, $b = 7.8$ and $C = 80°$.
2. In triangle ABC, $b = 64.5$, $c = 38.1$ and $A = 58°34'$. Find a.
3. In triangle ABC, $a = 17$, $c = 10$ and $B = 115°$. Find b, and hence find A and C.
4. WE5 Find the smallest angle in the triangle with sides 6 cm, 4 cm and 8 cm. (*Hint:* The smallest angle is opposite the smallest side.)
5. In triangle ABC, $a = 356$, $b = 207$ and $c = 296$. Find the largest angle.
6. In triangle ABC, $a = 23.6$, $b = 17.3$ and $c = 26.4$. Find the size of all the angles.

7. WE6 Two rowers set out from the same point. One rows N30°E for 1500 m and the other rows S40°E for 1200 m. How far apart are the two rowers?

8. Maria cycles 12 km in a direction N68°W and then 7 km in a direction of N34°E.

a. How far is she from her starting point?

b. What is the bearing of the starting point from her finishing point?

9. A garden bed is in the shape of a triangle, with sides of length 3 m, 4.5 m and 5.2 m.

a. Calculate the smallest angle.

b. Hence, find the area of the garden. (*Hint:* Draw a diagram, with the longest length as the base of the triangle.)

10. A hockey goal is 3 m wide. When Sophie is 7 m from one post and 5.2 m from the other, she shoots for goal. Within what angle, to the nearest degree, must the shot be made if it is to score a goal?

11. An advertising balloon is attached to two ropes 120 m and 100 m long. The ropes are anchored to level ground 35 m apart. How high can the balloon fly?

12. A plane flies in a direction of N70°E for 80 km and then on a bearing of S10°W for 150 km.

a. How far is the plane from its starting point?

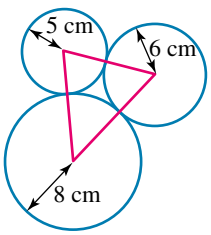
b. What direction is the plane from its starting point?

13. Ship A is 16.2 km from port on a bearing of 053°T and ship B is 31.6 km from the same port on a bearing of 117°T. Calculate the distance between the two ships.

14. A plane takes off at 10.00 am from an airfield and flies at 120 km/h on a bearing of N35°W. A second plane takes off at 10.05 am from the same airfield and flies on a bearing of S80°E at a speed of 90 km/h. How far apart are the planes at 10.25 am?

Communicating, reasoning and problem solving

15. Three circles of radii 5 cm, 6 cm and 8 cm are positioned so that they just touch one another. Their centres form the vertices of a triangle. Find the largest angle in the triangle.

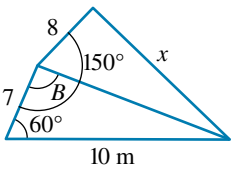


16. For the shape shown, determine:

a. the length of the diagonal

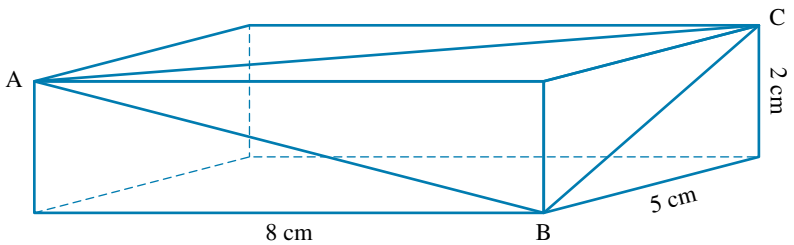
b. the magnitude (size) of angle B

c. the length of x.



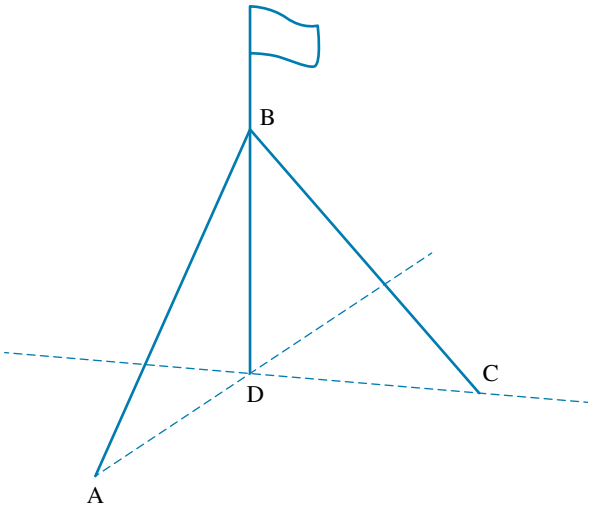
17. From the top of a vertical cliff 68 m high, an observer notices a yacht at sea. The angle of depression to the yacht is 47°. The yacht sails directly away from the cliff, and after 10 minutes the angle of depression is 15°. How fast does the yacht sail?

18. Find the measure of angles CAB, ABC and BCA. Give your answers correct to 2 decimal places.



19. A vertical flag pole DB is supported by two wires AB and BC. AB is 5.2 metres long, BC is 4.7 metres long and B is 3.7 metres above ground level. Angle ADC is a right angle.
a. Find the distance from A to C.
b. Calculate the angle between AB and BC.

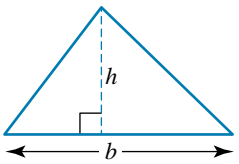
20. In what situations would you use the sine rule rather than the cosine rule?



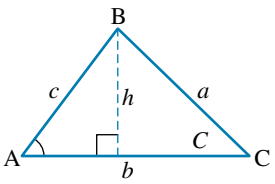
20.4 Area of triangles [Stage 5.3]

20.4.1 The triangle area formula

- The area of any triangle is given by the rule area $= \frac{1}{2}bh$, where b is the base and h is the perpendicular height of the triangle.



- However, often the perpendicular height is not given and must be calculated first. In the triangle ABC, b is the base and h is the perpendicular height of the triangle.



- Using the trigonometric ratio for sine:

$$\sin A = \frac{h}{c}$$

Transposing the equation to make h the subject, we obtain:

$$h = c \sin A$$

- Therefore, the area of triangle ABC becomes:

$$\text{area} = \frac{1}{2}bc \sin A$$

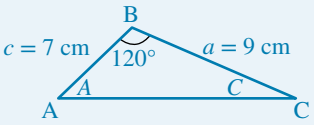
- Depending on how the triangle is labelled, the formula could read:

$$\text{area} = \frac{1}{2}ab\sin C \quad \text{area} = \frac{1}{2}ac\sin B \quad \text{area} = \frac{1}{2}bc\sin A$$

- The area formula may be used on any triangle provided that two sides of the triangle and the included angle (that is, the angle between the two given sides) are known.

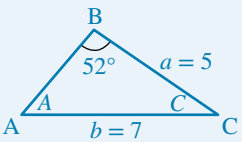
WORKED EXAMPLE 7

Find the area of the triangle shown.

THINK	WRITE/DRAW
1 Draw a labelled diagram of the triangle, call it ABC and fill in the given information.	 Let $a = 9$ cm, $c = 7$ cm, $B = 120°$.
2 Check that the criterion for the area rule has been satisfied.	The area rule can be used since two side lengths and the included angle have been given.
3 Write the appropriate rule for the area.	$\text{Area} = \frac{1}{2}ac\sin B$
4 Substitute the known values into the rule.	$= \frac{1}{2} \times 9 \times 7 \times \sin 120°$
5 Evaluate. Round the answer to 2 decimal places and include the appropriate unit.	$\approx 27.28 \text{ cm}^2$

WORKED EXAMPLE 8

A triangle has known dimensions of $a = 5$ cm, $b = 7$ cm and $B = 52°$. Find A and C and hence the area.

THINK	WRITE/DRAW
1 Draw a labelled diagram of the triangle, call it ABC and fill in the given information.	 Let $a = 5$, $b = 7$, $B = 52°$.
2 Check whether the criterion for the area rule has been satisfied.	The area rule cannot be used, as the included angle has not been given.
3 Write the sine rule to find A.	To find angle A: $\frac{a}{\sin A} = \frac{b}{\sin B}$
4 Substitute the known values into the rule.	$\frac{5}{\sin A} = \frac{7}{\sin 52°}$
5 Transpose the equation to make sin A the subject.	$5\sin 52° = 7\sin A$ $\sin A = \frac{5\sin 52°}{7}$
6 Evaluate.	$A = \sin^{-1}\left(\frac{5\sin 52°}{7}\right)$ $\approx 34.254\,151\,87°$
7 Round the answer to degrees and minutes.	$\approx 34°15'$

8 Determine the value of the included angle, C, using the fact that the angle sum of any triangle is 180°.	$C \approx 180° - (52° + 34°15')$ $= 93°45'$
9 Write the appropriate rule for the area.	$\text{Area} = \frac{1}{2}ab\sin C$
10 Substitute the known values into the rule.	$\approx \frac{1}{2} \times 5 \times 7 \times \sin 93°45'$
11 Evaluate. Round the answer to 2 decimal places and include the appropriate unit.	$\approx 17.46 \text{ cm}^2$

20.4.2 Heron's formula

- If the lengths of all the sides of the triangle are known but none of the angles are known, the cosine rule can be used to find an angle and then the area can be calculated. Alternatively, **Heron's formula** could be used to find the area.
- Heron's formula gives the area of a triangle as:

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

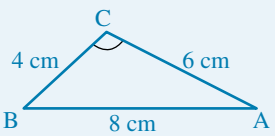
where s is the *semi-perimeter* of the triangle; that is,

$$s = \frac{1}{2}(a + b + c)$$

Note: The proof of this formula is beyond the scope of this course.

WORKED EXAMPLE 9

Find the area of the triangle with sides of 4 cm, 6 cm and 8 cm.

THINK	WRITE/DRAW
1 Draw a labelled diagram of the triangle, call it ABC and fill in the given information.	 Let $a = 4$, $b = 6$, $c = 8$.
2 Determine which area rule will be used.	As three side lengths have been given, use Heron's formula.
3 Write the rule for Heron's formula.	$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$
4 Write the rule for s, the semi-perimeter of the triangle.	$s = \frac{1}{2}(a + b + c)$
5 Substitute the given values into the rule for the semi-perimeter.	$= \frac{1}{2}(4 + 6 + 8)$ $= 9$
6 Substitute all of the known values into Heron's formula.	$\text{Area} = \sqrt{9(9-4)(9-6)(9-8)}$
7 Evaluate.	$= \sqrt{9 \times 5 \times 3 \times 1}$ $= \sqrt{135}$ $\approx 11.618\,950\,04$
8 Round the answer to 2 decimal places and include the appropriate unit.	$\approx 11.62 \text{ cm}^2$

DISCUSSION

Given the three different formulas for calculating the area of a triangle, which values of a triangle need to be given to determine the area?

ACTIVITY: COSINE, SINE AND AREA RULES APPLIED TO RIGHT-ANGLED TRIANGLES

The cosine rule states that $c^2 = a^2 + b^2 - 2ab\cos C$. We also know that $\cos 90^\circ = 0$. Therefore, if $C = 90^\circ$:

$$c^2 = a^2 + b^2 - 2ab(0)$$
$$c^2 = a^2 + b^2$$

Hence, when the cosine rule is applied to a right-angled triangle, you are left with Pythagoras' theorem for right-angled triangles.

Investigate what happens when the sine and area rules are applied to right-angled triangles.

learnON RESOURCES — ONLINE ONLY

Interactivity: The area of triangles (int-6483)

Interactivity: Using Heron's formula to find the area of a triangle (int-6475)

Digital doc: WorkSHEET Cosine rule and area of triangles (doc-14632)

Exercise 20.4 Area of triangles

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–8, 12, 14, 16, 18, 20, 23	Questions: 1–6, 8, 9, 11, 13, 15, 17, 19, 21, 23	Questions: 1, 4, 5, 8–10, 12–14, 16–25

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE7 Find the area of the triangle ABC with $a = 7$, $b = 4$ and $C = 68°$.

2. Find the area of the triangle ABC with $a = 7.3$, $c = 10.8$ and $B = 104°40'$.

3. Find the area of the triangle ABC with $b = 23.1$, $c = 18.6$ and $A = 82°17'$.

4. A triangle has $a = 10$ cm, $c = 14$ cm and $C = 48°$. Find A and B and hence the area.

5. WE8 A triangle has $a = 17$ m, $c = 22$ m and $C = 56°$. Find A and B and hence the area.

6. A triangle has $b = 32$ mm, $c = 15$ mm and $B = 38°$. Find A and C and hence the area.

7. MC In a triangle, $a = 15$ m, $b = 20$ m and $B = 50°$. The area of the triangle is:

A. 86.2 m^2 **B.** 114.9 m^2 **C.** 149.4 m^2 **D.** 172.4 m^2

8. WE9 Find the area of the triangle with sides of 5 cm, 6 cm and 8 cm.

9. Find the area of the triangle with sides of 40 mm, 30 mm and 5.7 cm.

10. Find the area of the triangle with sides of 16 mm, 3 cm and 2.7 cm.

11. MC A triangle has sides of length 10 cm, 14 cm and 20 cm. The area of the triangle is:

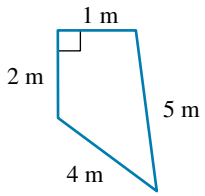
A. 41 cm^2 **B.** 65 cm^2 **C.** 106 cm^2 **D.** 137 cm^2

12. A piece of metal is in the shape of a triangle with sides of length 114 mm, 72 mm and 87 mm. Find its area using Heron's formula.

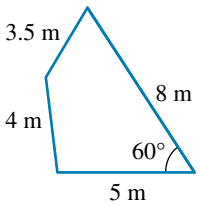
13. A triangle has the largest angle of 115°. The longest side is 62 cm and another side is 35 cm. Find the area of the triangle.

14. A triangle has two sides of 25 cm and 30 cm. The angle between the two sides is 30°. Find:
a. its area
b. the length of its third side
c. its area using Heron's formula.

15. The surface of a fish pond has the shape shown in the diagram below. How many goldfish can the pond support if each fish requires 0.3 m^2 surface area of water?



16. Find the area of the quadrilateral shown at right.

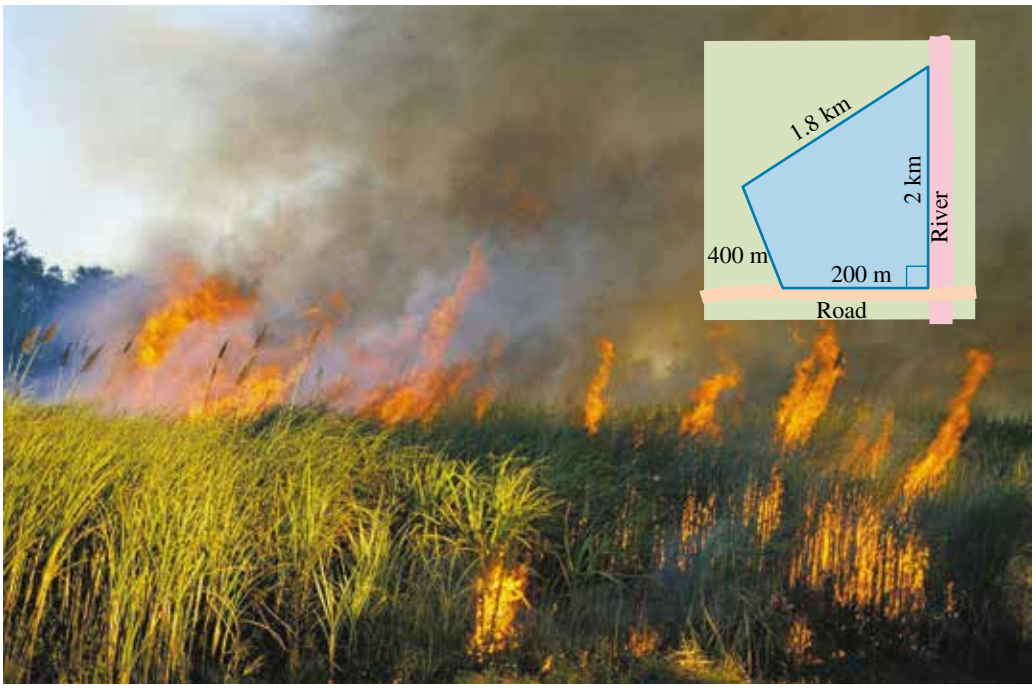


Communicating, reasoning and problem solving

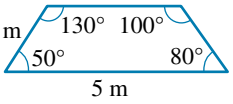
17. A parallelogram has diagonals of length 10 cm and 17 cm. An angle between them is 125°. Find:
a. the area of the parallelogram
b. the dimensions of the parallelogram.

18. A lawn is to be made in the shape of a triangle with sides of length 11 m, 15 m and 17.2 m. How much grass seed, to the nearest kilogram, needs to be purchased if it is sown at the rate of 1 kg per 5 m^2?

19. A bushfire burns out an area of level grassland shown in the diagram. (*Note:* This is a sketch of the area and is not drawn to scale.) What is the area, in hectares, of the land that is burned?



20. An earth embankment is 27 m long and has a vertical cross-section as shown in the diagram. Find the volume of earth needed to build the embankment.



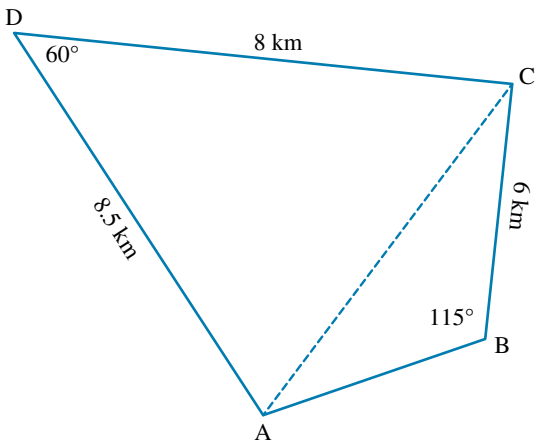
21. MC A parallelogram has sides of 14 cm and 18 cm and an angle between them of 72°. The area of the parallelogram is:

A. 118.4 cm^2 **B.** 172.4 cm^2 **C.** 239.7 cm^2 **D.** 252 cm^2

22. MC An advertising hoarding is in the shape of an isosceles triangle with sides of length 15 m, 15 m and 18 m. It is to be painted with two coats of purple paint. If the paint covers 12 m^2 per litre, the amount of paint needed, to the nearest litre, would be:

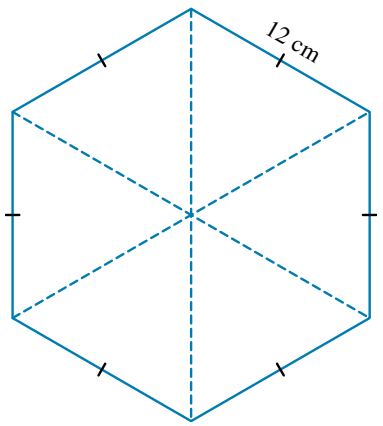
A. 9 L **B.** 18 L **C.** 24 L **D.** 36 L

23. A surveyor measured the boundaries of a property as shown. The side AB could not be measured because it crossed through a marsh. The owner of the property wanted to know the total area and the length of the side AB. Answer the following, giving all lengths correct to 2 decimal places and angles to the nearest degree.



a. Find the area of the triangle ACD.
b. Find the distance AC.
c. Find the angle CAB.
d. Find the angle ACB.
e. Find the length AB.
f. Find the area of the triangle ABC.
g. What is the area of the property?

24. A regular hexagon has sides of length 12 centimetres. It is divided into six smaller equilateral triangles. Find the area of the hexagon, giving your answer correct to 2 decimal places.

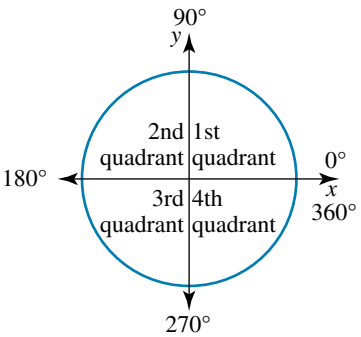


25. List three formulas for finding the area of a triangle.

20.5 The unit circle [Stage 5.3]

20.5.1 Definition of the unit circle

- A **unit circle** is a circle with a radius of 1 unit.
- The unit circle is divided into 4 quadrants.
- All angles in quadrant 1 are between 0° and 90°. All angles in quadrant 2 are between 90° and 180°, in quadrant 3 between 180° and 270°, and in quadrant 4 between 270° and 360°.
- The quadrants are numbered in the anticlockwise direction, and positive angles are measured anticlockwise from 0°. Negative angles are measured clockwise from 0°.

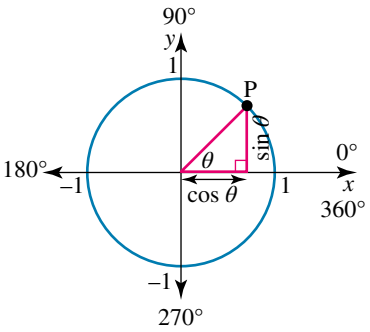


WORKED EXAMPLE 10

State the quadrant of the unit circle in which each of the following angles is found.
a 145° **b** 282°

THINK	WRITE
a The given angle is between 90° and 180°. State the appropriate quadrant.	**a** 145° is in quadrant 2.
b The given angle is between 270° and 360°. State the appropriate quadrant.	**b** 282° is in quadrant 4.

- If a right-angled triangle containing angle θ is constructed in quadrant 1 of the unit circle, then the value of sin θ can be found by measuring the length of the opposite side and the value of cos θ by measuring the length of the adjacent side.
- The point of intersection of the radius (which is one of the arms of angle θ) with the unit circle is P. From the diagram at right, observe that cos θ represents the x-coordinate of point P and sin θ represents its y-coordinate. This observation provides us with the technique for finding sine and cosine of any angle in the unit circle, as shown in the diagram at right.



- To find the value of sine, cosine or tangent of any angle θ from the unit circle, follow these steps:
  1. Download the unit circle template (doc-22898) from your learnON resources.
  2. Construct the required angle so that its vertex is at the origin and the angle itself is measured from 0° (as marked on the x-axis) in an anticlockwise direction. Label the point of intersection of the radius and the unit circle, P.
  3. Use a ruler to find the coordinates of point P.
  4. For P, $(x, y) = (\cos\theta, \sin\theta)$.
  5. From previous study of trigonometric ratios, it has been established that $\tan\theta = \dfrac{\text{Opposite side}}{\text{Adjacent side}}$ $\dfrac{y}{x} = \dfrac{\sin\theta}{\cos\theta} = \tan\theta$, where x and y are coordinates of P.

20.5.2 The 4 quadrants of the unit circle

ACTIVITY: INVESTIGATING THE SINE, COSINE AND TANGENT RATIOS USING THE UNIT CIRCLE

Use the unit circle template (doc-22898) to investigate the following:

1. If P has coordinates (x, y) and is located on the unit circle, what is the highest value that the x-coordinate can take? Hence, what is the largest value that the cosine of an angle can take?
2. What is the lowest value that the x-coordinate can take? Hence, what is the smallest value that the cosine of an angle can take?
3. What is the highest value that the y-coordinate can take? Hence, what is the largest value that the sine of an angle can take?
4. What is the lowest value that the y-coordinate can take? Hence, what is the smallest value that the sine of an angle can take?
5. Note that P could be in any of the four quadrants (depending on the size of the angle). Hence, its coordinates could take either positive or negative values, or zero.
   Copy and complete the following table to summarise whether sine, cosine and tangent are positive or negative for angles in each of the four quadrants.

	1st quadrant	2nd quadrant	3rd quadrant	4th quadrant
$\sin\theta$	positive (+)		negative (−)	
$\cos\theta$	positive (+)	negative (−)		
$\tan\theta$	positive (+)			negative (−)

6. Copy and complete the following sentences.
   Sine is positive in the _____ and _____ quadrants and is negative in the _____ and _____ quadrants.
   Cosine is positive in the _____ and _____ quadrants and is negative in the _____ and _____ quadrants.
7. Use the identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ to work out whether the tangent of an angle in each quadrant is positive or negative. Copy and complete the following sentence.
   Tangent is positive in the _____ and _____ quadrants and is negative in the _____ and _____ quadrants.

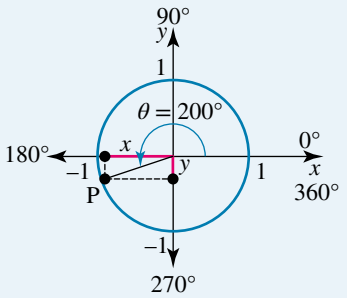
WORKED EXAMPLE 11

Find the approximate value of each of the following using the unit circle.
a $\sin 200°$ **b** $\cos 200°$ **c** $\tan 200°$

THINK

On the unit circle, construct an angle of 200°. Label the point corresponding to the angle of 200° on the circle P. Highlight the lengths, representing the x- and y-coordinates of point P.

WRITE/DRAW



a The sine of the angle is given by the y-coordinate of P. Find the y-coordinate of P by measuring the distance along the y-axis. State the value of sin 200°. (*Note:* The sine value will be negative as the y-coordinate is negative.)

a $\sin 200° = -0.3$

b The cosine of the angle is given by the x-coordinate of P. Find the x-coordinate of P by measuring the distance along the x-axis. State the value of cos 200°. (*Note:* Cosine is also negative in quadrant 3, as the x-coordinate is negative.)

b $\cos 200° = -0.9$

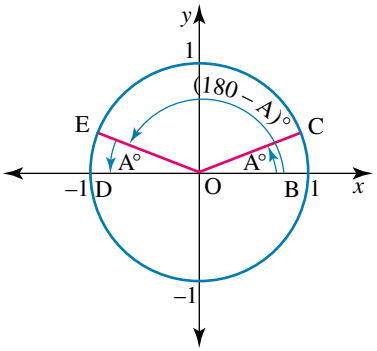
c $\tan 200° = \dfrac{\sin 200°}{\cos 200°}$

c $\dfrac{-0.3}{-0.9} = \dfrac{1}{3} = 0.3333$

- The approximate results obtained in Worked example 11 can be verified with the aid of a calculator:
  $\sin 200° = -0.342020143$, $\cos 200° = -0.93969262$ and $\tan 200° = 0.3640$.
- Rounding these values to 1 decimal place would give −0.3, −0.9 and 0.4 respectively, which match the values obtained from the unit circle.

20.5.3 The special relationship between the sine, cosine and tangent of supplementary angles

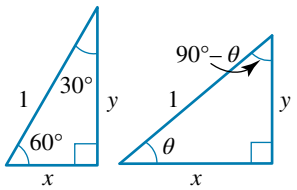
- In the diagram, triangle OBC is reflected in the y-axis to form triangle ODE. Clearly, $\angle BOC = \angle = A°$. It follows that the obtuse angle BOE equals $(180 - A)°$.
  From the diagram, $|BC| = |ED|$ and $|OB| = |OD|$.
  $ED = \sin(180° - A)$ and is positive.
  $OD = \cos(180° - A)$ and is negative.
  Thus, $\sin(180° - A) = \sin A$ and $\cos(180° - A) = -\cos A$.
  It follows that $\tan(180° - A) = \dfrac{\sin(180° - A)}{\cos(180° - A)} = \dfrac{\sin A}{-\cos A} = -\tan A$.



20.5.4 Complementary angles

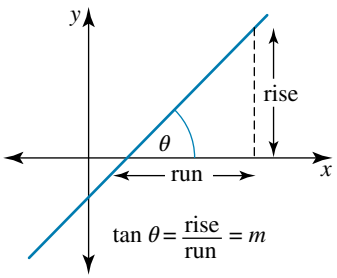
- Complementary angles add to 90°. Therefore, 30° and 60° are complementary angles, and θ and $(90° - \theta)$ are also complementary angles.
- The sine of an angle is equal to the cosine of its complement. Therefore, $\sin 60° = \cos 30°$.
- We say that sine and cosine are complementary functions.

$$\sin A = \cos(90° - A)$$
$$\cos A = \sin(90° - A)$$



20.5.5 Gradient and the angle made by the line with the x-axis

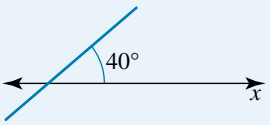
If the angle a line makes with the positive direction of the x-axis is known, the gradient may be found using trigonometry applied to the triangle shown below.

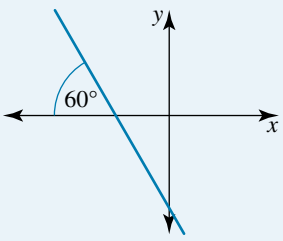


WORKED EXAMPLE 12

a Find the gradient (accurate to 3 decimal places) of a line making an angle of 40° to the positive x-axis, as shown in the figure at right.

b Find the gradient of the line shown at right. Express your answer to 2 decimal places.





THINK

a Since the angle the line makes with the positive x-axis is given, the formula $m = \tan\theta$ can be used.

WRITE/DRAW

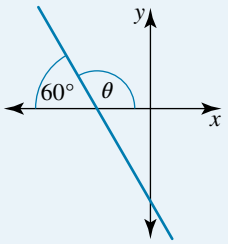
a $m = \tan\theta$
$= \tan 40°$
$= 0.839$

b 1 The angle given is not the one between the graph and the positive direction of the x-axis. Calculate the required angle θ.

2 Use $m = \tan\theta$ to calculate m to 2 decimal places.

b $\theta = 180° - 60°$
$= 120°$

$m = \tan\theta$
$= \tan 120°$
$= -1.73$



learnON RESOURCES — ONLINE ONLY

Digital doc: WorkSHEET: The unit circle template (doc-22898)

Exercise 20.5 The unit circle

Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1, 4–15, 18	Questions: 1–6, 8, 10, 12, 15, 17, 18	Questions: 1–25

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. WE10 State which quadrant of the unit circle each of the following angles is in.
a. 60° **b.** 130° **c.** 310° **d.** 260°
e. 100° **f.** 185° **g.** 275° **h.** 295°

2. MC If $\theta = 43°$, the triangle drawn to show this would be in:
A. quadrant 1 **B.** quadrant 2 **C.** quadrant 3 **D.** quadrant 4

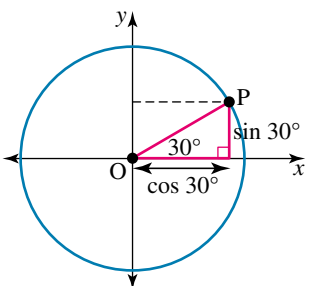
3. MC If $\theta = 295°$, the triangle drawn to show this would be in:
A. quadrant 1 **B.** quadrant 2 **C.** quadrant 3 **D.** quadrant 4

4. WE11 Find the value of each of the following using the unit circle.
a. sin 20° **b.** cos 20° **c.** cos 100° **d.** sin 100°
e. sin 320° **f.** cos 320° **g.** sin 215° **h.** cos 215°

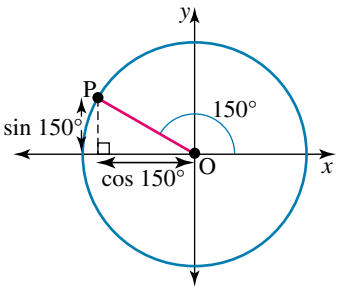
5. Use the unit circle to find each of the following.
a. sin 90° **b.** cos 90° **c.** sin 180° **d.** cos 180°
e. sin 270° **f.** cos 270° **g.** sin 360° **h.** cos 360°

Questions 6–13 refer to the digital document 'The unit circle template' (doc-22898). You can find this WorkSHEET in your online resources.

6. On the unit circle template, point P and its coordinates have been marked on the circle. Use this point to construct a triangle in quadrant 1 as shown.
a. Find cos 30°. (Remember that the length of the adjacent side of the triangle is cos 30°.)
b. Find sin 30°. (This is the length of the opposite side of the triangle.)
c. Check your answers to parts **a** and **b** by finding these values with a calculator.



7. Using your unit circle template, measure 150° with a protractor and mark the point P on the circle. Use this point to draw a triangle in quadrant 2 as shown.



a. What angle does the radius OP make with the negative x-axis?

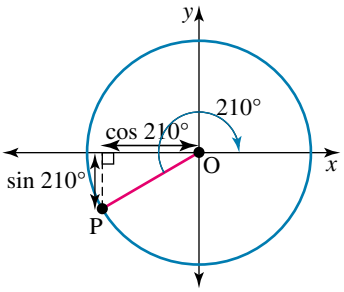
b. Remembering that $x = \cos\theta$, use your circle to find the value of cos 150°.

c. How does cos 150° compare to cos 30°?

d. Remembering that $y = \sin\theta$, use your circle to find the value of sin 150°.

e. How does sin 150° compare with sin 30°?

8. Using your unit circle template, measure 210° with a protractor and mark the point P on the circle. Use this point to draw a triangle in quadrant 3 as shown.



a. What angle does the radius OP make with the negative x-axis?

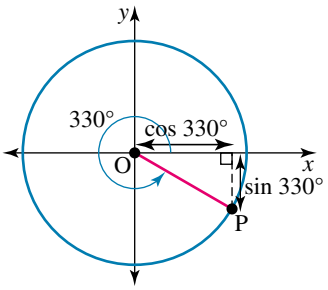
b. Use your circle to find the value of cos 210°.

c. How does cos 210° compare to cos 30°?

d. Use your circle to find the value of sin 210°.

e. How does sin 210° compare with sin 30°?

9. Using your unit circle template, measure 330° with a protractor and mark the point P on the circle. Use this point to draw a triangle in quadrant 4 as shown.



a. What angle does the radius OP make with the positive x-axis?

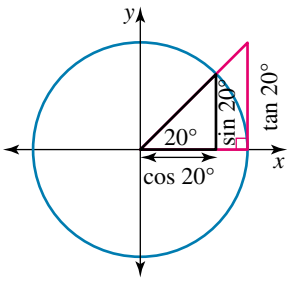
b. Use your circle to find the value of cos 330°.

c. How does cos 330° compare to cos 30°?

d. Use your circle to find the value of sin 330°.

e. How does sin 330° compare with sin 30°?

10. Using your unit circle template, draw an appropriate triangle for the angle of 20° in quadrant 1.



a. Find sin 20°.

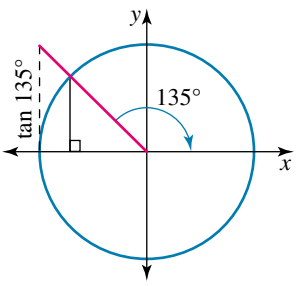
b. Find cos 20°.

c. Draw a tangent and extend the hypotenuse of the triangle to meet the tangent as shown. Accurately measure the length of the tangent between the x-axis and the point where it meets the hypotenuse and, hence, state the value of tan 20°.

d. What is the value of $\dfrac{\sin 20°}{\cos 20°}$?

e. How does tan 20° compare with $\dfrac{\sin 20°}{\cos 20°}$?

11. Using your unit circle template, draw an appropriate triangle for the angle of 135° in quadrant 2.



a. Find sin 135°, using sin 45°.

b. Find cos 135°, using cos 45°.

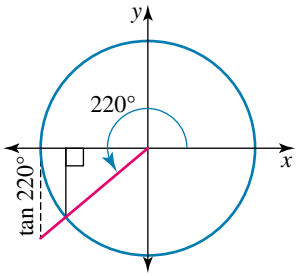
c. Draw a tangent and extend the hypotenuse of the triangle to meet the tangent as shown. Accurately measure the length of the tangent to where it meets the hypotenuse to find the value of tan 135°.

d. What is the value of $\dfrac{\sin 135°}{\cos 135°}$?

e. How does tan 135° compare with $\dfrac{\sin 135°}{\cos 135°}$?

f. How does tan 135° compare with tan 45°?

12. Using your unit circle template, draw an appropriate triangle for the angle of 220° in quadrant 3.



a. Find sin 220°.

b. Find cos 220°.

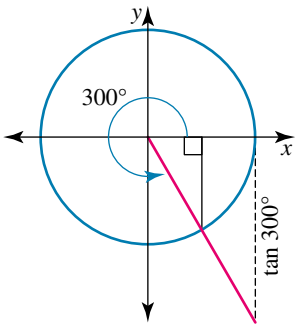
c. Draw a tangent and extend the hypotenuse of the triangle to meet the tangent as shown. Find tan 220° by accurately measuring the length of the tangent to where it meets the hypotenuse.

d. What is the value of $\frac{\sin 220°}{\cos 220°}$?

e. How does tan 220° compare with $\frac{\sin 220°}{\cos 220°}$?

f. How does tan 220° compare with tan 40°? (Use a calculator.)

13. Using your unit circle template, draw an appropriate triangle for the angle of 300° in quadrant 4.



a. Find sin 300°.

b. Find cos 300°.

c. Draw a tangent and extend the hypotenuse of the triangle to meet the tangent as shown. Find tan 300° by accurately measuring the length of the tangent to where it meets the hypotenuse.

d. What is the value of $\frac{\sin 300°}{\cos 300°}$?

e. How does tan 300° compare with $\frac{\sin 300°}{\cos 300°}$?

f. How does tan 300° compare with tan 60°? (Use a calculator.)

14. MC In a unit circle, the length of the radius is equal to:

A. $\sin\theta$ **B.** $\cos\theta$ **C.** $\tan\theta$ **D.** 1 **E.** $\sqrt{2}$

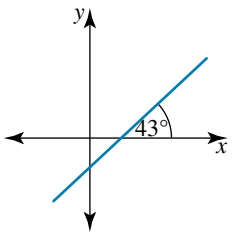
15. WE12a Find the gradients (accurate to 3 decimal places) of lines making the following angles with the positive x-axis.

a. 50° **b.** 72° **c.** 10° **d.** −30°

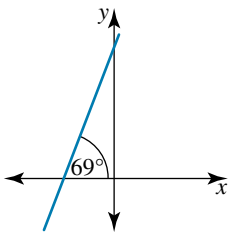
e. 150° **f.** 0° **g.** 45° **h.** 89°

16. WE12b Find the gradient of each line in the figures below. Give your answers accurate to 2 decimal places.

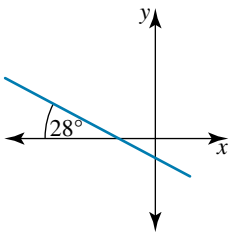
a.



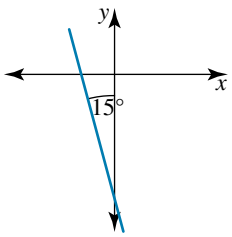
b.



c.



d.



Communicating, reasoning and problem solving

17. If $\sin x° = p$, $0 \leq x \leq 90°$, write each of the following in terms of p.

a. $\cos x°$ **b.** $\sin(180 - x)°$ **c.** $\cos(180 - x)°$

18. Simplify $\sin(180 - x)° - \sin x°$.

19. The point P $(\cos\theta, \sin\theta)$ lies in the first quadrant on the circumference of a unit circle. Find the new coordinates of P under the following transformations.

a. Dilated by a factor of 2 horizontally and then 3 vertically

b. Rotated 180° anticlockwise about the origin

c. Reflected in both axes

20. Using the unit circle, show that:

a. $\sin A = \cos(90° - A)$ **b.** $\cos A = \sin(90° - A)$

21. If $\sin x° = p$, $0 \leq x \leq 90°$, write each of the following in terms of p.

a. $\tan x°$ **b.** $\sin(90 - x)°$ **c.** $\cos(90 - x)°$

22. Simplify:

a. $\dfrac{\sin(180° - \theta)\cos(270° + \theta)}{\tan(-\theta)}$, given that $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

b. $\dfrac{\sin^2(90° - \theta) - \cos^2(270° + \theta)}{\cos(90° - \theta) + \cos(-\theta)}$.

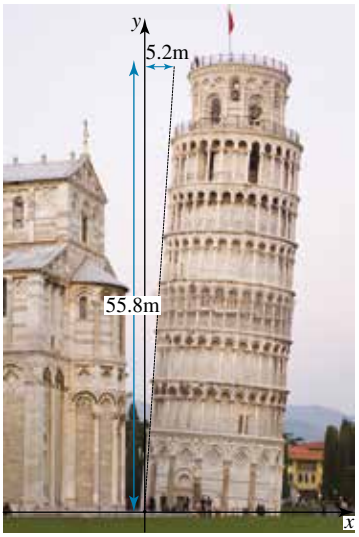
23. Simplify:

a. $\dfrac{\sin(90° - x)\cos(90° - x)}{\cos^2(x)}$

b. $\sin^2(180° - x)\,\text{cosec}(270° + x) + \cos^2(360° - x)\sec(180° - x)$

where $\sec(x) = \dfrac{1}{\cos(x)}$ and $\text{cosec}(x) = \dfrac{1}{\sin(x)}$.

24. A set of axes is placed as shown over the Leaning Tower of Pisa. What angle does the tower make with the ground?

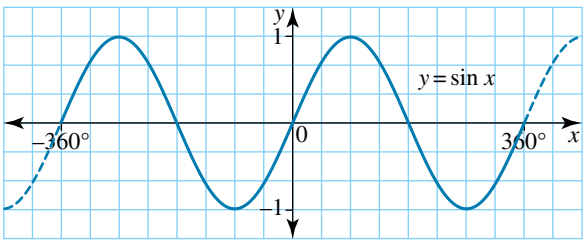


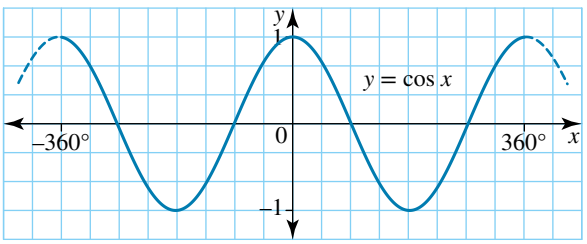
25. What is the length of the diameter of the unit circle?

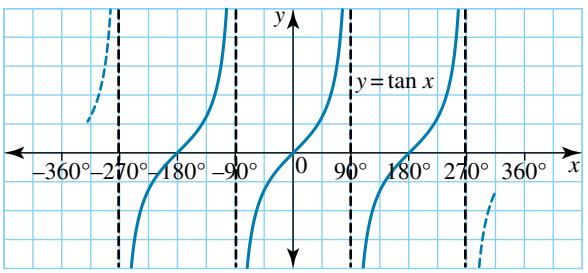
20.6 Trigonometric functions [Stage 5.3]

20.6.1 Sine, cosine and tangent graphs

- The graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$ are shown at right.
- Trigonometric graphs repeat themselves continuously in cycles, and hence they are called **periodic functions**.
- The period of the graph is the distance between repeating peaks or troughs. In the graphs shown above, the period between the repeating peaks for $y = \sin x$ and $y = \cos x$ is 360°. The graph of $y = \tan x$ shows a periodic function, but it is not of the same form as $y = \sin x$ and $y = \cos x$. The period of the basic tan graph is 180°, and asymptotes occur at $x = 90°$ and intervals of 180°.
- The **amplitude** of a periodic graph is half the distance between the maximum and minimum values of the function. In the graphs of $y = \sin x$ and $y = \cos x$ shown above, the distance is half of 2 units (the distance between −1 and 1); hence, the amplitude is 1 unit. Amplitude can also be described as the amount by which the graph goes above and below its mean value. In the above examples, the mean value lies along the x-axis. The amplitude of $y = \tan x$ is undefined.







WORKED EXAMPLE 13

Sketch the graphs of a $y = 2\sin x$ and b $y = \cos 2x$ for $0° \leq x \leq 360°$.

THINK

a 1 The graph must be drawn from 0° to 360°.

2 Compared to the graph of $y = \sin x$, each value of $\sin x$ has been multiplied by 2. Therefore, the amplitude of the graph must be 2.

3 Label the graph $y = 2\sin x$.

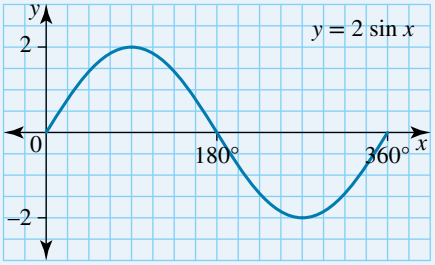
b 1 The graph must be drawn from 0° to 360°.

2 Compared to the graph of $y = \cos x$, each value of x has been multiplied by 2.

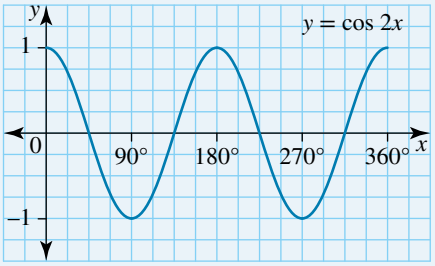
3 Therefore, the period of the graph must become 180°. Label the graph $y = \cos 2x$.

WRITE/DRAW

a



b



- For the graph of $y = a\sin nx$ or $y = a\cos nx$, the amplitude is a and the period becomes $\frac{360°}{n}$.
- If the graph has a negative value of a, the amplitude is the positive value of a; that is, the amplitude is always $|a|$.
- For the graph of $y = a\tan nx$, the amplitude is undefined and the period is $\frac{180°}{n}$.
- Dynamic geometry software can be used to investigate the chord, secant and tangent properties of circles.

WORKED EXAMPLE 14

For each of the following graphs, state:
i the amplitude **ii the period.**

a $y = 2\sin 3x$ **b** $y = \cos\frac{x}{3}$ **c** $y = \tan 2x$

THINK

a The value of a is 2.

The period is $\frac{360°}{n}$.

b The value of a is 1.

The period is $\frac{360°}{n}$.

c The tangent curve has infinite amplitude.

The period is $\frac{180°}{2}$.

WRITE

a i Amplitude = 2

ii Period $= \frac{360}{3}$
$= 120°$

b i Amplitude = 1

ii Period $= \frac{360}{\frac{1}{3}}$
$= 1080°$

c i Amplitude infinite

ii Period $= \frac{180°}{2}$
$= 90°$

DISCUSSION

Describe how the value of each trigonometric ratio changes as the angle increases from 0° to 360°.

learnON RESOURCES — ONLINE ONLY

Interactivity: Graphs of trigonometric functions (int-4821)

Interactivity: Trigonometric functions (int-2796)

Exercise 20.6 Trigonometric functions

Individual pathways

PRACTISE
Questions:
1–19, 22, 24, 27

CONSOLIDATE
Questions:
1–21, 23–27

MASTER
Questions:
1–20, 22, 24, 26–29

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. Using your calculator (or the unit circle if you wish), complete the following table.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\sin x$													
x	390°	420°	450°	480°	510°	540°	570°	600°	630°	660°	690°	720°	
$\sin x$													

2. On graph paper, rule x- and y-axes and carefully mark a scale along each axis. Use 1 cm = 30° on the x-axis to show x-values from 0° to 720°. Use 2 cm = 1 unit along the y-axis to show y-values from −1 to 1. Carefully plot the graph of $y = \sin x$ using the values from the table in question **1**.

3. How long does it take for the graph of $y = \sin x$ to complete one full cycle?

4. Using your graph of $y = \sin x$ from question **2**, find the value of y for each of the following.

a. $x = 42°$ **b.** $x = 130°$ **c.** $x = 160°$ **d.** $x = 200°$
e. $x = 180°$ **f.** $x = 70°$ **g.** $x = 350°$ **h.** $x = 290°$

5. Using your graph of $y = \sin x$ from question **2**, find the value of x for each of the following.

a. $y = 0.9$ **b.** $y = -0.9$ **c.** $y = 0.7$
d. $y = -0.5$ **e.** $y = -0.8$ **f.** $y = 0.4$

6. Using your calculator (or the unit circle if you wish), complete the following table.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$\cos x$													
x	390°	420°	450°	480°	510°	540°	570°	600°	630°	660°	690°	720°	
$\cos x$													

7. On graph paper, rule x- and y-axes and carefully mark a scale along each axis. Use 1 cm = 30° on the x-axis to show x-values from 0° to 720°. Use 2 cm = 1 unit along the y-axis to show y-values from −1 to 1. Carefully plot the graph of $y = \cos x$ using the values from the table in question **6**.

8. If you were to continue the graph of $y = \cos x$ that you drew in question **7**, what shape would you expect it to take?

9. Is the graph of $y = \cos x$ the same as the graph of $y = \sin x$? How does it differ? What features are the same?

10. Using your graph of $y = \cos x$ from question **7**, find a value of y for each of the following.

a. 48° **b.** 170° **c.** 180° **d.** 340°
e. 240° **f.** 140° **g.** 40° **h.** 165°

11. Using your graph of $y = \cos x$ from question **7**, find a value of x for each of the following.

a. $y = -0.5$ **b.** $y = 0.8$ **c.** $y = 0.7$
d. $y = -0.6$ **e.** $y = 0.9$ **f.** $y = -0.9$

12. Using your calculator (or the unit circle if you wish), complete the following table.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
tan x													
x	390°	420°	450°	480°	510°	540°	570°	600°	630°	660°	690°	720°	
tan x													

13. On graph paper, rule x- and y-axes and carefully mark a scale along each axis. Use 1 cm = 30° on the x-axis to show x-values from 0° to 720°. Use 2 cm = 1 unit along the y-axis to show y-values from −2 to 2. Carefully plot the graph of $y = \tan x$ using the values from the table in question **12**.

14. If you were to continue the graph of $y = \tan x$ that you drew in question **13**, what shape would you expect it to take?

15. Is the graph of $y = \tan x$ the same as the graphs of $y = \sin x$ and $y = \cos x$? How does it differ? What features are the same?

16. Using your graph of $y = \tan x$ from question **13**, find a value of y for each of the following.

a. 60° **b.** 135° **c.** 310° **d.** 220°
e. 500° **f.** 590° **g.** 710° **h.** 585°

17. Using your graph of $y = \tan x$ from question **13**, find a value of x for each of the following.

a. $y = 1$ **b.** $y = 1.5$ **c.** $y = -0.4$
d. $y = -2$ **e.** $y = 0.2$ **f.** $y = -1$

18. WE12 Sketch the following graphs

a. $y = \cos x$ for $x \in [-180°, 180°]$ **b.** $y = \sin x$ for $x \in [0°, 720°]$
c. $y = \sin 2x$ for $x \in [0°, 360°]$ **d.** $y = 2 \cos x$ for $x \in [-360°, 0°]$

19. WE13 For each of the graphs in question **18**, state

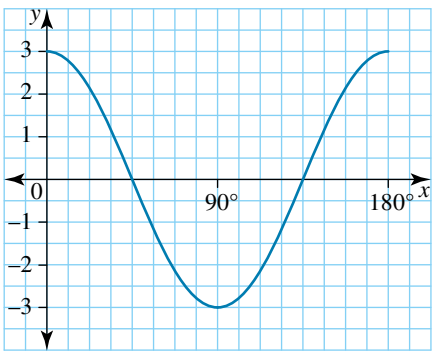
i. the period **ii.** the amplitude.

20. For each of the following, state:

i. the period **ii.** the amplitude.

a. $y = 3 \cos 2x$ **b.** $y = 4 \sin 3x$ **c.** $y = 2\cos\frac{x}{2}$
d. $y = \frac{1}{2}\sin\frac{x}{4}$ **e.** $y = -\sin x$ **f.** $y = -\cos 2x$

21. MC Parts **a** to **c** refer to the graph at right.



a. The amplitude of the graph is:

A. 180° **B.** 90° **C.** 3
D. −3 **E.** 6

b. The period of the graph is:

A. 180° **B.** 360° **C.** 90°
D. 3 **E.** −3

c. The equation of the graph could be:

A. $y = \cos x$ **B.** $y = \sin x$ **C.** $y = 3\cos\frac{x}{3}$
D. $y = 3 \cos 2x$ **E.** $y = 3 \sin 2x$

22. Sketch the following graphs, stating the period and amplitude of each. (Digital technology may be used to help sketch these graphs.)

a. $y = 2\cos\frac{x}{3}$ for $x \in [0°, 360°]$

b. $y = -3\sin 2x$ for $x \in [0°, 360°]$

c. $y = 3\sin\frac{x}{2}$ for $x \in [-180°, 180°]$

d. $y = -\cos 3x$ for $x \in [0°, 360°]$

e. $y = 5\cos 2x$ for $x \in [0°, 180°]$

f. $y = -\sin 4x$ for $x \in [0°, 180°]$

23. Use technology to sketch the graphs of each of the following for $0° \leq x \leq 360°$.

a. $y = \cos x + 1$ **b.** $y = \sin 2x - 2$ **c.** $y = \cos(x - 60°)$ **d.** $y = 2\sin 4x + 3$

Communicating, reasoning and problem solving

24. a. Sketch the graph of $y = \cos 2x$ for $x \in [0°, 360°]$.

i. What is the minimum value of y for this graph?

ii. What is the maximum value of y for this graph?

b. Using the answers you obtained in part **a**, write the maximum and minimum values of $y = \cos 2x + 2$.

c. What would be the maximum and minimum values of the graph of $y = 2\sin x + 3$? Explain how you obtained these values.

25. a. Complete the table below for $y = \tan x$.

x	0°	30°	60°	90°	120°	150°	180°
$y = \tan x$							

b. Sketch the graph of $y = \tan x$ for $[0°, 180°]$.

c. What happens at $x = 90°$?

d. For the graph of $y = \tan x$, $x = 90°$ is called an asymptote. Write when the next asymptote would occur.

e. State the period and amplitude of $y = \tan x$.

26. a. Sketch the graph of $y = \tan 2x$ for $[0°, 180°]$.

b. When do the asymptotes occur?

c. State the period and amplitude of $y = \tan 2x$.

27. The height of the tide above the mean sea level on the first day of the month is given by the rule

$$h = 3\sin(30t°)$$

where t is the time in hours since midnight.

a. Sketch the graph of h versus t.

b. What was the height of the high tide?

c. Calculate the height of the tide at 8 am.

28. The temperature inside a house t hours after 3 am is given by the rule

$$T = 22 - 2\cos(15t°) \text{ for } 0 \leq t \leq 24$$

where t is the temperature in degrees Celsius.

a. What is the temperature inside the house at 6 am?

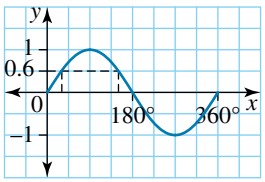
b. Sketch the graph of T versus t.

c. What is the warmest and coolest that it gets inside the house over the 24-hour period?

29. For the graph of $y = a\tan nx$, what would be the period and amplitude?

20.7 Solving trigonometric equations [Stage 5.3]

20.7.1 Solving trigonometric equations graphically



- Because of the periodic nature of circular functions, there are infinitely many solutions to trigonometric equations.
- Equations are usually solved within a particular domain (x-values) to restrict the number of solutions.
- The sine graph below shows the solutions between 0° and 360° for the equation $\sin x° = 0.6$.
- From the graph, it can clearly be seen that there are two solutions to this equation, which are approximately $x = 37°$ and $x = 143°$.
- The smaller the period, the greater the number of solutions there will be, within a particular domain.
- It is difficult to obtain accurate answers from a graph. More accurate answers can be obtained using technology.

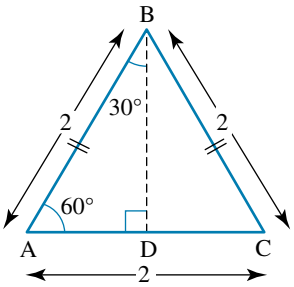
DISCUSSION

How do the size of the period of a trigonometric function and the size of the domain relate to the number of solutions within that domain?

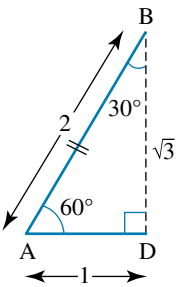
20.7.2 Solving trigonometric equations using exact values

- Most of the trigonometric values that we will deal with in this chapter are approximations. However, angles of 30°, 45° and 60° have exact values of sine, cosine and tangent.
- Consider an equilateral triangle, ABC, of side length 2 units.

  If the triangle is perpendicularly bisected, then two congruent triangles are obtained: ABD and CBD. From triangle ABD it can be seen that altitude BD creates a right-angled triangle with angles of 60° and 30° and base length (AD) of 1 unit. Altitude BD is obtained using Pythagoras' theorem.



$$\begin{aligned}(AB)^2 &= (AD)^2 + (BD)^2\\ 2^2 &= 1^2 + (BD)^2\\ 4 &= 1 + (BD)^2\\ 4 - 1 &= (BD)^2\\ (BD)^2 &= 3\\ BD &= \sqrt{3}\end{aligned}$$



Using triangle ABD and the three trigonometric ratios, the following exact values are obtained:

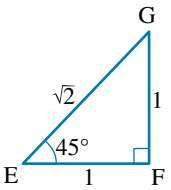
$$\sin B = \frac{\text{opp}}{\text{hyp}} \Rightarrow \sin 30° = \frac{1}{2} \qquad \sin A = \frac{\text{opp}}{\text{hyp}} \Rightarrow \sin 60° = \frac{\sqrt{3}}{2}$$

$$\cos B = \frac{\text{adj}}{\text{hyp}} \Rightarrow \cos 30° = \frac{\sqrt{3}}{2} \qquad \cos A = \frac{\text{adj}}{\text{hyp}} \Rightarrow \cos 60° = \frac{1}{2}$$

$$\tan B = \frac{\text{opp}}{\text{adj}} \Rightarrow \tan 30° = \frac{1}{\sqrt{3}} \text{ or } \frac{\sqrt{3}}{3} \qquad \tan A = \frac{\text{opp}}{\text{adj}} \Rightarrow \tan 60° = \frac{\sqrt{3}}{1} \text{ or } \sqrt{3}$$

Consider a right-angled isosceles triangle EFG whose equal sides are of 1 unit. The hypotenuse EG is obtained by using Pythagoras' theorem.

$$\begin{aligned}(EG)^2 &= (EF)^2 + (FG)^2\\ &= 1^2 + 1^2\\ &= 2\\ EG &= \sqrt{2}\end{aligned}$$



Using triangle EFG and the three trigonometric ratios, the following exact values are obtained:

$$\sin E = \frac{\text{opp}}{\text{hyp}} \Rightarrow \sin 45° = \frac{1}{\sqrt{2}} \text{ or } \frac{\sqrt{2}}{2}$$

$$\cos E = \frac{\text{adj}}{\text{hyp}} \Rightarrow \cos 45° = \frac{1}{\sqrt{2}} \text{ or } \frac{\sqrt{2}}{2}$$

$$\tan E = \frac{\text{opp}}{\text{adj}} \Rightarrow \tan 45° = \frac{1}{1} \text{ or } 1$$

Summary of exact values

θ	30°	45°	60°
$\sin\ \theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\ \theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\ \theta$	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

WORKED EXAMPLE 15

a $\sin x = \frac{\sqrt{3}}{2}, x \in [0°, 360°]$ **b** $\cos 2x = -\frac{1}{\sqrt{2}}, x \in [0°, 360°]$

THINK

a 1 The inverse operation of sine is $\sin^{-1}$.

2 The first solution in the given domain from the 'Summary of exact values' table is $x = 60°$.

3 Since sine is positive in the first and second quadrants, another solution must be $x = 180° - 60° = 120°$.

b 1 The inverse operation of cosine is $\cos^{-1}$.

2 From the table of values, $\cos^{-1}\frac{1}{\sqrt{2}} = 45°$.

3 Cosine is negative in the second and third quadrants, which gives the first two solutions to the equation as: $180° - 45°$ and $180° + 45°$.

WRITE

a $x = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

There are two solutions in the given domain, $x = 60°$ and $x = 120°$.

b $2x = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$

$2x = 135°, 225°$

4 Solve for x by dividing by 2.

$x = 67.5°, 112.5°$

5 Since the domain in this case is [0°, 360°]and the period has been halved, there must be 4 solutions altogether. The other 2 solutions can be found by adding the period onto each solution.

$\text{The period} = \frac{360°}{2} = 180°.$

$x = 67.5° + 180°, 112.5° + 180°$

$x = 67.5°, 112.5°, 247.5°, 292.5°$

learnON RESOURCES — ONLINE ONLY

Interactivity: Solving trigonometric equations graphically (int-4822)

Interactivity: Exact values of trigonometric functions (int-4816)

Interactivity: Trigonometry (int-2797)

Exercise 20.7 Solving trigonometric equations

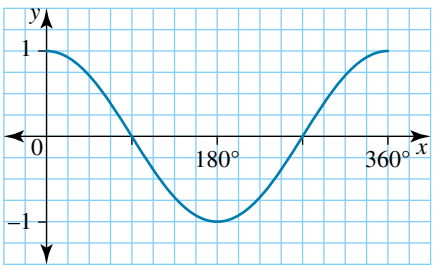
Individual pathways

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–6	Questions: 1–7	Questions: 1–9

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

Understanding and fluency

1. Use the graph shown to find approximate answers to the following equations for the domain $0 < x < 360°$. Check your answers using a calculator.



i. $\cos x = 0.9$

ii. $\cos x = 0.3$

iii. $\cos x = -0.2$

iv. $\cos x = -0.6$

2. Solve the following equations for the domain $0 < x < 360°$.

a. $\sin x = \frac{1}{2}$ **b.** $\sin x = \frac{\sqrt{3}}{2}$ **c.** $\cos x = -\frac{1}{2}$ **d.** $\cos x = -\frac{1}{\sqrt{2}}$

e. $\sin x = 1$ **f.** $\cos x = -1$ **g.** $\sin x = -\frac{1}{2}$ **h.** $\sin x = -\frac{1}{\sqrt{2}}$

i. $\cos x = \frac{\sqrt{3}}{2}$ **j.** $\cos x = -\frac{\sqrt{3}}{2}$ **k.** $\sin x = 1$ **l.** $\cos x = 0$

3. WE14 Solve the following equations for the given values of x.

a. $\sin 2x = \frac{\sqrt{3}}{2}, x \in [0°, 360°]$ **b.** $\cos 2x = -\frac{\sqrt{3}}{2}, x \in [0°, 360°]$

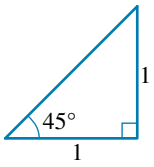
c. $\tan 2x = \frac{1}{\sqrt{3}}, x \in [0°, 360°]$ **d.** $\sin 3x = \frac{-1}{\sqrt{2}}, x \in [-180°, 180°]$

e. $\sin 4x = -\frac{1}{2}, x \in [0°, 180°]$ **f.** $\cos 4x = -\frac{1}{2}, x \in [0°, 180°]$

g. $\tan 3x = -1, x \in [0°, 90°]$ **h.** $\cos 3x = 0, x \in [0°, 360°]$

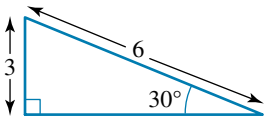
Communicating, reasoning and problem solving

4. a. For the triangle shown, calculate



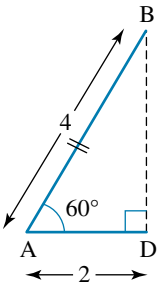
i. sin 45°
ii. cos 45°
iii. tan 45°

b. For the triangle shown, calculate



i. sin 30°
ii. cos 30°
iii. tan 30°

c. For the triangle shown, calculate



i. sin 60°
ii. cos 60°
iii. tan 60°

5. Sam measured the depth of water at the end of the Intergate jetty at various times on Friday 13 August 2010. The table below provides her results.

Time	6 am	7	8	9	10	11	12 pm	1	2	3	4	5	6	7	8	9
Depth	1.5	1.8	2.3	2.6	2.5	2.2	1.8	1.2	0.8	0.5	0.6	1.0	1.3	1.8	2.2	2.5

a. Plot the data.
b. Determine:
i. the period **ii.** the amplitude.
c. Sam fishes from the jetty when the depth is a maximum. Specify these times for the next 3 days.
d. Sam's mother can moor her yacht when the depth is above 1.5 m. During what periods can she moor the yacht on Sunday 16 January?

6. Solve:
a. $\sqrt{3}\sin(x°) = \cos(x°)$ for $0° \leq x \leq 360°$
b. $2\sin(x°) + \cos(x°) = 0$ for $0° \leq x \leq 360°$.

7. Solve $2\sin^2(x°) + 3\sin(x°) - 2 = 0$ for $0° \leq x \leq 360°$.

8. Explain why sine and cosine functions can be used to model situations that occur in nature such as tide heights and sound waves.

9. The grad (^{g}) is another measurement used when measuring the size of angles. A grad is equivalent to $\frac{1}{400}$ of a full circle. Write each of the following as grads (1grad is written as 1^g).

a. 90° **b.** 180° **c.** 270° **d.** 360°

20.8 Review

Investigation | Rich task

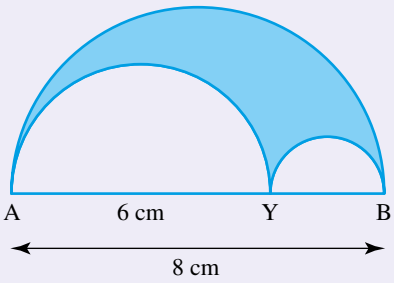
What's an arbelos?

As an introduction to this task, you are required to complete the following construction. The questions that follow require the application of measurement formulas and an understanding of semicircles related to this construction.

1. Constructing an arbelos

- Rule a horizontal line AB, 8 cm long.
- Find the midpoint of the line and construct a semicircle on top of the line with AB as the diameter.

- Mark Y as a point on AB such that AY = 6 cm.
- Find the midpoint of AY and draw a small semicircle inside the larger semicircle with AY as the diameter.
- Find the midpoint of YB and construct a semicircle (also inside the larger semicircle) with a diameter YB.



The shape enclosed by the three semicircles is known as an arbelos. This is an Ancient Greek word that means 'shoemaker's knife', as the shape resembles the blade of a knife used by cobblers. The point Y is not fixed and can be located anywhere along the diameter of the larger semicircle, which can also vary in size.

2. Perimeter of an arbelos

The perimeter of an arbelos is the sum of the arc length of the three semicircles. Perform the following calculations, leaving each answer in terms of π.

a. Calculate the arc length of the semicircle with diameter AB.

b. Calculate the arc length of the semicircle with diameter AY.

c. Calculate the arc length of the semicircle on diameter YB.

d. Compare the largest arc length with the two smaller arc lengths. What do you conclude?

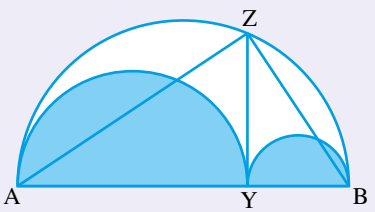
3. We can generalise the arc length of an arbelos. The point Y can be located anywhere on the line AB, which can also vary in length. Let the diameter AB be d cm, AY be d_1 cm and YB be d_2 cm. Prove that your conclusion from question **2d** holds true for any value of d, where $d_1 + d_2 = d$.

4. Area of an arbelos

The area of an arbelos may be treated as the area of a composite shape.

a. Using your original measurements, calculate the area of the arbelos you drew in question **1**. Leave your answer in terms of π.

The area of the arbelos can also be calculated using another method. We can draw the common tangent to the two smaller semicircles at their point of contact and extend this tangent to the larger semicircle. It is said that the area of the arbelos is the same as the area of the circle constructed on this common tangent as diameter.



YZ is the common tangent.

Triangles AYZ, BYZ and AZB are all right-angled triangles. We can use Pythagoras' theorem, together with a set of simultaneous equations, to determine the length of the tangent YZ.

b. Complete the following.

In ΔAYZ, $AZ^2 = AY^2 + YZ^2$

$= 6^2 + YZ^2$

In ΔBYZ, $BZ^2 = BY^2 + YZ^2$

$= \ldots\ldots\ldots\ldots\ldots\ldots + YZ^2$

Adding these two equations,

$AZ^2 + BZ^2 = \ldots\ldots\ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots\ldots\ldots$

But, in ΔAZB, $AZ^2 + BZ^2 = AB^2$

$\ldots\ldots\ldots\ldots\ldots\ldots + \ldots\ldots\ldots\ldots\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots$

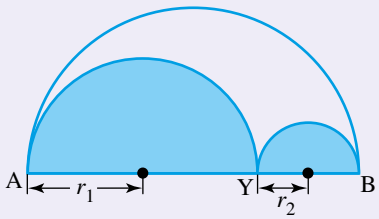
Therefore, $YZ = \ldots\ldots\ldots\ldots\ldots\ldots$ (Leave your answer in surd form.)

c. Now calculate the area of the circle with diameter YZ. Is your answer the same as that calculated in question **4a**?

The area of an arbelos can be generalised.

Let the radii of the two smaller semicircles be r_1 and r_2.

5. Develop a formula for the area of the arbelos in terms of r_1 and r_2. Demonstrate the use of your formula by checking your answer to question **4a**.



learnON RESOURCES — ONLINE ONLY

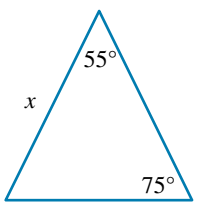
Digital doc: Investigation — What's an arbelos? (doc-15954)
Digital doc: Code puzzle — What is the periodic table? (doc-15955)
Digital doc: Summary Topic 20 (doc-22918)
Interactivity: Word search Topic 20 (int-2883)
Interactivity: Crossword Topic 20 (int-2884)
Interactivity: Sudoku Topic 20 (int-3895)

Exercise 20.8 Review questions

To answer questions online and to receive immediate feedback and fully worked solutions for every question, go to your learnON title at www.jacplus.com.au. *Note:* Question numbers may vary slightly.

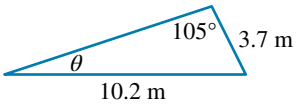
Understanding and fluency

1. Find the value of x, correct to 1 decimal place.

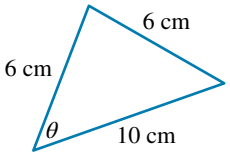


2. Find the value of θ, correct to the nearest minute.

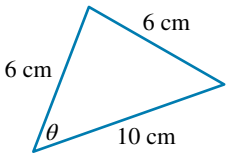
3. Find all unknown sides and angles of triangle ABC, given $a = 25$ m, $A = 120°$ and $B = 50°$.



4. Find the value of x, correct to 1 decimal place.



5. Find the value of θ, correct to the nearest degree.



6. A triangle has sides of length 12 m, 15 m and 20 m. Find the magnitude (size) of the largest angle.

7. A triangle has two sides of 18 cm and 25 cm. The angle between the two sides is 45°. Find:

a. its area **b.** the length of its third side

c. its area using Heron's formula.

8. If $\theta = 290°$, the triangle to show this would be drawn in which quadrant?

9. On the unit circle, draw an appropriate triangle for the angle 110° in quadrant 2.

a. Find sin 110° and cos 110°, correct to 2 decimal places.

b. Find tan 110°, correct to 2 decimal places.

10. MC The value of sin 53° is equal to:

A. cos 53° **B.** cos 37° **C.** sin 37° **D.** tan 53° **E.** tan 37°

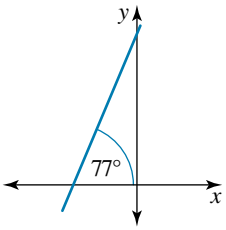
11. Simplify $\dfrac{\sin 53°}{\sin 37°}$.

12. Draw a sketch of $y = \sin x$ from $0° \leq x \leq 360°$.

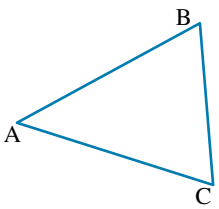
13. Draw a sketch of $y = \cos x$ from $0° \leq x \leq 360°$.

14. Draw a sketch of $y = \tan x$ from $0° \leq x \leq 360°$.

15. Calculate the gradient of the line shown. Give your answer correct to 3 decimal places.



16. Label this triangle so that $\dfrac{x}{\sin 46°} = \dfrac{y}{\sin 68°}$.

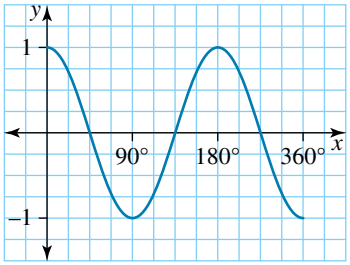


17. State the period and amplitude of each of the following graphs.

a. $y = 2 \sin 3x$

b. $y = -3 \cos 2x$

c.



18. Sketch the following graphs.

a. $y = 2\sin x, x \in [0°, 360°]$ **b.** $y = \cos 2x, x \in [-180°, 180°]$

19. Use technology to write the solutions to the following equations for the domain $0° \leq x \leq 360°$ to 2 decimal places.

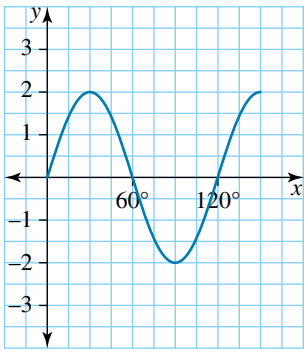
a. $\sin x = -0.2$ **b.** $\cos 2x = 0.7$ **c.** $3\cos x = 0.1$ **d.** $2\tan 2x = 0.5$

20. Solve each of the following equations.

a. $\sin x = -\frac{1}{2}, x \in [0°, 360°]$ **b.** $\cos x = \frac{\sqrt{3}}{2}, x \in [0°, 360°]$

c. $\cos x = \frac{1}{\sqrt{2}}, x \in [0°, 360°]$ **d.** $\sin x = \frac{1}{\sqrt{2}}, x \in [0°, 360°]$

21. MC The equation that represents the graph at right could be:



A. $y = 3\sin 2x$

B. $y = 2\cos 3x$

C. $y = 3\cos 2x$

D. $y = 2\sin 2x$

E. $y = 2\sin 3x$

22. a. Use technology to help sketch the graph of $y = 2\sin 2x - 3$.

b. Write the period and the amplitude of the graph in part **a**.

23. Sketch the graphs of each of the following, stating

i. the period **ii.** the amplitude.

a. $y = 2\cos 2x, x \in [0°, 360°]$ **b.** $y = 3\sin 4x, x \in [0°, 180°]$

c. $y = -2\cos 3x, x \in [-60°, 60°]$ **d.** $y = 4\sin 2x, x \in [-90°, 90°]$

24. Solve each of the following equations for the given values of x.

a. $\cos 2x = \frac{\sqrt{3}}{2}, x \in [0°, 360°]$ **b.** $\sin 3x = \frac{1}{2}, x \in [-90°, 90°]$

c. $\sin 2x = \frac{1}{\sqrt{2}}, x \in [0°, 360°]$ **d.** $\cos 3x = -\frac{1}{\sqrt{2}}, x \in [0°, 360°]$

e. $\sin 4x = 0, x \in [0°, 180°]$ **f.** $\tan 4x = 21, x \in [0°, 180°]$

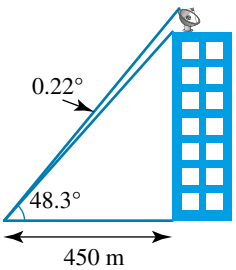
25. Solve the following for $x \in [\,0°,\ 360°\,]$.

a. $2\cos x - 1 = 0$ **b.** $2\sin x = -\sqrt{3}$ **c.** $-\sqrt{2}\cos x + 1 = 0$ **d.** $\sqrt{2}\sin x + 1 = 0$

26. Sketch the graph of $y = \tan 2x, x \in [0°, 180°]$. Write the period, amplitude and the equations of any asymptotes.

Communicating, reasoning and problem solving

27. A satellite dish is placed on top of an apartment building as shown in the diagram. Find the height of the satellite dish.



28. Australian power points supply voltage, V, in volts, where $V = 240\sin(18000t)$ and t is measured in seconds.

a. Complete the table below and sketch the graph, showing the fluctuations in voltage over time.

t	0.000	0.005	0.010	0.015	0.020	0.025	0.030	0.035	0.040
V									

b. Find the times at which the maximum voltage output occurs.

c. How many seconds are there between times of maximum voltage output?

d. How many periods (or cycles) are there per second?

Answers

Topic 20 Trigonometry

Exercise 20.2 The sine rule

1. 44°58′, 77°2′, 13.79

2. 39°18′, 38°55′, 17.21

3. 70°, 9.85, 9.4

4. 33°, 38.98, 21.98

5. 19.12

6. $C = 51°$, $b = 54.66$, $c = 44.66$

7. $A = 60°$, $b = 117.11$, $c = 31.38$

8. $B = 48°26'$, $c = 103°34'$, $c = 66.26$; or $B = 131°34'$, $c = 20°26'$, $c = 23.8$

9. 24.17

10. B, C

11. $A = 73°15'$, $b = 8.73$; or $A = 106°45'$, $b = 4.12$

12. 51.9 or 44.86

13. $C = 110°$, $a = 3.09$, $b = 4.64$

14. $B = 38°$, $a = 3.36$, $c = 2.28$

15. $B = 33°33'$, $c = 121°27'$, $c = 26.24$; or $B = 146°27'$, $c = 8°33'$, $c = 4.57$

16. 43.62 m

17. a. 6.97 m **b.** 4 m

18. a. 13.11 km **b.** N20°47′ W

19. a. 8.63 km **b.** 6.48 km/h **c.** 9.90 km

20. 22.09 km from A and 27.46 km from B

21. C

22. B

23. Yes, she needs 43.5 m altogether.

24. a. 7.3 km **b.** 282°3′

25. $h = 7.5$ cm

26. The sine rule can be used on right-angled triangles. The sine rule is the relationship between an angle of a right-angled triangle, and the length of the side opposite this angle and the length of the hypotenuse. If any one of these three is unknown, the remaining two can be used to determine the unknown.

Exercise 20.3 The cosine rule

1. 7.95

2. 55.22

3. 23.08, 41°53′, 23°7′

4. 28°57′

5. 88°15′

6. $A = 61°15'$, $b = 40°$, $c = 78°45'$

7. 2218 m

8. a. 12.57 km **b.** S35°E

9. a. 35°6′ **b.** 6.73 m^2

10. 23°

11. 89.12 m

12. a. 130 km **b.** S22°12′E

13. 28.5 km

14. 74.3 km

15. 70°49′

16. a. 8.89 m **b.** 77° **c.** $x = 10.07$ m

17. 1.14 km/h

18. $\angle CAB = 34.65°$, $\angle ABC = 84.83°$ and $\angle BCA = 60.52°$

19. a. 4.6637 m **b.** 55.93°

20. In right-angled triangles, if the length of the hypotenuse is known, the sine rule is used to determine angles or lengths of *opposite* sides, whereas the cosine rule is used to determine angles or lengths of *adjacent* sides.

Exercise 20.4 Area of triangles

1. 12.98

2. 38.14

3. 212.88

4. $A = 32°4'$, $b = 99°56'$, area $= 68.95\ cm^2$

5. $A = 39°50'$, $b = 84°10'$, area $= 186.03\ m^2$

6. $A = 125°14'$, $b = 16°46'$, area $= 196.03\ mm^2$

7. C

8. $14.98\ cm^2$

9. $570.03\ mm^2$

10. $2.15\ cm^2$

11. B

12. $3131.41\ mm^2$

13. $610.38\ cm^2$

14. a. $187.5\ cm^2$ **b.** $15.03\ cm^2$ **c.** $187.47\ cm^2$

15. 17 goldfish

16. $22.02\ m^2$

17. a. Area $= 69.63\ cm^2$

b. Dimensions are 12.08 cm and 6.96 cm.

18. 17 kg

19. 52.2 hectares

20. $175\ m^3$

21. C

22. B

23. a. $29.44\ km^2$ **b.** 8.26 km **c.** 41° **d.** 24° **e.** 3.72 km

f. $10.11\ km^2$ **g.** $39.55\ km^2$

24. $374.12\ cm^2$

25. Area $= \frac{1}{2}ab\sin C$, area $= \frac{1}{2}ac\sin B$, area $= \frac{1}{2}bc\sin A$

Exercise 20.5 The unit circle

1. a. 1st **b.** 2nd **c.** 4th **d.** 3rd

e. 2nd **f.** 3rd **g.** 4th **h.** 4th

2. A

3. D

4. a. 0.35 **b.** 0.95 **c.** −0.17 **d.** 0.99

e. −0.64 **f.** 0.77 **g.** −0.57 **h.** −0.82

5. a. 1 **b.** 0 **c.** 0 **d.** −1

e. −1 **f.** 0 **g.** 0 **h.** 1

6. a. 0.87 **b.** 0.50

7. a. 30° **b.** −0.87 **c.** $\cos 150° = -\cos 30°$

d. 0.5 **e.** $\sin 150° = \sin 30°$

8. a. 30° **b.** −0.87 **c.** cos 210° = −cos 30°
d. −0.50 **e.** sin 210° = −sin 30°

9. a. 30° **b.** 0.87 **c.** cos 330° = cos 30°
d. −0.50 **e.** sin 330° = −sin 30°

10. a. 0.34 **b.** 0.94 **c.** 0.36 **d.** 0.36 **e.** They are equal.

11. a. 0.71 **b.** −0.71 **c.** −1
d. −1 **e.** They are equal. **f.** tan 135° = −tan 45°

12. a. −0.64 **b.** −0.77 **c.** 0.84
d. 0.83 **e.** They are approximately equal. **f.** tan 220° = tan 40°

13. a. −0.87 **b.** 0.5 **c.** −1.73
d. −1.74 **e.** They are approximately equal. **f.** tan 300° = −tan 60°

14. D

15. a. 1.192 **b.** 3.078 **c.** 0.176 **d.** −0.577
e. −0.577 **f.** 0 **g.** 1 **h.** 57.290

16. a. 0.93 **b.** 2.61 **c.** −0.53 **d.** −3.73

17. a. $\sqrt{1-p^2}$ **b.** p **c.** $-\sqrt{1-p^2}$

18. 0

19. a. $(2\cos\theta, 3\sin\theta)$ **b.** $(-\cos\theta, -\sin\theta)$ **c.** $(-\cos\theta, -\sin\theta)$

20. a., b. Check with your teacher

21. a. $\dfrac{p}{\sqrt{1-p^2}}$ **b.** $\sqrt{1-p^2}$ **c.** p

22. a. $-\frac{1}{2}\sin(2\theta)$ **b.** $\cos(\theta) - \sin(\theta)$

23. a. $\tan(x)$ **b.** $-\sec(x)$

24. 84.7°

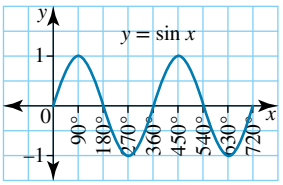
25. The diameter of the unit circle is 2, as it has a radius of 1 and the diameter is twice the radius.

Exercise 20.6 Trigonometric functions

1.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
sin x	0	0.5	0.87	1	0.87	0.5	0	−0.5°	−0.87	−1	−0.87	−0.5	0
x	390°	420°	450°	480°	510°	540°	570°	600°	630°	660°	690°	720°	
sin x	0.5	0.87	1	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0	

2.



3. 360°

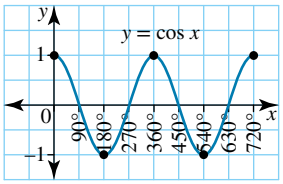
4. a. 0.7 **b.** 0.8 **c.** 0.35 **d.** −0.35
e. 0 **f.** 0.9 **g.** −0.2 **h.** −0.9

5. a. 64°, 116°, 424°, 476° **b.** 244°, 296°, 604°, 656° **c.** 44°, 136°, 404°, 496°
d. 210°, 330°, 570°, 690° **e.** 233°, 307°, 593°, 667° **f.** 24°, 156°, 384°, 516°

6.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
cos x	1	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0	0.5	0.87	1
x	390°	420°	450°	480°	510°	540°	570°	600°	630°	660°	690°	720°	
cos x	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0	0.5	0.87	1	

7.



8. The graph would continue with the cycle.

9. It is a very similar graph with the same shape; however, the sine graph starts at (0, 0), whereas the cosine graph starts at (0, 1).

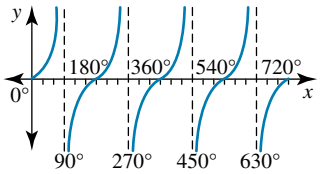
10. a. 0.7 **b.** −0.98 **c.** −1 **d.** 0.9
e. −0.5 **f.** −0.8 **g.** 0.8 **h.** −0.96

11. a. 120°, 240°, 480°, 600° **b.** 37°, 323°, 397°, 683° **c.** 46°, 314°, 406°, 674°
d. 127°, 233°, 487°, 593° **e.** 26°, 334°, 386°, 694° **f.** 154°, 206°, 514°, 566°

12.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
tan x	0	0.58	1.73	undef.	−1.73	−0.58	0	0.58	1.73	undef.	−1.73	−0.58	0
x	390°	420°	450°	480°	510°	540°	570°	600°	630°	660°	690°	720°	
tan x	0.58	1.73	undef.	−1.73	−0.58	0	0.58	1.73	undef.	−1.73	−0.58	0	

13. $y = \tan x$



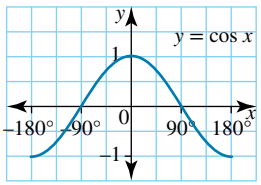
14. The graph would continue repeating every 180° as above.

15. The graph of $y = \tan x$ is quite different from the graphs of $y = \sin x$ and $y = \cos x$. $y = \tan x$ has undefined values (asymptotes) and repeats every 180° rather than 360°. It also gives all y-values, rather than just values between −1 and 1.

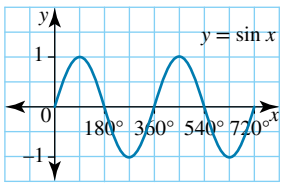
16. a. 1.7 **b.** −1 **c.** −1.2 **d.** 0.8
e. −0.8 **f.** 1.2 **g.** −0.2 **h.** 1

17. a. 45°, 225°, 405°, 585° **b.** 56°, 236°, 416°, 596° **c.** 158°, 338°, 518°, 698°
d. 117°, 297°, 477°, 657° **e.** 11°, 191°, 371°, 551° **f.** 135°, 315°, 495°, 675°

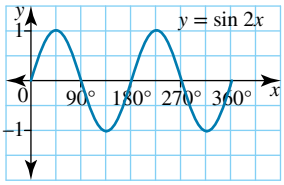
18. a.



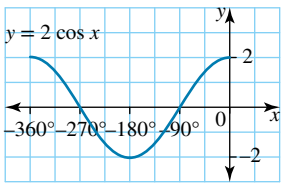
b.



c.



d.



19. a. i. 360° **ii.** 1
b. i. 360° **ii.** 1
c. i. 180° **ii.** 1
d. i. 360° **ii.** 2

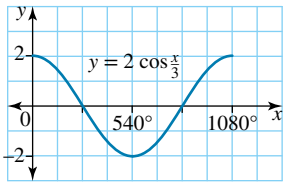
20. a. i. 180° **ii.** 3
b. i. 120° **ii.** 4
c. i. 720° **ii.** 2
d. i. 1440° **ii.** $\frac{1}{2}$

e. i. 360° **ii.** 1

f. i. 180° **ii.** 1

21. a. C **b.** A **c.** D

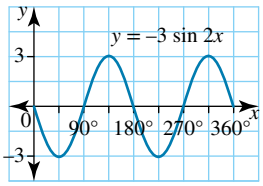
22. a.



Period = 1080°

Amplitude = 2

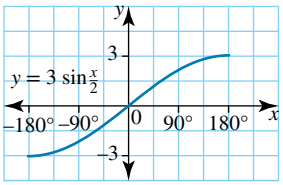
b.



Period = 180°

Amplitude = 3

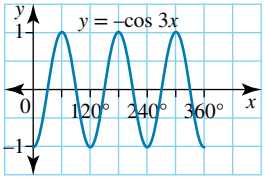
c.



Period = 720°

Amplitude = 2

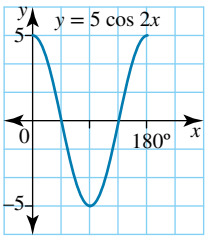
d.



Period = 120°

Amplitude = 1

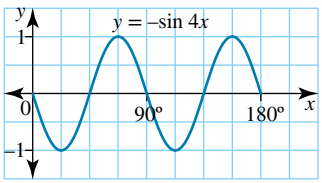
e.



Period = 180°

Amplitude = 5

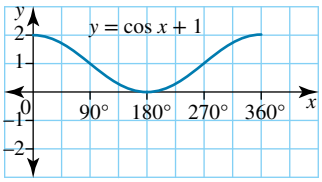
f.



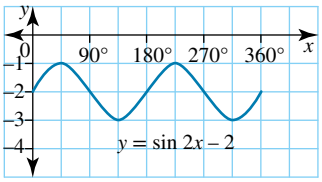
Period = 90°

Amplitude = 1

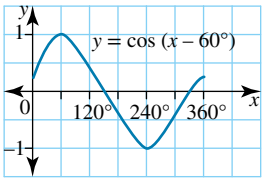
23. a.



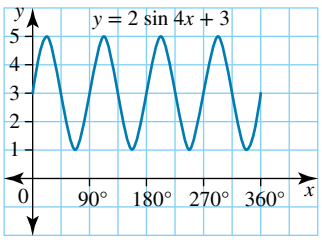
b.



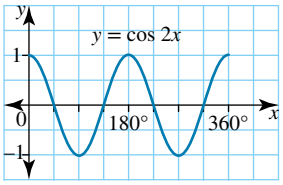
c.



d.



24. a.



i. –1 **ii.** 1

b. i. 3 **ii.** 1

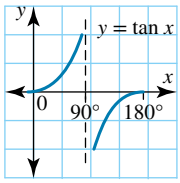
c. Max. value of $\sin x = 1$; hence, max. value of $y = 2 \times 1 + 3 = 5$.

Min. value of $\sin x = -1$; hence, min. value of $y = 2 \times -1 + 3 = 1$.

25. a.

x	0°	30°	60°	90°	120°	150°	180°
y	0	0.58	1.73	Undef.	–1.73	–0.58	0

b.

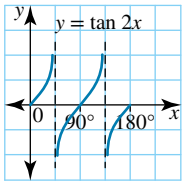


c. At $x = 90°$, y is undefined.

d. $x = 270°$

e. The period is 180°; the amplitude is undefined.

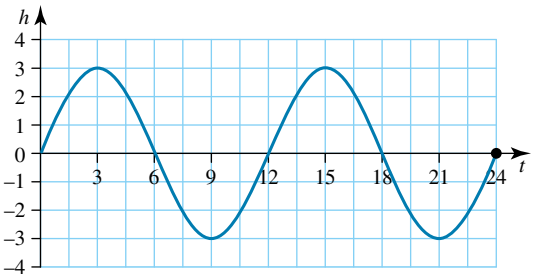
26. a.



b. $c = 45°$ and $x = 135°$

c. The period is 90° and the amplitude is undefined.

27. a.

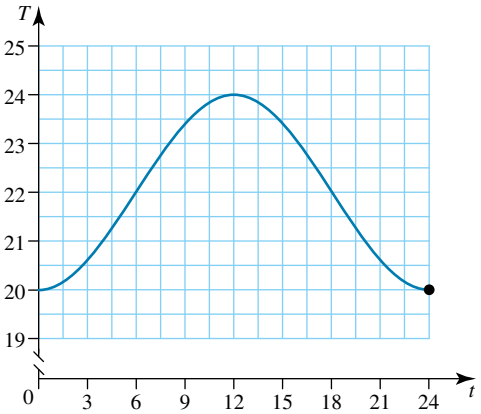


b. 3 metres

c. −2.6 metres

28. a. 22°C

b.



c. Coolest: 20°C; warmest: 24°C

29. The period is $\frac{180°}{n}$ and the amplitude is undefined.

Exercise 20.7 Solving trigonometric equations

1. Calculator answers:

i. 25.84°, 334.16° **ii.** 72.54°, 287.46° **iii.** 101.54°, 258.46° **iv.** 126.87°, 233.13°

2. a. 30°, 150° **b.** 60°, 120° **c.** 120°, 240° **d.** 135°, 225° **e.** 90° **f.** 180°

g. 210°, 330° **h.** 225°, 315° **i.** 30°, 330° **j.** 150°, 210° **k.** 90° **l.** 90°, 270°

3. a. 30°, 60°, 210°, 240°

b. 75°, 105°, 255°, 285°

c. 15°, 105°, 195°, 285°

d. −165°, −135°, −45°, −15°, 75°, 105°

e. 52.5°, 82.5°, 142.5°, 172.5°

f. 30°, 60°, 120°, 150°

g. 45°

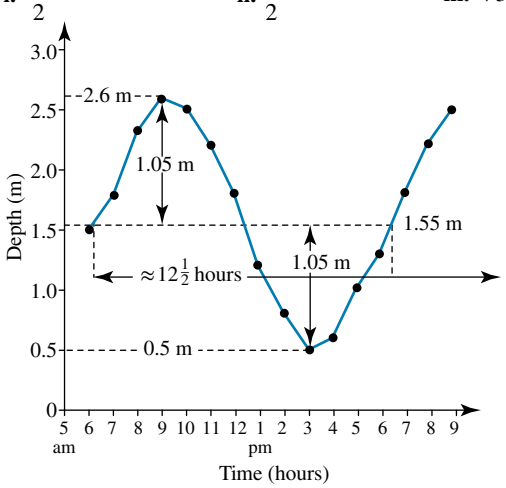
h. 30°, 90°, 150°, 210°, 270°, 330°

4. a. i. $\frac{1}{\sqrt{2}}$ **ii.** $\frac{1}{\sqrt{2}}$ **iii.** 1

b. i. $\frac{1}{2}$ **ii.** $\frac{\sqrt{3}}{2}$ **iii.** $\frac{1}{\sqrt{3}}$

c. i. $\frac{\sqrt{3}}{2}$ **ii.** $\frac{1}{2}$ **iii.** $\sqrt{3}$

5. a.



b. i. $12\frac{1}{2}$ h **ii.** 1.05 m

c. 10.00 am, 10.30 pm, 11.00 am, 11.30 pm, noon

d. Until 1.45 am Sunday, 8 am to 2.15 pm and after 8.30 pm

6. a. $x = 30°, 210°$ **b.** $x = 153.43°, 333.43°$

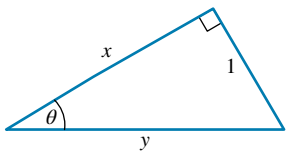
7. $x = 30°, 150°$

8. Many natural phenomenon such as tide heights and sound waves repeat in continuous periodic cycles just as sine and cosine functions do. So it is logical that sine and cosine functions are used to model natural wave phenomena.

9. a. 100^g **b.** 200^g **c.** 300^g **d.** 400^g

Investigation | Rich task

Take one triangle containing the angle θ and label its sides.



This gives

$\tan(\theta) = \frac{1}{x}$

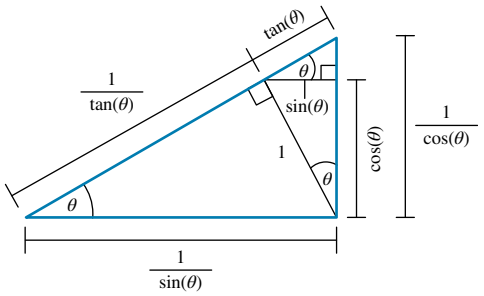
$\Rightarrow x = \frac{1}{\tan(\theta)}$

and

$\sin(\theta) = \frac{1}{y}$

$\Rightarrow y = \frac{1}{\sin(\theta)}$.

Continue until you get the following lengths.



$\text{Area}_1 = \frac{1}{2}\sin(\theta)\cos(\theta)$

$\text{Area}_2 = \frac{90 - \theta}{360}\pi$

$\text{Area}_3 = \frac{1}{2}\left(\frac{1}{\cos(\theta)} - \cos(\theta)\right)\sin(\theta)$

$\text{Area}_4 = \frac{1}{2} \times \frac{1}{\tan(\theta)} - \frac{90 - \theta}{360}\pi$

$\text{Area}_5 = \frac{3\pi}{4}$

Exercise 20.8 Review questions

1. 14.15 cm

2. 20°31′

3. $b = 22.11$ m, $c = 5.01$ m, $C = 10°$

4. 3.64 cm

5. 34°

6. 94°56′

7. a. 159.10 cm^2 **b.** 17.68 cm **c.** 159.10 cm^2

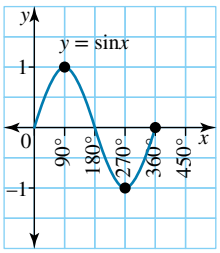
8. 4th quadrant

9. a. 0.94, –0.34 **b.** –2.75

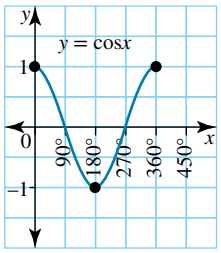
10. B

11. tan 53°

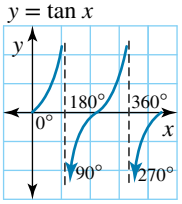
12.



13.

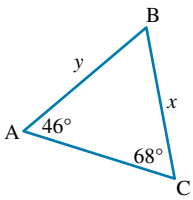


14. $y = \tan x$



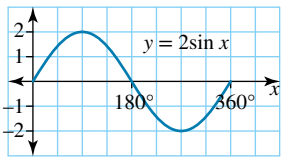
15. 4.331

16.

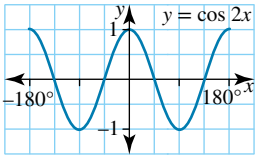


17. a. Period = 120°, amplitude = 2 **b.** Period = 180°, amplitude = 3 **c.** Period = 180°, amplitude = 0.5

18. a.



b.



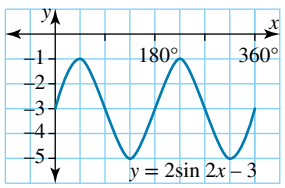
19. a. $x = 191.54,\ 348.46$ **b.** $x = 22.79,\ 157.21,\ 202.79,\ 337.21$

c. $x = 88.09,\ 271.91$ **d.** $x = 7.02,\ 97.02,\ 187.02,\ 277.02$

20. a. 210°, 330° **b.** 30°, 330° **c.** 45°, 315° **d.** 45°, 135°

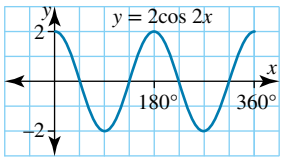
21. E

22. a.



b. Period = 180°, amplitude = 2

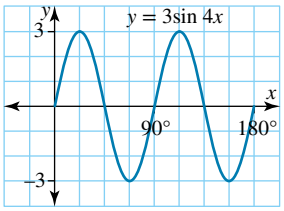
23. a.



i. Period = 180°

ii. Amplitude = 2

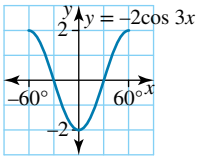
b.



i. Period = 90°

ii. Amplitude = 3

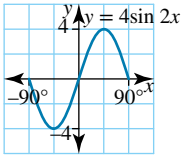
c.



i. Period = 120°

ii. Amplitude = 2

d.



i. Period = 180°

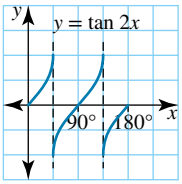
ii. Amplitude = 4

24. a. 15, 165°, 195°, 345° **b.** –70°, 10°, 50° **c.** 22.5°, 67.5°, 202.5°, 247.5°

d. 45°, 75°, 165°, 195°, 285°, 315° **e.** 0°, 45°, 90°, 135°, 180° **f.** 33.75°, 78.75°, 123.75°, 168.75°

25. a. 60°, 300° **b.** 240°, 300° **c.** 45, 315° **d.** 225, 315

26.



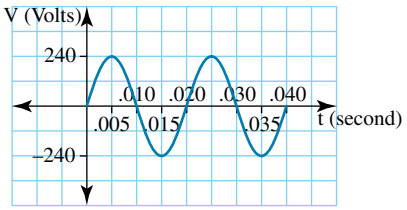
Period = 90°, amplitude is undefined.

Asymptotes are at $x = 45°$ and $x = 135°$.

27. 3.92 m

28. a.

t	*V*
0.000	0
0.005	240
0.010	0
0.015	–240
0.020	0
0.025	240
0.030	0
0.035	–240
0.040	0



b. Maximum voltage occurs at $t = 0.005$ s and 0.025 s.

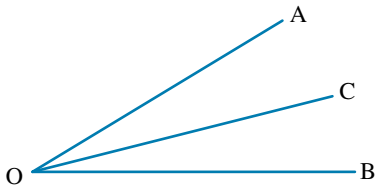
c. 0.02 s

d. 50 cycles per second

GLOSSARY

Addition Law of probability: if A and B are mutually exclusive events, then $\text{P}(A \text{ or } B) = \text{P}(A) + \text{P}(B)$ or $\text{P}(A \cup B) = \text{P}(A) + \text{P}(B)$.

Adjacent angles: angles at a point that share a common ray and a common vertex



Adjacent side [trigonometry]: the side of a right-angled triangle that is adjacent to the reference angle

Algebraic equations: equations that contain pronumerals (letters)

Algebraic expression: an expression formed by numbers and algebraic symbols using arithmetic operations. For example, $4x + 3y - 2$ is an algebraic expression.

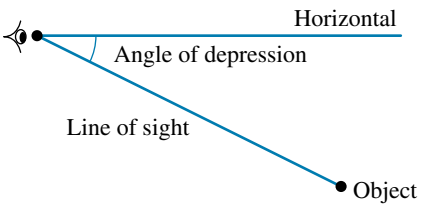
Algebraic fractions: fractions that contain pronumerals (letters)

Algebraic term: an algebraic expression that forms a 'separable' part of some other algebraic expression. For example, in the expression $4x + 2y - 3$, $4x$ and $2y$ are algebraic terms, while -3 is a constant term.

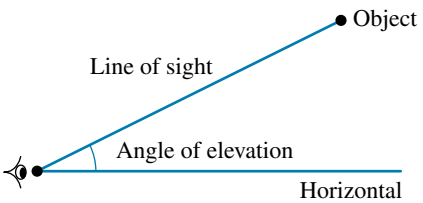
Alternate angles: angles on alternate sides of a transversal. On parallel lines, alternate angles are equal.

Amplitude: half the distance between the maximum and minimum values of a function

Angle of depression: the angle measured down from the horizontal line (through the observation point) to the line of vision



Angle of elevation: the angle measured up from the horizontal line (through the observation point) to the line of vision

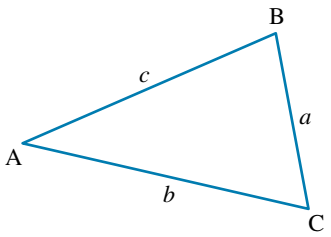


Arc (of a circle): a portion of the circumference of a circle

Area: the amount of flat surface enclosed by a shape. It is measured in square units, such as square metres, m^2, or square kilometres, km^2.

Area of triangle (using sine): if the perpendicular height of a triangle is not known, but two sides and the included angle are known, the area of any triangle ABC can be calculated using the rule:

Area $= \frac{1}{2}ab \sin C$, Area $= \frac{1}{2}ac \sin B$ or Area $= \frac{1}{2}bc \sin A$



Associative Law: a method of combining two numbers or algebraic expressions is associative if the result of the combination of these objects does not depend on the way in which the objects are grouped. Addition and multiplication obey the Associative Law, but subtraction and division are not associative.

Asymptote: a line that a graph approaches but never meets

Average: *see* **Mean**

Back-to-back stem-and-leaf plot: a stem-and-leaf plot that compares two related sets of data

Bar graph: a graph drawn in a similar way to a column graph, with horizontal bars instead of vertical columns. Categories are graphed on the vertical axis and the frequencies (numbers) on the horizontal axis.

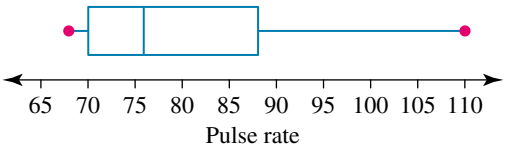
Base: the digit at the bottom of numbers written in index form. For example, in 6^4, the base is 6. This tells us that 6 is multiplied by itself four times.

Bi-modal: describes data whose distribution has two modes

Bisect: cut into two equal parts

Bivariate data: data where each piece is represented by two variables

Boxplot (box-and-whisker plot): a graphical representation of the five-number summary; that is, the lowest score, lower quartile, median, upper quartile and highest score, for a particular set of data

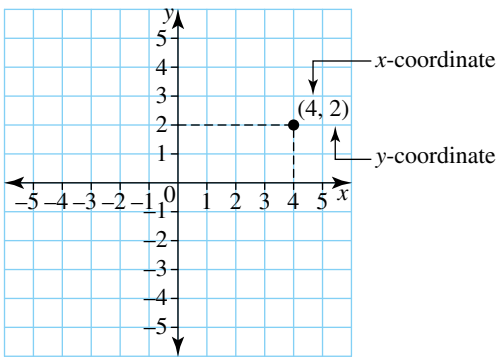


Brackets: also called grouping symbols

By eye (line of best fit): drawing a line of best fit by carefully looking at the data so that there is an equal number of points above and below the line

Capacity: the maximum amount of fluid that can be contained in an object. It is usually applied to the measurement of liquids and is measured in units such as millilitres (mL), litres (L) and kilolitres (kL).

Cartesian coordinate system: the position of any point in the Cartesian plane can be represented by an ordered pair of numbers (x, y). These are called the coordinates of the point.



Cartesian plane: the area formed by a horizontal line with a scale (x-axis) joined to a vertical line with a scale (y-axis). The point of intersection of the lines is called the origin.

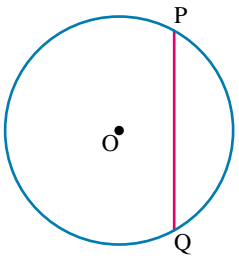
Categorical (data): data that cannot be measured or counted but can be categorised; for example, eye colour or television programs

Census: collection of data from a population (e.g. all Year 10 students) rather than a sample

Centre (of circle): middle point of a circle, equidistant (equal in distance) from all points on its circumference

Centre of enlargement: the point from which the enlargement of an image is measured

Chord: straight line from one point on the circumference of a circle to another point on the circumference



Circle: a set of points that lie a fixed distance (the radius) from a fixed point (the centre)

Circle (equation): the general equation of a circle with centre (0, 0) and radius is $x^2 + y^2 = r^2$.

Circumcentre: the centre of a circle drawn so that it touches all three vertices of a triangle

Circumcircle: a circle drawn so that it touches all three vertices of a triangle

Circumference: distance around the outside of a circle. It is given by the rule $2\pi r$ or πD, where r is the radius and D is the diameter of the circle.

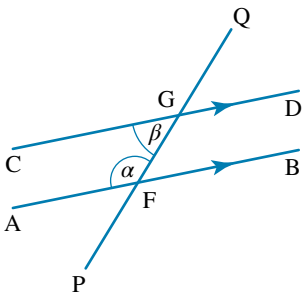
Class interval: a subdivision of a set of data. For example, students' heights may be grouped into class intervals of 150 cm – 154 cm, 155 cm – 159 cm.

Closure Law: when an operation is performed on an element (or elements) of a set, the result produced must also be an element of that set.

Coefficient: the number part of a term, generally written in front of the pronumeral

Coincident (lines): lines that lie on top of each other

Co-interior angles: angles that lie on the same side of a transversal that cuts across a pair of lines. For a pair of parallel lines, co-interior angles are supplementary (add to 180°).



Collinear points: points that all lie on the same straight line

Column graph: a graph in which equal width columns are used to represent the frequencies (numbers) of different categories

Common factor (common divisor): a factor that is common to each element of the set; for example, $3x$ is a common factor of the elements $9x^2$ and $12x$

Commutative Law: a method of combining two numbers or algebraic expressions is commutative if the result of the combination does not depend on the order in which the objects are given. For example, the addition of 2 and 3 is commutative, since $2 + 3 = 3 + 2$. However, subtraction is not commutative, since $2 - 3 \neq 3 - 2$.

Compass bearings: directions measured in degrees from the north–south line in either a clockwise or anticlockwise direction. To write the compass bearing we need to state whether the angle is measured from the north or south, the size of the angle and whether the angle is measured in the direction of east or west; for example, N27°W, S32°E.

Complement (of a set): the complement of a set, A, written A', is the set of elements that are in ξ but not in A

Complementary angles: two angles that add to 90°; for example, 24° and 66° are complementary angles

Complementary events: events that have no common elements and together make up the sample space. If A and A' are complementary events, then $P(A) + P(A') = 1$.

Completing the square: a procedure used to transform an algebraic expression into a perfect square

Composite figure: a figure made up of more than one basic shape

Compound interest: the interest earned by investing a sum of money (the principal) when each successive interest payment is added to the principal for the purpose of calculating the next interest payment. The formula used for compound interest is: $A = P(1 + R)^n$, where A is the amount to which the investment grows, P is the principal or initial amount invested, R is the interest rate per compounding period (as a decimal) and n is the number of compounding periods. The compound interest is calculated by subtracting the principal from the amount: $CI = A - P$.

Compounded value: the value of the investment with accrued interest included

Compounding period: the period of time over which interest is calculated

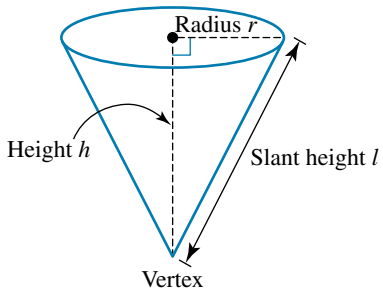
Concave polygon: a polygon with at least one reflex interior angle

Concyclic (points): points that lie on the circumference of a circle

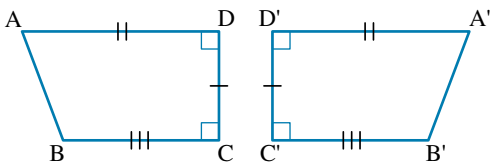
Conditional probability: where the probability of an event is conditional (depends) on another event occurring first. For two events A and B, the conditional probability of event B, given that event A occurs, is denoted by $P(B|A)$ and can be calculated using the formula:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, P(A) \neq 0.$$

Cone: a solid formed by taking a circular base and a point not in the plane of the circle, called the vertex, which lies above or below the circle, and joining the vertex to each point on the circumference of the circle.



Congruent figures: figures that are identical; that is, they have exactly the same shape and size



Congruent triangles: triangles that have the same shape and size. There are five standard congruence tests for triangles: SSS (side, side, side), SAS (side, included angle, side), ASA (two angles and one side), AAS (two angles and a non-included side) and RHS (right angle, hypotenuse, side).

Conjugate surds: surds that, when multiplied together, result in a rational number. For example, $(\sqrt{a} + \sqrt{b})$ and $(\sqrt{a} - \sqrt{b})$ are conjugate surds, because $(\sqrt{a} + \sqrt{b}) \times (\sqrt{a} - \sqrt{b}) = a - b$.

Constant: a term or expression whose value does not vary

Constant of proportionality (or variation) k: used to prove that a proportionality relationship (direct or inverse) exists between 2 or more variables (or quantities)

Continuous (data): numerical data that can take any value within a certain range. They are generally associated with measuring; for example, the heights of students.

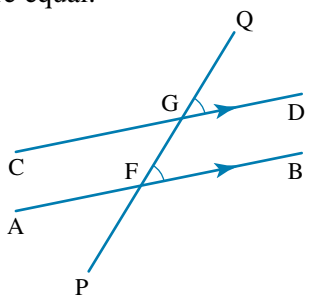
Converse: the reverse of a statement

Convex polygon: a polygon with no interior reflex angles

Coordinates (x, y): two numbers that give the position of a point on the Cartesian plane. The first number is the x-coordinate and the second number is the y-coordinate.

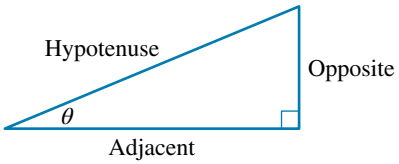
Correlation: a measure of the relationship between two variables. Correlation can be classified as linear, non-linear, positive, negative, weak, moderate or strong.

Corresponding angles: angles that are in corresponding positions with respect to a transversal. On parallel lines, corresponding angles are equal.

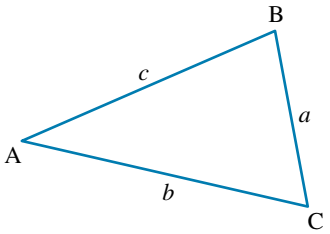


Cosine (cos) ratio: the ratio of the adjacent side to the hypotenuse in a right-angled triangle.

So, $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$.



Cosine rule: in any triangle ABC, $c^2 = a^2 + b^2 - 2ab\cos C$.



Counting numbers: the non-negative integers; that is, one of the numbers 0, 1, 2, 3, …

Cross-section: the shape (plane section) produced when a solid is cut through by a plane parallel to the base. For example, the cross-section of a cone is a circle.

Cube: a solid with 6 faces. All faces are squares of the same size.

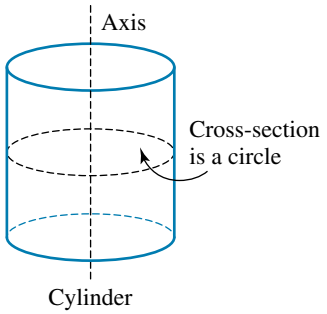
Cubic functions: the basic form of a cubic function is $y = ax^3$. These functions can have 1, 2 or 3 roots.

Cumulative frequency: the total of all frequencies up to and including the frequency for a particular score in a frequency distribution

Cyclic quadrilateral: a quadrilateral that has all four vertices on the circumference of a circle. That is, the quadrilateral is inscribed in the circle.

Cyclical pattern: a pattern that displays fluctuations that repeat but will usually take longer than a year to repeat

Cylinder: a solid that has parallel circular discs of equal radius at the ends. The centres of the discs form the axis of the cylinder.



Data: various forms of information

Decimal number system: the base 10, place-value system most commonly used for representing real numbers

Degree (angle): a unit used to measure the size of an angle

Degree (of a polynomial): the degree of a polynomial in x is the highest power of x in the expression.

Denominator: the lower number of a fraction that represents the number of equal fractional parts a whole has been divided into

Dependent events: successive events in which one event affects the occurrence of the next

Dependent variable: this variable is graphed on the y-axis.

Depreciation: the reduction in the value of an item as it ages over a period of time. The formula used is $A = P(1 - R)^n$, where A is the depreciated value of the item, P is its initial value, R is the percentage the item depreciates each year (expressed as a decimal) and n is the number of years the item has depreciated.

Deviation: the difference between a data value and the mean

Diameter: the straight line from one point on the circumference of a circle to another on the circumference, passing through the centre.

Digital footprints: information about people that can be collected via the internet or by digital devices

Dilated [quadratics]: occurs when graphs are made thinner or wider

Dilation [geometry]: occurs when figures are made larger (enlarged) or smaller (reduced) in proportion

Direct proportion: describes a particular relationship between two variables (or quantities); that is, as one variable increases so does the other variable. The graph of the relationship is a straight line, passing through the origin and the rule used to relate the two variables is $y = kx$.

Direction (correlation): if the points in a scatterplot show an upward pattern, the direction is said to be positive; if the pattern is downward, the direction is negative.

Discrete data: numerical data in which the information can take only certain exact values, usually whole numbers. They are associated with counting.

Discriminant: referring to the quadratic equation $ax^2 + bx + c = 0$, the discriminant is given by $\Delta = b^2 - 4ac$. It is the expression under the square-root sign in the quadratic formula and can be used to determine the number and type of solutions of a quadratic equation.

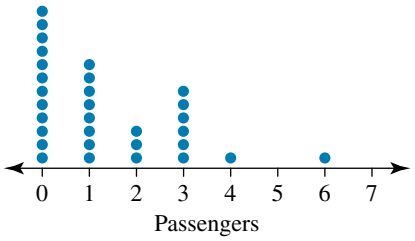
Distance formula: the distance between two points A (x_1, y_1) and B (x_2, y_2) is given by the formula $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Distributive Law: the product of one number with the sum of two others equals the sum of the products of the first number with each of the others; for example, $4(6 + 2) = 4 \times 6 + 4 \times 2$. It is also applicable to algebra; for example, $3x(x + 4) = 3x^2 + 12x$.

Dividing by a fraction: when dividing by a fraction, multiply by the reciprocal of the fraction, then simplify the expression. For example, $6 \div \frac{3}{2} = 6 \times \frac{2}{3} = 4$.

Domain (of a function or relation): the set of all allowable values of x

Dot plot: this graphical representation uses one dot to represent a single observation. Dots are placed in columns or rows, so that each column or row corresponds to a single category or observation.



Eighth Index Law: terms with fractional indices can be written as surds. For example, $a^{\frac{1}{n}} = \sqrt[n]{a}$ and $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.

Element: an element of a set is a member of that set; for example, 5 is an element of the set of counting numbers.

Elimination method: a method used to solve simultaneous equations. This method combines the two equations into a third equation involving only one of the variables.

Ellipse: a plane figure in the shape of an oval

Enlargement (dilation): a scaled-up (or down) version of a figure in which the transformed figure is in proportion to the original figure; that is, the two figures are similar

Equally likely outcomes: outcomes in a probability experiment that have the same chance of occurring

Equating: the process of writing one expression as equal to another

Equation: a statement that asserts that two expressions are equal in value. An equation must have an equal sign. For example, $x + 4 = 12$.

Equilateral triangle: a triangle with all sides equal in length, and all angles equal to 60°

Equivalent fractions: fractions that can be reduced to the same basic fraction; that is, fractions that have the same value, for example, $\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{4}{12}$

Estimate: information about a population extrapolated from a sample of the population

Evaluate: determine a value for an expression

Event: a set of favourable outcomes in each trial of a probability experiment

Event space: a list of all the possible outcomes obtained from a probability experiment. It is written as ξ or S, and the list is enclosed in a pair of curled brackets $\{\}$. It is also called the sample space.

Expanding (algebra): this is the process of multiplying everything inside the brackets by what is directly outside the brackets. Expanding is the opposite of factorising.

Expected frequency: the number of times a particular event is expected to occur when a chance experiment is repeated a number of times

Experimental probability: the probability of an event based on the outcomes of experiments, simulations or surveys

Exponent: *see* **Index**

Exponential decay: a quantity that decreases by a constant percentage in each fixed period of time. This growth can be modelled by exponential functions of the type $y = ka^x$, where $0 < a < 1$.

Exponential functions: relationships of the form $y = a^x$, where $a \neq 1$, are called exponential functions with base a.

Exponential growth: a quantity that grows by a constant percentage in each fixed period of time. This growth can be modelled by exponential functions of the type $y = ka^x$, where $a > 1$.

Expression: this is a collection of two or more numbers or variables, connected by operations. For example, $12 - 2$, $2a + 3b$. Expressions do not contain an equal sign.

Extrapolation: the process of predicting a value of a variable outside the range of the data

Factor: a factor of a given number is a whole number that divides it exactly. For example, 1, 2, 3, 4, 6 and 12 are the factors of 12.

Factor theorem: if $P(x)$ is a polynomial, and $P(a) = 0$ for some number a, then $P(x)$ is divisible by $(x - a)$.

Factorising: breaking down a number or expression into smaller factors that can be numeric or algebraic. The process of factorising an algebraic expression involves changing it from a sum (or difference) into a product of factors.

Fifth Index Law: to remove brackets containing a product, raise every part of the product to the index outside the brackets. So, $(ab)^m = a^m b^m$.

Finite: a fixed number or amount. For example, the decimal 0.25 has a fixed number of decimal places.

First Index Law: when terms with the same base are multiplied, the indices are added. So, $a^m \times a^n = a^{m+n}$.

Five-number-summary: a method for summarising a data set using five statistics: the minimum value, the lower quartile, the median, the upper quartile and the maximum value

FOIL: a diagrammatic method of expanding a pair of brackets. The letters in FOIL represent the order of the expansion: First, Outer, Inner and Last.

Form (correlation): the general shape of a pattern. For example, if the points in a scatterplot show an approximate linear pattern, it is described as having linear form.

Formula: a literal equation that uses algebra to state a relationship

Fourth Index Law: to remove brackets, multiply the indices inside the brackets by the index outside the brackets. Where no index is shown, assume that it is 1. So, $(a^m)^n = a^{mn}$.

Fraction: numbers represented in the form $\frac{a}{b}$, where a and b are whole numbers and b is not equal to zero.

Frequency: the number of times a particular score appears

Frequency polygon: a special type of line graph, which uses the same scaled axes as the histogram. The midpoints of the tops of the histogram columns are joined by straight line intervals. The polygon is closed by drawing lines at each end down to the score- or x-axis.

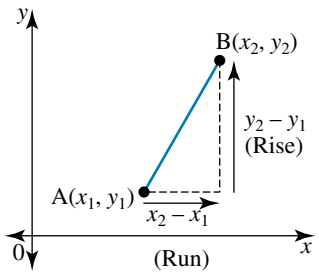
Frequency table: a means of organising a large set of data. It shows the number of scores (frequencies) that belong to each group or class interval.

Function: a process that takes a set of x-values and produces a related set of y-values. For each distinct x-value, there is only one related y-value. A function is usually defined by a formula for $f(x)$ in terms of x; for example, $f(x) = x^2$.

Future value: the future value of a loan or investment

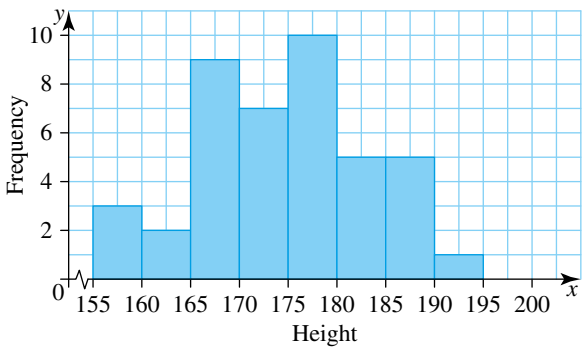
General upward or downward trend: a graph that overall goes up or down over time

Gradient (slope) m: this is a measure of the steepness of a line or plane. The gradient of a line is given by $m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}$ and is constant anywhere along that line.



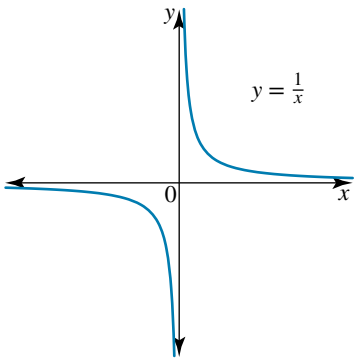
Heron's formula: this formula is used to find the area of a triangle when all three sides are known. The formula is $A = \sqrt{s(s-a)(s-b)(s-c)}$, where a, b and c are the lengths of the sides and s is the semi-perimeter or $s = \frac{a+b+c}{2}$.

Histogram: a special type of column graph, in which no gaps are left between columns and each column straddles an x-axis score. The x-axis scale is continuous and usually a half-interval is left before the first column and after the last column.



Horizontal: a line is said to be horizontal if it is parallel to the horizon of the Earth. Horizontal lines have a gradient of zero and are parallel to the x-axis.

Hyperbola: the graph of $y = \frac{1}{x}$ has the shape of a hyperbola with asymptotes on the x- and y-axes.



Hypotenuse: the longest side of a right-angled triangle. It is the side opposite the right angle.

Identity Law: when 0 is added to an expression or the expression is multiplied by 1, the value of the variable does not change. For example, $x + 0 = x$ and $x \times 1 = x$.

Image (similar figures): the enlarged (or reduced) figure produced

Improper fraction: a fraction in which the numerator is greater than the denominator

Independent events: successive events that have no effect on each other

Independent variable: this is the x-axis (or horizontal) variable

Index (power or exponent): the number expressing the power to which a number or pronumeral is raised. For example, in the expression 3^2, the index is 2. Plural: *indices*.

Inequality: when one algebraic expression or one number is greater than or less than another

Inequality signs: signs used in inequations. They are $<$ (less than), $>$ (greater than), $\leq$ (less than or equal to) and $\geq$ (greater than or equal to).

Inequations: similar to equations, but contain an inequality sign instead of an equal sign. For example, $x = 3$ is an equation, but $x < 3$ is an inequation.

Infinite: never-ending; for example, the decimal 0.3 is non-terminating and therefore its number of decimal places cannot be counted.

Integers (Z): These include the positive and negative whole numbers, as well as zero; that is, . . ., -3, -2, -1, 0, 1, 2, . . .

Intercepts: points where a curve crosses the x- or y-axis

Interpolation: the process of predicting a value of a variable from within the range of the data

Interquartile range: the difference between the upper (or third) quartile, Q_3, and the lower (or first) quartile, Q_1; that is, $IQR = Q_3 - Q_1$. It is the range of approximately the middle half of the data.

Intersection (of sets): region that represents the common elements of two or more sets. $A \cap B$ denotes the intersection of sets A and B.

Inverse Law: when the additive inverse of a number or pronumeral is added to itself, the sum is equal to 0. When the multiplicative inverse of a number or pronumeral is multiplied by itself, the product is equal to 1. So, $x + (-x) = 0$ and $x \times \frac{1}{x} = 1$.

Inverse operation: the operation that reverses the effect of the original operation. Addition and subtraction are inverse operations; multiplication and division are inverse operations.

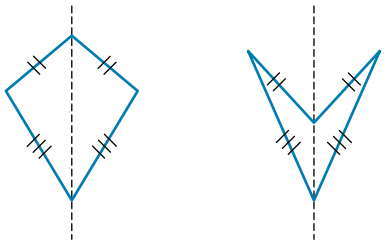
Inverse proportion: describes a particular relationship between two variables (or quantities); that is, as one variable increases, the other decreases. The rule used to relate the two variables is $y = \frac{k}{x}$.

Irrational numbers (I): numbers that cannot be written as fractions. Examples of irrational numbers include surds, π and non-terminating, non-recurring decimals.

Isosceles triangle: a triangle with two sides equal in length

Iterations: repeated calculations

Kite: a quadrilateral with two pairs of adjacent sides equal. A kite may be convex or non-convex.



Lay-by: a method used to purchase an item whereby the purchaser makes regular payments to the retailer, who retains the item until the complete price is paid

Like terms: terms that contain exactly the same pronumeral (letter) part; for example, $3ab$ and $7ab$ are like terms but $5a$ is not.

Line of best fit: a straight line that best fits the data points of a scatterplot that appear to follow a linear trend. It is positioned on the scatterplot so that there is approximately an equal number of data points on either side of the line, and so that all the points are as close to the line as possible.

Line segment: a line segment or interval is a part of a line with end points.

Linear equation: an equation involving pronumerals of degree 1. The general form of a linear equation in one variable is $ax + b = 0$.

Linear graph: consists of an infinite number of points that can be joined to form a straight line

Literal equation: an equation that includes several pronumerals or variables

Logarithm: the power to which a given positive number b, called the base, must be raised in order to produce the number x. The logarithm of x, to the base b, is denoted by $\log_b x$. Algebraically: $\log_b x = y \leftrightarrow b^y = x$; for example, $\log_{10} 100 = 2$ because $10^2 = 100$.

Logarithmic equation: an equation that requires the application of the laws of indices and logarithms to solve

Logarithm laws:

Law 1: $\log_a x + \log_a y = \log_a (xy)$

Law 2: $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$

Law 3: $\log_a x^n = n \log_a x$

Law 4: $\log_a 1 = 0$

Law 5: $\log_a a = 1$

Law 6: $\log_a \left(\frac{1}{x}\right) = -\log_a x$

Law 7: $\log_a a^x = x$

Lower (or first) quartile: the score that marks the end of the first quarter in an ordered set of data. It is denoted by Q_L or Q_1. It is calculated by finding the median of the lower half of the data.

Lowest common denominator (LCD): the lowest number that denominators of all fractions considered in a problem will divide equally into

Lowest common multiple (LCM): the lowest multiple that two or more numbers have in common

Many-to-many relation: a relation in which one range value may yield more than one domain value and vice versa

Many-to-one relation: a function or mapping that takes the same value for at least two different elements of its domain

Maximal domain: the limit of the x-values that a function can have

Maximum vertex: the highest point of a parabola that is inverted

Mean: one measure of the centre of a set of data. It is given by $\text{mean} = \frac{\text{sum of all scores}}{\text{number of scores}}$ or $\bar{x} = \frac{\Sigma x}{n}$.

When data are presented in a frequency distribution table, $\bar{x} = \frac{\Sigma(f \times x)}{n}$.

Measures of central tendency: mean, median and mode

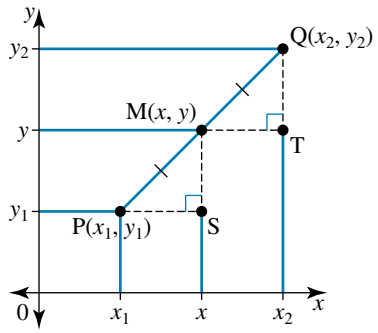
Measures of spread: range, interquartile range, standard deviation

Median: one measure of the centre of a set of data. It is the middle score for an odd number of scores arranged in numerical order. If there is an even number of scores, the median is the mean of the two middle scores when they are ordered. Its location is determined by the rule $\frac{n+1}{2}$.

For example, the median value of the set 133456899 is 5, while the median value for the set 13345689910 is the mean of 5 and 6 (5.5).

Metric system: the system of measurement based on the metre

Midpoint: the midpoint of a line segment is the point that divides the segment into two equal parts. The coordinates of the midpoint M between the two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by the formula $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.



Minimum vertex: the lowest point of a parabola which is upright

Minutes (angle): units of angular measurement, where 1 degree (1°) = 60 minutes (60′)

Mixed number (numeral): a number that consists of a whole number part and a fractional part, for example $2\frac{1}{3}$

Modal class: the term used when analysing grouped data. It is the class interval with the highest frequency.

Mode: one measure of the centre of a set of data. It is the score that occurs most often. There may be no mode (all scores occur once), one mode or more than one mode (two or more scores occur equally frequently).

Moderate (correlation): describes a relationship in which the points on the scatterplot are reasonably close together and approximate a linear pattern

Monic: a monic pronumeral or monic expression is one in which the coefficient of the leading term is 1. For example, $x^2 + 4x - 3$ is monic, but $6x^2 + 4x - 3$ is not.

Multiple: a number that is the product of a given number and any whole number greater than zero. For example, the multiples of 3 are $3, 6, 9, 12, 15, \ldots$

Multiplication Law of probability: if events A and B are independent, then: $P(A \text{ and } B) = \text{P}(A) \times \text{P}(B)$ or $P(A \cap B) = \text{P}(A) \times P(B)$.

Mutually exclusive events: events that cannot occur together. On a Venn diagram, two mutually exclusive events will appear as disjoint sets.

Natural numbers: the set of positive integers, or counting numbers; that is, the set $1, 2, 3, \ldots$

Negatively skewed: showing larger amounts of data as the values of the data increase

Nominal (data): a type of categorical data in which the information is divided into subgroups; for example, eye colour (hazel, blue, green)

Non-recurring decimals: decimals that have no repeating digits or pattern, for example 5.4827862…

Non-terminating decimals: decimals that have an infinite number of decimal places

Null Factor Law: if $a \times b = 0$, then either $a = 0$ or $b = 0$ or both $a = 0$ and $b = 0$; used when solving quadratic equations

Numerator: the upper number of a fraction that represents the number of equal fractional parts

Numerical (data): data that can be measured or counted

Object (similar figures): the original figure is classed as the object

Ogive (cumulative frequency polygon): a graph formed by joining the top right-hand corners of the columns of a cumulative frequency histogram

One-to-many relation: a relation in which there may be more than one range value for one domain value but only one domain value for each range value

One-to-one relation: refers to the relationship between two sets such that every element of the first set corresponds to one and only one element of the second set

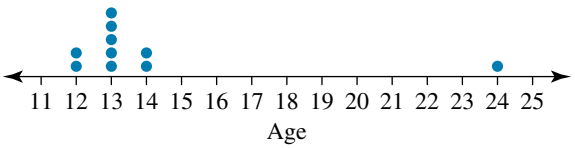
Opposite side [trigonometry]: the side of a right-angled triangle that is opposite the reference angle

Ordinal (data): a type of categorical data in which the information is in some type of ranked order; for example, first, second, third, . . .

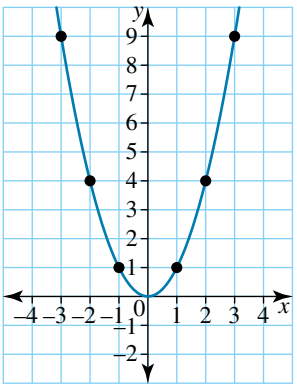
Origin: the point on the Cartesian plane where the x- and y-axes intersect

Outcome: the result obtained when a probability experiment is conducted

Outlier: a piece of data that is considerably different from the rest of the values in a set of data; for example, 24 is the outlier in the set of ages $\{12, 12, 13, 13, 13, 13, 13, 14, 14, 24\}$.

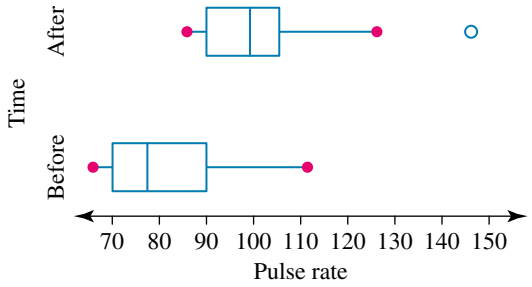


Parabola: the graph of a quadratic function has the shape of a parabola. For example, the typical shape is that of the graph of $y = x^2$.



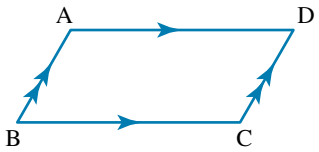
Parallel: parallel lines in a plane never meet, no matter how far they are extended. Parallel lines have the same gradient.

Parallel boxplots: two or more boxplots drawn on the same scale to visually compare the five-number summaries of the data sets. These boxplots compare the pulse rates of the same group of people before and after exercise.



Parallel dot plots: two or more dot plots drawn on the same scale to visually compare data sets

Parallelogram: a quadrilateral with both pairs of opposite sides parallel



Percentage: a fraction whose denominator is 100; for example, $28\% = \frac{28}{100}$

Percentile: the value below which a given percentage of all scores lie. For example, the 20th percentile is the value below which 20% of the scores in the set of data lie.

Perimeter: the distance around the boundary of a 2-dimensional shape

Periodic functions: functions that have graphs that repeat themselves continuously in cycles, for example, graphs of $y = \sin x$ and $y = \cos x$. The period of the graph is the distance between repeating peaks or troughs.

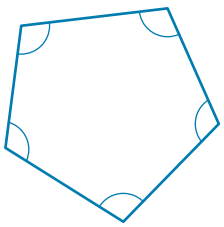
Perpendicular: perpendicular lines are at right angles to each other. The product of the gradients of two perpendicular lines is -1.

Pi (π): the Greek letter π represents the ratio of the circumference of any circle to its diameter. The number π is irrational, with an approximate value of $\frac{22}{7}$, and a decimal value of $\pi = 3.141\,59\ldots$

Pie chart (graph): *see* **Sector graph**

Plotting: placing points on a Cartesian plane using their coordinates

Polygon: a plane figure bounded by line segments



Polynomial: an expression containing only non-negative integer powers of a variable

Population: the whole group from which a sample is drawn

Positively skewed: showing larger amounts of data as the values of the data decrease

Power: *see* **Index**

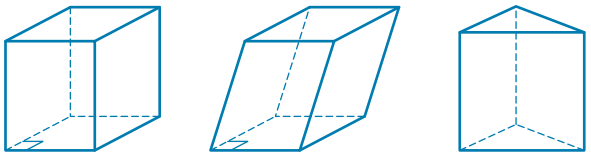
Primary data: data collected by the user

Prime factor: a prime number that divides a given number exactly; for example, the prime factors of 42 are 2, 3 and 7

Prime number: a number that has only two different factors — itself and one. For example, 3 is a prime number because its only factors are 1 and 3. Note that 1 is not a prime number because its two factors, 1×1, are the same.

Principal: an amount of money borrowed or invested

Prism: A solid comprising two congruent parallel faces (bases) and the (lateral) faces that connect them. The lateral faces are parallelograms. If they are all right-angled (i.e. rectangles) the prism is a right prism; if they are not all right-angled, then the prism is an oblique prism.



Right rectangular prism | Oblique rectangular prism | Right triangular prism

Probability: the likelihood or chance of a particular event (result) occurring.

$$\text{P(event)} = \frac{\text{number of favourable outcomes}}{\text{number of possible outcomes}}.$$

The probability of an event occurring ranges from 0 (impossible — will not occur) to 1 (certainty — will definitely occur).

Product: the result of a multiplication

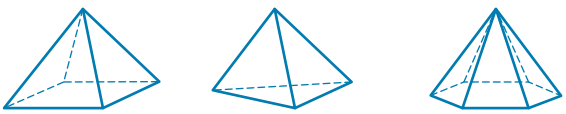
Pronumerals: letters used in place of numbers

Proof: an argument that shows why a statement is true

Proportion: corresponding elements are in proportion if there is a constant ratio; for example,

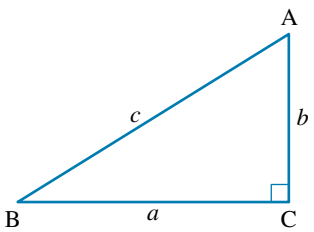
$$\frac{\text{circumference}}{\text{diameter}} = \pi \text{ for all circles}$$

Pyramid: a group of solids with any polygon as the base. Its other faces are triangles that meet at a common vertex. Pyramids are named according to their base. For example, a pyramid with a square base is a square pyramid.



Square-based pyramid | Triangular-based pyramid | Hexagonal-based pyramid

Pythagoras' theorem: in any right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides. This is often expressed as $c^2 = a^2 + b^2$.



Quadrant: a sector with an arc equal to a quarter of a circle (and therefore centre angle of 90°)

Quadrant 1: the quarter of the unit circle where the value of the angle being considered is between 0° and 90°. That is, the x- and y-coordinates are both positive.

Quadrants of a Cartesian plane: four regions of the Cartesian plane produced by the intersection of the x- and y-axes

Quadratic equation: the general form of the quadratic equation is $ax^2 + bx + c = 0$.

Quadratic formula: gives the roots of the quadratic equation $ax^2 + bx + c = 0$. It is expressed as $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Quadratic trinomial: an algebraic expression that contains three terms, in which the highest power of the pronumeral is a squared term; for example, $4x^2 - 3x + 7$

Quadrilateral: a 4-sided polygon

Quantitative data: data that can be counted (discrete data) or measured (continuous data), for example, the number of students enrolled in a school (discrete), the heights in centimetres of the students in a class (continuous)

Quartic functions: the basic form of a quartic function is $y = ax^4$. If the value of a is positive, the curve is upright, whereas a negative value of a results in an inverted graph. A maximum of 4 roots can result.

Quartiles: values that divide an ordered set into four (approximately) equal parts. There are three quartiles — the first (or lower) quartile Q_1, the second quartile (or median) Q_2 and the third (or upper) quartile Q_3.

Quotient: the result of dividing one number or algebraic expression by another

Radius: the straight line from a circle's centre to any point on its circumference; plural: *radii*

Random number: a number whose value is governed by chance, and cannot be predicted in advance

Random pattern: a pattern that does not show any regular fluctuation

Range: the difference between the highest and lowest scores in a set of data; that is,
range = highest score − lowest score

Range (of a function or relation): the set of all allowable values of y

Rate: a particular kind of ratio where the two quantities are measured in different units; for example, km/h, $/g

Ratio: the comparison of two or more quantities of the same kind. A ratio has no units.

Rational numbers (Q): numbers that can be written as fractions, where the denominator is not zero

Rationalising the denominator: a method used to express the denominator as a rational number. Both the numerator and denominator of a fraction are multiplied by the surd (or conjugate surd) contained in the denominator.

Real numbers (R): the set of all rational and irrational numbers

Reciprocal: a number by which a given number is multiplied to result in 1

Rectangle: a parallelogram with four right angles

Rectangular prism (cuboid): a solid that has six rectangular faces and a uniform cross-section

Recurring decimals: These decimals have one or more digits repeated continuously; for example, 0.999. . . . They can be expressed exactly by placing a dot or horizontal line over the repeating digits; for example, $8.343\,434 = 8.\dot{3}\dot{4}$ or $8.\overline{34}$.

Reflection [quadratics]: one parabola is a mirror image of the other.

Regular polygon: a polygon with sides of the same length and interior angles of the same size

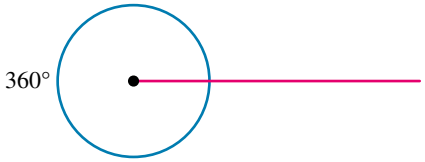
Relation: a set of ordered pairs

Relative frequency: represents the frequency of a particular score divided by the total sum of the frequencies. It is given by the rule:

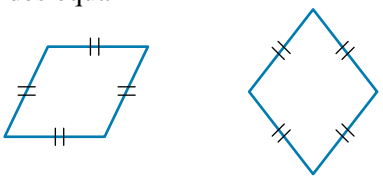
$$\text{relative frequency of a score} = \frac{\text{frequency of the score}}{\text{total sum of frequencies}}.$$

Remainder theorem: if a polynomial $P(x)$ is divided by $x - a$, where a is any real number, the remainder is $P(a)$.

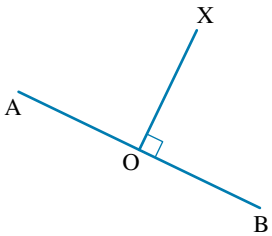
Revolution (angle): the size of a revolution is 360°.



Rhombus: a parallelogram with all sides equal



Right angle: the size of a right angle is 90°.



Right prism: a solid object with a uniform cross-section and flat sides at right angles to the base

Sample: part of a population chosen so as to give information about the population as a whole

Sample space: *see* **Event space**

Sampling: obtaining data from a small group of subjects (often people) within a larger population. This smaller group should be representative of the larger population.

Scale factor: the ratio of the corresponding sides in similar figures, where the enlarged (or reduced) figure is referred to as the image and the original figure is called the object. scale factor $= \dfrac{\text{image length}}{\text{object length}}$

Scalene triangle: a triangle with no two sides equal in length

Scatter plot: a graphical representation of bivariate data that displays the degree of correlation between two variables. Each piece of data on a scatterplot is shown by a point. The x-coordinate of this point is the value of the independent variable and the y-coordinate is the corresponding value of the dependent variable.

Scientific notation: a method of expressing a number as the product of a power of 10, and a decimal that has just one digit to the left of the decimal point; for example, 54267 would be written as 5.4267×10^4 and 0.0054267 as 5.4267×10^{-3}.

Seasonal pattern: a pattern that displays fluctuations that repeat at the same time over a particular time interval (such as a week, month or quarter) and usually last less than a year

Secant: a chord of a circle that is extended beyond the circumference on one side

Second Index Law: when terms with the same base are divided, the indices are subtracted. So, $a^m \div a^n = a^{m-n}$.

Secondary data: data collected by others

Sector: part of a circle bounded by two radii and an arc

Sector graph: a type of graph mostly used to represent categorical data. A circle is used to represent all the data, with each category being represented by a sector of the circle, whose size is proportional to the size of that category compared to the total.

Segment: a region of a circle between a chord and the circumference. The smaller segment is called the minor segment and the larger one is called the major segment.

Semicircle: part (half) of a circle bounded by a diameter and an arc joining the ends of the diameter

Set: a collection of similar elements

Seventh Index Law: a term with a negative index can be expressed with a positive index using this law. So, $a^{-n} = \dfrac{1}{a^n}$ and $\dfrac{1}{a^{-n}} = a^n$.

Similar figures: figures that have identical shape but different size. The corresponding angles in similar figures are equal in size, and the corresponding sides are in the same ratio, called a scale factor.

Similar triangles: triangles that have identical shape but different size. There are four standard tests to determine whether two triangles are similar: AAA (angle, angle, angle), SAS (side, angle, side), SSS (side, side, side) and RHS (right angle, hypotenuse side).

Simple interest: the interest accumulated when the interest payment in each period is a fixed fraction of the principal. The formula used is $I = \frac{P \times r \times T}{100}$, where I is the interest earned (in \$) when a principal of \$$P$ is invested at an interest rate of r% p.a. for a period of T years.

Simple random sampling: a survey that ensures all subjects have an equal chance of inclusion in the sample

Simplify: to write an expression in its simplest form by the use of algebraic or arithmetical techniques

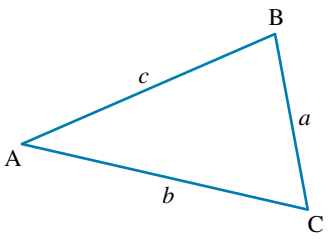
Simultaneous: occurring at the same time

Simultaneous equations (linear): two (or more) linear graphs that have the same solution

Sine (sin) ratio: the ratio of the opposite side to the hypotenuse in a right-angled triangle.

So, $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$.

Sine rule: in any triangle ABC, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

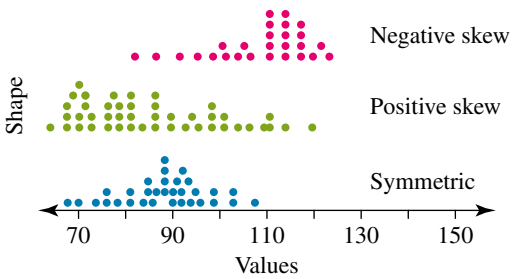


Single variable data: data where each piece is represented by one variable

Sixth Index Law: to remove brackets containing a fraction, multiply the indices of both the numerator and denominator by the index outside the brackets. So, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.

Sketch: the drawing of a graph highlighting its special features; for example, the y-intercept and gradient or the x- and y-intercepts of a straight line. Sketches are not drawn on graph paper and scales are not shown along the axes.

Skewed: if a distribution's shape is not symmetric, it can be described as being positively skewed (tailing off to the upper end of the distribution) or negatively skewed (tailing off to the lower end of the distribution).



Slant height (of cone): the distance from any point on the circumference of the circular base of the cone to the vertex of the cone

Solid: a 3-dimensional object

Speed: the rate that describes how quickly distance changes over a period of time. It is given by the formula speed $= \frac{\text{distance}}{\text{time}}$.

Sphere: a solid for which all points on the surface are equidistant from the centre

Standard deviation: a measure of the variability of spread of a data set. It gives an indication of the degree to which the individual data values are spread around the mean. It is given by the formula $\sigma = \sqrt{\frac{\Sigma(x - \bar{x})^2}{n}}$, where $\bar{x}$ is the mean of the data values and n is the number of data values.

Stem-and-leaf plot: a display that provides simultaneously a rank order of individual scores and the shape of the distribution. The stem is used to group the scores and the leaves indicate the individual scores within each group. The stem-and-leaf plot for the pulse rates of a group of students is shown.

	Pulse rate
6	8 8 8 9
7	0 1 1 4 6 6 8
8	2 6 8 8
9	0 6
10	4
11	0

A back-to-back stem-and-leaf plot has two sets of data displayed — one on either side of the common stem.

Strength (correlation): an indication of how closely the points on a scatterplot fit a straight line

Strong (correlation): describes a relationship in which the points on a scatterplot are close together and show a definite linear pattern

Subset: a smaller set within another set. It is denoted by the symbol $\subset$.

Substitution: the replacement of a variable by a number. For example, substituting $x = 2$ in the expression $5x$ gives $5 \times 2 = 10$.

Substitution method: a method used to solve simultaneous equations. It is useful when one (or both) of the equations has one of the variables as the subject.

Subtended (angle): an angle standing on an arc of a circle, with its vertex on the circumference or at the centre of the circle

Supplementary angles: angles that add to 180°

Summary statistics: measures such as mean, mode, median and range, used in analysing a set of data

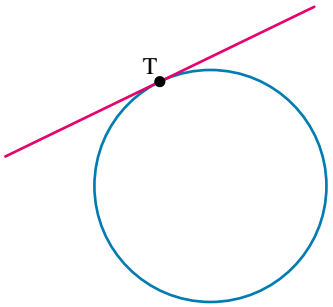
Surds: roots of numbers that do not have an exact answer, so they are irrational numbers. Surds themselves are exact numbers; for example, $\sqrt{6}$ or $\sqrt[3]{5}$.

Symmetrical (object): the identical size, shape and arrangement of parts of an object on opposite sides of a line or plane

Symmetrical (data): showing equal amounts of data on either side of a central point

System of equations: a set of two or more equations with the same variables

Tangent (to a circle): a straight line that touches the circumference of a circle or a curve at one point only



Tangent (tan) ratio: the ratio of the opposite side to the adjacent side in a right-angled triangle.

So, $\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$.

Terminating decimals: decimals that have a fixed number of places; for example, 0.6 and 2.54

Theorem: a statement that can be demonstrated to be true

Theoretical probability: given by the rule $\text{P(event)} = \dfrac{\text{number of favourable outcomes}}{\text{number of possible outcomes}}$ or $\text{P}(E) = \dfrac{n(E)}{n(S)}$, where $n(E) =$ number of times or ways an event, E, can occur and $n(S) =$ number of elements in the sample space or number of ways all outcomes can occur, given all the outcomes are equally likely

Third Index Law: any term (excluding 0) with an index of 0 is equal to 1. So, $a^0 = 1$.

Three dimensions: length, width and height; the spatial dimensions formed by three lines at right angles to each other

3-dimensional: a shape that occupies space (a solid). That is, one that has dimensions in three directions — length, width and height.

Time series: a sequence of measurements taken at regular intervals (daily, weekly, monthly and so on) over a certain period of time. They are used for analysing general trends and making predictions for the future.

Total surface area (TSA): the area of the outside surface of a 3-dimensional figure

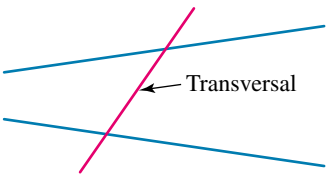
Transcendental number: a non-recurring decimal that is not the root of any polynomial equation with rational coefficients. A transcendental number is an irrational number but not an algebraic number. Pi (π) is an example of a transcendental number.

Transformations [quadratics]: changes that occur to the basic parabola $y = x^2$ in order to obtain another graph. Examples of transformations are translations, reflections or dilations. Transformations can also be applied to non-quadratic functions.

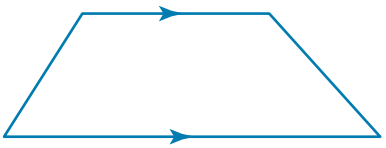
Translated [quadratics]: moving a parabola horizontally (left/right) or vertically (up/down)

Transposing: changing the order of terms in an equation

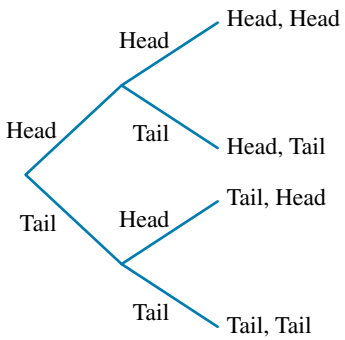
Transversal: a line that meets two or more other lines in a plane



Trapezium: a quadrilateral with only one pair of parallel sides



Tree diagrams: branching diagrams that list all the possible outcomes of a probability experiment. This diagram shows the outcomes when a coin is tossed twice.



Trend: a general tendency in a set of data

Trend line: the line of best fit that is drawn on a time series graph, which is used to forecast future values

Trial: the number of times a probability experiment is conducted

Triangle: a 3-sided polygon

Trigonometric ratios: three different ratios of one side of a triangle to another. The three ratios are the sine, cosine and tangent ratios.

Trinomial: an expression consisting of three terms; for example, $x^2 + 3x - 5$

True bearings: directions that are written as the number of degrees (3 digits) from north in a clockwise direction, followed by the word *true* or *T*; for example, due east would be 090° true or 090°T

Truncated cone: a cone with its top cut off

Turning point: a point at which a graph changes direction (either up or down). For a parabola, *see* **Vertex**.

2-dimensional: a description of a plane shape. The dimensions are given in two directions, such as length and width or length and height.

Two-step chance experiment: a probability experiment that involves two trials

Two-way tables: a table that lists all the possible outcomes of a probability experiment in a logical manner

Hair colour	Hair type		Total
Red	1	1	2
Brown	8	4	12
Blonde	1	3	4
Black	7	2	9
Total	17	10	27

Uniform cross-section: a solid has a uniform cross-section if cross-sections taken parallel to its base are always the same size and shape. Cross-sections parallel to the base of prisms are uniform, whereas cross-sections parallel to the base of pyramids are not.

Union (of sets): represents the combination of elements of two or more sets. $A \cup B$ denotes the union of sets A and B.

Unit circle: a circle with its centre at the origin and having a radius of 1 unit

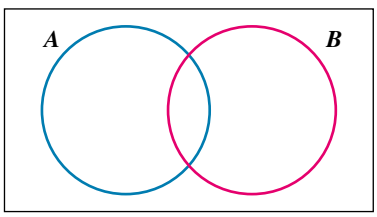
Unit fraction: a fraction that has a numerator of one; for example, $\frac{1}{2}$, $\frac{1}{5}$, $\frac{1}{10}$

Univariate data: data relating to a single variable

Universal set (ξ): the largest set that contains all possible elements of the data considered

Upper quartile: the score that marks the end of the third quarter in an ordered set of data. It is denoted by Q_U or Q_3. It is calculated by finding the median of the upper half of the data.

Venn diagrams: a series of circles, representing sets, within a rectangle, which represents the universal set. They show the relationships between the sets.



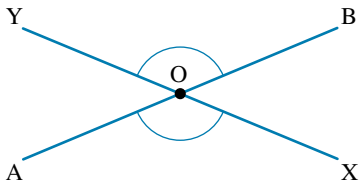
Vertex: plural: *vertices*; the point at which the graph of a quadratic function (parabola) changes direction (either up or down)

Vertex form: the equation of a parabola in the form $y = (x - h)^2 + k$ gives the vertex, (h, k).

Vertical: vertical lines are parallel to the y-axis and have an undefined (infinite) gradient.

Vertical line test (function): the graph of a function cannot be crossed more than once by any vertical line.

Vertically opposite angles: when two lines intersect, four angles are formed at the point of intersection, and two pairs of vertically opposite angles result. Vertically opposite angles are equal.



Volume: the amount of space a 3-dimensional object occupies. The units used are cubic units, such as cubic centimetres (cm^3) and cubic metres (m^3).

Weak (correlation): describes a relationship in which the points on a scatterplot are far apart

***x*-intercept:** the point where a graph intersects the x-axis

***y*-intercept:** the point where a graph intersects the y-axis. In the equation of a straight line, $y = mx + b$, the constant term, b, represents the y-intercept of that line.

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