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NELSON QMATHS

for the Australian Curriculum

Stephen Swift | Ross Brodie | Stephen Corcoran | Sue Garner

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Nelson QMaths for the Australian Curriculum 10+10A
1st Edition
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Publishing editor: George Hook
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Senior designers: Kar Heng Goh and Sarah Anderson
Permissions researcher: Corrina Tauschke
Cover images: Robert Essel NYC/Corbis (right); Shutterstock.com/Martin Kraft (left)
Back cover: Steve Parish/Steve Parish Publishing/Corbis
Production controller: Erin Dowling
Typeset by: Cenveo Publisher Services

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National Library of Australia Cataloguing-in-Publication Data
Nelson QMaths for the Australian Curriculum 10+10A /
Stephen Swift ... [et al.]

9780170194860 (pbk.)
Includes index.
For secondary school age.

Mathematics--Textbooks.
Mathematics--Study and teaching (Secondary)

Swift, Stephen.

510

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Printed in China by China Translation & Printing Services.
1 2 3 4 5 6 7 16 15 14 13 12

Stephen Swift started teaching Mathematics, Science and Computing in 1973 and has taught at all levels from Years 7 to 12 in several states, in urban and country schools until retiring in 2010 from the role of Mathematics Head of Department at Wellington Point State High School in Brisbane.

Ross Brodie has worked as a classroom teacher and Mathematics Head of Department at a number of regional and metropolitan secondary schools. He has taught Mathematics at Years 8 to 12. Ross brings a wide range of experience from education and other sectors to the writing of mathematics student books.

Stephen Corcoran is an experienced teacher who has been Head of Mathematics in four schools. He is very familiar with current trends in mathematics education. He received a National Excellence in Teaching award for inspiring students and encouraging their abilities through innovative mathematics programs.

Sue Garner is assistant Head of Mathematics at Ballarat Grammar, specialising in both Mathematical Methods (CAS) and Specialist Mathematics. Sue has also assessed senior examinations in both Mathematical Methods (CAS) and Specialist Mathematics, and has taught in secondary schools in Melbourne, Ballarat and the UK. She has published extensively in Mathematics and has lectured at The University of Ballarat and at The University of Melbourne.

The **Digital Resources Team** includes Sue Garner, Alan Preston, Allason MacNamara, Dawn Fernandez, Francesca Moloney, Frank Moya, Greg Neal, Ian Bull, Roger Walter and Tobias Cooper.

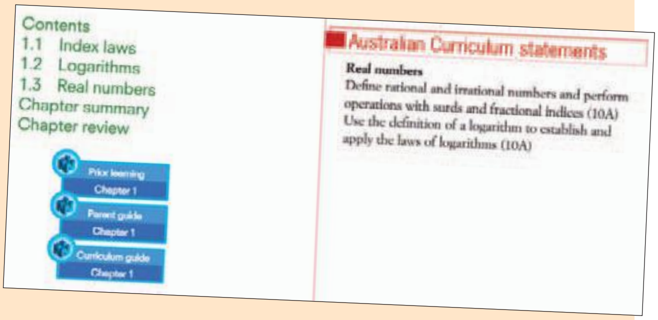
Contents

7 Advanced algebra and functions 310

About this book

At the beginning of each chapter

- Clear outline of chapter contents
- Strong links to the curriculum
- Prior learning provides a quick check of understanding for the individualisation of instructional sequence

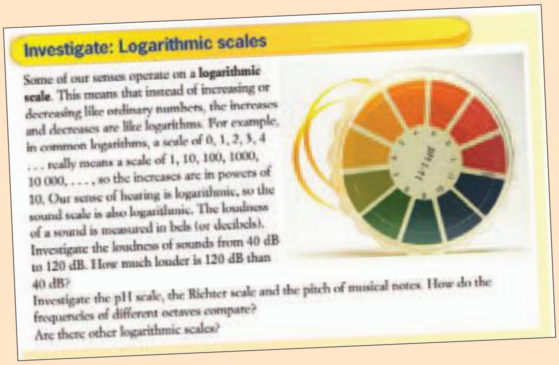


Contents
1.1 Index laws
1.2 Logarithms
1.3 Real numbers
Chapter summary
Chapter review

Prior learning Chapter 1
Parent guide Chapter 1
Curriculum guide Chapter 1

Australian Curriculum statements

Real numbers
Define rational and irrational numbers and perform operations with surds and fractional indices (10A)
Use the definition of a logarithm to establish and apply the laws of logarithms (10A)



Investigate: Logarithmic scales

Some of our senses operate on a **logarithmic scale**. This means that instead of increasing or decreasing like ordinary numbers, the increases and decreases are like logarithms. For example, in common logarithms, a scale of 0, 1, 2, 3, 4 . . . really means a scale of 1, 10, 100, 1000, 10 000, . . . , so the increases are in powers of 10. Our sense of hearing is logarithmic, so the sound scale is also logarithmic. The loudness of a sound is measured in bels (or decibels). Investigate the loudness of sounds from 40 dB to 120 dB. How much louder is 120 dB than 40 dB?
Investigate the pH scale, the Richter scale and the pitch of musical notes. How do the frequencies of different octaves compare?
Are there other logarithmic scales?

- Parent guide provides vital information for parents
- Curriculum guide provides a rationale in words that students will understand
- Clear sectioning that is consistent with the exercises
- Many sections begin with a concrete investigation, with teacher support materials provided
- Intriguing aids for students provided

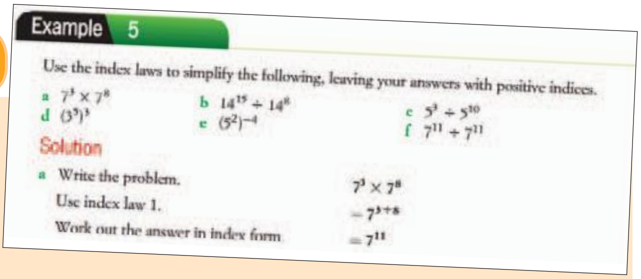
In each chapter

- Examples are clearly set out with explanations on the left and expected student answers in bold on the right
- Colour-coded examples make it easy for teachers to direct students

  Green – nearly all students will master

  Gold – most students will master

  Blue – some students should master
- Examples are sequenced in a natural learning progression that promotes chunking and effective use of working memory
- Clear examples enable parents to assist their student with confidence that they are 'doing it the way the teacher explained it'
- Important boxes highlight key terms and concepts



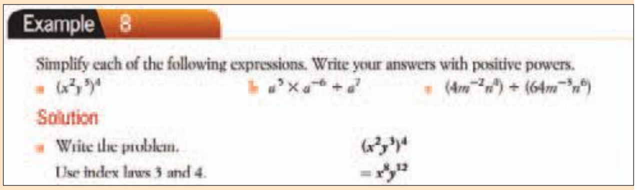
Example 5

Use the index laws to simplify the following, leaving your answers with positive indices.

a $7^3 \times 7^8$ b $14^{15} \div 14^8$ c $5^3 \div 5^{10}$
d $(3^5)^3$ e $(5^2)^{-4}$ f $7^{11} \div 7^{11}$

Solution

a Write the problem. $7^3 \times 7^8$
Use index law 1. $= 7^{3+8}$
Work out the answer in index form $= 7^{11}$



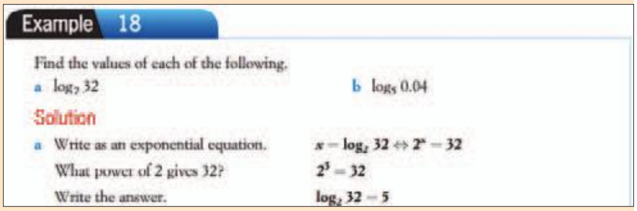
Example 8

Simplify each of the following expressions. Write your answers with positive powers.

a $(x^2y^3)^4$ b $a^5 \times a^{-6} \div a^7$ c $(4m^{-2}n^8) \div (64m^{-5}n^6)$

Solution

a Write the problem. $(x^2y^3)^4$
Use index laws 3 and 4. $= x^8y^{12}$



Example 18

Find the values of each of the following.

a $\log_2 32$ b $\log_5 0.04$

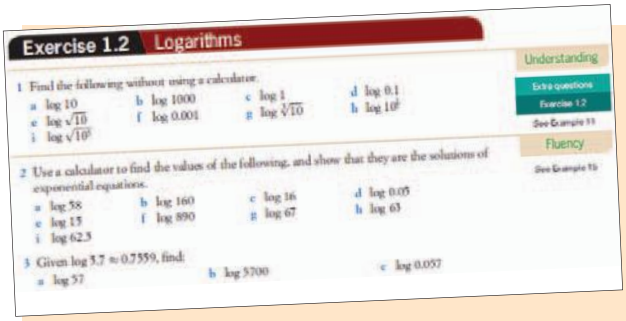
Solution

a Write as an exponential equation. $x = \log_2 32 \Leftrightarrow 2^x = 32$
What power of 2 gives 32? $2^5 = 32$
Write the answer. $\log_2 32 = 5$

Important!

Change of base theorem

For $b, c \neq 0$, $\log_b x = \dfrac{\log_c x}{\log_c b}$



Exercise 1.2 Logarithms

Understanding

Extra questions Exercise 1.2

See Example 11

1 Find the following without using a calculator.

a log 10 b log 1000 c log 1 d log 0.1

e log $\sqrt{10}$ f log 0.001 g log $\sqrt[3]{10}$ h log 10^2

i log $\sqrt{10^3}$

Fluency

See Example 15

2 Use a calculator to find the values of the following, and show that they are the solutions of exponential equations.

a log 58 b log 160 c log 16 d log 0.05

e log 15 f log 890 g log 67 h log 63

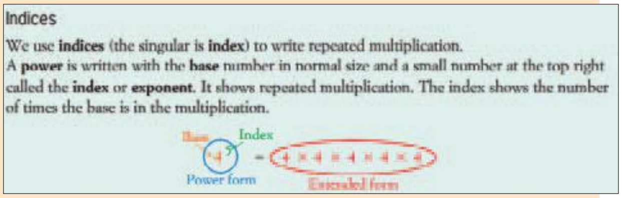
i log 62.5

3 Given log 5.7 ≈ 0.7559, find:

a log 57 b log 5700 c log 0.057

- Initial exercise questions focus on understanding or fluency and refer directly back to examples
- Questions are categorised according to the four proficiency strands – understanding, fluency, problem solving and reasoning
- Questions and parts are colour coded to be consistent with the examples

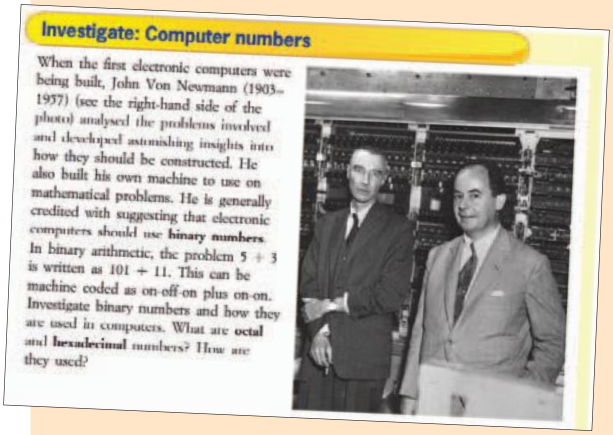
- Colour is used pedagogically to enhance learning



Indices

We use **indices** (the singular is **index**) to write repeated multiplication.
A **power** is written with the **base** number in normal size and a small number at the top right called the **index** or **exponent**. It shows repeated multiplication. The index shows the number of times the base is in the multiplication.

$4^5 = 4 \times 4 \times 4 \times 4 \times 4$

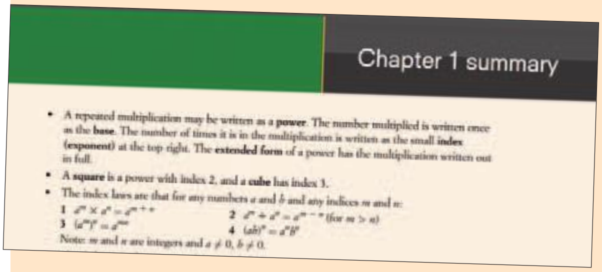


Investigate: Computer numbers

When the first electronic computers were being built, John Von Newmann (1903–1957) (see the right-hand side of the photo) analysed the problems involved and developed astonishing insights into how they should be constructed. He also built his own machine to use on mathematical problems. He is generally credited with suggesting that electronic computers should use **binary numbers**. In binary arithmetic, the problem 5 + 3 is written as 101 + 11. This can be machine coded as on-off-on plus on-on. Investigate binary numbers and how they are used in computers. What are **octal** and **hexadecimal** numbers? How are they used?

- There is an abundance of carefully designed investigations proven to aid deep understanding

At the end of each chapter



Chapter 1 summary

- A repeated multiplication may be written as a **power**. The number multiplied is written once as the **base**. The number of times it is in the multiplication is written as the small **index (exponent)** at the top right. The **extended form** of a power has the multiplication written out in full.
- A **square** is a power with index 2, and a **cube** has index 3.
- The index laws are that for any numbers a and b and any indices m and n:

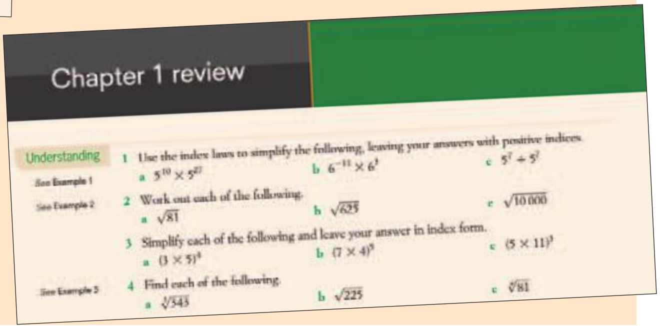
1 $a^m \times a^n = a^{m+n}$ 2 $a^m \div a^n = a^{m-n}$ (for $m > n$)

3 $(a^m)^n = a^{mn}$ 4 $(ab)^n = a^n b^n$

Note: m and n are integers and $a \neq 0$, $b \neq 0$.

- A chapter summary that re-states all important concepts

- A comprehensive review set out with example references and consistent with exercise layout
- Answers to all questions are given at the back of the book



Chapter 1 review

Understanding

See Example 1

1 Use the index laws to simplify the following, leaving your answers with positive indices.

a $5^{10} \times 5^{27}$ b $6^{-11} \times 6^3$ c $5^7 \div 5^7$

See Example 2

2 Work out each of the following.

a $\sqrt{81}$ b $\sqrt{625}$ c $\sqrt{10\,000}$

3 Simplify each of the following and leave your answer in index form.

a $(3 \times 5)^4$ b $(7 \times 4)^5$ c $(5 \times 11)^3$

See Example 3

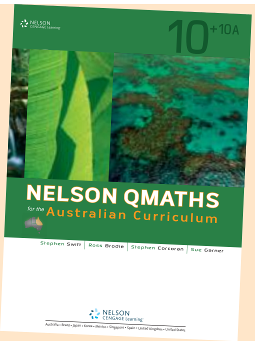
4 Find each of the following.

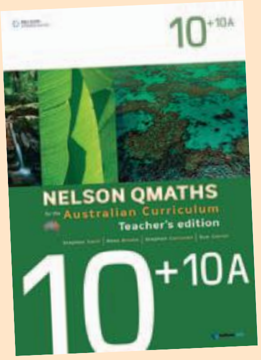
a $\sqrt[3]{343}$ b $\sqrt{225}$ c $\sqrt[4]{81}$

About the series

Other books in the series

- There are five books in this Years 7 to 10 series, which has been written specifically for the Australian Mathematics Curriculum, including two books for Year 10 (see back cover).
- There is also a printed Teacher's Edition of each textbook with additional guidance and advice on implementing the Australian Mathematics Curriculum in the classroom.
- Accompanying each printed textbook is a digital textbook called the NelsonNetBook (see opposite page).





Icons in the textbook

- In the outside margins of your textbook you will find numerous icons for resources that can be accessed through the NelsonNet website or through the NelsonNetBook.

Alternative method Moving the decimal	Explains different ways of solving a maths problem	Puzzle sheet Find the number	Makes learning maths methods and skills fun
Animated example Stem-and-leaf plots	Shows how to tackle difficult problems	Quiz Expressions	Tests and marks your knowledge of a topic
CAS calculator exercise Quadratic equations	Develops your CAS calculator skills	Teacher notes Restaurant tables	Gives guidance for your teacher
Curriculum guide Chapter 6	Explains what you need to know and be able to do	Technology GeoGebra: Exterior angle of a triangle	Provides a spreadsheet or GeoGebra activity
Extra questions Exercise 6.1	Allows you to do additional practice and drill	TLF learning object Exploring order of operations (L6543)	Provides interactive ways of learning about maths*
Maths clip Statistics	Gives the 'big picture' view of your maths topic	Video tutorial Algebraic expressions	Fully explains maths methods and skills
Maths dictionary	Provides an illustrated explanations of all terms	Weblink Algebra masterclass	Links to a website that makes maths relevant
Parent guide Chapter 6	Provides guidance for your parents to help you	Worked solutions Exercise 6.1	Gives solutions steps for selected problems
Prior learning Chapter 6	Finds out what you already know and can do	Worksheet Writing a rule	Supplies tasks to be done digitally or by using a pen

* Your teacher will explain how to access The Learning Federation learning objects.

NelsonNet Website

- Go to www.nelsonnet.com.au to log in or find out more.
- The first screen you will see on the student website is called 'Chapter resources'.
- Click on a chapter and a list of different resources types will appear.
- Click on a resource type and a list of specific resources will pop up.
- Clicking on one of those resources will open a pdf file, start a video, or some special maths software.
- Use the blue tabs on the left of the screen to access calculator resources and the maths dictionary.

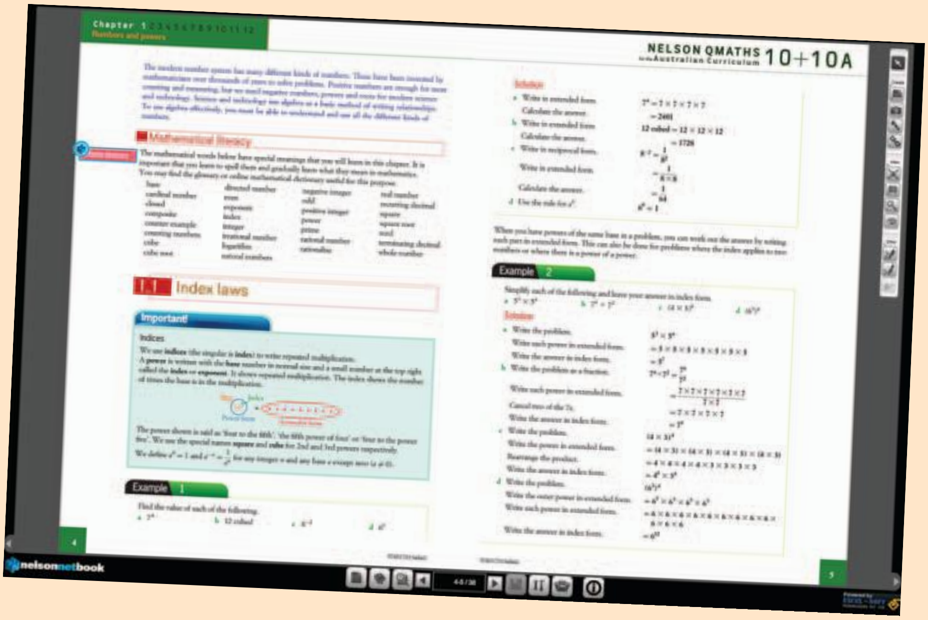
Nelson QMaths for the Australian Curriculum Year 9

Chapter resources

- Chapter 1: Statistics
- Chapter 2: Number and indices
- Chapter 3: Measurement
  - Animated examples (1)
  - CAS ClassPad Exercises (1)
  - CAS TI-Nspire Exercises (1)
  - Curriculum guides (1)
  - Extra questions (4)
  - Learning Federation objects (1)
  - Parent guides (1)
  - Prior learning (1)
  - Puzzle sheets (3)
  - Quizzes (1)
  - Teaching notes (1)
  - Technology (1)
  - Technology worksheets (2)
  - Video tutorials (6)
  - Weblinks (1)
  - Worked solutions (4)
  - Worksheets (15)
- Chapter 4: Algebra
- Chapter 5: Ratio and proportion
- Chapter 6: Congruence and similarity
- Chapter 7: Probability
- Chapter 8: Equations
- Chapter 9: Trigonometry
- Chapter 10: Coordinates and graphs
- Chapter 11: Angles and shapes
- Chapter 12: Money and business

NelsonNetBook

- This is a web-based ebook that you can customise to meet your own learning needs.
- The icons with the blue NelsonNet logo are 'hotspots'. Click on an icon to open that resource.
- The tools on the vertical toolbar allow you to personalise pages in a variety of ways, including voice recordings, drawings and links to favourite websites. They also enable you to zoom in and out.
- The tools on the horizontal toolbar allow you to navigate around your ebook and change settings.
- Administration tools allow your teacher to manage classes and share annotations with you.



9780170194860

Acknowledgements

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Metric system

Symbols

millimetre — mm
centimetre — cm
metre — m
kilometre — km
hectare — ha
millilitre — mL
litre — L
kilolitre — kL
megalitre — ML
milligram — mg
gram — g
kilogram — kg
tonne — t

Length

10 mm = 1 cm
100 cm = 1 m
1000 m = 1 km

Area

100 mm^2 = 1 cm^2
10 000 cm^2 = 1 m^2
10 000 m^2 = 1 ha
100 ha = 1 km^2

Capacity

1000 mL = 1 L
1000 L = 1 kL
1000 kL = 1 ML

Mass

1000 mg = 1 g
1000 g = 1 kg
1000 kg = 1 t

Symbols and abbreviations

∴	therefore
=	is equal to
≠	is not equal to
≈, ∼, ≑, ≃	is approximately equal to
>	is greater than
≯	is not greater than
≥	is greater than or equal to
<	is less than
≮	is not less than
≤	is less than or equal to
∝	is proportional to
$5.3\overline{7214}$ or $5.3\dot{7}21\dot{4}$	shows 5.3721472147214 …
1 : 100	Scale of 1 to 100
BC	before Christ
BCE	before the common era
AD	Anno Domini
CE	common era
QED	*quod erat demonstrandum*
RTP	Required to prove
LHS	Left-hand side
RHS	Right-hand side
log	common logarithm
$\log_b x$	logarithm of x to the base b
$\sum$	the sum of
$\triangle ABC$	triangle ABC
$\angle ABC$	angle ABC
≡, ≅	is congruent to
⦀, ∼	is similar to
$A \leftrightarrow P$	A corresponds to P
	right angle
	supplementary angles
	angles at a point
	angles in a triangle
	angles in a quadrilateral
	angles in an isosceles triangle
	alternate angles
∥	is parallel to

⊥	is perpendicular to
%	percentage
$\frac{a}{b}$	fraction, division $a \div b$ or ratio $a : b$
±	plus or minus
$\sqrt{\ }$	square root
$\sqrt[3]{\ }$	cube root
$\sqrt[n]{\ }$	nth root
45°	45 degrees
$\bar{x}$	mean of sample, average
μ	mean of population
Q_1, $Q1$	first quartile
Q_3, $Q3$	third quartile
σ, σ_n	standard deviation of population
IQR	interquartile range
s, σ_{n-1}	standard deviation of sample
m	gradient
c	y-intercept
π	pi ≈ 3.141592653589793238 …
{}, Ø	null set, empty set
⊆	subset of
⊂	proper subset of
∈	is a member of
∉	is not a member of
∪	union of sets
∩	intersection of sets
⇒	it follows that
sin θ	sine ratio of angle θ
tan θ	tangent ratio of angle θ
cos θ	cosine ratio of angle θ
$P(A \mid B)$	conditional probability of A given B
	complementary angles
	vertically opposite angles
	exterior angle of a triangle
	opposite angles in a parallelogram
	corresponding angles
	co-interior (allied) angles

Greek alphabet

A, α alpha
B, β beta
Γ, γ gamma
Δ, δ delta
E, ε epsilon
Z, ζ zeta
H, η eta
Θ, θ theta
I, ι iota
K, κ kappa
Λ, λ lambda
M, μ mu
N, ν nu
Ξ, ξ xi
O, ο omicron
Π, π pi
P, ρ rho
Σ, σ sigma
T, τ tau
Y, υ upsilon
Φ, ϕ phi
X, χ chi
Ψ, ψ psi
Ω, ω omega

Number and algebra

1

Numbers and powers

Prior learning
Chapter 1
MAT10NAPL00012

Parent guide
Chapter 1
MAT10NAPG00012

Curriculum guide
Chapter 1
MAT10NACU00012

Australian Curriculum statements

Real numbers
Define rational and irrational numbers and perform operations with surds and fractional indices. (10A)
Use the definition of a logarithm to establish and apply the laws of logarithms. (10A)

Patterns and algebra
Simplify algebraic products and quotients using index laws. (10)

Video tutorial
Numbers and powers
MAT10NAVT00001

Weblink
Scale of the Universe
MAT10NAWB00001

The modern number system has many different kinds of numbers. These have been invented by mathematicians over thousands of years to solve problems. Positive numbers are enough for most counting and measuring, but we need negative numbers, powers and roots for modern science and technology. Science and technology use algebra as a basic method of writing relationships. To use algebra effectively, you must be able to understand and use all the different kinds of numbers.

Mathematical literacy

Maths dictionary
MAT10ASDI00001

The mathematical words below have special meanings that you will learn in this chapter. It is important that you learn to spell them and gradually learn what they mean in mathematics. You may find the glossary or online mathematical dictionary useful for this purpose.

base
cardinal number
closed
composite
counter example
counting numbers
cube
cube root
directed number
even
exponent
index
integer
irrational number
logarithm
natural numbers
negative integer
odd
positive integer
power
prime
rational number
rationalise
real number
recurring decimal
square
square root
surd
terminating decimal
whole number

1.1 Index laws

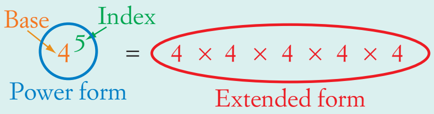
Weblink
Math is fun: Exponents
MAT10NAWB00001

TLF Learning object
Exploring powers of 10 (L6548)
MAT10NAIN00001

Important!

Indices

We use **indices** (the singular is **index**) to write repeated multiplication.
A **power** is written with the **base** number in normal size and a small number at the top right called the **index** or **exponent**. It shows repeated multiplication. The index shows the number of times the base is in the multiplication.



$$4^5 = 4 \times 4 \times 4 \times 4 \times 4$$

The power shown is said as ‘four to the fifth’, ‘the fifth power of four’ or ‘four to the power five’. We use the special names **square** and **cube** for 2nd and 3rd powers respectively.

We define $a^0 = 1$ and $a^{-n} = \dfrac{1}{a^n}$ for any integer n and any base a except zero ($a \neq 0$).

Example 1

Find the value of each of the following.

a 7^4 **b** 12 cubed **c** 8^{-2} **d** 6^0

Solution

a Write in extended form. $7^4 = 7 \times 7 \times 7 \times 7$

Calculate the answer. $= 2401$

b Write in extended form. $12 \text{ cubed} = 12 \times 12 \times 12$

Calculate the answer. $= 1728$

c Write in reciprocal form. $8^{-2} = \frac{1}{8^2}$

Write in extended form. $= \frac{1}{8 \times 8}$

Calculate the answer. $= \frac{1}{64}$

d Use the rule for a^0. $6^0 = 1$

When you have powers of the same base in a problem, you can work out the answer by writing each part in extended form. This can also be done for problems where the index applies to two numbers or where there is a power of a power.

Example 2

Simplify each of the following and leave your answer in index form.

a $5^3 \times 5^4$ **b** $7^6 \div 7^2$ **c** $(4 \times 3)^4$ **d** $(6^3)^4$

Solution

a Write the problem. $5^3 \times 5^4$

Write each power in extended form. $= 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5$

Write the answer in index form. $= 5^7$

b Write the problem as a fraction. $7^6 \div 7^2 = \frac{7^6}{7^2}$

Write each power in extended form. $= \frac{7 \times 7 \times 7 \times 7 \times \cancel{7} \times \cancel{7}}{\cancel{7} \times \cancel{7}}$

Cancel two of the 7s. $= 7 \times 7 \times 7 \times 7$

Write the answer in index form. $= 7^4$

c Write the problem. $(4 \times 3)^4$

Write the power in extended form. $= (4 \times 3) \times (4 \times 3) \times (4 \times 3) \times (4 \times 3)$

Rearrange the product. $= 4 \times 4 \times 4 \times 4 \times 3 \times 3 \times 3 \times 3$

Write the answer in index form. $= 4^4 \times 3^4$

d Write the problem. $(6^3)^4$

Write the outer power in extended form. $= 6^3 \times 6^3 \times 6^3 \times 6^3$

Write each power in extended form. $= 6 \times 6 \times 6 \times 6 \times 6 \times 6 \times 6 \times 6 \times 6 \times 6 \times 6 \times 6$

Write the answer in index form. $= 6^{12}$

You can prove the index laws from the definition of an index.

Important!

Index laws

For any bases a and b and any indices m and n:

1 $a^m \times a^n = a^{m+n}$
2 $a^m \div a^n = a^{m-n}$ (for $m > n$)
3 $(a^m)^n = a^{mn}$
4 $(ab)^n = a^n b^n$

Note: m and n are integers and $a \neq 0$, $b \neq 0$.

The opposite of finding a power is finding a **root**.

Important!

Roots

The **surd** symbol $\sqrt{\ }$ is used to show roots. The nth root of a number x is shown as $\sqrt[n]{x}$ and is the number y so that $y^n = x$. For example, $\sqrt[4]{16} = 2$ because $2^4 = 16$.
The **square root** ($\sqrt[2]{\ }$) is normally written without the index, so $\sqrt{25} = 5$.
The **cube root** ($\sqrt[3]{\ }$) is the opposite of finding a cube, so $(\sqrt[3]{216}) = 6$.

We define $\sqrt[n]{a} = a^{\frac{1}{n}}$ for any positive integer n.

It makes sense to write a root as a fractional power, because of the index law that $(a^m)^n = a^{mn}$.
This works for $\sqrt[4]{16} = 2$ as $(16^{\frac{1}{4}})^4 = 16^{\frac{1}{4}\times 4} = 16^1 = 16$, and $2^4 = 16$ so $\sqrt[4]{16} = 16^{\frac{1}{4}}$.

Puzzle sheet
Indices squaresaw
MAT10NAPS00002

Example 3

Find each of the following.

a $\sqrt{121}$ **b** $\sqrt[5]{1024}$ **c** $81^{\frac{1}{2}}$ **d** $32^{\frac{1}{5}}$

Solution

a Think of the square that gives 121. $\mathbf{11^2 = 121}$

Write the answer. $\mathbf{\sqrt{121} = 11}$

b Try some possibilities. $\mathbf{3^5 = 3 \times 3 \times 3 \times 3 \times 3 = 243}$

$\mathbf{4^5 = 4 \times 4 \times 4 \times 4 \times 4 = 1024}$

Write the answer. $\mathbf{\sqrt[5]{1024} = 4}$

c Write in surd form. $81^{\frac{1}{2}} = \sqrt{81}$

Think of the square that gives 81. $9^2 = 81$

Write the answer. $81^{\frac{1}{2}} = 9$

d Try a possibility. $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$

Write the answer. $32^{\frac{1}{5}} = 2$

The power key on your calculator could be shown as $x^{\blacksquare}$, a^b or y^x. The *n*th root could be shown as $\sqrt[n]{\ }$, $\sqrt[x]{\ }$ or $x^{1/n}$. Many calculators also have square and square root keys shown as x^2 and $\sqrt{\ }$. In most cases, roots are obtained using the 2nd function key.
The order in which you need to enter powers may vary between calculators.

Example 4

Use a calculator to find:

a 16^3 **b** $\sqrt{127}$ **c** 7^{-3} **d** $\sqrt[4]{500}$ **e** $5^{1.7}$

Solution

a Enter as 16 y^x 3 =

16³ 4096

Write the answer. $16^3 = 4096$

b Enter as $\sqrt{\ }$ 127 =

√127 11.26942767

Round and write the answer. $\sqrt{127} \approx 11.269$

c Enter as 7 y^x (–) 3 =

7^{-3} 0.00292

Round and write the answer. $7^{-3} \approx 0.002\,92$

d Enter as 4 $\sqrt[x]{\ }$ 500 =

$\sqrt[4]{500}$ 4.728708045

Round and write the answer. $\sqrt[4]{500} \approx 4.729$

e Enter as 5 y^x 1.7 =

$5^{1.7}$ 15.42584657

Round and write the answer. $5^{1.7} \approx 15.426$

Worksheet
Numbers and indices
MAT10NAWK00001

Puzzle sheet
Powers and roots
MAT10NAPS00001

Example 5

Use the index laws to simplify the following, leaving your answers with positive indices.

a $7^3 \times 7^8$ **b** $14^{15} \div 14^8$ **c** $5^3 \div 5^{10}$
d $(3^5)^3$ **e** $(5^2)^{-4}$ **f** $7^{11} \div 7^{11}$

Solution

a Write the problem. $7^3 \times 7^8$
Use index law 1. $= 7^{3+8}$
Work out the answer in index form. $= 7^{11}$

b Write the problem. $14^{15} \div 14^8$
Use index law 2. $= 14^{15-8}$
Work out the answer in index form. $= 14^7$

c Write the problem. $5^3 \div 5^{10}$
Use index law 2. $= 5^{3-10}$
Work out the answer in index form. $= 5^{-7}$
Change to a positive index. $= \frac{1}{5^7}$

d Write the problem. $(3^5)^3$
Use index law 3. $= 3^{5\times 3}$
Work out the answer in index form. $= 3^{15}$

e Write the problem. $(5^2)^{-4}$
Use index law 3. $= 5^{2\times -4}$
Work out the answer in index form. $= 5^{-8}$
Change to a positive index. $= \frac{1}{5^8}$

f Write the problem. $7^{11} \div 7^{11}$
Use index law 2. $= 7^{11-11}$
Work out the answer in index form. $= 7^0$
Simplify using the definition $a^0 = 1$. $= 1$

Some expressions need to be simplified using more than one index law. The normal order of operations still applies, although you may have to insert brackets where they are only implied in the original expression.

Example 6

Simplify each of the following expressions and leave in index form.

a $(5^2 \times 3^{-3})^4$ **b** $\dfrac{5^3 \times 3^2}{(7^{-4} \times 5^2)^3}$ **c** $\left(\dfrac{7}{4}\right)^2 \div \left(\dfrac{7^{-2}}{5}\right)^3$

Solution

a Write the problem. $(5^2 \times 3^{-3})^4$

Use index law 4. $= (5^2)^4 \times (3^{-3})^4$

Use index law 3. $= 5^8 \times 3^{-12}$

Write the answer with positive powers. $= \dfrac{5^8}{3^{12}}$

b Write the problem. $\dfrac{5^3 \times 3^2}{(7^{-4} \times 5^2)^3}$

Move the denominator to the top and change the sign of the power. $= 5^3 \times 3^2 \times (7^{-4} \times 5^2)^{-3}$

Use index law 4. $= 5^3 \times 3^2 \times (7^{-4})^{-3} \times (5^2)^{-3}$

Use index law 3 to remove the brackets. $= 5^3 \times 3^2 \times 7^{12} \times 5^{-6}$

Apply index law 1 to the common bases. $= 3^2 \times 5^{-3} \times 7^{12}$

Write the answer as a fraction with positive powers. $= \dfrac{3^2 \times 7^{12}}{5^3}$

c Write the problem. $\left(\dfrac{7}{4}\right)^2 \div \left(\dfrac{7^{-2}}{5}\right)^3$

Use index law 3. $= \dfrac{7^2}{4^2} \div \dfrac{7^{-6}}{5^3}$

Invert the 2nd fraction as a result of changing the division sign to a multiplication sign. $= \dfrac{7^2}{4^2} \times \dfrac{5^3}{7^{-6}}$

Use index law 2 on the common base of 7 and change base 4 to its lowest prime base. $= \dfrac{7^{2+6}}{(2^2)^2} \times \dfrac{5^3}{1}$

Tidy up and use index law 3. $= \dfrac{7^8 \times 5^3}{2^4}$

CAS TI-Nspire exercise
Numbers and powers
MAT10NATI00012

CAS ClassPad exercise
Numbers and powers
MAT10NACP00012

Example 7

Simplify each of the following and leave in index form.

a $30^4 \times 9 \times 15^{-2}$

b $\dfrac{27^3 \times 9^5}{(3^{-4} \times 9^3)^2}$

Solution

a Write the problem. $30^4 \times 9 \times 15^{-2}$

Rewrite the bases as products of prime factors. $= (2 \times 3 \times 5)^4 \times (3 \times 3) \times (3 \times 5)^{-2}$

Use index law 3 and tidy up where possible. $= 2^4 \times 3^4 \times 5^4 \times 3^2 \times 3^{-2} \times 5^{-2}$

Use index law 1 with common bases. $= 2^4 \times 3^{4+2-2} \times 5^{4-2}$

Work out the answer in index form. $= 2^4 \times 3^4 \times 5^2$

b Write the problem. $\dfrac{27^3 \times 9^5}{(3^{-4} \times 9^3)^2}$

Move the denominator to the top and change the sign of the power. $= 27^3 \times 9^5 \times (3^{-4} \times 9^3)^{-2}$

Use index laws 3 and 4 to remove the brackets. $= 27^3 \times 9^5 \times 3^8 \times 9^{-6}$

Rewrite the bases as products of prime factors. $= (3 \times 3 \times 3)^3 \times (3 \times 3)^5 \times 3^8 \times (3 \times 3)^{-6}$

Tidy up where possible inside the brackets. $= (3^3)^3 \times (3^2)^5 \times 3^8 \times (3^2)^{-6}$

Use index law 3. $= 3^9 \times 3^{10} \times 3^8 \times 3^{-12}$

Use index law 1. $= 3^{9+10+8-12}$

Work out the answer in index form. $= 3^{15}$

Animated example
Index laws
MAT10NAAE00001

CAS TI-Nspire exercise
Numbers and powers
MAT10NATI00012

CAS ClassPad exercise
Numbers and powers
MAT10NACP00012

Puzzle sheet
Index laws
MAT10NAPS00042

The index laws can also be used with algebraic expressions.

Example 8

Simplify each of the following expressions. Write your answers with positive powers.

a $(x^2y^3)^4$ **b** $a^5 \times a^{-6} \div a^7$ **c** $(4m^{-2}n^4) \div (64m^{-3}n^6)$

Solution

a Write the problem. $(x^2y^3)^4$

Use index laws 3 and 4. $= x^8y^{12}$

b Write the problem. $a^5 \times a^{-6} \div a^7$

Use index laws 1 and 2. $= a^{5+-6-7}$

$= a^{-8}$

Write as a fraction with a positive power. $= \dfrac{1}{a^8}$

c Write the problem. $(4m^{-2}n^{4}) \div (64m^{-3}n^{6})$

Change the 64 to 4^3 so that the numbers have the same base. $= (4m^{-2}n^{4}) \div (4^{3}m^{-3}n^{6})$

Use index laws 1 and 2. $= 4^{1-3}m^{-2+3}n^{4-6}$

Simplify. $= 4^{-2}mn^{-2}$

Write as a fraction with positive powers. $= \dfrac{m}{4^{2}n^{2}}$

Example 9

Simplify and express with positive indices.

a $\dfrac{4h^{-6} \times 3h^{2}}{h^{-3} \times 6h^{5}}$

b $\dfrac{2^{m-n} \times 3^{m-n} \times 6^{n}}{12^{m}}$

Solution

a Write the expression. $\dfrac{4h^{-6} \times 3h^{2}}{h^{-3} \times 6h^{5}}$

Simplify numerical part and apply index laws. $= \dfrac{12}{6} \times h^{({}^{-}6+2-{}^{-}3-5)}$

Evaluate and simplify. $= 2h^{-6}$

Write with a positive exponent. $= \dfrac{2}{h^{6}}$

b Write the expression. $\dfrac{2^{m-n} \times 3^{m-n} \times 6^{n}}{12^{m}}$

Express bases as products of prime factors. $= \dfrac{2^{m-n} \times 3^{m-n} \times (2 \times 3)^{n}}{(2^{2} \times 3)^{m}}$

Apply index laws 3 and 4. $= \dfrac{2^{m-n} \times 3^{m-n} \times 2^{n} \times 3^{n}}{2^{2m} \times 3^{m}}$

Apply index laws. $= 2^{(m-n+n-2m)} \times 3^{(m-n+n-m)}$

Evaluate exponents. $= 2^{-m} \times 3^{0}$

Apply the definitions of powers. $= \dfrac{1}{2^{m}} \times 1$

Evaluate and simplify. $= \dfrac{1}{2^{m}}$

Important!

Powers with radical signs

The definition of powers can be extended to rational indices by defining:

$$a^{\frac{p}{q}} = \sqrt[q]{(a^{p})} = (\sqrt[q]{a})^{p}$$

where p is an integer, q is a positive integer and a is positive.

Example 10

Simplify the following using index notation.

a $x^3 \times \sqrt{x}$ **b** $\sqrt[6]{8a^4} \times \sqrt{8} \times a^{-\frac{8}{3}}$

Solution

a	Write the expression.	$x^3 \times \sqrt{x}$
	Apply the definition of a fractional index.	$= x^3 \times x^{\frac{1}{2}}$
	Apply index laws.	$= x^{3+\frac{1}{2}}$
	Evaluate exponents.	$= x^{\frac{7}{2}}$
b	Write the expression.	$\sqrt[6]{8a^4} \times \sqrt{8} \times a^{-\frac{8}{3}}$
	Express bases as products of prime factors.	$= \sqrt[6]{2^3a^4} \times \sqrt{2^3} \times a^{-\frac{8}{3}}$
	Apply the definition of a fractional index.	$= (2^3a^4)^{\frac{1}{6}} \times (2^3)^{\frac{1}{2}} \times a^{-\frac{8}{3}}$
	Apply index laws 3 and 4.	$= 2^{\frac{3}{6}}a^{\frac{4}{6}} \times 2^{\frac{3}{2}} \times a^{-\frac{8}{3}}$
	Simplify exponents and apply index laws.	$= 2^{\left(\frac{1}{2}+\frac{3}{2}\right)} \times a^{\left(\frac{2}{3}+-\frac{8}{3}\right)}$
	Evaluate exponents and simplify.	$= 2^2 \times a^{-2}$
	Evaluate and write with a positive exponent.	$= \dfrac{4}{a^2}$

Exercise 1.1 Index laws

Understanding

Extra questions
Exercise 1.1
MAT10NAEQ00001

See Example 1
See Example 2

1 Work out each of the following.

a 8^3 **b** 4^{-5} **c** 11^0
d 7^{-2} **e** 13^{-1} **f** 7^{-3}
g 12^{-2} **h** 115^0 **i** 5^{-4}

2 Simplify each of the following and leave your answer in index form.

a $(4 \times 5)^3$ **b** $(6 \times 4)^7$ **c** $(7 \times 11)^4$
d $(6 \times 7)^8$ **e** $(11 \times 13)^0$ **f** $(2 \times 7)^{12}$

See Example 2

3 Simplify each of the following and leave your answer in index form.

a $(5^4)^3$ **b** $(7^2)^5$ **c** $(3^7)^4$
d $(11^{12})^4$ **e** $(1^4)^{16}$ **f** $(5^5)^5$

See Example 3

4 Find each of the following.

a $\sqrt[3]{125}$ **b** $\sqrt{169}$ **c** $\sqrt[4]{16}$
d $\sqrt[3]{216}$ **e** $\sqrt[6]{729}$ **f** $\sqrt[4]{256}$
g $\sqrt[7]{10\ 000\ 000}$ **h** $\sqrt{289}$ **i** $\sqrt[3]{27\ 000}$

See Example 4

5 Use a calculator to find each of the following.

a 11^{-3} **b** $\sqrt{58}$ **c** $\sqrt[4]{79}$
d 2.9^4 **e** $\sqrt{29.73}$ **f** $\sqrt[3]{257.8}$
g 6.7^{-1} **h** 7.89^{-3} **i** $\sqrt[5]{1.05}$

6 Use a calculator to find each of the following. *See* Example 4

a $5^{1.8}$ **b** $6^{1.7}$ **c** $81^{0.5}$
d $7^{1.6}$ **e** $1.4^{2.6}$ **f** $3.77^{4.01}$

7 Use the index laws to simplify the following, leaving your answers with positive indices. *See* Example 5

a $4^3 \times 4^7$ **b** $7^6 \times 7^0$ **c** $15^3 \times 15^3$
d $6^4 \times 6^{15}$ **e** $10^{14} \times 10^{12}$ **f** $9^5 \times 9^3 \times 9^8$
g $14^3 \times 14^{12} \times 14^1$ **h** $3^6 \times 3^6 \times 3^5$ **i** $25^{13} \times 25^3 \times 25^4$

8 Use the index laws to simplify the following, leaving your answers with positive indices. *See* Example 5

a $8^5 \div 8^3$ **b** $3^6 \div 3^2$ **c** $7^{15} \div 7^0$
d $11^{28} \div 11^{17}$ **e** $6^{12} \div 6^{12}$ **f** $15^7 \div 15^6$
g $14^{29} \div 14^{23}$ **h** $17^{15} \div 17^{13}$ **i** $12^{13} \div 12^9$

9 Use the index laws to simplify the following, leaving your answers with positive indices.

a $(7 \times 5)^4$ **b** $(6 \times 3)^6$ **c** $(9 \times 11)^4$
d $(6 \times 8)^7$ **e** $(12 \times 13)^8$ **f** $(2 \times 7)^{12}$

Fluency

10 Use the index laws to simplify the following, leaving your answers with positive indices.

a $7^{10} \times 7^{25}$ **b** $6^{-7} \times 6^3$ **c** $7^8 \div 7^8$
d $(4^{-9})^5$ **e** $(8^{-7})^{-1}$ **f** $(6^{-3} \times 5^6)^3$
g $(7^{-1} \times 8^4)^{-5}$ **h** $(11^{-3} \times 7^4)^{-2} \times 7^5$ **i** $8^6 \times 8^{-4} \div 8^{-3}$

11 Find each of the following. *See* Example 3

a $25^{\frac{1}{2}}$ **b** $125^{\frac{1}{3}}$ **c** $343^{\frac{1}{3}}$
d $243^{\frac{1}{5}}$ **e** $64^{\frac{1}{2}}$ **f** $64^{\frac{1}{6}}$
g $216^{\frac{1}{3}}$ **h** $81^{\frac{1}{4}}$ **i** $625^{\frac{1}{4}}$

12 Use a calculator to find each of the following, correct to 2 decimal places.

a $50^{\frac{1}{2}}$ **b** $236^{\frac{1}{3}}$ **c** $28^{\frac{1}{3}}$ **d** $560^{\frac{1}{5}}$
e $95^{\frac{1}{2}}$ **f** $2^{\frac{1}{6}}$ **g** $4^{1.8}$ **h** $3^{1.8}$
i $16^{0.5}$ **j** $5^{1.6}$ **k** $1.4^{2.6}$ **l** $3.75^{4.1}$

Round all answers to 4 decimal places where appropriate.

13 Express with positive indices.

a a^{-2} **b** 4^{-3} **c** g^{-7} **d** 5^{-4}
e y^{-6} **f** $(cd^2)^{-4}$ **g** $4h^{-4}$ **h** $(4h)^{-4}$
i $2k^{-1}$ **j** $7p^{-4}$ **k** $8q^{-6}$ **l** $(4v^{-2})^{-1}$

14 Express without denominators.

a $\frac{1}{y^4}$ **b** $\frac{1}{4^3}$ **c** $\frac{1}{h^9}$ **d** $\frac{t^4}{k^6}$
e $\frac{y^2}{p^6}$ **f** $\frac{4j^8}{i^6}$ **g** $\frac{3y^{-2}}{m^5}$ **h** $\frac{2m^{-1}}{z^4}$
i $\frac{1}{x^{-8}}$ **j** $\frac{2x^4}{y^{-4}}$ **k** $\frac{3k^{-3}}{b^{-1}}$ **l** $\frac{20r^{-9}}{5w^{-6}}$

15 Express as roots (using radical signs).

a $6^{\frac{1}{2}}$
b $z^{\frac{1}{4}}$
c $216^{\frac{1}{3}}$
d $y^{\frac{3}{5}}$
e $49^{\frac{3}{2}}$
f $v^{\frac{7}{3}}$
g $8^{2\frac{1}{3}}$
h $16^{1\frac{1}{4}}$
i $m^{2\frac{1}{2}}$
j $u^{-\frac{1}{4}}$
k $\left(\frac{1}{9}\right)^{-\frac{1}{2}}$
l $\left(p^{-3}\right)^{\frac{1}{5}}$

16 Express the following as powers.

a $\sqrt{k}$
b $\sqrt[4]{3}$
c $\sqrt{5}$
d $\sqrt[5]{4}$
e $\sqrt[3]{w}$
f $\sqrt{c^3}$
g $\left(\sqrt[3]{64}\right)^4$
h $\sqrt[4]{e^3}$
i $\sqrt[9]{28^4}$
j $\sqrt[5]{q^6}$
k $\left(\sqrt{8}\right)^5$
l $\left(\sqrt[6]{16}\right)^7$

17 Evaluate the following.

a $16^{\frac{3}{4}}$
b 9^0
c $8^{\frac{1}{3}}$
d 36^0
e $36^{\frac{1}{2}}$
f $27^{\frac{2}{3}}$
g 17^0
h $81^{\frac{1}{2}}$
i $49^{\frac{3}{2}}$
j $36^{\frac{6}{4}}$
k $64^{\frac{2}{3}}$
l $32^{\frac{6}{5}}$
m $8^{-\frac{2}{3}}$
n $\left(\frac{3}{4}\right)^{-2}$
o $\left(\frac{1}{16}\right)^{-\frac{1}{4}}$
p 18^0
q $\left(\frac{9}{16}\right)^{-\frac{3}{2}}$
r $216^{-\frac{2}{3}}$
s $25^{1\frac{1}{2}}$
t $\left(6\frac{1}{4}\right)^{-\frac{1}{2}}$
u $\left(2\frac{1}{4}\right)^{\frac{3}{2}}$
v $4^{\frac{1}{3}} \times \sqrt[3]{16}$
w $(0.01)^{-\frac{5}{2}}$
x $1000^{-\frac{4}{3}}$

Problem solving

See Example 6

18 Simplify each of the following expressions and leave them in index form.

a $(4^2 \times 11^{-2})^3$
b $(3^{-2} \times 7^4)^2$
c $(5^2 \times 16^{-3})^{-3}$
d $\dfrac{8^3 \times 5^2}{(5^4 \times 2^2)^3}$
e $\dfrac{243^{-3} \times 5^2}{(5^{-3} \times 27^{-2})^3}$
f $\dfrac{5^{-2} \times 7^{-2}}{(5^{-4} \times 7^2)^{-5}}$
g $\left(\dfrac{7}{4^4}\right)^3 \times \left(\dfrac{4^2}{7}\right)^4$
h $\left(\dfrac{7}{8}\right)^2 \div \left(\dfrac{7^{-2}}{16}\right)^3$
i $\left(\dfrac{7}{5}\right)^6 \div \left(\dfrac{7^{-2}}{5^{-2}}\right)^{-3}$

See Example 7

19 Simplify each of the following expressions and leave them in index form.

a $18^2 \times 12 \times 27^{-2}$
b $21^3 \times 7^{-3} \times 49^{-2}$
c $15^4 \times 125 \times 50^{-2} \times 15^{-3}$
d $\dfrac{5^3 \times 2^{20}}{(5^{-4} \times 16^3)^2}$
e $\dfrac{3^{-3} \times 5^5}{(5^{-4} \times 15^3)^{-3}}$
f $\dfrac{5^{-9} \times 10^8}{(20^{-4} \times 25^3)^{-2}}$

See Example 8

20 Simplify each of the following expressions. Write your answers with positive powers.

a $(x^4y^3)^3$
b $(xy^{-5})^4$
c $(x^{-12}y^{-1})^{-3}$
d $b^{14} \times b^3 \div b^{17}$
e $a^{-11} \times a^{-5} \div a^{-9}$
f $c^{14} \div c^{-6} \times c^{-12}$
g $(27m^{-3}n^4) \div (81m^{-2}n)$
h $(25p^4q^{-3}) \div (625p^4q^{-7})$
i $(1024r^{-2}s^4t^3) \div (64r^{-3}s^6t)$

Worked solution
Exercise 1.1
MAT10NAWS00034
See Example 9

21 Simplify each of the following expressions and leave them in index form.

a $r^{-6} \times r^4$
b $4y^{-5} \times 3y^{-4}$
ⓒ $\dfrac{12p^6 \times 6p^{-2}}{8p^{-1} \times 3p}$
d $\dfrac{4j^5 \times 5j^{-4}}{10j^7}$
e $\dfrac{2m^3 \times m^{-10}}{12m^6}$
f $\dfrac{5k^{-3}}{2k^4} \div \dfrac{10k^5}{k^{-2}}$
g $3g^{-3}h^4 \times 5g^4h^{-6}$
h $5k^3m^{-6} \times (k^{-3}m^{-4})^{-5}$
i $\dfrac{8v^{-4}w^2 \times 4v^{10}w^{-8}}{2vw^{-9} \times v^{-6}w^3}$
j $\dfrac{12p^{-4}q^{-3} \times (2pq^4)^2}{(3p^{-1}q^4)^{-3} \times (18p^4)^2}$
k $\left(\dfrac{e^2g^4}{e^{-5}g}\right)^3 \times \dfrac{4e^{-3}g^4}{3e^6g^6}$
ⓛ $\dfrac{c^7n^8}{n^{-5}} \div \dfrac{(8c^{-2}n^3)^{-1}}{4^{-2}c^{-2}n^3}$

22 Simplify and express with positive indices.

ⓐ $a^4 \times \sqrt{a}$ **b** $b^2 \times \sqrt{b^3}$ **c** $3n^{-3} \times 2\sqrt{n^5}$

d $6g^5 \times 2\sqrt[3]{g^4}$ ⓔ $4\sqrt[3]{w^2} \times 3\sqrt{p^5w^6}$ **f** $5\sqrt[3]{m^3n^{-2}} \times 2\sqrt{m^5n^6}$

g $7\sqrt[4]{s^2u^6} \times \sqrt{9s^{-8}u^3}$ **h** $8\sqrt[3]{p^{12}q^{-8}} \times 3\sqrt[4]{p^{-10}q^9}$

Worked solution

Exercise 1.1

MAT10NAWS00034

See Example 10

23 Simplify and express with positive indices.

ⓐ $\dfrac{2^{n+2} \times 2^{n-3}}{2^{2n-5}}$ **b** $\dfrac{5^{2k} \times 5^{k-1}}{5^{2k-2}}$ **c** $\dfrac{3^{j+1} \times 2^{j+3}}{6^j}$

ⓓ $\dfrac{6^{2n-m} \times 9^{n+m}}{36^{n-m}}$ **e** $\dfrac{15^{4i-5} \times 21^{2i+1}}{35^{2i-2} \times 45^{2i-3}}$ **f** $\dfrac{8^{2f-3} \times 16^{4-f}}{4^{f+2}}$

Reasoning

Worked solution

Exercise 1.1

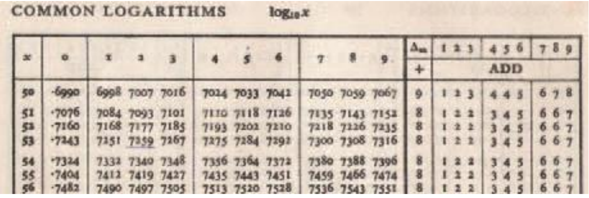
MAT10NAWS00034

See Example 9

1.2 Logarithms

Investigate: Calculation using logarithms

Before the invention of cheap electronic calculators in the early 1970s, people performed calculations like 0.7246 × 315 000 and $\pi \times (4.65)^2$ using **logarithms**. The excerpt below shows part of a logarithm table that was used in this kind of calculation. Students learned to use logarithms to do multiplications, divisions, powers and roots of numbers with more than 2 significant figures.



COMMON LOGARITHMS $\log_{10}x$

x	0	1	2	3	4	5	6	7	8	9	Δ_m	1	2	3	4	5	6	7	8	9
											+	ADD								
50	·6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	9	1	2	3	4	4	5	6	7	8
51	·7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	8	1	2	2	3	4	5	6	6	7
52	·7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	8	1	2	2	3	4	5	6	6	7
53	·7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	8	1	2	2	3	4	5	6	6	7
54	·7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	8	1	2	2	3	4	5	6	6	7
55	·7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8	1	2	2	3	4	5	6	6	7
56	·7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	8	1	2	2	3	4	5	6	6	7

A table like this was used to find the logarithm of 5.326 by looking across the 53 line to the 2 column and adding on the number in the 'differences' column for 6. So you would work out 7259 + 5 = 7264. This means that the logarithm of 5.326 is 0.7264. Your calculator has a log or $\log_{10}$ button on it that will give the same answer.

When your grandparents' generation were your age they were taught to work out 5.326 × 1.497 by using a table to find the logarithms of both numbers, and then add them. Use your calculator to find the logarithms of 5.326 and 1.497, correct to 4 decimal places, and add your answers. You should get 0.9016. Your grandparents would then use an 'antilogarithm' table to find the answer. You can do this with the 10^x or y^x button. You should get $100^{0.9016} \approx 7.973$. Check that $5.326 \times 1.497 \approx 7.973$.

Your grandparents changed every number in a calculation to scientific notation and did the number parts of the multiplications using logarithms separately from the powers of ten.

Try doing these using logarithms:

2.675 × 1.769 6.973 × 4.748 8.746 × 3.074

Each calculation like these, using logarithms, took between 1 and 5 minutes, depending how practiced you were.

Find out how to do division using logarithms.

It is still beneficial to learn about logarithms because they are useful for solving some types of equations, and the logarithm function itself is an important mathematical tool.

Important!

Common logarithms

The **common logarithm** of the number a is the solution of the equation $10^x = a$.
We write the common logarithm of the number a as $x = \log_{10} a$ or just $x = \log a$.
The common logarithm is also called the **logarithm to the base 10** (or **log base 10**).

$$x = \log_{10} a \Leftrightarrow 10^x = a$$

Calculators usually have two logarithm buttons, log and ln. We won't use ln in this book, but we can use the calculator to solve exponential equations of the form $y = a^x$ using base 10.

Example 11

a Find the value of log 100 without using a calculator.
b Find the value of $\log \sqrt[5]{10}$ without using a calculator.

Solution

a	Express as $\log_{10}$.	$\mathbf{\log 100 = \log_{10} 100}$
	Use $x = \log_{10} a \Leftrightarrow 10^x = a$.	$\boldsymbol{x = \log_{10} 100 \Leftrightarrow 10^x = 100}$
	Use the fact that $10^2 = 100$.	$\boldsymbol{10^x = 10^2}$
	Equate the indices.	$\boldsymbol{x = 2}$
	State the result.	$\mathbf{\log 100 = 2}$
b	Express as $\log_{10}$ and in index form.	$\boldsymbol{\log \sqrt[5]{10} = \log_{10} 10^{\frac{1}{5}}}$
	Use $x = \log_{10} a \Leftrightarrow 10^x = a$.	$\boldsymbol{x = \log_{10} 10^{\frac{1}{5}} \Leftrightarrow 10^x = 10^{\frac{1}{5}}}$
	Equate the indices.	$\boldsymbol{x = \frac{1}{5}}$
	State the result.	$\boldsymbol{\log \sqrt[5]{10} = \frac{1}{5}}$

The laws of common logarithms follow from the laws of indices. The most useful of these laws are shown below.

Important!

Laws of common logarithms

$$\log ab = \log a + \log b \qquad \log 1 = 0$$

$$\log \frac{a}{b} = \log a - \log b \qquad \log 10 = 1$$

$$\log a^n = n \log a \qquad \log \frac{1}{a} = -\log a$$

Example 12

Use $\log 5.326 = 0.7264$ to find the following.

a $\log 532.6$ **b** $\log 0.005\,326$

Solution

a	Write 532.6 in scientific notation.	$\mathbf{532.6 = 5.326 \times 10^2}$
	Change to logarithms.	$\mathbf{\log 532.6 = \log (5.326 \times 10^2)}$
	Use $\log ab = \log a + \log b$	$\mathbf{= \log 5.326 + \log 10^2}$
	Use $\log 5.326 = 0.7264$ and $\log 10^2 = 2$	$\mathbf{= 0.7264 + 2}$
	Complete the calculation.	$\mathbf{\log 532.6 = 2.7264}$
b	Write 0.005 326 in scientific notation.	$\mathbf{0.005\,326 = 5.326 \times 10^{-4}}$
	Change to logarithms.	$\mathbf{\log 0.005\,326 = \log (5.326 \times 10^{-3})}$
	Use $\log ab = \log a + \log b$	$\mathbf{= \log 5.326 + \log 10^{-3}}$
	Use $\log 5.326 = 0.7264$ and $\log 10^{-3} = {}^{-}3$	$\mathbf{= 0.7264 - 3}$
	Complete the calculation.	$\mathbf{\log 0.005\,326 = {}^{-}2.2736}$

Example 13

Use your calculator to find the logarithm of 234 and use it to find the logarithms of the following.

a 2.34 **b** 0.000 234 **c** 2 340 000

Solution

	Enter as log 234 =	log(234) 2.369215857
	Write your answer.	$\mathbf{\log 234 \approx 2.3692}$
a	Write 2.34 in terms of 234.	$\mathbf{2.34 = 234 \div 100}$
	Change to logarithms.	$\mathbf{\log 2.34 = \log (234 \div 10^2)}$
	Use $\log \frac{a}{b} = \log a - \log b$	$\mathbf{= \log 234 - \log 10^2}$
	Use $\log 234 = 2.3692$ and $\log 10^2 = 2$	$\mathbf{= 2.3692 - 2}$
	Complete the calculation.	$\mathbf{\log 2.34 = 0.3692}$
b	Write 0.000 234 in terms of 234.	$\mathbf{0.000\,234 = 234 \div 1\,000\,000}$
	Change to logarithms.	$\mathbf{\log 0.000\,234 = \log (234 \div 10^6)}$
	Use $\log \frac{a}{b} = \log a - \log b$.	$\mathbf{= \log 234 - \log 10^6}$
	Use $\log 234 = 2.3692$ and $\log 10^6 = 6$	$\mathbf{= 2.3692 - 6}$
	Complete the calculation.	$\mathbf{\log 0.000\,234 = {}^{-}3.6308}$

CAS TI-Nspire exercise
Numbers and powers
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CAS ClassPad exercise
Numbers and powers
MAT10NACP00012

c Write 2 340 000 in terms of 234.	$\mathbf{2\ 340\ 000 = 234 \times 10\ 000}$
Change to logarithms.	$\mathbf{\log 2\ 340\ 000 = \log (234 \times 10^4)}$
Use $\log ab = \log a + \log b$	$\mathbf{= \log 234 + \log 10^4}$
Use $\log 234 = 2.3692$ and $\log 10^4 = 4$	$\mathbf{= 2.3692 + 4}$
Complete the calculation.	$\mathbf{\log 2\ 340\ 000 = 6.3692}$

The symbol ⇒ is used in maths before a step that logically follows from the last one. It means 'it follows that' or 'it implies'.

Example 14

Given that $\log 2.8 \approx 0.4472$ and $\log 52.9 \approx 1.7235$, find:

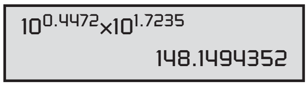
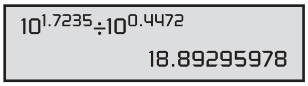
a $\log (2.8 \times 52.9)$ **b** $\log (52.9 \div 2.8)$

Using the 10^x button on your calculator, find:

c 2.8×52.9 **d** $52.9 \div 2.8$

e Set out $96.56 \div 24.76$ as it would have been done using logarithms.

Solution

a Write the expression.	$\mathbf{\log (2.8 \times 52.9)}$
Use $\log ab = \log a + \log b$	$\mathbf{= \log 2.8 + \log 52.9}$
Use $\log 2.8 \approx 0.4472$ and $\log 52.9 \approx 1.7235$	$\mathbf{= 0.4472 + 1.7235}$
Complete the calculation.	$\mathbf{= 2.1707}$
b Write the expression.	$\mathbf{\log (52.9 \div 2.8)}$
Use $\log \frac{a}{b} = \log a - \log b$	$\mathbf{= \log 52.9 - \log 2.8}$
Use $\log 2.8 \approx 0.4472$ and $\log 52.9 \approx 1.7235$	$\mathbf{= 1.7235 - 0.4472}$
Complete the calculation.	$\mathbf{= 1.2763}$
c Write the expression.	$\mathbf{2.8 \times 52.9}$
$\log 2.8 \approx 0.4472 \Rightarrow 10^{0.4472} \approx 2.8$ and	
$\log 52.9 \approx 1.7235 \Rightarrow 10^{1.7235} \approx 52.9.$	$\mathbf{= (10^{0.4472} \times 10^{1.7235})}$
Enter 10^x 0.4472 × 10^x 1.7235 =	 $10^{0.4472}$x$10^{1.7235}$ 148.1494352
Write your answer.	$\mathbf{2.8 \times 52.9 \approx 148.1}$
d Write the expression.	$\mathbf{52.9 \div 2.8}$
$\log 2.8 \approx 0.4472 \Rightarrow 10^{0.4472} \approx 2.8$ and	
$\log 52.9 \approx 1.7235 \Rightarrow 10^{1.7235} \approx 52.9.$	$\mathbf{= (10^{1.7235} \div 10^{0.4472})}$
Enter 10^x 1.7235 ÷ 10^x 0.4472 =	 $10^{1.7235}$÷$10^{0.4472}$ 18.89295978
Write your answer.	$\mathbf{52.9 \div 2.8 \approx 18.9}$

e Set out the addition of logs in columns.	**Number**	**Logs**
Find the log of 96.56 to 4 decimal points.	**96.56**	**≈ 1.9848**
Find the log of 24.76 to 4 decimal points.	**24.76**	**≈ 1.3938**
Subtract for division.		**0.5940**
Enter as 10^x 0.5940 =	**Antilog 0.5940 ≈ 3.9**	
Write the solution.	**96.56 ÷ 24.76 ≈ 3.9**	

Investigate: Logarithmic scales

Some of our senses operate on a **logarithmic scale**. This means that instead of increasing or decreasing like ordinary numbers, the increases and decreases are like logarithms. For example, in common logarithms, a scale of 0, 1, 2, 3, 4 . . . really means a scale of 1, 10, 100, 1000, 10 000, . . . , so the increases are in powers of 10. Our sense of hearing is logarithmic, so the sound scale is also logarithmic. The loudness of a sound is measured in bels (or decibels). Investigate the loudness of sounds from 40 dB to 120 dB. How much louder is 120 dB than 40 dB?

Investigate the pH scale, the Richter scale and the pitch of musical notes. How do the frequencies of different octaves compare?

Are there other logarithmic scales?

Example 15

Find the value of log 47 using a calculator and show that it is the solution of an exponential equation.

Solution

Use a calculator to calculate log 47.	**$\log 47 \approx 1.6721$**
Use $x = \log_{10} a \Leftrightarrow 10^x = a$.	**$1.6721 \approx \log 47 \Leftrightarrow 10^{1.6721} \approx 47$**
Use a calculator to check the equation.	**$10^{1.6721} \approx 47$**
State the result.	**So log 47 is the solution of the equation $10^x = 47$.**

Example 16

Show that $\log ab = \log a + \log b$ for any positive numbers a and b.

Solution

Assign variables to $\log a$ and $\log b$.	**Let $\log a = x$ and $\log b = y$**
Use the definition of a common logarithm.	**$a = 10^x$ and $b = 10^y$**
Write ab as a product of powers.	**$ab = 10^x \times 10^y$**
Simplify.	**$= 10^{x+y}$**
Use the definition of a common logarithm.	**$\log ab = x + y$**
Replace x and y with $\log a$ and $\log b$.	**$= \log a + \log b$**
Write the result.	**$\log ab = \log a + \log b$**

Common logarithms were once used to simplify calculations by changing the multiplication and division of large numbers to the addition and subtraction of their logarithms. Nowadays, we use a calculator instead because it is quicker and more accurate. However, we can still use logarithms to help solve exponential equations.

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Numbers and powers
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CAS ClassPad exercise
Numbers and powers
MAT10NACP00012

Example 17

Use common logarithms to solve the equation $1.8^x = 15$.

Solution

Write the equation to be solved.	**$1.8^x = 15$**
Take the logarithms of both sides.	**$\log 1.8^x = \log 15$**
Use the property $\log a^n = n \log a$.	**$x \log 1.8 = \log 15$**
Divide by log 1.8.	**$x = \dfrac{\log 15}{\log 1.8}$**
Calculated values of logs.	**$\approx \dfrac{1.176}{0.2553}$**
Evaluate.	**≈ 4.607**

Common logarithms are not as important as they used to be, as they are no longer used for calculations with large and small numbers. Logarithms can actually be defined for any base, not just base 10.

Important!

Logarithms

The **logarithm to the base** b of the number x, where b and x are positive and $b \neq 1$, is the solution of the equation $b^y = x$. It is written as $y = \log_b x$ and is the power to which b must be raised to produce the number x.

Example 18

Find the values of each of the following.

a $\log_2 32$

b $\log_5 0.04$

Solution

a	Write as an exponential equation.	$\boldsymbol{x = \log_2 32 \Leftrightarrow 2^x = 32}$
	What power of 2 gives 32?	$\boldsymbol{2^5 = 32}$
	Write the answer.	$\boldsymbol{\log_2 32 = 5}$
b	Change 0.04 to a fraction.	$\boldsymbol{0.04 = \frac{1}{25}}$
	Write as an exponential equation.	$\boldsymbol{x = \log_5 \frac{1}{25} \Leftrightarrow 5^x = \frac{1}{25}}$
	What power of 5 gives $\frac{1}{25}$?	$\boldsymbol{5^{-2} = \frac{1}{25}}$
	Write the answer.	$\boldsymbol{\log_5 0.04 = {}^{-}2}$

Puzzle sheet

Logarithms 1

MAT10NAPS00059

Puzzle sheet

Logarithms 2

MAT10NAPS00060

The laws of logarithms to other bases are the same as the laws of common logarithms.

Important!

Laws of logarithms

For any base $b \neq 1$.

$$\log_b xy = \log_b x + \log_b y \qquad \log_b 1 = 0$$

$$\log_b \frac{x}{y} = \log_b x - \log_b y \qquad \log_b b = 1$$

$$\log_b x^n = n \log_b x \qquad \log_b \frac{1}{x} = -\log x$$

Investigate: Logarithms and exponentials

Use your calculator to complete the following table.

x	$^{-}0.4$	$^{-}0.2$	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6
10^x											

Use graph paper to plot the graph and join the points with a smooth line.
Now use your calculator to complete the following table.

x	0.4	0.6	1	1.6	2.5	4	6	10	16	25	40
$\log x$											

Use graph paper to plot the graph and join the points with a smooth line.
Look at your graphs. What can you say about the graphs of 10^x and $\log x$?
Now complete the following tables and plot the graphs. Join the points with a smooth line.

x	$^{-}3$	$^{-}2$	$^{-}1$	0	1	2	3	4	5
2^x									

x	0.125	0.25	0.5	1	2	4	8	16	32
$\log_2 x$									

What can you say about the graphs of 2^x and $\log_2 x$?

You can use the laws of logarithms to simplify simple expressions involving logarithms with the same base.

Example 19

Video tutorial
Logarithm laws
MAT10MGVT10001

Simplify each of the following expressions.

a $\log_2 5 - \log_2 10$ **b** $2 \log_3 6 - \log_3 4$ **c** $\log_5 8 + \log_5 3 - \log_5 4$

Solution

a Write the expression. $\mathbf{\log_2 5 - \log_2 10}$

Use the log law $\log_b \frac{x}{y} = \log_b x - \log_b y$. $= \mathbf{\log_2 \frac{5}{10}}$

Simplify. $= \mathbf{\log_2 \frac{1}{2}}$

Use the log law $\log_b \frac{1}{x} = -\log_b x$. $= \mathbf{-\log_2 2}$

Use the log law $\log_b b = 1$. $= \mathbf{^{-}1}$

b Write the expression. $\mathbf{2 \log_3 6 - \log_3 4}$

Use the log law $\log_b x^n = n \log_b x$ $= \mathbf{\log_3 6^2 - \log_3 4}$

Simplify. $= \mathbf{\log_3 36 - \log_3 4}$

Use the log law $\log_b \frac{x}{y} = \log_b x - \log_b y$. $= \mathbf{\log_3 \frac{36}{4}}$

Simplify. $= \mathbf{\log_3 9}$

Evaluate using $3^2 = 9$. $= \mathbf{2}$

c Write the expression. $\mathbf{\log_5 8 + \log_5 3 - \log_5 4}$

Simplify using the first two log laws. $= \mathbf{\log_5 \frac{8 \times 3}{4}}$

Simplify. $= \mathbf{\log_5 6}$

Expressions such as $\log_5 6$ are easily evaluated using the **change of base theorem**.

Important!

Change of base theorem

For $b,\ c \neq 0$, $\log_b x = \dfrac{\log_c x}{\log_c b}$

This is shown by writing $\log_b x = y$ so that $x = b^y$
Taking logs to the base c of both sides gives $\log_c x = \log_c b^y$
Using the third log law gives $\log_c x = y \log_c b = \log_b x \times \log_c b$
Dividing by $\log_c b$ then gives the required expression.

Example 20

CAS TI-Nspire exercise
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CAS ClassPad exercise
Numbers and powers
MAT10NACP00012

Evaluate $\log_4 12$.

Solution

Write the expression. $\mathbf{\log_4 12}$

Use the change of base theorem. $= \mathbf{\dfrac{\log_{10} 12}{\log_{10} 4}}$

Evaluate with your calculator and round off. $\approx \mathbf{1.792}$

Exercise 1.2 Logarithms

Understanding

Extra questions
Exercise 1.2
MAT10NAEQ00034
See Example 11

1 Find the following without using a calculator.

a $\log 10$ **b** $\log 1000$ **c** $\log 1$
d $\log 0.1$ **e** $\log \sqrt{10}$ **f** $\log 0.001$
g $\log \sqrt[3]{10}$ **h** $\log 10^{\frac{3}{2}}$ **i** $\log \sqrt{10^5}$

Fluency

See Example 15

2 Use a calculator to find the values of the following, and show that they are the solutions of exponential equations.

a $\log 58$ **b** $\log 160$ **c** $\log 16$
d $\log 0.05$ **e** $\log 15$ **f** $\log 890$
g $\log 67$ **h** $\log 63$ **i** $\log 62.5$

3 Given $\log 5.7 \approx 0.7559$, find:

a $\log 57$ **b** $\log 5700$ **c** $\log 0.057$

4 Given $\log 2.84 \approx 0.4533$, find:

a $\log 28.4$ **b** $\log 2840$ **c** $\log 0.284$

5 Given $\log 9.735 \approx 0.9883$, find:

a $\log 97\,350$ **b** $\log 0.9735$ **c** $\log 0.000\,973\,5$

6 Given $\log 1.23 \approx 0.0899$, find:

a $\log 123$ **b** $\log 12\,300$ **c** $\log 0.123$

See Example 18

7 Find the values of each of the following.

a $\log_3 81$ **b** $\log_6 216$ **c** $\log_5 3125$
d $\log_2 0.125$ **e** $\log_5 0.008$ **f** $\log_4 0.003\,906\,25$
g $\log_2 1024$ **h** $\log_4 8$ **i** $\log_4 0.125$

See Example 20

8 Evaluate each of the following.

a $\log_3 8$ **b** $\log_8 3$ **c** $\log_2 25$ **d** $\log_{1.4} 300$ **e** $\log_{0.88} 0.5$

See Example 19

9 Simplify each of the following expressions.

a $\log_3 18 + \log_3 4.5$ **b** $\log_4 8 - \log_4 128$ **c** $\log_5 30 + \log_5 3 - \log_5 2$
d $\log_2 30 + \log_2 3 - \log_2 9$ **e** $2\log_4 6 + 3\log_4 2 - \log_4 9$

Problem solving

10 a Use a calculator to complete the following table of values.

x	5	10	15	20	25	30	35	40	45	50
$\log x$										
x	55	60	65	70	75	80	85	90	95	100
$\log x$										

b Draw a graph of the function $y = \log x$ for $5 \le x \le 100$.

c Use the fact that $y = \log_{10} x \Leftrightarrow 10^y = x$ and the values in the completed table to draw the graph of $y = 10^x$ for $5 \le x \le 100$.

d Comment on the shapes of the graphs of $y = \log_{10} x$ and $y = 10^x$.

11 a Use a calculator to complete the following table of values.

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\log x$										
x	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
$\log x$										

b Draw a graph of the function $y = \log x$ for $0.1 \le x \le 2$.

c Use the fact that $y = \log_{10} x \Leftrightarrow 10^y = x$ and the values in the completed table to draw the graph of $y = 10^x$ for $0.1 \le y \le 2$.

d Comment on the shapes of the graphs of $y = \log_{10} x$ and $y = 10^x$.

12 a Use a calculator to complete the following table of values.

x	0.001	0.005	0.01	0.05	0.1	0.5	1	5	10	50	100	500	1000
$\log x$													

b Draw a graph of the function $y = \log x$ for $0.001 \le x \le 1000$.

c Use the fact that $y = \log_{10} x \Leftrightarrow 10^y = x$ and the values in the completed table to draw the graph of $y = 10^x$ for $0.001 \le y \le 1000$.

d Comment on the shapes of the graphs of $y = \log_{10} x$ and $y = 10^x$.

13 **a** Find the values of log 3.5 and log 350.

b Find the relationship between log 3.5, log 350 and log 100.

14 **a** Find the values of log 18, log 3 and log 6.

b Find the relationship between log 18, log 3 and log 6.

15 Use a graphics calculator to draw the graphs of $y = \log x$ and $y = 10^x$ on the same axes, and comment on the shapes of the graphs.

⑯ **a** What are the domain and range of $y = 10^x$?

b What are the domain and range of $y = \log x$?

Worked solution
Exercise 1.2
MAT10NAWS00035
See Example 12

⑰ Use $\log 4.523 = 0.6554$ to find the following:

ⓐ log 45.23 **b** log 4523 **c** log 0.04523 **ⓓ** log 0.000 0452 300

Worked solution
Exercise 1.2
MAT10NAWS00035
See Example 13

⑱ Use your calculator to find the logarithm of 317 and use it to find the logarithms of the following.

a 31.7 **b** 0.000 317 **c** 317 000 **d** 0.0317 **e** 31 700 000

Worked solution
Exercise 1.2
MAT10NAWS00035
See Example 14

19 Given that $\log 3.1 \approx 0.4914$ and $\log 54.6 \approx 1.7372$, find:

a $\log (3.1 \times 54.6)$ **b** $\log (54.6 \div 3.1)$

Using the 10^x button on your calculator, find:

c 3.1×54.6 **d** $54.6 \div 3.1$

e Set out $116.72 \div 23.81$ as it would have been done using logarithms.

f Set out 16.34×43.67 as it would have been done using logarithms.

See Example 17

20 Use common logarithms to find the solutions of the following exponential equations.

a $15^x = 75$ **b** $15^x = 5$ **c** $1.04^x = 2$ **d** $0.85^x = 0.2$

e $1.45^x = 4$ **f** $3.54^x = 16.75$ **g** $2.67^{2x} = 28.5$ **h** $0.753^{3x} = 0.1$

i $0.753^{3x} = 2.5$ **j** $4^{3x-2} - 37 = 0$

Reasoning

See Example 16

21 Show that $\log \frac{a}{b} = \log a - \log b$.

22 Show that $\log a^x = x \log a$.

23 Show that $\log 10 = 1$ and $\log 1 = 0$.

24 Show that $\log \frac{1}{a} = -\log a$.

25 Show that $\log(a \times 10)^n = n + \log a$.

1.3 Real numbers

Important!

Rational and irrational numbers

Rational numbers can always be expressed as terminating or recurring decimals.
Formally, rational numbers are those that can be expressed as a ratio of integers: i.e. $\frac{a}{b}$ where $b \neq 0$.
An **irrational number** is not rational. It cannot be written as a terminating or recurring decimal. An irrational number cannot be expressed as a fraction. A **surd** is an irrational number that is a root of a rational number, such as $\sqrt{5}$.
The rational and irrational numbers completely fill the number line and, together, make the real numbers.

Example 21

State whether each of the following numbers is probably rational or irrational.

a 0.3333 . . . **b** 3.0128 **c** 1.256 386 42 . . . **d** 4.256 565 . . .

Solution

a This looks like it is actually **0.$\dot{3}$**.	**0.3333 . . . is probably rational.**
b This is a terminating decimal.	**3.0128 is rational.**
c This does not recur or terminate.	**1.256 386 42 . . . is probably irrational.**
d This looks like it is actually $4.2\overline{56}$.	**4.256 565 . . . is probably rational.**

Investigate: Finding roots

Work in pairs and, with a calculator, try to find a number that squares to give 5, using the following procedure.

1 You know that $2^2 = 4$ and $3^2 = 9$, and that $\sqrt{4} < \sqrt{5} < \sqrt{9}$, so $\sqrt{5}$ must be between 2 and 3.

2 Try $2.5^2 = 6.25$. It is too big. The square is 1.25 out.
$2.1^2 = 4.41$ misses 5 by 0.59.
Try other numbers, including numbers with more than 1 decimal place.
Record each number you try and how far out you are.

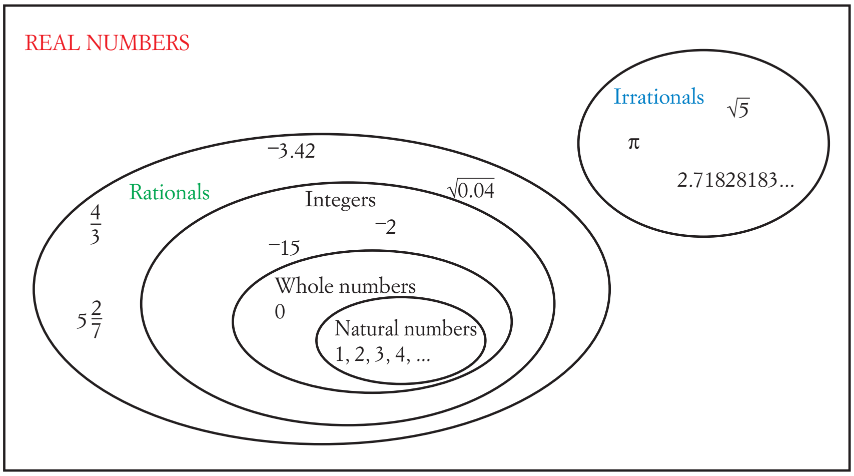
3 After 5 minutes, check which pair of people are closest.
Try this investigation with other surds.

Important!

Kinds of numbers

You already know quite a few kinds of numbers. The **positive integers, counting** or **natural numbers** are 1, 2, 3, 4, 5, 6, 7, . . . and so on. You know that **even** numbers divide exactly by 2, and **odd** numbers don't. **Prime** numbers have exactly two different factors, themselves and 1. **Composite** numbers have more than two factors.

When we include zero and the **negative integers** we get the **integers**. The counting numbers are sometimes called **cardinal numbers** and the integers are often called **whole numbers**.



If we multiply two counting numbers, we get another one. Mathematicians say that the counting numbers are **closed** for multiplication. Not every subtraction of counting numbers gives a counting number. We say that the counting numbers are **not closed** for subtraction.

If you *always* get another number from the same set from some operation, the set of numbers is closed for that operation.

To show that something is *not true*, all you have to do is find one example to show that it doesn't work. This is called a **counter-example**.

An even number can always be written in the form $2n$, where n is an integer. Similarly, any multiple of 3 can be written as $3n$ and any number that has a remainder of 4 when divided by 5 can be written as $5n + 4$.

Example 22

Show that integers not divisible by 3 are not closed under addition.

Solution

Try adding non-multiples of 3.	**$2 + 5 = 7$, which *is not* divisible by 3.**
Try another case.	**$2 + 4 = 6$, which is a multiple of 3.**
Use the relevant fact.	**Since there are 2 numbers not divisible by 3 that add up to a number divisible by 3, numbers not divisible by 3 are not closed under addition.**

Investigate: Computer numbers

When the first electronic computers were being built, John Von Newmann (1903–1957) (on the right-hand side of the photo) analysed the problems involved and developed astonishing insights into how they should be constructed. He also built his own machine to use on mathematical problems. He is generally credited with suggesting that electronic computers should use **binary numbers**. In binary arithmetic, the problem 5 + 3 is written as 101 + 11. This can be machine coded as on-off-on plus on-on. Investigate binary numbers and how they are used in computers. What are **octal** and **hexadecimal** numbers? How are they used?

To show that something is true, it has to be proven for every possible case.

Example 23

Show that odd numbers are closed under multiplication.

Solution

An odd number is a multiple of 2, plus 1.	**If a is odd then $a = 2n + 1$ for some integer n.**
To *prove* that it is closed, you have to show that the product of *any* two odd numbers is odd.	**Let a and b be any two odd numbers.**
Write the odd numbers in a useful form.	**Choose integers n and m so that $a = 2n + 1$ and $b = 2m + 1$.**
Write the product.	$a \times b = a(2m + 1)$
Expand using the distributive law.	$= 2am + a$
Write the last a as $2n + 1$.	$= 2am + 2n + 1$
Use the common factor of 2.	$= 2(am + n) + 1$
State the kind of number it is.	**$2(am + n) + 1$ is an odd number because it is in the form $2p + 1$, where p is an integer.**
State the result.	**Odd numbers are closed under multiplication.**

9780170194860

Example 24

Video tutorial
Simplifying surds
MAT10MGVT10002

Technology worksheet
Excel worksheet: Simplifying surds quiz
MAT10NACT00019

Simplify these surds.

a $\sqrt{8}$ **b** $4\sqrt{45}$ **c** $\sqrt{108}$

Solution

a Write the surd. $\boldsymbol{\sqrt{8}}$

Write as a product with a perfect square. $= \boldsymbol{\sqrt{4} \times \sqrt{2}}$

Work out $\sqrt{4}$. $= \boldsymbol{2\sqrt{2}}$

b Write the surd. $\boldsymbol{4\sqrt{45}}$

Write as a product with a perfect square. $= \boldsymbol{4 \times \sqrt{9} \times \sqrt{5}}$

Work out $\sqrt{9}$. $= \boldsymbol{4 \times 3 \times \sqrt{5}}$

Simplify and write the answer. $= \boldsymbol{12\sqrt{5}}$

c Write the surd. $\boldsymbol{\sqrt{108}}$

Use the largest possible perfect square. $= \boldsymbol{\sqrt{36} \times \sqrt{3}}$

Workout $\sqrt{36}$. $= \boldsymbol{6\sqrt{3}}$

Note that in Example **24** part **c** you could write $\sqrt{108} = \sqrt{4} \times \sqrt{27} = \sqrt{4} \times \sqrt{9} \times \sqrt{3}$ and get the same answer.

You can also reverse the process and write a simplified surd as an entire surd.

Example 25

Write $5\sqrt{6}$ as an entire surd.

Solution

Write the surd. $\boldsymbol{5\sqrt{6}}$

Write 5 as $\sqrt{25}$. $= \boldsymbol{\sqrt{25} \times \sqrt{6}}$

Multiply. $= \boldsymbol{\sqrt{150}}$

Expressions involving surds can be simplified by multiplying or dividing the surds.

Example 26

Simplify each of the following.

a $\sqrt{3}\times\sqrt{11}$ **b** $\sqrt{6}\times\sqrt{10}$ **c** $4\sqrt{5}\times\sqrt{15}$

d $\sqrt{33}\div\sqrt{11}$ **e** $\dfrac{\sqrt{120}}{\sqrt{10}}$ **f** $\dfrac{4\sqrt{45}}{8\sqrt{5}}$

Solution

a Write the problem. $\sqrt{3}\times\sqrt{11}$

Multiply. $=\sqrt{33}$

b Write the problem. $\sqrt{6}\times\sqrt{10}$

Multiply. $=\sqrt{60}$

Write as a product with a perfect square. $=\sqrt{4}\times\sqrt{15}$

Simplify. $=2\sqrt{15}$

c Write the problem. $4\sqrt{5}\times\sqrt{15}$

Separate $\sqrt{15}$. $=4\times\sqrt{5}\times\sqrt{5}\times\sqrt{3}$

Use $\sqrt{5}\times\sqrt{5}=5$. $=4\times5\times\sqrt{3}$

Simplify. $=20\sqrt{3}$

d Write the problem. $\sqrt{33}\div\sqrt{11}$

Divide. $=\sqrt{3}$

e Write the problem. $\dfrac{\sqrt{120}}{\sqrt{10}}$

Divide. $=\sqrt{12}$

Write as a product with a perfect square. $=\sqrt{4}\times\sqrt{3}$

Simplify. $=2\sqrt{3}$

f Write the problem. $\dfrac{4\sqrt{45}}{8\sqrt{5}}$

Simplify $\sqrt{45}$. $=\dfrac{4\times\sqrt{9}\times\sqrt{5}}{8\sqrt{5}}$

Work out $\sqrt{9}$ and cancel. $=\dfrac{{}^{1}\cancel{4}\times3\times\cancel{\sqrt{5}}^{1}}{{}_{2}\cancel{8}\times\cancel{\sqrt{5}}_{1}}$

Simplify. $=\frac{3}{2}$ or $1\frac{1}{2}$

You can add or subtract surds if the number in the surds is the same. It is similar to adding and subtracting like terms in algebra. If necessary, you can simplify the surds first.

Puzzle sheet
Surds
MAT10NAPS00043

Example 27

Simplify the following if possible.

a $3\sqrt{7}+2\sqrt{7}$ **b** $5\sqrt{12}-\sqrt{27}$ **c** $\sqrt{5}+\sqrt{15}$

Solution

a Treat as $3x + 2x$.

$$\mathbf{3\sqrt{7}+2\sqrt{7}=5\sqrt{7}}$$

b Write the problem.

$$\mathbf{5\sqrt{12}-\sqrt{27}}$$

Simplify $\sqrt{12}$ and $\sqrt{27}$.

$$\mathbf{=5\times\sqrt{4}\times\sqrt{3}-\sqrt{9}\times\sqrt{3}}$$

$$\mathbf{=10\sqrt{3}-3\sqrt{3}}$$

Treat as $10x - 3x$.

$$\mathbf{=7\sqrt{3}}$$

c $\sqrt{15}$ cannot be simplified, so leave.

$\mathbf{\sqrt{5}+\sqrt{15}}$ **cannot be simplified.**

You could write Example **27** part **c** as $\sqrt{5}(1+\sqrt{3})$ but this is not simpler.
It is not usual to leave an answer with surds on the bottom of a fraction. The fraction should be **rationalised** by multiplying the top and bottom of the fraction (which is just the reverse of cancelling down) by any surds on the bottom of the fraction. This is called rationalising the denominator.

Example 28

Simplify the following by rationalising the denominator.

a $\frac{1}{\sqrt{2}}$ **b** $\frac{1}{\sqrt{12}}$ **c** $\frac{15}{\sqrt{5}}$ **d** $\frac{\sqrt{2}}{\sqrt{5}}$ **e** $\frac{5\sqrt{2}}{3\sqrt{5}}$

Solution

a Write the problem.

$$\mathbf{\frac{1}{\sqrt{2}}}$$

Multiply top and bottom by $\sqrt{2}$.

$$\mathbf{=\frac{1}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}}$$

Simplify your answer. $\sqrt{2}\times\sqrt{2}=2$

$$\mathbf{=\frac{\sqrt{2}}{2}}$$

b Write the problem.

$$\mathbf{\frac{1}{\sqrt{12}}}$$

Multiply top and bottom by $\sqrt{12}$.

$$\mathbf{=\frac{1}{\sqrt{12}}\times\frac{\sqrt{12}}{\sqrt{12}}}$$

$$\mathbf{=\frac{\sqrt{12}}{12}}$$

Simplify your answer using $\sqrt{12}=2\sqrt{3}$.

$$\mathbf{=\frac{{}^{1}\cancel{2}\sqrt{3}}{\cancel{12}^{6}}}$$

Write your answer.

$$\mathbf{=\frac{\sqrt{3}}{6}}$$

c Write the problem.

$$\frac{15}{\sqrt{5}}$$

Multiply top and bottom by $\sqrt{5}$.

$$= \frac{15}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$

$$= \frac{15\sqrt{5}}{5}$$

Cancel.

$$= \frac{{}^{3}\cancel{15}\sqrt{5}}{\cancel{5}^{1}}$$

Write your answer.

$$= 3\sqrt{5}$$

d Write the problem.

$$\frac{\sqrt{2}}{\sqrt{5}}$$

Multiply top and bottom by $\sqrt{5}$.

$$= \frac{\sqrt{2}}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$

Write your answer. It cannot be simplified further.

$$= \frac{\sqrt{10}}{5}$$

e Write the problem.

$$\frac{5\sqrt{2}}{3\sqrt{5}}$$

Multiply top and bottom by $\sqrt{5}$.

$$= \frac{5\sqrt{2}}{3\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$

$$= \frac{5\sqrt{10}}{15}$$

Cancel.

$$= \frac{{}^{1}\cancel{5}\sqrt{10}}{\cancel{15}^{3}}$$

Write your answer.

$$= \frac{\sqrt{10}}{3}$$

Exercise 1.3 Real numbers

Understanding

Extra questions
Exercise 1.3
MAT10NAEQ00035

1 For each of the following, state whether it is a surd.

a $\sqrt{5}$ **b** $\sqrt{1}$ **c** $\sqrt{9}$ **d** $\sqrt{21}$
e $\sqrt{4}$ **f** $\sqrt{40}$ **g** $\sqrt{0.9}$ **h** $\sqrt{0}$
i $\sqrt{0.04}$ **j** $\sqrt{12.1}$ **k** $\sqrt{1.21}$ **l** $\sqrt{250}$

See Example 24

2 Simplify each of the following.

a $\sqrt{12}$ **b** $\sqrt{20}$ **c** $\sqrt{36}$ **d** $\sqrt{32}$
e $\sqrt{27}$ **f** $\sqrt{28}$ **g** $\sqrt{54}$ **h** $\sqrt{72}$
i $\sqrt{96}$ **j** $\sqrt{120}$ **k** $\sqrt{112}$ **l** $\sqrt{392}$

See Example 25

3 Write each of the following as an entire surd.

a $2\sqrt{6}$ **b** $3\sqrt{5}$ **c** $4\sqrt{3}$ **d** $5\sqrt{7}$
e $6\sqrt{2}$ **f** $8\sqrt{3}$ **g** $3\sqrt{11}$ **h** $7\sqrt{10}$
i $5\sqrt{13}$ **j** $6\sqrt{14}$ **k** $4\sqrt{5}$ **l** $3\sqrt{10}$

Fluency

4 State whether each of the following is probably rational or irrational.

See Example 21

a 12.5 **b** 0.111 … **c** 1.57183 …
d 2.181 81 … **e** 1.123 42 … **f** 1.356
g 7.272 73 … **h** 3.522 22 … **i** 156.5
j 0.856 66 … **k** 124.387 219 … **l** 414.214 12 …

5 Simplify the following.

See Example 26

a $\sqrt{3}\times\sqrt{2}$ **b** $\sqrt{4}\times\sqrt{7}$ **c** $\sqrt{5}\times\sqrt{6}$ **d** $\sqrt{3}\times\sqrt{11}$
e $\sqrt{11}\times\sqrt{11}$ **f** $\sqrt{28}\times\sqrt{21}$ **g** $\sqrt{10}\times\sqrt{20}$ **h** $\sqrt{24}\times\sqrt{18}$

6 Simplify these products.

a $2\sqrt{3}\times 3\sqrt{5}$ **b** $8\sqrt{7}\times 6\sqrt{6}$ **c** $4\sqrt{13}\times 5\sqrt{2}$ **d** $3\sqrt{5}\times 4\sqrt{5}$
e $2\sqrt{3}\times 7\sqrt{3}$ **f** $4\sqrt{5}\times 3\sqrt{10}$ **g** $5\sqrt{7}\times 2\sqrt{14}$ **h** $4\sqrt{3}\times 5\sqrt{12}$

7 Simplify these quotients.

a $\sqrt{33}\div\sqrt{3}$ **b** $\dfrac{\sqrt{24}}{\sqrt{6}}$ **c** $\sqrt{45}\div\sqrt{15}$ **d** $\dfrac{\sqrt{240}}{\sqrt{20}}$
e $\dfrac{\sqrt{50}}{\sqrt{5}}$ **f** $3\sqrt{28}\div\sqrt{7}$ **g** $\dfrac{4\sqrt{80}}{2\sqrt{10}}$ **h** $\dfrac{8\sqrt{36}}{4\sqrt{12}}$
i $\dfrac{15\sqrt{72}}{20\sqrt{18}}$ **j** $25\sqrt{120}\div 15\sqrt{15}$

8 Simplify if possible.

See Example 27

a $\sqrt{5}+3\sqrt{5}$ **b** $4\sqrt{3}+6\sqrt{3}$ **c** $5\sqrt{13}-2\sqrt{13}$
d $8\sqrt{5}-4\sqrt{5}$ **e** $12\sqrt{7}-6\sqrt{7}$ **f** $4\sqrt{10}+3\sqrt{20}$
g $4\sqrt{6}+7\sqrt{5}-3\sqrt{5}$ **h** $3\sqrt{2}+6\sqrt{5}-2\sqrt{2}+4\sqrt{5}$

Problem solving

9 Show that integers not divisible by 3 are not closed under subtraction.

10 Show that the set {1, 2, 3} is not closed under addition.

See Example 22

11 Show that the set $\{^{-}1, 0, 1\}$ is not closed under addition.

12 Show that prime numbers are not closed under either addition or multiplication.

13 Show that multiples of 5 are not closed under division.

14 Show that numbers that have a remainder of 2 when divided by 3 are not closed under addition.

15 Show that numbers that have a remainder of 2 when divided by 3 are not closed under subtraction.

16 Simplify the following.

a $\sqrt{2}+\sqrt{8}$ **b** $\sqrt{27}-\sqrt{3}$ **c** $\sqrt{5}+\sqrt{125}$ **d** $\sqrt{11}+\sqrt{44}$
e $3\sqrt{12}+\sqrt{27}$ **f** $9\sqrt{3}-2\sqrt{48}$ **g** $8\sqrt{10}-\sqrt{250}$ **h** $9\sqrt{7}+3\sqrt{28}$

17 Simplify the following.

a $5\sqrt{2}+6\sqrt{8}+\sqrt{32}$ **b** $4\sqrt{27}-\sqrt{128}+\sqrt{2}$
c $2\sqrt{28}+\sqrt{20}+\sqrt{125}$ **d** $\sqrt{40}+\sqrt{72}-3\sqrt{32}+\sqrt{250}$
e $3\sqrt{45}-\sqrt{72}+6\sqrt{8}-2\sqrt{20}$ **f** $4\sqrt{63}+3\sqrt{20}-2\sqrt{28}+3\sqrt{80}$

Reasoning

18 A square floor tile has an area of 15 cm^2. Find an expression for its side length.

See Example 23

19 Show that odd numbers are not closed under addition.

20 Show that the set $\{^-1, 1\}$ is closed under multiplication.

Worked solution
Exercise 1.3
MAT10NAWS00036

㉑ Show that the set $\{^-1, 0, 1\}$ is closed under multiplication.

㉒ Show that odd numbers are not closed under subtraction.

23 Show that even numbers are closed under multiplication.

24 Show that even numbers are not closed under division.

25 A new operation called malition (symbol #) is made up. Two numbers are malited by adding their product and sum. For example, $3 \# 4 = (3 \times 4) + (3 + 4) = 19$.

a Are even numbers closed under malition?

b Are odd numbers closed under malition?

c Are integers closed under malition?

26 Make up an operation that is closed for prime numbers.

27 Simplify the following.

a $3\sqrt{20} - 3\sqrt{50} - 3\sqrt{27} - 2\sqrt{45} - \sqrt{72} - \sqrt{12}$

b $\sqrt{3} - 2\sqrt{80} - 4\sqrt{2} - 2\sqrt{108} + \sqrt{125} - 4\sqrt{2}$

c $\sqrt{20} + 4\sqrt{8} - \sqrt{12} + 4\sqrt{5} - \sqrt{50} + 2\sqrt{75}$

d $2\sqrt{80} + 2\sqrt{216} + 4\sqrt{108} - 2\sqrt{5} - 2\sqrt{216} + 3\sqrt{12}$

28 Simplify the following.

a $3\sqrt{216} - 3\sqrt{20} - 2\sqrt{32} + \sqrt{6} + 3\sqrt{80} + \sqrt{18} + 3\sqrt{24} + 3\sqrt{180}$

b $2\sqrt{96} - 4\sqrt{2} + \sqrt{48} + 3\sqrt{24} + 4\sqrt{32} + 4\sqrt{12} + 3\sqrt{24} + \sqrt{50}$

c $2\sqrt{338} + 3\sqrt{432} + \sqrt{54} + 3\sqrt{450} + 4\sqrt{48} - 4\sqrt{150} + 2\sqrt{32} - 4\sqrt{300}$

d $4\sqrt{720} - 4\sqrt{600} - 4\sqrt{32} - 4\sqrt{20} + 3\sqrt{6} + 3\sqrt{450} + 3\sqrt{320} - 3\sqrt{1350}$

See Example 28

29 Simplify the following by rationalising the denominator.

a $\dfrac{1}{\sqrt{5}}$ **b** $\dfrac{1}{\sqrt{8}}$ **c** $\dfrac{4}{\sqrt{2}}$

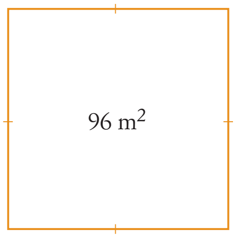
d $\dfrac{5}{\sqrt{3}}$ **e** $\dfrac{\sqrt{8}}{\sqrt{2}}$ **f** $\dfrac{\sqrt{6}}{\sqrt{2}}$

g $\dfrac{\sqrt{6}}{\sqrt{10}}$ **h** $\dfrac{5\sqrt{3}}{2\sqrt{7}}$ **i** $\dfrac{\sqrt{6a}}{\sqrt{10a^2}}$

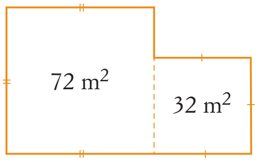
Worked solution
Exercise 1.3
MAT10NAWS00036

㉚ Find an expression for the length of fencing needed to enclose each of these yards.

a



b



Chapter 1 summary

Quiz

Numbers and powers

MAT10NAQZ00001

- A repeated multiplication may be written as a **power**. The number multiplied is written once as the **base**. The number of times it is in the multiplication is written as the small **index (exponent)** at the top right. The **extended form** of a power has the multiplication written out in full.
- A **square** is a power with index 2, and a **cube** has index 3.
- The index laws are that for any numbers a and b and any indices m and n:

  1 $a^m \times a^n = a^{m+n}$ **2** $a^m \div a^n = a^{m-n}$ (for $m > n$)

  3 $(a^m)^n = a^{mn}$ **4** $(ab)^n = a^n b^n$

  Note: m and n are integers and $a \neq 0$, $b \neq 0$.
- The definition of powers can be extended to:

  $a^0 = 1$ and $a^{-n} = \frac{1}{a^n}$ for any integer n and any base a except zero ($a \neq 0$).

  $a^{\frac{p}{q}} = \sqrt[q]{(a^p)} = (\sqrt[q]{a})^p$ where p is an integer, q is a positive integer and a is positive.
- The nth **root** of a number x is written using the **surd** symbol as $\sqrt[n]{x}$. It is the new number y that must be raised to the nth power to give x, so $y^n = x$. The **square root**, $\sqrt{x}$, is written without the index number. The third root, $\sqrt[3]{}$, is called the **cube root**.
- Logarithms have the following laws:

  $\log_b xy = \log_b x + \log_b y$ $\log_b \frac{x}{y} = \log_b x - \log_b y$ $\log_b x^n = n \log_b x$

  $\log_b 1 = 0$ $\log_b b = 1$ $\log_b \frac{1}{x} = -\log_b x$
- The **change of base theorem** says that $\log_b x = \dfrac{\log_c x}{\log_c b}$
- Logarithms may be used to solve exponential equations arising from exponential functions.
- A **recurring decimal** is a decimal that has one or more digits that repeat forever. We show that digits are repeating by putting a bar (or dots) over the repeating digits. A **terminating decimal** is a decimal that is not recurring, but comes to an end.
- A **rational number** can be written as a fraction. It is also a terminating or recurring decimal.
- An **irrational number** is not rational. It cannot be written as a terminating or recurring decimal. An irrational number cannot be expressed as a fraction.
- The rational and irrational numbers together make the **real** numbers and fill the number line.
- The **positive integers, counting** or **natural numbers** are 1, 2, 3, 4, 5, 6, 7, … . **Even** numbers divide exactly by 2, and **odd** numbers don't. **Prime** numbers have exactly two different factors: themselves and 1. **Composite** numbers have more than two factors.
- A **surd** is an irrational number that is a root of a rational number, such as $\sqrt{5}$.
- Surds can be simplified by writing the number as a product of a perfect square and another number. The reverse process is used to write an **entire surd**.
- Surds can be multiplied or divided to simplify expressions. Surds can be added or subtracted by treating them as algebraic variables.
- If you *always* get another number of the same kind from some operation, those numbers are **closed** under that operation.
- **Rationalising** the denominator is a process of simplifying fractions that contain surds in their denominators by multiplying the top and bottom of the fraction by any surds on the bottom of the fraction. This makes the denominator rational.

Chapter 1 review

Understanding

See Example 1

1 Use the index laws to simplify the following, leaving your answers with positive indices.

a $5^{10} \times 5^{27}$ **b** $6^{-11} \times 6^{3}$ **c** $5^{7} \div 5^{7}$

See Example 2

2 Work out each of the following.

a $\sqrt{81}$ **b** $\sqrt{625}$ **c** $\sqrt{10\,000}$

3 Simplify each of the following and leave your answer in index form.

a $(3 \times 5)^4$ **b** $(7 \times 4)^5$ **c** $(5 \times 11)^3$

See Example 3

4 Find each of the following.

a $\sqrt[3]{343}$ **b** $\sqrt{225}$ **c** $\sqrt[4]{81}$

5 Use the index laws to simplify the following, leaving your answers with positive indices.

a $7^5 \div 7^3$ **b** $23^{13} \times 23^{3} \times 23^{-4}$ **c** $(11^5)^{12}$

Fluency

See Example 3

6 Use a calculator to find each of the following, correct to 2 decimal places.

a $40^{\frac{1}{2}}$ **b** $736^{\frac{1}{3}}$ **c** $39^{\frac{1}{4}}$

See Example 6

7 Simplify each of the following expressions and leave them in index form.

a $(7^2 \times 5^{-3})^3$ **b** $\dfrac{729^{-3} \times 5^2}{(5^{-3} \times 27^{-2})^3}$ **c** $\left(\dfrac{49}{11}\right)^2 \div \left(\dfrac{7^{-2}}{22}\right)^4$

See Example 7

8 Simplify each of the following expressions and leave them in index form.

a $16^3 \times 4^{-3} \times 48^{-2}$ **b** $\dfrac{5^{-3} \times 3^5}{(3^{-4} \times 15^3)^{-4}}$ **c** $\dfrac{5^{-8} \times 20^8}{(10^{-4} \times 25^3)^{-2}}$

See Example 8

9 Simplify each of the following expressions. Write your answers with positive powers.

a $(xy^{-2})^{-4}$ **b** $a^{-12} \times a^5 \div a^{-2}$ **c** $(36p^4q^{-3}) \div (216p^4q^{-2})$

See Example 10

10 Simplify the following.

a $x^3 \times x^8$ **b** $20k^{12} \div 4k^3$ **c** $(3a^4b^2)^3$

d $\dfrac{6t^4 \times 9t^7}{12t^5}$ **e** $\dfrac{14^5 \times 9^2}{21^4 \times 49}$

11 Simplify the following.

a $\dfrac{24p^3q \times 3p^5q^{-3}}{27p^{-6}q^4}$ **b** $\dfrac{9^4 \times 12^{-2} \times 8^2}{18^3}$

c $81^{-\frac{3}{4}}$ **d** $5d^0$

See Example 21

12 State whether each of the following is probably rational or irrational.

a 0.666 … **b** 67.501 **c** 1.236 067 977 …

d 4.192 192 … **e** 6.283 185 307 … **f** 1.7706

See Examples 18, 19

13 Evaluate each of the following.

a $\log_2 32$ **b** $\log_5 0.04$ **c** $\log_7 50$

See Example 20

14 Simplify $\log_7 21 + \log_7 98 - \log_7 6$.

15 Express $4\sqrt{7}$ as an entire surd.

See Example 25

16 Simplify each of the following.

See Examples 24, 26, 27

a $\sqrt{8}$ **b** $\sqrt{63}$ **c** $\sqrt{6} \times \sqrt{12}$

d $2\sqrt{15} \times \sqrt{24}$ **e** $\dfrac{6\sqrt{30}}{3\sqrt{5}}$ **f** $8\sqrt{6} - 5\sqrt{7} + 4\sqrt{7} - 3\sqrt{6}$

Problem solving

17 Show that numbers that have a remainder of 2 when divided by 3 are not closed under multiplication.

See Example 22

18 Simplify $5\sqrt{2} + 3\sqrt{18} - 2\sqrt{32}$.

19 Find the values of each of the following.

See Example 18

a $\log_3 243$ **b** $\log_2 0.015\,625$ **c** $\log_4 32$

20 Use $\log 5.677 = 0.7541$ to find the following.

See Example 12

a $\log 56.77$ **b** $\log 56\,770$ **c** $\log 0.005\,677$ **d** $\log 0.000\,005\,67$

21 Use your calculator to find the logarithm of 415 and use it to find the logarithms of the following.

See Example 13

a 4150 **b** 0.000 415 **c** 41 500 000 **d** 0.000 000 000 415

22 Given that $\log 7.5 \approx 0.8751$ and $\log 63.7 \approx 1.8041$, find

See Example 14

a $\log (7.5 \times 63.7)$ **b** $\log (63.7 \div 7.5)$

Using the 10^x button on your calculator, find:

c 7.5×63.7 **d** $63.7 \div 7.5$

e Set out $125.36 \div 19.87$ as it would have been done using logarithms.

f Set out 23.55×35.22 as it would have been done using logarithms.

Reasoning

23 Show that $\log_b x + \log_b y = \log_b xy$

24 Show that $\log_b x^n = n \log_b x$

See Example 16

25 Show that $\log_b b^n = n$

26 Show that even numbers are closed under addition.

See Example 23

27 Simplify the following.

a $4\sqrt{2} + 4\sqrt{3} - 2\sqrt{20} - 2\sqrt{72} - 2\sqrt{12} + 2\sqrt{45}$

b $2\sqrt{24} + 2\sqrt{2} + 2\sqrt{125} - 2\sqrt{24} - 2\sqrt{8} - 3\sqrt{80}$

c $4\sqrt{27} + 3\sqrt{5} + 4\sqrt{54} - \sqrt{75} - 3\sqrt{20} - \sqrt{96} + 3\sqrt{75} + 2\sqrt{5}$

d $2\sqrt{50} - \sqrt{54} - 2\sqrt{5} + 4\sqrt{72} + 3\sqrt{216} - \sqrt{320} + \sqrt{450} + 2\sqrt{1176}$

28 A box has a square base, a volume of 51 840 cm^3 and a height of 30 cm. Find an expression for the side length of the box.

Statistics and probability

2

Statistics

Contents

Prior learning
Chapter 2
MAT10SPPL00013

Parent guide
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Australian Curriculum statements

Chance

Investigate reports of studies in digital media and elsewhere for information on their planning and implementation. (10A)

Data representation and interpretation

Determine quartiles and interquartile range. (10)

Construct and interpret box plots and use them to compare data sets. (10)

Compare shapes of box plots to corresponding histograms and dot plots. (10)

Use scatter plots to investigate and comment on relationships between two numerical variables. (10)

Investigate and describe bivariate numerical data where the independent variable is time. (10)

Evaluate statistical reports in the media and other places by linking claims to displays, statistics and representative data. (10)

Calculate and interpret the mean and standard deviation of data and use these to compare data sets. (10A)

Use information technologies to investigate bivariate numerical data sets. Where appropriate use a straight line to describe the relationship allowing for variation. (10A)

Video tutorial
Statistics
MAT10SPVT00002

Weblink
Kaggle: Data science competitions
MAT10SPWB00002

TLF Learning object
Australia's population dataset (M007102)
MAT10SPIN00002

The modern world is an information world. Governments, credit bureaus, banks and marketeers collect and use information about how we live, where we live, what we buy and what we desire. You can tap into information at home and elsewhere at any time using the Internet. We use statistics to collect, display and analyse information.

Statistics is used to work out the needs of particular parts of the Australian population, as well as what is likely to happen in the future. In medicine, statistics is used to investigate diseases and their treatment. Governments and other agencies use statistics to decide where to put new developments, and businesses use them to design ad campaigns and to target consumer groups for sales of particular products. Investment firms, trusts and fund managers use statistics to decide what stocks they should buy on the share market and how much they are willing to pay for particular shares.

Mathematical literacy

Maths dictionary
MAT10ASDI00001

The mathematical words below have special meanings that you will learn in this chapter. It is important that you learn to spell them and gradually learn what they mean in mathematics. You may find the glossary or online mathematical dictionary useful for this purpose.

back-to-back
bias
bimodal
bivariate
box-and-whisker plot
boxplot
categorical
central tendency
class centre
continuous
cumulative frequency
discrete
extrapolation
fair
first quartile
five number summary
interpolation
interquartile range
line of best fit
line of regression
lower quartile
mean
mean deviation
median
mode
negative relationship
negative skew
nominal
non-compliant
numerical
ordinal
outlier
population
positive relationship
positive skew
quartile
range
relationship
sample
scatter plot
side-by-side
skewed
spread
standard deviation
strong relationship
survey
symmetrical
third quartile
two means method
two-way table
univariate
upper quartile
weak relationship

2.1 Calculating and interpreting data measures

There are many ways to measure numerical data, but fewer methods of measuring categorical data.

Important!

Data types

Categorical data places the information in categories such as colour, sex and nationality. There are two kinds of categorical data.

Nominal data has no numeric meaning or order.
Ordinal data has a natural order, but does not represent a measurement.
Numerical data consists of numbers. There are two kinds of numerical data.
Continuous data can take any value between the smallest and largest values, so we may need to show how accurate it is.
Discrete data can only have particular values. In most cases, the values are whole numbers.

Important!

Data measures

The **mode** is the score with the highest frequency. It is the most common score, so it occurs more often than any other score. Where there are two scores with equal highest frequencies, we say the distribution is **bimodal**. The mode can be used for any type of data.
The **median** is the *middle* score. To find the median, all scores are arranged in order from smallest to largest and the middle score is chosen. If there are an even number of scores, then the average of the middle two scores is used. When there are a large number of scores, the progressive total of frequencies (the **cumulative frequency**) is used to work out the median.
The median is the $\frac{n+1}{2}th$ score. The median can be used for ordinal, discrete or continuous data.
The **mean** ($\overline{x}$) is the **average** score. It is calculated by adding all the scores and dividing by the number of scores. It can be used for discrete or continuous data, but may give unfeasible values for discrete data.
For numeric data, the **range** is the difference between the highest and lowest scores. For categorical data it is the number of different scores.
The **quartiles** of a frequency distribution divide it into 4 equal sections.
The **first quartile** (**lower quartile**) is the score that has a quarter (25%) of the scores below it. Its symbol is $\boldsymbol{Q1}$ or $\boldsymbol{Q_1}$.
The **second quartile** is the score that has two quarters (50%) of the scores below it. It is another name for the median. It can be written as $\boldsymbol{Q2}$ or $\boldsymbol{Q_2}$.
The **third quartile** (**upper quartile**) is the score that has three quarters (75%) of the scores below it. Its symbol is $\boldsymbol{Q3}$ or $\boldsymbol{Q_3}$.
The **interquartile range** (**IQR**) is the difference between the third and first quartiles. It can be worked out for ordinal and numeric data.

$$\mathbf{IQR} = \boldsymbol{Q3} - \boldsymbol{Q1}$$

The mode, mean and median are all measures of central tendency, while the range and interquartile range are **measures of spread**.

The mathematical symbol Σ, which means 'the sum of' can be used to write the formula for the mean as $\overline{x} = \frac{\Sigma x}{n} = \frac{\Sigma x}{\Sigma f}$, where x stands for a score, n is the number of scores and f stands for a frequency.

Example 1

The hair colours of a class of Year 10 students in Ireland were as follows.
Dark Brown, Red, Blonde, Dark Brown, Blonde, Celtic Bronze, Black, Blonde, Dark Brown, Light Brown, Light Brown, Dark Brown, Black, Light Brown, Blonde, Black, Blonde, Light Brown, Blonde, Blonde, Red, Dark Brown, Dark Brown, Blonde, Light Brown, Blonde, Strawberry Blonde
Find the mode and range.

Solution

Make a frequency table of the colours.

Colour	Frequency
Black	3
Blonde	9
Celtic Bronze	1
Dark Brown	6
Light Brown	5
Red	2
Strawberry Blonde	1
Total	**27**

State the colour with the highest frequency. **The mode is Blonde.**

How many different colours are there? **The range is 7.**

Example 2

A class of Year 10 students were ask to rank takeaway chicken, chips, hamburgers, kebabs and pizza in order of their preference from 1 to 5, with 1 being the best. The results for pizza are as follows:
1, 2, 1, 2, 1, 5, 2, 2, 4, 4, 4, 1, 4, 1, 1, 3, 4, 3, 3, 3, 3, 4, 2, 2, 2, 3

a Find the median and mode for pizza.
b Which is the best measure of pizza's popularity?
c Is this sample sufficient to make good conclusions?

Solution

a Arrange the results in order. **1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 4, 4, 4, 4, 4, 4, 5**

There are more 2s than anything else. **The mode is second preference.**

The middle 2 scores are 2 and 3. **Median** $= \frac{2+3}{2} = 2.5$

b The data is ordinal. **The best measure is the median. Pizza is 2nd or 3rd in popularity.**

c Only the pizza results are given. **The results are incomplete, and the sample is quite small.**

The preferences in Example 2 are not really numbers. Preferences are ordinal data, so it is not sensible to calculate a mean.

A table with frequencies can be used to find the mean and median. In that case, the total of all the scores is the total of the score × frequency column. This is written as Σxf.

Example 3

The marks of some Year 10 students for an English essay were as follows.

10, 16, 12, 13, 10, 13, 10, 18, 11, 15, 9, 17, 7, 14, 11, 13, 10, 19, 11, 15, 8, 16, 12, 19, 10, 13, 9

Find the mean, median, mode and range for the essay marks.

Solution

Make a frequency table with score × frequency and cumulative frequency columns.

Score x	Frequency f	xf	Cumulative frequency
7	1	7	1
8	1	8	2
9	2	18	4
10	5	50	9
11	3	33	12
12	2	24	14
13	4	52	18
14	1	14	19
15	2	30	21
16	2	32	23
17	1	17	24
18	1	18	25
19	2	38	27
Totals	27	341	

10 has the highest frequency. **The mode is 10.**

There are 27 scores, so the 14th score has 13 either side of it. **The median is 12.**

Find the total. $\Sigma xf = 341$

There are 27 scores to find the mean of. **Mean,** $\bar{x} = \frac{\Sigma x}{\Sigma f} = \frac{341}{27} \approx 12.63$

The lowest mark is 7 and the highest is 19. **Range** $= 19 - 7 = 12$

CAS TI-Nspire exercise
Statistics
MAT10SPTI00002

CAS ClassPad exercise
Statistics
MAT10SPCP00002

Technology worksheet
Excel worksheet: Finding the mean for grouped data
MAT10SPCT00001

Puzzle sheet
Mean, median and mode
MAT10SPPS00044

Strictly speaking, it is only possible to find quartiles exactly when the total frequency is a multiple of 4. As a result, there is some disagreement among statisticians about how quartiles should be calculated. Different methods may give slightly different values for the interquartile range.

Important!

Finding quartiles

To find the first quartile, find the median of the first half of the scores (*below* the median). For a frequency table, this is the $\frac{n+1}{4}$th score for n odd and the $\frac{n+2}{4}$th score for n even. To find the third quartile, find the median of the second half of the scores (*above* the median). For a frequency table, this is the $\frac{3n+3}{4}$th score for n odd and the $\frac{3n+2}{4}$th score for n even.

Example 4

Video tutorial
Interquartile range
MAT10SPVT10003

Find the median, quartiles and interquartile range for each of the following sets of scores.

a 13, 19, 14, 17, 14, 16, 12, 21, 15, 16, 15, 24, 14, 18, 14, 20, 14, 18, 13, 18, 14, 26, 15, 22

b 8, 17, 11, 17, 11, 15, 10, 12, 11, 13, 12, 12, 7, 14, 7, 17, 8, 20, 12

c

x	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
f	1	1	0	2	0	2	3	2	3	1	1	3	4	4	1	1	3	1	0	1

d

x	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
f	1	2	2	3	2	2	11	1	3	2	4	1	0	4	3

Solution

a Put the 24 scores in order and divide in two.	**12, 13, 13, 14, 14, 14, 14, 14, 14, 15, 15, 15 | 16, 16, 17, 18, 18, 18, 19, 20, 21, 22, 24, 26**
Find the median.	**Median, $Q2 = \frac{15+16}{2} = 15.5$**
Divide the scores before the median in two.	**12, 13, 13, 14, 14, 14 | 14, 14, 14, 15, 15, 15**
Find the first quartile.	**First quartile, $Q1 = \frac{14+14}{2} = 14$**
Divide the scores after the median in two.	**16, 16, 17, 18, 18, 18 | 19, 20, 21, 22, 24, 26**
Find the third quartile.	**Third quartile, $Q3 = \frac{18+19}{2} = 18.5$**
Find the interquartile range.	**IQR $= Q3 - Q1 = 18.5 - 14 = 4.5$**
Write the results.	**Median $= 15.5$, $Q1 = 14$, $Q3 = 18.5$, IQR $= 4.5$**

b Put the 19 scores in order and divide in two.

7, 7, 8, 8, 10, 11, 11, 11, 12, 12, 12, 12, 13, 14, 15, 17, 17, 17, 20

Find the median.

Median, $Q2 = 12$

Divide the scores before the median in two.

7, 7, 8, 8, 10, 11, 11, 11, 12

Find the first quartile.

First quartile, $Q1 = 10$

Divide the scores after the median in two.

12, 12, 13, 14, 15, 17, 17, 17, 20

Find the third quartile.

Third quartile, $Q3 = 15$

Find the interquartile range.

IQR $= Q3 - Q1 = 15 - 10 = 5$

Write the results.

Median $= 12$, $Q1 = 10$, $Q3 = 15$, IQR $= 5$

c Rearrange the table so that the scores are listed vertically and include a cumulative frequency column.

$\Sigma f = 34$, so there are 34 scores.

Score x	Frequency f	Cumulative frequency
11	1	1
12	1	2
13	0	2
14	2	4
15	0	4
16	2	6
17	3	9
18	2	11
19	3	14
20	1	15
21	1	16
22	3	19
23	4	23
24	4	27
25	1	28
26	1	29
27	3	32
28	1	33
29	0	33
30	1	34
Total	34	

The median is the $\frac{34+1}{2} = 17.5$th score.

The 17th and 18th scores are both 22, so median $= 22$

34 is even, so use $\frac{n+2}{4}$ for $Q1$.

$Q1$ is the $\frac{34+2}{4} = 9$th score.

Find $Q1$.

The 9th score is the last 17, so $Q1 = 17$

34 is even, so use $\frac{3n+2}{4}$ for $Q3$.

$Q3$ is the $\frac{3 \times 34 + 2}{4} = 26$th score.

Find $Q3$.

The 26th score is the third 24, so $Q3 = 24$

Find the interquartile range.

IQR $= Q3 - Q1 = 24 - 17 = 7$

Write the results.

Median $= 22$, $Q1 = 17$, $Q3 = 24$, IQR $= 7$

d Rearrange the table so that the scores are listed vertically and include a cumulative frequency column.

$\Sigma f = 41$, so there are 34 scores.

Score x	Frequency f	Cumulative frequency
9	1	1
10	2	3
11	2	5
12	3	8
13	2	10
14	2	12
15	11	23
16	1	24
17	3	27
18	2	29
19	4	33
20	1	34
21	0	34
22	4	38
23	3	41
Total	41	

The median is the $\frac{41+1}{2} = 21\text{st}$ score.

The 21st score is the third last 15, so median = 15

41 is odd, so use $\frac{n+1}{4}$ for $Q1$.

$Q1$ is the $\frac{41+1}{4} = 10.5\text{th}$ score.

Find $Q1$.

The 10th score is the second 13 and the 11th score is the first 14, so

$$Q1 = \frac{13+14}{2} = 13.5$$

41 is odd, so use $\frac{3n+3}{4}$ for $Q3$.

$Q3$ is the $\frac{3 \times 41 + 3}{4} = 31.5\text{th}$ score.

Find $Q3$.

The 31st and 32nd scores are the middle 19s, so $Q3 = \frac{19+19}{2} = 19$

Find the interquartile range.

$\text{IQR} = Q3 - Q1 = 19 - 13.5 = 5.5$

Write the results.

Median = 15, $Q1 = 13.5$, $Q3 = 19$, IQR = 5.5

When you find the mean from a grouped distribution you need to use a value in the group to represent all values in the group. It is usual to use the class centre; the middle of the group.

Example 5

Use the table below of 100 m sprint times for a group of Year 10 students to find the average time.

Time (s)	10–14	15–19	20–24	25–29	30–34	35–39
Frequency	3	23	18	14	5	2

Solution

Rearrange the table vertically and put in the true score groups and class centres.

Extend the table to include xf.

Add the frequencies.

Add the score × frequency column.

Time (s)	**Centre x**	**Frequency f**	xf
9.5–14.5	12	3	36
14.5–19.5	17	23	391
19.5–24.5	22	18	396
24.5–29.5	27	14	378
29.5–34.5	32	5	160
34.5–39.5	37	2	74
	Totals	65	1435

Calculate the mean.

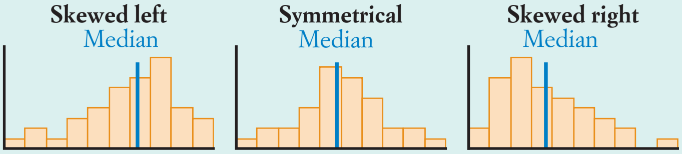
$$\textbf{Mean} = \frac{\mathbf{1435}}{\mathbf{65}} \approx \mathbf{22.1\ s}$$

Examination of the minimum, first quartile, median, and maximum of a set of data can indicate important information about the distribution.

Important!

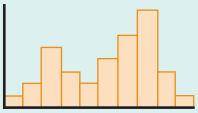
Five number summary and distribution shapes

The five numbers; the minimum, first quartile, median, third quartile and maximum (Q0, Q1, Q2, Q3, Q4) of a frequency distribution are called the **five number summary**.
A **symmetrical** distribution is roughly the same shape on either side of the middle. The mean, median and mode are about the same.
A **skewed** distribution is spread out more on one side of the median than the other.
A distribution that is **skewed to the left** is spread out more below the median than above the median. The mean is below the median and the mode usually above the median.
If the scores are spread more on the right, it is **skewed to the right**. The mean is above the median and the mode usually below the median.



Some distributions have two high parts. These are called **bimodal** distributions, although they don't always have two actual modes.

Bimodal



You can use the five number summary to see if a distribution is likely to be skewed or symmetrical. A skewed distribution will have greater distances on one side of the median than the other.

Example 6

a Find the five number summary for 30, 39, 30, 36, 34, 37, 25, 39, 21, 35, 33, 38, 33, 35, 34, 36, 30, 36, 35, 37, 32, 38, 33, 40, 35, 40, 22, 37, 33, 37.

b Comment on the five number summary for **a**.

c Find the five number summary for 7, 15, 10, 17, 7, 15, 5, 17, 10, 20, 8, 13, 7, 15, 6, 17, 10, 15, 8, 12, 6, 14, 12, 14, 12.

d Comment on the five number summary for **c**.

Solution

a Write the scores in order.

21, 22, 25, 30, 30, 30, 32, 33, 33, 33, 33, 34, 34, 35, 35, 35, 35, 36, 36, 36, 37, 37, 37, 37, 38, 38, 39, 39, 40, 40

Divide into 4 parts.

21, 22, 25, 30, 30, 30, 32, 33, 33, 33, 33, 34, 34, 35, 35 | 35, 35, 36, 36, 36, 37, 37, 37, 37, 38, 38, 39, 39, 40, 40

Write the five number summary.

The five number summary is 21, 33, 35, 37, 40.

b Look at the position of the median.

35 − 21 = 14 and 40 − 35 = 5, so the bottom half of the range extends further than the top half.

Comment on the distribution.

The distribution is probably skewed to the left.

c Write the scores in order.

5, 6, 6, 7, 7, 7, 8, 8, 10, 10, 10, 12, 12, 12, 13, 14, 14, 15, 15, 15, 15, 17, 17, 17, 20

Divide into 4 parts.

5, 6, 6, 7, 7, 7 | 8, 8, 10, 10, 10, 12, 12, 12, 13, 14, 14, 15, 15 | 15, 15, 17, 17, 17, 20

Write the five number summary.

The five number summary is 5, 7.5, 12, 15, 20.

d Look at the position of the median.

20 − 12 = 8 and 12 − 5 = 7, so the distribution is fairly balanced either side of the median.

Comment on the distribution.

The distribution is probably symmetrical.

Investigate: Use of technology

You can use digital technology to find the mean, median, mode, range, quartiles and interquartile range of a set of data. Scientific calculators can usually be used for at least the mean. Some will calculate all of these measures. Unfortunately, different technologies use quite different methods for data entry and retrieval of the information. They may also use different methods of calculation of the quartiles. The following small sets of data can be used to work out which method your calculator uses to find the quartiles.
Use both data sets to find which method your calculator uses.

Set A: 4, 8, 12, 16, 20		Set B: 4, 8, 12, 16, 20, 24			$Q1$		$Q3$	
$Q1$	$Q3$	$Q1$	$Q3$	**Method**	**n odd**	**n even**	**n odd**	**n even**
6	18	7	21	1	$(n+1)/4$	$(n+1)/4$	$(3n+3)/4$	$(3n+3)/4$
8	16	8	20	2	$(n+3)/4$	$(n+2)/4$	$(3n+1)/4$	$(3n+2)/4$
6	18	8	20	3	$(n+1)/4$	$(n+2)/4$	$(3n+3)/4$	$(3n+2)/4$
4	16	8	20	4	$(n+1)/4$	$(n+1)/4$	$(3n+3)/4$	$(3n+3)/4$
8	16	9	19	5	$(n+3)/4$	$(n+3)/4$	$(3n+1)/4$	$(3n+1)/4$

In this book, method 3 is used to calculate quartiles.

For typical numerical data, the mean is used as the measure of the centre because it is calculated using all the data. The range and interquartile range use only a few data points.

Important!

Mean deviation and standard deviation

The **mean deviation** of a set of data is the average of the absolute values of differences of each data item from the mean. The formula is $\frac{\sum |x - \bar{x}|}{n}$.
The **standard deviation** of a set of data is the square root of the average of the squares of the differences of each data item from the mean.

The formula is

$$\sigma = \sqrt{\frac{\sum (x - \bar{x})^2}{n}} = \sqrt{\frac{\sum x^2}{n} - (\bar{x})^2}.$$

The symbol for standard deviation is the Greek letter sigma, σ.

The use of the absolute value in the mean deviation makes calculation more difficult, so although this is intuitively more attractive it is rarely used. Your calculator may use the symbol *s* instead of sigma. The second formula is easier to use than the first one.

Example 7

Find the mean and standard deviation of these scores: 4, 7, 3, 5, 4, 8, 3, 10.

Solution

Find the total.

Total $= 4 + 7 + 3 + 5 + 4 + 8 + 3 + 10 = 44$

Find the mean.

$$\bar{x} = \frac{\sum x}{n} = \frac{44}{8} = 5.5$$

Method 1

Find the values of $(x - \bar{x})$.

$(x - \bar{x})$ **are** $^{-}1.5, 1.5, ^{-}2.5, ^{-}0.5, \ldots$

Find the value of $\sum (x - \bar{x})^2$.

$$\sum (x - \bar{x})^2 = (^{-}1.5)^2 + 1.5^2 + (^{-}2.5)^2 + \ldots$$
$$= 46$$

Divide by 8.

$$\frac{\sum (x - \bar{x})^2}{n} = \frac{46}{8} = 5.75$$

Find the square root.

$$\sigma = \sqrt{\frac{\sum (x - \bar{x})^2}{n}} = \sqrt{5.75} \approx 2.4$$

Method 2

Find the total of x^2.

$$\sum x^2 = 16 + 49 + 9 + 25 + 16 + 64 + 9 + 100$$
$$= 288$$

Divide by 8.

$$\frac{\sum x^2}{n} = \frac{288}{8} = 36$$

Subtract $(\bar{x})^2$.

$$\frac{\sum x^2}{n} - (\bar{x})^2 = 36 - 5.5^2 = 5.75$$

Find the square root.

$$\sigma = \sqrt{\frac{\sum x^2}{n} - (\bar{x})^2} = \sqrt{5.75} \approx 2.4$$

Write the answers.

The mean is 5.5 and the standard deviation is about 2.4.

If you are using a frequency table, you need to multiply the values by the frequencies. A frequency table should be extended to include xf and x^2f columns if you use it to work out the mean and standard deviation. Σf is used instead of n, Σxf instead of Σx, and $\Sigma x^2 f$ instead of Σx^2 in the formulas.

Example 8

Find the mean and standard deviation for the following table of learner driver test scores.

Score	20	21	22	23	24	25	26	27	28	29	30
Frequency	6	1	4	4	7	5	11	16	15	15	6

Solution

Rearrange the table so that the scores are listed vertically and put in xf and x^2f columns.

Score x	**Frequency** f	**Score × freq** xf	**Score^2 × freq** x^2f
20	6	120	2400
21	1	21	441
22	4	88	1936
23	4	92	2116
24	7	168	4032
25	5	125	3125
26	11	286	7436
27	16	432	11 664
28	15	420	11 760
29	15	435	12 615
30	6	180	5400
Total	90	2367	62 925

Write the formula for the mean.

$$\bar{x} = \frac{\sum xf}{\sum f}$$

Substitute and calculate.

$$= \frac{2367}{90} = 26.3$$

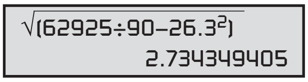
Write the formula for the standard deviation.

$$\sigma = \sqrt{\frac{\sum x^2 f}{\sum f} - (\bar{x})^2}$$

Substitute and calculate.

$$= \sqrt{\frac{62\,925}{90} - (26.3)^2}$$

Enter as [√] [(] 62925 [÷] 90 [−] 26.3 [x^2] [)] [=]



√(62925÷90−26.3²)

2.734349405

Write the answers.

The mean is 26.3 and the standard deviation is about 2.73.

The mean and median do not make sense for nominal data, even if it is coded by numbers. While the mean uses all the scores, its value can be dramatically changed by one or two very large or small scores. The median is not changed much by large or small scores.

You should use the measure of central tendency that suits your purpose. The mode is the *most popular* score, the median has *half the values on either side* and the mean is based on the *total*.

The median should be used for numeric data where a few very large or very small values may distort the value of the mean.

Example 9

Which measure of central tendency should you use for incomes in each of the following cases?

a You want to know whether the people in a town can afford to build a public pool.

b You want to know whether many people in a town will be able to afford to buy paintings for their homes.

c You want to know what tax rate is paid by the most people in a town.

Solution

a In this case, it is the income of the whole town that is important.	**The mean income should be used because it is about all the people.**
b In this case, it is the income of a proportion of the people that is important.	**The median income should be used to work out how many people will have spare money.**
c The tax rate is determined by the income bracket.	**The mode should be used to work out the most common income bracket.**

It is important for you to know that people often think something with a number must be true. A media report that says a survey showed that 64% of Australians were opposed to the use of nuclear energy could be very significant. However, if the survey was badly done or a very small sample it would not necessarily be true. The result by itself does not tell you whether you should take any notice of it. Larger samples are more reliable than smaller samples.

Example 10

A current affairs TV program interviewed robbery victims who were upset by the sentences the courts gave their attackers. They asked people to SMS two numbers to say Yes or No to mandatory sentences of 5 years for armed robbery. The next night they reported that 70% of the people who phoned in supported the proposal. Coincidentally, a phone survey of 2000 people spread across the state on the same subject by a polling company reported that only 38% supported mandatory sentencing, with an uncertainty of 4%. The TV host said their own survey had received 5320 SMSs, and they had made sure no-one could ring twice. Explain the differences in the results and suggest which is most likely to be correct.

Solution

Only voters can directly influence laws.	**The surveys should be directed to people of voting age, as their views determine elections and laws.**
Phone surveys are not as good as interviews or personally delivered forms.	**Neither method is particularly good as you do not really know who is answering a phone call, so you don't know if the person is an adult.**
Groups with a particular view may coordinate survey responses if they can.	**A ring-up TV survey is liable to be targeted by pressure groups making organised responses, so it is possible for the results to be very biased.**
Polling companies choose a representative sample.	**The group of people picked by the polling organisation is probably an unbiased group.**
The uncertainty is worked out from the size of the sample.	**The 4% uncertainty assumes that the group of people that answered are mostly those picked.**
Write a conclusion.	**Only people who are very interested will ring up the TV station, so the polling survey is more likely to be correct.**

Exercise 2.1 Calculating and interpreting data measures

Understanding

Extra questions
Exercise 2.1
MAT10SPEQ00004
See Example 1

1 Find the mode and range for each of the following sets of data.

a Yellow, Yellow, Black, Green, White, Green, Blue, Black, White, Green, Yellow, Green, Yellow, White, Blue, Green, Blue, Orange, Blue, Red, Red, Yellow, Yellow, Red

b Germany, Germany, Greece, Greece, Germany, Greece, Holland, India, Australia, Italy, Australia, Vietnam, Vietnam, Australia, Australia, Germany, New Zealand, Australia, Holland

c Mountain, Desert, Riverine, Plains, Coastal, Forest, Coastal, Desert, Desert, Desert, Coastal, Desert, Desert, Forest, Riverine, Plains, Forest, Desert, Riverine, Mountain, Forest, Plains, Plains

d Oliver, Jack, Noah, Noah, Ethan, Joshua, Cooper, William, Ethan, William, William, Lachlan, Oliver, Cooper, Jack, Joshua, Lachlan, Thomas, Noah, Lachlan, Jack, Jack, Noah, Oliver, Jack, Cooper, Thomas, Lachlan

e Charlotte, Olivia, Isabella, Ruby, Olivia, Ruby, Chloe, Isabella, Olivia, Charlotte, Olivia, Ruby, Ruby, Chloe, Isabella, Olivia, Chloe, Charlotte

See Example 2

2 Find the median and mode for each of the following sets of ordinal data.

a 6, 5, 2, 6, 3, 4, 6, 4, 3, 4, 4, 3, 4, 2, 2, 3, 7, 2, 4, 2, 4, 7, 4

b B, C, C, C, E, C, C, E, E, E, A, C, D, B, A, C, C, C, D, D, C, C, D, C, E

c N, A, A, SA, D, N, A, D, N, N, A, D, A, A, SD, A, A, SD, N, A, A, A, A, A, A, A, A, N, N, A, SA, N, N, N, N, A, D, N, A, A

d **, **, ****, ****+, ****, ****, ****, *, ***, ***, ***+, ****, **, ***+, ***+, ***, ***+, *, *, ****+, **, ***, ***, **, ***+, ***, ***+, ****+, ***, **, **, ***, *, ***, **

e PG, PG, M, M, X, PG, MA15+, X, PG, M, M, MA15+, M, G, M, PG, X, PG, M, PG, G, MA15+, MA15+, X, PG, PG, PG, G, M, M

See Example 3

3 Find the mean, median, mode and range for each of the following sets of data.

a 7, 3, 2, 1, 9, 10, 3, 10, 11

b 8, 11, 8, 16, 7, 19, 9, 13, 7, 14, 8, 10

c

Score	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Frequency	1	3	3	2	1	4	5	6	4	3	3	2	2	2	1	0	0	2

d

Score	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49
Frequency	6	3	6	5	6	13	12	5	5	1	8	2	4	4	2	1	1	3	1	1

e

Score	12	13	14	15	16	17	18	19
Frequency	4	19	18	10	6	6	3	4

See Example 4

4 Find the median, quartiles and interquartile range for each of the following sets of data.

a 26, 28, 22, 30, 23, 27, 23, 28, 22, 27, 19, 27, 26, 28, 21, 28, 21, 27, 25, 27, 24

b 4, 6, 2, 7, 5, 7, 6, 10, 2, 7, 3, 6, 4, 7, 4, 8, 5, 10, 5, 8, 6, 6, 2, 7, 3

c 30, 37, 33, 35, 30, 37, 32, 41, 31, 38, 33, 35, 34, 35, 34, 35, 33, 35, 32, 35

d 14, 15, 14, 16, 14, 18, 14, 17, 15, 15, 14, 18, 14, 15, 13, 16, 12, 18, 14, 16, 15, 15, 12, 17, 12, 16, 15, 16

e 41, 48, 41, 46, 43, 46, 44, 47, 30, 45, 40, 51, 43, 45, 37, 46, 39, 48, 41, 46, 43, 48, 41, 49, 41, 48

5 Find the median, quartiles and interquartile range for each of the following sets of data.

a

Score	2	3	4	5	6	7	8	9	10
Frequency	2	6	6	5	5	15	13	5	2

b

Score	15	16	17	18	19	20	21	22	23	24	25
Frequency	1	1	0	3	3	9	7	17	19	3	1

c

Score	3	4	5	6	7	8	9
Frequency	1	8	12	17	12	2	2

d

Score	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Frequency	2	3	3	3	1	7	6	3	7	8	2	6	7	6	5	2	2

e

Score	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
Frequency	2	5	2	8	7	12	3	3	2	5	4	2	3	3	3	3	1	2	2	0	1

9780170194860

6 Find the mean and standard deviation of the following sets of data.

See Example 7

- **a** 13, 15, 13, 19, 14, 19
- **b** 7, 9, 13, 2, 6, 6, 8, 12
- **c** 55, 48, 41, 44, 47, 53, 53, 55
- **d** 12, 16, 12, 16, 14, 15, 14
- **e** 97, 103, 100, 107, 92, 109, 100, 102, 92

7 Find the mean and standard deviation of the following sets of data.

See Example 8

a

Score	7	8	9	10	11	12	13	14
Frequency	2	8	3	6	5	2	1	3

b

Score	25	26	27	28	29	30	31	32	33	34	35
Frequency	2	5	2	4	10	9	7	5	3	2	1

c

Score	12	13	14	15	16	17	18	19	20
Frequency	2	0	5	4	3	5	7	10	3

d

Score	2	3	4	5	6	7	8
Frequency	3	9	1	2	1	0	4

e

Score	100	101	102	103	104	105	106	107	108
Frequency	2	0	3	0	2	4	5	7	2

Fluency

8 Find the five number summary for each of the following distributions.

See Example 6

- **a** 30, 31, 30, 32, 28, 31, 29, 32, 27, 33, 27, 31, 30, 41, 29, 36, 26
- **b** 12, 12, 9, 18, 6, 15, 4, 13, 6, 16, 6, 16, 2, 15, 5, 12, 3, 15, 6, 15, 11, 15, 11, 20, 4, 14, 9
- **c** 15, 20, 14, 24, 13, 35, 11, 21, 14, 24, 11, 21, 15, 16, 14, 23, 7, 17, 14, 16, 11, 22
- **d** 8, 14, 11, 14, 9, 16, 8, 14, 9, 14, 9, 13, 13, 16, 11, 14, 8, 13, 9, 15, 1
- **e** 55, 60, 37, 60, 50, 61, 51, 73, 55, 62, 50, 60, 57, 58, 54, 68, 58, 65, 57, 59, 52, 61, 52, 59, 57

9 Find the five number summary for each of the following distributions.

a

Score	30	31	32	33	34	35	36	37	38	39
Frequency	3	1	2	7	11	7	4	2	3	1

b

Score	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Frequency	1	0	7	8	9	7	7	5	0	4	3	1	2	1

c

Score	8	9	10	11	12	13	14	15	16	17	18
Frequency	2	5	8	7	4	4	8	4	3	4	4
Score	19	20	21	22	23	24	25	26	27	28	29
Frequency	2	0	0	1	1	0	1	0	0	1	1

d

Score	101	102	103	104	105	106	107	108	109
Frequency	3	3	7	10	7	3	4	2	1

e

Score	23	24	25	26	27	28	29	30	31	32	33	34	35	36
Frequency	1	1	1	3	9	6	6	11	5	5	4	5	3	1

See Example 9

10 Which data measure(s) should be used in each of the following cases?

a You want to know what you are likely to earn after a few years working in a local furniture factory.

b A caterer is going to set up a sandwich bar in a local industrial estate. In their first week, what size of iced coffee should they stock in their drinks fridge?

c A security company is doing weekly payrolls in cash on-site for an orchardist to pay his casual pickers, who earn a variety of amounts.

d A band is ordering tour t-shirts to sell at their concerts. They only want to bother with one size.

e How much can you expect families to spend at a theme park?

11 5 friends were playing Yahtzee. At the end of the game Peter came first with a score of 233, Simon came second with 186, Paul and Andrew came equal third with 165 and David came last with 108. Find the ranges of the names, places and scores.

See Example 5

12 The age distribution of the population of Queensland in 2003 is shown below. Find the average age of Queenslanders.

Age	0–9	10–19	20–29	30–39	40–49	50–59
Number	515 990	543 501	524 769	563 335	558 129	477 056

Age	60–69	70–79	80–89	90–99	100–109
Number	299 655	205 678	96 896	15 456	574

13 The results of a Chemistry class' titrations of 2 M HCl against 100 mL of 0.5 M NaOH are shown below. Find the average quantity of HCl used.

HCl (mL)	23.6–24	24.1–24.5	24.6–25	25.1–25.5	25.6–26	26.1–26.5
f	2	4	5	7	3	0

HCl (mL)	26.6–27	27.1–27.5	27.6–28	28.1–28.5	28.6–29
f	0	1	2	1	2

Problem solving

14 The table shows the mass of fruit from each plant in a new variety of tomatoes.

a What is the typical weight of fruit?

b How much fruit would you expect from 5000 plants?

c Is the sample sufficient to make good conclusions?

Mass of fruit (kg)	**Frequency**
3	1
4	9
5	11
6	8
7	5
8	7
9	4
10	1
11	3
12	1

Worked solutions
Exercise 2.1
MAT10SPWS00037

⑮ The rally lengths of tennis games in the state titles were as follows. Find the mean, median, mode and range of the rally lengths.

Number of shots played	1	2	3	4	5	6	7	8	9
Frequency	80	63	95	42	47	39	24	13	4

a An average game has 6 rallies. How many rally shots would you expect in 5 games?
b An average set has 20 games. How many rally shots would there be in an typical set?
c Is this sample sufficient to make good conclusions?

16 A real estate company is doing a report for its shareholders, predicting the year ahead. What measure of house prices should they use to work out each of the following?
a Earnings from commissions on sales.
b Typical prices of houses they will offer for sale.
c Amounts earned by their salespeople, who work on commission.
d Amount earned from their 'prestige properties' division.
e Amount earned by their 'family specialist' division.

17 The mean, median and mode of the incomes of the 97 employees, including the management at a large warehouse-type hardware store were \$1240, \$945 and \$830. The lowest wage was \$730.
a What would be the effect of a 10% increase in everyone's wages?
b What would be the effect of a \$100 increase?
c Estimate the five number summary for the original wages.

18 The auction prices of some paintings are shown below.

Price (\$)	300–499	500–699	700–899	900–1099	1100–1299	1300–1499
Number	1	4	12	14	4	5

a Find the average price.
b How much money would you expect to change hands if 165 paintings were sold?

Reasoning

19 The information below shows the ratings of accommodation in the Brisbane CBD. Explain the typical rating.

★★★★, ★★★★★, ★★★★, ★★★★,
★★★★★, ★★★★, ★★★★, ★★★★,
★★★★, ★★★★, ★★★★, ★★★★★,
★★★★, ★★★★★, ★★★★, ★★★★,
★★★★, ★★★★, ★★★★, ★★★★,
★★★★, ★★★★

20 Explain the positions of the median, mean and mode when a distribution is skewed to the right.

㉑ 20 people have a median height of 170 cm and a mean height of 169.4 cm. They are joined by a basketball team of 6 players with an average height of 180 cm. Explain what happens to the mean and median.

Worked solutions
Exercise 2.1
MAT10SPWS00037

22 Comment on the five number summaries for each of the following distributions.

- **a** 6, 16, 7, 16, 6, 13, 5, 17, 6, 13, 9, 13, 8, 10, 9, 10, 6, 20, 7, 12, 7, 15, 8, 17
- **b** 164, 172, 163, 179, 163, 170, 163, 183, 164, 179, 163, 165, 163, 167, 164, 171, 164, 168, 162, 178
- **c** 8, 13, 3, 13, 10, 10, 0, 16, 0, 11, 10, 14, 8, 13, 8, 11, 9, 11, 10, 11, 7, 13, 8, 19
- **d** 39, 50, 36, 48, 41, 47, 37, 51, 41, 45, 45, 47, 38, 50, 43, 47, 43, 47, 40, 49, 44, 46, 35, 49, 44, 48, 35, 46, 44, 47, 45, 51
- **e** 14, 15, 15, 16, 15, 18, 14, 20, 13, 18, 13, 17, 15, 19, 13, 19, 15, 17

See Example 10

23 The 'Top 40' hits from different radio stations can be different. One radio station compiles its 'Top 40' hits from a survey of the sales of single CDs in a music store in the city that regularly sells about 3000 singles each week. A chain of stores in suburban shopping centres sells about the same number of singles, but gets a different 'Top 40' that a suburban radio station uses. Explain how the stations can get different 'Top 40's in the same city.

Worked solutions
Exercise 2.1
MAT10SPWS00037

㉔ Polling companies try to predict the way that voters will react at an election. Different polls commonly get different results, and frequently the results they get are different to actual election results. There are 150 electorates in Australia of almost equal size. One polling company does a phone poll of 3000 people chosen at random from the electoral rolls, spread evenly through Australia. Another does a poll of 100 electorates in urban areas and also uses 3000 people. The polls are done 3 days before the election, and they both predict a victory (by 20 seats and 10 seats respectively) for the party that is actually defeated by 15 seats at the actual election. Explain how these surveys properly conducted by reputable organisations can get the result completely wrong.

25 The Australian Bureau of Statistics regularly checks the earnings of Australians. It has found that when it uses information from employers or employees it gets different results, even when it surveys employees of the employers that it surveys. Explain how this could be so.

2.2 Constructing and interpreting data displays

Statistical data displays are used for different reasons. Some are best suited to communicating the data, while others are more suitable for comparison or analysis.

Weblink

Visualising how a population reaches 7 billion

MAT10SPWB00002

TLF Learning object

Graph investigator (L5903)

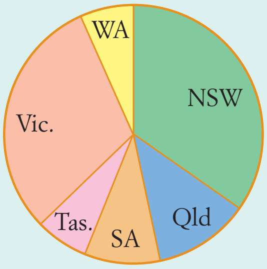
MAT10SPIN00002

Important!

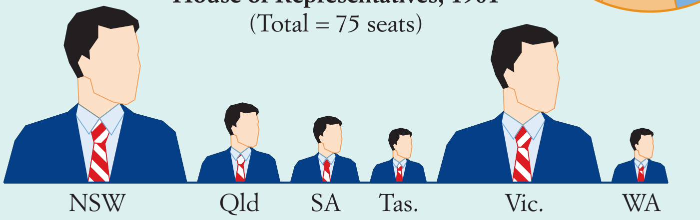
Types of data displays

Circle graphs (**pie graphs** or **sector graphs**), **picture graphs** (**pictograms** or **pictographs**) and **divided bar charts** give you an excellent impression of overall relationships between different aspects of statistical information. In a circle graph, the angles of the sectors show the parts of a total amount. In a divided bar graph, the lengths of the sections show the parts of the total amount.

House of Representatives, 1901
(Total = 75 seats)

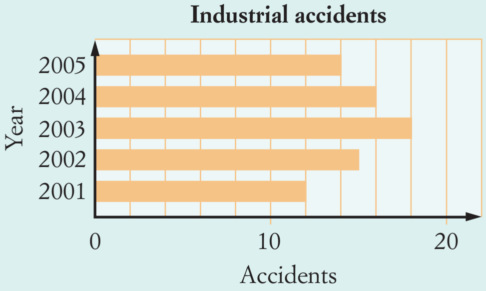


House of Representatives, 1901
(Total = 75 seats)

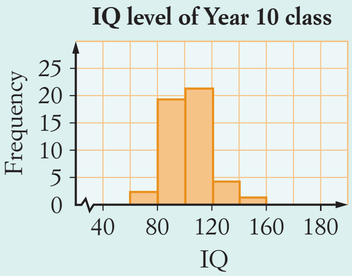


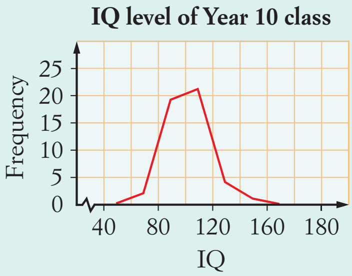
House of Representatives, 1901
(Total = 75 seats)

You can read **bar charts**, **column graphs** and **line graphs** quite accurately, so they are useful for making predictions or finding precise relationships.

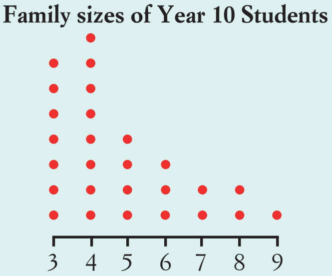


Histograms and **polygons** are used to show frequencies of grouped data. There is no gap between the columns because the data could be continuous. The score groups (such as 25–29, 30–34, etc.) are called classes. The actual upper and lower boundaries of a class are called the **upper class limit** and **lower class limit**. You put the points of **frequency polygons** in the centres of class groups. It is usual to show frequencies of zero before the first class and after the last class. Polygons can also be used for discrete data.





You use **dot plots** and **stem-and-leaf plots** to get a quick idea of the nature of a distribution. You can make them while you are still collecting data and add new data to them as you go.

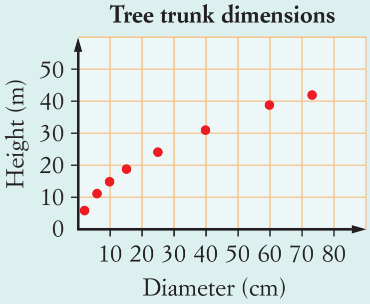


Golden-oldies concert audience ages

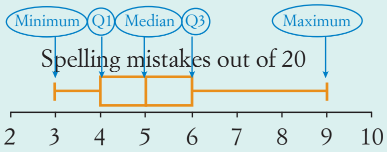
Key: 5 | 9 = 59 years old

2	5
3	2 9
4	3 6
5	4 5 9 9
6	0 2 4 7 7 8 9
7	3 6 7
8	1

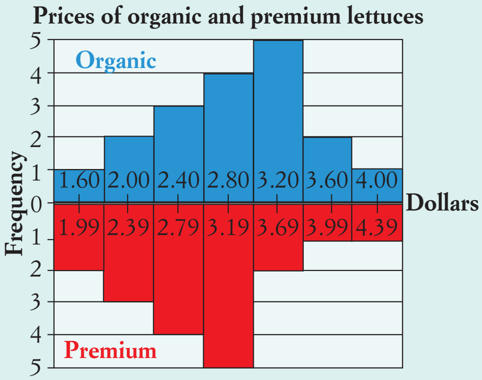
A **scatter plot** (**scatter graph**) is used to show bivariate data, particularly where the variables could be related. For convenience, the **scales** of a graph or chart usually go up in 1s, 2s, 4s, 5s, 10s and so on.



A **box-and-whisker plot** (**boxplot**) is used to show the spread of data. You put a rectangle extending from the lower quartile to the upper quartile, divided at the median. Then draw lines from the quartiles to the maximum and minimum. You can make these clearer with short vertical lines. Put the scale underneath.



Back-to-back (**side-to-side** or **parallel**) data displays are used to compare different distributions.



Reaction times (Key: 6|2 = 0.26 s = 2|6)

Left hand		Right hand
9 9 6 6	2	3 6 6 7 8 9 9 9 9
9 8 7 7 6 5 5 4 4 4 2 2 1 1	3	1 1 2 2 2 2 5 5 6 6 7 9
8 8 5 5 3 3 2 2	4	0 0 4
7	5	4 4
	6	6

Example 11

The table below shows the temperature of a beaker of hot water as measured at 2-minute intervals by students working in groups.

Time (min)	0	2	4	6	8	10	12	14	16
Temperature (°C)	60	58	56	54	45	51	50	48	47

a Draw a scatter graph of the results.

b Does the temperature fall evenly?

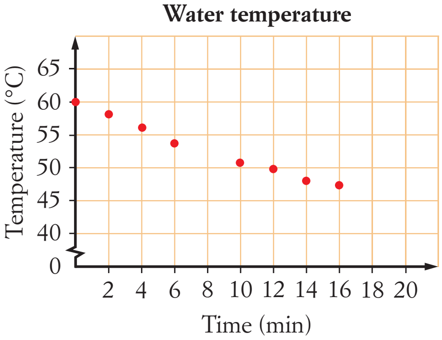
CAS TI-Nspire exercise
Statistics
MAT10SPTI00002

CAS ClassPad exercise
Statistics
MAT10SPCP00002

Worksheet
Statistics 1
MAT10SPWK00002

Solution

a The temperature for 8 minutes has clearly been misread. Discard it as non-compliant and draw the graph.



b Overall, it looks as if the temperature falls about $1\frac{1}{2}$ °C every 2 minutes.

The temperature appears to have fallen evenly over the time it was measured.

Example 12

a Draw a box-and-whisker plot of the following data.
26, 27, 23, 27, 24, 27, 23, 28, 24, 27, 22, 27, 26, 27, 25, 28, 26, 26, 26, 26

b Comment on the distribution.

Video tutorial
Box-and-whisker plots
MAT10SPVT10004

Technology worksheet
Excel worksheet: Five number summary
MAT10SPCT00002

Solution

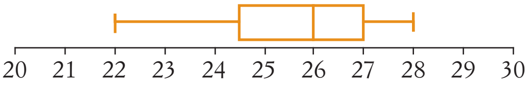
a Write the data in order and divide into 4 parts.

22, 23, 23, 24, 24 | 25, 26, 26, 26, 26 | 26, 26, 27, 27, 27 | 27, 27, 27, 28, 28

Write the 5 number summary.

The 5 number summary is 22, 24.5, 26, 27, 28

Draw the boxplot.



b *Q*1 is further from the median than *Q*3, and the minimum is further from the median than the maximum.

The distribution is negatively skewed.

Example 13

Technology: GeoGebra
Boxplot and dot plot
MAT10SPCT00002

Compare a boxplot to the dot plot and histogram of this data.
5, 9, 6, 10, 5, 6, 3, 6, 5, 10, 3, 9, 5, 8, 5, 13, 5, 12, 6, 13, 6, 14, 5, 8, 5, 13, 5

Solution

Arrange in order and divide into 4 groups.

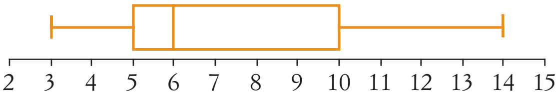
3, 3, 5, 5, 5, 5, 5, 5, 5, 5, 5, 6, 6, 6, 6, 6, 8, 8, 9, 9, 10, 10, 12, 13, 13, 13, 14

Write the 5 number summary.

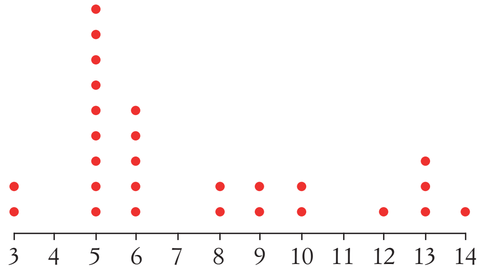
The 5 number summary is 3, 5, 6, 10, 14

Draw a boxplot.

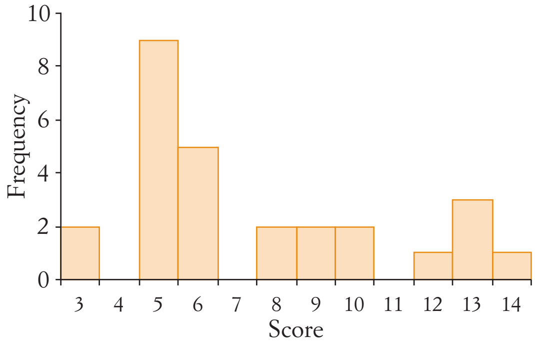
Box-and-whisker



Draw a dot plot.



Draw a histogram.



Compare the shapes.

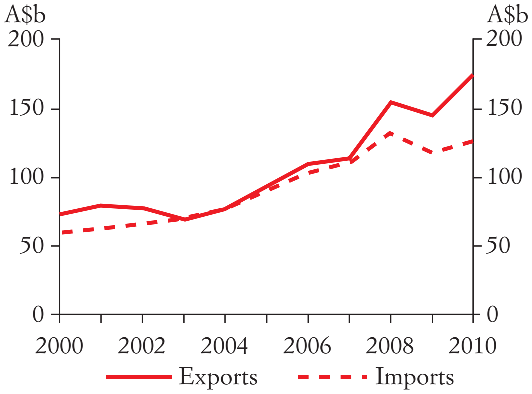
The dot plot and histogram spread out to the same extent as the boxplot and have their highest parts between the first quartile and the median, where the box-and-whisker plot has its shortest section.

Back-to-back or parallel plots are used when a population divided naturally into two parts, such as male and female, there are different populations or information is obtained at different times about the same population.

Investigate: Trade with Asia

The following graph appeared in the report *Australia's trade with East Asia*, published by the Department of foreign Affairs and Trade.

Australia's trade in goods and services with East Asia



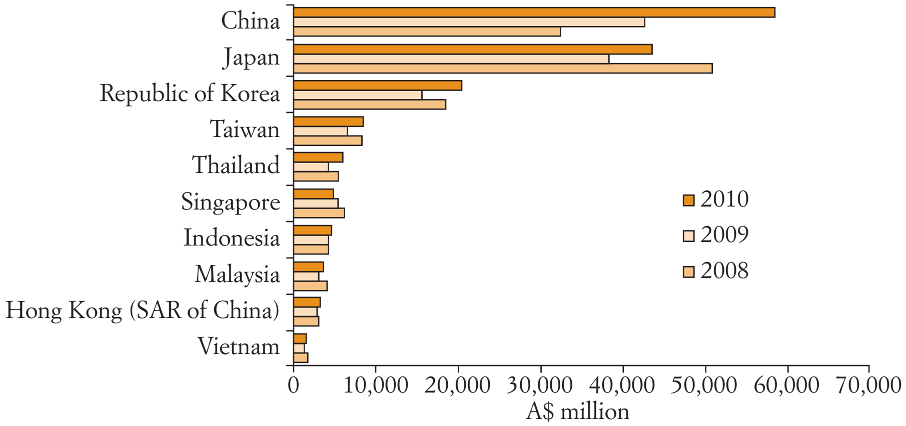
Based on ABS trade data on DFAT STARS database and ABS catalogue 5368.0.

In the report it says that:

- The value of total trade in goods and services increased 14.9 per cent to $302.9 billion in 2010 – since 2005, total trade values have increased by an average of 10.2 per cent per annum.
- In comparison, Australia's total trade with the world rose 9.6 per cent to $552.4 billion.
- East Asia accounted for 54.8 per cent of Australia's total trade in goods and services.

The whole report is available at www.dfat.gov.au. Use the internet to look up the report and investigate the basis of the statistics that are quoted in the report. You may also need to investigate the statistics that relate to the report compiled by the ABS.

The graph below of Merchandise Exports also appears in the report.



What does it show about changes of Australian exports to Asia?

In both the DFAT and ABS statistics, some data that has been collected is not available. It says that it is 'commercial in confidence' or just 'confidential in ABS trade statistics'. What does this mean, and why is some data withheld? Try to find the value of Australian wheat and gold exports during the last 20 years.

Example 14

a Make a back-to-back stem-and-leaf plot of the following information about the Maths and Science marks of the same Year 10 class.

Maths: 50, 70, 46, 71, 37, 72, 40, 70, 53, 72, 45, 56,
55, 55, 35, 61, 34, 67, 39, 69, 22, 84, 52, 57

Science: 41, 63, 55, 59, 42, 79, 21, 74, 44, 77, 48, 92,
46, 81, 54, 56, 24, 66, 28, 86, 53, 71, 39, 83

b What can you say about the marks in Maths and Science?

c Comment on the means, medians and ranges of the distributions.

Solution

a The lowest mark is 21 and the highest is 92, so make the stems go from 2 to 9. The units will be the ‘leaves’. Record the Maths marks on the left and the Science marks on the right.

Maths and Science marks

Maths Key: 6|5 = 56

Science Key: 5|6 = 56

Maths		Science
2	2	1 4 8
9 7 5 4	3	9
6 5 0	4	1 2 3 6 8
7 6 5 5 3 2	5	5 3 4 5 6 9
9 7 1	6	3 6
2 2 1 0 0	7	1 4 7 9
4	8	1 3 6
	9	2

b The centres, spreads and shapes of the plots are about the same.

The class performed as well in Maths as they did in Science, although there were more high scores in Science.

c Find the total of the Maths results.

Total = 1312

Find the mean of the Maths results.

$$\text{Maths mean} = \frac{1312}{24} \approx 54.7$$

Use the plot to find the median.

$$\text{Maths median} = \frac{55 + 55}{2} = 55$$

Find the range.

$$\text{Maths range} = 84 - 22 = 62$$

Find the total of the Science results.

Total = 1382

Find the mean of the Science results.

$$\text{Science mean} = \frac{1382}{24} \approx 57.6$$

Use the plot to find the median.

$$\text{Science median} = \frac{55 + 56}{2} = 55.5$$

Find the range.

$$\text{Science range} = 92 - 21 = 71$$

Compare the results.

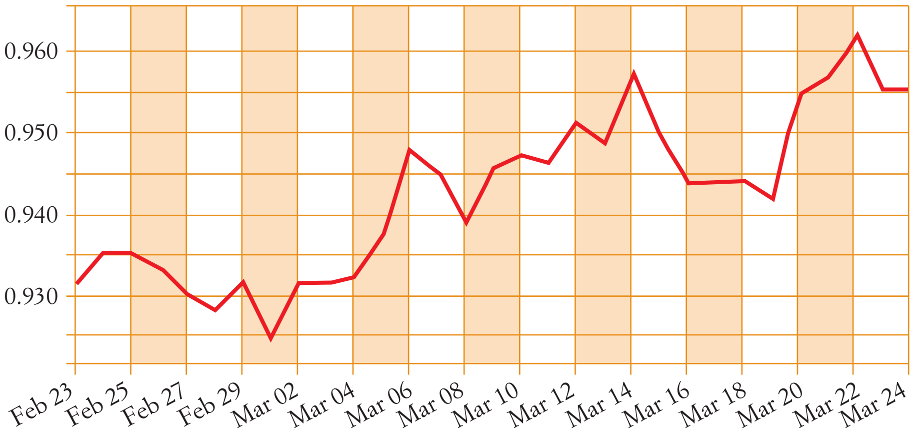
The medians for Maths and Science were almost the same, but the Science average was a little higher and the Science’s range was greater. Looking more closely at the distributions, *Q3* was higher for Science, which explains the higher mean.

A graph may be drawn in a way that emphasises particular features. The temperature of a hospital patient is normally drawn with a vertical scale between about 36°C and 41°C. Other graphs drawn with non-zero origins may be misleading. Graphs over short time periods, graphs using missing or selected data and graphs with uneven scales can also be misleading. Sometimes graphs may be misleading because of uneven scales or use of 2D or 3D representation, although in these cases it is likely to be deliberate.

Example 15

The graph below shows the exchange rate between the Australian and US currencies in the month from 23 February 2012 to 24 March 2012. The graph was in a report from Exchange-Rates.org.

Australian Dollars (AUD) to 1 US Dollar (USD)



min = 0.92521 (March 1) avg = 0.94263 max = 0.96205 (March 22)

a What does the graph (apparently) show?
b How could this graph be misleading?

Solution

a What is the trend?	**The graph shows a fairly steady increase in the value of the US dollar.**
b What is the time period for the graph?	**The graph shows only one month**
What is the vertical scale?	**The graph shows only part of the scale**
What other data could be used?	**The graph compares the US dollar only to the Australian dollar.**
Summarise the ways it could be misleading.	**The graph shows only a short time period, only compares the value of the US dollar with the Australian dollar and has a very large vertical scale that exaggerates changes.**

Exercise 2.2 Constructing and interpreting data displays

Understanding

Extra questions
Exercise 2.2
MAT10SPEQ00005

1 Make a stem-and-leaf plot of the following times for drivers to stop at a red light.

4.2 3.0 1.9 3.7 4.6 2.6 3.5 3.8 4.1 3.7 3.4 3.1 2.7 0.2 2.1
1.8 4.3 3.4 3.6 2.2 3.1 5.0 2.6 24 1.9 2.6 2.7 2.8 3.4 2.3

2 Draw a stemplot of the following marks that students said they got for a university entrance exam.

56 40 38 40 72 92 63 88 69 78 66 87 60 72 64
44 60 320 35 36 52 58 46 84 75 57 69 63 46 51
95 66 84 42 54 56 37 62 50 46 50 91 88

See Example 11

3 Draw scatter plots for each of the following tables.

a

x	7	22	15	12	25	8	18	5	10
y	10	36	18	20	42	12	28	2	12

b

x	18	27	10	12	15	22	30	5	8	20	24
y	3.2	3.8	17	12.8	8	6.8	5	17	15.2	5	2.6

c

x	25	7	15	25	22	18	5	9	12	13	6
y	62	13	27	57	53	46	7	19	18	21	5

See Example 12

4 Draw box-and-whisker plots for each of the sets of data.

a 26, 27, 24, 28, 25, 28, 24, 28, 24, 27, 24, 30, 25, 28, 26, 28, 26, 27, 23, 28, 24, 30, 27, 30

b 5, 7, 5, 8, 5, 9, 5, 9, 7, 8, 4, 8, 6, 8, 4, 9, 7, 10, 4, 8, 7, 7, 2, 9, 6, 9, 6

c 23, 76, 57, 86, 53, 78, 36, 67, 20, 79, 34, 61, 11, 80, 57, 67, 32, 72, 49, 97, 43, 65, 25, 62, 30

Fluency

See Example 13

5 Compare box-and-whisker plots of each of the following sets of data to histograms and dot plots of the same data.

a 14, 19, 13, 18, 14, 15, 12, 16, 14, 15, 15, 21, 10, 19, 15, 19, 11, 17, 13, 16, 11, 18, 11, 17, 14, 15, 13, 21, 15, 15, 15, 17, 14, 16, 10, 17

b 100, 102, 98, 104, 99, 102, 101, 103, 99, 104, 101, 104, 100, 103, 98, 102, 101, 103, 99, 102, 102, 102, 100, 102, 96, 104, 102, 103, 100

c 21, 28, 22, 24, 21, 26, 21, 24, 22, 26, 22, 28, 23, 28, 21, 23, 23, 25, 22, 23, 20, 24, 22

Problem solving

6 The data below shows the numbers of customers and total sales in a music store over a period of 2 weeks.

Day	M	T	W	Th	F	S
Customers	41	52	53	59	70	63
Sales ($)	1705	2620	2525	3005	3150	2555
Day	M	T	W	Th	F	S
Customers	38	48	56	62	69	65
Sales ($)	1990	1880	2240	3210	2755	2435

a Draw a column graph of the sales over the two weeks.

b Can you make any conclusion from the graph?

c Draw a scatter graph of the customers and sales.

d What can you conclude from the scatter graph?

⑦ The table below shows the minimum and maximum temperatures in a Queensland town over a period of 2 weeks.

Worked solutions
Exercise 2.2
MAT10NAWS00038

Min. (°C)	16	18	20	22	18	19	20	17	15	18	22	23	19	17
Max. (°C)	25	28	34	34	27	31	31	28	27	27	36	34	29	26

a Draw a scatter plot to show the results.

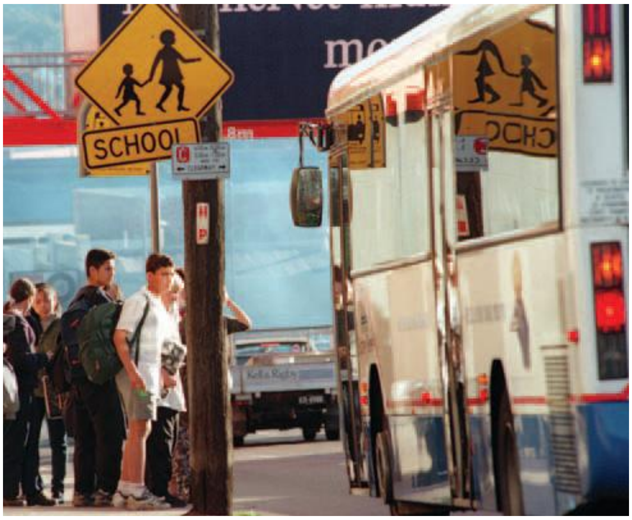
b Does there seem to be a relationship between the minimum and the maximum?

8 The information below shows the times taken to travel to school reported by some Year 10 students in Victoria and Western Australia.

See Example 14

a Use a back-to-back stem-and-leaf plot to compare the results.

b Compare the distributions using the means, medians and ranges.



Victorian students: 15, 18, 50, 40, 1, 15, 10, 22, 10, 5, 35, 10, 45, 20, 15, 5, 15, 25, 10, 10, 9, 25, 10, 37, 4, 10, 20, 20

Western Australian students: 30, 10, 15, 20, 20, 10, 3, 5, 10, 20, 2, 10, 10, 3, 36, 50, 20, 10, 5, 15

9 The information below shows the 2011 weekly running costs of a range of small and medium cars from the RAC WA, rounded to the nearest dollar.

a Use a back-to-back stemplot (with costs in \$5 groups) to compare them.

b Use the means, medians and ranges to compare the distributions.

Small cars \$cost/week: 42, 49, 48, 51, 47, 49, 46, 42, 56, 51, 39, 49, 61, 44, 32, 42, 41
Medium cars \$cost/week: 52, 51, 60, 54, 57, 60, 40, 60, 52, 45, 55, 56, 62

10 Use a box-and-whisker plot to comment on the following times taken by some Year 10 students to answer an online questionnaire. The times were automatically recorded by the website.
13, 31, 17, 14, 22, 22, 28, 20, 8, 31, 12, 11, 15, 21, 10, 20, 16, 19, 8,
18, 21, 16, 22, 16, 18, 26, 11, 26, 18, 10, 23, 17, 6, 12, 24, 19, 16, 19, 27

⑪ Use a box-and-whisker plot to comment on the following times that people said they spend watching TV over a week.
4, 14, 5, 13, 6, 22, 5, 8, 22, 3, 3, 3, 10, 1, 0, 8, 8, 10, 15, 15, 2, 1, 5, 5, 11, 1, 6, 15, 5,
10, 2, 1, 1, 2, 7, 3, 4, 4, 3, 18, 21, 1, 8, 2, 10, 3, 2, 2, 8, 8, 15, 23, 4, 2, 2, 20, 7, 5, 21

Worked solutions
Exercise 2.2
MAT10NAWS00038

Reasoning

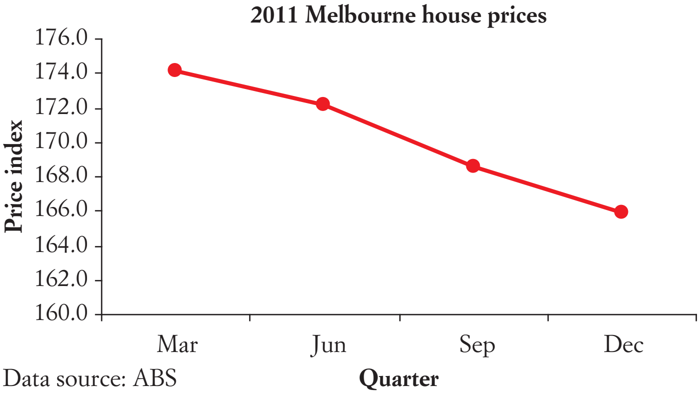
12 Many websites automatically record data. The online sales recorded each day by such an Australian hobby shop site was displayed as a box-and-whisker plot each week for the online manager. Explain whether you think it would provide good feedback.

Worked solutions
Exercise 2.2
MAT10NAWS00038

⑬ A customer service toll-free line had some automated and some operator parts. One of the automated parts allowed people to state whether they were happy with the service provided. Customers were asked to 'stay on line' after their query was answered to answer this survey question. They could say they were Very happy, Happy, Satisfied, Not satisfied or Very unhappy. Explain whether a boxplot of the responses on a scale from 1 to 5 would be a good way to show the results.

See Example 15

14 The graph below shows the price index for established houses in Melbourne.

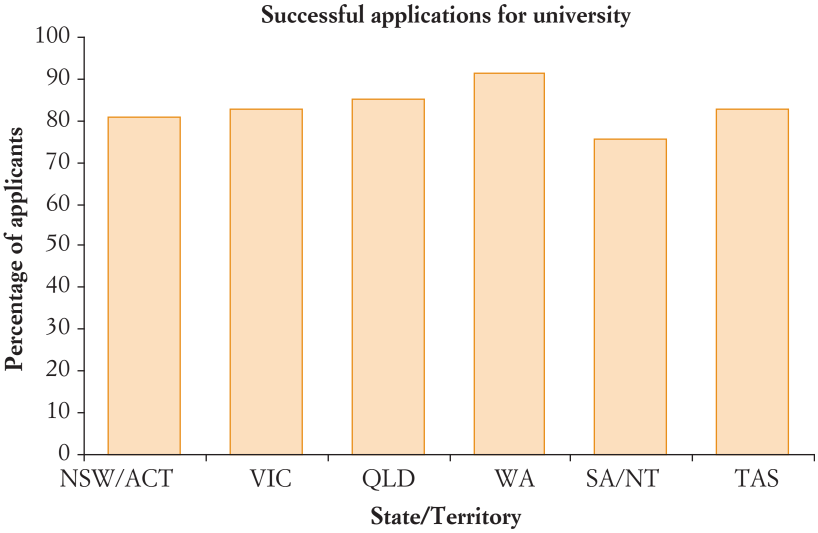


Data source: ABS

a What does the data apparently show?

b How might this graph be misleading?

15 The graph below shows the proportion of students gaining places at universities in different parts of Australia. A report says that 'There are more successful applications in Western Australia than any other state'. Explain why the graph does not support this statement.



9780170194860

2.3 Analysing data

The first step in analysing data is really to determine whether the data has been collected in a reasonable way. If the data is poor, then any conclusions or decisions made from it will also be of poor quality. You should remember the terms below from your work in previous years.

Important!

Data quality

A **fair sample** is collected from a sample that is like the rest of the population. A **biased sample** is not typical of the population. **Biased methods** and **biased questions** are likely to influence results in a particular direction. **Fair** ones are not likely to influence results in a particular way.
Non-compliant data is obviously wrong, like someone's age being 164.
Outliers are data items that are a long way from the rest of the data, but that could be correct. A simple criterion for an outlier is that it lies beyond 1.5 IQRs below the first quartile or above the third quartile.

If data is collected in a biased way, contains many non-compliant items or lots of outliers, or is collected using very small samples, then it is likely to be of poor quality.

Example 16

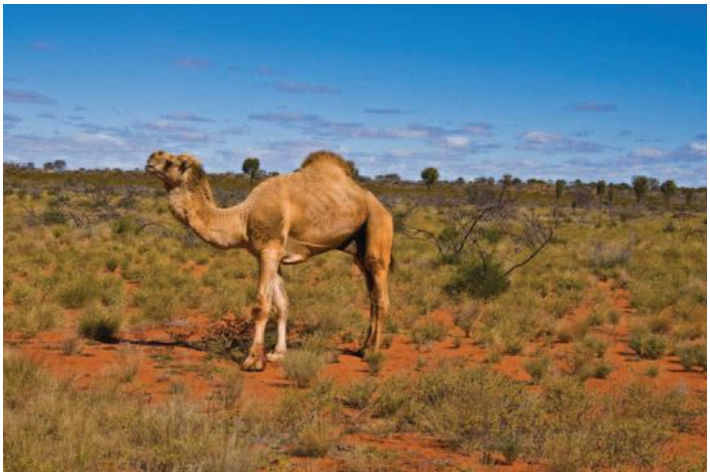
A survey of shoppers about their holiday plans is conducted as a competition by leaving survey forms in fast food outlets, with prizes for the first 3 names drawn. Respondents are asked questions about their income, family, desired holiday destinations and the kinds of holidays they prefer. They complete the forms, fill in their name, phone number, email and address on the back and put the entry in boxes at the shops. Explain why this is likely to give poor quality data.

Solution

Many people will not bother doing it.	**There will be a non-response bias – only people who are interested in the prizes will do the survey.**
People are often reluctant to give their true income, or may not even know it.	**Responses are unlikely to be accurate.**
People are also reluctant to give their contact details.	**Responses are unlikely to be able to be checked.**
People who really want the prizes may complete multiple forms.	**Multiple entries will cause bias.**
Only people who buy fast food will be in the shops.	**The survey will be biased towards those who often buy fast food.**
Summarise the problems.	**The survey answers are unlikely to be accurate, and the sample is quite biased in a number of different ways.**

Investigate: Unintended consequences

Research using statistics can be used to predict what will happen in the future. This can lead to actions to change what is anticipated, or to avoid particular results. Unfortunately, the results of actions are not always what we mean them to be. For example, camels were imported into Australia to carry materials and supplies for work on the railway line to Alice Springs. When they were no longer needed, they were released and have since become a problem in central Australia. Work in groups to consider what the unintended effects of each of the following actions were or might be.



- Keeping cats as pets in Australia
- Importation of *Bufo marinus* (Cane toads) from South America to Queensland to control cane beetles
- The one-child policy in China
- Planting of eucalyptus tree for shade in California and Italy
- Closure of phosphate mines in Nauru
- Use of fluorocarbons and chlorofluorocarbons in refrigerators
- Prohibition of the use of alcohol in the United States between 1919 and 1933

Example 17

Use the interquartile range to find if any of the following sets of positive data have outliers.

a 25, 33, 30, 30, 29, 31, 26, 30, 24, 32, 22, 17, 28, 30, 30

b 31, 91, 55, 59, 52, 82, 43, 62, 30, 57, 24, 96, 29, 70, 34, 73, 8, 80, 45, 72, 53, 59, 39, 75, 29

c 11, 12, 9, 12, 11, 13, 12, 14, 11, 14, 7, 13, 9, 23, 9, 18, 9, 15, 12, 16

Solution

a Arrange in order. **17, 22, 24, 25, 26, 28, 29, 30, 30, 30, 30, 30, 31, 32, 33**

Divide into 4 parts. **17, 22, 24, 25, 26, 28, 29, 30, 30, 30, 30, 30, 31, 32, 33**

Find $Q1$, $Q3$ and IQR. **$Q1 = 25$, $Q3 = 30$, IQR = 5**

Find the outlier limits. **Lower limit for outliers $= 25 - 1.5 \times 5 = 18.5$**

Upper limit for outliers $= 30 + 1.5 \times 5 = 37.5$

State the result. **17 is an outlier because it is more than 1.5 times the IQR below $Q1$.**

b Arrange in order. **8, 24, 29, 29, 30, 31, 34, 39, 43, 45, 52, 53, 55, 57, 59, 59, 62, 70, 72, 73, 75, 80, 82, 91, 96**

Divide into 4 parts. **8, 24, 29, 29, 30, 31 | 34, 39, 43, 45, 52, 53, 55, 57, 59, 59, 62, 70, 72 | 73, 75, 80, 82, 91, 96**

Find $Q1$, $Q3$ and IQR. **$Q1 = 32.5$, $Q3 = 72.5$, IQR = 40**

Find the outlier limits. **Lower limit for outliers $= 32.5 - 1.5 \times 40 < 0$**

Upper limit for outliers $= 32.5 + 1.5 \times 40 = 112.5$

State the result. **There are no outliers.**

c Arrange in order **7, 9, 9, 9, 9, 11, 11, 11, 12, 12, 12, 12, 13, 13, 14, 14, 15, 16, 18, 23**

Divide into 4 parts. **7, 9, 9, 9, 9 | 11, 11, 11, 12, 12 | 12, 12, 13, 13, 14 | 14, 15, 16, 18, 23**

Find $Q1$, $Q3$ and IQR. **$Q1 = 10$, $Q3 = 14$, IQR = 4**

Find the outlier limits. **Lower limit for outliers $= 10 - 1.5 \times 4 = 4$**

Upper limit for outliers $= 14 + 1.5 \times 4 = 20$

State the result. **23 is an outlier because it is more than 1.5 times the IQR above $Q3$.**

Bivariate data has two values for each item. For example, collecting the ages and heights of children together makes a bivariate data set. Each child has two values that are tied together. Bivariate data can have both values categorical, one categorical and one numeric, or both numeric. When one or both values are categorical, a **two-way table** is often the best way to analyse the data.

Worksheet
Statistics 2
MAT10SPWK00003

Puzzle sheet
Statistical tables
MAT10SPPS00006

Example 18

The table below shows the results of a survey of Year 10 students about smoking and colds. It is claimed that this survey shows that smoking causes you to have more colds. Some information is missing.

		Number of colds last year					
		0	**1**	**2**	**3**	**4**	**Total**
Smoking	**Smokers**	0	3		6	2	
	Non-smokers	4		8	4	1	27
	Total	4	13		10	3	43

a Complete the missing parts of the table.
b Did smokers or non-smokers have more colds?
c Calculate the percentages of smokers and non-smokers who had more than 2 colds.
d Does smoking appear to affect health?
e Are the conclusions of this survey likely to be reliable?
f Is the assertion that 'smoking causes more colds' supported by the data?

Solution

a Complete the 'Total' column on the right. **Total smokers = 43 − 27 = 16**

Complete the '1 cold' column. **Non-smokers with 1 cold = 13 − 3 = 10**

Check the non-smokers total. **4 + 10 + 8 + 4 + 1 = 27** ✓OK

Complete the 'Smokers' row. **Smokers with 2 colds = 16 − 11 = 5**

Complete the '2 colds' column. **5 + 8 = 13**

Check the 'Total' row. **4 + 13 + 13 + 10 + 3 = 43** ✓OK

Show the complete table.

		Number of colds last year					
		0	**1**	**2**	**3**	**4**	**Total**
Smoking	**Smokers**	0	3	5	6	2	16
	Non-smokers	4	10	8	4	1	27
	Total	4	13	13	10	3	43

b Work out the numbers who had more than 2 colds. **Smokers with colds = 16 − 0 = 16**

Non-smokers with = 27 − 4 = 23

Write the answer. **More non-smokers than smokers had 2 colds.**

c Find the percentage of smokers with more than 2 colds.

$$\textbf{Smokers with more than 2 colds} = \frac{8}{16} \times 100\% = 50\%$$

Find the percentage of non-smokers with more than 2 colds.

$$\textbf{Non-smokers with more than 2 colds} = \frac{5}{27} \times 100\% \approx 18.5\%$$

Write the answer. **50% of smokers and 18.5% of non-smokers had more than 2 colds.**

d Use the table and the percentages.	**All the smokers had colds but some non-smokers didn't, as well as a much greater percentage of smokers having multiple colds, so smoking does appear to affect health.**
e Half the cells have frequencies less than 5.	**Conclusions about smoking and getting no colds have too small a sample.**
Consider joining parts together.	**The lowest frequency for 'more than 2 colds' and '2 or less colds' is 5 (for non-smokers).**
Assess the reliability.	**The conclusions are actually based on 'more than 2 colds' or not, so they do appear to be reasonable.**
f The survey is fairly small and only involves Year 10 students.	**A limited survey like this might provide the basis for a larger study. Although it does support the assertion, it is not large enough to be conclusive.**

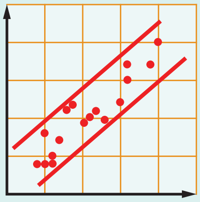
Scatter plots are the best way to analyse numerical bivariate data. Variables that have a linear relationship will make a straight line when plotted. Most statistical variables do not make a true straight line, but may still be related. For example, you know that tall people are usually heavier than short people, but this is not always true. A person is not necessarily heavier than someone shorter than them, but they are more likely to be heavier than lighter.

Important!

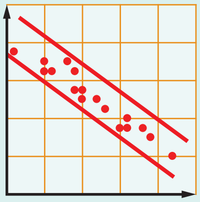
Scatter plots

If the points on a scatter plot lie between sloping parallel lines, you can say that there is a relationship between the variables.
If the lines have a positive slope you say that it is a **positive relationship**. The variables generally change in the same way; when one goes up, so does the other.
If the lines have a negative slope you say that it is a **negative relationship**. The variables generally change in the opposite way; when one goes up, the other goes down.
The parallel lines are close together for a strong relationship. If they are well separated, you say there is only a weak relationship.

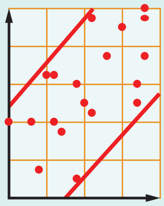
Strong positive relationship



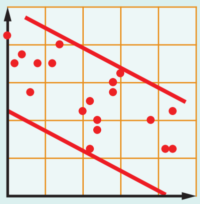
Strong negative relationship



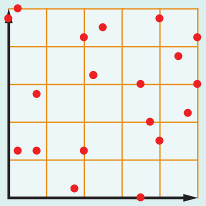
Weak positive relationship



Weak negative relationship



No relationship



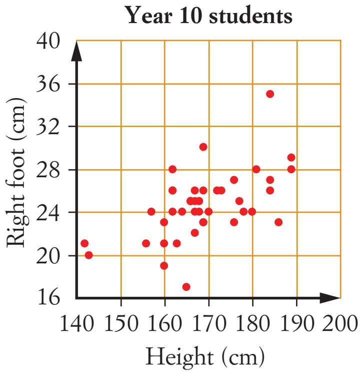
Example 19

Worksheet
Statistics 1
MAT10SPWK00002

The information below shows the heights and right foot lengths of some Year 10 students. Use a scatter plot to comment on the relationship. The data is in the form (height, foot length), in cm.
(157, 24), (167, 25), (156, 21), (172, 26), (163, 21), (181, 28), (180, 24), (167, 24), (169, 30), (177, 25), (162, 24), (165, 17), (167, 22), (189, 29), (168, 25), (169, 23), (184, 35), (162, 26), (170, 24), (142, 21), (169, 26), (167, 26), (186, 23), (178, 24), (160, 21), (166, 25), (169, 23), (168, 24), (173, 26), (164, 24), (143, 20), (162, 28), (189, 28), (166, 25), (176, 23), (184, 26), (168, 25), (184, 27), (160, 23), (162, 26), (160, 19), (176, 27), (166, 25)

Solution

Plot the points on graph paper, choosing the scales so that the points nearly fill the space.



Look at the pattern and make a conclusion.

There appears to be a weak relationship between height and the length of the right foot for Year 10 students.

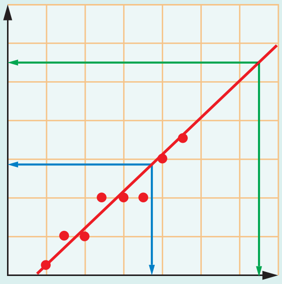
Strong relationships can be used to predict one variable from the other.

Important!

Technology worksheet
Excel worksheet: Line of best fit
MAT10SPCT00003

Prediction from a scatter plot

A **line of best fit** is a line that shows the general pattern of points on a graph.
Interpolation is the prediction of a value within the range of data.
Extrapolation is the prediction of a value outside the range of the data.



Example 20

CAS TI-Nspire exercise
Statistics
MAT10SPTI00002

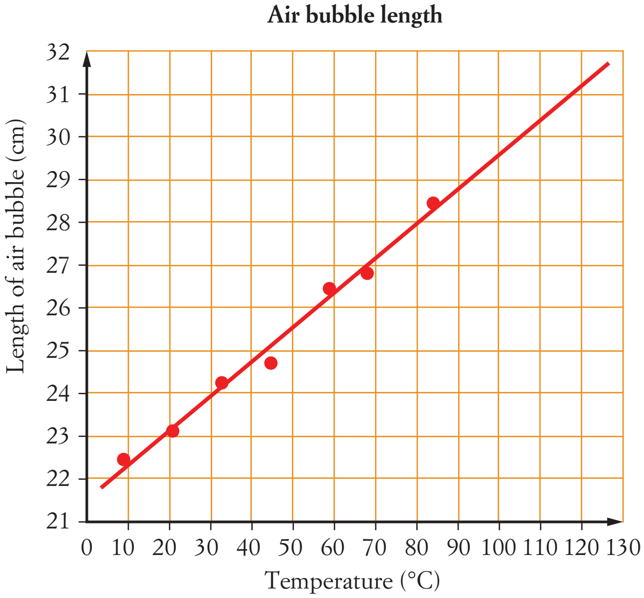
CAS ClassPad exercise
Statistics
MAT10SPCP00002

Students heated a thin glass tube with one end sealed and the other bent down in an oil bath. They measured the length of a bubble of air trapped in the tube at different temperatures and recorded the following results. Plot the points and predict the length if the oil could be safely heated to 120°C.

Temperature (°C)	9	21	33	45	59	68	84
Length (cm)	22.4	23.1	24.2	24.7	26.4	26.8	28.4

Solution

Plot the points and draw a line of best fit past 120°.



Read the value from the line of best fit.

If it were possible, the air bubble would have a length of about 31.2 cm at 120°C.

In Example **20** it is possible to find the equation of the line of best fit to summarise the relationship. This could be used to make predictions instead of reading from the graph. This is very important in Science.

Example 21

a Find the equation of the relationship in Example **20**.
b At what temperature would the air in the tube have a volume of zero?

Solution

a Extend the line of best fit through the axis to find the length intercept.

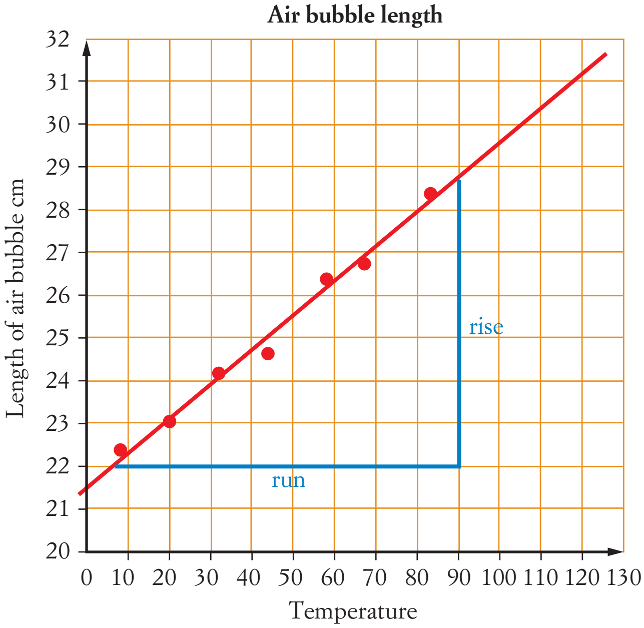
The length intercept is $c = 21.5$ cm

Choose two points on the line of best fit to find the gradient.

rise = 6.8

run = 83

slope $m = \frac{6.8}{83} \approx 0.082$



Use $y = mx + c$. **The equation is $L = 0.082T + 21.5$, where T is in °C and L in cm.**

b The air in the tube will have zero volume when its length is 0. Substitute $L = 0$.

$$\mathbf{0 = 0.082T + 21.5}$$

Solve for L.

$$\mathbf{0.082T = {}^{-}21.5}$$

Divide by 0.082.

$$\mathbf{T = \frac{{}^{-}21.5}{0.082} \approx {}^{-}262}$$

Write the answer. **The volume of air would be zero at about $^{-}262$ °C.**

Of course, the answer in Example **21** is a bit strange. Even if the experiment was accurate, what would a volume of zero mean anyway?
Instead of drawing a line of best fit by hand, you can use other methods.

Important!

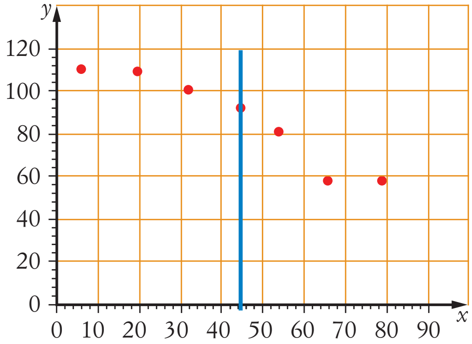
Two means method

1. Divide the data into two halves (on the horizontal axis).
2. Find the mean of the points in the lower half.
3. Find the mean of the points in the upper half.
4. Use the two means to calculate a line of best fit.

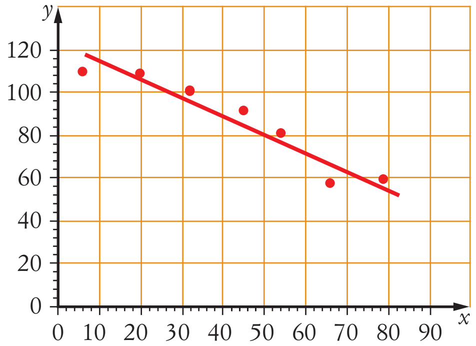
Example 22

Use the two means method to find the equation of a line of best fit for the data below.
(6, 110), (20, 109), (32, 101), (45, 92), (54, 81), (66, 58), (79, 59)

Solution

Draw a scatter plot. Divide the data in half with a vertical line.	
Choose the points below the middle.	**The points below the centre are (6, 110), (20, 109) and (32, 101).**
Find the means of the x- and y-coordinates.	**First mean** $= \left(\frac{6+20+32}{3}, \frac{110+109+101}{3}\right) \approx (19.3, 106.7)$
Choose the points above the middle.	**The points above the centre are (54, 81), (66, 58), (79, 59).**
Find the means of the x- and y-coordinates.	**Second mean** $= \left(\frac{54+66+79}{3}, \frac{81+58+59}{3}\right) \approx (66.3, 66)$
Find the rise.	**Rise** $= 66 - 106.7 = {}^{-}40.7$
Find the run.	**Run** $= 66.3 - 19.3 = 47$
Find the slope.	$m = \frac{{}^{-}40.7}{47} \approx {}^{-}0.87$
Write the line.	$y = mx + c = {}^{-}0.87x + c$
Put in one mean.	$106.7 = {}^{-}0.87 \times 19.3 + c$
Solve for c.	$c = 106.7 + 0.87 \times 19.3 \approx 123$
Write the answer.	**The equation of a line of best fit is** $y = {}^{-}0.87x + 123$.

The line of best fit found by the 'two means' method in Example **22** is plotted below on the graph with the original points.

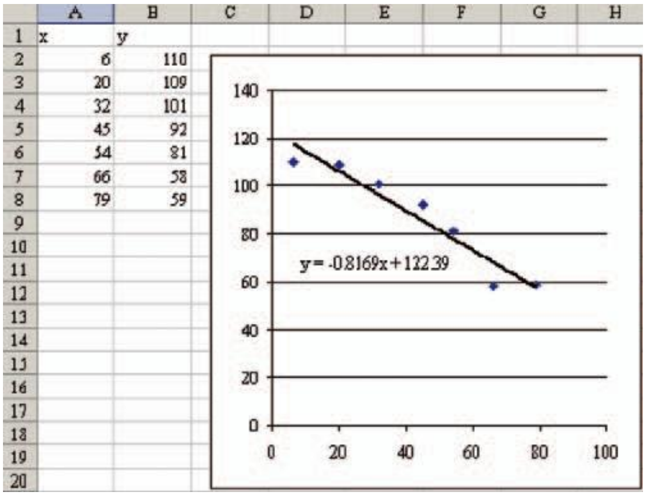


As can be seen, the line does appear to be a good line of best fit, passing through the middle of the points.

Investigate: Line of regression

The line of best fit can be calculated directly from the points. For historical reasons, it is called the **line of regression**. The total distance of the points from the line of regression is smaller than from any other line. It is a better fit to the points than that found with the two means method. It can be found using spreadsheets, graphics calculators, CAS calculators such as the TInspire and Classpad, and even some scientific calculators.
To find the line of regression of the points (6, 110), (20, 109), (32, 101), (45, 92), (54, 81), (66, 58) and (79, 59) using EXCEL, they should be entered as a table. Then follow these instructions:

- highlight the values of *x* and *y*
- click on the graph wizard and choose a scatter plot
- when the graph is drawn, click on one of the points on the graph
- the spreadsheet will allow you to add the trendline
- choose the first type (straight line)
- click on the options tab of 'Add Trendline'
- set 'show equation'
- click on OK
- you should get a display as shown on the right.



x	y
6	110
20	109
32	101
45	92
54	81
66	58
79	59

Investigate: Australian biodiversity

Australia is recognised as one of the most biologically diverse areas of the world. However, many species have become extinct in a short time in Australia. Many of these extinctions are the result of human disturbance. It is even possible that the extinction of Australian mega fauna was the first extinction in Australia caused by humans. However, the rate of extinction since European settlement has been very high. Australia has the worst mammal extinction record in the world – 27 mammals have become extinct in the last 200 years. No other country or continent has such a tragic record of mammal extinctions.

- Investigate extinctions of mammals in Australia
- Investigate extinctions of birds in Australia
- Investigate extinctions of plants in Australia
- How are the extinctions and the reduction of biodiversity spread over time?

A strong relationship is not necessarily a straight line.

Example 23

Worksheet
Data 2
MAT10SPWK00033

The table below shows the amount of cement powder used in different mixes for concrete blocks and the compressive force (in newtons) needed to break the block.

Cement (g/kg)	30	80	90	120	150	180	210
Breaking force (N)	46	76	90	96	106	108	110

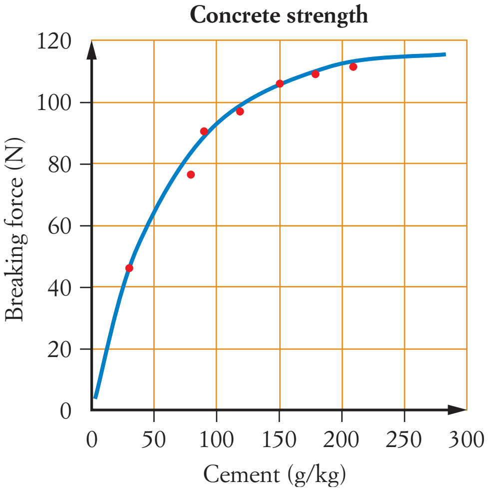
a Draw a scatter plot of the information.

b Use the graph to predict the breaking forces needed for 60 g/kg and 250 g/kg cement.

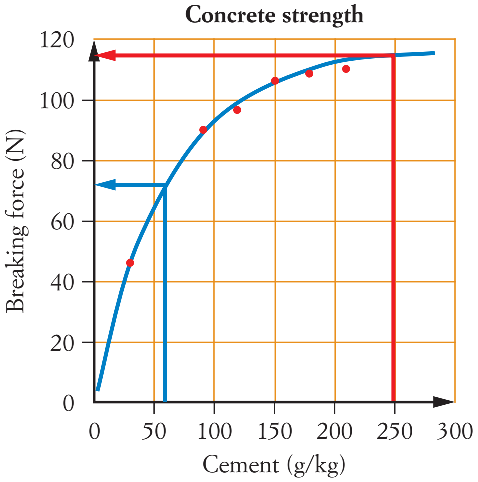
Solution

a Plot the points with the cement axis extended enough to make the required predictions.

Put in a line of best fit.



b Use vertical and horizontal lines to read the required values from the line of best fit.



Write the answers to an appropriate accuracy.

The breaking forces would be about 70 N for 60 g/kg cement and about 115 N for 250 g/kg cement.

Parallel boxplots are a quick and straightforward way to compare sets of data. The boxplots are drawn on the same scale, but above each other. You need to label the boxplots so you and other people can distinguish between them.

Example 24

Worksheet
Data 1
MAT10SPWK00032

Animated example
Analysing data
MAT10SPAE00002

a Draw parallel boxplots for the results below of two Year 10 classes in English.
Class 1: 56, 70, 55, 80, 51, 66, 29, 69, 59, 80, 47, 84, 41, 70, 30, 78, 50, 60, 92, 42, 67, 64, 72, 60, 77
Class 2: 43, 53, 33, 65, 44, 53, 33, 64, 37, 67, 47, 52, 15, 67, 23, 57, 49

b Compare the results of the classes.

Solution

a Arrange the results of class 1 in order and divide into 4 parts.

29, 30, 41, 42, 47, 50 | 51, 55, 56, 59, 60, 60, 64, 66, 67, 69, 70, 70, 72 | 77, 78, 80, 80, 84, 92

Work out the 5 number summary.

The 5 number summary is 29, 50.5, 64, 74.5, 92.

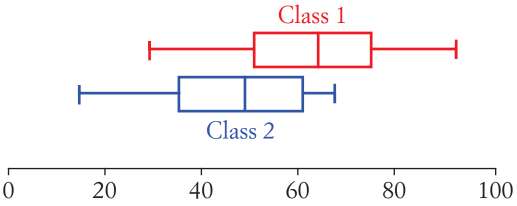
Arrange the results of class 2 in order and divide into 4 parts.

15, 23, 33, 33 | 37, 43, 44, 47, 49, 52, 53, 53, 57 | 64, 65, 67, 67

Work out the 5 number summary.

The 5 number summary is 15, 35, 49, 60.5, 67.

Use the 5 number summaries to draw the box-and-whisker-plots on top of each other.



b The box is further up the scale for class 1 than class 2.

Class 1 generally did better than class 2.

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Boxplots provide a good way to compare data from different groups, but the statistics do not use all the data. You can compare data from different groups using the mean and standard deviation, which both use all the data. The mean and standard deviation can also be calculated using many different digital technologies, including scientific calculators. The larger the sample you use, the closer the mean and standard deviation will be to the mean and standard deviation of the population. Larger samples are more likely to contain extreme values than smaller samples. Since they are just as likely to be high values as low values, this is unlikely to make any difference to the mean. However, it could make a difference to the standard deviation. The standard deviation value you get from a sample is likely to be less than the value you get using the whole population because of the inclusion of all extreme values. Mathematicians have worked out a way of compensating for this effect. Instead of using the formula for the standard deviation of the population, when you calculate the standard deviation using a sample, you should use the formula for the sample standard deviation to get a truer estimate of the population standard deviation.

Important!

Sample and population statistics

The statistics for a population are called **parameters** and have special symbols.
The mean of a population is shown as μ to distinguish it from the sample mean $\overline{x}$.
The number in a population is shown as N and in a sample as n.
The standard deviation of a population is shown as σ or σ_n and the standard deviation of a sample is shown as s or σ_{n-1}. The formulas for the standard deviation of a population and the estimate of the standard deviation from a sample are

$$\sigma = \sigma_n = \sqrt{\frac{\sum (x-\mu)^2}{N}}$$

and

$$s = \sigma_{n-1} = \sqrt{\frac{\sum (x-\overline{x})^2}{n-1}}.$$

You should be familiar with how your scientific calculator can be used to calculate the mean and standard deviation of data. Unless otherwise instructed, you should use your calculator to find the standard deviation.

Example 25

The results of students in 3 different classes are shown below. Use means and standard deviations to compare them.

Maths: 19, 17, 13, 17, 12, 26, 16, 9, 6, 15, 16, 14, 23, 16, 15, 18, 18, 19, 21, 15, 10, 17, 13
Computing: 20, 25, 12, 22, 22, 15, 10, 13, 12, 7, 6, 26, 13, 7, 6, 17, 23, 12, 16, 19, 8, 6, 12, 16, 23
Music: 17, 19, 21, 16, 17, 6, 19, 16, 18, 21, 20, 18, 17, 16, 17, 22, 26, 20, 14, 26, 16

Solution

The data is from the whole class.	**Find the population standard deviations.**
Use your calculator to find the mean and SD of the Maths class.	**For Maths, mean ≈ 15.9, SD ≈ 4.3**

Use your calculator for the Computing class.	**For Computing, mean ≈ 14.7, SD ≈ 6.3**
Use your calculator for the Music class.	**For Music, mean ≈ 18.2, SD ≈ 4.1**
Compare the means.	**The Music class generally performed much better than the Maths or Computing classes, and the computing class did the worst.**
Compare the standard deviations.	**The results in the computing class are quite widely spread, while those in the Music class are a bit more concentrated than the Maths class.**
Write a general conclusion.	**The results could reflect the tests used, or could mean that students taking Music are more interested than those doing Maths or Computing.**

Investigate: Aboriginal and Torres Strait Islander age distribution

The table below shows the age distributions of Aboriginal and Torres Strait Islander people and non-Indigenous Australians.

Indigenous and non-Indigenous populations in Australia by age, 30 June 2006

Age group	0–4	5–9	10–14	15–19	20–24	25–29	30–34
ATSI	64 426	65 136	64 687	54 943	44 779	36 866	36 283
non-ATSI	1 245 656	1 275 073	1 335 467	1 360 262	1 427 079	1 367 064	1 453 521
Age group	35–39	40–44	45–49	50–54	55–59	60–64	65–69
ATSI	34 760	30 251	25 073	19 812	14 423	9 689	6 477
non-ATSI	1 490 096	1 502 303	1 468 357	1 342 491	1 257 071	979 653	774 942
Age group	70–74	75–79	80–84	85 and over	Total		
ATSI	4291	2634	1394	1119	517 043		
non-ATSI	626 539	549 915	404 354	320 994	20 180 837		

Source: ABS

- Use the table to find the 5 number summaries of the age distributions for each group. Your teacher might want you to estimate the positions of the quartiles within the age groups.
- Draw parallel boxplots of the age distributions.
- Calculate the mean and standard deviation of both sets of results, using the centres of groups.
- What can you conclude?
- What implications does this have for Government health policy?

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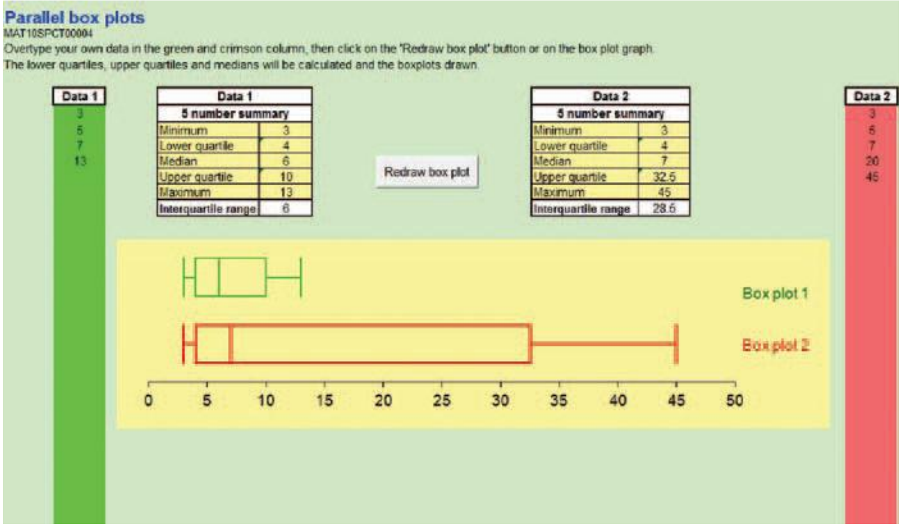
Technology Parallel box plots

Technology worksheet

Excel worksheet:
Parallel boxplots

MAT10SPCT00004

You can use an interactive spreadsheet to draw parallel box-and-whisker plots like the one in Example **24**. You can download the spreadsheet from NelsonNet, or your teacher may have already downloaded it to the school computers for you to use.



Parallel box plots
MAT10SPCT00004
Overtype your own data in the green and crimson column, then click on the 'Redraw box plot' button or on the box plot graph.
The lower quartiles, upper quartiles and medians will be calculated and the boxplots drawn.

Data 1
3
5
7
13

Data 1	
5 number summary	
Minimum	3
Lower quartile	4
Median	6
Upper quartile	10
Maximum	13
Interquartile range	6

Data 2	
5 number summary	
Minimum	3
Lower quartile	4
Median	7
Upper quartile	32.5
Maximum	45
Interquartile range	28.5

Data 2
3
6
7
20
45

The mean and standard deviation are commonly used to compare sets of data. These both use all the data, so most people think they are the best way to compare sets of data. However, a few very high or very low values will make a big difference to the values of these measures. You can compare values from different distributions by checking how many standard deviations they are from the mean.

Example 26

Most of Andrew's Maths class was in the same group for English. In Maths he scored 63 and in English he received 68. The mean and standard deviation of the Maths results were 54 and 12, while those for English were 52 and 20. Which subject did he perform best in?

Solution

Find the distance above the mean for Maths.	**Maths difference** $= 63 - 54 = 9$
Find the number of standard deviations above the mean for Maths.	**Maths SDs** $= \frac{9}{12} = 0.75$
Find the distance above the mean for English.	**English difference** $= 68 - 52 = 16$
Find the number of standard deviations above the mean for English.	**English SDs** $= \frac{16}{20} = 0.8$
Write a conclusion.	**Andrew performed slightly better in English than in Maths.**

Investigate: Examining media reports

Read the following media report from the ABC.

Indigenous people 'living longer'

May 25, 2009 15:00:00

NEW figures released by the Australian Bureau of Statistics show Indigenous life expectancy is longer than previously thought.

The ABS announced today that the life expectancy gap between Indigenous and non-Indigenous people is about 10 years - down from previous estimates of almost 17 years.

But the new results are not necessarily a reflection of better health amongst Indigenous people.

ABS assistant director of demography, Matthew Montgomery, says a new method, which better accounts for Indigenous deaths, has been used to compile the latest figures.

"ABS certainly intends to use that direct method into the future because it provides us with a better estimate," he said.

"The previous estimates, really there was some uncertainty, [because] we had to make quite a bit of assumptions about the method.

"The current method allows us to stand back and let the data speak for itself."

The new data shows Indigenous life expectancy across Australia for men and women is 67.2 years and 72.9 years respectively - almost 10 years below non-Indigenous life expectancy.

The Northern Territory has the worst Indigenous life expectancy of any state or territory, at just 61.5 years for men and 69.2 years for women.

NSW recorded the highest life expectancy for Indigenous men and women, with 69.9 years and 75 years respectively.

- What is meant by 'life expectancy'?
- Does the report mean that Aboriginal health has improved?
- Find out the life expectancy of all Australians.
- How is life expectancy worked out?
- Does the information in the article support the headline that Indigenous Australians are living longer?
- Does life expectancy have any commercial value or is it only of interest to governments?

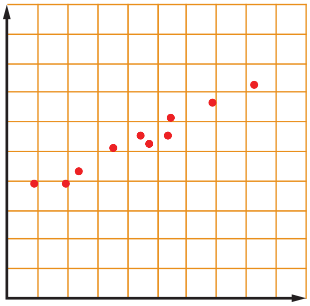
Exercise 2.3 Analysing data

Understanding

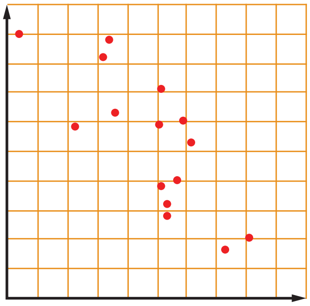
Extra questions
Exercise 2.3
MAT10SPEQ00006

1 State whether any relationship is indicated by each of the following scatter plots, and if so, the kind of relationship that is shown.

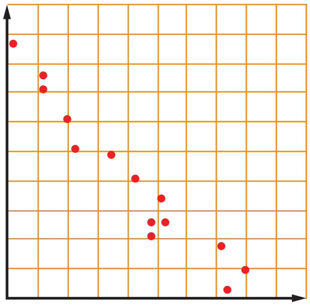
a



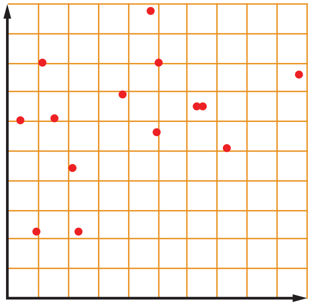
b



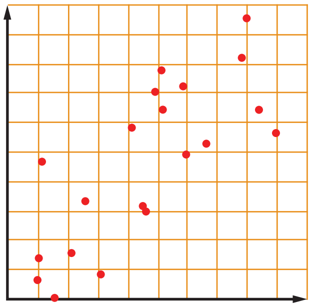
c



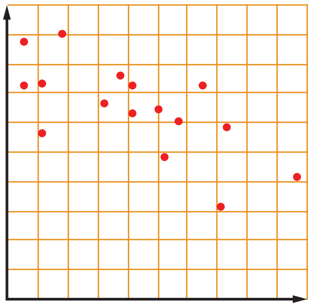
d



e



f



Fluency

See Example 17

2 Use the interquartile range to determine if any of the following sets of positive data have outliers.

a 8, 12, 10, 12, 8, 10, 9, 20, 8, 12, 10, 12, 7, 14, 9, 10, 5, 12, 5, 10

b 21, 30, 24, 35, 23, 27, 26, 33, 26, 28, 24, 26, 23, 27, 24, 27, 26, 27, 16, 34, 21, 26, 22, 27, 20

c 21, 26, 20, 26, 23, 27, 25, 26, 24, 27, 21, 25, 25, 27, 20, 25, 19, 26, 23, 25, 23, 29, 21, 27

d 12, 15, 13, 15, 13, 19, 12, 19, 13, 15, 11, 14, 5, 16, 12, 17, 10, 16, 11

e 46, 62, 49, 51, 48, 51, 45, 52, 46, 56, 48, 51, 50, 51, 45, 54, 47, 54, 47, 52, 49, 54, 46, 52, 48, 51, 36, 51, 49, 54, 44, 50

See Example 19

3 Draw scatter plots for each of the following sets of points and state whether a relationship is shown. If a relationship is shown, state the nature of the relationship.

a (10, 53), (11, 49), (14, 49), (22, 39), (23, 46), (28, 33), (35, 40), (37, 34), (43, 24), (43, 27), (50, 26), (53, 21), (57, 16)

b (6, 11), (8, 14), (13, 43), (13, 41), (18, 49), (22, 1), (28, 12), (30, 53), (36, 27), (38, 55), (41, 22), (50, 3), (50, 22), (59, 17), (34, 9), (17, 35), (37, 21), (41, 9), (39, 52)

c (1, 10), (9, 22), (15, 23), (21, 20), (24, 38), (24, 26), (26, 51), (30, 28), (37, 45), (37, 55), (45, 36), (46, 47), (48, 53), (57, 49), (15, 22), (8, 32), (17, 17), (1, 12), (30, 30), (30, 36), (29, 27)

See Example 24

4 Draw parallel boxplots of each of the following sets of data.

a **Set 1:** 11, 16, 9, 14, 12, 13, 10, 12, 7, 16, 9, 14, 5, 15, 12, 14, 11, 12

Set 2: 8, 9, 7, 11, 6, 9, 9, 9, 7, 16, 7, 12, 8, 15, 8, 15, 5, 9

b **Set 1:** 23, 26, 23, 28, 25, 29, 24, 25, 25, 25, 25, 26, 22, 26, 23, 27, 23, 28, 24, 29, 23, 25

Set 2: 25, 26, 22, 31, 24, 28, 20, 27, 25, 32, 16, 25, 21, 27, 24, 27, 24, 29, 20, 28, 25, 29, 23, 26

c **Set 1:** 6, 10, 3, 6, 4, 9, 4, 7, 5, 10, 5, 8, 2, 8, 3, 9, 5, 6, 5, 9, 5, 7, 4, 6, 4

Set 2: 5, 9, 7, 8, 3, 8, 3, 9, 8, 9, 7, 9, 6, 8, 7, 9, 7, 8, 4, 9, 7, 8, 6, 10, 7, 8, 6

See Examples 20, 23

5 Make scatter plots of the following sets of data and draw lines of best fit.

a

x	30	35	40	45	50	55	60	65	70	75
y	5	3.5	4	6.5	7	7.5	6	6.5	7	9.5

b

x	30	35	40	45	50	55	60	65	70	75
y	7	4.5	4	5.5	5	2.5	2	1.5	3	2.5

c

x	2	4	6	8	10	12	14	16	18	20
y	2.1	3.2	3.6	3.8	3.4	3.3	3.6	3.2	2.1	1.4

See Example 21

6 Find the equations of the lines of best fit in question **5** parts **a** and **b**.

See Example 22

7 Use the two means method to find the equations of lines of best fit for the data sets below.

a (3, 111), (9, 97), (12, 95), (17, 90), (21, 72), (26, 67), (27, 51), (30, 56)

b (2, 15), (10, 21), (13, 21), (17, 30), (19, 31), (22, 48), (25, 45), (33, 53), (35, 63)

Problem solving

See Example 18

8 The table below shows the numbers of accidents that drivers of different age groups were involved in. According to the report, it shows that 'older drivers are safer than younger drivers'.

		Accidents in last 2 years					
		1	2	3	4	5	Total
Age group	17–21		24		14	12	
	22–26	16	14		9	9	60
	27–31	9		7	4	2	29
	32–41	4	2	2	0	0	8
	Over 41	4	2	0	0	0	6
	Total	49		42	27	23	

a Complete the missing parts of the table

b Are older or younger drivers involved in more accidents?

c Find the percentages of drivers with accidents in each age-group who have had 4 or more accidents in the last 2 years.

d Do older or younger drivers seem more prone to multiple accidents?

e Is the conclusion supported by the data?

9 The table below shows the 'reading ages' of boys and girls in Year 10.

		Reading age						
		12	13	14	15	16	17	Total
Sex	Male	14	17		31	24		122
	Female		10	17		29		
	Total	21		41	67			240

a Complete the missing parts of the table

b Are there more boys or girls who are good readers in Year 10?

c Do boys or girls appear to be better readers in Year 10?

d Are your conclusions of this survey likely to be reliable?

e Are your conclusions supported by the data?

10 The data below shows the heights and arms spans of a sample of students in Year 10.
(157, 167), (167, 165), (156, 153), (172, 173), (163, 164), (30, 162), (181, 185), (180, 180), (167, 167), (169, 169), (190, 88), (177, 176), (162, 156), (165, 165), (167, 160), (189, 189), (168, 167), (169, 159), (184, 182), (175, 174), (162, 162), (170, 160), (142, 155), (169, 168), (167, 167), (186, 183), (133, 166), (178, 174), (160, 153), (166, 167), (169, 173), (168, 166), (173, 172), (164, 166), (143, 153), (162, 155)

a Make a scatter plot.

b Does there appear to be any relationship?

11 The data below shows the ages and heights of some Australian students on 1/1/2011.
(9, 142), (15, 175), (13, 167), (12, 150), (14, 172), (10, 134), (12, 145), (15, 163), (13, 162), (11, 134), (11, 138), (14, 157), (12, 144), (14, 171), (12, 155), (9, 144), (14, 165), (12, 168), (13, 156), (13, 150), (12, 164), (10, 147), (16, 157), (14, 163), (19, 173), (11, 136), (17, 174), (15, 160), (17, 156), (17, 165), (15, 176), (13, 152), (12, 158), (8, 135), (15, 172), (16, 167), (16, 168), (11, 149), (13, 158), (12, 150), (10, 146), (18, 175), (13, 159), (16, 150), (17, 186), (8, 250), (10, 146), (14, 169), (16, 174), (11, 148)

- **a** Make a scatter plot.
- **b** Does there appear to be any relationship?
- **c** What would you expect to find if you collected ages and heights for university students?

⑫ Two sprinters have the following training times over 100 m (in seconds).

Fred:	12.3	12.4	12.5	12.3	12.4	12.8	13.1	12.1	12.7	12.5
Sam:	12.1	12.0	12.0	16.8	12.1	12.2	12.4	12.1	12.3	12.4

Worked solutions
Exercise 2.3
MAT10NAWS00039

- **a** Draw parallel box-and-whisker plots for both sprinters on the same scale.
- **b** Compare the sprinters using the means, medians, quartiles and ranges.
- **c** Which sprinter should be chosen for the school track-and-field team? Give reasons.

13 The rents per week, in dollars, of houses in Brisbane, advertised one Saturday morning, were as follows:

Inner southern suburbs: 275, 340, 385, 330, 290, 235, 345, 300, 400, 265, 235, 250, 310, 300, 340, 330

Inner northern suburbs: 500, 450, 325, 390, 280, 230, 360, 275, 400, 285, 298, 480, 340, 390, 290, 470, 600, 290

- **a** Draw parallel box-and-whisker plots for both areas on the same scale.
- **b** Compare rental costs in inner northern and inner southern suburbs.
- **c** Compare the areas using the means, medians, quartiles and ranges.

14 The marks of some students for Physics and Chemistry are shown below:

Physics: 60, 81, 74, 45, 44, 44, 34, 59, 48, 74, 43, 48, 62, 46, 59, 64, 64, 55, 55, 45, 71, 59, 51, 55, 67

Chemistry: 80, 65, 56, 67, 70, 53, 76, 68, 43, 59, 43, 61, 43, 80, 74, 53, 52, 58, 49, 67, 62, 37, 65, 58, 46, 65, 65

- **a** Draw parallel boxplots for the data.
- **b** Compare the results.
- **c** Compare the classes using the means, medians, quartiles and ranges.

Worked solutions
Exercise 2.3
MAT10NAWS00039

⑮ The germination rate for parsnip seeds stored for different times is shown below.

Time (weeks)	6	8	10	12	14	16
Germination (%)	75	83	71	67	67	55

a Draw a scatter plot to show the results and put in a line of best fit.

b What happens to the germination rate as time goes by?

c What would the germination rate be if the seed was stored for only 1 week?

d What would the germination rate be if the seed was stored for 20 weeks?

See Example 23

16 The table below shows the amount of potassium iodide (KI) used up in a reaction and the time taken for the reaction (with other quantities kept constant).

KI amount (g)	5	10	15	20	25	30	35	40	45	50
Time (s)	19.5	8.5	5.2	4	5	2.3	1.4	1	2.7	1.5

a Draw a scatter plot of the information and add a line of best fit.

b Use the graph to predict the time of reaction if 70 g of KI is used.

17 The records for the women's high-jump are shown below.

Year	1932	1943	1956	1958	1960	1961	1974	1977	1984
Record (m)	1.65	1.71	1.75	1.80	1.85	1.90	1.95	2.00	2.07

a Draw a scatter plot of the information and add a line of best fit.

b Use the graph to predict the record at the Sydney Olympic Games in 2000. Find out whether this prediction was correct.

18 The birth rate in Australia (births per 1000 people) is shown in the table below.

Year	1860	1870	1880	1890	1900	1910	1920	1930	1940	1950	1960	1970	1980	1990	2000	2010
Rate	42.6	38.7	35.3	35	27.3	26.7	25.5	19.9	17.9	23.3	22.4	20.6	15.3	15.4	13	13.3

a Draw a scatter plot to show the information.

b Use the two means method to find the equation of a line of best fit.

c Predict the birth rate in 2050.

d Does the data and graph support a conclusion that Australia's birth rate is steadily declining?

See Example 25

19 60 students from each of Years 8, 10 and 12 were asked how much sleep they usually had on a school night. Their answers were as follows.

Year 8: 7, 8, 7, 9, 10, 9, 9, 8, 11, 9, 10, 9, 9, 10, 7, 9, 11, 8, 10, 8, 8, 7, 9, 11, 8, 7, 9, 9, 8, 7, 6, 9, 7, 6, 8, 8, 7, 9, 6, 9, 9, 11, 10, 8, 10, 9, 9, 10, 11, 10, 7, 10, 10, 8, 7, 7, 10, 8, 10, 9

Year 10: 7, 8, 9, 9, 6, 12, 9, 9, 8, 7, 9, 7, 7, 8, 8, 7, 9, 9, 9, 10, 8, 10, 8, 9, 7, 10, 6, 10, 8, 8, 7, 8, 7, 7, 8, 8, 11, 9, 10, 11, 7, 8, 6, 9, 8, 9, 10, 5, 5, 8, 7, 9, 10, 8, 8, 10, 11, 7, 6, 8

Year 12: 7, 8, 8, 8, 8, 7, 9, 7, 9, 6, 9, 9, 8, 9, 8, 8, 7, 8, 10, 8, 8, 7, 8, 10, 8, 7, 9, 7, 7, 7, 8, 9, 7, 6, 7, 7, 7, 9, 9, 7, 10, 8, 8, 6, 8, 8, 8, 8, 8, 11, 7, 7, 10, 6, 7, 8, 9, 8, 9, 7

Compare the amounts of sleep of the students using the mean and standard deviation.

20 The times of the top songs over periods of 10 years are shown below.

1950–1959: 2:35, 2:26, 2:26, 2:54, 1:55, 2:22, 2:19, 2:43, 4:06, 2:31

2000–2009: 2:13, 3:33, 3:48, 3:52, 4:14, 3:23, 3:48, 3:48, 3:54, 3:33

Compare the times of old hit songs with recent hit songs using the mean and standard deviation.

Worked solutions
Exercise 2.3
MAT10NAWS00039

See Example 26

㉑ The mean height and standard deviation of female Year 10 students is 164.5 cm and 9.0 cm, while for male students they are 174.0 and 9.2 cm respectively. Is a female student of height 155 cm or a male student of height 163 cm more unusual?

22 Use the figures in question **21** to determine which you would be more likely to find in a Year 10 class: a girl over 180 cm or a girl under 145 cm.

Reasoning

23 Explain why the information in question **8** is not useful in determining whether older or younger drivers are more accident prone.

See Example 16

24 A marketing firm placed survey forms inside the plastic wrapping of adult board games asking about the kinds of games people liked. The makers of the board game wanted to find out how to expand their market and appeal to a wider range of people. The survey could be sent back using a prepaid envelope. Explain whether you think this was a good way to do the survey.

25 To find the kinds of child care holiday programs that children want, students in Year 3 at a school near the childcare centre were asked to pick from a list of 10 different activities they would like to do in the holidays. The centre planned to offer local parents a special deal to try to replace business lost when people went away with their children for the holidays. Explain whether you think this method was a good way to obtain the information.

26 Explain whether or not the data in question **15** could be used to set a 'use-by' date for parsnip seed.

27 Explain why records like those in question **17** are not very good for predicting future records.

28 Use the results of question **18** to determine when there will be no births in Australia, and explain the problems with this prediction.

Chapter 2 summary

Quiz
Statistics
MAT10SPQZ00002

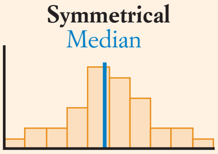
- **Categorical** data separates information into different (named) categories: **nominal** data has no numeric meaning or order, **ordinal** data has a natural order, but does not represent a measurement.
- **Numerical** data consists of numbers: **continuous** data can take any value between the smallest and largest values; **discrete** data can only have particular values. In most cases, the values are whole numbers.
- The **mean** ($\overline{x}$) is the **average** score. It is the total divided by the number of scores.
- For grouped frequency distributions, the class centre is used to represent all the scores in a group to calculate the mean.
- The **median** is the middle score. It is found by arranging all scores in order to choose the middle score. **Cumulative frequency** is the progressive total of frequencies and can be used to work out the median.
- The **mode** is the score with the highest frequency. A bimodal distribution has two scores with equal highest frequencies.
- The mean, median and mode are called **measures of central tendency**.
- The mathematical symbol Σ means 'the sum of' and can be used to write the formula for the mean as $\overline{x} = \frac{\Sigma x}{n} = \frac{\Sigma x}{\Sigma f}$, where x stands for a score, n is the number of scores and f stands for a frequency.
- The mean is most often used as the typical score. However, if there are a few high or low scores that would distort the mean, you should use the median. The mode should be used in cases where the most common score is needed.
- The **range** is the difference between the highest and lowest scores and measures the **spread** of the data.
- The **quartiles** of a frequency distribution divide it into 4 equal sections.
- The **first quartile** (**lower quartile**) is the score that has a quarter (25%) of the scores below it. Its symbol is $Q1$ or Q_1. For a finite distribution of n scores, it is the median of the scores below the median, or the $\frac{n+1}{4}$th score for n odd and the $\frac{n+2}{4}$th score for n even.
- The **second quartile** is the score that has two quarters (50%) of the scores below it. It is another name for the median. It can be written as $Q2$ or Q_2.
- The **third quartile** (**upper quartile**) is the score that has three quarters (75%) of the scores below it. Its symbol is $Q3$ or Q_3. For a finite distribution of n scores, it is the median of the scores above the median, or the $\frac{n+1}{4}$th score for n odd and the $\frac{n+2}{4}$th score for n even.
- The **interquartile range** (***IQR***) is the difference between the third and first quartiles.

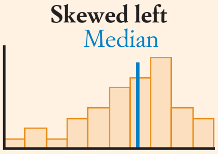
$$\mathrm{IQR} = Q3 - Q1$$

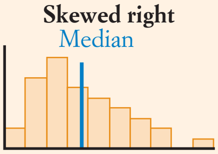
- The **five number summary** is the minimum, first quartile, median, third quartile and maximum ($Q0$, $Q1$, $Q2$, $Q3$, $Q4$) of a frequency distribution.

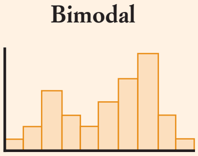
Chapter 2 summary

- **A symmetrical** distribution is roughly the same shape on either side of the middle.
- A distribution is **skewed to the left (negatively skewed)** if the scores spread out more below the median than above the median. If the scores are spread more on the right, it is **skewed to the right (positively skewed)**.
- **Bimodal** distributions have two high parts, although they don't always have two actual modes.









- The **mean deviation** of a set of data is the average of the absolute values of differences of each data item from the mean. The formula is

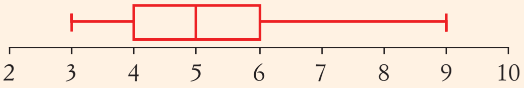
$$\frac{\sum |x - \overline{x}|}{n}.$$

- The **standard deviation** (symbol σ) of a set of data is the square root of the average of the squares of the differences of each data item from the mean. The formula is

$$\sigma = \sqrt{\frac{\sum (x - \overline{x})^2}{n}} = \sqrt{\frac{\sum x^2}{n} - (\overline{x})^2}.$$

- A **box-and-whisker plot** (**boxplot**) uses the 5 number summary to show how data is distributed. You extend a rectangle from the lower quartile to the upper quartile, divided at the median and draw horizontal lines from the quartiles to the maximum and minimum. A scale is shown under the drawing.

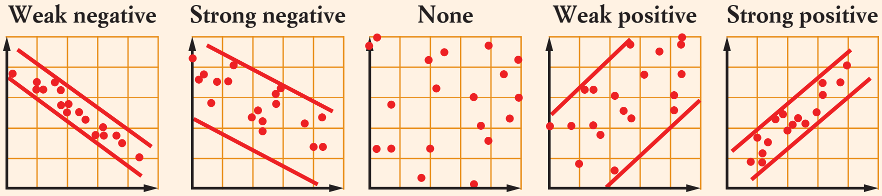
Spelling mistakes out of 20



- **Back-to-back** or **side-by-side** data displays are used to compare different distributions.
- A **fair sample** is like the whole population. A **biased sample** is not typical of the population.
- **Biased methods** and **biased questions** are likely to influence results in a particular direction. Fair ones are not likely to influence results in a particular way.
- **Non-compliant data** is obviously wrong, like someone's age being 164.
- An **outlier** is a score that has a very different value to the rest of a set of data. It can be defined as being more than 1.5 times the interquartile range below the first quartile or above the third quartile.
- Data where each item has two measurements that can be represented as variables is called **bivariate data**.
- A **scatter plot** of bivariate data has values that are plotted as coordinates. One variable is on the horizontal axis and the other one is on the vertical axis. It is used to look for relationships between variables.

Chapter 2 summary

- If the points on a scatter plot lie between sloping parallel lines, you can say that there is a **relationship** between the variables. If the lines have a positive slope you say that it is a **positive** relationship, but for a **negative** relationship they slope down. The variables generally change in the same way in a positive relationship but in opposite ways in a negative relationship.
- The parallel lines are close together for a **strong** relationship but are well separated when the relationship is **weak**.



- A **line of best fit** is a line that shows the general pattern of points on a graph.
- The equation of a line of best fit can be calculated using the means of the top and bottom halves of the data.
- A **line of regression** is the line that best represents the data in a scatter plot. It is usually calculated using digital technologies.
- **Interpolation** is the prediction of a value within the range of the data.
- **Extrapolation** is the prediction of a value outside the range of the data.

Chapter 2 review

Understanding

1 Find the mode and range of the following set of data. *See* Example 1

Macaroni, Spaghetti, Fettuccini, Lasagne, Lasagne, Macaroni, Spaghetti, Fusilli, Spaghetti, Spaghetti, Macaroni, Fettuccini, Lasagne, Spaghetti, Fettuccini, Macaroni, Lasagne, Macaroni, Lasagne, Gnocchi, Spaghetti, Fettuccini, Spaghetti, Gnocchi

2 Find the median and mode of the following set of data. *See* Example 2

Very poor, Very poor, OK, Poor, Very poor, Very good, Very poor, Poor, OK, OK, OK, Poor, Good, Very poor, Poor, Very poor, Very poor, Poor, Very poor, Good, Good, Very good, Very good, Very poor, Good, Good, Very good

3 Find the mean, median, mode and range of each of the following sets of data. *See* Example 3

a 5, 13, 10, 12, 10, 13, 11, 14, 10, 13, 7, 12, 7, 13, 10, 13, 8, 12, 9, 12, 5

b

Score	25	26	27	28	29	30	31	32	33	34	35	36	37
Frequency	5	1	4	1	3	9	16	13	17	9	5	5	2

4 Make a stem-and-leaf plot of the times below that Year 10 students say got up on Monday morning.

7:00, 5:30, 6:30, 7:30, 7:30, 8:00, 8:30, 8:00, 7:45, 7:00, 7:30, 7:00, 7:15, 6:15, 7:30, 7:00, 7:45, 7:30, 6:45, 7:30, 7:30, 7:00, 6:15, 6:30, 8:00, 7:00, 6:00, 8:00, 6:30, 6:30

5 Make scatter plots of the following sets of data. *See* Example 11

a

x	15	20	25	30	35	40	45	50	55
y	45	39	38	36	34	37	31	29	32

b

x	3	7	11	15	19	23	27	31	35	39
y	26	38	68	67	80	78	66	52	31	8

6 Find the median, quartiles and interquartile range of each of the following sets of data. *See* Example 4

a 3, 7, 4, 9, 3, 9, 6, 8, 7, 7, 2, 10, 7, 9, 7, 7, 5, 7, 4, 8

b

Score	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Frequency	2	2	1	0	5	1	7	7	3	5	4	9	4	7	4	4	3	1	3

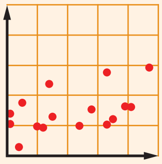
7 Draw box-and-whisker plots for each of these sets of data. *See* Example 12

a 8, 9, 9, 12, 10, 13, 14, 8, 4, 5, 7, 10, 7, 9, 8, 7, 5, 7

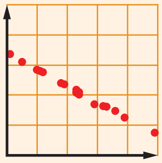
b 26, 25, 24, 20, 24, 15, 19, 26, 27, 31, 21, 26, 16, 29, 22, 22, 15, 29

8 State whether any relationship is indicated by each of the following scatter plots, and if so, the kind of relationship that is shown.

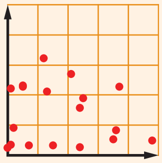
a



b



c



9 Find the mean and standard deviation of 4, 14, 2, 16, 6, 15, 8, 13. *See* Example 7

Chapter 2 review

Fluency

See Example 10

10 The following table shows the masses of a group of people.

Mass (kg)	40–49	50–59	60–69	70–79	80–89	90–99	100–109	110–119
Frequency	4	7	16	14	6	2	1	4

a Draw a histogram of the results.
b Comment on the shape of the distribution.

See Example 6

11 Find the five number summary for each of the following distributions.
a 45, 54, 49, 50, 44, 54, 46, 52, 41, 53, 42, 56, 43, 53, 39, 50, 47, 55, 47, 54, 42
b

Score	12	13	14	15	16	17	18	19	20
Frequency	1	2	7	15	26	12	13	4	1

See Example 8

12 Which data measure(s) should be used in each of the following cases?
a A car yard wants to know how much money it will make from selling 15 cars.
b An up-market credit card provider wants to know if plumbers earn enough to be worth targeting in a marketing campaign.

See Example 13

13 Compare a box-and-whisker plot of the following set of data to a histogram and dot plot of the same data.
10, 10, 7, 11, 6, 11, 7, 12, 7, 14, 10, 10, 9, 13, 6, 12, 6, 10, 8, 7, 6, 14, 6, 12, 5, 11, 8, 14, 5, 11, 6, 10, 8, 12, 9

See Example 17

14 Use the interquartile range to find if any of the following sets of positive data have outliers.
a 8, 11, 9, 11, 9, 12, 9, 12, 4, 11, 7, 13, 9, 12, 9, 13, 10, 11, 10, 14
b 13, 27, 13, 23, 19, 23, 15, 25, 16, 21, 15, 27, 12, 22, 19, 27, 17, 23, 12, 26, 15, 25, 19, 20, 17

See Example 19

15 Draw scatter plots for each of the following sets of points and state whether a relationship is shown. If a relationship is shown, state the nature of the relationship.
a (11, 57), (24, 65), (35, 75), (48, 83), (60, 98), (73, 110), (21, 64), (24, 63), (29, 71), (28, 74), (25, 63), (40, 74), (5, 46), (17, 61), (12, 50), (6, 50), (22, 59), (20, 54), (15, 58)
b (20, 75), (19, 71), (37, 9), (49, 5), (24, 52), (35, 49), (31, 14), (35, 46), (13, 96), (28, 12), (34, 29), (4, 91), (15, 68), (2, 112), (25, 32), (21, 62), (26, 43), (50, 3), (50, 22), (59, 17), (34, 9), (17, 35), (37, 21), (41, 9), (39, 52)

See Example 22

16 Draw parallel boxplots of the following sets of data.
Set 1: 8, 8, 4, 9, 8, 9, 5, 13, 7, 9, 5, 18, 7, 11, 3, 12, 4, 18, 5, 11, 5, 19, 5, 14
Set 2: 7, 15, 12, 15, 8, 17, 5, 14, 11, 14, 12, 13, 12, 17, 2, 14, 9, 15, 8, 15, 13, 13, 10, 16, 7, 15

See Example 5

17 The following table shows the masses of a group of people. Find their average mass.

Mass (kg)	40–49	50–59	60–69	70–79	80–89	90–99	100–109	110–119
Frequency	4	7	16	14	6	2	1	4

18 Make scatter plots of the following sets of data and draw lines of best fit.

See Examples 20, 23

a

x	1.3	2.5	3.1	4.4	5.4	6.6
y	12.6	18.3	27.4	38.3	51.1	68.1

b

x	1.5	3.9	6	10.7	11.3	12.4	15.8	18.8
y	4.3	6.5	10.1	10.9	14.1	18.7	17.6	24.7

Problem solving

19 The numbers of damaged apples in cases of 40 apples were recorded as follows.

Number of damaged apples	0	1	2	3	4	5
Number of cases	12	10	7	5	4	2

a What is the typical number of damaged apples in a case?

b How many damaged apples would you expect to get in 74 cases?

20 Two greyhounds have the following race times over the same meetings (in seconds).

See Example 14

Houndog: 53, 59, 61, 51, 59, 53, 58, 44, 47, 48, 73, 73, 70, 66, 60, 60, 50, 71, 37, 64

Chaser: 64, 53, 60, 61, 66, 69, 82, 68, 64, 68, 76, 59, 57, 58, 71, 83, 68, 77, 64, 76

a Make a back-to-back stemplot for the data.

b Compare the greyhounds using the stemplot.

c Compare the greyhounds using the means, medians and ranges and comment on your results.

21 The table shows the heights and errors in a spelling test of a group of 15-year-olds.

See Example 18

		Errors in spelling					
		1	**2**	**3**	**4**	**5**	**Total**
Height (cm)	**150–159**	0	1		5	1	10
	160–169	8	6		2	9	34
	170–179						
	180–189	2	3		2	2	11
	Over 189	2	0		1	1	4
	Total	23	12		17	22	94

a Complete the missing parts of the table.

b Do taller students appear to make more or fewer spelling errors than shorter students?

22 The following table shows the numbers of new sports utilities sold in Australia.

See Examples 19–21

Year	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010
Number ('000s)	113	126	149	163	187	194	185	216	208	205	252

a Draw a scatter plot to show the results.

b What kind of relationship is shown, if any?

c Put in a line of best fit and predict the number of sales in 2014.

d What is the equation of the line of best fit?

23 Two classes of students sat the same exam, one first thing in the morning and the other last thing in the afternoon. Their results are shown below:

Morning class: 3, 15, 10, 12, 14, 15, 15, 9, 10, 9, 13, 15, 11, 13, 12,
14, 13, 13, 14, 15, 17, 14, 5, 13, 8, 9, 17, 11, 11, 17

Afternoon class: 3, 11, 8, 6, 7, 9, 3, 11, 11, 8, 12, 12, 3, 12, 9,
5, 5, 5, 13, 5, 4, 12, 10, 13, 16, 12, 12, 17, 9, 6

a Make parallel box-and-whisker plots for the classes.

b Does either class appear to have had an advantage from the time of the exam?

c Compare the morning and afternoon distributions using the means, medians, quartiles and ranges and comment.

24 **a** Use the two means method and the information in question **22** to find the equation that shows the number of sports utilities sold in Australia.

b Predict the number of sports utilities that will be sold in 2020.

See Example 25 **25** Two students were arguing about whether History or Geography was the 'hardest' subject. To settle the argument, they decided to compare the results of classes in History and Geography. They obtained these results for recent tests from two classes.

History: 28, 29, 37, 21, 37, 26, 26, 15, 14, 20, 19, 21, 31, 31, 23, 11, 26, 20, 17, 2, 13, 10, 24

Mean	21.7826087		
Median	21	Minimum	2
Mode	26	Maximum	37
Range	35	*Q*1	16
IQR	11	*Q*3	27
SD(pop)	8.474804201	SD(sample)	8.665273

Geography: 24, 23, 27, 25, 20, 21, 20, 19, 18, 30, 17, 28, 19, 29,
21, 23, 20, 17, 15, 21, 22, 22, 18, 19, 20, 11, 25

Mean	21.25925926		
Median	21	Minimum	11
Mode	20	Maximum	30
Range	19	*Q*1	19
IQR	4.5	*Q*3	23.5
SD(pop)	4.2215724	SD(sample)	4.301991

The History test was out of 50 and the Geography test was out of 40. Change the results to percentages and compare them using mean and standard deviation. Does this settle the argument?

See Example 26 **26** The prices of bananas in fruit shops had an average of $4.60/kg with a standard deviation of $1.05. The prices of oranges had a mean of $3.80/kg with a standard deviation of $0.85. One fruit shop had bananas and oranges equally priced at $4.15/kg. Was the price of bananas or the price of oranges more unusual in this shop?

Reasoning

27 Explain the positions of the median and the mean in a negatively skewed distribution.

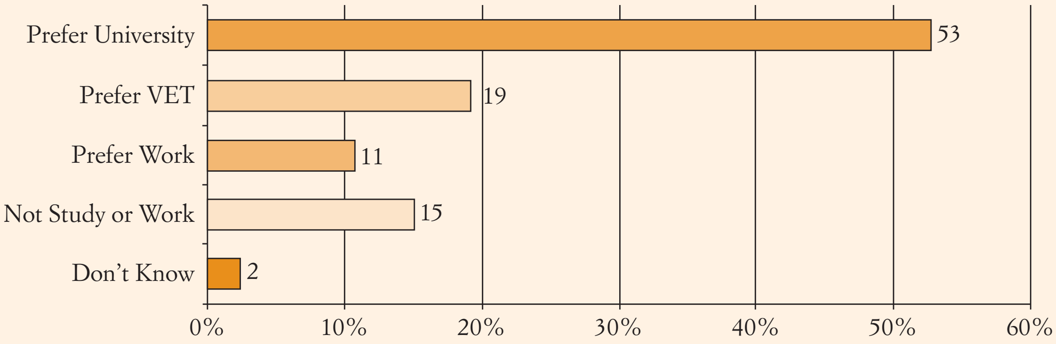
28 A class of 20 Year 10 students has an average mass of 72 kg. The median mass is 68 kg. 3 more students join the class. These 3 have an average mass of 80 kg and a median mass of 78 kg. Explain what will happen to the mean and median for the class.

29 What are other explanations for the results in question **23**?

30 The 5 number summary of a distribution is 12, 17, 19, 20, 22. Comment on the distribution and explain how you would estimate the mean.

31 The graph below shows the results of a survey of students completing Year 12 by the Department of Education, Employment and Workplace Relations.

See Example 15



A media report uses the information under the headline 'Australian school leavers prefer University.' Explain if this headline is misleading.

32 Two market research companies undertake surveys for different local chocolate makers to find the sizes of Easter eggs that they should make for the next Easter season. One maker rings 1000 people from different areas of the city and asks them to state the mass of chocolate they would want to have in Easter eggs. The other company takes sample eggs to 100 selected houses in the suburbs. The first company gets higher weights than the second. Explain which you think would be the most reliable.

33 How reliable is the prediction for 2014 in question **22**?

Number and algebra

3

Algebra

9780170194860

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Patterns and algebra

Factorise algebraic expressions by taking out a common algebraic factor. (10)

Simplify algebraic products and quotients using index laws. (10)

Apply the four operations to simple algebraic fractions with numerical denominators. (10)

Substitute values into formulas to determine an unknown. (10)

Investigate the concept of a polynomial and apply the factor and remainder theorems to solve problems. (10A)

9780170194860

Video tutorial
Algebra
MAT10NAVT00003

Weblink
What is Algebra, really?
MAT10NAWB00003

Algebra has a long history. As far back as 2000 BCE, the ancient Babylonians developed a form of algebra. Evidence of equations recorded on clay tablets shows that the Babylonians used algebra to solve problems.

Like much of our mathematics, algebra came into modern use through the Moslem culture of more than a thousand years ago. Muhammad ibn Mūsā al-Khwārizmī (about 780–850 CE) of the 'House of Wisdom' in Baghdad used the phrase al-jabr to describe some basic operations in the solution of equations. It has become the word algebra and his name is immortalised in the word 'algorithm' that describes the arithmetic methods he detailed in his book Al-Kitāb al-mukhtaṣar fī ḥisāb al-jabr wa-l-muqābala.

In algebra, unknown quantities are represented by variables. Variables are used in rules or equations that can then be used to model difficult problems.

Mathematical literacy

Maths dictionary
MAT10ASDI00001

The mathematical words below have special meanings that you will learn in this chapter. It is important that you learn to spell them and gradually learn what they mean in mathematics. You may find the glossary or online mathematical dictionary useful for this purpose.

algebraic expression
algebraic fraction
arithmetic expression
binomial brackets
coefficient
collecting like terms
common factor
constant
degree
difference of two squares
distributive law
evaluate
expand
expression
factorise
formula
highest common factor
leading term
like terms
lowest common denominator
perfect square
polynomial
quadratic
simplification
simplify
square of a difference
square of a sum
subject
substitution
term
transpose
variable

3.1 Expressions and polynomials

You have already done quite a bit of algebra in previous years. You should already be familiar with most of the terms below.

Important!

Variables, expressions and formulas

A **variable** is a letter or symbol that stands for a number. A **constant** is a number.
An **expression** has variables and/or numbers connected by arithmetic operations like +, ÷ and powers. An expression with numbers only is an **arithmetic expression**, while one with variables is an **algebraic expression**.
The numbers that are multiplied by the variables are called **coefficients**, the parts separated from the rest by + or − are called **terms** and a number on its own is called a **constant term**.
In algebra, you usually leave out the multiply symbol × between a constant and a variable or between variables. So the expression $5x + 8$ means $5 \times x + 8$ and has 2 terms, $5x$ and the constant term 8. x is the variable and 5 is its coefficient.
To **evaluate** an algebraic expression you use **substitution**. You put in values for the variables (**substitute** values) and work out the answer. When $x = {}^{-}3$ is substituted into the expression $5x + 8$, its value is $^{-}7$.
A **formula** is an equation with a variable on the left and an algebraic expression on the right. The variable on the left is the **subject** of the formula. The value of the subject variable is calculated by evaluating the expression on the right using given values. When the values $A = 12$ and $h = 8$ are substituted into the formula $V = \frac{1}{3}Ah$ you get an answer of 32 for the subject V. This formula can be **transposed** so that the subject is h. It then becomes $h = \frac{3V}{A}$.

In mathematics, expressions may be **simplified** by various methods. **Collection of like terms** is the simplest method of simplifying of algebraic expressions.

Example 1

Simplify the following if possible.

a $6pq - 6qt - 2ptqp + 4tq + 8$ **b** $m^2 + 2m^3 - 3m^2 - 4m^3$

Solution

a Write the expression. $\boldsymbol{6pq - 6qt - 2ptqp + 4tq + 8}$

Use the commutative law and index notation. $\boldsymbol{= 6pq - 6qt - 2p^2qt + 4qt + 8}$

The only like terms are $^{-}6qt$ and $4qt$. $\boldsymbol{= 6pq - 2qt - 2p^2qt + 8}$

b Write the expression. $\boldsymbol{m^2 + 2m^3 - 3m^2 - 7m^3}$

m^2 means $1m^2$, and m^3 is different.
Collect like terms. $\boldsymbol{= {}^{-}2m^2 - 5m^3}$

CAS TI-Nspire exercise
Algebra
MAT10NATI00013

CAS ClassPad exercise
Algebra
MAT10NACP00013

Simplification can involve the use of the index laws, particularly when an expression involves the same base in different parts.

Example 2

Simplify each of the following.

a $8eq^2 \times 5e^3q^3$ **b** $10n^2x^5 \div 15n^4x^4$ **c** $4b^4u^2r^{-3} \times 10b^3u^{-4}r^4$

d $18k^{-3}p^5u^3 \div 6k^4p^{-2}u$ **e** $3x \times (yz^2)^5 \times 5x^{-3} \times (2y^{-3})^{-3}$

Solution

a	Write with index 1 for the first e.	$8e^1q^2 \times 5e^3q^3$
	Add indices and multiply coefficients.	$= 8 \times 5 \times e^{1+3} \times q^{2+3}$
	Simplify.	$= 40e^4q^5$
b	Write in fraction format, with positive indices.	$\frac{10n^2x^5}{15n^4x^4}$
	Cancel coefficients, subtract denominator indices.	$= \frac{{}^{2}\cancel{10}n^{2-4}x^{5-4}}{{}_{3}\cancel{15}}$
	Simplify.	$= \frac{2n^{-2}x^1}{3}$
	Move the negative index to the denominator and write x^1 as x.	$= \frac{2x}{3n^2}$
c	Write the problem.	$4b^4u^2r^{-3} \times 10b^3u^{-4}r^4$
	Add indices and multiply coefficients. Write the variables in alphabetical order.	$= 4 \times 10 \times b^{4+3} \times r^{-3+4} \times u^{2+-4}$
	Simplify.	$= 40b^7ru^{-2}$
	Move the negative index to the denominator and write r^1 as just r.	$= \frac{40b^7r}{u^2}$
d	Write in fraction format, with the power of 1 written explicitly.	$\frac{18k^{-3}p^5u^3}{6k^4p^{-2}u^1}$
	Cancel coefficients, subtract denominator indices.	$= 3 \times k^{-3-4} \times p^{5--2} \times u^{3-1}$
	Simplify.	$= 3k^{-7} \times p^7 \times u^2$
	Move the negative index to the denominator to leave with positive indices.	$= \frac{3p^7u^2}{k^7}$
e	Write the problem.	$3x \times (y^4z^2)^5 \times 5x^{-3} \times (2y^{-3})^{-3}$
	Simplify the powers of powers.	$= 3x \times y^5z^{10} \times 5x^{-3} \times 2^{-3}y^9$
	Add indices and multiply coefficients where possible. Write the variables in alphabetical order.	$= 15 \times 2^{-3}x^{-2}y^{14}z^{10}$
	Write in fraction format with positive indices and evaluate 2^3.	$= \frac{15y^{14}z^{10}}{8x^2}$

In some cases, you may need to **collect like terms** in the numerators or denominators when simplifying multiplications or divisions of expressions.

Example 3

Simplify each of the following.

a $\dfrac{12x - 4x}{5xz - xz} \times (3a + 2a)$ **b** $\dfrac{11pqr + 9pqr}{24mrw - 9mrw} \times \dfrac{18wm - 10wm}{4v + 3v}$

Solution

a Write the second part in fraction format and insert brackets. $\dfrac{(12x - 4x)}{(5xz - xz)} \times \dfrac{(3a + 2a)}{1}$

Collect like terms. $= \dfrac{8x}{4xz} \times \dfrac{5a}{1}$

Cancel the numbers and the x. $= \dfrac{2}{z} \times \dfrac{5a}{1}$

Simplify. $= \dfrac{10a}{z}$

b Write the problem. $\dfrac{11pqr + 9pqr}{24mrw - 9mrw} \times \dfrac{18wm - 10wm}{4v + 3v}$

Collect like terms. $= \dfrac{20pqr}{15mrw} \times \dfrac{8wm}{7v}$

Cancel the numbers, m, w and r. $= \dfrac{4pq}{3} \times \dfrac{8}{7v}$

Simplify. $= \dfrac{32pq}{21v}$

Investigate: Space stations

Astronauts in a space station need to be able to move around in comfort. They work best under the same gravitational conditions as on Earth. Mathematicians discovered that, if the space station rotated at the appropriate speed, the gravity of Earth could be simulated on board. This is the formula that they worked out to let them simulate Earth's gravity:

$$N = \frac{42}{\pi}\sqrt{\frac{5}{r}}$$

where N = rotations per minute of the space station and r = the radius of the space station in metres.

Use the formula to answer these questions.

1 A space station with a radius of 11 m is launched. Calculate the number of rotations per minute that the space station must make to simulate Earth's gravity.

2 An American space station has the same gravity on board as Earth's. It has a radius of 35 m. How many rotations per minute does the space station make?

3 A Russian space lab has a radius of 6.2 m. In orbit it rotates 15 times every minute.
  a Would the people on board have trouble moving about? Explain.
  b If a problem exists, how could it be corrected?

Research the International Space Station. Does it use artificial gravity? Can you think of reasons why they wouldn't use artificial gravity on a space station?

Example 4

CAS TI-Nspire exercise
Algebra
MAT10NATI00013

CAS ClassPad exercise
Algebra
MAT10NACP00013

Substitute the values given in each of the formulas below to find the values of the subject.

a $s = ut + \frac{1}{2}at^2$, for $u = 10$, $t = 3$ and $a = 8$

b $T = \sqrt{2P + 4B}$, for $P = 1.5$ and $B = 5.5$

Solution

a Write the formula.	$\boldsymbol{S = ut + \frac{1}{2}at^2}$
Substitute $u = 10$, $t = 3$ and $a = 8$.	$\boldsymbol{= 10 \times 3 + 0.5 \times 8 \times 10 \times 10}$
Evaluate.	$\boldsymbol{= 430}$
b Write the formula.	$\boldsymbol{T = \sqrt{2P + 4B}}$
Substitute $P = 1.5$ and $B = 5.5$.	$\boldsymbol{= \sqrt{2 \times 1.5 + 4 \times 5.5}}$
Evaluate.	$\boldsymbol{= 5}$

Example 5

For a right cone, $V = \frac{1}{3}\pi r^2 h$. Substitute $V = 201$ cm^3, $r = 4$ cm in the formula and solve the resulting equation for h.

Solution

Substitute the values in the formula.	$\boldsymbol{201 = \frac{\pi \times 4^2 \times h}{3}}$
Multiply both sides by 3	$\boldsymbol{16\pi \times h = 603}$
Use your calculator to divide by (16π).	$\boldsymbol{h = 11.996...}$
Round and write the answer.	$\boldsymbol{h \approx 12}$ **cm**

If you want to find the value of a variable that is needed to give a particular value for a formula, you can rearrange the formula first. If you wanted to find multiple values, this would save time anyway. You use inverse operations like you have to solve equations in previous years. Remember that they are done in reverse to the usual order of operations.

Example 6

Make x the subject of each of the following formulas.

a $dx + c = p$ **b** $\frac{x}{g} + d = bc$

Solution

a	Write the formula.	$dx + c = p$
	Take c from both sides to isolate the x term.	$dx + c - c = p - c$
	Simplify	$dx = p - c$
	Divide both sides by d to isolate x.	$\frac{dx}{d} = \frac{p - c}{d}$
	Simplify.	$x = \frac{p - c}{d}$
b	Write the formula.	$\frac{x}{g} + d = bc$
	Take d from both sides to isolate the x term.	$\frac{x}{g} = bc - d$
	Multiply both sides by g to isolate x.	$x = g(bc - d)$

Video tutorial
Changing the subject of a formula
MAT10NAVT10005

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Algebra
MAT10NATI00013

CAS ClassPad exercise
Algebra
MAT10NACP00013

One of the most important kinds of mathematical functions is the **polynomial**. What do you think 'poly – nomial' means? Polynomials are important because other functions can be approximated using polynomials, at least over limited domains. While polynomials may have many variables, we will look only at polynomials with one variable.

Important!

Polynomials

A **polynomial** is a sum of terms that are each the product of a constant and positive integer powers of the variables. A polynomial in one variable can be written as

$$P(x) = a_nx_n + a_{n-1}x_{n-1} + a_{n-2}x_{n-2} + \ldots + a_2x_2 + a_1x + a_0,$$

where the coefficients a_n, a_{n-1}, a_{n-2}, . . ., a_2, a_1, a_0 are real numbers. In the work on polynomials this year, we will assume they are integers.
The **degree** of the polynomial is the highest power n, and the **leading term** is the term that contains the highest power.

A linear function can be considered as a polynomial of degree 1. Polynomials of degree 2 are called **quadratics**.

Example 7

State the leading term and degree of any of the following that are polynomials.

a $3m^3 - 5m^2 + m^7 - 5m + 2$

b $4x^4 + 5x^3 - 4x^2 + 2x^{-1} - 8$

c $5 - 3p + 4p^2 - 7p^5 - p^3$

Solution

a All terms are products of powers of m and constants.

$3m^3 - 5m^2 + m^7 - 5m + 2$ is a polynomial.

Arrange in standard form.

$= m^7 + 3m^3 - 5m^2 - 5m + 2$

State the leading term and degree.

The leading term is m^7 and it is of degree 7.

b $2x^{-1}$ is not a positive integer power.

$4x^4 + 5x^3 - 4x^2 + 2x^{-1} - 8$ is not a polynomial.

c All terms are products of powers of p and constants.

$5 - 3p + 4p^2 - 7p^5 - p^3$ is a polynomial.

Arrange in standard form.

$= {}^{-}7p^5 - p^3 + 4p^2 - 3p + 5$

State the leading term and degree.

The leading term is $^{-}7p^5$ and it is of degree 5.

As you know, the value of a function is calculated by substituting a number for the variable. This also applies to polynomials. Some people prefer to write brackets around negative numbers when they are raised to a power to distinguish between, say, the square of a negative number and the negative of a square. For many calculators, this is actually necessary to ensure that it is evaluated correctly.

Example 8

Worksheet
Algebra 5
MAT10NAWK00034

CAS TI-Nspire exercise
Algebra
MAT10NATI00013

CAS ClassPad exercise
Algebra
MAT10NACP00013

$P(e) = e^4 - 5e^2 + 6e - 7$. Find $P(^{-}3)$.

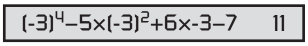
Solution

Write the polynomial with $e = {}^{-}3$.

$P(^{-}3) = (^{-}3)^4 - 5 \times (^{-}3)^2 + 6 \times {}^{-}3 - 7$

Enter as

((–) 3) a^b 4 – 5 × ((–) 3) x^2 + 6 × (–) 3 – 7 =.



(-3)^4–5×(-3)^2+6×-3–7 11

Investigate: Combining polynomials

What happens when you add or subtract polynomials?

- Add the following polynomials.
  1 $3x^4 - 5x^2 + 7$ and $4x^3 + 2x^2 - x - 7$
  2 $3x^5 - 5x^7 + 4x^3 + 9$ and $5x^7 - 3x^4 + x^3 - 3x - 5$
- What kinds of expressions do you get?
- What can you say about the expressions you get when you add polynomials?
- What about subtracting polynomials?

Exercise 3.1 Expressions and polynomials

Understanding

Extra questions

Exercise 3.1

MAT10NAEQ00036

See Example 4

See Example 7

1 Find the value of A in each of the following formulas if $x = 3$, $y = 2$ and $z = 5$.

a $A = 2x + 3y$
b $A = 3x - 4y$
c $A = 5(x + 2y)$
d $A = 3(2x - z)$
e $A = x^2 + y^2$
f $A = z^2 - y^2$
g $A = 3(x + y)^2$
h $A = x(y + z)^2$
i $A = 3xz - 2y^2$

2 State the leading term and degree of any of the following that are polynomials.

a $^{-}2v^5 - 9v^4 + 2v^3 + 5v^2 + 6v - 4 + 9v^8 + 6v^7 + v^6$
b $4z^5 - 8z^4 - 4z^8 - 6z^2 - z + 8z^6 - 6z^3 - \sqrt{9}$
c $7x^3 - \dfrac{3x^2 - 1}{4} + 4$
d $6 - 2y^6 + 6y^3 - 5y - 5 + 2y^4 + 8y^2 - 3y$
e $2e^2 + 4e^3 - 2\sqrt{e}$
f $7n^7 - n^6 + n^5 + 6n^4 - 6n^3 - 9n^2 + 4n + 4$
g $4w^3 - 3w + 6$
h $7 + 9w - 5w^4 - 8x^{\frac{3}{2}} + 2w^2$
i $^{-}2w^5 - 6w^4 - w^3 - 9w^2 + 9w - 9w^8 + 9w^6 - 3$
j $2u - 5 + 6u^5 + 5u^6 - 9u^4 + 6u^3 + 5u^2$

Fluency

See Example 1

3 Simplify if possible.

a $7c^2 + 4r + 3c^2 - 9r$
b $12m^2 - 7g - 8m^2 + 5g$
c $4x^2 - 3x + 2 + 6x^2 - 5x$
d $3ac^2 + 4ce^2 - 7ac^2 - 4ce$
e $7b^2 - 3ab + 4 - 10b^2 + 5ab - 8$
f $3a^2 + 2ac - a^2 - 5ac$
g $4mn + 2m \times 3n^2 - m \times n$
h $3m \times 2m + 5m^3 - 4mn + mn$
i $g^2 + 5g^3 + g \times 3g - 7$
j $ab \times 5ab - 3ab^2 + 7ab$
k $3p \times 2q \times 4r + p^2 \times 8qr$
l $p \times q + q \times r + r \times p$
m $8pq^2 - 6pq + 4p^2q + q^2p$
n $2m \times 3f \times 4r^2 - 4m \times 5$
o $6y \times 5xyz - 3xyz^2 + xy^2z$
p $16a \div 4 \times 5a + a^2 - 6a^3$
q $^{-}4a \times 3b + {}^{-}6a \times {}^{-}4b - ab$
r $7t - 4t^3 + 6t \times 4t - 8t \times 6$

4 Find the values of the following formulas for the values given.

a $M = 5c + 2d$ when $c = 2$ and $d = 5$
b $p = 4e - 3q$ when $e = 8$ and $q = 3$
c $a = m(3m - 4)$ when $m = 5$
d $d = 2s(3y - s)$ when $s = 6$ and $y = 4$

e $h = \dfrac{7g - 4r}{3}$ when $g = 6$ and $r = 9$
f $v = \dfrac{2w + 5z}{4w - 3z}$ when $w = 3$ and $z = 2$
g $y = 4x^2 - 5x + 3$ when $x = 3$
h $s = 4ut - t^2 + 5u$ when $u = 6$ and $t = 7$

See Example 5

5 Substitute the given values in each of the following formulas and solve the resulting equation for the remaining variable.

a $A = bh$, $A = 45\text{ m}^2$, $h = 3\text{ m}$
b $s = vt$, $s = 100\text{ m}$, $v = 8\text{ m/s}$
c $V = bhw$, $V = 450\text{ cm}^3$, $h = 9\text{ m}$, $w = 10\text{ m}$
d $V = \pi r^2 h$, $V = 600\text{ cm}^3$, $r = 5\text{ cm}$
e $A = \dfrac{(a + b)h}{2}$, $A = 400\text{ m}^2$, $a = 15\text{ m}$, $b = 23\text{ m}$
f $s = ut +, \frac{1}{2}at^2$, $a = 9.8\text{ m/s}^2$, $s = 50\text{ m}$, $t = 3\text{ s}$
g $E = \frac{1}{2}mc^2$, $E = 1.2 \times 10^9\text{ J}$, $c = 3 \times 10^8\text{ m/s}$ (answer in kg)
h $PV = nRT$, $P = 1$, $V = 22.4$, $T = 293$, $n = 1$ (answer in atm-L/mole/K)

See Example 6

6 Transpose the following formulas so that the letter shown in brackets is the subject.

a $v = u(2 - 3at)$ (t)
b $G = k(I - ct)$ (c)
c $m = \dfrac{b - a}{c - a}$ (b)
d $a = \dfrac{u + at}{s}$ (u)
e $\dfrac{a}{b} = \dfrac{c}{d}$ (d)
f $v = \sqrt{\dfrac{a - b}{c}}$ (a)
g $a^2 = 2b + c^2$ (b)
h $d = \sqrt{bc} - 5$ (c)

See Example 2

7 Simplify each of the following.

a $4a^2c^2k^3 \times 9a^3c^2k^3$
b $3hux^3 \times 2hux^3$
c $6k^3p^3r^3 \times 5k^3p^2r^3$
d $e^3ny \times 8e^3n^2$
e $5cn \times 4cn^2w$
f $4a^2g^3n^2 \times 3a^2g^3n^2$
g $2ah^2k^3 \times 7ah^3k^2$
h $5a^2h^3v^2 \times 2a^2hv$
i $9e^3p^2t^2 \times 7ept$
j $6h^2ru \times 7h^2ru^2$
k $8bhk^2 \times b^3h^3$
l $5bq^2t^3 \times 9b^3q^3t$
m $63k^2p^2t^2 \div 54k^3p$
n $2e^2u^2v^3 \div 5eu^3v$
o $24a^3g^2 \div 21a^3g^3u^3$
p $5b^2p^2 \div 45b^2p^3u^2$
q $15c^2q^2 \div 10c^2q^2v$
r $42ab^2h \div 35a^2b^2h^3$
s $24c^2g^2k^2 \div 4c^2g^2k$
t $12h^2u^3y^3 \div 14h^2uy^2$
u $12c^3u^3v^2 \div 32cu$
v $60a^3c^3h^2 \div 10a^2c^3h^3$
w $72g^2nu \div 64g^3nu$
x $45m^3qu^3 \div 54m^2q^3u$
y $54b^2c^2 \div 9bcn^3$
z $15q^2v^2 \div 25qv$

8 Simplify each of the following.

a $9h^7n^2u^6 \div 6h^2n^9u^8$
b $4r^3t^{10}w^2 \div 5r^3t^2w$
c $27e^3p^6t^4 \div 45e^{10}p^{10}t^7$
d $2a^{12}c^9 \times 1a^7c^9$
e $4b^5q^6 \times 5b^9q^5v^3$
f $25a^8pu^7 \div 10a^{12}p^9u^3$
g $4n^4p^{11}v^9 \times n^8p^7v^{10}$
h $4a^3t^{10}y^2 \times 3a^{12}t^2y^9$
i $16m^{12}q^7r^9 \div 8mq^2r^4$
j $40h^9m^{11}u^{10} \div 10h^6m^9u^8$
k $3b^8c^4h^9 \times 5b^{12}c$
l $2a^2b^8p^6 \times 1a^4b^6p^8$
m $12k^{12}u^8w^7 \div 3ku^5w^9$
n $b^{10}h^{11}r^7 \times 1b^7h^8r^6$
o $16k^9n^{10}p^7 \div 20k^8n^8p^{11}$
p $15a^8g^6m^9 \div 12a^5g^{12}m^{11}$
q $2a^2c^9 \times 1a^5c^4g^{11}$
r $3r^5t^4u^6 \div 15r^{11}t^{12}u^3$
s $24b^5e^3n^8 \div 8b^6e^5$
t $21e^7m^2u^{11} \div 35e^8m^8u^{12}$
u $28b^4m^4t^7 \div 35b^{11}m^5t^6$
v $16q^{11}t^8w^7 \div 24q^{12}t^9w^8$
w $4b^8h^8m^3 \times 3b^{12}h^8m^4$
x $5b^{11}g^{12} \times 4b^{11}g^6v^{12}$
y $e^6p^{12}q \times 5e^8p^4q^4$
z $4g^2n^{12}v^5 \times 3g^7n^6v^6$

9 Simplify each of the following.

a $a^{-4}b^5 \times {}^{-}5a^2b^2g^4$
b $15h^{-1}k^{-3}x^3 \div 20h^3k^4x^2$
c $56a^{-4}c^5m^{-5} \div 8a^3c^4m^4$
d $24ce^3h^{-4} \div {}^{-}54c^{-5}eh^{-2}$
e ${}^{-}3a^{-3}n^{-1} \times 4a^5np^3$
f ${}^{-}5k^{-3}p^{-3}r^{-5} \div {}^{-}5kp^5r^{-3}$
g ${}^{-}3b^{-4}r^{-3}t^4 \div 27b^{-5}t^3$
h $b^5 \times {}^{-}9b^2p^{-2}q^2$
i $40a^5c^{-5}m^3 \div {}^{-}56a^{-2}c^2m^2$

j ${}^{-}21b^{-1}n^{4} \div {}^{-}7b^{2}n^{-1}r$
k $4ae^{-3}k \times a^{-4}e^{-1}k^{-4}$
l $8c^{3}q \div {}^{-}20c^{3}q^{5}t^{-4}$
m ${}^{-}3b^{-5}h^{3}k^{-5} \div 2b^{-2}h^{5}k^{5}$
n $8b^{5}k^{-1}p^{-3} \div {}^{-}40b^{-3}k^{-3}p^{3}$
o $8m^{5}pr^{2} \times 9m^{2}p^{4}r^{3}$
p $2a^{3}c^{-4}v^{4} \times a^{3}c^{3}v$
q $6c^{-1}g^{-3}p^{3} \times {}^{-}7c^{5}g^{3}p^{-3}$
r ${}^{-}12m^{2}q^{4}t^{-5} \div 28m^{-4}q^{-1}$
s $7b^{5}e^{2}k^{4} \div {}^{-}4b^{-5}e^{-1}k^{3}$
t ${}^{-}20b^{5}h^{5}v^{4} \div 5b^{-1}h^{-3}v^{2}$
u ${}^{-}2a^{-2}e^{-1}m^{-5} \times 3a^{-1}e^{5}m^{4}$
v $7e^{2}n \times 8e^{3}m^{-1}n^{2}$
w $5g^{-3}k^{-2}t \times 7g^{4}k^{-3}t^{-4}$
x $42e^{-5}g^{-5}m^{-5} \div 7e^{3}g^{5}m^{3}$
y $20c^{4}g^{4}h^{5} \div {}^{-}4c^{-4}g^{-3}$
z $5g^{-3}k^{3}c^{4} \times 6g^{4}k^{-3}c^{-5}$

10 Simplify each of the following.

a ${}^{-}35a^{-1}(e^{-1}k)^{3} \div {}^{-}15a^{3}e^{-2}$
b $3a(c^{-2}u^{2})^{-3} \times {}^{-}4(a^{-2}c^{-3})^{-2}\,u^{-3}$
c ${}^{-}6(n^{2}r^{3}u^{3})^{-2} \times 7(n^{-3}r^{-3}u^{2})^{-1}$
d $28a^{-3}(k^{-2})^{3} \div 12a^{-2}(k^{-2}u^{-1})^{-2}$
e ${}^{-}8(c^{-2}t^{3})^{2} \div {}^{-}24(c^{-2}t^{-2})^{-3}u^{-2}$
f $36a^{-3}(k^{-1}t^{-2})^{-1} \div 30(a^{-3}k^{-2})^{-3}t^{-1}$
g $7(c^{2}k^{-2})^{4}m \times {}^{-}6c^{2}(k^{3}m^{-2})^{-1}$
h ${}^{-}2h^{-1}(p^{3})^{3} \times {}^{-}7h^{-1}(p^{-2}q^{2})^{3}$
i $15c^{-2}(gm)^{-3} \div {}^{-}27(c^{2}g^{3})^{-2}m^{-2}$
j $3b^{3}(p^{3}r^{-1})^{-2} \times 8(bpr^{3})^{-1}$
k ${}^{-}3(a^{2}g^{3}k^{-2})^{4} \times 8(a^{-3}g^{2})^{-3}k^{2}$
l ${}^{-}5a^{-1}(c^{-2}e)^{3} \div 2(a^{-2}c^{-3})^{-4}$

11 Simplify each of the following.

a ${}^{-}12e^{7}k^{-8}u^{12}z^{3} \times -e^{-2}k^{-8}u^{10}z^{-6}$
b $24m^{-4}n^{9}v \div 20m^{7}v^{-12}w^{-11}$
c $12a^{-2}m^{-8}u^{2}v^{-2} \times {}^{-}5a^{11}u^{-12}v^{5}$
d $70kn^{7}w^{-11}z^{-10} \div {}^{-}60k^{11}n^{10}w^{-12}z^{4}$
e ${}^{-}6a^{-10}e^{10}z \div {}^{-}24a^{11}e^{-5}z^{7}$
f ${}^{-}7g^{11}mr^{-11} \times 5g^{-3}m^{3}r^{-11}v^{-11}$
g ${}^{-}15a^{10}k^{8}z^{6} \div {}^{-}25a^{4}k^{-7}y^{6}z^{5}$
h $3q^{12}u^{-4}v^{9}w^{-11} \times 10q^{-3}u^{-1}vw^{7}$
i ${}^{-}21g^{3}k^{10}w^{-11}x^{12} \div {}^{-}30g^{-2}k^{-3}w^{-12}x^{-5}$
j $4g^{10}m^{-4}n^{-1} \times 7g^{-12}m^{-9}n^{-10}u^{-9}$
k ${}^{-}8b^{-3}h^{5}k^{-9} \div {}^{-}20b^{11}eh^{3}k^{-11}$
l ${}^{-}12n^{12}p^{2}r^{3}x^{4} \times {}^{-}11n^{11}r^{-11}x^{-6}$
m ${}^{-}12a^{2}eq^{9} \times {}^{-}13a^{8}eq^{10}$
n $7b^{-3}m^{-5}w^{-10}z^{8} \times 9b^{-9}m^{4}w^{5}z^{-10}$
o $14e^{6}k^{2}m^{-12}t^{-5} \div {}^{-}91e^{10}k^{2}m^{8}t^{-6}$
p $4c^{4}q^{-5}r^{8} \times {}^{-}7c^{9}q^{-7}$
q $12a^{2}k^{-11}m^{7}u^{-3} \times a^{-12}m^{4}u^{-11}$
r ${}^{-}54h^{-11}q^{-1}v^{10}x^{-5} \div {}^{-}30h^{6}q^{-2}v^{-5}x^{8}$
s $72n^{10}q^{-7}t^{8}u^{-9} \div {}^{-}66n^{12}t^{3}$
t ${}^{-}66a^{-9}n^{7}r^{-1}v^{7} \div {}^{-}6a^{5}n^{3}v$
u $90c^{-10}m^{-6}p^{-6} \div 9c^{7}m^{3}p^{-8}q^{-6}$
v ${}^{-}13e^{12}q^{-9}u^{-8} \div -e^{-12}p^{7}q^{-2}u^{-10}$
w $32a^{9}y^{-8}z^{-3} \div {}^{-}104a^{4}n^{9}z^{-9}$
x $48g^{3}h^{-12}m^{-10}u^{4} \div 42g^{-11}h^{6}m^{-5}$
y ${}^{-}7c^{-10}e^{11}n^{-7}q^{-7} \times 13c^{6}n^{10}q^{-12}$
z $70k^{-9}n^{3}ty^{-6} \div 130k^{11}n^{-8}t^{11}y^{10}$

"SOMETIMES IT DOES, SOMETIMES IT DOESN'T."

See Example 3

12 Simplify each of the following.

a $\dfrac{9u - 7u}{9vu + 6vu} \times v(qu + 5qu)$

b $\dfrac{5x - 3x}{10w - 7w} \times \dfrac{3w + w}{3wx + 2wx}$

c $\dfrac{10n - 4n}{4k^2 + 3k^2} \times \dfrac{8tk - 6tk}{2n^2 + n^2}$

d $\dfrac{3xy + 7xy}{2x^2 + 5x^2} \times \dfrac{9mx - 7mx}{7ay - 2ay}$

e $\dfrac{9ae - 7ae}{9wz - 6wz} \times \dfrac{7ez + 8ez}{8gw + 6gw}$

f $\dfrac{15kx^2 - 13kx^2}{15uw + 11uw} \times \dfrac{13w^2 + 7w^2}{13x^2 - 11x^2}$

See Example 8

13 $P(x) = 8x^3 - 2x^2 - x + 5$, $Q(x) = {}^{-}2x^5 - 8x^3 - 2x + 7$ and $R(x) = 5x^6 - 9x^4 + 6x^2 + 4x$. Find each of the following.

a $P(1)$ **b** $P(^{-}3)$ **c** $P(3)$ **d** $P(2)$
e $P(^{-}2)$ **f** $P(4)$ **g** $Q(3)$ **h** $Q(4)$
i $Q(^{-}3)$ **j** $Q(2)$ **k** $Q(^{-}1)$ **l** $R(^{-}2)$
m $R(4)$ **n** $R(1)$ **o** $R(^{-}3)$ **p** $R(^{-}4)$

Worked solutions
Exercise 3.1
MAT10NAWS00040

14 $P(x) = 8x^3 - 2x^2 - x + 5$, $Q(x) = {}^{-}2x^5 - 8x^3 - 2x + 7$, $R(x) = 5x^6 - 9x^4 + 6x^2 + 4x$, $S(x) = 4x^7 - 5x^4 + 8x^2 - 6$ and $T(x) = 9x^4 - 9x^3 + 2x - 2$. Find each of the following.

ⓐ $P(x) + Q(x)$ **b** $R(x) + S(x)$ **c** $T(x) + Q(x)$
d $P(x) + R(x)$ **e** $S(x) + T(x)$ **f** $P(x) - Q(x)$
g $T(x) - R(x)$ **h** $S(x) - Q(x)$ **i** $P(x) - S(x)$
j $Q(x) - R(x)$

Problem solving

Worked solutions
Exercise 3.1
MAT10NAWS00040

15 Find the value the nominated variable in each of the following formulas.

a u if $v = u + at$ and $v = 12$, $a = 2$ and $t = 3$

b m if $k = m(1 - ct)$ and $k = 28$, $c = 4$ and $t = 2$

c a if $s = ut + \dfrac{1}{2}at^2$ and $s = 26$, $u = 8$ and $t = 2$

d E if $I = \dfrac{E - V}{R}$ and $I = 4.2$, $V = 7$ and $R = 2.5$

e h if $d = 8\sqrt{\dfrac{h}{a}}$ and $d = 24$ and $a = 5$

ⓕ g if $B = \sqrt{2gh}$ and $B = 20$ and $h = 20$

g t if $s = \dfrac{(u + v)t}{2}$ and $s = 15$, $u = 3$ and $v = 5$

h c if $a^2 = b^2 + c^2$ and $a = 24$ and $b = 10$

i w if $v = \dfrac{w - u}{gr}$ and $v = 14$, $u = 6.8$, $g = 9.8$ and $r = 2.5$

j r if $p = ghr + B$ and $p = 980$, $g = 9.8$, $h = 15$ and $B = 24.5$

Reasoning

Worked solutions
Exercise 3.1
MAT10NAWS00040

16 Explain the difference between a formula and an equation.

⑰ Show that when 2 polynomials of degree 4 are added or subtracted, the result can be a polynomial of a lower degree.

3.2 Expansion and factorisation

As you saw last year, you use the **distributive** law to expand brackets.

Important!

The distributive law

The product of the sum (or difference) of two numbers with another number is the same as the sum (or difference) of the products of that number with each separately.
This can be written in symbols as:

$$a \times (b + c) = a \times b + a \times c$$

and

$$a \times (b - c) = a \times b - a \times c.$$

The distributive law can be summarised as: $a(b \pm c) = ab \pm ac$.

When you are **expanding brackets**, you may want to write subtractions as addition of a negative. This helps you to avoid mistakes with signs.

Example 9

Simplify each of the following.

a $4(a + 2)$ **b** $2m(4 - 3m)$ **c** ${}^{-}3k(4k - 7)$

Solution

a	Write the problem.	$\mathbf{4(a + 2)}$
	Use the distributive law.	$= \mathbf{4 \times a + 4 \times 2}$
	Simplify.	$= \mathbf{4a + 8}$
b	Write the subtraction as adding a negative.	$\mathbf{2m(4 + {}^{-}3m)}$
	Use the distributive law.	$= \mathbf{2m \times 4 + 2m \times {}^{-}3m}$
	Simplify.	$= \mathbf{8m + {}^{-}6m^2}$
	Write in the usual way.	$= \mathbf{8m - 6m^2}$
c	Write the subtraction as adding a negative.	$\mathbf{{}^{-}3k(4k + {}^{-}7)}$
	Use the distributive law.	$= \mathbf{{}^{-}3k \times 4k + {}^{-}3k \times {}^{-}7}$
	Simplify.	$= \mathbf{{}^{-}12k^2 + 21k}$
	Write with the positive term first.	$= \mathbf{21k - 12k^2}$

CAS TI-Nspire exercise
Algebra
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Algebra
MAT10NACP00013

You may also need to use the index laws when expanding brackets.

Example 10

Simplify each of the following.

a $6b^5k^4(6b^6 - 5v^3)$

b $^-5b^5z^{-6}(2b^{-2} - 9z^4)$

Solution

a Write the problem.	$6b^5k^4(6b^6 - 5v^3)$
Use the distributive law.	$= 6b^5k^4 \times 6b^6 + 6b^5k^4 \times {}^-5v^3$
Simplify and use the index law.	$= 36b^{5+6}k^4 - 30b^5k^4v^3$
	$= 36b^{11}k^4 - 30b^5k^4v^3$
b Write the subtraction as adding a negative.	$^-5b^5z^{-6}(2b^{-2} + {}^-9z^4)$
Use the distributive law.	$= {}^-5b^5z^{-6} \times 2b^{-2} + {}^-5b^5z^{-6} \times {}^-9z^4$
Simplify, using the index laws.	$= {}^-10b^{5+{}^-2}z^{-6} + 45b^5z^{-6+4}$
Simplify the indices and write the positive term first.	$= 45b^5z^{-2} - 10b^3z^{-6}$

Investigate: Multiplying polynomials

What happens when you multiply polynomials?

- Multiply the following pairs of polynomials.
  1 $3x^4 - 5x^2 + 7$ and $4x^3 - x$
  2 $3x^5 + 9$ and $5x^7 - 3x$
- What kinds of expressions do you get?
- What can you say about the expressions you get when you multiply polynomials?

Example 11

$P(m) = 9m^7 + 5m^5 - 8m + 4$ and $Q(n) = 6n^5 - 6n^4 + 5n$. Find $P(x) \times Q(x)$.

Solution

Write P with the variable x.	$P(x) = 9x^7 + 5x^5 - 8x + 4$
Write Q with the variable x.	$Q(x) = 6x^5 - 6x^4 + 5x$
Write the product.	$P(x) \times Q(x) = (9x^7 + 5x^5 - 8x + 4\ )(6x^5 - 6x^4 + 5x)$
Expand the first brackets.	$= 9x^7(6x^5 - 6x^4 + 5x) + 5x^5(6x^5 - 6x^4 + 5x)$ $- 8x(6x^5 - 6x^4 + 5x) + 4(6x^5 - 6x^4 + 5x)$
Expand the remaining brackets.	$= 54x^{12} - 54x^{11} + 45x^8 + 30x^{10} - 30x^9 + 25x^6 - 48x^6$ $+ 48x^5 - 40x^2 + 24x^5 - 24x^4 + 20x$
Collect like terms and write in the standard order.	$= 54x^{12} - 54x^{11} + 30x^{10} - 30x^9 + 45x^8 - 23x^6 + 72x^5$ $- 24x^4 - 40x^2 + 20x$

Two brackets multiplied together are called **binomial brackets**. These can be expanded in several different ways.

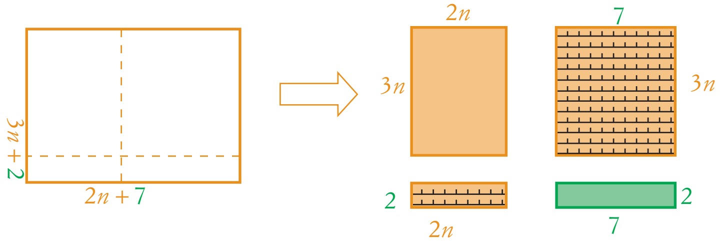
Investigate: Binomial brackets

Technology: GeoGebra
Simplifying expressions
MAT10NATC00003

Weblink
Expanding brackets using the grid method
MAT10NAWB00003

The Area method

You expand the binomial brackets $(3n + 2)(2n + 7)$ by considering the area of a rectangle with a height of $(3n + 2)$ and width $(2n + 7)$.
Then divide the height and width into their parts to make smaller rectangles as shown below.



Area $= (3n + 2)(2n + 7)$
Total area $= 3n \times 2n + 3n \times 7 + 2 \times 2n + 2 \times 7$
The areas are the same, so $(3n + 2)(2n + 7) = 6n^2 + 21n + 4n + 14 = 6n^2 + 25n + 14$

The Table method

You use a table instead of a rectangle, but still multiply widths and heights. Then you put the answers in the table as shown below.

	$2n$	$+7$
$3n$	$6n^2$	$21n$
$+2$	$4n$	14

Then add the answers to give $(3n + 2)(2n + 7) = 6n^2 + 21n + 4n + 14 = 6n^2 + 25n + 14$

The Distributive Law method

You just apply the distributive law twice, so this is a purely algebraic method.

$$\begin{aligned} &(3n+2)(2n+7) \\ &= 3n(2n+7) + 2(2n+7) \\ &= 3n \times 2n + 3n \times 7 + 2 \times 2n + 2 \times 7 \\ &= 6n^2 + 21n + 4n + 14 \\ &= 6n^2 + 25n + 14 \end{aligned}$$

The FOIL method

For this method, you just multiply the terms in each bracket in the order **F**irst terms, **O**uter terms, **I**nner terms and **L**ast terms.
For $(3n + 2)(2n + 7)$, you get:
First terms: $3n \times 2n = 6n^2$
Outer terms: $3n \times 7 = 21n$

Inner terms: $2 \times 2n = 4n$
Last terms: $2 \times 7 = 14$
Try each method for yourself.
Even though there is no such thing as a negative length, you can use negative lengths on the sides of the Area model for this investigation.
Your teacher might want you to use one method more than the other, but generally different people find they prefer different models.

The FOIL method only works with binomial brackets. If there are more than 2 terms in one of the brackets, this method doesn't work. The distributive law method is the most general method, and applies to expansions of all kinds of brackets.

Example 12

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Algebra
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Algebra
MAT10NACP00013

Simplify each of the following by expanding the brackets.

a $(x + 2)(x + 3)$ **b** $(a + 5)(a - 4)$ **c** $(k - 5)(k - 4)$

Solution

a Write the problem. $(x + 2)(x + 3)$

Expand using your chosen method. $= x^2 + 2x + 3x + 6$

Simplify. $= x^2 + 5x + 6$

b Write the problem. $(a + 5)(a - 4)$

Expand using your chosen method. $= a^2 - 4a + 5a - 20$

Simplify. $= a^2 + a - 20$

c Write the problem. $(k - 5)(k - 4)$

Expand using your chosen method, being very careful with signs. $= k^2 - 4k - 5k + 20$

Simplify. $= k^2 - 9k + 20$

The methods work just as well when there is a second variable. If the variables are not in the same order in both brackets, you should change the order before starting the expansion.

Example 13

Simplify each of the following.

a $(3x + 2y)(4x - 3y)$ **b** $(4 - 5c)(2c - 3)$ **c** $(3p - 4m)(3m + 4p)$

Solution

a Write the problem. $(3x + 2y)(4x - 3y)$

Expand using your chosen method. $= 12x^2 - 9xy + 8xy - 6y^2$

Simplify. $= 12x^2 - xy - 6y^2$

9780170194860

b Write the problem. $(4 - 5c)(2c - 3)$

Change the order in the first bracket. $= (^-5c + 4)(2c - 3)$

Expand using your chosen method. $= {}^-10c^2 + 15c + 8c - 12$

Simplify, and write the positive term first. $= 23c - 10c^2 - 12$

c Write the problem. $(3p - 4m)(3m + 4p)$

Changing the order in the second bracket is easier than the first bracket. $= (3p - 4m)(4p + 3m)$

Expand using your chosen method. $= 12p^2 + 9mp - 16mp - 12m^2$

Simplify. $= 12p^2 - 7mp - 12m^2$

When you are expanding several sets of brackets, you will probably have to collect like terms to finish the simplification.

Example 14

Simplify each of the following.

a $3a(2a + 5) - 2(a - 8)$

b $2e(a + 3e - 6h) - h(9a - 4e + 4h) - 5a(7a - 3e + 8h)$

c $(8r - 9m)(r - 4m) - (9r - m)(6r + m)$

d $8m^{-5}p(9h^{-1}m^5p^{-6} - h^2m^{-3} + 9m^{-3}p^4) - 7h^{-1}m^{-3}(h^3m^{-5}p - 4m^3p^{-5} - 5hp^4)$

Solution

a Write the problem. $3a(2a + 5) - 2(a - 8)$

Expand the brackets, being careful with signs. $= 6a^2 + 15a - 2a + 16$

Simplify. $= 6a^2 + 13a + 16$

b Write the problem. $2e(a + 3e - 6h) - h(9a - 4e + 4h) - 5a(7a - 3e + 8h)$

Expand the brackets, being careful with signs. $= 2ae + 6e^2 - 12eh - 9ah + 4eh - 4h^2 - 35a^2 + 15ae - 40ah$

Simplify. $= 6e^2 - 35a^2 - 4h^2 + 17ae - 49ah - 8eh$

c Write the problem. $(8r - 9m)(r - 4m) - (9r - m)(6r + m)$

Expand the brackets, being careful with signs. $= 8r^2 - 32mr - 9mr + 36m^2 - 54r^2 - 9mr + 6mr + m^2$

Simplify. $= 37m^2 - 44mr - 46r^2$

d Write the problem. $8m^{-5}p(9h^{-1}m^5p^{-6} - h^2m^{-3} + 9m^{-3}p^4) - 7h^{-1}m^{-3}(h^3m^{-5}p - 4m^3p^{-5} - 5hp^4)$

Expand the brackets, being careful with signs and powers. $= 72h^{-1}p^{-5} - 8h^2m^{-8}p + 72m^{-8}p^5 - 7h^2m^{-8}p + 28h^{-1}p^{-5} + 35m^{-3}p^4$

Simplify. $= 100h^{-1}p^{-5} - 15h^2m^{-8}p + 35m^{-3}p^4 + 72m^{-8}p^5$

Investigate: Special expansions

Expand each of the following expressions.

$(x+3)(x-3)$, $(x-4)(x+4)$, $(4x+1)(4x-1)$, $(5x-1)(5x+1)$ and $(3x+7)(3x-7)$

Write a rule for the expansions of $(x+y)(x-y)$ and $(x-y)(x+y)$.

Now expand the following expressions. Note that you can write $(x+1)^2 = (x+1)(x+1)$.

$(x+1)^2$, $(x-3)^2$, $(x+4)^2$, $(5x-3)^2$ and $(7x+5)^2$

Write rules for the expansions of $(x+y)^2$ and $(x-y)^2$.

Important!

Special expansions

The special expansion $(x+y)(x-y) = x^2 - y^2$ is called the **difference of two squares**.

The special expansion $(x+y)^2 = x^2 + 2xy + y^2$ is called the **square of a sum**.

The special expansion $(x-y)^2 = x^2 - 2xy + y^2$ is called the **square of a difference**.

The term **perfect square** may be used to mean either the square of a sum or the square of a difference.

Example 15

Video tutorial
Special binomial products
MAT10NAVT10006

Expand each of the following expressions.

a $(m+4)^2$ **b** $(p-4)^2$ **c** $(r-6)(r+6)$

Solution

a Identify the product. **$(m+4)^2$ is a square of a sum**

Apply the pattern. $\boldsymbol{(m+4)^2 = m^2 + 2 \times m \times 4 + 4^2}$

Simplify. $\boldsymbol{= m^2 + 8m + 16}$

b Identify the product. **$(p-4)^2$ is a square of a difference**

Apply the pattern. $\boldsymbol{(p-4)^2 = p^2 - 2 \times p \times 4 + 4^2}$

Simplify. $\boldsymbol{= p^2 - 8p + 16}$

c Identify the product. **$(r-6)(r+6)$ is a difference of squares**

Apply the pattern. $\boldsymbol{(r-6)(r+6) = r^2 - 6^2}$

Simplify. $\boldsymbol{= r^2 - 36}$

Example 16

Puzzle sheet
Expanding binomials
MAT10NAPS00007

Expand each of the following expressions.

a $(3a+5c)(3a-5c)$ **b** $(4q-7r)^2$ **c** $(7y+5)^2$

Solution

a	Identify the product.	$(3a + 5c)(3a - 5c)$ **is a difference of squares**
	Apply the pattern.	$(3a - 5c)(3a + 5c) = (3a)^2 - (5c)^2$
	Simplify.	$= 9a^2 - 25c^2$
b	Identify the product.	$(4q - 7r)^2$ **is a square of a difference**
	Apply the pattern.	$(4q - 7r)^2 = (4q)^2 - 2 \times 4q \times 7r + (7r)^2$
	Simplify.	$= 16q^2 - 56qr + 49r^2$
c	Identify the product.	$(7y + 5)^2$ **is a square of a sum**
	Apply the pattern.	$(7y + 5)^2 = (7y)^2 + 2 \times 7y \times 5 + 5^2$
	Simplify.	$= 49y^2 + 70y + 25$

Investigate: Higher power special expansions

Polynomials with more than one variable may have terms where the sum of the powers of the variables is always the same. These polynomials are called **homogeneous polynomials**. for example, $3x^2y^2 + 2xy^3 - 5x^3y$ is homogeneous (of degree 4) because the sum of the powers of variables in each term is 4.

Consider the products of $(x - y)$ with the homogeneous polynomials in x that consist of all the possible terms of a particular degree in x and y with coefficients equal to 1.

Make a table of the products as the degree n increases.

n	Product	Expansion
1	$(x-y)(x+y)$	$x^2 - y^2$
2	$(x-y)(x^2+xy+y^2)$	
3	$(x-y)(x^3+x^2y+xy^2+y^3)$	

What do you find?

Can you prove it?

Now consider the same type of products with $(x + y)$, but with alternating signs in the homogeneous polynomial.

n	Product	Expansion
1	$(x+y)(x-y)$	$x^2 - y^2$
2	$(x+y)(x^2-xy+y^2)$	
3	$(x+y)(x^3-x^2y+xy^2-y^3)$	

What do you find?

Can you prove it?

The opposite of expansion of brackets is **factorisation**. In the simplest kind of factorisation, you find a common factor of the terms and use the distributive law in reverse to make brackets.

Important!

Factorisation

Factorisation is the opposite process to multiplying out brackets. In the **common factor** method, the highest common factor of the terms is put outside the brackets, leaving the other factors of each term inside the brackets.

Sometimes the terms containing variables in an expression are all negative. In this case, it is normal to use a negative common factor. In other cases you should make the common factor positive.

Example 17

Factorise each of the following.

a $12p + 18$ **b** $28u - 35v + 42$ **c** $24 - 30w - 36v$

Solution

a	Write the problem.	$\mathbf{12}\boldsymbol{p} + \mathbf{18}$
	Write each term as a product of the HCF, 6.	$= \mathbf{6} \times \mathbf{2}\boldsymbol{p} + \mathbf{6} \times \mathbf{3}$
	Use the distributive law in reverse.	$= \mathbf{6(2}\boldsymbol{p}\mathbf{ + 3)}$
b	Write the problem.	$\mathbf{28}\boldsymbol{u} - \mathbf{35}\boldsymbol{v} + \mathbf{42}$
	Write each term as a product of the HCF, 7.	$\mathbf{7} \times \mathbf{4}\boldsymbol{u} - \mathbf{7} \times \mathbf{5}\boldsymbol{v} + \mathbf{7} \times \mathbf{6}$
	Use the distributive law in reverse.	$= \mathbf{7(4}\boldsymbol{u} - \mathbf{5}\boldsymbol{v} + \mathbf{6)}$
c	Write the problem.	$\mathbf{24} - \mathbf{30}\boldsymbol{w} - \mathbf{36}\boldsymbol{v}$
	Choose ${}^{-}6$ as the HCF, since all the variable terms are negative.	${}^{-}\mathbf{6} \times {}^{-}\mathbf{4} + {}^{-}\mathbf{6} \times \mathbf{5}\boldsymbol{w} + {}^{-}\mathbf{6} \times \mathbf{6}\boldsymbol{v}$
	Use the distributive law in reverse.	$= {}^{-}\mathbf{6(}{}^{-}\mathbf{4} + \mathbf{5}\boldsymbol{w} + \mathbf{6}\boldsymbol{v}\mathbf{)}$
	Re-arrange the terms in the brackets to start with a positive term.	$= {}^{-}\mathbf{6(5}\boldsymbol{w} + \mathbf{6}\boldsymbol{v} - \mathbf{4)}$

The highest common factor might involve one or more variables, and it could also include powers.

Example 18

Factorise each of the following.

a $24mnp + 18mp - 2mpq$ **b** $56q^2h^3 - 8eq^5h^7 - 16e^3q^5h^4$

Solution

a	Write the problem.	$\mathbf{24}\boldsymbol{mnp} + \mathbf{18}\boldsymbol{mp} - \mathbf{2}\boldsymbol{mpq}$
	Think of the terms as products of the HCF, $2mp$.	$= \mathbf{2}\boldsymbol{mp} \times \mathbf{12}\boldsymbol{n} + \mathbf{2}\boldsymbol{mp} \times \mathbf{9} - \mathbf{2}\boldsymbol{mp} \times \boldsymbol{q}$
	Use the distributive law in reverse.	$= \mathbf{2}\boldsymbol{mp}\mathbf{(12}\boldsymbol{n} + \mathbf{9} - \boldsymbol{q}\mathbf{)}$

b The HCF includes 8, q^2 and h^3.

Think of the terms as products of the HCF, $8q^2h^3$.

Use the distributive law in reverse.

$$\begin{aligned}&56q^2h^3 - 8eq^5h^7 - 16e^3q^5h^4\\ &= 8q^2h^3 \times 7 - 8q^2h^3 \times eq^3h^4 - 8q^2h^3 \times 2e^3q^3h\\ &= 8q^2h^3(7 - eq^3h^4 - 2e^3q^3h)\end{aligned}$$

When you factorise expressions with terms that have negative powers, it is usual to make sure that all the powers inside the brackets are positive. This means that you will include the most negative powers of each variable in the highest common factor.

Example 19

Factorise $9x^9y^{-5}z^3 - 12x^4y^{-2}z^{-1}$.

Solution

Write the problem and think of powers in the HCF.

$$9x^9y^{-5}z^3 - 12x^4y^{-2}z^{-1}$$

Think of the terms as products of the HCF, $3x^4y^{-5}z^{-1}$.

$$= 3x^4y^{-5}z^{-1} \times 3x^5z^4 - 3x^4y^{-5}z^{-1} \times 4y^3$$

Use the distributive law in reverse.

$$= 3x^4y^{-5}z^{-1}(3x^5z^4 - 4y^3)$$

Write the negative powers as a denominator.

$$= \frac{3x^4(3x^5z^4 - 4y^3)}{y^5z}$$

Exercise 3.2 Expansion and factorisation

Understanding

Extra questions

Exercise 3.2

MAT10NAEQ00037

1 Simplify each of the following.

See Example 9

a $a(a + 5)$ **b** $m(m - p)$ **c** $g(2g + 3f)$

d $z(4z - 3w)$ **e** $x(3x + 2y)$ **f** $c(8c - 7)$

g $-m(5m - 3n)$ **h** $-f(2f - 5h)$ **i** $-d(2d + 3e)$

j $^{-}4c(2c - 7)$ **k** $^{-}3n(2n + 6)$ **l** $^{-}5k(3c - 2k)$

2 Simplify each of the following.

See Example 12

a $(x + 4)(x + 3)$ **b** $(1 + b)(2 + b)$ **c** $(a + 4)(a + 5)$

d $(6 + a)(2 + a)$ **e** $(y + 1)(y + 6)$ **f** $(a + 1)(a + 2)$

g $(m + 5)(m + 7)$ **h** $(a + 2)(a + 6)$ **i** $(n + 2)(n + 4)$

3 Factorise each of the following.

See Example 17

a $^{-}70q - 30$ **b** $75q - 40$ **c** $28y - 35$

d $44a - 56$ **e** $^{-}72v - 81$ **f** $130c - 143$

g $^{-}90c - 75$ **h** $91t - 117$ **i** $^{-}45u - 165$

j $^{-}18n - 26$ **k** $^{-}28w - 210$ **l** $70r + 126$

m $48b + 20$ **n** $90g + 63$ **o** $^{-}22x - 12$

p $78k + 143$ **q** $^{-}60u - 156$ **r** $20p + 5$

s $^{-}66v - 77$ **t** $80h + 120$ **u** $^{-}143x - 55$

v $33e + 39$ **w** $77m - 35$ **x** $140t + 154$

y $^{-}7a - 35$ **z** $55k - 10$

See Example 13

4 Simplify each of the following.

a $(a + 1)(2a + 3)$
b $(2a + 1)(a + 3)$
c $(3x + 2)(x + 4)$
d $(2y + 3)(4y + 5)$
e $(5g + 1)(7g - 2)$
f $(3a + 2)(6a - 2)$
g $(7m - 1)(7m + 1)$
h $(3x - 2)(5x - 3)$
i $(5y + 4)(5y + 4)$
j $(4y + 3)(3y + 4)$
k $(7n + 5)(7n - 5)$
l $(4 - 2c)(2 + 4c)$

See Example 15

5 Complete the simplification of each of the following using special expansions.

a $(a + b)^2 = a^2 + 2ab + \dots$
b $(x + y)^2 = x^2 + \dots + y^2$
c $(w - r)^2 = w^2 - 2rw + \dots$
d $(a + 2)^2 = a^2 + 4a + \dots$
e $(x - 3)^2 = x^2 - \dots + 9$
f $(m - 5)^2 = \dots - 10m + 25$
g $(g + 7)^2 = g^2 + 14g + \dots$
h $(6 + b)^2 = 36 + \dots + b^2$
i $(3 - n)^2 = \dots - 6n + n^2$
j $(k + 8)^2 = k^2 + 16k + \dots$

Fluency

6 Simplify each of the following.

a $3(a^2 - 3a - 6)$
b $7(x - 3y + 4z)$
c $4(x^2 + 5x + 7)$
d $6(2m - n + 3p)$
e $3m(m - 3n - 2p)$
f $2g(4 - 3g + a)$
g $(x + 5)(x - 5)$
h $(y - 7)(y + 3)$
i $(x + 12)(x + 5)$
j $(k - 9)(k + 11)$
k $(x + 6)(x - 5)$
l $(2 + n)(9 - n)$

See Example 14

7 Simplify each of the following.

a $4(x + 5) + 9(x + 3)$
b $7(2a + 3) + 6(3a - 1)$
c $10(3d + 5) - 7$
d $c(c + 9) - c(c - 8)$
e $y(4y - 3) - y(4 - y)$
f $h(2h - 3) - h(5 + 2h)$
g $2x(3x^2 - 2x - 4) + 6x(x^2 - 2x + 3)$
h $9a(a^2 + 2a - 8) + 3a(a^2 - 4a - 2)$
i $3x(5x^2 - x + 2) - 2x(x^2 + 2x + 4)$
j $6a(4 - 2a + 3a^2) - 5a(2a^2 - 7 - 3a)$

8 Factorise each of the following.

a ${}^{-}144b - 156e - 12$
b $156p - 36u - 36$
c $42x + 45k + 18$
d $144m - 156t + 108$
e $35z + 77m - 28$
f ${}^{-}90m - 63e - 72$
g ${}^{-}18b - 33g - 6$
h $24q + 26w + 30$
i $48x - 20e + 24$
j ${}^{-}84t - 35x - 91$
k $130p - 90n - 110$
l ${}^{-}60w - 44c - 52$
m $12a - 45h + 3$
n $5p - 55r + 35$
o $84q - 54e + 30$
p ${}^{-}104c - 8p - 16$
q $120x - 180m + 195$
r $40t + 88a - 104$
s $105h + 135y + 135$
t $110k - 120m + 20$
u $5y - 75r - 40$
v $26h - 4x - 24$
w $42v - 182n - 112$
x $40r - 60m - 75$
y $40c + 52m + 36$
z $16k - 120x - 48$

See Example 10

9 Simplify each of the following.

a $6h^{-6}m(2h^{-4} - 9m^4)$
b ${}^{-}3n^{-5}x^3(6x^{-5} + 5z^3)$
c $6t^3w^{-3}(8x^4 - 5t^{-6})$
d $2a^{-1}c(6a^{-6} - 7w^{-6})$
e $3b^{-1}x^{-6}(5 - 7x^4)$
f $2b^6n^{-1}(4n^6 - 3b)$
g $5a^{-1}k^6(8a^{-3} - 5k^{-2})$
h $8a^2b^6(6b^{-1} + 7k^7)$
i ${}^{-}k^6t^4(4x^{-1} + 3)$
j $4b^{-1}c^{-5}(2b^{-7} + 7z^{-1})$
k $3g^{-2}p^4(8r^{-1} + p^4)$
l $7m^{-2}q^{-4}(6 + 7m^5)$
m $5k^3m^{-5}(5k^3 + 3m^{-2})$
n $5h^{-6}k^{-2}(7k^4 + 3p^7)$
o ${}^{-}5k^{-7}r^4(9v + 5k^{-5})$
p $8e^6h^{-3}(3e^{-5} - 5p^5)$
q $7h^{-3}r^{-1}(4z^{-2} + 5r^{-3})$
r ${}^{-}4a^{-6}y^{-3}(9y^{-6} + 4a^{-7})$
s $9k^2(2e^2 + k^{-3})$
t $5u^{-3}(4u^{-6} + 1)$
u $6e^2n^{-3}(5v - 8e^{-3})$
v $5r^6t^6(7r^2 - 2y^{-5})$
w ${}^{-}8c^{-5}t(2z^6 + 9)$
x $-n^{-3}t^3(8t + 7n^{-2})$
y $8m^{-2}w^4(9m^3 + 4w^2)$
z $2g^5t^{-4}(2t^7 + 9u^2)$

9780170194860

10 Simplify each of the following.

a $(a-3)(4+2a)$
b $(5-2x)(3-x)$
c $(3-5a)(4+a)$
d $(k-4)(2k-5)$
e $(^{-}2m+1)(3-m)$
f $(3x+1)(7-2x)$
g $(2p+q)(^{-}2p-q)$
h $(2n-5m)(3m+n)$
i $(2x+3y)(3x+2y)$
j $(4x-3y)(2y-7x)$
k $(8a-5b)(7b-2a)$
l $(3y+x)(-x-y)$
m $(8u+12e)(8u-12e)$
n $(8g+5v)(9g+4v)$
o $(8e+11t)(10e+11t)$
p $(11a-9x)(7a+5x)$
q $(11x-2r)(15x-1r)$
r $(14c+3z)(9c+1z)$
s $(14x-13m)(14x-3m)$
t $(8x+9p)(7x+15p)$
u $(2v+1c)(3v-4c)$
v $(12p-5z)(5p+4z)$
w $(7y-13r)(15y+1r)$
x $(2m-15q)(11m-12q)$
y $(2z+13u)(7z-15u)$
z $(14z-1a)(5z+3a)$

11 Simplify each of the following.

a $(a+b)^2$
b $(x-y)^2$
c $(t+2)^2$
d $(g+3)^2$
e $(9-h)^2$
f $(a+6)^2$
g $(m+11)^2$
h $(10-b)^2$
i $(f-g)^2$
j $(u+9)^2$
k $(8+v)^2$
l $(y-13)^2$

12 Simplify each of the following.

a $(a+2)(a-2)$
b $(x-3)(x+3)$
c $(e+5)(e-5)$
d $(n+4)(n-4)$
e $(j-9)(j+9)$
f $(q+4)(q-4)$
g $(h+2)(h-2)$
h $(5+w)(w-5)$
i $(a+r)(a-r)$
j $(p+t)(t-p)$
k $(y-g)(g+y)$
l $(a-p)(a+p)$

13 Simplify each of the following.

a $^{-}7c(6c+5q-5x)-q(c-4q-6x)-6x(9c+7q-2x)$
b $5n(7n+4t+7x)+x(6n-t+4x)+8t(7n+8t-6x)$
c $9e(5a-7e+6m)+3a(5a+1e+5m)+m(4a-e-6m)$
d $u(p+1u-7y)+5y(3p+u-5y)+p(3p-5u+4y)$
e $^{-}8y(k+5x+9y)+6x(5k-7x-9y)+2k(4k-9x+6y)$
f $^{-}2p(8a-5e-7p)-2a(7a-9e+6p)+e(7a+4e+8p)$
g $^{-}6a(3a+2m-6t)-4m(5a-6m+t)-5t(7a-6m+9t)$
h $8e(9e-2u+8y)-4y(3e-u+9y)+3u(4e-7u-4y)$
i $-r(5q+4r+u)-9q(5q-7r+u)+9u(7q-8r-2u)$
j $^{-}8w(6k-7w+7z)-5z(4k-3w-4z)-7k(8k-9w-6z)$
k $^{-}9v(7p+9r+8v)-9r(5p+1r-4v)+5p(2p+1r+6v)$
l $^{-}6u(7b+2m-6u)+3b(3b-5m+3u)+5m(5b-2m-8u)$
m $3g(9g-8h-2x)+3h(7g+9h+9x)-5x(9g-8h+3x)$
n $9e(2e-3g-4v)-9v(9e-5g-6v)+6g(6e-5g-7v)$
o $5p(c+9p+9r)-c(9c-1p+4r)-8r(4c+p+7r)$
p $^{-}4u(4b-5u+5v)-v(7b+8u-6v)+7b(b-4u-2v)$
q $2e+7h-2m-6h(8e+9h-3m)-9e(2e-7h+m)$
r $48a+5e+8z+a(5a+2e+z)-6e(a+4e+9z)$
s $3e(2e-5h-6q)-2h(3e+7h-3q)-8q(7e-4h-8q)$
t $^{-}2b(9b+4p+6y)+2y(2b+7p+4y)-6p(b-4p+8y)$
u $3u(5q+6u+5w)+9q(6q-1u-7w)+7w(8q+3u-5w)$
v $7b(a+5b+8u)+5u(4a+7b-6u)-a(a-6b+9u)$

w $28a + 5c + 5h + 5c(7a + 2c - 9h) - 4a(a - 2c - 7h)$

x ${}^{-}5p(7h + 8n - p) + 3h(9h - 1n + 7p) + 7n(6h - 1n - 7p)$

y ${}^{-}5a(3a + 7c - 6n) - c(6a - 7c + 7n) + 2n(a - 3c + 8n)$

z $q(4q - 5t + 7x) + 5x(9q - 5t - 6x) - 6t(7q - 4t + 5x)$

14 Simplify each of the following.

a $(6u - t)(6u - 7t) + (7u - 8t)(7u - 6t)$

b $(5y + 4g)(7y + 3g) - (3y + 4g)(8y + 9g)$

c $(5p - 8n)(6p - 5n) - (p - 6n)(3p + 2n)$

d $(3b - 2p)(6b + p) - (7b - 9p)(8b - p)$

e $(3c + 2g)(9c + 7g) - (2c - 3g)(8c + 9g)$

f $(6t - 7a)(9t - 7a) - (t - a)(5t - 8a)$

g $(8c + q)(2c - q) + (6c + 5q)(3c - 7q)$

h $(3p + 5v)(7p - 8v) + (9p + 5v)(6p + v)$

i $(8q + r)(2q - 7r) + (7q - 2r)(q - 7r)$

j $(5h + 7y)(7h - 4y) + (6h + 5y)(4h + 7y)$

k $(7z + 2h)(9z + h) - (3z + 5h)(5z - 9h)$

l $(4r - 7n)(7r + 4n) - (5r - 8n)(r - 2n)$

m $(9h + 7q)(5h + 6q) + (5h + 8q)(h + 9q)$

n $(7v + 4t)(4v + 3t) - (8v + 3t)(3v + 4t)$

o $(3m - 2y)(8m + 5y) - (7m + 3y)(5m - 2y)$

p $(5x + b)(9x - 2b) - (9x - b)(3x + 7b)$

q $(a - 6b)(7a - b) + (3a + 2b)(6a - b)$

r $(4c - p)(7c - 9p) - (8c - p)(c - 4p)$

s $(7a - 6h)(9a - 7h) - (9a + 2h)(7a - 4h)$

t $(3g - 7h)(5g + 9h) + (3g - 5h)(2g + 9h)$

u $(2n - 3c)(5n - 6c) + (n + 6c)(n - 5c)$

v $(4q - a)(9q - 8a) + (5q - 4a)(8q + 7a)$

w $(4h + 9p)(7h + 9p) - (3h - 5p)(7h - 9p)$

x $(9e + 4z)(7e + 4z) - (7e + 9z)(e - 5z)$

y $(8c + 3w)(8c + 7w) - (3c + 2w)(6c - w)$

z $(3u + 2h)(u - 3h) + (3u - 5h)(u - 4h)$

15 Simplify each of the following.

a $8q^{-7}r^{7}(9q^{-3}r^{-2}v^{2} - 8q^{2}v + 7r^{-3}v^{-2}) - 6r^{-2}v^{2}(q^{-7}r^{6}v^{-4} + 4q^{-10}r^{7} + 9r^{7}v)$

b $6k^{7}r^{4}(2k^{-2}r^{-4}u^{-1} + 2r^{-2}u^{-4} - 7k^{-1}u^{3}) - 9r^{-4}u^{-1}(3k^{7}r^{6}u^{-3} + 4k^{5}r^{4} - 5r^{4}u^{3})$

c $8k^{-3}q^{4}(9k^{3}q^{2}t - 7kt^{6} + 5q^{3}t^{2}) + q^{2}t(7k^{-3}q^{5}t + 9q^{4}t^{6} - 8k^{0}q^{4})$

d $8t^{6}u^{-6}(7g^{-6}tu^{12} - 6g^{-1}t^{2} + 5t^{2}u^{3}) - 7g^{-6}t^{2}(7g^{5}t^{6}u^{-6} + 6g^{-6}u^{3} + 9t^{5}u^{6})$

e $u^{-7}y^{3}(4ru^{2}y^{-10} - r^{-6}u^{2} - 2u^{2}y^{6}) + 7ru^{2}(4r^{-7}u^{-7}y^{3} - 6u^{-7}y^{-7} + 9r^{3}y^{6})$

f $3h^{3}q(7b^{-5}h^{3}q^{2} - h^{3}q^{2} - 3b^{-2}h^{3}) - 6b^{-5}h^{3}(6b^{3}h^{3}q - bq^{2} + h^{3}q^{3})$

g $2a^{-4}n^{7}(6a^{4}n^{5}v^{-5} - a^{-5}v + 6n^{4}v^{5}) + 8n^{5}v^{-5}(9a^{-4}n^{6}v^{10} - 7a^{0}n^{7} - 2n^{7}v)$

h ${}^{-}3m^{-6}t^{5}(4m^{3}t^{6}z^{-6} - 8t^{3}z^{6} + 5m^{-6}z^{5}) + 8t^{6}z^{-6}(3m^{-6}t^{2}z^{12} - m^{-3}t^{5} - t^{5}z^{5})$

i $4g^{2}k^{-2}(4g^{3}k^{-3}u^{2} + g^{2}u - k^{3}u^{-3}) + 5k^{-3}u^{2}(3g^{2}k^{4}u^{-5} - 3k^{-2}u - g^{5}k^{-2})$

j ${}^{-}8u^{5}w^{-1}(7b^{2}u^{-9}w^{6} + b^{5}u^{-7} - 9u^{-7}w) - 9b^{2}u^{-7}(3b^{3}u^{5}w^{-1} - 2b^{-1}w - 4u^{3}w^{5})$

k $7t^{6}w^{-4}(2q^{4}t^{7}w^{10} - q^{11}t^{6} + 5t^{6}w^{-2}) - 7q^{4}t^{6}(2q^{7}t^{6}w^{-4} - 2t^{7}w^{6} + q^{-4}w^{-2})$

l $q^{-5}t^{-5}(4g^{-3}q^{-2} + 6q^{-4}t^{5} + 9g^{-6}q^{-4}) - 6g^{-3}q^{-4}(5g^{-3}q^{-5}t^{-5} - 3g^{-5}t^{5} - 9q^{-3}t^{-5})$

m ${}^{-}6g^{2}n^{2}(g^{2}n^{-2}p^{2} + g^{2}p^{4} - 7n^{2}p^{-2}) - 7n^{-2}p^{2}(2g^{2}n^{6}p^{-4} - g^{4}n^{2} - 3n^{2}p^{4})$

n ${}^{-}8a^{4}w^{2}(9a^{-4}w^{2}y + w^{-4}y^{2} + 8ay^{4}) + 9w^{2}y(3a^{4}w^{-4}y - 8a^{0}w^{2} - 3w^{2}y^{4})$

o $7e^{5}n^{4}(3e^{-4}n^{6}t^{-1} - 8e^{-1}t^{-6} + 9n^{-4}t^{6}) + 9n^{6}t^{-1}(6e^{5}n^{-6}t^{7} + 5n^{4}t^{-6} - 7en^{4})$

p ${}^{-}6n^{-3}t(6h^{3}n^{4}t^{-4} + 5h^{8}n^{-4} + 9n^{-4}t^{3}) + 5h^{3}n^{-4}(9h^{5}n^{-3}t + 2ht^{3} + 8n^{5}t^{-3})$

q $5m^{2}p(8a^{-1}m^{6}p + 7a^{4}m^{3} + 2m^{3}p^{-2}) + 6a^{-1}m^{3}(a^{5}m^{2}p + 9m^{5}p^{2} - ap^{-2})$

r $9p^{-4}y^{4}(9g^{-4}py^{-8} - 7p^{-6}y^{3} + 5g^{-2}p^{-6}) - 9g^{-4}p^{-6}(8g^{2}p^{-4}y^{4} - 3g^{4}y^{3} + 6p^{2}y^{-4})$

s $9h^{6}y^{-3}(8h^{4}y^{3}z - 3hz^{6} - 2y^{4}z^{3}) - 3y^{3}z(h^{6}y^{-2}z^{2} + 9h^{10}y^{-3} + y^{-3}z^{6})$

t $k^{-7}n^{7}(8k^{4}n^{7}q^{6} + 3n^{4}q^{7} + 5k^{6}q^{3}) - 5n^{7}q^{6}(3k^{-7}n^{4}q - 8k^{-3}n^{7} + 7n^{7}q^{3})$

u $6k^{5}q^{-2}(9k^{-2}q^{-4}r - 4kr^{6} - 5q^{-2}r^{-4}) + 9q^{-4}r(k^{5}qr^{-5} + 6q^{-2}r^{6} + 4k^{3}q^{-2})$

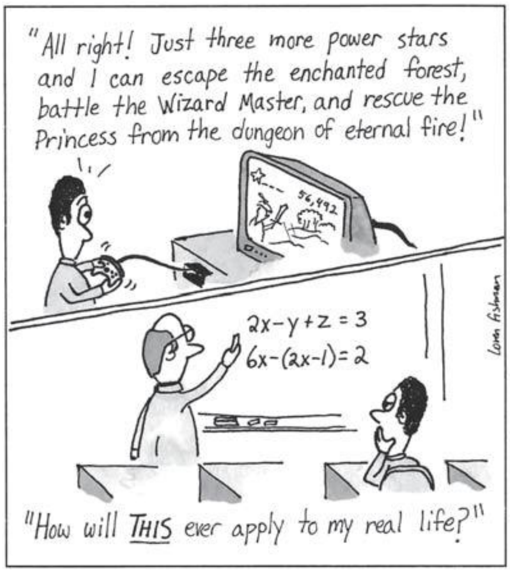
v $p^{5}x^{2}(6a^{7}p^{4}x^{3} + a^{9}p^{7}) + 2a^{7}p^{7}(5a^{2}p^{5}x^{2} - 2a^{2}x^{7} + 5p^{2}x^{5})$

w ${}^{-}6m^{3}r^{2}(4a^{-7}m^{-7}r + 3a^{-5}m^{-6} - 7m^{-6}r^{-6}) - 9a^{-7}m^{-6}(4a^{2}m^{3}r^{2} + 6m^{2}r^{3} + a^{2}r^{-6})$

x ${}^{-}7k^{4}w(5e^{4}k^{-6}w^{3} - 4k^{-5}w^{2} + e^{7}k^{-5}) - 4e^{4}k^{-5}(7e^{3}k^{4}w - 3ew^{2} - 4k^{3}w^{4})$

y $2e^{5}r^{-7}(3e^{6}r^{-2}v + 4ev^{-3} + 6r^{6}v^{-2}) - 7r^{-2}v(9e^{5}rv^{-3} - 4e^{11}r^{-7} - 3r^{-7}v^{-3})$

z $3a^{2}b^{-6}(3a^{-4}b^{-5}u^{-1} - 7b^{-4}u^{-5} - 7a^{-1}u^{6}) - 9b^{-5}u^{-1}(3a^{2}b^{-5}u^{-4} - 7a^{-2}b^{-6} - 7b^{-6}u^{6})$



16 Factorise each of the following. *See* Example 18

- **a** $20a^{2} - 18ap + 18au$
- **b** $126bk + 140by + 210b^{2}$
- **c** $3ey + 11e^{2} + 7eh$
- **d** $^{-}8c^{2} - 15cv - 10ct$
- **e** $169by + 182bv + 195b^{2}$
- **f** $70gh + 168g^{2} + 140gz$
- **g** $10a^{2} - 10av + 150az$
- **h** $^{-}99cn - 154cw - 110c^{2}$
- **i** $16at - 120a^{2} - 40aq$
- **j** $84b^{2} - 7bu - 49bh$
- **k** $10cr - 12cp - 12c^{2}$
- **l** $^{-}12ev - 24e^{2} - 108ew$
- **m** $5k^{2} + 45kr - 65kv$
- **n** $77xy - 49xz - 56x^{2}$
- **o** $12bu + 2b^{2} + 14bm$
- **p** $^{-}75n^{2} - 20nt - 55np$
- **q** $^{-}99cz - 77cm - 121c^{2}$
- **r** $^{-}36np - 39n^{2} - 12nq$
- **s** $48a^{2} + 88ae + 48ay$
- **t** $30ah - 22az - 28a^{2}$
- **u** $50ev - 80e^{2} - 110em$
- **v** $132a^{2} - 12ay - 156am$
- **w** $^{-}13kw - 3kp - 6k^{2}$
- **x** $4kp - 36k^{2} - 60kx$
- **y** $98u^{2} - 168uv - 126ux$
- **z** $15gq - 105gu - 120g^{2}$

17 Factorise each of the following.

- **a** $^{-}168e^{3}k^{6}q^{3} - 182e^{2}k^{6}q^{5} - 196e^{7}k^{2}q^{7}$
- **b** $44w^{3}y^{3}z^{10} - 143w^{4}yz^{6} + 66w^{6}y^{2}z^{5}$
- **c** $^{-}35m^{6}n^{5}u^{7} - 63m^{9}n^{6}u^{5} - 70m^{4}n^{5}u^{9}$
- **d** $30g^{6}r^{3}y^{4} - 28g^{3}r^{6}y^{3} - 18g^{6}r^{8}y$
- **e** $^{-}28c^{5}y^{3}z^{6} - 49c^{8}y^{5}z^{3} - 35c^{6}y^{3}z^{6}$
- **f** $70h^{2}m^{9}u - 110h^{4}m^{3}u^{3} + 110hm^{7}u^{4}$
- **g** $30g^{6}p^{4}q^{5} + 14gp^{5}q^{7} - 24g^{6}p^{4}q^{6}$
- **h** $4a^{2}r^{9}z^{2} - a^{7}r^{4}z^{3} + 3a^{6}r^{7}z$
- **i** $20a^{3}m^{4}p^{3} + 24a^{7}m^{8}p^{2} - 28a^{2}m^{8}p^{6}$
- **j** $110b^{5}m^{2}q^{4} - 121bm^{6}q^{5} + 110b^{3}m^{7}q^{2}$
- **k** $120an^{5}x^{4} + 8a^{6}n^{10}x - 32a^{2}n^{5}x^{2}$
- **l** $66c^{5}p^{13}v^{3} - 55c^{5}p^{5}v^{7} - 77cp^{10}v^{4}$
- **m** $77b^{6}e^{2}h^{3} + 70be^{4}h^{5} - 105b^{6}e^{2}h^{7}$
- **n** $75e^{5}r^{7}z^{4} + 120e^{10}r^{4}z^{6} + 150e^{8}r^{8}z^{2}$
- **o** $22h^{7}x^{4}y^{8} - 55h^{9}x^{6}y^{4} - 66h^{4}x^{4}y^{5}$
- **p** $105a^{4}hu^{7} + 7a^{2}h^{3}u^{4} - 28a^{6}h^{6}u^{2}$
- **q** $^{-}35e^{2}tz^{10} - 56e^{5}t^{5}z^{5} - 28e^{3}tz^{8}$
- **r** $26h^{7}q^{10}u^{5} + 12h^{3}u^{9} + 14h^{2}qu^{9}$
- **s** $^{-}130e^{5}v^{2}x^{3} - 140e^{4}v^{5}x^{5} - 120e^{9}v^{2}x^{5}$
- **t** $5a^{5}k^{9}t^{7} + 12a^{8}k^{5}t^{6} + 4a^{8}k^{10}t^{3}$
- **u** $52c^{6}x^{5}y^{5} - 39c^{8}x^{6}y - 130c^{5}x^{7}y^{6}$
- **v** $105e^{4}u^{2}z^{7} - 60eu^{3}z^{5} + 105e^{6}u^{5}z^{3}$
- **w** $7t^{2}xz^{7} - 3t^{4}x^{2}z^{5} + 9t^{7}xz^{10}$
- **x** $^{-}88g^{6}t^{9}x^{4} - 99g^{2}tx^{5} - 121g^{2}tx^{6}$
- **y** $^{-}16g^{3}xz^{4} - 8g^{3}xz^{7} - 32g^{6}xz^{8}$
- **z** $^{-}91p^{8}r^{5} - 104b^{5}p^{3}r^{6} - 91b^{4}p^{5}r^{3}$

18 $P(n) = 2n^{3} + 7n$ and $Q(y) = 7y^{4} - y^{3} + 3y - 6$. Find $P(x) \times Q(x)$. *See* Example 11

19 $P(n) = 6n^2 - 3n + 7$ and $Q(x) = -x^6 - x^2 + 3x$. Find $P(x) \times Q(x)$.

20 $P(e) = {}^{-}9e^4 + 9e$ and $Q(k) = {}^{-}7k^4 - 5k^3 - 1$. Find $P(x) \times Q(x)$.

21 $P(z) = 7z^4 - 6z^3 + 3$ and $Q(e) = 5e^6 + 5e^5 - e^4 - 9e^3 + 8e - 1$. Find $P(x) \times Q(x)$.

See Example 16

22 Simplify each of the following.

a $(2a + 3)^2$ **b** $(3x + 1)^2$ **c** $(5m - 2)^2$

d $(3h - 7)^2$ **e** $(8u - 3)^2$ **f** $(2 - 9k)^2$

g $(5 + 4d)^2$ **h** $(9k - 2)^2$ **i** $(2a + 5m)^2$

j $(4g + 3s)^2$ **k** $(7h + 2k)^2$ **l** $(8d - q)^2$

23 Simplify each of the following.

a $(2b + 1)(2b - 1)$ **b** $(3c - 2)(3c + 2)$ **c** $(5m - 3)(5m + 3)$

d $(1 + 2x)(1 - 2x)$ **e** $(4m + 1)(4m - \mathrm{l})$ **f** $(8g - 3)(8g + 3)$

g $(2a - p)(2a + p)$ **h** $(c - 3d)(c + 3d)$ **i** $(8d + 2)(8d - 2)$

j $(8r - 5y)(8r + 5y)$ **k** $(7y + 4d)(7y - 4d)$ **l** $(6 - 3p)(6 + 3p)$

24 Simplify each of the following.

a $(a + b - 4)(a + b + 4)$ **b** $(x + y + 1)(x + y - 1)$

c $(c - 2d + 3)(c - 2d - 3)$ **d** $(2x - y + 3)(2x - y + 3)$

e $(m - n + 2)(m - n - 2)$ **f** $(a + b + 4)(-a + b + 4)$

g $(2c + d + 3)(2c - d - 3)$ **h** $(2r - 3t + s)(2r + 3t - s)$

i $\left(\frac{2}{x} - \frac{1}{y} + 1\right)\left(\frac{2}{x} + \frac{1}{y} - 1\right)$ **j** $\left(\frac{1}{a} - \frac{1}{b} + 2\right)\left(\frac{1}{a} + \frac{1}{b} - 2\right)$

See Example 19

25 Factorise each of the following.

a $3a^4g^4t^6 - 13a^{-3}g^7t^{12} - 13a^{-2}g^2t^8$ **b** $165q^{-1}v^{-3}z^8 - 135v^{-2}z^5 + 210q^{-4}v^{-1}z^7$

c ${}^{-}98c^{12}u^9y^{-3} - 21c^{10}u^9y^{-4} - 28c^6u^9y$ **d** $18m^9q^{-6}u^2 - 8m^3q^{-1}u^5 + 16m^9q^{-2}u$

e $13m^{-1}t^{13}z^5 + 195m^{-2}t^4z^{11} + 52m^{-3}t^5z^6$ **f** $42h^{-1}p^5q^{-7} - 15hp^{-6}q^{-4} + 3h^{-6}p^{-2}q^{-3}$

g ${}^{-}24e^{-2}h^{-3}p^{-7} + 10e^{-6}h^{-2}p^{-2} - 22e^{-4}h^{-3}p^{-2}$ **h** $45h^{-2}q^7z + 126h^2q^7 - 36hq^{12}z^{-4}$

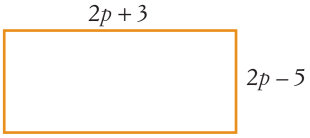
i ${}^{-}20a^6g^8x^{-3} - 130a^9g^{12}x^{-6} - 150a^5g^{11}x^{-3}$ **j** ${}^{-}225n^{-5}q^{-6}r^3 + 30n^{-7}r^8 + 105n^{-6}q^{-6}r^8$

Problem solving

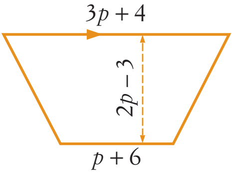
26 Find expressions for the areas of the following shapes and simplify your expressions.

Worked solutions
Exercise 3.2
MAT10NAWS00041

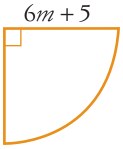
a



b



c



27 **a** Expand $\left(n + \frac{1}{2}\right)^2$.

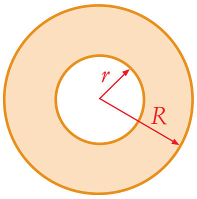
b Use your result to find the value of the following, without using a calculator.

i $\left(3\frac{1}{2}\right)^2$ **ii** $\left(19\frac{1}{2}\right)^2$

Worked solutions
Exercise 3.2
MAT10NAWS00041

Reasoning

28 An *annulus* is the area enclosed by concentric circles, as shown by the shaded area on the right.
Show that the area is given by $A = \pi(R - r)(R + r)$.



Worked solutions
Exercise 3.2
MAT10NAWS00041

3.3 Algebraic fractions

Important!

Algebraic fractions

These are fractions that involve algebraic expressions, rather than only numbers. The algebraic part may be in the denominator or numerator. Algebraic fractions are added, subtracted, multiplied and divided in the same way as ordinary fractions.

Remember that when you add and subtract fractions, you must use a common denominator. You should use the lowest common denominator (LCD).

Example 20

Video tutorial
Algebraic fractions
MAT10NAVT10007

Simplify each of the following.

a $\frac{a}{4}+\frac{b}{6}$ **b** $\frac{g}{2}-\frac{g}{5}$ **c** $\frac{m}{7}\times\frac{n}{3}$ **d** $\frac{p}{3}\div\frac{q}{5}$

Solution

a Write the problem and think of $\frac{1}{4}+\frac{1}{6}$.

$$\frac{a}{4}+\frac{b}{6}$$

Do like $\frac{1}{4}+\frac{1}{6}=\frac{3\times1}{3\times4}+\frac{2\times1}{2\times6}$ with LCD 12.

$$=\frac{3\times a}{3\times4}+\frac{2\times b}{2\times6}$$

Write over the common denominator.

$$=\frac{3a+2b}{12}$$

b Write the problem and think of $\frac{1}{2}-\frac{1}{5}$.

$$\frac{g}{2}-\frac{g}{5}$$

Do like $\frac{5\times1}{5\times2}-\frac{2\times1}{2\times5}$ with LCD 10.

$$=\frac{5\times g}{5\times2}-\frac{2\times g}{2\times5}$$

Write over the common denominator.

$$=\frac{5g-2g}{10}$$

Simplify.

$$=\frac{3g}{10}$$

c Write the problem and think of $\frac{2}{7}\times\frac{5}{3}$.

$$\frac{m}{7}\times\frac{n}{3}$$

Multiply the tops and multiply the bottoms.

$$=\frac{m\times n}{7\times3}$$

Simplify.

$$=\frac{mn}{21}$$

d Write the problem and think of $\frac{2}{3}\div\frac{7}{5}$.

$$\frac{p}{3}\div\frac{q}{5}$$

Change to multiplication and turn the fraction you are dividing by upside down.

$$=\frac{p}{3}\times\frac{5}{q}$$

Multiply the tops and multiply the bottoms.

$$=\frac{p\times5}{3\times q}$$

Simplify.

$$=\frac{5p}{3q}$$

Example 21

Animated example
Algebraic fractions
MAT10NAAE00003

Simplify $\frac{6h+1}{8}+\frac{4h-7}{12}$.

Solution

Write the problem and think of $\frac{1}{8}+\frac{1}{12}$.

$$\frac{6h+1}{8}+\frac{4h-7}{12}$$

Do like $\frac{1}{8}+\frac{1}{12}=\frac{3\times1}{3\times8}+\frac{2\times1}{2\times12}$ with LCD 24.

$$=\frac{3\times(6h+1)}{3\times8}+\frac{2\times(4h-7)}{2\times12}$$

Write over the common denominator.

$$=\frac{3(6h+1)+2(4h-7)}{24}$$

Expand the brackets.

$$=\frac{18h+3+8h-14}{24}$$

Collect like terms.

$$=\frac{26h-11}{24}$$

Example 22

CAS TI-Nspire exercise
Algebra
MAT10NATI00013

CAS ClassPad exercise
Algebra
MAT10NACP00013

Simplify $\frac{9m+2n}{6}-\frac{8m-5n}{5}$.

Solution

Write the problem and think of $\frac{1}{6}-\frac{1}{5}$.

$$\frac{9m+2n}{6}-\frac{8m-5n}{5}$$

Do like $\frac{1}{6}-\frac{1}{5}=\frac{5\times1}{5\times6}-\frac{6\times1}{6\times5}$ with LCD 30.

$$=\frac{5\times(9m+2n)}{5\times6}-\frac{6\times(8m-5n)}{6\times5}$$

Write over the common denominator.

$$=\frac{5(9m+2n)-6(8m-5n)}{30}$$

Expand the brackets.

$$=\frac{45m+10n-48m+30n}{30}$$

Collect like terms.

$$=\frac{^{-}3m+40n}{30}$$

Write the positive term first.

$$=\frac{40n-3m}{30}$$

Remember that when you multiply fractions, you multiply the numerators and multiply the denominators.

Example 23

Simplify $\frac{5p+3q}{6}\times\frac{3p-q}{5}$.

Solution

Write the problem with brackets.

$$\frac{(5p+3q)}{6}\times\frac{(3p-q)}{5}$$

Multiply the tops and multiply the bottoms.

$$=\frac{(5p+3q)(3p-q)}{6\times5}$$

Expand the brackets.

$$=\frac{15p^2+4pq-3q^2}{30}$$

Remember that to divide by a fraction, you multiply by its reciprocal.

Example 24

Simplify $\frac{2k+3}{5} \div \frac{k-4}{3}$.

Solution

Write the problem with brackets. $\frac{(2k+3)}{5} \div \frac{(k-4)}{3}$

Change to multiplication and turn the fraction you are dividing by upside down. $= \frac{(2k+3)}{5} \times \frac{3}{(k-4)}$

Multiply the tops and multiply the bottoms. $= \frac{3(2k+3)}{5(k-4)}$

Expand the brackets. $= \frac{6k+9}{5k-20}$

You can use index laws when multiplying and dividing algebraic fractions.

Example 25

Simplify each of these:

a $\frac{3x^4y^3}{4z^5} \times \frac{6x^5z^8}{9y^7}$

b $\frac{a^3c}{4b^6} \div \frac{8a^4}{3bc^2}$

Solution

a Write the problem $\frac{3x^4y^3}{4z^5} \times \frac{6x^5z^8}{9y^7}$

Separate the numeric and variable parts. $= \frac{3 \times 6 \times x^4y^3x^5z^8}{4 \times 9 \times y^7z^5}$

Simplify the numeric part and use index laws for the variables. $= \frac{x^{4+5}y^{3-7}z^{8-5}}{2}$

Simplify the indices. $= \frac{x^9y^{-4}z^3}{2}$

Change the negative index to positive on the bottom. $= \frac{x^9z^3}{2y^4}$

b Write the problem. $\frac{a^3c}{4b^6} \div \frac{8a^4}{3bc^2}$

Change to multiplication by the inverse. $\frac{a^3c}{4b^6} \times \frac{3bc^2}{8a^4}$

Separate the numeric and variable parts. $= \frac{3 \times a^3cbc^2}{4 \times 8 \times b^6a^4}$

Simplify the numeric part and use index laws for the variables (remember $c = c^1$). $= \frac{3 \times a^{3-4}c^{1+2}b^{1-6}}{32}$

Simplify the indices. $= \frac{3 \times a^{-1}c^3b^{-5}}{32}$

Change the negative indices to positive on the bottom. $= \frac{3c^3}{32ab^5}$

Exercise 3.3 Algebraic fractions

Understanding

Extra questions
Exercise 3.3
MAT10NAEQ00009

See Example 20

1 Simplify each of the following.

a $\frac{3a}{4}+\frac{2y}{3}$ **b** $\frac{2g}{9}+\frac{m}{3}$ **c** $\frac{7m}{5}+\frac{n}{10}$

d $\frac{2c}{9}-\frac{3k}{2}$ **e** $\frac{9c}{8}-\frac{r}{10}$ **f** $\frac{3g}{4}-\frac{7y}{5}$

g $\frac{5b}{6}\times\frac{n}{8}$ **h** $\frac{c}{12}\times\frac{p}{24}$ **i** $\frac{e}{14}\times\frac{k}{4}$

j $\frac{4g}{9}\div\frac{7u}{4}$ **k** $\frac{9e}{2}\div\frac{6h}{7}$ **l** $\frac{7m}{8}\div\frac{9v}{7}$

m $\frac{7p}{4}+\frac{p}{12}$ **n** $\frac{q}{10}+\frac{q}{20}$ **o** $\frac{2a}{7}+\frac{a}{6}$

p $\frac{9m}{8}-\frac{m}{12}$ **q** $\frac{5q}{6}-\frac{2q}{9}$ **r** $\frac{c}{12}-\frac{c}{3}$

s $\frac{9b}{4}\times\frac{7b}{6}$ **t** $\frac{5c}{3}\times\frac{c}{12}$ **u** $\frac{3a}{4}\times\frac{7a}{2}$

v $\frac{a}{4}\div\frac{6a}{5}$ **w** $\frac{5c}{4}\div\frac{5c}{3}$ **x** $\frac{m}{15}\div 4m$

y $\frac{13b}{15}-\frac{13b}{10}$ **z** $\frac{9b}{2}-\frac{14b}{9}$

Fluency

See Example 21

2 Simplify each of the following.

a $\frac{8h-9}{7}+\frac{9h-5}{6}$ **b** $\frac{g-4}{6}+\frac{4g-5}{10}$ **c** $\frac{9k+8}{5}+\frac{8k+5}{2}$

d $\frac{h+7}{7}+\frac{8h-9}{4}$ **e** $\frac{4g-7}{9}+\frac{8g-9}{2}$ **f** $\frac{6b+1}{6}+\frac{4b-5}{12}$

g $\frac{4g-1}{5}+\frac{8g+5}{3}$ **h** $\frac{7n+3}{3}+\frac{8n-7}{6}$ **i** $\frac{2a+1}{12}+\frac{6a-1}{6}$

j $\frac{2a-7}{8}+\frac{6a-5}{10}$ **k** $\frac{7b-6g}{8}+\frac{8b+5g}{10}$ **l** $\frac{a+e}{3}+\frac{3a+8e}{9}$

m $\frac{2h-9y}{2}+\frac{4h+y}{3}$ **n** $\frac{8r+5t}{3}+\frac{2r+7t}{4}$ **o** $\frac{3g-x}{10}+\frac{4g+9x}{8}$

p $\frac{4g-7m}{5}+\frac{7g-4m}{2}$ **q** $\frac{8q-9r}{10}+\frac{4q+r}{5}$ **r** $\frac{7h-5t}{3}+\frac{h+t}{12}$

s $\frac{9g-8x}{5}+\frac{5g+8x}{10}$ **t** $\frac{6c-e}{4}+\frac{6c-e}{6}$

See Example 22

3 Simplify each of the following.

a $\frac{2r+7}{12}-\frac{6r-7}{18}$ **b** $\frac{7r+3}{4}-\frac{2r-9}{5}$ **c** $\frac{6e+1}{4}-\frac{6e-5}{6}$

d $\frac{5k-6}{6}-\frac{8k-7}{3}$ **e** $\frac{9v-1}{8}-\frac{2v-7}{4}$ **f** $\frac{n+9}{3}-\frac{4n+9}{15}$

g $\frac{7c+4}{3}-\frac{6c+1}{8}$ **h** $\frac{7h-1}{6}-\frac{8h-7}{5}$ **i** $\frac{5h-9}{2}-\frac{4h+5}{5}$

j $\frac{9k-2}{12}-\frac{2k-1}{3}$ **k** $\frac{e+8v}{9}-\frac{8e+5v}{6}$ **l** $\frac{7c+9t}{4}-\frac{3c-5t}{6}$

m $\frac{2u+3v}{10}-\frac{u-3v}{6}$ **n** $\frac{k+7t}{6}-\frac{4k-5t}{3}$ **o** $\frac{9g+7r}{5}-\frac{8g+9r}{10}$

p $\frac{9c+2p}{8}-\frac{6c-7p}{10}$ **q** $\frac{2c+5p}{2}-\frac{4c-p}{10}$ **r** $\frac{2e+3t}{8}-\frac{9e-7t}{5}$

s $\frac{p-3q}{12}-\frac{2p-3q}{2}$ **t** $\frac{9g-2v}{9}-\frac{4g+9v}{6}$

4 Simplify each of the following. *See* Example 23

a $\frac{3q-7}{8}\times\frac{7q-4}{5}$ **b** $\frac{e+8}{7}\times\frac{6e+7}{5}$ **c** $\frac{8a-7}{9}\times\frac{4a+3}{5}$

d $\frac{2e-3}{4}\times\frac{3e+7}{12}$ **e** $\frac{7g-6}{3}\times\frac{5g-2}{2}$ **f** $\frac{7c+9}{9}\times\frac{5c+8}{6}$

g $\frac{b-9}{5}\times\frac{4b+9}{2}$ **h** $\frac{3a+1}{9}\times\frac{a-9}{12}$ **i** $\frac{6e+5}{9}\times\frac{9e-1}{6}$

j $\frac{3b-5}{8}\times\frac{6b-5}{5}$ **k** $\frac{3p+5q}{3}\times\frac{7p+5q}{6}$ **l** $\frac{8b-7c}{5}\times\frac{4b-5c}{4}$

m $\frac{3n-4q}{12}\times\frac{8n-q}{18}$ **n** $\frac{7c+6m}{3}\times\frac{c+5m}{4}$ **o** $\frac{5n-2p}{2}\times\frac{n-8p}{6}$

p $\frac{6a+7e}{7}\times\frac{a+5e}{4}$ **q** $\frac{5c+6g}{6}\times\frac{c+2g}{4}$ **r** $\frac{5g-3n}{4}\times\frac{9g+7n}{8}$

s $\frac{3p-2q}{12}\times\frac{6p-7q}{2}$ **t** $\frac{9g+2u}{6}\times\frac{8g+u}{8}$

5 Simplify each of the following. *See* Example 24

a $\frac{5n-9}{6}\div\frac{2n-3}{7}$ **b** $\frac{9p-7}{2}\div\frac{6p+5}{7}$ **c** $\frac{9p+1}{9}\div\frac{8p-7}{8}$

d $\frac{2a+1}{8}\div\frac{9a-7}{7}$ **e** $\frac{4c+3}{8}\div\frac{7c+4}{5}$ **f** $\frac{4h+1}{3}\div\frac{7h-4}{7}$

g $\frac{8u-3}{4}\div\frac{4u-5}{3}$ **h** $\frac{7e+9}{4}\div\frac{8e-3}{5}$ **i** $\frac{9a-2}{7}\div\frac{7a-2}{5}$

j $\frac{2a-7}{6}\div\frac{3a+5}{5}$ **k** $\frac{5n-6p}{9}\div\frac{9n+5p}{7}$ **l** $\frac{8c-7h}{2}\div\frac{9c+4h}{9}$

m $\frac{9e+4p}{3}\div\frac{5e+3p}{7}$ **n** $\frac{e+9n}{8}\div\frac{6e+7n}{3}$ **o** $\frac{9a+7h}{7}\div\frac{2a-7h}{8}$

p $\frac{9a+8e}{5}\div\frac{6a+e}{3}$ **q** $\frac{4g+7v}{3}\div\frac{6g+v}{8}$ **r** $\frac{g-9h}{6}\div\frac{8g-7h}{7}$

s $\frac{3h+5p}{5}\div\frac{9h-8p}{7}$ **t** $\frac{b-3r}{3}\div\frac{8b+5r}{5}$

6 Simplify each of the following. *See* Example 25

a $\frac{8b^5g}{12k^6y^2}\times\frac{2bk^6}{10g^6y}$ **b** $\frac{6n^5r^5}{3u^2}\div\frac{8n}{2r^3u^7}$ **c** $\frac{m^3u^6}{4v^7w^3}\div\frac{8u^4}{3mv^2w^5}$ **d** $\frac{5b^2c^7}{12q^5w}\times\frac{9b^5q^6w^2}{10c^7}$

7 Simplify each of the following.

a $\frac{a+6k}{2}-\frac{a-2k}{7}+\frac{4a-5k}{21}$ **b** $\frac{2c-k}{10}+\frac{2c+7k}{2}-\frac{3c-k}{5}$

c $\left(\frac{7a-8k}{3}-\frac{5a-4k}{9}\right)\times\frac{2a+3k}{4}$ **d** $\left(\frac{8h-7p}{2}-\frac{7h+2p}{4}\right)\times\frac{2h-5p}{6}$

e $\frac{9k+7w}{3}-\left(\frac{k-6w}{4}-\frac{3k+8w}{12}\right)$ **f** $\frac{4a-3e}{8}-\left(\frac{a-2e}{12}-\frac{a-2e}{2}\right)$

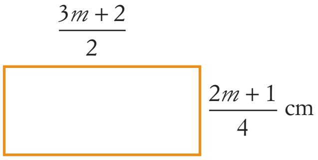
g $\frac{7a-8k}{9}+\frac{5a-4k}{3}\times\frac{2a+3k}{4}$ **h** $\frac{8h-7p}{6}-\frac{7h+2p}{4}\times\frac{2h-5p}{2}$

Problem solving

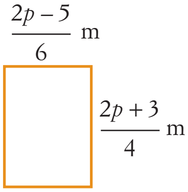
Worked solutions
Exercise 3.3
MAT10NAWS00042

⑧ Find the perimeters of the following shapes.

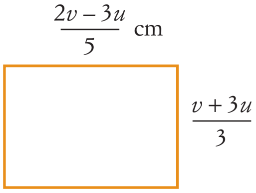
ⓐ



b



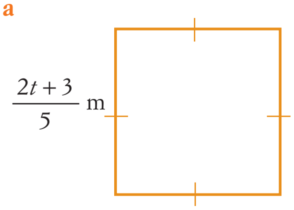
c



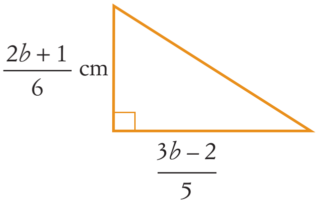
Worked solutions
Exercise 3.3
MAT10NAWS00042

9 Find the areas of the following shapes.

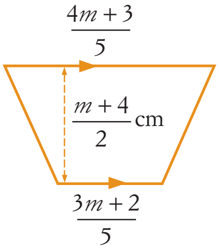
a



b



ⓒ



Reasoning

10 The simplifications below may have errors. For each one, if there is an error, identify the line in which the *first* error occurs and state the correct answer.

a

$$\begin{aligned} &\frac{9q+2r}{6}+\frac{7q+5r}{8} \\ &=\frac{8(9q+2r)}{24}+\frac{6(7q+5r)}{24} && \text{(A)} \\ &=\frac{72q+16r+42q+5r}{24} && \text{(B)} \\ &=\frac{114q+21r}{24} && \text{(C)} \end{aligned}$$

b

$$\begin{aligned} &\frac{4b+5h}{4}-\frac{6b+h}{3} \\ &=\frac{3(4b+5h)}{12}-\frac{4(6b+h)}{12} && \text{(A)} \\ &=\frac{12b+15h-24b+4h}{12} && \text{(B)} \\ &=\frac{15h-12b}{12} && \text{(C)} \end{aligned}$$

c

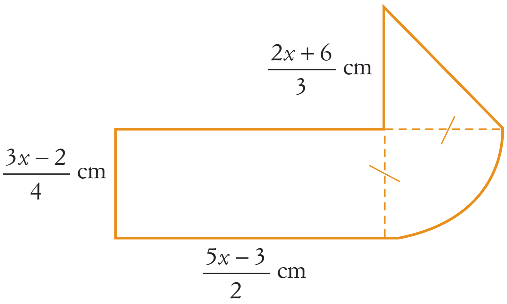
$$\begin{aligned} &\frac{9h-8r}{4}\times\frac{4h+5r}{5} \\ &=\frac{(9h-8r)(4h+5r)}{20} && \text{(A)} \\ &=\frac{36h^2-32hr+45hr-40r^2}{20} && \text{(B)} \\ &=\frac{36h^2+13hr-40r^2}{20} && \text{(C)} \end{aligned}$$

d

$$\begin{aligned} &\frac{3k-7u}{5}\div\frac{5k+u}{9} \\ &=\frac{3k-7u}{5}\times\frac{9}{5k+u} && \text{(A)} \\ &=\frac{9(3k-7u)}{5(5k+u)} && \text{(B)} \\ &=\frac{27k-7u}{25k+5u} && \text{(C)} \end{aligned}$$

Worked solutions
Exercise 3.3
MAT10NAWS00042

⑪ Explain the steps in finding the area of the shape below.



Chapter 3 summary

Quiz
Algebra
MAT10NAQZ00003

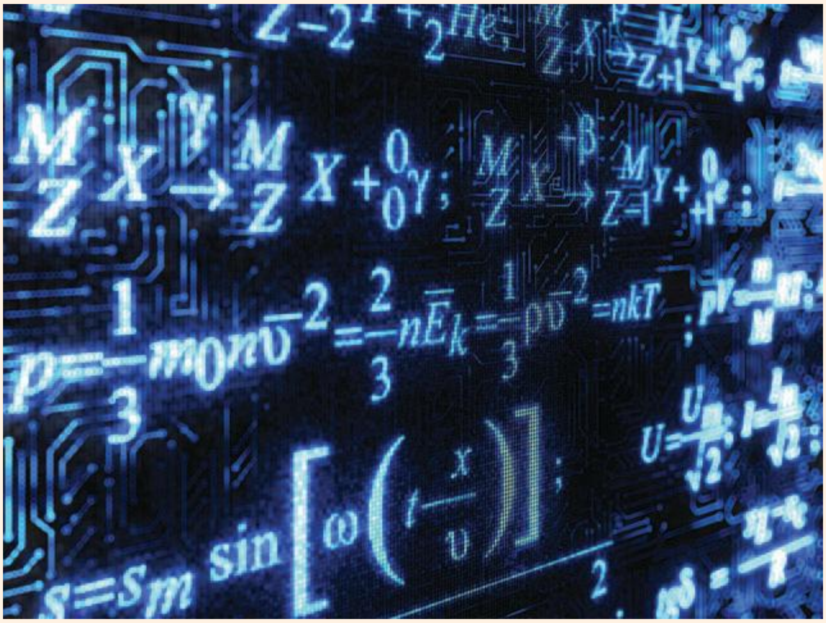
Worksheet
Algebra 2
MAT10NAWK00007

Worksheet
Algebra 6
MAT10NAWK00035

- A **variable** is a letter or symbol that stands for a number. A **constant** is a number.
- An **expression** has variables and/or numbers connected by arithmetic operations like +, ÷ and powers. An expression with numbers only is an **arithmetic expression**, while one with variables is an **algebraic expression**.
- The numbers in an expression that are multiplied by variables are called **coefficients**, the parts separated from the rest by + or − are called **terms** and a number on its own is called a **constant term**.
- To **evaluate** an algebraic expression you **substitute** values for the variables and work out (**evaluate**) the answer.
- A **formula** is an equation with a variable on the left and an algebraic expression on the right. The variable on the left is the **subject** of the formula. A formula is **transposed** using inverse operations to change the subject of the formula. A formula is evaluated by substituting values for the variables.
- Algebraic terms with exactly the same variables are called **like terms**. Like terms can be added or subtracted to simplify the expression. This is called **collecting like terms**.
- When you multiply or divide algebraic terms, the coefficients and each different variable are multiplied or divided separately.
- A **polynomial** is a sum of terms that are each the product of a constant and positive integer powers of the variables. A polynomial in one variable can be written as
$$P(x) = a_n x_n + a_{n-1} x_{n-1} + a_{n-2} x_{n-2} + \ldots + a_2 x_2 + a_1 x + a_0$$
where the coefficients $a_n, a_{n-1}, a_{n-2}, \ldots, a_2, a_1, a_0$ are real numbers.
- The highest power in a polynomial is called the **degree** of the polynomial, and the term that contains the highest power is called the **leading term**.
- A polynomial of degree 2 is called a **quadratic**.
- The **distributive law** states that the product of the sum (or difference) of two numbers with another number is the same as the sum (or difference) of the products of that number with each one separately. This can be written in symbols as:
$$a \times (b + c) = a \times b + a \times c \quad \text{and} \quad a \times (b - c) = a \times b - a \times c.$$
- **Expansion of brackets** means the removal of brackets using the distributive law. It is also called **simplification**.
- **Binomial products** are double brackets like $(x + 2)(x - 5)$. They can be expanded using the distributive law several times.
- The expansion $(x - y)(x + y) = x^2 - y^2$ is called the difference of two squares.
- The expansion $(x + y)^2 = x^2 + 2xy + y^2$ is called the **square of a sum.**
- The expansion $(x - y)^2 = x^2 - 2xy + y^2$ is called the **square of a difference**.
- The **highest common factor** of two algebraic terms is the largest expression that will divide exactly into the given terms.

Chapter 3 summary

- **Factorisation** is the opposite process to multiplying out brackets. In the common factor method, the highest common factor of the terms is put outside the brackets, leaving the other factors of each term inside the brackets.
- An **algebraic fraction** is a fraction that has at least one algebraic expression. They are simplified in the same way as normal fraction operations. The top line of an algebraic fraction sometimes factorises to make a factor that will cancel with the denominator.



9780170194860

Chapter 3 review

Understanding

1 Simplify if possible.

See Example 1

a $4p + 8p - 5p$ **b** $3h + t - 5h + 3t$ **c** $4xy + 3x^2 - 3y - xy - 5x^2$

2 Simplify each of the following.

See Examples 9, 12, 13

a $m(m - 3)$ **b** ${}^{-}2k(4m - 3k)$ **c** $(x + 2)(x + 5)$ **d** $(4e - 5)(e + 3)$

3 Factorise: **a** $30b + 18$ **b** ${}^{-}9m - 21$

See Example 17

4 $a = 3$, $b = 5$ and $c = {}^{-}2$. Find the value of $D = ab - 5c$.

See Example 4

5 Simplify each of the following.

See Example 18

a $\frac{9q}{8} + \frac{8w}{3}$ **b** $\frac{4p}{9} - \frac{p}{6}$ **c** $\frac{14b}{9} \times \frac{13b}{12}$ **d** $\frac{14r}{9} \div \frac{4t}{13}$

6 State the leading term and degree of any of the following that are polynomials.

See Example 7

a $5m^6 + 6m^5 + 5m^4 - 5m^3 + 3m^2 - 7m$ **b** $7u^7 - 6u^{-5} + 3u^4 + u$

c ${}^{-}5x^9 + x^5 - 5x^4 - 2x^3 + 6x^2 + 6$

Fluency

7 Simplify each of the following.

See Example 2

a $54n^2tu^2 \div 63ntu^3$ **b** $6g^3k^2r^3 \times 7gk$

8 Transpose the following formulas so that the letter shown in brackets is the subject.

See Example 6

a $A = 2\pi rs \quad (r)$ **b** $Z = \frac{x - u}{s} \quad (u)$

9 Simplify each of the following.

See Example 14

a $3(4t + 3) + 4(2t + 1)$ **b** $v(3v + 7) - 4v(5v - 6)$

c $3mp^2t(mt - 3mp + 2pt)$ **d** $3k(5 - 4k + 2k^2) - 3k(4k^2 - 5 - k)$

10 Simplify each of the following.

See Example 10

a ${}^{-}8b^2c^{-3}(5g^5 + 14b^{-5})$

b $a(7a + 4c - 3x) - 9x(8a - c - 4x) - c(3a - 7c + 5x)$

c $(7g + p)(3g + 5p) - (8g - 5p)(5g + 4p)$

d $4m^{10}u^{-3}(3g^{10}m^{-11}u^{13} + m^9 - 8m^9u^3) + 2g^{10}m^9(7g^{-10}m^{10}u^{-3} - 2g^{-3}u^3)$

11 Simplify each of the following using special expansions.

See Example 15

a $(4 - a)(4 + a)$ **b** $(h + 3)^2$

12 Factorise each of the following.

See Examples 17, 18

a $42gn + 196gw - 210g^2$ **b** $63ar^{10}w^6 - 135a^4r^5w^5 + 99a^2r^9w^4$

c $24n^3y^3 + 88n^{-5}y^{-2}z^{-3} + 40n^{-6}y^3z^{-2}$

13 Simplify each of the following.

See Examples 21–25

a $\frac{3q + 8r}{7} + \frac{8q - 3r}{6}$ **b** $\frac{5p + 9}{4} - \frac{3p + 1}{5}$

c $\frac{8b - 9v}{8} \times \frac{5b - 4v}{2}$ **d** $\frac{p + 1}{6} \div \frac{8p - 9}{5}$

e $\frac{48g^4h^4}{25k^7} \div \frac{15kh^2}{8g^2}$ **f** $\frac{x + u}{10} \times \left(\frac{7u - 5x}{12} - \frac{6u - 5x}{6}\right)$

14 Simplify each of the following.

a $^{-}35b^3t^{-1}u^{-3} \div {}^{-}45b^5t^{-4}u^{-2}$ **b** $5e^{-1}h^{-4}r^2 \times {}^{-}8e^4h^{-5}r^{-2}$

c $54e^{-11}t^{-5}u^4y^{-12} \div {}^{-}63e^{-4}t^6u^{-10}y^{-7}$ **d** $^{-}5(g^3p^{-1})^3x \times {}^{-}4g^{-5}(p^4x^{-3})^{-2}$

e $9b^7gt^{12}v^{-12} \times {}^{-}7b^6g^{-11}t^{-9}v^4$

See Example 8

15 $P(x) = 3x^4 + 5x^3 + 6x^2 + 6x + 2$ and $Q(x) = 8x^6 - 4x^3 - 6x + 4$. Find each of the following.

a $P(^{-}3)$ **b** $P(4)$ **c** $P(3)$ **d** $Q(2)$ **e** $Q(^{-}2)$ **f** $Q(^{-}1)$

16 $P(x) = {}^{-}9x^4 - 2x^3 + x^2 + 6x$, $Q(x) = {}^{-}9x^3 + 5x^2 - 6x + 3$ and $R(x) = 6x^5 - 5x^3 - 9x^2 - 8$. Find each of the following.

a $P(x) + R(x)$ **b** $R(x) - Q(x)$ **c** $R(x) + Q(x)$

See Example 3

17 Simplify $\dfrac{9ev^2 - 4ev^2}{10w^2v - w^2v} \times \dfrac{2e^2w^2v + 4e^2w^2v}{7wve - 5wve}$.

See Example 11

18 $P(z) = 7z^5 + 6z^4 + 6$ and $Q(y) = 4y^3 - 8y^2 + 9y - 5$. Find $P(x) \times Q(x)$.

See Example 16

19 Simplify each of the following.

a $(7 - 3q)^2$ **b** $(6y - 5)(6y + 5)$ **c** $(5h + 2v)^2$

d $(m + n - 6)(m + n + 6)$

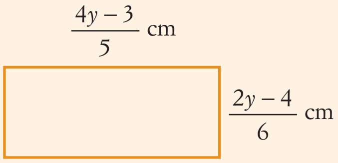
Problem solving

20 Find the value of the nominated variable in each of the following formulas.

a n if $PV = nRT$ and $P = 1100$, $V = 4.2$, $R = 70.2$ and $T = 324$

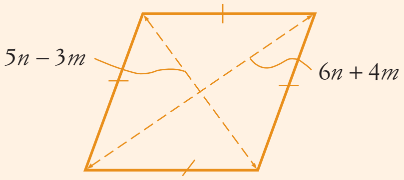
b a if $v = 5$, $u = 3$, $s = 6$ and $v = \sqrt{u^2 + 2as}$

21 Find an expression for the perimeter of this shape.

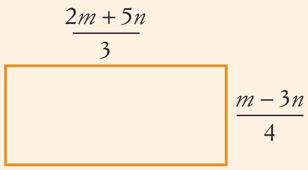


22 Find expressions for the areas of the shapes below and simplify your expressions.

a



b



Reasoning

23 The simplifications below may have errors. For each one, if there is an error, identify the line in which the *first* error occurs and state the correct answer.

a
$$\begin{aligned}&\frac{5m+n}{6}-\frac{3m-4n}{4}\\&=\frac{2(5m+n)}{12}-\frac{3(3m-4n)}{12} &&\text{(A)}\\&=\frac{10m+2n-9m-12n}{12} &&\text{(B)}\\&=\frac{m-10n}{12} &&\text{(C)}\end{aligned}$$

b
$$\begin{aligned}&\frac{3b-2c}{4}\times\frac{4b-3c}{3}\\&=\frac{3(3b-2c)}{12}\times\frac{4(4b-3c)}{12} &&\text{(A)}\\&=\frac{(9b-6c)(16b-12c)}{144} &&\text{(B)}\\&=\frac{144b^2-204bc-72c^2}{144} &&\text{(C)}\\&=\frac{12(12b^2-17bc-6c^2)}{144} &&\text{(D)}\\&=\frac{12b^2-17bc-6c^2}{12} &&\text{(E)}\end{aligned}$$

24 Expand $(10n+5)^2$ and use your expansion to find a short cut for the calculation of 15^2, 25^2, 35^2, . . .

25 Show that when 2 polynomials of degree 4 are added, the result can be a polynomial of any degree up to and including 4.

26 Show that when polynomials of degrees n and m are multiplied, the result is a polynomial of degree $n+m$.

Measurement and geometry

4

Area and volume

Prior learning
Chapter 4
MAT10MGPL00015

Parent guide
Chapter 4
MAT10MGPG00015

Curriculum guide
Chapter 4
MAT10MGCU00015

Australian Curriculum statements

Using units of measurement
Solve problems involving surface area and volume of right pyramids, right cones, spheres and related composite solids. (10A).

Video tutorial
Area and volume
MAT10MGVT00004

Weblink
Shape, space and measures
MAT10MGWB00004

Measuring length and calculating area and volume are important skills for everyday living. Imagine that you need to buy new carpet for your bedroom. How much carpet will you need to buy?
Another area problem you may need to deal with is calculating the number of cans of paint to buy to cover the walls of your room. Suppose that you want to build a garden bed. How much soil will you need? How much fertilizer will you need to add to the soil? You could use volume to find out how much cement mix it will take to pour a walkway or how much sand is needed to fill a sandbox.
Before measuring devices such as rulers were invented, parts of the human body were used to measure distances. This meant that people had ready access to a measuring device that was accurate enough for most practical uses.

Most ancient civilisations used lengths like a foot (about 30 cm) to measure short distances. They used something like one marching pace or stride (about 90 cm) for longer distances. Carpenters and traders measured lengths of timber using the cubit (about 45 cm)—the distance from the elbow to the tip of the longest finger. Can you imagine how complicated it would be to order the timber for a building project in cubits?

Today, measuring length and calculations involving measurements have been simplified because we use units from the metric system to express these quantities.

Mathematical literacy

Maths dictionary
MAT10ASDI00001

The mathematical words below have special meanings that you will learn in this chapter. It is important that you learn to spell them and gradually learn what they mean in mathematics.
You may find the glossary or online mathematical dictionary useful for this purpose.

apex
area
base
capacity
centimetre
circle
cone
cubic centimetre
cubic metre
cubic millimetre
curved surface area
cylinder
edges
face
hectare
hemisphere
Heron's formula
hypotenuse
kilometre
kite
litre
metre
metric system
millimetre
net
parallelogram
prism
pyramid
Pythagoras' theorem
rectangle
rhombus
slant height
sphere
square millimetre
square centimetre
square metre
square kilometre
surface area
three-dimensional
triangle
trapezium
two-dimensional
volume
vertices
vertex

4.1 Solving area problems

In the **metric system**, the standard unit of length is the **metre** (**m**). Other common units of length are shown below.

You often need to change from one unit to another. When changing to a *larger* unit, *divide*. When changing to a *smaller* unit, *multiply*. The arrows in the diagram below show the directions in which you move the decimal point when changing from one unit to another.

$$\begin{tikzcd}
\mathbf{mm} \arrow[r, bend left, "\div 10"] & \mathbf{cm} \arrow[l, bend left, "\times 10"] \arrow[r, bend left, "\div 100"] & \mathbf{m} \arrow[l, bend left, "\times 100"] \arrow[r, bend left, "\div 1000"] & \mathbf{km} \arrow[l, bend left, "\times 1000"]
\end{tikzcd}$$

The **area** of a flat shape measures the size of the region enclosed by the boundary of the shape. In the metric system, small areas are measured in:

- **square millimetres** (**mm^2**)—the area of a square 1 mm long and 1 mm wide, or
- **square centimetres** (**cm^2**)—the area of a square 1 cm long and 1 cm wide.

Larger areas are measured in:

- **square metres** (**m^2**)—the area of a square 1 m long and 1 m wide.

Very large areas are measured in:

- **hectares** (**ha**)—the area of a square 100 m long and 100 m wide, or
- **square kilometres** (**km^2**)—the area of a square 1 km long and 1 km wide.

Important!

Metric area unit conversions

$1 \text{ cm}^2 = 10 \text{ mm} \times 10 \text{ mm} = 100 \text{ mm}^2$

$1 \text{ m}^2 = 100 \text{ cm} \times 100 \text{ cm} = 10\,000 \text{ cm}^2 = 1\,000\,000 \text{ mm}^2$

$1 \text{ ha} = 100 \text{ m} \times 100 \text{ m} = 10\,000 \text{ m}^2$

$1 \text{ km}^2 = 1000 \text{ m} \times 1000 \text{ m} = 1\,000\,000 \text{ m}^2 = 100 \text{ ha}$

If you are changing to a *larger* unit, *divide*. If changing to a *smaller* unit, *multiply*.

$$\begin{tikzcd}
\mathbf{mm^2} \arrow[r, bend left, "\div 100"] & \mathbf{cm^2} \arrow[l, bend left, "\times 100"] \arrow[r, bend left, "\div 10\,000"] & \mathbf{m^2} \arrow[l, bend left, "\times 10\,000"] \arrow[r, bend left, "\div 10\,000"] & \mathbf{ha} \arrow[l, bend left, "\times 10\,000"] \arrow[r, bend left, "\div 100"] & \mathbf{km^2} \arrow[l, bend left, "\times 100"]
\end{tikzcd}$$

Example 1

Convert the measurements to the units shown in brackets.

a 26 500 m^2 (ha)

b 3.215 km^2 (ha)

Solution

a Write the measurement. **26 500 m^2**

Changing to a *larger* unit, so *divide*. **= 26 500 ÷ 10 000 ha**

= 2.65 ha

b Write the measurement. **3.215 km^2**

Changing to a *smaller* unit, so *multiply*. **= 3.215 × 100 ha**

= 321.5 ha

For some plane shapes there are rules that can be used to calculate areas. The most common ones are shown over the page.

Teacher notes

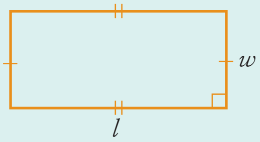
Area rules

MAT10MGTN00012

Important!

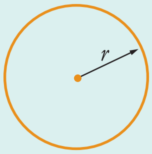
Area rules

Rectangle



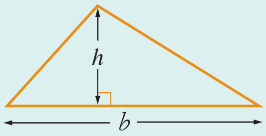
$A = l \times w$

Circle



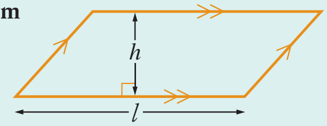
$A = \pi r^2$

Triangle



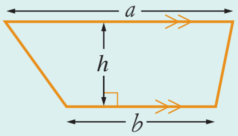
$A = \frac{1}{2}bh$

Parallelogram



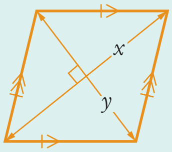
$A = l \times h$

Trapezium (trapezoid)

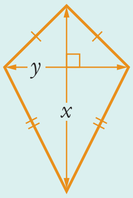


$A = \frac{1}{2} \times (a + b)h$

Rhombus



Kite



$A = \frac{1}{2}xy = \frac{1}{2}$(product of diagonals)

Example 2

CAS TI-Nspire exercise

Area and volume

MAT10MGTI00004

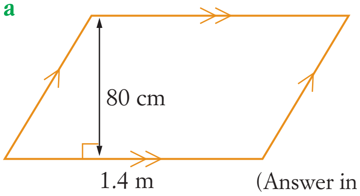
CAS ClassPad exercise

Area and volume

MAT10MGCP00004

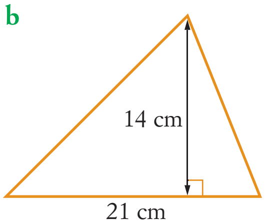
Find the areas of these shapes.

a



(Answer in m^2.)

b



Solution

a Write the area rule for a parallelogram.	$\boldsymbol{A = l \times h}$
Substitute the given measurements.	$= \mathbf{1.4\ m \times 80\ cm}$
Change both units to metres.	$= \mathbf{1.4\ m \times 0.8\ m}$
Evaluate.	$= \mathbf{1.12\ m^2}$
State the result.	**The parallelogram has an area of 1.12 m^2.**

b Write the area rule for a triangle. $A = \frac{1}{2}bh$

Substitute the given measurements and cancel. $= \frac{1}{\cancel{2}_1} \times 21 \times \cancel{14}^7$

Evaluate. $= 147 \text{ cm}^2$

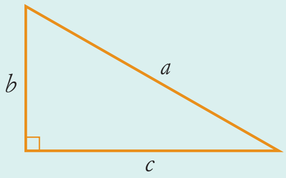
State the result. **The triangle has an area of 147 cm^2.**

The height of a triangle must be known to calculate its area. The height of a triangle is not always given. **Pythagoras' theorem** can be used to help calculate areas involving right-angled triangles.

Important!

Pythagoras' theorem

For a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.
For this triangle: $a^2 = b^2 + c^2$



Teacher notes
Proof of Pythagoras' theorem
MAT10MGTN00014

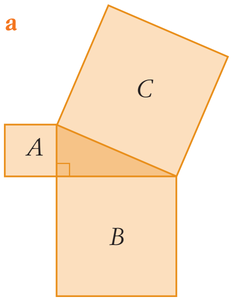
Pythagorean triples (or **triads**) such as (3, 4, 5), (5, 12, 13) and (7, 24, 25) can sometimes be used as a 'short cut' in calculations with right-angled triangles.

Investigate: Pythagoras' theorem

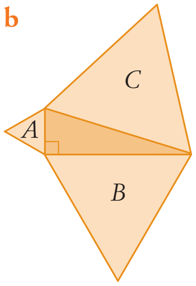
Pythagoras' theorem tells us that in diagram **a** below:

Area of A + area of B = area of C

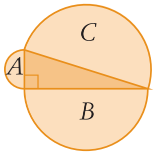
a



b



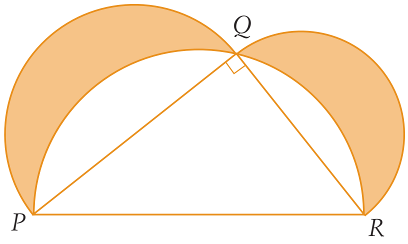
c



1 Diagram **b** shows a right-angled triangle with an equilateral triangle constructed on each side. Is it true to say that: area of A + area of B = area of C?
2 Diagram **c** shows a right-angled triangle with a semicircle constructed on each side. Is it true to say that: area of A + area of B = area of C?
3 Investigate other demonstrations of Pythagoras' theorem.

Lunes

Triangle PQR is drawn so that PR is the diameter of a semicircle and Q lies on the circumference, as shown here.
Any triangle drawn in this way will be right-angled.
Semicircles are constructed using PQ and QR as diameters, as shown.
The shaded areas formed are called **lunes**.

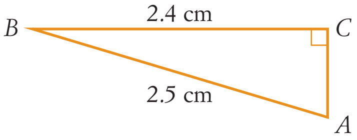


How does the sum of the areas of the lunes compare with the area of $\triangle PQR$?

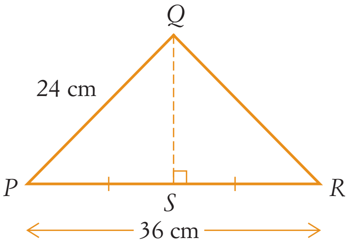
Example 3

Find the area of each of these triangles.

a



b



Solution

a Apply Pythagoras' theorem to the triangle. $BA^2 = CA^2 + BC^2$

Substitute for known sides. $2.5^2 = CA^2 + 2.4^2$

Evaluate. $6.25 = CA^2 + 5.76$

Reverse the equation. $CA^2 + 5.76 = 6.25$

Subtract 5.76 from both sides. $CA^2 = 6.25 - 5.76$

$= 0.49 \text{ cm}^2$

Take the square root. $CA = 0.7 \text{ cm}$

Write the rule for the area of a triangle. $A = \frac{1}{2}bh$

Substitute for known values and cancel. $= \frac{1}{{}_{1}\cancel{2}} \times \cancel{2.4}^{1.2} \times 0.7$

Evaluate. $= 0.84 \text{ cm}^2$

State the result. **The area of $\triangle ACB$ is 8.4 cm^2.**

b The height of $\triangle PQR$ is QS. QS needs to be calculated.

Apply Pythagoras' theorem to $\triangle PQS$. $PQ^2 = QS^2 + PS^2$

Substitute for known sides. $\left(PS = \frac{1}{2}PR\right)$ $24^2 = QS^2 + 18^2$

Evaluate. $576 = QS^2 + 324$

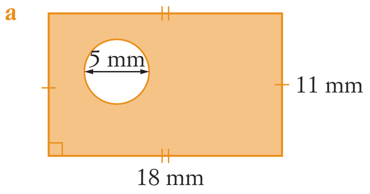
Reverse the equation.	$QS^2 + 324 = 576$
Subtract 5.76 from both sides.	$QS^2 = 576 - 324$
	$= 252$
Keep the square root on your calculator.	$= 15.874\,5\ldots$
Write the rule for the area of a triangle.	$A = \frac{1}{2}bh$
Substitute for known values and cancel.	$= \frac{1}{{}_{1}\cancel{2}} \times \cancel{36}^{18} \times 15.874\,5\ldots$
Use the value on your calculator.	$= 285.741\ldots\ \text{cm}^2$
State the result.	**The area of $\triangle PQR$ is about 286 cm^2.**

Some complex shapes are formed by combining simple plane shapes. When finding the areas of these figures, mark on your diagram how the shape has been formed.

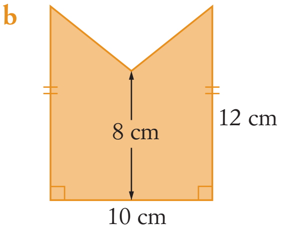
Example 4

Find the areas of the following shaded regions.

a



b



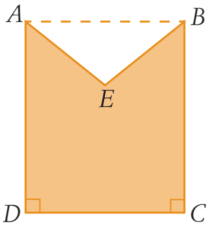
Solution

a The shaded region is a rectangle with a circular unshaded region.

Begin with the rectangle.	**Rectangle area** $= l \times w$
Substitute for l and w.	$= 18 \times 11$
Evaluate, including units.	$= 198\ \text{mm}^2$
Write the area rule for a circle.	**Circle area** $= \pi r^2$
Use $D = 2r$ to substitute for r.	$= \pi \times 2.5^2$
Keep the value on your calculator.	$= 19.634\ldots\ \text{mm}^2$
Find the shaded area.	**Shaded area = rectangle area − circle area**
Use the value on your calculator.	$= 198\ \text{mm}^2 - 19.634\ldots\ \text{mm}^2$
Evaluate.	$= 178.365\ldots\ \text{mm}^2$
Round off.	$\approx 178\ \text{mm}^2$

b There are *many* ways to work out this area. Here is *one*.

Name the vertices of the region.



Begin with rectangle *ABCD*. $\textbf{Area of } ABCD = l \times w$

Substitute. $= 12 \times 10$

Evaluate. $= 120 \text{ cm}^2$

Find the area of △*ABE*. $A = \frac{1}{2}bh$

Find the height, then substitute and cancel. $= \frac{1}{\cancel{2}_1} \times \cancel{10}^5 \times 4$

Evaluate. $= 20 \text{ cm}^2$

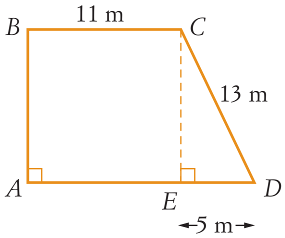
Calculate the shaded region. $\textbf{Shaded area} = \textbf{area of } ABCD - \textbf{area of } \triangle ABE$

Substitute. $= 120 \text{ cm}^2 - 20 \text{ cm}^2$

Evaluate. $= 100 \text{ cm}^2$

Example 5

Calculate the area of this figure.



Solution

First find *CE* using Pythagoras' theorem. $13^2 = CE^2 + 5^2$

Evaluate. $169 = CE^2 + 25$

Reverse the equation and subtract 25 from both sides. $CE^2 = 169 - 25$

$= 144 \text{ m}^2$

Take the square root of both sides. $CE = 12 \text{ m}$

Now find the area of rectangle *ABCE*. $\textbf{Area of } ABCE = l \times w$

Substitute. $= 11 \times 12$

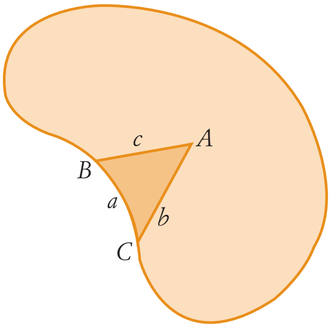
Evaluate. $= 132 \text{ m}^2$

Find the area of $\triangle CDE$.	$A = \frac{1}{2}bh$
Substitute and cancel.	$= \frac{1}{\cancel{2}_1} \times 5 \times \cancel{12}^6$
Evaluate.	$= 30 \text{ m}^2$
Calculate the total area.	**Total area = area of *ABCE* + area of $\triangle CDE$**
Substitute.	$= 132 \text{ m}^2 + 30 \text{ m}^2$
Evaluate.	$= 162 \text{ m}^2$

Investigate: Finding areas of irregular shapes

Before you begin this activity, your teacher will have already marked out a large, irregular shape (like a green on a golf course). The shape will be about 30 m long and 20 m wide. Work in groups of four or five and do the following.

1. Discuss a method of estimating the area of the shape.
2. Use your method to work out the area. Record your estimate and the method you used.
3. Meet back at the classroom and have all groups write their estimates on the whiteboard. Each group should discuss the method that they used.



You can use the following method to find the area.
Select a point near the centre of the region and mark out a triangular section like $\triangle ABC$ in the diagram shown here. The area of the triangle is given by this rule:

Area $= \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{1}{2}(a+b+c)$

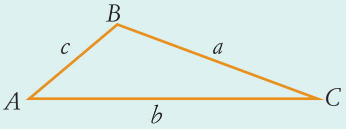
This is called **Heron's formula**. The total area can be found by dividing up the whole region into triangles and adding all of their areas.

1. Use Heron's formula to find the area you estimated earlier.
2. Compare your answer with the initial estimate you made and with the calculations of the other groups. Comment on what you find.
3. Get a local contour map and mark out an area such as a forest or other irregular shape. Find the area of the shape using Heron's formula. (Don't forget to look for the scale!)

Important!

Heron's formula

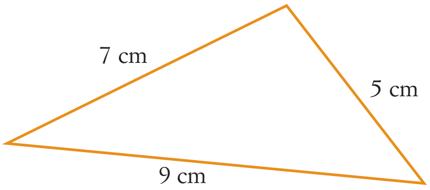
For a triangle ABC with sides a, b and c, the area is given by $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$ is the **semi-perimeter**.



You can use Heron's formula to calculate the area of a triangle.

Example 6

Find the area of this triangle.



Solution

We don't know the height of the triangle, so we need to use Heron's formula.

Calculate s, half the perimeter. $\quad \boldsymbol{s = \frac{1}{2}(a+b+c)}$

Substitute for the side lengths. $\quad \boldsymbol{= \frac{1}{2}(9+5+7)}$

Evaluate. $\quad \boldsymbol{= 10.5 \text{ cm}}$

State Heron's formula. $\quad \textbf{Area} \boldsymbol{= \sqrt{s(s-a)(s-b)(s-c)}}$

Substitute for s and the side lengths. $\quad \boldsymbol{= \sqrt{10.5(10.5-9)(10.5-5)(10.5-7)}}$

Use a calculator. $\quad \boldsymbol{\approx 17.4 \text{ cm}^2}$

Extra questions
Exercise 4.1
MAT10MGEQ00038

Exercise 4.1 Solving area problems

A calculator should be used where appropriate in this exercise. For calculations involving π, use the π key on your calculator and answer correct to 1 decimal place.

Understanding

See Example 1

1 Convert the measurements below to the units shown in brackets.

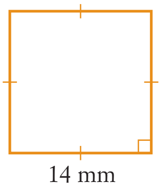
- **a** 7.3 ha (m^2)
- **b** 48 000 m^2 (km^2)
- **c** 57 ha (km^2)
- **d** 0.025 km^2 (ha)
- **e** 0.008 27 ha (m^2)
- **f** 0.003 42 km^2 (m^2)

2 Find the area of each of the following.

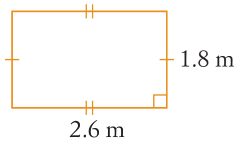
- **a** Square floor tile of side 12 cm
- **b** Rectangular birthday card 18 cm by 11 cm
- **c** Rectangular table top 1.5 m by 0.9 m
- **d** Australian flag 2 m by 1.3 m
- **e** Parallelogram of base 12.5 mm and height 5.6 mm
- **f** Rhombus with diagonals of 1.7 m and 2.8 m
- **g** Kite with diagonals of 26 cm and 44 cm

3 Find the areas of the following shapes.

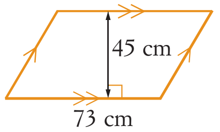
a



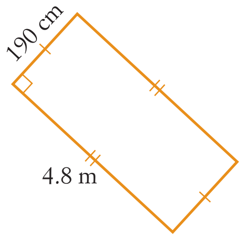
b



c

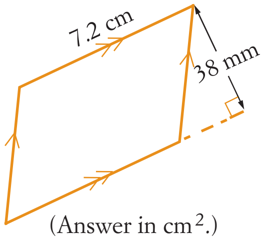


d



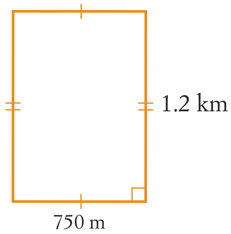
(Answer in m^2.)

e



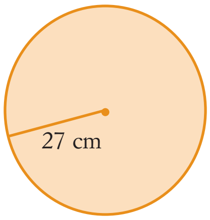
(Answer in cm^2.)

f

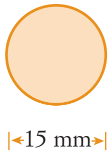


(Answer in km^2.)

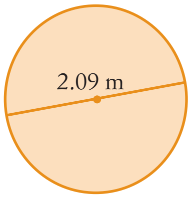
g



h



i



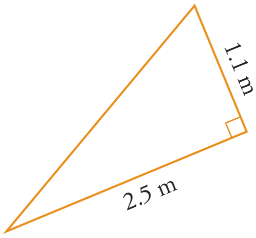
4 Find the area of each of the following.

- **a** Triangular scarf with base 1.5 m and height 0.8 m
- **b** Circular lid with a radius of 25 mm
- **c** Triangular wall tile with base 25 cm and height 12 cm
- **d** Trapezoidal table top with parallel sides 1.2 m and 1.8 m and height 80 cm
- **e** Compact disc with a diameter of 12 cm
- **f** Circular tablecloth with a radius of 1.5 m
- **g** Kite with diagonals of 95 cm and 46 cm
- **h** Rhombus shaped rug with diagonals of 3.8 m and 2.1 m

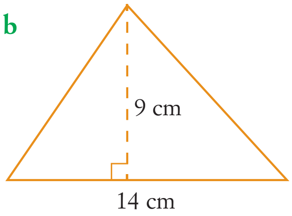
See Example 2

5 Find the areas of the following shapes.

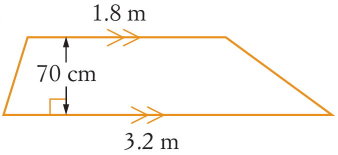
a



b

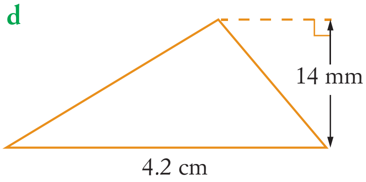


c



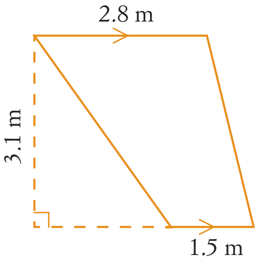
(Answer in m².)

d



(Answer in cm².)

e



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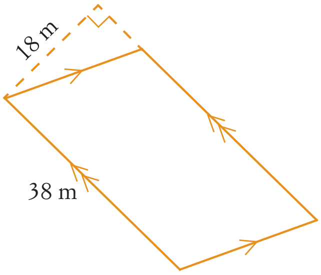

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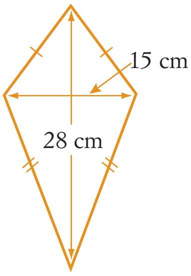

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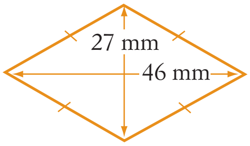

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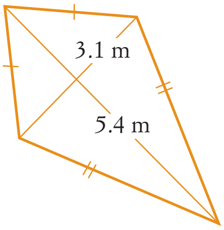
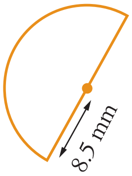


Fluency

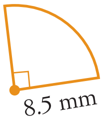
6 Find the areas of the following shapes.

a

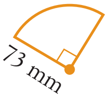
b



c



d



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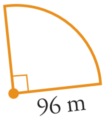

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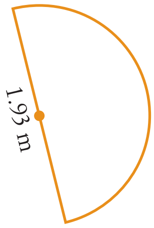
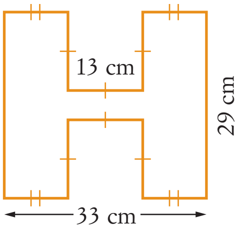
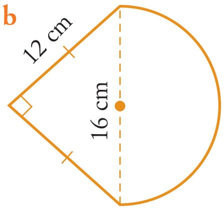


7 Work out the missing measurements, then find the areas of these shapes.

a



b



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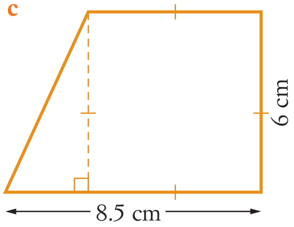
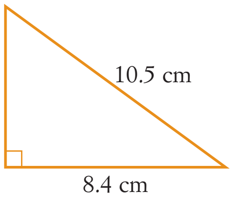


Worked solutions
Exercise 4.1
MAT10MGWS00010
See Example 3

8 Find the areas of these triangles.

ⓐ



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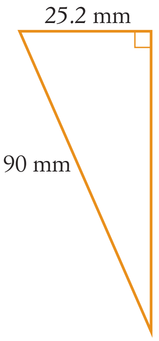
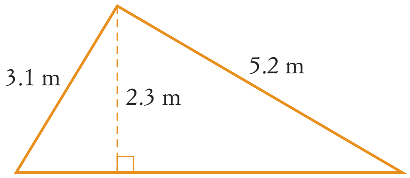


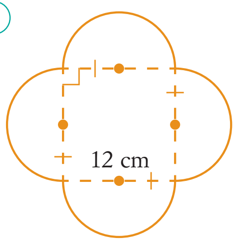
c



Worked solutions
Exercise 4.1
MAT10MGWS00010
See Example 3

9 Find the areas of the following compound shapes.

ⓐ



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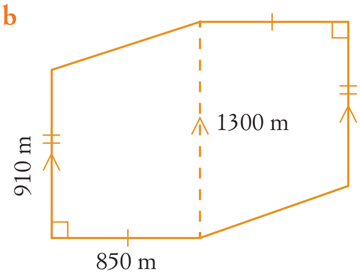

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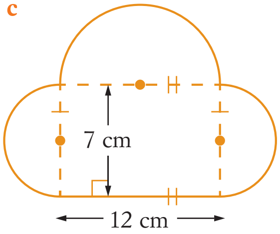
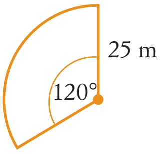


10 Find the areas of the following shapes.

a



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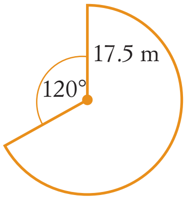

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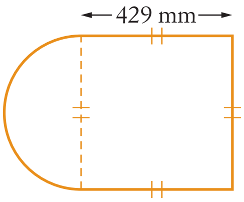
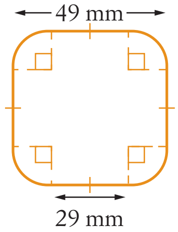


d



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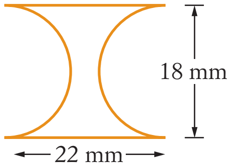

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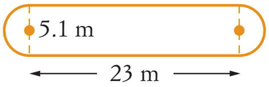
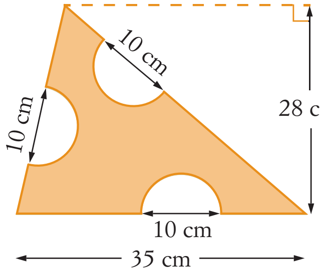


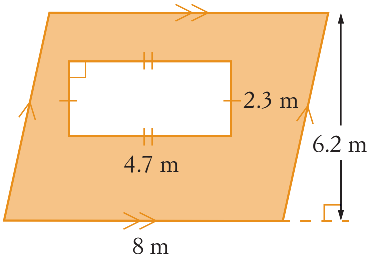
Worked solutions
Exercise 4.1
MAT10MGWS00010
See Example 5

11 Find the areas of the following shaded regions.

ⓐ

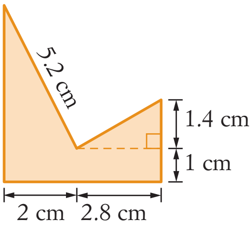


b



12 Find the areas of the following shapes.

a



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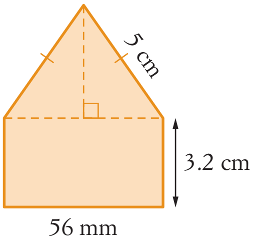
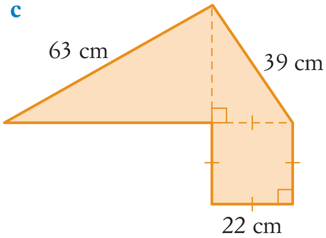


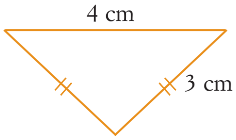
c



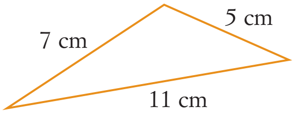
See Example 6

13 Use Heron's formula to find the area of each triangle, correct to 1 decimal place.

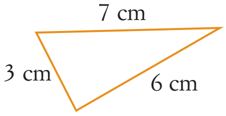
a



b



c

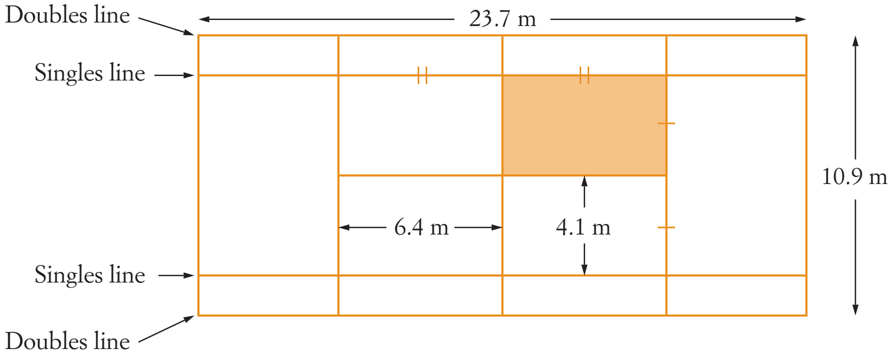


Problem solving

14 A circular plate has an area of 707 cm^2. What is its radius?

15 A square piece of metal has an area of 110.25 cm^2. What are its dimensions?

16 Here is a diagram of a full-sized tennis court.



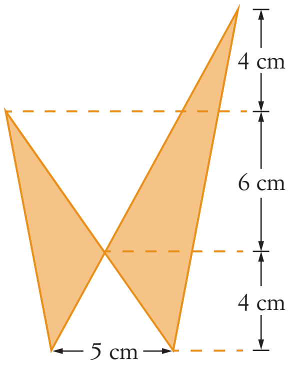
a What area (shaded) does a player have to serve into?

b Find the total area for play in a singles match.

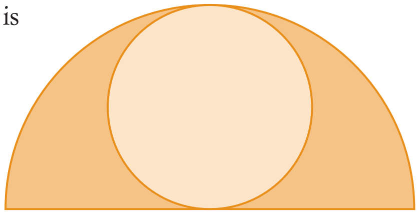
c Find the extra area that is available for a doubles match.

17 The minute hand of a clock is 10 cm long. It is 3 cm longer than the hour hand. In one full revolution, how much more of the face is covered by the minute hand than by the hour hand?

18 Find the shaded area shown here.



19 Prove that the shaded area of the semicircle shown here is equal to the area of the smaller circle.



4.2 Surface area

In the previous section of this chapter we found the areas of a range of **two-dimensional (2D)** shapes. Two-dimensional shapes are flat and have length and width but no depth. **Three-dimensional (3D)** shapes occupy space and have length, width and depth.

Solids are 3D figures. The **faces** of a 3D shape are its separate outside surfaces. The lines where the faces meet are called **edges** and the corners where the edges meet are called **vertices.** The cross-section and/or the faces are used to name 3D shapes.

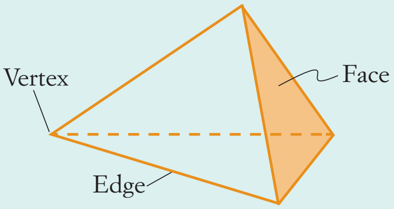
Important!

3D shapes

The **faces** of a 3D shape separate the inside from the outside.

The lines where faces meet are called **edges**.

The corners where edges meet are called **vertices** (the singular form is **vertex**).



Teacher notes
Naming 3D shapes
MAT10MGTN00013

Weblink
K-6 Geometric Shapes
MAT10MGWB00004

Worksheet
Making hedrons
MAT10MGWK00038

Important!

Classifying 3D Shapes

A prism has constant polygonal cross-sections in one direction. All the cross-sections parallel to one face (the **base**) are the same shape and size as the base.

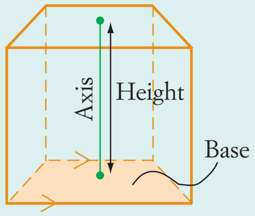
A **pyramid** has a base that is a polygon. The other faces are triangles that all meet at a **vertex** called the **apex**. The cross-sections parallel to the base are different sizes, so it is not a prism.

The **axis** of a prism or pyramid is a line passing through the centres of all the cross-sections parallel to its base. The height of a prism or pyramid is measured at right angles (90°) to the base. **Slant** prisms and pyramids have the axis at an angle to the height. The axis *is* the height for **right** prisms and pyramids.

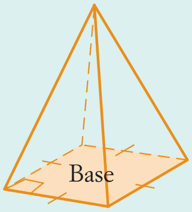
A **parallelepiped** is a slant parallelogram-based prism. Its faces are all parallelograms.

A **cylinder** is like a prism with a circular base. A **cone** is like a pyramid with a circular base.

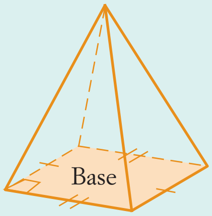
A **sphere** is a ball shape. A **hemisphere** is half a sphere.



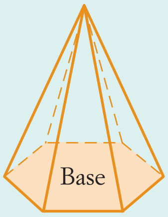
Trapezoidal prism



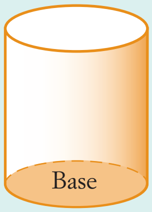
Square pyramid



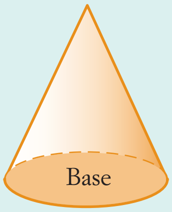
Rectangular pyramid



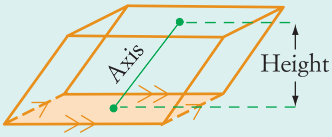
Hexagonal pyramid



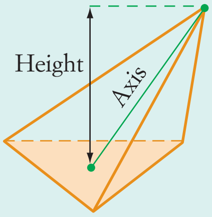
Cylinder



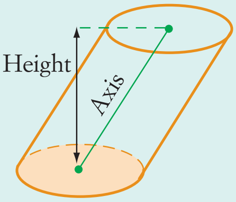
Cone



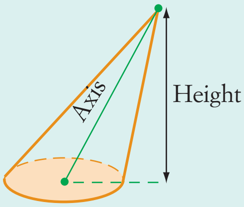
Parallelopiped



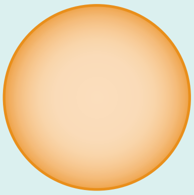
Slant triangular pyramid



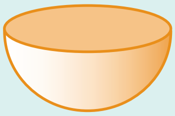
Slant cylinder



Slant cone



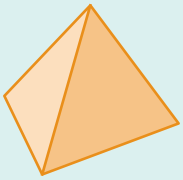
Sphere



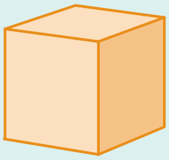
Hemisphere

Other 3D shapes made from polygons are called **hedrons**. Hedrons must have flat faces. They are named using Greek words for the number of faces: **tetra** = **4, hexa** = **6, octa** = **8, deca** =**10, dodeca** =**12** and **icosa** = 20.

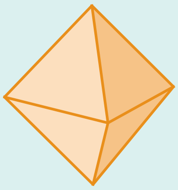
There are just five **Platonic solids.** These are regular hedrons in which every face is the same and every vertex is the same.



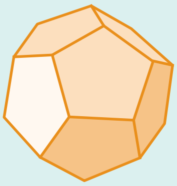
Regular tetrahedron



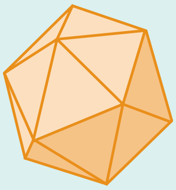
Cube or regular hexahedron



Regular octahedron



Regular dodecahedron



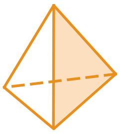
Regular icosahedron

Example 7

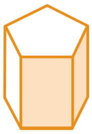
Name the following shapes.

a

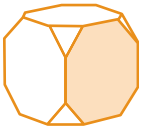
b



c



d

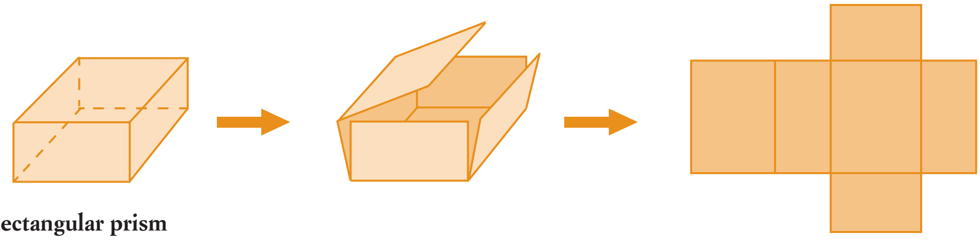


Solution

a	This is like a prism but has a circular base.	**Cylinder.**
b	This is a pyramid with a triangular base.	**Triangular pyramid**
	It is also a hedron with 4 faces.	**or tetrahedron.**
c	This is a prism with a base that is a pentagon.	**Pentagonal prism**
d	This is a hedron with 14 faces.	**14-sided hedron.**

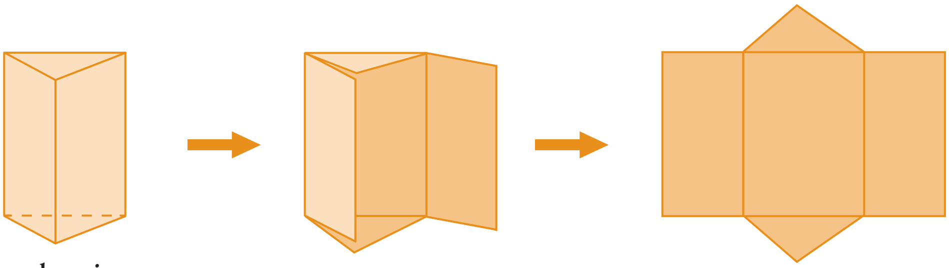
Some people would call Example **7d** a **truncated cube**. It is not regular because it has a mixture of different shaped faces.

The **net** of a 3D shape is the 2D shape that can be unfolded from the 3D shape, or folded up to make the 3D shape.



Rectangular prism

Net



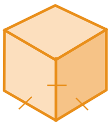
Triangular prism

Net

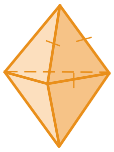
Example 8

Draw nets of the following shapes.

a



b

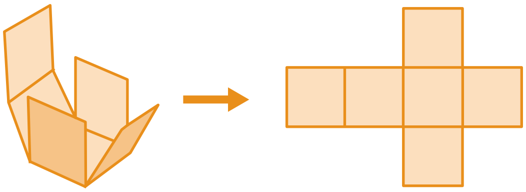


Solution

a This solid is a cube.

All the faces are squares.

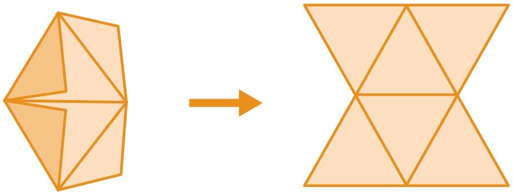
Imagine unfolding the cube.



b 3 triangles meet at the top and bottom vertices, but 4 meet at the others. It is an irregular hexahedron.

Imagine unfolding the hexahedron.

The faces are all equilateral triangles.



The area of a 3D shape is known as the **surface area**.

Important!

Surface area

The surface area of a 3D shape is the total area of all its faces.

The net of a 3D shape is useful in finding its surface area.

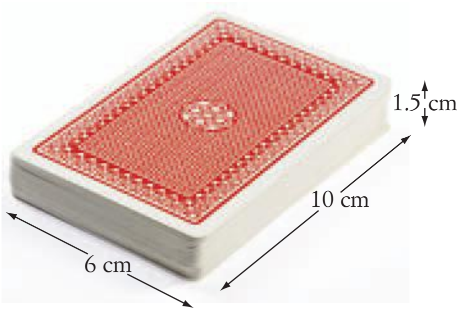
Example 9

Worksheet
Area and volume 1
MAT10MGWK00010

Puzzle sheet
Surface area
MAT10MGPS00009

Find the surface area of each of the following shapes.

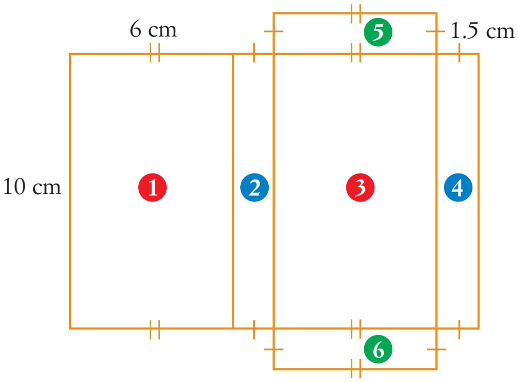
a



b

Solution

a First draw a net of the shape.
Write in the dimensions you know.
Number the faces.
There are 6 rectangular faces altogether.
Rectangle 1 = rectangle 3
Rectangle 2 = rectangle 4
Rectangle 5 = rectangle 6



Find the area of face 1 or 3. Substitute for l and w.

$$\text{Face 1} = l \times w = 10\text{ cm} \times 6\text{ cm} = 60\text{ cm}^2$$

Find the area of face 2 or 4. Substitute for l and w.

$$\text{Face 2} = l \times w = 10\text{ cm} \times 1.5\text{ cm} = 15\text{ cm}^2$$

Find the area of face 5 or 6. Substitute for l and w.

$$\text{Face 5} = l \times w = 6\text{ cm} \times 1.5\text{ cm} = 9\text{ cm}^2$$

Write the rule for surface area (SA).

$$\text{SA} = \text{sum of the areas of all faces}$$

Use the fact that some rectangles are equal.

$$= 2 \times \text{face 1} + 2 \times \text{face 2} + 2 \times \text{face 5}$$

Substitute areas previously calculated.

$$= (2 \times 60 + 2 \times 15 + 2 \times 9)\text{ cm}^2$$

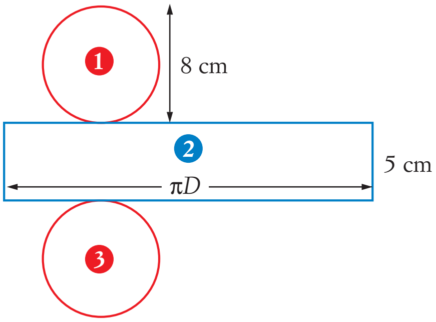
Evaluate.

$$= 168\text{ cm}^2$$

b First draw a net of the shape.

Write in the dimensions you know.
Number the faces.

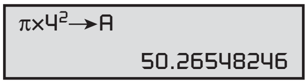
There are only 3 faces: a rectangle and 2 equal-sized circles.



Find the area of the circles. **Face 1** $= \pi r^2$

Substitute for known values. $= \pi \times (4\text{ cm})^2$

Enter as π × 4 x^2 STO A



```
π×4²→A
          50.26548246
```

Round, but store the full value in A. **Face 1** $\approx$ **50.3 cm²**

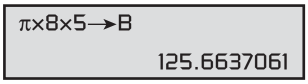
Find the area of the rectangle. **Face 2** $= l \times w$

Substitute for l and w. Here, l is the circumference of the circle.

$= \pi D \times 5\text{ cm}$

$= \pi \times 8\text{ cm} \times 5\text{ cm}$

Enter as π × 8 × 5 STO B



```
π×8×5→B
          125.6637061
```

Round, but store the full value in B. **Face 2** $\approx$ **125.7 cm²**

Write the rule for surface area (SA). **SA = sum of the areas of all faces**

Use the fact that 2 faces are equal. $= 2 \times \text{face 1} + \text{face 2}$

Substitute for areas previously calculated. $\approx 2 \times 50.3 + 125.7\text{ cm}^2$

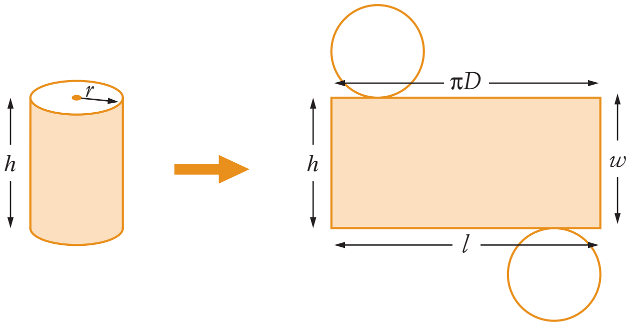
Enter as 2 × A + B =

```
2×A+B
          226.1946711
```

Round to the nearest cm². $\approx 226\text{ cm}^2$

It is usually more convenient to use a rule for working out the surface area of a cylinder. The curved area of a cylinder, without the top and bottom, is called the **curved surface area** (**CSA**). From the previous example, you can see that the curved surface area is calculated as follows:

$$
\begin{aligned}
\text{CSA} &= l \times w \\
&= \pi D \times h \\
&= \pi \times 2r \times h \\
&= 2\pi rh
\end{aligned}
$$



Important!

Curved surface area of a cylinder

Curved surface area (CSA) of a cylinder = $2\pi rh$

Example 10

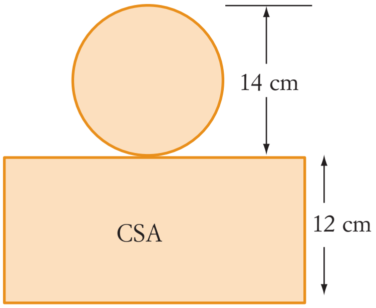
A cylinder is 12 cm high and has a radius of 7 cm. Find the surface area of the cylinder if it has a base but no lid.

Solution

First draw a net of the shape.

Write in the dimensions you know.

There are only 2 faces: the CSA of the cylinder and a circular end.



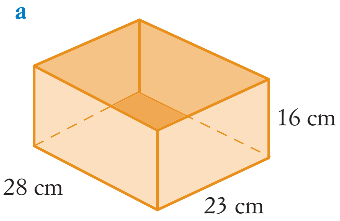
Write in the rule for the curved surface area.	**CSA** $= 2\pi rh$
Substitute known information.	$= 2 \times \pi \times 7\text{ cm} \times 12\text{ cm}$
Enter as 2 × π × 7 × 12 STO A	2×π×7×12→A 527.7875658
Round, but store the full value in A.	**CSA** $\approx$ **527.8 cm^2**
Calculate the area of the base.	**Area of base** $= \pi r^2$
Substitute known information.	$= \pi \times (7\text{ cm})^2$
Enter as π × 7 x^2 STO B	π×7²→B 153.93804
Round, but store the full value in B.	**Area of base** $\approx$ **153.9 cm^2**
Find the surface area (SA).	**SA** = **CSA** + **area of base**
Substitute calculated values.	$\approx$ **527.8 cm^2 + 153.9 cm^2**
Enter as A + B =	A+B 681.7256058
Round to the nearest cm^2.	$\approx$ **682 cm^2**

A net may be used to form an open solid or a closed solid. A sealed tissue box is an example of a **closed solid**. A can with the lid removed is an example of an **open solid**. The surface area of an open solid usually refers only to the outer surface.

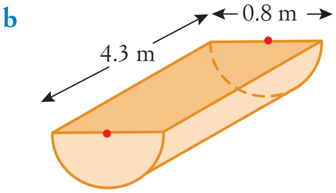
9780170194860

Example 11

Calculate the surface area of each of the following solids.



Open box



Closed half-cylinder

Solution

a Draw the net for the open box.

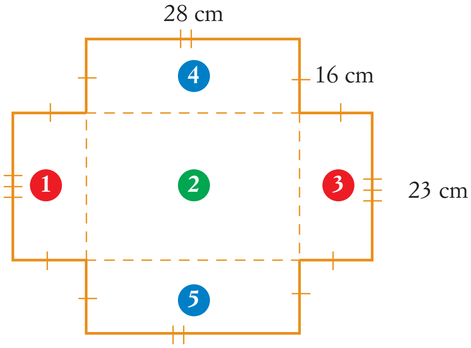
Write in the dimensions you know.

Number the faces.

There are 5 rectangular faces altogether.

Rectangle 1 = rectangle 3

Rectangle 4 = rectangle 5



Find the area of face 1 or 3.

Substitute for l and w.

Face 1 $= l \times w$

$= 23\text{ cm} \times 16\text{ cm}$

$= 368\text{ cm}^2$

Find the area of face 4 or 5.

Substitute for l and w.

Face 4 $= l \times w$

$= 28\text{ cm} \times 16\text{ cm}$

$= 448\text{ cm}^2$

Find the area of face 2.

Substitute for l and w.

Face 2 $= l \times w$

$= 28\text{ cm} \times 23\text{ cm}$

$= 644\text{ cm}^2$

Write the rule for surface area (SA).

Use the fact that some rectangles are equal.

Substitute areas previously calculated.

Evaluate.

SA = sum of the areas of all faces

$= 2 \times \text{face } 1 + \text{face } 2 + 2 \times \text{face } 4$

$= (2 \times 368 + 644 + 2 \times 448)\text{ cm}^2$

$= 2276\text{ cm}^2$

b First draw a net of the shape.

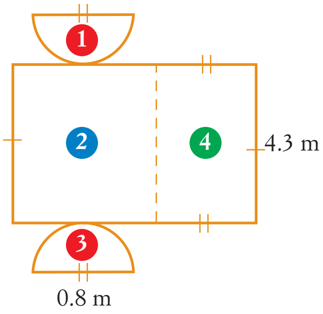
Write in the dimensions you know. Number the faces.

There are 4 faces: 2 rectangles and 2 equal-sized semicircles.

Faces 1 and 3 are the semicircular ends of the half-cylinder.

Face 2 is the curved surface area (CSA) of the half-cylinder.

Face 4 is the top of the half-cylinder.



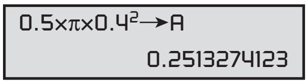
Find the area of the semicircles.

Face 1 $= \frac{1}{2}\pi r^2$

Cancel and substitute for known values. Here $r = \frac{1}{2} \times 0.8 = 0.4$

$= \frac{1}{2} \times \pi \times 0.4^2$

Enter as 0.5 × π × 0.4 x^2 STO A


0.5×π×0.4²→A
0.2513274123

Face 1 = **0.251 327 . . . m²**

Write the rule for the curved surface area of the half-cylinder.

Face 2 $= \frac{1}{2} \times 2\pi rh$

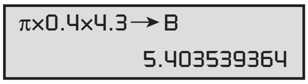
Cancel.

$= \pi rh$

Substitute known information.

$= \pi \times 0.4 \text{ m} \times 4.3 \text{ m}$

Enter as π × 0.4 × 4.3 STO B


π×0.4×4.3→B
5.403539364

Face 2 = **5.403 53 . . . m²**

Find the area of the remaining rectangle.

Face 4 $= l \times w$

Substitute for l and w. Here, l is the diameter of the semicircle and w is the height of the half-cylinder.

$= 0.8 \text{ m} \times 4.3 \text{ m}$

Face 4 = **3.44 m²**

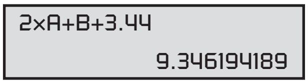
Write the rule for surface area (SA).

SA = sum of the areas of all faces

Use the fact that 2 faces are equal.

= 2 × face 1 + face 2 + face 4

Enter as 2 × A + B + 3.44


2×A+B+3.44
9.346194189

$\approx 9.35 \text{ m}^2$

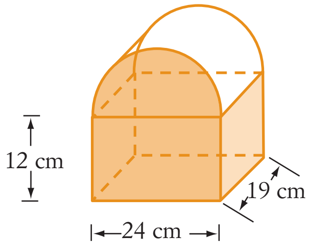
For part **b** of the previous example it should be noted that the areas of faces 1, 2 and 3 were not rounded before the calculation of the surface area. This ensures that the final answer is more accurate than it would have been if these values had been rounded.

Many solids are formed by combining two or more different solids.

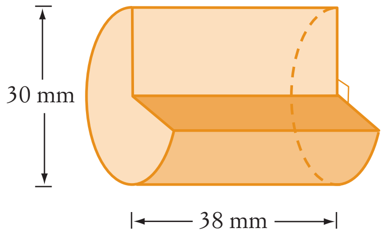
Example 12

Calculate the surface area of each of the following solids correct to one decimal place.

a



b



Solution

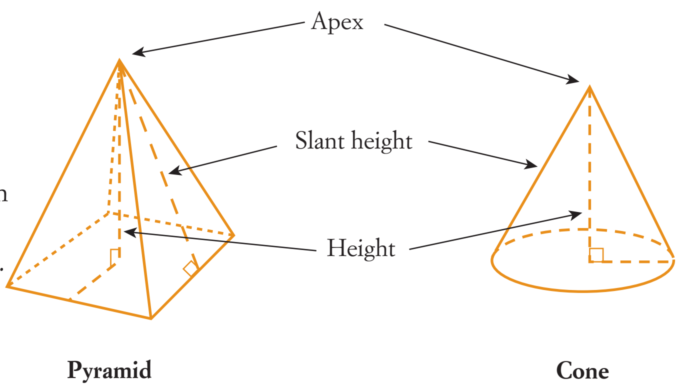
a This solid consists of a half-cylinder mounted on a rectangular prism.

The surface area of the shape will include the 3 faces of the half-cylinder and 5 faces of the rectangular prism.

Calculate the curved surface of the half-cylinder. CSA is the curved surface of a cylinder.	**Curved surface of half-cylinder** $= \frac{1}{2}$ **CSA**
Substitute the rule for CSA.	$= \frac{1}{2} \times 2\pi r h$
Substitute for known values. The diameter is 24 cm.	$= \frac{1}{2} \times 2\pi \times 12 \times 19$
Cancel and evaluate.	$= 716.283 \ldots \text{ cm}^2$
The other two faces are semicircles.	**Area of semicircular faces** $= \pi r^2$
Substitute for known values.	$= \pi \times 12 \times 12$
Evaluate.	$= 452.389 \ldots \text{ cm}^2$
Now calculate the area of the 5 rectangular faces of the prism.	**Area of rectangular faces** $= (24 \times 19) + 2 \times (12 \times 24) + 2 \times (19 \times 12)$
Evaluate.	$= 1488 \text{ cm}^2$
Calculate the surface area.	$\text{SA} = 716.283 \ldots + 452.389 \ldots + 1488$
Evaluate.	$= 2656.67 \ldots \text{ cm}^2$
Round off and state the result.	**The surface area of the solid is approximately 2656.7 cm^2.**

b This solid consists of a closed cylinder with a portion removed.	
Calculate the portion of the cylinder that has been removed.	**Portion removed** $= \frac{90^\circ}{360^\circ} = \frac{1}{4}$
The surface area of the solid consists of $\frac{3}{4}$ of the CSA for a full cylinder, two ends that are each $\frac{3}{4}$ of a full circle and two equal rectangular faces.	
Calculate the curved surface of the solid.	**Curved surface of solid** $= \frac{3}{4}$ **CSA**
Substitute the rule for CSA.	$= \frac{3}{4} \times 2\pi rh$
Substitute for known values. The diameter is 30 mm.	$= \frac{3}{{}_{1}\cancel{2}\cancel{4}} \times \cancel{2}^{1}\pi \times 15 \times \cancel{38}^{19}$
Cancel and evaluate.	$= 2686.06 \ldots \text{ mm}^2$
The two end faces are three-quarter circles.	**Area of end faces** $= 2 \times \frac{3}{4} \times \pi r^2$
Substitute for known values and cancel.	$= \frac{3}{2} \times \pi \times 15 \times 15$
Evaluate.	$= 1060.28 \ldots \text{ mm}^2$
Now calculate the area of the rectangular faces of the solid.	**Area of rectangular faces** $= 2 \times (15 \times 38)$
Evaluate.	$= 1140 \text{ mm}^2$
Calculate the surface area.	**SA** $= 2686.06 \ldots + 1060.28 \ldots + 1140$
Evaluate.	$= 4886.34 \ldots \text{ mm}^2$
Round off and state the result.	**The surface area of the solid is approximately 4886.3 mm^2.**

A **pyramid** is a solid made up of a base and triangular faces that meet at an **apex**, which is also called the **vertex** of the pyramid. The **slant height** is the perpendicular distance from the vertex down a face, as shown in the diagram below. A **cone** is a solid with a circular base and an apex. The height, h, of the cone is measured perpendicular to the base and the slant height is the distance from the edge of the base to the apex along the face of the cone.

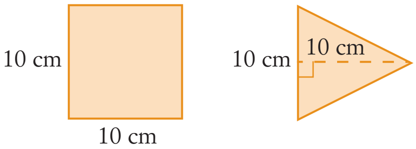


Investigate: Surface areas of pyramids and cones

You will need cardboard, scissors, tape, a protractor, a compass and a ruler.

Pyramids

1 First, make a square pyramid using the dimensions shown here.



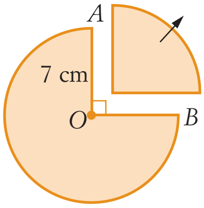
2 Next, construct a pyramid using only equilateral triangles.

3 Now use the pyramids you have made to answer the following.

- **a** What are the differences between a triangular pyramid and a square pyramid?
- **b** How are these two types of pyramid similar?
- **c** What would you call a pyramid with a six-sided base?

Cones

1 Draw a circle with a radius of 7 cm. Mark in two radii, *OA* and *OB*, so that ∠*AOB* is 90°, as shown. Cut out the 90° sector and put it to one side. Join up the cut edges with some tape to form a cone.



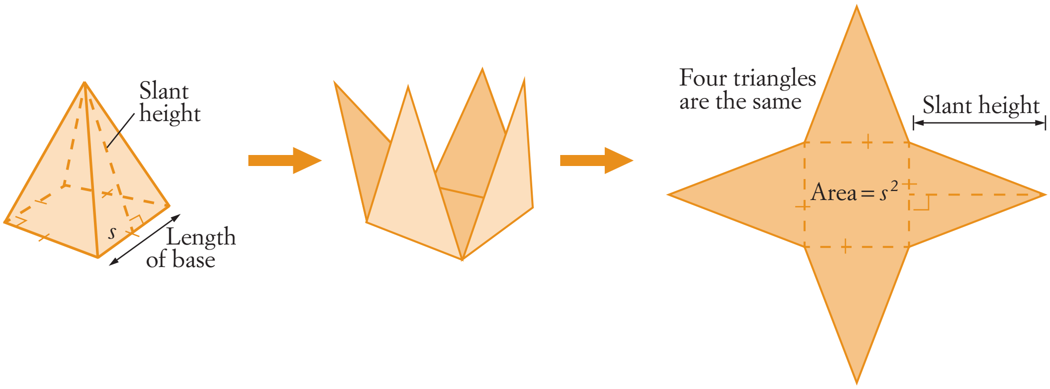
2 Repeat this procedure, but this time take out a larger sector. How are the two cones different? How are they similar?

3 Make a list of household objects that have a conical shape. (They may only be part of a cone.)

4 Find out what the *Venturi effect* has to do with cones.

5 What are the advantages and disadvantages of making paper cups in a conical shape?

When you work out the surface area of a square pyramid, you need to know the length of the base and the slant height.



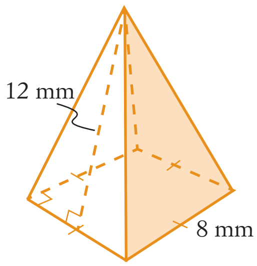
Example 13

Video tutorial

Surface area of a pyramid

MAT10MGVT10008

Find the surface area of this square pyramid.



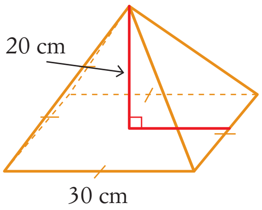
Solution

The base is a square.	**Area of base** $= l^2$
Substitute for *l*.	$= 8\text{ mm} \times 8\text{ mm}$
Evaluate.	$= 64\text{ mm}^2$
Each side is a triangle.	**Area of side** $= \frac{1}{2}bh$
Substitute for *b* and *h*.	$= \frac{1}{2} \times 8 \times 12\text{ mm}^2$
Evaluate.	$= 48\text{ mm}^2$
Write a rule for the surface area.	**Surface area** $=$ **area of base** $+ 4 \times$ **area of side**
Substitute.	$= 64\text{ mm}^2 + 4 \times 48\text{ mm}^2$
Evaluate.	$= 256\text{ mm}^2$

If the slant height of a pyramid is not given, then it needs to be calculated before the surface area can be found.

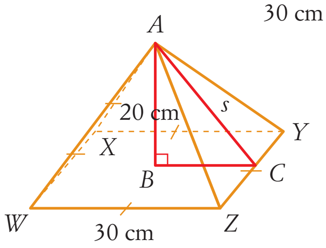
Example 14

Calculate the surface area of this square-based pyramid.



Solution

Draw in the slant height and label the diagram.

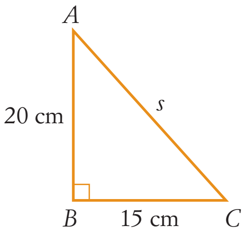


Make a sketch of the triangle formed by the height of the pyramid and the slant height.

AB = height of the pyramid

BC = half the length of the base

AC = slant height of the pyramid



Use Pythagoras' theorem to calculate the slant height. $AC^2 = AB^2 + BC^2$

Substitute for known values. $= 20^2 + 15^2$

Evaluate. $= 400 + 225$

$AC^2 = 625$

Take square roots of both sides. $AC = 25$

Calculate the area of the triangular faces. In this case, the 4 triangular faces are equal. **Area of faces** $= 4 \times$ **area of** $\triangle AYZ$

Use the rule for the area of a triangle. $= 4 \times \frac{1}{2}bh$

Substitute for known values, cancel and evaluate. $= \overset{2}{\cancel{4}} \times \frac{1}{\underset{1}{\cancel{2}}} \times 30 \times 25$

$= 1500 \text{ cm}^2$

Calculate the area of the base. **Area of base** $= l \times w$

Substitute for known values. $= 30 \times 30$

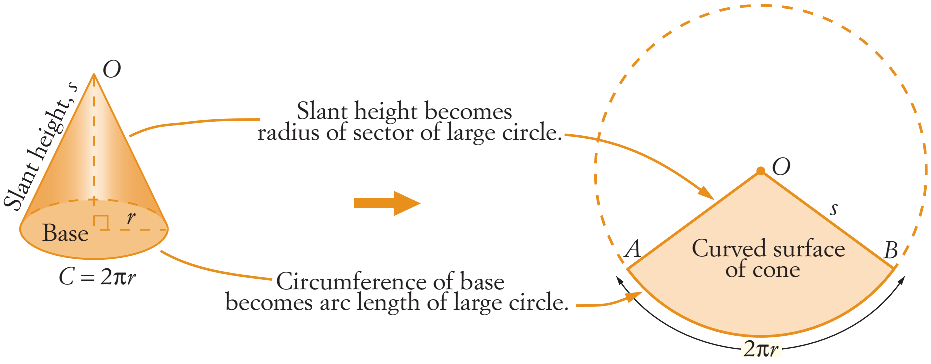
$= 900 \text{ cm}^2$

Calculate the surface area. **SA = area of triangular faces + area of base**

Substitute. $= 1500 + 900$

Evaluate. $= 2400 \text{ cm}^2$

The net of a cone consists of the curved surface and the circular base as shown here. The base is a circle with radius r. The curved surface is the sector of a circle with radius, s, and arc length AB. The length of AB must be the same as the circumference of the base, i.e. $2\pi r$.



We can develop a formula for the area of a cone using this information.

$$\begin{aligned}\frac{\text{Area of sector } AOB}{\text{Area of large circle}} &= \frac{\text{arc length } AB}{\text{circumference of large circle}}\\ \frac{\text{Area of sector } AOB}{\pi s^2} &= \frac{2\pi r}{2\pi s} = \frac{r}{s}\\ \text{Area of sector } AOB &= \frac{r}{s} \times \pi s^2\\ &= \pi rs\end{aligned}$$

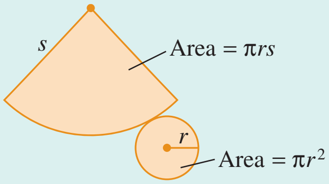
The total surface area of the cone is the area of the base and the area of the sector *ABC*.

Important!

Surface area of a cone

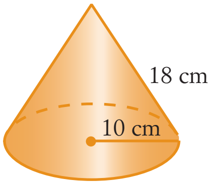
Surface area of a cone $= \pi r^2 + \pi rs$

where $r =$ radius of the base of the cone and
$s =$ slant height of the cone



Example 15

Find the surface area of this cone.



Worksheet
Area and volume 5
MAT10MGWK00036

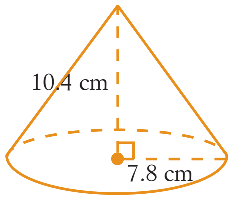
Solution

Write the rule for surface area.	$\mathbf{SA} = \pi rs + \pi r^2$
Substitute known values.	$= \pi \times 10\text{ cm} \times 18\text{ cm} + \pi \times (10\text{ cm})^2$
Evaluate.	$= 565.486\ldots\text{ cm}^2 + 314.159\ldots\text{ cm}^2$
Round to the nearest cm^2.	$\approx 880\text{ cm}^2$

As was the case with a pyramid, if the slant height of a cone is not given, then it needs to be calculated before the surface area can be found.

Example 16

Calculate the surface area of this cone.



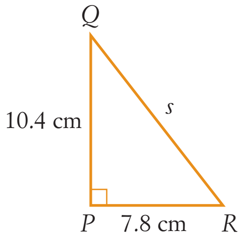
Solution

Make a sketch of the triangle formed by the height of the cone and the slant height.

QP = height of the pyramid

PR = radius of the base

QR = slant height of the cone (s)



Use Pythagoras' theorem to calculate the slant height.	$QR^2 = QP^2 + PR^2$
Substitute for known values.	$= 10.4^2 + 7.8^2$
Evaluate.	$= 108.16 + 60.84$
	$= 169$
Take square roots of both sides.	$QR = 13$ **cm**
Use the rule for the area of the cone.	**Surface area of cone** $= \pi r^2 + \pi rs$
Substitute for known values.	$= \pi \times 7.8^2 + \pi \times 7.8 \times 13$
Evaluate.	$= 509.691 \ldots$ **cm**^{2}
Round to 1 decimal place.	≈ 509.7 **cm**^{2}

Investigate: Surface area of a sphere

Work in groups of four or five students. Each group will need an orange, some paper towels and a knife to cut the orange in half.
Finding a rule for the surface area of a sphere can be done using some sophisticated mathematics. It can also be approximated using some basic reasoning.

- Squeeze and mould your orange to make it shaped like a sphere as much as possible.
- Cut your orange in half. The circle formed by the edge of the orange is called the **great circle** of the orange. This is the circle with the same radius as the radius of the sphere.
- Trace a great circle of your orange on a paper towel.

9780170194860

- Discuss with the members of your group how many of these great circles you think you can cover with pieces of your orange's peel.
- Trace four more great circles on your paper towels.
- Now carefully tear off pieces of the orange peel, each about to 4 or 5 cm^2.
- Use the pieces of peel you have torn off to cover as many circles as possible. Place the pieces, one at a time, in the great circle you drew. Be careful not to overlap pieces or leave gaps. You may need to tear off some smaller pieces to entirely cover each circle without gaps or overlaps.
- Record the number of great circles covered by the orange peel.

Work as a group to use your results to answer the following questions. Make sure that you record your answers and can explain your conclusions because each group member needs to be prepared to explain your group's thinking to the whole class.

1 Write a mathematical expression for the area covered for each great circle. What does r in your expression stand for?
2 How many circles did you cover in all?
3 Based on your results, write a rule or equation that gives the surface area of your orange.
4 Do you think your equation will work for finding the surface area of any sphere? Explain your reasoning.

Now summarise your group's findings and conclusions.
Your teacher will ask all groups to share their findings and conclusions in turn.
Don't forget to clean up!

From the previous investigation, you should have seen that the surface area of a sphere is $4\pi r^2$. A hemisphere is half a sphere and may be either open or closed. An open hemisphere has no circular base while a closed hemisphere does. You can also use the formula for the surface area of a sphere to calculate the surface of a hemisphere – with or without a base.

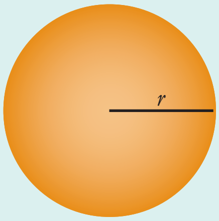
$$\begin{aligned}\text{SA of an open hemisphere} &= \tfrac{1}{2}\ \text{SA of a sphere}\\ &= \tfrac{1}{2} \times 4\pi r^2\\ &= 2\pi r^2\end{aligned}$$

$$\begin{aligned}\text{SA of a closed hemisphere} &= \text{SA of an open hemisphere} + \text{area of circular base}\\ &= 2\pi r^2 + \pi r^2\\ &= 3\pi r^2\end{aligned}$$

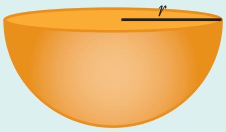
Important!

Surface area of a sphere

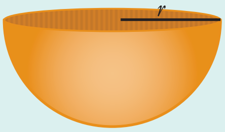
SA of a sphere
$= 4\pi r^2$



SA of a closed hemisphere
$= 3\pi r^2$



SA of an open hemisphere
$= 2\pi r^2$



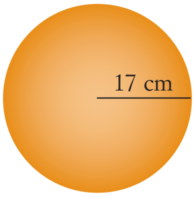
Video tutorial
Surface area of a cone and sphere
MAT10MGVT10009

Worksheet
Area and volume 6
MAT10MGWK00037

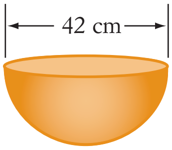
Example 17

Calculate the surface areas of each of the following.

a



b



Closed hemisphere

Solution

a	Write the rule for the SA of a sphere.	$\mathbf{SA = 4\pi r^2}$
	Substitute for r.	$\mathbf{= 4 \times \pi \times 17^2}$
	Evaluate.	$\mathbf{= 3631.68\ldots}$
	Round off.	$\mathbf{\approx 3631.7\ cm^2}$
b	Write the rule for the SA of a sphere.	$\mathbf{SA = 3\pi r^2}$
	Substitute for r. Diameter = 42 cm.	$\mathbf{= 3 \times \pi \times 21^2}$
	Evaluate.	$\mathbf{= 4156.32\ldots}$
	Round off.	$\mathbf{\approx 4156.3\ cm^2}$

Exercise 4.2 Surface area

Understanding

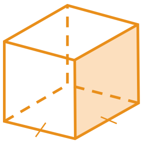
Extra questions
Exercise 4.2
MAT10MGEQ00039

See Example 7

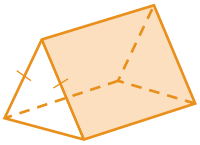
A calculator should be used where appropriate in this exercise. For calculations involving π, use the π key on your calculator and answer correct to 1 decimal place.

1 Name the following shapes.

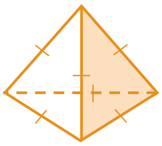
a



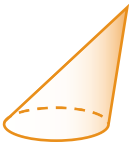
b



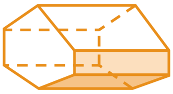
c



d

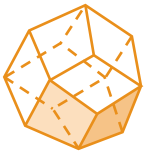


e

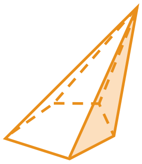


f

g

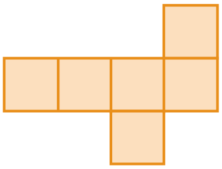


h

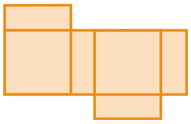


2 Identify the shapes whose nets are shown below.

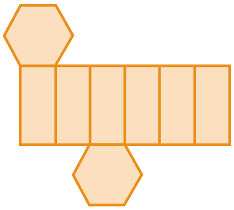
a



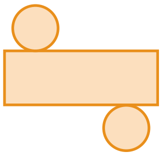
b



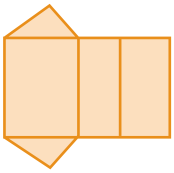
c



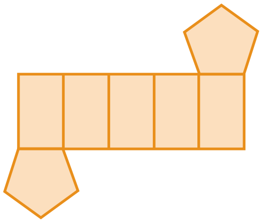
d



e



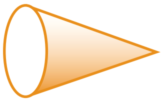
f



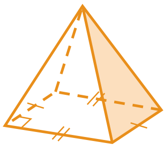
3 Draw nets for the following shapes.

See Example 8

a

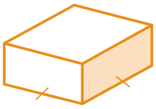


b

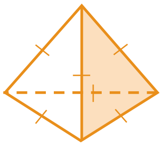


c

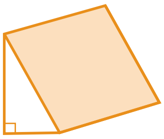
d



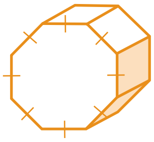
e



f



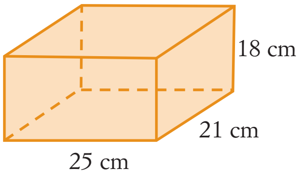
g



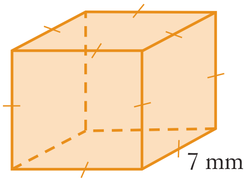
See Example 9

4 Draw nets of the following shapes, including all necessary measurements.

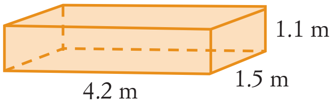
a



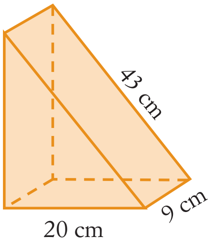
b



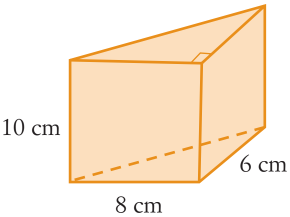
c



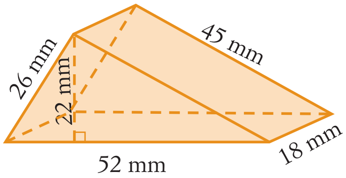
d



e

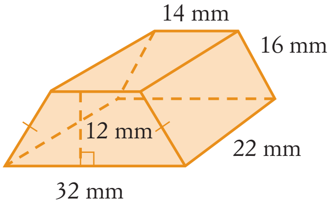


f

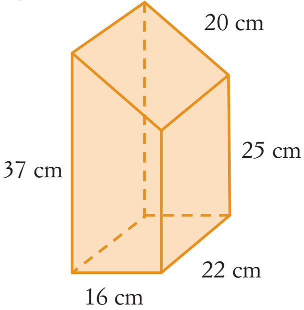


5 Draw nets of the following shapes, including all necessary measurements.

a



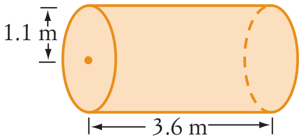
b



See Examples 9, 11

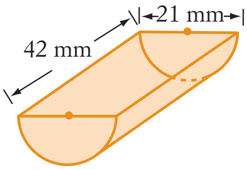
6 Draw nets of the following shapes, including all necessary measurements.

a



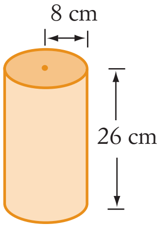
Closed cylinder

b



Open half-cylinder

c

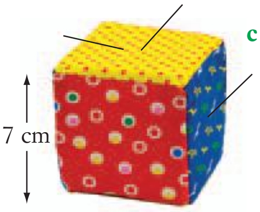


Cylinder (open both ends)

7 Draw nets of the following shapes, including all necessary measurements.

a

b



c

8 Draw nets of the following shapes, including all necessary measurements.

a

b

c

Fluency

9 Use the nets you drew in question **4** to find the surface area of each shape.

See Example 9

10 Use the nets you drew in question **5** to find the surface area of each shape.

11 Use the nets you drew in question **6** to find the surface area of each shape.

12 Use the nets you drew in question **7** to find the surface area of each shape.

13 Use the nets you drew in question **8** to find the surface area of each shape.

14 Calculate the surface area of rectangular prisms with the following dimensions.

a length 12 cm, width 8 cm, height 17 cm
b length 24 mm, width 19 mm, height 11 mm
c length 2.3 m, width 1.3 m, height 0.9 m
d length 35 cm, width 17 cm, height 42 cm
e length 1.7 m, width 46 cm, height 1.4 m
f length 22 cm, width 76 mm, height 19 cm

See Example 10

15 Calculate the curved surface area of the cylinders with the following dimensions. Give your answers correct to two decimal places.

a radius 7 cm, height 11 cm
b diameter 32 mm, height 42 mm
c radius 1.7 m, height 3.3 m
d radius 5.7 cm, height 12.4 cm
e diameter 63 cm, height 2.3 m
f radius 85 mm, height 22.7 cm

Worked solutions
Exercise 4.2
MAT10MGWS00044

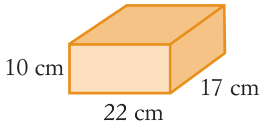
See Example 11

16 Calculate the surface area of the closed cylinders with the following dimensions. Give your answers correct to two decimal places.

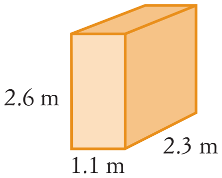
ⓐ radius 15 cm, height 22 cm
b radius 2.8 m, height 3.1 m
c diameter 61 mm, height 89 mm
d radius 41 cm, height 86 cm
e radius 92 cm, height 1.7 m
f diameter 33 mm, height 7.9 cm

17 Calculate the surface area of each of the following solids.

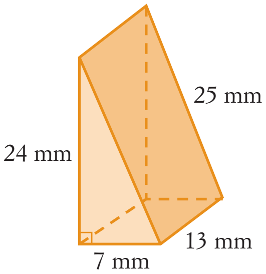
a



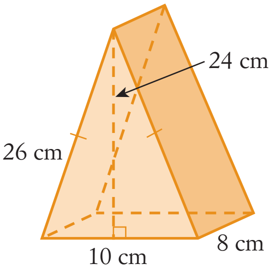
b



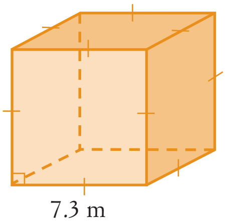
c



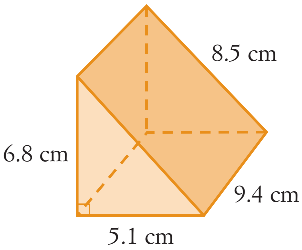
d



e



f



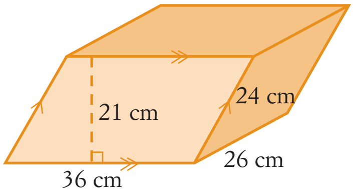
Worked solutions

Exercise 4.2

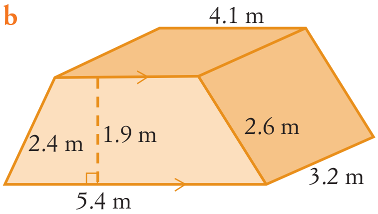
MAT10MGWS00044

18 Calculate the surface area of each of the following solids.

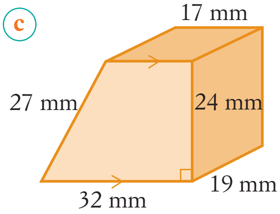
a



b

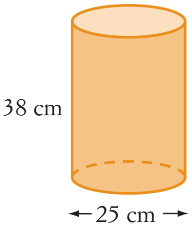


ⓒ

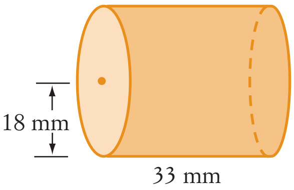


19 Calculate the surface area of each of the following solids.

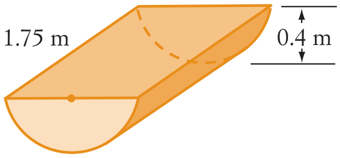
a **Closed cylinder**



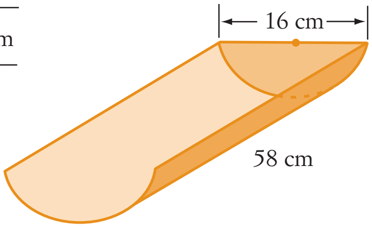
b **Cylinder, open both ends**



c **Closed half-cylinder**



d **Open half-cylinder, one end open**



20 Calculate the surface area of each of the following pyramids.

- **a** square base measuring 20 cm by 20 cm and a slant height of 24 cm
- **b** square base measuring 25 mm by 25 mm and a slant height of 38 mm
- **c** rectangular base measuring 2.2 m by 4.1 m and a slant height of 3 m

21 Calculate the surface area of each of the following cones.

- **a** radius 7 cm, slant height 24 cm
- **b** diameter 9 mm, slant height 12 mm
- **c** diameter 5 m, slant height 6 m
- **d** radius 6.8 cm, slant height 17.2 cm

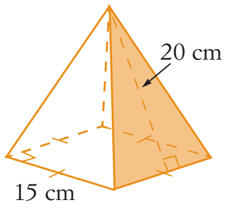
22 Calculate the surface area of each of the following.

- **a** sphere with a radius of 5 cm
- **b** sphere with a diameter of 28 mm
- **c** open hemisphere with a radius of 3.2 m
- **d** closed hemisphere with a diameter of 43 cm
- **e** open hemisphere with a diameter of 1.9 m
- **f** closed hemisphere with a diameter of 63 mm

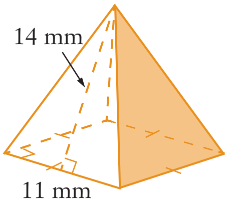
23 Find the surface area of each of these square pyramids.

See Example 13

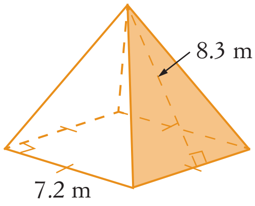
a



b

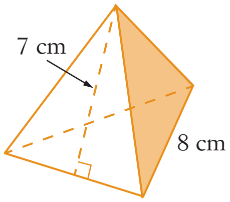


c

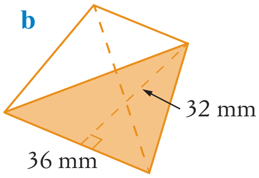


24 Here are two regular tetrahedrons. In each solid, each face is the same. Find the surface area of each solid.

a

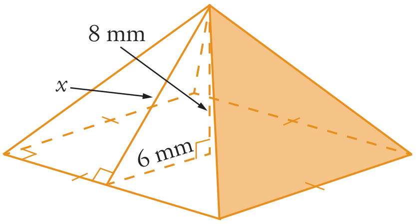


b



25 This is a square pyramid.

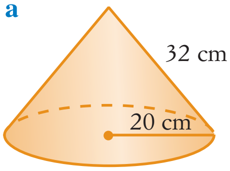
- **a** Use Pythagoras' theorem to find the slant height of the pyramid, x.
- **b** Find the surface area of the pyramid.



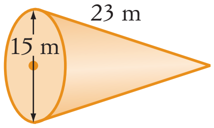
26 Find the surface areas of these cones.

See Example 15

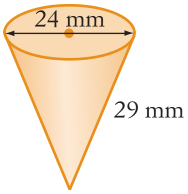
a



b



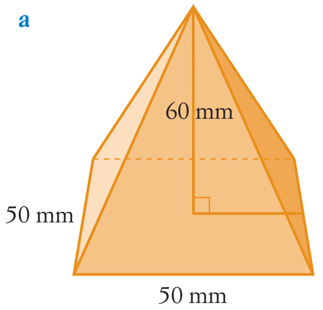
c



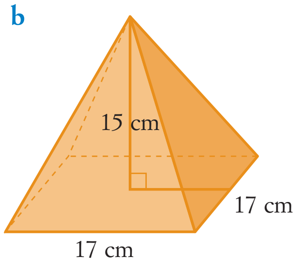
See Example 14

27 Calculate the surface area of each of the following pyramids.

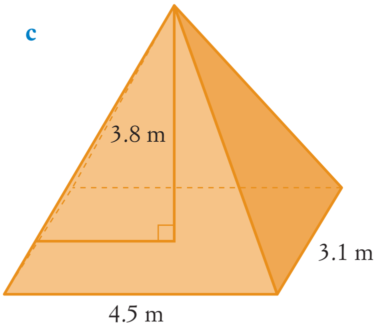
a



b



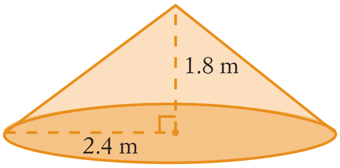
c



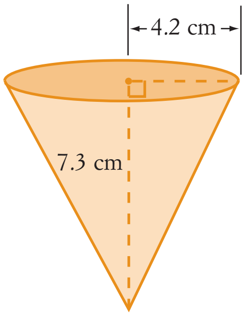
See Example 16

28 Calculate the surface area of each of the following cones.

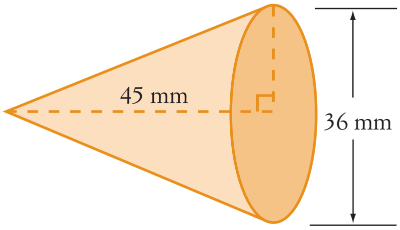
a



b



c

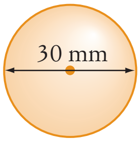


See Example 17

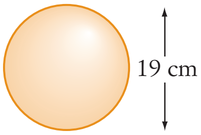
29 Find the surface area of each of these spheres.

a

b

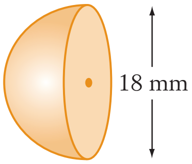


c



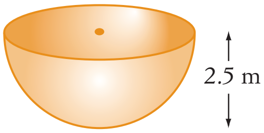
30 Find the surface areas of these hemispheres.

a



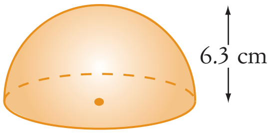
Closed

b



Open

c



Closed

Problem solving

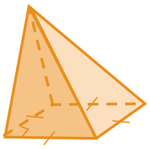
31 Sketch the following shapes.

a Cube
b Right trapezoidal prism
c Slant rectangular prism
d Slant cone
e Right square pyramid
f Slant rectangular pyramid

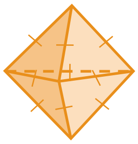
32 Draw nets for the following shapes.

See Example 7

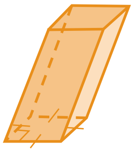
a



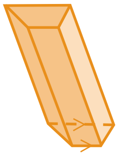
b



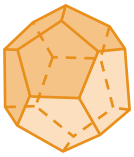
c



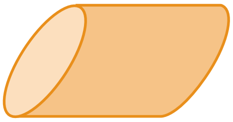
d



e

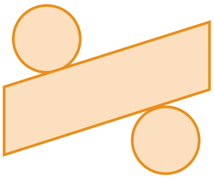


f

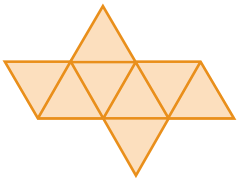


33 Identify the shapes whose nets are shown below.

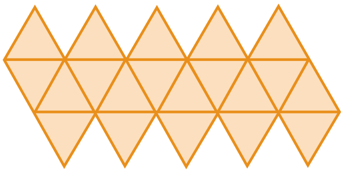
a



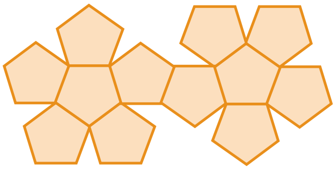
b



c



d



34 The surfaces of each of the following solids need to be coated with a special paint. In each case, calculate the total area that will be coated.

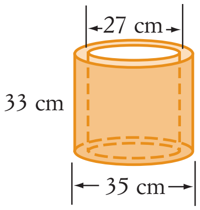
Worked solutions

Exercise 4.2

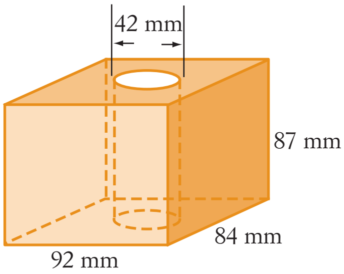
MAT10MGWS00044

See Example 12

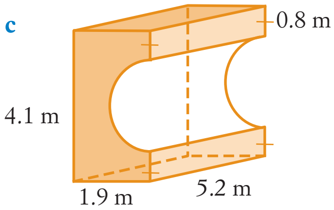
a



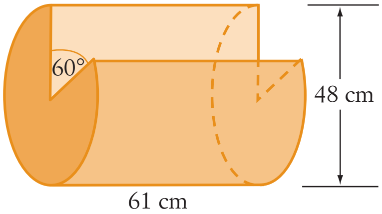
b



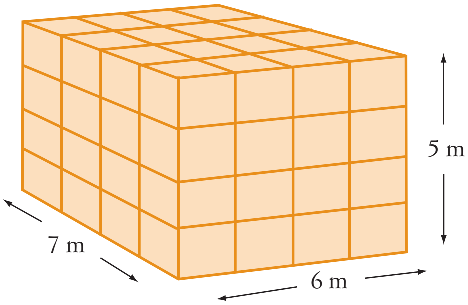
c



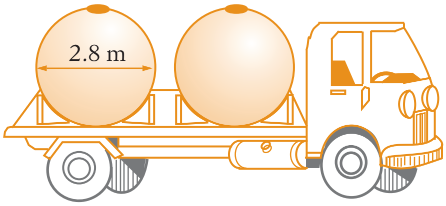
d



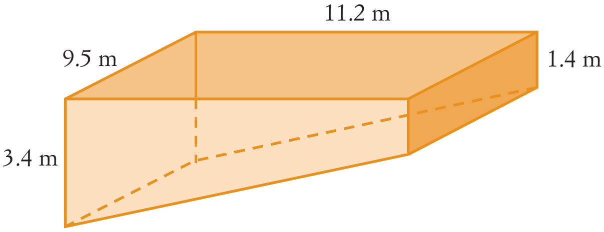
35 This pile of hay bales has to be covered to protect it from the weather. What area of tarpaulin is needed?



36 This truck carries liquid oxygen in two identical spherical containers. Find the total surface area of the spheres.

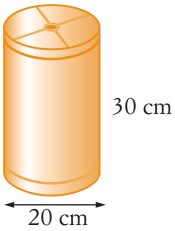


37 An in-ground swimming pool is in the shape of a trapezoidal prism as shown below.

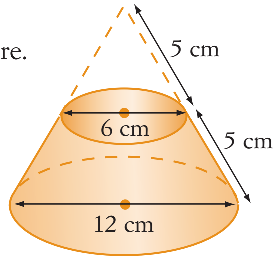


- **a** Calculate the length of the sloping bottom edge of the pool.
- **b** Calculate the surface area of the inside of the pool.
- **c** Find the cost of coating the inside of the pool with paint that comes in 4 L tins if the tins cost $125 and each litre of paint covers 8 m^2.

38 This lampshade needs to be re-covered. Find the area of material required if 5% wastage must be allowed for seams and overlaps.



39 This figure is called a **frustum**. It is formed when a cone is cut parallel to its base. Find the surface area of the frustum shown here.

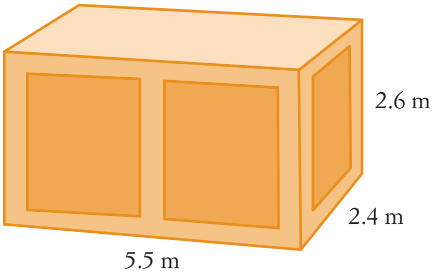


Reasoning

40 This container is used to transport goods by sea. It has to be painted all over.

a Find the area to be painted.

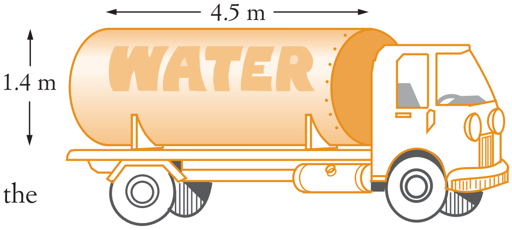
b Find the cost of painting the container if a contractor quotes \$57/m^2 for materials and labour.



41 This water tanker is used to help compact new road base. The water tank on the truck needs repainting.

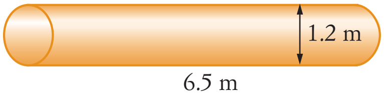
a Find the area to be painted (including the ends).

b Find the number of litres of paint needed if the coverage is 9 m^2/L.



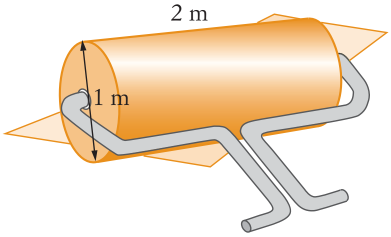
42 A rectangular sheet of steel has to be bent and welded to form this pipe.

a Find the dimensions of the smallest sheet of steel that could be used.

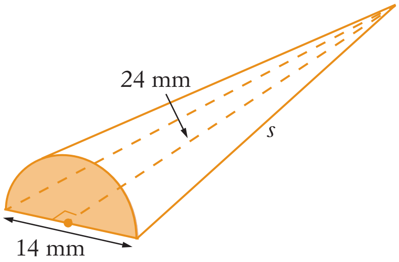


b Find the cost of constructing the pipe if 1.35 m^2 must be allowed for overlap and materials and labour cost \$75/m^2.

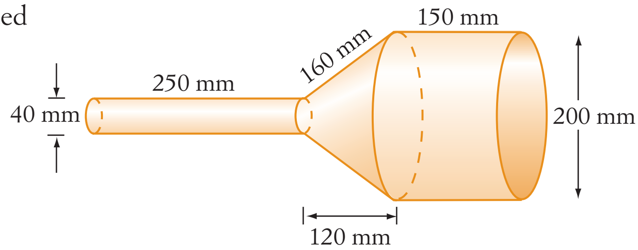
43 This roller is used to roll a cricket pitch 20 m long and 2 m wide. How many revolutions will the roller make in rolling the pitch?



44 Here is a cone that has been sliced in half. Find the surface area of the figure.



45 This water funnel is to be constructed of copper sheeting. What area of sheeting will be required, if 5% extra must be allowed for seams and overlap?



4.3 Volumes of shapes

TLF Learning object
Turn up the volume
(R11505)
MAT10MGIN00004

TLF Learning object
Inside a cubic metre
(L164)
MAT10MGIN00004

The **volume** of an object is the amount of space that it occupies. Volume is measured in **cubic** units. The units we usually use for volume are **cubic millimetres (mm^3)**, **cubic centimetres (cm^3)** and **cubic metres (m^3)**. **Capacity** is another word for volume but it usually refers to the amount of space inside a container. Capacity can be measured using volume units, but it is usually measured in **litres (L)**. Multiples of litres are shown below.

Important!

Volume and capacity units

1 millilitre (mL) = 1 cm^3
1 litre (L) = 1000 mL = 1000 cm^3
1 kilolitre (kL) = 1000 L = 1 m^3
1 megalitre (ML) = 1000 kL = 1000 m^3

You can use the following when converting units. If you are changing to a *larger* unit, *divide*. If changing to a *smaller* unit, *multiply*.

$$\begin{tikzcd} \textbf{mL (cm}^3\textbf{)} \arrow[r, bend left, "\div 1000"] & \textbf{L} \arrow[l, bend left, "\times 1000"] \arrow[r, bend left, "\div 1000"] & \textbf{kL (m}^3\textbf{)} \arrow[l, bend left, "\times 1000"] \arrow[r, bend left, "\div 1000"] & \textbf{ML} \arrow[l, bend left, "\times 1000"] \end{tikzcd}$$

Example 18

Convert the following measurements to the units shown in brackets.

a 34 mL (cm^3)
b 1.405 m^3 (L)
c 750 m^3 (ML)
d 0.004 55 kL (cm^3)

Solution

a Write the measurement. **34 mL**
These are equal units. = **34 cm^3**

b Write the measurement. **1.405 m^3**
Changing to a *smaller* unit, so *multiply*. = **1.405 × 1000 L**
Evaluate. = **1405 L**

c Write the measurement. **750 m^3**
Since 1 m^3 = 1 kL, change to kL first. = **750 kL**
Changing to a *larger* unit, so *divide*. = **750 ÷ 1000 ML**
Evaluate. = **0.75 ML**

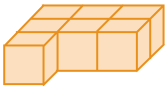
d Write the measurement.	**0.004 55 kL**
Changing to a *smaller* unit, so *multiply*.	$= \mathbf{0.004\,55 \times 1000\ L}$
Evaluate.	$= \mathbf{4.55\ L}$
Changing to a *smaller* unit, so *multiply*.	$= \mathbf{4.55 \times 1000\ mL}$
Evaluate.	$= \mathbf{4550\ mL}$
Use $1\text{ cm}^3 = 1\text{ mL}$.	$= \mathbf{4550\ cm^3}$

Investigate: Volume formulas

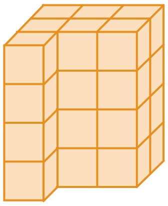
You will need centicubes, an open cylindrical container (such as a jam tin), light cardboard, scissors, a compass, some tape, some sand and a ruler.
Work in groups of three to five.

Prisms

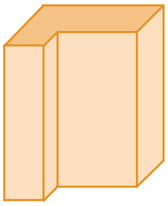
1 A centicube has an edge length of 1 cm, so the volume is 1 cm^3 and the area of each face is 1 cm^2. Use some centicubes to form a shape with a polygon base, similar to the one shown here.



2 Use more centicubes to make copies of this shape and place them one on top of the other to form a prism, as shown here.



3 Find the volume of the prism by counting the centicubes and write down the result.



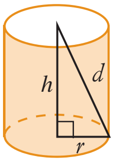
4 Work out the area of the polygon base and the height of the prism and record these.

5 Draw the prism you have formed without marking in the centicubes, as shown here. Mark in the area of the base and the height.

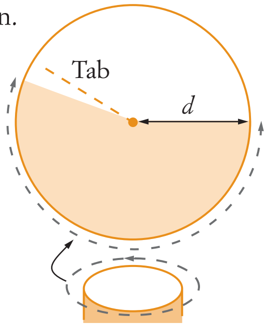
6 Complete this rule for the volume of the prism.
Volume = . . . × . . .

Cones

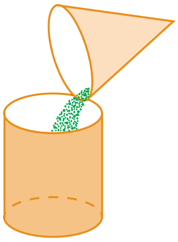
1. Measure the radius, r, and height, h, of the cylinder.
2. Measure d as shown on the diagram on the right or work it out using Pythagoras' theorem. The measurement d is the radius of part of a circle that will form the curved surface of a cone.



3. Draw a circle with radius d on cardboard.
4. Measure or calculate the circumference of the cylinder. This will be the length of the arc needed to form the cone.
5. Mark this distance around the edge of the circle you have drawn. Now cut out the sector, adding a small tab for taping.



6. Form the cone by taping it up. Check to see that the cone fits neatly into the cylinder.
7. Now fill the cone with sand and empty it into the cylinder. How many times do you need to do this to fill the cylinder?
8. The formula for the volume of a cylinder is $V = \pi r^2 h$.
9. Use this information to work out a formula for the volume of a cone. Compare your formula with others in the class.



Square-based pyramids

Try using the same procedure to find a formula for the volume of a square-based pyramid.

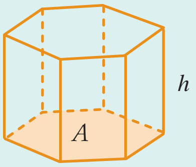
9780170194860

The volumes of common shapes can be worked out using formulas.

Important!

Volume formulas

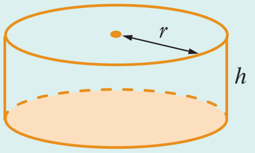
Prism



Volume = area of base × height

$V = ah$

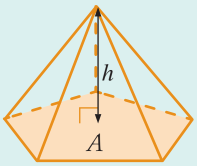
Cylinder



Volume = area of base × height

$V = \pi r^2 h$

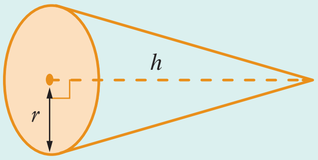
Pyramid



Volume = $\frac{1}{3}$ area of base × height

$V = \frac{1}{3} Ah$ or $V = \frac{Ah}{3}$

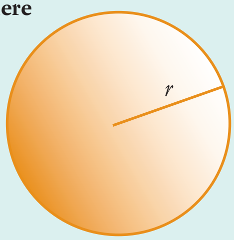
Cone



Volume = $\frac{1}{3}$ area of base × height

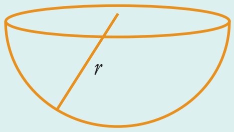
$V = \frac{1}{3} \pi r^2 h$ or $V = \frac{\pi r^2 h}{3}$

Sphere



Volume = $\frac{4}{3}\pi r^3$

Hemisphere

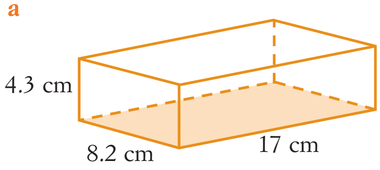


Volume = $\frac{2}{3}\pi r^3$

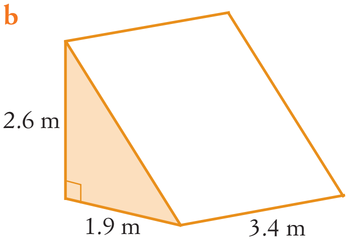
Example 19

Find the volume of each of these prisms.

a



b



Solution

a The base is a rectangle. $A = l \times w$

Substitute the values. $= 17 \times 8.2$

Evaluate. $= 139.4 \text{ cm}^2$

Write the rule for the volume of a prism. $V = Ah$

Substitute the values. $= 139.4 \text{ cm}^2 \times 4.3 \text{ cm}$

Evaluate. $= 599.42 \text{ cm}^3$

Write the answer. **The volume is about 600 cm^3.**

b The base is a triangle. $A = \frac{1}{2}bh$

Substitute the values. $= \frac{1}{2} \times 1.9 \times 2.6$

Evaluate. $= 2.47 \text{ m}^2$

Write the rule for volume of a prism. $V = Ah$

Substitute the values. $= 2.47 \text{ m}^2 \times 3.4 \text{ m}$

Evaluate. $= 8.398 \text{ m}^3$

Round to 2 significant figures. **The volume is about 8.4 m^3.**

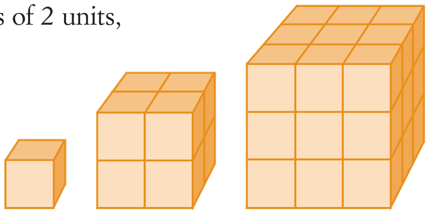
Investigate: Cubes

Task 1

You will need about 80 identical cubic blocks, such as centicubes, to complete this task.

We can consider each cube as having a length of 1 unit.

1 Use the blocks to make up larger cubes with edges of 2 units, 3 units and so on. Sketch each of the larger cubes.



2 For each cube, with edge x units, separate the cube into x sets of blocks. This needs to be done so that there are an odd number of blocks in each set and so that the odd numbers are consecutive. Record the results in a table similar to the one shown below.

Edge length, x	**Number of unit cubes**	**Consecutive odd sum**
1	$1^3 = 1$	1
2	$2^3 = 8$	$3 + 5$
3	$3^3 = 27$	$7 + \ldots + \ldots$
4	…	…

3 Extend the table and describe any patterns that you see.

Task 2

Here is another activity you can undertake using identical blocks (centicubes). You will also need some paint that washes off. As an alternative, you could use a good sketch.

1 Build or sketch a cube made up of 8 identical blocks (edge length 2 blocks). Colour the outer faces of the cube a different colour from the remaining faces of the blocks.

2 How many of the blocks have:

- **a** three faces coloured differently?
- **b** two faces coloured differently?
- **c** one face coloured differently?
- **d** all faces unchanged (no faces coloured differently)?

3 Build and colour cubes with edge lengths of 3, 4, 5, etc. blocks, and repeat observations **a** to **d**. Record your results in a table similar to the one shown below.

		a	**b**	**c**	**d**
		Number of faces coloured differently			
Edge length	**Total blocks**	**3**	**2**	**1**	**0**
2	8	8	0	0	0
3	27	8	12	…	…
4	64	8	…	…	…
5	…	…	…	…	…

4 What patterns can you see in the numbers in columns **a**, **b**, **c** and **d**?

5 Divide each of the numbers in column **c** by 6. What do you see?

Task 3

1 Use the cubes formed in Task 1 to complete the following table.

Edge length, x	1	2	3	4	…
Volume, V	1	8	27	…	…
Surface area, A	6	24	…	…	…
$V : A$	0.16	0.3	…	…	…

2 Draw a graph of volume (V) (horizontal axis) against $10 \times V : A$ (vertical axis).

3 What do you conclude?

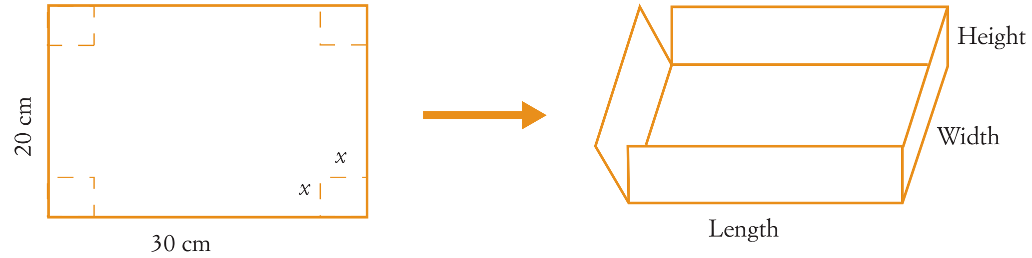
Technology Maximum volume

Puzzle sheet

Volume and capacity

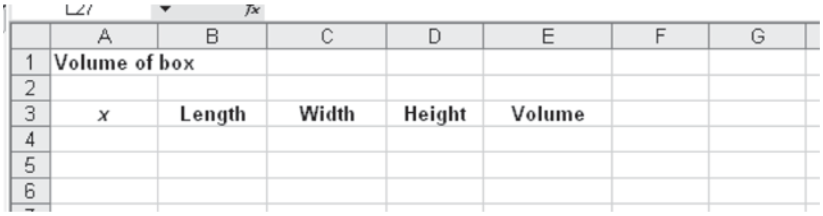
MAT10MGPS00046

A sheet of cardboard measures 30 cm by 20 cm.
If equal sized squares are cut from each corner, the cardboard can be folded and fastened to form an open box.



Let's see how the volume of the box changes as the size of the cut out corner varies.

1 Set up a spreadsheet as shown here.



	A	B	C	D	E	F	G
1	Volume of box						
2							
3	x	Length	Width	Height	Volume		
4							
5							
6							

2 The cells A4, A5, A6, ... will contain the various lengths of the sides of the square corners that are cut out as shown in the diagram above. Enter 0 in cell A4. What is the greatest value of x? Why?

3 Enter '=A4+1' in cell A5 and then copy the formula into cells A6 to A14.

4 Cells B4 to B14 will contain the various lengths of the box. Work out the formula that should be entered into cell B4. Copy the formula into cells B5 to B14.

5 Cells C4 to C14 will contain the various widths of the box. Work out the formula that should be entered into cell C4. Copy the formula into cells C5 to C14.

6 Cells D4 to D14 will contain the various heights of the box. What measurement is the same as the height of the box? Enter a formula into cell D4 and copy it into cells D5 to D14.

7 Cells E4 to E14 will contain the volumes of the box. Enter a formula into cell E4 and copy it into cells E5 to E14.

8 What are the dimensions of the box with the largest volume?

Example 20

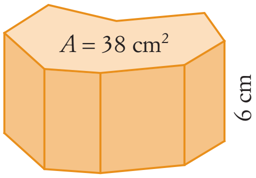
Technology worksheet

Excel worksheet: Volume of a box

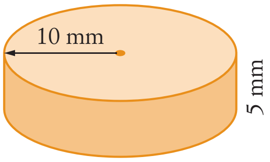
MAT10MGCT00007

Find the volume of each of these solids to the nearest whole number.

a



b



Solution

a Write the prism formula. $V = Ah$

Substitute the values. $= 38\text{ cm}^2 \times 6\text{ cm}$

Evaluate. $= 228\text{ cm}^3$

Write the answer. **The volume is 228 cm^3.**

b Write the cylinder formula. $V = \pi r^2 h$

Substitute $r = 10$ mm and $h = 5$ mm. $= \pi \times (10\text{ mm})^2 \times 5\text{ mm}$

Use your calculator for π. $= 1570.79\ldots$

Round off. $\approx 1571\text{ mm}^3$

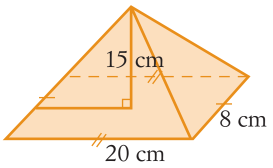
Write the answer. **The volume is about 1571 mm^3.**

Example 21

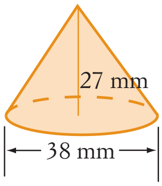
Worksheet
Area and volume 2
MAT10MGWK00011

Find the volume of each of these solids.

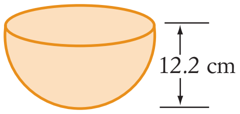
a



b



c



Solution

a Write the pyramid formula. $V = \frac{1}{3}Ah$

The base is a rectangle. $= \frac{1}{3} \times l \times w \times h$

Substitute $l = 20$ cm, $w = 8$ cm and $h = 15$ cm. $= \frac{1}{3} \times 20 \times 8 \times 15\text{ cm}^3$

Evaluate. $= 800\text{ cm}^3$

Write the answer. **The volume is 800 cm^3.**

b Write the cone formula. $V = \frac{1}{3}\pi r^2 h$

Substitute the values. $D = 38$ mm. $= \frac{1}{3} \times \pi \times 18^2 \times 27$

Evaluate. $= 9160.88\ldots\text{ mm}^3$

Round off and write the answer. **The volume is about 9161 mm^3.**

c Write the hemisphere formula. $V = \frac{2}{3}\pi r^3$

Substitute the values. $= \frac{2}{3} \times \pi \times 12.2^3$

Evaluate. $= 3803.10\ldots\text{ cm}^3$

Round off and write the answer. **The volume is about 3803 cm^3.**

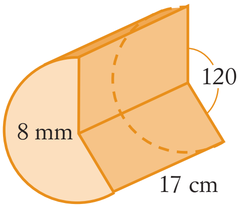
It is possible to calculate the volume of solids that are formed by combining two or more different solids.

Example 22

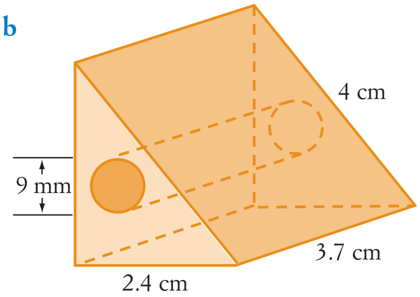
Animated example
Volumes of shapes
MAT10MGAE00004

Calculate the volume of each of the following solids correct to one decimal place.

a



b



Solution

a This solid consists of a cylinder with a 120° sector removed. This means that the remaining part is a 240° sector.

Calculate the area of the cross-section.

Area of cross-section

$$= \frac{240°}{360°} \times \pi r^2$$

Cancel.

$$= \frac{\cancel{240°}^{2}}{\cancel{360°}^{3}} \times \pi \times r^2$$

Substitute for *r*.

$$= \frac{2}{3} \times \pi \times 8^2$$

Evaluate.

$$= 134.041\ldots \text{ cm}^2$$

Use the rule for the volume of prism.

$$V = Ah$$

Substitute.

$$= 134.041\ldots \times 17 \text{ cm}^3$$

Evaluate.

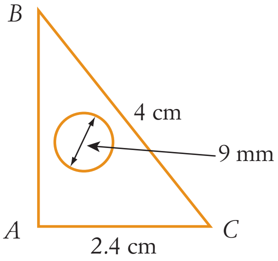
$$= 2278.70\ldots \text{ cm}^3$$

Round off and state the result.

The volume of the solid is approximately 2278.7 cm^3.

b The cross-section is a triangle with a circle removed.

Draw a diagram to represent the cross-section.



Use Pythagoras' theorem to calculate the height of the triangle.

$$BC^2 = BA^2 + AC^2$$

Substitute for known values.

$$4^2 = BA^2 + 2.4^2$$

Evaluate.

$$16 = BA^2 + 5.76$$

Reverse the equation.

$$BA^2 + 5.76 = 16$$

9780170194860

Subtract 6.25 from both sides.	$BA^2 = 10.24$
Take the square root of both sides.	$BA = 3.2$ **cm**
Calculate the area of the triangle.	$A = \frac{1}{2}bh$
Substitute for known values.	$= \frac{1}{2} \times 2.4 \times 3.2$
Evaluate.	$= 3.84$ **cm**^{2}
Calculate the area of the circle.	$A = \pi r^2$
Substitute using $r = 0.9$ cm.	$= \pi \times 0.45^2$
Evaluate.	$= 0.63617 \ldots$ **cm**^{2}
Calculate the area of the cross-section.	**Area of cross-section** **= area of triangle − area of circle**
Substitute for the areas.	$= 3.84 - 0.63617 \ldots$ **cm**^{2}
Evaluate.	$= 3.2038 \ldots$ **cm**^{2}
Calculate the volume of the solid.	$V = Ah$
Substitute.	$= 3.2038 \ldots \times 3.7$ **cm**^{3}
Evaluate.	$= 11.8541 \ldots$ **cm**^{3}
Round off and state the result.	**The volume of the solid is approximately 11.9 cm^3.**

Exercise 4.3 Volumes of shapes

Understanding

Extra questions

Exercise 4.3

MAT10MGEQ00040

See Example 18

A calculator should be used where appropriate in this exercise. For calculations involving π, use the π key on your calculator and answer correct to 1 decimal place.

1 Convert each of the following to the unit indicated.

a 2400 mL to L
b 3.2 L to cm^3
c 7.5 L to mL
d 730 mL to L
e 41 cm^3 to mL
f 8.2 m^3 to L
g 432 mL to cm^3
h 3.24 L to m^3
i 2320 cm^3 to L

2 Convert each of the following to the unit indicated.

a 5400 mL to kL
b 0.027 m^3 to cm^3
c 4300 cm^3 to kL
d 354 ML to m^3
e 0.025 m^3 to L
f 0.84 kL to cm^3

3 The following table relates to the dimensions of a range of rectangular prisms. Complete the table.

	Length	Width	Height	Volume
a	15 cm	12 cm	8 cm	…
b	4 m	3 m	…	120 m^3
c	…	11 mm	22 mm	4114 mm^3
d	26.4 cm	17.5 cm	14.4 cm	…
e	6.7 m	…	1.5 m	32.16 m^3

Fluency

4 Find the volume of:

a a cube with a side of 3 cm

b a rectangular prism measuring 4 cm by 8 cm by 6 cm

c a rectangular prism measuring 1.2 m by 0.9 m by 0.6 m

d a rectangular prism measuring 33 cm by 12 cm by 50 cm

e a rectangular block of cheese measuring 13 cm by 8 cm by 3 cm.

5 Find the volumes of cylinders with the following dimensions.

a radius 17 cm, height 22 cm

b diameter 20 mm, height 15 mm

c diameter 2.4 m, height 1.9 m

d radius 8.3 cm, height 5.2 cm

e radius 0.75 m, height 3.6 m

f diameter 14 mm, height 37 mm

6 Find the volume of:

a a conical pile of sugar with a height of 3.4 m and a radius of 4.8 m

b a square based pyramid with a base measuring 13 cm by 13 cm and a height of 11 cm

c a rectangular based pyramid with a base measuring 2.1 m by 1.7 m and a height of 3.3 m

d a conical funnel with a height of 26 cm and a diameter of 18 cm

e a rectangular based pyramid with a base measuring 42 mm by 21 mm and a height of 11 mm.

7 Find the volume of:

a a tennis ball with a radius of 3.5 cm

b a golf ball with a diameter of 43 mm

c a basketball with a radius of 12.7 cm

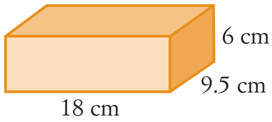
d a spherical meteorological balloon with a diameter of 2.7 m

e half an orange with a diameter of 9 cm.

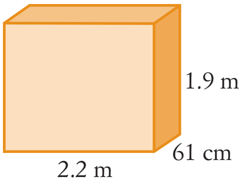
See Example 19

8 Find the volumes of the following prisms.

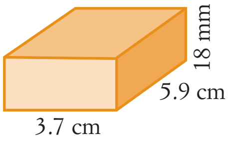
a



b



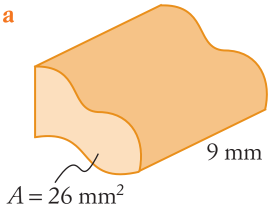
c



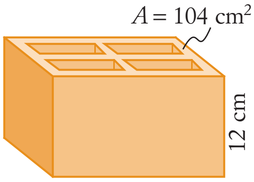
See Examples 19, 20

9 Find the volumes of the following solids.

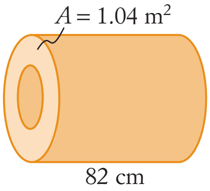
a



b

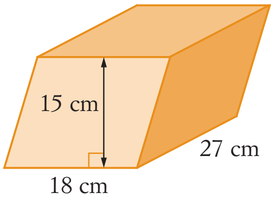


c

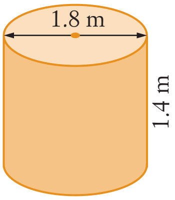


(Answer in m^3.)

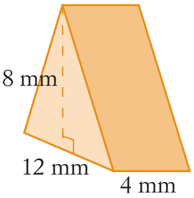
d



e



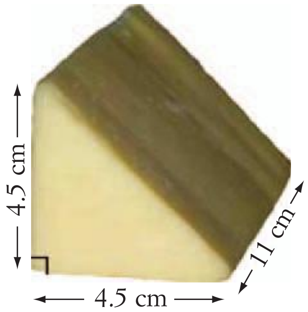
f



10 Find the volume of each of the following.

a

b

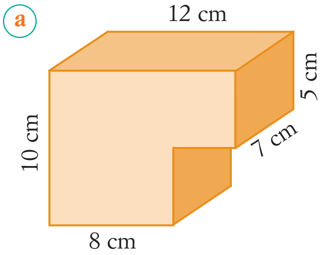


c

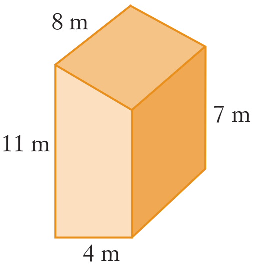
11 Find the volumes of the following solids.

Worked solutions
Exercise 4.3
MAT10MGWS00012

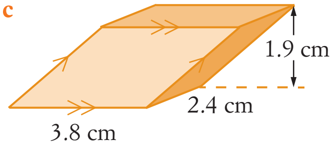
ⓐ



b

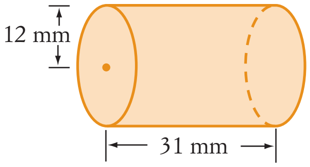


c

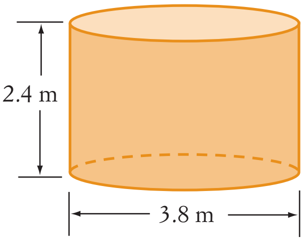


12 Find the volumes of the following solids.

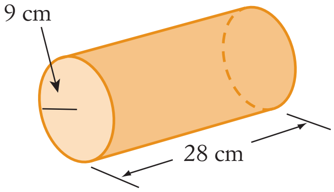
a



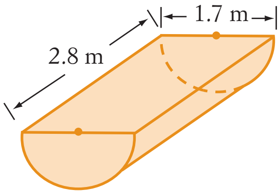
b



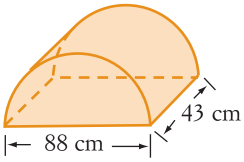
c



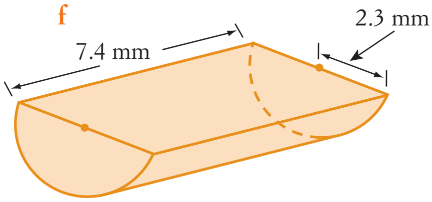
d



e



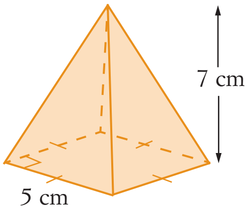
f



13 Find the volumes of the following solids.

Worked solutions
Exercise 4.3
MAT10MGWS00012
See Example 21

a



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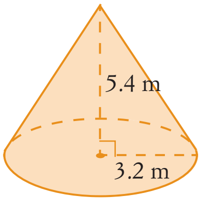

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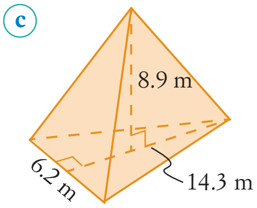
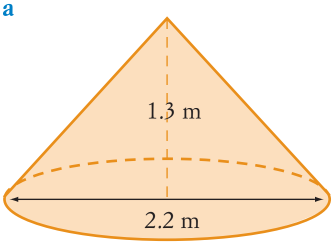
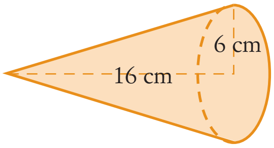


14 Find the volume of each of the following cones.

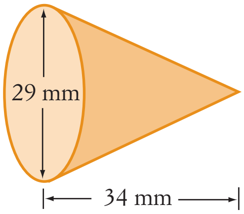
a



b



c



Worked solutions

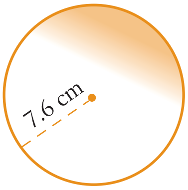
Exercise 4.3

MAT10MGWS00012

See Example 21

15 Find the volume of each of the following.

a



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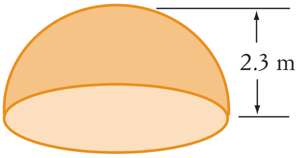
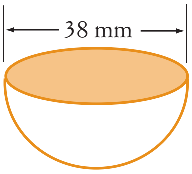


c

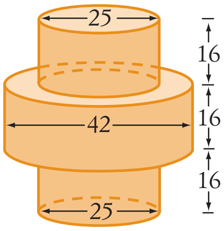


Problem solving

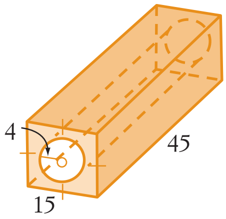
See Example 22

16 Each of the following solids is made of plastic and manufactured using an infusion moulding process. Calculate the volume of plastic in each. All dimensions are shown in centimetres.

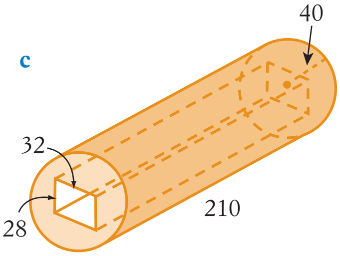
a



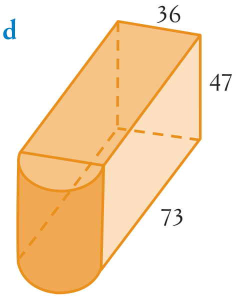
b



c

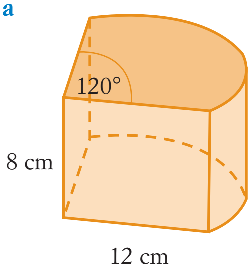


d

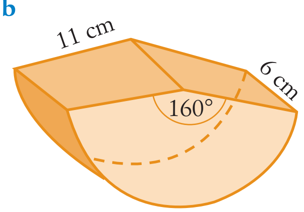


17 Find the volume of each of the following solids.

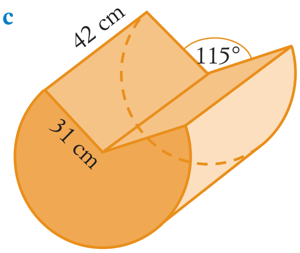
a



b

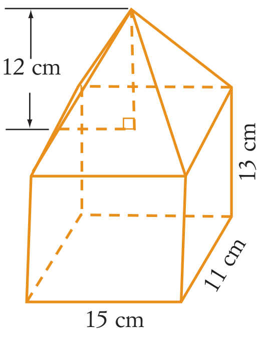


c

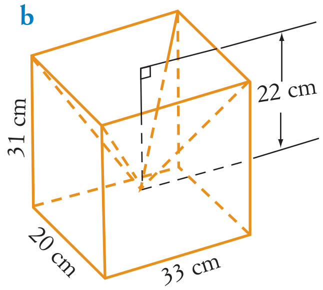


18 Find the volume of each of the following solids.

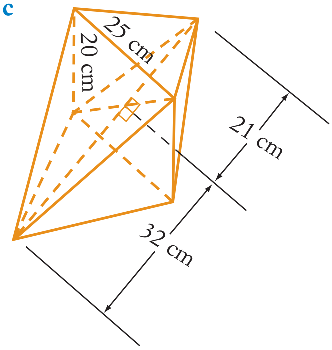
a



b

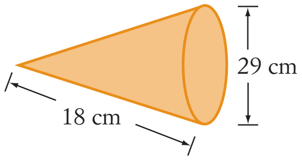


c

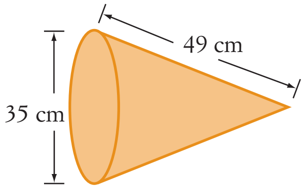


19 Calculate the volume of each of the following cones.

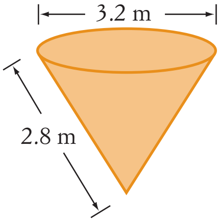
a



b

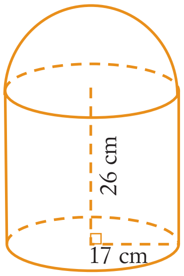


c

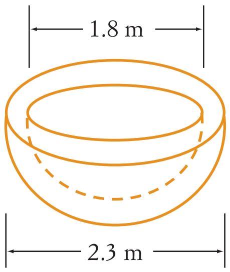


20 Calculate the volume of each of the following solids.

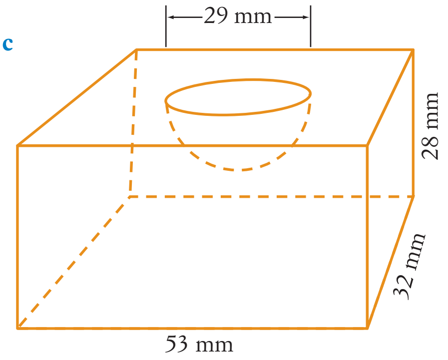
a



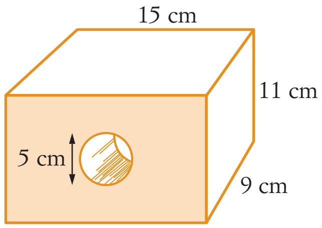
b



c



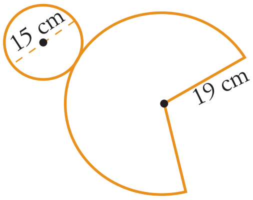
21 This metal block has had a circular hole drilled through it. Calculate the volume of metal remaining in the block.



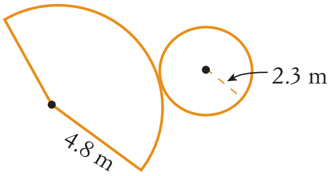
22 For each net shown below:

i draw the cone **ii** calculate the volume

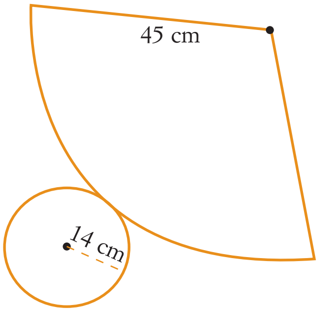
a



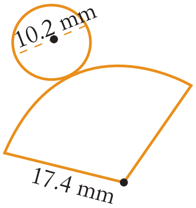
b



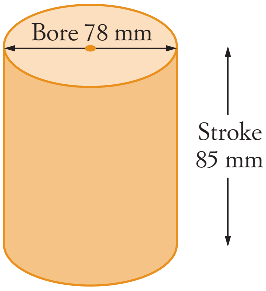
c



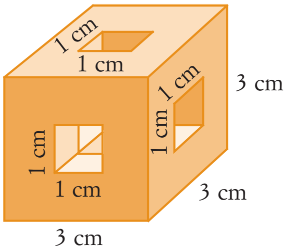
d



23 The engine capacity of a motor vehicle is the total capacity of the cylinders in the engine. Calculate the engine capacity of a four-cylinder car if each cylinder has a bore (diameter) of 78 mm and a stroke (length) of 85 mm. Answer in cm^3.



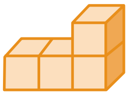
24 This cubic block has holes that pass all the way through it. Each hole is 1 cm × 1 cm × 3 cm. Find the total surface area (internal and external) of the block.

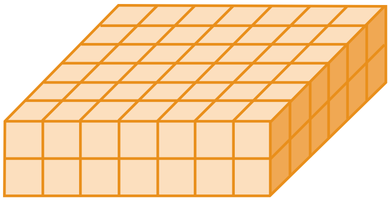


Reasoning

25 Each  represents 1 cm^3.

What is the greatest number of these solids:

 that can be fitted into the solid on the right?

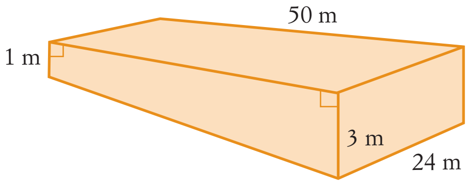


(The shapes must be wholly contained within the solid, but may form part of the solid's surface.)

26 **a** Find the volume of this swimming pool in cubic metres.

b What is the capacity of the pool in kilolitres?

c The pool is filled using a pipe with a cross-sectional area of 0.02 m^2. How long will this take if the water flows through the pipe at 2 m per second?



27 Tennis balls with a radius of 3.5 cm are packed into a rectangular carton for shipment to a sports store. The carton has a base measuring 35 cm by 70 cm and is 70 cm high. The balls are packed in layers in the carton.

a What is the maximum number of tennis balls that will fit in each layer?

b When the carton is full, how many tennis balls will it hold?

c What percentage of space in the carton is occupied by the tennis balls?

28 The planets in our solar system can be considered to be spherical. The Earth has a diameter of approximately 12 742 km.

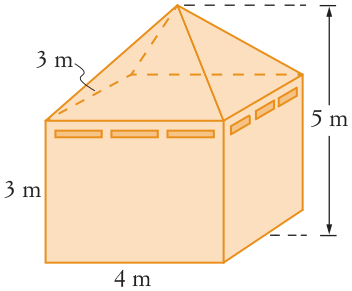
a Jupiter has a diameter of approximately 142 981 km. How many times would the volume of the Earth fit into the volume of Jupiter?

b If the volume of the Earth is 17.823 times the volume of Mercury, what is Mercury's diameter?

29 This is a tent used for the fumigation of imported goods suspected of carrying pests from overseas.

a Calculate the volume of the tent.

b The gas used for fumigation is dense and it pushes out the air through special vents when the tent is prepared for use. If the gas can be delivered at the rate of 0.035 m^3/s, how long will the tent take to fill?

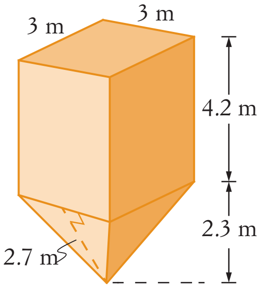


30 This grain hopper is used to store wheat.

a What volume of wheat is the hopper able to hold?

b A quarter of the wheat is released from the full hopper into a railway wagon. How far will the level of the grain fall?

c The outside of the hopper has to be painted. It needs two coats of paint. How much will the painting cost if a contractor will do the job for \$17/m^2?

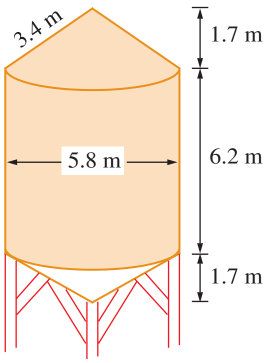


31 This grain silo is made up of a cylinder and two identical shallow cones.

a How many cubic metres of barley can this silo hold?

b Some barley is removed from the full silo and the level of grain drops by 1 m. What percentage of the grain has been removed?

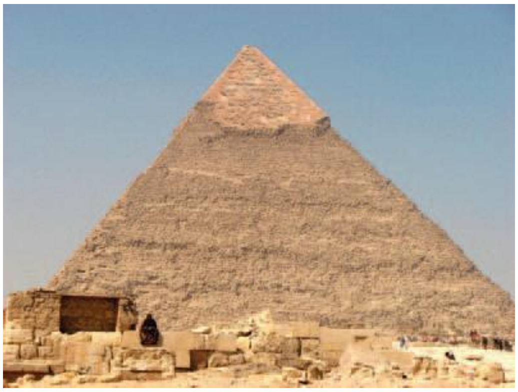
c A farmer wishes to build a silo to these dimensions. How many square metres of sheet metal are required if 5% must be allowed for overlaps and wastage?



32 The Great Pyramid in Egypt has been reduced in size from its original dimensions by weather and a number of other causes. It is known that it originally had a volume of 2.57 million m^3 and a square base of side length 230 m.

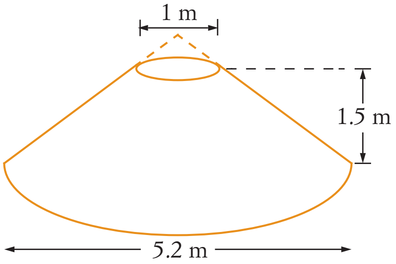
a What was its original height?

b Its volume has been reduced by about 6% over the years. Assuming that the base is still 230 m square, what is its height now?



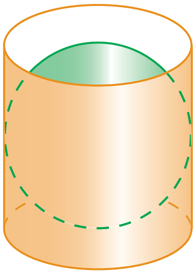
33 The (Aztec) Great Pyramid of Cholula in Mexico is 66 m tall, with a base that covers 20.25 ha. Compare its volume with that of the Great Pyramid in Egypt (see question **32**). Which is the larger, and what is the difference in their volumes?

34 A floating pontoon is held in place by a concrete 'anchor' in the shape of a truncated cone (frustum), as shown here. The density of concrete is known to be 2400 kg/m^3. If the pontoon requires an anchor of mass 80 t to keep it in place, will the concrete 'anchor' described here be sufficient?



35 This sphere fits snugly inside the cylinder.

Find the ratio $\frac{\text{volume of sphere}}{\text{volume of cylinder}}$.



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Chapter 4 summary

Quiz

Area and volume

MAT10MGQZ00004

- The metric system has the **metre** (**m**) as its standard unit of length. Other units of length are the **kilometre** (**km**), the **centimetre** (**cm**) and the **millimetre** (**mm**). If you are changing to a larger unit, divide. If changing to a smaller unit, multiply.
- The **area** of a flat shape is the size of the region enclosed by its boundary. The units of area in the metric system are the **square millimetre (mm^2)**, the **square centimetre (cm^2)**, the **square metre (m^2)**, the **hectare (ha)** and the **square kilometre (km^2)**. If you are changing to a *larger* unit, *divide*. If changing to a *smaller* unit, *multiply*.
- There are rules that can be used to calculate the areas of some plane shapes.

$$\text{Area of rectangle} = \text{length} \times \text{width or } A = lw$$

$$\text{Area of circle} = \pi \times \text{radius}^2 \text{ or } A = \pi r^2$$

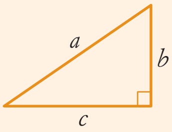
$$\text{Area of triangle} = \tfrac{1}{2} \times \text{base} \times \text{height or } A = \tfrac{1}{2}bh$$

$$\text{Area of parallelogram} = \text{base} \times \text{height or } A = bh$$

$$\text{Area of trapezium} = \text{average of parallel sides} \times \text{height or } A = \tfrac{1}{2}(a+b)h$$

$$\text{Area of rhombus or kite} = \tfrac{1}{2}(\text{product of diagonals})$$

- **Pythagoras' theorem** can be used to help calculate the lengths of sides in right-angled triangles. Pythagoras' theorem states that, for a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides. For this triangle:



$$a^2 = b^2 + c^2$$

- **Heron's formula** can be used to calculate the area of a triangle using the lengths of its three sides. Heron's formula for a triangle with sides a, b and c is:

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}, \text{ where } s = \tfrac{1}{2}(a+b+c)$$

- **Two-dimensional (2D) shapes** are flat and have length and width but no depth. **Three-dimensional (3D) shapes** occupy space and have length, width and depth. The **faces** of a 3D shape separate the interior from the exterior. The lines where faces meet are called **edges** and the corners where edges meet are called **vertices** (singular **vertex**).
- The **net** of a 3D shape is the 2D shape that can be folded to form the 3D shape.
- The **surface area (SA)** of a 3D shape is the total area of all its faces.
- The **curved surface area (CSA)** of a cylinder $= 2\pi rh$.
- A **pyramid** is a solid made up of a base and triangular faces that meet at an **apex**, which is also called the **vertex** of the pyramid. The height, h, of a pyramid is the perpendicular distance from the apex to the base. The **slant height**, s, of a pyramid is the height of the triangular face measured along the face.
- A **cone** is a solid with a circular base and an apex. The height, h, of the cone is measured perpendicular to the base and the slant height, s, is the distance from the edge of the base to the apex along the face of the cone. The surface area of a cone is given by $\pi r^2 + \pi rs$ where r is the radius of the cone.
- The surface area of a **sphere** $= 4\pi r^2$. A **hemisphere** is half a sphere. The surface area of a closed hemisphere $= 3\pi r^2$ and the surface area of an open hemisphere $= 2\pi r^2$.

Chapter 4 summary

- The **volume** of an object is the amount of space that it occupies. **Capacity** is another word for volume but it usually refers to the amount of space inside a container. The units we usually use for volume are the **cubic millimetre (mm^3)**, **cubic centimetre (cm^3)** and **cubic metre (m^3)**. Capacity can be measured using volume units, but it is usually measured in **litres (L)**. If you are changing to a *larger* unit, *divide*. If you are changing to a *smaller* unit, *multiply*.
- The volumes of common shapes can be worked out using formulas.

$$\text{Volume of prism} = \text{area of base} \times \text{height or } V = Ah$$

$$\text{Volume of cylinder} = \text{area of base} \times \text{height or } V = \pi r^2 h$$

$$\text{Volume of pyramid} = \tfrac{1}{3} \times \text{area of base} \times \text{height or } V = \tfrac{1}{3} Ah$$

$$\text{Volume of cone} = \tfrac{1}{3} \times \text{area of base} \times \text{height or } V = \tfrac{1}{3} \pi r^2$$

$$\text{Volume of sphere} = \tfrac{4}{3} \pi r^3$$

$$\text{Volume of hemisphere} = \tfrac{2}{3} \pi r^3$$

Chapter 4 review

Understanding

1 Convert the measurements below to the units shown in brackets.

See Example 1

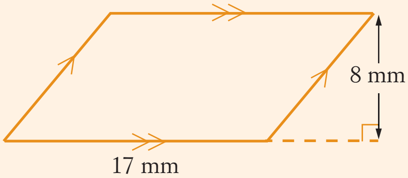
a 56 700 m^2 (km^2)
b 3.87 ha (m^2)
c 57 ha (km^2)
d 0.025 km^2 (ha)

2 Find the area of each of the following.

a a square of side 17 cm
b a rectangle measuring 24 mm by 15 mm
c a parallelogram with a base of 21.4 m and a height of 15.8 m
d a rhombus with diagonals of 28 cm and 16 cm
e a kite with diagonals of 51 mm and 13 mm

3 Find the area of this figure.

See Example 2

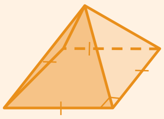


4 Identify the shapes shown below.

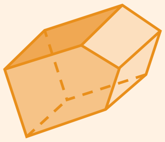
See Example 7

a

b



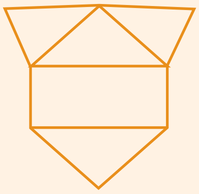
c



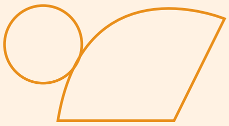
5 Identify the shapes whose nets are shown below.

See Example 8

a



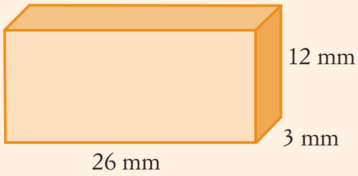
b



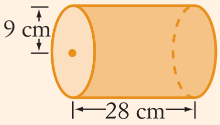
6 Draw a net for this shape, including any dimensions shown.

See Examples 9, 10

a

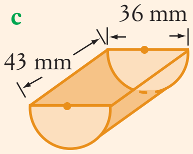


b



Closed cylinder

c



Open half-cylinder

7 Convert each of the following to the unit indicated.

See Example 18

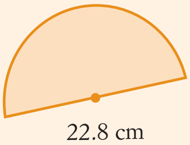
a 2.75 L to mL
b 5.43 m^3 to L
c 271 mL to cm^3
d 3600 mL to kL
e 0.049 m^3 to cm^3
f 2800 cm^3 to kL

Chapter 4 review

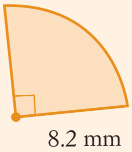
Fluency

8 Find the areas of the following shapes.

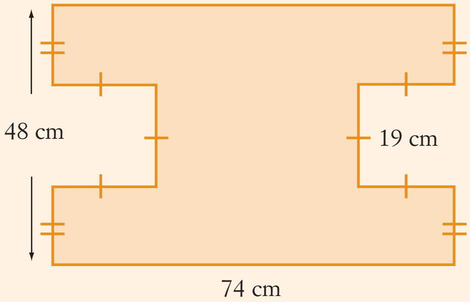
a



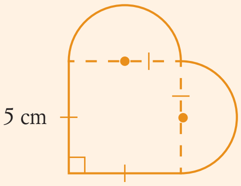
b



9 Work out the missing measurements, then find the areas of this shape.

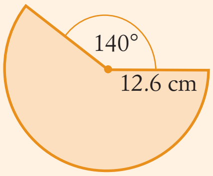


10 Find the area of this shape.

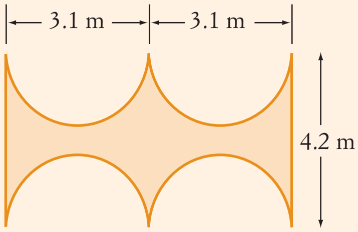


11 Find the areas of the following shapes.

a



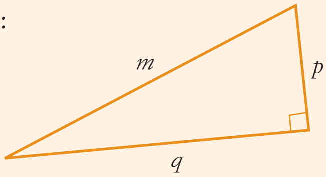
b



12 Use Pythagoras' theorem and the triangle on the right to:

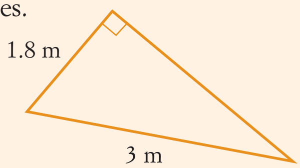
a complete $m^2 = \ldots + \ldots$

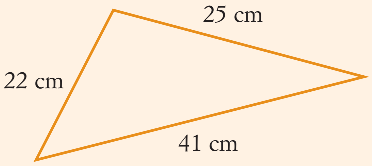
b find p if $m = 5$ and $q = 4.8$.



See Example 2, 6

13 Find the area of these triangles.





See Example 9

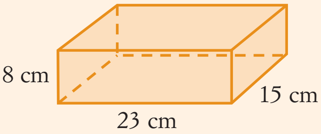
14 Calculate the surface area of the solid shown in question **6a**.

15 Calculate the surface area of the solid shown in question **6b**.

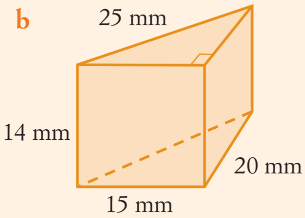
16 Calculate the surface area of the solid shown in question **6c**.

17 Calculate the surface area for each of these solids.

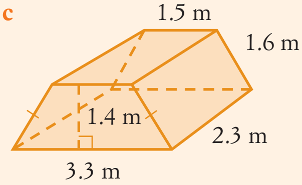
a



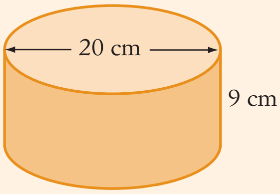
b



c



18 **a** Draw a net for this shape, including any dimensions shown.
b Calculate the surface area.



19 Calculate the curved surface area of the cylinders with the following dimensions. Give your answers correct to two decimal places.

See Example 10

a radius 12 cm, height 9 cm
b diameter 18 mm, height 11 mm

20 Calculate the surface area of the closed cylinders with the following dimensions. Give your answers correct to two decimal places.

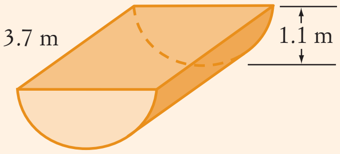
a diameter 9 cm, height 14 cm
b radius 1.6 m, height 2.5 m

21 Calculate the surface area of each the following solids.

See Examples 9, 11

a **Closed cylinder**

b **Closed half-cylinder**



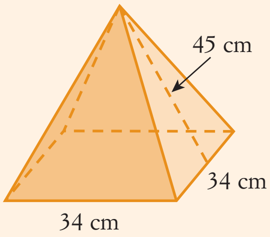
22 Calculate the surface area of each of the following.

a square based pyramid measuring 12 cm by 12 cm with a slant height of 14 cm
b cone with diameter 18 mm, slant height 14 mm
c sphere with a radius of 1.7 m
d closed hemisphere with a diameter of 26 cm

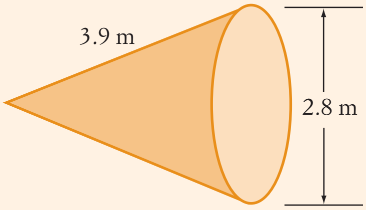
See Examples 14, 16, 17

23 Calculate the surface area of each of the following.

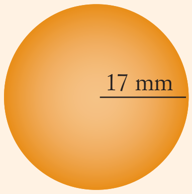
a



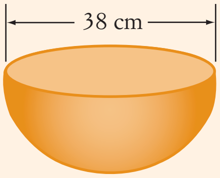
b



c



d



Closed hemi-sphere

24 Find the volume of:

a a cube with a side of 3 cm

b a rectangular prism measuring 4 cm by 8 cm by 6 cm

c a cylinder with a radius 17 cm, height 22 mm

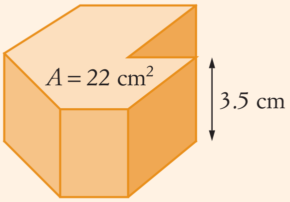
d a cone with a height of 3.4 m and a radius of 4.8 m

e a rectangular based pyramid with a base measuring 2.1 m by 1.7 m and a height of 3.3 m

f a sphere with a diameter of 3.5 cm

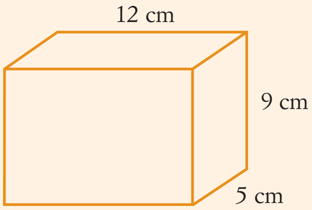
See Example 20

25 Find the volume of this prism.



See Example 19

26 Find the volume of this object.

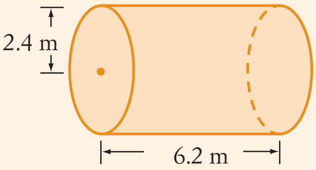


27 Find the volume of a box measuring 1.3 m by 2.2 m by 0.8 m.

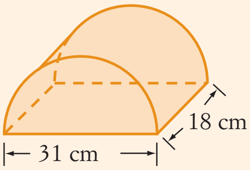
28 Find the volumes of the following solids.

See Example 21

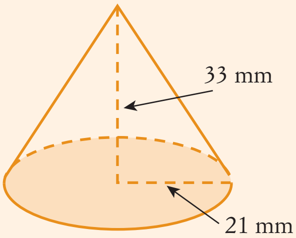
a



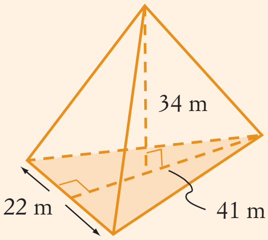
b



c

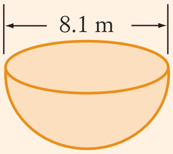


d



e

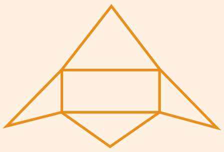
f



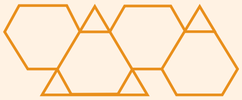
Problem solving

29 Identify the shapes whose nets are shown below.

a

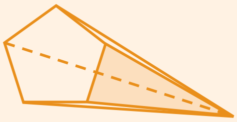


b

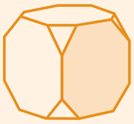


30 Draw nets for the following shapes.

a



b

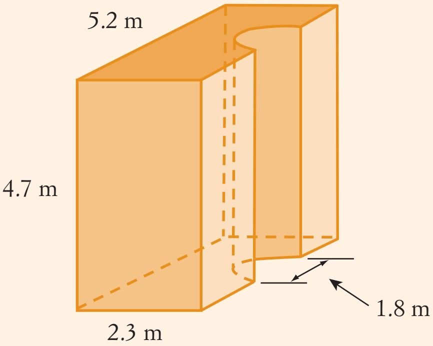


31 A circular disc has an area of 908 mm^2. What is its radius?

32 A circle has a diameter of 16 m. What is the perimeter of a 60° sector?

33 The surfaces of this solid need to be painted. Calculate the total area that will be painted.

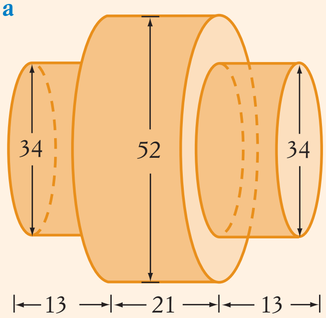
See Example 12



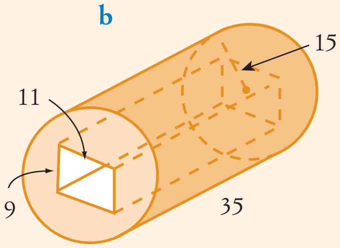
See Example 22

34 Each of the following solids is made of plastic and manufactured using an infusion moulding process. Calculate the volume of plastic in each. All dimensions are shown in centimetres.

a

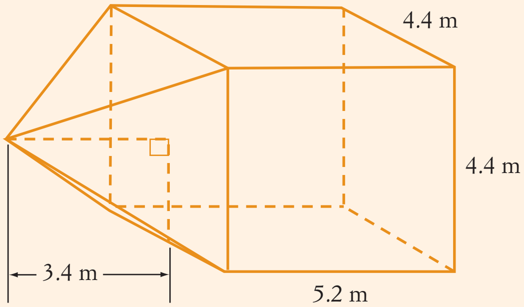


b

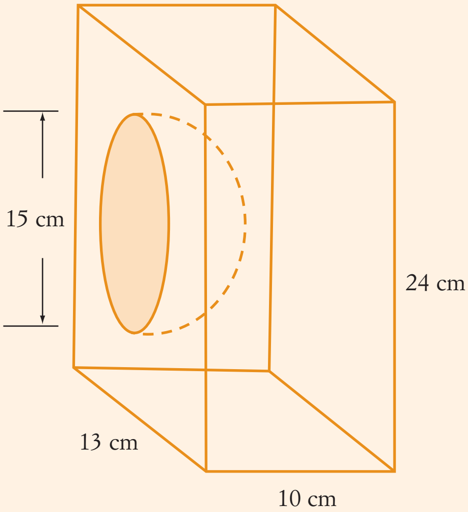


35 Find the volume of each of the following solids.

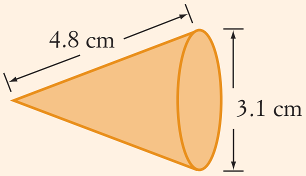
a



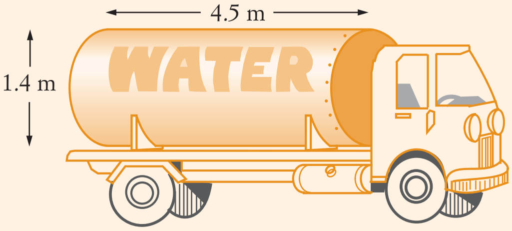
b



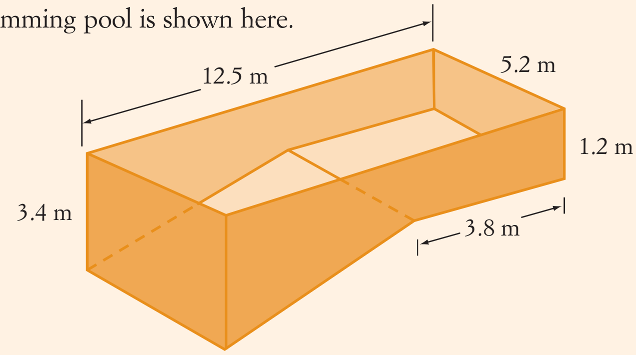
36 Calculate the volume of this cone.



37 This water tanker is able to carry a maximum load of 5 tonnes up a steep hill. Will it be able to make it up the hill if the tanker is full of water? (1 cm^3 of water has a mass of 1 g.)

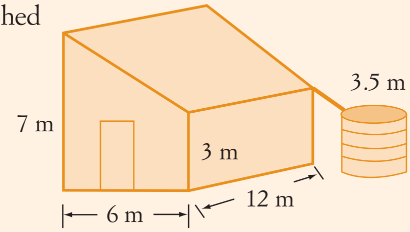


38 A newly constructed in-ground swimming pool is shown here.



- **a** The interior surface of the swimming pool requires one coat of a special sealant which costs \$78.50/L. If each litre of the sealant covers 8 m^2, calculate the cost of the sealant required to complete the job.
- **b** Calculate the volume of the swimming pool.
- **c** A hose with a flow rate of 225 L per minute is used to fill the pool. How long will this take if the swimming pool is filled to a depth 15 cm from the top?

39 A shed has the dimensions shown here. The roof of the shed drains into a rainwater tank with a diameter of 3.5 m. If 8 mm of rain falls, by how much (to the nearest millimetre) does the level of water in the tank rise?



Number and algebra

5

Coordinate geometry

Contents

Prior learning
Chapter 5
MAT10NAPL00016

Parent guide
Chapter 5
MAT10NAPG00016

Curriculum guide
Chapter 5
MAT10NACU00016

Australian Curriculum statements

Linear and non-linear relationships

Solve problems involving parallel and perpendicular lines. (10)

Explore the connection between algebraic and graphical representations of relations such as simple quadratics, circles and exponentials using digital technology as appropriate. (10)

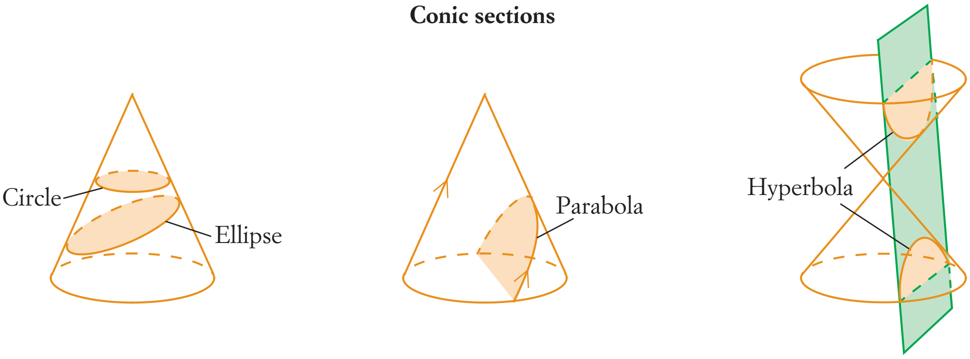
Describe, interpret and sketch parabolas, hyperbolas, circles and exponential functions and their transformations. (10A)

Video tutorial
Coordinate geometry
MAT10NAVT00005

We can use maps and street directories to describe and locate places we want to find. It is also important to be able to describe and locate positions in mathematics. Many cars are now fitted with global positioning systems to locate places. These systems rely on mathematical descriptions of positions and satellite technology.

In mathematics, positions are described using coordinates. You are already familiar with using Cartesian coordinates to plot and describe the position of points on the Cartesian plane. René Descartes (1596–1650) is credited with developing the Cartesian system, which still forms the foundation for graphing many centuries after his death.

Coordinates are also used in maps and they have important uses in science. The use of coordinates is essential in the study of vectors. They form the basis for much of the physics of movement. Without coordinates, it would be very difficult to study speed, acceleration, force and mechanics. Pierre de Fermat, who knew Descartes, greatly extended this work and discovered the shapes and equations of many different curves, including the curves that are obtained by cutting a slice through a cone – the conic sections.


Conic sections

Mathematical literacy

Maths dictionary
MAT10ASDI00001

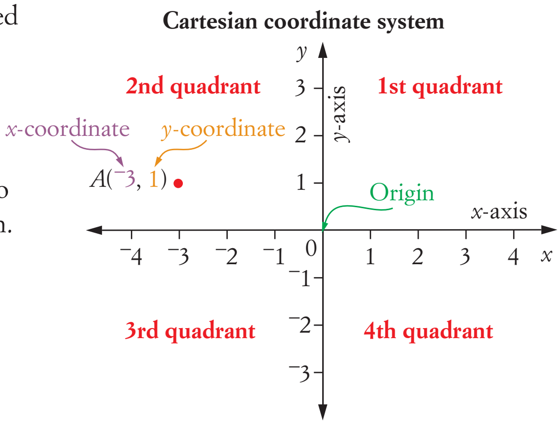
The mathematical words below have special meanings that you will learn in this chapter. It is important that you learn to spell them and gradually learn what they mean in mathematics. You may find the glossary or online mathematical dictionary useful for this purpose.

- asymptote
- branch
- Cartesian coordinates
- Cartesian plane
- circle
- distance formula
- exponential function
- falling
- gradient
- gradient–intercept form
- gradient–point form
- hyperbola
- intercepts
- linear equation
- linear function
- linear model
- line of best fit
- maximum
- midpoint
- minimum
- model
- origin
- parabola
- parallel lines
- perpendicular lines
- quadrants
- quadratic function
- rectangular coordinates
- rising
- roots
- slope
- standard form
- trendline
- turning point
- vertex
- *x*-axis
- *x*-coordinate
- *x*-intercept
- *y*-axis
- *y*-intercept
- *y*-coordinate
- zeros

5.1 Distances and midpoints in the plane

You have already seen that points can be plotted on a number lattice known as the **Cartesian plane**.

This is done using numbers on the x-axis and y-axis called **coordinates**. The x-coordinate is always first. A series of points may be plotted to give a shape, or to show the graph of a function. The diagram on the right shows the main features of the Cartesian coordinate system.



Cartesian coordinates are also called **rectangular coordinates**.

We can calculate the distance between two points on the Cartesian plane using Pythagoras' rule.

Suppose there are two points $A(1, 2)$ and $B(6, 14)$.

Once these have been plotted, extra lines need to be drawn in to form the right-angled triangle ABC shown on the right.

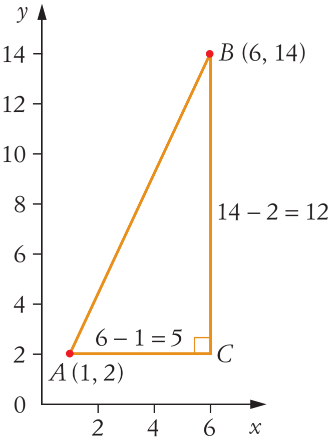
In $\triangle ABC$:

$$AC = 6 - 1 = 5$$
$$BC = 14 - 2 = 12$$

Using Pythagoras' theorem:

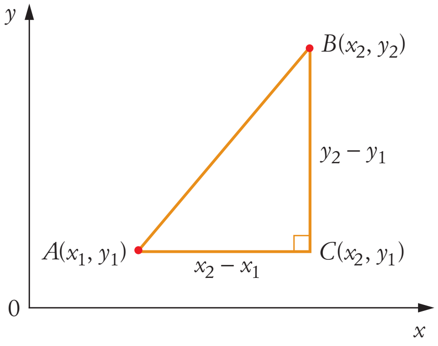
$$AB^2 = AC^2 + BC^2$$
$$AB^2 = 5^2 + 12^2$$

So $$AB = \sqrt{5^2 + 12^2} = \sqrt{169} = 13$$



Now suppose $A = (x_1, y_1)$ and $B = (x_2, y_2)$, as shown in the diagram on the right.

$$AC = x_2 - x_1$$
$$BC = y_2 - y_1$$
$$AB^2 = AC^2 + BC^2$$
$$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$
$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



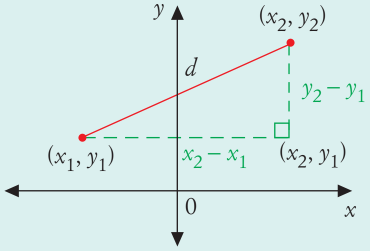
This is known as the **distance formula**.

Important!

Distance formula

The distance between two points (x_1, y_1) and (x_2, y_2) is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Example 1

Video tutorial
Distance, midpoint and gradient formulas
MAT10NAVT00010

Find the distance between points (3, 7) and ($^{-}4$, 10).

Solution

Identify the two points.	**Let $(x_1, y_1) = (3, 7)$ and $(x_2, y_2) = (^{-}4, 10)$.**
Write the distance formula.	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Substitute for x_1, y_1, x_2 and y_2.	$= \sqrt{(^{-}4 - 3)^2 + (10 - 7)^2}$
Evaluate the brackets.	$= \sqrt{(^{-}7)^2 + (3)^2}$
	$= \sqrt{58}$
Evaluate and round off.	$\approx$ **7.62 units**

Example 2

Puzzle sheet
Finding coordinates for given segment lengths
MAT10NAPS00048

If two points $X(^{-}3, 2)$ and $Y(y, ^{-}1)$ are $\sqrt{34}$ units apart, find y.

Solution

State the distance formula.	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Let $(x_1, y_1) = (^{-}3, 2)$ and $(x_2, y_2) = (y, ^{-}1)$.	$\sqrt{34} = \sqrt{(y - {}^{-}3)^2 + (^{-}1 - 2)^2}$
Square both sides of the equation.	$34 = (y + 3)^2 + 9$
Subtract 9 from both sides and reverse the equation.	$(y + 3)^2 = 25$
Take square roots.	$y + 3 = \pm 5$
	$y = -3 \pm 5$
	$y = 2$ **or** -8

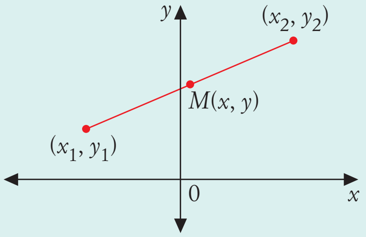
We can calculate the **midpoint** of two points using averages. The x-coordinate of the midpoint will be the average of the x-coordinates, and the y-coordinate will be the average of the y-coordinates.

Important!

Midpoint

The **midpoint**, M, of two points (x_1, y_1) and (x_2, y_2) is given by

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$



Example 3

Find the midpoint of $(3, 6)$ and $(^-5, 8)$.

Solution

Identify the two points. **Let $(x_1, y_1) = (3, 6)$ and $(x_2, y_2) = (^-5, 8)$.**

Write the midpoint formula. $$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Calculate the x-coordinate of M. $$x_M = \left(\frac{x_1 + x_2}{2}\right) = \frac{3 + {^-5}}{2} = {^-1}$$

Calculate the y-coordinate of M. $$y_M = \left(\frac{y_1 + y_2}{2}\right) = \frac{6 + 8}{2} = 7$$

State the result. **The midpoint of $(3, 6)$ and $(^-5, 8)$ is $(^-1, 7)$.**

Example 4

The midpoint of A and B is $(3, {^-4})$. If $A = (5, 1)$, find B.

Solution

Identify the two points. **Let $A = (x_1, y_1) = (5, 1)$ and $B = (x_2, y_2)$.**

Write the midpoint formula. $$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Substitute for x_1, y_1, x_2 and y_2. $$= \left(\frac{5 + x_2}{2}, \frac{1 + y_2}{2}\right)$$

Use the fact that $M = (3, {^-4})$. $$3 = \frac{5 + x_2}{2}$$

Multiply both sides by 2. $$3 \times 2 = \frac{5 + x_2}{2} \times 2$$

Cancel and reverse the equation. $$5 + x_2 = 6$$

Solve for x_2. $$x_2 = 1$$

Use the fact that $M = (3, {}^{-}4)$. $\quad {}^{-}\mathbf{4} = \dfrac{\mathbf{1} + \boldsymbol{y_2}}{\mathbf{2}}$

Multiply both sides by 2. $\quad {}^{-}\mathbf{4} \times \mathbf{2} = \dfrac{\mathbf{1} + \boldsymbol{y_2}}{\mathbf{2}} \times \mathbf{2}$

Cancel and reverse the equation. $\quad \mathbf{1} + \boldsymbol{y_2} = {}^{-}\mathbf{8}$

Solve for y_2. $\quad \boldsymbol{y_2} = {}^{-}\mathbf{9}$

State the result. $\quad$ ***B* is (1, ⁻9).**

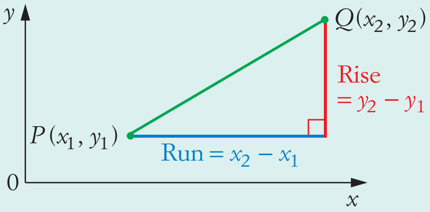
An important property of a segment or line is its steepness. This is called the **slope** or **gradient**. It measures how much the segment or line goes up for each step along the x-axis. We can work out the gradient using the **rise** (vertical distance) and **run** (horizontal distance) of a line between two points.

Important!

Gradient

If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are points on a straight line:

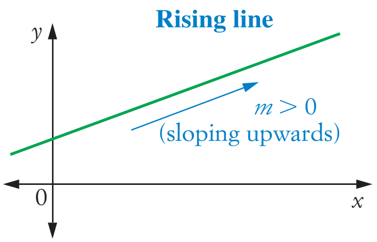
- the **rise** from P to Q is the y-distance $y_2 - y_1$; and
- the **run** from P to Q is the x-distance $x_2 - x_1$.

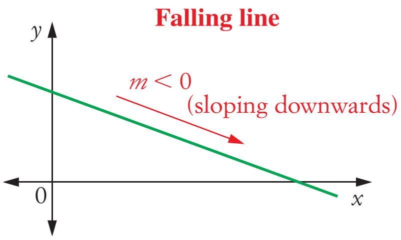


We use the symbol m to represent gradient (slope).
The gradient of the line PQ has the symbol m_{PQ} and is worked out using the formula

$$\text{Gradient} = \frac{\text{rise}}{\text{run}} \quad \text{or} \quad m_{PQ} = \frac{y_2 - y_1}{x_2 - x_1}$$

Lines that rise from left to right are described as **rising** and have a *positive gradient* ($m > 0$). **Falling** lines fall from left to right and have a *negative gradient* ($m < 0$).

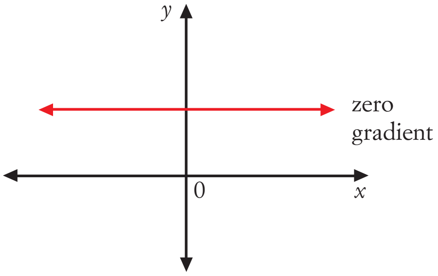

Rising line


Falling line

For a horizontal or flat line, the vertical rise is 0.

So, gradient $= \dfrac{0}{\text{run}} = 0$.

This means a horizontal line has a **zero gradient**.

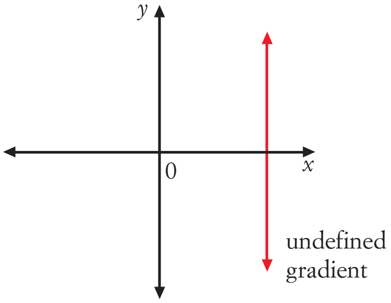


For a vertical line, the run is 0.

So, gradient $= \frac{\text{rise}}{0}$.

But division by zero is undefined.

A vertical line has an **undefined gradient**. Some people say that a vertical line has an infinite gradient, but this is imprecise.



Example 5

Worksheet
Gradient, midpoint and distance
MAT10NAWK00014

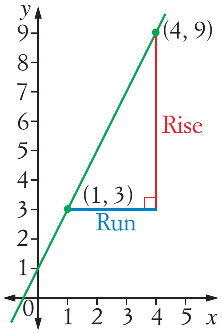
Work out the gradient of the line through (1, 3) and (4, 9).

Solution

Sketch a diagram.

Gradient $= \frac{\text{rise}}{\text{run}}$

So we need to work out the vertical and horizontal distances.



Calculate the rise. **Rise** $= 9 - 3 = 6$

Calculate the run. **Run** $= 4 - 1 = 3$

Write the rule for gradient, substitute for known values and evaluate. **Gradient** $= \frac{\text{rise}}{\text{run}} = \frac{6}{3} = 2$

State the result. **The gradient of the line is 2.**

You may also wish to calculate the gradient without sketching a diagram.

Example 6

Work out the gradient of the segment joining ($^-2$, 5) and (7, $^-4$).

Solution

Identify the points. **Let ($^-2$, 5) = (x_1, y_1) and (7, $^-4$) = (x_2, y_2).**

Write the rule for gradient $m = \frac{y_2 - y_1}{x_2 - x_1}$

Substitute for known values. $= \frac{^-4 - 5}{7 - {}^-2}$

Evaluate. $= \frac{^-9}{9} = {}^-1$

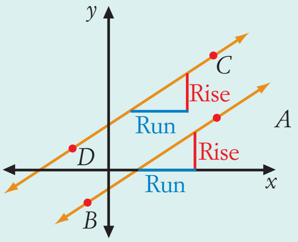
State the result. **The gradient of the segment is $^-1$.**

When parallel lines are graphed, they have the same slope.

Important!

Parallel lines

The gradients of parallel lines are equal.
If $AB \parallel CD$, $m_{AB} = m_{CD}$.



The rules for calculating lengths, midpoints and gradients can be useful in problem solving.

Example 7

Show that the following points are collinear: $A(1, 2)$, $B(^-2, 4)$, $C(^-5, 6)$

Solution

Collinear points lie on the same line. This means that the gradients of AB and BC will be equal.

Calculate the gradient of AB.	$m_{AB} = \frac{4-2}{^-2-1} = ^-\frac{2}{3}$
Calculate the gradient of BC.	$m_{BC} = \frac{6-4}{^-5-^-2} = ^-\frac{2}{3}$
State the result.	**$A(1, 2)$, $B(^-2, 4)$ and $C(^-5, 6)$ are collinear as $m_{AB} = m_{BC}$.**

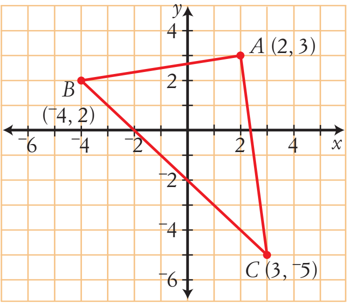
Example 8

Determine if the triangle whose vertices are $A(2, 3)$, $B(^-4, 2)$ and $C(3, ^-5)$ is scalene, isosceles or equilateral.

Solution

Sketch a diagram.

You need to compare the lengths of the sides of the triangle.



State the distance formula.	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Calculate the length of *AB* using $A(2, 3) = (x_1, y_1)$ and $B(^{-}4, 2) = (x_2, y_2)$.	$AB = \sqrt{(^{-}4 - 2)^2 + (2 - 3)^2} = \sqrt{37}$
Calculate the length of *BC* using $B(^{-}4, 2) = (x_1, y_1)$ and $C(3, ^{-}5) = (x_2, y_2)$.	$BC = \sqrt{(3 - ^{-}4)^2 + (^{-}5 - 2)^2} = \sqrt{98}$
Calculate the length of *AC* using $A(2, 3) = (x_1, y_1)$ and $C(3, ^{-}5) = (x_2, y_2)$.	$AC = \sqrt{(3 - 2)^2 + (^{-}5 - 3)^2} = \sqrt{65}$
All sides of $\triangle ABC$ are of different lengths.	**$\triangle ABC$ is scalene.**

Example 9

Show that the points (2, 3), (4, $^{-}1$), (1, $^{-}2$) and ($^{-}1$, 2) are the vertices of a parallelogram.

Solution

Draw a diagram first.

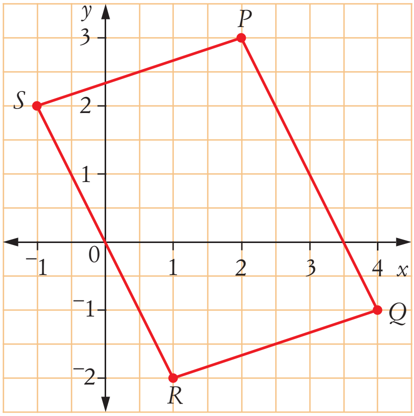
Label the vertices *P*, *Q*, *R*, *S* as shown here.

To show that *PQRS* is a parallelogram, we need to show that opposite sides are parallel.

So we need to show that

$PQ \parallel SR$ and $SP \parallel RQ$.

This means that we need to work out the gradient of each side.



Write the gradient formula.	$m = \dfrac{y_2 - y_1}{x_2 - x_1}$
Calculate m_{PQ}.	$m_{PQ} = \dfrac{^{-}1 - 3}{4 - 2} = {}^{-}2$
Calculate m_{SR}.	$m_{SR} = \dfrac{^{-}2 - 2}{1 - ^{-}1} = {}^{-}2$
State the result.	**$m_{PQ} = m_{SR}$, so $PQ \parallel SR$.**
Calculate m_{SP}.	$m_{SP} = \dfrac{3 - 2}{2 - ^{-}1} = \dfrac{1}{3}$
Calculate m_{RQ}.	$m_{RQ} = \dfrac{^{-}1 - ^{-}2}{4 - 1} = \dfrac{1}{3}$
State the result.	**$m_{SP} = m_{RQ}$, so $SP \parallel RQ$.**
Write the conclusion.	***PQRS* is a parallelogram because opposite sides are parallel.**

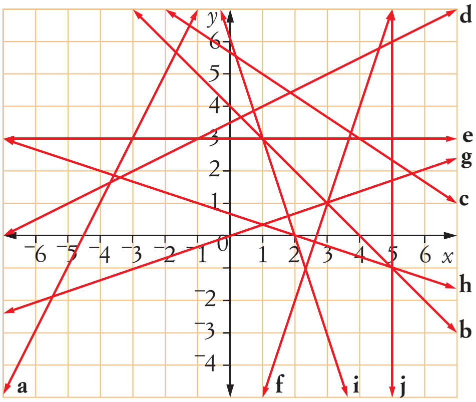
In the previous example we could also show that *PQRS* is a parallelogram by using the fact that the diagonals of a parallelogram bisect each other. We would do this by showing that the midpoints of *PR* and *SQ* are the same.

Exercise 5.1 Distances and midpoints in the plane

Understanding

Extra questions
Exercise 5.1
MAT10NAEQ00041

1 Classify the segments shown below as either rising, falling or neither.

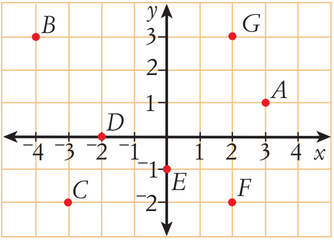


2 Without using a sketch, state the quadrant of the Cartesian plane that each of the following points lie in.

a $(1, 7)$ **b** $(^-4, 2)$ **c** $(^-9, ^-14)$ **d** $(^-9, 15)$
e $(^-4, ^-9)$ **f** $(12, ^-1)$ **g** $(^-4, 11)$ **h** $(3, 13)$

3 Use the diagram on the right to find the coordinates of the midpoint of the line segment:

a *GF* **b** *BG* **c** *CF* **d** *EF*
e *GE* **f** *BD* **g** *BC* **h** *BF*



Fluency

See Example 1

4 Find the distance between these points.

a $(3, 6)$ and $(5, 8)$ **b** $(^-2, 4)$ and $(4, 0)$
c $(7, ^-4)$ and $(^-5, 1)$ **d** $(8, 6)$ and $(^-7, 13)$
e $(^-3, ^-5)$ and $(6, 4)$ **f** $(6, 2)$ and $(^-3, ^-7)$
g $(4, 1)$ and $(^-2, ^-4)$ **h** $(^-3, 5)$ and $(^-1, ^-7)$
i $(6, 1)$ and $(3, ^-7)$ **j** $(5, ^-5)$ and $(^-3, 8)$

See Example 3

5 Find the midpoint of these points.

a $(3, ^-2)$, and $(7, 4)$ **b** $(1, 5)$ and $(^-3, ^-4)$
c $(4, 5)$ and $(^-4, 7)$ **d** $(3, ^-3)$ and $(^-2, 4)$
e $(^-3, 6)$ and $(^-7, 4)$ **f** $(^-3, 9)$ and $(4, ^-6)$
g $(^-7, 0)$ and $(0, 5)$ **h** $(0, 0)$ and $(9, ^-4)$
i $(^-3, ^-9)$ and $(^-6, ^-7)$ **j** $(4, 2)$ and $(^-3, ^-8)$

6 Find the gradient of each line shown in question **1**.

See Example 5

7 Find the slope of the line through each of these pairs of points.

a $(3, 5)$ and $(5, 11)$
b $(1, 6)$ and $(4, 0)$
c $(^-1, ^-3)$ and $(2, 3)$
d $(3, ^-6)$ and $(4, 1)$
e $(4, 2)$ and $(^-4, 1)$
f $(0, 3)$ and $(5, ^-7)$
g $(^-3, 4)$ and $(3, ^-2)$
h $(6, 6)$ and $(7, 7)$
i $(^-2, 3)$ and $(^-6, 3)$
j $(^-4, 0)$ and $(0, ^-3)$
k $(^-4, ^-5)$ and $(^-4, 6)$
l $(2, 4)$ and $(4, 1)$

Worked solutions
Exercise 5.1
MAT10NAWS00046

8 For the intervals joining each of these pairs of points find:

i the length (leave in surd form if necessary)
ii the midpoint
iii the gradient

See Examples 1, 3, 5

a $(1, 3)$ and $(6, 7)$
b $(4, 4)$ and $(7, 1)$
c $(^-3, 1)$ and $(^-5, 6)$
d $(8, 2)$ and $(^-5, 3)$
e $(^-7, 5)$ and $(0, 2)$
f $(1, 9)$ and $(^-8, 2)$
g $(9, ^-6)$ and $(^-5, 8)$
h $(11, 2)$ and $(6, ^-3)$
i $(^-5, ^-1)$ and $(^-4, ^-7)$
j $(^-3, ^-1)$ and $(^-8, 2)$
k $(^-5, 0)$ and $(0, ^-4)$
l $(^-6, 1)$ and $(6, ^-1)$

9 Determine whether or not the following sets of points are collinear.

See Example 7

a $A(5, 6)$, $B(^-4, ^-6)$ and $C(^-1, ^-2)$
b $X(^-1, 1)$, $Y(^-3, ^-3)$ and $Z(^-6, ^-9)$
c $P(^-5, ^-1)$, $Q(10, 4)$ and $R(^-2, 0)$
d $J(12, 9)$, $K(0, ^-1)$ and $L(^-6, ^-6)$
e $W(3, 1)$, $R(0, ^-2)$ and $S(^-3, ^-5)$
f $F(^-2, 6)$, $G(2, 0)$ and $H(4, ^-3)$

Problem solving

10 Find a given that:

See Example 2

a $P(a, a)$ is $\sqrt{32}$ from the origin.
b $M(10, 4)$ and $N(7, a)$ are 5 units apart.
c $R(a, 3)$ and $T(2, ^-1)$ are $\sqrt{20}$ units apart.
d $A(a, ^-2)$ is equidistant from $B(^-2, ^-5)$ and $C(^-5, ^-4)$.
e $D(^-2, 0)$ and $E(a, ^-3)$ are $\sqrt{18}$ units apart.

11 Find b given that the line joining:

a $D(^-2, ^-1)$ and $F(3, b)$ is parallel to a line with gradient $\frac{4}{5}$
b $Q(^-4, b)$ and $T(b, 3)$ is parallel to a line with gradient $\frac{1}{6}$.

12 The midpoint of AB is $(^-5, 3)$. If A is $(2, 9)$, find B.

Worked solutions
Exercise 5.1
MAT10NAWS00046

13 The midpoint of PQ is $(8, 7)$. If P is $(5, 10)$, find Q.

14 The midpoint of RT is $(^-4, ^-5)$. If T is $(^-10, 3)$, find R.

See Example 4

15 A pentagon $ABCDE$ has vertices $(3, 4)$, $(7, ^-1)$, $(8, ^-4)$, $(3, ^-2)$ and $(^-3, 5)$. Find the lengths of its sides.

16 A quadrilateral has vertices $(1, ^-1)$, $(2, 2)$, $(5, 2)$ and $(2, ^-3)$. Find the lengths of its sides and the lengths of its diagonals.

17 Name the shape formed by points $(^-1, 1)$, $(3, 3)$, $(1, ^-1)$ and $(^-3, ^-3)$.

See Example 8

18 Show that points $(^-2, ^-3)$, $(2, 4)$ and $(5, 1)$ are the vertices of an isosceles triangle.

19 A triangle has vertices $(^-2, 3)$, $(1, 1)$ and $(^-1, ^-2)$.

a Find the lengths of its sides.
b What type of triangle is it?

Worked solutions
Exercise 5.1
MAT10NAWS00046

20 Points $M(3, 4)$, $N(5, {}^{-}2)$, $O(4, {}^{-}5)$ and $P({}^{-}2, 1)$ are the vertices of a quadrilateral.

- **a** Find the midpoints of the sides of *MNOP*.
- **b** What type of quadrilateral is formed when the midpoints of *MNOP* are joined? Give reasons for your answer.

21 A quadrilateral has vertices $G({}^{-}1, 1)$, $H(2, 3)$, $I(3, {}^{-}1)$ and $J(0, {}^{-}3)$.

- **a** Find the midpoints of the diagonals.
- **b** What type of quadrilateral is *GHIJ*?

See Example 9

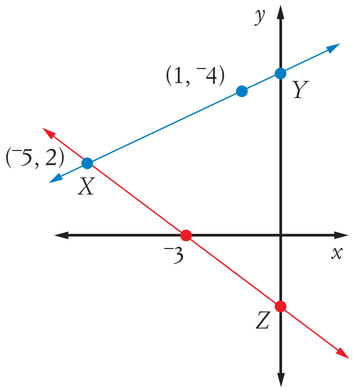
22 A quadrilateral has vertices $(3, 1)$, $(1, {}^{-}3)$, $({}^{-}3, {}^{-}5)$ and $({}^{-}1, {}^{-}1)$.

- **a** Find the lengths of its sides.
- **b** What type of quadrilateral is it?

23 Show that points $({}^{-}2, 0)$, $(2, 2)$, $(3, {}^{-}1)$ and $({}^{-}1, {}^{-}3)$ are the vertices of a parallelogram..

Reasoning

24 Calculate the area of $\triangle XYZ$ shown on the right.



25 The diagonals of a parallelogram bisect each other. *ABCD* is a parallelogram where $A = ({}^{-}1, {}^{-}1)$, $B = (3, 0)$ and $C = (1, {}^{-}3)$ and *AC* is a diagonal. Use midpoints to find point *D*

26 The vertices of a quadrilateral are $D(1, 5)$, $E(3, {}^{-}2)$, $F({}^{-}2, {}^{-}5)$ and $G({}^{-}4, 2)$. What type of quadrilateral is *DEFG*? (Give reasons for your answer.)

27 Points $({}^{-}1, 0)$, $(0, {}^{-}2)$ and $(4, 1)$ are three vertices of a parallelogram. Use midpoints to find the three possible positions of the fourth vertex.

28 The vertices of a quadrilateral are $P(3, {}^{-}4)$, $Q({}^{-}3, {}^{-}2)$, $R({}^{-}7, 4)$ and $S(5, 6)$. What type of quadrilateral is formed when the midpoints of the sides are joined? Explain your answer.

29 $A(3, 4)$, $B(1, {}^{-}3)$ and $C({}^{-}2, 1)$ are the vertices of a triangle.

- **a** Find the midpoints *D* and *E* of *AB* and *AC* respectively.
- **b** Show that $DE \parallel BC$.
- **c** Show that $DE = \frac{1}{2}BC$.

5.2 Straight-line graphs

Linear functions have graphs that are straight lines. The equation of a linear function is called a **linear equation**. When you draw the graph of a linear equation, you really only need two points, but it is a good idea to plot a third one as a check.

Important!

Linear equations

The equation of a straight line (a **linear equation**) in the Cartesian system may have an x term, a y term and a number.
The following are all examples of linear equations.

$$2x - 3y + 7 = 0 \qquad y = \frac{2x}{3} - 5 \qquad 4y = 9 \qquad 3x = 11 - 5y \qquad x = \frac{3}{7}$$

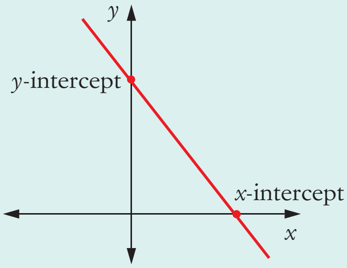
Equations involving higher powers of variables (e.g. x^2, y^3) or roots of variables (e.g. $\sqrt{x}$, $\sqrt[3]{y}$) are not linear equations, nor are equations with variables forming part of the denominator of a fraction $\left(\text{e.g.} \frac{5}{2x}, \frac{7}{y}\right)$.

The graph of a linear function will always cut at least one of the Cartesian axes.

Important!

Intercepts

The **intercepts** of a straight line are the points where it crosses the axes of the Cartesian plane.
The x-**intercept** is where the line crosses the x-axis and the y-**intercept** is where it crosses the y-axis.



The term 'intercept' is also used to refer to the distance from the origin to the intersection of the line and the axis. The x-intercept is the value of x when $y = 0$. The y-intercept is the value of y when $x = 0$.

CAS TI-Nspire exercise
Coordinate geometry
MAT10NATI00014

CAS ClassPad exercise
Coordinate geometry
MAT10NACP00014

Example 10

Draw $y = 3x - 4$ for $^-2 \le x \le 5$ and use the graph to find the intercepts.

Solution

Since $y = 3x - 4$ is a linear equation, we will find three points.

Construct a table of values for x and y using any three x values from $^-2$ to 5.

$y = 3x - 4$

x	$^-2$	1	5
y			

Now calculate the corresponding y values.

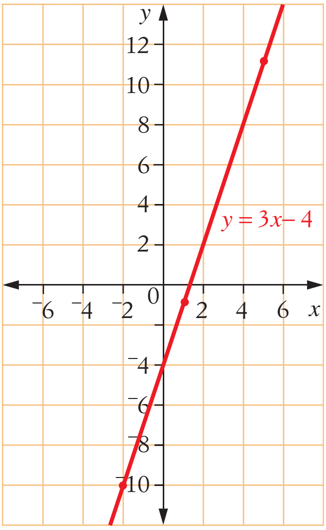
x	$y = 3x - 4$	(x, y)
$^-2$	$3 \times {^-2} - 4 = {^-6} - 4 = {^-10}$	$(^-2, {^-10})$
1	$3 \times 1 - 4 = 3 - 4 = {^-1}$	$(1, {^-1})$
5	$3 \times 5 - 4 = 15 - 4 = 11$	$(5, 11)$

x	$^-2$	1	5
y	$^-10$	$^-1$	11

Plot the points and draw the straight line.

Find where the line cuts the axes.

You need to measure the x-intercept.



State the result.

The x-intercept is about (1.3, 0).

The y-intercept is (0, $^-4$).

In the previous example, the exact value of the x-intercept can be calculated as follows.

For the x-intercept, $y = 0$. $\qquad 0 = 3x - 4$

Reverse the equation. $\qquad 3x - 4 = 0$

Add 4 to both sides. $\qquad 3x = 4$

Divide both sides by 3. $\qquad x = \frac{4}{3} = 1\frac{1}{3}$

So, the x-intercept is $\left(1\frac{1}{3}, 0\right)$.

It is usually easier to rearrange an equation into the form ‘$y = \ldots$’ before you calculate the y values that correspond to given x values.

Example 11

Find the intercepts of $2x + 3y - 8 = 0$.

Solution

Write the equation.	$\mathbf{2x + 3y - 8 = 0}$
Let $y = 0$ to find the x-intercept.	$\mathbf{2x + 3 \times 0 - 8 = 0}$
Solve for x.	$\mathbf{2x = 8}$ $\mathbf{x = 4}$
State the result.	**The x-intercept is (4, 0).**
Let $x = 0$ to find the y-intercept.	$\mathbf{2 \times 0 + 3y - 8 = 0}$
Solve for y.	$\mathbf{3y = 8}$ $\mathbf{y = \frac{8}{3} = 2\frac{2}{3}}$
State the result.	**The y-intercept is $\left(0, 2\frac{2}{3}\right)$.**

To work out whether a point is on a line or not, we can graph the equation of the line and check. However, we won't always have time to draw a graph, so we need a faster way. We can substitute the x- and y-coordinates of the point into the equation. If they work (**satisfy** the equation), the point is on the line. If they don't satisfy the equation, the point is not on the line.

Example 12

Is $A(^-1, 2)$ on the line $3x - 2y + 7 = 0$?

Solution

Write the equation.	$\mathbf{3x - 2y + 7 = 0}$
Substitute the coordinates of $A(^-1, 2)$ into the LHS of the equation and evaluate.	$\mathbf{3 \times {}^-1 - 2 \times 2 + 7}$ $\mathbf{= {}^-3 - 4 + 7}$ $\mathbf{= {}^-7 + 7}$ $\mathbf{= 0}$
This is the same as the RHS (0), so the coordinates satisfy the equation.	**$A(^-1, 2)$ is on line $3x - 2y + 7 = 0$.**

An equation has an 'equals' sign. You have seen that when we plot the graph of an equation, we get a line that shows all the points that satisfy the equation.

Important!

Inequations

An **inequation** has a <, >, ≤ or ≥ symbol instead of an equals sign. When we try to show the points that satisfy the inequation, we get an entire area of the plane instead of just a line. This is usually shown by shading on one side of the boundary line—the side that satisfies the inequation.

For inequations involving ≤ or >, a *solid line* is drawn to show that the points on the line *are included*.

For inequations involving < or >, a *dashed line* is drawn to show that the points on the line *are not included*.

Example 13

Worksheet
Matching inequations with their graphs
MAT10NAWK00017

CAS TI-Nspire exercise
Coordinate geometry
MAT10NATI00014

CAS ClassPad exercise
Coordinate geometry
MAT10NACP00014

Draw the graph of $y \leq x - 2$.

Solution

First you need to draw the graph of $y = x - 2$.

Use a solid line because we want to include the points on the line.

Now choose a point either above or below the line you have drawn, say $(2, -1)$.

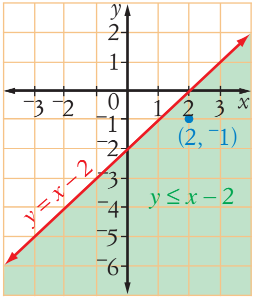
Substitute the x and y values into the inequation.

When $x = 2$ and $y = -1$:

$$y \leq x - 2 \Rightarrow -1 \leq 2 - 2$$
$$-1 \leq 0$$

This is true, so the chosen point lies in the region $y \leq x - 2$.

Shade in that region.



Investigate: Equations of lines

Technology worksheet
Excel worksheet: Drawing linear graphs
MAT10NACT00009

TLF Learning object
Grapher (L3531)
MAT10NAIN00005

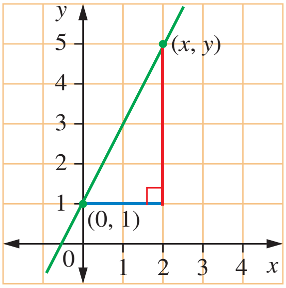
1 Draw graphs of the lines represented by the following equations.

a $y = 2x + 5$
b $y = -x - 3$
c $y = {}^{-}3x + 4$
d $y = 3x - 1$
e $y = \frac{1}{2}x - 3$
f $y = {}^{-}\frac{2}{3}x + 1$

2 Work out the gradient and y-intercept for each.
3 Compare the values of the gradient and y-intercept with the equations for the lines.
4 What do you notice? Does this always work? Can you explain why?

Suppose that a straight line has a slope of 2 and y-intercept (0, 1), so $c = 1$. We can draw the graph of the line as shown below with the point (x, y) somewhere on the line.

Let $(x_1, y_1) = (0, 1)$ and $(x_2, y_2) = (x, y)$.



Here $$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$2 = \frac{y - 1}{x - 0}$$

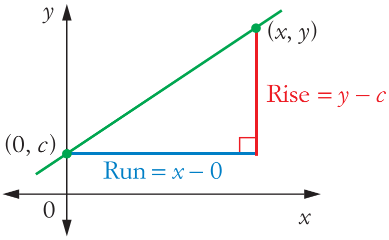
So $$\frac{y - 1}{x} = 2$$

$$y - 1 = 2x$$

$$y = 2x + 1$$

In the above investigation, you should have seen that the equation of a straight line can be written using its gradient and y-intercept. The gradient (slope) is the coefficient of the x term and the y-intercept is the constant term. We usually use m as the symbol for gradient and c as the symbol for the y-intercept. This can be shown to be true in all cases as follows.

In general, suppose that a straight line has slope m and y-intercept $(0, c)$. We can draw a graph of the general case using the point (x, y) and the other information as shown below.



Here $$m = \frac{\text{rise}}{\text{run}} = \frac{y - c}{x - 0}$$

So $$\frac{y - c}{x} = m$$

$$y - c = mx$$

$$y = mx + c$$

This is called the **gradient–intercept form** of the equation of a line.

Another way of writing the equation of a line, called the **standard form**, is with no fractions and all terms on one side. The standard form of the equation $y = 2x + 1$ is $2x - y + 1 = 0$.

Important!

Forms of the equation of a line

The **gradient–intercept** (or **slope–intercept**) **form** of the equation of a straight line is

$$y = mx + c$$

where m is the gradient and c is the y-intercept.

The **standard** (or **general**) **form** of the equation of a straight line is

$$ax + by + c = 0$$

where a, b and c are constants (numbers).

Puzzle sheet
Equations in gradient form
MAT10NAPS00011

Example 14

Write the standard form equation of the line with:

a gradient 7 and y-intercept $^{-}2$

b gradient $^{-}\frac{3}{4}$ and y-intercept (0, 5).

Solution

a Write the equation in gradient–intercept form. $\boldsymbol{y = mx + c}$

We know that $m = 7$ and $c = {}^{-}2$. $\boldsymbol{y = 7x + {}^{-}2}$

Simplify. $\boldsymbol{y = 7x - 2}$

To write the equation in standard form, subtract $7x$ from and add 2 to both sides. $\boldsymbol{y - 7x + 2 = 0}$

b Write the equation in gradient–intercept form. $\boldsymbol{y = mx + c}$

We know that $m = {}^{-}\frac{3}{4}$ and $c = 5$. $\boldsymbol{y = {}^{-}\frac{3}{4}x + 5}$

Multiply all terms by 4 (the denominator of the x term). $\boldsymbol{y \times 4 = {}^{-}\frac{3}{4}x \times 4 + 5 \times 4}$

Simplify. $\boldsymbol{4y = {}^{-}3x + 20}$

To write the equation in standard form, add $3x$ to and subtract 20 from both sides. $\boldsymbol{4y + 3x - 20 = 0}$

Video tutorial
The gradient–intercept formula
MAT10NAVT10011

CAS TI-Nspire exercise
Coordinate geometry
MAT10NATI00014

CAS ClassPad exercise
Algebra
MAT10NACP00014

Example 15

What are the gradient and the y-intercept of each of these lines?

a $y = {}^{-}4x + 9$

b $3x + 2y - 6 = 0$

Solution

a Write the equation. $\boldsymbol{y = {}^{-}4x + 9}$

Compare with the gradient–intercept form. $\boldsymbol{y = mx + c}$

You can see that $m = {}^{-}4$ and $c = 9$. **Gradient = $^{-}$4 and y-intercept = (0, 9).**

b We first need to write the equation in gradient–intercept form.

Write the equation. $\boldsymbol{3x + 2y - 6 = 0}$

Subtract $3x$ from and add 6 to both sides. $\boldsymbol{2y = 6 - 3x}$

Divide throughout by 2 (the coefficient of y). $\boldsymbol{\frac{2y}{2} = \frac{6}{2} - \frac{3x}{2}}$

Simplify. $\boldsymbol{y = 3 - \frac{3}{2}x}$

Write the equation with the x term first. $\boldsymbol{y = {}^{-}\frac{3}{2}x + 3}$

Compare with the gradient–intercept form. $\boldsymbol{y = mx + c}$

You can see that $m = {}^{-}\frac{3}{2}$ and $c = 3$. **Gradient = $^{-}\frac{3}{2}$ and y-intercept = (0, 3).**

We can also work out the equation of a line using the gradient and one point on the line. Suppose that a straight line has slope $m = {}^{-}3$ and passes through the point $({}^{-}2, 4)$. The equation uses (x, y) for points on the line.

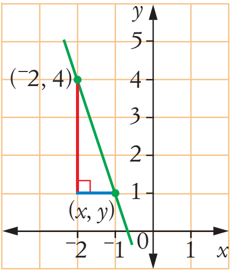
Let $(x_1, y_1) = ({}^{-}2, 4)$ and $(x_2, y_2) = (x, y)$.

Here $$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$^{-}3 = \frac{y - 4}{x - {}^{-}2}$$

So $$\frac{y - 4}{x - {}^{-}2} = {}^{-}3$$

$$y - 4 = {}^{-}3(x - {}^{-}2)$$



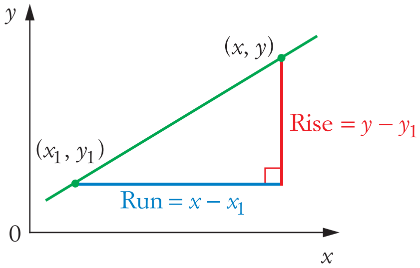
In standard form, this equation is written as $3x + y + 2 = 0$.

In general, suppose that a straight line has slope m and passes through the point (x_1, y_1). The equation uses (x, y) for points on the line.

Here $$m = \frac{\text{rise}}{\text{run}} = \frac{y - y_1}{x - x_1}$$

So $$\frac{y - y_1}{x - x_1} = m$$

$$y - y_1 = m(x - x_1)$$



This is called the **gradient–point form** of the equation of a line.

Important!

Gradient–point form of the equation of a line

The **gradient–point form** of the equation of a line is given by

$$y - y_1 = m(x - x_1)$$

where m is the gradient and (x_1, y_1) is a point on the line.

Example 16

Work out the standard form equation of a straight line with slope $\frac{2}{3}$ that passes through the point $({}^{-}2, 1)$.

Solution

Write the gradient–point equation of a line.	$\boldsymbol{y - y_1 = m(x - x_1)}$
Substitute using $(x_1, y_1) = ({}^{-}2, 1)$ and $m = \frac{2}{3}$.	$\boldsymbol{y - 1 = \frac{2}{3}(x - {}^{-}2)}$
Simplify.	$\boldsymbol{y - 1 = \frac{2}{3}(x + 2)}$
Multiply throughout by 3.	$\boldsymbol{3(y - 1) = 2(x + 2)}$
Expand both sides.	$\boldsymbol{3y - 3 = 2x + 4}$
	$\boldsymbol{3y - 2x - 7 = 0}$
Rearrange into standard form.	$\boldsymbol{2x - 3y + 7 = 0}$

Video tutorial
The gradient–point formula
MAT10NAVT10012

It is also possible to calculate the equation of a line if you know two points through which it passes.

CAS TI-Nspire exercise
Coordinate geometry
MAT10NATI00014

CAS ClassPad exercise
Coordinate geometry
MAT10NACP00014

Example 17

Find the equation of a line passing through (1, 3) and $(4, {}^{-}3)$.

Solution

Identify the coordinates.	**Let** $(x_1, y_1) = (1, 3)$ **and** $(x_2, y_2) = (4, {}^{-}3)$.
First find the gradient.	$m = \dfrac{y_2 - y_1}{x_2 - x_1}$ $= \dfrac{{}^{-}3 - 3}{4 - 1} = \dfrac{{}^{-}6}{3} = {}^{-}2$
Now you know the gradient and a point through which the line passes.	
Write the gradient–point formula.	$y - y_1 = m(x - x_1)$
Substitute using $(x_1, y_1) = (1, 3)$ and $m = {}^{-}2$.	$y - 3 = {}^{-}2(x - 1)$
Expand.	$y - 3 = {}^{-}2x + 2$
Rearrange into standard form.	$2x + y - 5 = 0$

In Example **17**, you could have used point $(4, {}^{-}3)$ instead of (1, 3) to obtain the equation of the line.

Parallel and perpendicular lines

Technology: GeoGebra
Perpendicular lines
MAT10NATC00005

The gradients of parallel and perpendicular lines are closely related.

Suppose two lines *AB* and *DE* are parallel. Then the angles they make with the *x*-axis are the same, as shown in this diagram.

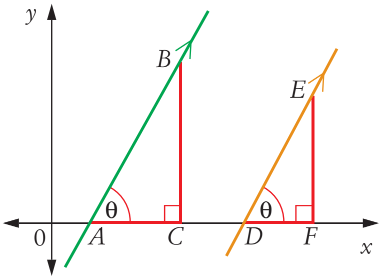
If we draw in the extra lines *BC* and *EF*, we can work out the gradients.

$$\triangle ABC \;|||\; \triangle DEF$$

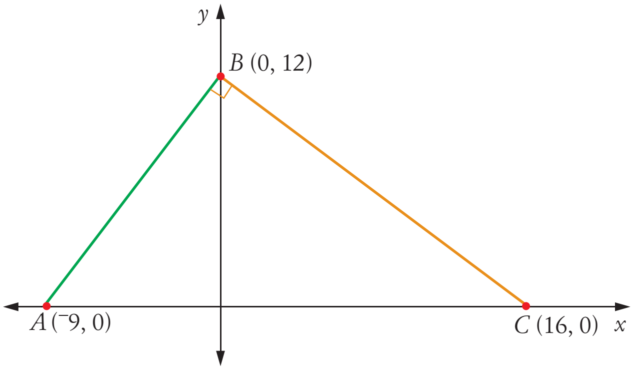
So $$\frac{BC}{AC} = \frac{EF}{DF}$$

But $$m_{AB} = \frac{BC}{AC} \text{ and } m_{DE} = \frac{EF}{DF}$$

So $$m_{AB} = m_{DE}$$



Suppose two lines AB and BC are perpendicular. Then they make a right angle at the intersection B, as shown here.



First calculate the gradient of AB.

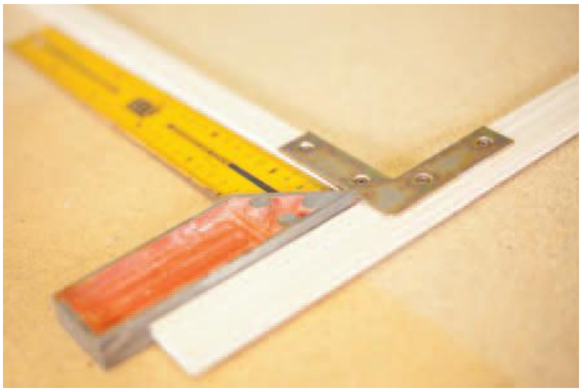
$$\begin{aligned} m_{AB} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{12 - 0}{0 - {}^{-}9} \\ &= \tfrac{4}{3} \end{aligned}$$

Now calculate the gradient of BC.

$$\begin{aligned} m_{BC} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{0 - 12}{16 - 0} \\ &= {}^{-}\tfrac{3}{4} \end{aligned}$$

So $m_{AB} \times m_{BC} = \frac{4}{3} \times {}^{-}\frac{3}{4} = {}^{-}1$

It is possible to show that this result applies to all pairs of lines that are perpendicular.



Important!

Parallel and perpendicular lines

The gradients m_1 and m_2 of **parallel lines** are the same.

For parallel lines: $m_1 = m_2$

The gradients m_1 and m_2 of **perpendicular lines** have a product of $^{-}1$. They are negative reciprocals.

For perpendicular lines: $m_1 \times m_2 = {}^{-}1$

Teacher notes

Perpendicular lines

MAT10NATN00002

Example 18

Worksheet

Parallel and perpendicular lines

MAT10NAWK00015

Show that $2x - y + 8 = 0$ and $4x = 2y + 5$ are parallel.

Solution

If we can show that the gradients are the same, the lines must be parallel. Therefore, we need to find the gradient of each line. This can be done by writing the equations of the lines in gradient–intercept form.

Write the equation of the first line.	$\mathbf{2x - y + 8 = 0}$
Rearrange into the form $y = mx + c$.	$\mathbf{y = 2x + 8}$
Compare with $y = mx + c$ to find the gradient.	**Gradient of $2x - y + 8 = 0$ is 2.**
Write the equation of the second line.	$\mathbf{4x = 2y + 5}$
Rearrange into the form $y = mx + c$.	$\mathbf{y = 2x - 2\frac{1}{2}}$
Compare with $y = mx + c$ to find the gradient.	**Gradient of $4x = 2y + 5$ is 2.**
The gradients of the two lines are equal.	**The lines are parallel.**

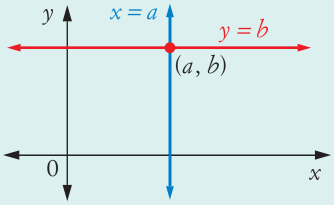
It is not possible to calculate the gradient of a line parallel to the y-axis. Lines parallel to the x-axis have a slope of 0. The equations of these lines can be written as long as you know any point on the lines.

Important!

Lines parallel to the axes

The horizontal line parallel to the x-axis through (a, b) is $y = b$.

The vertical line parallel to the y-axis through (a, b) is $x = a$.



Example 19

Write the equation of:

a the vertical line through $(^-4, 7)$

b the horizontal line through $(^-2, 6)$.

Solution

a The y value can be anything but the x value must be $^-4$. Write the equation. **The equation is $x = {}^-4$.**

b The x value can be anything but the y value must be 6. Write the equation. **The equation is $y = 6$.**

Example 20

Find the equation of the line through $(^-2, 5)$ that is:

a parallel to $2x + 3y - 4 = 0$ **b** perpendicular to $2x + 3y - 4 = 0$.

Solution

a Write the equation.	$\mathbf{2x + 3y - 4 = 0}$
Rearrange into gradient–intercept form.	$\mathbf{y = {^-}\tfrac{2}{3}x + 1\tfrac{1}{3}}$
Compare with $y = mx + c$.	$\mathbf{m = {^-}\tfrac{2}{3}}$
A parallel line will have the same gradient, m.	
Write the gradient–point equation of the line.	$\mathbf{y - y_1 = m(x - x_1)}$
We know that $m = {^-}\tfrac{2}{3}$ and $(x_1, y_1) = (^-2, 5)$.	$\mathbf{y - 5 = {^-}\tfrac{2}{3}(x - {^-}2)}$
Multiply throughout by 3.	$\mathbf{3(y - 5) = {^-}2(x + 2)}$
Expand both sides.	$\mathbf{3y - 15 = {^-}2x - 4}$
Rearrange into standard form.	$\mathbf{2x + 3y - 11 = 0}$
b Write the rule for perpendicular lines.	$\mathbf{m_1 \times m_2 = {^-}1}$
Substitute the gradient of $2x + 3y - 4 = 0$.	$\mathbf{{^-}\tfrac{2}{3} \times m_2 = {^-}1}$
Multiply both sides by ${^-}\tfrac{3}{2}$.	$\mathbf{m_2 = \tfrac{3}{2}}$
A line perpendicular to $2x + 3y - 4 = 0$ will have the gradient m_2.	
Write the gradient–point equation of the line.	$\mathbf{y - y_1 = m(x - x_1)}$
We know $m = \tfrac{3}{2}$ and $(x_1, y_1) = (^-2, 5)$.	$\mathbf{y - 5 = \tfrac{3}{2}(x - {^-}2)}$
Multiply throughout by 2.	$\mathbf{2(y - 5) = 3(x + 2)}$
Expand both sides.	$\mathbf{2y - 10 = 3x + 6}$
Rearrange into standard form.	$\mathbf{3x - 2y + 16 = 0}$

Puzzle sheet

Gradients of parallel and perpendicular lines

MAT10NAPS00012

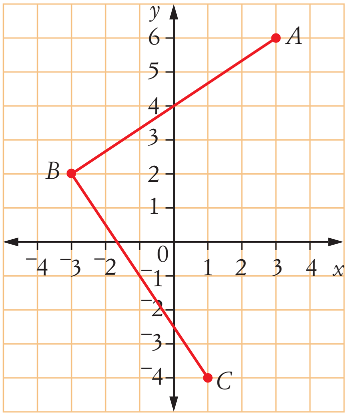
Example 21

Plot the points $A(3, 6)$, $B(^-3, 2)$ and $C(1, ^-4)$ and show that $\angle ABC = 90°$.

Solution

Plot the points.

Join AB and BC.



Write the rule for gradient.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

For AB, let $(x_1, y_1) = (3, 6)$ and $(x_2, y_2) = (^-3, 2)$.

$$m_{AB} = \frac{2 - 6}{^-3 - 3} = \frac{^-4}{^-6} = \frac{2}{3}$$

For BC, let $(x_1, y_1) = (^-3, 2)$ and $(x_2, y_2) = (1, ^-4)$.

$$m_{BC} = \frac{^-4 - 2}{1 - ^-3} = \frac{^-6}{4} = ^-\frac{3}{2}$$

Find the product of the gradients.

$$\begin{aligned} m_{AB} \times m_{BC} &= \frac{2}{3} \times ^-\frac{3}{2} \\ &= ^-1 \end{aligned}$$

So $m_1 \times m_2 = ^-1$.

AB and BC are perpendicular, so $\angle ABC = 90°$.

The work you have done on gradients, distance and midpoints can be used to investigate more complicated problems.

You will usually find it very helpful to draw a diagram first.

Example 22

Points $K(10, {}^{-}3)$, $L(8, 8)$ and $M(4, 5)$ are plotted and joined. Show that $\triangle KLM$ is right-angled and find its area.

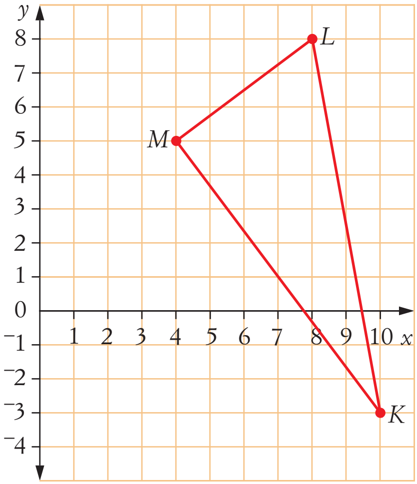
Solution

Draw a diagram first.

It looks as though there is a right angle at *M*.

We need to show that *ML* and *MK* are perpendicular.

This means that we need to find the gradients of *ML* (m_{ML}) and *MK* (m_{MK}) and to calculate the value of $m_{ML} \times m_{MK}$.



Write the gradient formula. $\boldsymbol{m = \dfrac{y_2 - y_1}{x_2 - x_1}}$

Calculate m_{ML}. $\boldsymbol{m_{ML} = \dfrac{8 - 5}{8 - 4} = \dfrac{3}{4}}$

Calculate m_{MK}. $\boldsymbol{m_{MK} = \dfrac{{}^{-}3 - 5}{10 - 4} = \dfrac{{}^{-}8}{6} = \dfrac{{}^{-}4}{3}}$

Find the product of m_{ML} and m_{MK}. $\boldsymbol{m_{ML} \times m_{MK} = \dfrac{3}{4} \times \dfrac{{}^{-}4}{3} = {}^{-}1}$

State the result. ***ML* is perpendicular to *MK*, so △*KLM* is right-angled.**

To find the area of △*KLM* we need to use the distance formula to find the lengths of *ML* and *MK*.

Write the distance formula. $\boldsymbol{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$

Calculate the length of *ML*.

$$\boldsymbol{\begin{aligned} ML &= \sqrt{(8 - 4)^2 + (8 - 5)^2} \\ &= \sqrt{16 + 9} \\ &= 5 \text{ units} \end{aligned}}$$

Calculate the length of *MK*.

$$\boldsymbol{\begin{aligned} MK &= \sqrt{(10 - 4)^2 + ({}^{-}3 - 5)^2} \\ &= \sqrt{36 + 64} \\ &= 10 \text{ units} \end{aligned}}$$

Write the rule for the area of a triangle. $\textbf{Area} = \frac{1}{2}bh$

Substitute the values of *ML* and *MK*. $= \frac{1}{2} \times 5 \times 10$

$= 25$ **square units**

State the result. **The area of △*KLM* is 25 square units.**

We can **model** real patterns and relationships using algebraic equations, tables of values or graphs. Often, models will provide only approximations of real-life situations. However, this will usually be close enough for most practical purposes. Relationships that can be represented by functions of the form $y = mx + c$ are called **linear models**.

Example 23

The height, h cm, of a seedling is given by $h = 2d + 10$, where d is the number of days since it was planted.

a Complete a table of values for the function for $0 \leq d \leq 8$.
b Use the table to draw a graph of the height function.
c What type of function is the height function?
d Compare the height function with the gradient–intercept form of a linear function ($y = mx + c$) and explain what the values 2 and 10 represent.

Solution

a Draw up the table of values for d and h using any three values of d from 0 to 8.

$h = 2d + 10$

d	1	5	8
h			

Calculate the corresponding values of h.

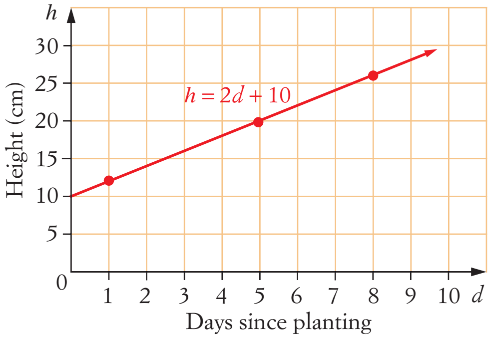
d	$h = 2d + 10$	**(d, h)**
1	$2 \times 1 + 10 = 12$	(1, 12)
5	$2 \times 5 + 10 = 20$	(5, 20)
8	$2 \times 8 + 10 = 26$	(8, 26)

d	1	5	8
h	12	20	26

9780170194860

b Plot the points and join them. Label the axes and add a title for the graph.

Seedling growth



c The graph is a straight line. **The function $h = 2d + 10$ is linear.**

d Write the height function. **$h = 2d + 10$**

Write the gradient–intercept form. **$y = mx + c$**

Compare the variables. **$y \leftrightarrow h$ and $x \leftrightarrow d$**

For the height function, the height (h) depends on the number of days after planting (d), so d is the independent variable and h is the dependent variable. **The horizontal axis shows the independent variable and the vertical axis shows the dependent variable.**

Compare the x-coefficients and the constant terms. **In the height function, 2 is the gradient of the graph and 10 is the intercept on the vertical (h) axis, i.e. the height when the seedling was planted ($d = 0$).**

In the previous example, the gradient of the graph for the height function was positive. This means that the value of h (height) increases as d (the number of days) increases.

Example 24

A small business buys \$20 000 worth of computer equipment. For tax purposes, the value, \$$V$, of the equipment decreases (depreciates) over time according to the equation:

$$V = 20\ 000 - 4000t$$

where t is the number of years since purchase.

a Complete a table of values for the function for $0 \leq t \leq 6$.

b Use the table to draw a graph of the value function.

c When will the computer equipment be worthless for tax purposes?

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Solution

a Draw up the table of values for t and V using any three values of t from 0 to 6.

$V = 20\,000 - 4000t$

t	1	3	6
V			

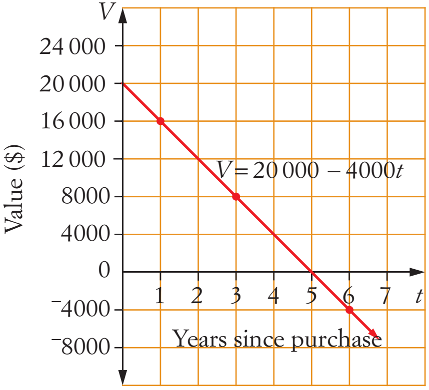
Calculate the corresponding values of V.

t	$V = 20\,000 - 4000t$	(t, V)
1	$20\,000 - 4000 \times 1$ $= 16\,000$	(1, 16 000)
3	$20\,000 - 4000 \times 3$ $= 8000$	(3, 8000)
6	$20\,000 - 4000 \times 6$ $= {}^{-}4000$	(6, ⁻4000)

t	1	3	6
V	16 000	8000	⁻4000

b Plot the points and join them. Label the axes and add a title for the graph.

Computer equipment depreciation



c Find the value of t when $V = 0$.

State the result.

$t = 5$ years

The equipment will be worthless for tax purposes after 5 years.

In real-life situations, observed data usually does not exactly fit the linear model $y = mx + c$. When real data is graphed, we may need to draw a **line of best fit** for the plotted points in order to see the relationship.

Important!

Line of best fit

The line of best fit for data points:

- represents most of the points as closely as possible
- should have approximately the same number of points above the line as below
- is drawn so that the perpendicular distances between the points and the line are as small as possible.

The line of best fit may be used to make predictions of values of variables. **Interpolation** is the prediction of values within the range of data. **Extrapolation** is the prediction of values outside the range of data.

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Example 25

Technology worksheet

Excel worksheet: Line of best fit

MAT10NACT00003

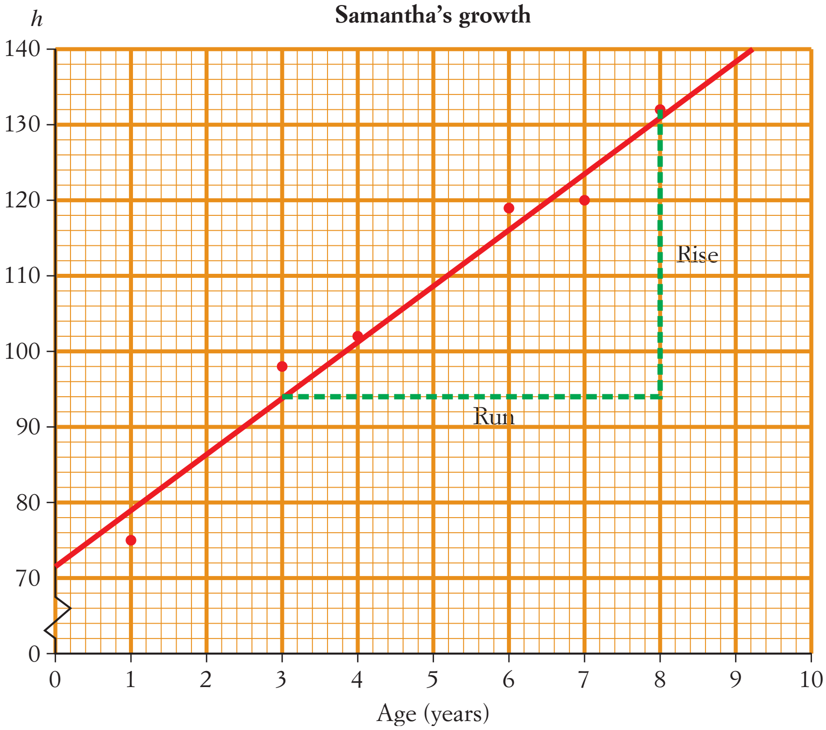
Samantha's height was measured over a period of 8 years. The table below shows the data.

Age, *a* (years)	1	3	4	6	7	8
Height, *h* (cm)	75	98	102	119	120	131

a Plot the data and draw in a line of best fit.
b Use the graph to estimate Samantha's height when she was born.
c Find the equation of the line of best fit.
d Use the equation of the line of best fit to estimate Samantha's height when she turns 12.
e Use the equation to estimate Samantha's height when she turns 30 and comment on the result.

Solution

a Draw up a set of axes and plot the points. Label the axes and add a title for the graph. Use a transparent ruler to draw in the line of best fit.



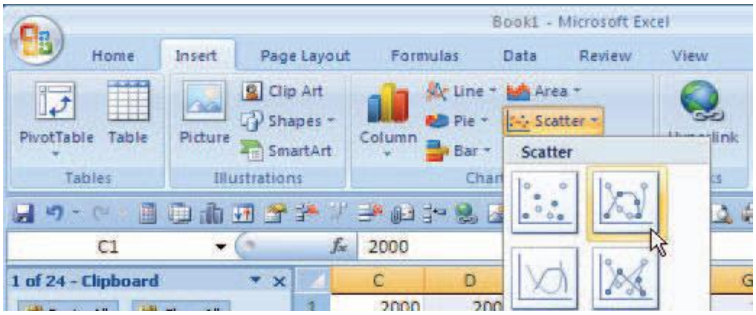
b	Find the point where the line of best fit cuts the vertical axis. Use the scale to calculate the value of h when $a = 0$.	$h \approx 71$ **cm when** $a = 0$
	Write the result.	**Samantha's height was about 71 cm when she was born.**
c	This is a linear function, so it can be written in the form $y = mx + c$, where m is the gradient and c is the intercept on the vertical axis.	
	For this height function, the independent variable is age (a) and the dependent variable is height (h).	$y \leftrightarrow h$ **and** $x \leftrightarrow a$
	We can read the intercept on the vertical (h) axis from the graph.	**Vertical axis intercept = (0, 71)**
	We need to find two points on the line of best fit to calculate its gradient. Locate two points on the line.	**(3, 94) and (8, 131) are on the line.**
	Write the rule for the gradient.	$\text{gradient} = \dfrac{\text{rise}}{\text{run}}$
	Substitute using the chosen points.	$= \dfrac{131 - 94}{8 - 3}$
	Evaluate. The graph is rising, so the gradient is positive.	$= \dfrac{37}{5}$ $= 7.4$
	Write the equation based on $y = mx + c$.	$h = 7.4a + 71$
d	Substitute $a = 12$.	$h = 7.4 \times 12 + 71$
	Evaluate.	$= 159.8$ **cm**
	State the result.	**Samantha will be about 160 cm high when she turns 12.**
e	Substitute $a = 30$.	$h = 7.4 \times 30 + 71$
	Evaluate.	$= 293$ **cm**
	State the result and comment on it.	**The graph predicts that Samantha will be about 293 cm high when she turns 30. This is not possible. The answer obtained using the formula shows that there are limitations to how useful models of this kind can be. We all stop growing after a certain age but the formula doesn't take this into account.**

Technology Lines of best fit

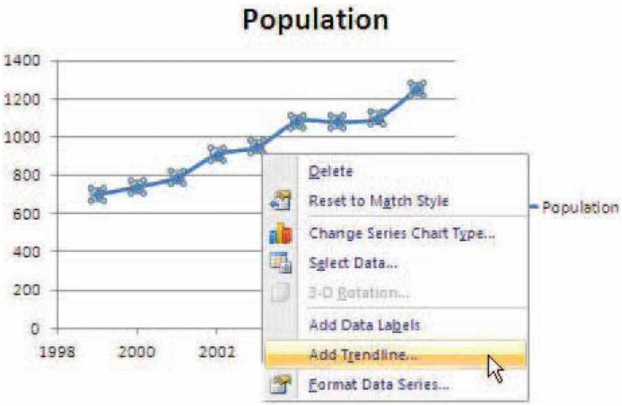
Let's see how Excel can be used to insert a line of best fit for plotted data.
The population of a small town is recorded as follows.

Year	1999	2000	2001	2002	2003	2004	2005	2006	2007
Population	700	740	790	910	950	1080	1080	1100	1250

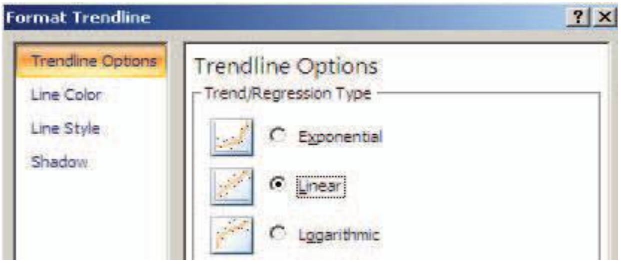
Enter the information from the table into a spreadsheet. Type 'Year' into cell A1 and 'Population' into cell A2. Then enter the years from the first row of the table into cells B1 to J1 and the population values into cells B2 to J2.
Select cells B1 to J2, then under the 'Insert' tab, click on 'Scatter' and select 'Scatter with smooth lines and markers'.



Now place the cursor on one of the data points and press the right-hand button ('right click') on the mouse. Select the Add Trendline option. A **trendline** is actually a line of best fit.

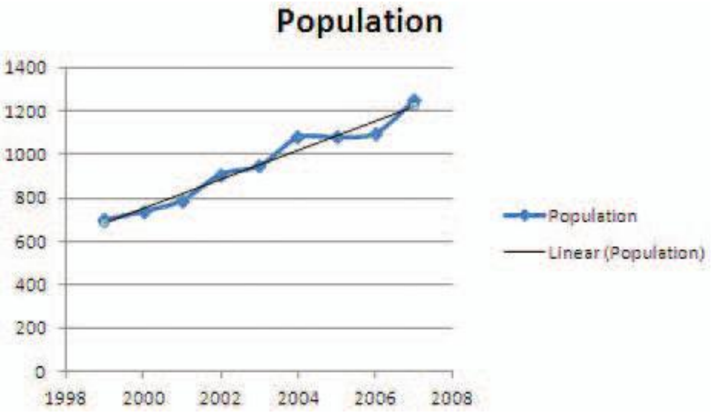


Select 'Linear' and then click on 'Close'.



○ Exponential
● Linear
○ Logarithmic

The trendline will then be inserted as shown below.



If you right-click on the trendline and select the Format Trendline option, you will be able to experiment with the look of the trendline.

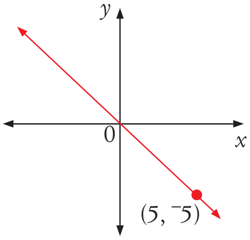
Exercise 5.2 Straight-line graphs

Understanding

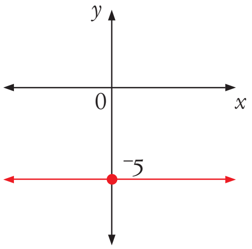
Extra questions
Exercise 5.2
MAT10NAEQ00042

1 The graphs below are not drawn to scale.

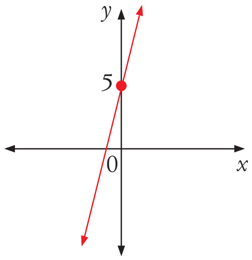
a



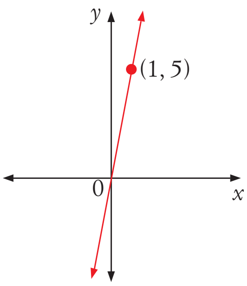
b



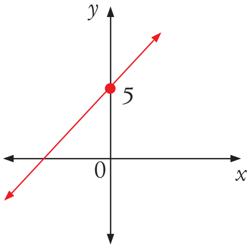
c



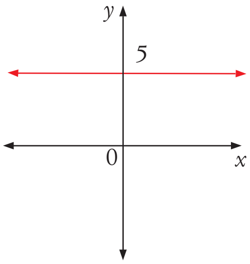
d



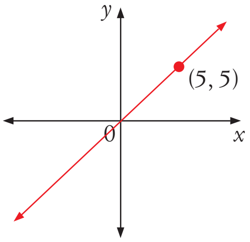
e



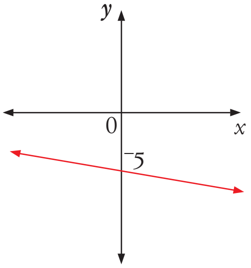
f



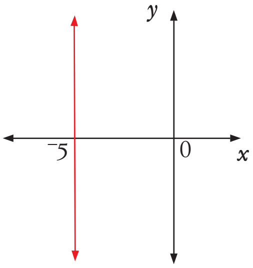
g



h



i



9780170194860

Choose the most appropriate equation for each one from the following list.

i $y = x + 5$
ii $y = 5$
iii $x = 5$
iv $y = 5 - x$
v $y = 5x$
vi $y = x - 5$
vii $y = {}^{-}5$
viii $y = 5x + 5$
ix $y = x$
x $x = {}^{-}5$
xi $y = {}^{-}x$
xii $y = {}^{-}\frac{1}{2}x - 5$

2 Draw each of the following and state the intercepts from the graph if possible.

See Example 10

a $y = 2x - 3$ for ${}^{-}3 \le x \le 6$
b $y = {}^{-}3$ for ${}^{-}4 \le x \le 4$
c $x = 1$ for ${}^{-}3 \le x \le 3$
d $y = 2x + 1$ for ${}^{-}5 \le x \le 5$
e $y = -x + 2$ for ${}^{-}6 \le x \le 6$
f $y = {}^{-}2x - 3$ for ${}^{-}6 \le x \le 3$
g $y = 3x - 1$ for ${}^{-}2 \le x \le 3$
h $x = {}^{-}3$ for ${}^{-}3 \le x \le 3$
i $y = 3x + 4$ for ${}^{-}4 \le x \le 2$
j $y = -x - 3$ for ${}^{-}6 \le x \le 6$

3 Work out whether each of the points is on the given line.

See Example 12

a	$3x + 2y + 8 = 0$	$({}^{-}4, 2)$	$(3, 4)$	$({}^{-}2, 1)$	$(2, {}^{-}7)$	$(4, 0)$
b	$y = x - 3$	$({}^{-}1, 2)$	$(1, {}^{-}2)$	$(3, 0)$	$({}^{-}2, 5)$	$({}^{-}3, {}^{-}6)$
c	$y = 2x + 6$	$(0, {}^{-}6)$	$(2, 8)$	$(1, 14)$	$({}^{-}2, 2)$	$(3, 12)$
d	$y = {}^{-}3x + 2$	$(1, 1)$	$({}^{-}1, 3)$	$(3, {}^{-}7)$	$(0, {}^{-}2)$	$({}^{-}4, 14)$
e	$3x - y - 6 = 0$	$({}^{-}4, 6)$	$(0, {}^{-}3)$	$(2, 0)$	$(10, {}^{-}12)$	$({}^{-}4, {}^{-}18)$

4 Without drawing a graph, find the intercepts for each of the following.

See Example 11

a $3x - 2y + 6 = 0$
b $4x - 3y = 12$
c $x + 3y - 4 = 0$
d $2x + 2y + 5 = 0$
e $6x - 2y + 3 = 0$
f $5x + 3y = 15$

5 Find the equation of the line parallel to the x-axis that passes through each point.

See Example 19

a $({}^{-}2, 4)$
b $(3, 5)$
c $(2, {}^{-}1)$
d $(0, 5)$
e $(4, 0)$
f $({}^{-}1, {}^{-}4)$
g $(6, 1)$
h $({}^{-}4, 7)$
i $(7, {}^{-}3)$
j $({}^{-}8, {}^{-}8)$
k $(3, 3)$
l $(5, 1)$

6 Find the equation of the line parallel to the y-axis that passes through each point.

a $({}^{-}2, 4)$
b $(3, 5)$
c $(2, {}^{-}1)$
d $(0, 5)$
e $(4, 0)$
f $({}^{-}3, {}^{-}4)$
g $(6, 1)$
h $({}^{-}4, 7)$
i $(7, {}^{-}3)$
j $({}^{-}8, {}^{-}8)$
k $(3, 3)$
l $(5, 1)$

Fluency

7 For each of the following functions, draw the line, then:

i choose two points and calculate the gradient
ii find the intercepts if possible.

a $y = 3x - 2$
b $y = x + 4$
c $y = {}^{-}2x + 3$
d $x = {}^{-}1$
e $y = {}^{-}\frac{1}{4}x + 1$
f $2y = 3x - 4$
g $2y = 4x + 1$
h $3y = {}^{-}2x - 3$
i $5y = {}^{-}7x + 8$
j $4x - y + 8 = 0$
k $x + 3y - 4 = 0$
l $4x - 3y = 0$

8 Draw a graph for each of the following.

See Example 13

a $y \ge x + 1$
b $y \le x - 3$
c $y > x + 4$
d $y \ge 2x - 3$
e $y < 3x + 1$
f $y > 5 - x$

9 Draw a graph for each of the following.

a $x + 2y \ge 5$
b $2x \le 3y - 8$
c $x + y + 3 < 0$
d $3x \ge 4y + 6$
e $x - 2y + 4 < 0$
f $5x < 4y$

10 Write in gradient–intercept form the equation of the line with:

a gradient 2 and y-intercept (0, 4) **b** slope $^-3$ and y-intercept (0, 1)
c y-intercept $^-6$ and slope 5 **d** gradient 3 and y-intercept $^-2$
e y-intercept (0, $^-3$) and slope 0 **f** slope $^-4$ and y-intercept 0.

See Example 14

11 Write in standard form the equation of the line with:

a gradient 4 and y-intercept $^-1$ **b** gradient $\frac{2}{3}$ and y-intercept (0, $^-2$)
c y-intercept $^-1$ and slope $^-\frac{4}{3}$ **d** y-intercept (0, 2) and gradient $\frac{5}{2}$
e gradient $3\frac{1}{3}$ and y-intercept $^-4$ **f** y-intercept and slope $^-1\frac{3}{4}$.

See Example 15

12 Without drawing a graph, work out the gradient and y-intercept of each line.

a $y = 2x + 5$ **b** $y = 5 - 3x$ **c** $y = {}^-x + 1$
d $y = 4 - 7x$ **e** $y = x + 2$ **f** $y = {}^-8x$
g $x - 2y + 1 = 0$ **h** $7x + y - 14 = 0$ **i** $x - 3y + 6 = 0$
j $3y - 8 = {}^-4x$ **k** $7x = 3y$

See Example 16

13 Find the equation, in both gradient–intercept and standard form, of the line:

a with gradient 2, passing through (2, 5) **b** through ($^-1$, 4) with slope $^-1$
c through ($^-3$, $^-2$) with slope 1 **d** with gradient $^-4$, passing through ($^-1$, $^-4$)
e through (4, 3) with gradient $^-\frac{3}{4}$ **f** through ($^-3$, $^-4$) with slope $\frac{7}{3}$

Worked solutions
Exercise 5.2
MAT10NAWS00047

⑭ Find the equation of the line passing through (2, 3) and (2, $^-1$).

15 Find the equation of the line with slope $^-\frac{2}{3}$ that passes through (0, 4), in both gradient–intercept and standard forms.

16 Work out the general and slope-intercept forms of the equation of the line with:

a x-intercept 2 and y-intercept 1 **b** x-intercept $^-3$ and y-intercept 4
c x-intercept 4 and y-intercept $^-2$ **d** x-intercept $^-1$ and y-intercept 4
e x-intercept (3, 0) and y-intercept (0, $^-8$) **f** intercepts ($^-5$, 0) and (0, $^-3$)
g intercepts (0, $^-1$) and (5, 0) **h** intercepts (0, 6) and ($^-4$, 0)
i both intercepts $^-4$ **j** both intercepts 2.

17 A line passes through (0, 4) and (2, 8).

a What is the slope of the line? **b** What is the y-intercept?
c Write its equation in standard form.

See Example 17

18 Find the equation of the line through each of these pairs of points, in both general and slope–intercept forms.

a (4, 6) and (5, 3) **b** (3, 1) and ($^-3$, $^-5$) **c** (0, 3) and (2, 7)
d ($^-1$, $^-3$) and ($^-4$, 6) **e** (3, 4) and (6, $^-8$) **f** ($^-2$, 5) and (2, 1)
g (2, 1) and ($^-3$, $^-4$) **h** (4, 2) and (6, 10) **i** (3, $^-4$) and ($^-5$, 1)
j ($^-2$, $^-6$) and ($^-4$, $^-2$) **k** (3, 6) and ($^-2$, $^-4$) **l** (6, $^-1$) and (4, $^-5$)

19 Use the most appropriate method to find the general form of the equation of each line.

a Gradient 2 and y-intercept $^-4$ **b** Slope $^-\frac{2}{3}$ and passing through (4, $^-3$)
c Passing through (5, 1) and ($^-4$, $^-5$) **d** Passing through (2, 1) and (2, 8)
e Slope 4 and intercept (0, 2) **f** Slope $^-2$ and intercept (5, 0)

20 State whether each of the following pairs of gradients, m_1 and m_2, represent lines that are parallel, perpendicular or neither parallel nor perpendicular.

a $m_1 = 2$ and $m_2 = {}^{-}\frac{1}{2}$ **b** $m_1 = 0.4$ and $m_2 = \frac{2}{5}$ **c** $m_1 = 2$ and $m_2 = {}^{-}2$
d $m_1 = \frac{1}{4}$ and $m_2 = 4$ **e** $m_1 = 1$ and $m_2 = {}^{-}1$ **f** $m_1 = \frac{1}{8}$ and $m_2 = 0.125$

21 For each line with gradient m_1 shown below, find the gradient m_2, of the line perpendicular to it.

a $m_1 = 3$ **b** $m_1 = {}^{-}1\frac{1}{3}$ **c** $m_1 = 0.2$
d $m_1 = {}^{-}3.1$ **e** $m_1 = \frac{a}{b}$ **f** $m_1 = {}^{-}\frac{1}{h}$

22 Find b given that the line joining:

a $Z({}^{-}1, 3)$ and $K(2, b)$ is perpendicular to a line with gradient $\frac{3}{2}$
b $M({}^{-}5, b)$ and $P(b, 2)$ is perpendicular to a line with gradient ${}^{-}\frac{1}{3}$

Worked solutions
Exercise 5.2
MAT10NAWS00047
See Example 18

23 Without drawing graphs, work out whether the following pairs of lines are parallel.

a $2x + 3y + 8 = 0$ and $6y = 5 - 4x$ **b** $2x - y - 4 = 0$ and $y = 2x + 6$
c $5y - 2x = 3$ and $2y - 5x + 6 = 0$ **d** $3y = 4x - 9$ and $y - \frac{4}{3}x + 4 = 10$
e $y = 4 - 2\frac{1}{2}x$ and $2y + 5x - 2 = 0$ **f** $3y + 5x = 9$ and $5y - 3x + 6 = 0$

Worked solutions
Exercise 5.2
MAT10NAWS00047
See Example 20

24 Find the equations of the lines through:

a $(5, {}^{-}4)$ that are parallel and perpendicular to $y = 4x + 7$
b $({}^{-}2, 5)$ that are parallel and perpendicular to $y = \frac{3}{5}x - 4$
c $({}^{-}2, 5)$ that are parallel and perpendicular to $3x + 2y + 5 = 0$
d $({}^{-}1, {}^{-}5)$ that are parallel and perpendicular to $x - 4y + 3 = 0$.

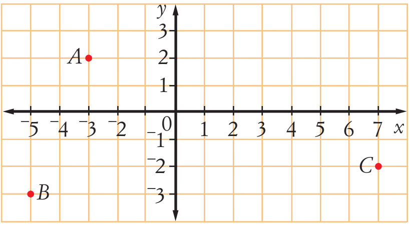
25 Show that $2x + 3y + 8 = 0$ and $6y = 5 - 4x$ are parallel.

26 Show that $4x + 5y - 6 = 0$ and $10x = 8y - 5$ are perpendicular.

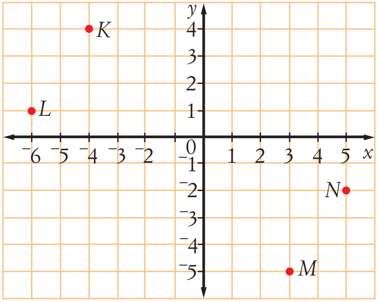
Problem solving

See Example 22

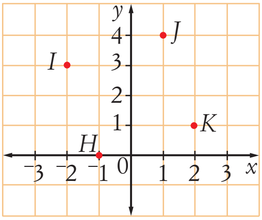
27 Show that $A({}^{-}3, 2)$, $B({}^{-}5, {}^{-}3)$ and $C(7, {}^{-}2)$ form a right-angled triangle and find its area.



28 Show that $K({}^{-}4, 4)$, $L({}^{-}6, 1)$, $M(3, {}^{-}5)$ and $N(5, {}^{-}2)$ are the vertices of a rectangle and find its area.



29 Show that $H(^-1, 0)$, $I(^-2, 3)$, $J(1, 4)$ and $K(2, 1)$ are the vertices of a square.



See Example 21 **30** $J = (^-3, ^-1)$, $K = (^-1, 0)$ and $L = (1, ^-4)$. Show that $\angle JKL = 90°$.

See Example 22 **31** $\triangle PMQ$ has vertices at $P(0, 3)$, $M(3, 1)$ and $Q(^-4, ^-3)$. Show that it is a right-angled triangle.

32 The vertices of a quadrilateral are $P(^-6, 5)$, $Q(^-2, 4)$, $R(^-5, 1)$ and $S(2, ^-3)$. What type of quadrilateral is $PQRS$? (Give reasons for your answer.)

See Examples 23, 24 **33** The volume, V, in litres, of petrol remaining in a portable generator is given by:

$$V = 8 - \tfrac{3}{2}h$$

where h is the number of hours after the generator starts with a full tank.

a Construct a table of values for the function for $0 \leq h \leq 6$.

b Use the table to draw a graph of the function.

c What type of function is it?

d Use the graph to find when there is 3 L of petrol left in the generator.

e How many litres of petrol are left in the generator after 2 hours?

f What does the value of the function at $h = 0$ represent?

g What does the value of the function where the graph crosses the h-axis represent?

34 The cost, C, in dollars, of hiring a clown for a birthday party is given by:

$$C = 50 + 40h$$

where h is the number of hours for which the clown is hired.

a Draw a graph of the function for $0 \leq h \leq 8$.

b What type of function is it?

c Use the graph to find the cost of hiring the clown for 3 hours.

d For how long could the clown be hired with \$200 if the time may be charged in fractions of an hour?

e What does the value 50 in the function $C = 50 + 40h$ represent?

f What does the value 40 in the function $C = 50 + 40h$ represent?

35 A minibus can carry 20 passengers and a coach can carry 70 passengers. A total of 250 pupils are to be taken on an excursion. Write the number of minibuses used as x and the number of coaches as y.

a How many pupils can be carried on x minibuses and y coaches?

b Write an equation for the number of minibuses and the number of coaches that could be used to carry 250 pupils.

c Draw the graph for this equation.

d What are the exact values of x and y that can be used for 250 pupils?

e Are any other possibilities suggested by the graph?

f On the same graph, draw lines to show the equations for 150 pupils and 400 pupils.

g What do you notice about the graphs?

36 A coffee wholesaler spends a maximum of \$20 000 a week on coffee. He can buy coffee from Africa for \$15/kg and coffee from Papua New Guinea for \$9/kg.

- **a** Write an equation for the number of kilograms of coffee from Africa (a) and the number of kilograms of coffee from Papua New Guinea (p) that he can buy for \$20 000.
- **b** Construct a table of values for a and p and use it to draw a graph of the equation.
- **c** If the supply of Papua New Guinea coffee one week is restricted to 800 kg, use the graph to work out how much African coffee the wholesaler can buy.

37 A movie theatre is filled to capacity with 600 people. When the movie finishes, people start leaving at the rate of 80 people per minute.

- **a** Express P, the number of people in the theatre, as a function of t, the number of minutes after the movie ends.
- **b** Draw a graph of the function.
- **c** Use the graph to find how long after the movie ends the theatre will still be half-full.

Reasoning

38 Find the equation of the line that is the perpendicular bisector of the segment joining the points $K(6, 10)$ and $L(^{-}18, 4)$.

See Example 25

39 The heights and weights of a group of people considered to be close to the 'ideal' weight for their height are measured and recorded as follows.

Height, *h* (cm)	140	147	153	159	163	173	179	182	187	193
Weight, *w* (kg)	47	48	56	55	63	66	68	75	74	80

- **a** Plot the data and draw in the line of best fit—the 'ideal' height-for-weight line.
- **b** Use the graph to interpolate the 'ideal' weight for a person who is 170 cm tall.
- **c** Use the graph to extrapolate the 'ideal' height of a person who weighs 85 kg.
- **d** Find the equation of the line of best fit.
- **e** Use the equation of the line of best fit to check the answers you found in parts **b** and **c**.

40 The air temperature, T, in °C, at various heights above sea level is recorded as follows.

Height, *h* (m)	0	1400	2500	4400	6000	6900	9400
Air temperature, *T* (°C)	15	9	$^{-}7$	$^{-}14$	$^{-}20$	$^{-}32$	$^{-}43$

- **a** Plot the data and draw in the line of best fit.
- **b** Use the graph to find the likely air temperature at 5000 m above sea level.
- **c** Use the graph to find the likely height above sea level if the air temperature is $^{-}55$°C.
- **d** Find the equation of the line of best fit.
- **e** Use the equation of the line of best fit to find the likely air temperature at 8000 m above sea level.
- **f** Use the equation of the line of best fit to find the likely height above sea level if the air temperature is $^{-}60$°C.

5.3 Modelling curves

So far, the work of this chapter has focused on linear functions and their equations. We will now look at some nonlinear functions. You have previously looked at a class of functions called **quadratics**. **Quadratic functions** have the form $y = ax^2 + bx + c$ and the graph of a quadratic function is called a **parabola**.

When you draw the graph of a linear function, a minimum of two points is required. When drawing the graphs of nonlinear functions more points are required in order to represent it on the Cartesian plane.

Investigate: Graphing quadratics

1 Copy and complete the table of values for $y = x^2$.

x	$^{-}3$	$^{-}2$	$^{-}1$	$^{-}0.5$	$^{-}0.25$	0	0.25	0.5	1	2	3
y											

2 Draw x- and y-axes on a sheet of grid paper. Use the scale 1 cm : 1 unit for both axes and label the x-axis from $^{-}4$ to 4 and the y-axis from $^{-}1$ to 10.

3 Plot each of the points from the table of values on the grid paper.

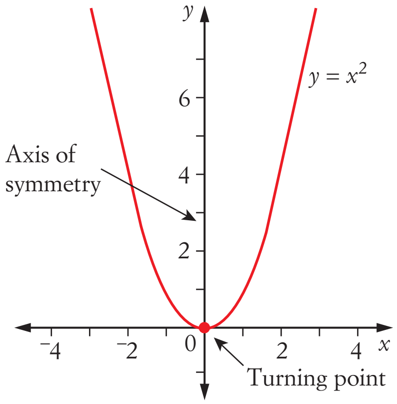
4 Join the points with a smooth curve.

5 Describe the shape of the graph.

Parabolas

In the previous investigation you drew a basic parabola – the graph of the function $y = x^2$. The main features of this parabola are described below.

- It is symmetrical and the y-axis is the axis of symmetry. The left and right sides are mirror images of each other.
- The point (0, 0) is called the **turning point** or **vertex** of this parabola.
- The 'U' shape of this parabola is described as being **concave upwards**.



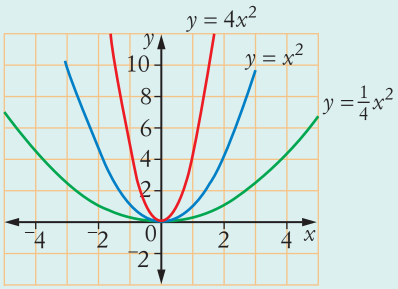
Important!

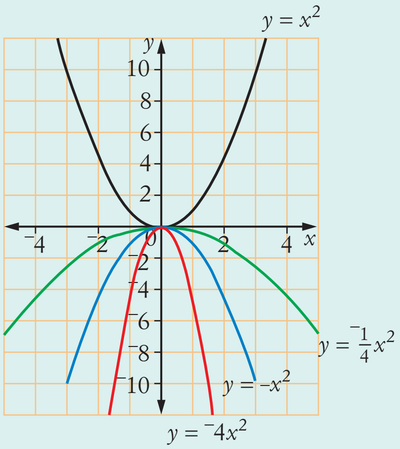
The graph of the parabola $y = ax^2$

For the graph of $y = ax^2$, the value of a (the coefficient of x^2) determines whether the parabola is:

- 'wide' or 'narrow'
- concave upwards or downwards.

As the size of a increases, the graph of the parabola becomes 'narrower' and as the size of a decreases, the graph 'widens'. When $a > 0$, the parabola is concave upwards and when $a < 0$, the parabola is concave downwards (it is reflected in the x-axis). The parabolas below illustrate this.



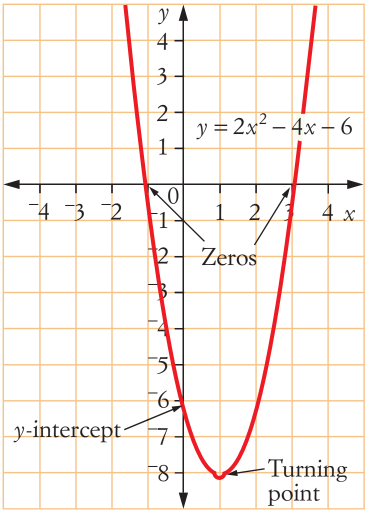


Technology worksheet
Excel worksheet: Investigating parabolas 1
MAT10NACT00010

Technology worksheet
Excel worksheet: Power functions
MAT10NACT00020

The general form of a quadratic function is $y = ax^2 + bx + c$. The graph of the quadratic $y = 2x^2 - 4x + 6$ as shown on the right can be used to illustrate the main features of the graphs of quadratic functions.

- Parabolas are symmetrical.
- The **zeros** $^-1$ and 3 are where the graph crosses the x-axis. The zeros are also known as the **roots** of the graph. Not all parabolas have zeros. The root is the value of x that makes $y = 0$.
- The ***y*-intercept** $(0, {}^-6)$ is where the graph crosses the y-axis. All parabolas have a y-intercept.
- The **turning point** or **vertex** of this parabola is a **minimum** (at $1, {}^-8$).
- The turning point of a parabola must be a **minimum** or a **maximum.** A minimum is the point where the gradient of the parabola changes from negative to positive. A maximum is the point where the gradient of the parabola changes from positive to negative.



TLF Learning object
Eaglecat: Parabola (L10095)
MAT10NAIN00005

Weblink
Parabola
MAT10NAWB00005

TLF Learning object
Graphing functions (R11253)
MAT10NAIN00005

Example 26

CAS TI-Nspire exercise
Coordinate geometry
MAT10NATI00014

CAS ClassPad exercise
Coordinate geometry
MAT10NACP00014

Draw the graph of $y = x^2 + 2x + 6$ for $^-3 \le x \le 2$ and identify its main features.

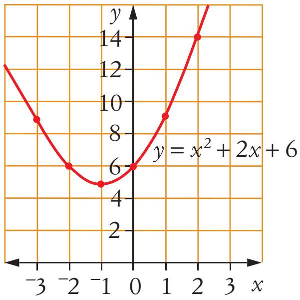
Solution

Draw up a table of values for $^-3 \le x \le 2$ and calculate the corresponding y values.

$\mathbf{y = x^2 + 2x + 6}$

x	$^-3$	$^-2$	$^-1$	0	1	2
x^2	9	4	1	0	1	4
$+ 2x$	$^-6$	$^-4$	$^-2$	0	2	4
$+ 6$	6	6	6	6	6	6
y	**9**	**6**	**5**	**6**	**9**	**14**

Plot the points and join them with a smooth curve. Continue the curve so that it is symmetrical.



Look at the graph to identify the main features.

There is a turning point at ($^-1$, 5). The turning point is a minimum.
The y-intercept is (0, 6).
There are no zeros.

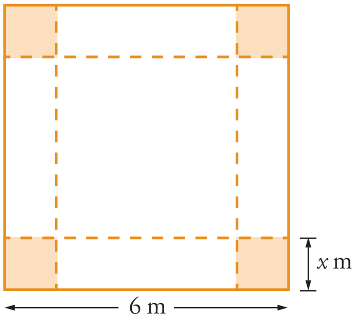
State the result.

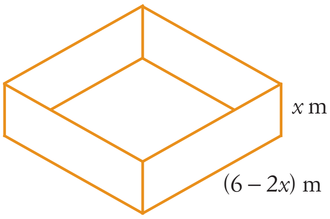
Minimum at ($^-1$, 5) and y-intercept at (0, 6).

You can also use spreadsheets to investigate nonlinear functions such as quadratics.

Example 27

A square sheet of metal is 6 m long. It has a square of side x m cut from each corner so that it can be formed into a box as shown.





The volume of the box is given by $V = x(6 - 2x)^2$.
Draw a graph of the volume of the box and use the graph to estimate its volume when $x = 1.8$ m.

Solution

Draw up a table of values for $0 \le x \le 3$ and calculate the corresponding y values.

$$V = x(6 - 2x)^2$$

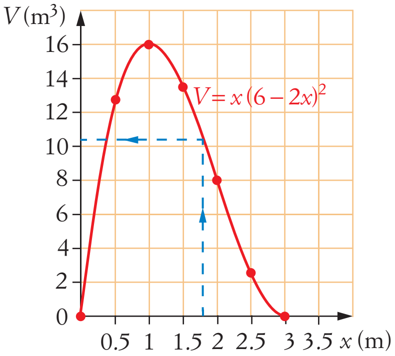
x	0	0.5	1	1.5	2	2.5	3
$(6 - 2x)^2$	0	25	16	9	4	1	0
V	0	12.5	16	13.5	8	2.5	0

Plot the points and join them with a smooth curve.

Locate 1.8 on the x-axis.

Draw a line vertically up until it reaches the graph.

Draw a line horizontally across to the V-axis and read the value.



State the result.

The volume of the box is about 10.4 m^3 when x = 1.8 m.

The equation of the volume of the box in Example **27** is $V = x(6 - 2x)^2$. If this equation is expanded, it becomes $V = 36x - 24x^2 + 4x^3$.
This equation is a model that is a **cubic** function because the highest power of x is 3.

Circles

A **circle** is a curve in which the distance from the centre to the circumference (the radius) is always the same.
Suppose a circle has a radius of 4 and a centre at $C(^-1, 2)$, as shown here. If we use $P(x, y)$ for a point on the circumference, we can use the distance formula to find the equation of a circle.

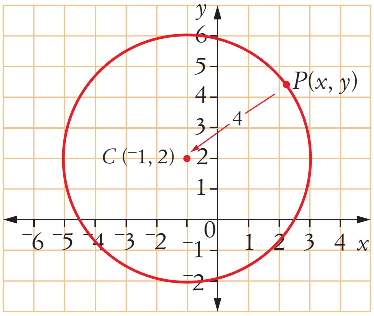
$$PC = \sqrt{(x - {}^-1)^2 + (y - 2)^2}$$

But $$PC = 4$$

So $$\sqrt{(x - {}^-1)^2 + (y - 2)^2} = 4$$

Squaring, we get:

$$(x + 1)^2 + (y - 2)^2 = 16$$



This is the equation of a circle with a radius of 4 and a centre at $(^-1, 2)$.
We can generalise this to state the equation of any circle.

Important!

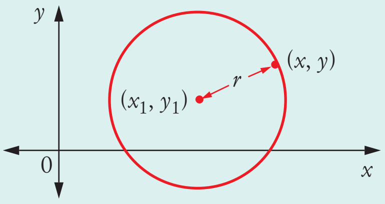
Equation of a circle

The **equation of a circle** with centre (x_1, y_1) and radius r is given by:

$$(x - x_1)^2 + (y - y_1)^2 = r^2$$

The equation of a circle with centre (0, 0) and radius r is given by:

$$x^2 + y^2 = r^2$$



Example 28

Puzzle sheet

Circle equations

MAT10NAPS00047

A circle has centre $(3, {}^{-}2)$ and radius 5.

a Find the equation of the circle.

b Sketch the circle.

c Determine whether point (6, 2) is on the circle.

d Determine whether point $({}^{-}2, 1)$ is on the circle.

Solution

a Write the general equation of a circle.

$$\mathbf{(x - x_1)^2 + (y - y_1)^2 = r^2}$$

Substitute using $(x_1, y_1) = (3, {}^{-}2)$ and $r = 5$.

$$\mathbf{(x - 3)^2 + (y - {}^{-}2)^2 = 5^2}$$

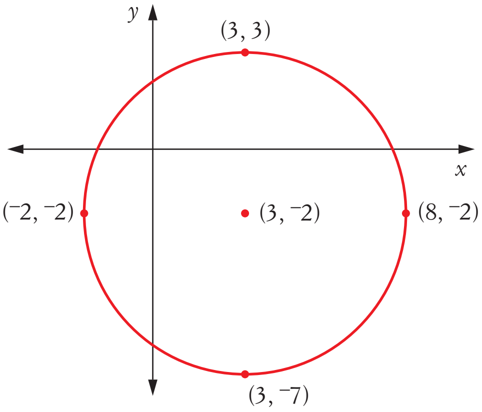
Simplify.

$$\mathbf{(x - 3)^2 + (y + 2)^2 = 25}$$

b Work out the coordinates needed.

With radius 5 the circle will reach down to $(3, {}^{-}7)$, up to (3, 3), back to $({}^{-}2, {}^{-}2)$ and across to (8, 2).

Draw the axes, mark the centre and extreme points and sketch the circle.



c If (6, 2) lies on the circle, $x = 6$ and $y = 2$ will satisfy the equation of the circle.

Write the equation of the circle.

$$\mathbf{(x - 3)^2 + (y + 2)^2 = 25}$$

Substitute using $(x, y) = (6, 2)$ in the LHS.

$$\begin{aligned}\textbf{LHS} &= \mathbf{(6 - 3)^2 + (2 + 2)^2}\\ &= \mathbf{9 + 16}\\ &= \mathbf{25}\end{aligned}$$

Evaluate.

LHS = RHS, so state the result.

(6, 2) lies on $(x - 3)^2 + (y + 2)^2 = 25$.

9780170194860

d Write the equation of the circle.	$(x - 3)^2 + (y + 2)^2 = 25$
Substitute using $(x, y) = (^-2, 1)$ in the LHS.	**LHS** $= (^-2 - 3)^2 + (1 + 2)^2$
Evaluate.	$= 25 + 9$
	$= 34$
LHS ≠ RHS, so state the result.	**$(^-2, 1)$ does not lie on the circle.**

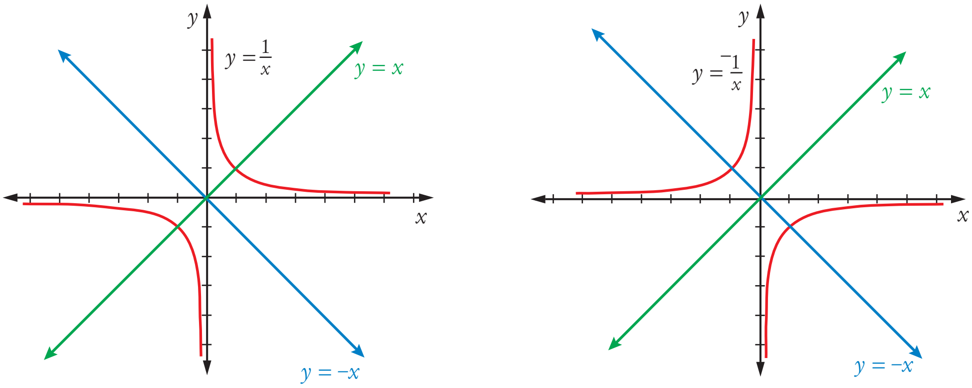
Hyperbolas

Important!

Equation of a hyperbola

The **equation of a rectangular hyperbola** is $y = \frac{a}{x}$, where a is a constant and $a \neq 0$.

We can use the graphs of the hyperbolas $y = \frac{1}{x}$ and $y = \frac{^-1}{x}$ shown below to examine the major features of hyperbolas.



The graphs have similar features that are described below.

- The graph has two parts called **branches**.
- The graph never intersects the x-axis or y-axis, so there is no x-intercept or y-intercept.
- The graph has two axes of symmetry: $y = x$ and $y = -x$.
- As x becomes very large, y gets close to 0 and the graph gets very close to the x-axis. This means that the x-axis is an **asymptote**.
- As x gets very close to 0, y becomes very large and the graph gets very close to the y-axis. This means that the y-axis is also an **asymptote**.
- When a is positive, the branches of the hyperbola are in the 1st and 3rd quadrants of the Cartesian plane. When a is negative, the branches of the hyperbola are in the 2nd and 4th quadrants.

Example 29

The equation of a hyperbola is $y = \frac{8}{x}$.

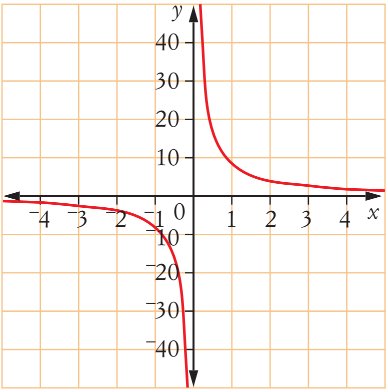
a Draw a graph of the hyperbola for $^-4 \le x \le 4$.
b Determine whether point (1.6, 6) is on a branch of the hyperbola, above it or below it.

Solution

a Complete a table of values using values of x between $^-4$ and 4.

As $8 > 0$, the branches of the hyperbola are in the 1st and 3rd quadrants.

Plot the points and join them with a smooth curve.

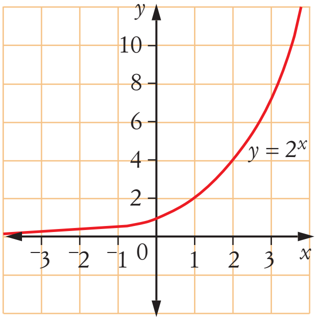


b If (1.6, 6) lies on the hyperbola, $x = 1.6$ and $y = 6$ will satisfy $y = \frac{8}{x}$.

Substitute using $x = 1.6$ in the RHS.	**RHS** $= \frac{\mathbf{8}}{\mathbf{1.6}}$
Evaluate.	$= \mathbf{5}$
Substitute using $y = 6$ in the LHS.	**LHS** $= \mathbf{6}$
LHS $\neq$ RHS, so state the result.	**(1.6, 6) does not lie on** $\boldsymbol{y = \frac{8}{x}}$.
Compare (1.6, 6) and (1.6, 5).	**(1.6, 6) is above (1.6, 5)**
State the result.	**(1.6, 6) is above** $\boldsymbol{y = \frac{8}{x}}$.

Exponential functions

Functions that have the general form $y = a^x$ are known as **exponential functions**. The graph of $y = 2^x$ is shown on the right. We can use it to list the features of the graph of an exponential function as follows.



- The slope of the graph is always positive.
- The graph intersects the y-axis at $x = 0$.
- There is no axis of symmetry.
- As x gets smaller, y gets close to the x-axis and the graph becomes flat. The x-axis is an **asymptote**.
- As x gets larger, y gets much larger and the graph becomes very steep.

Example 30

a Draw a graph of the exponential function $y = 3^x$ for $^{-}2 \leq x \leq 2$.
b Use the graph to estimate, correct to one decimal place, the value of x for which $3^x = 4$.

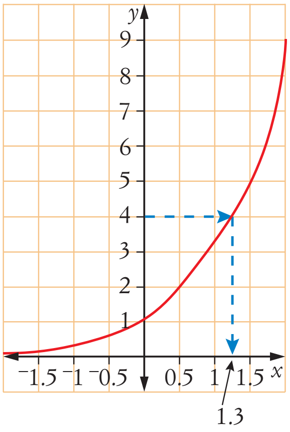
Weblink
The most important video
MAT10NAWB00005

Solution

a Complete a table of values, using your calculator, for values of x between $^{-}2$ and 2.

x	$^{-}2$	$^{-}1$	0	1	2
$y = 3^x$	0.111. . .	0.333. . .	1	3	9

Plot the points and join them with a smooth curve.



b Locate the point on the graph where $y = 4$. Find the corresponding x value.

When $y = 4$, $x \approx 1.3$.

State the result.

$3^x = 4$ when $x \approx 1.3$.

It is obvious that the graph of $y = 3^x + 2$ will be the same general shape as the graph of $y = 3^x$ but it will be moved up by 2 units, so the asymptote will be $y = 2$ instead of $y = 0$. An exponential function can be sketched by finding a few points and using the general shape.

Example 31

Sketch the graph of $y = 4^x + 1$

Animated example
Modelling curves
MAT10NAAE00005

Solution

Find the values for $^{-}1$, 0 and 1 as a guide.

x	$^{-}1$	0	1
$y = 4^x + 1$	1.25	2	5

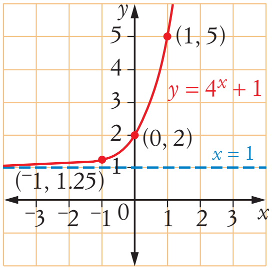
State the asymptote.

The asymptote will be $y = 1$.

State the general shape.

The shape will be like, say, $y = 2^x$, but steeper.

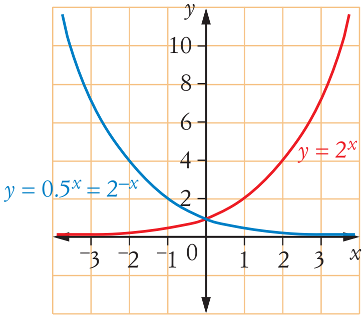
Show the important points, the asymptote and sketch the graph.



Consider the function $y = 0.5^x$. This exponential function is closely related to $y = 2^x$. $0.5 = \frac{1}{2} = 2^{-1}$, so $0.5^x = 2^{-x}$. The values of $y = 0.5^x = 2^{-x}$ are related to those of $y = 2^x$ as shown in the table below. You can check the table on your calculator.

x	$^{-}3$	$^{-}2$	$^{-}1$	0	1	2	3
$y = 2^x$	$\frac{1}{8} = 0.125$	$\frac{1}{4} = 0.25$	$\frac{1}{2} = 0.5$	1	2	4	8
$y = 2^{-x}$	8	2	4	1	$\frac{1}{2} = 0.5$	$\frac{1}{4} = 0.25$	$\frac{1}{8} = 0.125$

Both graphs are shown on the same set of axes below.



The graph of $y = 2^{-x}$ is the reflection of $y = 2^x$ in the y-axis. Its features are shown below.

- The slope of the graph is always negative.
- The graph intersects the y-axis at $x = 0$.
- There is no axis of symmetry.
- As x gets larger, y gets close to the x-axis and the graph becomes flat. The x-axis is an asymptote.
- As x gets smaller, y gets much larger and the graph becomes very steep.

An exponential function with a base less than 1 (a negative exponent) can also be sketched using the general shape and a few points.

Example 32

Sketch the graph of $y = 0.25^x - 2$.

Solution

Find the values for $^{-}1$, 0 and 1 as a guide.

x	$^{-}1$	0	1
$y = 0.25^x - 2$	2	$^{-}1$	$^{-}1.75$

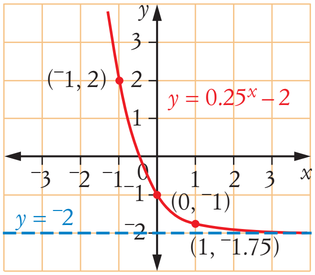
State the asymptote.

The asymptote will be $y = {^{-}2}$.

State the general shape.

The shape will be like, say, $y = 2^{-x}$, but steeper.

Show the important points, the asymptote and sketch the graph.



In this section you have seen how the graph of a parabola and other functions can be stretched, translated and reflected. This is known as transforming the graph of a function. Some of the important transformations are summarised below.

Important!

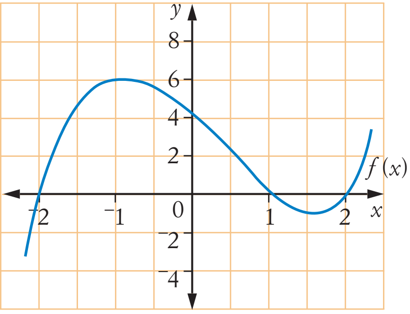
Transforming the graph of a function

Compared with the graph of a function, $y = f(x)$:

- the graph of $y = f(x) + c$ is a vertical translation. If $c > 0$, the translation is up and if $c < 0$, the translation is down.
- the graph of $y = -f(x)$ is a reflection in the x-axis. The zeros of $y = -f(x)$ are the same as the zeros of $y = f(x)$.
- the graph of $y = f(-x)$ is a reflection in the y-axis.
- the graph of $y = kf(x)$ is a vertical stretch or compression. The value of k is the stretch or compression factor. If $k > 0$, the graph is stretched and if $k < 0$, the graph is compressed. The zeros of $y = kf(x)$ are the same as the zeros of $y = f(x)$.

Example 33

The graph of $y = f(x)$ is shown below.



Use this graph to sketch graphs of the following functions.

a $y = f(x) + 3$

b $y = -f(x)$

c $y = 2f(x)$

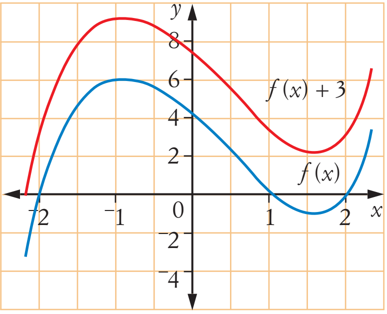
d $y = f(-x)$

Solution

a First sketch $y = f(x)$.

The graph of $y = f(x) + 3$ is a vertical translation 3 units up compared with the graph of $y = f(x)$.

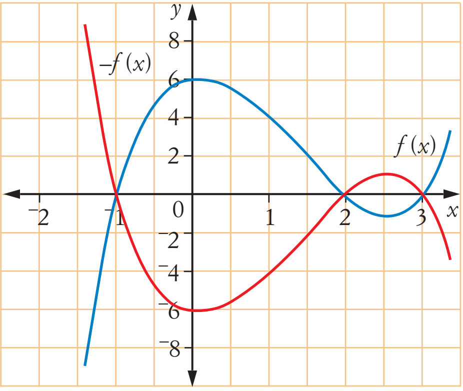
Sketch the translation.



b Sketch a graph of $y = f(x)$.

The graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ in the x-axis.

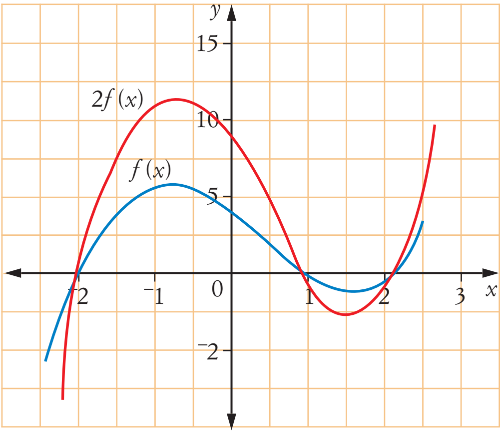
Sketch the reflection.



c Sketch a graph of $y = f(x)$.

The graph of $y = 2f(x)$ is a vertical stretch of $y = f(x)$. The stretch factor is 2.

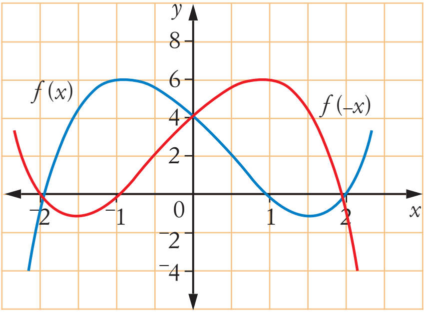
Sketch the graph of $y = 2f(x)$.



d Sketch a graph of $y = f(x)$.

The graph of $y = f(-x)$ is a reflection of $y = f(x)$ in the y-axis.

Sketch the reflection.



Exercise 5.3 Modelling curves

Your teacher may ask you to use a graphics calculator for some of these questions.

Understanding

Extra questions
Exercise 5.3
MAT10NAEQ00043

1 For each of the following equations, state whether the graph is a straight line, a parabola, a hyperbola, a circle, exponential or none of these.

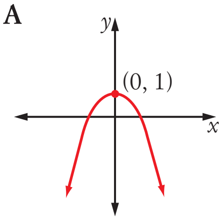
a $y = \frac{3}{x}$ **b** $y = {}^{-}3$ **c** $x^2 + y^2 = 49$
d $x = 17$ **e** $y = 3x^2$ **f** $y = \frac{{}^{-}4}{x}$
g $y = 5^x$ **h** $x^2 + y^2 = 36$ **i** $y = 4x - 2$
j $y = 2x^2 + 5$ **k** $3x - 2y - 15 = 0$ **l** $y = x^3 + 2x + 12$
m $y = x$ **n** $y = 3^x$ **o** $y = 22 - 4x^2$

Worksheet
Matching parabolas with their equations
MAT10NAWK00016

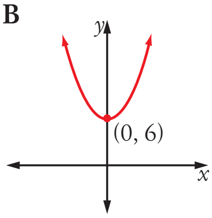
2 Match each of these equations with a parabola from those shown below.

a $y = 1 + x^2$ **b** $y = 10 - x^2$ **c** $y = x^2 - 6$
d $y = {}^{-}10 - x^2$ **e** $y = x^2$ **f** $y = 6 - x^2$
g $y = x^2 + 6$ **h** $y = -x^2 + 1$ **i** $y = x^2 - 10$
j $y = -x^2$ **k** $y = -x^2 - 6$ **l** $y = x^2 + 10$

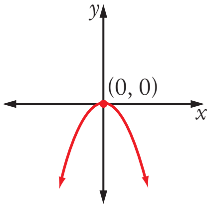
A



B



C



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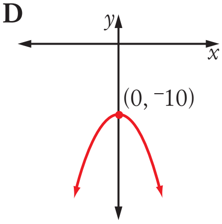
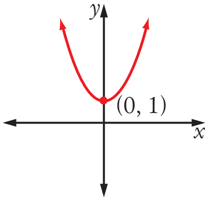
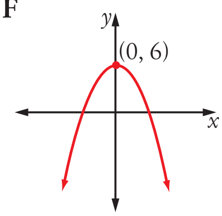


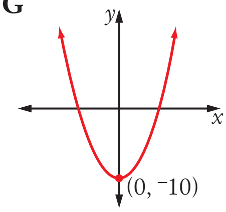
E



F



G



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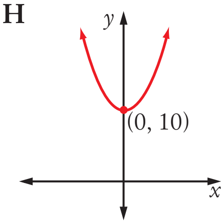
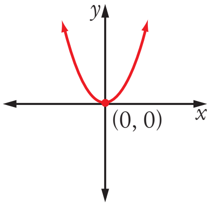
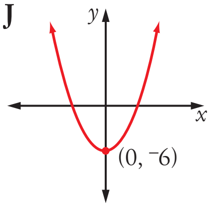


I



J



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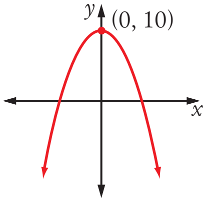
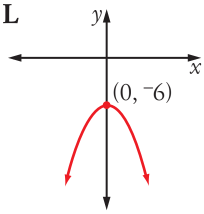


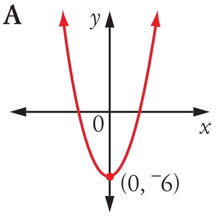
L



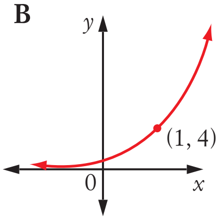
3 Match each of these equations with a graph from those shown below.

a $y = 2^x$ **b** $y = \frac{^-9}{x}$ **c** $x^2 + y^2 = 9$
d $y = 2 - x^2$ **e** $y = -x$ **f** $y = x^2 + 4$
g $y = \frac{1}{x}$ **h** $x^2 + y^2 = 16$ **i** $y = 3x^2$
j $y = 4^x$ **k** $y = x^2 - 6$ **l** $y = \frac{9}{x}$

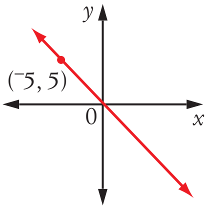
A



B



C



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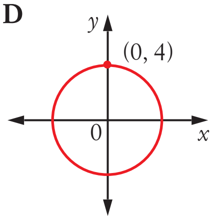

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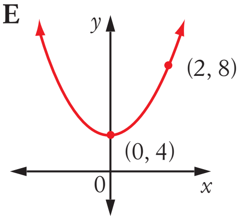

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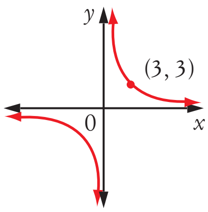

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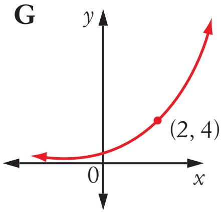

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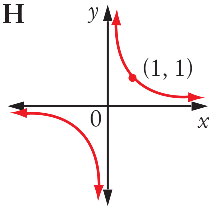

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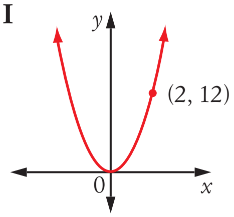

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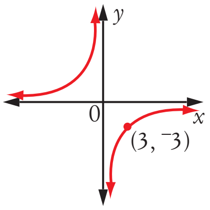

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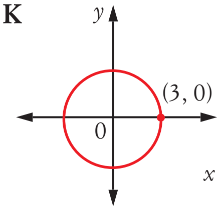

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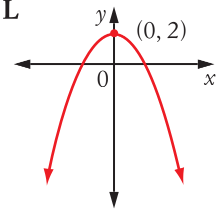


4 For each pair of parabolas described below, which one will be 'wider' when compared to the other?

a $y = x^2$ or $y = 4x^2$ **b** $y = -x^2$ or $y = {}^-2x^2$
c $y = x^2$ or $y = \frac{1}{2}x^2$ **d** $y = 2x^2$ or $y = \frac{1}{5}x^2$
e $y = {}^-3x^2$ or $y = \frac{1}{3}x^2$ **f** $y = 0.25x^2$ or $y = 4x^2$

Fluency

5 **a** Sketch the graphs of the following parabolas on the same set of axes.

i $y = x^2$ **ii** $y = (x + 2)^2$ **iii** $y = (x - 2)^2$

b Describe how the value of a effects the graph of the parabola whose equation is $y = (x + a)^2$.

6 **a** Draw graphs of the following quadratics for $^{-}4 \le x \le 4$ on the same set of axes.

i $y = x^2$ **ii** $y = x^2 + 3$ **iii** $y = x^2 + {}^{-}2$

b For each of the parabolas in part **a**:

i what is the equation of the axis of symmetry?

ii what are the coordinates of the vertex?

iii what is the minimum value?

c Describe how the constant a changes the graph of the parabola with the equation $y = x^2 + a$.

Worked solutions
Exercise 5.3
MAT10NAWS00048
See Example 28

7 Write down the equation of the circle with:

a centre (4, 5) and radius 3 **b** centre ($^{-}2$, 4) and radius 8

c centre (3, $^{-}6$) and radius 5 **d** centre (0, 0) and radius 9

e centre ($^{-}3$, $^{-}5$) and radius 2

Worked solutions
Exercise 5.3
MAT10NAWS00048

8 For circles with the following equations, find:

i the coordinates of the centre **ii** the length of the radius

a $x^2 + y^2 = 16$ **b** $x^2 + y^2 = 1$

c $(x + 2)^2 + (y - 7)^2 = 25$ **d** $(x + 5)^2 + (y + 3)^2 = 9$

e $(x - 6)^2 + (y + 9)^2 = 121$ **f** $(x - 4)^2 + (y - 8)^2 = 64$

g $(x + 3)^2 + (y + 2)^2 = 100$ **h** $(x + 10)^2 + (y - 4)^2 = 225$

9 Sketch each of the following circles on suitable axes.

a $x^2 + (y + 4)^2 = 16$ **b** $(x - 3)^2 + y^2 = 36$ **c** $(x + 2)^2 + (y - 1)^2 = 81$

d $(x - 3)^2 + (y - 3)^2 = 49$ **e** $(x + 1)^2 + (y + 5)^2 = 25$ **f** $(x - 8)^2 + (y + 8)^2 = 64$

See Example 30

10 Consider the exponential function $y = 2^x$.

a Draw the graph of $y = 2^x$ for $^{-}3 \le x \le 3$.

b Use the graph to estimate, correct to one decimal place, the value of:

i 2^0 **ii** 2^1 **iii** $2^{0.5}$ **iv** $2^{1.6}$

c Use the graph to estimate, correct to one decimal place, the value of x if:

i $2^x = 1$ **ii** $2^x = 4$ **iii** $2^x = 7$ **iv** $2^x = 0.8$

11 Consider the exponential function $y = 3^x$.

a Draw the graph of $y = 3^x$ for $^{-}2 \le x \le 2$.

b Use the graph to estimate, correct to one decimal place, the value of:

i 3^0 **ii** $3^{1.5}$ **iii** $3^{0.6}$ **iv** $3^{1.8}$

c Use the graph to estimate, correct to one decimal place, the value of x if:

i $3^x = 3$ **ii** $3^x = 5$ **iii** $3^x = 0.4$ **iv** $3^x = 2.7$

See Example 31

12 Sketch each of the following on suitable axes.

a $y = 3^x - 2$ **b** $y = 2^x + 3$ **c** $y = 4^x - 1$

d $y = 2^x - 3$ **e** $y = 3^x + 4$ **f** $y = 5^x + 1$

Worked solutions
Exercise 5.3
MAT10NAWS00048

See Example 26

13 Draw a graph of each of the following quadratic functions. For each graph:

i show the zeros (if possible), the y-intercept and the turning point

ii identify the turning point as a maximum or minimum.

ⓐ $y = x^2 - 4x - 5$ **b** $y = -x^2 - x + 12$
c $y = x^2 - 6x + 5$ **d** $y = 3x^2 + 24x + 36$
e $y = 2x^2 - 14x + 20$ **f** $y = {}^{-}2x^2 + 4x + 6$
g $y = -x^2 + 2x - 1$ **h** $y = x^2 - 6x + 9$
i $y = 2x^2 - 4x + 2$ **j** $y = 3x^2 + 6x + 21$

14 Each of the parabolas described below has an equation in the form $y = x^2 + c$ or $y = -x^2 + c$. Use the description to write the equation of the parabola in each case.

a a vertex at (0, 0) and concave upwards

b a maximum at (0, 0) and concave downwards

c the axis of symmetry is $x = 0$ and there is a minimum at $y = {}^{-}2$

d concave downwards and a vertex at (0, 3)

e a vertex at $(0, \frac{1}{2})$ and concave upwards

f the y-axis is the axis of symmetry and there is a maximum at $y = {}^{-}\frac{2}{3}$

g a turning point at $(0, {}^{-}6)$ and concave downwards

15 Which, if any, of the following points lie on each of the circles in question **7**?

$A(4, 8)$ $B(3, {}^{-}2)$ $C(6, 4)$ $D(1, 4)$ $E({}^{-}5, {}^{-}5)$

See Example 32

16 Sketch each of the following on suitable axes.

a $y = 0.2^x - 2$ **b** $y = 0.5^x + 3$ **c** $y = 0.4^x - 1$
d $y = 2^{-x} - 1$ **e** $y = 5^{-x} + 2$ **f** $y = 3^{-x} + 1$

See Example 29

17 Sketch the graphs of the following hyperbolas on separate sets of axes. Label one point on each branch of the hyperbolas that you draw.

a $y = \frac{2}{x}$ **b** $y = \frac{{}^{-}4}{x}$ **c** $y = \frac{8}{x}$ **d** $y = \frac{5}{x}$

e $y = \frac{{}^{-}1}{x}$ **f** $y = \frac{{}^{-}20}{x}$ **g** $y = \frac{16}{x}$ **h** $y = \frac{12}{x}$

18 Consider the branch of the hyperbola $y = \frac{12}{x}$ in the first quadrant. Determine if each of the following points is on, above or below the hyperbola.

a (3, 4) **b** (1, 10) **c** (0.6, 24)
d (1.6, 7.5) **e** (2.5, 5)

19 Consider the branch of the hyperbola $y = \frac{{}^{-}20}{x}$ in the fourth quadrant. Determine if each of the following points is on, above or below the hyperbola.

a $(3.8, {}^{-}6)$ **b** $(1.6, {}^{-}12.5)$ **c** $(3.2, {}^{-}6.25)$
d $(1.2, {}^{-}16)$ **e** $(2.6, {}^{-}8)$

20 **a** Copy and complete the following table for the equation $y = \frac{1}{x-3}$.

x	$^{-}1$	0	1	2	2.8	3.2	4	5	6
y									

b Use the points to draw a graph of $y = \frac{1}{x-3}$.

c What type of curve is $y = \frac{1}{x-3}$?

d What are the equations of the asymptotes of the graph of $y = \frac{1}{x-3}$?

Problem solving

21 **a** Sketch the graphs of the following hyperbolas on the same set of axes.

i $y = \frac{3}{x}$ **ii** $y = \frac{^{-}3}{x}$

b Describe how the two graphs are related.

22 **a** Sketch the graphs of the following hyperbolas on the same set of axes.

i $y = \frac{1}{x}$ **ii** $y = \frac{1}{x} + {}^{-}3$ **iii** $y = \frac{1}{x} + 2$

b Describe how the constant a changes the graph of the hyperbola with the equation $y = \frac{1}{x} + a$.

23 An object is dropped from the top of a tall building. The height (h) in metres of the object at any time (t) seconds is given by:

$$h = 105 - 4.8t^2$$

a Draw a graph of the equation for $0 \leq t \leq 5$.

b How tall is the building?

c How far has the object fallen after 2 seconds?

d How long does it take for the object to hit the ground?

24 The distance (d) that a car travels after the brakes are applied at various speeds (s) is given by:

$$d = \frac{3s^2}{500}$$

where d is in metres and s is in km/h.

Draw a graph of the equation for $0 \leq s \leq 100$ and use it to answer the following questions.

a How far does the car travel after the brakes are applied at a speed of 38 km/h?

b How far does the car travel after the brakes are applied at a speed of 75 km/h?

c What was the speed of the car if it took 30 m to stop after the brakes were applied?

25 The equation $(x - 15)^2 + (y + 8)^2 = 289$ is a circle.

- **a** Is the origin on, inside or outside the circle?
- **b** Is the point (4, 1) on, inside or outside the circle?
- **c** Is the point (1, 4) on, inside or outside the circle?
- **d** How can you easily tell whether a point is inside or outside a given circle?

26 The distance, d metres, that a ball falls from its starting position after t seconds is

$$d = \frac{t^3}{8}$$

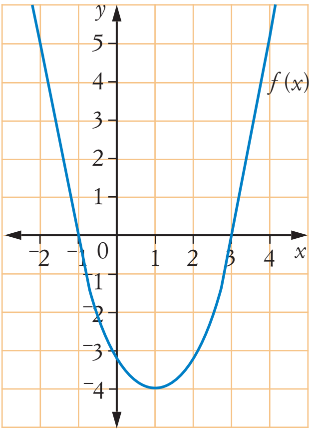
Draw a graph of the equation for $0 \le t \le 4$, using intervals of 0.5 s, and use it to answer the following questions.

- **a** How far has the ball fallen after 2.4 s?
- **b** How far has the ball fallen after 3.8 s?
- **c** When is the ball 6 m from its starting position?

27 Consider the function $y = x^2(2 - x)$.

- **a** Draw a graph of the function for $^{-}1 \le x \le 2.5$.
- **b** What is the maximum value of y for $0 \le x \le 2$?
- **c** What is the x value that gives the maximum calculated in part **b**?

28 The graph of $f(x) = x^2 - 2x - 3$ is shown on the right.

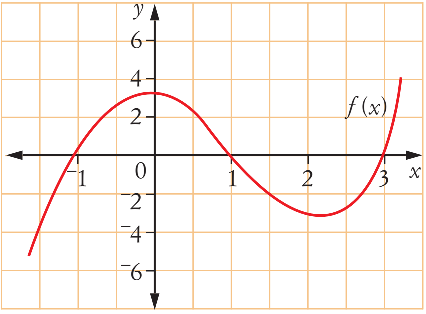


Use this graph to sketch graphs of the following functions.

- **a** $y = f(x) + 4$
- **b** $y = 2f(x)$
- **c** $y = \frac{1}{3}f(x)$
- **d** $y = f(-x)$
- **e** $y = -f(x)$

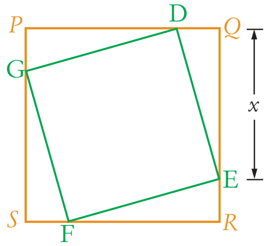
See Example 33

29 The graph of $y = f(x)$ is shown on the right. Use this graph to sketch graphs of the following functions.



- **a** $y = f(x) - 4$
- **b** $y = -f(x)$
- **c** $y = 3f(x)$
- **d** $y = \frac{1}{2}f(x)$
- **e** $y = f(-x)$

30 A square $DEFG$ is drawn inside a larger square $PQRS$ that has a side length of 1 unit, as shown here.



- **a** If $PD = QE = RF = SG = x$ units, show that

$$DQ = (1 - x) \text{ units}$$

- **b** Show that the area of the smaller square is given by

$$A = 2x^2 - 2x + 1$$

- **c** Draw a graph of this equation.
- **d** Use your graph to find the value of x at which the area of the smaller square is a minimum. What is the minimum area of the smaller square?

31 A special fabric is used as a fire retardant. A manufacturer is asked to produce rectangular sheets of the fabric with an area of 24 m^2.

- **a** Write an equation for the area of a sheet, using x for the length and y for the width of the sheet.
- **b** Draw a graph of this equation.
- **c** Use the graph to determine the width of the sheet when the length is 6.5 m.

32 A group of environmentalists set up a project to track the size of the population of possums in a native forest. The size of the population is given by:

$$P = 50 \times 1.06^n$$

where n is the number of months after the project was established.

- **a** What was the population of possums when the project started?
- **b** How many possums were present after 12 months?
- **c** How many possums were present after 30 months?
- **d** Draw a graph of P against n for $0 \leq n \leq 50$.
- **e** How long did it take for the population to reach 500?

33 A population of cockroaches is discovered in a warehouse. The size of the population is given by:

$$C = 200 \times 1.12^n$$

where n is the number of weeks after the cockroaches were discovered.

a How many cockroaches were in the warehouse when they were discovered?

b How many cockroaches were present after 10 weeks?

c How many cockroaches were present after 20 weeks?

d Draw a graph of C against n for $0 \leq n \leq 30$.

e How long did it take for the population to reach 3000?

34 A hobby farmer wants to build a rectangular enclosure for chickens. He has enough material to build an enclosure with a perimeter of 30 m.

a If the length of the enclosure is l and the width is w, complete the expression:

$l = \ldots - \ldots$

b Write an equation for A in terms of w.

c Draw a graph of A against w for $0 \leq w \leq 30$.

d What are the dimensions of the enclosure that will allow a maximum area?

Chapter 5 summary

Quiz

Coordinate geometry

MAT10NAQZ00005

- On the **Cartesian plane,** points are identified by their ***x*-coordinate** and ***y*-coordinate.** The ***x*-axis** and ***y*-axis** cross at right angles at the **origin** to form four **quadrants**. **Cartesian (rectangular) coordinates** are shown in round brackets with the x-coordinate first.
- The graph of a function, $f(x)$, is shown by substituting x values into the function and plotting $(x, f(x))$ on the Cartesian plane. A table of values is often used.
- A **linear function** has a graph that is a straight line. The rule for a linear function is called a **linear equation**. In the Cartesian system a linear equation may have an x term, a y term and a number (constant).
- The distance between two points (x_1, y_1) and (x_2, y_2) is given by the **distance formula**:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- The **midpoint**, M, of two points (x_1, y_1) and (x_2, y_2) is $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
- The points where the graph of a function crosses the axes are called the **intercepts.** (The term 'intercept' can also refer to the distance from the origin to the point where the graph cuts the axis.)
- The **gradient (slope)** of a straight line has the symbol m and is given by the formula:

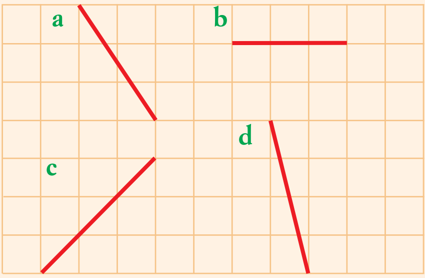
$$\text{Gradient} = m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

  where (x_1, y_1) and (x_2, y_2) are points on the line.
- Upward-sloping (**rising**) lines have a positive gradient and downward-sloping (**falling**) lines have a negative gradient. **Parallel lines** have the same gradient.
- The **standard form** of the equation for a straight line is $ax + by + c = 0$. It has no fractions and all the terms are on one side of the equals sign.
- The **gradient–intercept** form is $y = mx + c$, where m is the gradient and c is the y-intercept.
- The equation of a horizontal line through (a, b), parallel to the x-axis, is $y = b$. The equation of a vertical line through (a, b), parallel to the y-axis, is $x = a$.
- The **gradient–point form** of the equation of a line is $y - y_1 = m(x - x_1)$, where m is the gradient and (x_1, y_1) is a point on the line.
- An **inequation** has a $<$, $>$, $\leq$ or $\geq$ symbol. The graph of an inequation is a region (area) of the Cartesian plane. A solid boundary line is drawn for inequations involving $\leq$ or $\geq$ and a dashed line is drawn for inequations involving $<$ or $>$.
- Lines with gradients m_1 and m_2 are **perpendicular** if $m_1 \times m_2 = {}^{-}1$ and are **parallel** if $m_1 = m_2$.
- A **model** is a representation of real-life patterns and relationships. A relationship that can be represented by a function of the form $y = mx + c$ is called a **linear model**.
- For real-life data, we may need to draw a **line of best fit (trendline)** for the plotted points in order to see the relationship.
- A **quadratic function** has the form $y = ax^2 + bx + c$ and forms a graph that has a shape called a **parabola**. A parabola is symmetrical about the **turning point** or **vertex,** which may be a **maximum** or **minimum.** The **zeros** or **roots** of a quadratic are where the graph cuts the x-axis.
- The **equation of a circle** with centre (x_1, y_1) and radius r is $(x - x_1)^2 + (y - y_1)^2 = r^2$.
- The **equation of a rectangular hyperbola** is $y = \frac{a}{x}$ where a is a constant and $a \neq 0$. A hyperbola has two parts called **branches**.
- Functions that have the general form $y = a^x$ are called **exponential functions**.

Chapter 5 review

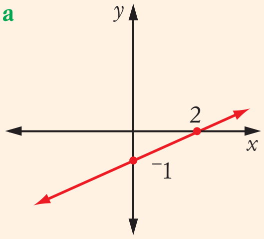
Understanding

1 Use the graphs below to calculate the gradient of each of the segments.



2 The graphs below are not drawn to scale.

a



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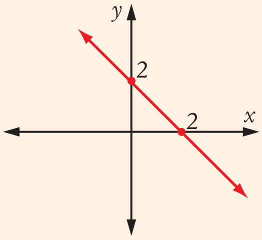

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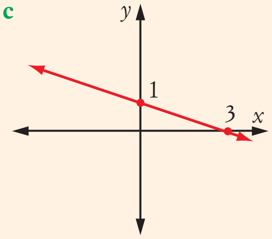

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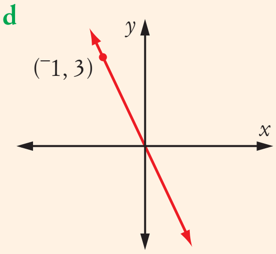


Choose the most appropriate equation for each one from the following list.

A $y = {}^{-}\frac{1}{3}x + 1$ **B** $y = {}^{-}3x$ **C** $y = \frac{1}{2}x - 1$ **D** $y = -x + 2$

3 Draw $y = {}^{-}1$ for ${}^{-}3 \leq x \leq 3$.

4 Draw $y = 3x - 4$ for ${}^{-}2 \leq x \leq 2$.

5 State whether each of the following is a linear equation.

a $3y = 7 - 5x$ **b** $\frac{x}{5} + 2y = 8$ **c** $y = \frac{3}{x} + 4$ **d** $y = x^2 - 4$

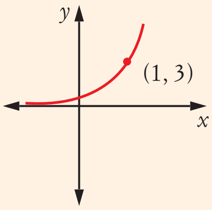
See Example 10 **6** Draw the line $y = 2x - 3$ for ${}^{-}3 \leq x \leq 3$ and find the intercepts.

See Example 19 **7** Determine whether (5, 6) is on the line $y = 2x - 3$.

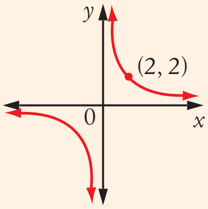
8 Find the equation of the line parallel to the x-axis passing through (⁻4, 7).

9 The graphs below are not drawn to scale.
Match the equation with the graph for quadratic, circular, hyperbolic and exponential functions.

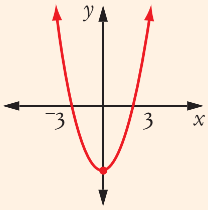
a



b



c



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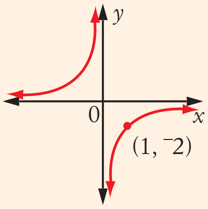

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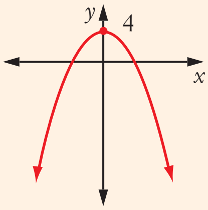

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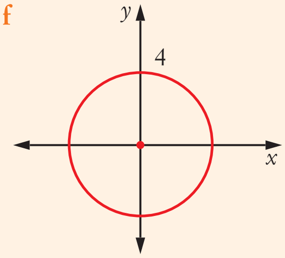


Choose the most appropriate equation for each one from the following list.

i $y = \frac{-2}{x}$

ii $y = x^2 - 9$

iii $y = 3^x$

iv $x^2 + y^2 = 16$

v $y = -x^2 + 4$

vi $y = \frac{4}{x}$

Fluency

10 Find the distance between points $(^{-}2, 4)$ and $(3, ^{-}8)$.

See Example 1

11 Find the midpoint of the line joining $(^{-}2, 4)$ and $(3, ^{-}8)$.

See Example 3

12 Find the slope of the line passing through $(^{-}2, 4)$ and $(1, ^{-}2)$.

See Examples 5, 6

13 Find the equation of the line with a gradient of 3 and y-intercept $(0, ^{-}1)$.

See Example 14

14 Without drawing a graph, find the gradient and y-intercept of the line $x - 2y + 4 = 0$.

See Example 15

15 A line with gradient 3 passes through the point $(^{-}2, 5)$. Find the equation of the line in gradient–intercept and standard forms.

See Example 16

16 Find the equation of the line passing through $(^{-}2, 3)$ with gradient $^{-}\frac{2}{3}$, in standard form.

17 A line passes through the points $(2, ^{-}1)$ and $(^{-}4, 5)$. Find the equation of the line in gradient–intercept and standard forms.

See Example 17

18 Without drawing a graph, determine whether $3x - 4y + 2 = 0$ and $8x + 6y + 1 = 0$ are parallel, perpendicular or neither.

19 State whether each of the following pairs of gradients, m_1 and m_2, represent lines that are parallel, perpendicular or neither.

a $m_1 = 3$ and $m_2 = \frac{1}{3}$

b $m_1 = 0.25$ and $m_2 = \frac{1}{4}$

c $m_1 = 2$ and $m_2 = ^{-}0.5$

20 For each line with gradient m_1 shown below, find the gradient m_2, of the line perpendicular to it.

a $m_1 = 4$ **b** $m_1 = 2\frac{1}{2}$ **c** $m_1 = {}^{-}0.4$

See Example 20 **21** Find the equation of a line parallel to $y = 3x + 4$, passing through $({}^{-}1, {}^{-}5)$.

22 Find the equations of the lines parallel and perpendicular to $y = 3x + 4$, passing through $({}^{-}1, {}^{-}5)$, in standard form.

See Example 28 **23** Find the equation of a circle with radius 4 and centre $({}^{-}1, 3)$.

See Example 30 **24**
a Sketch the graph of the exponential function $y = 4^x$ for ${}^{-}2 \leq x \leq 2$.
b Use the graph to estimate, correct to one decimal place, the value of $4^{0.8}$.
c Use the graph to estimate, correct to one decimal place, the value of x if $4^x = 6$.

See Example 13 **25** Graph the inequation $y < 2x - 1$.

See Example 26 **26**
a Draw the graph of $y = x^2 - x - 6$ and use it to identify the main features of the parabola.
b Identify the turning point as a maximum or minimum.

See Example 29 **27** Sketch the graph of the hyperbola with equation $y = \frac{10}{x}$ and label one point on the graph.

Problem solving

28 The midpoint of AB is $(4, {}^{-}1)$. If $A = (6, {}^{-}3)$, find point B.

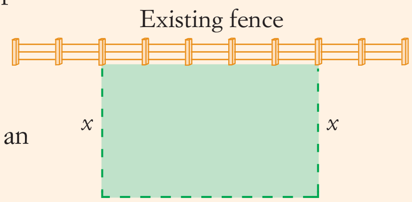
See Example 4 **29** Show that the points $(5, {}^{-}4)$, $({}^{-}4, {}^{-}1)$, $({}^{-}6, 6)$ and $(3, 3)$ are the vertices of a parallelogram.

See Example 9 **30** Show that the points $(6, 1)$, $(4, 4)$, $(5, {}^{-}2)$ and $({}^{-}3, {}^{-}5)$ are the vertices of a rectangle.

See Examples 23, 24 **31** The speed of a car after passing a checkpoint is given by the function $S = 10 + 3T$, where S is the speed in m/s and T is the time in seconds after passing the checkpoint.
a Draw a graph of the function for $0 \leq T \leq 10$.
b Use the graph to find when the speed of the car will reach 30 m/s.
c Use the graph to find the speed of the car after 4 seconds.

See Examples 23, 24 **32** The owner of a fast-food outlet has a total of $200 to spend on hamburger patties and frankfurters for hot dogs each day. Each hamburger patty costs $1.60, while each frankfurter costs 60c.
a Write an equation for the number of hamburger patties (h) and frankfurters (f) that the owner could buy.
b Complete a table of values for h and f and use it to draw a graph of the equation.
c If the owner knows that she usually sells 70 hamburgers a day, how many frankfurters should she buy?

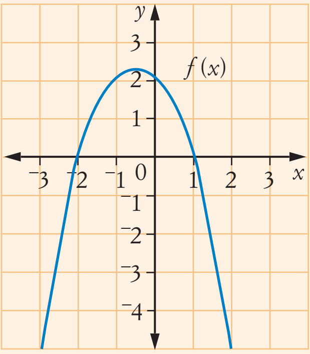
33 A hobby farmer wants to fence off a rectangular pen for chickens using an existing fence as one boundary for the pen, as shown. She has only 16 m of fencing material and wants to use it all to form the pen.



a If the side of the chicken pen is x metres long, form an equation for the area, A, of the pen.
b Draw a graph of the area.
c What is the maximum area of the pen?
d Find the dimensions of the pen that give the maximum area.

34 The graph of $f(x) = 2 - x - x^2$ is shown below.

See Example 31



Use this graph to sketch graphs of the following functions.

a $y = f(x) + 3$ **b** $y = 3f(x)$ **c** $y = -f(x)$ **d** $y = f(-x)$

---

35 The hearing ability, h, measured in hearing units, of people of various ages, a, was measured as follows.

Reasoning

See Example 25

Age, *a* (years)	8	24	43	51	62	77
Hearing ability, *h*	96	69	64	49	32	28

a Plot the points and draw a line of best fit for the data.
b Find the equation of the line of best fit.
c Use the equation of the line of best fit to calculate the likely hearing ability of a 30-year-old.
d At what age would you expect a person's hearing ability to reach 20 hearing units?

Statistics and probability

6 Probability

9780170194860

Contents

Prior learning
Chapter 6
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Australian Curriculum statements

Chance

Describe the results of two- and three-step chance experiments, both with and without replacements, assign probabilities to outcomes and determine probabilities of events. Investigate the concept of independence. (10)

Use the language of 'if … then, 'given', 'of', 'knowing that' to investigate conditional statements and identify common mistakes in interpreting such language. (10)

Investigate reports of studies in digital media and elsewhere for information on their planning and implementation. (10A)

Video tutorial
Probability
MAT10SPVT00006

Weblink
Jonny Heeley's masterclass on probability
MAT10SPWB00006

Radioactive materials are an important part of our modern world. They are used in medicine to treat cancer and to trace blood in its journey around the body to treat cardiovascular diseases. The theory of radioactivity uses mathematical probability theory. Indeed, much of modern physics relies on probability to explain the atomic and subatomic theories about matter. While it is certain that uranium will decay to give other elements, it is chance that determines which atoms of the uranium decay. The probabilities are worked out using energy states within the uranium nucleus. Calculation of the statistics from the probabilities produces the theory of atomic decay. Probability and statistics are both important to understanding the certainties of physics, as well as their uses in our society as a whole.

Mathematical literacy

Maths dictionary
MAT10ASDI00001

The mathematical words below have special meanings that you will learn in this chapter. It is important that you learn to spell them and gradually learn what they mean in mathematics. You may find the glossary or online mathematical dictionary useful for this purpose.

and
assuming
certain
complement
conditional probability
dependent
element
event
expected frequency
experimental probability
favourable outcome
frequency
given
if … then
impossible
independent
intersection
multiplication principle
mutually exclusive
null set
or
outcome
probability
relative frequency
sample point
sample space
simplified tree diagram
theoretical probability
tree diagram
trial
two-way table
union
universal set
Venn diagram

6.1 Probabilities of single events

You already know how to calculate probabilities from frequencies. You should also know that although you can use probability to predict the proportion of times something will happen in the long run, it will not tell you what will definitely happen next. The probability of tossing a tail the next time you toss a normal coin is always 0.5. It doesn't matter if you have tossed 20 heads in a row, the probability is still 0.5. While there is natural variation, the greater the number of trials, the closer the theoretical and experimental probabilities will be.

Investigate: Lotto simulation

Teacher notes
Lotto simulation
MAT10SPTN00003

CAS TI-Nspire exercise
Probability
MAT10SPTI00006

CAS ClassPad exercise
Probability
MAT10SPCP00006

Australia, like many countries, has various Lotto games where you choose 4 or more numbers up to 36, 40, 50, 70 or 80. They are marketed as Powerball, Gold Lotto, Lotto Strike, Keno, Oz Lotto, Australian Soccer Pools, Intralot, Super 66, Cash 3 and so on. If some or all of your numbers 'come up', you win a prize. Most people who play these games do so in the hope of winning the major prize or jackpot. The probability of winning the major prize varies with the format but it is usually less than 0.000 001. The game below, *Class Lotto*, only uses ten numbers.

Rules for *Class Lotto*

1 Each person in the class chooses 4 numbers from 0 to 9 to put on their entry form.
2 You need to put the game number on your form before it is collected.
3 The entry forms are collected and the draw is done.
4 Your teacher will use a ten-sided die or a computer to pick the winning numbers.
5 The winner (if there is one) is the person whose numbers come up.

Class Lotto entry form Game number: _____
Name: ________________________________
Circle four numbers:
0 1 2 3 4 5 6 7 8 9

- Play the game until someone in the class wins.
- Then keep going until someone wins again.
- Would you need to go this long again until you got another winner?
- Try it out to see how long it takes.
- What is the experimental probability of winning *Class Lotto*?
- Compare the experimental probability with the theoretical probability (your teacher will tell you what it is).

The probability of winning *Class Lotto* is about 5000 to 20 000 times as good as winning any of the commercial games. This means that, in the long run, you would win *Class Lotto* at least 5000 times before you won a commercial lotto game.

Important!

Probability

Experimental probability uses real data to work out chances or likelihoods. Each item of data is called a **trial**. **Theoretical probability** is based on a list that contains all the possible outcomes. The list is called the **sample space**. Every outcome in the sample space is listed exactly as many times as it is possible for it to occur. A single outcome is called an **element** or **sample point**.

An **event** is a collection of outcomes, whether they come from trials or a sample space. This set could have any number of elements, from none up to the total number.

The outcomes in an event for which you are calculating the probability are called **favourable outcomes**. The number of elements in an event is written as n**(event)** or is the **frequency** of the event.

The probability of an event is a number from 0 to 1, written as P**(event)** and worked out using the formulas below, provided that all elements of the sample space have an equal chance of occurring.

$$P(\text{event}) = \frac{\text{Frequency of event}}{\text{Total Frequency}} = \frac{\text{Number of favourable outcomes}}{\text{Number of trials}}$$

$$\text{or } P(\text{event}) = \frac{n(\text{event})}{n(\text{sample space})}$$

Experimental probabilities are also called **relative frequencies**. Probabilities can be written as fractions, decimals or percentages. An event which has a probability of 0 can never happen and is said to be **impossible**. An event with a probability of 1 always happens and is said to be **certain**.

A probability measures the expected proportion of the time an event will occur.

Expected frequency is the product of the probability and the future number of attempts.

An event E is composed of elements in the sample space. The **complement** is the event consisting of the rest of the sample space, and is shown as E′ or $\overline{\text{E}}$. The sum of the probability of an event and its complement is 1, so $P(\text{E}) + P(\text{E}') = 1$, $P(\text{E}') = 1 - P(\text{E})$ and $P(\text{E}) = 1 - P(\text{E}')$.

Example 1

The masses of 100 g chocolate bars were measured as they came off the production line. The results (in grams) were as follows.

100	99	99	103	100	100	101
102	102	100	101	100	101	101
102	99	99	98	100		

a What is the probability of a bar being underweight?
b How many underweight bars would you expect to get in a carton of 2000 bars?
c How many bars would you need to buy to end up with 30 overweight bars?

Solution

a	Check the numbers.	**There are 19 bars, 5 underweight and 8 overweight.**
	Write the formula.	**Probablity** $= \frac{\text{favourable outcomes}}{\text{trials}}$
	Substitute values.	$= \frac{5}{19} \approx 0.263$
b	Multiply by the number of trials.	**Expected frequency** $= \frac{5}{19} \times 2000 \approx 526$
	Write the answer.	**There would be about 526 underweight bars from 2000.**
c	Write an equation for the number of trials needed.	$30 = \frac{8}{19} \times t$
	Solve to find *t*.	$t = \frac{30 \times 19}{8} \approx 71$
	Write the answer in a sentence.	**You would need about 71 bars to find 30 overweight ones.**

Example 2

Worksheet

Single event probabilities

MAT10SPWK00018

Peter, Louisa, Jessica, David and Hannah play 'eenie, meenie, minie, mo' to decide which two will sit in the front on a roller coaster ride.

a List the sample space for the front seats.
b What is the probability that two girls will be at the front?
c What is the probability that there will be at least one boy at the front?

Solution

a Use the first letters of their names to list the sample space. It doesn't matter who is first.

Sample space = {PL, PJ, PD, PH, LJ, LD, LH, JD, JH, DH}

b There are 3 combinations with 2 girls: LJ, LH and JH.

$$P(\text{two girls}) = \frac{3}{10} = 0.3 = 30\%$$

Write the answer.

The probability of two girls in front is 30%.

c 'At least one boy' is the complement of 'two girls'.

$$P(\text{at least one boy in front}) = 1 - 0.3 = 0.7$$

Write the answer.

The probability of at least one boy being at the front is 70%.

Investigate: Archery targets

Worksheet
Area problems
MAT10SPWK00039

An archery target is divided into 5 colours with a thin line in the middle of each colour. Points are given for arrows that land in each zone as shown in the diagram on the right. The inner ring in the middle of the 10 part of the yellow circle is only used for tie-breaks and still scores 10. The whole target is 122 cm or 80 cm in diameter.

If you hit the target, what is the probability of scoring 6?

- Assuming the target is 80 cm in diameter, what is the area of the whole target?
- What are the diameter and radius of the 6 point circle?
- What is the area of the 6 point circle?
- What is the probability of scoring 6?
- What are the probabilities of the other scores, assuming that you do hit the target?

In the investigation above, there are an infinite number of places where an arrow can land on a target. It is still sensible to work out probability in situations like this where there are an infinite number of possible events, so you cannot list a sample space. Mathematically defining *probability* for infinite sets is quite difficult. However, you can still calculate probability as a *ratio*. You need to be able to find the fraction of the favourable events to all the events using area or some other method.

Example 3

Ten-cent coins (diameter 23.6 mm) are thrown onto a small table that has been divided into squares of side length 4 cm. If the coin lands completely inside a square, you get 50c back.

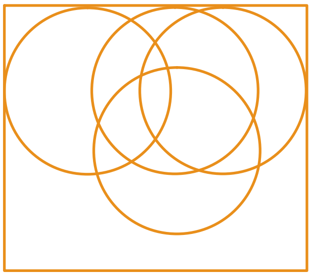
a What is the probability you win?

b Your friends say you'll probably lose. Are they right?

Solution

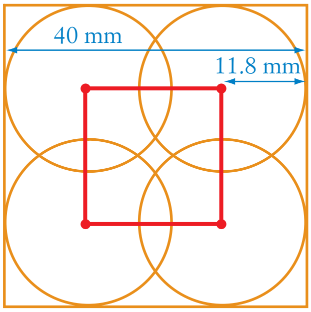
a Draw a square and show some places where the coin could land and be completely inside.

It is immediately clear that the centre of the circle is restricted to a much smaller square.



Draw a diagram showing the square where the coin's centre could land.

For the coin to be completely inside the square, its centre must land within a smaller square that is 40 − 2 × 11.8 mm wide.



Calculate the area that the centre can land in to be completely inside.

Area inside $= \mathbf{16.4 \times 16.4\ mm^2}$
$= \mathbf{269\ mm^2}$

Calculate the area of the whole square.

Area of square $= \mathbf{40 \times 40}$
$= \mathbf{1600\ mm^2}$

Find the probability of landing completely inside as the ratio of the areas.

$$\boldsymbol{P}(\textbf{inside}) = \frac{\mathbf{269}}{\mathbf{1600}} \simeq \mathbf{0.168}$$

Write the answer.

The probability of winning is about 17%.

b To make money, you'd have to win 1 in 5 times.

17% < 20%, so your friend is right.

If coins landed too close to the edge of the table in Example **3**, they would fall off. However, in sideshow alley games like this, it wouldn't make any difference to the probability.

Probabilities are used in betting, but they are given as odds. Of course, the odds used by bookmakers, TAB, casinos, poker machines, keno games and so on are *not fair*. They need to make money from the people who bet to pay for venues, equipment, staff, and other costs and to provide a profit.

Important!

Odds

Odds are given as a ratio. Odds of $n : m$ mean that for every m dollars you bet, you get n dollars plus the amount of your bet if you win. For **odds on**, the ratio is given in the reverse order.

Example 4

For each of the following bets, calculate the amount you would get back if you won.

a \$5 at 6 : 1 **b** \$4 at 3 : 2 **c** \$8 at 2 : 1 on

Solution

a Calculate the amount won. **Amount won** $= \$5 \times \frac{6}{1} = \$30$

Calculate the amount received. **Amount back for win** $= \$30 + \$5 = \$35$

b Calculate the amount won. **Amount won** $= \$4 \times \frac{3}{2} = \$6$

Calculate the amount received. **Amount back for win** $= \$6 + \$4 = \$10$

c Calculate the amount won, remembering that odds on means the ratio is reversed. **Amount won** $= \$8 \times \frac{1}{2} = \$4$

Calculate the amount received. **Amount back for win** $= \$4 + \$8 = \$12$

Fair odds are based on probability. That is, in the long run you would expect to come out even, neither winning nor losing. Consider the case of a spinner with 15 equal sectors where you won if it landed on either of 2 sectors. Then the probability of winning would be $\frac{2}{15}$. You would expect to win about 2 times out of every 15 times you played. If you did play 15 times and won twice, then you would get back your \$1 bet each time, so you would need to win another \$13 to come out even. In other words, you would need to win \$6.50 each time you won to break even. The odds would be the ratio 6.5 : 1 = 13 : 2. Notice that 2 is the denominator in the probability and 13 = denominator − numerator. Using this as a rule will obviously always produce fair odds.

Important!

Fair odds

For a situation where your probability of winning is the fraction $\frac{p}{q}$, the fair odds are $(q - p) : p$.

Example 5

Puzzle sheet
Fair odds
MAT10SPPS00049

What are the fair odds for betting on a situation where you have a 12% chance of winning?

Solution

Write 12% as a fraction. $12\% = \frac{12}{100} = \frac{3}{25}$

Find the difference between the denominator and the numerator. $25 - 3 = 22$

Write the odds. **The fair odds are 22 : 3.**

Exercise 6.1 Probabilities of single events

Understanding

See Example 4

1 How much would you get back if you won each of the following bets?

a \$15 at 5 : 1 **b** \$20 at 2 : 1 on **c** \$10 at 4 : 1
d \$30 at 4 : 3 on **e** \$40 at 5 : 4

See Example 5

2 What are the fair odds for each of the following situations?

a The probability of winning is 0.3. **b** The probability of winning is 10%.
c The probability of winning is $\frac{3}{11}$. **d** The probability of wining is $\frac{7}{36}$.
e The probability of winning is 25%.

Fluency

Extra questions
Exercise 6.1
MAT10SPEQ00044

3 In June, Brisbane has an average of 8.7, Sydney 12.5, Melbourne 15.4, Adelaide 14.9, Perth 15.1, Darwin 0.6 and Hobart 14.5 rainy days.

a What is the probability of a rainy day in June in each capital?

b If June is typical of the winter, how many rainy days would you expect in Melbourne over winter?

c How many days in Brisbane in winter would you expect to be rain-free?

See Example 1

4 In a particular oyster bed off the Western Australian coast, a diver found 7 pearls in 58 oysters.

a What was the probability of finding a pearl in an oyster?

b What was the probability of not getting a pearl?

c How many pearls would you expect to find in 400 oysters from this bed?

d How many oysters would you have to open to get 400 pearls?

5 There are 52 cards in a normal pack. 'Cutting' the pack selects one card, which is then replaced.

a What is the probability of cutting a heart?

b What is n(jack)?

c What is P(jack)?

d What is the probability of cutting a picture card?

e What is the probability of cutting an ace?

f What is the probability of cutting a number card?

g If the cards were cut 60 times, how many court cards (K, Q, J) would you expect to get?

See Example 2

6 Esmeralda has 3 sun-caps (red, yellow and blue) and 2 colours of zinc cream (pink and yellow) for her face.

a Show the sample space of combinations of caps and cream, with zinc cream first.

b What is the probability that Esmeralda has a matching sun-cap and zinc cream, if she puts on both at random?

See Example 3

7 A game has a table divided into 5 cm squares for throwing Australian 20 cent coins. You win a dollar if you land the coin completely inside a square. Australian 20 cent coins are 28.5 mm in diameter.

a What is the probability of winning?

b Is the game set up for the operator to win?

8 Find the probability for each of the following fair odds.

a 4 : 1 on **b** 10 : 1 **c** 6 : 1 **d** 3 : 2 on **e** 7 : 5

Problem solving

⑨ Insurance companies use records of deaths to work out the rates for life insurance. The table on the right shows the numbers of deaths expected from 50 000 Australian men and 50 000 Australian women at certain ages in 1 year.

Age at death	Number of deaths	
	Males	Females
20	32	13
30	41	19
40	67	39
50	161	94
60	348	205

Source: From ABS mortality tables

Worked solution
Exercise 6.1
MAT10SPWS00049

a What is the probability that a 30-year-old man will die in the next year?

b From 53 000 40-year-old men, how many would you expect to die in the next year?

c Work out a recommended premium for $100 000 life insurance coverage for a 50-year-old man for 1 year, assuming that 40% of the premium is taken by costs.

d Work out the premium for a 60-year-old woman on the same basis.

e A leading insurance company quotes a premium of $23.44 per month on the internet for $200 000 cover on a 40-year-old female non-smoker. Use the probability of death from the table to decide whether this is a reasonable premium.

f Another insurer quotes $79.05 per fortnight on an internet calculator for $200 000 cover for a 60-year-old male non-smoking electrician. Use the probability of death from the table to decide whether this is a reasonable premium.

10 A drawer contains 3 green socks and 2 yellow socks. Two socks are taken out at random.

a List the sample space using the abbreviations G1, G2, G3, Y1 and Y2 for the socks.

b Which event is more likely: getting matching-colour socks or unmatched socks?

c What is the probability of getting 2 green socks?

d What is the probability of getting 2 yellow socks?

e What are the fair odds for matching colours?

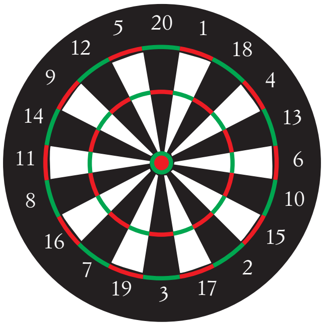
⑪ A dartboard is 451 mm in diameter, and the scoring area has a diameter of 340 mm. The inner bull has a diameter of 12.7 mm and the outer bull has a diameter of 31.8 mm. The outside of the treble ring has a diameter of 214 mm and the double and treble rings are both of width 9 mm. Beginners generally struggle to consistently hit the dartboard. Assuming that you hit the board:

a What is the probability of scoring?

b What is the probability of getting a triple?

c What is the probability of getting a double 3?

d What are the fair odds for a double?



Worked solution
Exercise 6.1
MAT10SPWS00049

Worked solution
Exercise 6.1
MAT10SPWS00049

Reasoning

⑫ In question **7**, explain why it doesn't matter how big the table is. What about if you could have another go if your coin fell off? Explain your answer.

6.2 Probabilities of combined events

Many probability questions are concerned with combined events. The simplest ways of considering combined events are by using **two-way tables**, **tree diagrams** and **Venn diagrams**.

Important!

Use of probability diagrams

You will find two-way tables most useful for situations with two dimensions. Situations with two variables are of this kind.
Tree diagrams are most useful for situations in stages. If one part occurs before another, or you can consider it like that, then a tree diagram is a useful way to visualise the problem.
You would obviously use Venn diagrams when a situation can be thought of in terms of combinations of sets. They are particularly useful for situations involving ‘and/or’ combinations.

Example 6

TLF Learning object

Performance of diagnostic tests (R12187)

MAT10SPIN00006

The measured reaction times and the time (in hours) students said they spent watching TV on school nights are recorded in the table below.

	Reaction time (seconds)		
TV hours	**0.2–0.29**	**0.3–0.39**	**0.4+**
0–5	37	102	65
6–15	29	80	57
> 15	11	29	24

a What is the probability that a student who watches more than 15 hours of TV on school nights has a reaction time under 0.3 seconds?

b What is the probability that a student who watches less than 6 hours of TV on school nights has a reaction time under 0.3 seconds?

c Compare the probabilities of reaction times under 0.3 seconds of students who watch less than 6 hours of TV on school nights with those who watch more than 15 hours of TV on school nights.

d Do these results support the idea that parents should restrict their children’s TV watching?

Solution

a Write the number of trials. $n(\text{watch more than 15 hours of TV}) = 64$

Find the probability. $P(< 0.3 \text{ s with more than 15 hrs}) = \frac{11}{64} \approx 0.1719$

b Write the number of trials. $n(\text{watch less than 6 hours of TV}) = 204$

Find the probability. $P(< 0.3 \text{ s with less than 6 hrs}) = \frac{37}{204} \approx 0.1814$

c 17% and 18% are nearly the same. **There is little difference between the reaction times under 0.3 seconds of students who watch less than 6 hours and those who watch more than 15 hours of TV on school nights.**

d Reaction times are not affected. **If you are only concerned about reaction times, the results do not support restrictions, but most parents would have other concerns.**

Tree diagrams can be extended to cope with three stage events.

Example 7

Video tutorial
Tree diagrams
MAT10SPVT10013

Puzzle sheet
Combined events: Tree diagrams
MAT10SPPS00015

CAS TI-Nspire exercise
Probability
MAT10SPTI00006

CAS ClassPad exercise
Probability
MAT10SPCP00006

Three coins are tossed together.

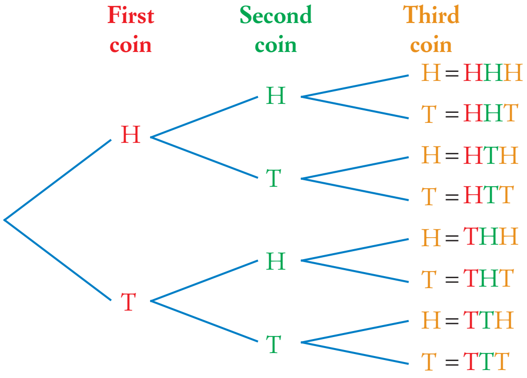
a Use a tree diagram to show the sample space.

b What is the probability of getting at least 2 heads?

Solution

a It makes no difference whether the coins are tossed at the same time or one after the other. You can draw the tree diagram as if they are tossed in succession.

There are 8 elements in the sample space.



b Write the formula. $P(\text{at least 2 heads}) = \frac{n(\text{at least 2 heads})}{n(\text{sample space})}$

Include 2 heads and 3 heads. $= \frac{4}{8} = \frac{1}{2} = 0.5$

When people say '2 heads' in situations like Example **7**, they usually mean 'exactly 2 heads'. This is different to 'at least 2 heads', and different again to '2 heads in a row'. You can see from the tree diagram above that $P(\text{exactly 2 heads}) = \frac{3}{8}$ and $P(\text{2 heads in a row}) = \frac{1}{4}$. It is not really necessary to show the elements of the sample space on the end of the tree diagram, so the HHH, HHT, HTH, etc. can be omitted on the end of the tree diagram.

When there are more than a few possibilities, you will find the size of a tree diagram becomes quite large. You can combine branches, provided you keep track of the number of branches at each node.

Important!

Simplified tree diagrams

A simplified tree diagram shows repeated branches on a normal tree diagram as single branches. You write a fraction in the combined branch. The numerator shows the number of repeated branches that have been combined. The denominator shows the total number of branches.

Animated example

Probabilities of combined events

MAT10SPAE00006

Puzzle sheet

Selections without replacement

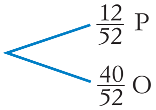
MAT10SPPS00050

Example 8

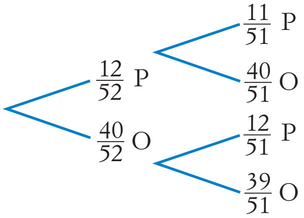
Three cards are dealt from a normal 52-card pack after it is shuffled. Use a simplified tree diagram to find the probability that they are all picture cards (jacks, queens or kings).

Solution

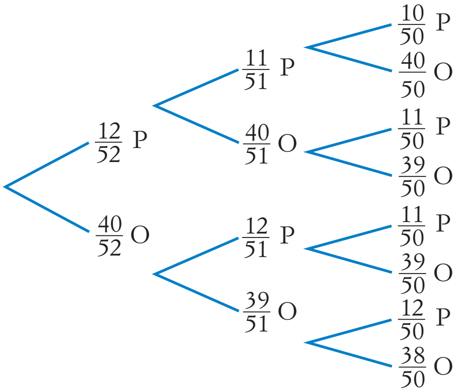
For the first card, combine the 12 picture card branches and the 40 other card branches from the 52 cards at the node.



This is a non-replacement question, so the second card is chosen from 51 cards.
On the picture branch, 11 are picture cards and there are still 40 non-pictures.
On the other branch there are still 12 picture cards but only 39 others.



There are 50 cards left to pick from.
On the first node there are 10 picture cards and 40 others left.
On the second node there are 11 picture cards and 39 others left.
On the third node there are also 11 picture cards and 39 others left.
On the last node there are 12 picture cards and 38 others left.



You want the P-P-P combined branches. These really have 12 × 11 × 10 branches from the total of 52 × 51 × 50 branches altogether in an unsimplified diagram.

$$P(\mathbf{3\ pictures}) = \frac{12 \times 11 \times 10}{52 \times 51 \times 50}$$

Work out the answer.

$$= \frac{11}{1105} \approx 0.009955$$

Round and write the answer.

The probability of getting 3 picture cards is just under 1%.

It does make a difference whether the cards are replaced or not. In Example **8**, the probabilities in the second stage are affected by the result at the first stage, but in Example **7** the first stage does not affect the second.

Important!

Dependent and independent events

Two events are **independent** if the probability of one is *not* affected by the occurrence of the other. Dependent events are not **independent**.

Investigate: Babies' names

Teacher notes
Qld v. NSW
MAT10SPTN00004

In NSW, the probability that a newborn baby will be named Michael is obviously independent of the probability that a newborn in Queensland whose parents are not known to the parents of the NSW couple will be named Michael. Suppose that on a particular day 25 babies are born at a NSW hospital and 23 are born in a Queensland hospital. Three of the NSW babies are named Michael and 2 of the Queensland babies are named Michael. What is the probability that if one baby from each of NSW and Queensland is chosen at random they will both be named Michael?

- In how many ways can you choose a baby named Michael from Queensland?
- For each of the ways that a baby named Michael is chosen from Queensland, how many ways can a baby named Michael be chosen from NSW?
- How many ways altogether does this mean that babies named Michael can be chosen from both Queensland and NSW?
- For each of the ways that a baby named Michael is chosen from Queensland, how many ways can a baby *not* named Michael be chosen from NSW?
- How many ways altogether does this mean that a baby named Michael can be chosen from Queensland and one not named Michael can be chosen from NSW?
- In how many ways can you choose a baby *not* named Michael from Queensland?
- For each of the ways that a baby *not* named Michael is chosen from Queensland, how many ways can a baby named Michael be chosen from NSW?
- How many ways altogether does this mean that a baby *not* named Michael can be chosen from Queensland and one named Michael can be chosen from NSW?
- For each of the ways that a baby *not* named Michael is chosen from Queensland, how many ways can a baby *not* named Michael be chosen from NSW?
- How many ways altogether does this mean that a baby *not* named Michael can be chosen from Queensland and one *not* named Michael can be chosen from NSW?

Puzzle sheet

Combined events: Two-way tables

MAT10SPPS00014

- Complete the two-way table below.

		New South Wales		
		Michael	Not Michael	Total
Queensland	Michael			
	Not Michael			
	Totals			575

Work out the following.

- The probability of a baby named Michael being chosen from Queensland.
- The probability of a baby named Michael being chosen from NSW.
- The probability of a baby named Michael being chosen from Queensland and a baby named Micahel being chosen from NSW.
- The product of the probability of a baby named Michael being chosen from Queensland and the probability of a baby named Michael being chosen from NSW.

What do you find?

From the investigation above, it is clear that the probability of a combination of independent events can easily be calculated from the individual probabilities.

Important!

Probability of a combination of independent events

The probability of two independent events A and B both occurring is the product of their individual probabilities.
For *independent* events A and B, $P(\text{A and B}) = P(\text{A}) \times P(\text{B})$.

Example 9

Two dice are rolled together. What is the probability they both land on 1?

Solution

The results for the dice are independent.	$\boldsymbol{P(\text{1st is 1 and 2nd is 1}) = P(\text{1st is 1}) \times P(\text{2nd is 1})}$
Substitute the values.	$= \frac{1}{6} \times \frac{1}{6}$
Calculate the answer.	$= \frac{1}{36}$

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The commonly used English words ‘and’ and ‘or’ are often used as if they mean the same thing. In mathematics the difference between them is very important.

Important!

The meanings of ‘and’, ‘or’

In mathematics, **‘… and …’** means *both*, but **‘… or …’** means *at least one*.
‘and’ corresponds to the **intersection** of sets, while ‘or’ corresponds to the **union** of sets.
The intersection of two sets A and B, $A \cap B$, is the set that contains the elements in both A and B.
The union of two sets A and B, $A \cup B$, is the set that contains the elements in at least one of A or B.

Example 10

Video tutorial
Venn diagrams
MAT10SPVT10014

a Draw a Venn diagram to show the numbers from 1 to 15 inclusive. Show set A = {odd numbers} and B = {prime numbers}. Use your diagram to find the following probabilities for a number from 1 to 15.

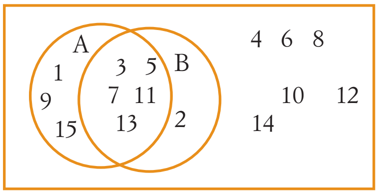
b It is odd and prime.

c It is odd or prime.

d It is even or prime.

Solution

a Show all the numbers in the universal set (the numbers from 1 to 15) in the rectangle. Put the numbers in the appropriate parts of the Venn diagram. Remember that 2 is the first prime number.



b Write the formula using set notation.

$$P(\textbf{odd and prime}) = \frac{n(A \cap B)}{n(U)}$$

Substitute values (there are 5 in the intersection).

$$= \frac{5}{15} = \frac{1}{3}$$

c Write the formula.

$$P(\textbf{odd or prime}) = \frac{n(A \cup B)}{n(U)}$$

Substitute values.

$$= \frac{9}{15} = \frac{3}{5}$$

d Write the formula.

$$P(\textbf{even or prime}) = \frac{n(\overline{A} \cup B)}{n(U)}$$

Substitute values.

$$= \frac{12}{15} = \frac{4}{5}$$

The link between and, or, intersection and union allows you to work out a formula for combinations of probabilities.

Important!

Probability of a combination involving and/or

The probability of the combined event (A or B) is given by the formula below.

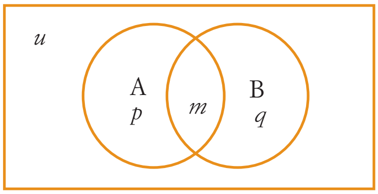
$$P(\text{A or B}) = P(\text{A}) + P(\text{B}) - P(\text{A and B})$$

Technology: GeoGebra
Probability and Venn diagrams
MAT10SPTC00006

This is easily shown with a Venn diagram of events A and B with a sample space (universal set) having u members.

Suppose there are m elements in $\text{A} \cap \text{B}$, p elements in A that are not in B and q elements in B that are not in A. Show all the numbers in the diagram.

Obviously $n(\text{A} \cup \text{B}) = p + m + q$



Then $P(\text{A or B}) = \dfrac{n(\text{A} \cup \text{B})}{n(\text{U})} = \dfrac{m + p + q}{u}$

Now add 0 to the top line in the form $m - m$.

Then you get $P(\text{A or B}) = \dfrac{m + p + q + m - m}{u}$

$$P(\text{A or B}) = \frac{(m + p) + (q + m) - m}{u} = \frac{n(\text{A}) + n(\text{B}) - n(\text{A} \cap \text{B})}{u} = \frac{n(\text{A})}{u} + \frac{n(\text{B})}{u} - \frac{n(\text{A} \cap \text{B})}{u}$$

But $\dfrac{n(\text{A})}{u} = P(\text{A})$, $\dfrac{n(\text{B})}{u} = P(\text{B})$ and $\dfrac{n(\text{A} \cap \text{B})}{u} = P(\text{A and B})$, so

$P(\text{A or B}) = P(\text{A}) + P(\text{B}) - P(\text{A and B})$ **QED**

Example 11

When they were children, 20% of adults had measles, 30% had chickenpox and 12% had both. What is the probability that an adult chosen at random had childhood measles or chickenpox?

Solution

Write the formula.	$\boldsymbol{P(\text{A or B}) = P(\text{A}) + P(\text{B}) - P(\text{A and B})}$
Substitute values.	$\boldsymbol{= 0.2 + 0.3 - 0.12 = 0.38}$
Write the answer.	**The probability is 38%.**

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Exercise 6.2 Probabilities of combined events

Understanding

Extra questions
Exercise 6.2
MAT10SPEQ00017

1 Two normal coins are tossed together. What is the probability that they are both heads?

2 Two weighted coins are tossed together. They have a 60% chance of landing with heads up. What is the probability that they are both heads?

See Example 9

3 What is the probability that 3 normal coins tossed together will all be heads?

4 What is the probability that 4 normal coins tossed together will all be heads?

5 If 3 normal dice are tossed, what is the probability of getting a total of 18?

6 Three dice have the dot for the ‘one’ drilled out and lead inserted so that the chance of getting a six increases to 25%.

- a What is the probability that when two dice are tossed you get a double six?
- b What is the probability that when all three dice are tossed the total is 18?

See Example 11

7 A number is chosen at random from the numbers 1 to 50. What is the probability that it is a multiple of 3 or 4?

8 A card is chosen at random from a normal pack. What is the probability that it is a number card (not A, K, Q, J) or a heart?

Fluency

See Example 6

9 A bag contains 2 yellow and 3 black marbles. Two marbles are drawn at random, with replacement.

- a Use a tree diagram to show the sample space.

Find the probability that the marbles are:

b both yellow c both black d different colours.

10 Repeat question **9** without replacement.

11 Two red, two blue and two yellow marbles are placed in a jar. Two marbles are then chosen at random.

- a If the first is replaced before the second is chosen, what is the probability that:
  - i they will be the same colour?
  - ii they will be different colours?
  - iii at least one marble will be yellow?
- b If there is no replacement, what is the probability that:
  - i they will be the same colour?
  - ii they will be different colours?
  - iii at least one marble will be blue?

See Example 10

12 a Draw a Venn diagram to show the names of the days of the week and months of the year. Let A = {Names containing the letter ‘t’}, B = {Names starting with ‘m’} and C = {Names with at least two vowels}.

Use your diagram to find the following probabilities for the name of a day of the week or month of the year.

- b It has at least two vowels and a ‘t’.
- c It has at least two vowels or a ‘t’.
- d It starts with ‘m’ and has a ‘t’

e It starts with ‘m’ or has a ‘t’.
f It starts with ‘m’ and has at least two vowels.
g It starts with ‘m’ or has at least two vowels.
h It does not have ‘t’ or start with ‘m’
i It does not have at least two vowels or start with ‘m’.
j It does not have at least two vowels or a ‘t’.

See Example 7

13 A family is selected at random from families having 3 children.
a Use a tree diagram to show the sample space for the genders of the children.
b What is the probability that 2 of the children are girls?
c What is the probability that at least 2 of the children are girls?
d What is the probability that all of the children are boys?

See Example 8

14 A caterer prepares 4 salad, 3 ham and 5 fish paste sandwiches for mixed plates. A guest takes 2 sandwiches at random from a plate no one else has touched yet.

a Use a simplified tree diagram to show the sample space.
b What is the probability that they are both ham sandwiches?
c What is the probability that they are both salad sandwiches?
d What is the probability that they are both fish paste sandwiches?
e What is the probability that one is ham and the other is a salad sandwich?
f What is the probability that one is ham and the other is a fish paste sandwich?
g What is the probability that one is fish paste and the other is a salad sandwich?
h What is the probability that they are both the same kind of sandwich?
i What is the probability that they are different kinds of sandwich?

15 A card is cut from a normal 52-card pack and the suit is noted. The card is replaced and the process is twice repeated.
a Draw a simplified tree diagram to show the sample space for spades.
b Find the probability that all 3 cards are spades.
c Find the probability of getting at most 1 spade.

16 A multiple-choice test has 4 questions, each having 5 responses. Only one of the responses is correct in each question.
a Draw a simplified tree diagram to show correct and incorrect responses.
b Find the probability of getting all 4 questions correct by chance.
c Find the probability of getting at least 2 questions correct by chance.
d If 50 people did the test, how many would you expect to pass by chance?

17 A university English literature library has 15 000 books: 2000 of them are about Shakespeare’s works, 5000 of them are about plays and 1300 are about Shakespeare’s plays. A book is selected at random by a silverfish.
a What is the probability that it is about Shakespeare’s plays?

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b What is the probability that it is about Shakespeare or plays?

c What is the probability that it is about neither Shakespeare nor plays?

18 A collection of 'silver' coins consists of three 20-cent, ten 10-cent, five 5-cent and twelve 50-cent coins. Two coins are chosen at random.

a Use a simplified tree diagram to show the sample space.

b What is the probability of getting two coins that are the same?

c What is the probability of getting different coins?

d What is the probability of getting a 50-cent coin?

e What is the probability of getting a 20-cent coin?

f What is the probability of getting a 50-cent and a 20-cent coin?

g What is the probability of getting a 20-cent coin or a 50 cent coin?

h What is the probability of getting a 20-cent coin or a 50 cent coin, but not both?

i What is the probability of ending up with more than 25 cents?

j What is the probability of ending up with less than 25 cents?

19 Three coins are tossed. Find the fair odds for 3 heads, 2 heads, 2 tails and 3 tails.

Problem solving

20 The table below shows the results of a survey of Year 10 students about smoking and colds.

		Number of colds last year					
		0	**1**	**2**	**3**	**4**	**Total**
Smoking	**Smokers**	0	3	5	6	2	16
	Non-smokers	4	10	8	4	1	27
	Total	4	13	13	10	3	43

a What is the probability that a smoker had more than 2 colds?

b What is the probability that a non-smoker had more than 2 colds?

c Compare the probabilities that smokers and non-smokers have a lot of colds.

d Do you thank the sample size is sufficient to make conclusions about smoking and colds?

e Do the results support the conclusion that smokers get more colds than others?

21 The table below shows the numbers of accidents in which drivers of different age groups were involved.

		Accidents in last 2 years					
		1	**2**	**3**	**4**	**5**	**Total**
Age group	**17–21**	16	24	21	14	12	87
	22–26	16	14	12	9	9	60
	27–31	9	7	7	4	2	29
	32–41	4	2	2	0	0	8
	Over 41	4	2	0	0	0	6
	Total	49	49	42	27	23	190

a What is the probability that a driver under 27 had exactly one accident in the last 2 years?

b What is the probability that a driver aged 27 or more had exactly one accident in the last 2 years?

c Compare the probabilities of younger and older drivers having accidents.

d Is the conclusion 'Younger drivers have more accidents' supported by the data?

Worked solution
Exercise 6.2
MAT10SPWS00050

㉒ The table below shows the age-specific deaths of the Indigenous (ATSI) and non-Indigenous (non-ATSI) populations of Australia in 2008.

%s	0	1–14	15–24	25–34	35–44	45–54	55–64	65+ years	Total
ATSI	105	47	124	170	266	366	405	848	2331
Non-ATSI	792	310	782	1183	2112	4598	8925	80 287	98 989

Source: Source: ABS

In the following, assume that the 2008 figures apply to all years.

a What is the probability of an Indigenous person living to 65?

b What is the probability of a non-Indigenous person living to 65?

c Compare the likelihood of Indigenous and non-Indigenous people living to age 65 and comment on the result.

d Do the figures in this table support the claim that Indigenous Australians are more likely to die young than other Australians?

23 Two multiple choice questions have answers labelled *A*, *B*, *C* and *D*. Only one answer is correct in each question.

a Work out the probabilities of getting 0, 1 or 2 questions right by just guessing (say *A* is the answer to both questions).

b From 100 people doing the test, how many would you expect to guess each of 0, 1 or 2 questions correctly?

Worked solution
Exercise 6.2
MAT10SPWS00050

㉔ What is the probability that a number made up of the digits 4, 7, 8 and 3, each used once only, is more than 4500?

25 Two cards are dealt from a well-shuffled pack.

a What is the probability that both cards will be clubs?

b What is the probability that both cards will be the same suit?

c What is the probability that the cards will be different suits?

d What is the probability that at least one card will be a spade?

e From 400 such trials, how many times would you expect to get the same suit?

26 David and Peter regularly play tennis. Over a long period, David has won 40% f the games they play. What is the probability of him winning at least 2 of the next three games they play?

27 In the game of poker, a flush is all 5 cards of the same suit.

a What is the chance of being dealt a flush in hearts?

b What is the probability of being dealt a flush?

28 In the game 'shooting dice' two dice are used and the total of the dice determines the results. The player with the dice is called the shooter. On the shooter's first roll of a round, if they roll a 7 or 11, they win the round. If they roll a 2, 3 or 12 they lose the round. If they roll anything else this is the 'point'. Then they keep rolling until they get the point again or they get a seven. If they get a 7 they lose the round ('seven- out') and the next player becomes the shooter. If they get the same number again (the point) they win the round. The same shooter keeps rolling until they 'seven out'.

- **a** What are the fair odds for the shooter to win immediately?
- **b** What are the fair odds for the shooter to lose immediately?
- **c** What is the probability of a point?

Worded solution
Exercise 6.2
MAT10SPWS00050

㉙ Twenty people are trapped in a lift: 8 have blue eyes and 9 have brown eyes; 5 have blond hair and 7 have black hair; 1 person has red hair and the rest are brunettes. The red-haired person has green eyes, there are 2 blue-eyed blonds and 2 hazel-eyed brunettes. Three black-haired people have blue eyes. One person is chosen at random to climb up the shaft to get help. Find each of the following probabilities.

- **a** It is someone with blue eyes and blond hair.
- **b** It is someone with blue eyes or blond hair.
- **c** It is someone with black hair and brown eyes.
- **d** It is someone with black hair or brown eyes.
- **e** It is someone with brunette hair or brown eyes.

Reasoning

30 A stack of timber at a yard has planks of lengths 3.6, 3.9, 4.2, 4.5, 4.8 and 5.1 m, with ends facing outwards. There are 10 planks of each length. Find the probability of selecting 3 lengths greater than 4.5 m long by chance and explain your reasoning.

31 There are 20 bags of sugar on a shelf and 4 of them have breaks in the bags, so the sugar leaks. Explain how to find the probability of getting 3 good bags by chance.

32 In a production line, the rotor, bearings and case are put together to make an electric motor. The rotor incorporates the commutator and the case has the field coils. The tolerance of manufacture of the parts means that the probability that the rotor will not fit the case properly is 3%, the probability that the bearings will not fit the case properly is 2% and the rotor will not fit the bearings properly is 1%. The motor will still work if any of these things happen, but it will be a 'lemon' and shake when it runs and will wear out very quickly. Explain why the probability of getting a lemon is not 6%.

6.3 Conditional probability

Weblink
Conditional probability and independent events
MAT10SPWB00006

Investigate: Probability of two events both occurring

You can use simplified tree diagrams to find the probability of two particular items selected from a group of items, whether the items are replaced or not. Consider the case of two marbles being chosen from a collection of 3 red and 2 green marbles. This can be done with or without replacement.

- Use a tree diagram to calculate the probability of the first marble being red and the second marble being green if the first is replaced before the second is drawn.
- Use a tree diagram to calculate the probability of the first marble being red and the second marble being green if the first is *not* replaced before the second is drawn.
- What is the probability of the first marble being red?
- For replacement, what is the probability of the second marble being green?
- Compare this with the probability of the first marble being green.
- For non-replacement, what is the probability of the second marble being green?
- How is this different from the probability of the first marble being green?

In many practical situations, you will know that events are not independent. It is important to be able to deal with situations where we have dependent events. **Conditional probability** allows you to calculate probabilities for dependent events.

Important!

Conditional probability

The conditional probability of event A given event B, written as $P(A|B)$, is the probability that A occurs, assuming that B occurs. In $P(A|B)$, the sample space is restricted to event B. The words **given**, **assuming**, **knowing**, **if … then**, **provided**, are often used in conditional probability.

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Example 12

In draw poker, you are dealt 5 cards and get to throw some away and obtain new ones before you see who has the highest hand. The discards are usually shuffled back into the pack before the new cards are dealt out to the players who discarded. If you are dealt 4 hearts and a spade and discard the spade, what is the probability of getting another heart to make a flush?

Solution

How many cards are left in the restricted sample space?	**There are 48 unknown cards that could be in the reshuffled pack.**
How many of the remaining cards are favourable?	**There are 9 hearts left in the unknown cards.**
Find the conditional probability.	$\boldsymbol{P(\text{5th heart} \mid \text{4 hearts}) = \frac{9}{48} = \frac{3}{16} = 0.1875}$
Write the answer.	**There is an 18.75% chance of making a flush.**

Example 13

The table below shows the results of a survey of the times taken by Year 10 students to get to school and the transport they used.

a What is the probability that a student who walks takes less than 15 minutes to get to school?

b Consider the sample and methods and comment on the reliability of the results.

	0–4 min	**5–9 min**	**10–14 min**	**15–19 min**	**20–29 min**	**30–39 min**	**40–59 min**	**60+ min**
Bike	0	0	2	1	0	0	0	0
Bus	2	1	7	4	16	22	14	5
Car	11	25	13	10	11	6	4	1
Other	0	1	0	1	1	1	0	2
Train/tram	0	0	1	2	1	2	6	0
Walk	11	5	3	2	5	1	0	0

Source: Census at school

Solution

a What is the conditional probability? **We want $P(\text{less than } 15 \mid \text{walk})$.**

Focus on the important line of the table.

	0–4 min	**5–9 min**	**10–14 min**	**15–19 min**	**20–29 min**	**30–39 min**	**40–59 min**	**60+ min**
Bike	0	0	2	1	0	0	0	0
Bus	2	1	7	4	16	22	14	5
Car	11	25	13	10	11	6	4	1
Other	0	1	0	1	1	1	0	2
Train/tram	0	0	1	2	1	2	6	0
Walk	11	5	3	2	5	1	0	0

What is the restricted sample space?	**Walking** $= 11 + 5 + 3 + 2 + 5 + 1 = 27$
How many *of these* take less than 15 minutes?	$n(\text{under } 15) = 11 + 5 + 3 = 19$
Find the probability.	$P(\text{under 15 min}\mid\text{walking}) = \frac{19}{27} \approx 0.7037$
Write the answer.	**A Year 10 student who walks to school has a probability of about 70% of getting there in under 15 minutes.**
b The sample is not random, as only students in some schools complete the survey. However, thousands of students throughout Australia do the survey.	**The results may be unreliable, but could give a reasonable indication.**

Common errors in calculation of conditional probability include doing the problem backwards, not restricting the sample space correctly and using all of event A instead of just the part that is inside event B for the calculation of $P(A|B)$.

Example 14

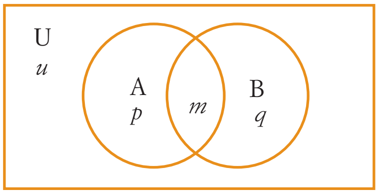
A student incorrectly used the table in Example **13** and calculated that the probability that someone who travelled by bus took at least an hour to get to school was 0.625. What is the correct answer and what did the student do incorrectly?

Solution

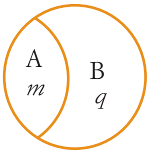
What is the conditional probability?	**We want** $P(60+\mid\text{bus})$**.**
What is the restricted sample space?	**Bus** $= 2 + 1 + 7 + 4 + 16 + 22 + 14 + 5$ $= 71$
How many *of these* take 60+ minutes?	$n(60+) = 5$
Find the probability.	$P(60+\mid\text{bus}) = \frac{5}{71} \approx 0.07042$
Write 0.625 as a fraction.	$0.625 = \frac{5}{8}$
What did the student do?	**The student used 60+ as the sample space.**
Write the answer.	**The correct probability is about 0.07; the student actually found the probability that someone who took at least an hour travelled by bus.**

As shown in Example **14**, it is very important when calculating conditional probabilities to be clear about the restricted sample space and what part of this is favourable.

Conditional probability is easily related to the probabilities of A, B and $A \cap B$ using a Venn diagram. Using the same setup we used previously, assume that a sample space has u elements, there are m elements in A ∩ B, p elements in A that are not in B and q elements that are in B but not in A. The Venn diagram of this situation is shown below.



Full sample space



Restricted sample space

For this situation, $P(\mathrm{A}|\mathrm{B})$ can be viewed as a probability with the sample space restricted to the set B, since we are assuming that B occurs. This is shown on the right of the main Venn diagram above. The number of elements in the restricted set A is now m, so $P(\mathrm{A}|\mathrm{B}) = \frac{m}{m+q}$.

We can multiply this by 1 in the form $1 = \frac{u}{u}$, giving the following.

$$\begin{aligned} P(\mathrm{A}|\mathrm{B}) &= \frac{m}{m+q} \\ &= \frac{m}{m+q} \times \frac{u}{u} \\ &= \frac{mu}{u(m+q)} \\ &= \frac{m}{u} \times \frac{u}{m+q} \\ &= \frac{m}{u} \div \frac{m+q}{u} \end{aligned}$$

But $P(\mathrm{A\ and\ B}) = \frac{m}{u}$ and $P(\mathrm{B}) = \frac{m+q}{u}$.

This means that we have $P(\mathrm{A}|\mathrm{B}) = P(\mathrm{A\ and\ B}) \div P(\mathrm{B})$ or $P(\mathrm{A}|\mathrm{B}) = \frac{P(\mathrm{A\ and\ B})}{P(\mathrm{B})}$.

It doesn't make any difference whether you consider $P(\mathrm{A}|\mathrm{B})$ or $P(\mathrm{B}|\mathrm{A})$, the analysis is still the same, so $P(\mathrm{B}|\mathrm{A}) = \frac{P(\mathrm{A\ and\ B})}{P(\mathrm{A})}$. This also leads to a mathematical definition of independence, since you can rewrite the formula as

$$P(\mathrm{A\ and\ B}) = P(\mathrm{A}|\mathrm{B}) \times P(\mathrm{B}) \text{ or } P(\mathrm{A\ and\ B}) = P(\mathrm{B}|\mathrm{A}) \times P(\mathrm{A})$$

Important!

Conditional probability and independence

The events A and B are independent if and only if $P(\mathrm{A}|\mathrm{B}) = P(\mathrm{A})$. It follows that $P(\mathrm{B}|\mathrm{A}) = P(\mathrm{B})$. For dependent events A and B, the conditional probability of A given B is given by the formula $P(A|B) = \frac{P(\mathrm{A\ and\ B})}{P(\mathrm{B})}$. This is also expressed as $P(\mathrm{A\ and\ B}) = P(\mathrm{A}|\mathrm{B}) \times P(\mathrm{B})$ for practical use.

Many situations involving the use of tree diagrams can be dealt with using conditional probability instead. When you want the probability of two events both occurring, you can use the conditional probability formula for $P(\mathrm{A\ and\ B})$.

Example 15

Video tutorial
Conditional probability
MAT10SPVT10015

What is the probability of being dealt two aces in a row from a shuffled pack of cards?

Solution

Write the desired probability as an 'and' statement.

$$\boldsymbol{P}(\text{2 aces}) = \boldsymbol{P}(\text{2nd card is an ace and 1st card is an ace})$$

Use the conditional probability formula.

$$= \boldsymbol{P}(\text{2nd card is an ace}|\text{1st card an ace}) \times \boldsymbol{P}(\text{1st is an ace})$$

There are 4 aces in a pack of 52 cards.

$$\boldsymbol{P}(\text{1st card is an ace}) = \frac{4}{52} = \frac{1}{13}$$

After an ace is taken out there are 3 aces left out of 51 cards.

$$\boldsymbol{P}(\text{2nd card is an ace}|\text{1st card an ace}) = \frac{3}{51} = \frac{1}{17}$$

Use the formula.

$$\boldsymbol{P}(\text{2 aces}) = \frac{1}{13} \times \frac{1}{17} = \frac{1}{221} \approx 0.004\ 255$$

Write the answer.

The probability of being dealt 2 aces in a row from a normal pack of cards is about 0.43%.

The rules for finding the probabilities of both of two events, *P*(A and B), are basically the same for independent and conditional probability. The probabilities are multiplied.

Important!

Multiplication principle for probability

The probability of two events both occurring, *P*(A and B) is the product of two probabilities. For independent events it is the simple product of the each of the probabilities. For dependent events, the probability of one is multiplied by the conditional probability of the other occurring.
The principle can be applied to multiple events.

Example 16

A bag contains 3 red, 2 green and 5 blue marbles. What is the probability that, when three marbles are shaken out, the first is red, the second green and the third blue?

Solution

For the first marble, there are 3 red out of 10 marbles.

$$\boldsymbol{P}(\text{1st R}) = \frac{3}{10}$$

If the first is red, there are 2 green out of 9 marbles.

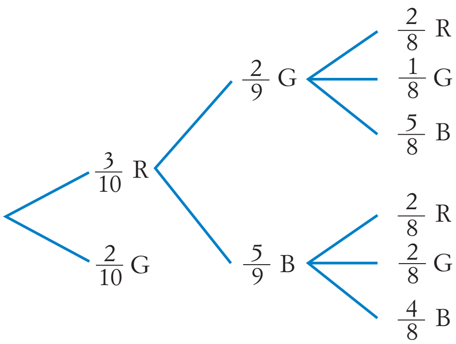
$$\boldsymbol{P}(\text{G}|\text{R}) = \frac{2}{9}$$

If the first two are red and green, there are 5 blue out of 8 marbles left.

$$\boldsymbol{P}(\text{B}|\text{G and R}) = \frac{5}{8}$$

Use the probability formula for (G and R) and B.	**P(B and (R and G))** **$= P$(B|R and G) $\times$ P(R and G)**
Use the probability formula for G and R and B.	**$= P$(B|R and G) $\times$ P(G|R) $\times$ P(R)**
Substitute values.	$= \frac{5}{8} \times \frac{2}{9} \times \frac{3}{10} = \frac{1}{24}$
Write the answer.	**The required probability is $\frac{1}{24}$.**

If you draw part of a simplified tree diagram for Example **13**, you get the following.



You can see that for red-green-blue you get $P(\text{R-G-B}) = \frac{3 \times 2 \times 5}{10 \times 9 \times 8}$.
How would you work out the probability of getting red, green and blue marbles by taking out three marbles without regard to order? From the above, you can see that $P(\text{R-B-G}) = \frac{3 \times 5 \times 2}{10 \times 9 \times 8}$.
Altogether, there would be 6 different ways of taking out red, green and blue marbles without regard to order: RGB, RBG, GRB, GBR, BRG and BGR. In each case, the numerator is $3 \times 5 \times 2$ and the denominator is $10 \times 9 \times 8$, so the overall probability is $6 \times \frac{1}{24} = \frac{1}{4}$. Of course, this only applies when the different ways of picking the 3 marbles have the same probabilities.

Example 17

Another bag contains 4 red, 3 green and 3 blue marbles. What is the probability that when three marbles are shaken out there is one of each colour?

Solution

Find the probability of red-green-blue.	$P(\text{RGB}) = \frac{4}{10} \times \frac{3}{9} \times \frac{3}{8} = \frac{1}{20}$
Find the probability of green-blue-red.	$P(\text{GBR}) = \frac{3}{10} \times \frac{3}{9} \times \frac{4}{8} = \frac{1}{20}$
Use symmetry.	**RGB, GBR, RBG, GRB, BRG and BGR all have probability $\frac{1}{20}$.**
Calculate the required probability.	$P(\text{one of each colour}) = 6 \times \frac{1}{20} = \frac{3}{10}$
State the answer.	**The probability they are all different is 0.3.**

Example 18

A drawer contains unpaired socks. There are 5 grey, 4 white and 3 black socks. If you picked out a pair in the dark, what is the probability that they do not make a pair?

Solution

Find the probability of a grey pair.	$\boldsymbol{P(\text{grey pair}) = \frac{5}{12} \times \frac{4}{11} = \frac{5}{33}}$
Find the probability of a white pair.	$\boldsymbol{P(\text{white pair}) = \frac{4}{12} \times \frac{3}{11} = \frac{1}{11}}$
Find the probability of a black pair.	$\boldsymbol{P(\text{black pair}) = \frac{3}{12} \times \frac{2}{11} = \frac{1}{22}}$
Find the probability of a pair.	$\boldsymbol{P(\text{pair}) = \frac{5}{33} + \frac{1}{11} + \frac{1}{22} = \frac{19}{66}}$
Use the complement.	$\boldsymbol{P(\text{no pair}) = 1 - \frac{19}{66} = \frac{47}{66} \approx 0.7121}$
Write the answer.	**The probability that they are not a pair is about 71%.**

In Example **18** it would be a mistake to assume that the probability of grey-white, $P(\text{GW}) = \frac{5}{33}$ would be the same as WG, WB, BW, BG and GB and simply multiply that by 6.
In Example **16**, the different orders correspond to the 180 ways of getting red, green and blue marbles from the 720 branches on a full tree diagram. The overall probability would be $\frac{180}{720} = \frac{1}{4}$.
You could get the same answer by adding the probabilities of RGB, RBG, GRB, GBR, BRG and BGR. These 6 methods of getting red, green and blue marbles without regard to order are completely separate. In Example **18** the probability of getting a matched pair is obviously the sum of the ways of getting white, grey and black pairs because they correspond to the 38 branches for pairs in the 132 branches in the full tree diagram. The overall probability is $\frac{38}{132} = \frac{19}{66}$.

Important!

Addition principle for probability

The probability of an event made up from two separate events is the total of the probabilities of both events.

$$\text{A} \cap \text{B} = \varnothing \Rightarrow P(\text{A or B}) = P(\text{A}) + P(\text{B})$$

This can be extended to multiple events.

Notice that since the intersection of the sets must be null to use the addition principle, it only applies to mutually exclusive sets, and is really the formula $P(\text{A} \cup \text{B}) = P(\text{A}) + P(\text{B}) - P(\text{A} \cap \text{B})$ for $\text{A} \cap \text{B} = \varnothing$.
The addition principle can be used in many areas of probability.

Example 19

A high school has an enrolment of 720 pupils. Ninety pupils came from primary school A, while two-fifths of the pupils came from primary school B. What is the probability that a pupil came from one of the primary schools A or B?

Solution

Find the probability of coming from school A. $P(\mathbf{A}) = \frac{90}{720} = \frac{1}{8}$

Write the probability of coming from school B. $P(\mathbf{B}) = \frac{2}{5}$

Clearly, coming from A or B are mutually exclusive, so they can be added. $P(\mathbf{A} \text{ or } \mathbf{B}) = \frac{1}{8} + \frac{2}{5} = \frac{21}{40} = 0.525$

Write the answer. **The probability of a student being from either primary school is 52.5%.**

Example 20

A lot of the nice coloured sticks in a pick-up-sticks set have been used for other things, so there are only 8 black, 5 white and 7 grey sticks left. When they were dropped on a table, two sticks are not tangled with the others, so were put aside. What is the probability that they were different colours?

Solution

The sticks could be BW, WB, GW, WG, GB or BG. Find $P(\text{BW})$ first. $P(\mathbf{BW}) = \frac{8}{20} \times \frac{5}{19} = \frac{2}{19}$

Use symmetry. $P(\mathbf{BW} \text{ or } \mathbf{WB}) = 2 \times \frac{2}{19} = \frac{4}{19}$

Now find $P(\text{GW})$. $P(\mathbf{GW}) = \frac{7}{20} \times \frac{5}{19} = \frac{7}{76}$

Use symmetry. $P(\mathbf{GW} \text{ or } \mathbf{WG}) = 2 \times \frac{7}{76} = \frac{7}{38}$

Now find $P(\text{BG})$ or $P(\text{GB})$. $P(\mathbf{BG} \text{ or } \mathbf{GB}) = 2 \times \frac{8}{20} \times \frac{7}{19} = \frac{28}{95}$

State the condition for the addition rule. **BW, WB, GW, WG, BG and GB are mutually exclusive events, so the addition rule applies.**

Find the required probability. $P(\text{different colours}) = \frac{4}{19} + \frac{7}{38} + \frac{28}{95} = \frac{131}{190} \approx 0.6895$

State the answer. **The probability they were different colours is about 69%.**

In problems like Example **20**, you must take care when finding all the possibilities and be sure that they are really mutually exclusive.

Exercise 6.3 Conditional probability

Understanding

Extra questions
Exercise 6.3
MAT10SPEQ00046

See Example 12

1 Two normal dice are thrown and their total is calculated.
   - a Given that their total is 7, what is the probability that one of the dice shows 2?
   - b Given that one of the dice shows 2, what is the probability that the total is 7?
   - c Given that their total is 8, what is the probability of a double?
   - d Given a double, what is the probability of getting a total of 8?
   - e Given a double, what is the probability that the total is 10?
   - f Given a total of 10, what is the probability of a double?
   - g Given that their total is 5, what is the probability that one of the dice shows 3?
   - h Given that one of the dice shows 3, what is the probability that the total is 5?

2 A box of coloured crayons has 4 red, 2 black, 4 yellow, 2 blue, 3 green and 5 purple crayons left in it. Students are allowed to use 2 crayons at a time, so a student takes 2 crayons at random.
   - a Given that one crayon is black, what is the probability that the other crayon is black?
   - b If one crayon is yellow, what is the probability that the other is purple?
   - c Having a red crayon, what is the probability that the other is also red?
   - d Assuming that one crayon is green, what is the probability that the other is black?
   - e What is the probability that the second crayon is yellow if you know that the first is red?
   - f Providing that the first crayon is purple, what is the probability that the other is yellow?
   - g If one crayon is green, what is the probability that the other is red?
   - h If one is red, what is the probability that the other is green?
   - i If one is black, what is the probability that the other is blue?
   - j If one is purple, what is the probability that the other is also purple?

3 Instead of taking 2 crayons, the student in question **2** took 3 crayons.
   - a If two of the crayons are green, what is the probability that the other is yellow?
   - b If two of the crayons are yellow and green, what is the probability that the other is green?
   - c If 2 of the crayons are yellow, what is the probability that the other is also yellow?
   - d If 2 of the crayons are red, what is the probability that the other is also red?
   - e If 2 of the crayons are red and yellow, what is that the probability that the other is yellow?
   - f If 2 of the crayons are red and yellow, what is that the probability that the other is red?
   - g If 2 of the crayons are red, what is the probability that the other is yellow?
   - h If 2 of the crayons are yellow, what is the probability that the other is red?
   - i If one crayon is red, what is the probability that the other two are blue?
   - j If one crayon is blue, what is the probability that the other two are red?

4 Two cards are dealt from a normal shuffled pack. Find the probabilities that they are both: *See* Example 15

a kings **b** the same value **c** picture cards **d** number cards
e non-number cards **f** spades **g** the same suit

5 A bag contains 4 black, 2 yellow, 3 white and 6 red marbles. Three marbles are taken out at random. Find each of the following probabilities for the order of the marbles. *See* Example 16

a Red, yellow, black **b** Yellow, yellow, black **c** Black, yellow, yellow
d Yellow, black, yellow **e** Black, black, yellow **f** Yellow, black, black
g Black, yellow, black **h** White, white, white **i** Black, black, black
j Red, red, red

6 A bag contains 6 black, 3 white and 6 red marbles. Three marbles are taken out at random. Find the probability that they are different colours. *See* Example 17

7 If only two marbles are shaken out of the bag in question **6**, what is the probability that they are different colours? *See* Example 20

8 If three marbles are shaken out of the bag in question **5**, what is the probability that they are different colours?

Fluency

9 Find the probability of drawing one card to an inside straight in draw poker. That is, if you have something like 7, 8, 10, J and discard a 4, what is the probability of drawing a 9 to get a straight?

10 In draw poker, what is the probability of drawing to an outside straight? That is, if you keep 8, 9, 10, J, what is the probability of drawing either a 7 or a queen to get a straight?

11 Using the table from Example **13**, find each of the following probabilities. *See* Example 13

a The probability that a student who went to school by car took less than 5 minutes to get there.

b The probability that a student who took less than 5 minutes to get to school went by car.

c The probability that a student who went by train or tram took at least 40 minutes to get to school.

d The probability that a student who took at least 40 minutes to get to school went by train or tram.

e The probability that a student who took the bus took less than 10 minutes to get to school.

f The probability that a student who took less than 10 minutes to get to school went by bus.

g The probability that a student who walked to school took less than 15 minutes to get there.

h The probability that a student who took less than 15 minutes to get to school walked there.

i The probability that a student who caught the bus took at least half an hour to get to school.

j The probability that a student who took at least half an hour to get to school went by bus.

12 At a dinner party, knives and forks from 3 different party sets jumbled in a drawer were used to set the table. The first set had 5 knives but only 3 forks left, the second had 3 forks and 6 knives and the last one had 4 knives and 4 forks left. Find the probability that if you set them out at random, the first place would not have matching knives and forks. *See* Example 18

13 A Holden car club had a rally run from the Melbourne CBD to the Dandenongs with 40 cars. 60% of the cars were Holdens, and another 8 had the club badge on the grill of the car. What was the probability that a car in the rally was a Holden or had the club badge on the car? *See* Example 19

Problem solving

Worked solution
Exercise 6.3
MAT10SPWS00051

⑭ You are dealt a draw poker hand of A A K 7 3. You could try for 2 pairs by keeping the aces and king to get A A K K x or A A K x x. You could also try for 3 aces (A A A x y) by throwing the king as well. Which is more likely?

15 The table below shows the results of a large survey of students' eye and hair colours.

		Hair colour			
		Black	**Brown**	**Red**	**Blond**
Eye colour	**Brown**	68	119	26	7
	Blue	20	74	17	94
	Hazel	15	54	14	10
	Green	5	29	14	16

a Find the probability that someone with black hair has brown eyes.

b Find the probability that someone with blond hair has blue eyes.

c Find the probability that someone with red hair has green eyes.

d Use your results to comment on the stereotypes that people with black hair have brown eyes, blond-haired people have blue eyes and redheads have green eyes.

ⓔ People with brown eyes are said to have black hair. Use probability to comment on this stereotype.

f The data was taken from an internet site. Comment on the method and sample size.

Worked solution
Exercise 6.3
MAT10SPWS00051

16 When you fly from Brisbane to Alice Springs, you can go direct, through Sydney, through Cairns, through Melbourne or through Melbourne and Adelaide. Coming back, there is the same choice of routes. If you book for particular times of arrival and departure, the routes booked are essentially random as nearly all flights to and from Alice arrive and depart about the middle of the day.

What is the probability that if you book for about midday to travel to Alice Springs one day and come back 5 days later, your flights will not be over the same route?

17 Students went to a skating rink for a sports afternoon. Unfortunately the school there in the morning had put the skating boots back by just throwing them into the store. There were 20 pairs of sizes 5 to 9, and 10 pairs each of sizes 10 to 12, 3 and 4. Work out the probability that when your teacher gave out the boots at random, you got a matching pair – even if they were the wrong size.

Reasoning

See Example 14

Worked solution
Exercise 6.3
MAT10SPWS00051

⑱ A student incorrectly worked out that the probability of getting a total of 4 from rolling two dice when one of the dice showed a 3 was $\frac{3}{11}$. Find the correct answer and explain what the student did wrong.

19 Using the table in question **15**, a student worked out that the probability of someone with brown hair having blue eyes was $\frac{74}{205}$. Another student said it was $\frac{37}{291}$ and a third said it was $\frac{74}{119}$. The teacher said they were all wrong. Find the correct answer and explain their errors.

20 A student incorrectly claimed that the probability of getting a matching pair when 2 cards are dealt from a 'fruit pack' with 10 apple, 10 orange, 8 banana, 6 peach and 6 pear picture cards was $\frac{1}{4}$. Explain the correct answer and what the student did wrong.

9780170194860

Chapter 6 summary

Quiz

Probability

MAT10SPQZ00006

- **Experimental probability** uses real data to work out probability. Each item of data is called a **trial**.
- **Theoretical probability** is based on a list containing all the possible outcomes called the **sample space**. Every outcome in the sample space is listed exactly as many times as it occurs. A single outcome is called an **element** or **sample point**.
- An **event** is a collection of outcomes, whether they come from trials or a sample space. The outcomes in an event for which you are calculating the probability are called **favourable outcomes**. The number of elements in an event is written as ***n*(event)** and is the **frequency** of the outcome.
- The probability of an event is a number from 0 to 1, written as ***P*(event)** worked out using the formulas below.

$$P(\text{event}) = \frac{\text{Frequency of event}}{\text{Total Frequency}} = \frac{\text{Number of favourable outcomes}}{\text{Number of trials}}$$

$$\text{or } P(\text{event}) = \frac{n(\text{event})}{n(\text{sample space})}$$

- Experimental probabilities are also called **relative frequencies**.
- Probabilities can be written as fractions, decimals or percentages. An event which has a probability of 0 can never happen and is called **impossible**. An event with a probability of 1 always happens and is called **certain**.
- **Expected frequency** is the product of the probability and the number of attempts made in the future.
- For any event E, the **complement** of the event is shown as E′ or $\overline{\text{E}}$. The sum of the probability of an event and its complement is 1. $P(\text{E}) + P(\text{E}') = 1$, so $P(\text{E}') = 1 - P(\text{E})$ and $P(\text{E}) = 1 - P(\text{E}')$.
- **Two-way tables** are most useful for situations with two variables. **Tree diagrams** are most useful for events in stages. **Venn diagrams** are most useful for combinations of sets, particularly and/or situations.
- **A simplified tree diagram** shows repeated branches on a normal tree diagram as single branches, with a fraction showing the number of branches that have been combined, out of the total number at that join.
- Two events are **independent** if the probability of one is *not* affected by the occurrence of the other. For independent events A and B, $P(\text{A and B}) = P(\text{A}) \times P(\text{B})$.
- Dependent events are not **independent**.
- The **intersection** of two sets A and B, $A \cap B$, is the set that contains the elements in both A and B.
- The **union** of two sets A and B, $A \cup B$, is the set that contains the elements in at least one of A or B.
- In mathematics, '**… and …**' means *both*, but '**… or …**' means *at least one*. 'and' corresponds to the intersection of sets, while 'or' corresponds to the union.
- The probabilities of (A or B) and (A and B) are related by $P(\text{A or B}) = P(\text{A}) + P(\text{B}) - P(\text{A and B})$.
- The **conditional probability** of event A given event B, written as $P(\text{A}|\text{B})$, is the probability that A occurs, assuming that B occurs. In $P(\text{A}|\text{B})$, the sample space is restricted to event B.
- The words **given**, **assuming**, **knowing**, **if … then**, **provided**, are often used in conditional probability.

Chapter 6 summary

- Events A and B are independent if and only if $P(\mathrm{A}|\mathrm{B}) = P(\mathrm{A})$. It follows that $P(\mathrm{B}|\mathrm{A}) = P(\mathrm{B})$.
- For dependent events A and B, the conditional probability of A given B is given the formula $P(\mathrm{A}|B) = \dfrac{P(\mathrm{A \text{ and } B})}{P(\mathrm{B})}$. This is also expressed as $P(\mathrm{A \text{ and } B}) = P(\mathrm{A}|\mathrm{B}) \times P(\mathrm{B})$ for practical use.
- The probability of $P(\mathrm{A \text{ and } B})$ is the product of two probabilities. For independent events it is the simple product of the each of the probabilities. For dependent events, the probability of one is multiplied by the conditional probability of the other occurring. This is called the **multiplication principle**.
- For mutually exclusive events A and B, $P(\mathrm{A \text{ or } B}) = P(\mathrm{A}) + P(\mathrm{B})$. This is called the **addition principle**. In symbols, $\mathrm{A} \cap \mathrm{B} = \varnothing \Rightarrow P(\mathrm{A \text{ or } B}) = P(\mathrm{A}) + P(\mathrm{B})$

Chapter 6 review

Understanding

1 When a cricket bat is spun up into the air, it has equal chances of landing with the flat side up or down. What is the probability that if a cricket bat is spun up three times, it will land flat side down all three times? *See* Example 9

2 40% of the milk sold in a supermarket is ordinary white homogenised milk.
- **a** Find the probability that two customers buying milk have bought ordinary white milk.
- **b** What is the probability that they have both bought other kinds of milk?
- **c** What is the probability that one has ordinary milk and the other doesn't?

3 In the card game Pontoon, an ace counts as 1 or 11, picture cards count as 10 and number cards count as their number value. You get your cards 1 at a time, and after the first 2 can decide whether to keep going or not. You try to get as close to a total of 21 as possible without busting. Busting means going over 21 and if you bust you lose. Pontoon means getting a total of 21 with only 2 cards and you then win automatically, otherwise your total has to be bigger than the dealer's total to win. *See* Example 12
- **a** If your first card is a king, what is the probability of getting pontoon?
- **b** If your first card is an ace, what is the probability of getting pontoon?
- **c** If the first two cards you get are a 7 and a 10, what is the probability of getting 20 or 21 on your third card?
- **d** If the first two cards you are dealt are a 6 and a king, what is the probability of busting on your third card?
- **e** If the first two cards you get are both 9s, what is the probability of busting on your third card?
- **f** What is the largest number to have less than a 50% chance of busting from on your third card?

4 If you made a bet of $28 at 7 : 2 on, how much would you get if you won? *See* Example 4

5 What are the fair odds for rolling a 6 on a normal die? *See* Example 5

6 A number is chosen from 1 to 25. What is the probability that it is divisible by 2 or 3? *See* Example 11

7 There are 3 raspberry, 4 lime and 5 pineapple iceblocks left in the fridge. Two iceblocks are taken out at random. Find each of the following probabilities for the iceblocks. *See* Example 15
- **a** They are both raspberry.
- **b** They are both lime.
- **c** They are both pineapple.
- **d** They are different flavours.

8 A sports bag contains 5 basketballs, 2 volleyballs and 4 netballs. Three balls are tipped out. Find the probabilities for each of the following orders of balls tipped out. *See* Example 16
- **a** Basketball, volleyball, netball
- **b** Volleyball, basketball, basketball
- **c** Netball, netball, volleyball
- **d** Basketball, netball, basketball

9 A box has a mixture of 20 Zener diodes, 20 ordinary resistors and 10 accurate gold-banded resistors, which are all about the same size. What is the probability that if 3 are taken out, they are all different? *See* Example 17

See Example 20

10 Daniel has collected golf balls from a water trap. He has 4 Bridgestone, 6 Nike, 4 Titleist, 8 Callaway and 6 Top Flite balls. He doesn't bother picking any particular brands when he sells them.

- **a** If a customer buys 2 balls, what is the probability that they are different brands?
- **b** If a customer buys 3 balls, what is the probability that they are all different brands?

Fluency

See Example 1

11 Two classes of Year 10 students were asked about the money they received last week. 14 said they didn't get any, 10 said they got pocket money, 18 said they got paid for work and the other 12 said they got money from something else.

- **a** What was the probability of a Year 10 student doing paid work?
- **b** What was the probability of not working?
- **c** If you asked another 120 students, how many would you expect to get pocket money?

12 The two classes in question **11** had 24 boys and 30 girls. If you choose a student at random, what is the probability that the student is female?

See Example 6

13 A box of fluoro lamps has two 9 W and two 15 W lamps in it. Two lamps are taken out in the dark because both lamps in the room have blown.

- **a** Use a tree diagram to show the sample space.

Find the probabilities of each of the following.

- **b** The lamps are both 9W.
- **c** The lamps are different.

14 Repeat question **13** if the first lamp is replaced before the second is taken out.

See Example 10

15
- **a** Draw a Venn diagram to show the years from 2005 to 2025. Let A = {leap years} and B = {Years that include the digit 1}. Use your diagram to find the following probabilities for an arbitrary year.
- **b** It is a leap year.
- **c** It includes the digit 1.
- **d** It is a leap year that includes the digit 1.
- **e** It is a leap year or it includes the digit 1.
- **f** It does not include the digit 1 but is a leap year.

See Example 2

16 In her pencil case, Tingting has 2 blue, 2 red and one black biros. She takes out 2 biros without looking.

- **a** Show the sample space for the biros taken out, using B1 and B2 for blue and just B for black.
- **b** What is the probability that she gets a red and a blue biro?

See Example 7

17 If a couple have blood types A and O, then their children can also have blood types A or O. Such a couple have 4 children.

- **a** Use a tree diagram to show the sample space for the blood types of the children.
- **b** What is the probability that 3 of the children have blood type A?
- **c** What is the probability that at least 2 of the children have blood type A?
- **d** What is the probability that they are all of type O?

18 A bag contains 3 black, 5 red and 6 green counters. Two counters are shaken from the bag. *See* Example 8

- a Draw a simplified tree diagram to show the sample space.
- b Find the probability that both counters are red.
- c Find the probability of getting a red counter and then a black counter.
- d Find the probability that the first is red and the second is green.
- e Find the probability that one is red and the other is green.
- f What is the probability that the counters are different?

19 The table below shows where Year 10 students said they got money from last week. *See* Example 13

Year 10 money source

	Work	None	Pocket Money	Chores	Gift	Other
Female	9	6	6	2	1	20
Male	8	5	4	1	1	3

- a What is the probability that a boy got money from work?
- b What is the probability that a student who received pocket money was female?
- c What is the probability that a student received no money?
- d What is the probability that a girl received no money?

20 A draughts board has squares that are 3 cm in length and pieces that are 2 cm in diameter. In a game, you throw a draughts piece onto the board but if it goes completely off the board, you do it again. What is the probability that it crosses the line between 2 or more squares? *See* Example 3

21 A shop specialising in sports shoes has 21 kinds of cross-trainers, 15 kinds of sprinting runners, 12 kinds of long distance runners, 16 kinds of soccer boots, 12 kinds of rugby boots, 18 kinds of Aussie Rules boots, 6 kinds of bowling shoes and 60 other kinds of sports shoes. A pair of shoes is chosen at random.

- a What is the probability that they are football boots?
- b What is the probability that they are suitable for running or bowling?

22 A Year 10 class has 10 Australian-born students, 7 from Asia, 4 from Europe and 5 from Africa. Two students are chosen at random to represent the class.

- a Use a simplified tree diagram to show the sample space.
- b What is the probability that the students are both Asians?
- c What is the probability that they are from different continents?
- d What is the probability that they are European and African?
- e What is the probability that they are European or African?

23 A class set of school Atlases had first, second and third editions. There were 4 first editions, 10 second editions and 11 third editions in the set. Assuming that the editions were mixed up, what is the probability that two students sitting next to each other got different editions?

See Example 19

24 An exhibition of paintings has 25 Australian artists and 15 overseas artists in the exhibition. There are 8 paintings by 21st century artists and 2 of these are Australian. 15 of the artists exhibited are 20th century and the rest are 19th century impressionists. All of the 19th century artists worked in France for most of their life and of the 20th century artists, all but 4 of the 10 Australians also worked in France for a while. Three of the 20th century artists were French and 2 were Americans working in France. Neither of the 21st century Australians have been overseas and of the foreign artists, 3 are Americans who have never left the USA. The other 21st century artists are French or have worked in France.
If you picked an artist at random, what is the probability that the artist has or had worked in France?

Problem solving

25 Students in a typical Year 10 class said that they got up at the following times yesterday morning.
7:30, 7:00, 6:30, 5:30, 7:30, 7:00, 7:30, 8:00, 6:30, 7:30, 7:00, 7:30, 7:00, 6:30, 8:00, 8:30, 7:30, 7:30, 7:00, 7:30, 6:00, 6:30, 6:30, 7:00, 7:00, 7:30, 6:30, 6:30, 8:00, 8:00
One student is chosen at random.

a What is the probability that the student got up at 7:00 a.m.?

b From 200 Year 10 students, how many would you expect to get up at 8:00 a.m.?

c How many students would you need to ask to find 40 who got up before 7:00 a.m.?

26 The table below shows the heights and errors in spelling of a group of 16-year-olds.

		Errors in spelling				
		1	**2**	**3**	**4**	**5**
Height (cm)	**150–159**	0	1	3	5	1
	160–169	8	6	9	2	9
	170–179	11	2	6	7	9
	180–189	2	3	2	2	2
	Over 189	2	0	0	1	1

a What is the probability that a student shorter than 170 cm made more than 2 errors?
b What is the probability that a student at least 170 cm tall made more than 2 errors?
c Compare the probability of spelling errors of shorter and taller students.

27 A box of car fuses has 4 ten-amp and 2 five-amp fuses left in it. Two fuses are shaken out of the box.
a List the sample space for the 2 fuses taken out.
b What is the probability that they will both be the same current rating?

28 A test has 3 true-false questions and 2 multiple choice questions with options A, B, C, D and E. Only one answer is correct for each question.
a What is the probability of guessing all 5 questions correctly?
b What is the probability of guessing 2 of the true-false questions and 1 of the multiple choice correctly?
c What is the probability of guessing 1 of the true-false and both multiple choice questions correctly?
d What is the probability of getting them all wrong?

29 What is the probability that a 3-digit number using the digits 4, 3 and 2 is greater than 320?

30 The table below shows the results of a large survey of Year 10 students who received money in the fifth week of Term 2.

Year 10 money and source

	Work	**Pocket Money**	**Chores**	**Gift**	**Other**
\$1–\$9	1	2	2	0	4
\$10–\$19	1	11	3	0	1
\$20–\$49	10	13	6	3	8
\$50–\$99	24	10	2	2	4
\$100–\$199	20	3	0	3	1
\$200+	11	1	0	0	3

Note: 46 of the students surveyed received no money from any source.

a What is the probability that a student who worked got less than \$100?
b What is the probability that a student who got less than \$100 worked for it?
c What is the probability that a student who got money from a gift received less than \$100?
d Compare the probability of getting more than \$99 from gifts or working.

31 Rebecca, Simone and Tamsin regularly play badminton. Rebecca beats Simone 60% of the time and Tamsin 70% of the time. Simone beats Tamsin 60% of the time. They play each other twice in an afternoon. What is the probability that Tamsin wins more than half her 4 games, assuming that they all play to their usual standard?

32 A beginner at pool can generally hit the ball he is aiming for so that its centre goes within 10 cm of the where he wants it to, as long the white ball is less than 30 cm from the coloured ball and the coloured ball is less than 40 cm from the point at which it is being aimed. On a particular table the balls are 60 mm in diameter and the pockets have an effective width of 90 mm. Provided that the ball is not hit too hard, and overlaps the edge of the pocket by less than 10 mm, it will go in. What is the probability that the beginner will sink a ball he is aiming for if the white and the coloured ball are within 30 cm and 40 cm as outlined?

33 A class of Year 10 students are asked about their intentions in studying Science in Year 11. Of the 24 students asked, 14 say they are going to do Biology, 10 are going to take Chemistry, and 8 are going to do Physics. There are 2 students who are going to take all three subjects. There are 2 students doing only Physics and another 2 taking only Chemistry. There is one student not taking Chemistry who is doing both Physics and Biology. Find each of the following probabilities if you select a student at random.

a The student is taking none of the three subjects.
b The student is doing only Biology.
c The student is taking Chemistry and Physics.
d The student is doing Biology or Physics.

34 A local flower show had prizes for roses, dahlias, bromeliads, orchids and cacti. The best in show prize is selected from all the species. There were 20 rose categories, 16 dahlia categories, 12 bromeliad categories, 18 orchid categories and 10 cactus categories. Best in show had 1st, 2nd and 3rd prizes, chosen from the first prize winners in all 76 categories.

a What is the probability that they were all roses?
b What is the probability that the first two prizes were orchids and the third was a rose?
c What is the probability that the prizes were awarded in order as orchid, dahlia, dahlia?
d Given that the first prize was a Dahlia, what is the probability that the 2nd and 3rd were both bromeliads?
e Given that the first was a rose, what is the probability that the 2nd and 3rd were both orchids?

Reasoning

35 In a school library, DVDs recorded from Educational broadcasts have call numbers on the edges facing out from the shelves, so you normally look up the call number on the computer to access a recording. On a rainy day, one of the sports teachers is looking for a recording about basketball or netball and knows which shelf these are on, so doesn't bother looking up the call number. There are actually 8 basketball recordings, 6 netball recordings, 12 football recordings, 9 cricket recordings and 20 recordings of other sports on that shelf. If they have been jumbled up, explain the probability that the first 3 recordings picked out by the teacher are basketball or netball.

36 A student said that you could get 4 heads, 3 heads and a tail, 2 heads and 2 tails, 1 tail and 3 heads or 4 tails from tossing 4 coins, so the probability of getting 4 heads is 20%. Explain what is wrong with this reasoning and find the correct answer.

37 Another student said that the probability that if two dice were tossed and one of the dice showed a 4, the chances of getting a total of 7 from throwing two dice was $\frac{6}{12} = 50\%$ because there were 6 ways of getting a total of 7, 6 ways the first dice could show 4, and 6 ways the second die could show 4. Explain the correct answer and what is wrong with this reasoning.

38 Using the table in question **30**, someone said that a student who got pocket money had a probability of 40% of getting at least $10 but less than $20. What is the correct answer and what is wrong with this reasoning?

39 In question **20**, explain the difference it would make to the probability if you had to rethrow the piece when it landed partly off the board. Remember that a draughts board has 8 squares each way.

40 In question **32**, a class member said there was an 80% chance that a student intending to take Chemistry also intended to do Physics. Explain if this was correct, and if not, what is the right answer?

Number and algebra

7

Advanced algebra and functions

9780170194860

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MAT10NAPL00018

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Australian Curriculum statements

Patterns and algebra

Expand binomial products and factorise monic quadratic expressions using a variety of strategies. (10)

Investigate the concept of a polynomial and apply the factor and remainder theorems to solve problems. (10A)

Linear and non-linear relationships

Explore the connection between algebraic and graphical representations of relations such as simple quadratics, circles and exponentials using digital technology as appropriate. (10)

Describe, interpret and sketch parabolas, hyperbolas, circles and exponential functions and their transformations. (10A)

Apply understanding of polynomials to sketch a range of curves and describe the features of these curves from their equation. (10A)

Factorise monic and non-monic quadratic expressions and solve a wide range of quadratic equations derived from a variety of contexts. (10A)

Video tutorial
Advanced algebra and functions
MAT10NAVT00007

Changing the format of algebraic expressions allows you to see connections that are otherwise hidden. This is one of the key skills of algebra. Expansion and factorisation are the main methods used, apart from simplification of terms. These are essential in solving nonlinear equations and in many other aspects of mathematics in your senior years. Expansion and factorisation do not have many direct applications, but they are essential enabling skills that allow you to do lots of other things.

Mathematical literacy

Maths dictionary
MAT10ASDI00001

Weblink
Algebra help
MAT10NAWB00007

The mathematical words below have special meanings that you will learn in this chapter. It is important that you learn to spell them and gradually learn what they mean in mathematics. You may find the glossary or online mathematical dictionary useful for this purpose.

- coefficient
- common factor
- cross method
- decomposition method
- degree
- difference of two squares
- dividend
- division identity
- double zero
- dummy variable
- expansion
- expression
- factor
- factor theorem
- factorise
- grouping factorisation
- integer
- leading term
- monic quadratic
- non-monic quadratic
- parabola
- perfect square
- polynomial
- quadratic
- quotient
- remainder
- remainder theorem
- single zero
- sketch
- square of a difference
- square of a sum
- synthetic division
- term
- trinomial
- triple zero
- turning point
- zeros

7.1 Algebraic patterns and groups

You have already learnt some special expansions that allow you to quickly simplify some expressions. They can be used for factorisation as well as for simplification.

Important!

Special expansions

The special expansion $(x + y)(x - y) = x^2 - y^2$ is called the **difference of two squares**.
The special expansion $(x + y)^2 = x^2 + 2xy + y^2$ is called the **square of a sum**.
$(x - y)^2 = x^2 - 2xy + y^2$ is called the **square of a difference**.
The last two expressions are sometimes just called **perfect squares**.

When you use the difference of two squares it doesn’t matter whether you put the subtraction bracket or the addition bracket first.

Example 1

Factorise each of the following expressions.

a $p^2 - 16$ **b** $4x^2 - 9y^2$ **c** $25 - (m + n)^2$

Puzzle sheet
Difference of two perfect squares
MAT10NAPS00019

Solution

a Write the problem. $\quad \boldsymbol{p^2 - 16}$

Write in the difference of two squares pattern. $\quad \boldsymbol{= p^2 - 4^2}$

Use the difference of two squares. $\quad \boldsymbol{= (p + 4)(p - 4)}$

b Write the problem. $\quad \boldsymbol{4x^2 - 9y^2}$

Write in the difference of two squares pattern. $\quad \boldsymbol{= (2x)^2 - (3y)^2}$

Use the difference of two squares. $\quad \boldsymbol{= (2x - 3y)(2x + 3y)}$

c Write the problem. $\quad \boldsymbol{25 - (m + n)^2}$

Write in the difference of two squares pattern. $\quad \boldsymbol{= 5^2 - (m + n)^2}$

Use the difference of two squares. $\quad \boldsymbol{= [5 - (m + n)][5 + (m + n)]}$

Simplify the internal brackets. $\quad \boldsymbol{= (5 - m - n)(5 + m + n)}$

For perfect squares, remember that you must have double the product of the terms that are squared in the centre. Remember also that the sign of this term determines whether it is a sum or a difference.

Example 2

Factorise each of the following expressions.

a $m^2 + 2mn + n^2$ **b** $p^2 - 2p + 1$
c $9c^2 - 6cg + g^2$ **d** $4x^2 + 20x + 25$

Puzzle sheet
Perfect squares
MAT10NAPS00020
CAS TI-Nspire exercise
Advanced algebra and functions
MAT10NATI00015
CAS ClassPad exercise
Advanced algebra and functions
MAT10NACP00015

Solution

a Write the problem. $\quad \boldsymbol{m^2 + 2mn + n^2}$

Write as $(\text{first})^2 \pm 2 \times \text{first} \times \text{second} + (\text{second})^2$ $\quad \boldsymbol{= m^2 + 2 \times m \times n + n^2}$

Use $x^2 + 2xy + y^2 = (x + y)^2$. $\quad \boldsymbol{= (m + n)^2}$

b Write the problem. $\quad \boldsymbol{p^2 - 2p + 1}$

Write as $(\text{first})^2 \pm 2 \times \text{first} \times \text{second} + (\text{second})^2$ $\quad \boldsymbol{= p^2 - 2 \times p \times 1 + 1^2}$

Use $x^2 - 2xy + y^2 = (x - y)^2$. $\quad \boldsymbol{= (p - 1)^2}$

c Write the problem. $\quad \boldsymbol{9c^2 - 6cg + g^2}$

Write as $(\text{first})^2 \pm 2 \times \text{first} \times \text{second} + (\text{second})^2$ $\quad \boldsymbol{= (3c)^2 - 2 \times 3c \times g + g^2}$

Use $x^2 - 2xy + y^2 = (x - y)^2$. $\quad \boldsymbol{= (3c - g)^2}$

d Write the problem. $\quad \boldsymbol{4x^2 + 20x + 25}$

Write as $(\text{first})^2 \pm 2 \times \text{first} \times \text{second} + (\text{second})^2$ $\quad \boldsymbol{= (2x)^2 + 2 \times 2x \times 5 + 5^2}$

Use $x^2 + 2xy + y^2 = (x + y)^2$. $\quad \boldsymbol{= (2x + 5)^2}$

Investigate: Graphs of perfect squares

You have already seen that the graphs of $y = x^2$ and $y = -x^2$ are parabolas. What about the graphs of perfect squares?
For the function $y = x^2 - 6x + 9$, do the following.

- Factorise $x^2 - 6x + 9$.
- Complete the table of values below.

x	$^{-}2$	$^{-}1$	0	1	2	3	4	5	6	7	8
y											

- Plot the points and join with a smooth curve.
- What shape is formed?
- Compare the graph to that of $y = x^2$.
- Write a conclusion.

Repeat the steps above for the functions $y = x^2 + 4x + 4$, $y = 2x - x^2 - 1$ and $y = -x^2 - 4x - 4$.
What can you conclude about the graphs of perfect squares?

Important!

Graphs of perfect squares

The graph of the perfect square $y = (x - a)^2$ is a parabola with its turning point at $(a, 0)$.
It is shifted a units to the right compared to $y = x^2$.
The graph of the perfect square $y = (x + a)^2$ is a parabola with its turning point at $(-a, 0)$.
It is shifted a units to the left compared to $y = x^2$.
$y = -x^2$, $y = -(x - a)^2$ and $y = -(x + a)^2$ are upside down (reflected in the x-axis) compared to $y = x^2$.

Example 3

Puzzle sheet
Factorisation
MAT10NAPS00045

Use factorisation to help sketch the graph of $y = -8x - x^2 - 16$.

Solution

Write the function in the perfect square order.

$$\begin{aligned} \mathbf{y} &= \mathbf{-8x - x^2 - 16} \\ &= \mathbf{-(x^2 + 8x + 16)} \end{aligned}$$

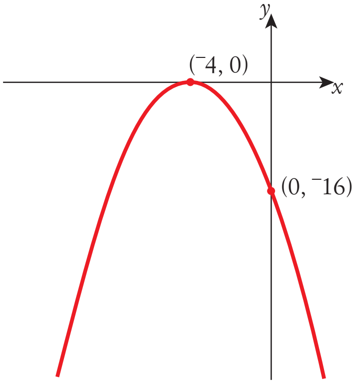
Factorise the function.

$$= \mathbf{-(x + 4)^2}$$

State the relationship to $y = x^2$.

$y = -(x + 4)^2$ is shifted 4 units to the left and upside down compared to $y = x^2$. Its turning point is $(^{-}4, 0)$.

Sketch the graph, including the y-intercept at $(0, {}^{-}16)$.



In some cases you can group terms together to make a common factor that is a group. This is called **grouping factorisation** and is common when there are 4 terms.

Example 4

Factorise each of the following expressions.

a $mx - nx + 3m - 3n$

b $5a^2 - 4ab + 4b - 5a$

Solution

a Write the problem. $\quad \mathbf{mx - nx + 3m - 3n}$

Factorise the first and last pairs of terms. $\quad \mathbf{= x(m - n) + 3(m - n)}$

Use $(m - n)$ as a common factor. $\quad \mathbf{= (x + 3)(m - n)}$

b Write the problem. $\quad \mathbf{5a^2 - 4ab + 4b - 5a}$

Make a common factor of $(5a - 4b)$ with the pairs. $\quad \mathbf{= a(5a - 4b) - 1(5a - 4b)}$

Use $(5a - 4b)$ as a common factor. $\quad \mathbf{= (a - 1)(5a - 4b)}$

You should check that both parts in Example **4** can also be done by rearranging the terms. You can rearrange part **a** as $mx + 3m - nx - 3n$ to get the common factor $(x + 3)$. You can rearrange part **b** as $5a^2 - 5a - 4ab + 4b$ to get the common factor $(a - 1)$. It doesn't matter which way around you do it, so you can pick whichever way that seems easier to you.

Exercise 7.1 Algebraic patterns and groups

Understanding

Extra questions
Exercise 7.1
MAT10NAEQ00019

See Example 1

1 Copy and complete each of the following factorisations.

a $x^2 - y^2 = (x + y)(\ldots\ldots)$
b $16 - a^2 = (4 + a)(\ldots\ldots)$
c $64 - b^2 = (\ldots\ldots)(8 - b)$
d $w^2 - 49 = (w + 7)(\ldots\ldots)$
e $p^2 - q^2 = (\ldots\ldots)(p - q)$
f $a^2 - 25 = (\ldots\ldots)(a - 5)$
g $c^2 - 81 = (\ldots\ldots)(c - 9)$
h $x^2 - 121 = (\ldots\ldots)(x - 11)$
i $36 - d^2 = (6 + d)(\ldots\ldots)$
j $100 - m^2 = (10 + m)(\ldots\ldots)$

See Example 2

2 Copy and complete each of the following factorisations.

a $m^2 + 2mn + n^2 = (\ldots + n)^2$
b $x^2 - 2xy + y^2 = (x - \ldots)^2$
c $a^2 + 4a + 4 = (\ldots + 2)^2$
d $w^2 + 6w + 9 = (w + \ldots)^2$
e $h^2 - 2hk + k^2 = (\ldots - k)^2$
f $b^2 - 10b + 25 = (b - \ldots)^2$
g $n^2 + 12n + 36 = (n + \ldots)^2$
h $m^2 - 12m + 36 = (m - \ldots)^2$
i $x^2 - 14x + 49 = (x - \ldots)^2$
j $k^2 + 18k + 81 = (k + \ldots)^2$

3 Factorise each of the following.

a $a^2 - m^2$
b $x^2 - 1$
c $c^2 - 9$
d $y^2 - 16$
e $25 - w^2$
f $49 - a^2$
g $b^2 - 100$
h $g^2 - 81$
i $169 - n^2$
j $x^2 - 64$

4 Factorise each of the following.

a $a^2 + 2ab + b^2$
b $m^2 - 2mn + n^2$
c $h^2 - 6h + 9$
d $x^2 + 20x + 100$
e $p^2 + 10p + 25$
f $g^2 - 12g + 36$
g $x^2 - 14x + 49$
h $h^2 + 6h + 9$
i $25 - 10n + n^2$
j $p^2 + 8p + 16$
k $9 - 6x + x^2$
l $y^2 + 18y + 81$

See Example 4

5 Factorise each of the following.

a $6b + ab + 6m + am$
b $^-7g - dg + 7y + dy$
c $^-5a - ac + 5b + bc$
d $m^2 + mn + mq + nq$
e $ab - 3b + a^3 - 3a^2$
f $m^2 + 2m^3 - 6m - 3$
g $ax + ay + bx + by$
h $xy - xz + ay - az$
i $ad + bd - ap - bp$

Fluency

6 Factorise each of the following.

a $4a^2 - 25b^2$
b $16x^2 - 9y^2$
c $49m^2 - 16n^2$
d $9a^2 - 64b^2$
e $4x^2 - 121y^2$
f $36x^2 - 25y^2$
g $144m^2 - 25n^2$
h $64p^2 - 49q^2$
i $81a^2 - 16b^2$

7 Factorise each of the following.

a $4x^2 - 4x + 1$
b $9x^2 + 6x + 1$
c $16g^2 - 8g + 1$
d $4a^2 - 4ab + b^2$
e $9p^2 + 6pq + q^2$
f $25x^2 - 10xy + y^2$
g $x^2 + 16bx + 64b^2$
h $b^2 + 22bc + 121c^2$
i $a^2 - 18ab + 81b^2$
j $a^2 - 16ab + 64b^2$
k $9h^2 + 6hk + k^2$
l $16g^2 + 8g + 1$
m $4m^2 - 4mn + n^2$
n $16p^2 + 8pq + q^2$
o $25x^2 - 10x + 1$
p $49y^2 - 56y + 16$

Worked solutions
Exercise 7.1
MAT10NAWS00019

8 Factorise each of the following.

a $(2a)^2 + 12a + 3^2$
b $(3x)^2 - 30x + 5^2$
c $(4m)^2 - 24m + 3^2$
d $9n^2 + 12n + 4$
e $25k^2 - 40k + 16$
f $4a^2 - 28a + 49$
g $4m^2 - 24m + 36$
h $81x^2 + 36x + 4$
i $16a^2 - 40a + 25$
j $64m^2 + 112m + 49$

Worked solutions
Exercise 7.1
MAT10NAWS00019

9 Factorise each of the following.

a $2xy^2 - y^2 - 2x + 1$
b $m^2n^2 + 1 - n - m^2n$
c $hx + gx + jx + hz + gz + jz$
d $ad - ac + cg + cf - df - dg$
e $a^2 - b^2 + 3a - 3b$
f $4x^2 + x - 16x - 4$

g $ac + 4d + ad + 4c$ **h** $x^2y + 3xy - 4x - 12$
i $a^2b - 1 - a + ab$ **j** $6a^3 - 12a^2 - 4a + 8$

Worked solutions
Exercise 7.1
MAT10NAWS00019

10 Factorise each of the following.

a $(a - 3)^2 - 1$ **b** $(a + b)^2 - 9$ **c** $(x - 7)^2 - 16$
d $4 - (2m - 3)^2$ **e** $4 - (4a - 7)^2$ **f** $(x - 3)^2 - 25$
g $36(x - 1)^2 - 25(x + 2)^2$ **h** $64(a + 1)^2 - 9(a - 3)^2$ **i** $4(3x - 5)^2 - 49(2x - 3)^2$
j $(x - y)^2 - (x + 2y)^2$ **k** $(a + b)^2 - (a - 2b)^2$

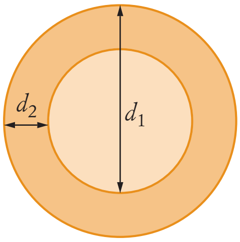
Problem solving

See Example 3

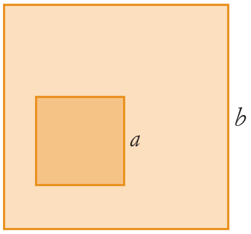
11 Use factorisation to help sketch the graphs of each of the following.

a $y = x^2 - 2x + 1$ **b** $y = -x^2 - 6x - 9$ **c** $y = x^2 + 2x + 1$
d $y = x^2 + 6x + 9$ **e** $y = x^2 - 8x + 16$ **f** $y = x^2 + 8x + 16$
g $y = 10x - x^2 - 25$ **h** $y = 2x - x^2 - 1$

12 An **annulus** is the shape left when a circle is cut out of the centre of another circle, as shown in the diagram. Find a formula for the area of the annulus in terms of d_1 and d_2 shown on the diagram. (*Hint:* Start with the areas of the circles in terms of the radii.)



13 This diagram shows two squares whose sides are a metres and b metres.



a What is the area of the larger square?
b What is the area of the smaller square?
c Write an expression for the area of the shaded region.
d Factorise the expression you wrote in part **c**.
e If $a = 37$ m and $b = 137$ m, use both of the expressions you wrote to find the area of the shaded region. Which calculation was easier?

Reasoning

14 Show that the volume of concrete used to make a concrete pipe of outer diameter D, inner diameter d and length L is given by $V = \frac{1}{4}\pi L(D + d)(D - d)$.

7.2 Quadratic expressions

Quadratic functions have the format $y = ax^2 + bx + c$ and their graphs are parabolas.

Important!

Quadratic expression

A **quadratic expression** in the variable x has the format $ax^2 + bx + c$, where a, b and c are real numbers. Quadratic expressions are often just called quadratics. They are also called **trinomials** since they have three terms. A quadrilateral where $a = 1$, $x^2 + bx + c$, is called a **monic quadratic**.

A quadratic is actually a polynomial of order 2.

You have already seen that quadratics are produced by the expansion of binomial brackets, such as $(a + 5)(a - 4) = a^2 + a - 20$ or $(5c - 4)(2c - 3) = 10c^2 - 23c + 12$. It is not immediately clear how you can do the reverse, factorisation of quadratics.

Example 5

Video tutorial
Factorising quadratic expressions 1
MAT10NAVT10016

Complete each of the following factorisations and check by expansion.

a $p^2 + 8p + 15 = (p + 3)(p + \ldots)$
b $m^2 + m - 6 = (m + 3)(m - \ldots)$
c $x^2 - 7x + 6 = (x - 1)(x - \ldots)$
d $a^2 - 6a - 16 = (a - 8)(a + \ldots)$

Solution

a	Write the partial answer.	$(p + 3)(p + \ldots)$
	$15 = 3 \times 5$, so try 5.	$= (p + 3)(p + 5)$
	Expand the brackets.	$= p(p + 5) + 3(p + 5)$
	Keep going.	$= p^2 + 5p + 3p + 15$
	Simplify.	$= p^2 + 8p + 15$ ✓OK
	Write the whole answer.	$p^2 + 8p + 15 = (p + 3)(p + 5)$
b	Write the partial answer.	$(m + 3)(m - \ldots)$
	$^{-}6 = 3 \times {}^{-}2$, so try $^{-}2$.	$= (m + 3)(m - 2)$
	Expand the brackets.	$= m(m - 2) + 3(m - 2)$
	Keep going.	$= m^2 - 2m + 3m - 6$
	Simplify.	$= m^2 + m - 6$ ✓OK
	Write the whole answer.	$m^2 + m - 6 = (m + 3)(m - 2)$
c	Write the partial answer.	$(x - 1)(x - \ldots)$
	$6 = {}^{-}1 \times {}^{-}6$, so try $^{-}6$.	$= (x - 1)(x - 6)$
	Expand the brackets.	$= x(x - 6) - 1(x - 6)$
	Keep going.	$= x^2 - 6x - x + 6$
	Simplify.	$= x^2 - 7x + 6$ ✓OK
	Write the whole answer.	$x^2 - 7x + 6 = (x - 1)(x - 6)$
d	Write the partial answer.	$(a - 8)(a + \ldots)$
	$^{-}16 = {}^{-}8 \times 2$, so try 2.	$= (a - 8)(a + 2)$
	Expand the brackets.	$= a(a + 2) - 8(a + 2)$
	Keep going.	$= a^2 + 2a - 8a - 16$
	Simplify.	$= a^2 - 6a - 16$ ✓OK
	Write the whole answer.	$a^2 - 6a - 16 = (a - 8)(a + 2)$

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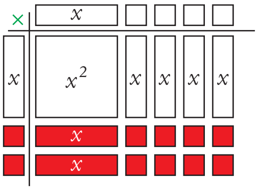
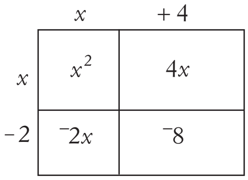
Investigate: Factorisation of quadratics I

Teacher notes
Factorisation of quadratics I
MAT10NATN00005

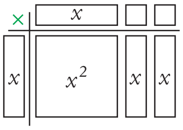
Weblink
Factorising with concrete materials
MAT10NAWB00007

You may have seen previously that you can model expansion of binomial brackets with squares to represent x^2 and oblongs of the same length to represent x. By colouring them on the back, you can also show $-x^2$ and $-x$. You can represent numbers as small squares of the same width as the oblongs to represent 1. You show $(x - 2)(x + 4)$ using this method as below. The *area* method is shown beside it.

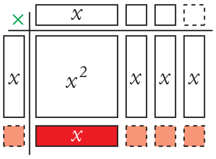




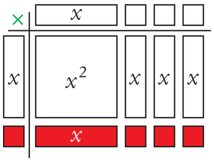
The expansion of $(x - 2)(x + 4)$ is $x^2 + 2x - 8$, as can be seen from either model.
How can you start with the expansion and end up with the model, using the squares and oblongs?
Looking at the first 2 terms of $x^2 + 2x - 8$, $x^2 + 2x = x(x + 2)$.
Make a model of $x(x + 2) = x^2 + 2x$.
This gives you the x^2 term and *some* of the xs.
It doesn't give any units.



To improve the model you add zero in the form $x - x$, with the positive x on the top and the negative x down the side.



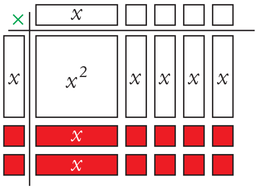
Inside the lines you now have $x^2 - x + 3x$
To make the model correct you need to put a unit on the top, a negative unit down the side and some units inside the lines.
Now count the units down the side, on the top and inside the lines.
Does the number of units inside the lines match the number of units in the problem?



Units table:

Side	Top	Inside	OK?
$^{-}1$	3	$^{-}3$	✗

Add another zero in the form $x - x$, putting the negative x down the side and the positive x on the top.
You now have $x^2 - 2x + 4x$ inside the lines.
Put in extra units to make the expansion work.



Units table:

Side	Top	Inside	OK?
$^{-}1$	3	$^{-}3$	✗
$^{-}2$	4	$^{-}8$	✓

Count the units down the side, on the top and inside the lines.

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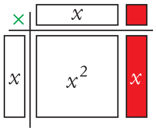
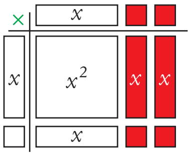
Does it match?
Yes!
The factorisation is, unsurprisingly, $x^2 + 2x - 8 = x^2 - 2x + 4x - 8 = (x - 2)(x + 4)$
Obviously, any quadratic with signs like the one above can be done in the same way. The signs in a quadratic can be distributed in any of 4 ways: $ax^2 + bx - c$, $ax^2 - bx + c$, $ax^2 + bx + c$ or $ax^2 - bx - c$.

Let's look at factorising $x^2 - x - 12$.
Start with a model of $x^2 - x = x(x - 1)$.
In this case, you need to add zero again, but will have to put the negative on the top and the positive down the side.



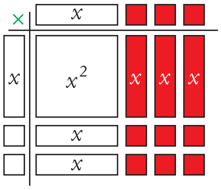
This will make $x^2 - 2x + x$ inside the lines. You will also need to put in extra units to make the model work. Draw up a table of the units.
Do the units match the units in the quadratic?



Units table:

Side	Top	Inside	OK?
1	$^-2$	$^-2$	✗

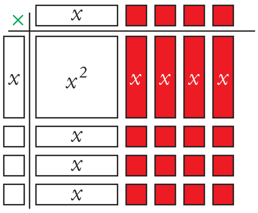
No, so keep going by adding another zero.



Units table:

Side	Top	Inside	OK?
1 2	$^-2$ $^-3$	$^-2$ $^-6$	✗ ✗

You now have $x^2 - 3x + 2x$ inside the lines. What about the units? Do they match the quadratic?
No, you need $^-12$.
Go again.

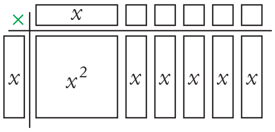


Units table:

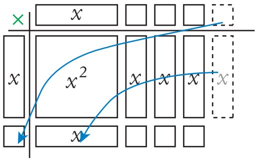
Side	Top	Inside	OK?
1 2 3	$^-2$ $^-3$ $^-4$	$^-2$ $^-6$ $^-12$	✗ ✗ ✓

You now have $x^2 - 4x + 3x$ inside the lines. Do the units match the quadratic?
Yes! There are $^-12$.
The factorisation is $x^2 - x - 12 = x^2 - 4x + 3x - 12 = (x - 4)(x + 3)$

Now let's look at $x^2 + 5x + 6$.
Make a model of $x^2 + 5x = x(x + 5)$
This gives the x^2 term and *all* of the *x*s.
You need some units. Adding $x - x$ will make the units inside the lines all negative.



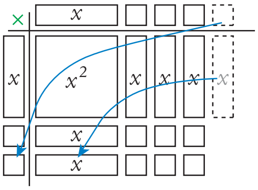
Move one of the *x*s from the top to the side instead of adding zero.
Put in the units to make the model work.



Units table:

Side	Top	Inside	OK?
1	4	4	✗

Do the units match the 6 in the quadratic?
No, so move another one.



Units table:

Side	Top	Inside	OK?
1 2	4 3	4 6	✗ ✓

Do the units match the 6 in the quadratic? Yes!
The factorisation is $x^2 + 5x + 6 = x^2 + 2x + 3x + 6 = (x + 2)(x + 3)$.
Try working through the model for $x^2 - 8x + 12$. You need to remember that when the units are negative in *both* on the top and down the side, they will be *positive* inside the lines.
Starting with $x^2 - 8x = x(x - 8)$ you will have to move $-x$ from the top to the side.
You should end up with the last line of the units table being |$^{-}2$ |$^{-}6$ | 12 |✓ |.
What is the factorisation?
Your teacher will probably want you to do some more models before you go on.

In the investigation above, you will have noticed that the last column and row numbers in the units table always *add up* to the coefficient of *x* and *multiply* to give the constant. The column number is the number in the first bracket and the row number is the number in the second bracket. If there is a factorisation, this always works when x^2 has a coefficient of 1.
The numbers *u* and *v* that give $x^2 + bx + c = (x + u)(x + v)$ are such that $uv = c$ and $u + v = c$.

Example 6

Factorise each of the following.

a $a^2 + 7a + 12$ **b** $x^2 + 9x + 8$

Solution

a Write the problem. $\mathbf{a^2 + 7a + 12}$

You want two numbers with product 12 and sum 7. $\mathbf{2 \times 6 = 12}$ **but** $\mathbf{2 + 6 \neq 7}$

Try some other numbers. $\mathbf{3 \times 4 = 12}$ **and** $\mathbf{3 + 4 = 7}$ ✓OK

Rewrite the expression using 3*a* and 4*a*. $\mathbf{a^2 + 7a + 12 = a^2 + 3a + 4a + 12}$

Factorise each pair. $\mathbf{= a(a + 3) + 4(a + 3)}$

Use $(a + 3)$ as the common factor. $\mathbf{= (a + 3)(a + 4)}$

b Write the problem.	$x^2 + 9x + 8$
You want two numbers with product 8 and sum 9.	$2 \times 4 = 8$ but $2 + 4 \neq 9$
Try some other numbers.	$1 \times 8 = 8$ and $1 + 8 = 9$ ✓OK
Rewrite the expression using $1x$ and $8x$.	$x^2 + 9x + 8 = x^2 + x + 8x + 8$
Factorise each pair.	$= x(x + 1) + 8(x + 1)$
Use $(x + 1)$ as the common factor.	$= (x + 1)(x + 8)$

When you rewrite the middle term in the quadratic, it is always easier to *write the smallest term first*. If one term is negative and the other is positive, the smallest is the negative term.

Technology worksheet

Excel worksheet: Factorising trinomials

MAT10NACT00011

CAS TI-Nspire exercise

Advanced algebra and functions

MAT10NATI00015

CAS ClassPad exercise

Advanced algebra and functions

MAT10NACP00015

Example 7

Factorise each of the following.

a $x^2 + x - 6$ **b** $a^2 - 2a - 15$ **c** $y^2 - 6y + 8$

Solution

a Write the problem.	$x^2 + x - 6$
You want two numbers with product $^{-}6$ and sum 1.	$2 \times {}^{-}3 = {}^{-}6$ but $2 + {}^{-}3 = {}^{-}1$
Swap signs to change the sign of $^{-}1$.	$^{-}2 \times 3 = {}^{-}6$ and $^{-}2 + 3 = 1$ ✓OK
Rewrite the expression with the negative first.	$x^2 + x - 6 = x^2 - 2x + 3x - 6$
Factorise each pair.	$= x(x - 2) + 3(x - 2)$
Use $(x - 2)$ as the common factor.	$= (x - 2)(x + 3)$
b Write the problem.	$a^2 - 2a - 15$
You want two numbers with product $^{-}15$ and sum $^{-}2$.	$^{-}5 \times 3 = {}^{-}15$ and $^{-}5 + 3 = {}^{-}2$ ✓OK
Rewrite the expression with the negative first.	$a^2 - 2a - 15 = a^2 - 5a + 3a - 15$
Factorise each pair.	$= a(a - 5) + 3(a - 5)$
Use $(a - 5)$ as the common factor.	$= (a - 5)(a + 3)$
c Write the problem.	$y^2 - 6y + 8$
You want two numbers with product 8 and sum $^{-}6$.	$^{-}2 \times {}^{-}4 = {}^{-}8$ and $^{-}2 + {}^{-}4 = {}^{-}6$ ✓OK
Rewrite the expression with the smallest first.	$y^2 - 6y + 8 = y^2 - 4y - 2y + 8$
Factorise each pair, being careful with signs.	$= y(y - 4) - 2(y - 4)$
Use $(y - 4)$ as the common factor.	$= (y - 4)(y - 2)$

You will have noticed that once you have found the right two numbers, then with quadratics like those in Examples **6** and **7** you can go straight to the answer.

Investigate: Graphs of quadratics

In the last section of the text, you looked at sketching perfect squares. How can you use factorisation to help sketch quadratics?
For the function $y = x^2 - 5x - 6$, do the following.

- Factorise $x^2 - 5x - 6$.
- Complete the table of values below.

x	$^-2$	$^-1$	0	1	2	3	4	5	6	7	8
y											

- Plot the points and join with a smooth curve.
- What shape is formed?
- Compare the graph to that of $y = x^2$.
- Where is the turning point?
- What are the zeros (x-intercepts)?
- Write a conclusion.

Repeat the steps above for the functions $y = x^2 + x + 6$, $y = 2x - x^2 - 8$ and $y = -x^2 - 5x - 4$.
What can you conclude about the graphs of quadratics?

Technology worksheet

Excel worksheet: Investigating parabolas 2

MAT10NACT00012

Technology: GeoGebra

Parabolas

MAT10NATC00007

TLF Learning object

Grapher (L3531)

MAT10NAIN00007

TLF Learning object

Graphing functions (R11253)

MAT10NAIN00007

Important!

Graphs of quadratics

The graph of the quadratic $f(x) = (x + a)(x + b)$ is a parabola with its **turning point** at $\left(\frac{-a-b}{2}, f\left(\frac{-a-b}{2}\right)\right)$. It is shifted $\frac{a+b}{2}$ units to the left and $f\left(\frac{-a-b}{2}\right)$ units up or down according to the sign compared to $y = x^2$.

If $\frac{a+b}{2}$ is negative, then the graph is shifted to the right. If the whole function is negative (i.e., the coefficient of x^2 is negative) the graph is upside down compared to $y = x^2$. The **zeros** of the function, where it passes through the x-axis, are at $(-a, 0)$ and $(-b, 0)$.

Example 8

Worksheet
Parabolas
MAT10NAWK00019

Use factorisation to help sketch the graph of $y = -6x - x^2 - 5$.

Solution

Rearrange the function to write as a quadratic.

$$\begin{aligned} y &= -6x - x^2 - 5 \\ &= -(x^2 + 6x + 5) \end{aligned}$$

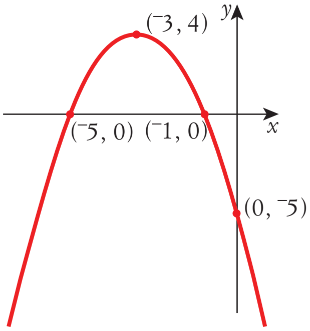
Use $1 \times 5 = 5$ and $1 + 5 = 6$ to factorise the quadratic.

$$= -(x + 1)(x + 5)$$

Use the characteristics of the factorised function to state important points.

The graph is shifted $\frac{1+5}{2} = 3$ units to the left. It has its turning point at $({}^{-}3, 4)$ and zeros at $({}^{-}5, 0)$ and $({}^{-}1, 0)$. The y-intercept is at $(0, {}^{-}5)$.

Sketch the graph.

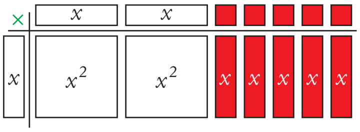


What happens with quadratics where the coefficient of x^2 is not 1 (**non-monic** quadrilaterals)?

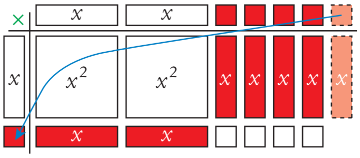
Investigate: Factorisation of quadratics II

Teacher notes
Factorisation of quadratics II
MAT10NATN00006

You can use the previous model to factorise quadratics where the coefficient of x is not 1, but it is a little more tedious. Let's try factorising $2x^2 - 5x - 3$.
Looking at the first 2 terms of $2x^2 - 5x - 3$, $2x^2 - 5x = x(2x - 5)$.
Make a model of $x(2x - 5) = 2x^2 - 5x$.
This gives you the x^2 terms and all the xs.
It doesn't give any units.



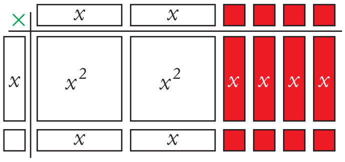
Move one of the units from the top to the side.
You have to put some units inside the lines to make the model work.
But these will be positive as $^{-}1 \times {}^{-}1 = 1$.



Units table:

Side	Top	Inside	OK?
$^{-}1$	$^{-}4$	4	✗

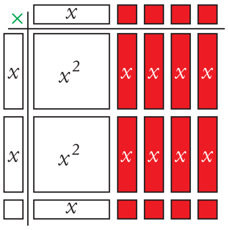
There are now too many $-x$s. To make the units inside the lines negative, change the sign of the unit at the side.



Units table:

Side	Top	Inside	OK?
$^{-}1$	$^{-}4$	4	✗
1	$^{-}4$	$^{-}4$	✗

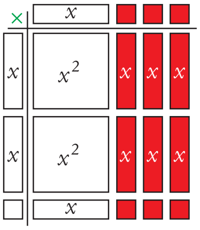
Now there aren't enough negative xs and there are too many negative units. If you move more to the side, it gets worse.
Try swapping an x from the top to the side.



Units table:

Side	Top	Inside	OK?
$^{-}1$	$^{-}4$	4	✗
1	$^{-}4$	$^{-}4$	✗
1	$^{-}4$	$^{-}4$	✗

Now there are too many $-x$s *and* too many $^{-}1$s.
Try taking one of the units off the top.



Units table:

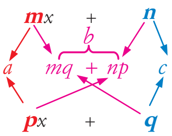
Column	Row	Inside	OK?
$^{-}1$	$^{-}4$	4	✗
1	$^{-}4$	$^{-}4$	✗
1	$^{-}4$	$^{-}4$	✗
1	$^{-}3$	$^{-}3$	✓

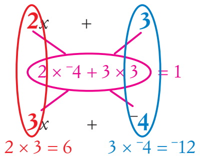
That works!
But the units table by itself would not have shown you the steps to get to this.
To get the correct model, you do need to use a bit of trial and error.
The factorisation is $2x^2 - 6x + x - 3 = (2x + 1)(x - 3)$.
Your teacher may want you to try some more models.

While $(2x + 3)(3x - 4) = 6x^2 + x - 12$, factorising $6x^2 + x - 12$ is difficult.
Assume that $ax^2 + bx + c$ factorises to give $(mx + n)(px + q)$.
$(mx + n)(px + q) = mpx^2 + mqx + npx + nq = mpx^2 + (mq + np)x + nq$
so $ax^2 + bx + c = mpx^2 + (mq + np)x + nq$
This means that $a = mp$, $c = nq$ and $b = mq + np$.

Cross method

To use the **cross method** you arrange numbers that might be m, n, p and q in a square as shown on the right.
The correct arrangement for $6x^2 + x - 12$ is shown beside it on the far right.
The end products give a and c.
The cross products give b.





Decomposition method

We can multiply a and c to get $ac = mp \times nq = mpnq = mq \times np$.
Write $s = mq$ and $t = np$.

Then $st = ac$ and $s + t = b$,
so $ax^2 + bx + c = ax^2 + sx + tx + c$
will be the expansion of $(mx + n)(px + q)$.

For the **decomposition method**, you need to find two numbers s and t that multiply to give ac and add to give b.

Then you decompose bx into $sx + tx$ to factorise the quadratic using grouping.

For $6x^2 + x - 12$, $ac = {}^{-}72$ and $b = 1$.
${}^{-}8 \times 9 = {}^{-}72$ and ${}^{-}8 + 9 = 1$, so
$s = {}^{-}8$ and $t = 9$, so use $x = {}^{-}8x + 9x$

$$\begin{aligned} 6x^2 + x - 12 &= 6x^2 - 8x + 9x - 12 \\ &= 2x(3x - 4) + 3(3x - 4) \\ &= (2x + 3)(3x - 4) \end{aligned}$$

Example 9

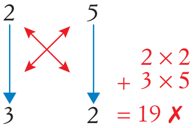
Factorise $6p^2 + 17p + 10$.

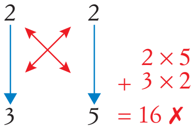
Solution

Cross method

Make a cross with numbers arranged vertically that multiply to give 6 and 10 and try the cross products.

Try 2×3 and 5×2 first.

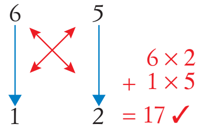




Try them with the 5 and 2 switched over.

Try 6 × 1 and 5 × 2.

It gives the right answer for b.



Write the answer. $\mathbf{6p^2 + 17p + 10 = (6p + 5)(p + 2)}$

Decomposition method

What numbers do you need? **$ac = 6 \times 10 = 60$ and $b = 17$, so you need to find u and v so that $uv = 60$ and $u + v = 17$.**

Try some out until you find the right ones. **$6 \times 10 = 60$ but $6 + 10 \neq 17$ ✗**

$5 \times 12 = 60$ and $5 + 12 = 17$ ✓

Write the quadratic using 5 and 12. $\mathbf{6p^2 + 17p + 10 = 6p^2 + 5p + 12p + 10}$

Factorise each pair. $\mathbf{= p(6p + 5) + 2(6p + 5)}$

Use $(6p + 5)$ as the common factor. $\mathbf{= (6p + 5)(p + 2)}$

The decomposition method is the exact reverse of the expansion of brackets, while the cross method uses more obvious trial and error to find the correct factorisation. The quadratic $4x^2 + 7x + 2$ does not factorise. It is harder to see that a quadratic will not factorise from the cross method than from decomposition. Your teacher may have a preference as to which method you use, or may leave it up to you. If you use the decomposition method, it is easiest to put the *smallest* of u and v *first*, just as with the simpler quadratics.

Example 10

Video tutorial
Factorising quadratic expressions 2
MAT10NAVT10017

CAS TI-Nspire exercise
Advanced algebra and functions
MAT10NATI00015

CAS ClassPad exercise
Advanced algebra and functions
MAT10NACP00015

Factorise each of the following.

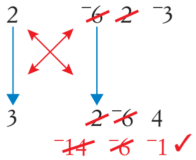
a $6x^2 - x - 12$ **b** $3x^2 - 11x + 10$ **c** $4x^2 - 3x - 7$

Solution

Cross method

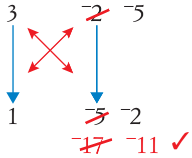
a Make a cross with numbers that multiply to give 6 and $^{-}12$.

Work out the cross products and write the answers underneath the cross, moving across as you try new ones.



Write the answer. $\mathbf{6x^2 - x - 12 = (2x - 3)(3x + 4)}$

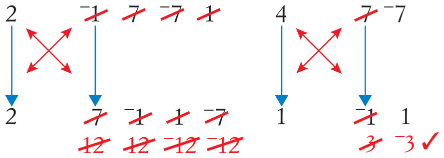
b Make a cross with numbers that multiply to give 3 and 10. The cross product is negative so the numbers that give 10 have to both be negative.



Write the answer. $\mathbf{3x^2 - 11x + 10 = (3x - 5)(x - 2)}$

c Make a cross with numbers that multiply to give 4 and $^{-}7$. Try different numbers and do the cross products as before.

2 × 2 doesn't work. Try 4 × 1 on a new cross.



Write the answer. $4x^2 - 3x - 7 = (4x - 7)(x + 1)$

Decomposition method

a What numbers do you need? **$ac = 6 \times {}^{-}12 = {}^{-}72$ and $b = {}^{-}1$, so you need to find u and v so that $uv = {}^{-}72$ and $u + v = {}^{-}1$.**

Try some out. $8 \times {}^{-}9 = {}^{-}72$ and $8 + {}^{-}9 = {}^{-}1$ ✓

Write the quadratic using $^{-}9$ and 8. $6x^2 - x - 12 = 6x^2 - 9x + 8x - 12$

Factorise each pair. $= 3x(2x - 3) + 4(2x - 3)$

Use $(2x - 3)$ as the common factor. $= (2x - 3)(3x + 4)$

b What numbers do you need? **$ac = 3 \times 10 = 30$ and $b = {}^{-}11$, so you need to find u and v so that $uv = -30$ and $u + v = {}^{-}11$.**

They must both be negative. $^{-}3 \times {}^{-}10 = 30$ but $^{-}3 + {}^{-}10 \neq {}^{-}11$ ✗

$^{-}6 \times {}^{-}5 = 30$ and $^{-}6 + {}^{-}5 = {}^{-}11$ ✓

Write the quadratic using $^{-}6$ and $^{-}5$. $3x^2 - 11x + 10 = 3x^2 - 6x - 5x + 10$

Factorise each pair (watch signs). $= 3x(x - 2) - 5(x - 2)$

Use $(x - 2)$ as the common factor. $= (x - 2)(3x - 5)$

c What numbers do you need? **$ac = 4 \times {}^{-}7 = {}^{-}28$ and $b = {}^{-}3$, so you need to find u and v so that $uv = {}^{-}28$ and $u + v = {}^{-}3$.**

Try them out. Whoops, swap signs. $7 \times {}^{-}4 = {}^{-}28$ but $7 + {}^{-}4 \neq {}^{-}3$ ✗

$^{-}7 \times 4 = {}^{-}28$ and $^{-}7 + 4 = {}^{-}3$ ✓

Write the quadratic using $^{-}7$ and 4. $4x^2 - 3x - 7 = 4x^2 - 7x + 4x - 7$

Factorise each pair, using 1 as the common factor for the second bracket. $= x(4x - 7) + 1(4x - 7)$

Use $(4x - 7)$ as the common factor. $= (4x - 7)(x + 1)$

You can factorise quadratic expressions with two variables in exactly the same way as you have been factorising quadratic functions with one variable.

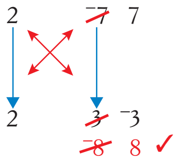
Example 11

Animated example
Quadratic expressions
MAT10NAAE00007

Factorise each of the following.

a $v^2 - 3vw - 10w^2$ **b** $4p^2 + 8pq - 21q^2$

Solution

a Write the problem.	$\mathbf{v^2 - 3vw - 10w^2}$
You want numbers with product $^{-}10$ and sum $^{-}3$.	$^{-}2 \times 5 = {}^{-}10$ but $^{-}2 + 5 = 3$ $^{-}5 \times 2 = {}^{-}10$ and $^{-}5 + 2 = {}^{-}3$ ✓OK
Write the answer.	$v^2 - 3vw - 10w^2 = (v - 5w)(v + 2w)$
b Cross method	
Make a cross with numbers that multiply to give 4 and $^{-}21$. Work out the cross products and write the answers underneath the cross, moving across as you try new ones.	
Write the answer.	$4p^2 + 8pq - 21q^2 = (2p + 7q)(2p - 3q)$
Decomposition method	
What numbers do you need?	$ac = 4 \times {}^{-}21 = {}^{-}84$ and $b = 8$, so you need to find u and v so that $uv = {}^{-}84$ and $u + v = 8$.
Try some out.	$^{-}7 \times 12 = {}^{-}84$ but $^{-}7 + 12 \neq 8$ ✗ $^{-}6 \times 14 = {}^{-}84$ and $^{-}6 + 14 = 8$ ✓
Write the quadratic using $^{-}6$ and 14.	$4p^2 + 8pq - 21q^2$ $= 4p^2 - 6pq + 14pq - 21q^2$
Factorise each pair.	$= 2p(2p - 3q) + 7q(2p - 3q)$
Use $(2p - 3q)$ as the common factor.	$= (2p - 3q)(2p + 7q)$

Exercise 7.2 Quadratic expressions

Understanding

Extra questions
Exercise 7.2
MAT10NAEQ00020

See Example 5

1 Complete each of the following factorisations and check by expansion.

a $a^2 + 5a + 4 = (a + 4)(a + \ldots)$
b $x^2 + 6x + 5 = (x + \ldots)(x + 1)$
c $y^2 + 7y + 10 = (y + 2)(y + \ldots)$
d $m^2 + 10m + 9 = (m + 1)(m + \ldots)$
e $p^2 - 5p + 4 = (p - 4)(p - \ldots)$
f $b^2 - 7b + 12 = (b - \ldots)(b - 4)$
g $a^2 - 3a - 4 = (a - \ldots)(a + 1)$
h $x^2 - 2x - 8 = (x + 2)(x - \ldots)$
i $y^2 + y - 12 = (y - 3)(y + \ldots)$
j $a^2 + 8a - 20 = (a - \ldots)(a + 10)$

See Example 6

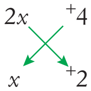
2 Factorise each of the following.

a $g^2 + 8g + 12$
b $h^2 + 5h + 6$
c $k^2 + 13k + 42$
d $x^2 + 5x + 6$
e $t^2 + 7t + 6$
f $z^2 + 11z + 30$
g $t^2 + 15t + 54$
h $y^2 + 7y + 12$
i $a^2 + 11a + 28$
j $w^2 + 12w + 35$

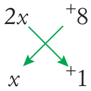
If you are going to use the ‘cross’ method, answer question **3**.

3 **a** Which ‘cross’ diagram will give the factors of $2x^2 + 17x + 8$?

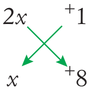
A



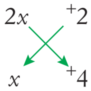
B



C

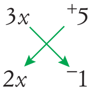


D

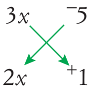


b Which ‘cross’ diagram will give the factors of $6x^2 + 7x - 5$?

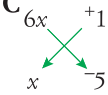
A



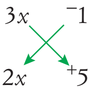
B



C

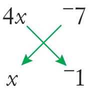


D

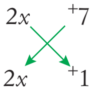


c Which ‘cross’ diagram will give the factors of $4x^2 - 16x + 7$?

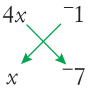
A



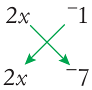
B



C

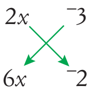


D

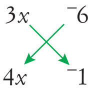


d Which ‘cross’ diagram will give the factors of $12x^2 - 17x + 6$?

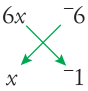
A



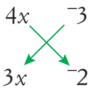
B



C

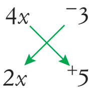


D

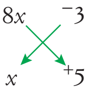


e Which ‘cross’ diagram will give the factors of $8x^2 - 2x - 15$?

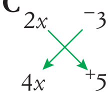
A



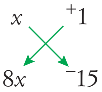
B



C



D



If you are going to use the decomposition method, answer questions **4** and **5**.

4 Find two numbers that:

a multiply to give 15 and add to give 8
b multiply to give $^{-}27$ and add to give $^{-}6$
c multiply to give $^{-}40$ and add to give $^{-}3$
d add to give 12 and multiply to give 32
e add to give $^{-}5$ and multiply to give $^{-}36$
f multiply to give 64 and add to give $^{-}16$
g add to give 2 and multiply to give $^{-}15$
h add to give $^{-}3$ and multiply to give $^{-}54$.

5 Each quadratic below has been ‘decomposed’. Complete the factorisations.

a $2x(x - 1) + 7(x - 1)$
b $3a(2a + 3) - 2(2a + 3)$
c $4x(3x - 5) + 3(3x - 5)$
d $5y(y - 4) + 6(y - 4)$
e $7x(2x + 1) + 3(2x + 1)$
f $3x(3x - 1) - 1(3x - 1)$
g $5a(4a + 3) - 2(4a + 3)$
h $5x(2x - 7) + (2x - 7)$
i $6a(a - 3) - (a - 3)$
j $4x(2x - 9) + 3(2x - 9)$

Fluency

See Examples 6, 7

6 Factorise each of the following.

a $a^2 + 4a + 3$
b $x^2 + 9x + 8$
c $y^2 + 7y + 6$
d $m^2 + 5m + 6$
e $x^2 + 6x + 8$
f $p^2 + 9p + 20$
g $p^2 + 10p + 16$
h $g^2 + 8g + 15$
i $x^2 + 10x + 24$
j $x^2 + 13x + 30$
k $a^2 + 14a + 40$
l $m^2 + 17m + 42$
m $x^2 - 3x + 2$
n $a^2 - 4a + 3$
o $y^2 - 5y + 6$

p $x^2 - 10x + 9$
q $y^2 - 7y + 10$
r $a^2 - 10a + 16$
s $x^2 - 10x + 21$
t $w^2 - 11w + 18$
u $a^2 - 11a + 24$
v $a^2 - 18a + 32$
w $y^2 - 16y + 15$
x $b^2 - 12b + 35$

7 Factorise each of the following.

a $b^2 - b - 6$
b $a^2 - a - 20$
c $x^2 - x - 12$
d $y^2 - y - 42$
e $c^2 - 2c - 3$
f $a^2 - 2a - 8$
g $n^2 - 2n - 15$
h $m^2 - 7m - 8$
i $a^2 - 3a - 10$
j $x^2 - 3x - 10$
k $y^2 - 4y - 12$
l $b^2 - 6b - 27$
m $p^2 + 5p - 6$
n $k^2 + 3k - 4$
o $a^2 + a - 6$
p $c^2 + c - 20$
q $x^2 + 3x - 40$
r $y^2 + 5y - 6$
s $a^2 + 4a - 12$
t $b^2 + 6b - 7$
u $k^2 + 6k - 16$
v $p^2 + 4p - 32$
w $a^2 + 2a - 35$
x $x^2 + 5x - 36$

8 Factorise each of the following.

See Example 11

a $e^2 - 14en + 45n^2$
b $w^2 + 8wt - 105t^2$
c $a^2 - 8ag + 15g^2$
d $b^2 + 3bv - 108v^2$
e $p^2 + 3pe - 4e^2$
f $c^2 - 8cm - 33m^2$
g $z^2 - 10zm - 11m^2$
h $y^2 + 6yn - 55n^2$
i $g^2 + 7ge - 78e^2$
j $g^2 + 2gp - 99p^2$
k $h^2 + 21hw + 98w^2$
l $z^2 + 29zq + 210q^2$
m $z^2 - 11zu - 60u^2$
n $x^2 + 19xz + 84z^2$
o $q^2 + 3qn - 28n^2$
p $p^2 - 2pr - 3r^2$
q $x^2 - 14xv + 48v^2$
r $k^2 + 4kp + 3p^2$
s $h^2 - 3hz - 108z^2$
t $c^2 - 23cp + 130p^2$
u $z^2 - 6zr - 40r^2$
v $t^2 - 6tp - 27p^2$
w $g^2 + 10gq - 24q^2$
x $k^2 - 14km + 24m^2$

9 Factorise each of the following.

See Example 9

a $2x^2 + 9x + 9$
b $2a^2 + 9a + 10$
c $2x^2 + 13x + 15$
d $3m^2 + 14m + 8$
e $3x^2 + 20x + 12$
f $3b^2 + 13b + 12$
g $5x^2 + 22x + 8$
h $5c^2 + 14c + 9$
i $4x^2 + 12x + 9$
j $4b^2 + 13b + 3$
k $5p^2 + 7p + 2$
l $5h^2 + 11h + 2$
m $6a^2 + 7a + 2$
n $6x^2 + 13x + 2$
o $3y^2 + 8y + 4$
p $6x^2 + 23x + 20$
q $12n^2 + 29n + 14$
r $8x^2 + 18x + 9$
s $12y^2 + 35y + 25$
t $16w^2 + 24w + 9$
u $15x^2 + 28x + 12$
v $32r^2 + 52r + 15$
w $12a^2 + 22a + 8$
x $42x^2 + 73x + 28$

10 Factorise each of the following.

See Example 10

a $2x^2 - 9x - 5$
b $5a^2 - a - 4$
c $2x^2 - 7x - 15$
d $2m^2 - m - 1$
e $3x^2 - 2x - 1$
f $6b^2 - 5b - 1$
g $2x^2 - 5x - 3$
h $2h^2 - 3h - 2$
i $6y^2 - 11y - 2$
j $5x^2 - 9x - 2$
k $9n^2 - 17n - 2$
l $10x^2 - 19x - 2$
m $2a^2 - 7a - 4$
n $7x^2 + 13x - 2$
o $10y^2 + 19y - 2$
p $4x^2 + 15x - 4$
q $5x^2 + 14x - 3$
r $2b^2 + 5b - 3$
s $3x^2 + 8x - 3$
t $4w^2 + 11w - 3$
u $5x^2 + 19x - 4$
v $6r^2 + 23r - 4$
w $2x^2 + 7x - 4$
x $2a^2 + 9a - 5$

11 Factorise each of the following.

a $5x^2 + 39x - 8$
b $3a^2 - 10a - 8$
c $7b^2 - 16b - 15$
d $5x^2 + 7x - 6$
e $3y^2 + 19y + 6$
f $5x^2 - 24x + 27$
g $7f^2 - 54f + 35$
h $11k^2 - 89k + 8$
i $12a^2 + 32a - 35$
j $20y^2 - 13y - 84$
k $24x^2 + 110x + 121$
l $6d^2 - d - 77$

m $2x^2 + x - 10$
n $30m^2 - 31m - 44$
o $6x^2 - 13x - 5$
p $6c^2 + 7c - 20$
q $9x^2 + 6x + 1$
r $18a^2 + 21a - 4$
s $12p^2 + 20p - 25$
t $15x^2 + 7x - 2$
u $8a^2 - 10a - 3$
v $7x^2 - 27x - 4$
w $9n^2 + 35n - 4$
x $6x^2 + x - 77$

12 Factorise each of the following.

a $27z^2 + 12zn - 32n^2$
b $27b^2 + 60bg + 28g^2$
c $84p^2 - 16pe - 5e^2$
d $5v^2 - 34vk - 99k^2$
e $40b^2 + 94bn + 55n^2$
f $15m^2 + 2mp - 8p^2$
g $98x^2 + 189xu + 55u^2$
h $72p^2 - 198pn + 121n^2$
i $48x^2 - 46xg - 77g^2$
j $30v^2 - 229vn + 30n^2$
k $24m^2 - 110mh + 75h^2$
l $45z^2 - 57zm - 4m^2$
m $112b^2 + 150bh + 27h^2$
n $63y^2 - 34ya - 80a^2$
o $24m^2 - 52mr + 24r^2$
p $104n^2 - 67na - 10a^2$
q $143u^2 + 87ug - 90g^2$
r $110e^2 - 41eu - 126u^2$
s $180x^2 - 153xp - 14p^2$
t $195t^2 - 92tc - 7c^2$
u $36r^2 - 72rk + 11k^2$
v $3x^2 + 41xb - 14b^2$
w $99z^2 - 170zn + 39n^2$
x $8t^2 + 27tp + 9p^2$

Problem solving

Worked solutions
Exercise 7.2
MAT10NAWS00020
See Example 8

⑬ How many integer values of b are there for which $x^2 + bx + 6$ can be factorised using integer values?

14 How many integer values of b are there for which $x^2 + bx - 12$ can be factorised using integer values?

15 Use factorisation to help sketch the graph of each of the following.

a $y = x^2 - 2x - 3$
b $y = -x^2 - 4x - 3$
c $y = x^2 + 6x + 8$
d $y = x^2 - 6x + 8$
e $y = x^2 + 4x + 3$
f $y = x^2 - 5x + 4$
g $y = 5x - x^2 - 6$
h $y = x^2 - x - 6$

Reasoning

Worked solutions
Exercise 7.2
MAT10NAWS00020

⑯ How many whole number values of c between 1 and 100 can you find so that $x^2 - x - c$ can be factorised using whole numbers?

17 How many whole number values of c between 1 and 100 can you find so that $x^2 + 2x - c$ can be factorised using whole numbers?

18 How many integer values of a can you find, strictly between 100 and 200, so that $ax^2 + x - 6$ can be factorised using integers?

Worked solutions
Exercise 7.2
MAT10NAWS00020

⑲ For what values of c are there an odd number of values of b for which $x^2 + bx + c$ can be factorised using integer values?

7.3 Other expressions and functions

All the expressions and functions you have looked at so far in this chapter have had just two factors. In most cases, each factor has been a group so you have ended up with binomial brackets. Combinations of the types you have already looked at may give more than two factors or enable you to factorise expression with higher powers.

Common factors should always be removed before any other factorisation is done.

Example 12

Factorise each of the following.

a $3t^2 - 48$ **b** $2m^3n - 8mn^3$ **c** $5x^3 - 30x^2 + 45x$

Solution

a Write the expression. $\mathbf{3t^2 - 48}$

Take out the common factor. $\mathbf{= 3(t^2 - 16)}$

Use the difference of two squares. $\mathbf{= 3(t - 4)(t + 4)}$

b Write the expression. $\mathbf{2m^3n - 8mn^3}$

Take out the common factor. $\mathbf{= 2mn(m^2 - 4n^2)}$

Use the difference of two squares. $\mathbf{= 2mn(m + 2n)(m - 2n)}$

c Write the expression. $\mathbf{5x^3 - 30x^2 + 45x}$

Take out the common factor. $\mathbf{= 5x(x^2 - 6x + 9)}$

Use perfect squares. $\mathbf{= 5x(x - 3)^2}$

Quadratics will not always be given in the usual order, so check any three-term expressions to see if they are quadratics.

Example 13

Express each of the following as products of the simplest possible factors.

a $6x^2 - 30x + 24$ **b** $5 + 6a^2 - 13a$ **c** $8ab^2 - 20ab - 12a$

Solution

a Write the expression. $\mathbf{6x^2 - 30x + 24}$

Take out the common factor. $\mathbf{= 6(x^2 - 5x + 6)}$

Use $^-3 \times {}^-2 = 6$ and $^-3 + {}^-2 = {}^-5$
to factorise the quadratic. $\mathbf{= 6(x - 3)(x - 2)}$

b Write the expression. $\mathbf{5 + 6a^2 - 13a}$

Put in the normal quadratic order. $\mathbf{= 6a^2 - 13a + 5}$

Factorise as a quadratic. $\mathbf{= (2a - 1)(3a - 5)}$

c Write the expression. $\mathbf{8ab^2 - 20ab - 12a}$

Take out the common factor. $\mathbf{= 4a(2b^2 - 5b - 3)}$

Factorise as a quadratic. $\mathbf{= 4a(b - 3)(2b + 1)}$

CAS TI-Nspire exercise

Advanced algebra and functions

MAT10NATI00015

CAS ClassPad exercise

Advanced algebra and functions

MAT10NACP00015

Occasionally, you can rewrite powers as squares. You may be able to substitute a **dummy variable** for an expression to simplify a factorisation. In this case, remember to substitute the original expression back in and simplify.

Example 14

Factorise each of the following.

a $(e-3)^2-25$ **b** a^4-16

c x^4-5x^2-36 **d** $2(b-1)^2+5(b-1)+2$

Solution

a Write the expression.	$(e-3)^2-25$
Use the dummy variable $p=(e-3)$.	$=p^2-25$
Factorise as a difference of squares.	$=(p-5)(p+5)$
Substitute the dummy variable $p=(e-3)$.	$=[(e-3)-5][(e-3)+5]$
Simplify the internal brackets.	$=(e-3-5)(e-3+5)$
Complete the simplification.	$=(e-8)(e+2)$
b Write the expression.	a^4-16
Express as a difference of squares.	$=(a^2)^2-4^2$
Factorise as a difference of squares.	$=(a^2+4)(a^2-4)$
Factorise (a^2-4) as a difference of squares.	$=(a^2+4)(a+2)(a-2)$
c Write the expression.	x^4-5x^2-36
Express as a quadratic in x^2.	$=(x^2)^2-5x^2-36$
Use $^{-}9\times4=36$ and $^{-}9+4={}^{-}5$ to factorise the quadratic.	$=(x^2-9)(x^2+4)$
Factorise (x^2-9) as a difference of squares.	$=(x-3)(x+3)(x^2+4)$
d Write the expression.	$2(b-1)^2+5(b-1)+2$
Use the dummy variable $p=(b-1)$.	$=2p^2+5p+2$
Factorise as a quadratic.	$=(2p+1)(p+2)$
Substitute the dummy variable $p=(b-1)$.	$=[2(b-1)+1][(b-1)+2]$
Simplify the internal brackets.	$=(2b-2+1)(b-1+2)$
Complete the simplification.	$=(2b-1)(b+1)$

You may be able to use a special pattern or reorganisation to do a grouping factorisation.

Example 15

Factorise each of the following.

a $a^3+4-4a-a^2$ **b** $a^2-2ab+b^2-c^2$

Solution

a Write the expression.	$a^3+4-4a-a^2$
Rearrange to make groups.	$=a^3-a^2-4a+4$

Factorise each pair.	$= \boldsymbol{a^2(a-1) - 4(a-1)}$
Use $(a-1)$ as a common factor.	$= \boldsymbol{(a-1)(a^2-4)}$
Factorise (a^2-4) as a difference of squares.	$= \boldsymbol{(a-1)(a-2)(a+2)}$
b Write the expression.	$\boldsymbol{a^2 - 2ab + b^2 - c^2}$
Use perfect squares for the first 3 terms.	$= \boldsymbol{(a-b)^2 - c^2}$
Use the dummy variable $p = (a-b)$.	$= \boldsymbol{p^2 - c^2}$
Use the difference of squares.	$= \boldsymbol{(p-c)(p+c)}$
Substitute the dummy variable $p = (a-b)$.	$= \boldsymbol{[(a-b)-c][(a-b)+c]}$
Simplify the internal brackets.	$= \boldsymbol{(a-b-c)(a-b+c)}$

You may remember the definition of a polynomial from earlier in the year.

Important!

Polynomials

A **polynomial** is a sum of terms that are each the product of a constant and positive integer powers of the variables. A polynomial in one variable can be written as

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0,$$

where the coefficients $a_n, a_{n-1}, a_{n-2}, \ldots, a_2, a_1, a_0$ are real numbers. In the work on polynomials this year, we will assume they are integers.
The **degree** of the polynomial is the highest power n, and the **leading term** is the term that contains the highest power.

You may already have done some addition, subtraction and multiplication of polynomials. Division is done in a similar way to long division of whole numbers and similar rules apply. When you divide 22 by 5, you get an answer of 4 with 2 left over. You can write this as $22 = 4 \times 5 + 2$. You can always write the result of a division of integers this way, so it is called the **division identity** for integers.

Important!

Division identity for polynomials

For any polynomials $a(x)$ and $b(x)$, where $b(x)$ is not a constant, there exist unique polynomials $q(x)$ and $r(x)$ such that $a(x) = q(x) \times b(x) + r(x)$, where the degree of $r(x)$ is less than the degree of $b(x)$.
$q(x)$ and $r(x)$ are called the **quotient** and **remainder**, $b(x)$ is the **divisor** and $a(x)$ is the **dividend**.

The division identity really just says that you will always get an answer when you divide one polynomial by another. $b(x)$ cannot be a constant because its degree would then be zero, so the degree of $r(x)$ could not be strictly less than that of $b(x)$.

Puzzle sheet
Finding the quotient
MAT10NAPS00052

Puzzle sheet
The remainder theorem
MAT10NAPS00053

CAS TI-Nspire exercise
Advanced algebra and functions
MAT10NATI00015

CAS ClassPad exercise
Advanced algebra and functions
MAT10NACP00015

Example 16

Find the quotient and remainder when $4x^3 - 9x^2 + 6x - 1$ is divided by $x - 2$.

Solution

Set out the division like a long division.

$$x-2\,\overline{)4x^3-9x^2+6x-1}$$

Divide x into $4x^3$. What do you multiply x by to get $4x^3$? Put the answer over the next power of x ($4x^2$).

$$\begin{array}{rl} & 4x^2 \\ x-2 & \overline{)4x^3-9x^2+6x-1} \end{array}$$

Multiply your answer by $x - 2$ and put it under the first two terms.

$$\begin{array}{rl} & 4x^2 \\ x-2 & \overline{)4x^3-9x^2+6x-1} \\ & 4x^3-8x^2 \end{array}$$

Subtract the product.

Be careful of signs (${}^-9 - {}^-8 = {}^-1$).

Bring down the next term.

$$\begin{array}{rl} & 4x^2 \\ x-2 & \overline{)4x^3-9x^2+6x-1} \\ & \underline{-4x^3-8x^2} \downarrow \\ & \qquad {}^-1x^2+6x \end{array}$$

Divide again and multiply by $x - 2$.

Put the answer underneath.

$$\begin{array}{rl} & 4x^2-x \\ x-2 & \overline{)4x^3-9x^2+6x-1} \\ & \underline{4x^3-8x^2} \\ & \qquad {}^-1x^2+6x \\ & \qquad \underline{-1x^2+2x} \downarrow \\ & \qquad\qquad 4x-1 \end{array}$$

Subtract and bring down the next term.

Divide again, and multiply by $x - 2$.

Put the answer underneath.

Subtract the product.

There are no more terms to bring down, so you have finished the division.

$$\begin{array}{rl} & 4x^2-x+4 \\ x-2 & \overline{)4x^3-9x^2+6x-1} \\ & \underline{4x^3-8x^2} \\ & \qquad {}^-1x^2+6x \\ & \qquad \underline{-1x^2+2x} \\ & \qquad\qquad 4x-1 \\ & \qquad\qquad \underline{-4x-8} \\ & \qquad\qquad\qquad 7 \end{array}$$

Write the answer in the form of the division algorithm and give the required information.

$4x^3 - 9x^2 + 6x - 1 = (4x^2 - x + 4)(x - 2) + 7$ so the quotient is $4x^2 - x + 4$ and the remainder is 7.

It is easy to get lost in the details of the long division of polynomials. The actual calculations in Example **16** were as follows.

1 The coefficient of the leading term of the dividend was written as the first numeric answer
2 It was multiplied by the ${}^-2$ in the divisor
3 This was subtracted from the coefficient of the next term
4 The coefficient of the answer was multiplied by the ${}^-2$ in the divisor

5 This was subtracted from the coefficient of the next term
6 There were no more terms, so the process stopped.

You can summarise this as shown below, using a straight line instead of the normal).

$$\begin{array}{r|rrrr} {}^{-}2 & 4 & {}^{-}9 & 6 & {}^{-}1 \\ & & & & \\ \hline & & & & \end{array}$$

1
$$\begin{array}{r|rrrr} {}^{-}2 & 4 & {}^{-}9 & 6 & {}^{-}1 \\ & & {}^{-}8 & & \\ \hline & 4 & & & \end{array}$$

2
$$\begin{array}{r|rrrr} {}^{-}2 & 4 & {}^{-}9 & 6 & {}^{-}1 \\ & & {}^{-}8 & & \\ \hline & 4 & {}^{-}1 & & \end{array}$$

3
$$\begin{array}{r|rrrr} {}^{-}2 & 4 & {}^{-}9 & 6 & {}^{-}1 \\ & & {}^{-}8 & 2 & \\ \hline & 4 & {}^{-}1 & & \end{array}$$

4
$$\begin{array}{r|rrrr} {}^{-}2 & 4 & {}^{-}9 & 6 & {}^{-}1 \\ & & {}^{-}8 & 2 & \\ \hline & 4 & {}^{-}1 & 4 & \end{array}$$

5
$$\begin{array}{r|rrrr} {}^{-}2 & 4 & {}^{-}9 & 6 & {}^{-}1 \\ & & {}^{-}8 & 2 & {}^{-}8 \\ \hline & 4 & {}^{-}1 & 4 & \end{array}$$

6
$$\begin{array}{r|rrrr} {}^{-}2 & 4 & {}^{-}9 & 6 & {}^{-}1 \\ & & {}^{-}8 & 2 & {}^{-}8 \\ \hline & 4 & {}^{-}1 & 4 & \end{array}$$

For **synthetic division**, the sign of the constant in the divisor is changed and the operation of subtraction is changed to addition. This gives the following for the division in Example **16**. The last subtraction gives the remainder.

$$\begin{array}{r|rrrr} 2 & 4 & {}^{-}9 & 6 & {}^{-}1 \\ & & & & \\ \hline & & & & \end{array}$$

1
$$\begin{array}{r|rrrr} 2 & 4 & {}^{-}9 & 6 & {}^{-}1 \\ & & 8 & & \\ \hline & 4 & & & \end{array}$$

2
$$\begin{array}{r|rrrr} 2 & 4 & {}^{-}9 & 6 & {}^{-}1 \\ & & 8 & & \\ \hline & 4 & {}^{-}1 & & \end{array}$$

3
$$\begin{array}{r|rrrr} 2 & 4 & {}^{-}9 & 6 & {}^{-}1 \\ & & 8 & {}^{-}2 & \\ \hline & 4 & {}^{-}1 & & \end{array}$$

4
$$\begin{array}{r|rrrr} 2 & 4 & {}^{-}9 & 6 & {}^{-}1 \\ & & 8 & {}^{-}2 & \\ \hline & 4 & {}^{-}1 & 4 & \end{array}$$

5
$$\begin{array}{r|rrrr} 2 & 4 & {}^{-}9 & 6 & {}^{-}1 \\ & & 8 & {}^{-}2 & 8 \\ \hline & 4 & {}^{-}1 & 4 & \end{array}$$

6
$$\begin{array}{r|rrrr} 2 & 4 & {}^{-}9 & 6 & {}^{-}1 \\ & & 8 & {}^{-}2 & 8 \\ \hline & 4 & {}^{-}1 & 4 & 7 \end{array}$$

Since the last number (remainder) is 7, the other numbers are the coefficients of the quotient.
It is easier to think of the divisor as $x - a$, and use the this a in the synthetic division.
If one of the terms of the polynomial has a coefficient of 0, you must put this term in the division, for *both* long division or synthetic division. This is just like putting the zero in the number 407.

Example 17

Use synthetic division to find the quotient and remainder for $(2x^4 - 3x^2 + 5x + 2) \div (x + 3)$

Solution

Write the divisor in the form $x - a$.

$$\mathbf{x + 3 = x - {}^{-}3}$$

Set out the synthetic division, with the x^3 coefficient of 0 included.

$$\begin{array}{r|rrrrr} {}^{-}3 & 2 & 0 & {}^{-}3 & 5 & 2 \\ & & {}^{-}6 & 18 & {}^{-}45 & {}^{-}80 \\ \hline \end{array}$$

Write the first coefficient under the line.

$$\begin{array}{r|rrrrr} & 2 & & & & \end{array}$$

Multiply the first number under the line by the value of a and put it under the second coefficient. Add the numbers together.

$$\begin{array}{r|rrrrr} {}^{-}3 & 2 & 0 & {}^{-}3 & 5 & 2 \\ & & {}^{-}6 & 18 & {}^{-}45 & {}^{-}80 \\ \hline & 2 & {}^{-}6 & & & \end{array}$$

Continue until you run out of coefficients.

$$\begin{array}{r|rrrrr} {}^{-}3 & 2 & 0 & {}^{-}3 & 5 & 2 \\ & & {}^{-}6 & 18 & {}^{-}45 & 120 \\ \hline & 2 & {}^{-}6 & 15 & {}^{-}40 & 122 \end{array}$$

Read the answer from the bottom line and put in the powers of x.

$\mathbf{2x^4 - 3x^2 + 5x + 2}$
$\mathbf{= (2x^3 - 6x^2 + 15x - 40)(x - {}^{-}3) + 122,}$
so the quotient is $\mathbf{2x^3 - 6x^2 + 15x - 40}$ **and the remainder is 122.**

If a polynomial is divisible by a linear expression, then it would have no remainder and you could factorise the expression with the quotient and divisor. There is a way to tell if there is a remainder without first dividing.

Important!

Remainder and factor theorems

The **remainder theorem** says that when a polynomial $p(x)$ is divided by $x - a$, the remainder is $p(a)$.
The **factor theorem** says that $p(a) = 0$ if and only if $x - a$ is a factor of the polynomial $p(x)$.

The proof of the remainder theorem is fairly straightforward.
According to the division identity, there exist polynomials $q(x)$ and $r(x)$ such that $p(x) = q(x) \times (x - a) + r(x)$, where the degree of $r(x)$ is less than the degree of $x - a$.
But the degree of $x - a$ is 1, so the degree of $r(x)$ must be 0 and $r(x) = r$ for some real number r.
Thus $p(x) = q(x) \times (x - a) + r$ for some polynomial $q(x)$ and real number r.
Substituting $x = a$ gives $p(a) = q(a) \times (a - a) + r = q(a) \times 0 + r = 0 + r = r$.
Thus $r = p(a)$. **QED**
The factor theorem follows immediately from the meanings of factor and remainder.

Example 18

Show that $x - 1$ and $x + 2$ are both factors of $x^4 - 2x^3 - 4x^2 + 7x - 2$.

Solution

Choose a name for the polynomial.	**Let** $p(x) = x^4 - 2x^3 - 4x^2 + 7x - 2$
Find the value of $p(1)$.	$p(1) = 1^4 - 2 \times 1^3 - 4 \times 1^2 + 7 \times 1 - 2$
Calculate each term.	$= 1 - 2 - 4 + 7 - 2$
Calculate the answer.	$= 0$
Use the factor theorem.	**Since** $p(1) = 0$, $x - 1$ **is a factor of** $p(x)$.
Find the value of $p({}^{-}2)$.	$p({}^{-}2) = ({}^{-}2)^4 - 2 \times ({}^{-}2)^3 - 4 \times ({}^{-}2)^2 + 7 \times ({}^{-}2) - 2$
Calculate each term.	$= 16 + 16 - 16 - 14 - 2$
Calculate the answer.	$= 0$
Use the factor theorem.	**Since** $p({}^{-}2) = 0$, $x - {}^{-}2 = x + 2$ **is a factor of** $p(x)$.
State the final answer.	**Since** $p(1) = p({}^{-}2) = 0$, $x - 1$ **and** $x + 2$ **are both factors of** $p(x) = x^4 - 2x^3 - 4x^2 + 7x - 2$.

Since $x - 1$ and $x + 2$ are both factors of $x^4 - 2x^3 - 4x^2 + 7x - 2$, you can write that there must be a polynomial $q(x)$ such that $x^4 - 2x^3 - 4x^2 + 7x - 2 = q(x)(x - 1)(x + 2)$. Since the polynomials on each side are the same, their degrees must be the same, so $q(x)$ is a quadratic.

Expanding, $(x - 1)(x + 2) = x^2 + x - 2$, so $x^4 - 2x^3 - 4x^2 + 7x - 2 = (ax^2 + bx + c)(x^2 + x - 2)$ for the quadratic $q(x) = ax^2 + bx + c$.
The leading term and the constant term are given by $x^4 = ax^2 \times x^2$ and ${}^{-}2 = c \times {}^{-}2$.
Thus it follows that $a = 1$ and $c = 1$, so $x^4 - 2x^3 - 4x^2 + 7x - 2 = (x^2 + bx + 1)(x^2 + x - 2)$.
Only the value of b remains to be found.
The x term is given by $7x = bx \times {}^{-}2 + 1 \times x = {}^{-}2bx + x$, so $b = {}^{-}3$.
You could also use the x^3 or x^2 term to find b, but the x^2 term would involve more multiplications.
This gives $x^4 - 2x^3 - 4x^2 + 7x - 2 = (x^2 - 3x + 1)(x - 1)(x + 2)$.
Since $x^2 - 3x + 1$ cannot be factorised, $x^4 - 2x^3 - 4x^2 + 7x - 2 = (x^2 - 3x + 1)(x - 1)(x + 2)$ is a complete factorisation.
When you are looking for factors of a polynomial $p(x)$, you should work systematically through $1, {}^{-}1, 2, {}^{-}2, 3, {}^{-}3, \ldots$ until you find a value for which $p(a) = 0$.

Example 19

Factorise $x^3 - 7x^2 - 73x - 65$.

Solution

Choose a name for the polynomial. **Let $m(x) = x^3 - 7x^2 - 73x - 65$**

Find the value of $m(1)$. **$m(1) = 1 - 7 - 73 - 65 = {}^{-}144 \neq 0$**

Find the value of $m({}^{-}1)$. **$m({}^{-}1) = -1 - 7 + 73 - 65 = 0$**

Success! **Since $m({}^{-}1) = 0$, $x + 1$ is a factor.**

Divide $x^3 - 7x^2 - 73x - 65$ by $x + 1$.

$$\begin{array}{r}
x^2 - 8x - 65 \\
x + 1\overline{\smash{)}\,x^3 - 7x^2 - 73x - 65} \\
-\,\underline{x^3 - x^2} \\
-\,8x^2 - 73x \\
-\,\underline{8x^2 - 8x} \\
-\,65x - 65 \\
-\,\underline{65x - 65} \\
0
\end{array}$$

There is no remainder, of course.

Write $m(x)$ in factored form. **Thus $m(x) = (x^2 - 8x - 65)(x + 1)$**

Use ${}^{-}13 \times 5 = {}^{-}65$ and ${}^{-}13 + 5 = {}^{-}8$ to factorise the quadratic. **$= (x - 13)(x + 5)(x + 1)$**

Write the answer in the form of the question. **$x^3 - 7x^2 - 73x - 65 = (x - 13)(x + 5)(x + 1)$**

Video tutorial
Factorising polynomials
MAT10NAVT10018

Puzzle sheet
Factorising cubic functions
MAT10NAPS00051

Just as factorisation of a quadratic can be used to help sketch a parabola, factorisation of a polynomial can help you to sketch the polynomial by giving you (some) zeros of the function. You can also use the y-intercept and the behaviour of the function as x becomes large. You need to know about the shapes of the simplest polynomials – those that have only a leading term.

Investigate: Shapes of simple power functions

You have already studied the shapes of linear and quadratic functions. You can model quadratics as transformations of $y = x^2$. What are the shapes of the higher powers of x? There is a spreadsheet called 'Power functions' on the NelsonNet website.
Load the spreadsheet and use it to investigate the shapes of the following power functions.

1 $y = x^3$, $y = 4x^3$, $y = -x^3$ and $y = {}^{-}3x^3$.
2 $y = x^4$, $y = 2x^4$, $y = -x^4$ and $y = {}^{-}4x^4$.
3 $y = x^5$, $y = 3x^5$, $y = -x^5$ and $y = {}^{-}2x^5$.
4 $y = x^6$, $y = 4x^6$, $y = -x^6$ and $y = {}^{-}2x^6$.

How are they similar?
How are they different?

Technology worksheet

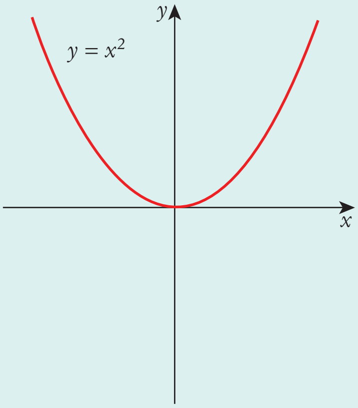
Excel worksheet: Power functions

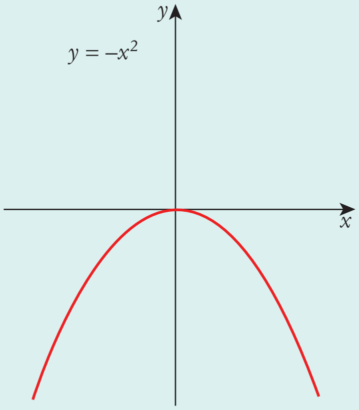
MAT10NACT00020

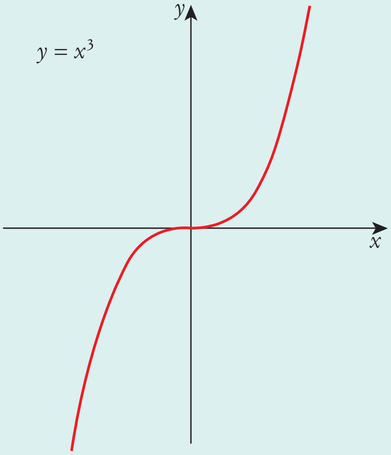
Important!

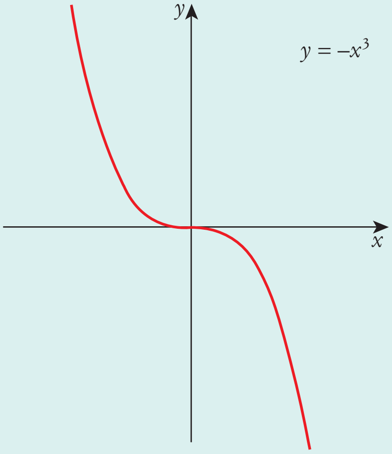
Shapes of simple power functions

Higher power functions $y = x^n$ with even powers have shapes similar to $y = x^2$, but are steeper.
Higher power functions $y = x^n$ with odd powers have shapes similar to $y = x^3$, but are steeper.









As the value of x becomes large in magnitude, the magnitude of the leading term of a polynomial becomes much greater than that of other terms, because it is at least x times larger. For example, when $x = 10\ 000$, x^4 is 10 000 times larger than x^3. This means that the polynomial looks similar to a simple power function for large values of x, so its behaviour is also similar.

If a factorised polynomial has repeated factors, the behaviour of the zeros corresponding to those factors will be like the zeros of the corresponding power function. A factor that occurs twice is said to give a **double zero**, while a factor that occurs three times corresponds to a **triple zero**. Zeros from single factors are called **single zeros**. When there are multiple zeros of a polynomial finding the signs of the polynomial between the zeroes helps you to sketch the graph. Of course, you should also check the y-intercept.

Example 20

Sketch the graph of $p(x) = (x + 2)(x - 1)^2(x - 2)^3(x - 4)$

Solution

State the zeros.	**$p(x)$ has zeros at $x = {}^-2$, $x = 1$, $x = 2$ and $x = 4$.**
State their types.	**$^-2$ and 4 are single zeros, 1 is a double zero and 2 a triple zero.**
Find the signs of $p(x)$ between the zeros.	**$p(0) = 64 > 0$, $p(1.5) \approx 0.27 > 0$, $p(3) = {}^-20 < 0$**
State the y-intercept.	**$p(0) = 64$**
State the behaviour as x becomes large in magnitude.	**When x is large and positive, $g(x)$ is like x^7, large and positive.** **When x is large and negative, $g(x)$ is like x^7, large and negative.**

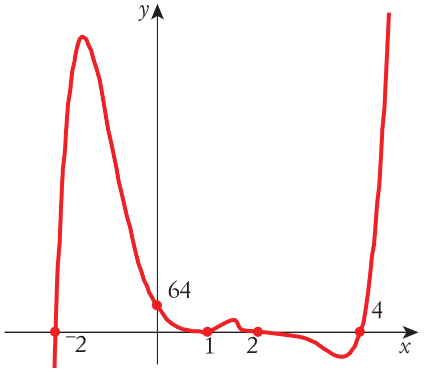
Place the known points on the graph.

The graph must be positive between $^-2$ and 1, positive between 1 and 2, and negative between 2 and 4.

1 is a double zero and 2 is a triple zero.

Join the points with a smooth curve.

In this case, as in many of the graphs, the scale has to be exaggerated near the multiple zeros to show the behaviour.



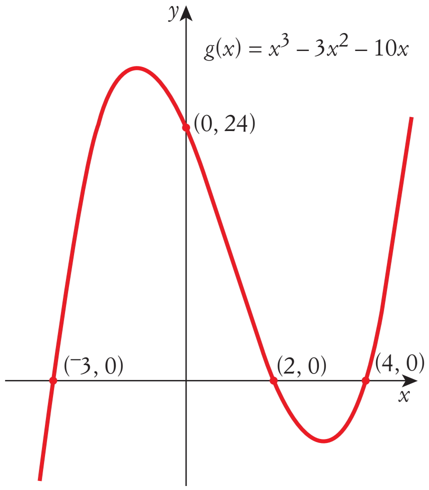
Example 21

CAS TI-Nspire exercise
Advanced algebra and functions
MAT10NATI00015

CAS ClassPad exercise
Advanced algebra and functions
MAT10NACP00015

Use factorisation to help sketch the graph of $g(x) = x^3 - 3x^2 - 10x + 24$

Solution

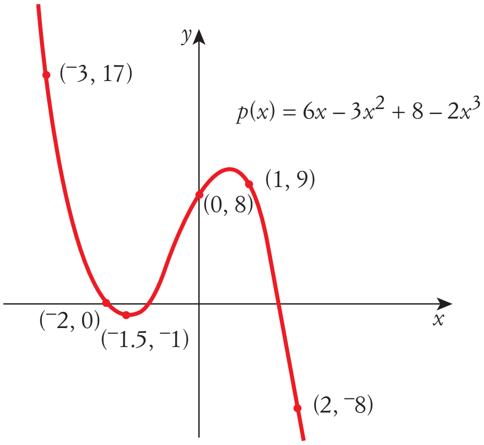
Find the value of $g(1)$.	$\mathbf{g(1) = 1 - 3 - 10 + 24 = 2 \neq 0}$
Find the value of $g(^{-}1)$.	$\mathbf{g(^{-}1) = -1 - 3 + 10 + 24 = 30 \neq 0}$
Find the value of $g(2)$.	$\mathbf{g(2) = 8 - 12 - 20 + 24 = 0}$
State the result.	**Since $g(2) = 0$, $x - 2$ is a factor of $g(x)$.**
Divide $g(x)$ by $x - 2$. Remember to change the sign if you use synthetic division. The remainder is 0, which is correct.	$\begin{array}{r\|rrrr} 2 & 1 & ^{-}3 & ^{-}10 & 24 \\ & & 2 & ^{-}2 & 24 \\ \hline & 1 & ^{-}1 & ^{-}12 & 0 \end{array}$
Write $g(x)$ in the factored form.	**Thus $g(x) = (x^2 - x - 12)(x - 2)$**
Factorise the quadratic.	$\mathbf{= (x - 4)(x + 3)(x - 2)}$
Write the zeros of $g(x)$.	**$g(x)$ has zeros at $x = {}^{-}3$, $x = 2$ and $x = 4$**
Find the y-intercept.	$\mathbf{g(0) = 24}$
State the behaviour as x becomes large in magnitude.	**When x is large and positive, $g(x)$ is like x^3, large and positive.** **When x is large and negative, $g(x)$ is like x^3, large and negative.**
Place the known points on the graph. Join the points with a smooth curve.	

When you are factorising polynomials, you will not always be able to get only linear factors. If you are sketching the graph of a polynomial and have a quadratic factor, you will have to find some additional points to find the behaviour resulting from the quadratic factor.

Example 22

Use factorisation to help sketch the graph of $p(x) = 6x - 3x^2 + 8 - 2x^3$.

Solution

Find the value of $p(1)$.	$\boldsymbol{p(1) = 6 - 3 + 8 - 2 = 9 \neq 0}$
Find the value of $p(^{-}1)$.	$\boldsymbol{p(^{-}1) = {}^{-}6 - 3 + 8 + 2 = 1 \neq 0}$
Find the value of $p(2)$.	$\boldsymbol{p(2) = 12 - 12 + 8 - 16 = {}^{-}8 \neq 0}$
Find the value of $p(^{-}2)$.	$\boldsymbol{p(^{-}2) = {}^{-}12 - 12 + 8 + 16 = 0}$
State the result.	**Since $p(^{-}2) = 0$, $x + 2$ is a factor of $p(x)$.**
Divide $p(x)$ by $x + 2$ using any method.	$\boldsymbol{p(x) = (2x^2 - x - 4)(x + 2)}$
The quadratic $2x^2 - x - 4$ will not factorise, so state the known zero.	**$p(x)$ has a zero at $x = {}^{-}2$**
Find the y-intercept.	$\boldsymbol{p(0) = 8}$
Find some other points.	$\boldsymbol{p(^{-}3) = 17, p(^{-}1\tfrac{1}{2}) = {}^{-}1, p(^{-}1) = 1,}$ $\boldsymbol{p(1) = 9, p(2) = {}^{-}8}$
State your conclusion from the other points.	**There must be zeros between $^{-}1\frac{1}{2}$ and $^{-}1$ and also between 1 and 2.**
State the behaviour as x becomes large in magnitude.	**When x is large and positive, $g(x)$ is like $^{-}2x^3$, large and negative.** **When x is large and negative, $g(x)$ is like $^{-}2x^3$, large and positive.**
Place the known points on the graph. Join the points with a smooth curve.	

In Example **22** you could find the positions of zeros corresponding to the quadratic factor using the quadratic formula. In some cases, such as $(x^2 + 2x + 5)$, a quadratic factor will not correspond to any real zeros.
Although there is a maximum shown between 0 and 1, its exact coordinates are not clear. To get a better idea of the function, you would need to use a graphics calculator or a CAS calculator such as the TInspire or Casio Classpad. In some cases there will be extra zeros, but in other cases there will not be any extra zeros.

Exercise 7.3 Other expressions and functions

Understanding

Extra questions
Exercise 7.3
MAT10NAEQ00049
See Example 18

1 Show that $x - 2$ is a factor of $p(x) = 3x^3 - 2x^2 - 7x - 2$.

2 Show that $x + 3$ is a factor of $q(x) = 5x^3 + 14x^2 + 9$.

3 Show that $x - 1$ is a factor of $r(x) = 2x^4 - 3x^3 + 4x^2 - 1x - 2$.

4 Show that $x + 2$ and $x - 2$ are factors of $m(x) = x^4 + 3x^3 + 1x^2 - 12x - 20$.

5 Show that $x + 1$ and $x - 3$ are factors of $n(x) = 2x^4 - 5x^3 - 8x^2 + 11x + 12$.

Fluency

See Example 12

6 Factorise each of the following.

a $3x^2 - 12y^2$
b $5a^2 - 45b^2$
c $8m^2 - 32n^2$
d $3a^3b - 27ab^3$
e $6x^3y - 24xy^3$
f $7p^3q - 63pq^3$
g $28n^3 - 7n$
h $27x^3 - 3x$
i $16h^3 - 4h$
j $18x^2 - 2$

See Example 13

7 Factorise each of the following expressions.

a $2n^2 - 12nk + 16k^2$
b $7n^2 - 91ny + 252y^2$
c $5q^2 - 65qt + 210t^2$
d $4u^2 + 4ut - 48t^2$
e $3w^2 + 42wm + 147m^2$
f $8n^2 + 24nw - 224w^2$
g $7z^2 - 42zw + 56w^2$
h $6a^2 - 78ac + 240c^2$
i $3t^2 + 6th - 72h^2$
j $8a^2 - 16am - 120m^2$

8 Factorise each of the following.

a $48w^2 - 56wc - 392c^2$
b $128m^2 - 624mx + 216x^2$
c $12q^2 - 45qm + 27m^2$
d $42c^2 + 74cm - 56m^2$
e $16p^2 - 100pr + 126r^2$
f $49a^2 - 203am - 210m^2$
g $36n^2 + 108nq + 81q^2$
h $56c^2 + 42cv - 245v^2$
i $16t^2 - 142tw - 18w^2$
j $216h^2 + 12hq - 140q^2$

See Examples 16, 17

9 Find the quotient and remainder for each of the following divisions. Your teacher may want you to try using different methods.

a $x^3 - x^2 - x - 2$ divided by $x + 2$
b $3x^3 - 2x^2 - 2x - 1$ divided by $x + 2$
c $x^3 - 3x^2 - 2x$ divided by $x - 1$
d $2x^4 + 2x^2 + x - 3$ divided by $x + 1$
e $2x^4 + 2x^3 - 3x^2 - 2x + 1$ divided by $x + 3$
f $x^4 + 2x^2 + 3x$ divided by $x - 2$
g $x^4 + x^3 + 2x^2 + 3x$ divided by $x - 3$
h $2x^5 - x^4 - 3x^3 + x^2 - 2$ divided by $x - 2$
i $3x^5 + x^4 - 3x^3 - 3x^2 + x + 2$ divided by $x + 1$
j $2x^5 - x^4 - 2x^2 + 2x + 1$ divided by $x - 2$

Problem solving

See Example 14

10 Factorise each of the following expressions.

a $(a - 3)^2 - 1$
b $(a + b)^2 - 9$
c $(x - 7)^2 - 16$
d $32 - 8(2m - 3)^2$
e $36 - 9(4a - 7)^2$
f $5(x - 3)^2 - 125$
g $36(x - 1)^2 - 25(x + 2)^2$
h $64(a + 1)^2 - 9(a - 3)^2$
i $4(3x - 5)^2 - 49(2x - 3)^2$
j $3(x - y)^2 - 3(x + 2y)^2$
k $2(a + b)^2 - 2(a - 2b)^2$
l $a^4 - b^4$
m $x^4 - 5x^2 - 36$
n $a^4 - 2a^2 - 63$
o $80a^4 - 5n^4$

11 Factorise the following.

See Example 15

a $5a^3 + 4a + 5a^2 + 4$

b $b^3 - 5b^2 + 2b - 10$

c $5m^3 - m^2 + 15m - 3$

d $a^2 - 9 + ab - 3b$

e $3x^2 - 12y^2 + x - 2y$

f $6 + a^3 + 6a + a^2$

g $x^2 + 2xy + y^2 - a^2$

h $y^2 - a^2 - 6a - 9$

i $9x^2 - p^2 + 2p - 1$

j $36k^2 + 12k + 1 - 25h^2$

12 Factorise each of the following.

Worked solutions
Exercise 7.3
MAT10NAWS00054

a $4x^2y^2 - x^2 + 2xy - y^2$

b $9a^2b^2 - x^2 - 2px - p^2$

c $x^2 + 2xy + y^2 - z^2$

d $m^2 - 2mn + n^2 - p^2$

e $z^2 - x^2 - 2xy - y^2$

f $a^2 - 2ab + b^2 - 4$

g $p^2 + 2pq + q^2 - 25$

h $a^2 - 4 - 2ab + b^2$

i $m^2 + 9 - 6m - 16k^2$

j $m^2 + 25 - 10m - 16c^2$

k $r^2 - s^2 - 16 - 8s$

l $a^2 + 12cm - 4c^2 - 9m^2$

m $16x^2 + y^2 - z^2 - 8xy$

n $a^4 - b^2 + 25 - 10a^2$

13 Factorise each of the following.

Worked solutions
Exercise 7.3
MAT10NAWS00054

a $4a^2 - 28a + 48$

b $10x^2 + 50x + 60$

c $20m^2 - 125$

d $12a^4 - 75a^2$

e $8m^2 - 32n^2p^2$

f $(a - b)^2 + 5(a - b) + 4$

g $4(a + b)^2 - (a - b)^2$

h $x^4 + x^2 - 20$

i $6ay^2 + 5ay + a$

j $64 - 64y - 48y^2$

k $12x^2y^4 - 14xy^3 - 6y^2$

l $3(2 - h)^2 + 11(2 - h) + 6$

m $9x^4 - 2x^2 - 32$

14 Factorise each of the following.

See Example 19

a $2x^3 - 1x^2 - 7x + 6$

b $x^3 + 4x^2 - 7x - 10$

c $3x^3 + 1x^2 - 8x + 4$

d $2x^3 + 5x^2 - 8x - 20$

e $2x^4 - 8x^3 + 13x^2 - 13x + 6$

f $3x^4 + 17x^3 + 27x^2 + 7x - 6$

g $x^4 + 7x^3 + 8x^2 - 28x - 48$

h $x^4 + 2x^3 - 9x^2 - 16x - 6$

i $x^4 - 1x^3 - 7x^2 + 1x + 6$

j $2x^4 - 1x^3 - 16x^2 + 17x + 6$

15 Sketch each of the following.

See Example 20

a $p(x) = (x -1)(x + 2)(x - 3)$

b $q(x) = -(x + 1)(x - 2)(x - 3)$

c $g(x) = (x - 1)(x + 2)^3(x + 3)$

d $h(x) = (x - 2)(x - 3)(x - 4)^2$

e $k(x) = -(x + 2)(x + 1)(x + 3)^3$

f $m(x) = -(x + 1)^2(x - 5)(x - 3)^3$

16 Use factorisation to help sketch each of the following. Your teacher will tell you what technology to use for each one.

See Examples 21, 22

a $p(x) = x^3 + 3x^2 - 6x - 8$

b $p(x) = x^2 - x^3 + 14x - 24$

c $p(x) = 10x + 8 - x^3 - 5x^2$

d $p(x) = x^4 - 4x^3 - 23x^2 + 54x + 72$

Reasoning

17 Use factorisation to find short cuts for the following calculations.

Worked solutions
Exercise 7.3
MAT10NAWS00054

a 21^2 **b** 19^2 **c** 31^2 **d** 29^2 **e** 101^2

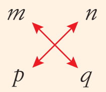
Chapter 7 summary

Quiz

Advanced algebra and functions

MAT10NAQZ00007

- The **difference of two squares** is the special expansion $(x + y)(x - y)= x^2 - y^2$.
- The **square of a sum** is the expansion $(x + y)^2 = x^2 + 2xy + y^2$ and the **square of a difference** is the expansion $(x - y)^2 = x^2 - 2xy + y^2$. These are often called **perfect squares**.
- The difference of two squares and perfect squares are used in reverse to factorise expressions.
- A **grouping factorisation** is common when there are 4 terms and works by rearranging terms, if necessary, to obtain a grouped term as a common factor.
- A **quadratic expression** in the variable x has the format $ax^2 + bx + c$, where a, b and c are real numbers. Quadratic expressions are often just called quadratics or **trinomials** because they have three terms.
- If a quadratic with a unit coefficient of x^2 can be factorised you use the numbers that add to give b and multiply to give c. The numbers u and v that give $x^2 + bx + c = (x + u)(x + v)$ are such that $uv = c$ and $u + v = c$.
- You can use factorisation to help sketch quadratics. The perfect square $y = (x + a)^2$ is a parabola with its turning point at $(-a, 0)$. It is shifted a units to the left compared to $y = x^2$. If a is negative, it is shifted to the right.
- The graph of a quadratic that can be factorised to $f(x) = (x + a)(x + b)$ is a parabola with its turning point at $\left(\frac{-a-b}{2}, f\left(\frac{-a-b}{2}\right)\right)$. It is shifted $\frac{a+b}{2}$ units to the left and $f\left(\frac{-a-b}{2}\right)$ units up or down according to the sign compared to $y = x^2$. If $\frac{a+b}{2}$ is negative, then the graph is shifted to the right. If the whole function is negative (i.e., the coefficient of x^2 is negative) the graph is upside down compared to $y = x^2$. The zeros of the function, where it passes through the x-axis, are at $(-a, 0)$ and $(-b, 0)$.
- For coefficients of x^2 other than 1, there are two methods of finding the factors of a quadratic. Factors always *add up* to give the coefficient of x and *multiply* to give the constant.
- In the **cross method** of factorisation of a quadratic $ax^2 + bx + c$ to find the factors $(mx + n)(px + q)$, you use a square to find the factors m, p, n and q as shown on the right.



$mp = a,$

$nq = c,$

$mq + np = b$

- In the **decomposition method** of factorising quadratics, you find numbers u and v such that the product $uv = ac$ and the sum $u + v = b$. The term bx is then decomposed as $bx = ux + vx$ to make a grouping factorisation.
- To factorise general expressions, first remove common factors. If there are three terms, look for a quadratic or perfect square. If there are four terms, look for a grouping factorisation. You may also be able to write higher powers as squares or use a **dummy variable** to make it easier to factorise an expression.
- A **polynomial** is a sum of terms that are each the product of a constant and positive integer powers of the variables. $P(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0$, where the coefficients $a_n, a_{n-1}, a_{n-2}, \ldots, a_2, a_1, a_0$ are real numbers.
- The **degree** of a polynomial is the highest power n, and the **leading term** is the term that contains the highest power.

Chapter 7 summary

- The **division identity** for polynomials states that for any polynomials $a(x)$ and $b(x)$, where $b(x)$ is not a constant, there exist unique polynomials $q(x)$ and $r(x)$ such that $a(x) = q(x) \times b(x) + r(x)$, where the degree of $r(x)$ is less than the degree of $b(x)$.

  $q(x)$ and $r(x)$ are called the **quotient** and **remainder**, $b(x)$ is the **divisor** and $a(x)$ is the **dividend**.
- You can divide one polynomial into another using a **long division** technique.
- You can divide a polynomial by a linear expression of the form $x - a$ through **synthetic division**, which simplifies the process to the multiplication and addition of the coefficients by a.
- The **remainder theorem** says that when a polynomial $p(x)$ is divided by $x - a$, the remainder is $p(a)$.
- The **factor theorem** says that $p(a) = 0$ if and only if $x - a$ is a factor of the polynomial $p(x)$.
- You can sketch a polynomial using its zeros, y-intercept and the fact that when x becomes large in magnitude it behaves like the leading term.

Chapter 7 review

Understanding

See Examples 1, 2

1 Copy and complete each of the following factorisations.

a $m^2 - p^2 = (m + \ldots)(m - \ldots)$ **b** $x^2 - 6x + 9 = (x - \ldots)^2$ **c** $g^2 + 10g + 25 = (g + \ldots)^2$

2 Factorise each of the following.

a $r^2 - 16$ **b** $h^2 + 12h + 36$ **c** $k^2 - 14k + 49$

See Example 4

3 Factorise each of the following.

a $x(3x - y) + 2y(3x - y)$ **b** $ab + ac + bd + cd$ **c** $x^2 - 3xy - 6xz + 18yz$

See Example 5

4 Complete each of the following factorisations and check by expansion.

a $m^2 + 7m + 12 = (m + 4)(m + \ldots)$ **b** $x^2 - 6x - 16 = (x - \ldots)(x + 2)$

See Example 6

5 Factorise each of the following.

a $k^2 + 10k + 16$ **b** $g^2 + 11g + 28$

See Example 18

6 Show that $x + 2$ and $x - 3$ are factors of $p(x) = 3x^4 - x^3 - 15x^2 - 17x - 30$.

Fluency

7 Factorise each of the following.

a $25g^2 - 36t^2$ **b** $16u^2 - 56u + 49$ **c** $9m^2 + 6mn + n^2$

8 Factorise each of the following.

a $7m^3 - 7m^2 + 4am - 4a$ **b** $36 - (m - n)^2$

See Examples 6, 7

9 Factorise each of the following.

a $x^2 + x - 42$ **b** $p^2 - 2p - 15$ **c** $k^2 - 10k + 16$ **d** $a^2 + 13a + 36$

See Example 9

10 Factorise each of the following.

a $y^2 + 11yk + 18k^2$ **b** $m^2 + 8mz + 7z^2$ **c** $n^2 + nz - 132z^2$

See Examples 12, 13

11 Factorise each of the following expressions.

a $5x^3 - 80xy^2$ **b** $2m^2 + 6ma - 56a^2$

c $3q^2 - 3qt - 216t^2$ **d** $7g^2 - 14gm - 168m^2$

See Example 11

12 Factorise each of the following.

a $g^2 - 13gp + 30p^2$ **b** $48u^2 + 14ug - 49g^2$ **c** $15t^2 + 62tb + 63b^2$

See Examples 9, 10

13 Factorise each of the following.

a $10u^2 + 41u + 40$ **b** $28b^2 + 9b - 9$

c $56a^2 - 51a - 27$ **d** $4a^2 - 35a + 49$

See Examples 16, 17

14 Find the quotient and remainder for each of the following divisions. Your teacher may want you to use different methods for each one.

a $3x^4 + 2x^3 - 2$ divided by $x + 1$

b $2x^5 - 3x^4 + 2x^3 + 3x^2 - 6$ divided by $x - 2$

Problem solving

See Examples 3, 8

15 Use factorisation to help sketch the graphs of each of the following.

a $y = x^2 + 10x + 25$ **b** $y = x^2 - 3x - 4$

16 How many integer values of b are there for which $x^2 + bx + 12$ can be factorised using integer values?

17 How many integer values of b are there for which $x^2 + bx - 20$ can be factorised using integer values?

18 Factorise each of the following expressions.

See Examples 14, 15

a $63 - 7(3a + 5)^2$
b $3x^2 - 12 + xy - 2y$
c $3x^4y^2 - 12x^2y^3 + 15x^3y^4$
d $16p^4 - 81q^2$
e $6(p - 4)^2 - 11(p - 4) - 10$

19 Factorise each of the following.

See Example 19

a $2x^4 + 5x^3 - 2x^2 + 1x - 6$
b $3x^4 + 7x^3 - 1x^2 - 7x - 2$

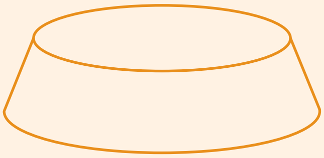
20 Use factorisation to help sketch each of the following. Your teacher will tell you what technology to use for each one.

See Examples 20, 21

a $p(x) = x^4 - 15x^2 + 10x + 24$
b $p(x) = 2x^3 - 5x^2 - 2x - 3$

Reasoning

21 A *frustum* is formed by cutting the top off a cone parallel to the base. Show that for a frustum with height h, base radius R and top radius r, the angle θ between the base and the sloping side is given by $\tan\theta = \dfrac{h}{R - r}$. Hence find expressions for the height of the full cone, the height of the part cut off and show that the volume of the frustum is $V = \pi h(R^2 + Rr + r^2)$.



22 How many whole number values of c between 1 and 100 can you find so that $x^2 - 3x - c$ can be factorised using whole numbers?

23 How many integer values of a can you find, strictly between 100 and 200, so that $ax^2 + 2x - 10$ can be factorised using integers?

Measurement and geometry

8 Geometry

Contents

Prior learning
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Australian Curriculum statements

Geometric reasoning

Formulate proofs involving congruent triangles and angle properties. (10)

Apply logical reasoning, including the use of congruence and similarity, to proofs and numerical exercises involving plane shapes. (10)

Prove and apply angle and chord properties of circles. (10A)

Video tutorial
Geometry
MAT10MGVT00008

Geometry is about shapes, lines, space and angles and the relationships between them. The geometry of flat surfaces was first studied in ancient Greece by the mathematician Euclid and this field of study is now known as Euclidean geometry in recognition of his work. Euclid was the founder of deductive geometry – the process of arriving at new geometric facts from previously known facts by using logical reasoning. In geometry, a written logical argument is called a proof. Geometry has many uses. It is used whenever we are dealing with the size, shape, volume, or position of an object. As a school subject, it helps students to develop logical reasoning skills. Geometry forms the basis of computer-aided design (CAD), which is now used to draw almost all building plans. Geometry also underpins the field of 2D and 3D computer animation that is widely used in the gaming and film industries.

Mathematical literacy

Maths dictionary
MAT10ASDI00001

The mathematical words below have special meanings that you will learn in this chapter. It is important that you learn to spell them and gradually learn what they mean in mathematics. You may find the glossary or online mathematical dictionary useful for this purpose.

- acute-angled triangle
- allied angles
- alternate angles
- angle sum of a quadrilateral
- angles at a point
- angles at the circumference subtended by the same chord or arc
- angle subtended at the centre of a circle
- anticlockwise
- arc
- centre of magnification
- centre of rotation
- chord
- clockwise
- cointerior angles
- complementary angles
- congruent
- congruent figures
- convex figure
- corresponding angles
- deductive geometry
- diagonally opposite angles of a parallelogram
- diameter of a circle
- dilation
- enlargement
- equilateral triangle
- exterior angle of a cyclic quadrilateral
- exterior angle of a triangle
- irregular polygon
- isosceles triangle
- kite
- line of reflection
- magnification
- obtuse-angled triangle
- opposite angles of a cyclic quadrilateral
- parallelogram
- plane shape
- polygon
- quadrilateral
- reduction
- rectangle
- reflection
- regular polygon
- rhombus
- right-angled triangle
- rotation
- scalene triangle
- similar figures
- square
- subtend
- sum of angles in a triangle
- supplementary angles
- tangent and radius at the same point on a circle
- transformation
- translation
- transversal
- trapezium
- two-dimensional
- vertex
- vertically opposite angles

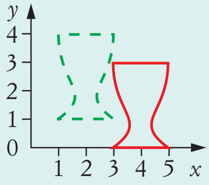
8.1 Properties of shapes

Plane shapes (**two-dimensional** or **2D figures**) can be drawn on a flat surface.
A **transformation** changes a shape (an **object**) or its position to form an **image**.

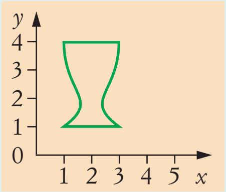
Important!

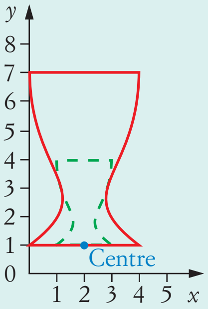
Transformations

A **translation** is a sliding movement. The vertices of the object are all changed by the same amount to form the image.
A **rotation** is a turn **(clockwise** or **anticlockwise)** through an angle. Each point stays the same distance from the **centre of rotation.**
A **dilation** is a change of size. The **enlargement** or **reduction** is measured by the scale factor or **magnification.** The **centre of magnification** keeps the same coordinates.
When a shape is **reflected,** all points swap to the other side of the **line of reflection**.

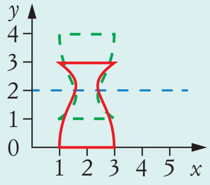


Translation

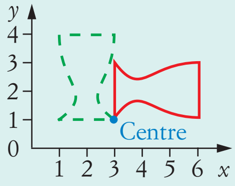




Dilation



Reflection



Rotation

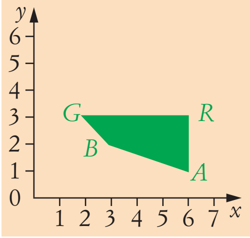
Example 1

Puzzle sheet
Transformations
MAT10MGPS00024

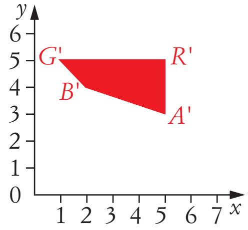
The diagram below shows transformations of the figure *GRAB*.

a Use coordinates to describe each transformation.

b Fully describe each transformation in words.



i



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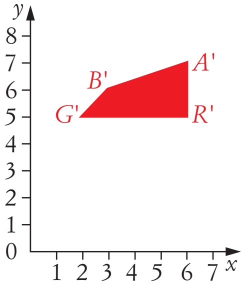

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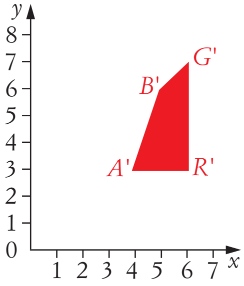

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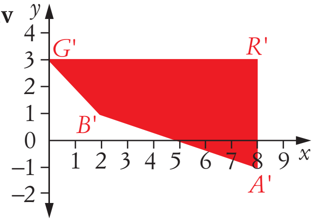


Solution

a Write the coordinates of the original figure and the coordinates of the new figure.

i $G(2, 3) \to G'(1, 5), R(6, 3) \to R'(5, 5)$
$A(6, 1) \to A'(5, 3), B(3, 2) \to B'(2, 4)$

ii $G(2, 3) \to G'(2, 5), R(6, 3) \to R'(6, 5)$
$A(6, 1) \to A'(6, 7), B(3, 2) \to B'(3, 6)$

iii $G(2, 3) \to G'(6, 7), R(6, 3) \to R'(6, 3)$
$A(6, 1) \to A'(4, 3), B(3, 2) \to B'(5, 6)$

iv $G(2, 3) \to G'(0, 3), R(6, 3) \to R'(8, 3)$
$A(6, 1) \to A'(8, {}^{-}1), B(3, 2) \to B'(2, 1)$

b **i** The figure has slid up 2 and left 1.
A translation up 2 and left 1.

ii The figure is reflected upwards.
A reflection in the line $y = 4$.

iii The figure has turned, but R is the same.
A rotation 90° clockwise about R.

iv The figure is twice as big, and all points are twice as far from (4, 3) as they were.
A dilation with factor 2 and centre (4, 3).

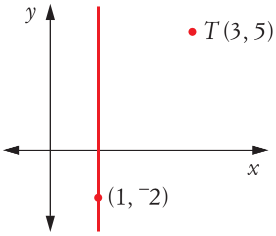
Example 2

What will be the coordinates of $T(3, 5)$ after a reflection in the vertical line passing through $(1, {}^{-}2)$?

Solution

Make a sketch.

T is 2 units to the right of the vertical line passing through $(1, {}^{-}2)$.



T will swap from 2 units to the right of (1, 5) to 2 units to the left of (1, 5), as this is the point opposite T across the line of reflection.

$\mathbf{T(3, 5) \to T'({}^{-}1, 5)}$

A combination of transformations is the situation where an initial transformation is followed by another transformation.

Example 3

$\triangle PQR$ has vertices $P(2, 1)$, $Q(5, 1)$ and $R(5, 2)$.
T_1 is a reflection in the line $x = -y$.
T_2 is a rotation of 90° clockwise about the origin (0, 0).
Find a single transformation that is the same as:

a T_1 followed by T_2

b T_2 followed by T_1

Solution

a Draw a set of axes and mark in $\triangle PQR$. Draw the line $x = -y$.

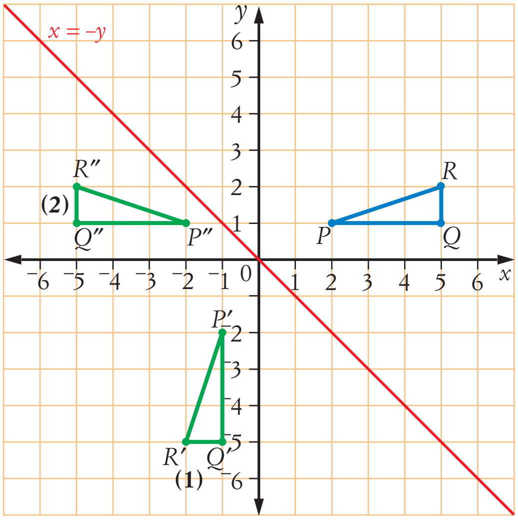
Do T_1 by reflecting each vertex of $\triangle PQR$ the same perpendicular distance from above

$x = -y$ to below $x = -y$.

T_1 transforms $\triangle PQR$ to position (1): $\triangle P'Q'R'$.

Do T_2 by rotating each vertex of $\triangle P'Q'R'$ 90° clockwise about (0, 0).

T_2 transforms $\triangle P'Q'R'$ to position (2): $\triangle P''Q''R''$.



Compare the position of $\triangle PQR$ with the position of $\triangle P''Q''R''$.

Each vertex of $\triangle PQR$ is the same distance to the left of the y-axis as the corresponding vertex of $\triangle P''Q''R''$ is to the right the y-axis.

Identify the transformation.

T_1 followed by T_2 is the same as a reflection in the y-axis.

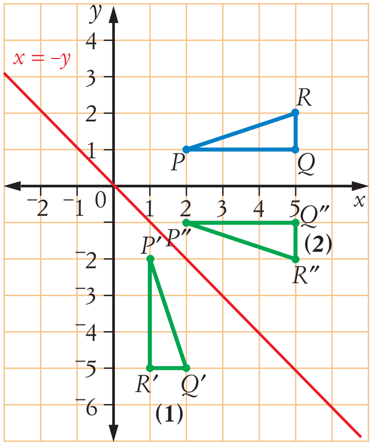
b Draw a new set of axes and mark in $\triangle PQR$. Draw the line $x = -y$.

Do T_2 by rotating each vertex of $\triangle PQR$ 90° clockwise about (0, 0).

T_2 transforms $\triangle PQR$ to position (1): $\triangle P'Q'R'$.

Do T_1 by reflecting each vertex of $\triangle P'Q'R'$ the same distance from the left of $x = -y$ to the right of $x = -y$.

T_1 transforms $\triangle P'Q'R'$ to position (2): $\triangle P''Q''R''$.



Compare the position of $\triangle PQR$ with the position of $\triangle P''Q''R''$.

Each vertex in $\triangle PQR$ is the same distance above the x-axis as the corresponding vertex in $\triangle P''Q''R''$ is below the x-axis.

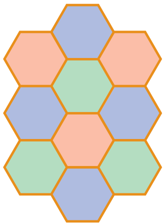
Identify the transformation.

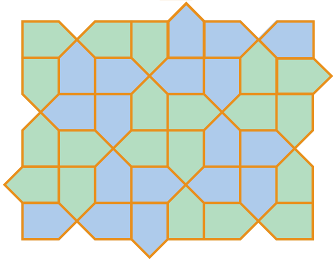
T_2 followed by T_1 is the same as a reflection in the x-axis.

From the outcomes of Example **3** you can see that the order of transformations is important. If we perform two transformations on an object in a different order, the final image may be different. This means that transformations are not **commutative** operations.

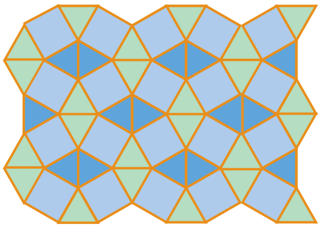
Investigate: Tessellations

A **tessellation** is a pattern that covers an area with a repeating pattern. The pattern formed is symmetrical, but the 'tiles' that make the pattern are not necessarily symmetrical.
A **regular tessellation** is made using regular polygons of the same size.





A **semi-regular tessellation** is made using combinations of (different) regular polygons.



An **irregular tessellation** is made using congruent irregular polygons.
You can see examples of tessellations in floor and wall tiling and in artwork.

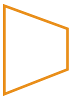
1 Locate examples of tessellations in magazines or on the Internet and classify them as regular, irregular or semi-regular.

2 Make a chart to display the various examples of tessellations that you have found.

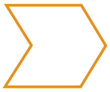
3 Create separate tessellations on an A4 sheet of paper using each of the following shapes (use colours).

a

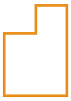
b



c

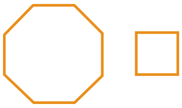


d

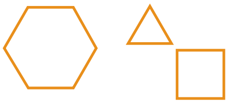


4 Create a semi-regular tessellation on an A4 sheet of paper with each group of shapes, using colours as well.

a



b



Teacher notes

Quasi-tessellations

MAT10MGTN00007

You can create drawings that are like tessellations, but have recognisable shapes included, by changing the desired shapes into approximate polygons to begin with. It is simplest to use a quadrilateral.

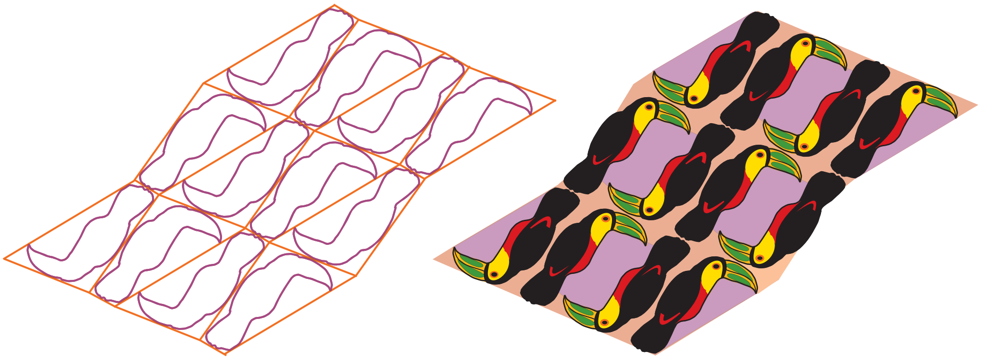
5 Make an outline of the shape you want.

6 Fit the outline into a quadrilateral.

7 On an A3 sheet of paper make a tessellation of copies of the quadrilateral (with outline).

8 Fill in the spaces with some neutral colours.

9 Colour your shapes.



Your teacher may wish to display some of your finished products in the classroom.

The most important plane shapes have straight sides.

Technology: GeoGebra
Naming polygons
MAT10MGTC00008

Important!

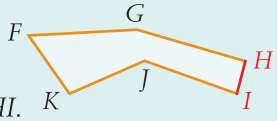
Polygons

Polygons are closed figures with straight sides. A polygon is classified by its number of sides and the nature of its angles and sides.

Number of sides	Name of polygon
3	**Tri**angle
4	**Quad**rilateral
5	**Pent**agon
6	**Hex**agon
7	**Hept**agon

Number of sides	Name of polygon
8	**Oct**agon
9	**Non**agon
10	**Dec**agon
12	**Dodec**agon
20	**Icos**agon

Regular polygons have equal sides and equal angles. In an **irregular polygon**, at least one of the angles or sides is different from the others.
All the vertices of a **convex** polygon point outwards. **Concave** polygons have at least one inward-pointing vertex, so at least one internal angle is greater than 180°.
Polygons are named by writing the **vertices** (corners) in order. These are normally shown by capital letters. The sides of a polygon are named using the two vertices at the ends.
FGHIJK is a concave hexagon. It is irregular. The red side is named *HI*.



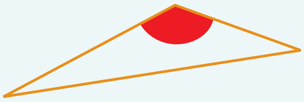
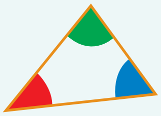
Triangles and quadrilaterals are shapes that are commonly used in geometry. It is important for you to know their properties.

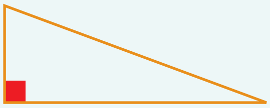
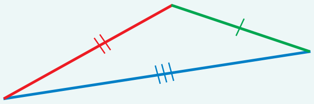
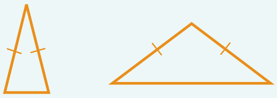
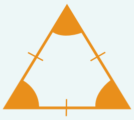
Important!

Triangles and some quadrilaterals have special names.

Triangles

Triangles can be classified by their angles or by their sides. The sides of triangles are often named using the lower case letter of the opposite vertex.

Name	Property	Example
Obtuse-angled triangle	An obtuse angle (between 90° and 180°)	
Acute-angled triangle	All acute angles (less than 90°)	

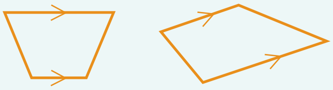
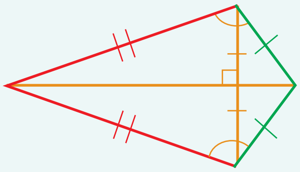
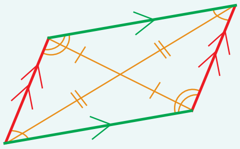
Name	Property	Example
Right-angled triangle	A right angle (90°)	
Scalene triangle	All sides different	
Isosceles triangle	Two sides equal	
Equilateral triangle	All sides equal and all angles equal (60°) (regular)	

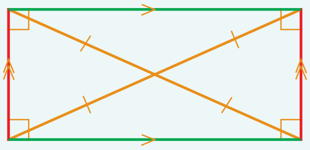
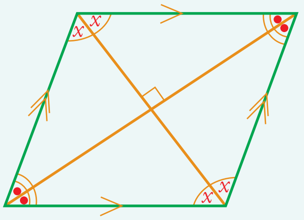
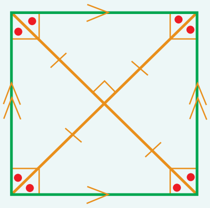
Quadrilaterals

It is possible to define the special quadrilaterals as follows.
A **trapezium** is a quadrilateral with at least one pair of opposite sides parallel.
A **parallelogram** is a quadrilateral with both pairs of opposite sides parallel.
A **rectangle** is a parallelogram with one angle a right angle.
A **square** is a rectangle with two adjacent sides equal in length.
A **rhombus** is a parallelogram with two adjacent sides equal in length.
A **kite** is a convex quadrilateral with two pairs of equal adjacent sides.
In addition to these definitions, special quadrilaterals have a number of other properties.

Weblink
Sum of interior angles of a quadrilateral
MAT10MGWB00008

Technology: GeoGebra
Making quadrilaterals
MAT10MGTC00012

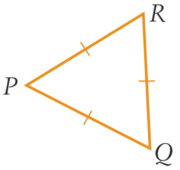
Name	Property	Example
Trapezium (trapezoid)	One pair of parallel sides	
Kite	Two pairs of equal adjacent sides One pair of opposite angles equal One axis of symmetry Diagonals intersect at right angles One diagonal bisected by the other	
Parallelogram	Opposite sides parallel Opposite sides equal Opposite angles equal Diagonals bisect each other Has rotational symmetry	

Name	Property	Example
Rectangle	Opposite sides parallel Opposite sides equal All angles are right angles Diagonals are equal in length Diagonals bisect each other Two axes of symmetry Has rotational symmetry	
Rhombus	Opposite sides parallel All sides equal Opposite angles equal Diagonals bisect each other at right angles Diagonals bisect the angles of the rhombus Two axes of symmetry Has rotational symmetry	
Square	Opposite sides parallel All sides equal All angles are right angles Diagonals equal in length Diagonals bisect each other at right angles Diagonals bisect the angles of the square Four axes of symmetry Has rotational symmetry	

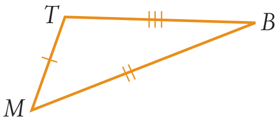
Example 4

Name and classify the following shapes.

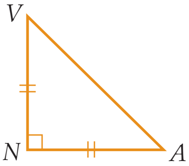
a



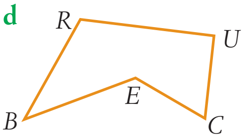
b



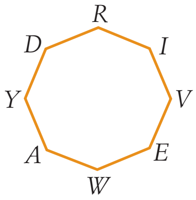
c



d



e



Solution

a This shape has 3 equal sides. **PRQ is an acute-angled equilateral triangle.**

b This shape has 3 different sides and an angle between 90° and 180°. **MTB is an obtuse-angled scalene triangle.**

c This shape has 3 sides, 2 of which are equal, and one angle is a right angle (90°). **VAN is a right-angled isosceles triangle.**

d This shape has 5 sides and an inward-pointing vertex at E. **$BRUCE$ is an (irregular) concave pentagon.**

e This shape has 8 sides and it looks as if all the sides and all the angles are equal. **$DRIVEWAY$ is a regular (convex) octagon.**

Example 5

a Is a rectangle a square?

b Is a square a parallelogram?

c A parallelogram has equal diagonals. What kind of quadrilateral is it?

d Give three possible properties, any one of which will ensure that a parallelogram is a rhombus.

Worksheet

Quadrilaterals: True or false

MAT10MGWK00020

Solution

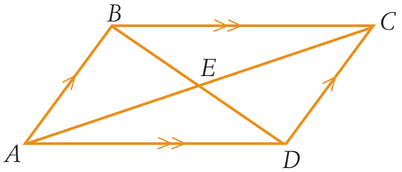
a A rectangle has opposite sides equal but adjacent sides are not necessarily equal.

A square has all sides equal. **A rectangle is not a square.**

b A square has both pairs of opposite sides equal and parallel.

A parallelogram has both pairs of opposite sides parallel. **A square is a parallelogram.**

c Draw the situation.



The diagonals are equal and bisect each other. **From the information, $EA = EB = EC = ED$**

What does this mean about the triangles? **$\triangle ABE$, $\triangle BCE$, $\triangle CDE$ and $\triangle DAE$ are all isosceles so $\angle EBC = \angle ECB$ and $\angle ECD = \angle EDC$.**

Use the parallel lines. **$\angle ABD = \angle EDC$ (alternate angles), so $\angle ABD = \angle ECD$**

Now consider $\angle ABC$ and $\angle BCD$. **$\angle ABC = \angle EDC + \angle EBC = \angle ECD + \angle ECB = \angle BCD$**

Use the fact they are cointerior. **But $\angle ABC + \angle BCD = 180°$**

State the conclusion. **Thus $\angle ABC = \angle BCD = 90°$ and the parallelogram must be a rectangle.**

d A parallelogram has opposite sides equal	**If two adjacent sides are equal they must all be equal so it is a rhombus.**
The diagonals bisect each other.	**If the diagonals are perpendicular then all the triangles formed are congruent (SAS), so it must be a rhombus.**
The angles between a diagonal and side are alternate angles.	**If a diagonal bisects the corner angle, then the triangle it forms is isosceles, so its sides are equal and the shape must be a rhombus.**

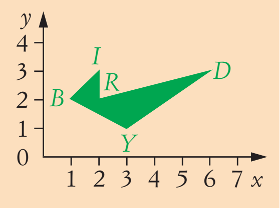
Exercise 8.1 Properties of shapes

Understanding

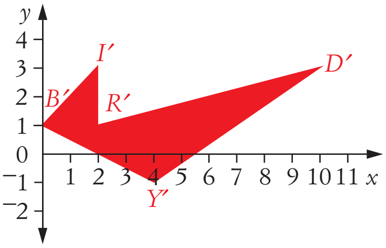
Extra questions
Exercise 8.1
MAT10MGEQ00022

See Example 1

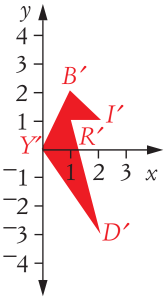
1 Use coordinates to describe each of the transformations shown below.



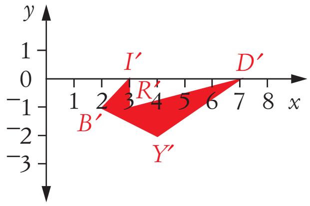
a



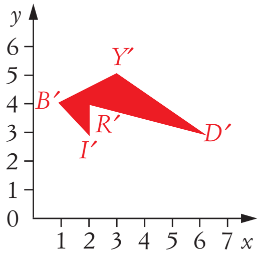
b



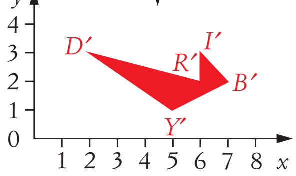
c



d



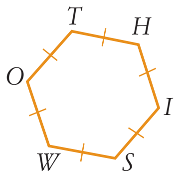
e



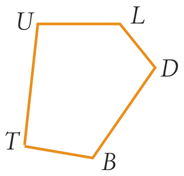
See Example 4

2 Name and classify the following shapes.

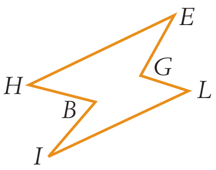
a



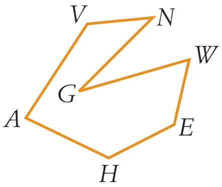
b



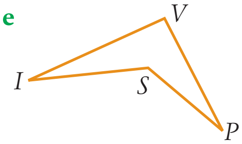
c



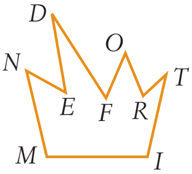
d



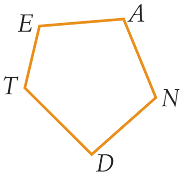
e



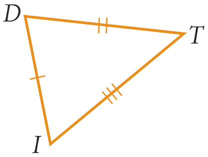
f



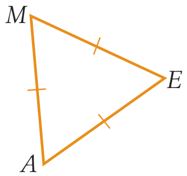
g



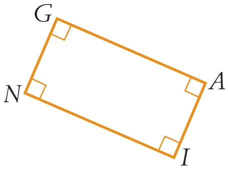
h



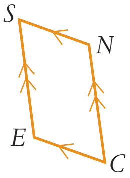
i



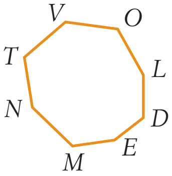
j



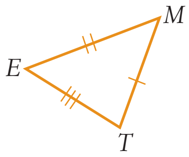
k



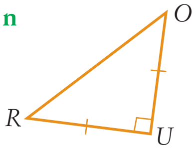
l



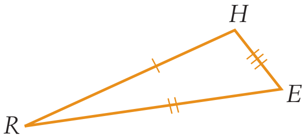
m



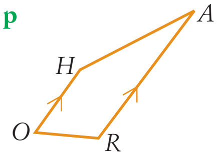
n



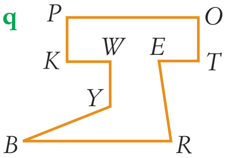
o



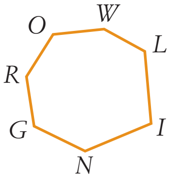
p



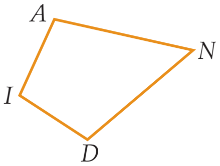
q



r

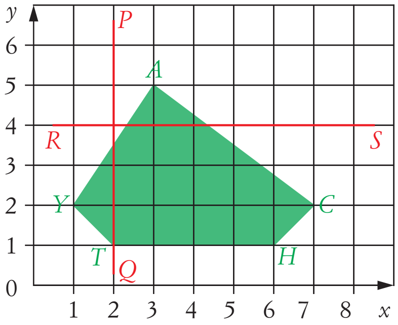


s



Fluency

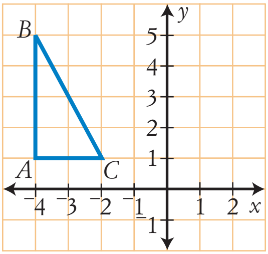
3 Copy the shape shown onto grid paper and show a:



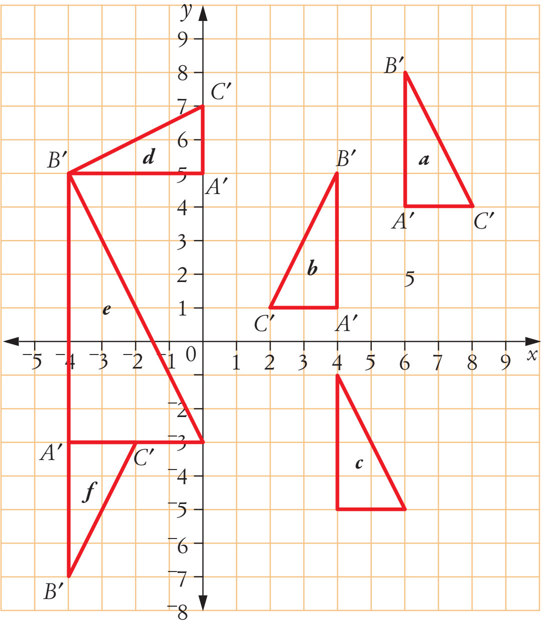
- **a** translation 3 up and 4 left
- **b** translation 2 down and 3 right
- **c** rotation 90° clockwise with centre (2, 1)
- **d** rotation 90° anticlockwise with centre (7, 2)
- **e** reflection in the line *PQ*
- **f** reflection in the line *RS*
- **g** magnification with factor 3 and centre (2, 3)
- **h** reduction with factor 0.5 and centre (1, 2).

4 The shape *ABC* shown here has been transformed by a single transformation to produce images ***a***, ***b***, ***c***, ***d***, ***e*** and ***f*** shown below.

See Example 1



Describe each transformation in words.



Worked solutions
Exercise 8.1
MAT10MGWS00055

See Example 2

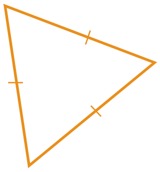
5 Fully describe each of the transformations in question **1** in words.

6 Find the coordinates of $P(1, 3)$, $Q(4, 6)$, $R(5, 3)$ and $S(2, 2)$ after a:

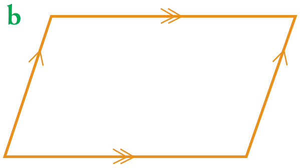
- **a** translation 1 left and 2 down
- **b** translation 3 up and 2 right
- **c** reflection in the line $x = 3$
- **d** reflection in the line $y = 1$
- **e** rotation 180° clockwise with centre (3, 2)
- **f** rotation 90° anticlockwise with centre (1, 3)
- **g** magnification with factor 2 and centre (3, 5)
- **h** reduction with factor $\frac{1}{2}$ and centre (2, 4).

7 Use the properties of triangles and quadrilaterals to name each of the following shapes.

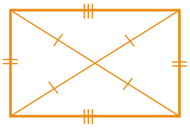
a



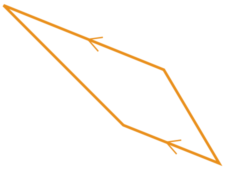
b



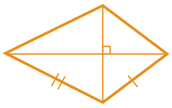
c



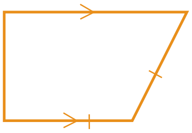
d



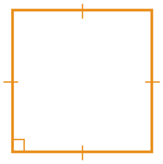
e



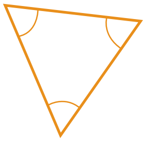
f



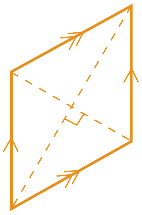
g



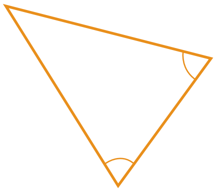
h



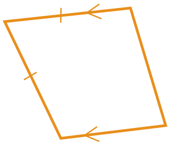
i



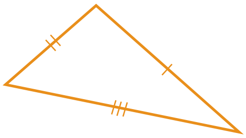
j



k



l



See Example 5

8 State whether each of the following is true (T) or false (F).

- **a** The diagonals of a square, a rhombus and a rectangle bisect each other.
- **b** The angles of a rectangle are equal to 90°.
- **c** A kite and an isosceles triangle both have only one axis of symmetry.
- **d** An isosceles triangle is an equilateral triangle.
- **e** A rhombus has two axes of symmetry.
- **f** Opposite sides of a kite are equal.
- **g** The sides opposite the equal angles of an isosceles triangle are equal.
- **h** Adjacent sides of a parallelogram are equal.
- **i** A square is a rhombus.
- **j** A scalene triangle has rotational symmetry.

k The diagonals of a square are equal.

l The diagonals of a rhombus bisect each other at right angles.

m The diagonals of a kite are equal.

n An equilateral triangle has only two axes of symmetry.

Problem solving

Worked solutions
Exercise 8.1
MAT10MGWS00055

Worked solutions
Exercise 8.1
MAT10MGWS00055

⑨ An object with vertices $A(1, ^-3)$, $B(^-4, 1)$, $C(^-3, 7)$, $D(3, 7)$ and $E(4, 3)$ is reflected in the line $y = x$. Find the coordinates of the image $A'B'C'D'E'$.

10 An object with vertices $X(^-3, 1)$, $Y(^-3, 4)$ and $Z(^-1, 4)$ is subjected to a number of separate transformations. The coordinates of the image of XYZ after each transformation are given below. In each case, fully describe the transformation made to XYZ to produce $X'Y'Z'$.

- **ⓐ** $X(^-3, 1) \rightarrow X'(^-2, 8)$, $Y(^-3, 4) \rightarrow Y'(1, 8)$, $Z(^-1, 4) \rightarrow Z'(1, 6)$
- **b** $X(^-3, 1) \rightarrow X'(5, 6)$, $Y(^-3, 4) \rightarrow Y'(5, 9)$, $Z(^-1, 4) \rightarrow Z'(7, 9)$
- **c** $X(^-3, 1) \rightarrow X'(^-7, ^-7)$, $Y(^-3, 4) \rightarrow Y'(^-7, ^-4)$, $Z(^-1, 4) \rightarrow Z'(^-5, ^-4)$
- **d** $X(^-3, 1) \rightarrow X'(3, 1)$, $Y(^-3, 4) \rightarrow Y'(3, 4)$, $Z(^-1, 4) \rightarrow Z'(1, 4)$
- **e** $X(^-3, 1) \rightarrow X'(^-3, ^-1)$, $Y(^-3, 4) \rightarrow Y'(^-3, ^-4)$, $Z(^-1, 4) \rightarrow Z'(^-1, ^-4)$
- **f** $X(^-3, 1) \rightarrow X'(^-7, 1)$, $Y(^-3, 4) \rightarrow Y'(^-7, 4)$, $Z(^-1, 4) \rightarrow Z'(^-9, 4)$
- **g** $X(^-3, 1) \rightarrow X'(^-3, 11)$, $Y(^-3, 4) \rightarrow Y'(^-3, 8)$, $Z(^-1, 4) \rightarrow Z'(^-1, 8)$
- **h** $X(^-3, 1) \rightarrow X'(^-3, ^-5)$, $Y(^-3, 4) \rightarrow Y'(^-3, 4)$, $Z(^-1, 4) \rightarrow Z'(3, 4)$

Reasoning

See Example 3

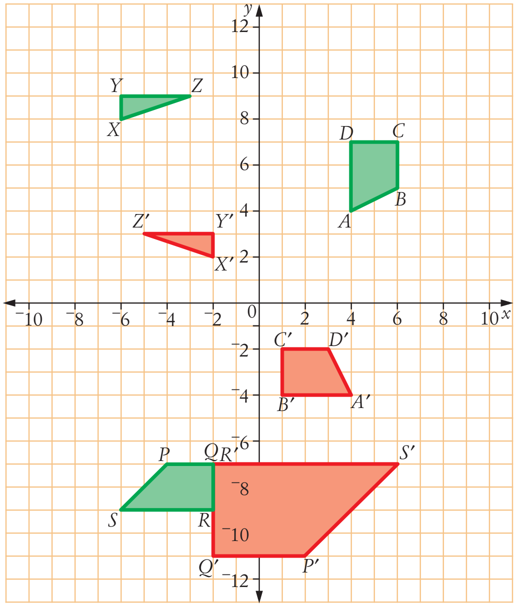
11 A shape PQR has vertices at $P(2, 1)$, $Q(5, 2)$ and $R(5, 1)$.
R is a reflection in the x-axis.
M is a reflection in the line $y = -x$.

- **a** What single transformation is equivalent to R followed by M?
- **b** What single transformation is equivalent to M followed by R?

12 A shape PQR has vertices at $P(2, 1)$, $Q(5, 2)$ and $R(5, 1)$.
T_1 is a rotation of 180° about $O(0, 0)$.
T_2 is a reflection in the y-axis.

- **a** What single transformation is equivalent to T_1 followed by T_2?
- **b** What single transformation is equivalent to T_2 followed by T_1?

13 Objects $ABCD$, XYZ and $PQRS$ are shown in the diagram below. In each case, the object has undergone one transformation followed by another to form images $A'B'C'D'$, $X'Y'Z'$ and $P'Q'R'S'$ respectively. Fully describe the two transformations that have been used in each case.



14 Use $\triangle XYZ$ with vertices at $X(1, 1)$, $Y(3, 1)$ and $Z(3, 2)$ to consider the following transformations.

T_0: a rotation of 360° about $O(0, 0)$
T_1: a reflection in the line $y = x$
T_2: a rotation of 90° anticlockwise about $O(0, 0)$
T_3: a reflection in the y-axis
T_4: a rotation of 180° about $O(0, 0)$
T_5: a reflection in the line $y = -x$
T_6: a rotation of 90° clockwise about $O(0, 0)$
T_7: a reflection in the x-axis

What single transformation is equivalent to:

a T_1 followed by T_2?
b T_2 followed by T_1?
c T_4 followed by T_2?
d T_2 followed by T_4?
e T_5 followed by T_4?
f T_7 followed by T_6?

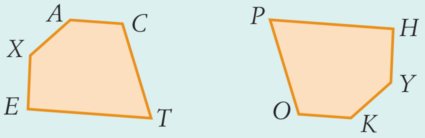
8.2 Similarity and congruence

You looked at a number of transformations in the previous section of this book. When an object is subjected to a translation, reflection or rotation, its image is exactly the same shape and size – the object and image are identical. However, when an object is subjected to a dilation, the image is the same shape but a different size. It is either enlarged or reduced by a scale factor. In geometry there are special terms to describe these situations.

Important!

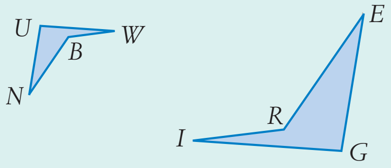
Similarity and congruence

Congruent shapes are exact copies of each other. The figures are exactly the same shape and size. The symbol for congruency is ≡. **Similar** figures are exactly the same shape but not necessarily the same size. The symbol for similarity is |||.



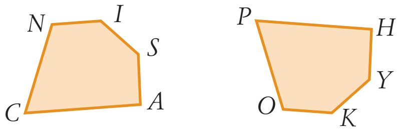
ACTEX ≡ KOPHY

When naming similar or congruent figures, corresponding vertices are named in the same order (as above). For congruent and similar figures, the **corresponding angles** in each figure are the same. The sides opposite corresponding angles are called **corresponding** sides. In similar shapes, corresponding angles are equal *and* the ratio of corresponding sides is equal. This ratio is called the **scale factor**. The lengths of one shape are multiplied by the scale factor to work out the corresponding lengths in the other shape. In congruent figures they are the same, so the scale factor is 1.

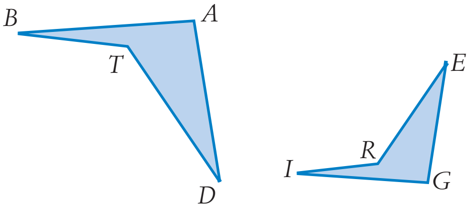


UWBN ||| GIRE

Even when a shape must be flipped over to form a mirror image, it is considered to be congruent or similar to another. This is shown in the following diagrams.



INCAS ≡ *KOPHY*



ABTD ||| *GIRE*

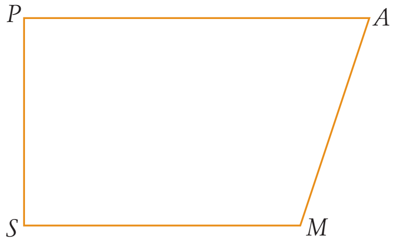
Example 6

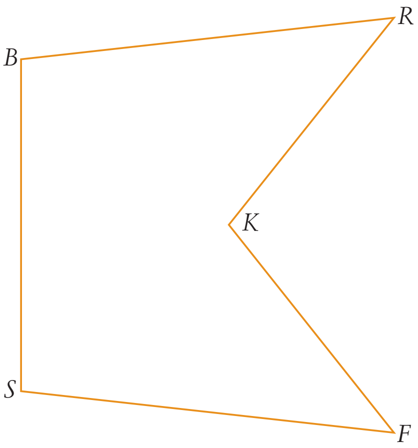
TLF Learning object

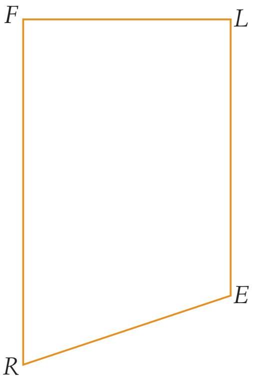
Measures: Similar shapes (L2309)

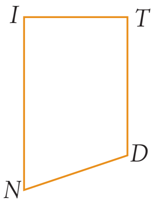
MAT10MGIN00008

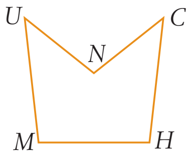
Measure the sides and angles of the following shapes. Name any shapes that are either congruent or similar. (Don't include congruent shapes as similar shapes.) Work out the scale factors of similar shapes.











Solution

Compare shapes by measuring corresponding angles and lengths of corresponding sides.

SPAM and *LFRE* are mirror images that are exactly the same shape and size. **SPAM ≡ LFRE**

MUNCH and *BRKFS* are the same shape, but different sizes. **MUNCH ||| BRKFS**

BR = 54 mm = 3 × 18 mm = 3 × *MU*. **The scale factor is 3.**

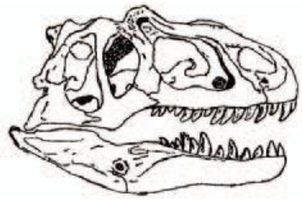
SPAM and *TIND* are mirror images that are the same shape but different sizes. **SPAM ||| TIND**

SP = 30 mm and *TI* = 15 mm.	**The scale factor is $\frac{1}{2}$ (0.5).**
SPAM ≡ *LFRE*, so *LFRE* and *TIND* are also similar. Each side in *TIND* is half the length of the corresponding side in *LFRE*.	***LFRE* ||| *TIND* (scale factor = 0.5)**

A scale drawing can be used as a two-dimensional representation of an object. The scale drawing and the view of the object are similar figures.

Example 7

Here is a scale drawing of the skull of a dinosaur called Allosaurus fragilis – a large carnivorous predator. The scale used is 1 : 20. What is the actual width of the dinosaur's skull?

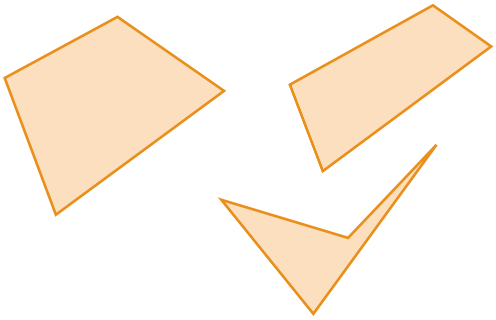


Solution

Measure the width of the skull on the drawing.	**Drawing skull width = 4 cm**
Write the scale.	**Scale = 1 : 20 = $\frac{1}{20}$**
1 : 20 means that 1 cm on the drawing represents 20 cm on the object.	$\frac{1}{20} = \frac{\text{drawing distance}}{\text{object distance}}$
Substitute for drawing skull width.	$\frac{1}{20} = \frac{4 \text{ cm}}{\text{object distance}}$
Cross multiply.	**Object distance × 1 = 4 cm × 20**
Evaluate.	**= 80 cm**
State the result.	**The dinosaur's skull is 80 cm wide.**

Investigate: Angles and sides

Of these three quadrilaterals, two have the same sides and two have the same angles, but none are congruent or similar.



Draw each of the following pairs so that they are neither congruent nor similar, if you can.

1 Two convex quadrilaterals with the same sides in the same order

2 Two pentagons with the same sides in the same order

3 Two pentagons with the same angles in the same order

4 Two triangles with the same sides

5 Two triangles with the same angles

Discuss your findings as a class group. Are there any general rules?

You can use the fact that figures are similar to find unknown sides.

CAS TI-Nspire exercise
Geometry
MAT10MGTI00008

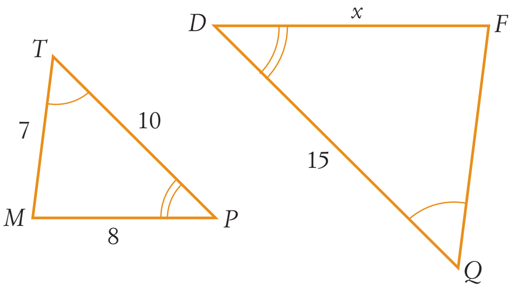
CAS ClassPad exercise
Geometry
MAT10MGCP00008

Technology worksheet
Excel worksheet: Area of similar shapes
MAT10MGCT00013

Puzzle sheet
Similar triangles
MAT10MGPS00025

Example 8

The triangles shown are similar. Find the value of *x*.



Solution

Name the vertices in corresponding order. $\triangle MTP \ ||| \ \triangle FQD$

Find the scale factor using corresponding sides.

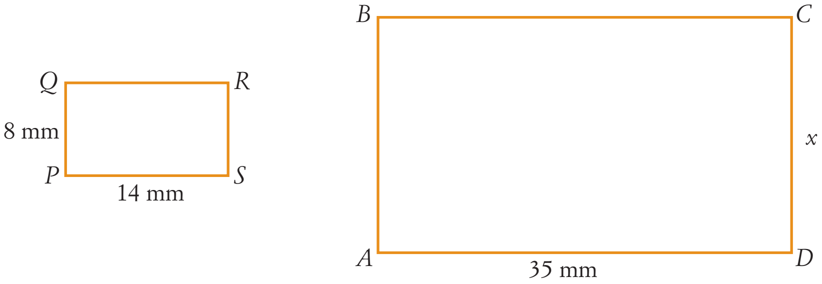
$$\textbf{Scale factor} = \frac{DQ}{PT} = \frac{15}{10} = 1.5$$

Use the scale factor to find *x* from the corresponding side.

$$x = 1.5 \times MP = 1.5 \times 8 = 12$$

Example 9

If *PQRS* ||| *ABCD*, calculate the value of *x*.



Solution

Identify corresponding sides. $PQ \leftrightarrow AB,\ QR \leftrightarrow BC,\ RS \leftrightarrow CD$ and $SP \leftrightarrow DA$

Write a statement using the ratio of corresponding sides. $\frac{PQ}{AB} = \frac{SP}{DA}$

Substitute for known values. $\frac{8}{x} = \frac{14}{35}$

Cross multiply. $8 \times 35 = 14 \times x$

Reverse the equation. $14 \times x = 8 \times 35$

Divide both sides by 14 and cancel. $\frac{{}^{1}\cancel{14} \times x}{\cancel{14}_{1}} = \frac{{}^{4}\cancel{8} \times \cancel{35}^{5}}{\cancel{14}_{\cancel{7}_{1}}}$

Evaluate. $= 20$

State the result including the unit. $x = 20$ **mm**

Similar triangles can also be used to solve real world problems.

Example 10

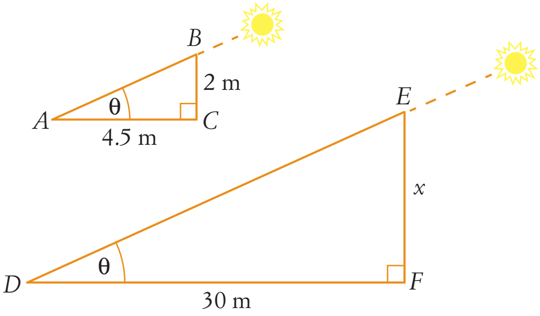
Alternative methods
Similar triangles
MAT10MGAM00001

An electricity pole casts a shadow of 30 m. At the same time, a 2 m stick held vertically casts a 4.5 m shadow. Find the height of the pole.

Solution

Draw a diagram, putting in all the information.

The angle of the Sun will be the same in both cases, because it is at the same time.



The two angles are the same, so the third is also the same. $\triangle ABC\ |||\ \triangle DEF$ (AAA)

Find the scale factor, using corresponding sides. Scale factor $= \frac{DF}{AC} = \frac{30}{4.5}$

$= 6\frac{2}{3}$

Use the scale factor to find x from the corresponding side. $x = 6\frac{2}{3} \times BC$

$= 6\frac{2}{3} \times 2$

≈ 13.3 **m**

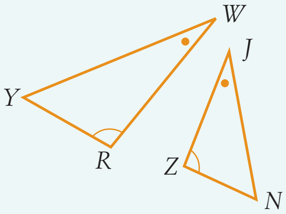
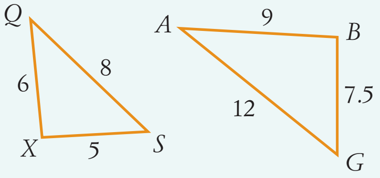
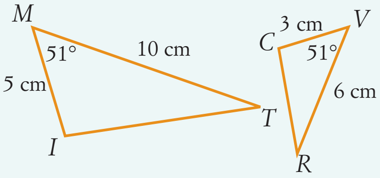
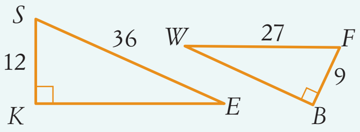
Write the answer. **The pole is about 13.3 m high.**

All polygons can be divided into triangles, so congruent and similar triangles are used extensively in geometric proofs. An important relationship used in these proofs is the fact that the sum of the internal angles of a triangle is 180°. This means that as long as two angles of a triangle are known, the third can be easily calculated. How?

There are four tests (proofs) of triangle similarity.

Important!

Triangle similarity tests

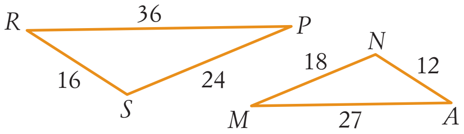
Relationship	**Example**
Angle/angle/angle (AAA) or **Angle/angle (AA)** The angles of each of the triangles are the same. Two angles the same is good enough because the third must then be the same anyway, so some people just use AA as the test. In the example, $\triangle YWR \mathrel{\vert\vert\vert} \triangle NJZ$ (AAA).	
Side/side/side (SSS) The three corresponding sides of the triangles are in the same ratios. In the example, $\triangle QSX \mathrel{\vert\vert\vert} \triangle AGB$ (SSS).	
Side/angle/side (SAS) Two corresponding sides of the triangles are in the same ratio and the enclosed angles are equal. In the example, $\triangle IMT \mathrel{\vert\vert\vert} \triangle CVR$ (SAS).	
Right angle/hypotenuse/side (RHS) The hypotenuse and a corresponding side are in the same ratio in two right-angled triangles. In the example, $\triangle SKE \mathrel{\vert\vert\vert} \triangle FBW$ (RHS).	

In proving that the ratios of corresponding sides are equal it doesn't matter whether you use the ratios between corresponding sides within the triangles or corresponding sides between the triangles (the scale factor).

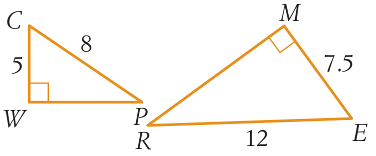
Example 11

Prove similarity for any of the following pairs of triangles which are similar.

a



b



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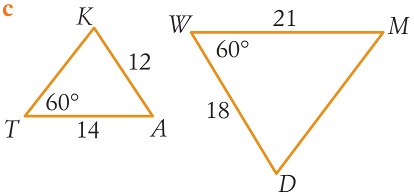


Solution

a It might be SSS. The scale factor is the easiest for this one. **$RP : PS : SR = 36 : 24 : 16$**

Try multiplying by the scale factor $\frac{3}{4}$. **$= 27 : 18 : 12$**

Replace with the side names. **$= AM : MN : NA$**

State the result and the test used. **$\triangle RPS \mathrel{|||} \triangle AMN$ (SSS)**

b It might be RHS. Do the angle first. **$\angle PWC = \angle MER = 90°$**

Do the ratio for the first triangle. **$WC : CP = 5 : 8$**

Simplify the ratio for the other triangle. **$ME : ER = 7.5 : 12 = 15 : 24 = 5 : 8$**

State the ratio equality. **$WC : CP = 5 : 8 = ME : ER$**

State the result and the test used. **$\triangle PWC \mathrel{|||} \triangle MER$ (RHS)**

c It might be SAS. **In $\triangle KAT$, the 60° angle is opposite a known side, but in $\triangle WMD$ it is between the known sides.**

The positions are not corresponding. **The triangles are not necessarily similar.**

In part **a** of Example **11** you could compare the ratios of the sides within the triangles. For $\triangle RPS$ the ratios are $\frac{RP}{PS} = \frac{36}{24} = 3 : 2$, $\frac{RS}{SP} = \frac{16}{24} = 2 : 3$ and $\frac{RS}{RP} = \frac{16}{36} = 4 : 9$ while for $\triangle AMN$ the ratios are $\frac{AM}{NM} = \frac{27}{18} = 3 : 2$, $\frac{AN}{MN} = \frac{12}{18} = 2 : 3$ and $\frac{AN}{NM} = \frac{12}{27} = 4 : 9$ so the ratios of corresponding sides are equal. In fact, two comparisons for each triangle are enough, because then the third must be the same anyway.

In part **b**, you could also use the scale factor for the sides, which works out to be $1\frac{1}{2}$.

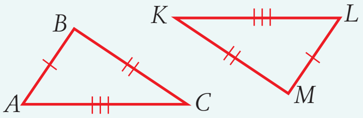
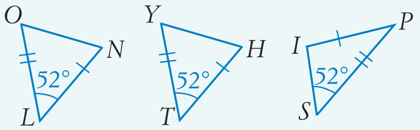
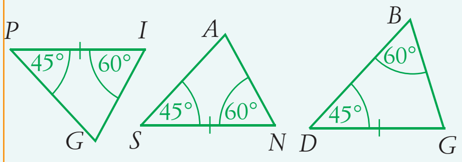
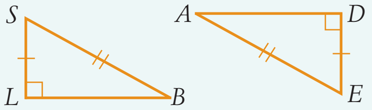
Notice in Example **11** that the triangle names are written with the angles in the same order.

There are four tests for congruent triangles.

TLF Learning object
Congruent triangles (L3517)
MAT10MGIN00008

Important!

Triangle congruency tests

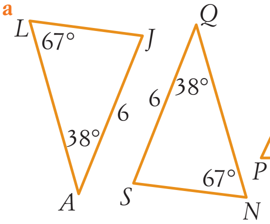
Relationship	Example
Side/side/side (SSS) The three corresponding sides are equal.	 $\triangle ABC \equiv \triangle LMK$ (SSS)
Side/angle/side (SAS) Two sides and the *enclosed* angle are equal. (The angle must be between the sides.)	 $\triangle OLN \equiv \triangle YTH$ (SAS)
Angle/side/angle (ASA) or angle/angle/side (AAS) Two angles and the *corresponding* side are equal. (The side must be in the same position in relation to the angles in each triangle.)	 $\triangle PIG \equiv \triangle SNA$ (ASA)
Right angle/hypotenuse/side (RHS) The hypotenuse and a corresponding side are equal in two right-angled triangles.	 $\triangle SBL \equiv \triangle EAD$ (RHS)

Note that, in the second example above, $\triangle IPS$ may not be congruent with $\triangle OLN$ and $\triangle YTH$ because the sides do not enclose the angle. In the third example, $\triangle BGD$ has the known side in a different position relative to the angles, compared with $\triangle PIG$ and $\triangle SNA$, so we can't say that $\triangle BGD$ is congruent with either of the other triangles.

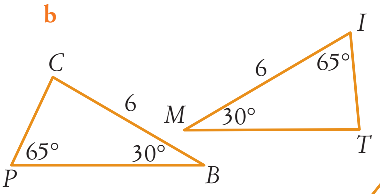
Example 12

Prove congruence for any of the following pairs of triangles that are congruent.

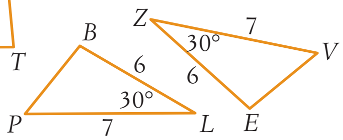
a



b



c



Solution

a It looks like ASA. Do an angle. $\angle JLA = 67° = \angle SNQ$

Do the side. $AJ = 6 = QS$

Do the other angle. $\angle LAJ = 38° = \angle NQS$

State the result and the test used. $\triangle JLA \equiv \triangle SNQ$ **(ASA)**

b In $\triangle PCB$ 6 is opposite 65°, but in $\triangle JMT$ it is opposite 85°. **Two angles are equal but the corresponding side is not. The triangles are not necessarily congruent.**

c It looks like SAS. Do a side. $BL = 6 = EZ$

Do the angle. $\angle BLP = 30° = \angle EZV$

Do the other side. $LP = 7 = ZV$

State the result and the test used. $\triangle BLP \equiv \triangle EZV$ **(SAS)**

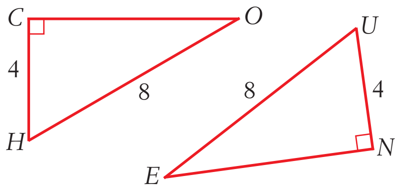
In Example **12**, the triangles in part **b** *could* be congruent but we cannot be sure they are because we cannot prove it. It would be equally wrong to say they are not congruent as to say they are congruent because we cannot prove it either way.

In geometry, you can use **logical reasoning** to show that certain things are true. This involves making statements supported by reasons for making these statements. The reason is shown in brackets to the right of the statement. The conclusion of logical reasoning is a **deduction** based on the reasons or argument that has been developed.

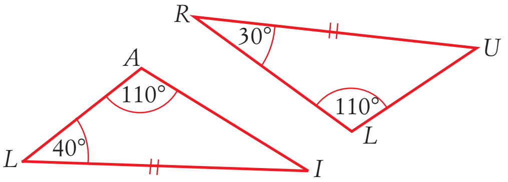
Example 13

Use logical reasoning to determine if the following pairs of triangles are congruent.

a



b



Solution

a Compare the sides and angles in each triangle.

$\angle HCO = \angle UNE$ **(both equal 90°)**

$HO = UE$ **(given)**

$HC = UN$ **(given)**

The hypotenuse and a corresponding side in the two right-angled triangles are equal, so the RHS congruency test is satisfied. $\triangle HCO = \triangle UNE$ **(RHS)**

b Compare the sides and angles in each triangle.

$\angle IAL = \angle RLU$ (given)

$\angle ALI = \angle LUR = 40°$ (angles in a $\triangle$)

$LI = UR$ (given)

Two angles and a corresponding side are equal, so the ASA congruency test is satisfied.

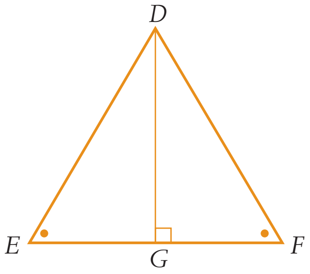
$\triangle ALI = \triangle LUR$ **(AAS)**

Example 14

In this diagram $\angle DEF = \angle DFE$ and $DG \perp EF$.

a Use logical reasoning to show that $\triangle DEG \equiv \triangle DFG$.

b Hence show that G is the midpoint of EF.



Solution

a Compare corresponding angles and sides of the triangles.

$\angle DEF = \angle DFE$ (given)

$\angle DGE = \angle DGF$ (both equal 90°)

DG **is a common side** (given)

Two angles and a corresponding side are equal, so the ASA congruency test is satisfied.

$\triangle DEG = \triangle DFG$ **(ASA)**

b EG and FG are corresponding sides of congruent triangles.

$EG = FG$ **(corresponding sides)**

Make the deduction.

G **is the midpoint of** EF**.**

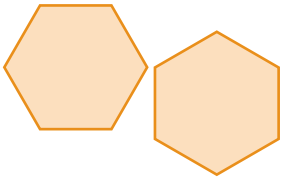
Exercise 8.2 Similarity and congruence

Understanding

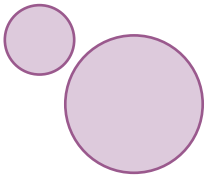
Extra Questions
Exercise 8.2
MAT10MGEQ00051
See Example 6

1 State whether each pair of shapes is congruent, similar or neither. Work out the scale factor for similar shapes.

a



b



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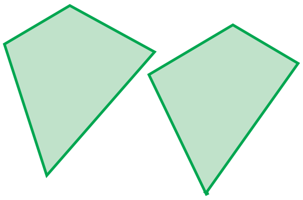
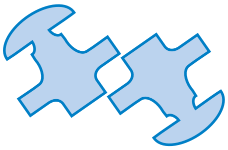


d



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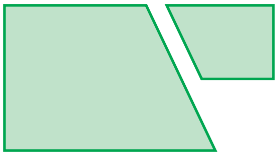

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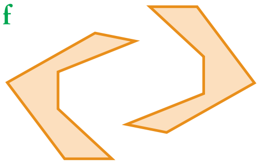
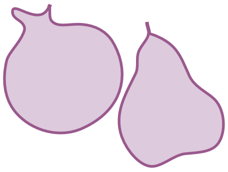


g



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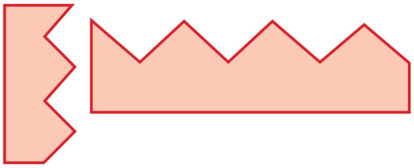

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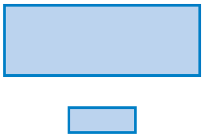
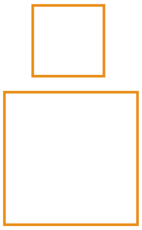


2 Use direct measurement to identify any pairs of figures that are similar.

a



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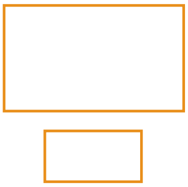

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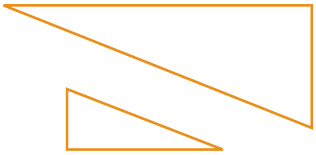

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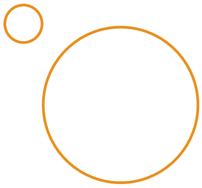

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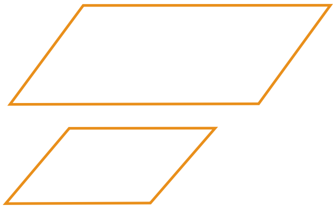

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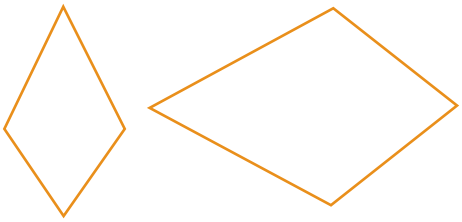
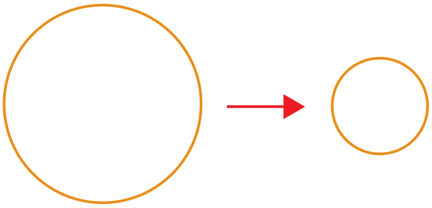


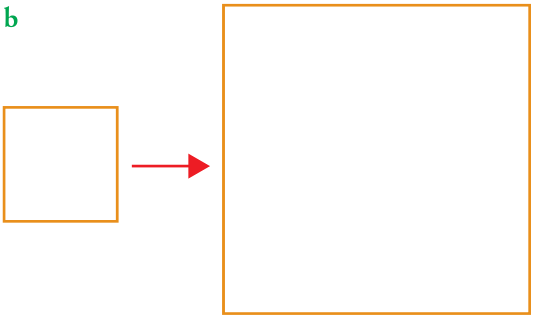
h

3 Each of the following pairs of shapes is similar. By measuring with a ruler, find the scale factor of the enlargement or reduction for each of these pairs of diagrams.

a



b



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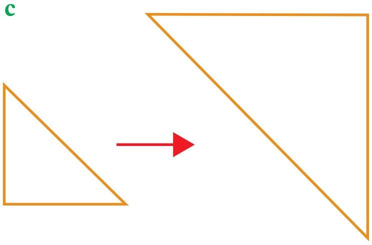
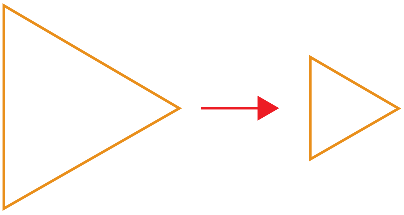


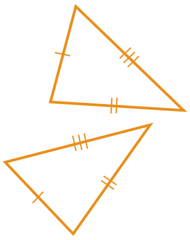
d



4 Each of the following pairs is congruent. Which congruence test (SSS, SAS, AAS or RHS) does each pair of triangles satisfy?

See Example 12

a



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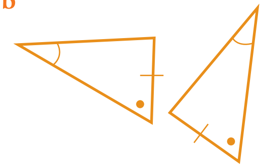

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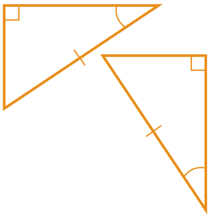

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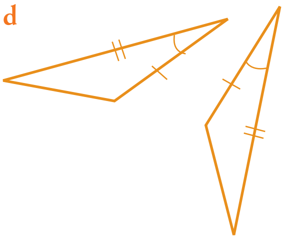

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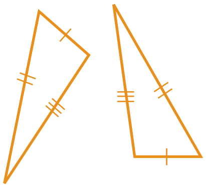

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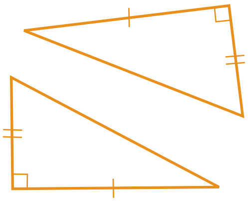

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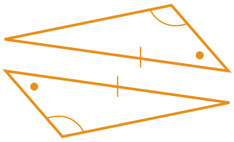

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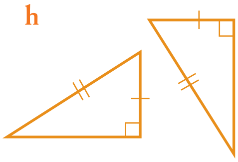

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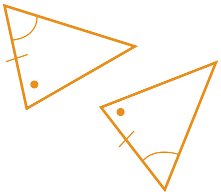
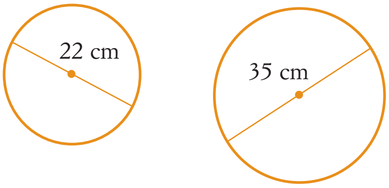


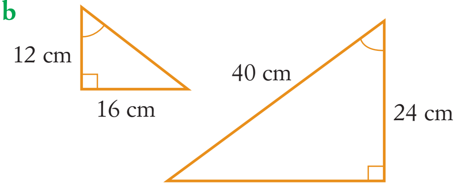
Fluency

5 Without measuring angles or sides, determine if the figures in each of the following pairs are similar.

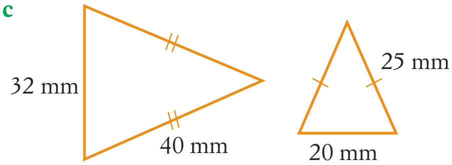
a



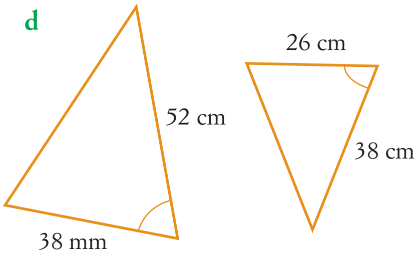
b



c



d



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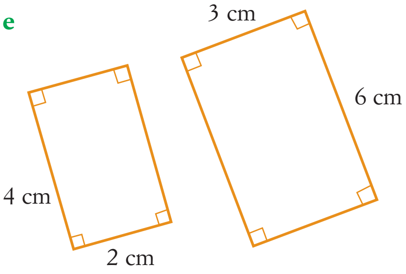
e



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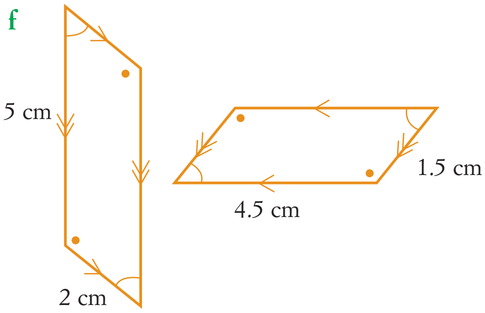
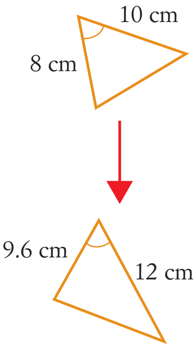


6 For each case below, determine if the shapes are similar. For similar shapes, calculate the scale factor.

a



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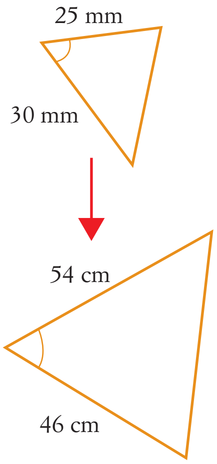

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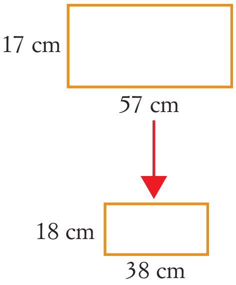

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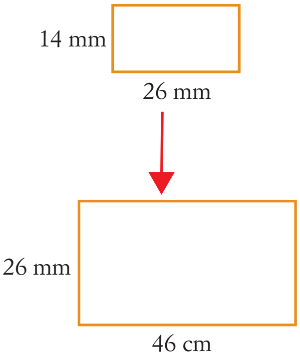


7 **a** Draw $\triangle ABC$ with $AB = 60$ mm, $BC = 80$ mm and $CA = 40$ mm.

b Draw $\triangle HQP$ with $HQ = 90$ mm, $QP = 120$ mm and $HP = 60$ mm.

c Are the matching sides of the triangles you drew in parts **a** and **b** in the same ratio?

d Are the matching angles of the triangles equal?

e Are $\triangle ABC$ and $\triangle HQP$ similar? Why?

8 **a** Draw $\triangle DAF$ with $\angle D = 70°$, $\angle F = 50°$, and $DF = 60$ mm.

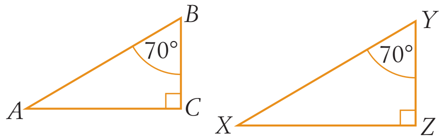
b Now draw $\triangle KUT$ with $\angle K = 70°$, $\angle T = 50°$, and $KT = 90$ mm.

c Is $\triangle DAF$ similar to $\triangle KUT$? Why?

9 State whether each pair of triangles below is similar. For pairs of similar triangles, state the similarity test used.

See Example 11

a



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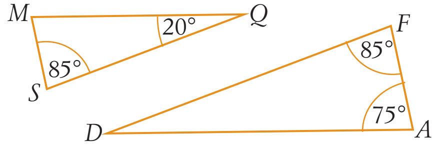

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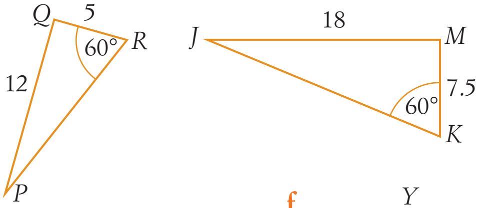
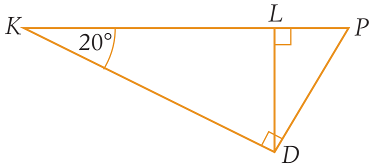


d



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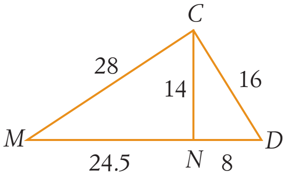

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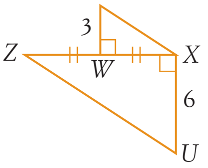
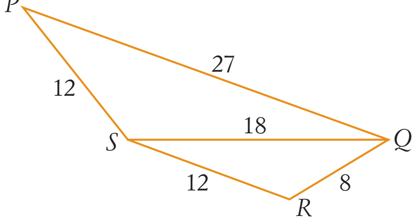
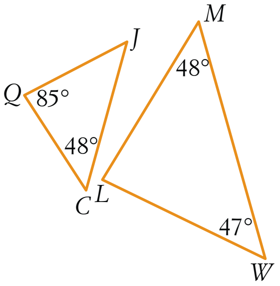


g

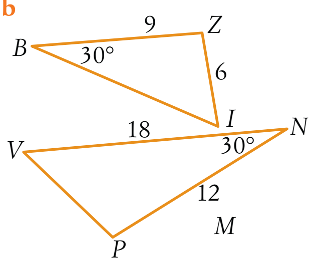


10 Prove similarity for any of the following pairs of triangles which are similar.

a



b



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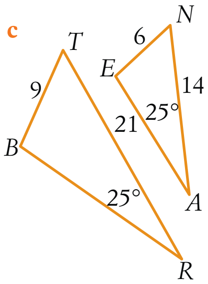

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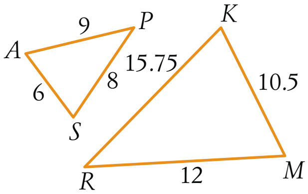

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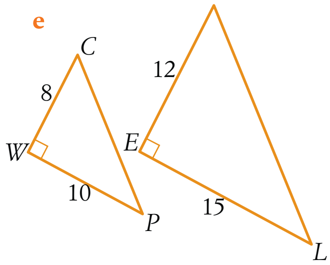

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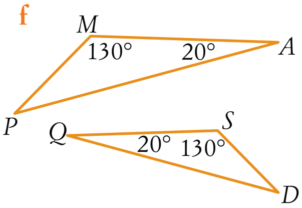

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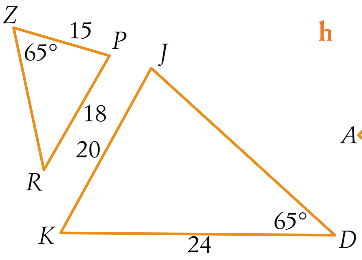

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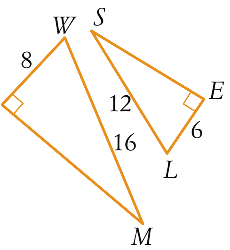
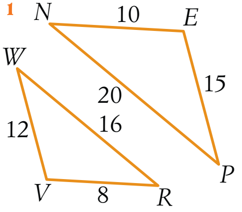
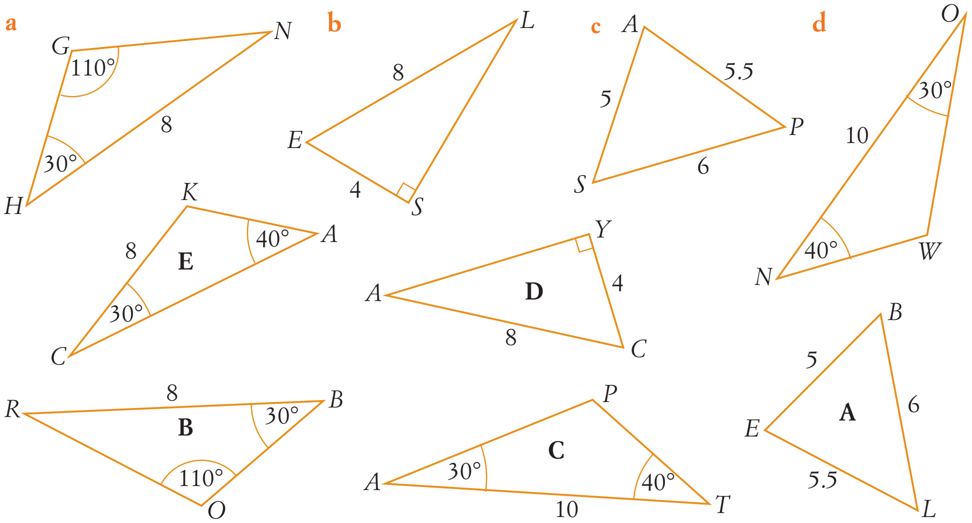


i



See Examples 12, 13

11 Use logical reasoning to match each of triangles **a** to **d** with a congruent triangle from **A** to **E** below. State which test (SSS, SAS, AAS or RHS) you used to prove them congruent.



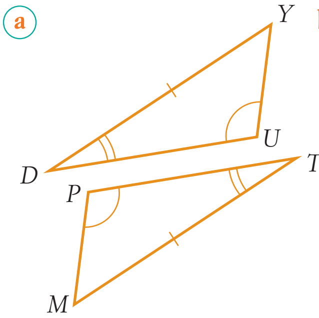
Worked solutions

Exercise 8.2

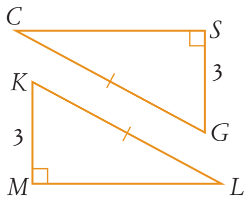
MAT10MGWS00056

12 Use logical reasoning to determine if the following pairs of triangles are congruent, include the congruency test used.

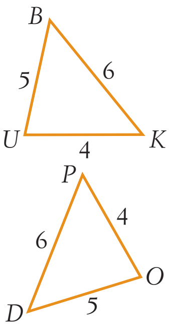
a



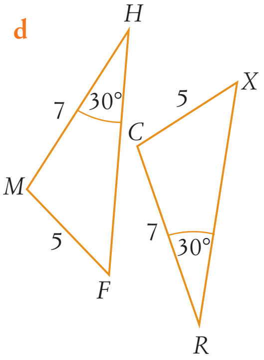
b



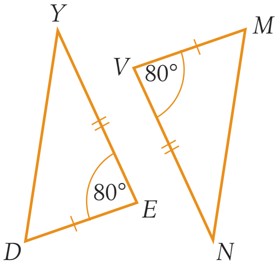
c



d



e



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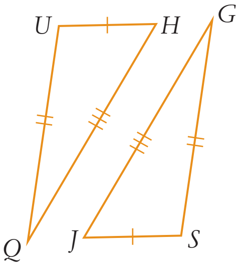

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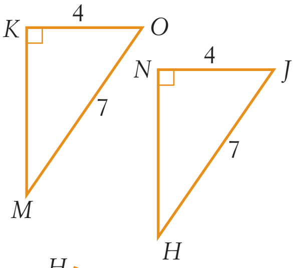

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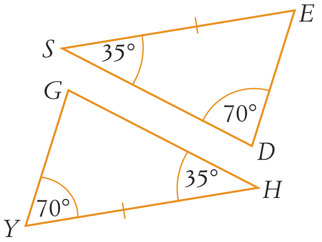

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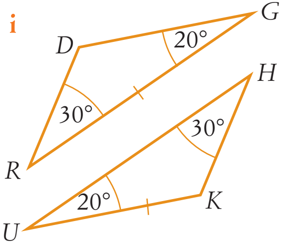
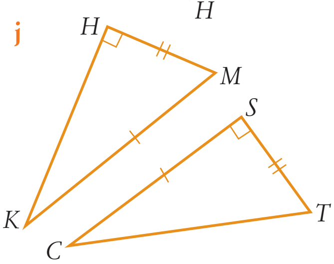
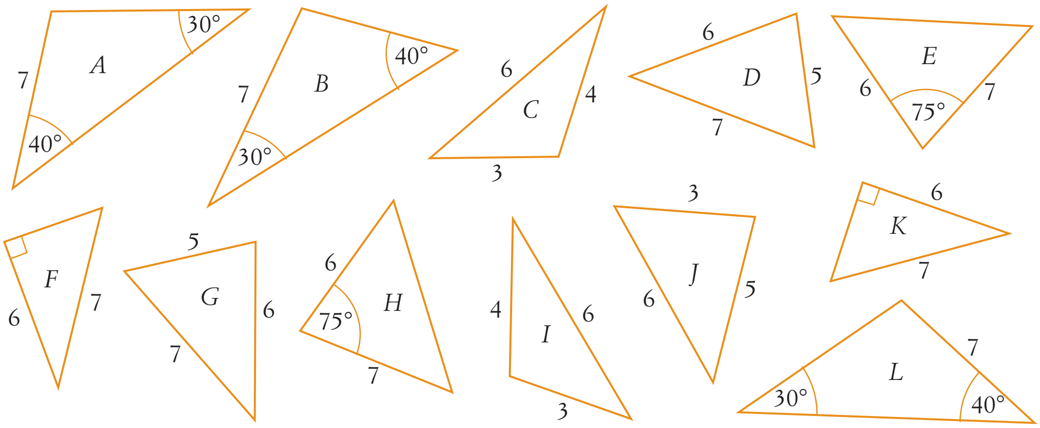


j



13 State which of the following pairs of triangles are congruent, include the congruency test used.



Problem solving

14 Answer true (T) or false (F) to each of the following statements.

a All squares are similar.

b The sides of similar figures are in the same ratio.

c All rhombuses are similar.

d All circles are similar.

e The angles of similar figures are equal.

f All parallelograms are similar.

g Any two isosceles triangles are always similar.

h All equilateral triangles are similar.

i All hexagons are similar.

j Since the angles of two rectangles are equal, they must be similar.

k All right-angled triangles are similar.

l All regular octagons are similar.

m If two triangles have two pairs of matching angles equal, they are similar.

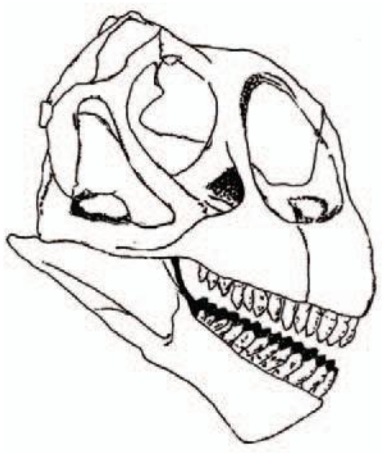
See Example 7

15 This is a scale drawing of the skull of a dinosaur called *Camarasaurus lentus* – a large herbivore that reached a length of about 15 m. The scale used is 1 : 25. Use the scale drawing to answer the following questions.

a What is the actual width of the dinosaur's skull?

b What is the actual height of the dinosaur's skull?

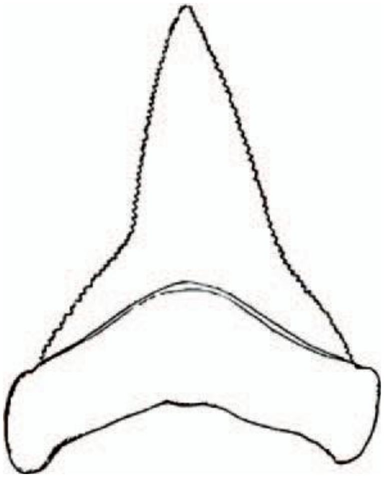
c What is the length of the longest tooth that is visible in the jaw of the dinosaur?



16 This is a scale drawing of the tooth of a shark. The scale used is 4 : 1. Use the scale drawing to answer the following questions.

a What is the actual width of the shark's tooth?

b What is the actual height of the shark's tooth?

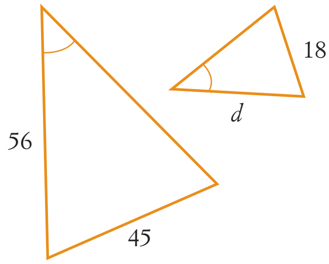


17 A photograph measures 105 mm wide by 60 mm high. An enlargement has a height of 20 cm. What is its width?

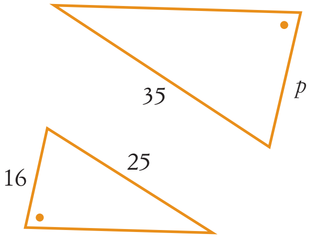
See Examples 8, 9

18 The pairs of figures below are similar. Calculate the values of the pronumerals.

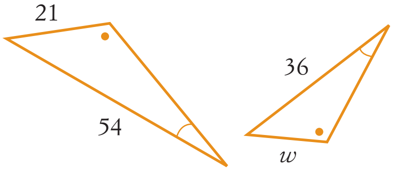
a



b



c



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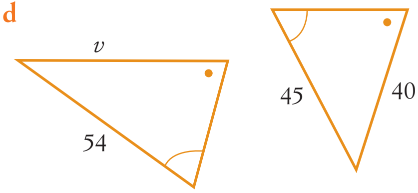

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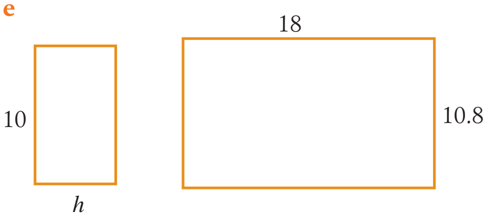
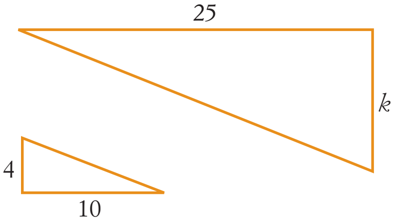
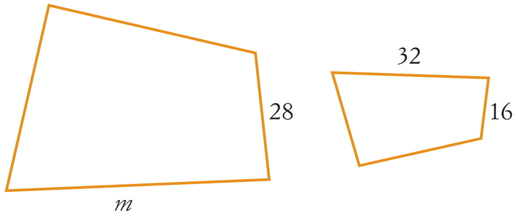


f



g



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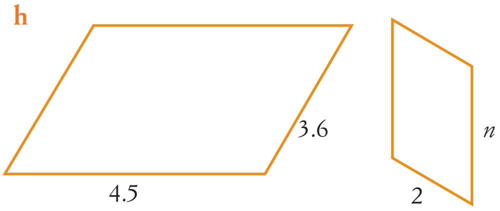
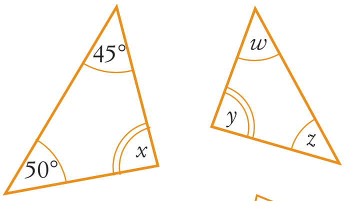
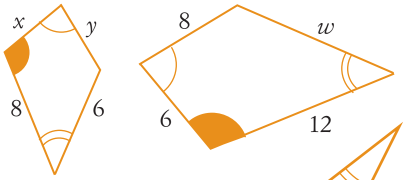


19 The pairs of figures below are similar. Calculate the values of the pronumerals.

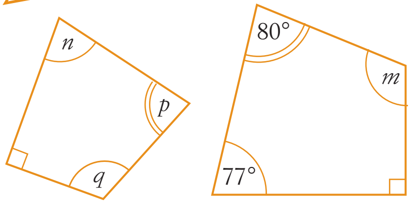
a



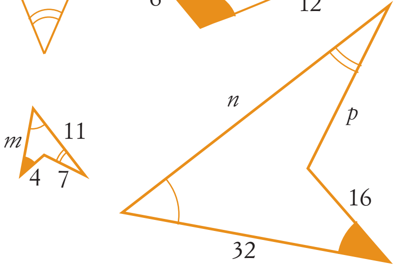
b



c

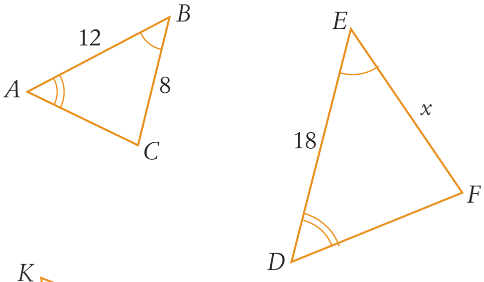


d

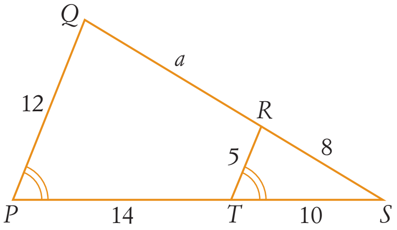


20 Use logical reasoning to find the unknowns in each of the following, giving reasons for each step.

a



b



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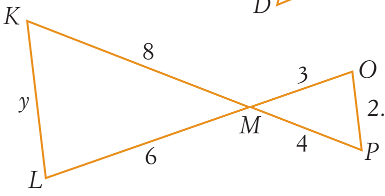

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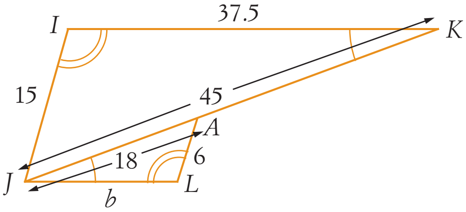
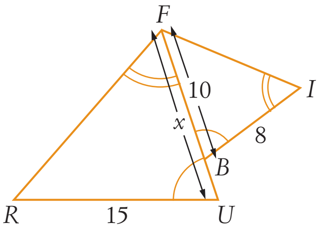


e



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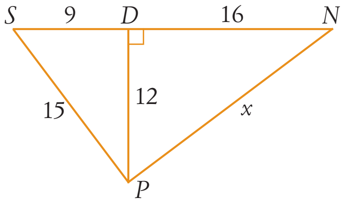
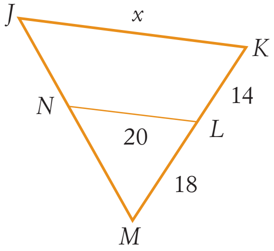
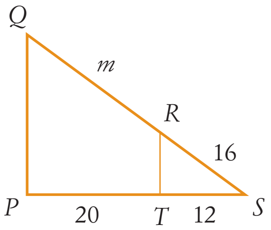


21 △MLN is similar to △MKJ. Find x, correct to one decimal place.



Worked solutions
Exercise 8.2
MAT10MGWS00056

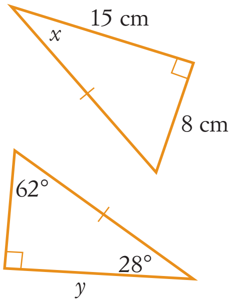
㉒ △QSP is similar to △RST. Find m, correct to one decimal place.



Worked solutions
Exercise 8.2
MAT10MGWS00056

23 For these pairs of congruent triangles, find the value of each pronumeral.

ⓐ



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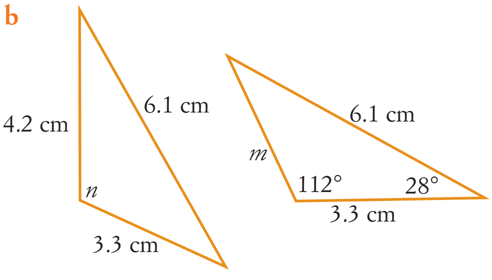

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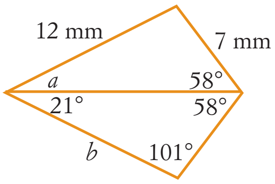

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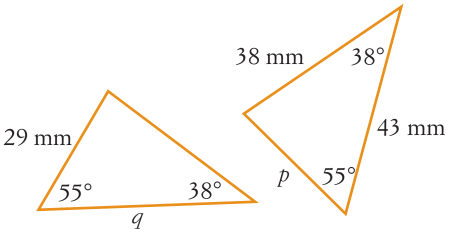

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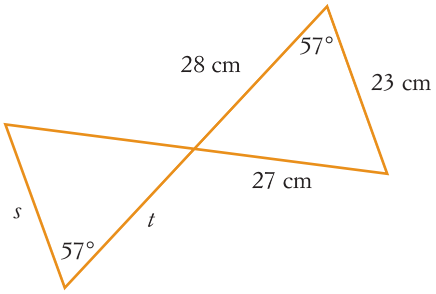

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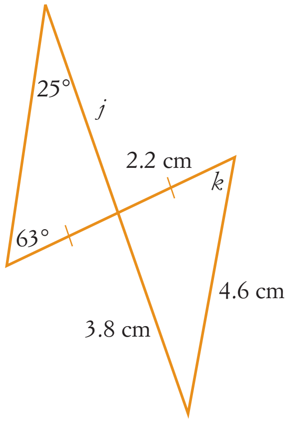
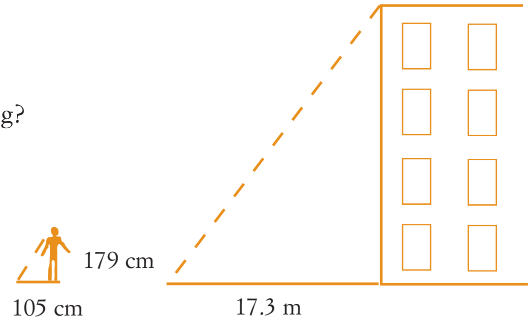


24 At 1:30 p.m., Jose casts a shadow 105 cm long. At the same time a building next to him casts a shadow 17.3 m long. If Jose is 179 cm tall, what is the height of the building?

See Example 10

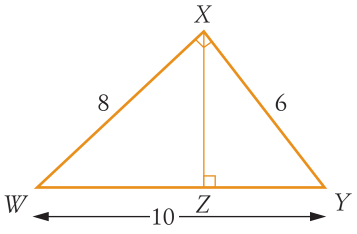


25 A right-angled triangle has sides that measure 14 cm, 48 cm and 50 cm. A similar triangle has a hypotenuse that measures 20 cm. Find the length of its shortest side.

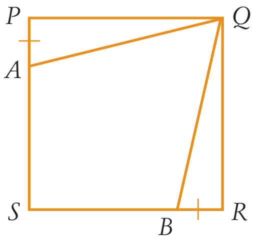
Reasoning

See Example 14

26 Use logical reasoning to show that the three triangles *WXZ*, *WYX* and *XYZ* are similar.



27 *PQRS* is a square and $PA = RB$. Use logical reasoning to show that $QA = QB$.



28 Donna held a ruler vertically 60 cm away from her face and found that, when she was 3.4 m from her friend, she could line up the top of the 30 cm ruler with the top of the friend's head and the bottom of the ruler with the friend's feet. How tall (to the nearest centimetre) is the friend?

29 A tree casts a shadow 18 m long. At the same time, a 3 m stick held vertically casts a shadow of 4 m. How high is the tree?

8.3 Geometric problems and proofs

Since the angles of a triangle add up to 180°, we can easily find the sums of angles of other polygons by dividing the polygon into triangles.

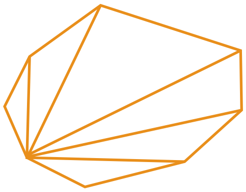
Example 15

Weblink
Interior angles of polygons
MAT10MGWB00008

Find the sum of the internal angles of an octagon.

Solution

Draw an octagon and divide the inside into triangles that have the same vertices as the octagon.



Count the triangles. **Octagon angles = 6 triangles**

Work out the total. **= 6 × 180°**

= 1080°

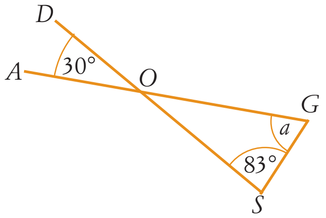
Write the answer. **The angles of an octagon add up to 1080°.**

In problems like the following examples, there is often more than one way to find the solution. In some cases, you will need to use more than one rule in steps to find a value. Sometimes you may have to use algebra.

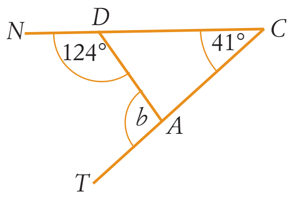
Example 16

Work out the angles shown by letters below. Give reasons for each step in words and include the code.

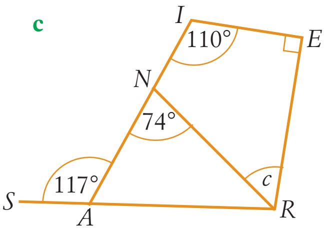
a



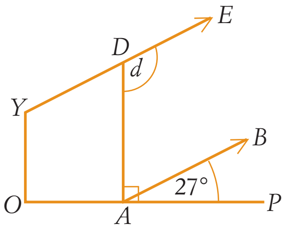
b



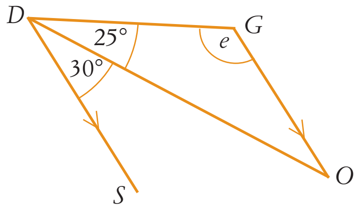
c



d



e



Solution

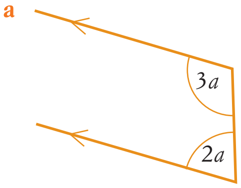
a	∠*GOS* is vertically opposite 30°.	$\angle \boldsymbol{GOS} = \mathbf{30^\circ}$
	The angle sum of a triangle is 180°.	$\boldsymbol{a} = \mathbf{180^\circ - (83^\circ + 30^\circ)}$ $= \mathbf{67^\circ}$
b	The exterior angle is the sum of the interior opposite angles.	$\mathbf{124^\circ = 41^\circ} + \angle \boldsymbol{DAC}$ $\angle \boldsymbol{DAC} = \mathbf{83^\circ}$
	b is supplementary with ∠*DAC* = 83°.	$\boldsymbol{b} + \mathbf{83^\circ = 180^\circ}$ $\boldsymbol{b} = \mathbf{97^\circ}$
c	∠*INR* is supplementary with 74°.	$\angle \boldsymbol{INR} + \mathbf{74^\circ = 180^\circ}$ $\angle \boldsymbol{INR} = \mathbf{106^\circ}$
	The angle sum of quadrilateral *NIER* is 360°.	$\boldsymbol{c} = \mathbf{360^\circ - (106^\circ + 110^\circ + 90^\circ)}$ $= \mathbf{54^\circ}$
d	∠*BAD* is complementary with 27°.	$\angle \boldsymbol{BAD} + \mathbf{27^\circ = 90^\circ}$ $\angle \boldsymbol{BAD} = \mathbf{63^\circ}$
	d is cointerior with ∠*BAD* = 63°.	$\boldsymbol{d} + \mathbf{63^\circ = 180^\circ}$ $\boldsymbol{d} = \mathbf{117^\circ}$
e	∠*DOG* is alternate to 30°.	$\angle \boldsymbol{DOG} = \mathbf{30^\circ}$
	The angle sum of a triangle is 180°.	$\boldsymbol{e} = \mathbf{180^\circ - (25^\circ + 30^\circ)}$ $= \mathbf{125^\circ}$

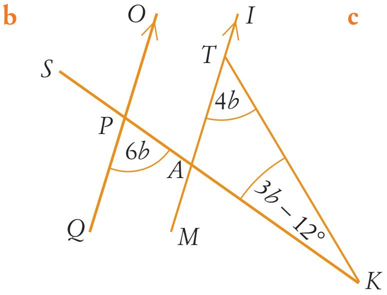
CAS TI-Nspire exercise
Geometry
MAT10MGTI00008

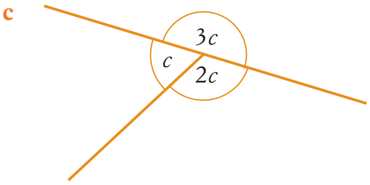
CAS ClassPad exercise
Geometry
MAT10MGCP00008

Example 17

Use algebra to find the values of the variables below. Give reasons and show your working.



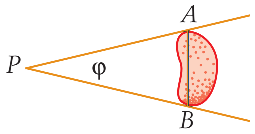


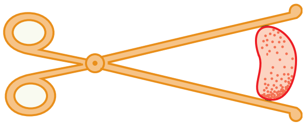


Solution

a	$3a$ and $2a$ are cointerior.	$3a + 2a = 180°$
	Simplify.	$5a = 180°$
	Divide by 5.	$a = 36°$
b	$\angle PAT$ is alternate to $6b$.	$\angle PAT = 6b$
	The exterior angle $\angle PAT = 6b$ is the sum of the interior opposite angles.	$6b = 4b + (3b - 12°)$
	Simplify.	$6b = 7b - 12°$
	Solve for b.	$b = 12°$
c	c, $2c$ and $3c$ are angles at a point.	$c + 2c + 3c = 360°$
	Simplify.	$6c = 360°$
	Divide by 6.	$c = 60°$

An angle at a point P can be **subtended** by an object. This means that the arms of the angle can be extended to just touch the ends of the object. The object is 'held' by the extended arms of the angle as if they were the arms of tongs. In the diagram below, the line AB subtends φ at P. Notice that $\varphi = \angle APB$.

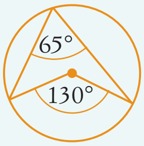
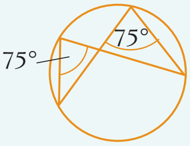
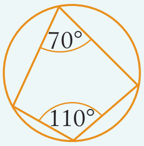
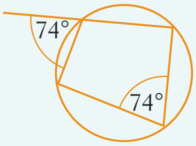
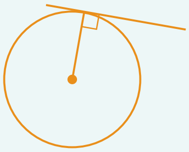




The term '**subtend**' is used in describing the relationships of angles in circles.

Important!

Angle relationships in circles

Relationship	Code	Example
The **angle subtended at the centre** of a circle is always twice the angle subtended at the circumference by the same arc or chord.		
The **diameter of a circle** subtends a right angle at the circumference.		
Angles at the circumference subtended by the same chord or arc are equal.		
A **cyclic quadrilateral** has its vertices in a circle. Its **opposite angles** add up to 180°.		
The **exterior angle of a cyclic quadrilateral** is equal to the interior opposite angle.		
The **tangent and the radius at the same point** on a circle form a right angle.		

Worksheet
Circle angle facts
MAT10MGWK00041

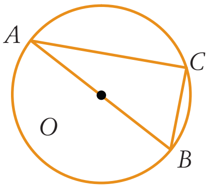
Most of the angle relationships in circles can be proved by constructing a radius or diameter to make isosceles triangles.

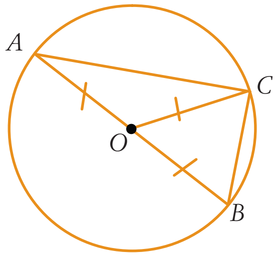
Example 18

Prove that the tangent and radius through the same point on the circumference of a circle intersect at right-angles.

Solution

Draw a diagram and state what is required.	**RTP** **For any point C on the circumference of a circle subtended by a diameter AB, $\angle ACB = 90^\circ$**





Draw an extra radius from C.	**Proof**
Show the radii as being equal.	**Construct the radius OC.**
State the isosceles triangles.	**Then $AO = BO = CO = r$**
	Thus $\triangle OAC$ is isosceles.
	Thus $\angle OAC = \angle OCA$
	Similarly $\angle OCB = \angle OBC$
Now relate the centre angles.	**But $\angle BOC = \angle OAC + \angle OCA$ (external angle of triangle)** **$= \angle OCA + \angle OCA$**
	So $\angle BOC = 2\angle OCA$
	Similarly $\angle AOC = 2\angle OCB$
Add the angles.	**Thus $\angle AOC + \angle BOC = 2\angle OCA + 2\angle OCB$**
Factorise.	**$= 2(\angle OCA + 2\angle OCB)$**
Now state the sums.	**But $\angle AOC + \angle BOC = 180^\circ$ (Straight line) and $\angle OCA + 2\angle OCB = 2\angle ACB$**
Substitute the sums.	**Thus $2\angle ACB = 180^\circ$**
Divide by 2.	**Thus $\angle ACB = 90^\circ$ QED**

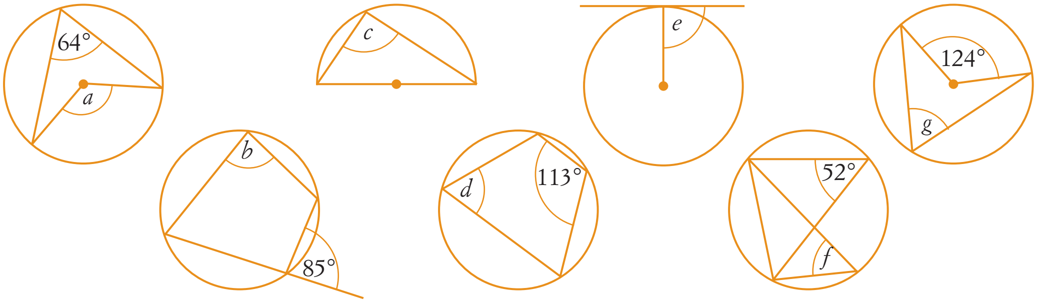
Example 19

Puzzle sheet

Finding angles in circles

MAT10MGPS00054

Work out the angles shown by letters below. Give reasons using the code.



Solution

a is subtended at the centre by the same arc as 64°.

$$a = 2 \times 64° = 128°$$

b is the interior opposite and 85° is the exterior angle of a cyclic quadrilateral.

$$b = 85°$$

c is subtended by the diameter of a circle.

$$c = 90°$$

d is opposite to 113° in a cyclic quadrilateral.

$$d + 113° = 180°$$
$$d = 67°$$

e is the angle between a tangent and radius at the same point.

$$e = 90°$$

f is subtended by the same chord as 52°.

$$f = 52°$$

g is subtended at the circumference by the same arc as 124°.

$$124° = 2 \times g$$
$$g = 62°$$

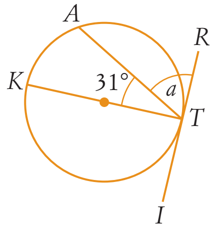
In the case of circle relationships, you may need to use more than one rule in steps, or algebra, to find a value, just as you did in previous types of problems.

Example 20

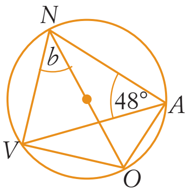
Video tutorial
Circle geometry
MAT10MGVT10020

Work out the angles shown by letters below. Give reasons for each step and include the code.

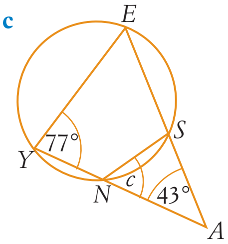
a



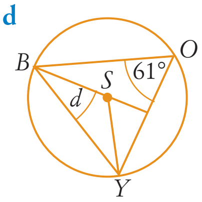
b



c



d



Solution

a ∠*KTR* is the angle between a tangent and radius.

$\angle KTR = 90^\circ$

a is complementary with 31°.

$a + 31^\circ = 90^\circ$

$a = 59^\circ$

b ∠*NAO* is subtended by the diameter of a circle.

$\angle NAO = 90^\circ$

∠*VAO* is complementary with 48°.

$\angle VAO + 48^\circ = 90^\circ$

$\angle VAO = 42^\circ$

b is subtended by the same chord as ∠*VAO*.

$b = 42^\circ$

c ∠*ASN* is the exterior angle of a cyclic quadrilateral.

$\angle ASN = 77^\circ$

The angle sum of a triangle is 180°.

$c = 180^\circ - (77^\circ + 43^\circ)$

$= 60^\circ$

d ∠*BSY* is subtended at the centre by the same chord as 61°.

$\angle BSY = 2 \times 61^\circ$

$= 122^\circ$

BS = *SY* = radius.

△*BSY* is isosceles.

d is a base angle in an isosceles triangle, which adds to 180°.

$2d + 122^\circ = 180^\circ$

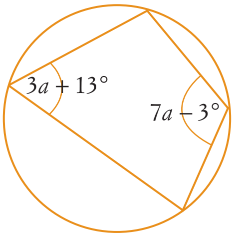
$d = 29^\circ$

Example 21

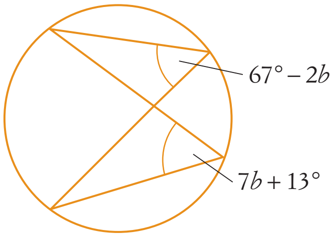
Worksheet
Angle problems with algebra
MAT10MGWK00040

Use algebra to find the values of the variables below. Give reasons and include the code.

a



b



Solution

a $3a + 13°$ and $7a - 3°$ are opposite angles in a cyclic quadrilateral. $\quad \mathbf{3a + 13° + 7a - 3° = 180°}$

Simplify. $\quad \mathbf{10a + 10° = 180°}$

Subtract 10° from both sides. $\quad \mathbf{10a = 170°}$

Divide by 10. $\quad \mathbf{a = 17°}$

b $67° - 2b$ is subtended by the same arc as $7b + 13°$. $\quad \mathbf{67° - 2b = 7b + 13°}$

Add $2b - 13°$ to both sides. $\quad \mathbf{54° = 9b}$

Reverse the equation and divide by 9. $\quad \mathbf{b = 6°}$

You have seen how logical reasoning can be used to deduce results in geometry. This is part of an area of study known as **deductive geometry**. In deductive geometry there is a standard and structured approach to logical reasoning known as **geometric proof**.

Important!

Geometric proof

A geometric proof is set out under three headings as follows.
To prove: A statement of what must be proved.
Construction: A diagram with any extra lines drawn in.
Proof: The logical steps, with the reasons for each step.
At the end of the proof the initials **QED** are added, to show that the proof is complete. These are an abbreviation for *quod erat demonstrandum*, which is Latin for 'which was to be proved'.

In the following examples, triangle congruency is used with the angle properties you have already studied. Construction lines are added to the diagram wherever appropriate.

Video tutorial

Congruent triangles proofs

MAT10MGVT10019

Animated example

Geometric problems and proofs

MAT10MGAE00008

Example 22

Prove that, if the diagonal of a quadrilateral bisects both angles, then the figure is a kite.

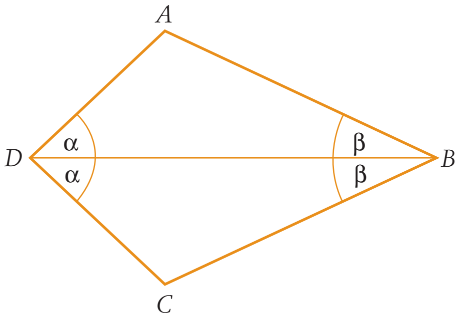
Solution

Draw a labelled sketch that can be used to represent the situation.
State the way that it is to be drawn.

The sketch will need to show a quadrilateral with a diagonal drawn in. The diagonal needs to bisect the angles of the quadrilateral.

Construction

Draw quadrilateral *ABCD* with the diagonal *BD*.
Show the angles as being bisected.



Re-state what has to be proved using the labelled diagram.

To prove

If the diagonal *BD* of a quadrilateral *ABCD* bisects angles *B* and *D*, then *ABCD* is a kite.

A kite has two pairs of equal adjacent sides. This means you have to prove that $AD = CD$ and $AB = CB$. This will be true if $\triangle DAB \equiv \triangle DCB$.

Proof

$\angle D$ is bisected.	**$\angle ADB = \angle CDB$ (given)**
BD is a common side.	**$BD = BD$ (common side)**
$\angle B$ is bisected.	**$\angle ABD = \angle CBD$ (given)**
Use ASA to prove that the triangles are congruent.	**So $\triangle DAB \equiv \triangle DCB$ (ASA)**
Corresponding sides are equal.	**$AD = DC$ and $AB = CB$**
ABCD has two pairs of adjacent sides equal.	***ABCD* is a kite. QED**

9780170194860

Example 23

Prove that the chord of a circle that is perpendicular to a radius is bisected by the radius.

Solution

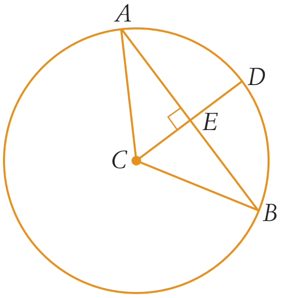
Construction

Draw a diagram to show all the information.

Draw a circle with centre *C* and draw a chord *AB*.

Show that the radius and chord are perpendicular by a right angle at E.

Draw a radius *CD* perpendicular to *AB* and name the intersection *E*.



To prove

Re-state what has to be proved using the labelled diagram.

If a chord *AB* is perpendicular to a radius *CD*, then the chord is bisected by the radius.

We need to prove that AB is bisected, so we need to show that $AE = EB$. This will be true if $\triangle AEC \equiv \triangle BEC$.

Proof

$AB \perp CD$	$\angle AEC = \angle BEC = 90°$ **(given)**
AC and BC are radii of the same circle.	$AC = BC$ **(both are radii)**
EC is a common side.	$EC = EC$ **(common side)**
Use RHS to prove that the triangles are congruent.	$\triangle AEC \equiv \triangle BEC$ **(RHS)**
Corresponding sides are equal.	$AE = EB$
	The chord *AB* is bisected. QED

Exercise 8.3 Geometric problems and proofs

Understanding

Extra questions
Exercise 8.3
MAT10MGEQ00052

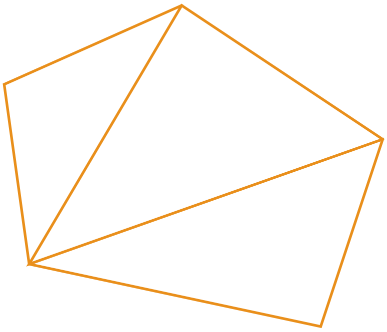
See Example 15

1 **a** Draw a pentagon and draw all the possible diagonals from one vertex.

b How many diagonals did you draw?

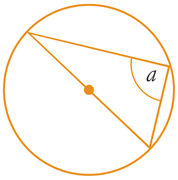
c How many triangles did you form?

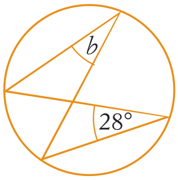
d Calculate the sum of the interior angles of a pentagon by finding the total angle sum of the triangles.

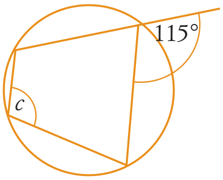


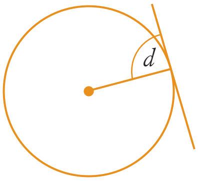
See Example 19

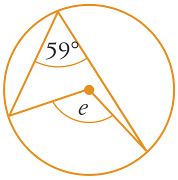
2 Work out the angles shown by letters below. Give reasons for each step in words and include the code.

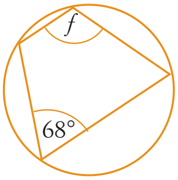


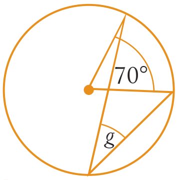


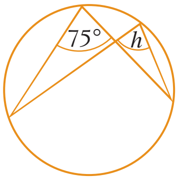


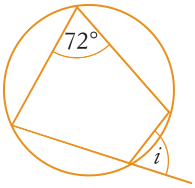


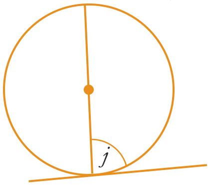




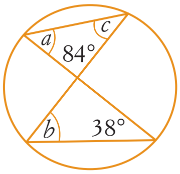


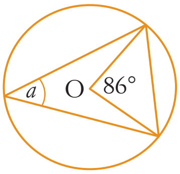


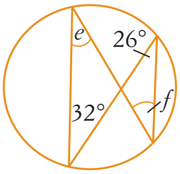


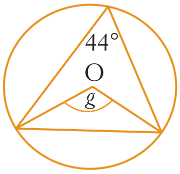


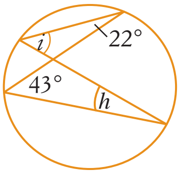
3 Work out the angles shown by letters below. Give reasons in words and include the code.

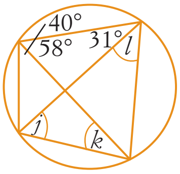










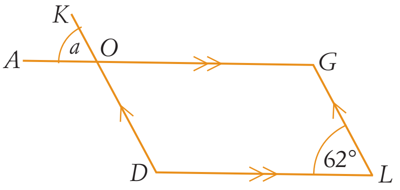


Fluency

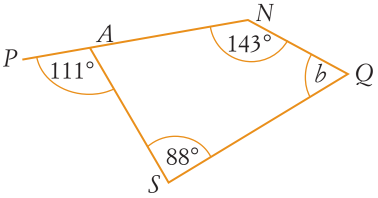
See Example 16

4 Work out the angles shown by letters below. Give reasons for each step in words and include the code.

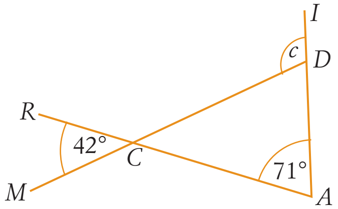
a



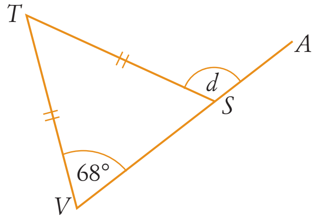
b



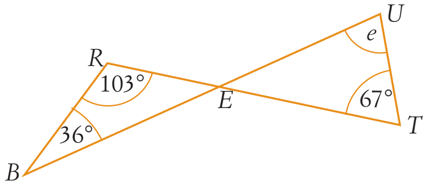
c



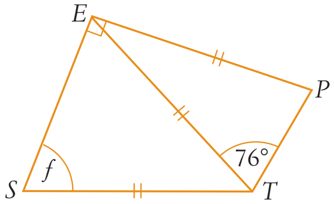
d



e



f



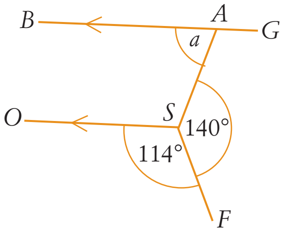
5 Work out the angles shown by letters below. Give reasons for each step in words and include the code.

Worked solutions

Exercise 8.3

MAT10MGWS00057

a



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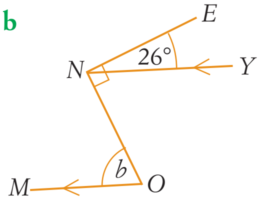

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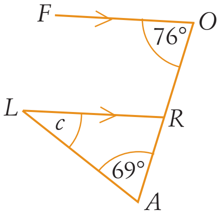

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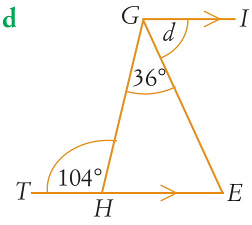
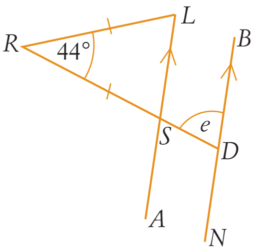
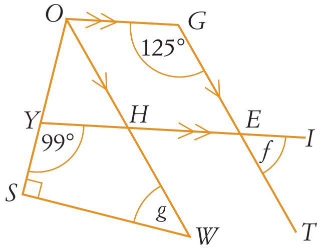


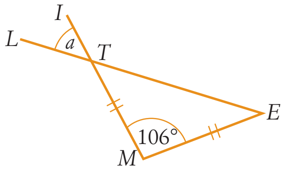
e



f



6 Work out the angles shown by letters in the following. Give reasons for each step in words and include the code.

a


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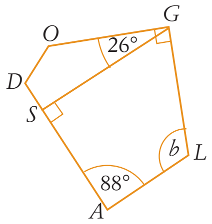

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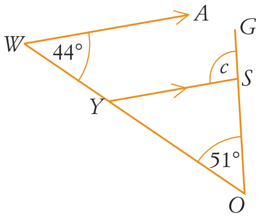

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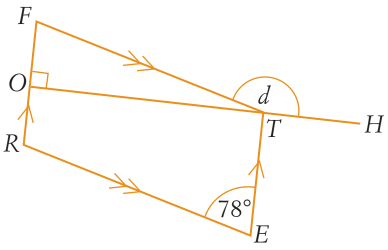
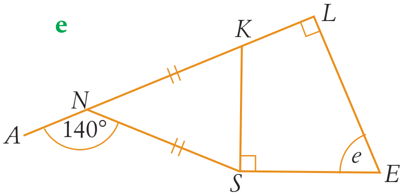


e


7 Find the sum of the internal angles of a polygon with:

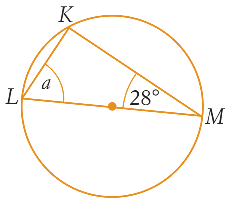
a 7 sides
b 12 sides
c 18 sides
d 21 sides
e 25 sides
f 30 sides

8 Find the number of sides for a polygon that has an angle sum of:

a 1620°
b 1980°
c 1260°
d 2520°
e 3600°
f 4140°

See Example 20

9 Work out the angles shown by letters below. Give reasons for each step in words and include the code.

a


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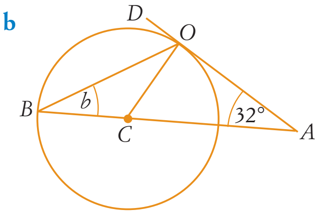
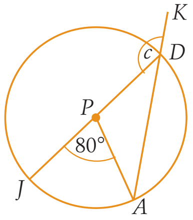
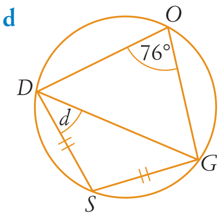
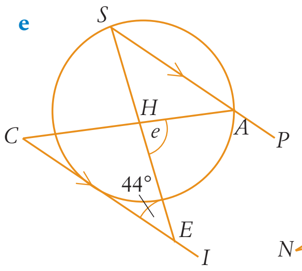
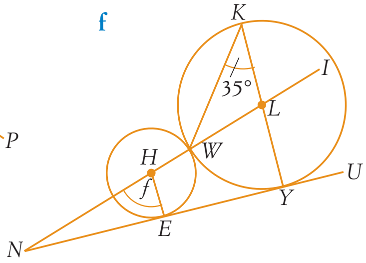


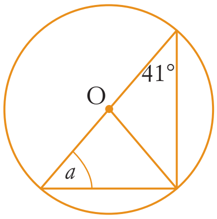
c


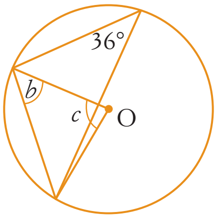
d


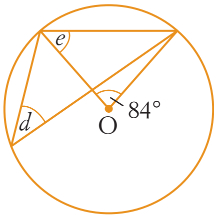
e


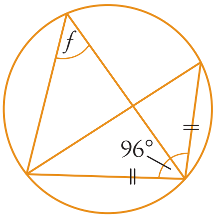
f


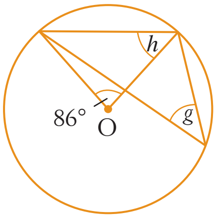
10 Work out the angles shown by letters in the following. Give reasons for each step in words and include the code.

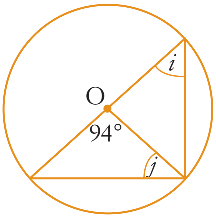


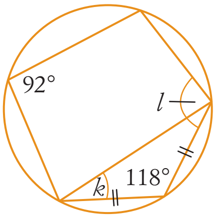


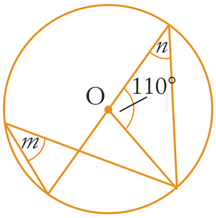


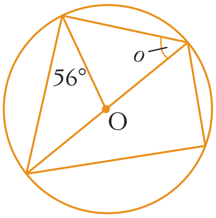








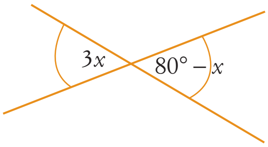


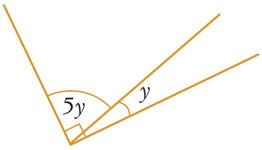


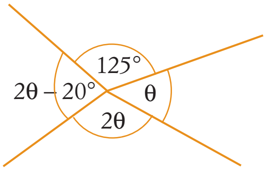
Problem solving

See Example 17

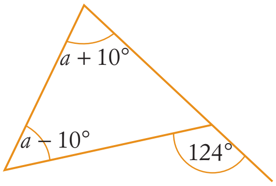
11 Use algebra to find the values of the variables below. Give reasons and show your working.

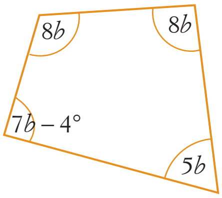




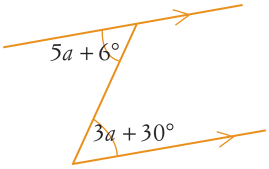


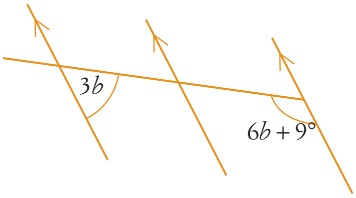
12 Use algebra to find the values of the variables below. Give reasons and show your working.

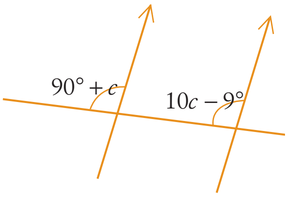




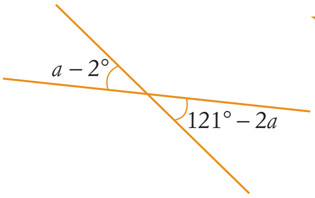
13 Use algebra to find the values of the variables below. Give reasons and show your working.

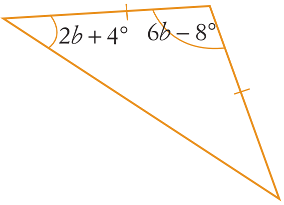


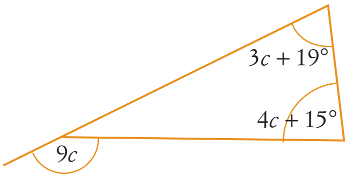


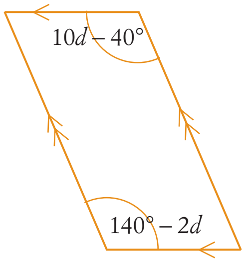


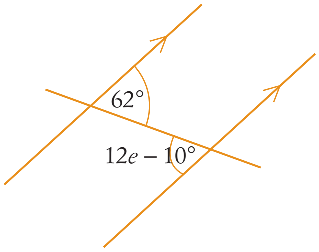
14 Use algebra to find the values of the variables below. Give reasons and show your working.

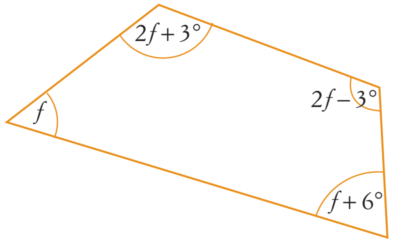












15 Find the size of each interior angle in a regular polygon that has:

a 28 sides **b** 8 sides **c** 12 sides **d** 45 sides

16 Find the number of sides of a regular polygon which has each of its interior angles equal to:

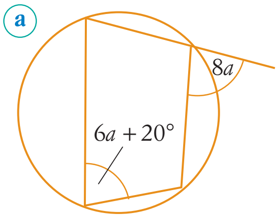
a 140° **b** 150° **c** 175° **d** 165°

Worked solutions
Exercise 8.3
MAT10MGWS00057

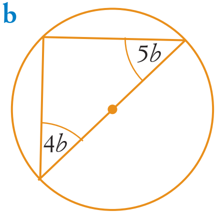
See Example 21

17 Use algebra to find the value of each variable. Give reasons and show your working.

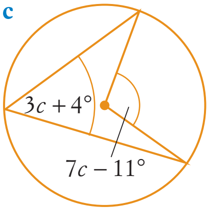
ⓐ



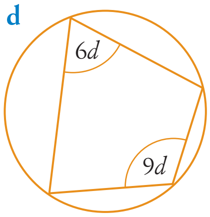
b



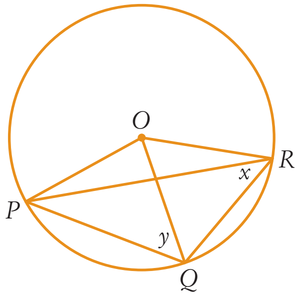
c



d

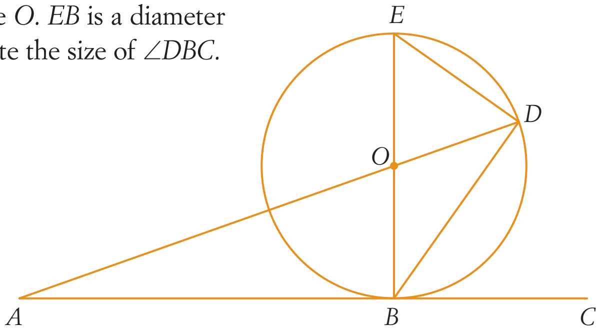


18 In the diagram, *O* is the centre of the circle.
If ∠*OPR* = 12° and ∠*QOR* = 80° calculate the values of *x* and *y*.



Worked solutions
Exercise 8.3
MAT10MGWS00057

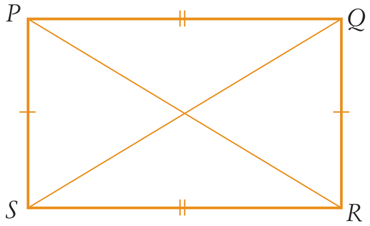
⑲ *AC* is a tangent to a circle with centre *O*. *EB* is a diameter of the circle. If ∠*OAB* = 36°, calculate the size of ∠*DBC*.



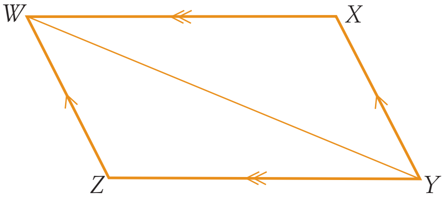
Reasoning

See Example 22

20 $PQRS$ is a rectangle. Prove that $\triangle PRS \equiv \triangle QSR$ and hence show that the diagonals of a rectangle are congruent.

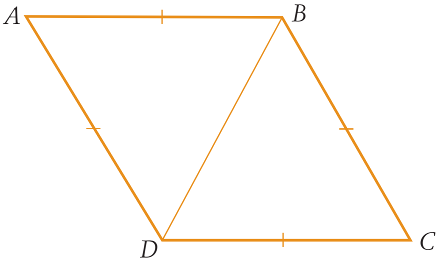


21 $WXYZ$ is a parallelogram. Prove that $\triangle WXY \equiv \triangle YZW$ and hence show that the opposite sides of a parallelogram are equal.



22 Use the results of the previous question to prove that the diagonals of a parallelogram bisect each other.

23 $ABCD$ is a rhombus. Prove that $\triangle ABD \equiv \triangle CBD$ and hence show that the opposite angles of a rhombus are bisected by the diagonal.



24 Prove that a quadrilateral with equal opposite sides is a parallelogram. (Hint: Show that alternate angles are equal to prove that the sides are parallel.)

25 Prove that the angle subtended by a chord at the centre is twice the angle subtended at the circumference of a circle by the same chord or arc.

See Example 18

26 Prove that angles subtended at the circumference of a circle by the same chord or arc are equal.

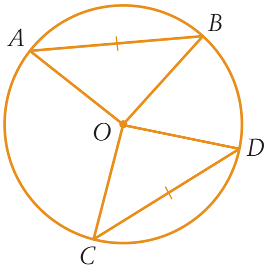
27 Prove that the perpendicular line segment from a given point to a line is to the shortest line from the point to the line (Hint: Use Pythagoras' theorem).

28 Prove the converse of question **27**.

29 Prove that the angle between a tangent and radius from the same point on the circumference of a circle is 90°.

30 If O is the centre of the circle and $AB = CD$, prove that $\angle AOB = \angle COD$.

See Example 23



31 Prove that a quadrilateral with one pair of sides equal and parallel must be a parallelogram.

32 Prove that the diagonals of a rhombus are perpendicular.

33 Prove that, if the diagonals of a quadrilateral bisect each other, then it must be a parallelogram.

34 Prove that the opposite sides of a parallelogram are equal.

35 Prove that the diagonals of a parallelogram bisect each other. (Use the result of question **34**.)

36 Prove that a parallelogram with equal diagonals must be a rectangle. (Use the result of question **35**.)

37 Prove that the diagonal between the unequal angles of a kite bisects the other diagonal.

Chapter 8 summary

Quiz

Geometry

MAT10MGQZ00008

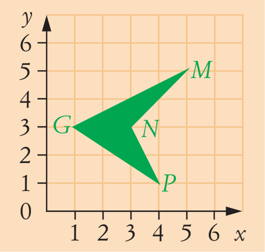
- **Plane shapes** or **two-dimensional (2D) figures** can be drawn on a flat surface.
- A **transformation** changes the position or appearance of a shape. Coordinates can be used to show a transformation. A **translation** is a sliding movement. A **rotation** is a turn, **clockwise** or **anticlockwise**, around the **centre of rotation**. A **reflection** swaps all the points across the **line of reflection**, like a mirror. A **dilation** is an enlargement or reduction. The **magnification** is the enlargement factor, while the **centre of magnification** stays in the same position.
- The most important plane shapes are **polygons,** which have straight sides. A polygon's name is determined by its number of sides. The corners are called **vertices. Regular** polygons have equal sides and equal angles. **Irregular** polygons have at least one angle or side that is different. **Convex** figures point outwards at the vertices but **concave** figures have at least one inward bend.
- **Equilateral** triangles have all sides equal, **isosceles** triangles have two, and **scalene** triangles have none. The angle names for triangles are **acute-, right-** and **obtuse-angled.**
- **Quadrilaterals** have four sides and are named by side and angle properties. The **square** and **rectangle** have 90° angles. A **trapezium (trapezoid)** has one pair of parallel sides. A **parallelogram** has two pairs of parallel sides. A kite has two pairs of adjacent sides equal. A **rhombus** and a square have four equal sides.
- **Congruent** (≡) figures are the same shape and size. There are four tests for **congruent triangles**: **SSS**, **SAS**, **ASA** (or **AAS**) and **RHS**.
- **Similar** (|||) figures are the same shape but not necessarily the same size. Similar triangles have corresponding angles equal and the ratios of corresponding sides are equal. There are four tests for **similar triangles**: **AAA**, **SSS**, **SAS** and **RHS**. The scale factor measures the relative sizes of similar shapes.
- If XY subtends the angle φ at point P, then $\angle XPY$ is the same angle.
- The **angle subtended at the centre** () of a circle by a chord or arc is twice the angle that the same chord or arc subtends at the circumference.
- **Angles at the circumference subtended by the same chord or arc** () are equal.
- The **diameter of a circle** () subtends a right angle at the circumference.
- The **tangent and radius** () at the same point on a circle are perpendicular.
- The **opposite angles of a cyclic quadrilateral** () add up to 180°.
- The **exterior angle of a cyclic quadrilateral** () is the same as the interior opposite angle.
- **Deductive geometry** uses logical steps set out in a formal way to prove geometric properties. The letters **QED** are put at the end of a proof to show that it is finished.

Chapter 8 review

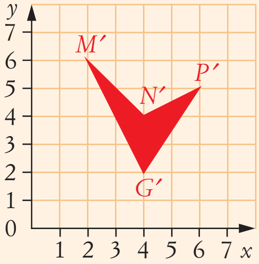
Understanding

See Example 1

1 Use coordinates to describe each of the transformations shown below.



a



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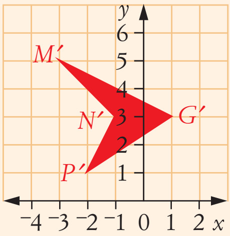
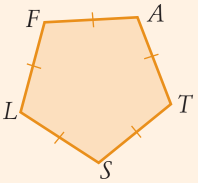


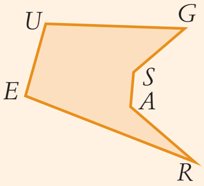
See Example 4

2 Name and classify the following shapes.

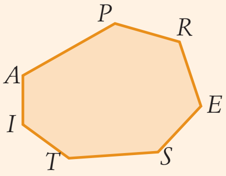
a



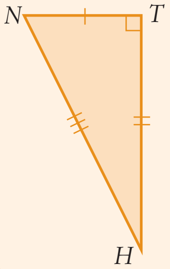
b



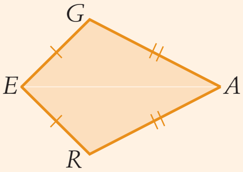
c



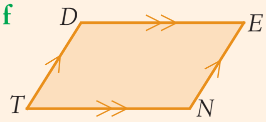
d



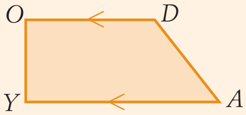
e



f



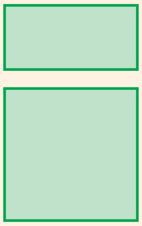
g



See Example 6

3 State whether each pair of shapes is congruent, similar or neither.

a



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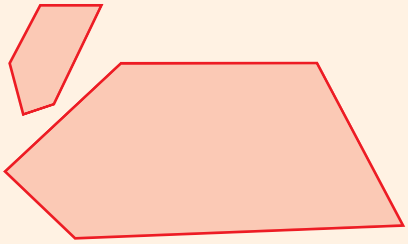

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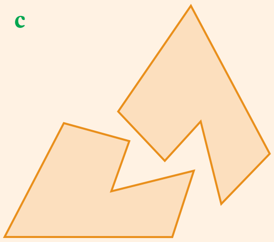
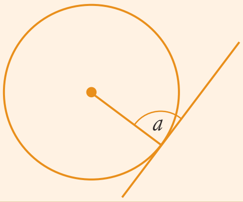
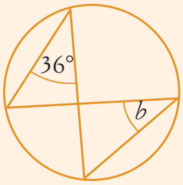
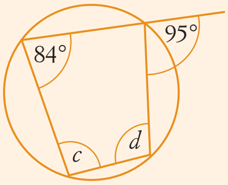


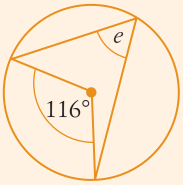
See Example 19

4 Work out the angles shown by letters below. Give reasons in words and include the code.









Fluency

5 Fully describe each of the transformations in question **1** in words.

See Example 2

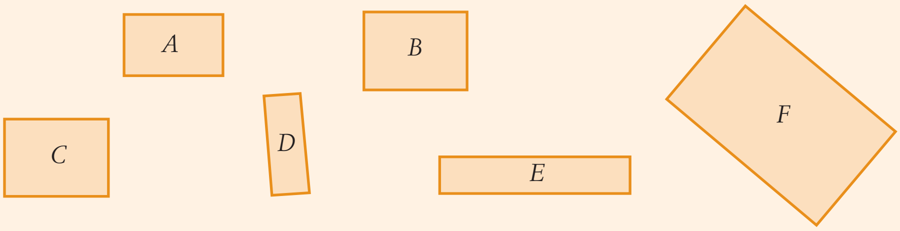
6 Find the coordinates of $M(3, 5)$ after a:

a translation 3 up and 4 left

b rotation 90° clockwise around (2, 3).

7 Work out the scale factor for any similar shapes found in question **3**.

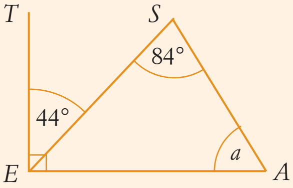
8 Name any similar and congruent shapes below. Work out the scale factors of similar shapes.



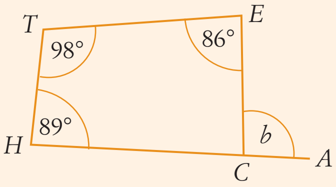
9 Find the angles shown by letters. Give reasons for each step in words and include the code.

See Example 16

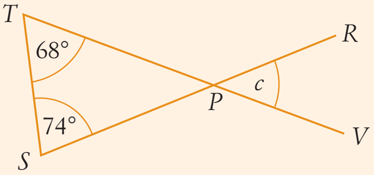
a



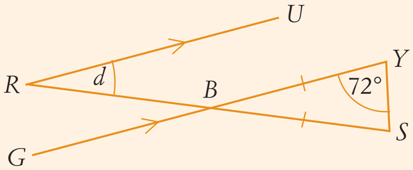
b



c



d



10 Find the sum of the internal angles of a polygon with 17 sides.

See Example 15

11 State if each of the following is true or false.

See Example 5

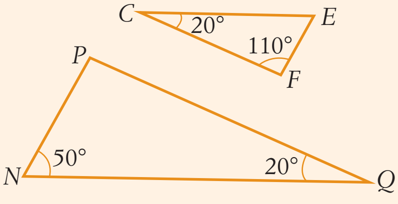
a A rhombus is a square
b All circles are similar
c The diagonals of squares intersect at 90°

12 Find the number of sides that a polygon with an angle sum of 2340° has.

13 State whether the triangles in each pair below are similar, giving the test for similarity if they are.

See Example 11

a



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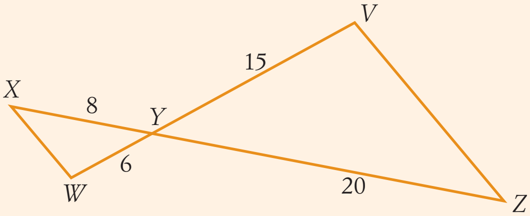
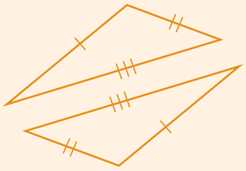


14 State whether each pair of triangles is congruent and state the test used.

See Example 12

a



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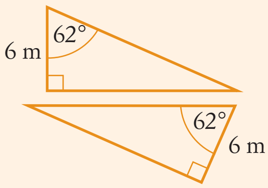
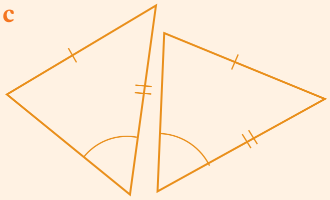
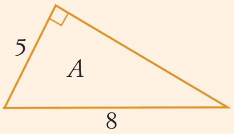
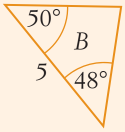


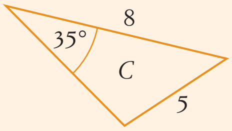
c

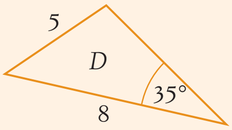


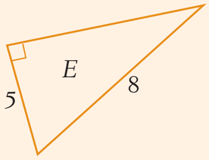
15 Prove congruence for any of the following pairs of triangles which are congruent.

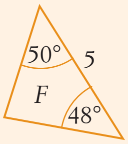






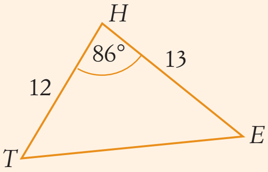




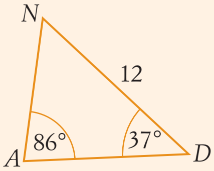


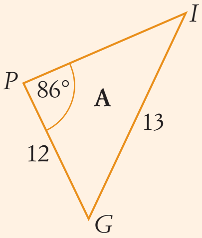
16 Match triangles **a** and **b** with a congruent triangle from **A**, **B** and **C** below and state the test used in each case.

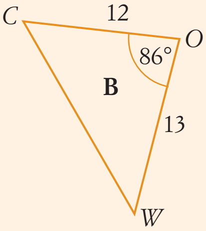
a

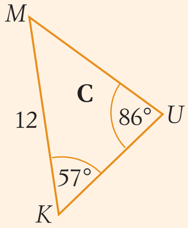


b



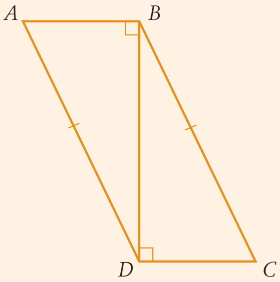






See Example 13

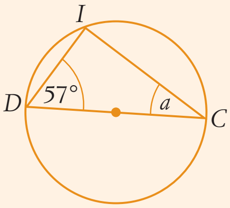
17 Use logical reasoning to determine if the following pair of triangles is congruent. Include the congruency test used.



See Example 19

18 Work out the angles shown by letters below. Give reasons for each step in words and include the code.

a



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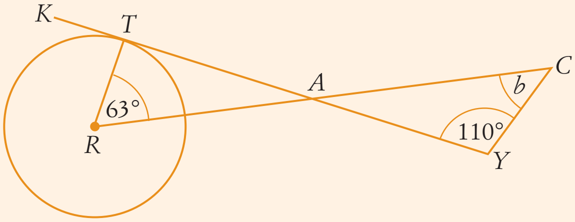


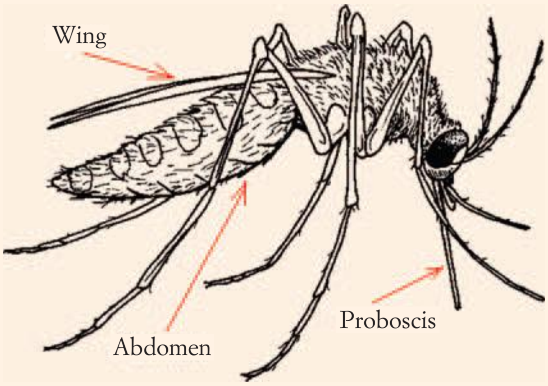
Problem solving

19 This scooter is 191 cm long.

See Example 7

a Calculate the scale for this photo.
b What is the actual diameter of the front wheel?
c What is the actual length of the seat?

20 This is a scale drawing of a mosquito. The scale used is 6 : 1. Use the scale drawing to answer the following questions.

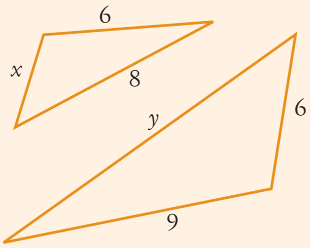


a What is the greatest actual width of the mosquito's abdomen?
b How long is the mosquito's proboscis?
c How long is the mosquito's wing?

21 Find the values of the unknowns in each of the following pairs of similar triangles.

See Example 8

a



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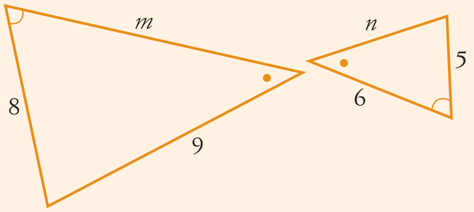
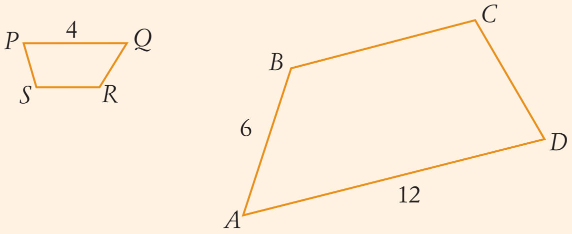


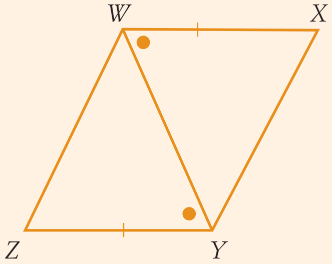
See Example 9

22 The shapes below are similar. Find the value of QR.



See Example 14

23 Use logical reasoning to show that $XY = ZW$.

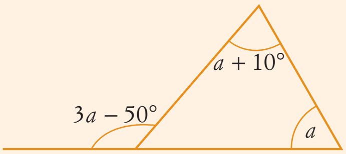


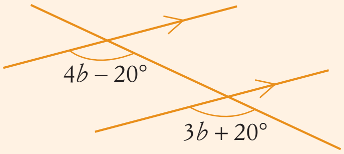
24 Find the size of the interior angles in a regular polygon that has 18 sides.

25 Find the number of sides in a regular polygon that has each of its interior angles equal to 162°.

See Example 17

26 Use algebra to find the values of these variables. Give reasons and show your working.





See Example 3

27 R is a reflection in the y-axis and M is a anticlockwise rotation about O(0, 0).

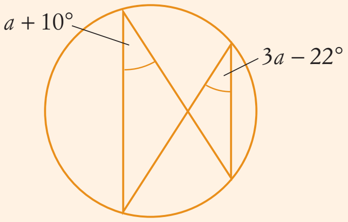
a What single transformation is equivalent to R followed by M?

b What single transformation is equivalent to M followed by R?

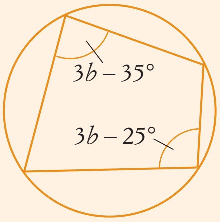
See Example 21

28 Use algebra to find the values of the variables below. Give reasons and show your working.

a



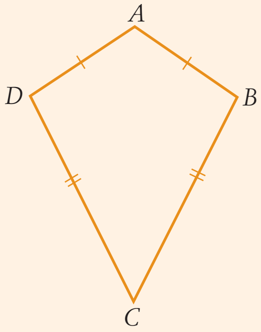
b



Reasoning

29 Construct the diagonal AC in the kite $ABCD$ in the diagram on the right. Prove that $\triangle ABC \equiv \triangle ADC$ and hence show that the angles between unequal sides of a kite are equal.

See Example 22



30 A tree casts a shadow 28 m long at the same time that a 4½ m stick casts a 6 m shadow. How tall is the tree?

See Example 10

31 Prove that the opposite angles of a cyclic quadrilateral are supplementary.

32 Prove that tangents from the same point to opposite sides of a circle are equal in length.

See Example 23

33 Prove that, if the diagonally opposite angles of a quadrilateral are equal, then the figure must be a parallelogram. (Hint: Use the angle sum of a quadrilateral.)

Number and algebra

9

Money and finance

Contents

Prior learning
Chapter 9
MAT10NAPL00020

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Australian Curriculum statements

Money and financial mathematics
Connect the compound interest formula to repeated applications of simple interest using appropriate digital technologies. (10)

Video tutorial
Money and finance
MAT10NAVT00009

Everyone borrows money at some time and nearly everybody invests money. If you have an account with a financial institution such as a bank, building society or credit union that pays interest you are actually investing money with the institution. In Australia, workers and employers invest money in their superannuation. In fact, superannuation funds are huge investors on behalf of their members, so if you are working for someone in Australia, you are automatically an investor. The amount you pay for the use of someone else's money when you have a loan is the interest. When you borrow money you would prefer the interest rate to be low, but when you invest money you would like it to be high. In this chapter, you will learn more about how interest works.

Mathematical literacy

Maths dictionary
MAT10ASDI00001

The mathematical words below have special meanings that you will learn in this chapter. It is important that you learn to spell them and gradually learn what they mean in mathematics. You may find the glossary or online mathematical dictionary useful for this purpose.

appreciation
average daily balance
cash-out
comparison rate
compound interest
daily rate
defaulting
flat-rate interest
growth factor
instalment
interest
interest rate
mortgage
nominal interest rate
on terms
pay-day lender
principal
reducing interest
repayment
rest
simple interest
vendor finance

9.1 Interest calculation

The extra amount you pay back for a loan, or the extra amount you get when you invest money is called **interest**. Interest is calculated in two different ways. While interest is calculated for loans and investments, it is easiest to consider them both as loans. When you invest money, you are effectively lending it to someone else.

Important!

Interest

Interest is the charge made for a loan. The ratio of the interest for one year to the amount borrowed is called the **interest rate** and is normally written as a decimal or percentage. It is also called the **nominal interest rate**.
The amount lent or borrowed is called the **principal**.
The amount that has to be paid back each week or month for a loan on **terms** is called the **instalment** or **repayment**.
Simple interest is calculated using the whole loan for the whole time it is borrowed. It is also called **flat-rate interest**.
Compound interest is calculated at regular intervals called **rests**, and added to the loan, with the result that the principal increases at each rest. If you make regular payments on a loan at compound interest, the interest is usually calculated before the payment is subtracted from the principal.

Pay-day lenders lend small amounts of cash for short periods of time. The amounts lent are less than banks and other financial institutions would lend.

Example 1

Annie borrowed \$1500 from a pay-day lender to tide her over and had to pay back \$1550 a fortnight later.

a What was the interest paid?
b What would it be for a year?
c What was the interest rate as a percentage?

Solution

a The interest is the extra amount paid. **Interest = \$1550 − \$1500 = \$50**

b There are 26 fortnights in a year. **Interest for 1 year = 26 × \$50 = \$1300**

c Calculate as a percentage of the loan. $$\textbf{Interest rate} = \frac{1300}{1500} \times 100\% \approx 86.7\%$$

Write the answer. **The interest rate is about 87%.**

When things are advertised on terms, the interest may not be given, is hidden in the fine print or is disguised as 'rental'.

Example 2

A 55″ 3D TV+Blu-ray home theatre pack is advertised at \$1999 after cashback or for weekly rental of \$29.25. The rental terms are given on another page as based on a 36-month contract.

a What is the total amount paid for 'rental'?
b How much is the interest for 1 year?
c What is the interest rate, assuming that you own the home theatre after the 3 years?

Solution

a There are 52 weeks in a year.	**Total paid** $= \$29.25 \times 52 \times 3 = \$4563$
b Calculate the total interest.	**Total interest** $= \$4563 - \$1999 = \$2564$
There are 3 years in the 36 months.	**Interest for 1 year** $= \$2564 \div 3 \approx \$854.67$
c Calculate the interest as a percentage of the value of the goods (the amount lent).	**Interest rate** $= \frac{854.\overline{6}}{1999} \times 100\% \approx 42.75\%$
State the nominal interest rate.	**The nominal interest rate is about 43%.**

In some cases, the interest paid is stated as a low percentage, but there are 'other charges' that inflate the amount paid.

Example 3

CAS TI-Nspire exercise
Money and finance
MAT10MGTI00009

CAS ClassPad exercise
Money and finance
MAT10MGCP00009

Puzzle sheet
Loan repayments: Simple interest
MAT10NAPS00055

A car dealer can arrange finance for Shona to buy a car for \$6700 over 2 years with monthly payments.
The stated interest rate is 12% but there is also an establishment cost of \$300, \$200 a year loan insurance and the car dealer gets a commission of \$250.

a What is the actual cost of the car?
b What is the real interest rate, with all charges taken into account?

Solution

a Calculate the interest.	**Interest for 1 year** $= 12\%$ **of** $\$6700$
Work out the percentage.	$= 0.12 \times \$6700$
	$= \$804$
Calculate the total interest.	**Total interest** $= 2 \times \$804$
	$= \$1608$
Calculate the other charges.	**Other charges** $= \$300 + 2 \times \$200 + \$250$
	$= \$950$
Find the total cost.	**Total amount** $= \$6700 + \$1608 + \$950$
	$= \$9258$
Write the answer.	**The actual cost of the car is \$9258.**
b Find the real interest.	**Real interest** $= \$9258 - \$6700$
	$= \$2558$
Find the real interest for 1 year.	**Real interest per year** $= \$2558 \div 2$
	$= \$1279$
Express as a percentage.	**Real interest rate** $= \frac{1279}{6700} \times 100\% \approx 19.1\%$
Write the answer.	**The real interest rate is about 19%.**

When you work out simple interest, you multiply the interest rate by the amount and then by the period of the loan. This can be written as a formula.

Important!

Simple interest formula

The simple interest I on the principal P at interest rate i (or r%) for n years is given by

$$I = Pin = \frac{Prn}{100}.$$

Worksheet
Simple interest table
MAT10NAWK00023

Puzzle sheet
Simple interest
MAT10NAPS00027

Example 4

Bobby borrowed \$5400 at 9% simple interest over 3 years. How much does it cost him in interest?

Solution

Write the formula.	$\boldsymbol{I = \frac{Prn}{100}}$
Write the values of the variables.	$\boldsymbol{P = \$5400, r = 9, n = 3}$
Substitute the values.	$\textbf{Interest} = \frac{\$4500 \times 9 \times 3}{100}$
Calculate the answer.	$= \$1215$
Write the answer.	**He has to pay \$1215 interest.**

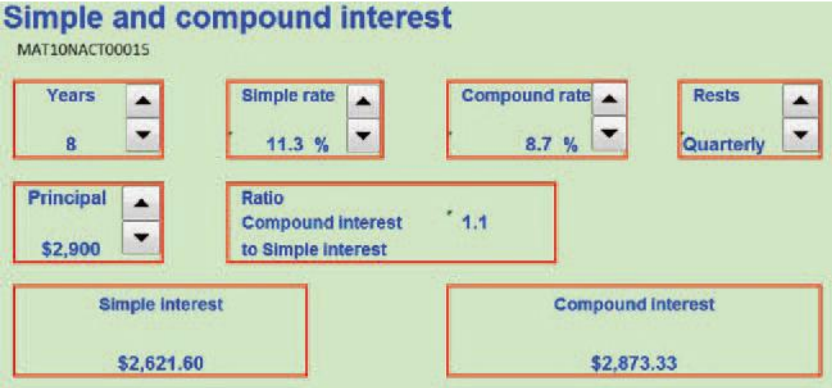
CAS TI-Nspire exercise
Money and finance
MAT10MGTI00009

CAS ClassPad exercise
Money and finance
MAT10MGCP00009

Because compound interest is added to the principal each time it is calculated, you need to work out the amount for each interest period (rest) separately. If no other information is given, then the rest is assumed to be a year.

Investigate: Compound and simple interest

You can use a spreadsheet to compare compound and simple interest.



Simple and compound interest
MAT10NACT00015

Years 8
Simple rate 11.3 %
Compound rate 8.7 %
Rests Quarterly
Principal \$2,900
Ratio Compound interest to Simple interest 1.1
Simple interest \$2,621.60
Compound interest \$2,873.33

Technology worksheet
Excel worksheet: Simple and compound interest calculator
MAT10NACT00015

Technology: GeoGebra
Compound and simple interest investments
MAT10NATC00009

Weblink
Accounting study guide
MAT10NAWB00009

The spreadsheet 'Simple and compound interest' can be found on the NelsonNet website. Download and open the spreadsheet and use it to find the simple interest rate that gives the same amount of interest as each of the following.

- 10% compounded annually for 3 years
- 10% compounded monthly for 3 years
- 10% compounded monthly for 6 years
- 15% compounded weekly for 4 years

Is there a simple way to work out a simple interest rate that gives the same interest as a compound interest rate?

CAS TI-Nspire exercise
Money and finance
MAT10MGTI00016

CAS ClassPad exercise
Money and finance
MAT10MGCP00016

Example 5

Work out the interest on \$2000 at 7% compound interest for 4 years.

Solution

Find the interest for the first year.

$$\textbf{Interest for 1 year} = 0.07 \times \$2000 = \$140$$

Add to the amount.

$$\textbf{Total after 1 year} = \$2140$$

Find the interest for the 2nd year.

$$\textbf{Interest for 2nd year} = 0.07 \times \$2140 = \$149.80$$

Add to the total.

$$\textbf{Total after 2 years} = \$2289.80$$

Find the interest for the 3rd year.

$$\textbf{Interest for 3rd year} = 0.07 \times \$2289.80 \approx \$160.29$$

Add to the total.

$$\textbf{Total after 3 years} \approx \$2450.09$$

Find the interest for the 4th year.

$$\textbf{Interest for 4th year} \approx 0.07 \times \$2450.09 \approx \$171.51$$

Add to the total.

$$\textbf{Total after 4 years} \approx \$2621.59$$

Find the total interest.

$$\textbf{Interest} \approx \$2621.59 - \$2000 = \$621.59$$

In Example **5**, you should keep full values in your calculator and round the final amount from the exact total (\$2621.59202). You should not use rounded values in calculations. In Australia, financial institutions are not allowed to round amounts in interest calculations.

You can save time in calculations by finding the amount at the end of each year as a percentage of the amount at the beginning of the year. You can think of this as a **growth factor**. You can also use the constant multiplier function on your calculator.

Example 6

Work out the compound interest on \$6000 at 6.5% for 4 years.

Solution

Find the growth factor.	**Growth factor** $= 100\% + 6.5\% = 106.5\%$ $= 1.065$
Find the amount after 1 year.	**Amount after 1 year** $= 1.065 \times \$6000$ $= \$6390$
Find the amount after 2 years.	**Amount after 2 years** $= 1.065 \times \$6390$ $= \$6805.35$
Find the amount after 3 years.	**Amount after 3 years** $= 1.065 \times \$6805.35$ $\approx \$7247.70$
Find the amount after 4 years.	**Amount after 4 years** $\approx 1.065 \times \$7247.70$ $\approx \$7718.80$
Find the interest.	**Interest** $= \$7718.80 - \$6000$ $= \$1718.80$

Puzzle sheet

Compound interest with annual rests

MAT10NAPS00028

In Examples **5** and **6** you should compare the compound interest to the simple interest that you would get with the same time and percentage.

The formula for compound interest is more complicated than that for simple interest.

In Example **6**, you should be able to see that the final amount could be worked out as $\$6000 \times (1.065)^4$. The formula works the same way.

Important!

Compound interest formula

The basic formula for the amount accumulated by compound interest is

$$A = P(1+i)^n = P\left(1+\frac{r}{100}\right)^n,$$

where A is the amount after n rests (years), P is the principal and i (or $r\%$) is the interest rate.

Example 7

Use the formula to find the compound interest on $2450 at 6.3% for 5 years.

Solution

Write the formula.	$A = P(1 + i)^n$
Write the values of the variables.	$P = \$2450,\ i = 0.063,\ n = 5$
Substitute the values.	**Amount** $= 2450(1 + 0.063)^5$
Simplify the brackets to the growth factor.	$= 2450 \times 1.063^5$
Find the amount.	≈ 3325.31
Find the interest.	**Interest** $\approx \$3325.31 - \$2450$ $= \$875.31$
Write the answer.	**The compound interest on $2450 at 6.3% for 5 years is $875.31.**

In cases where the rest period is less than a year, the compound interest rate has to be adjusted. It is simplest to adjust the rate and the period using the number of rests in a year.

CAS TI-Nspire exercise
Money and finance
MAT10MGTI00016

CAS ClassPad exercise
Money and finance
MAT10MGCP00016

Video tutorial
Compound interest
MAT10NAVT10021

Puzzle sheet
Compound interest with non-annual rests
MAT10NAPS00029

Example 8

Find the amount after 6 months when $894 is invested at 5.7% compounding each month.

Solution

Change the interest rate to monthly.	**Monthly rate** $= \frac{0.057}{12} = 0.00475$
Write as the growth factor.	**Monthly growth factor** $= 1.00475$
Find the amount after 1 month.	**Amount after 1 month** $= \$894 \times 1.00475$ $\approx \$898.25$
Do two months.	**Amount after 2 months** $\approx \$898.25 \times 1.00475$ $\approx \$902.51$
Do three months.	**Amount after 3 months** $\approx \$902.51 \times 1.00475$ $\approx \$906.80$
Do four months.	**Amount after 4 months** $\approx \$906.80 \times 1.00475$ $\approx \$911.11$
Do five months.	**Amount after 5 months** $\approx \$911.11 \times 1.00475$ $\approx \$915.44$
Do six months.	**Amount after 6 months** $\approx \$915.44 \times 1.00475$ $\approx \$919.78$

Enter on your calculator.

894 = × **1.00475** = = = = = =

The display for the last two times you press the equals key should look like this if you have used the constant multiplier correctly.

```
ANS×1.00475
        915.4351691
```

```
ANS×1.00475
        919.7834862
```

Write the answer.

\$894 amounts to \$919.78 after 6 months at 5.7% compounding monthly.

When you use the constant multiplier on your calculator, you need to be careful to count the number of times you press the equals sign.

Important!

The formula for the amount accumulated by compound interest for non-annual rests is

$$A = P\left(1+\frac{i}{k}\right)^{kt} = P\left(1+\frac{r}{100k}\right)^{kt},$$

where A is the amount after kt rests, P is the principal, i (or $r\%$) is the interest rate, k is the number of rests in a year and t is the number of years.

Example 9

Use the formula to find the interest on \$2450 invested at 6.3% compounding monthly for 5 years.

Solution

Write the formula.

$$A = P\left(1+\frac{i}{k}\right)^{kt}$$

Write the values of the variables.

$P = 2450$, $i = 0.063$, $k = 12$, $t = 5$

Substitute the values.

$$\text{Amount} = 2450\left(1+\frac{0.063}{12}\right)^{12\times 5}$$

Simplify.

$$= 2450 \times 1.00525^{60}$$

Calculate the answer.

$$\approx 3354.37$$

Enter as

```
2450(1+0.063÷12)^(12×5)        3354.370198
```

2450 (1 + 0.063 ÷ 12) ^ (12 × 5) =

Find the interest.

$$\text{Interest} = \$3354.37 - \$2450$$
$$= \$904.37$$

Write the answer.

The interest after 5 years is \$904.37.

Notice that the answer to Example **9** is more than Example **7**, so the interest from monthly rests is more than from annual rests at the same nominal interest rate. Compounding daily would increase the amount even more, but only by another $2.67.

Exercise 9.1 Interest calculation

Understanding

Extra questions
Exercise 9.1
MAT10NAEQ00025

See Example 4

1 Use the formula to calculate simple interest for each of the following.

- **a** $590 at 12% for 3 years
- **b** $2940 at 8% for 5 years
- **c** $6800 at 15% for 6 years
- **d** $458 at 9.6% for 4 years
- **e** $7320 at 24.7% for 2 years

See Example 6

2 Find the growth factor for each of the following cases of compound interest.

- **a** 8% compound interest
- **b** 6.4% compound interest
- **c** 3.8% compound interest
- **d** 12.4% compound interest
- **e** 9.6% compound interest

See Example 8

3 Find the growth factor for each of the following cases of compound interest.

- **a** 8.1% compound interest with monthly rests.
- **b** 6.5% compound interest with weekly rests.
- **c** 5.9% compound interest with quarterly rests.
- **d** 12.6% compound interest with monthly rests.
- **e** 9.1% compound interest with fortnightly rests.

Fluency

See Example 1

4 Find the interest rate for each of the following payday loans.

- **a** $300 for 5 days with charges of $75 and $1.95.
- **b** $400 for 9 days with a charge of $110.
- **c** $300 for 5 days with charges of $45 and $60.
- **d** $600 for 3 weeks with charges of $45 and an extra $30 per week.
- **e** $500 for 4 days with a charge of $27.50 per $100.

See Example 3

5 Find the real simple interest rate for each of the following vendor finance agreements.

- **a** $26 000 for a car with weekly payments of $342 a week over 3 years.
- **b** $7800 for a used car with stated interest of only 12% over 3 years, an establishment fee of $800 and commission of $590.

See Examples 5, 6

6 Calculate the amount and interest for each of the following.

- **a** $3000 at 12% compound interest for 3 years
- **b** $4900 at 8% compound interest for 4 years
- **c** $6200 at 6.4% compound interest for 4 years
- **d** $8400 at 9.6% compound interest for 3 years
- **e** $12 300 at 10% compound interest for 4 years

See Example 8

7 Calculate the amount and interest for each of the following.

- **a** $4000 at 9.6% compound interest for 6 months with monthly rests.
- **b** $3800 at 8.7% compound interest for 4 months with monthly rests.
- **c** $4600 at 10.8% compound interest for 7 months with monthly rests.

d \$8400 at 7.2% compound interest for 12 months with monthly rests.

e \$7200 at 5.7% compound interest for 9 months with monthly rests.

8 Use the formula to calculate the amount and interest for each of the following.

See Example 7

a \$5000 at 7.5% compound interest for 3 years

b \$7000 at 8.6% compound interest for 4 years

c \$12 300 at 9.2% compound interest for 5 years

d \$6200 at 5.8% compound interest for 3 years

e \$4920 at 4.9% compound interest for 3 years

9 Use the formula to calculate the amount and interest for each of the following.

See Example 9

a \$8000 at 8.4% compound interest for 5 years with monthly rests.

b \$4000 at 9.1% compound interest for 3 years with weekly rests.

c \$7000 at 9.3% compound interest for 4 years with monthly rests.

d \$9200 at 6.8% compound interest for 4 years with quarterly rests.

e \$5740 at 7.2% compound interest for 3 years with monthly rests.

Problem solving

⑩ You 'rent' a notebook computer worth \$399 for \$5.10 a week on a 36-month contract. At the end of the time you buy it for \$50.

a What is the total cost?

b How much is the interest for 1 year?

c What is the interest rate?

Worked solutions
Exercise 9.1
MAT10NAWS00025
See Example 2

⑪ You rented a 60 inch flat screen TV sold for \$1388 at a 'rent' of \$17.65 a week on a 36-month contract. At the end of the time you purchased it for \$400. What was the total cost and the real interest rate?

Worked solutions
Exercise 9.1
MAT10NAWS00025

⑫ Yoda bought a bedroom suite for \$2890 with 'nothing to pay' for 2 years. At the end of the 2 years, the interest is calculated back to the start of the time at 19%. There is also an 'accounting' fee of \$15 per month. She actually paid it off after 3 years. What was the real amount she paid and the real interest rate?

Worked solutions
Exercise 9.1
MAT10NAWS00025

Reasoning

13 Find the time taken for an amount to double using compound interest at rates of 7%, 10%, 14% and 20% using \$1000 as the initial amount. Can you give a rough rule for time taken for an amount to double under compound interest?

14 Calculate the return on \$1000 for 1 year at 5%, 10%, 15% and 20% compound interest, compounding annually, quarterly, monthly, weekly and daily and comment on the difference that compounding at shorter intervals makes to the interest gained.

9.2 Investing money

There are many different ways to invest money. Most investments are calculated at compound interest. Only very large investors on the short-term money market typically invest at simple interest. This kind of investment might be something like \$600 000 for 5 days. The market is used by banks and other financial institutions, as well as by wealthy individuals who have money that they are not due to use for a short time.

Example 10

\$700 000 is invested at 4.24% for 8 days. How much interest is earned?

Solution

The term is a fraction of a year.	**Time** $= \frac{8}{365} \approx$ **0.02192 years.**
Write the formula.	$I = Pin$
Write the values of the variables.	$P = \$700\,000,\ i = 0.0424,\ n \approx 0.02192$
Substitute the values.	**Interest** $\approx 700\,000 \times 0.0424 \times 0.02192$
Work out the answer.	≈ 650.52
Write the answer.	**The interest is \$650.52.**

Intermediate calculations such as the term in Example **10** should be kept in your calculator and used for the later calculations. Do not use the rounded amounts.
Most savings accounts calculate interest on minimum monthly balance. The interest may be paid monthly, 6-monthly or annually. This is effectively a sum of simple interest calculations.

Example 11

The interest for Sandy's savings account is paid at the end of June on minimum monthly balances at a rate of 3%. Use the minimum balances below to find Sandy's interest for the financial year.

Month	Jul	Aug	Sep	Oct	Nov	Dec
Min. balance	\$42.50	\$127.38	\$149.80	\$265.20	\$474.90	\$28.30
Month	Jan	Feb	Mar	Apr	May	Jun
Min. balance	\$248.90	\$521.40	\$672.90	\$783.60	\$985.68	\$32.40

Solution

Calculate the interest rate for a month.	**Monthly rate** $= \frac{0.03}{12} = 0.0025$
Find the interest for July.	**July interest** $= 0.0025 \times \$42.50 \approx \$0.11$
Find the interest for August.	**August interest** $= 0.0025 \times \$127.38 \approx \$0.32$
Find the interest for September.	**September interest** $= 0.0025 \times \$149.80 \approx \$0.37$
Find the interest for October.	**October interest** $= 0.0025 \times \$265.20 \approx \$0.66$

Find the interest for November.	**November interest $= 0.0025 \times \$474.90 \approx \$1.19$**
Find the interest for December.	**December interest $= 0.0025 \times \$28.30 \approx \$0.07$**
Find the interest for January.	**January interest $= 0.0025 \times \$248.90 \approx \$0.62$**
Find the interest for February.	**February interest $= 0.0025 \times \$521.40 \approx \$1.30$**
Find the interest for March.	**March interest $= 0.0025 \times \$672.90 \approx \$1.68$**
Find the interest for April.	**April interest $= 0.0025 \times \$783.60 \approx \$1.96$**
Find the interest for May.	**May interest $= 0.0025 \times \$985.68 \approx \$2.46$**
Find the interest for June.	**June interest $= 0.0025 \times \$32.40 \approx \$0.08$**
Add the (exact) amounts to find the total.	**Total $\approx \$0.11 + \$0.32 + \ldots = \$10.83$**
Write the answer.	**Sandy received \$10.83 interest for the year.**

Term deposits attract much greater interest than savings accounts, but usually you cannot get your money back until the end of the term. If you are allowed to take your money out early because of hardship you will receive no interest. Term deposits are normally offered for terms from about 3 months up to 5 years. You normally have to put a minimum amount of about \$5000 into a term deposit.

Example 12

Kim put \$7500 into a term deposit at 5.4% for 3 months. How much is it worth at the end of the term?

Solution

Calculate the growth factor.	**Growth factor $= 1 + \dfrac{0.054}{4} = 1.0135$**
Multiply the principal by the growth factor.	**Amount after 3 months $= \$7500 \times 1.0135$**
Calculate the amount.	**$= \$7601.25$**
Write the answer.	**It is worth \$7601.25 at the end of the term.**

When you set up a term deposit for two or more years, there are different ways the interest is paid. Some term deposits pay it into another account at fixed intervals such as a month or a quarter. Others will simply treat it as compounding interest, reinvesting it at the end of each year.

Example 13

Tom put an inheritance of $15 300 into a term deposit for 4 years at 5.8%. He could have the interest paid into another account at the end of each year, or have it reinvested in the term deposit.

a How much would he get each year it he chooses the first option?

b How much would the investment be worth after 4 years if he had it reinvested?

Solution

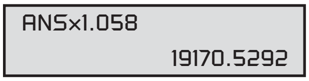
a Calculate the interest. **Interest = 0.058 × $15 300**

Work out the answer. **= $887.40**

Write the answer. **He would get $887.40 each year.**

b Work out the growth factor. **Growth factor = 1 + 0.058 = 1.058**

Enter as
15 300 = × 1.058 = = = = .



ANSx1.058
19170.5292

Write the answer. **It would be worth $19 170.53 after 4 years.**

It is easy to lose track of the number of times you press the equals sign when calculating compound interest, so use the formula for large numbers of rests.

Example 14

Inar invested $16 900 with a mortgage broker at 6.9% calculated monthly for 2 years. How much did she get back?

Solution

Write the formula.

$$A = P\left(1 + \frac{i}{k}\right)^{kt}$$

Write the values of the variables.

$$P = 16\ 900,\ i = 0.069,\ k = 12,\ t = 2$$

Substitute the values.

$$\text{Amount} = 16\ 900\left(1 + \frac{0.069}{12}\right)^{12 \times 2}$$

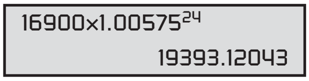
Simplify.

$$= 16\ 900 \times 1.00575^{24}$$

Calculate the answer.

$$\approx 3354.37$$

Enter as 16 900 × 1.00575 ^ 24 = .



$16900 \times 1.00575^{24}$
19393.12043

Write the answer. **She got back $19 393.12.**

Instead of putting money into a bank account or some other kind of investment, you can buy something that you think will increase in value. This is more risky than term deposits, but no investment is without risk. Generally, a higher return means a higher risk. If you invest in something with high returns, then you have a greater risk of losing your money.
"If something sounds too good to be true – it probably is!"

Things that increase in value over time are said to **appreciate**. You can often use the compound interest method to calculate their expected value.

Example 15

Jacquie is a numismatist. She believes that her coin collection, currently valued at $5400, will increase in value at a rate of 12% each year. What would she expect it to be worth 4 years from now?

Solution

Write the growth factor.	**Growth factor = 1.12**
Work out the value after 1 year.	**Value after 1 year = \$5400 × 1.12 = \$6048**
Work out the value after 2 years.	**Value after 2 years = \$6048 × 1.12 = \$6773.76**
Work out the value after 3 years.	**Value after 3 years = \$6773.76× 1.12 ≈ \$7586.61**
Work out the value after 4 years.	**Value after 4 years = \$7586.61 × 1.12 = \$8497.00**
You can enter it as 5400 × 1.12 = = = = .	`ANS×1.12   8497.004544`
Round for your answer.	**Her collection would be worth about \$8500.**

You could also do Example **15** using the compound interest formula.

Sometimes you will be interested in how long it takes for an investment to reach a particular value.

Example 16

How long would it take an investment of $3000 to reach a value of $5000 at 15% simple interest?

Solution

Work out the amount of interest needed.	**Interest = \$5000 − \$3000 = \$2000**
Write the simple interest formula.	$I = Pin$
Write the values of the variables.	$I = 2000,\ P = 3000,\ i = 0.15$
Substitute values in the formula.	$2000 = 3000 \times 0.15 \times n$
Simplify the equation.	$450n = 2000$
Divide both sides by 450.	$n = \frac{2000}{450} = 4.4\overline{4} \approx$ **4 years and 5.3 months**
Write the answer, allowing for different circumstances.	**It would take about 4 years and 5 months, or 6 years if it has to be a whole number of years.**

Example 17

How long would it take an investment of \$5000 to reach a value of \$8000 at 9% compound interest?

Solution

Work out the growth factor.	**Growth factor = 1.09**
Multiply by the growth factor.	**Amount after 1 year = \$5000 × 1.09 = \$5450**
Multiply by the growth factor again.	**Amount after 2 years = \$5450 × 1.09 = \$5940.50**
Multiply by the growth factor again.	**Amount after 3 years = \$5940.50 × 1.09 ≈ \$6475.15**
Multiply by the growth factor again.	**Amount after 4 years = \$6475.15 × 1.09 ≈ \$7057.91**
Multiply by the growth factor again.	**Amount after 5 years = \$7057.91 × 1.09 ≈ \$7693.12**
Multiply by the growth factor again.	**Amount after 6 years = \$7693.12 × 1.09 ≈ \$8385.50**
Write the answer, allowing for different circumstances.	**It would take about $5\frac{1}{2}$ years, or 6 years if it has to be a whole number of years.**

In Example **17** you would actually use the constant factor method on your calculator and count the number of times you had to press the = button. For more difficult problems involving compound interest terms you need to use logarithms to solve an equation.

Example 18

How long would it take an investment of \$6000 to reach a value of \$10 000 at 9% compounding monthly?

Solution

Work out the growth factor for months.	**Growth factor** $= 1 + \frac{0.09}{12} = 1.0075$
Write the compound interest formula.	$A = P\left(1 + \frac{i}{k}\right)^n$
Write the values of the variables.	$A = 10\ 000,\ P = 6000,\ 1 + \frac{i}{k} = 1.0075$
Substitute in the formula to find the number of rests.	$10\ 000 = 6000 \times 1.0075^n$
Divide both sides by 6000.	$1.0075^n = 1\frac{2}{3} = 1.\overline{6}$
Change to logarithms.	$\log 1.0075^n = \log 1.\overline{6}$
Use the log law that log $a^n = n$ log a	$n \log 1.0075 = \log 1.\overline{6}$
Divide by log 1.0075.	$n = \frac{\log 1.\overline{6}}{\log 1.0075}$
Calculate the answer.	≈ 68.4
Change the answer to years and months.	**= 5 years 8.4 months**
Write the answer.	**It will take 5 years 8.4 months or 5 years 9 months for a whole number of months.**

Exercise 9.2 Investing money

Fluency

Extra questions
Exercise 9.2
MAT10NAEQ00054

See Example 11

1 Find the interest earned in a financial year on a savings account that pays interest of 2.4% on minimum monthly balances if the balances are as shown in the table below.

Month	Jul	Aug	Sep	Oct	Nov	Dec
Balance	$267.59	$543.00	$555.43	$520.83	$551.17	$503.95
Month	Jan	Feb	Mar	Apr	May	Jun
Balance	$547.00	$569.18	$558.88	$507.61	$585.11	$298.01

2 Find the interest earned in a financial year on a savings account that pays interest of 3.3% on minimum monthly balances if the balances are as shown in the table below.

Month	Jul	Aug	Sep	Oct	Nov	Dec
Balance	$1701.44	$1868.21	$2303.47	$1802.33	$2261.56	$2472.81
Month	Jan	Feb	Mar	Apr	May	Jun
Balance	$202.61	$398.29	$1812.48	$491.75	$485.88	$307.84

3 Find the interest earned in a calendar year on a savings account that pays interest of 2.7% on minimum monthly balances if the balances are as shown in the table below.

Month	Jan	Feb	Mar	Apr	May	Jun
Balance	$2203.76	$381.81	$2256.72	$2380.21	$366.63	$677.40
Month	Jul	Aug	Sep	Oct	Nov	Dec
Balance	$478.36	$2378.85	$2223.04	$334.64	$583.62	$596.11

See Example 12

4 How much is a term deposit of $8300 at 5.6% for 6 months worth at the end of the term?

5 How much is a term deposit of $12 400 at 4.8% for 3 months worth at the end of the term?

6 How much is a term deposit of $10 500 at 6.1% for 9 months worth at the end of the term?

See Example 10

7 $850 000 is invested at 3.2% for 7 days. How much interest is earned?

8 $1 500 000 is invested at 3.7% for 6 days. How much interest is earned?

9 $590 000 is invested at 2.8% for 9 days. How much interest is earned?

See Example 13

10 Antoinette put $18 700 into a term deposit for 4 years at 6.3%. The interest was calculated and paid every six months. How much does she get paid into her savings account every 6 months from the term deposit?

11 Andrew has the income from a trust fund paid into his savings account every year. The trust fund is for $24 000 and is earning interest at the rate of 5.2%. How much does he get each year from the fund?

12 Carmen put $7200 into a term deposit at 6.25% for 3 years. How much does she get back when the term deposit matures (i.e. at the end of the term)?

13 Colin was given $6000 in settlement of an injury claim. He decided to put it on term deposit at 5.75% for 5 years. What does he get back at the end of the time?

See Example 14

14 What is an investment of \$6400 at 4.8% compounding monthly worth after 4 years?

15 What is an investment of \$13 900 at 6.8% compounding quarterly worth after 3 years?

16 What is an investment of \$8900 at 5.9% compounding weekly worth after 5 years?

Problem solving

See Example 16

17 How long would it take an investment of \$2500 to reach a value of \$4000 at 12% simple interest?

18 How long would it take an investment of \$4000 to reach a value of \$8000 at 15% simple interest?

19 How long would it take an investment of \$2800 to reach a value of \$6000 at 14.8% simple interest?

See Example 17

20 How long would it take an investment of \$2000 to reach a value of \$3500 at 12% compound interest?

21 How long would it take an investment of \$4500 to reach a value of \$9000 at 15% compound interest?

22 How long would it take an investment of \$1500 to reach a value of \$4300 at 8.9% compound interest?

See Example 18

23 How long would it take an investment of \$7200 to reach a value of \$9 000 at 6.9% compounding monthly?

24 How long would it take an investment of \$3200 to reach a value of \$6400 at 15% compounding monthly?

Worked solutions
Exercise 9.2
MAT10NAWS00059

㉕ How long would it take an investment of \$100 to reach a value of \$1000 at 7.5% compounding monthly?

See Example 15

26 An 'old masters' painting is appreciating at a rate of 8% each year. If it is worth \$240 000 now, what will it be worth in 5 years time?

27 A vintage car is thought to get more valuable each year. The increase is estimated at 11%. It is worth \$6800 now. What will it be worth in 10 years time?

28 A new car loses value instead of appreciating. It loses about 15% of its value each year. If you bought a new car now for \$34 000, what would it be worth in 3 years time?

Reasoning

29 Use a principal of \$1000 to find the simple interest rate that gives the same interest as compound interest of 12% over 3 years, logically explaining your steps.

Worked solutions
Exercise 9.2
MAT10NAWS00059

㉚ Find the simple interest rate that gives the same interest as 10% compounding semi-annually over 5 years, logically explaining your steps.

㉛ Find the interest rate that, compounding quarterly, gives the same interest as 15% simple interest over 3 years, logically explaining your steps.

Worked solutions
Exercise 9.2
MAT10NAWS00059

32 Find the interest rate that, compounding monthly, gives the same interest as 18% simple interest over 5 years, logically explaining your steps.

9.3 Borrowing money

There are many different ways to borrow money. Generally speaking, the easier it is to borrow, the higher the interest rate and other charges will be. Very few people are able to avoid borrowing money, at least for major purchases such as a car or house. Ultimately, you will be better off financially if you can avoid borrowing wherever possible. If you have to borrow money, you should take the time to become familiar with the rates and conditions.

Payday lenders offer very short term loans of up to a few weeks for small amounts from $100 to about $1500. They often disguise interest as fees and charges, so you need to compare all costs when obtaining credit. Most require you to allow them access to a savings account for repayments. It can be difficult for you to revoke this access. If you don't pay a loan according to the conditions, this is called **defaulting**, and usually attracts extra charges.

Example 19

Animated example
Borrowing money
MAT10NAAE00009

A cash loans business offers loans of up to $500 at no interest for up to 3 weeks, but charge a fee of $45 per loan and $30 per week for the period of the loan. If you default on the loan, they apply a default cost of $50, then roll over the loan and charges to a new loan for 1 week. On Thursday, Mandy agreed to repay a loan of $300 on her next payday, which was on Friday week. Unfortunately she had forgotten about the electricity bill, so couldn't pay it until her next payday, a fortnight later.

a Work out the total cost if she paid back the loan on time.
b Work out the cost by the time she had actually paid back the loan.
c Work out the interest rate.

Solution

a State the length of the loan.	**The loan is more than a week so counts as 2 weeks.**
Work out the total cost.	**Cost** $= \$45 + 2 \times \$30 = \$105$
b Work out the default cost.	**Second loan cost** $= \$50 + \$45 + \$30$ $= \$125$
Work out the 2nd default cost.	**Third loan cost** $= \$50 + \$45 + \$30$ $= \$125$
Find the total cost.	**Total cost** $= \$105 + \$125 + \$125$ $= \$355$
c Work out the actual loan time.	**Time of loan** $= 8$ **days** $+ 14$ **days** $= 22$ **days**
Work out the interest for 1 day.	**Interest for 1 day** $= \$355 \div 22$ $= \$16.13636\ldots$
Work out the interest for a year.	**Interest for 1 year** $= \$16.13636\ldots \times 365$ $= \$5889.7727\ldots$
Work out the interest rate.	**Interest rate** $= \dfrac{5889.7727\ldots}{300} \times 100\%$ $\approx 1963\%$

In Example **19**, you will notice that in the end, Mandy had to pay back more than double the amount she borrowed. If you cannot avoid defaulting on a payment, you should talk to the lender, preferably before it is due. Many lenders will try to help you by rescheduling payments. If you don't tell them about your problems they will assume the worst and apply the maximum penalties. Credit card companies usually allow you to get a cash advance when you buy goods (a **cash-out**), but charge you a high interest rate for this money. You may also have to pay an extra charge for the shop to allow you to use a credit card. Credit card interest is calculated on the **average daily balance** of your account, using the **daily rate** for the number of days in the billing period. This means that you actually pay interest on purchases or cash advances for the number of days left in the billing period for each purchase that you make.

Example 20

Dan bought some groceries worth $35 from his corner store on 20 June and also got a cash-out of $200. The shop charged 2.5% extra for using the credit card. His credit card's billing period is until the end of each month and is due 5 days later, on the 5th of the next month. The card cost $75/year and has a credit charge of 11% for purchases and 24% for cash advances.

a How much did the credit cost him?
b What was the overall interest rate for the groceries?
c What was the overall interest rate for the cash advance?

Solution

a	Work out the extra for groceries.	$\text{Groceries extra} = 0.025 \times \$35$ $\approx \$0.88$
	Work out the total for groceries.	$\text{Groceries total} \approx \$35 + \$0.88$ $= \$35.88$
	Work out the daily rate for purchases.	$\text{Purchases daily rate} = \frac{0.11}{365}$ $= 0.000301\ldots$
	There are 30 days in June.	$\text{Days left in billing period} = 30 - 20 = 10$
	Work out the interest for groceries.	$\text{Groceries interest} \approx \$35.88 \times 10 \times 0.000301\ldots$ $\approx \$0.11$
	Work out the groceries credit cost.	$\text{Total groceries credit cost} \approx 0.98$
	Work out the extra for the cash.	$\text{Cash extra} = 0.025 \times \$200 = \$5$
	Work out the total for the cash.	$\text{Cash total} = \$200 + \$5 = \$205$
	Work out the daily rate for cash.	$\text{Cash daily rate} = \frac{0.24}{365} = 0.0006575\ldots$
	Work out the interest for cash.	$\text{Cash interest} = \$205 \times 10 \times 0.000657\ldots$ $\approx \$1.35$
	Work out the cash credit cost.	$\text{Total cash credit cost} \approx \$5 + \$1.35$ $= \$6.35$
	Work out the total credit cost.	$\text{Total credit cost} \approx \$0.98 + \$6.35 = \$7.33$
	Write the answer.	**The total cost of the credit was $7.33.**

b Work out the cost for 1 day.	**Cost for 1 day** $\approx 0.98 \div 10$ $= 0.098$
Work out the cost for a year.	**Cost for a year** $\approx 0.098 \times 365$ ≈ 35.88
Work out the interest rate.	**Interest rate** $= \frac{35.88}{35} \times 100\%$ $\approx 102.5\%$
State the answer.	**The interest rate on the groceries was about 102.5%.**
c Work out the cost for 1 day.	**Cost for 1 day** $\approx 6.35 \div 10$ $= 0.635$
Work out the cost for a year.	**Cost for a year** $\approx 0.635 \times 365$ ≈ 231.70
Work out the interest rate.	**Interest rate** $\approx \frac{231.70}{200} \times 100\%$ $\approx 115.9\%$
State the answer.	**The interest rate on the cash advance was about 115.9%.**

The actual interest rate in Example **20** is very high because of the shop's charge for using the credit card. Credit card providers also charge very high fees for defaulting on payments. In addition, they normally keep charging interest at a higher rate on all purchases until the debt is paid in full.
If you want to avoid paying high charges for credit cards then:

- don't use them if you will be charged for credit,
- don't use them for cash advances and
- use them at the end of the billing period if you have to use them.

Because so many merchants now charge for the use of credit cards, it is better to use a debit card when making purchases. When you do this, you are using your own money so don't pay interest.
If you make a major purchase such as furniture or a car and can't pay cash, you should avoid using **vendor finance**. Vendor finance means that the shop or company you are buying from arranges the loan. As you have already seen in the first section of this chapter, the charges for vendor finance are usually quite high. A personal loan from a financial institution will almost always be cheaper. You need to make sure that you can afford the repayments. The costs for defaults on loans are always high. Legally, if you default on a debt, the lender can get a court to sell your possessions to get their money back. In this case, the court bailiffs can enter your home and take your things. However, reputable companies will usually try to help you avoid this.

TLF Learning object
Maths and the car: loan calculator (L1449)
MAT10NAIN00009

Example 21

Ying has a personal loan of \$12 000 to buy a car over 3 years at 11% flat rate interest. Work out the following.

a The total interest. **b** The total she has to pay. **c** Her monthly repayments.

Solution

a Write the formula. $I = \frac{Prn}{100}$

Write the values of the variables. $P = \$12\,000, r = 11, n = 3$

Substitute the values. $\text{Interest} = \frac{\$12\,000 \times 11 \times 3}{100}$

Calculate the answer. $= \$3960$

b Add the interest and the loan. $\text{Total to pay} = \$12\,000 + \$3960$

$= \$15\,960$

c Find the number of payments. $\text{Number of repayments} = 12 \times 3 = 36$

Divide to find the amount. $\text{Repayment} = \frac{\$15\,960}{36}$

$\approx \$443.33$

Write the answer. **Ying will have to pay \$443.33 a month for 3 years.**

Personal loans can be secured or unsecured. Example **21** would probably involve a secured loan. This means that if you default on the loan, the lender could repossess and sell the car to get their money back. Secured loans have lower interest rates than unsecured loans. You are usually required to have comprehensive insurance on a car purchased using a secured loan.

Example 22

David has an unsecured personal loan of \$7500 so he can pay for a landscape gardening course. The loan is at 16% simple interest over 2 years. What are his fortnightly repayments?

Solution

Write the formula. $I = Pin$

Write the values of the variables. $P = \$7500, i = 0.16, n = 2$

Substitute the values and work it out. $\text{Interest} = \$7500 \times 0.16 \times 2$

$= \$2400$

Work out the total to repay. $\text{Total to pay back} = \$7500 + \$2400$

$= \$9900$

There are 26 fortnights in a year. $\text{Number of payments} = 26 \times 2 = 52$

Divide to find the amount. $\text{Repayment} = \frac{\$9900}{52} \approx \$190.38$

Write the answer. **David will have to pay \$190.38 a fortnight.**

Buying a house is the biggest purchase that most people ever make. Banks and other institutions advertise their interest rates very aggressively. You need to look carefully at all the fees involved, which can include establishment fees of up to $2000, accounting fees of up to $20 a month, loan insurance, etc. The **comparison rate** is supposed to take all these costs into account to allow you to compare loans.

A loan for purchasing a house is called a **mortgage** if it involves using the property as security. Mortgages are calculated using compound interest, but because the payments you make exceed the interest payable, both the interest and the amount owed gradually decrease. This explains why it is often called **reducing interest**. Mortgage loans are for very long periods and are secured by the property itself. However, like any loan, if you default and the property is sold for less than what you owe, you will still have to pay the outstanding amount.

Example 23

Worksheet
Amount owing on a mortgage
MAT10NAWK00026

Weblink
Home loan repayment calculator
MAT10NAWB00009

Ezra and Kate have just borrowed $300 000 at 7.5% reducing interest to buy their first home. They have to pay $2100 a month for the mortgage. How much have they paid off after 6 months?

Solution

Find the monthly rate.	$\textbf{Monthly interest rate} = 7.5\% \div 12$ $= 0.625\%$
Find the 1st month's interest.	$\textbf{Interest} = 0.00625 \times \$300\ 000$ $= \$1875$
Find the amount owed.	$\textbf{Balance} = \$300\ 000 + \$1875 - \$2100$ $= \$299\ 775$
Find the 2nd month's interest.	$\textbf{Interest} = 0.00625 \times \$299\ 775$ $\approx \$1873.59$
Find the amount owed.	$\textbf{Balance} \approx \$299\ 775 + \$1873.59 - \$2100$ $\approx \$299\ 548.59$
Find the 3rd month's interest.	$\textbf{Interest} \approx 0.00625 \times \$299\ 548.59$ $\approx \$1872.18$
Find the amount owed.	$\textbf{Balance} \approx \$299\ 548.59 + \$1872.18 - \$2100$ $\approx \$299\ 320.77$
Find the 4th month's interest.	$\textbf{Interest} \approx 0.00625 \times \$299\ 320.77$ $\approx \$1870.75$
Find the amount owed.	$\textbf{Balance} \approx \$299\ 320.77 + \$1870.75 - \$2100$ $\approx \$299\ 091.53$
Find the 5th month's interest.	$\textbf{Interest} \approx 0.00625 \times \$299\ 091.53$ $\approx \$1869.32$
Find the amount owed.	$\textbf{Balance} \approx \$299\ 091.53 + \$1869.32 - \$2100$ $\approx \$298\ 860.85$
Find the 6th month's interest.	$\textbf{Interest} \approx 0.00625 \times \$298\ 860.85$ $\approx \$1867.88$
Find the amount owed.	$\textbf{Balance} \approx \$298\ 860.85 + \$1867.88 - \$2100$ $\approx \$298\ 628.73$

Find the amount paid off. **Amount paid off = $300 000 − $298 628.73**

= $1371.27

In Example **23**, although the couple have paid 6 × $2100 = $12 600 in the first six months, they have only reduced their mortgage by $1371.27. It will take them 30 years to pay the loan back and by that time they will have paid $756 000 altogether.

Banks, building societies and other lenders do some checking to make sure they will get their money back, but they don't have to check whether you can afford to pay the mortgage. In recent years, some lenders have badly miscalculated and lost a lot of money lending money to people who cannot pay. Many overseas banks have actually gone bust as a result of such poor lending practices such as lending money to people on the dole.

Calculation of mortgage payments is quite complicated, but the table below gives the costs for some typical loans. There are many mortgage calculators available on the internet.

Monthly payments for mortgages on $1000

Years	Interest rate							
	5%	5.5%	6%	6.5%	7%	7.5%	8%	8.5%
10	10.6066	10.8526	11.1021	11.3548	11.611	11.8702	12.1328	12.3986
15	7.9079	8.1708	8.4386	8.7111	8.9883	9.2701	9.5565	9.8474
20	6.5996	6.8789	7.1643	7.4557	7.753	8.0559	8.3644	8.6782
25	5.8459	6.1409	6.443	6.7521	7.0678	7.3899	7.7182	8.0523
30	5.3682	5.6779	5.9955	6.3207	6.653	6.9921	7.3376	7.6891

Example 24

Puzzle sheet
Mortgage repayments
MAT10NAPS00056

a What would a mortgage of $280 000 over 20 years at 6.5% cost each month?
b How much interest would be paid over the life of the mortgage?

Solution

a Use the 6.5% payment for 20 years in the table above. **Payment for $1000 = $7.4557 /month**

Multiply by 280 for $280 000. **Payment for loan = $7.4557 × 280 /month**

Find the answer. **≈ 2087.60**

Write the answer. **It would cost $2087.60 each month.**

b Multiply the payment by the number of them.

$$\textbf{Total paid} = \$2087.60 \times 12 \times 20$$

Calculate the answer.

$$= \$501\,024$$

Subtract the principal to get the interest.

$$\textbf{Interest} = \$501\,024 - \$280\,000$$
$$= \$221\,024$$

Write the answer.

The total interest paid would be $221 024.

How do you know whether you can afford to make payments on a loan without doing a detailed budget? A good rule of thumb is the 25% rule: you can probably afford to spend 25% of your income (after tax) on repayments. Some financial institutions lift this to 30% for high income earners.

Example 25

Erica and Martha have a combined income of $2120 a week after tax. They have commitments of $360 in existing payments. How much could they afford to borrow using each of the following loans?

a 20-year loan at 6.5% **b** 25-year loan at 6.5% **c** 30-year loan at 6.5%

Solution

a Change to yearly after-tax income.

$$\textbf{Yearly income} = \$2120 \times 52$$
$$= \$110\,240$$

Change to monthly after-tax income.

$$\textbf{Monthly income} = \$110\,240 \div 12$$
$$\approx \$9186.67$$

Divide by 4 to get 25%.

$$\textbf{Total payments} = 9186.67 \div 4$$
$$\approx \$2296.67$$

Subtract commitments.

$$\textbf{Affordable mortgage} = \$2296.67 - \$360$$
$$= \$1936.67$$

Use the 6.5% column for 20 years.

$$\textbf{Payment for \$1000} = \$7.4557$$

Divide to find the maximum loan.

$$\textbf{Maximum amount} = \$1936.67 \div \$7.4557$$

Round *down* to the nearest whole.

$$\approx 259$$

Write the answer.

They could afford a 20-year loan of $259 000.

b Use the 6.5% column for 25 years.

$$\textbf{Payment for \$1000} = \$6.7521$$

Divide to find the maximum loan.

$$\textbf{Maximum amount} = \$1936.67 \div \$6.7521$$

Round *down* to the nearest whole.

$$\approx 286$$

Write the answer.

They could afford a 25-year loan of $286 000.

c Use the 6.5% column for 30 years.

$$\textbf{Payment for \$1000} = \$6.3207$$

Divide to find the maximum loan.

$$\textbf{Maximum amount} = \$1936.67 \div \$6.3207$$

Round *down* to the nearest whole.

$$\approx 306$$

Write the answer.

They could afford a 30-year loan of $306 000.

Conservative lenders like banks will allow you to borrow up to about 90% of the value of a property. Mortgage lenders who allow you to borrow a greater proportion charge higher interest rates. You also need to allow enough to cover legal and transfer costs.

Exercise 9.3 Borrowing money

Fluency

Extra questions
Exercise 9.3
MAT10NAEQ00027

See Example 21

1 Constance has a personal loan for $13 000 to buy a car over 3 years at 11.5%. Work out the following.

a The total interest. **b** The total to repay. **c** The monthly repayments.

2 Mohammed borrowed $17 500 over 4 years at 10.8% flat rate interest. Work out the following.

a The total interest. **b** The total to repay. **c** The fortnightly repayments.

3 Zahra has a loan of $10 900 at 11.7% over 2 years. Work out the following.

a The total interest. **b** The total to repay. **c** The monthly repayments.

See Example 22

4 Find the interest payable and repayments on each of the following unsecured personal loans.

a $12 000 at 17% interest over 2 years with monthly repayments
b $16 600 at 16.8% interest over 3 years with fortnightly repayments
c $11 400 at 19.3% interest over 2 years with weekly repayments
d $22 000 at 15.8% interest over 4 years with monthly repayments
e $15 800 at 17.5% interest over 3 years with fortnightly repayments

See Example 24

5 For each of the following mortgage loans, use the table on page 434 to find the amount that must be paid each month to borrow $1000.

a A loan over 25 years at 7% **b** A loan over 30 years at 6.5%
c A loan over 20 years at 5.5% **d** A loan over 25 years at 7.5%
e A loan over 30 years at 8%

6 Use the table on page 434 to find the monthly payment for each of the following mortgages.

a $320 000 over 30 years at 8% interest
b $480 000 over 25 years at 7.5% interest
c $290 000 over 20 years at 6.5% interest
d $520 000 over 30 years at 7% interest
e $860 000 over 30 years at 6.5% interest

7 For each loan in question **6**, work out the amount of interest that is paid altogether.

Problem solving

See Example 19

8 Callum borrowed $400 from a payday lender for 4 days. The charge was $25 per $100 borrowed plus $2 a day.

a Work out the amount he has to pay back.
b Work out the interest rate.

9 Dorothea borrowed $500 from a cash loans provider for 6 days at a charge of $45 to set up the loan and $20 per hundred dollars or part thereof. She was told that if she paid it back late she would be charged for another loan, for the amount due and a penalty of $60, and given 5 days to pay the new loan.

9780170194860

a Work out the amount she has to pay back.
b Work out the interest rate if it is paid on time.
c Work out the amount she has to pay if she is late.
d Work out the interest rate if it is paid 5 days after it was due.

⑩ Andrew borrowed \$450 from an internet lender for 2 weeks at a rate of \$25 per hundred dollars or part thereof and \$10 per week of the loan. Defaulting would incur a penalty of \$80 and a new loan would be set up for a week for the total amount.
a Work out the amount he has to pay back.
b Work out the interest rate if it is paid on time.
c Work out the amount he has to pay if he is late.
d Work out the interest rate if it is paid a week (7 days) after it was due.

Worked solutions
Exercise 9.3
MAT10NAWS00027

11 Phoong's credit card costs \$90 a year and has an interest rate of 12.6% for purchases and 25.7% for cash advances. His billing period is from the 16th of one month to the 15th of the next and his account is due on the 22nd of each month. He used his credit card in an ATM to get \$400 cash on 20 October. Since the ATM was from another bank he was charged a flat fee of \$5 on the transaction. The same day he used his credit card to buy a takeaway that cost him \$23.65 and had to pay a fee of \$2 for using his credit card. Work out each of the following.
a The total cost of the cash advance (note that the \$5 fee is not subject to interest).
b The interest rate for the cash advance.
c The total cost of the takeaway.
d The interest rate on the takeaway.

See Example 20

12 Anja's credit card costs \$150 a year and has an interest rate of 10.8% on purchases and 20.5% on cash advances. Her billing period ends on the 12th of each month and the account is due on the 26th of the month. She has an 'interest-free' period of up to 45 days, which means that if she pays her account on or before the due date, she pays no interest on purchases. If she doesn't pay on time, she pays interest on purchases with a penalty of 5% on the outstanding amount, then outstanding amounts are charged at the cash rate until paid. Interest is always charged on cash advances. Interest is charged on the '45 days'. For the account due in November she had a total of \$690 in purchases, including \$12 in merchant credit charges and \$500 in cash advances, having paid 2.5% on the \$300 cash advance and \$4 for the other one. She paid this account on time. However, she didn't pay the account due in December until 3 January. This account had \$2424 (including \$15 in merchant credit charges) in purchases and \$900 in cash advances, plus merchant fees of \$18 for the cash advances.
a Explain how you decide to calculate the average daily balance in this case.
b How much did she pay on 28 November?
c What was the interest rate on her cash?
d How much was due on 26 December?
e How much did she have to pay on 3 January?
f What was the overall interest rate?

13 Simon bought \$84.60 worth of groceries on the 15th of the month and was going to get a cash-out of \$100 when he paid. However, when the checkout operator said "Credit card charge is 3%, OK?" he changed his mind and paid cash. His card doesn't have an annual fee but the interest charge for both purchases and cash advances is 26%. His billing period finishes on the 25th of the month.

a How much did he save by changing his mind?

b What would the interest rate have been if he had done as he planned?

See Example 23

14 Dick and Jane just bought a new house with a mortgage of $440 000 at a variable rate of 7.5% over 25 years. Use the table on page 434 to find out the monthly payment on their mortgage and work out how much they have paid off after the first 4 months.

Worked solutions
Exercise 9.3
MAT10NAWS00027

⑮ Peta and Konrad borrowed $360 000 to buy their first house on a mortgage at 6.5% over 25 years. They chose to make higher payments of $3136 a month because they were told this would reduce the term to only 15 years if they could keep it up.

a How much was left to pay after the first 6 months?

b How much extra were they paying each month?

c How much would they save over the term of the loan?

See Example 25

16 Carmelita and James have a combined income of $2920 a week after tax. They have commitments of $474 a month in existing payments. How much could they afford to borrow using each of the following loans?

a 20-year loan at 6% **b** 25-year loan at 6% **c** 30-year loan at 6%

17 Rajendra and Julia have a combined income of $2570 a week after tax. They have commitments of only $136 a month in existing payments. How much could they afford to borrow using each of the following loans?

a 15-year loan at 7.5% **b** 20-year loan at 7.5% **c** 30-year loan at 7.5%

18 Instead of a standard mortgage in question **14**, Dick and Jane could have chosen an 'easy-start' mortgage that had reduced payments of only $2000 a month for the first 6 months. If they had chosen this mortgage, what would they owe after the first 6 months? Approximately what would the payments be after the low start, assuming that the 25 years started after that?

Reasoning

Worked solutions
Exercise 9.3
MAT10NAWS00027

⑲ Alistair has a choice of credit cards. He normally uses a debit card and only intends to use his credit card in an emergency. If you were Alistair, explain which of the following you would choose.

A No annual fee, interest rate of 25% on purchases and cash advances.

B Annual fee of $40, interest rate of 12% on purchases and 24% on cash advances.

C Annual fee of $150, interest rate of 10% on purchases and 21% on cash advances.

D Annual fee of $250, interest rate of 10% on purchases and 21% on cash advances, with an interest-free period of 'up to 45 days'.

20 Tonya has the choice of credit cards given in question **19**, but she normally buys about $1500 worth of things a month and pays on time for her purchases about 75% of the time. She never gets cash-outs. When she is late, she always pays off her card the next month. Explain which card you think she should choose.

Chapter 9 summary

Quiz
Money and finance
MAT10NAQZ00009

- **Interest** is the charge made on a loan. The ratio of the interest for one year to the amount borrowed is called the **interest rate** and is normally written as a decimal or percentage. It is also called the **nominal interest rate**.
- The amount lent or borrowed is called the **principal.**
- The amount that has to be paid back each week or month for a loan on **terms** is called the **instalment** or **repayment**.
- **Simple interest** is calculated using the whole loan for the whole time it is borrowed. It is also called **flat rate interest**.
- **Compound interest** is calculated at regular intervals called **rests**, and added to the loan, with the result that the principal increases at each rest. If you make regular payments on a loan at compound interest, the interest is usually calculated before the payment is subtracted from the principal.
- Compound interest on a loan is often called **reducing interest**.
- A **payday lender** lends small amounts of cash for short periods of time, usually until the borrower's next pay-day.
- The **simple interest formula** for the interest I on the principal P at interest rate i (or $r\%$) for n years is $I = Pin = \frac{Prn}{100}$.
- The **growth factor** is the quantity $1 + \frac{i}{k}$ you multiply the amount accumulated at compound interest by at each rest to get the new amount, where i is the interest rate and k is the number of rests in a year.
- You can use the constant multiplier function on your calculator to help calculate compound interest.
- The **compound interest formula** is $A = P(1+i)^n = P\left(1 + \frac{r}{100}\right)^n$, where A is the amount after n years, P is the principal and i (or $r\%$) is the interest rate. For non-annual rests the formula is $A = P\left(1 + \frac{i}{k}\right)^{kt} = P\left(1 + \frac{r}{100k}\right)^{kt}$, where A is the amount after kt rests, k is the number of rests in a year and t is the number of years.
- **Appreciation** is the increase in value of things over time. The compound interest formula can often be used to predict the value of items that appreciate.
- Failure to pay something according to the contract such as a loan is called **defaulting**.
- A **cash-out** is money cash obtained with a credit or debit card when paying for purchases.
- The **average daily balance** of a credit account is the average of the amounts at the end of each day over the billing period.
- The **daily rate** of interest is the nominal interest rate divided by 365 (the days in a year).
- **Vendor finance** is the provision of a loan by the seller of goods.
- The **comparison rate** for a loan takes account of fees and charges, in addition to the stated interest rate and is supposed to make it easier to compare the rates of different providers.
- A **mortgage** is a secured loan for the purpose of buying property.

Chapter 9 review

Understanding

See Example 4

1 Use the formula for simple interest to calculate each of the following.

a \$760 at 13% for 4 years **b** \$2624 at 6.8% for 3 years

See Example 6

2 Find the growth factor for each of the following cases of compound interest.

a 7.5% compound interest **b** 9.7 compound interest

c 3.8% compound interest

See Example 8

3 Find the growth factor for each of the following cases of compound interest.

a 5.1% compound interest with monthly rests.

b 7.8% compound interest with weekly rests.

Fluency

See Example 1

4 Find the interest rate for each of the following payday loans.

a \$520 for 5 days with a charge of \$25 per \$100 or part thereof.

b \$380 for 9 days with a charge of \$50, plus \$30 a week or part thereof.

See Example 3

5 Find the real simple interest rate for a car available for \$19 990 or weekly payments of \$199.90 a week over 4 years.

See Example 11

6 Find the interest earned in a financial year on a savings account that pays interest of 2.1% on minimum monthly balances if the balances are as shown in the table below.

Month	Jul	Aug	Sep	Oct	Nov	Dec
Balance	\$2317.56	\$2256.27	\$470.29	\$412.39	\$2093.56	\$3064.87
Month	Jan	Feb	Mar	Apr	May	Jun
Balance	\$2185.10	\$265.82	\$310.88	\$400.70	\$222.39	\$418.17

See Example 12

7 How much is a term deposit of \$14 700 at 6.3% for 9 months worth at the end of the term?

See Example 21

8 Afua has a personal loan for \$17 000 to buy a car over 4 years at 11.5%. Work out the following.

a The total interest. **b** The total to repay. **c** The monthly repayments.

See Example 22

9 Find the interest payable and repayments on each of the following unsecured personal loans.

a \$7000 at 16.5% interest over 18 months with monthly repayments

b \$19 500 over 4 years at 18.6% interest with fortnightly repayments

See Examples 5, 6

10 Calculate the amount and interest for each of the following.

a \$8000 at 8% compound interest for 3 years

b \$7900 at 6.7% compound interest for 4 years

See Example 8

11 Calculate the amount and total interest for \$12 300 at 7.8% compound interest for 5 months with monthly rests.

See Example 10

12 \$940 000 is invested at 2.9% for 6 days on the short-term money market. How much interest is earned?

See Example 13

13 A term deposit of \$26 400 for 4 years pays 7.2%. The interest is calculated and paid into a savings account every quarter. How much is paid each quarter?

14 A term deposit of $21 000 over 3 years pays 5.75% interest. How much is it worth at the end of the term?

15 Use the table on page 434 to find the amount that must be paid each month on a mortgage to borrow $1000 at 8% interest over 25 years. *See* Example 24

16 Use the table on page 434 to find the monthly payment for a mortgage of $355 000 over 30 years at 6.5%. How much interest is paid altogether on the mortgage?

17 Use the formula to calculate the amount and interest for $6240 at 7.3% compound interest for 5 years. *See* Example 7

18 Use the formula to calculate the amount and interest for $24 000 at 6.5% compounding fortnightly for 8 years. *See* Example 9

19 What is an investment of $12 300 at 6.9% compounding monthly worth after 5 years? *See* Example 14

Problem solving

20 Derek 'rented' a lounge suite costing $2499 for $31.99 a week on a 48-month contract. Halfway through the contract he paid $300 to upgrade to a newer suite costing $2199, with a new 3-year contract for $25 a week. At the end of the contract he took up the offer to buy the suite for $150. *See* Example 2

- a What was the interest rate for the first suite, assuming that the $300 would have been enough to buy it outright after the 2 years?
- b What was the interest rate for the second contract?

21 How many years would it take an investment of $7200 to reach a value of at least $12 000 at 14% simple interest? *See* Example 16

22 Corey borrowed $360 from a payday lender for 5 days. The charge was $25 per $100 or part thereof borrowed plus an establishment fee of $20. Unfortunately he could not pay on time and had to pay a penalty of $75, and the loan and costs were rolled over into a new loan that was paid on his next payday, 7 days later. *See* Example 19

- a Work out the amount he had to pay back in the first instance.
- b Work out the interest rate for the first loan.
- c How much did he eventually have to pay back?
- d What was the overall interest rate?

See Example 17

23 How many years would it take an investment of $8500 to reach a value of at least $12 000 at 4.6% compound interest?

See Example 20

24 Wilhemina's credit card costs $120 a year and has an interest rate of 11.4% for purchases and 23.2% for cash advances. Her billing period ends on the 20th of each month. One month she bought a new outfit for $420 on the 2nd and got a cash advance of $200 on the 13th. She had to pay merchant charges of $5 and 3.5% respectively to use her credit card for these transactions.

Work out each of the following.

a The total cost of the new clothes. **b** The interest rate for the new clothes.

c The total cost of the cash advance. **d** The interest rate for the cash advance.

See Example 23

25 Carol bought a new house with a mortgage of $325 000 at a variable rate of 6% over 25 years. Use the table on page 434 to find out the monthly payment on her mortgage and work out how much she has paid off after the first 5 months.

See Example 25

26 Stevie has an income of $2420 a week after tax. He has commitments of $220 a month in existing payments. How much could he afford to borrow using each of the following loans?

a 20-year loan at 6.5% **b** 25-year loan at 6.5% **c** 30-year loan at 6.5%

See Example 18

27 How long would it take an investment of $12 000 to reach a value of $20 000 at 5.7% compounding monthly?

See Example 15

28 A stamp collection appreciates at 13% each year. If it is currently worth $12 500, what will it be worth in 5 years time?

Reasoning

29 Find the number of years it would take for an amount to at least triple with a compound interest rate of 10%, explaining your method.

30 Find the simple interest rate that will give the same return as an investment at 12% compounded over 4 years, clearly explaining your steps.

31 Find the interest rate that, compounding monthly, gives the same interest as 24% simple interest over 2 years, logically explaining your steps.

32 Colleen has a choice of the following credit cards. She uses her credit card for an average of $2400 in purchases and cash advances of about $1000 month. Explain which of the following you would recommend for her.

A No annual fee, interest rate of 24% on purchases and cash advances.
B Annual fee of $30, interest rate of 14% on purchases and 24% on cash advances.
C Annual fee of $120, interest rate of 12% on purchases and 21% on cash advances.
D Annual fee of $200, interest rate of 12% on purchases and 21% on cash advances, with an interest-free period of 'up to 45 days'.

Measurement and geometry

10 Trigonometry

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Contents

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MAT10MGPL00021

Parent guide
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Australian Curriculum statements

Pythagoras and trigonometry

Solve right-angled triangle problems including those involving direction and angles of elevation and depression. (10)

Establish the sine, cosine and area rules for any triangle and solve related problems. (10A)

Use the unit circle to define trigonometry functions, and graph them with and without the use of digital technologies. (10A)

Apply Pythagoras’ theorem and trigonometry to solving three-dimensional problems in right-angled triangles. (10A)

Video tutorial
Trigonometry
MAT10MGVT00010

Sumerian astronomers were the first to use angle measure, by dividing circles into 360 degrees. They also studied the ratios of sides of similar triangles. They were followed by the Babylonians who extended the study of the ratios of the sides of similar triangles to discover some of the major properties of these ratios. It was, however, the ancient Greeks who transformed trigonometry into a formal field of mathematics. In recognition of the work of the Greeks, the word trigonometry is derived from the Greek roots *trigon*—triangle and *metron*—measure.

Trigonometry is a branch of mathematics that uses the relationships between the sides and angles of triangles. The earliest applications of trigonometry were in the fields of astronomy, navigation and surveying. By using trigonometry, it became possible to calculate distances that could not be measured directly, such as the distance between the Earth and the Moon, or the height of a very tall structure.

The pyramids of Egypt and the megaliths of Stonehenge in England were built using a detailed knowledge of angles.

Applications of trigonometry are found in physics and virtually all branches of engineering, particularly in the study of periodic phenomena, such as sound vibrations and the flow of alternating current.

Mathematical literacy

Maths dictionary
MAT10ASDI00001

The mathematical words below have special meanings that you will learn in this chapter. It is important that you learn to spell them and gradually learn what they mean in mathematics. You may find the glossary or online mathematical dictionary useful for this purpose.

adjacent side
ambiguous case
angle of depression
angle of elevation
bearing
Cartesian plane
compass
compass rose
composite figures
cosine (cos)
cosine rule
direction
hypotenuse
line of sight
opposite side
origin
period
Pythagoras' theorem
Pythagorean triad
Pythagorean triple
right-angled
sine (sin)
sine rule
standard triangle
trigonometric (trig) ratio
tangent (tan)
three dimensions
two dimensions
unit circle

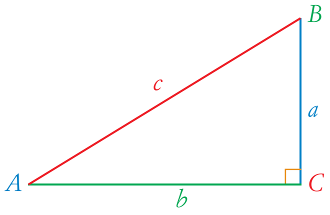
10.1 Pythagoras' theorem

There is an accepted naming convention for triangles. The vertices are named using capital letters, and the opposite sides have the same lower-case letter.

In $\triangle ABC$:

a is opposite $\angle A$, b is opposite $\angle B$ and c is opposite $\angle C$

and $AB = c$, $BC = a$ and $AC = b$.



It is also customary to indicate the right angle with a small square in the corner. In a right-angled triangle, the longest side is the **hypotenuse**. The hypotenuse is opposite the right angle.

In Chapter 4, we saw how Pythagoras' theorem could be used to find the lengths of unknown sides when calculating the areas of triangles and other shapes.

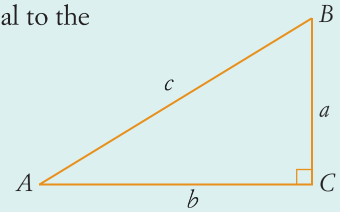
Important!

Pythagoras' theorem

In a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.
So for $\triangle ABC$:

$$AB^2 = BC^2 + AC^2$$

or $$c^2 = a^2 + b^2$$



TLF Learning object

Exploring the Pythagorean theorem (L6559)

MAT10MGIN00010

Teacher notes

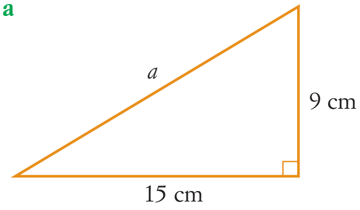
Proof of Pythagoras' theorem

MAT10MGTN00008

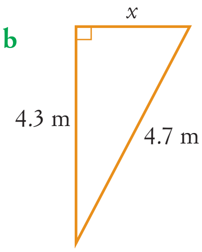
Example 1

Find the unknown side in each of the following triangles.

a



b



Solution

a Apply Pythagoras' theorem to the triangle. $a^2 = 15^2 + 9^2$

Evaluate. $= 225 + 81$

$= 306$

Take the square root of both sides. $a = 17.4928 \ldots$ cm

Round off and state the result. **The unknown side is about 17.5 cm.**

b Apply Pythagoras' theorem to the triangle. $4.7^2 = 4.3^2 + x^2$

Evaluate. $22.09 = 18.49 + x^2$

Reverse the equation. $x^2 + 18.49 = 22.09$

Subtract 18.49 from both sides. $x^2 = 22.09 - 18.49$

$= 3.6$

Take the square root of both sides. $x = 1.8973 \ldots$ m

Round off and state the result. **The unknown side is about 1.9 m.**

CAS TI-Nspire exercise

Trigonometry

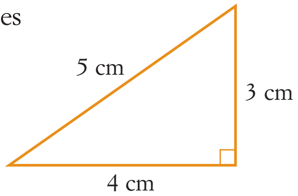
MAT10MGTI00017

CAS ClassPad exercise

Trigonometry

MAT10MGCP00017

There are some right-angled triangles in which the lengths of the sides are whole numbers. Consider the triangle shown here.
You can see that:



$$5^2 = 4^2 + 3^2$$
$$25 = 16 + 9$$

Teacher notes
Pythagorean triples
MAT10MGTN00009

Any three whole numbers that satisfy Pythagoras' theorem (e.g. 3, 4, 5) are known as a **Pythagorean triple** (or **triad**). It can easily be shown that any multiple of a Pythagorean triple is also a Pythagorean triple. For example:

$10^2 = 8^2 + 6^2$ because $100 = 64 + 36$, so 6, 8, 10 is a Pythagorean triple
$15^2 = 12^2 + 9^2$ because $225 = 144 + 81$, so 9, 12, 15 is a Pythagorean triple

and so on.

Puzzle sheet
Pythagorean triads
MAT10MGPS00030

Important!

Pythagorean triples (or triads)

Common Pythagorean triples include:

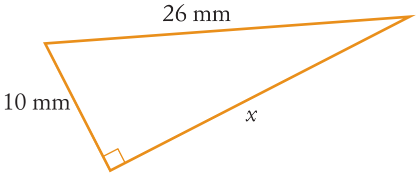
3, 4, 5   5, 12, 13   8, 15, 17   7, 24, 25

Any multiple of these Pythagorean triples will also be a Pythagorean triple.

Pythagorean triples can sometimes be used as a 'short cut' when solving triangles.

Example 2

Find the unknown side in this triangle.



Solution

Look at the factors of the side lengths.	
The shortest side is 5×2 and the hypotenuse is 13×2.	**There is a 5, 12, 13 triple.**
The sides are 2 times the triple 5, 12, 13.	$\boldsymbol{x = 12 \times 2 = 24}$ **mm**
State the result.	**The unknown side is 24 mm.**

You can use Pythagoras' theorem to check whether a triangle is right-angled. Builders often use Pythagoras' theorem to check whether corners are square.

Important!

Checking triangles

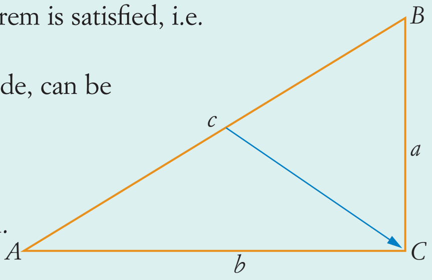
$\triangle ABC$ shown here is right-angled if Pythagoras' theorem is satisfied, i.e. if $c^2 = a^2 + b^2$.

In fact, the size of C, the angle opposite the longest side, can be checked as follows.

If $c^2 < a^2 + b^2$, $C < 90°$ and $\triangle ABC$ is acute-angled.

If $c^2 = a^2 + b^2$, $C = 90°$ and $\triangle ABC$ is right-angled.

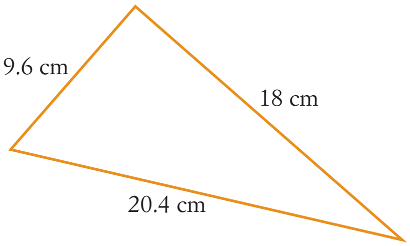
If $c^2 > a^2 + b^2$, $C > 90°$ and $\triangle ABC$ is obtuse-angled.



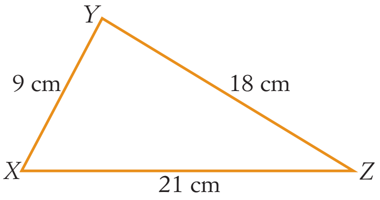
Example 3

Determine whether each of the following triangles is acute-, obtuse- or right-angled.

a



b



Solution

a If the triangle is right-angled, the longest side will be the hypotenuse and Pythagoras' theorem will apply, so $20.4^2 = 9.6^2 + 18^2$.

Find the sum of the squares of the shorter sides.

$$\mathbf{9.6^2 + 18^2 = 92.16 + 324}$$
$$\mathbf{= 416.16}$$

Find the square of the longest side.

$$\mathbf{20.4^2 = 416.16}$$

Compare the results.

$$\mathbf{20.4^2 = 9.6^2 + 18^2}$$

This satisfies Pythagoras' theorem. **The triangle is right-angled.**

b Find the square of the longest side.

$$\mathbf{21^2 = 441}$$

Find the sum of the squares of the shorter sides.

$$\mathbf{18^2 + 9^2 = 324 + 81}$$
$$\mathbf{= 405}$$

Compare the results.

$$\mathbf{21^2 > 18^2 + 9^2}$$

Write the value of Y. $\mathbf{Y > 90°}$

Identify the triangle. **$\triangle XYZ$ is obtuse-angled.**

Pythagoras' theorem has many applications to practical situations.

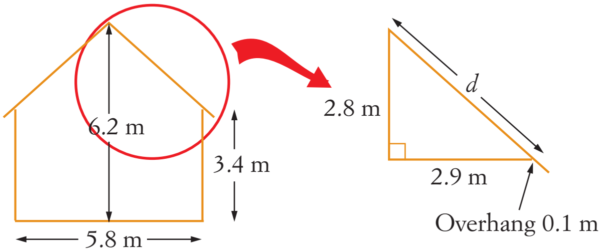
Example 4

The front of this machinery shed is 5.8 m wide. The sides are 3.4 m high and the highest point of the roof is 6.2 m above its base. The roof of the shed is made from sheets of iron that overhang by 10 cm on both sides.
How long are the sheets of iron?

Solution

Draw a diagram and add dimensions.

Change the overhang to the same units (m).



Apply Pythagoras' theorem to the triangle.	$d^2 = 2.9^2 + 2.8^2$
Evaluate.	$= 8.41 + 7.84$
	$= 16.25$
Take the square root of both sides.	$d = 4.031\ 1\ldots \approx 4.03$ m
The overhang needs to be added.	**Sheet length** $\approx 4.03 + 0.1$ **m**
Round off and state the result.	**The roof sheets are about 4.13 m (4130 mm) long.**

Exercise 10.1 Pythagoras' Theorem

In this exercise, answers should be rounded to 1 decimal place where necessary.

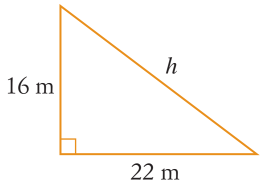
Fluency

Extra questions
Exercise 10.1
MAT10MGEQ00028

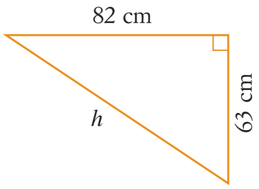
See Example 1

1 Calculate the hypotenuse in each of the following triangles.

a



b



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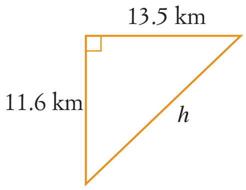

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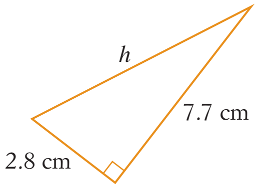

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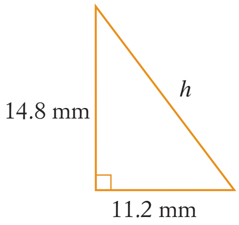

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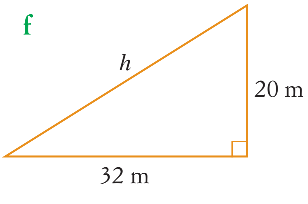
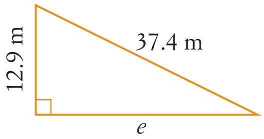


9780170194860

2 Calculate the unknown side in each of the following triangles.

a



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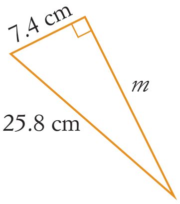

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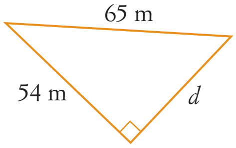

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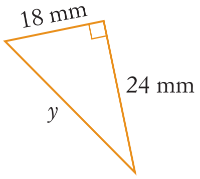

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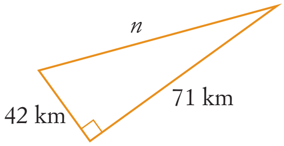

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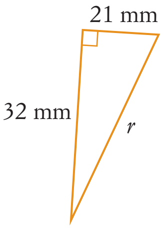

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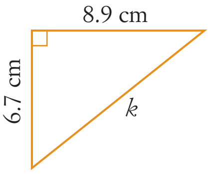

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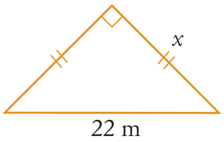
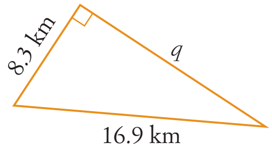


i



3 Use Pythagorean triples to find the unknown side in each of the following triangles.

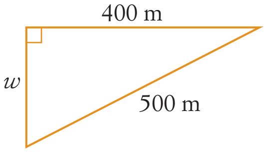
Worked solutions

Exercise 10.1

MAT10MGWS00061

See Example 2

a



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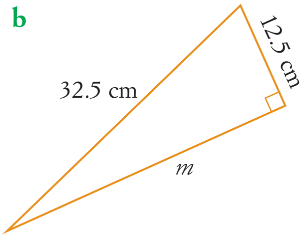

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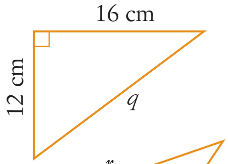

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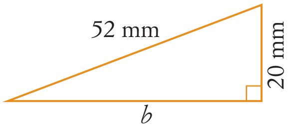

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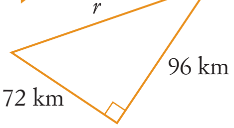
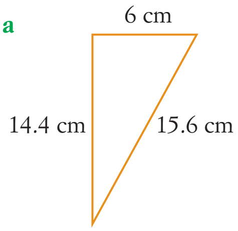


4 Determine whether the following triangles are acute-, obtuse- or right-angled.

See Example 3

a



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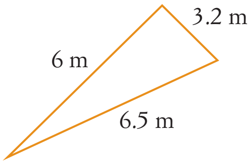

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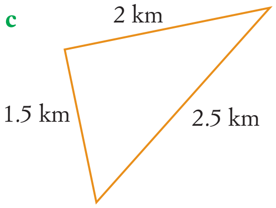

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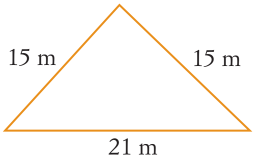

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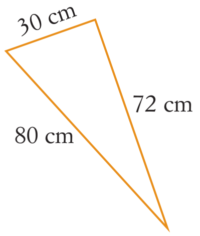

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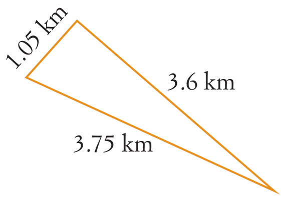
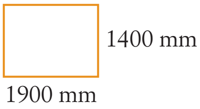


Problem solving

Worked solutions
Exercise 10.1
MAT10MGWS00061

5 Find the length of the diagonals in each of the following rectangles and squares.

a


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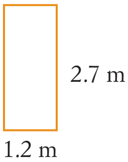
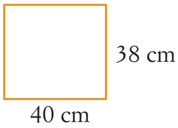


c


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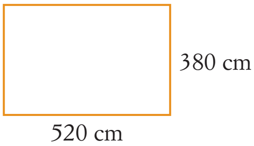

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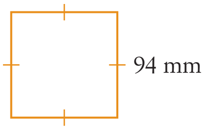

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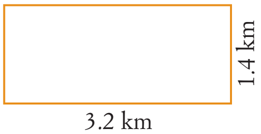
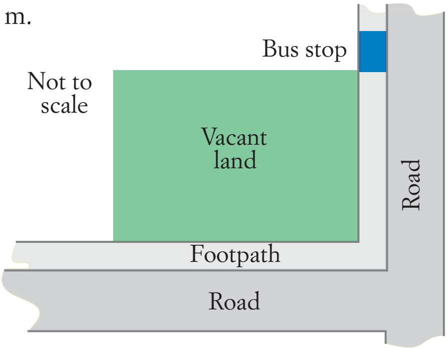


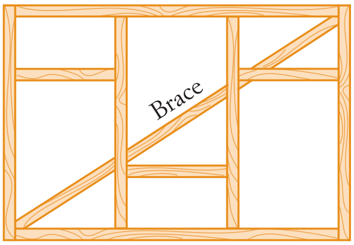
Worked solutions
Exercise 10.1
MAT10MGWS00061

6 A vacant corner block of land measures 32 m by 22 m. Instead of going around the footpath, many people walk diagonally across the unfenced block to reach a bus stop. How much further will they need to walk when the block is fenced?



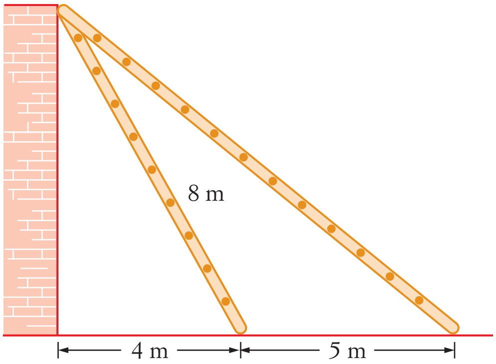
See Example 4

7 A frame is being constructed for a wall 3.5 m long and 2.4 m high. The wall requires a brace as shown here. How long should the brace be?

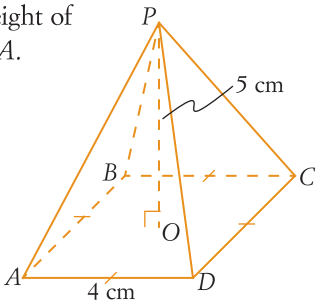


Puzzle sheet
Pythagorean two-step problems
MAT10MGPS00031

8 Two ladders are resting at the top of a wall as shown here. The shorter ladder is 8 m long and its bottom end is 4 m from the base of the wall. The end of the longer ladder is 5 m from the end of the shorter ladder on the ground. How long is the longer ladder?



9 *ABCDP* is a pyramid with a square base of side 4 cm. The height of the pyramid (*OP*) is 5 cm. Calculate the length of the edge *PA*.



Reasoning

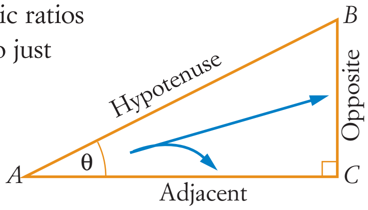
10 A cyclist travels 35 km north of her starting point and then travels 46 km due west. How far is she from her starting point 'as the crow flies'?

11 A rectangular room measuring 4 m by 8 m is to be used for a home theatre. The walls of the room are 3 m high. The projector is to be mounted on the ceiling halfway between the longer side walls and facing one of the 4 m wide walls where the screen will be mounted. The screen measures 1.6 m by 0.9 m and is set up with its centre in the centre of the 4 m wide wall. If the optimum distance for the projector is 7 m from the centre of the screen, how far along the ceiling from the wall on which the screen is mounted should the projector be installed?

10.2 Trigonometric ratios

TLF Learning object
Exploring trigonometry (L6561)
MAT10MGIN00010

You have previously done some work with trigonometric ratios in Year 9. We often shorten the word 'trigonometric' to just 'trig'. It's easier to say.
The diagram on the right shows a right-angled triangle, △*ABC*. An angle has been marked with the Greek letter θ (theta). Greek letters are often used to represent angles in trigonometry because of the enormous contribution that Greek mathematicians made to this field of study.



In △*ABC*:

- *AB* is called the **hypotenuse**. The hypotenuse is always the longest side. It is also the side opposite the right angle.
- *AC* is called the side **adjacent** to θ. (Adjacent means 'next to'.)
- *BC* is called the side **opposite** θ.

The three trig ratios that we will study this year are the sine, cosine and tangent ratios.

Important!

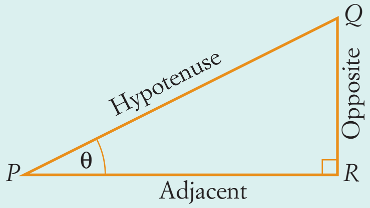
Trig ratios

In any right-angled triangle *PQR*,

$$\text{sine } \theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{QR}{PQ}$$

$$\text{cosine } \theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{PR}{PQ}$$

$$\text{tangent } \theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{QR}{PR}$$



The names of the trig ratios are usually shortened as follows:

sine ⟺ **sin** cosine ⟺ **cos** tangent ⟺ **tan**

The acronym ‘SOH CAH TOA’ may help you to remember these ratios.

Important!

sin = opp/hyp
S O H
Some Old Hams

cos = adj/hyp
C A H
Can’t Always Hide

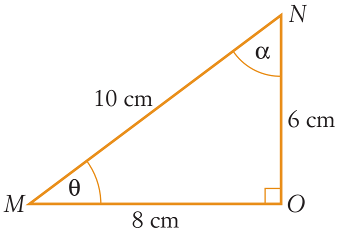
tan = opp/adj
T O A
Their Old Age

Example 5

For △*MNO*:

a name the side opposite θ

b express cos α as a ratio in simplest form.



Solution

a Imagine drawing a line from θ across the triangle. ***NO* is the side opposite θ.**

b Refer to the rule for the cos ratio.

$$\cos\alpha = \frac{\textbf{adjacent}}{\textbf{hypotenuse}}$$

Substitute information from the diagram.

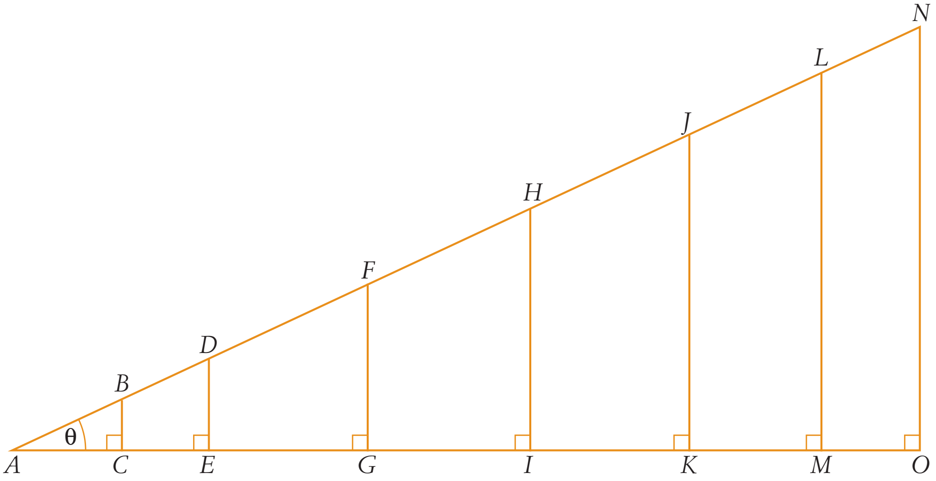
$$= \frac{\textbf{6 cm}}{\textbf{10 cm}}$$

Simplify.

$$= \tfrac{3}{5}$$

Investigate: Trig ratios

1 Look at the following diagram. How many right-angled triangles can you see?



2 Copy the table below and then complete it by measurement and calculation. Measure all lengths to the nearest mm and express any calculations correct to 3 decimal places.

$\triangle ABC$	$\triangle ADE$	$\triangle AFG$	$\triangle AHI$	$\triangle AJK$	$\triangle ALM$	$\triangle ANO$
$AB = \ldots$	$AD = \ldots$	$AF = \ldots$	$AH = \ldots$	$AJ = \ldots$	$AL = \ldots$	$AN = \ldots$
$AC = \ldots$	$AE = \ldots$	$AG = \ldots$	$AI = \ldots$	$AK = \ldots$	$AM = \ldots$	$AO = \ldots$
$BC = \ldots$	$DE = \ldots$	$FG = \ldots$	$HI = \ldots$	$JK = \ldots$	$LM = \ldots$	$NO = \ldots$
$\frac{BC}{AB} = \ldots$	$\frac{DE}{AD} = \ldots$	$\frac{FG}{AF} = \ldots$	$\frac{HI}{AH} = \ldots$	$\frac{JK}{AJ} = \ldots$	$\frac{LM}{AL} = \ldots$	$\frac{NO}{AN} = \ldots$
$\frac{AC}{AB} = \ldots$	$\frac{AE}{AD} = \ldots$	$\frac{AG}{AF} = \ldots$	$\frac{AI}{AH} = \ldots$	$\frac{AK}{AJ} = \ldots$	$\frac{AM}{AL} = \ldots$	$\frac{AO}{AN} = \ldots$
$\frac{BC}{AC} = \ldots$	$\frac{DE}{AE} = \ldots$	$\frac{FG}{AG} = \ldots$	$\frac{HI}{AI} = \ldots$	$\frac{JK}{AK} = \ldots$	$\frac{LM}{AM} = \ldots$	$\frac{NO}{AO} = \ldots$

3 Measure the size of θ using your protractor.

4 Locate the trig functions on your calculator. Your teacher will help you if you need help to use these functions. Use your calculator to find the values of sin θ, cos θ and tan θ.

5 Check back with the values in the table you have just completed. What do you notice?

6 What do the values in each row represent?

You saw in the previous investigation that the values of sin θ, cos θ and tan θ remain constant for any value of θ, no matter how long the sides of the triangle in which θ occurs. Because of this fact, it is possible to program calculators and computers with the values for sin θ, cos θ and tan θ. A table of trig ratios is provided on the following page for you to use, but it is more accurate to use the trigonometric functions on a scientific calculator. Different calculators use slightly different logic, so you will need to make sure that you can use the trig functions on *your own calculator*. Make sure that your calculator is in the *degree mode*.

Trigonometric ratios

Deg.	Sin	Cos	Tan	Deg.	Sin	Cos	Tan	Deg.	Sin	Cos	Tan
0	0.000	1.000	0.000	**30**	0.500	0.866	0.577	**60**	0.866	0.500	1.732
1	0.017	1.000	0.017	**31**	0.515	0.857	0.601	**61**	0.875	0.485	1.804
2	0.035	0.999	0.035	**32**	0.530	0.848	0.625	**62**	0.883	0.469	1.881
3	0.052	0.999	0.052	**33**	0.545	0.839	0.649	**63**	0.891	0.454	1.963
4	0.070	0.998	0.070	**34**	0.559	0.829	0.675	**64**	0.899	0.438	2.050
5	0.087	0.996	0.087	**35**	0.574	0.819	0.700	**65**	0.906	0.423	2.145
6	0.105	0.995	0.105	**36**	0.588	0.809	0.727	**66**	0.914	0.407	2.246
7	0.122	0.993	0.123	**37**	0.602	0.799	0.754	**67**	0.921	0.391	2.356
8	0.139	0.990	0.141	**38**	0.616	0.788	0.781	**68**	0.927	0.375	2.475
9	0.156	0.988	0.158	**39**	0.629	0.777	0.810	**69**	0.934	0.358	2.605
10	0.174	0.985	0.176	**40**	0.643	0.766	0.839	**70**	0.940	0.342	2.747
11	0.191	0.982	0.194	**41**	0.656	0.755	0.869	**71**	0.946	0.326	2.904
12	0.208	0.978	0.213	**42**	0.669	0.743	0.900	**72**	0.951	0.309	3.078
13	0.225	0.974	0.231	**43**	0.682	0.731	0.933	**73**	0.956	0.292	3.271
14	0.242	0.970	0.249	**44**	0.695	0.719	0.966	**74**	0.961	0.276	3.487
15	0.259	0.966	0.268	**45**	0.707	0.707	1.000	**75**	0.966	0.259	3.732
16	0.276	0.961	0.287	**46**	0.719	0.695	1.036	**76**	0.970	0.242	4.011
17	0.292	0.956	0.306	**47**	0.731	0.682	1.072	**77**	0.974	0.225	4.331
18	0.309	0.951	0.325	**48**	0.743	0.669	1.111	**78**	0.978	0.208	4.705
19	0.326	0.946	0.344	**49**	0.755	0.656	1.150	**79**	0.982	0.191	5.145
20	0.342	0.940	0.364	**50**	0.766	0.643	1.192	**80**	0.985	0.174	5.671
21	0.358	0.934	0.384	**51**	0.777	0.629	1.235	**81**	0.988	0.156	6.314
22	0.375	0.927	0.404	**52**	0.788	0.616	1.280	**82**	0.990	0.139	7.115
23	0.391	0.921	0.424	**53**	0.799	0.602	1.327	**83**	0.993	0.122	9.144
24	0.407	0.914	0.445	**54**	0.809	0.588	1.376	**84**	0.995	0.105	9.514
25	0.423	0.906	0.466	**55**	0.819	0.574	1.428	**85**	0.996	0.087	11.430
26	0.438	0.899	0.488	**56**	0.829	0.559	1.483	**86**	0.998	0.070	14.301
27	0.454	0.891	0.510	**57**	0.839	0.545	1.540	**87**	0.999	0.052	19.081
28	0.469	0.883	0.532	**58**	0.848	0.530	1.600	**88**	0.999	0.035	28.636
29	0.485	0.875	0.554	**59**	0.857	0.515	1.664	**89**	1.000	0.017	57.290
								90	1.000	0.000	n.d.

Note that all values in the trig table (except for the trig ratios for 0° and 90°, and the values for sin 30°, tan 45° and cos 60°) are approximate as each has been rounded off to 3 decimal places.

Example 6

Find the value of tan 58°, correct to 3 decimal places.

Solution

Using a calculator:

Enter as tan 58 = → tan(58) 1.600334529

Round off. **tan 58° ≈ 1.600**

Using the trig table:

Locate 58 in the degree column and read the tan value. **tan 58° ≈ 1.600**

If you know the value of a trig ratio for an angle, you can work out the angle.

Example 7

If tan β = 0.67, find the value of β.

Solution

Using a calculator:

Enter as SHIFT tan 0.67 = → $\tan^{-1}(0.67)$ 33.82208522

Round off. $\beta \approx \mathbf{33.8°}$

Using the trig table:

Locate values in the tan column close to 0.67 and read the degree values.

tan 33° = 0.649

tan 34° = 0.675

Select the degree value whose tan is closest to 0.67. $\beta \approx \mathbf{34°}$

It is easy to say 'find the angle whose sine is 0.5' but it isn't as easy to write. For this reason mathematicians created a shorthand way of writing this phrase.

Important!

Inverse trig notation

$\sin^{-1} x$ means the angle whose sine is x

$\cos^{-1} x$ means the angle whose cosine is x

$\tan^{-1} x$ means the angle whose tangent is x

It is important to remember that the $^{-1}$ in $\sin^{-1} x$ is not a power – it means the inverse or opposite.

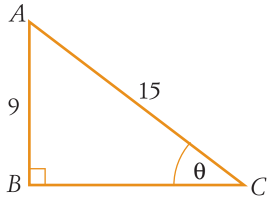
You can use the lengths of the sides of right-angled triangles and trig ratios to find the values of angles in the triangles.

CAS TI-Nspire exercise
Trigonometry
MAT10MGTI00017

CAS ClassPad exercise
Trigonometry
MAT10MGCP00017

Example 8

Find the value of θ in the triangle shown here.



Solution

AB and *AC* are the known sides of △*ABC*. *AB* is opposite θ and *AC* is the hypotenuse, so use sin.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$= \frac{AB}{AC}$$

Substitute the side lengths.

$$= \frac{9}{15}$$

Simplify.

$$= 0.6$$

Use your calculator to find θ.

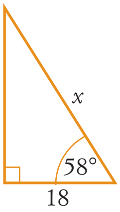
$$\theta \approx 36.9°$$

Trig ratios can also be used to calculate the lengths of sides in right-angled triangles.

Puzzle sheet
Trigonometry equations
MAT10MGPS00032

Example 9

Find the value of *x* in this triangle.



Solution

The known side, 18, is adjacent to the given angle, 58°, and *x* is the hypotenuse, so use cos.

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 58° = \frac{18}{x}$$

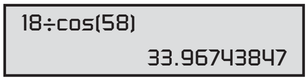
Multiply both sides by *x*.

$$x\cos 58° = 18$$

Divide both sides by cos 58°.

$$x = \frac{18}{\cos 58°}$$

Evaluate. Enter as: 18 ÷ cos 58 = →



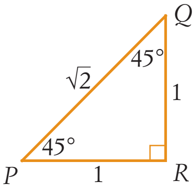
```
18÷cos(58)
             33.96743847
```

Round off and state the result.

$$x \approx 34$$

Exact trigonometric ratios of 30°, 45° and 60°

Consider the right-angled $\triangle PQR$ with $\angle P = \angle Q = 45°$.
If $PR = 1$ unit
then $QR = 1$ unit ($\triangle PQR$ is isosceles)
and $PQ = \sqrt{2}$ units (using Pythagoras' theorem)



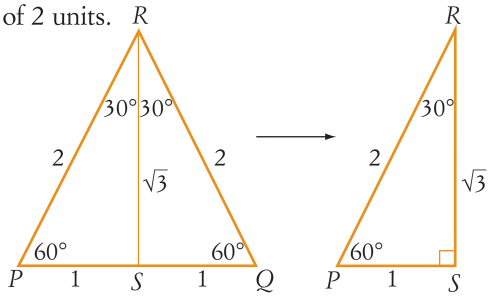
You can use the diagram to state the exact value for the trig ratios of 45°.

Important!

Exact trigonometric ratios of 45°

$$\sin 45° = \frac{1}{\sqrt{2}} \qquad \cos 45° = \frac{1}{\sqrt{2}} \qquad \tan 45° = 1$$

Consider the equilateral $\triangle PQR$ having a side length of 2 units.
RS is drawn perpendicular to PQ.
It can be shown that $\angle PRS = \angle QRS = 30°$ and
$PS = QS = 1$ unit



$$\begin{aligned} RS^2 &= PR^2 - PS^2 \text{(using Pythagoras' theorem)} \\ &= 2^2 - 1^2 \\ &= 3 \end{aligned}$$

$RS = \sqrt{3}$ units (taking square roots of both sides)

You can use the diagram to state the exact value for the trig ratios of 30° and 60°.

Teacher notes
Special trig ratios
MAT10MGTN00010

Important!

Exact trigonometric ratios of 30° and 60°

$$\sin 30° = \frac{1}{2} \qquad \cos 30° = \frac{\sqrt{3}}{2} \qquad \tan 30° = \frac{1}{\sqrt{3}}$$

$$\sin 60° = \frac{\sqrt{3}}{2} \qquad \cos 60° = \frac{1}{2} \qquad \tan 60° = \sqrt{3}$$

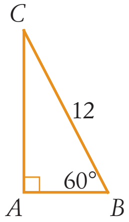
The triangles used to calculate the exact values of 30°, 45° and 60° are called **standard triangles**.

Example 10

CAS TI-Nspire exercise
Trigonometry
MAT10MGTI00017

CAS ClassPad exercise
Trigonometry
MAT10MGCP00017

Find the exact value of *AC*.



Solution

AC is opposite 60° and *BC* is the hypotenuse, so use sin.

$$\sin 60° = \frac{AC}{12}$$

Isolate *AC* on the LHS of the equation.

$$AC = 12 \sin 60°$$

Substitute the exact value of sin 60°.

$$= 12 \times \frac{\sqrt{3}}{2}$$

Evaluate.

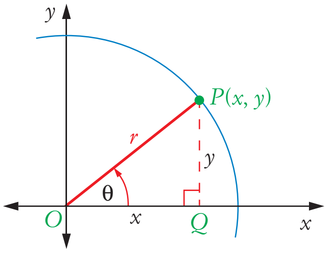
$$= 6\sqrt{3}$$

Now we will take a more detailed look at the values of trig functions. The diagram below shows a point $P(x, y)$ in the first quadrant of the Cartesian plane. By constructing the right-angled $\triangle OPQ$ as shown, we can write:

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \sin \theta = \frac{y}{r}$$

$$\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \cos \theta = \frac{x}{r}$$

$$\tan \theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \tan \theta = \frac{y}{x}$$

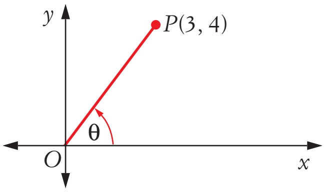


Example 11

P is the point (3, 4). *OP* makes an angle of θ with the *x*-axis as shown here.

a sin θ

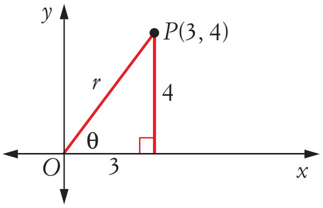
b cos θ



Solution

a Complete the right-angled triangle and label *OP* as *r*.

Because *P* is (3, 4), you can mark in the lengths of the other sides of the triangle as shown.



We need to know the length of the hypotenuse to calculate sin θ.

Use Pythagoras' theorem.

$$r^2 = 3^2 + 4^2$$

$$= 9 + 16$$

$$= 25$$

Take the square root of each side. $\mathbf{r = 5}$

Use the rule for sin θ where θ is drawn in the Cartesian plane. $\mathbf{\sin\ \theta = \frac{y}{r}}$

Substitute known values. $\mathbf{\sin\ \theta = \frac{4}{5}}$

b Use the rule for cos θ where θ is drawn in the Cartesian plane. $\mathbf{\cos\ \theta = \frac{x}{r}}$

Substitute known values. $\mathbf{\cos\ \theta = \frac{3}{5}}$

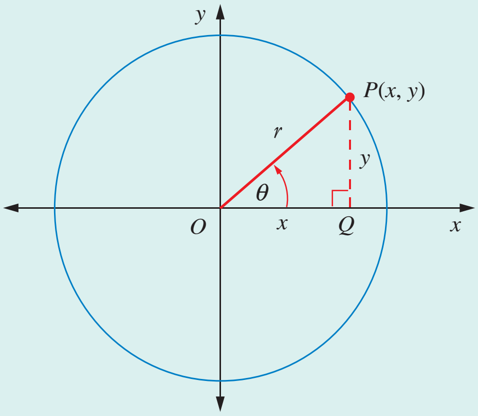
These definitions for sin θ, cos θ and tan θ are true for points in all four quadrants of the Cartesian plane (i.e. they work for angles greater than 90°).

Important!

Trigonometric ratios and the Cartesian plane

For any point $P(x, y)$ on a circle of radius r with centre at the origin:

$$\sin\theta = \frac{y}{r},\ \cos\theta = \frac{x}{r} \text{ and } \tan\theta = \frac{y}{x}$$



Technology: GeoGebra

Angles of any magnitude

MAT10MGTC00010

Weblink

The interactive Mathematics classroom: Trigonometry

MAT10MGWB00010

It is possible to express tan θ in terms of sin θ and cos θ.
You can see from the unit circle that:

$$\sin\theta = \frac{y}{r} \text{ and } \cos\theta = \frac{x}{r}$$

$$\text{So, } \frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}}$$

$$= \frac{y}{\cancel{r}} \times \frac{\cancel{r}}{x}$$

$$= \frac{y}{x}$$

$$\text{But } \tan\theta = \frac{y}{x}$$

$$\text{So, } \tan\theta = \frac{\sin\theta}{\cos\theta}$$

Important!

Definition of tan θ

For any angle θ

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

Example 12

Puzzle sheet

Trigonometry: Exact values

MAT10MGPS00057

If $\sin \beta = \frac{2}{\sqrt{5}}$ and $\cos \beta = \frac{1}{\sqrt{5}}$, find tan β.

Solution

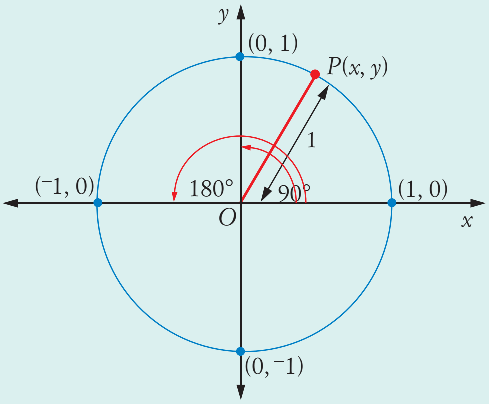
Apply the definition of tan θ to β.	$\mathbf{tan}\ \beta = \frac{\mathbf{sin}\ \beta}{\mathbf{cos}\ \beta}$
Substitute for known values.	$= \frac{\frac{2}{\sqrt{5}}}{\frac{1}{\sqrt{5}}}$
Express as a multiplication and cancel.	$= \frac{2}{\cancel{\sqrt{5}}} \times \frac{\cancel{\sqrt{5}}}{1}$
	$= \frac{2}{1}$
State the result.	$\mathbf{tan}\ \beta = 2$

We can use the **unit circle** to help us understand more about trig ratios.

Important!

Unit circle

The unit circle has its centre at the origin and a radius of 1.



Example 13

Use the unit circle to find the value of:

a cos 180° **b** tan 180° **c** sin 270°

Solution

a Use the definition for cos θ. $\cos\theta = \frac{x}{r}$

Locate 180° on the unit circle and substitute values. $\cos 180^\circ = \frac{{}^{-}1}{1}$

Evaluate. $\cos 180^\circ = {}^{-}1$

b Use the definition for tan θ. $\tan\theta = \frac{y}{x}$

Locate 180° on the unit circle and substitute values. $\tan 180^\circ = \frac{0}{{}^{-}1}$

Evaluate. $\tan 180^\circ = 0$

c Locate 270° on the unit circle and use the definition for sin θ. $\sin 270^\circ = \frac{{}^{-}1}{1}$

Evaluate. $\sin 270^\circ = {}^{-}1$

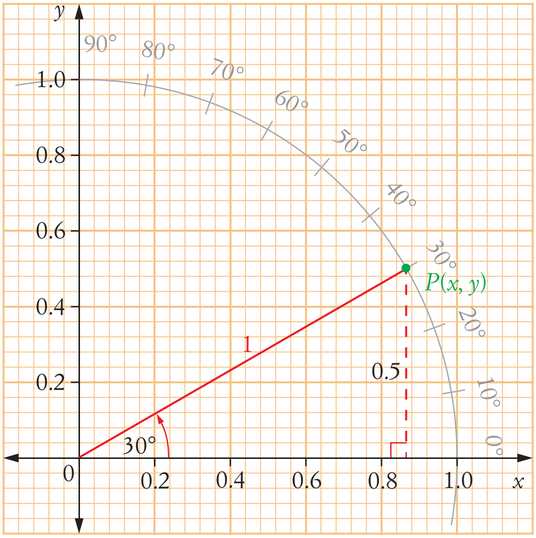
Use your calculator to check the values of the ratios in Example **13**.

Let's look at the unit circle in a little more detail. The diagram below shows the unit circle in the first quadrant.

Worksheet

Unit circle investigation

MAT10MGWK00042



In this case, the radius of the unit circle makes an angle of 30° with the *x*-axis.

Using the diagram,

$P(x, y) = (0.86, 0.50)$

Now,

$$\sin 30^\circ = \frac{y}{r} = \frac{0.5}{1} = 0.5$$

$$\cos 30^\circ = \frac{x}{r} = \frac{0.86}{1} = 0.86$$

The fact that the unit circle has a radius of 1 leads to a general result, as shown below.

Technology: GeoGebra
Angles of any magnitude
MAT10MGTC00010

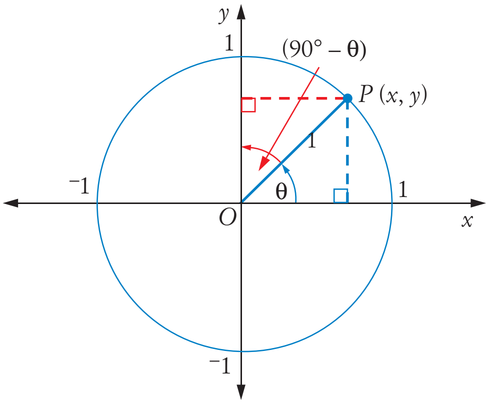
Important!

Sin and cos with the unit circle

For any unit circle:

$$\sin\theta = y\text{-coordinate} \qquad \cos\theta = x\text{-coordinate}$$

We can use the unit circle to investigate a number of trigonometric relationships.
Consider the unit circle shown here with $P(x, y)$, $\angle POA = \theta$ and $\angle POB = (90° - \theta)$.
Now, consider the following trig ratios.



$$\sin\theta = \frac{y}{1} = y \qquad \cos\theta = \frac{x}{1} = x$$

$$\tan\theta = \frac{y}{x} \qquad \sin(90° - \theta) = \frac{x}{1} = x$$

$$\cos(90° - \theta) = \frac{y}{1} = y \qquad \tan(90° - \theta) = \frac{x}{y}$$

This gives the following general result.

Important!

Trigonometric relationships for acute angles

$\sin\theta = \cos(90° - \theta)$

$\cos\theta = \sin(90° - \theta)$

$\tan\theta = \dfrac{1}{\tan(90° - \theta)}$

Example 14

If $\sin 28° = \cos y$, find y.

Solution

Use the relationship $\sin\theta = \cos(90° - \theta)$.	$\mathbf{\sin 28° = \cos(90° - 28°)}$ $\mathbf{= \cos 62°}$
Use the fact that $\sin 28° = \cos y$.	$\mathbf{y = 62°}$

Example 15

If $A + B = 90°$ and $\sin B = \frac{7}{25}$, find:

a $\cos A$ **b** $\cos B$

Solution

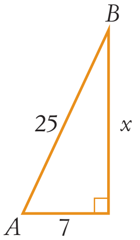
a A and B are complementary. $\boldsymbol{A = 90° - B}$

Use the relationship $\cos\theta = \sin(90° - \theta)$.

$$\boldsymbol{\cos A = \sin B}$$

$$\boldsymbol{\cos A = \tfrac{7}{25}}$$

b Sketch a right-angled triangle with $\sin B = \frac{7}{25}$.



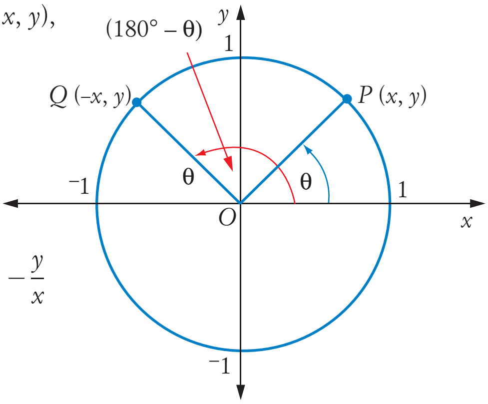
Use Pythagoras' theorem to calculate x. $\boldsymbol{x^2 = 25^2 - 7^2}$

Evaluate. $\boldsymbol{= 625 - 49}$

Take square roots of both sides. $\boldsymbol{x = 24}$

Calculate $\cos B$. $\boldsymbol{\cos B = \tfrac{24}{25}}$

Consider the unit circle shown here with $P(x, y)$, $Q(-x, y)$, $\angle POA = \theta$ and $\angle QOA = (180° - \theta)$.
Now, consider the following trig ratios.



$\sin\theta = \frac{y}{1} = y$ $\cos\theta = \frac{x}{1} = x$

$\tan\theta = \frac{y}{x}$ $\sin(180° - \theta) = \frac{y}{1} = y$

$\cos(180° - \theta) = \frac{-x}{1} = -x$ $\tan(180° - \theta) = \frac{y}{-x} = -\frac{y}{x}$

Important!

Trigonometric relationships for obtuse angles

- $\sin\theta = \sin(180° - \theta)$
- $\cos\theta = -\cos(180° - \theta)$
- $\tan\theta = -\tan(180° - \theta)$

where $0° \leq \theta \leq 90°$

Example 16

If x is an acute angle, calculate x in each of the following.

a $\tan 125° = -\tan x$ **b** $\sin 110° = \sin x$ **c** $\cos 140° = -\cos x$

Solution

a Use the trig relationship $\tan \theta = -\tan (180° - \theta)$.

$$\begin{aligned} x &= 180° - 125° \\ &= 55° \end{aligned}$$

b Use the trig relationship $\sin \theta = \sin (180° - \theta)$.

$$\begin{aligned} x &= 180° - 110° \\ &= 70° \end{aligned}$$

c Use the trig relationship $\cos \theta = -\cos (180° - \theta)$.

$$\begin{aligned} x &= 180° - 140° \\ &= 40° \end{aligned}$$

Example 17

Teacher notes
Special trig ratios
MAT10MGTN00010

Find the exact values of:

a $\tan 120°$ **b** $\cos 135°$

Solution

a Use the trig relationship $\tan \theta = -\tan (180° - \theta)$.

$$\begin{aligned} \tan 120° &= -\tan (180° - 120°) \\ &= -\tan 60° \end{aligned}$$

Refer to the exact trig values.

$$= -\sqrt{3}$$

b Use the trig relationship $\cos \theta = -\cos (180° - \theta)$.

$$\begin{aligned} \cos 135° &= -\cos (180° - 135°) \\ &= -\cos 45° \end{aligned}$$

Refer to the exact trig values.

$$= -\frac{1}{\sqrt{2}}$$

Investigate: The graphs of trigonometric functions

Technology worksheet
Excel worksheet: Trigonometric graphs
MAT10MGCT00016

Teacher notes
Blackline master for the unit circle
MAT10MGTN00016

In this investigation you are going to draw a large unit circle and use it to help draw the graphs of the trig ratios from 0° to 360°. You will need some sheets of graph paper (2 mm), a pair of compasses, a pair of dividers, a ruler and a protractor.

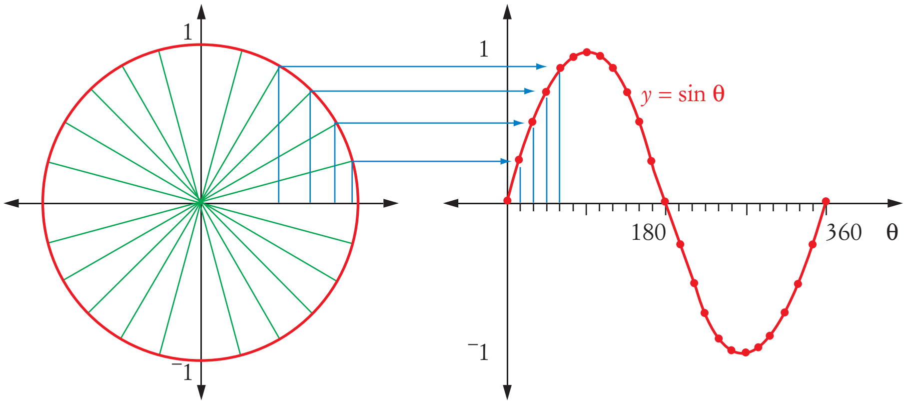
Graph of the sine function

1 Construct a unit circle on some graph paper, and on its circumference mark $P(15°)$, $P(30°)$, $P(45°)$, …, $P(345°)$ by drawing diameters as shown in the diagram on the opposite page.

2 Construct Cartesian axes with θ running from 0° to 360° and y from $^{-}1$ to 1 aligned with the unit circle. Mark the θ-axis in units of 15° as shown.

9780170194860

3 Now, sin θ = y = y-coordinate of P. So the graph of $y = \sin\theta$ can be drawn by transferring the y-coordinates of the points $P(\theta)$ on the unit circle to the Cartesian axes. This can be done physically as shown.



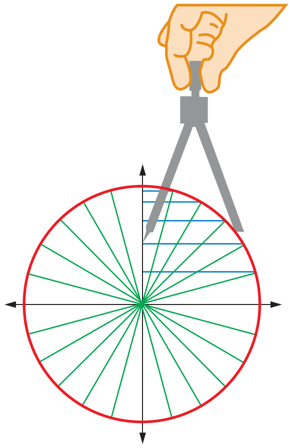
Weblink

Trigonometric graphs

MAT10MGWB00010

Graph of the cosine function

A similar method can be used to draw a graph of the cosine function, $y = \cos\theta$. However, the lengths of the x-coordinates must be used for cos, so it is necessary to transfer the lengths to the graph using a pair of dividers.



Graph of the tangent function

To draw the graph of the tangent function, we begin as for the sine and cosine except that the tangent, AT, is drawn to touch the unit circle at A. OP is extended to meet the tangent at T.

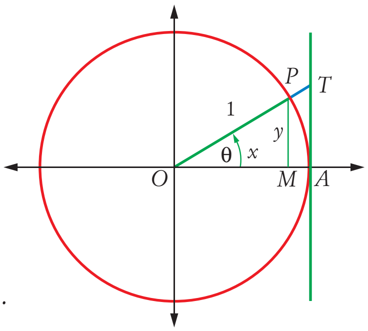
Now $\triangle OMP \,|||\, \triangle OAT$

So $\dfrac{y}{x} = \dfrac{AT}{1} = AT$

Thus $\tan\theta = AT$

So tan θ = length of tangent cut off by the extended radius.

That is, tan θ = y-coordinate of T.

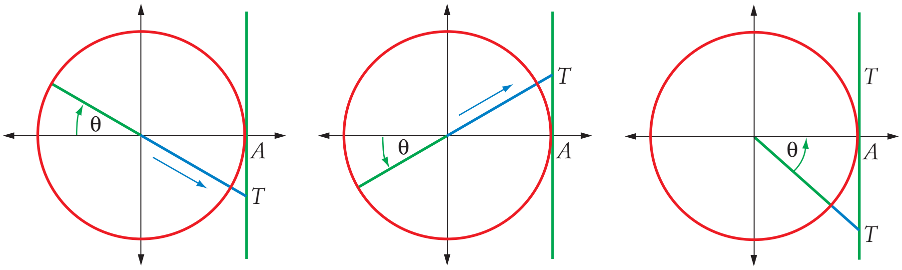


Teacher notes

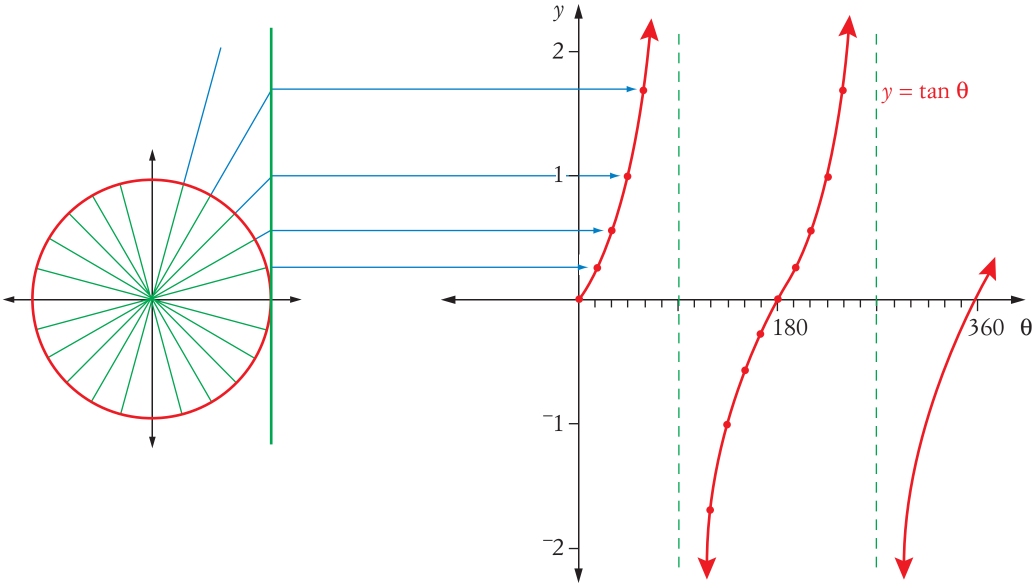
Blackline master for the unit circle

MAT10MGTN00016

In the fourth quadrant, this is negative. In the second and third quadrants, the radius is extended **backwards** to meet the tangent as shown.



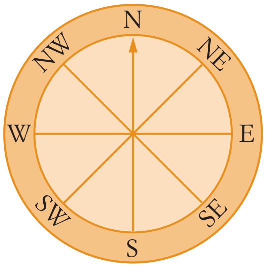
Plot the graph $y = \tan\theta$ between 0° and 360°.



Check the graphs you have drawn using a graphics calculator.

This activity above shows that the sine and cosine functions repeat their values every 360°. This is called the **period** of the functions. You can also see that the period of the tangent function is 180°.

Bearings and direction



A **compass** is used to find directions. Compasses are used by many different types of people. People who hike, sail or fly often rely on a compass for their safety.

A simplified compass face (or **compass rose**) is shown here.

The major directions, N (north), S (south), E (east) and W (west), are further divided by the directions NE (north-east), NW (north-west), SE (south-east) and SW (south-west). Because there are 360° in a circle, there are 90° between N and W, N and E, etc. Similarly, there are 45° between N and NE, E and SE, and so on. Captain Cook used a compass that was divided into many more directions, such as north-north-east and north-east by north. The divisions were about the same accuracy as a sailing ship could be steered.

Directions are given in terms of the basic compass points N, S, E and W.
There are two commonly accepted methods of stating directions. For instance, the direction NE can be stated as:

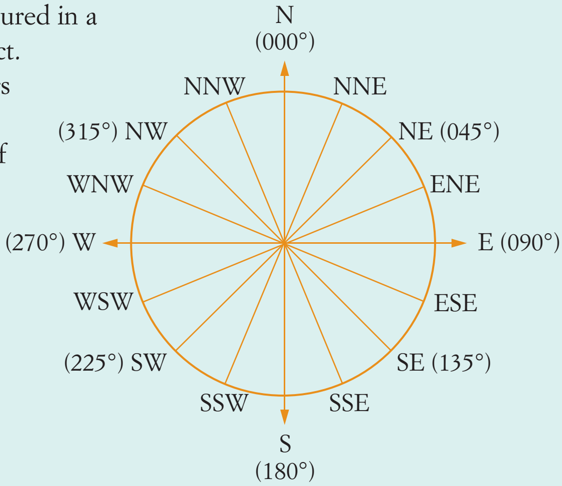
N 45° E OR 45° north of east
(Face N and turn 45° E.) (Face E and turn 45° N.)

We will use *the first method* in the remainder of this book.
You may have heard people say that they need to 'get their bearings' when they are not quite sure in which direction they should go. In navigation, the term **bearing** has a special meaning related to direction.

Important!

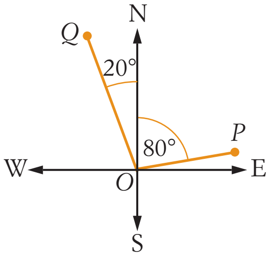
Bearings

The **bearing** of an object is the angle measured in a *clockwise* direction from *north* to the object. Bearings are written as three-digit numbers followed by the degree symbol (°). The diagram on the right shows the bearings of the major compass points.



Example 18

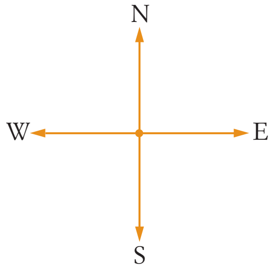
State the direction and bearing of P and Q from O as shown in the diagram below.



Solution

P is 80° east of north.	**P is in the direction N 80° E.**
P is 80° clockwise from north.	**The bearing of P is 080°.**
Q is 20° west of north.	**Q is in the direction N 20° W.**
Q is 340° clockwise from north.	**The bearing of Q is 340°.**

A simplified compass rose like the one shown here is usually drawn when working with bearings.



Example 19

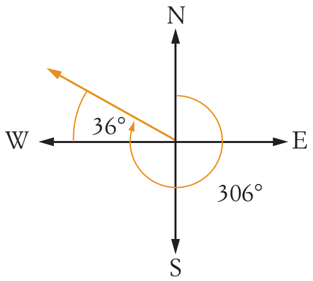
Sketch the bearing 306°.

Solution

Sketch a simplified compass rose.

$306° = 270° + 36°$

Draw in an arrow from the origin at an angle of 36° clockwise from west.



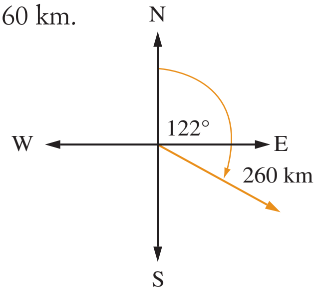
Example 20

Video tutorial
Bearings
MAT10MGVT10022

A plane leaves a runway and remains on a bearing of 122° for 260 km.

a How far south of the runway is the plane?

b What is the bearing of the runway from the plane?



Solution

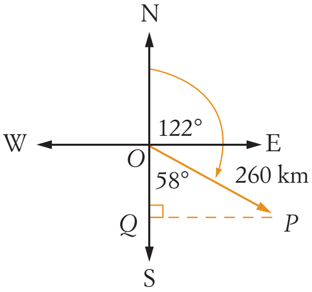
a Mark in the origin (O) and the position of the plane (P) on the sketch.

Draw in PQ to form the right-angled triangle OPQ.

We need to know the distance represented by OQ.

We know $OP = 260$ km.

$\angle POQ$ and 122° are supplementary, so $\angle POQ = 58°$.



For $\triangle POQ$, OP is the hypotenuse and OQ is adjacent to $\angle POQ$, so use the cos ratio.

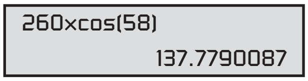
$$\cos 58^\circ = \frac{OQ}{260}$$

Multiply both sides by 260.

$$260 \times \cos 58^\circ = OQ$$

Reverse the equation and evaluate by entering:

260 × cos 58 = →



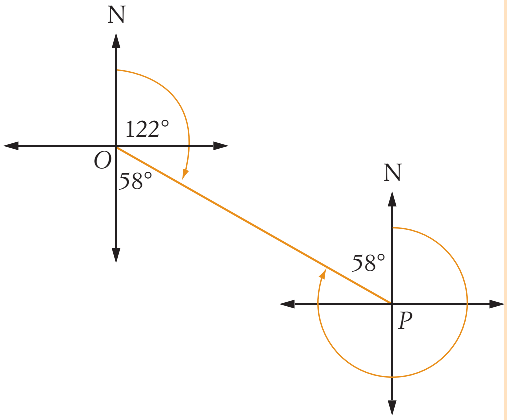
```
260×cos(58)
            137.7790087
```

Round off and state the result.

The plane is about 137.8 km south of the runway.

b Draw a compass rose with the centre at P.

The angle between OP and due north at P must be 58° as it is an alternate angle with the angle between OP and due south at O.



Calculate the bearing.

Bearing of the runway from the plane

$$= 360^\circ - 58^\circ$$

$$= 302^\circ$$

Exercise 10.2 Trigonometric ratios

Understanding

Extra questions

Exercise 10.2

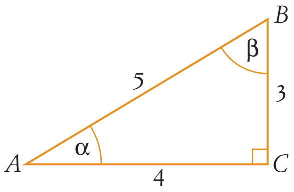
MAT10MGEQ00029

See Example 5

1 Use $\triangle ABC$ to write the following as fractions.

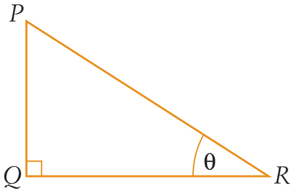
a $\sin \alpha$ **b** $\cos \beta$ **c** $\tan \beta$

d $\sin \beta$ **e** $\tan \alpha$ **f** $\cos \alpha$



2 For $\triangle PQR$, which trig ratio of θ is equal to:

a $\frac{PQ}{PR}$? **b** $\frac{QR}{PR}$? **c** $\frac{PQ}{QR}$?



See Example 6

3 Find the value of each of the following, correct to 3 decimal places.

a sin 20° **b** tan 45° **c** cos 80°
d cos 21° **e** sin 90° **f** tan 9°
g tan 17° **h** sin 33° **i** cos 77°
j cos 89°

4 Use the table of trigonometric ratios on page 456 to find the value of each of the following, correct to 3 decimal places.

a sin 20° **b** cos 42° **c** sin 17° **d** cos 45°
e sin 90° **f** cos 33° **g** sin 80° **h** cos 90°
i sin 45° **j** cos 55°

See Example 7

5 Find the value of θ, to the nearest degree.

a $\sin\theta = 0.970$ **b** $\tan\theta = 0.500$ **c** $\cos\theta = 0.150$
d $\tan\theta = 0.98$ **e** $\sin\theta = 0.15$ **f** $\cos\theta = 0.90$
g $\sin\theta = 0.123$ **h** $\cos\theta = 0.68$ **i** $\tan\theta = 0.034$

6 Use the table of trigonometric ratios on page 456 to find the value of θ (to the nearest degree).

a $\sin\theta = 0.970$ **b** $\sin\theta = 0.58$ **c** $\cos\theta = 0.123$
d $\cos\theta = 0.5$ **e** $\sin\theta = 0.15$ **f** $\cos\theta = 0.68$
g $\sin\theta = 0.252$ **h** $\cos\theta = 0.91$ **i** $\sin\theta = 0.034$

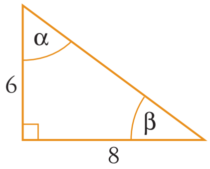
Fluency

See Example 8

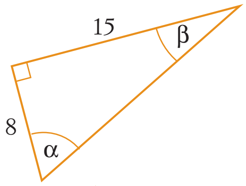
7 For each triangle below:

i state the values of sin α, cos α and tan α as ratios in simplest form
ii state the values of sin β, cos β and tan β as ratios in simplest form
iii find the values of α and β to the nearest degree.

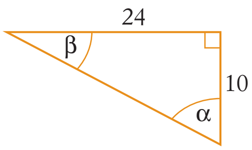
a



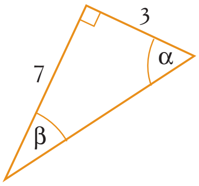
b



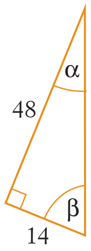
c



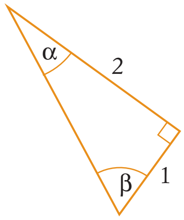
d



e



f



8 Calculate the following, correct to 3 decimal places.

a $20 \times \sin 30°$ **b** $15 \times \cos 70°$ **c** $\tan 45° \times 18$

d $\dfrac{10}{\sin 30°}$ **e** $\dfrac{25}{\cos 30°}$ **f** $\dfrac{32}{\cos 45°}$

9 Find the value of the unknown in each of the following, correct to 1 decimal place.

a $\tan 30° = \dfrac{a}{40}$ **b** $\sin 60° = \dfrac{b}{100}$ **c** $\sin 45° = \dfrac{d}{18}$

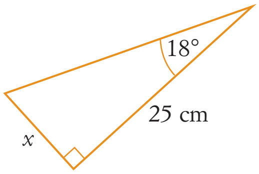
d $\sin 35° = \dfrac{18}{a}$ **e** $\cos 22° = \dfrac{72}{d}$ **f** $\cos 47° = \dfrac{48}{m}$

9780170194860

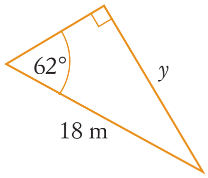
10 Find the value of the unknown side in each of these triangles, correct to 1 decimal place.

See Example 9

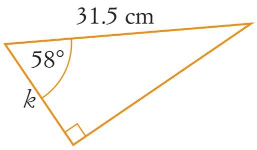
a



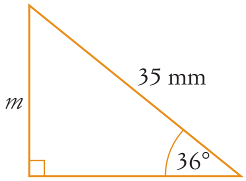
b



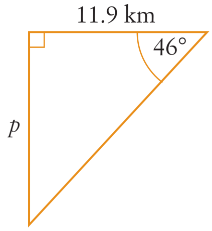
c



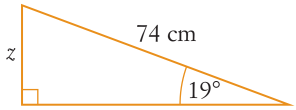
d



e

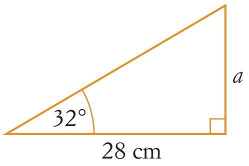


f

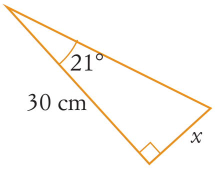


11 Find the value of the unknown side in each triangle, correct to 1 decimal place.

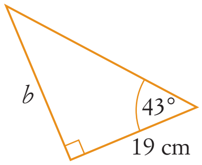
a



b



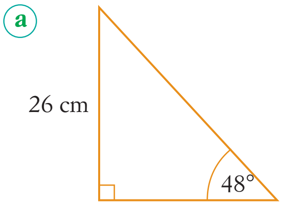
c



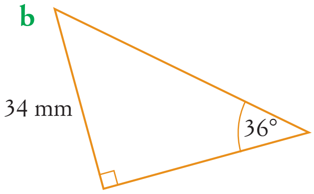
12 Find the value of the hypotenuse in each of these triangles, correct to 2 decimal places.

Worked solutions
Exercise 10.2
MAT10MGWS00062

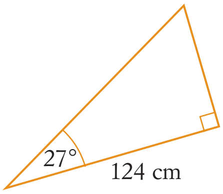
ⓐ



b



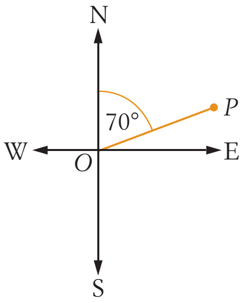
c



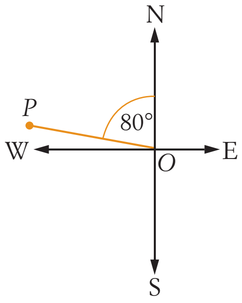
13 For each of the following, state the bearing of *P* from *O*.

See Example 18

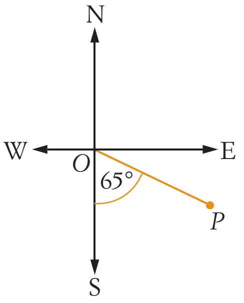
a



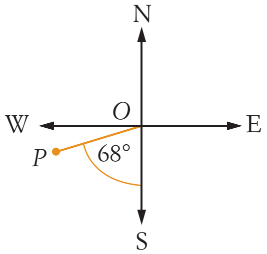
b



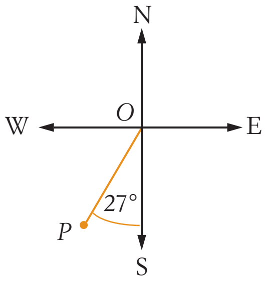
c



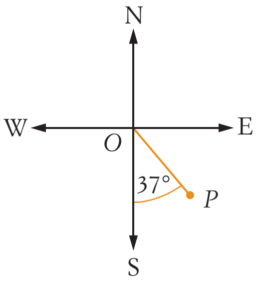
d



e



f



14 For each of the situations shown in the previous question, state the bearing of O from P.

See Example 19

15 Sketch the following bearings.

a 245° **b** 337° **c** 043° **d** 121°
e 208° **f** 114° **g** 316° **h** 261°

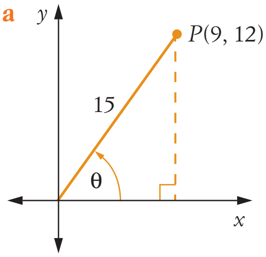
See Example 11

16 For each of the following:

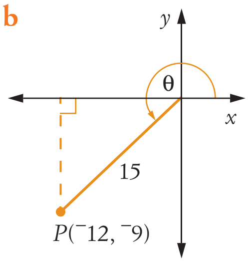
i state the values of x, y and r

ii use the values of x, y and r to write the values of sin θ, cos θ and tan θ.

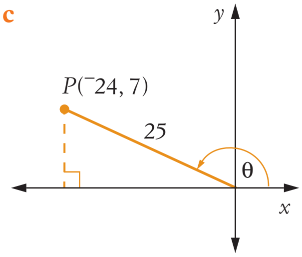
a



b



c



Worked solutions
Exercise 10.2
MAT10MGWS00062

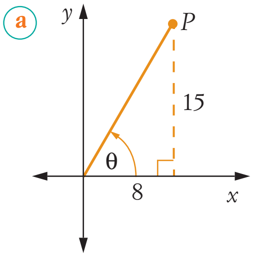
17 For each of the following:

i state the x-coordinate and y-coordinate of P

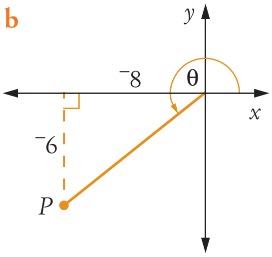
ii calculate r using Pythagoras' theorem

iii write the values of sin θ, cos θ and tan θ.

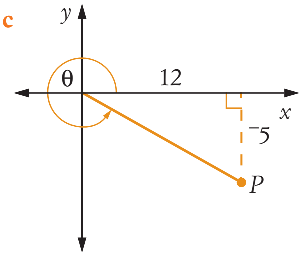
a



b



c



See Example 13

18 Use the unit circle shown on page 462 to calculate the values of the following trig ratios, if possible.

a sin 180° **b** cos 90° **c** tan 180° **d** cos 0°
e cos 270° **f** sin 360° **g** tan 270° **h** cos 180°

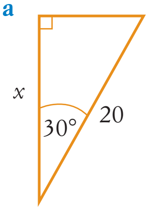
19 Use the diagram of the first quadrant of the unit circle on page 463 to find the values of the following. (Give approximate values where necessary.)

a sin 60° **b** cos 60° **c** cos 30° **d** sin 80°
e tan 45° **f** sin 10° **g** cos 80° **h** sin 70°

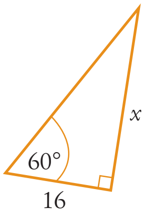
See Example 10

20 Find the exact value of x in each of the following.

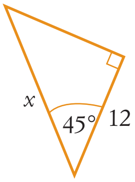
a



b

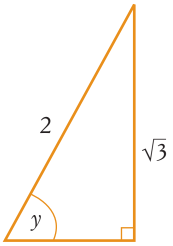


c

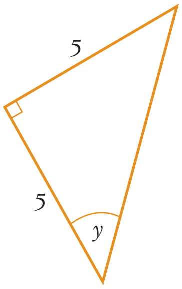


21 Without using a calculator or a trig table, find y (in degrees) in each of the following.

a



b



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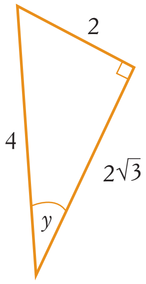
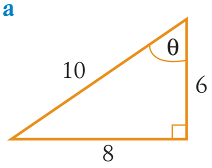


22 In the following triangles find the ratios for sin θ, cos θ and tan θ in their exact form and verify that $\frac{\sin\theta}{\cos\theta} = \tan\theta$ for each triangle. *See* Example 12

a



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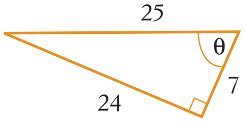

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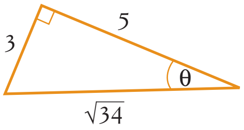


23 Use the result $\tan A = \frac{\sin\theta}{\cos\theta}$ in each of the following.

a If $\cos A = \frac{8}{17}$ and $\tan A = \frac{15}{8}$ find $\sin A$.

b Given $\tan A = \frac{9}{40}$ and $\sin A = \frac{9}{41}$, find $\cos A$.

c If $\tan A = \frac{5}{2}$ and $\cos A = \frac{2}{\sqrt{29}}$, find the exact value of $\sin A$.

d Given $\sin A = \frac{5}{\sqrt{106}}$ and $\tan A = \frac{5}{9}$, find the exact value of $\cos A$.

24 Find the value of the pronumeral in each of the following without using a calculator or a trig table. *See* Example 14

a $\sin 38° = \cos w$

b $\cos 68° = \sin x$

c $\sin 13.6° = \cos m$

d $\cos 81.4° = \sin p$

e $\sin 24°41' = \cos q$

f $\cos 57°17' = \sin k$

25 Answer the following without the aid of a calculator or a trig table. *See* Example 15

a If $\sin 53° \approx 0.8$, find $\cos 37°$.

b Given $\cos 27° \approx 0.89$, find $\sin 63°$.

c If $\sin x = \frac{9}{41}$ and $x + y = 90°$, find $\cos y$, $\cos x$ and $\sin y$.

d Given $\cos a = \frac{12}{13}$, find $\sin b$, $\sin a$ and $\cos b$ if a and b are complementary angles.

e If $P + Q = 90°$, find the exact values of $\cos Q$, $\sin P$ and $\sin Q$, given $\cos P = \frac{2}{5}$.

f Find the exact values of $\cos\alpha$, $\sin\alpha$ and $\cos\beta$ if $\sin\beta = \frac{3}{\sqrt{58}}$ and $\beta = 90° - \alpha$.

26 Simplify the following without using a calculator or a trig table.

a $\cos 34° - \sin 56°$

b $\frac{\cos 28°}{\sin 62°}$

c $\frac{\sin\beta}{\cos(90 - \beta)}$

27 If θ is an acute angle, find θ in each of the following. *See* Example 16

a $\sin 155° = \sin\theta$

b $\tan 102° = -\tan\theta$

c $\cos 121° = -\cos\theta$

d $\cos 167° = -\cos\theta$

e $\sin 98° = \sin\theta$

f $\tan 134° = -\tan\theta$

28 Find, correct to two decimal places, each of the following.

a $\cos 147°$ **b** $\sin 111°26'$ **c** $\tan 142.6°$
d $\tan 156°19'$ **e** $\cos 106.7°$ **f** $\sin 172.4°$
g $\sin 141°32'$ **h** $\tan 156.08°$ **i** $\cos 127°52'$

29 Express the following in terms of $\sin\theta$, $\cos\theta$ or $\tan\theta$ where $0° \le \theta \le 90°$.

a $\cos 130°$ **b** $\tan 122°$ **c** $\sin 108°$ **d** $\tan 163°$
e $\sin 142°$ **f** $\cos 172.8°$ **g** $\sin 150.6°$ **h** $\tan 98.2°$
i $\cos 102°$ **j** $\tan 134.4°$ **k** $\sin 118.5°$ **l** $\sin 149.7°$

See Example 17

30 Find the exact value of each of the following.

a $\sin 135°$ **b** $\sin 180°$ **c** $\tan 180°$ **d** $\sin 150°$ **e** $\tan 135°$
f $\cos 120°$ **g** $\cos 135°$ **h** $\sin 120°$ **i** $\tan 120°$ **j** $\cos 150°$

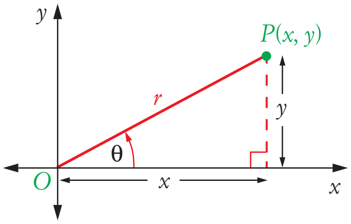
31 Find θ, to the nearest degree, in each of the following if x is obtuse.

a $\sin x = 0.63$ **b** $\tan x = {}^{-}\frac{4}{3}$ **c** $\cos x = {}^{-}0.412$
d $\cos x = {}^{-}\frac{5}{13}$ **e** $\sin x = 0.145$ **f** $\tan x = {}^{-}1$
g $\tan x = {}^{-}6.342$ **h** $\sin x = \frac{2}{5}$ **i** $\cos x = {}^{-}\frac{8}{17}$

Problem solving

32 In each of the following cases, P is a point on the plane. O is the origin and θ is the angle between OP and the x-axis. Draw a diagram to represent each case, then:

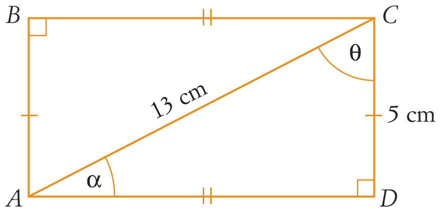
i find $\sin\theta$
ii use this value to calculate the value of θ, to the nearest degree.



a $P = (6, 8)$ **b** $P = (5, 12)$ **c** $P = ({}^{-}7, 24)$
d $P = (4, {}^{-}3)$ **e** $P = (24, 10)$ **f** $P = ({}^{-}9, {}^{-}12)$
g $P = (1, \sqrt{3})$ **h** $P = (2, \sqrt{5})$

Worked solutions
Exercise 10.2
MAT10MGWS00062

33 In rectangle $ABCD$, AC is 13 cm and the short sides are 5 cm long. Find the values of α and θ, to the nearest degree.

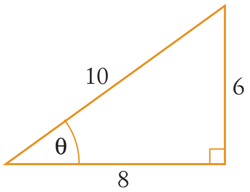


34 For each of the following triangles, find the value of:

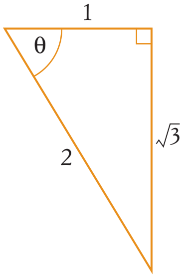
i $\sin\theta$ **ii** $\cos\theta$ **iii** $\sin^2\theta + \cos^2\theta$.

Hint: In part **iii**, $\sin^2\theta = (\sin\theta)^2$.

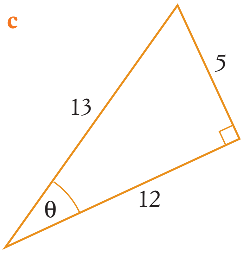
a



b



c

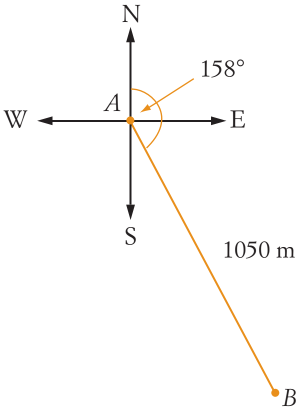


See Example 20

35 A hiker leaves a base camp (A) and remains on a bearing of 158° for 1050 m.

a How far east of the base camp is the hiker?

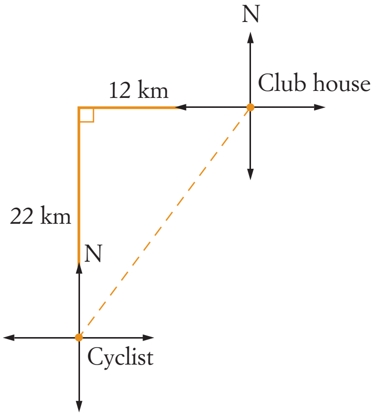
b What is the bearing of the base camp from the hiker?



36 A cyclist begins from a club house and travels due west for 12 km, then due south for 22 km.

a How far is the cyclist from the club house?

b What is the bearing of the cyclist from the club house?



Reasoning

37 Show that each of the following is true.

a $\cos x \times \tan x = \sin x$

b $\sin y \times \cos y \times \tan y = (\sin y)^2$

c $\dfrac{\sin a\ \cos a + (\sin a)^2}{\sin a\ \cos a + (\cos a)^2} = \tan a$

d $\dfrac{\cos(90° - b) + \sin b}{\sin b} = 2$

38 Two ships leave the same port. The first travels for 220 km on a bearing of 168° while the second ship travels for 186 km on a bearing of 256°.

a How far apart are the two ships?

b Calculate the bearing of the second ship from the first.

10.3 The sine and cosine rules

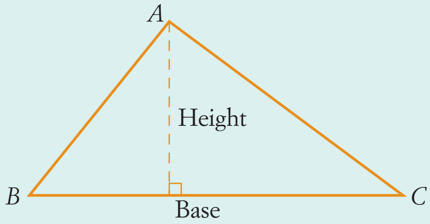
We can use trig ratios to help extend the work completed in earlier years.

Area of a triangle

In Year 9, you learnt that the area of a triangle can be worked out as:

Important!

Area of $\triangle ABC = \frac{1}{2}\text{base} \times \text{height}$

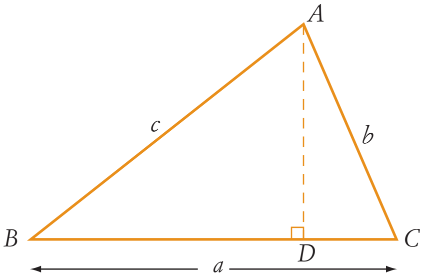


We can make another rule for the area as follows.
Draw the perpendicular height from A down to D.

$$\text{Then in } \triangle ADC, \sin C = \frac{AD}{b}$$

$$\text{So } AD = b \sin C$$

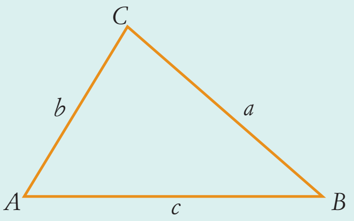
$$\begin{aligned}\text{Now area of } \triangle ABC &= \tfrac{1}{2}\text{ base} \times \text{height}\\ &= \tfrac{1}{2}a \times AD\\ &= \tfrac{1}{2}a \times b \sin C\end{aligned}$$



Even if C is an obtuse angle, since $\sin C = \sin(180° - C)$, the rule will work anyway.
It doesn't matter which way around we draw the triangle, so the full rule statement is:

Important!

$$\begin{aligned}\text{Area of } \triangle ABC &= \tfrac{1}{2}ab \sin C\\ &= \tfrac{1}{2}bc \sin A\\ &= \tfrac{1}{2}ac \sin B\end{aligned}$$



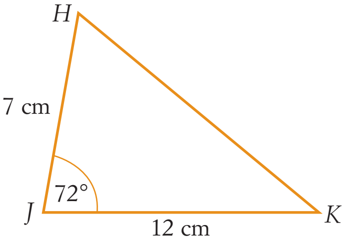
where the measures of the sides opposite the angles A, B and C are a, b and c respectively.

Example 21

For $\triangle HJK$, $h = 12$ cm, $k = 7$ cm and $\angle HJK = 72°$. Calculate the area of $\triangle HJK$.

Solution

Draw a diagram and show the information.



The rule for the area of a triangle is:

$\triangle ABC = \frac{1}{2}ab\sin C$

In this case:

$A \leftrightarrow H$ and $a \leftrightarrow h$
$B \leftrightarrow J$ and $b \leftrightarrow j$
$C \leftrightarrow K$ and $c \leftrightarrow k$

Write the rule. **Area of $\triangle HJK = \frac{1}{2}hk \sin J$**

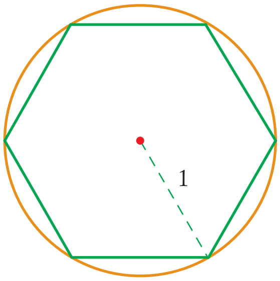
Substitute relevant values.

$$= \frac{1}{2} \times 12 \times 7 \times \sin 72°$$
$$= 39.944\ldots$$

Round and write the answer. **The area of $\triangle HJK$ is about 39.9 cm^2.**

Investigate: Areas of polygons

A regular hexagon is inscribed in a circle of radius 1 unit as shown here. Work in groups to calculate the area of the hexagon by calculating the areas of the triangles that make it up.



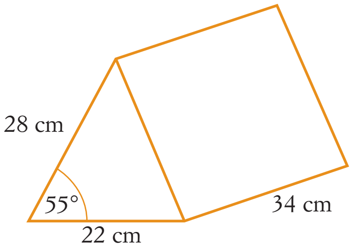
Carry out a similar procedure to calculate the area of a regular:

a octagon **b** decagon
c 30-gon **d** 50-gon

What happens to the area as the number of sides (n) becomes larger and larger? What does the area get close to?

Example 22

Find the volume of this prism.



Solution

From Chapter 4, you should remember that the volume of a prism is equal to the area of the end multiplied by the length or height. We put $a = 28$, $b = 22$, $C = 55°$ and $h = 34$.

Use the area of a triangle rule.	**Area of end** $= \frac{1}{2}ab \sin C$
Write the rule for the volume of a prism.	$V = Ah$
Substitute for A.	$= \frac{1}{2}ab \sin C \times h$
Substitute for known values.	$= \frac{1}{2} \times 28 \times 22 \times \sin 55° \times 34$
Evaluate.	$= 8578.160...$
Round off and write in words.	**The volume is about 8600 cm^3.**

Investigate: Comparing sides and angles in triangles

You will require a pair of compasses, a ruler and a protractor for this investigation.

1. Construct three separate triangles on a sheet of paper. One triangle should be acute-angled, one right-angled and the last one obtuse-angled. Make sure that the shortest side of any of the triangles is 5 cm long so that you can measure the angles.
2. Label the vertices of each triangle A, B and C and the sides opposite these angles a, b and c respectively.
3. Now measure each of the sides to the nearest millimetre and angles to the nearest degree and record your findings in a table like the one shown below.

Triangle	a	A	$\frac{a}{\sin A}$	b	B	$\frac{b}{\sin B}$	c	C	$\frac{c}{\sin C}$
Acute									
Right-angled									
Obtuse									

4. Calculate the values of $\frac{a}{\sin A}$, $\frac{b}{\sin B}$ and $\frac{c}{\sin C}$ correct to two decimal places and enter these values in the table.
5. Compare your calculated values of $\frac{a}{\sin A}$, $\frac{b}{\sin B}$ and $\frac{c}{\sin C}$ with those of others in your class and comment on your findings.

The result you established in the previous investigation is known as the **sine rule**.

9780170194860

The sine rule

The area of any $\triangle ABC$ may be written as:

$$\tfrac{1}{2}ab\sin C \text{ or } \tfrac{1}{2}bc\sin A \text{ or } \tfrac{1}{2}ca\sin B$$

$$\text{So } \tfrac{1}{2}\,ab\sin C = \tfrac{1}{2}\,bc\ \sin A = \tfrac{1}{2}\,ca\sin B$$

Multiplying by 2 gives $ab \sin C = bc \sin A = ca \sin B$

Dividing by abc, we have $\dfrac{\sin C}{c} = \dfrac{\sin A}{a} = \dfrac{\sin B}{b}$

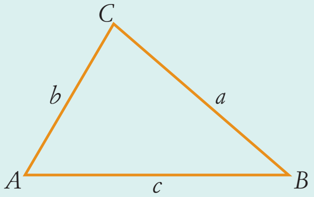
Inverting gives the **sine rule**:

Important!

The sine rule

In any $\triangle ABC$:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



A formal proof of the sine rule is given on NelsonNet.
We use the sine rule to solve non-right-angled triangles when we know two angles and one side or two sides and an opposite angle.

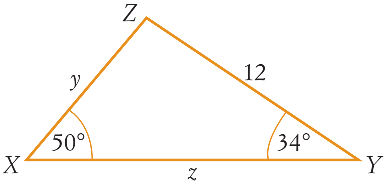
Teacher notes
The sine rule
MAT10MGTN00018

Example 23

In $\triangle XYZ$, $X = 50°$, $Y = 34°$ and $x = 12$. Find Z, y and z.

Solution

Sketch a triangle to show the information.



Write the sine rule for x, X, y and Y.

$$\frac{x}{\sin X} = \frac{y}{\sin Y}$$

Substitute for known values.

$$\frac{12}{\sin 50°} = \frac{y}{\sin 34°}$$

Isolate y and evaluate.

$$y = \frac{12 \sin 34°}{\sin 50°}$$
$$\approx 8.8$$

Now find Z.

$$Z + 50° + 34° = 180°$$
$$Z = 96°$$

Video tutorial
The sine rule
MAT10MGVT10024

CAS TI-Nspire exercise
Trigonometry
MAT10MGTI00017

CAS ClassPad exercise
Trigonometry
MAT10MGCP00017

Write the sine rule for x, X, z and Z.

$$\frac{x}{\sin X} = \frac{z}{\sin Z}$$

Substitute for known values.

$$\frac{12}{\sin 50^\circ} = \frac{z}{\sin 96^\circ}$$

Isolate z and evaluate.

$$z = \frac{12 \sin 96^\circ}{\sin 50^\circ}$$
$$\approx 15.6$$

Write the answer.

$$Z = 96^\circ,\ y \approx 8.8 \text{ and } z \approx 15.6.$$

The sine rule does not always give a unique answer for the angle in a triangle.
Since $\sin \theta = \sin (180^\circ - \theta)$, you must try *two possible answers* to check whether one or both answers are possible. Your calculator gives only the acute angle, so you must work out the other angle yourself.
For example, if $\sin \theta = 0.6$, your calculator will give $\sin^{-1} 0.6 \approx 36.87^\circ$.
Now $180^\circ - 36.87^\circ = 143.13^\circ$. Checking, you find that $\sin 36.87^\circ \approx 0.6$ and $\sin 143.13^\circ \approx 0.6$.
If both answers are possible, we say it is an example of the **ambiguous case**.

Example 24

In $\triangle ABC$, $a = 6$ cm, $b = 7.5$ cm and $A = 40^\circ$. Calculate the remaining angles and side and sketch the possible triangles.

Solution

Write the sine rule for a, A and b, B.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Substitute relevant values.

$$\frac{6}{\sin 40^\circ} = \frac{7.5}{\sin B}$$

Rearrange to isolate B.

$$\sin B = \frac{7.5 \sin 40^\circ}{6}$$

Evaluate

$$= 0.8034 \ldots$$

Use $\sin^{-1}$ on the calculator.

$$\text{So } B = 53.46 \ldots^\circ \text{ or } 126.54 \ldots^\circ$$

Keep exact values in your calculator.

Find the possible totals of A and B.

$$A + B \approx 40^\circ + 53.46 \ldots^\circ \text{ or } 40^\circ + 126.54 \ldots^\circ$$
$$= 93.46 \ldots^\circ \text{ or } 166.54 \ldots^\circ$$

Both are less than 180°, so both are possible.

Now find C.

$$C = 180^\circ - 93.46 \ldots^\circ \text{ or } 180^\circ - 166.54 \ldots^\circ$$
$$= 86.54 \ldots^\circ \text{ or } 13.46 \ldots^\circ$$

Use the sine rule to find the possible values of side c.

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

Substitute.

$$\frac{6}{\sin 40^\circ} = \frac{c}{\sin C}$$

Rearrange.

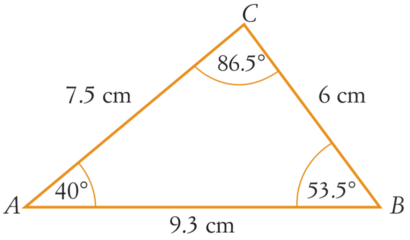
$$c = \frac{6 \sin C}{\sin 40^\circ}$$

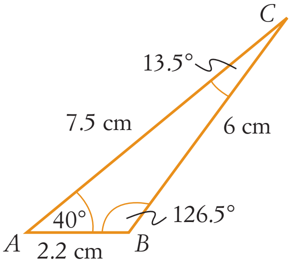
Use the exact values in your calculator. $= \dfrac{6\sin 86.54\ldots^\circ}{\sin 40^\circ}$ **or** $\dfrac{6\sin 13.46\ldots^\circ}{\sin 40^\circ}$

Evaluate. $= 9.3172\ldots$ **or** $2.1733\ldots$

This is an ambiguous case. Round and give both sets of answers. $B \approx 53.5^\circ$, $C \approx 86.5^\circ$ **and** $c \approx 9.3$ **cm or** $B \approx 126.5^\circ$, $C \approx 13.5^\circ$ **and** $c \approx 2.2$ **cm**

The possible triangles from the previous example are shown below.





The sine rule makes some calculations using trig ratios simpler than would otherwise be the case.

Example 25

Jermaine sees that the top of a sand dune is at an angle of 16° measured from the horizontal beach on which she is walking. On walking 40 m closer, she finds that the angle increases to 22°. What is the height of the sand dune?

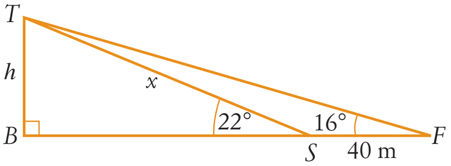
Solution

Make a sketch of the information.

Label sides and angles as needed.

The height is side *TB*.

Side *TS* is in between the known and unknown sides. Call it *x* and find it first.



Use the exterior angle of $\triangle SFT$ to find $\angle STF$.

$$22^\circ = 16^\circ + \angle STF$$
$$\angle STF = 6^\circ$$

Use the sine rule in $\triangle SFT$ to find an expression for *x*.

$$\frac{x}{\sin 16^\circ} = \frac{40}{\sin 6^\circ}$$

Isolate *x* but don't evaluate it.

$$x = \frac{40\sin 16^\circ}{\sin 6^\circ}$$

Now use $\triangle BST$ and the sine ratio to find *h*.

$$\sin 22^\circ = \frac{h}{x}$$

Isolate *h*.

$$h = x\sin 22^\circ$$

Substitute using the uncalculated value of *x*.

$$= \frac{40\sin 16^\circ \sin 22^\circ}{\sin 6^\circ}$$

Evaluate.

$$= 39.5129\ldots$$

Round and write the answer.

The dune is about 39.5 m high.

The cosine rule

The sine rule can be used in triangles where an angle and its opposite side are known. However, if the known angle is included between two sides or only the sides are known, the sine rule cannot be applied. In these cases, the **cosine rule** is used instead.

Important!

The cosine rule

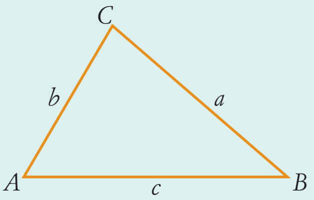
In any $\triangle ABC$

$$a^2 = b^2 + c^2 - 2bc \cos A \text{ or } \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$b^2 = a^2 + c^2 - 2ac \cos B \text{ or } \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$c^2 = a^2 + b^2 - 2ab \cos C \text{ or } \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

where the angles and sides are named in the usual way.



Formal proofs of the cosine rule for acute-angled, right-angled and obtuse-angled triangles are shown on NelsonNet.

Teacher notes
The cosine rule
MAT10MGTN00017

Video tutorial
The cosine rule
MAT10MGVT10025

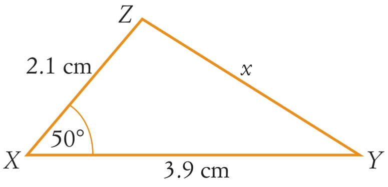
CAS TI-Nspire exercise
Trigonometry
MAT10MGTI00017

CAS ClassPad exercise
Trigonometry
MAT10MGCP00017

Example 26

In the triangle shown on the right, calculate:

a x **b** Y



Solution

a Write the cosine rule for $\triangle XYZ$.	$x^2 = y^2 + z^2 - 2yz \cos X$
Substitute values.	$= 2.1^2 + 3.9^2 - 2 \times 2.1 \times 3.9 \times \cos 50°$
Evaluate.	$= 9.0911 \ldots$
Use √ and keep the exact value for later use.	$x = 3.0151 \ldots$
Round and write the answer.	**x is approximately 3.02 cm.**
b You can now use the sine rule.	$\frac{x}{\sin X} = \frac{y}{\sin Y}$
Rearrange to isolate Y.	$\sin Y = \frac{y \sin X}{x}$
Substitute known values and use the value of x found in part **a**.	$= \frac{21 \sin 50°}{3.0151 \ldots}$
Evaluate.	$= 0.5335 \ldots$
Find Y using $\sin^{-1}$ and round off.	$Y \approx 32.2°$
Write the answer in words.	**Y is about 32.2°.**

Example 27

Video tutorial
The cosine rule
MAT10MGVT10025

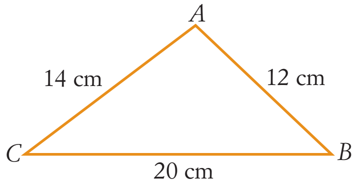
Puzzle sheet
The sine and cosine rules
MAT10MGPS00058

Calculate the largest angle in the triangle with sides 12 cm, 14 cm and 20 cm.

Solution

Make a rough sketch.

In triangles, the largest angle is opposite the longest side. So the triangle can be drawn as shown.



A is the largest angle, so use the form of the cosine rule involving cos *A*.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Substitute using the information from the diagram.

$$= \frac{14^2 + 12^2 - 20^2}{2 \times 14 \times 12}$$

$$= \frac{^{-}60}{336}$$

Keep the exact number in your calculator.

$$= {}^{-}0.1785\ \ldots$$

A is obtuse because cos *A* is negative. Most calculators will give the correct angle.

$$A = 100.286\ \ldots^\circ$$

Round and write the answer in words.

The largest angle is about 100.3° or 100°17′.

Example 28

A tower in a direction N 52° E is found to be at a range of 4.86 km. The range finder shows that another tower in a direction 106° is at a range of 7.96 km. Find the distance between the towers.

Solution

Sketch a diagram and name the points.

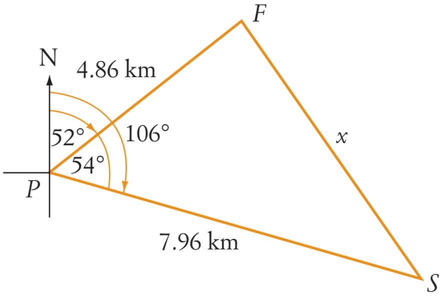
Name the observation point *P*.

Name the tower positions *F* and *S*.

Draw in the bearings.

The angle between the towers must be 54°.

We want to find the distance between the towers, *FS*.



Label *FS* as *x*.

Write the cosine rule for *FS* (*x*).

$$FS^2 = PF^2 + PS^2 - 2 \times PF \times PS \times \cos P$$

Substitute.

$$= 4.86^2 + 7.96^2 - 2 \times 4.86 \times 7.96 \times \cos 54^\circ$$

Keep exact value on calculator.	$x^2 = 41.5035 \ldots$
Use the exact value for the square root.	$x = \sqrt{41.5035 \ldots}$
Evaluate and round off.	≈ 6.44
Write the answer in words.	**Distance between the towers is about 6.44 km.**

Exercise 10.3 The sine and cosine rules

Fluency

Extra questions
Exercise 10.3
MAT10MGEQ00058

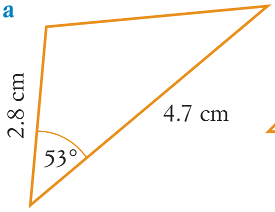
See Example 21

1 Find the area of each of the following triangles.

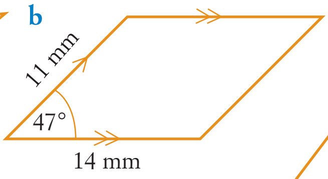
- **a** $\triangle ABC$ where $a = 5$, $b = 6$ and $C = 38°$
- **b** $\triangle XYZ$ where $x = 9$, $y = 5$ and $Z = 60°$
- **c** $\triangle PQR$ where $r = 15$, $q = 9$ and $P = 27.7°$
- **d** $\triangle MNO$ where $n = 17$, $m = 8$ and $O = 114°$
- **e** $\triangle FGH$ where $f = 21.65$, $h = 12.43$ and $G = 67°31'$

2 Calculate the area of each of the following shapes.

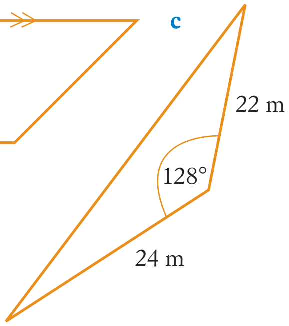
a



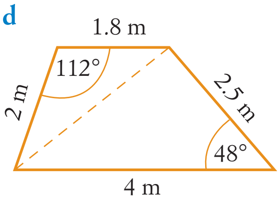
b



c



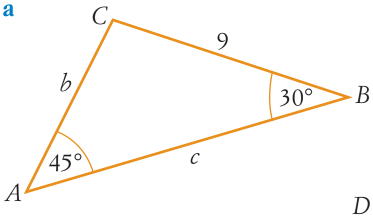
d



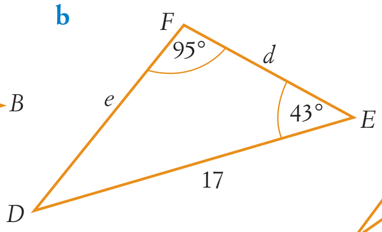
See Example 23

3 Use the sine rule to calculate the lengths of the unknown sides in each of the following triangles correct to 2 decimal places.

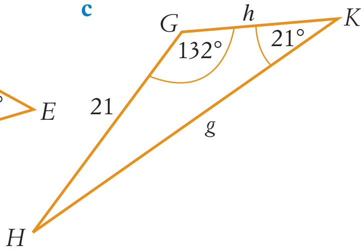
a



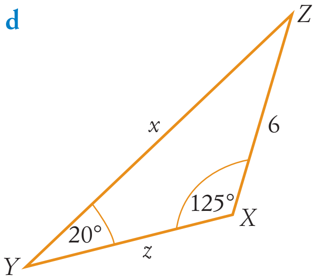
b



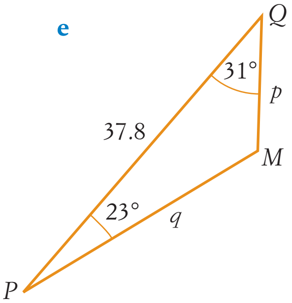
c



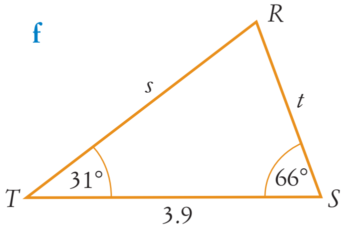
d



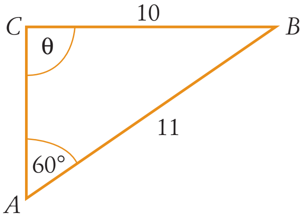
e

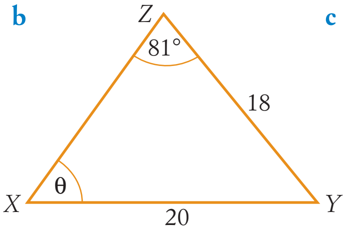


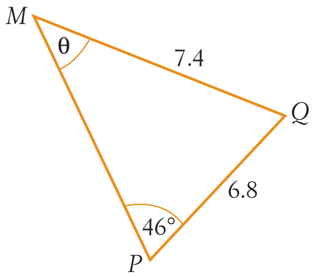
f

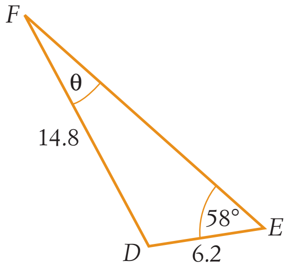


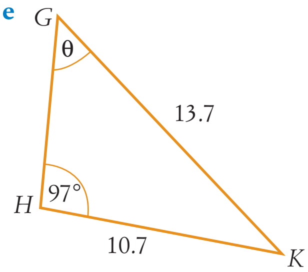
4 Calculate the value of θ in each of the following triangles correct to 1′. (Remember: 1° = 60′)

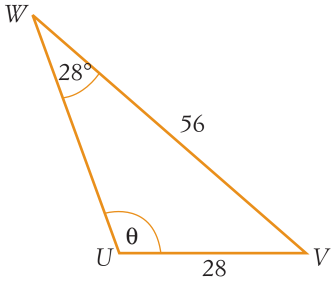
a


b


c


d


e


f


5 Find the length of the longest side of △*PQR* if $P = 31°42'$, $Q = 28°50'$ and $PR = 42$ mm.

6 Solve each of the following triangles. (Calculate the sizes of all unknown angles and sides, including both possibilities in ambiguous cases.)

- **a** △*ABC*, where $a = 20$ cm, $A = 47°$ and $B = 65°$
- **b** △*KLM*, where $k = 6$ cm, $m = 8$ cm and $K = 41.3°$
- **c** △*ABC*, where $b = 6.4$ m, $c = 4.7$ m and $B = 68°$
- **d** △*PQR*, where $q = 74$ mm, $Q = 42.3°$ and $R = 72.8°$
- **e** △*DEF*, where $d = 13.4$ cm, $e = 12.9$ cm and $D = 50.4°$
- **f** △*XYZ*, where $X = 38°5'$, $Y = 109°21'$ and $z = 47$ m
- **g** △*XYZ*, where $x = 2.3$ m, $z = 3.7$ m and $X = 37°43'$

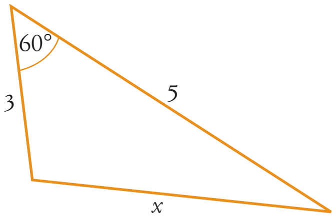
7 In △*DEF*, $f = 30$ cm, $d = 35$ cm and $F = 18°$. Find both possible values for e and D.

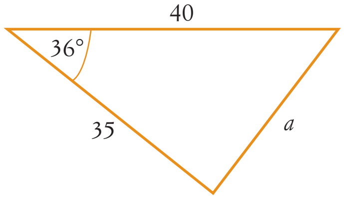
8 In △*ABC*, $BC = 18$ cm, $AC = 15$ cm and $B = 38°$. Find both possible values of A and make a sketch of each possible △*ABC*.

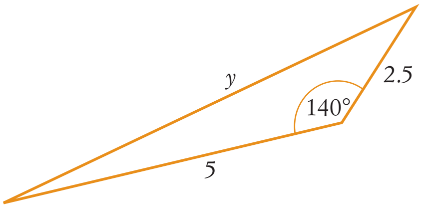
See Example 24

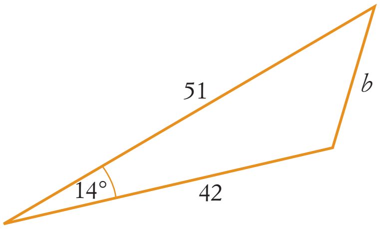
9 Use the cosine rule to calculate the unknown side lengths in the following triangles.

See Example 26

a


b


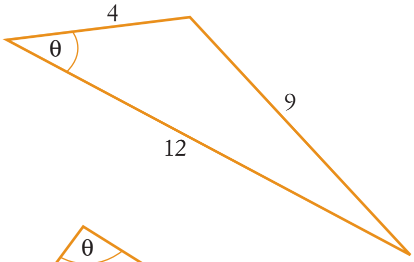
c


d


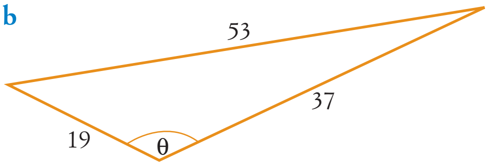
See Example 27

10 Use the cosine rule to find θ in each of the following triangles.

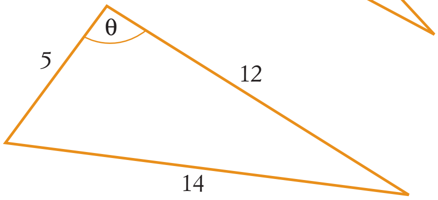
a



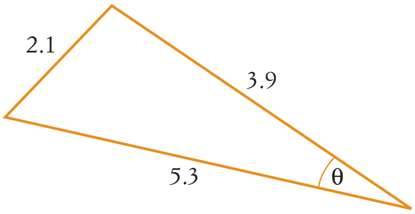
b



c



d



Worked solutions
Exercise 10.3
MAT10MGWS00063

11 Find the third side of each of the following triangles.

- **a** $\triangle XYZ$, where $z = 5$, $x = 8$ and $Y = 60°$
- **b** $\triangle PQR$, where $p = 2$, $q = 4$ and $R = 30°$
- **c** $\triangle ABC$, where $b = 10$, $c = 15$ and $A = 90°$
- **d** $\triangle XYZ$, where $x = 5.5$, $z = 6.5$ and $Y = 72°38'$
- **e** $\triangle ABC$, where $c = 8.9$, $b = 10.2$ and $A = 112°$

12 Find all unknown angles and sides in each of the following triangles.

- **a** $\triangle ABC$, where $a = 4$, $b = 8$ and $c = 5$
- **b** $\triangle XYZ$, where $x = 2$, $y = 3$ and $Z = 124°$
- **c** $\triangle DEF$, where $d = 6$, $f = 4$ and $e = 3$
- **d** $\triangle FGH$, where $f = 4.4$, $g = 6.7$ and $H = 72°$
- **e** $\triangle PQR$, where $p = 3\sqrt{6}$, $r = 2\sqrt{2}$ and $Q = 30°$

Worked solutions
Exercise 10.3
MAT10MGWS00063

13 In $\triangle ABC$, $a = 12$, $b = 9$ and $c = 8$. Calculate the greatest angle.

14 In $\triangle XYZ$, $x = 8.8$, $y = 11$ and $z = 17.6$. Calculate the smallest angle.

15 In $\triangle PQR$, $p = \sqrt{5}$, $q = 1$ and $r = 2$. Calculate the greatest angle.

Problem solving

Worked solutions
Exercise 10.3
MAT10MGWS00063

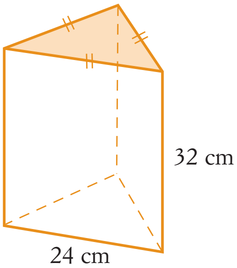
See Example 22

16 Two sides of a triangle are 4.5 cm and 6 cm and its area is 11.25 cm^2. Find the angle included by the sides.

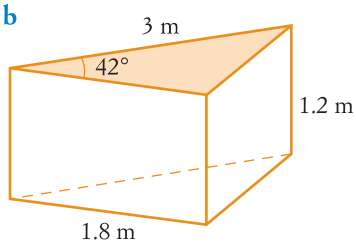
17 If α is the angle between two adjacent sides of a parallelogram whose lengths are p and q respectively, show that the area of the parallelogram is $pq \sin \alpha$.

18 Find the volumes of these prisms.

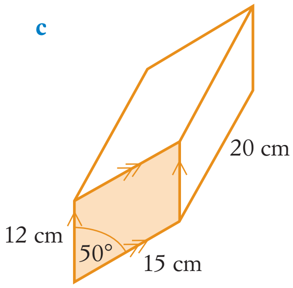
a



b



c



19 A triangle XYZ has side $XZ = 1.3$ m and angles $\angle ZXY = 42°$ and $\angle XYZ = 65°$. Find the perimeter of the triangle.

20 When Doug looks at the top of a tower, the angle of elevation is 27°. When Doug walks 20 m further away from the tower on level ground, the angle of elevation decreases to 20°. What is the height of the tower?

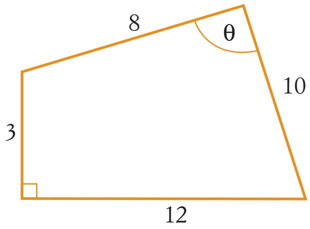
See Example 25

21 Anne is flying from Perth to Geraldton at a speed of 150 km/h. She thinks her altimeter, which shows a height of 15 000 feet (4600 m), is faulty. Looking ahead, she sees the top of a hill marked as 405 m on her map at an angle of depression of 13°. Two minutes later, the angle of depression has increased to 20.6°.

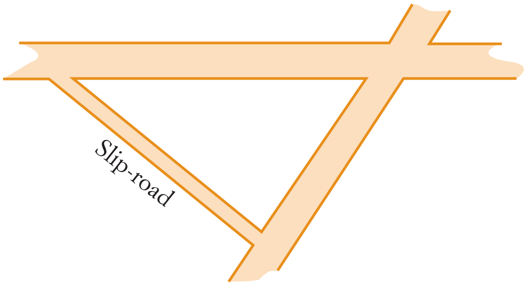
a Find her height above the hill and thus her altitude.

b What should the radio operator tell her about her altimeter?

22 Find the value of θ in the diagram on the right.



23 A fence is to be built on the perimeter of a triangular island where two roads and a slip-road meet. The angle between the two roads is 48°, and the distances along each road to the slip-road are 100 m and 150 m.
Find the length of the fence.



24 After a chat, Peter and Kyle walk away from each other. Peter walks 560 m in the direction N 42° W while Kyle walks 670 m in the direction N 32° E. Find their distance apart.

See Example 28

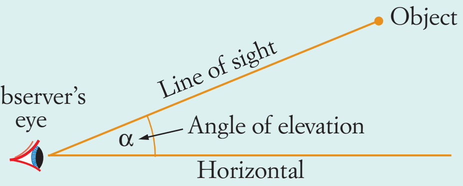
10.4 2D and 3D applications of trigonometry

Many practical problems in navigation, surveying and engineering can be solved using right-angled triangles. Trigonometry can be used to measure distances in these triangles that would be unmeasurable in any other way.
Angles of elevation and depression are commonly encountered in trig problems.

Important!

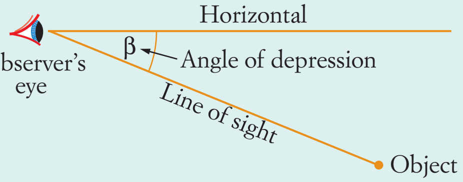
Angle of elevation

The **angle of elevation** of an object from an observer is the angle between the horizontal and the **line of sight** *up* to the object.



Angle of depression

The **angle of depression** of an object from an observer is the angle between the horizontal and the **line of sight** *down* to the object.



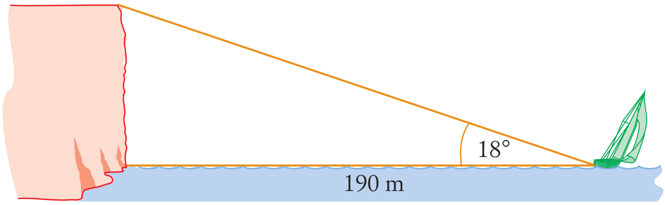
Example 29

Video tutorial

Angles of elevation and depression

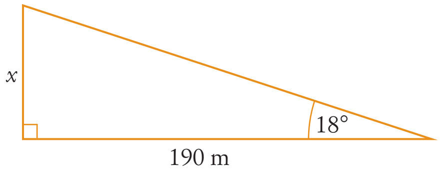
MAT10MGVT10023

The angle of elevation from a yacht to the top of a cliff is 18°. If the yacht is 190 m from the base of the cliff, find the height of the cliff.



Solution

Sketch a simple diagram using the information given.



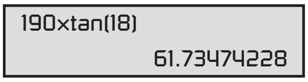
The known and unknown sides are adjacent to and opposite 18°. Use tan 18°.

$$\tan 18° = \frac{\text{opposite}}{\text{adjacent}} = \frac{x}{190}$$

Multiply both sides by 190 and reverse the equation.

$$x = 190 \times \tan 18°$$

Enter 190 × tan 18 = →



```
190xtan(18)
                61.73474228
```

State the result.

The cliff is about 61.7 m high.

When solving trigonometric problems, you must decide which rules to use. The following provides some ideas to guide you in choosing the correct (or easiest) method.

Important!

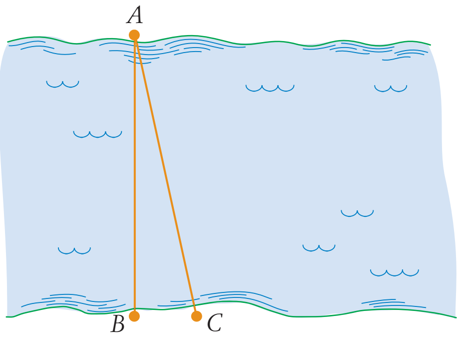
Solving triangles

When solving triangles:

- look for solutions using right-angled triangles first
- use the sine rule where you know one angle and the opposite side. Be careful of ambiguous cases
- use the cosine rule where the known angle is enclosed by two known sides or the three sides are known. After using the cosine rule once, the sine rule may be useful for other angles and sides
- if the known side is in a different triangle from an unknown side, the side common to the triangles should be found as an intermediate step

Example 30

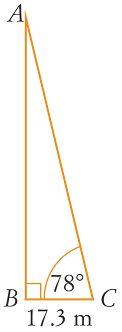
A survey locates points A, B and C on the banks of a river as shown here. BC is measured to be 17.3 m and $\angle ACB$ is found to be 78°. Calculate the width of the river (AB).



Solution

First sketch a diagram that includes all the supplied information.

We know the side adjacent to $\angle ACB$ and we want to find the opposite side.



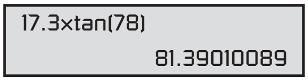
Use tan 78°.

$$\mathbf{\tan 78^\circ = \frac{opposite}{adjacent} = \frac{\mathit{AB}}{17.3}}$$

Multiply both sides by 17.3 and reverse the equation.

$$\mathbf{\mathit{AB} = 17.3 \times \tan 78^\circ}$$

Enter 17.3 × tan 78 = →.



```
17.3×tan(78)
                    81.39010089
```

State the result.

The river is about 81.4 m wide.

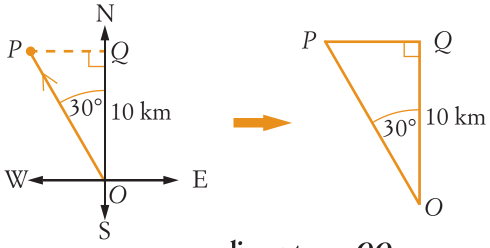
Example 31

A cyclist travels in the direction N 30° W until she is 10 km north of her starting point. Calculate how far the cyclist has travelled.

Solution

First sketch a diagram and highlight the information required.

We know the adjacent side (OQ) and want to find the hypotenuse (OP).



Use cos 30°.

$$\cos 30^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{OQ}{OP}$$

Substitute the known information.

$$\cos 30^\circ = \frac{10}{OP}$$

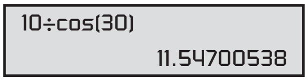
Multiply both sides by OP.

$$OP \times \cos 30^\circ = 10$$

Divide both sides by cos 30°.

$$OP = \frac{10}{\cos 30^\circ}$$

Enter 10 ÷ cos 30 = →



```
10÷cos(30)
           11.54700538
```

State the result.

The cyclist has travelled about 11.5 km.

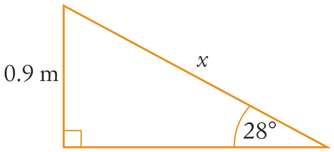
Example 32

An overhead sign is positioned 0.9 m from the top of a steel framework and is supported by a wire. If the angle between the wire and the sign is 28°, how much wire is needed?

9780170194860

Solution

Draw a simplified diagram using the information given. Let the length of the wire be *x* m. We know the opposite side and want to find the hypotenuse.



Use sin 28°.

$$\mathbf{\sin 28^\circ = \frac{opposite}{hypotenuse} = \frac{0.9}{\mathit{x}}}$$

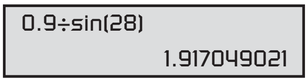
Multiply both sides by *x*.

$$\mathbf{\mathit{x} \sin 28^\circ = 0.9}$$

Divide both sides by sin 28°.

$$\mathbf{\mathit{x} = \frac{0.9}{\sin 28^\circ}}$$

Enter 0.9 ÷ sin 28 = →



```
0.9÷sin(28)
                1.917049021
```

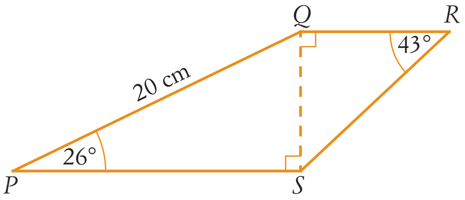
State the result.

About 1.9 m of wire is needed.

Some figures are composed of two or more right-angled triangles joined by a common side. These are sometimes called **composite figures**. When dealing with composite figures, you will usually have to find a length or angle from one triangle first and then transfer that information to the other triangle.

Example 33

Find the length of *SR* in this diagram.

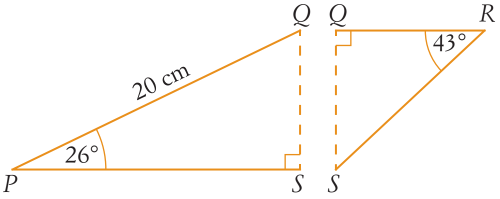


Solution

Draw the diagram as two separate triangles.

We need to find *QS* in △*PQS* before we can find *SR* in △*QRS*.

Look at △*PQS*. We know the hypotenuse and we want to find the opposite side, *QS*.



Use sin 26°.

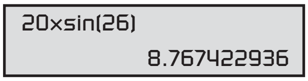
$$\mathbf{\sin 26^\circ = \frac{opposite}{hypotenuse} = \frac{\mathit{QS}}{20}}$$

Multiply both sides by 20 and reverse the equation.

$$\mathbf{\mathit{QS} = 20 \sin 26^\circ}$$

Enter 20 × sin 26 = →

Keep the value in your calculator.



```
20×sin(26)
                8.767422936
```

Look at △*QRS*. We know the side opposite 43° and we want to find the hypotenuse, *SR*.

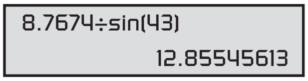
Use sin 43°. $\sin 43° = \frac{\text{opposite}}{\text{hypotenuse}}$

Substitute values. $\sin 43° = \frac{8.7674\ldots}{SR}$

Multiply both sides by *SR*. $SR \times \sin 43° = 8.7674\ldots$

Divide both sides by sin 43°. $SR = \frac{8.7674\ldots}{\sin 43°}$

Enter 8.7674... ÷ sin 43 = →



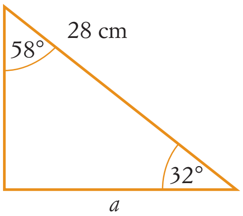
```
8.7674÷sin(43)
                12.85545613
```

State the result. **SR is about 12.9 cm long.**

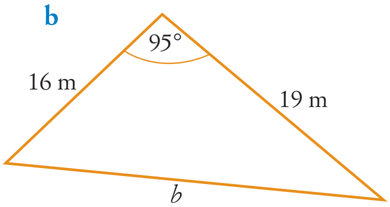
Example 34

Find the marked sides and angles in each of the triangles below.

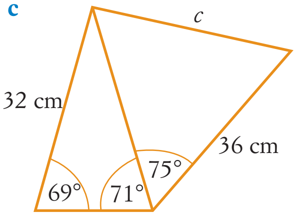
a



b



c



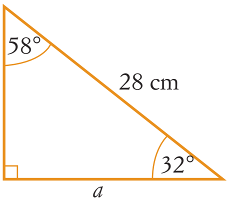
Solution

a The angles in a triangle add up to 180°.

So the third angle must be 90°.

The hypotenuse is 28 cm.

a is opposite 58°.



Use $\sin = \frac{\text{opposite}}{\text{hypotenuse}}$. $\sin 58° = \frac{a}{28}$

Rearrange to find *a*. $a = 28 \sin 58° = 23.7453\ldots$

Round and write the answer. **a is about 23.7 cm.**

b We need to find the third side, so use the cosine rule. $b^2 = a^2 + c^2 - 2ac \cos B$

Substitute for known values. $= 16^2 + 19^2 - 2 \times 16 \times 19 \times \cos 95°$

Evaluate. $= 256 + 361 + 52.9906\ldots$

Keep the exact number on your calculator. $= 669.9906\ldots$

Use √. $b = 25.8841\ldots$

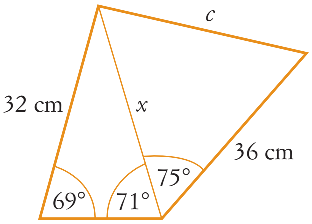
Round and write the answer. **b is about 25.9 m.**

c Three things are known in the first triangle.

c is in a second triangle.

The common side can be found.

Label this side as x.



Use the sine rule for x. $\quad \dfrac{x}{\sin 69°} = \dfrac{32}{\sin 71°}$

Rearrange to isolate x. $\quad x = \dfrac{32 \sin 69°}{\sin 71°}$

Keep the exact answer on your calculator. $\quad = 31.5959 \ldots$

Now use the cosine rule for c. $\quad c^2 = x^2 + 36^2 - 2 \times x \times 36 \times \cos 75°$

Substitute for x and evaluate, keeping exact figures on the calculator. $\quad = 1705.5151 \ldots$

Take the square root of both sides. $\quad c = 41.2978 \ldots$

Round and write the answer. **c is about 41.3 cm.**

The trigonometric problems solved so far in this chapter have involved using triangles only in two dimensions (2D). Problems involving triangles in three dimensions (3D) are a little more complex and need to be simplified before they can be solved using trigonometry.

Important!

Solving 3D trigonometric problems

To solve three-dimensional problems:

- use a line box drawing to help visualise the problem
- represent the 3D information by drawing it in the line box, including all relevant information
- label points on the diagram
- ‘unfold’ the diagram and draw the triangles flat
- label the unknown sides and angles
- use trigonometric ratios to find the unknowns
- write the solution to the problem in an appropriate form

Example 35

From a certain point, *A*, a mountain peak due north has an angle of elevation of 20°. From another point, *B*, 2 km east of *A* and on the same level as *A*, the bearing of the peak is N 40° W. Find the height of the peak above the level of *A* and *B*.

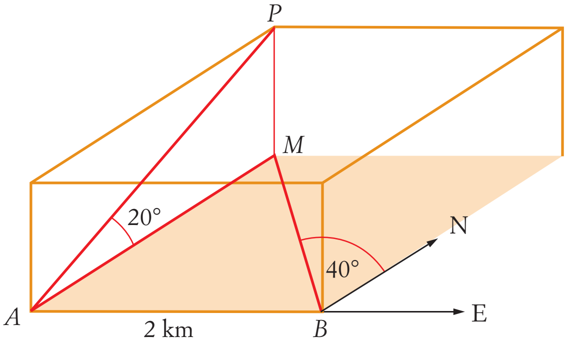
Solution

Draw a line box.

Make a sketch inside the box to create a 3D drawing.

Call the foot of the mountain *M* and the peak *P*.

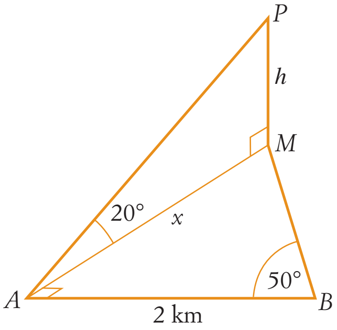
Show north and east at *B*. You don't have to show all of the major directions because this could make your diagram too confusing.



$\angle MBN = 40°$ and $\angle ABN = 90°$, so $\angle ABM = 50°$.

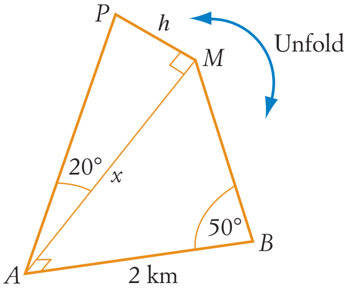
Now redraw the diagram without the parts of the box that you don't need.

Let the height of the mountain be *h* and the let *x* be the distance from *A* to the foot of the mountain directly below *P*.

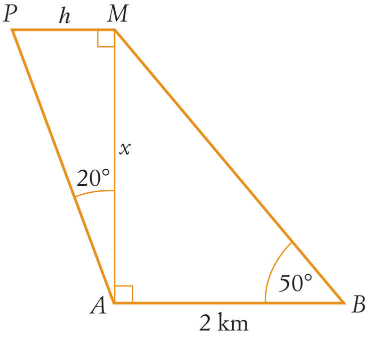


Now 'unfold' the diagram.

First lie $\triangle APM$ flat.



Then lie $\triangle MAB$ flat.



Use tan in △*MAB* to find *x*.	$\tan 50° = \frac{x}{2}$
Rearrange but don't evaluate.	$x = 2 \tan 50°$
Now use tan in △*PAM* to find *h*.	$\tan 20° = \frac{h}{x}$
Rearrange to isolate *h*.	$h = x \tan 20°$
Substitute for *x*.	$= 2 \tan 50° \times \tan 20°$
Evaluate.	$= 0.86752...$
Round off.	≈ 0.868 km
Write the answer.	**The mountain peak is about 868 m high.**

Example 36

Animated example
2D and 3D applications of trigonometry
MAT10MGAE00010

Hassan observes that the top of a transmission tower in the direction N 38° E is at an angle of elevation of 12°. Fatima is 375 m due east of Hassan, and she says that the bearing of the tower is 308°. Find the height of the tower.

Solution

Draw a line box.

Make a sketch inside the box.

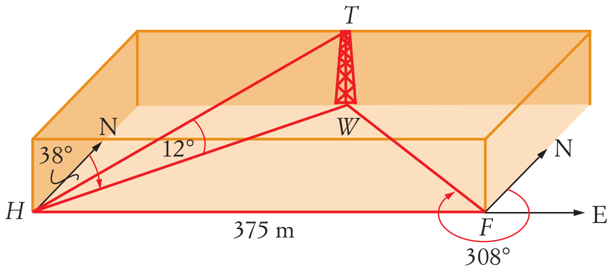
Call Hassan's position *H*.

Call Fatima's position *F*.

Show north and east.

Call the bottom of the tower *W*.

Call the top of the tower *T*.



$\angle WHN = 38°$ and $\angle FHN = 90°$, so $\angle FHW = 52°$.

The other bearing shows that $\angle WFH = 38°$.

$\angle TWH = 90°$.

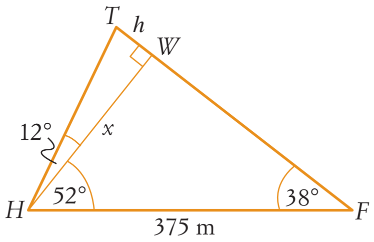
Unfold and draw the triangles flat.

Call the height *h*.

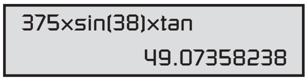
Call the common side *x*.

The angles in △*WHF* add up to 180°.

Thus $\angle HWF = 90°$.



Use sin in △*WHF* to find *x*.	$\sin 38° = \frac{x}{375}$
Rearrange but don't calculate.	$x = 375 \sin 38°$

Now use tan in △*THW* to find *h*.	$\tan 12° = \frac{h}{x}$
Rearrange and isolate *h*.	$h = x \tan 12°$
Substitute for *x*.	$= 375 \sin 38° \times \tan 12°$
Enter 375 × sin 38 × tan 12 = →	 375xsin(38)xtan 49.07358238
Round off.	$\approx$ **49.1 m**
Write the answer.	**The tower is about 49.1 m high.**

In the next example, one side has to be *duplicated* to unfold the diagram to make it flat. In addition, the required side cannot be found directly. After applying trigonometry you must solve a simple equation to find the height.

Example 37

Maria and Jack are standing on a level road 150 m apart. A church spire is due north of Maria and on a bearing of 070° from Jack. The top of the spire appears at an angle of elevation of 27° to Jack and 18° to Maria. Find the height of the spire above the road.

Solution

Make a sketch inside a line box.

Call Jack's position *J*.

Call Maria's position *M*.

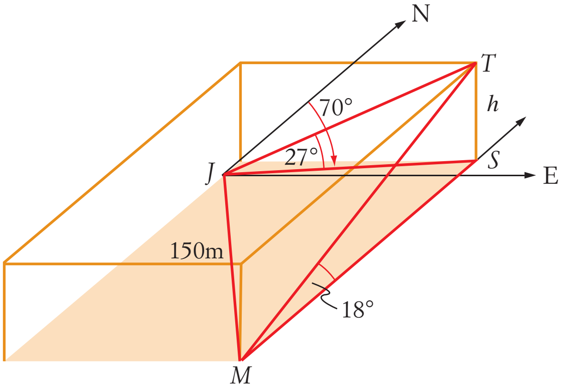
Show north and east.

Call the bottom of the spire *S*.

Call the top of the spire *T*.

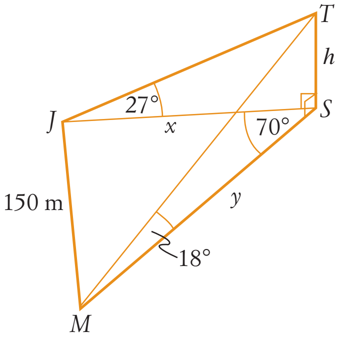
Call the height of the spire *h*.

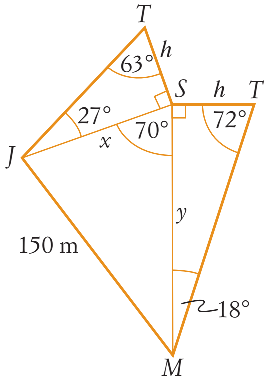
Call the distance of the spire from Jack *x* and its distance from Maria *y*.



∠*NJS* and ∠*JSM* are alternate,
so ∠*JSM* = 70°.
Unfold the diagram to make it flat, showing the side *TS* on two of the flat triangles.

By the angle sum of a triangle, ∠*JTS* = 63° and ∠*MTS* = 72°.





Use tan in $\triangle TSJ$ to find h.	$\tan 63° = \frac{h}{x}$
Use tan in $\triangle TSM$ to find h.	$\tan 72° = \frac{y}{h}$
Now use the cosine rule in $\triangle JMS$.	$150^2 = x^2 + y^2 - 2xy \cos 70°$
To proceed further you need x and y.	$x = h \tan 63°$ and $y = h \tan 72°$
Rearrange the previous expressions. Now substitute x and y.	$150^2 = (h \tan 63°)^2 + (h \tan 72°)^2 - 2(h \tan 63°)(h \tan 72°) \cos 70°$
Note: $\tan^2 \theta = (\tan \theta)^2$.	$= h^2 \tan^2 63° + h^2 \tan^2 72° - 2h^2 \tan 63° \tan 72° \cos 70°$
Factorise.	$= h^2(\tan^2 63° + \tan^2 72° - 2 \tan 63° \tan 72° \cos 70°)$
Evaluate.	$22\,500 = h^2 \times 9.1921 \ldots$
Rearrange.	$h^2 = 22\,500 \div 9.1921 \ldots$
Evaluate.	$= 2447.7350 \ldots$
Take the square root of both sides.	$h = 49.4745 \ldots$
Round and write the answer.	**The height of the spire is about 49.5 m.**

Exercise 10.4 2D and 3D applications of trigonometry

Fluency

Extra questions

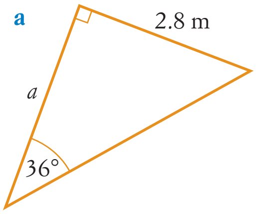
Exercise 10.4

MAT10MGEQ00030

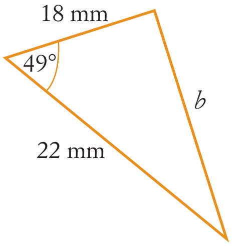
See Example 34

1 Find the marked angles and sides in each of the following triangles.

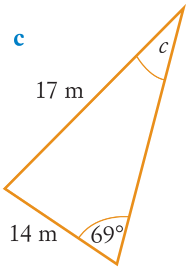
a



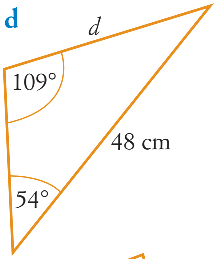
b



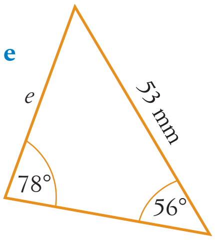
c



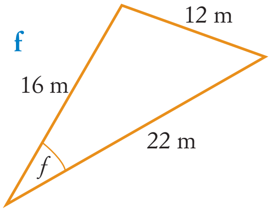
d



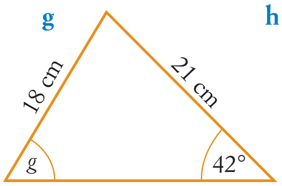
e



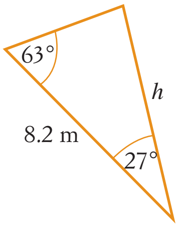
f



g

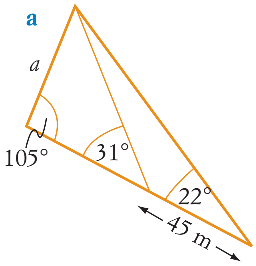


h

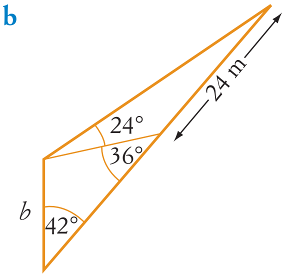


2 Find the marked sides in the following.

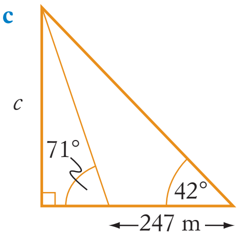
a



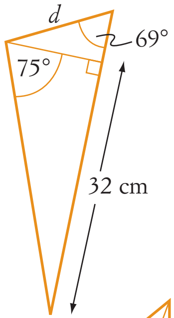
b



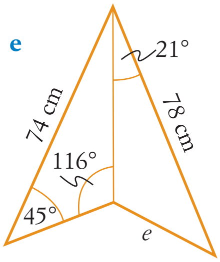
c



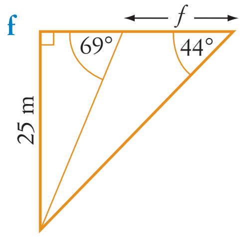
d



e



f



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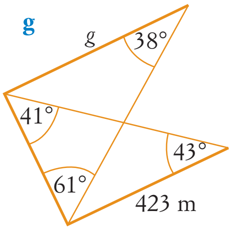
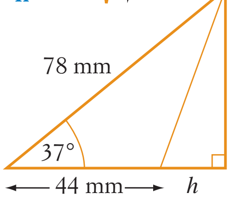


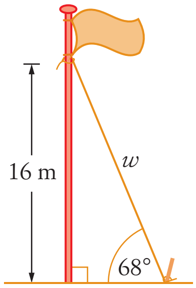
h



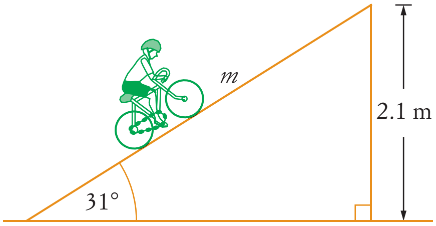
Problem solving

3 Find the unknown length in each of the following triangles, correct to 2 decimal places.

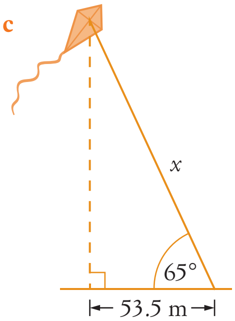
a



b



c



Worked solutions

Exercise 10.4

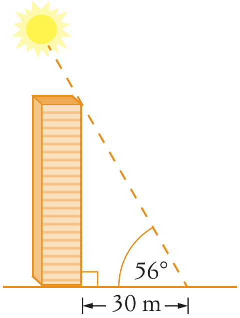
MAT10MGWS00064

See Example 29

④ The shadow of a building is 30 m long when the angle of elevation of the sun is 56°.

a Calculate the height of the building.

b How long will the shadow be when the angle of elevation of the sun becomes 32°?



5 An observer in a lighthouse, 19 m high, sights a boat at sea at an angle of depression of 18°. The base of the lighthouse is at sea-level. How far is the boat from the base of the lighthouse?

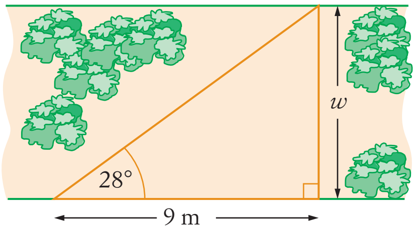
6 A helicopter flying at a height of 1800 m is directly overhead. After 15 minutes, it is still 1800 m above the ground, but its angle of elevation is 8°. Calculate:

a the distance travelled in that time, to the nearest 100 m

b the speed of the helicopter in km/h.

7 A nature strip was surveyed and measurements were recorded as shown here. What is the width of the nature strip?

See Example 30



⑧ Point P is 10 km west and 6 km south of O.

a Find θ.

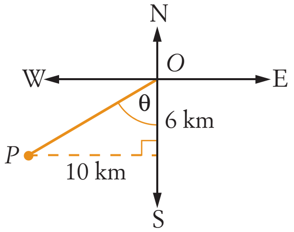
b Find the bearing from O to P.

Worked solutions

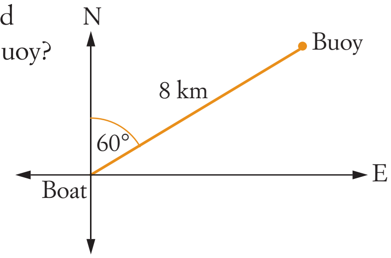
Exercise 10.4

MAT10MGWS00064

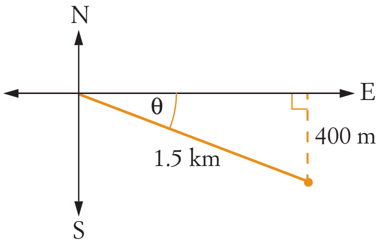
See Example 31



9 The bearing of a buoy from a boat is 060°. If the boat and the buoy are 8 km apart, how far east of the boat is the buoy?



10 A cyclist travels for 1.5 km in a straight line so that she is 400 m south of her starting point. What is the bearing of the cyclist from her starting point, to the nearest degree?

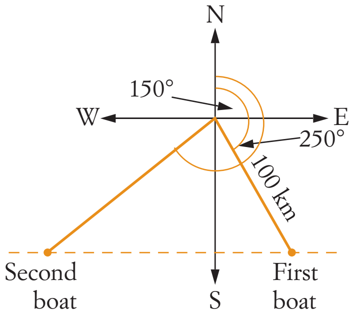


11 Two boats leave port at the same time. The first boat travels on a bearing of 150°, while the second boat travels on a bearing of 250°. After the first boat has travelled 100 km, it is due east of the second boat.

a How far east of its starting point is the first boat?

b How far south of its starting point is the second boat?

c What is the distance between the boats?

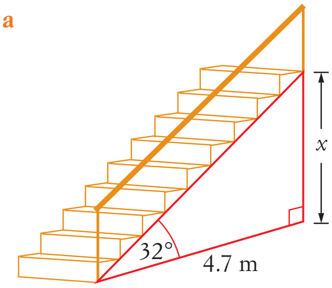


12 A plane flies for 500 km on a bearing of 205°. How far south of its starting point is it now?

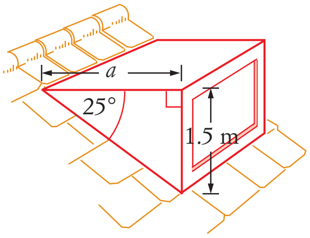
See Example 32

13 Find the unknown length in each of the following triangles, correct to 2 decimal places.

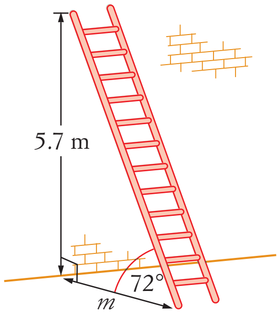
a



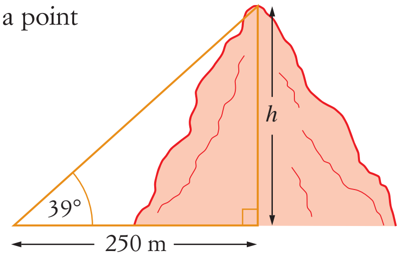
b



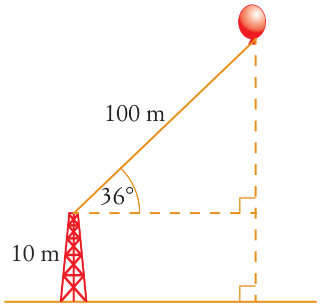
c



14 A surveyor measures the angle to the top of a peak from a point 250 m away to be 39°. How high is the peak?



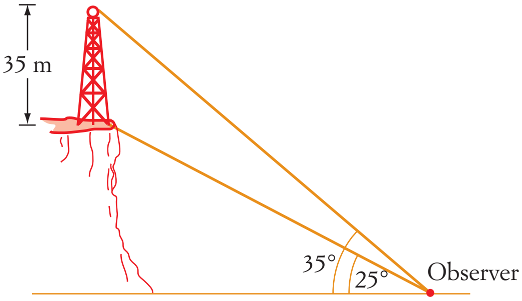
15 A balloon is tied to a string that makes an angle of 36° with the horizontal. The string is 100 m long, and the bottom end is tied to a tower 10 m above the ground. How high above the ground is the balloon?



16 A park has two flagpoles 20 m apart. One pole is 12 m high, the other is 21 m. Calculate the angle of depression from the top of the taller pole to the top of the shorter one.

17 While Ann is walking on a straight road running east-west from *A* to *B*, which are 10 km apart, she sees a church in the distance. The bearing from *A* to the church is N 48° E and from *B* the church is at a bearing of N 67° W. Find the distance from *A* to the church.

18 A beacon is mounted on a tower that sits on a vertical cliff. The beacon is 35 m higher than the top of the cliff. To an observer some distance from the base of the cliff, the beacon and the top of the cliff appear to be at angles of elevation of 35° and 25° respectively. How far is the observer from the base of the cliff?

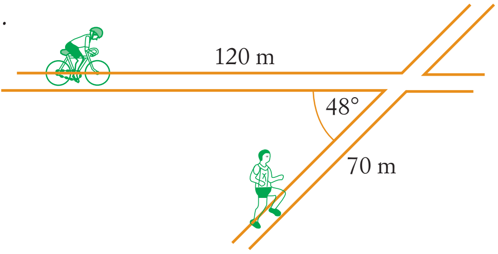


19 An aircraft flies on a course of 322° for 250 km and then changes course to 244° to an airstrip where it lands. If the airstrip is 480 km from the starting position, calculate:

a the distance travelled from the time that the course was changed until the plane landed

b the bearing from the airstrip to the starting position.

20 Two straight streets intersect at an angle of 48°. A cyclist approaching the intersection is 120 m from it at the same instant that a jogger is approaching the same intersection at a distance of 70 m. Find the most direct distance between the cyclist and the jogger at this time.



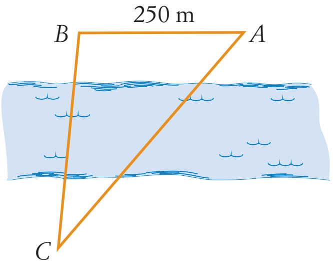
21 Two cyclists leave from the same shopping centre. The first travels along a straight road due west at 18 km/h. Thirty minutes later, the second leaves and proceeds along a straight road in a south-westerly direction at 24 km/h. How far apart are the two cyclists 30 minutes after the second departs from the shops?

22 Two ships are currently both at anchor. The first ship is 38 km from a port and on a bearing of 054° from the port. The second ship is 50 km from the same port and on a bearing of 115° from the port. Find the distance between the two ships and the bearing of the first ship from the second.

23 Two structures, *A* and *B*, on one side of a river are 250 m apart. There is another structure located at point *C*, on the other side of the river. Surveyors measure $\angle CAB$ and $\angle BCA$ to be 55° and 29° respectively. Find the distance from:

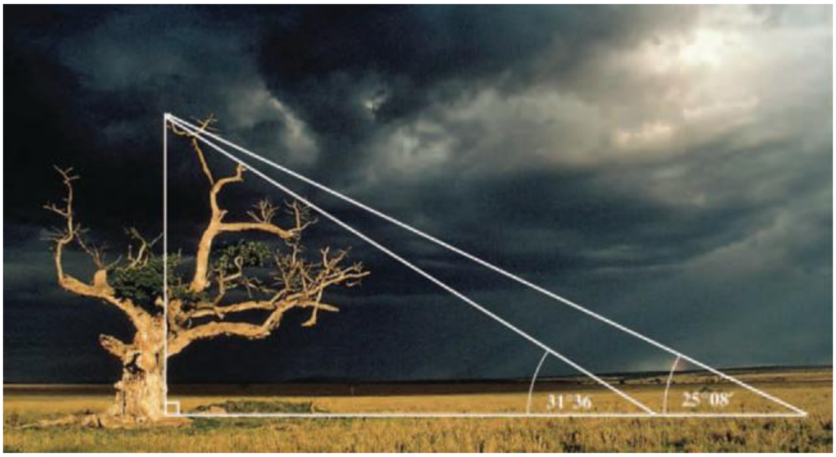
a *B* to *C*

b *A* to *C*



24 Two towns, Abbotsford and Bourkesville, are 65 km apart and are joined by a straight road. The bearing of Bourkesville from Abbotsford is 112°34′. The bearings of another town, Calen, from Abbotsford and Bourkesville are 182°51′ and 224°18′ respectively. Find the shortest distance from Calen to the road that joins Abbotsford and Bourkesville.

25 From a distance, the angle of elevation to the top of a tree is 25°08′. Ten metres closer, the angle of elevation is 31°36′. How high is the tree?



26 A vertical structure, 80 m tall, stands on top of a hill. From a point some distance from the base of the hill, the angle of elevation to the top of the structure is 11.4°. From the same point, the angle of elevation to the bottom of the structure is 8.2°. Find the height of the hill.

Reasoning

Worked solutions

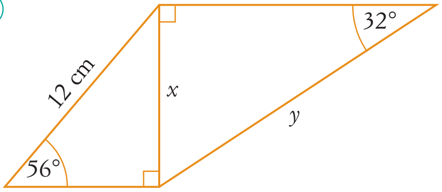
Exercise 10.4

MAT10MGWS00064

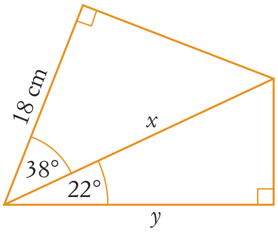
See Example 33

27 In each of the following, first find x and then find y, correct to 1 decimal place.

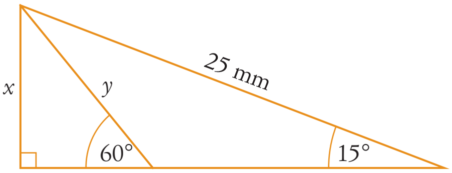
a



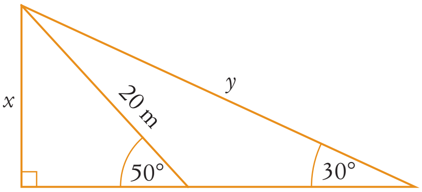
b



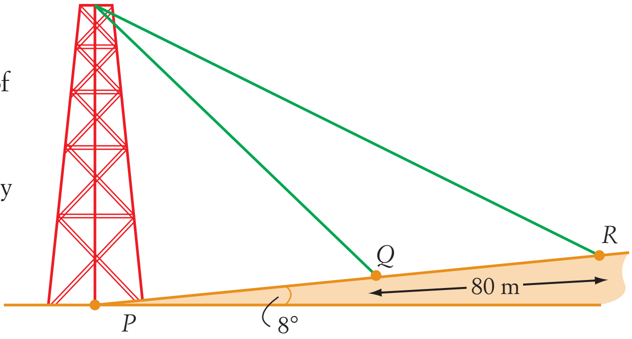
c



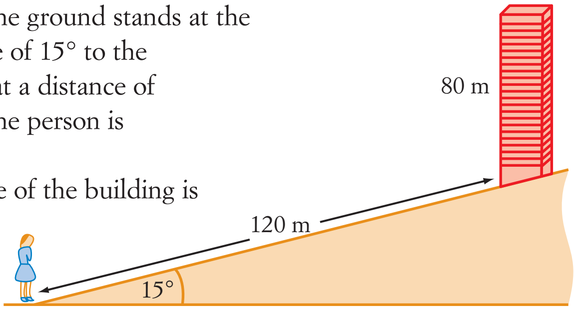
d



28 A tower is erected at the base of a road that is inclined at a constant angle of 8° to the horizontal. Angles of elevation to the top of the tower are measured at two points that are 80 m apart, some distance up the slope away from the tower. The angles are found to be 25° and 8° respectively. What is the height of the tower?



29 A person whose eye level is 1.8 m from the ground stands at the base of a road that is at a constant incline of 15° to the horizontal. There is a building 80 m tall at a distance of 120 m from the base of the road where the person is standing, as measured along the road.



a Find the vertical distance that the base of the building is above the ground where the person is standing.

b Calculate the angle of elevation of the top of the building from the person's eyes.

c Calculate the angle subtended by the top and bottom of the building at the person's eyes.

30 A low-impact mobile phone tower is on top of a building. The angles of elevation of the top and bottom of the tower are 16° and 10° respectively. Thirty metres closer to the building, the angle of elevation of the bottom of the tower has increased to 17°. Find the height of the tower above the building and the building's height above the ground.

31 A flagpole 12 m high is temporarily erected for an Australia Day ceremony. It is held in place by two wires: one 15 m long fastened due west of the pole and the other 17 m long fastened due south of the pole. Both supporting wires are attached to the top of the flagpole. Calculate: *See* Example 35

a the distance that each wire is from the pole at the point where it meets the ground

b the angle of inclination that each wire makes with the horizontal

c the distance between the points where the wires meet the ground.

d the angle between the wires.

32 From a point on a level straight road running due east, an observer finds that the summit of a mountain bears due north and that the angle of elevation is 11°30′. From a second point on the road 3.2 km due east of the first point, the observer finds that the bearing of the summit is N 25° W. Find the height of the summit of the mountain above the level of the road.

33 From a lighthouse 150 m above sea level, the lighthouse keeper observes a boat due east at an angle of depression of 25° and another boat due south at an angle of depression of 32°. *See* Example 36

a Find the distance that separates the boats.

b Find the bearing of the second boat from the first.

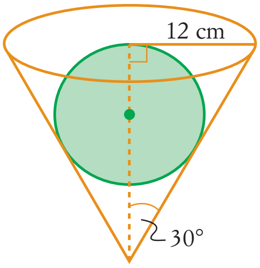
34 Frank is on the football ground and observes that the top of the stadium is at an angle of elevation of 24°. It is due east of his position. He gets passed the ball and runs 50 m towards the goals before losing it. He notices that the top of the stadium's angle of elevation has dropped to 13° and has a bearing of 102°. His freakish ability to measure angles does not endear him to the rest of the team, but how high is the stadium? *See* Example 37

35 An aircraft flying due east at 385 km/h at a constant height of 1500 m passes over a village at a particular instant. One minute later, a person who is 1.6 km south of the village sees the plane. In what direction and at what angle of elevation does the person see the plane?

36 A wall 2 m high and 60 m long stands in a vertical plane running east and west. Calculate the width and area of the shadow cast by the wall on level ground when the sun is due north at an elevation of 40°47′.

37 A straight stick 2 m long is stuck in the ground and is inclined at 60° to the horizontal, pointing in an easterly direction. Calculate the length and direction of the shadow of the stick when the sun is due north at an elevation of 30°. (Hint: Imagine a piece of string hanging vertically from the top of the stick to the ground. Consider the shadow of the string and its relation to the shadow of the stick.)

38 When the sun is north-west, the shadow of a pole 11 m high on a vertical wall 5 m from the pole reaches 3 m up the wall. The wall faces due north. Calculate the angle of elevation of the sun.



39 A cone with a radius of 12 cm has a semi-vertical angle of 30°. Find the radius of the largest sphere that can just fit inside the cone.

40 Two identical buildings 40 m tall are erected on a straight level road running east–west. From a point due south of one of the buildings, the angles of elevation to the tops of the buildings are 35° and 47° respectively. Find:

a the distance from the point to the base of each building

b the distance between the buildings.

41 Elena sees that the angle of elevation to the top of a tower, 80 m high, which is due south of her position is 37°. The top of another tower, 60 m high, which is due west of her, is at an angle of elevation of 28°. Calculate the horizontal distance between the two towers.

42 Miguel is 90 m due west of Consuella on a level road. He sees that the angle of elevation of a kite flown by their son is 30°, while Consuella sees that it is due north at an angle of elevation of 38°. What is the height of the kite?

Chapter 10 summary

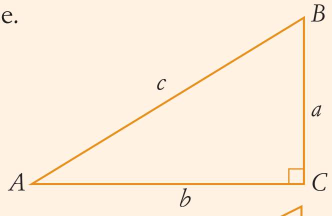
Quiz

Trigonometry

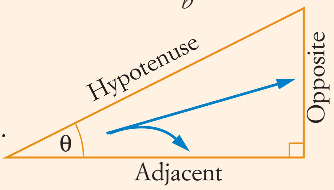
MAT10MGQZ00010

- In a right-angled triangle, the longest side is called the hypotenuse. It is opposite the right angle. Here, AB or c is the hypotenuse. **Pythagoras' theorem** applies to all right-angled triangles:

$$AB^2 = BC^2 + AC^2 \text{ or } c^2 = a^2 + b^2$$



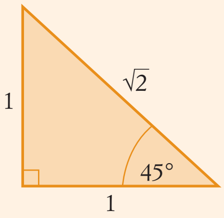
- Any three whole numbers that satisfy Pythagoras' theorem are called a **Pythagorean triple** (or **triad**), e.g. 3, 4, 5 and 5, 12, 13.

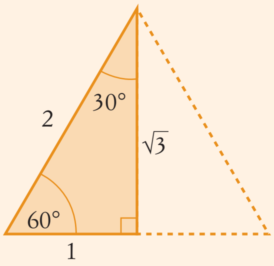


- **Trigonometric (trig)** ratios can be used to find the values of angles and the lengths of unknown sides in right-angled triangles.
- The **trig ratios** are defined as:

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite side}}{\text{adjacent}}$$

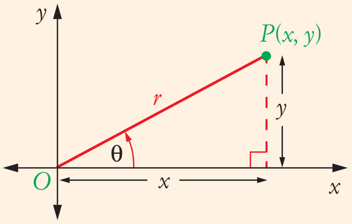
- It is possible to find exact values for the trigonometric ratios 30°, 45° and 60° using **standard triangles**.



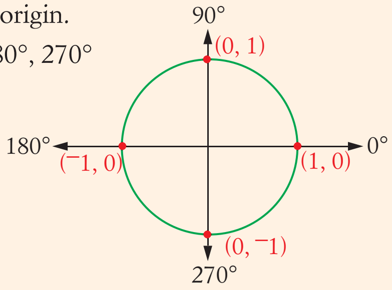


- The trig ratios can be defined using the Cartesian plane as shown on the right.

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$



- The **unit circle** has a radius of 1 unit and its centre at the origin.
- We can use the unit circle to find the trig ratios of 90°, 180°, 270° and 360° (0°).



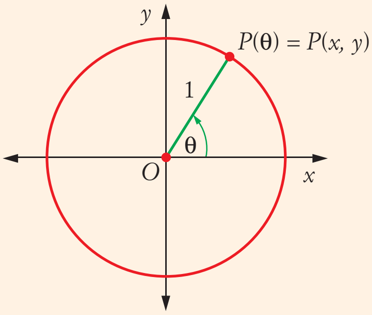
Chapter 10 summary

- The trigonometric ratios may be defined in terms of the unit circle as:

  sin $\theta = y = y$-coordinate of P

  cos $\theta = x = x$-coordinate of P

  $$\tan\theta = \frac{y}{x} = \frac{y\text{-coordinate of } P}{x\text{-coordinate of } P}$$



- The following trigonometric relations are important for acute angles

  $\sin\theta = \cos(90° - \theta)$

  $\cos\theta = \sin(90° - \theta)$

  $$\tan\theta = \frac{1}{\tan(90° - \theta)}$$

- The following trigonometric relations are important for obtuse angles

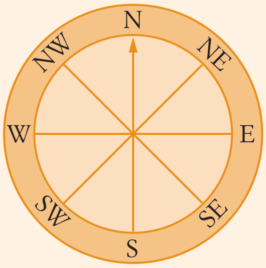
  $\sin\theta = \sin(180° - \theta)$

  $\cos\theta = -\cos(180° - \theta)$

  $\tan\theta = -\tan(180° - \theta)$

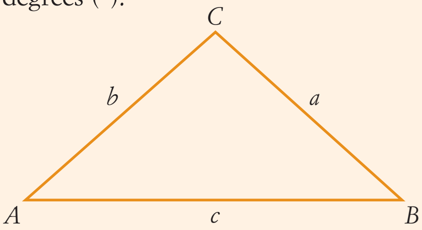
  where $0° \leq \theta \leq 180°$

- A **compass** is used to find **directions** and **bearings.** The major directions are N (north), S (south), E (east), W (west), NE (northeast), NW (north-west), SE (south-east) and SW (south-west).



- The **bearing** of an object is the angle measured in a clockwise direction from north to the object. A bearing is written as a three-digit number of degrees (°).
- The area of a non-right angled triangle is

  Area $\triangle ABC = \frac{1}{2}ab\sin C$



- The **sine rule** may be used to solve non-right-angled triangles where a side and an opposite angle are known. In any $\triangle ABC$,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Chapter 10 summary

- The **cosine rule** may be used to solve triangles if all three sides are known, or in cases where two sides and the enclosed angle are known. In any $\triangle ABC$,

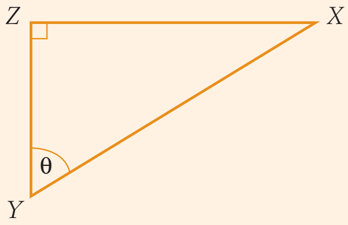
$$a^2 = b^2 + c^2 - 2bc \cos A \text{ or } \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

- When solving triangles:
  - look for solutions using right-angled triangles first
  - use the sine rule where you know one angle and the opposite side. Be careful of **ambiguous cases**.
  - use the cosine rule where the known angle is enclosed by two known sides or the three sides are known. After using the cosine rule once, the sine rule may be useful for other angles and sides.
  - if the known side is in a different triangle to an unknown side, the side common to the triangles should be found as an intermediate step
- The **angle of elevation** of an object from an observer is the angle between the horizontal and the line of sight *up* to the object. The **angle of depression** of an object from an observer is the angle between the horizontal and the line of sight *down* to the object.
- To solve **three-dimensional** problems:
  - a line box drawing may help to visualise the problem
  - make a 3D drawing including triangles and all relevant information
  - label points on the drawing
  - 'unfold' the diagram and draw the triangles flat
  - label the unknowns and intermediate sides and angles
  - use the methods of right-angled triangles to find the unknowns
  - write the solution to the problem in an appropriate form.

Chapter 10 review

Understanding

1 State Pythagoras' theorem for △*XYZ*.



2 Refer to △*XYZ* in question **1** to state:

a sin θ **b** cos θ **c** tan θ.

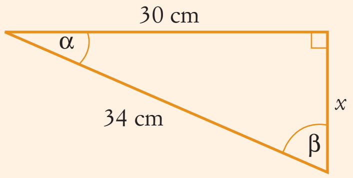
See Examples 6, 7

3 **a** Find the value of sin 31°, correct to 3 decimal places.
b Find θ, to the nearest degree, if sin θ = 0.438.

Fluency

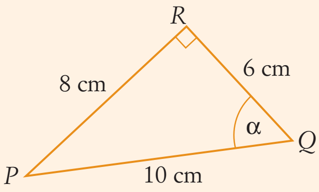
See Examples 1, 2

4 Use Pythagoras' theorem to find the unknown side in this triangle.



See Example 8

5 Use △*PQR* to:
a write the value of tan α in simplest form
b find the value of α to the nearest degree.



6 Calculate the values of α and β in the triangle shown in question **4**, to the nearest degree.

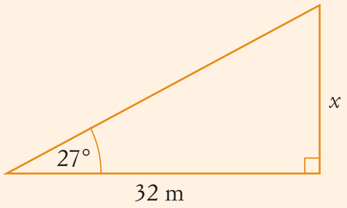
7 Find the values of the unknowns, correct to 1 decimal place.

a $a = 17 \tan 52°$ **b** $\tan 38° = \frac{b}{12}$

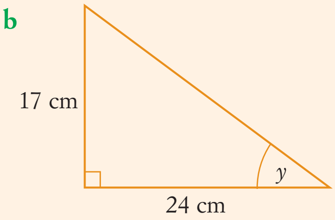
See Example 9

8 Find the values of the unknowns in the following triangles, correct to 1 decimal place.

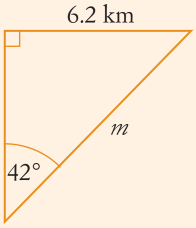
a



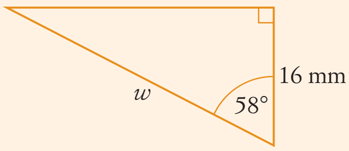
b



c

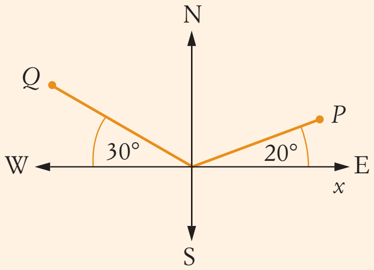


d



9 Write the bearings and directions of P and Q.

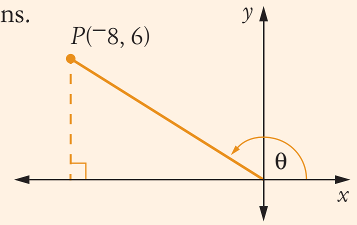
See Example 18



10 Use the diagram on the right to answer the following questions.

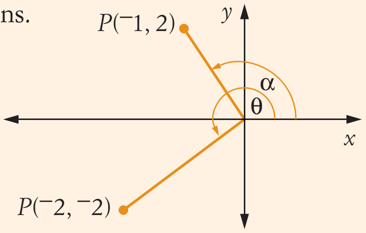
See Example 11

a What is the value of $\sin\theta$?
b What is the value of $\cos\theta$?
c What is the value of θ, to the nearest degree?



11 Use the diagram on the right to answer the following questions.

a What is the value of $\sin\theta$?
b What is the value of $\tan\theta$?
c What is the value of α, to the nearest degree?
d What is the value of θ?



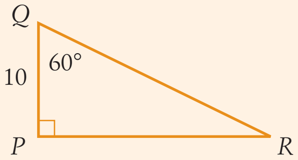
12 Use the unit circle to state the value of:

See Example 13

a $\sin 90°$ **b** $\cos 270°$ **c** $\tan 270°$ **d** $\sin 0°$

13 Find the exact value of PR.

See Example 10



14 Find the value of the pronumeral in each of the following where x is an acute angle.

See Examples 14, 16

a $\sin 47° = \cos x$ **b** $\tan 140° = -\tan x$

15 Find the exact value of $\cos 135°$.

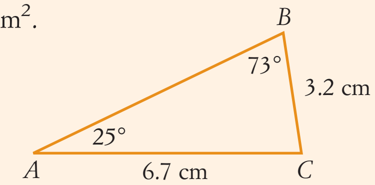
See Example 17

16 Find θ, to the nearest degree, if θ is obtuse and $\cos\theta = {}^{-}0.65$.

17 Find all possible values of A (correct to one decimal place) if $\sin A = 0.5738$.

18 Calculate the area of this triangle, correct to the nearest cm^2.

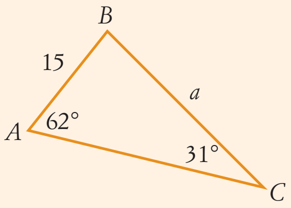
See Example 21



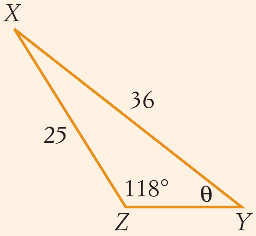
See Example 23

19 Find the value of the unknown in each of the following triangles correct to one decimal place.

a



b

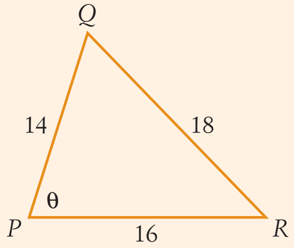


See Example 24

20 In $\triangle DEF$, $\angle D = 43°$, $d = 10$ and $f = 14$. Calculate the remaining angles and side and sketch the possible triangles.

See Example 27

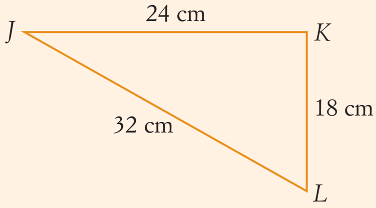
21 Find θ, correct to the nearest 0.1°.



Problem solving

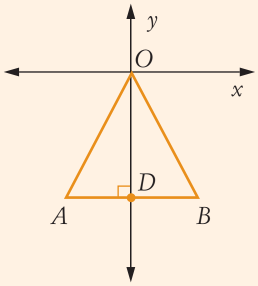
See Example 3

22 Is $\triangle JKL$ acute-, obtuse- or right-angled?



23 In the diagram below $\triangle OAB$ is equilateral, and symmetrical about the y-axis.

$OA = OB = AB = 2.$



Find the exact values of:

a AD

b OD

c $\angle AOD$

Use the diagram to find the value of

d sin 240°

e cos 240°

f tan 240°

g sin 300°

h cos ($^{-}60°$)

i tan ($^{-}60°$)

See Example 25

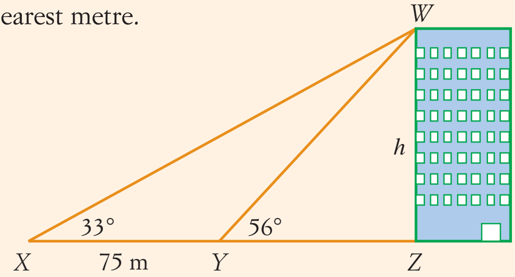
24 A surveyor on the beach finds that the angle of elevation of the top of a sand dune is 22°. After she walks along the beach towards the dune, the angle increases to 25°. On a detailed map of the area, the height of the dune is marked as 120 m. How far has the surveyor walked along the beach?

9780170194860

Reasoning

25 A spider and a fly are in the same room. The spider is hungry and the fly is asleep. The room is 15 m long, 6 m wide and 6 m high. The spider is on an end wall, 0.5 m from the ceiling and halfway between the side walls. The fly is on the other end wall, 0.5 m from the floor and halfway between the side walls. The spider wants to reach the fly but must crawl on the walls, ceiling and floor to get to its prey. What is the shortest route the spider might take? (Hint: The answer is not 21 m.)

26 The angle of elevation of a building is measured from two positions that are 75 m apart, as shown in the diagram.
Find the height, h, of the building, to the nearest metre.



27 Two groups of people set off to spotlight possums at night. One group spots a possum. The tree that has the possum is at a bearing of 332°. The angle of elevation of their spotlight beam is 38°. At this instant the other group, which is 100 m west of the first group, sights the same tree at a bearing of 62°. How high up in the tree is the possum?

See Example 37

Number and algebra

11

Equations

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Parent guide
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Curriculum guide
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Australian Curriculum statements

Linear and non-linear relationships

Solve problems involving linear equations, including those derived from formulas. (10)

Solve linear inequalities and graph their solutions on a number line. (10)

Solve linear simultaneous equations, using algebraic and graphical techniques including using digital technology. (10)

Solve linear equations involving simple algebraic fractions. (10)

Solve simple quadratic equations using a range of strategies. (10)

Solve simple exponential equations. (10A)

Factorise monic and non-monic quadratic expressions and solve a wide range of quadratic equations derived from a variety of contexts. (10A)

Pythagoras and trigonometry

Solve simple trigonometric equations. (10A)

Video tutorial
Equations
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The solution of equations is a key part of finding answers to many mathematical problems. There are many different types of mathematical equations, such as quadratic, polynomial, Diophantine, trigonometric, exponential, differential and vector equations. All kinds of mathematical equations rely on the skills developed in solving linear equations. This also applies to systems of mathematical equations and the solution of inequalities. In this chapter, you will extend your understanding of linear equations and look at the solution of simple inequalities and quadratic equations. You will also look at the simplest kind of equation system: simultaneous linear equations in 2 unknowns.

Mathematical literacy

Maths dictionary
MAT10ASDI00001

The mathematical words below have special meanings that you will learn in this chapter. It is important that you learn to spell them and gradually learn what they mean in mathematics. You may find the glossary or online mathematical dictionary useful for this purpose.

arccos *x*
arcsin *x*
coefficient
completing the square
constant
cos
$\cos^{-1}$
dummy variable
elimination method
equation
exponential equation
factorisation
growth factor
inequation
intersection
inverse operation
left-hand side
linear equation
linear inequality
logarithm
order of operations
period
periodic
principal value
QED
quadratic equation
quadratic formula
radian
right-hand side
satisfy
simultaneous equations
simultaneous solution
sin
$\sin^{-1}$
sinusoidal
solution
solve
substitution method
variable
zero product rule

11.1 Linear equations and inequations

In previous years, you may have solved equations using methods such as inspection, backtracking and modelling. Formal solution of equations is done by using **inverse operations**. This method allows you to solve more complex types of equations than other methods.

Important!

Equations and inverse operations

An **equation** is a mathematical sentence. It has an equals sign with expressions on both sides. The simplest algebraic equations are **linear** equations with *one variable*. When plotted for different values of the variable, a linear equation produces a straight line.

A **solution** of an algebraic equation is a value of the variable that makes the equation true. The process of finding the solution to an equation is called **solving** an equation. We say that a solution **satisfies** the equation.

Inverse operations are shown below.

Operation	**Symbol**	**Inverse operation**	**Symbol**
Addition	+	Subtract	–
Subtraction	–	Add	+
Multiplication	×	Divide	÷
Division	÷	Multiply	×

To solve an equation:

- aim to get the variable on its own on the left-hand side of the equals sign
- do inverse operations in the reverse order to the normal order of operations
- whatever you do to one side, do the same to the other side of the equation
- show your steps with the equals signs neatly under each other.

You can solve most equations with only one operation by inspection, so we will start with equations with two operations.

Example 1

Solve the following equations.

a $4a + 17 = 5$ **b** $\frac{m}{4} - 6 = 8\frac{1}{2}$ **c** $10 - 5n = {}^{-}27$

Solution

a	Write the equation.	$4a + 17 = 5$
	The side with the variable has a multiplication and an addition. Undo the addition before the multiplication.	$4a + 17 - 17 = 5 - 17$
	Simplify both sides.	$4a = {}^{-}12$
	Now undo the multiplication by dividing.	$\frac{4a}{4} = \frac{{}^{-}12}{4}$
	Simplify to get the answer.	$a = {}^{-}3$
b	Write the equation.	$\frac{m}{4} - 6 = 8\frac{1}{2}$
	Get rid of the – 6 by adding 6 before undoing division.	$\frac{m}{4} - 6 + 6 = 8\frac{1}{2} + 6$
	Simplify both sides.	$\frac{m}{4} = 14\frac{1}{2}$
	Now undo the division by multiplying by 4.	$\frac{m}{4} \times 4 = 14\frac{1}{2} \times 4$
	Simplify to get the answer.	$m = 58$
c	Write the equation.	$10 - 5n = {}^{-}27$
	Get rid of the 10 by subtracting 10 from both sides.	$10 - 5n - 10 = {}^{-}27 - 10$
	Simplify both sides.	${}^{-}5n = {}^{-}37$
	Now undo the multiplication by dividing by ⁻5.	$\frac{{}^{-}5n}{{}^{-}5} = \frac{{}^{-}37}{{}^{-}5}$
	Simplify to get the answer.	$n = 7\frac{2}{5}$

CAS TI-Nspire exercise
Equations
MAT10NATI00018

CAS ClassPad exercise
Equations
MAT10NACP00018

Weblink
The balance method
MAT10NAWB00011

In Example **1** part **c** some people would prefer to multiply the whole problem by $^{-}1$ first to change the $^{-}5n$ to $5n$, making the equation $5n - 10 = 27$. The final answer would still be $7\frac{2}{5}$. The next kind of equation has variables on both sides. Move all the variable terms to one side first. You can choose the side so that the coefficient of the variable ends up positive. Then you can do it like the previous ones.

Example 2

Solve the following equations.

a $5y + 6 = 9y - 21$ **b** $4 - 3y = 15 - 8y$

Solution

a Write the equation. $\mathbf{5y + 6 = 9y - 21}$

Move the variables to the right-hand side. $\mathbf{5y + 6 - 5y = 9y - 21 - 5y}$

Simplify. $\mathbf{6 = 4y - 21}$

Reverse the equation and add 21. $\mathbf{4y - 21 + 21 = 6 + 21}$

Simplify. $\mathbf{4y = 27}$

Divide by 4. $\mathbf{\frac{4y}{4} = \frac{27}{4}}$

Simplify to get the answer. $\mathbf{y = 6\frac{3}{4}}$

b Write the equation. $\mathbf{4 - 3y = 15 - 8y}$

Move the variables left and the constants right. $\mathbf{4 - 3y - 4 + 8y = 15 - 8y - 4 + 8y}$

Simplify. $\mathbf{5y = 11}$

Divide by 5. $\mathbf{\frac{5y}{5} = \frac{11}{5}}$

Simplify to get the answer. $\mathbf{y = 2\frac{1}{5}}$

When you have an equation with brackets, work in reverse order by multiplying out the brackets first.

Puzzle sheet
Solving linear equations 1
MAT10NAPS00035

Example 3

What are the roots of the following equations?

a $5 = 21 - 7(1 - x)$ **b** $3(2a - 5) = 2(a - 7)$

Solution

a Write the problem. $\mathbf{5 = 21 - 7(1 - x)}$

Multiply out the brackets, being careful with signs. $\mathbf{5 = 21 - 7 + 7x}$

Simplify and reverse the equation. $\mathbf{7x + 14 = 5}$

Subtract 14 from both sides. $\mathbf{7x + 14 - 14 = 5 - 14}$

Simplify. $\mathbf{7x = {}^{-}9}$

Divide by 7. $\frac{7x}{7} = \frac{^{-}9}{7}$

Simplify to get the answer. $x = {}^{-}1\frac{2}{7}$

b Write the equation. $3(2a - 5) = 2(a - 7)$

Multiply out the brackets, being careful with signs. $6a - 15 = 2a - 14$

Move the variables left and the constants right. $6a - 15 - 2a = 2a - 14 + 15$

Simplify $4a = 1$

Divide by 4. $\frac{4a}{4} = \frac{1}{4}$

Simplify to get the answer. $a = \frac{1}{4}$

You may be tempted to skip steps to 'save time' by doing them in your head. It actually takes longer if you do that, and you are more likely to make mistakes.

When you have an equation with fractions, get rid of the fractions first. Do this by multiplying *every term* by the lowest common denominator. If you have grouped terms in a numerator, leave it grouped with brackets. Multiply out the brackets *after* you've got rid of the fractions.

Example 4

Video tutorial
Equations with algebraic fractions
MAT10NAVT10026

Puzzle sheet
Solving linear equations 2
MAT10NAPS00036

Solve the following.

a $\frac{2m}{3} - \frac{m}{2} = 2$ **b** $\frac{2a + 4}{5} = \frac{2}{3}$ **c** $\frac{2n + 1}{3} - \frac{3n - 2}{2} = {}^{-}5$

Solution

a Write the problem. $\frac{2m}{3} - \frac{m}{2} = 2$

Multiply each term by the lowest common denominator, 6. $6 \times \frac{2m}{3} - 6 \times \frac{m}{2} = 6 \times 2$

Simplify by cancelling the denominators. $2 \times 2m - 3 \times m = 12$

Simplify each term. $4m - 3m = 12$

Simplify the variables. $m = 12$

b Write the equation with brackets around the grouped denominator. $\frac{(2a + 4)}{5} = \frac{2}{3}$

Multiply each term by the lowest common denominator, 15. $15 \times \frac{(2a + 4)}{5} = 15 \times \frac{2}{3}$

Simplify by cancelling the denominators. $3(2a + 4) = 5 \times 2$

Multiply out the brackets. $6a + 12 = 10$

Undo the + 12 by subtracting 12 from both sides. $6a + 12 - 12 = 10 - 12$

Simplify. $6a = {}^{-}2$

Divide by 6. $\frac{6a}{6} = \frac{^{-}2}{6}$

Simplify to get the answer. $a = {}^{-}\frac{1}{3}$

c Write the equation with brackets around the grouped numerators. $\frac{(2n+1)}{3} - \frac{(3n-2)}{2} = {}^{-}5$

Multiply each term by the lowest common denominator, 6. $6 \times \frac{(2n+1)}{3} - 6 \times \frac{(3n-2)}{2} = 6 \times {}^{-}5$

Simplify by cancelling the denominators. $2(2n+1) - 3(3n-2) = {}^{-}30$

Multiply out the brackets. $4n + 2 - 9n + 6 = {}^{-}30$

Simplify the LHS. ${}^{-}5n + 8 = {}^{-}30$

Subtract 8 from both sides. ${}^{-}5n + 8 - 8 = {}^{-}30 - 8$

Simplify. ${}^{-}5n = {}^{-}38$

Divide by ${}^{-}5$. $\frac{{}^{-}5n}{{}^{-}5} = \frac{{}^{-}38}{{}^{-}5}$

Simplify to get the answer. $n = 7\frac{3}{5}$

In some cases, equations with variables as denominators are actually linear equations. For these equations, you can still multiply by the lowest common denominator.

Example 5

Solve $\frac{5}{3a} - \frac{3}{5a} = \frac{4}{5}$.

Solution

Write the equation. $\frac{5}{3a} - \frac{3}{5a} = \frac{4}{5}$

Multiply each term by the lowest common denominator, 15*a*. $15a \times \frac{5}{3a} - 15a \times \frac{3}{5a} = 15a \times \frac{4}{5}$

Simplify by cancelling the denominators. $5 \times 5 - 3 \times 3 = 3a \times 4$

Simplify both sides and reverse the equation. $12a = 25 - 9 = 16$

Divide by 12. $\frac{12a}{12} = \frac{16}{12}$

Simplify to get the answer. $a = 1\frac{1}{3}$

When you substitute values into formulas with many variables, you will often end up with a linear equation to solve.

Example 6

A can manufacturer has to make cans of diameter 9 cm with a capacity of 415 mL. What height do they need to be?

Solution

Change the volume to cubic units. $415 \text{ mL} = 415 \text{ cm}^3$

Write the formula for the volume of a cylinder. $V = \pi r^2 h$

Substitute the values $V = 415$, $r = 4.5$. $415 = \pi \times 4.5^2 \times h$

Simplify and reverse the equation. $20.25\pi h = 415$

Divide by 20.25π.

$$\frac{20.25\pi h}{20.25\pi} = \frac{415}{20.25\pi}$$

Enter as 415 ÷ (20.25 × π) =

```
415÷(20.25×π)
          6.523387791
```

Write the answer. **The can needs to be about 6.5 cm high.**

Linear equations can be used to solve a wide range of problems. You should always start by choosing a name for the unknown. x is traditional, but you could use something else to avoid confusion with ×. When solving problems, you should check that your answer satisfies the conditions, not just the equation.

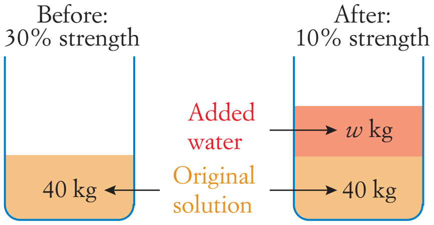
Example 7

A soft drink plant's basic sugar solution is at a concentration of 30% by mass. How much water must be added to 40 kg of the solution to make the strength of the solution 10% before adding the flavour?

Solution

Choose w for the mass of water that needs to be added.

Picture in your mind the solution and the added water before and after it is diluted.



Write your choice. **Let the mass of water added = w kg.**

Write down an expression for the amount of sugar. **Mass of sugar = 30% of 40 kg**

Evaluate.

$$\begin{aligned} &= 0.3 \times 40 \text{ kg} \\ &= 12 \text{ kg} \end{aligned}$$

Work out the final mass. **Total mass = $40 + w$ kg**

Write down an expression for the amount of sugar at the end. **Mass of sugar = 10% of $(40 + w)$ kg**

Write as an equation. $12 = 0.1 \times (40 + w)$

Multiply out the brackets. $12 = 4 + 0.1w$

Reverse the equation and subtract 4. $4 + 0.1w - 4 = 12 - 4$

Simplify. $0.1w = 8$

Multiply by 10 to get w.	$10 \times 0.1w = 10 \times 8$
Simplify to get the answer.	$w = 80$
Check the result.	**Amount of sugar = 12 kg**
Find the total amount of solution.	**Total mass = 40 kg + 80 kg = 120 kg**
Check the percentage.	**Concentration of sugar** $= \frac{12}{120} \times 100\% = 10\%$ **OK✓**
Write the answer.	**80 kg of water must be added to make it 10%.**

An equation has an equals sign that shows both sides are the same size. When you compare two expressions, one may be smaller, equal to or bigger than the other. When they are equal you get an equation. When they are not necessarily equal you get an **inequality**.

Important!

Linear inequalities

A **linear inequality** is like a linear equation, but instead of having an equals sign, it has one of the signs $>$, $<$, $\geq$ or $\leq$. It is also called an **inequation**.

Inequations can be simplified in the same way as you solve equations. As long as you do the same thing to both sides, you can do anything you like to an equation. You can add and subtract anything you like from both sides of an inequality. However, when you multiply or divide by a negative number you have to change the inequality.

Important!

Properties of inequalities

For any inequality $x > y$,

1 $x + a > y + a$ for any number a

2 $x - a > y - a$ for any number a

3 $ax > ay$ for any *positive* number a

4 $ax < ay$ for any *negative* number a

5 $\frac{x}{a} > \frac{y}{a}$ for any *positive* number a

6 $\frac{x}{a} < \frac{y}{a}$ for any *negative* number a.

The same laws hold for $\geq$, $<$ and $\leq$.

It is normal to show the solutions to an inequality as a graph on the number line.

Example 8

For each of the following, solve the inequality and show your solution on a number line.

a $m + 3 \leq 8$ **b** $4x > {}^-7$ **c** ${}^-5n \geq 12$

d $v - 5 < 7$ **e** $\frac{p}{{}^-3} < {}^-11$

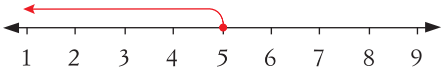
Solution

a Write the inequality. $\boldsymbol{m + 3 \leq 8}$

This is like $m + 3 = 8$, so take 3 from both sides. $\boldsymbol{m + 3 - 3 \leq 8 - 3}$

Simplify both sides. $\boldsymbol{m \leq 5}$

Put your answer on a number line. Remember to represent $\leq$ with a closed circle. Show the set going down from 5.

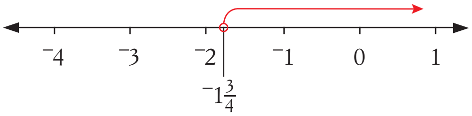


b Write the inequality. $\boldsymbol{4x > {}^-7}$

This is like $4x = {}^-7$, so divide both sides by 4. $\boldsymbol{\frac{4x}{4} > \frac{{}^-7}{4}}$

Simplify both sides. $\boldsymbol{x > {}^-1\frac{3}{4}}$

Put your answer on a number line. Remember to represent $>$ with an open circle. Show the set going up from ${}^-1\frac{3}{4}$.



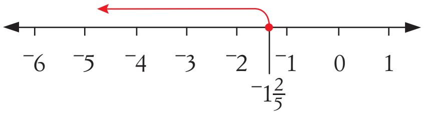
c Write the inequality. $\boldsymbol{{}^-5n \geq 12}$

This is like ${}^-5n = 12$, so divide both sides by ${}^-5$. $\boldsymbol{\frac{{}^-5n}{{}^-5} \leq \frac{12}{{}^-5}}$

Reverse the sign for division by a negative.

Simplify both sides. $\boldsymbol{n \leq {}^-1\frac{2}{5}}$

Put your answer on a number line. Remember to represent $\leq$ with a closed circle. Show the set going down from ${}^-1\frac{2}{5}$.

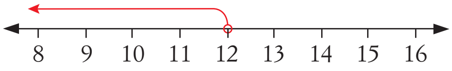


d Write the inequality. $\boldsymbol{v - 5 < 7}$

This is like $v - 5 = 7$, so add 5 to both sides. $\boldsymbol{v - 5 + 5 < 7 + 5}$

Simplify both sides. $\boldsymbol{v < 12}$

Put your answer on a number line. Remember to represent $<$ with an open circle. Show the set going down from 12.



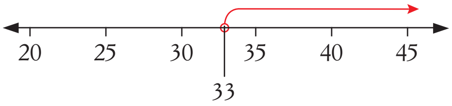
e Write the inequality. $\frac{p}{-3} < {}^{-}11$

This is like $\frac{p}{-3} = {}^{-}11$, so multiply by ${}^{-}3$. ${}^{-}3 \times \frac{p}{-3} > {}^{-}3 \times {}^{-}11$

Reverse the sign for multiplication by a negative.

Simplify both sides. $p > 33$

Put your answer on a number line. Remember to represent > with an open circle. Show the set going up from 33.



Inequalities with two operations are solved in the same way as linear equations.

Example 9

CAS TI-Nspire exercise
Equations
MAT10NATI00018

CAS ClassPad exercise
Equations
MAT10NACP00018

For each of the following, solve the inequality and show your solution on a number line.

a $7 - 3g \le 15$ **b** $4w - 9 < 2$

Solution

a Write the inequality. $7 - 3g \le 15$

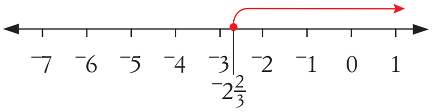
Subtract the 7 from both sides. $7 - 3g - 7 \le 15 - 7$

Simplify both sides. ${}^{-}3g \le 8$

Divide both sides by ${}^{-}3$ and *reverse the sign* for division by a negative. $\frac{{}^{-}3g}{-3} \ge \frac{8}{-3}$

Simplify to get the answer. $g \ge {}^{-}2\frac{2}{3}$

Show the answer on a number line.



b Write the problem. $4w - 9 < 2$

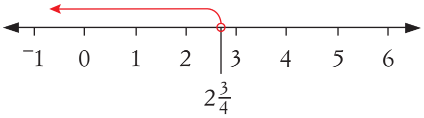
Add 9 to both sides. $4w - 9 + 9 < 2 + 9$

Simplify both sides. $4w < 11$

Divide both sides by 4. $\frac{4w}{4} < \frac{11}{4}$

Simplify to get the answer. $w < 2\frac{3}{4}$

Show the answer on a number line.



Linear inequalities with variables on both sides of the inequality, brackets or fractions can also be solved like linear equations.

Example 10

For each of the following, solve the inequality and show your solution on a number line.

a $4p + 5 > 7p - 8$

b $6(3r - 7) - 8 \leq 7(3r - 4) - 46$

c $\frac{m-7}{12} - \frac{m-1}{4} \geq 1$

Solution

a Write the inequality. $\mathbf{4p + 5 > 7p - 8}$

Move the variables right and the constants left. $\mathbf{4p + 5 - 4p + 8 > 7p - 8 - 4p + 8}$

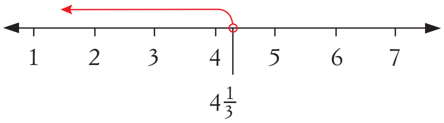
Simplify both sides. $\mathbf{13 > 3p}$

Reverse the inequality, including the > sign. $\mathbf{3p < 13}$

Divide by 3. $\mathbf{\frac{3p}{3} < \frac{13}{3}}$

Simplify to get the answer. $\mathbf{p < 4\frac{1}{3}}$

Show the answer on a number line.



b Write the inequality. $\mathbf{6(3r - 7) - 8 \leq 7(3r - 4) - 46}$

Expand the brackets. $\mathbf{18r - 42 - 8 \leq 21r - 28 - 46}$

Simplify both sides. $\mathbf{18r - 50 \leq 21r - 74}$

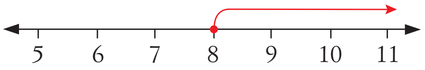
Move the variables right and the constants left. $\mathbf{18r - 50 - 18r + 74 \leq 21r - 74 - 18r + 74}$

Simplify both sides. $\mathbf{24 \leq 3r}$

Reverse the inequality. $\mathbf{3r \geq 24}$

Divide by 3. $\mathbf{r \geq 8}$

Show the answer on a number line.



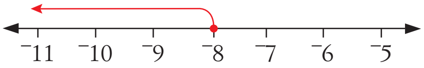
c Write the inequality with brackets around the grouped numerators. $\mathbf{\frac{(m-7)}{12} - \frac{(m-1)}{4} \geq 1}$

Multiply each term by the lowest common denominator, 12. $\mathbf{12 \times \frac{(m-7)}{12} - 12 \times \frac{(m-1)}{4} \geq 12 \times 1}$

Simplify by cancelling the denominators. $\mathbf{1(m - 7) - 3(m - 1) \geq 12}$

Multiply out the brackets. $\mathbf{m - 7 - 3m + 3 \geq 12}$

Simplify the LHS.	$^{-}2m - 4 \geq 12$
Add 4 to both sides.	$^{-}2m \geq 16$
Divide by $^{-}2$ and reverse the $\geq$ sign.	$\frac{^{-}2m}{^{-}2} \leq \frac{16}{^{-}2}$
Simplify to get the answer.	$m \leq {}^{-}8$
Show the answer on a number line.	



You can use simple inequalities to solve some types of problems.

Example 11

Anton wants to go in-line skating while on holiday. He has \$43 available and it costs \$7 an hour to hire the skates, pads and helmet. Hire is available in multiples of a quarter of an hour. He also has to leave a \$20 deposit. Write and solve an inequality to find the maximum time that he can spend skating.

Solution

Choose a variable for the time.	**Let the time skating** $= t$ **hours**
Write an expression for the money.	**Money needed** $= 7t + 20$
Write an inequality for the money.	$7t + 20 \leq 43$
Solve the inequality.	$7t + 20 - 20 \leq 43 - 20$
Simplify both sides.	$7t \leq 23$
Divide by 7.	$\frac{7t}{7} \leq \frac{23}{7}$
Simplify to find the answer.	$t \leq 3\frac{2}{7}$
Change $\frac{2}{7}$ hours to minutes.	$\frac{2}{7}\text{ h} = \frac{2}{7} \times 60 \approx 17$ **minutes**
State the maximum.	$t = 3$ **hours 15 minutes**
Check the answer.	**Money needed** $= \$7 \times 3\frac{1}{4} + \$20$ $= \$42.75$ ✓OK
Write the answer.	**Anton can hire skates for up to 3 h 15 minutes.**

You can prove properties of inequalities using the rules for addition, subtraction, multiplication and division of positive and negative numbers.
The symbol $\Rightarrow$ is used in maths before a step that logically follows from the last one. It means 'it follows that' or 'it implies'. A double headed one, like $\Longleftrightarrow$, means that the logic goes both ways so is taken to mean 'if and only if'. For powers and logarithms, we could write $\log_b a = x \iff a = b^x$, $\log_b a = x \Rightarrow a = b^x$ or $a = b^x \Rightarrow \log_b a = x$.

Example 12

If $x \le y$ and $a < 0$, prove that $\frac{x}{a} \ge \frac{y}{a}$.

Solution

Write the requirement.	**RTP:** $\boldsymbol{x \le y,\ a < 0 \Rightarrow \frac{x}{a} \le \frac{y}{a}}$
Put the Proof heading.	**Proof**
Write the inequality.	$\boldsymbol{x \le y}$
Write in terms of a number property.	$\boldsymbol{\Rightarrow x - y}$ **is negative or zero.**
Use the fact that the quotient of two negative numbers is positive.	$\boldsymbol{\Rightarrow \frac{x-y}{a}}$ **is positive or zero.**
Use fraction properties.	$\boldsymbol{\Rightarrow \frac{x}{a} - \frac{y}{a} \ge 0}$
Write as an inequality.	$\boldsymbol{\Rightarrow \frac{x}{a} \ge \frac{y}{a}}$ **QED**

Exercise 11.1 Linear equations and inequations

Understanding

Extra questions

Exercise 11.1

MAT10NAEQ00031

1 Solve each of the following equations.

a $a + 7.6 = {}^{-}12.5$
b $b - 4.3 = {}^{-}5.8$
c $5m = {}^{-}100$
d $7 + h = {}^{-}13$
e $\frac{k}{8} = {}^{-}7$
f $3.7 + y = {}^{-}9.8$
g $\frac{x}{10} = {}^{-}12$
h $h - 17 = {}^{-}12$
i $0.8n = {}^{-}4$
j $x + 7 = 4$
k $m - 5 = {}^{-}7$
l $12n = {}^{-}30$
m $\frac{n}{14} = {}^{-}2.5$
n $14d = {}^{-}35$
o $18.4 + a = 6.9$
p $6a - 6 = {}^{-}12$
q $\frac{e}{5} + 15 = 5$
r $2.2 - 3r = 5.8$
s $2m + 8.4 = 4.6$
t $1.4 - \frac{4.2d}{5} = 9.8$

2 For each of the following, solve the inequality and show your solution on a number line.

See Example 8

a $r + 9 \ge 6$
b $r + 3 > 0$
c $b + 1 \le {}^{-}8$
d $v + 1 > 4$
e $g + 8 \ge {}^{-}5$
f $h - 6 \ge 1$
g $w - 2 \le {}^{-}9$
h $m - 8 < {}^{-}4$
i $x - 7 \le 0$
j $e - 1 < 0$

3 For each of the following, solve the inequality and show your solution on a number line.

a ${}^{-}6a \le {}^{-}24$
b $7m \ge 28$
c $6w < 36$
d ${}^{-}5m \ge {}^{-}35$
e $3q > 24$
f $\frac{e}{-3} > {}^{-}3$
g $\frac{r}{9} \ge 5$
h $\frac{h}{8} > 1$
i $\frac{v}{9} \le {}^{-}7$
j $\frac{e}{-2} < 1$

4 For each of the following, solve the inequality and show your solution on a number line.

a $7b \le {}^{-}8$
b ${}^{-}5x > 2$
c ${}^{-}8e \le {}^{-}2$
d $8q > 0$
e ${}^{-}7z > {}^{-}6$
f $\frac{h}{{}^{-}3} \ge {}^{-}1$
g $\frac{e}{{}^{-}5} < {}^{-}4$
h $\frac{g}{{}^{-}5} \ge {}^{-}5$
i $\frac{z}{{}^{-}9} > 5$
j $\frac{t}{{}^{-}5} > {}^{-}7$

Fluency

5 Solve each of the following equations.

See Example 1

a $2a + 9 = 3$
b $4b + 16 = 4$
c $7m + 26 = {}^{-}9$
d $5 + 4n = {}^{-}23$
e $9k + 5 = {}^{-}40$
f $\frac{m}{3} + 5 = 2$
g $\frac{r}{2} + 11 = 5$
h $3 = 2h + 17$
i $\frac{w}{5} + 14 = 6$
j $\frac{3c}{2} + 22 = 10$
k $3b + 24 = 15$
l $\frac{4d}{3} + 13 = 5$

See Example 2

6 Find the roots of each of the following equations.

a $15a = 18a + 27$
b $12m = 5m - 35$
c $2a + 3 = 5a + 9$
d $5 - m = 19 + 6m$
e $4x + 9 = 2x - 21$
f $2g + 5 = 13 + 6g$
g $3k + 2 = 26 + 7k$
h $3w + 3 = 9w + 27$
i $10 - 3x = -x + 9$
j $4a - 3 = 2a - 9$
k $8m - 7 = 2m - 17$
l $15x - 9 = 12x - 15$
m $7r + 12 = {}^{-}18 - 3r$
n $11g - 2 = 10g - 12$
o $8 + 4w = 12 + 6w$
p $8m + 9 = 4m - 11$
q $5w - 3 = \frac{1}{2}w - 15$
r $4x - 6 = 1\frac{1}{2}x - 16$
s $6d - 25 = 8d + 7$
t ${}^{-}9 - 2b = 3b + 11$
u $14 + 2n = 2 - n$

See Example 3

7 Solve each of the following equations.

a $2(a + 4) = {}^{-}18$
b $3(3 - b) = 21$
c $6(n + 3) = {}^{-}24$
d $9(-m + 2) = 27$
e $8(y + 2) = {}^{-}32$
f ${}^{-}9(x + 4) = 54$
g $2({}^{-}3d + 1) = {}^{-}28$
h $3(3 - 4a) = 33$
i $5(3 - 2w) = 45$
j $8(6 - 3k) = 56$
k $12(3 - 2x) = 60$
l $5(3w - 6) = 8w - 2$
m $3(4x - 5) = 2x - 25$
n $5(2a + 1) = 3a - 23$
o $3(4k + 5) = 4(7 + 2k)$
p $3(b - 2) = 2(b + 3)$
q $2(5y + 2) = 4(2y + 5)$
r $5(2m + 7) = 3(2m + 3)$
s $2(n + 4) = 3(n - 2)$
t $6(2x + 1) = 5(2x + 3)$
u $7(a - 3) = 3(2a - 7)$

8 Solve each of the following equations.

a $33 = 7(1 + 2a) - a$
b $21 = 28 + 7(2 - b)$
c $5m + 2(10 - 2m) = 15$
d $9k + 6(5 - 2k) = 0$
e ${}^{-}16 = y - 3(y + 2)$
f ${}^{-}5 = 11x - 5(x + 4)$
g ${}^{-}6 = 3d + 5(d + 2)$
h $0 = 15 + 2w - 2(3 - w)$
i $5(n - 3) - 3(n + 4) = {}^{-}1$
j $4(1 - 2x) -3(2 - 3x) = 7 - 2x$

9 Solve each of the following equations. *See* Example 4

a $\frac{x}{2}+\frac{x}{4}=12$ **b** $\frac{a}{3}+\frac{a}{6}=7$ **c** $\frac{b}{4}+\frac{b}{3}=7$

d $\frac{2d}{5}+\frac{d}{2}=4\frac{1}{2}$ **e** $\frac{a}{2}-4=\frac{a}{6}$ **f** $\frac{x}{4}+2=\frac{x}{5}+1$

g $b+\frac{b}{7}=\frac{b}{2}+9$ **h** $\frac{m}{3}-4=\frac{3m}{5}$ **i** $\frac{3n}{5}-\frac{2n}{3}=n+16$

j $\frac{k}{5}+3=k-1$ **k** $\frac{2d}{5}+3=\frac{d}{2}$ **l** $\frac{2x}{3}+\frac{3}{4}=\frac{x}{3}$

10 Find the roots of these equations.

a $\frac{a+6}{6}=2$ **b** $\frac{b-7}{4}=3$ **c** $\frac{2h-3}{5}={}^{-}1$

d $\frac{k+3}{8}-\frac{k}{6}=0$ **e** $\frac{2n+3}{5}-\frac{n+2}{4}={}^{-}1$ **f** $\frac{b+3}{3}-\frac{b}{4}=1\frac{3}{4}$

11 For each of the following, solve the inequality and show your solution on a number line. *See* Example 9

a $5n-3>9$ **b** $2p-6>{}^{-}1$ **c** $3c+7>0$ **d** ${}^{-}6e-5\leq 0$

e $8k-9<4$ **f** $9y-9<{}^{-}7$ **g** ${}^{-}9c-6<9$ **h** $3k-1\leq {}^{-}4$

i ${}^{-}6z+2>5$ **j** ${}^{-}3u+8>0$

12 For each of the following, solve the inequality and show your solution on a number line. *See* Example 10

a ${}^{-}4q+1\leq 13-6q$ **b** $12b+8<9b+29$

c ${}^{-}4u-1>{}^{-}33-8u$ **d** $2e+8\geq 1-5e$

e $8(2p-6)-6>6(3p-10)-4$ **f** $6(2k-10)-9<4(2k-7)-29$

g $7(2b-4)-7\leq 9(3b-3)+57$ **h** $4(3w-9)-7\geq 5(w-4)-72$

i $\frac{8c+7}{3}-\frac{2c-9}{2}>{}^{-}1\frac{5}{6}$ **j** $\frac{a+4}{12}+\frac{2a-5}{18}\leq\frac{1}{4}$

k $\frac{g-3}{6}-\frac{3g+4}{7}\geq {}^{-}5$ **l** $\frac{g+8}{9}-\frac{6g+7}{5}>{}^{-}10$

13 Find the roots of these equations. *See* Example 5

a $\frac{1}{a}+\frac{3}{a}=4$ **b** $\frac{2}{3b}-\frac{2}{5b}=\frac{1}{10}$ **c** $\frac{6}{p-5}={}^{-}2$

d $\frac{12}{b+1}=3$ **e** $\frac{7}{2m-4}=1$ **f** $\frac{3}{5-2c}={}^{-}1$

g $\frac{8}{3g-5}={}^{-}4$ **h** $\frac{8}{5n-2}={}^{-}9$ **i** $\frac{-x}{3}-\frac{x+4}{6}={}^{-}10$

Problem solving

14 The length of a room is 4 m more than 3 times its width. If the perimeter is 28 m, how wide is the room? *See* Example 6

See Example 7

15 Trent jogged for 1 hour and walked for 2 hours. If he jogged three times as fast as he walked, and he travelled 20 km, how fast did he walk?

Worked solutions
Exercise 11.1
MAT10NAWS00031

⑯ Gail has \$8 worth of 5-cent and 10-cent coins. If the number of 10-cent coins is 20 less than twice the number of 5-cent coins, how many coins are there altogether?

17 A square garden has a 1 m wide walkway put around it. As a result, the area of the plot is decreased by 96 m^2. What were the dimensions of the original garden plot?

18 When three consecutive odd numbers are added and their total is doubled, the result is 186. What are the numbers?

Worked solutions
Exercise 11.1
MAT10NAWS00031

⑲ The temperature in the USA is always given in degrees Fahrenheit, while we always use Celsius. The formula for conversion of Fahrenheit to Celsius is $c = \frac{5}{9}(f - 32)$.

a Use the formula to write an equation to find the temperature in Fahrenheit that would have been reported in the US when Melbourne had a record temperature in 2009 of 46.4°.

b Find the temperature x that is the same on both scales. Find whether or not this temperature is possible in the United States and Australia.

20 1 kg of a 40% by mass solution of sodium hydroxide (NaOH) is to be diluted to 1 molar NaOH solution. This has 40 g of NaOH per litre of water. How much water must be added, and how much solution will be made?

See Example 11

21 Waldo is buying kiwifruit vines for his back fence. He has to plant at least one male plant for every 4 female plants to make sure he gets good fertilisation. The male plants only produce about a third of the fruit of the female plants, but they are harder to grow, so they cost \$12 each while the female plants only cost \$8 each. Write an inequality and find the maximum number of plants that Waldo can get for \$200, if he gets the minimum number of male plants.

22 Janet has \$29.50. Write an inequality and solve it to find the largest quantity of walnuts she can buy at \$8.40/kg.

23 A special applicator for in-situ wall cavity insulation production costs \$32, and the tubes of "compressed" insulation foam cost \$7 each. You only need one applicator for a job, but it has to be disposed of afterwards. Each tube of "compressed" insulation foam will do 3 square metres of wall cavity. What area of wall can be done for \$300?

24 A big tile/paver cutter costs \$120 a day to hire, and the blades for the cutter cost \$45 each. A landscaper hires the cutter and estimates that he needs to cut 450 pavers for a big job at a garden centre. Each blade will do a maximum of about 70 pavers. The landscaper reckons that if they leave all the cutting until the rest of the pavers are done, they will be able to cut the ones that need doing in one day. Write an inequality and find the minimum cost of the cutter and blades.

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Reasoning

See Example 12

25 If $x \geq y$ and $a < 0$, prove that $\frac{x}{a} \leq \frac{y}{a}$.

26 If $x < y$ and $a > 0$, prove that $\frac{x}{a} < \frac{y}{a}$.

27 If $x > 0$, prove that $x + \frac{a^2}{x} \geq 2a$ for any number a. Hint: Consider $(x - a)^2$.

28 If $x < 0$, prove that $x + \frac{a^2}{x} \leq 2a$ for any number a.

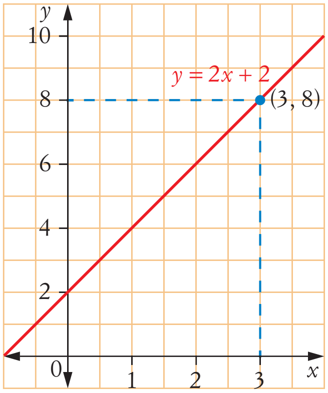
Worked solutions

Exercise 11.1

MAT10NAWS00031

11.2 Simultaneous equations

Look at the graph of the function $y = 2x + 2$.
Any ordered pair that lies on the graph satisfies the function.
There are an infinite number of solutions to this equation.
Here, the point (3, 8) lies on the graph. So (3, 8) must satisfy the equation $y = 2x + 2$.
Let's check by substituting.
When $x = 3$:
$y = 2 \times 3 + 2 = 6 + 2 = 8$
So (3, 8) does satisfy the equation $y = 2x + 2$.

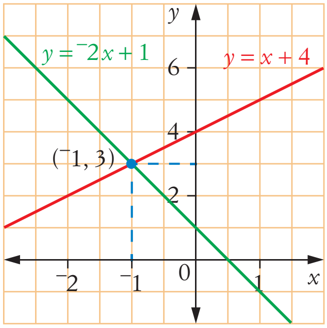


Now look at the two linear functions that are graphed below.
The graphs cross at the point $(^{-}1, 3)$. This means that $(^{-}1, 3)$ should satisfy both functions because it lies on the graph of each.
Let's check:
$(^{-}1, 3)$ satisfies $y = x + 4$ because, when $x = {}^{-}1$:

$$y = {}^{-}1 + 4 = 3$$

$(^{-}1, 3)$ also satisfies $y = {}^{-}2x + 1$ because, when $x = {}^{-}1$:

$$y = {}^{-}2 \times {}^{-}1 + 1 = 2 + 1 = 3$$



The solution $x = {}^{-}1$ and $y = 3$ to the equations $y = x + 4$ and $y = {}^{-}2x + 1$ satisfies both equations at the same time.
We can regard $y = 2x + 2$ as a function where y is the independent variable in order to sketch its graph. At the same time, we can think of it as an equation with two variables.
Similarly, $y = x + 4$ is both a function and an equation with two variables.
The point $(^{-}1, 3)$ is both the intersection of the lines and a solution of both equations, where $x = {}^{-}1$ and $y = 3$.

Important!

Simultaneous equations

Equations considered *at the same time* make an equation system called **simultaneous equations**.

You call a solution of these equations that satisfies all the equations at the same time a **simultaneous solution**. The simultaneous solution of two equations in two unknowns is the intersection of their lines on a graph.

CAS TI-Nspire exercise
Equations
MAT10NATI00018

CAS ClassPad exercise
Equations
MAT10NACP00018

Technology: GeoGebra
Simultaneous equations
MAT10NATC00011

Worksheet
Solving simultaneous equations by graphing
MAT10NAWK00028

Example 13

Find the solution of the following simultaneous equations by graphing.

$$2x - 3y = 3$$
$$x + 2y = 16$$

Solution

Write the equations and number them for convenience.

$$3y - 2x = 3 \qquad [1]$$
$$x + 2y = 16 \qquad [2]$$

Choose some convenient points to sketch the line for [1].

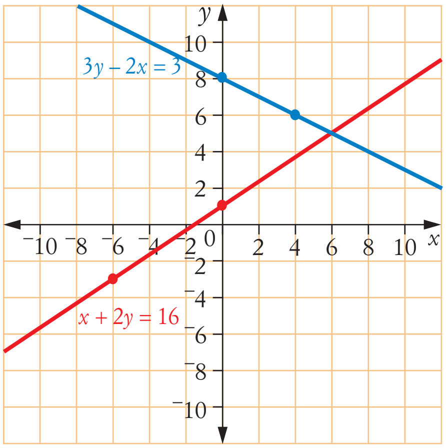
For $x = 0$, $y = 1$, so $(0, 1)$ is on the line.
For $x = {}^{-}6$, $y = {}^{-}3$, so $({}^{-}6, {}^{-}3)$ is on the line.

Choose some convenient points to sketch the line for [2].

For $x = 0$, $y = 0$, so $(0, 8)$ is on the line.
For $x = 4$, $y = 6$, so $(4, 6)$ is on the line.

Use the points to show both lines on graph paper.

The lines intersect at the point (6, 5).



Write the answer.

The simultaneous solution is $x = 6$ and $y = 5$.

You can also use graphs to find the solutions of nonlinear simultaneous equations. Of course, there may not be a solution if the lines do not intersect.

Example 14

Find the simultaneous solution to the equations $y = 3x$ and $y = x^2$.

Solution

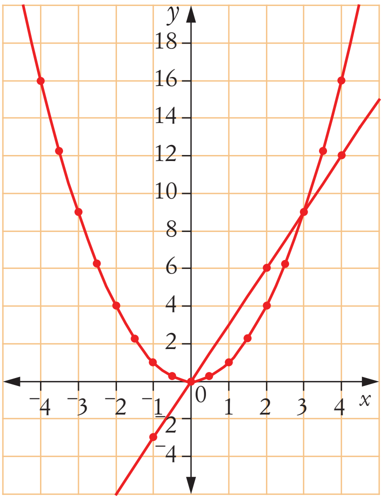
Create a table of values for $y = 3x$.

x	$^{-}1$	2	4
y	$^{-}3$	6	12

Create a table of values for $y = x^2$.

x	$^{-}4$	$^{-}3$	$^{-}2$	$^{-}1$	0	1	2	3	4
y	16	9	4	1	0	1	4	9	16

Draw both graphs on the same axes.



State the intersections. **The graphs intersect at (0, 0) and (3, 9).**

State the solution. **The simultaneous solutions are $x = y = 0$ or $x = 3$ and $y = 9$.**

It may take you a long time to use graphs to solve simultaneous equations. You may also find it difficult to find the exact point of intersection, so it is not always particularly accurate. It is often quicker and more accurate to use an algebraic method.

Important!

Substitution method

When you use the **substitution method** you use one equation to express one variable in terms of the other. Then you substitute this into the other equation to make a linear equation in one variable.

Solve this and use the expression to find the value of the substituted variable.

Check your solution in both *original* equations.

Technology worksheet

Excel worksheet: Solving simultaneous equations

MAT10NACT00017

Puzzle sheet

Simultaneous equations by substitution

MAT10NAPS00039

Example 15

Solve each of these pairs of simultaneous equations by substitution.

a $2b - a = 3$
$a + 3b = 7$

b $4x + 3y = 8$
$3x + 2y = 7$

Solution

a Write the equations and number them for convenience.

$$2b - a = 3 \qquad [1]$$
$$a + 3b = 7 \qquad [2]$$

It is easy to rearrange [1] to make a the subject.

$$a = 2b - 3 \qquad [3]$$

Use brackets and substitute [3] into [2].

$$(2b - 3) + 3b = 7$$

Get rid of the brackets.

$$2b - 3 + 3b = 7$$

Simplify and move the 3 to the RHS.

$$5b - 3 + 3 = 7 + 3$$

Simplify.

$$5b = 10$$

Divide by 5 to get b.

$$b = 2$$

Substitute b into [3] find a.

$$a = 2b - 3 = 2 \times 2 - 3 = 1$$

Check the values in [1].

$$2b - a = 2 \times 2 - 1 = 3 \ \checkmark\text{OK}$$

Substitute in [2].

$$a + 3b = 1 + 3 \times 2 = 7 \ \checkmark\text{OK}$$

Write the answer.

$$a = 1 \text{ and } b = 2$$

b Write the equations and number them for convenience.

$$4x + 3y = 8 \qquad [1]$$
$$3x + 2y = 7 \qquad [2]$$

Rearrange [2] to isolate y, the variable with the smallest coefficient.

$$y = \frac{7 - 3x}{2} \qquad [3]$$

Use brackets and substitute [3] into [1].

$$4x + 3 \times \frac{(7 - 3x)}{2} = 8$$

Get rid of the denominator by multiplying by 2.

$$2 \times 4x + 2 \times 3 \times \frac{(7 - 3x)}{2} = 2 \times 8$$

Simplify both sides.

$$8x + 3(7 - 3x) = 16$$

Multiply out the brackets.

$$8x + 21 - 9x = 16$$

Simplify again.

$$21 - x = 16$$

Move the x to the right and the 16 to the left.

$$21 - x + x - 16 = 16 + x - 16$$

Simplify and reverse the equation.

$$x = 5$$

Substitute in [3]

$$y = \frac{7 - 3 \times 5}{2} = {}^{-}4$$

Check the values in [1].

$$4x + 3y = 4 \times 5 + 3 \times {}^{-}4 = 8 \quad \checkmark\text{OK}$$

Check the values in [2].

$$3x + 2y = 3 \times 5 + 2 \times {}^{-}4 = 7 \quad \checkmark\text{OK}$$

Write the answer.

$$x = 5 \text{ and } y = {}^{-}4$$

Obviously, a pair of equations where one of the variables has a coefficient of 1 or $^{-}1$ is much easier to solve by substitution than a pair where no variable has a coefficient of 1. If the answer to the first variable is a fraction, then it is very easy to make a mistake.

Important!

Elimination method

In the **elimination method** you make the coefficients of one variable opposites by multiplying the equations by suitable constants. The coefficient in one equation is the same size but the opposite sign to the other equation. Then add the equations to make an equation with only one variable. Solve this and find the other value.
Check your solution in both *original* equations.

Example 16

Solve these simultaneous equations by elimination.

a $2a - b = 3$, $a + b = 3$

b $3x + y = 3$, $x + y = {}^{-}1$

c $3x + 2y = {}^{-}1$, $5x + 3y = {}^{-}3$

d $9p + 4q = 3$, $15p + 28q = {}^{-}11$

Solution

a Write the equations and number them for convenience.

$$\begin{aligned} 2a - b &= 3 \quad [1] \\ a + b &= 3 \quad [2] \end{aligned}$$

Since the coefficients of b are already opposites, just add the equations.

$$\begin{aligned} 2a - b &= 3 \quad [1] \\ a + b &= 3 \quad [2] \\ \hline 3a + 0 &= 6 \quad [3] \end{aligned}$$

Now solve equation [3]. $\frac{3a}{3} = \frac{6}{3}$

Simplify. $a = 2$

Substitute $a = 2$ into [2]. $2 + b = 3$

Solve for b. $b = 1$

Check the values in [1]. $2a - b = 2 \times 2 - 1 = 3$ ✓OK

Check the values in [2]. $a + b = 2 + 1 = 3$ ✓OK

Write the answer. $a = 2$ and $b = 1$

Video tutorial
Simultaneous equations
MAT10NAVT10027

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b Write the equations and number them for convenience.

$$\begin{aligned} 3x + y &= 3 && [1] \\ x + y &= {}^{-}1 && [2] \end{aligned}$$

Multiply equation [2] by ${}^{-}1$ to make the coefficients of y opposites and add the equations.

$$\begin{aligned} 3x + y &= 3 && [1] \\ \underline{-x - y} &= 1 && [2] \times {}^{-}1 \\ 2x \quad &= 4 && [3] \end{aligned}$$

Now solve equation [3].

$$x = 2$$

Substitute $x = 2$ into [2].

$$2 + y = {}^{-}1$$

Solve for y.

$$y = {}^{-}3$$

Check the values in [1].

$$3x + y = 3 \times 2 + {}^{-}3 = 3 \ \checkmark\text{OK}$$

Check the values in [2].

$$x + y = 2 + {}^{-}3 = {}^{-}1 \quad \checkmark\text{OK}$$

Write the answer.

$$x = 2 \text{ and } y = {}^{-}3$$

c Write the equations and number them for convenience.

$$\begin{aligned} 3x + 2y &= {}^{-}1 && [1] \\ 5x + 3y &= {}^{-}3 && [2] \end{aligned}$$

Multiply equation [1] by 3 and equation [2] by ${}^{-}2$ to make the coefficients of y opposites and add the equations.

$$\begin{aligned} 9x + 6y &= {}^{-}3 && [1] \times 3 \\ \underline{{}^{-}10x - 6y} &= 6 && [2] \times {}^{-}2 \\ -x \quad &= 3 && [3] \end{aligned}$$

Now solve equation [3].

$$x = {}^{-}3$$

Substitute $x = {}^{-}3$ into [1].

$$3 \times {}^{-}3 + 2y = {}^{-}1$$

Simplify.

$${}^{-}9 + 2y = {}^{-}1$$

Add 9 to both sides.

$${}^{-}9 + 2y + 9 = {}^{-}1 + 9$$

Simplify.

$$2y = 8$$

Solve for y.

$$y = 4$$

Check the values in [1].

$$\begin{aligned} 3x + 2y &= 3 \times {}^{-}3 + 2 \times 4 \\ &= {}^{-}9 + 8 \\ &= {}^{-}1 \quad \checkmark\text{OK} \end{aligned}$$

Check the values in [2].

$$\begin{aligned} 5x + 3y &= 5 \times {}^{-}3 + 3 \times 4 \\ &= {}^{-}15 + 12 \\ &= {}^{-}3 \quad \checkmark\text{OK} \end{aligned}$$

Write the answer.

$$x = {}^{-}3 \text{ and } y = 4$$

d Write the equations and number them for convenience.

$$\begin{aligned} 9p + 4q &= 3 && [1] \\ 15p + 28q &= {}^{-}11 && [2] \end{aligned}$$

Multiply equation [1] by 5 and equation [2] by ${}^{-}3$ to make the coefficients of p opposites and add the equations.

$$\begin{aligned} 45p + 20q &= 15 && [1] \times 5 \\ \underline{-45p - 84q} &= 33 && [2] \times {}^{-}3 \\ {}^{-}64q &= 48 && [3] \end{aligned}$$

Now solve equation [3].

$$q = \frac{48}{{}^{-}64} = {}^{-}\tfrac{3}{4}$$

Substitute $q = {}^{-}\frac{3}{4}$ into [1].

$$9p + 4 \times {}^{-}\tfrac{3}{4} = 3$$

Simplify. $\mathbf{9p - 3 = 3}$

Add 3 to both sides. $\mathbf{9p = 6}$

Solve for p. $\mathbf{p = \frac{6}{9} = \frac{2}{3}}$

Check the values in [1].

$$\mathbf{9p + 4q}$$
$$\mathbf{= 9 \times \frac{2}{3} + 4 \times \frac{^{-}3}{4}}$$
$$\mathbf{= 6 - 3 = 3} \quad ✓\text{OK}$$

Check the values in [2].

$$\mathbf{15p + 28q}$$
$$\mathbf{= 15 \times \frac{2}{3} + 28 \times \frac{^{-}3}{4}}$$
$$\mathbf{= 10 - 21 = {}^{-}11} \quad ✓\text{OK}$$

Write the answer. $\mathbf{p = \frac{2}{3} \text{ and } q = \frac{^{-}3}{4}}$

As with ordinary linear equations, you should get rid of fractions before solving simultaneous equations.

Example 17

Puzzle sheet

Simultaneous equations

MAT10NAPS00041

Solve the simultaneous equations $\frac{a-1}{2} - \frac{b}{3} = 1$ and $a - \frac{b+2}{5} = 4$.

Solution

Write the equations and number them for convenience.

$$\mathbf{\frac{a-1}{2} - \frac{b}{3} = 1} \qquad [1]$$
$$\mathbf{a - \frac{b+2}{5} = 4} \qquad [2]$$

Multiply [1] by the LCD, 6. $\mathbf{6 \times \frac{(a-1)}{2} - 6 \times \frac{b}{3} = 6 \times 1}$

Cancel the denominators. $\mathbf{3(a-1) - 2 \times b = 6}$

Multiply out the brackets. $\mathbf{3a - 3 - 2b = 6}$

Simplify. $\mathbf{3a - 2b = 9}$ [3]

Multiply [2] by 5. $\mathbf{5 \times a - 5 \times \frac{(b+2)}{5} = 5 \times 4}$

Cancel the denominator. $\mathbf{5a - 1(b+2) = 20}$

Multiply out the brackets. $\mathbf{5a - b - 2 = 20}$

Simplify. $\mathbf{5a - b = 22}$ [4]

Multiply [4] by $^{-}2$ to make the coefficients of b opposites and add the equations.

$$\mathbf{3a - 2b = 9} \qquad [3]$$
$$\underline{\mathbf{-10a + 2b = -44}} \qquad [4] \times {}^{-}2$$
$$\mathbf{-7a \quad = {}^{-}35} \qquad [5]$$

Solve equation [5]. $\mathbf{a = 5}$

Substitute $a = 5$ into [4]. $\mathbf{5 \times 5 - b = 22}$

Simplify.	$25 - b = 22$
Solve for *b*.	$b = 3$
Check the values in the *original* [1].	$\frac{a-1}{2} - \frac{b}{3} = \frac{5-1}{2} - \frac{3}{3} = 2 + 1 = 3$ ✓OK
Check the values in the *original* [2].	$a - \frac{b+2}{5} = 5 - \frac{3+2}{5} = 5 - 1 = 4$ ✓OK
Write the answer.	**The solution is $a = 5$ and $b = 3$.**

When you are solving problems involving simultaneous equations you follow the same basic steps as for all problem solving: create a mathematical model, solve the maths and check the solution.

Important!

Solving problems with simultaneous equations

1. Read the information carefully
2. Choose variable names for the unknown quantities
3. Write equations
4. Solve the equations using the most convenient method
5. Check the solution using the original information
6. State the solution in the same terms as the problem

Example 18

Animated example

Simultaneous equations

MAT10NAAE00011

A school play was attended by a total of 400 people. Adults were charged \$5 and children \$2. Ticket receipts were \$1100. How many adults attended?

Solution

Choose suitable variable names.	**Let the number of adults be *a* and the number of children be *c*.**
Write an equation for the attendance.	$a + c = 400$ [1]
Write an equation for the receipts.	$5a + 2c = 1100$ [2]
Rearrange [1] to make *c* the subject.	$c = 400 - a$ [3]
Use brackets to substitute [3] into [2].	$5a + 2(400 - a) = 1100$
Multiply out the brackets.	$5a + 800 - 2a = 1100$
Simplify and subtract 800.	$3a + 800 - 800 = 1100 - 800$
Simplify.	$3a = 300$
Solve for *a*.	$a = 100$
Substitute $a = 100$ into [1].	$100 + c = 400$
Solve for *c*.	$c = 300$

Check with the *original* information.	**Attendance = 100 + 300 = 400 ✓OK**
Check the receipts.	**Receipts = \$5 × 100 + \$2 × 300**
	= \$500 + \$600
	= \$1100 ✓OK
State the answer in terms of the question.	**100 adults attended the play.**

Investigate: Cramer's rule

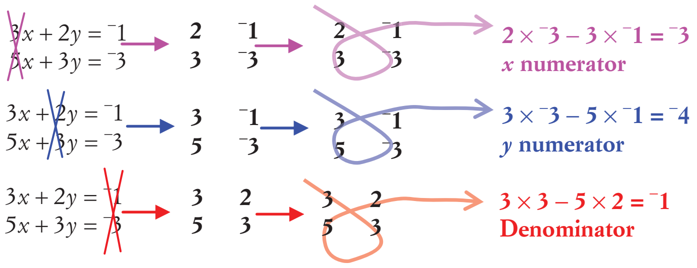
You can have more than two simultaneous equations with more than 2 unknowns. For example, the following system has 5 equations and 4 variables.

$$\begin{aligned} 8h + 14i + 9j + 9k &= 25 \\ 6h + 11i + 3j + 10k &= 13 \\ 4h + 6i - 12j + 11k &= {}^{-}11 \\ h + 2i + 3j &= 5 \\ 3h + 3i - 2j + 3k &= 5 \end{aligned}$$

You can check that the solution to this system is $h = 5$, $i = {}^{-}3$, $j = 2$ and $k = 1$.
Cramer's rule is an advanced formula that gives the solution to n equations in n variables. It is rarely used for large systems because the number of calculations is so large. Approximate methods are used instead. However, for 2 equations in 2 unknowns the rule can be applied in a simple way.
You need to put both equations in the form $ax + by = c$.
Then you use cross-multiplication and subtraction as shown below.
Consider the equations

$$\begin{aligned} 3x + 2y &= {}^{-}1 \\ 5x + 3y &= {}^{-}3 \end{aligned}$$

Progressively cross out a column and cross-multiply.



Change the sign of the x numerator to find the values of x and y.

$$x = \frac{-({}^{-}3)}{{}^{-}1}, \quad y = \frac{{}^{-}4}{{}^{-}1}$$

This gives $x = {}^{-}3$ and $y = 4$, as in Example **15** part **c**.
Check the result of Cramer's rule for Example **15** part **d**. You should get the same answer after cancelling down the fractions to their simplest form.

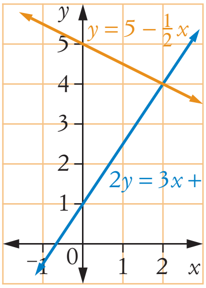
Exercise 11.2 Simultaneous equations

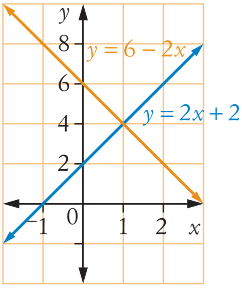
Understanding

Extra questions
Exercise 11.2
MAT10NAEQ00061

1 Use the following graphs to find simultaneous solutions to:

a $y = 5 - \frac{1}{2}x$ and $2y = 3x + 2$ **b** $y = 6 - 2x$ and $y = 2x + 2$



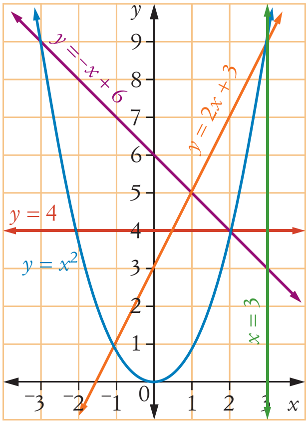


2 Find the solution of the following pairs of simultaneous equations by graphing.

a $y = x$ and $y = 2x + 1$
b $y = x + 3$ and $y = 2x + 4$
c $y = -x + 2$ and $y = 2x - 1$
d $y = 2x - 7$ and $y = -x - 1$
e $y = {}^{-}2x + 2$ and $y = 3x + 7$

3 Use the graphs drawn here to find simultaneous solutions for each of the following pairs.

a $y = x^2$ and $y = 6 - x$
b $y = 2x + 3$ and $y = x^2$
c $y = 4$ and $y = x^2$
d $y = x^2$ and $x = 3$



Fluency

See Example 13

4 Find the solution of the following pairs of simultaneous equations by graphing.

a $2x + y = 4$ and $x - y = 2$
b $3x - y = 7$ and $2x + y = 3$
c $10x - 5y = 50$ and $x + y = 2$
d $3x + 4y = {}^{-}2$ and $x - 4y = {}^{-}6$
e $4x + y = 23$ and $x + y = 8$

See Example 15

5 Find the solution of the following pairs of simultaneous equations by substitution.

a $x = y + 3$
$x + 2y = 9$

b $y = 3x$
$2x - y = {}^{-}6$

c $b = 2a - 3$
$a + 3b = 5$

d $3a + b = 8$
$b = a - 12$

e $x + 2y = 6$
$x = 3y - 9$

f $c = 3d - 1$
$5c - 7d = 19$

g $c = d + 1$
$2c + d = 14$

h $c = 7 - 2d$
$3c - 2d = 9$

i $x = 1 - y$
$3 = y + 2x$

6 Find the solution of the following pairs of simultaneous equations by elimination.

See Example 16

a $x + y = 5$
$x - y = 1$

b $2x + y = 12$
$3x - y = 13$

c $5m - 2n = 13$
$m + n = 4$

d $4a - b = 10$
$a + 3b = 9$

e $4m - n = 6$
$3m + 2n = {}^{-}1$

f $5x + y = 12$
$3x + 2y = 3$

g $3x + 2y = 6$
$2x - 3y = 17$

h $5a + 2b = {}^{-}8$
$a - 3b = {}^{-}5$

i $2m + 3n = {}^{-}1$
$m + 4n = 2$

7 Solve the following simultaneous equations by graphing the functions.

See Example 14

a $y = x^2$ and $y = x$

b $y = x^2$ and $y = 2x$

c $y = x^2$ and $y = x + 2$

d $y = x^2$ and $y = x + 6$

e $y = x^2$ and $y = {}^{-}2x + 3$

f $y = x^2$ and $y = 3x - 2$

8 Solve the following pairs of simultaneous equations.

See Example 17

a $\frac{x}{2} + y = 0$
$x - y = 6$

b $\frac{a}{3} + b = 4$
$a - b = 0$

c $m + \frac{n}{3} = 1$
$m - n = 5$

d $\frac{3x}{2} - \frac{y}{4} = 2$
$2x + y = 8$

e $\frac{3m}{2} - \frac{2n}{3} = 5$
$m + n = {}^{-}1$

f $\frac{c}{2} - \frac{d}{3} = \frac{4}{3}$
$\frac{c}{3} + \frac{d}{3} = 2$

g $x + \frac{y+1}{2} = 5$
$\frac{x-2}{2} - y = 0$

h $\frac{a-1}{2} + b = 1$
$\frac{b-1}{2} + a = 4$

i $\frac{y-x}{3} - \frac{x-y}{2} = {}^{-}5$
$\frac{x}{6} + \frac{2y}{3} = {}^{-}\frac{2}{3}$

Problem solving

See Example 18

9 The total cost of tickets for 2 adults and 3 children to see a show was \$20. The tickets for a family group with 3 adults and 5 children cost \$31. Find the cost of each type of ticket.

Worked solutions
Exercise 11.2
MAT10NAWS00066

⑩ The heights of water (h cm) in two containers after a certain time (t seconds) are given by the following equations:

Container A: $h = 2t$
Container B: $h = 3t - 20$

Find, graphically, the time when the two containers have the same height of water.

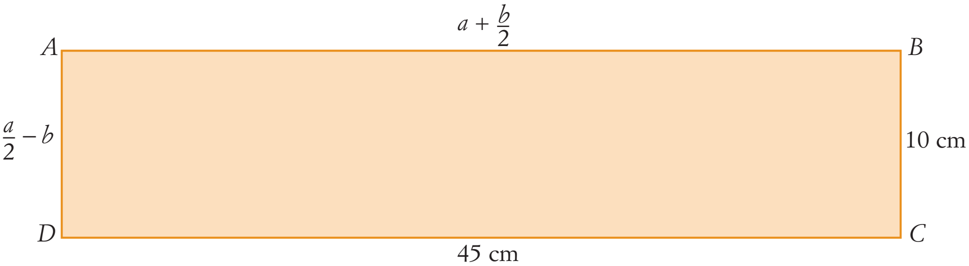
11 The length of a rectangular room is 2 m longer than its width. If the perimeter of the room is 68 m, find its length.

12 Two workers laid 250 bricks. If Sam had laid twice as many as he did and Peta had laid half as many as she did, there would have been 50 bricks left over. How many bricks did each lay?

Worked solutions
Exercise 11.2
MAT10NAWS00066

⑬ I bought three trays of strawberries and four trays of cherries for \$18 at the local fruit shop. My friend bought four trays of strawberries and three trays of cherries for \$17. How much did we pay for each tray of strawberries and cherries?

14 *ABCD* is a rectangle. Use this diagram to find the values of a and b.



15 Ross, Trish and Helen are comparing the number of DVDs that they own. Ross and Trish own the same number of DVDs but Helen has fewer than both of them. Ross has five less than six times the number of Helen's DVDs, while Trish has one more than three times Helen's number of DVDs. How many DVDs does Helen have?

Worked solutions
Exercise 11.2
MAT10NAWS00066

⑯ Jackie and Patrick are shearing at the same shed. One particular day, Jackie worked for 8 hours and Patrick worked for 9 hours. Between them they sheared 320 sheep. The next day they sheared 272 sheep, but Patrick knocked off after only 6 hours, while Jackie kept going for 8 hours again. Assuming that each man sheared at a consistent rate, find the number of sheep they could each shear in an hour.

11.3 Quadratic equations

You have already studied quadratic expressions. **Quadratic equations** are the result of putting a quadratic expression equal to zero.

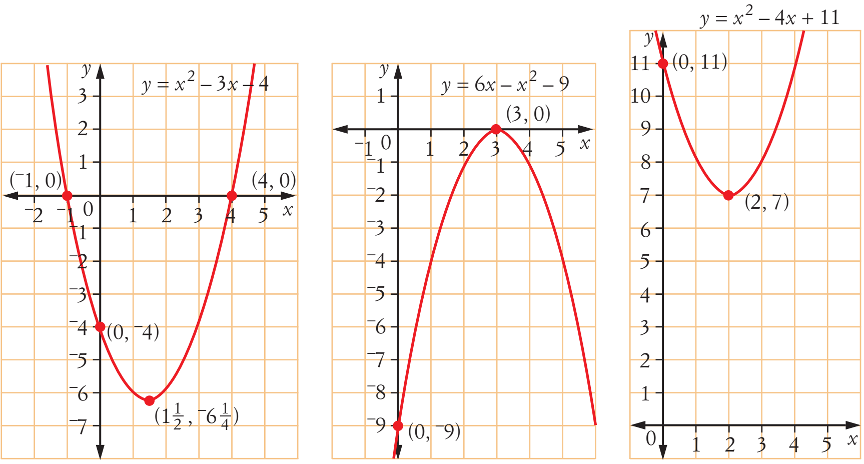
Important!

Quadratic equations

A quadratic equation is one that can be expressed in the form $ax^2 + bx + c = 0$, where x is the variable and a, b and c are constants.

You have already seen that quadratic functions produce parabolas and practiced drawing the graphs of quadratic functions. Consider the following quadratic equations: $x^2 - 3x - 4 = 0$, $6x - x^2 - 9 = 0$ and $x^2 - 4x + 11 = 0$. The corresponding quadratic functions are: $y = x^2 - 3x - 4$, $y = 6x - x^2 - 9$, and $y = x^2 - 4x + 11$. The first two expressions factorise to give $(x - 4)(x + 1)$ and $-(x - 3)^2$, but the third expression doesn't factorise. It is actually equal to $(x - 2)^2 + 7$.

You can use the methods from Chapters **5** and **7** to sketch the graphs as shown below.



The first graph has two zeros, the second only one and the third none. This means that the corresponding quadratic equations have two solutions, one solution and no solutions.

While you can find the solutions to quadratic equations by graphing, as above, it is quicker and more accurate to find them using algebra.

The first method just uses the square root to find solutions.

Example 19

Solve each of the following equations.

a $m^2 = 16$ **b** $3x^2 = 75$

Solution

a	Write the equation.	$m^2 = 16$
	Write the meaning of the equation.	**The square of m is 16.**
	Write the answers.	$m = {}^{-}4$ **or** $m = 4$
b	Write the equation.	$3x^2 = 75$
	Divide by 3.	$\frac{3x^2}{3} = \frac{75}{3}$
	Simplify.	$x^2 = 25$
	Write in the square root form.	$x = \pm\sqrt{25} = \pm 5$
	Separate the ± into + and −.	$x = {}^{-}5$ **or** $x = 5$

Instead of using the phrase 'the square of m is 16', you could write $m = \pm\sqrt{16}$. It is necessary to write the ± because, by itself, $\sqrt{\ }$ is taken to mean a *positive* number. You may need to isolate the square before you can do this.

Example 20

Video tutorial

Simple quadratic equations

MAT10NAVT10028

Solve each of the following equations.

a $(x - 8)^2 = 9$ **b** $(x + 3)^2 = 1$

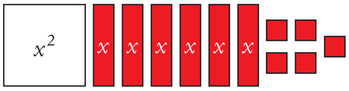
Solution

a	Write the equation.	$(x - 8)^2 = 9$
	Write in the square root form.	$x - 8 = \pm\sqrt{9} = \pm 3$
	Separate the ± into + and −.	$x - 8 = {}^{-}3$ **or** $x - 8 = 3$
	Solve each part.	$x - 8 + 8 = {}^{-}3 + 8$ **or** $x - 8 + 8 = 3 + 8$
	Complete the answers.	$x = 5$ **or** $x = 11$
b	Write the equation.	$(x + 3)^2 = 1$
	Write in the square root form.	$x + 3 = \pm\sqrt{1} = \pm 1$
	Separate the ± into + and −.	$x + 3 = {}^{-}1$ **or** $x + 3 = 1$
	Solve each part.	$x + 3 - 3 = {}^{-}1 - 3$ **or** $x + 3 - 3 = 1 - 3$
	Complete the answers.	$x = {}^{-}4$ **or** $x = {}^{-}2$

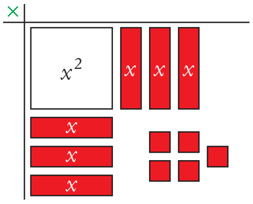
In principle, any quadratic equation can be solved by changing it to the form $(x \pm p)^2 = q$. You should remember that the expansion of a perfect square is given by $(x + y)^2 = x^2 + 2xy + y^2$ or for a difference by $(x - y)^2 = x^2 - 2xy + y^2$.
Thus $(x \pm p)^2 = x^2 \pm 2xp + p^2$ so $(x \pm p)^2 = q$ is the same as $x^2 \pm 2xp + p^2 = q$. This is used in reverse to change quadratic equations to the form $(x \pm p)^2 = q$.

Investigate: Completion of the square

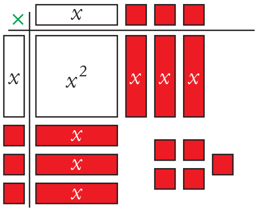
You can use a physical model to express any quadratic in the form $(x \pm p)^2 \pm q$. You will need some small squares to represent units, strips to represent x and a larger square to represent x^2. They need to be white on one side and coloured on the reverse to represent negatives. You may have used some in Chapter 7 to model factorisation of quadratics.
Consider the quadratic expression $x^2 - 6x - 5$.
You can model this as shown here.



Make a cross to show multiplication and arrange the x^2 and xs so there are an equal number of xs to the right and below the x^2. Leave the units to one side for the moment.



To get the x^2 and xs you will obviously need an x and three negative units at both the top and side.



But the negative units at the top and side multiply to give *positive* units inside. You will need to put in 9 positive units and another 9 negative units so the total is still $x^2 - 6x - 5$. This gives the model shown below.
The model now shows

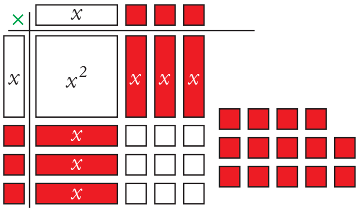
$x^2 - 6x + 9 - 14$.

The arrangement represents

$(x - 3)(x - 3) - 14$

which is $(x - 3)^2 - 14$.
This shows that

$x^2 - 6x - 5 = (x - 3)^2 - 14$.



Your teacher will probably give you some other problems to try modelling in the way shown above.

Important!

Completing the square

You can **complete the square** in a quadratic equation by:

1 dividing (if necessary) to make the coefficient of x^2 equal to 1
2 moving the constant to the RHS
3 adding half the square of the coefficient of x to both sides
4 factorising the LHS as a perfect square

Example 21

CAS TI-Nspire exercise
Equations
MAT10NATI00018

CAS ClassPad exercise
Equations
MAT10NACP00018

Solve the following quadratic equations by completing the square.

a $e^2 + 10e = 0$ **b** $x^2 - 6x + 5 = 0$ **c** $3k^2 + 6k - 24 = 0$

Solution

a	Write the equation.	$e^2 + 10e = 0$
	Start at step 3 by adding 5^2 to both sides.	$e^2 + 10e + 5^2 = 0 + 5^2$
	Factorise the LHS and simplify the RHS.	$(e + 5)^2 = 25$
	Write in the square root form.	$e + 5 = \pm\sqrt{25} = {}^{-}5$ or 5
	Separate the ± into + and −.	$e + 5 = {}^{-}5$ or $e + 5 = 5$
	Solve each part.	$e + 5 - 5 = {}^{-}5 - 5$ or $e + 5 - 5 = 5 - 5$
	Complete the answers.	$e = {}^{-}10$ or $e = 0$
b	Write the equation.	$x^2 - 6x + 5 = 0$
	Start at step 2 by taking 5 from both sides.	$x^2 - 6x + 5 - 5 = 0 - 5$
	Simplify.	$x^2 - 6x = {}^{-}5$
	Add the square of half of 6 to both sides.	$x^2 - 6x + 3^2 = {}^{-}5 + 3^2$
	Factorise the LHS and simplify the RHS.	$(x - 3)^2 = {}^{-}5 + 9 = 4$
	Write in square root form.	$x - 3 = \pm\sqrt{4} = {}^{-}2$ or 2
	Separate the ± into + and −.	$x - 3 = {}^{-}2$ or $x - 3 = 2$
	Solve each part.	$x - 3 + 3 = {}^{-}2 + 3$ or $x - 3 + 3 = 2 + 3$
	Complete the answers.	$x = 1$ or $x = 5$
c	Write the equation.	$3k^2 + 6k - 24 = 0$
	Make the coefficient of k^2 equal to 1 by dividing through by 3.	$\frac{3k^2}{3} + \frac{6k}{3} - \frac{24}{3} = \frac{0}{3}$
	Simplify.	$k^2 + 2k - 8 = 0$
	Add the constant 8 to both sides.	$k^2 + 2k - 8 + 8 = 0 + 8$
	Simplify.	$k^2 + 2k = 8$
	Add the square of half of 2 to both sides.	$k^2 + 2k + 1^2 = 8 + 1^2$

Factorise the LHS and simplify the RHS.	$(k+1)^2 = 9$
Write in square root form.	$k+1 = \pm\sqrt{9} = {}^{-}3 \text{ or } 3$
Separate the ± into + and −.	$k+1 = {}^{-}3 \text{ or } k+1 = 3$
Solve each part.	$k+1-1 = {}^{-}3-1 \text{ or } k+1-1 = 3-1$
Complete the answers.	$k = {}^{-}4 \text{ or } k = 2$

In some cases you will have a fraction involved in your answer.

Example 22

Solve $2m^2 + 5m - 12 = 0$ by completing the square.

Solution

Write the equation.	$2m^2 + 5m - 12 = 0$
Make the coefficient of m^2 equal to 1 by dividing through by 2.	$\frac{2m^2}{2} + \frac{5m}{2} - \frac{12}{2} = 0$
Simplify.	$m^2 + 2\frac{1}{2}m - 6 = 0$
Add the constant 6 to both sides.	$m^2 + 2\frac{1}{2}m - 6 + 6 = 0 + 6$
Simplify.	$m^2 + 2\frac{1}{2}m = 6$
Add the square of half of $\frac{5}{2}$ to both sides.	$m^2 + \frac{5}{2}m + \left(\frac{5}{4}\right)^2 = 6 + \left(\frac{5}{4}\right)^2$
Factorise the LHS and simplify the RHS.	$\left(m + \frac{5}{4}\right)^2 = 6 + \frac{25}{16} = \frac{96+25}{16} = \frac{121}{16}$
Write in square root form.	$m + \frac{5}{4} = \pm\sqrt{\frac{121}{16}} = \pm\frac{11}{4}$
Separate the ± into + and −.	$m + \frac{5}{4} = {}^{-}\frac{11}{4} \text{ or } m + \frac{5}{4} = \frac{11}{4}$
Solve each part.	$m + \frac{5}{4} - \frac{5}{4} = {}^{-}\frac{11}{4} - \frac{5}{4} \text{ or}$
	$m + \frac{5}{4} - \frac{5}{4} = \frac{11}{4} - \frac{5}{4}$
Complete the answers.	$m = \frac{{}^{-}16}{4} \text{ or } m = \frac{6}{4}$
Simplify to get the final answers.	$m = {}^{-}4 \text{ or } m = 1\frac{1}{2}$

In some cases it is necessary to leave part of the answer in square root form. You can also give an approximate answer using an approximate value of the square root.

Example 23

Solve $x^2 - 8x + 4 = 0$ by completing the square.

Solution

Write the equation.	$\mathbf{x^2 - 8x + 4 = 0}$
Complete the square.	$\mathbf{x^2 - 8x + 16 = {}^{-}4 + 16}$
Factorise.	$\mathbf{(x - 4)^2 = 12}$
Take the square root.	$\mathbf{x - 4 = {}^{-}\sqrt{12}}$ **or** $\mathbf{x - 4 = \sqrt{12}}$
Solve for x and simplify the surds.	$\mathbf{x = 4 - 2\sqrt{3}}$ **or** $\mathbf{x = 4 + 2\sqrt{3}}$
Find approximations.	$\mathbf{x \approx 0.5359}$ **or** $\mathbf{x \approx 7.464}$
Write the final answers.	$\mathbf{x = 4 - 2\sqrt{3} \approx 0.5359}$ **or** $\mathbf{x = 4 + 2\sqrt{3} \approx 7.464}$

In many cases, it is easier to solve a quadratic equation by factorisation than by completing the square. When solving by factorisation it is necessary to use the following rule.

Important!

Zero product rule

If $ab = 0$ then at least one of a or b must be zero. That is, $a = 0$ or $b = 0$.

Example 24

Solve each of the following.

a $m(m - 7) = 0$ **b** $(a + 7)(a - 3) = 0$

Solution

a Write the problem.	$\mathbf{m(m - 7) = 0}$
Use the zero product rule.	$\mathbf{m = 0}$ **or** $\mathbf{m - 7 = 0}$
Solve for m.	$\mathbf{m = 0}$ **or** $\mathbf{m = 7}$
b Write the equation.	$\mathbf{(a + 7)(a - 3) = 0}$
Use the zero product rule.	$\mathbf{a + 7 = 0}$ **or** $\mathbf{a - 3 = 0}$
Solve for a.	$\mathbf{a = {}^{-}7}$ **or** $\mathbf{a = 3}$

Example 25

Video tutorial
Quadratic equations by factorising
MAT10NAVT10029

Puzzle sheet
Factorising quadratic equations
MAT10NAPS00037

Solve each of the following by factorisation.

a $3m^2 = 6m$ **b** $x^2 + 5x + 6 = 0$ **c** $p^2 = 5p + 24$
d $n(n + 2) = 8$ **e** $2x^2 - x - 15 = 0$ **f** $(x - 1)^2 + 5(x - 1) - 6 = 0$

Solution

a	Write the equation.	$3m^2 = 6m$
	Put all terms on the LHS.	$3m^2 - 6m = 0$
	Simplify by dividing all terms by 3.	$m^2 - 2m = 0$
	Factorise.	$m(m - 2) = 0$
	Use the zero product rule.	$m = 0$ or $m - 2 = 0$
	Solve for m.	$m = 0$ or $m = 2$
b	Write the equation.	$x^2 + 5x + 6 = 0$
	Factorise.	$(x + 3)(x + 2) = 0$
	Use the zero product rule.	$x + 3 = 0$ or $x + 2 = 0$
	Solve for x.	$x = {}^-3$ or $x = {}^-2$
c	Write the equation.	$p^2 = 5p + 24$
	Put all terms on the LHS.	$p^2 - 5p - 24 = 0$
	Factorise.	$(p + 3)(p - 8) = 0$
	Use the zero product rule.	$p + 3 = 0$ or $p - 8 = 0$
	Solve for p.	$p = {}^-3$ or $p = 8$
d	Write the equation.	$n(n + 2) = 8$
	Multiply out the brackets.	$n^2 + 2n = 8$
	Put all terms on the LHS.	$n^2 + 2n - 8 = 0$
	Factorise.	$(n + 4)(n - 2) = 0$
	Use the zero product rule.	$n + 4$ or $n - 2 = 0$
	Solve for n.	$n = {}^-4$ or $n = 2$
e	Write the equation.	$2x^2 - x - 15 = 0$
	Find the product and sum numbers.	$ac = 2 \times {}^-15 = {}^-30$ and $b = {}^-1$
	Use $5 \times {}^-6 = {}^-30$ and $5 + {}^-6 = {}^-1$ to factorise.	$2x^2 + 5x - 6x - 15 = 0$
	Group and use the common factors.	$x(2x + 5) - 3(2x + 5) = 0$
	Complete the factorisation.	$(2x + 5)(x - 3) = 0$
	Use the zero product rule.	$2x + 5 = 0$ or $x - 3 = 0$
	Solve for x.	$x = {}^-2\frac{1}{2}$ or $x = 3$

f Write the equation.	$(x-1)^2+5(x-1)-6=0$
Substitute $y=(x-1)$.	$y^2+5y-6=0$
Factorise.	$(y+6)(y-1)=0$
Substitute $(x-1)$ back for y.	$[(x-1)+6][(x-1)-1]=0$
Remove the inner brackets.	$(x-1+6)(x-1-1)=0$
Simplify.	$(x+5)(x-2)=0$
Use the zero product rule.	$x+5=0$ or $x-2=0$
Solve for x.	$x={}^{-}5$ or $x=2$

Not all quadratics will factorise or the factorisation may not be obvious. In this case you could always try completing the square, but you may find that method tedious. The following rule is actually worked out by completing the square of the general equation $ax^2+bx+c=0$.

Teacher notes
The quadratic formula
MAT10NATN00011

Important!

The quadratic formula

The solutions to the quadratic equation $ax^2+bx+c=0$ where a, b and c are constants and $a \neq 0$ is

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Technology worksheet
Excel worksheet: The quadratic formula
MAT10NACT00018

Video tutorial
The quadratic formula
MAT10NAVT10030

Puzzle sheet
The quadratic formula
MAT10NAPS00038

Example 26

Solve each of the following equations using the quadratic formula.

a $x^2-3x+2=0$ **b** $6x^2+x-2=0$ **c** $x^2=1-7x$

Solution

a Write the problem.	$x^2-3x+2=0$
Compare with $ax^2+bx+c=0$.	$a=1, b={}^{-}3, c=2$
Write the formula.	$x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$
Substitute values.	$=\dfrac{-({}^{-}3)\pm\sqrt{({}^{-}3)^2-4\times1\times2}}{2\times1}$
Simplify the terms.	$=\dfrac{3\pm\sqrt{9-8}}{2}$
Separate the ± into + and −.	$=\dfrac{3-1}{2}$ or $\dfrac{3+1}{2}$
Write the final answers.	$x=1$ or $x=2$

b Write the equation. $6x^2 + x - 2 = 0$

Compare with $ax^2 + bx + c = 0$. $a = 6, b = 1, c = {}^-2$

Write the formula. $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Substitute values. $= \dfrac{-1 \pm \sqrt{1^2 - 4 \times 6 \times {}^-2}}{2 \times 6}$

Simplify the terms. $= \dfrac{{}^-1 \pm \sqrt{1 + 48}}{12}$

Separate the ± into + and −. $= \dfrac{{}^-1 - 7}{12}$ or $\dfrac{{}^-1 + 7}{12}$

Write the final answers. $x = {}^-\frac{2}{3}$ or $x = \frac{1}{2}$

c Write the problem. $x^2 = 1 - 6x$

Put all the terms on the LHS. $x^2 + 6x - 1 = 0$

Compare with $ax^2 + bx + c = 0$. $a = 1, b = 6, c = {}^-1$

Write the formula. $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Substitute values. $= \dfrac{{}^-6 \pm \sqrt{6^2 - 4 \times 1 \times {}^-1}}{2 \times 1}$

Simplify the terms. $= \dfrac{{}^-6 \pm \sqrt{36 + 4}}{2}$

Simplify the square root term. $= \dfrac{{}^-6 \pm \sqrt{40}}{2}$

Simplify the surd. $= \dfrac{{}^-6 \pm 2\sqrt{10}}{2}$

Factorise the numerator. $= \dfrac{{}^-2(3 \mp \sqrt{10})}{2}$

Cancel and separate the ± into + and −. $= {}^-3 - \sqrt{10}$ or ${}^-3 + \sqrt{10}$

Find approximations. $\approx {}^-6.162$ or 0.162

Write the final answers. $x = -\sqrt{10} - 3 \approx {}^-6.162$

or $x = \sqrt{10} - 3 \approx 0.162$

Using quadratic equations to model and solve problems requires the same approach as any other kind of problem solving. For example, you should always draw a diagram to show physical situations. You should always choose sensible variable names. The only thing that makes it different is the fact that the mathematical model is a quadratic equation. Sometimes a quadratic model may produce a false solution that has to be discarded when you check the solutions with the original information.

Worksheet

Problems involving quadratic equations

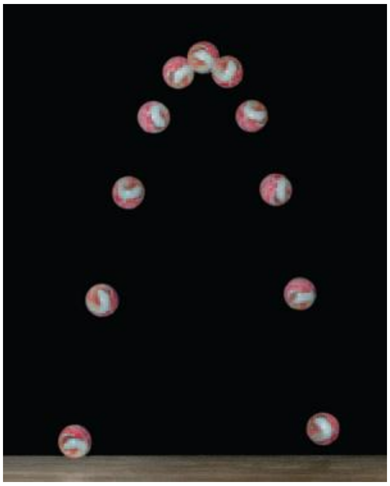
MAT10NAWK00029

Example 27

A ball is propelled upwards so that its height (h) in metres after t seconds is given by the formula $h = 30t - 5t^2$. Find the time taken for the ball to reach a height of 25 m.

Solution

Write the equation.	$h = 30t - 5t^2$
State the required value of h.	$h = 25$
Substitute in the equation.	$25 = 30t - 5t^2$
Rearrange the equation as a standard quadratic.	$5t^2 - 30t + 25 = 0$
Divide by 5 to simplify the equation.	$t^2 - 6t + 5 = 0$
Factorise the quadratic.	$(t - 1)(t - 5) = 0$
Use the zero product rule.	$t - 1 = 0$ or $t - 5 = 0$
Solve for t.	$t = 1$ or $t = 5$
Write the relevant original information.	$h = 30t - 5t^2$
Check the first answer.	At $t = 1$ s,
Calculate the height at $t = 1$ s.	$h = 30 \times 1 - 5 \times 1^2$ $= 30 - 5$ m $= 25$ m ✓OK
Check the second answer.	At $t = 5$ s,
Calculate the height at $t = 5$ s.	$h = 30 \times 5 - 5 \times 5^2$ $= 150 - 125$ m $= 25$ m ✓OK
State your answer.	**The ball takes 1 second to reach a height of 25 m on the way up and it is again at 25 m another 4 seconds later on the way down.**



Example 28

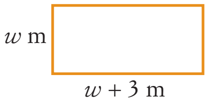
A rectangular room has a length that is 3 m longer than its width. If the area of the room is 54 m^2, what is the length?

Solution

Draw a diagram.

Label the width as w.

The length is then $w + 3$.



Write an expression for the area of the room.	$A = \text{width} \times \text{length} = w(w + 3)$
Substitute the value of A.	$w(w + 3) = 54$
Multiply out the brackets.	$w^2 + 3w = 54$
Rearrange the equation as a standard quadratic.	$w^2 + 3w - 54 = 0$
Factorise the quadratic.	$(w + 9)(w - 6) = 0$
Use the zero product rule.	$w + 9 = 0$ **or** $w - 6 = 0$
Solve for w.	$w = {}^{-}9$ **or** $w = 6$
Write the relevant original information.	**w is a width.**
Check the first answer.	**A negative width makes no sense.**
Check the second answer.	**If the width = 6 m, the length is 9 m, so the area is 54 m^2 ✓OK**
Write the answer to the problem.	**The length of the room is 9 m.**

Instead of solving simultaneous equations that include a quadratic using graphs, you can solve them using substitution. Use the linear equation to make a substitution formula and put that in the quadratic. At times, no solution may exist.

Example 29

Find the simultaneous solutions of the following pairs of equations.

a $2x + y - 7 = 0$
$y = x^2 - x - 5$

b $y = 2x + 1$
$x^2 = y - x - 4$

Solution

a Write and number the equations. y is not squared so it will be the easiest to substitute.

$$2x + y - 7 = 0 \quad [1]$$
$$y = x^2 - x - 5 \quad [2]$$

Rearrange [1] to make y the subject.

$$y = 7 - 2x \quad [3]$$

Substitute [3] into [2].

$$7 - 2x = x^2 - x - 5$$

CAS TI-Nspire exercise
Equations
MAT10NATI00018

CAS ClassPad exercise
Equations
MAT10NACP00018

Technology worksheet
Excel worksheet: Intersection of a parabola and straight line
MAT10NACT00021

Rearrange as a standard quadratic.	$x^2 + x - 12 = 0$ [4]
Factorise the quadratic.	$(x + 4)(x - 3) = 0$
Use the zero product rule.	$x + 4 = 0$ or $x - 3 = 0$
Solve for x.	$x = {}^{-}4$ or $x = 3$
Substitute in [3].	$y = 7 - 2 \times {}^{-}4$ or $y = 7 - 2 \times 3$
Evaluate.	$y = 15$ or $y = 1$
State the solutions.	**The solutions are $({}^{-}4, 15)$ or $(3, 1)$.**
Check $({}^{-}4, 15)$ in both equations.	[1] $2x + y - 7 = 2 \times {}^{-}5 + 15 - 7 = 0$ ✓ [2] $x^2 - x - 5 = ({}^{-}4)^2 - {}^{-}4 - 5 = 15 = y$ ✓OK
Check $(3, 1)$ in both equations.	[1] $2x + y - 7 = 2 \times 3 + 1 - 7 = 0$ ✓ [2] $x^2 - x - 5 = 3^2 - 3 - 5 = 1 = y$ ✓OK
Give the final answer.	**The simultaneous solutions are $x = {}^{-}4$ and $y = 15$ or $x = 3$ and $y = 1$.**
b Write and number the equations. y is not squared so it will be the easiest to substitute.	$y = 2x + 1$ [1] $x^2 = y - x - 4$ [2]
Substitute [1] into [2].	$x^2 = 2x + 1 - x - 4$
Rearrange as a standard quadratic.	$x^2 - x + 3 = 0$ [3]
[3] will not factorise.	**Use the quadratic formula to find x.**
Compare with $ax^2 + bx + c = 0$.	$a = 1, b = {}^{-}1, c = 3$
Write the formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Substitute values.	$= \frac{-({}^{-}1) \pm \sqrt{({}^{-}1)^2 - 4 \times 1 \times 3}}{2 \times 1}$
Simplify the terms.	$= \frac{1 \pm \sqrt{{}^{-}11}}{2}$
$\sqrt{{}^{-}11}$ is not a real number.	**This has no solutions on the real number line.**
State the final answer.	**There is no simultaneous solution.**

In part **b** above, each equation has an infinite number of possible solutions, but there is no simultaneous solution. In fact, the line $y = 2x + 1$ passes below the turning point of the quadratic.

Exercise 11.3 Quadratic equations

Understanding

1 Solve each of the following equations.

See Example 19

a $x^2 = 4$
b $a^2 = 49$
c $b^2 = 121$
d $25 = h^2$
e $x^2 = 81$
f $2a^2 = 8$
g $3x^2 = 27$
h $8b^2 = 200$
i $12x^2 = 300$
j $k^2 = \frac{9}{49}$
k $16n^2 = 81$
l $32m^2 = 18$

2 Solve each of the following equations.

See Example 20

Extra questions
Exercise 11.3
MAT10NAEQ00062

a $(z + 2)^2 = 9$
b $(h - 10)^2 = 25$
c $(e - 5)^2 = 81$
d $(k - 7)^2 = 100$
e $(c - 8)^2 = 100$
f $(q + 3)^2 = 16$
g $(q + 7)^2 = 9$
h $(b - 9)^2 = 144$
i $(y - 11)^2 = 144$
j $(m - 9)^2 = 81$

3 Solve each of the following equations.

See Example 24

a $x(x + 1) = 0$
b $(x + 2)x = 0$
c $(x - 3)(x + 1) = 0$
d $(f - 4)(f + 9) = 0$
e $(x - 10)(x + 3) = 0$
f $(n - 4)(n + 3) = 0$
g $(b + 8)b = 0$
h $(x - 3)^2 = 0$
i $(2m + 3)(m - 4) = 0$
j $(2k + 3)(k - 3) = 0$
k $(4w - 5)(5w + 7) = 0$
l $(10x + 1)(x - 2) = 0$

Fluency

4 Solve each of the following equations by completing the square.

See Examples 21, 22

a $v^2 - 2v - 3 = 0$
b $a^2 - 17a + 72 = 0$
c $r^2 - r - 30 = 0$
d $n^2 + 12n + 35 = 0$
e $w^2 + 2w - 15 = 0$
f $y^2 + 7y - 30 = 0$
g $k^2 - 23k + 130 = 0$
h $g^2 - 9g + 20 = 0$
i $c^2 - 4c - 165 = 0$
j $30n^2 + 61n + 30 = 0$
k $15b^2 - 13b - 6 = 0$
l $32m^2 - 108m + 81 = 0$
m $27x^2 - 60x - 32 = 0$

5 Solve each of the following equations by factorisation.

See Example 25

a $x^2 + 9x + 20 = 0$
b $a^2 + 8a + 12 = 0$
c $x^2 - 8x + 7 = 0$
d $m^2 - 3m - 28 = 0$
e $n^2 - 3n - 40 = 0$
f $a^2 + 2a - 63 = 0$
g $m^2 + 5m - 24 = 0$
h $d^2 - 18d + 81 = 0$
i $a^2 + 11a + 28 = 0$
j $x^2 + x = 0$
k $x^2 + 3x = 0$
l $a^2 - 2a = 0$
m $3d^2 + 6d = 0$
n $y^2 = 4y$
o $x^2 = 6x$
p $2x^2 + 10x + 12 = 0$
q $a^2 + a = 20$
r $x^2 - 4x = 45$
s $b^2 - 8b = {}^{-}7$
t $h^2 + 2h = 35$
u $5a + 6 = a^2$
v $u^2 + 6 = 5u$
w $2b = 8b^2$
x $d - 3d^2 = 0$
y $2n - 5n^2 = 0$

6 Solve each of the following equations using the quadratic formula.

See Example 26

a $2x^2 - 3x + 1 = 0$
b $3x^2 + 2x - 1 = 0$
c $4x^2 + 7x - 2 = 0$
d $5x^2 - 17x + 6 = 0$
e $3x^2 - 2x - 8 = 0$
f $2x^2 - 13x + 15 = 0$
g $2x^2 + 13x + 20 = 0$
h $2x^2 - 5x - 12 = 0$
i $3x^2 - 8x - 35 = 0$
j $5x^2 - 3x - 2 = 0$
k $4x^2 + 11x + 7 = 0$
l $2x^2 + 3x - 20 = 0$

7 Solve each of the following equations by completing the square.

See Example 23

a $g^2 + 10g + 17 = 0$
b $a^2 - 4a - 4 = 0$
c $t^2 + 4t - 6 = 0$
d $u^2 + 14u + 37 = 0$
e $n^2 - 6n - 2 = 0$

8 Solve each of the following equations by factorisation.

a $2x^2 + x - 1 = 0$
b $3x^2 + 5x + 2 = 0$
c $3x^2 - 14x + 15 = 0$
d $3x^2 - 11x + 6 = 0$
e $2c^2 - 5c - 33 = 0$
f $3a^2 + 2a - 21 = 0$
g $7a^2 + 25a + 12 = 0$
h $9h^2 + 36h + 11 = 0$
i $5x^2 - 7x - 6 = 0$
j $6a^2 - 7a + 1 = 0$
k $8m^2 - 26m + 15 = 0$
l $6m^2 + 7m - 3 = 0$
m $12h^2 - 8h - 15 = 0$
n $14m^2 - 55m + 21 = 0$
o $9x^2 - 12x + 4 = 0$
p $6m^2 - 13m + 5 = 0$
q $4c^2 - 19c - 30 = 0$
r $4b^2 + 21b - 18 = 0$
s $4a^2 - 4a = 3$
t $7b^2 + 48b = 7$
u $2m^2 - 10 = m$
v $2g^2 - 21 = 11g$
w $2c^2 - 11c = {}^{-}12$

9 Solve each of the following equations using the quadratic formula and leave your answers in surd form.

a $x^2 + 4x + 1 = 0$
b $x^2 - 7x + 2 = 0$
c $x^2 + 6x + 1 = 0$
d $x^2 - 6x + 3 = 0$
e $2x^2 - 4x - 5 = 0$
f $2x^2 + 6x + 1 = 0$
g $3x^2 + 6x + 2 = 0$
h $4x^2 - x - 1 = 0$
i $5x^2 - 10x + 3 = 0$
j $4x^2 - 3x - 2 = 0$
k $2x^2 + 4x - 1 = 0$
l $5x^2 + 2x - 1 = 0$

10 Solve each of the following equations using the quadratic formula and write your answers correct to 2 decimal places.

a $x^2 + 3x + 1 = 0$
b $a^2 - 7a + 1 = 0$
c $b^2 + 5b + 3 = 0$
d $3w^2 - 9w + 2 = 0$
e $2n^2 + 4n - 1 = 0$
f $2x^2 - 7x + 4 = 0$
g $x + 4 = x^2$
h $2x^2 + x = 2$
i $6 = 3n^2 + 2n$
j $3w^2 + 9w = {}^{-}5$
k $5c + 2 = 3c^2$

See Example 29

11 Solve the following pairs of simultaneous equations algebraically.

a $y = x^2$, $y = x + 6$
b $y = -x^2 - 1$, $y + 3 = x$
c $y = x^2$, $y - 3x = 4$
d $y + 3x = x^2 + 4$, $y + 12 = 5x$
e $y = 2x + 1$, $y = 2x^2 - 2x + 3$
f $y + 2x^2 = 6$, $y + 9x = 1$
g $y = 5x^2 - 5x + 3$, $y = 2x + 1$
h $y + 5 = 2x^2 + x$, $y = {}^{-}8x + 6$
i $y = 5x + 5$, $y + 7x = x^2 + 40$

Problem solving

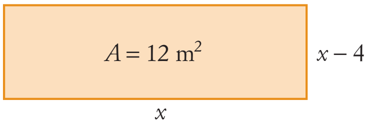
12 Solve each of the following equations.

a $(x - 1)^2 + 3(x - 1) + 2 = 0$
b $(x + 2)^2 + 4(x + 2) + 3 = 0$
c $(y + 1)^2 - 5(y + 1) - 6 = 0$
d $(x - 5)^2 - 7(x - 5) - 60 = 0$
e $(b + 3)^2 + 4(b + 3) - 45 = 0$
f $3(x - 2)^2 + 7(x - 2) + 2 = 0$
g $4(w + 4)^2 - 19(w + 4) - 5 = 0$
h $2(x - 7)^2 + 13(x - 7) - 24 = 0$
i $4(m - 6)^2 - 12(m - 6) - 7 = 0$
j $8(3 + x)^2 - 18(3 + x) + 9 = 0$

See Example 27

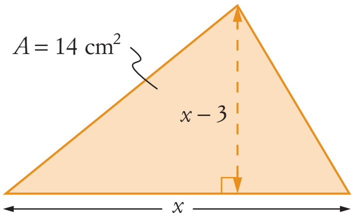
13 **a** The sum of a positive whole number and its square is 56. What is the number?
b The sum of a positive integer and its square is 110. What is the number?
c When a number is subtracted from its square, the result is 90. What is the number?

14 Find the dimensions of this rectangle if its area is 12 m^2.

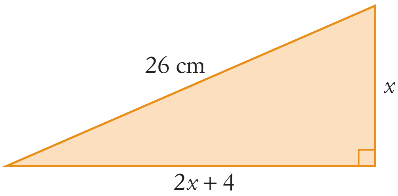


See Example 28

15 The area of this triangle is 14 cm^2.
What is the length of its base?

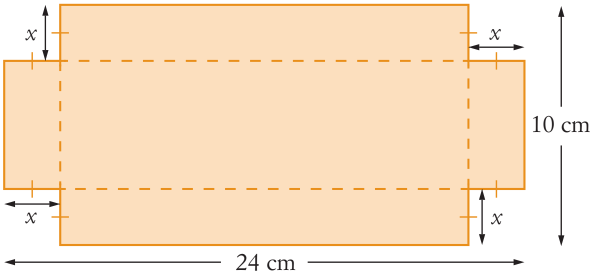


16 Use Pythagoras' theorem to find the dimensions of this right-angled triangle.



Worked solutions
Exercise 11.3
MAT10NAWS00067

17 This shape is the net for a box that is formed when the net is folded along the dotted lines. What are the dimensions of the box if the area of its base is 120 cm^2?



18 The sum of the first n integers is given by the formula $S = \frac{1}{2}n(n + 1)$.

Find the number of integers that must be added to obtain a sum of:

a 45 **b** 78 **c** 120

Worked solutions
Exercise 11.3
MAT10NAWS00067

19 A ball thrown upwards reached a height, h metres, after t seconds according to the formula $h = 38t - 5t^2$. At what time did the ball first reach a height of 48 m?

20 Peta decides to make a box to hold some newspaper clippings for an assignment. She starts with a rectangular piece of cardboard that is 3 times as long as it is wide. Peta then cuts out squares of side 10 cm from each corner and folds up the ends to form a box. She calculates that the box has a volume of 1200 cm^3. What are the dimensions of the box?

Reasoning

21 **a** Complete the square for the quadratic equation $x^2 + 6x + 10 = 0$.
b Hence show that there are no real answers on the number line.
c Use part **a** to find the turning point of $y = x^2 + 6x + 10 = 0$.

22 Complete the square for the quadratic equation $x^2 + bx + c = 0$ and show that the result is consistent with the quadratic formula.

Worked solutions
Exercise 11.3
MAT10NAWS00067

23 Complete the square for the quadratic equation $ax^2 + bx + c = 0$.

24 Show that any quadratic equation $ax^2 + bx + c = 0$, where $b^2 < 4ac$, can have no real answers on the number line.

11.4 Exponential equations

You looked at the relationship between exponentials and logarithms in Chapter **1** and did some work with exponential functions in Chapter **5**. Exponential functions are extremely important in the study of growth and decay. Given unlimited food and space, a population of animals will grow exponentially. This means that the initial growth of any population is also expected to be exponential. Many physical phenomena are also exponential in nature. For example, radioactive decay, the closing of a spring door with a hydraulic damper, the cooling of a cup of coffee and atmospheric pressure at different heights can all be modelled using exponential functions.

Important!

Exponential equations

An **exponential equation** is one where the unknown is involved in the exponent of a power.

Example 30

Solve the equation $5^x = 125\sqrt{5}$

Solution

Write the equation.	$5^x = 125\sqrt{5}$
Write the RHS as powers of 5.	$5^x = 5^3 \times 5^{\frac{1}{2}}$
Use the index law $a^n \times a^m = a^{n+m}$	$5^x = 5^{3\frac{1}{2}}$
Write the answer.	$x = 3\frac{1}{2}$

The solution of an exponential equation normally involves the conversion of the exponent to a coefficient using logarithms. You should remember from Chapter **1** that the exponential statement $y = a^x$ is the same as the logarithmic statement $x = \log_a y$. You may also remember solving some simple exponential equations in Chapter **1**.

Puzzle sheet
Exponential equations
MAT10NAPS00040

Example 31

Solve the equation $2^{5x+1} = 16$.

Solution

Write the equation.	$2^{5x+1} = 16$
Change to a logarithmic equation.	$5x + 1 = \log_2 16$
Find the value of $\log_2 16$.	Since $2^4 = 16$, $\log_2 16 = 4$
Substitute into the equation.	$5x + 1 = 4$
Solve the simplified equation.	$5x = 3$
Divide by 5.	$x = \frac{3}{5} = 0.6$

You should check that 0.6 *is* the solution to $2^{5x+1} = 16$ by substituting $x = 0.6$ into 2^{5x+1}. Most people find it easier to use common logarithms to solve exponential equations.

Example 32

Solve the equation $27 \times 1.065^t = 40.5$.

Solution

Write the equation. $\mathbf{27 \times 1.065^t = 40.5}$

Divide by 27. $\mathbf{1.065^t = \frac{40.5}{27} = 1.5}$

Take logarithms of both sides. $\mathbf{\log 1.065^t = \log 1.5}$

Use the law $\log a^n = n \log a$. $\mathbf{t \log 1.065 = \log 1.5}$

Divide both sides by log 1.065. $\mathbf{t = \frac{\log 1.5}{\log 1.065}}$

Enter as log (1.5) ÷ log (1.065) =

```
log(1.5)÷log(1.065)
                6.438529594
```

Write the answer. $\mathbf{t \approx 6.439}$

Compound interest problems where the time is the variable produce exponential equations.

Example 33

How long would it take \$310 to reach \$400 at 6.3% interest, compounding daily?

Solution

Change to a daily interest rate, and store the result in your calculator memory. $\mathbf{i = \frac{0.063}{365}}$

Enter as 0.063 ÷ 365 STO A.

```
0.063÷365→A
            0.00017260273
```

Write the compound interest formula. $\mathbf{A = P(1 + i)^n}$

Write the values. $\mathbf{A = 400,\ P = 310,\ i \approx 0.000172603}$

Substitute values into the formula. $\mathbf{400 = 310(1 + i)^n}$

Divide by 310 and store the answer in your calculator memory. $\mathbf{(1 + 0.0001726)^n \approx \frac{400}{310}}$

Enter as 400 ÷ 310 STO B.

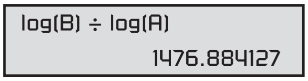
```
400÷310→B
              1.290322581
```

Put 1 + 0.000172603 into memory.

Enter as 1 + A STO A.

```
1+A→A
              1.000172603
```

Write what you now have. $\mathbf{1.00017\ldots^n = 1.2903\ldots}$

Take logarithms of both sides.	$\log 1.00017\ldots^{n} = \log 1.2903\ldots$
Use the law $\log a^{n} = n \log a$.	$n \log 1.00017\ldots = \log 1.2903\ldots$
Divide by $\log 1.00017\ldots$	$n = \dfrac{\log 1.2903\ldots}{\log 1.00017\ldots}$
Enter as log (B) ÷ log (A)	 log(B) ÷ log(A) 1476.884127
Change to years and days.	$1477 \div 365 = 4$ R 17
Write the answer.	**It would take 4 years and 17 days to reach \$400.**

Physical phenomena where the *rate* of change of a quantity can be expressed as a percentage of the quantity itself have exponential relationships with time because they are like compound interest. If the quantity is increasing, the growth factor is more than 1, but if the quantity is decreasing, the growth factor is less than 1.

Example 34

The population of rabbits released on a small island off the coast which previously had none increases by 40% each year. Only 10 rabbits were released. The island will be overrun with rabbits if the population reaches 2000. How long would this take?

Solution

Work out the growth factor.	**Growth factor** $= 1 + 40\% = 1.4$
Write the population n after 1 year.	$n(1) = 10 \times 1.4$
Write the population n after 2 years.	$n(2) = 10 \times 1.4 \times 1.4 = 10 \times 1.4^{2}$
Write the population n after 3 years.	$n(3) = 10 \times 1.4^{2} \times 1.4 = 10 \times 1.4^{3}$
Write the population n after 4 years.	$n(4) = 10 \times 1.4^{3} \times 1.4 = 10 \times 1.4^{4}$
Write the population n after t years.	$n(t) = 10 \times 1.4^{t}$
Substitute $n = 2000$.	$2000 = 10 \times 1.4^{t}$
Divide both sides by 10.	$1.4^{t} = 200$
Take logarithms of both sides.	$\log 1.4^{t} = \log 200$
Use the law $\log a^{n} = n \log a$.	$t \log 1.4 = \log 200$
Divide both sides by $\log 1.4$.	$t = \dfrac{\log 200}{\log 1.4}$
Calculate the answer.	$= 15.7466\ldots$
Write your answer in terms of the question.	**It would take about 16 years for the rabbits to overrun the island.**

Investigate: Macquarie Island Pest Eradication project

The Australian government is trying to eradicate introduced species on some islands. For example, the National Parks and Wildlife Service is eradicating rabbits, rats and mice on Macquarie Island off southern Tasmania. They have devastated the habitat and caused the extinction of at least one native bird (the Macquarie Island rail). Macquarie Island is home to the royal penguin and a breeding colony for burrowing seabirds including petrels and albatrosses. The reduction of vegetation and consumption of eggs by the introduced pests poses a serious threat to these birds.

Investigate this eradication project and suggest how mathematical modelling of breeding and populations can help.

Example 35

A red-hot garden fork (without its handle) at a temperature of 450°C is plunged into water at a temperature of 20°C to temper it. The iron cools rapidly, so that the difference between its temperature and that of the water decreases by 40% each minute. Model the temperature of the fork and find how long it takes until it can be handled (less than about 50°C).

Solution

Work out the temperature difference.	**Temperature difference $T(0)$** **$= 450° - 20° = 430°\text{C}$**
Write the new temperature difference as a percentage of the old difference.	**New difference = 60% of old difference**

Write the difference after 1 minute.	$T(1) = 60\%$ of $T(0)$ $= T(0) \times 0.6$
Write the difference after 2 minutes.	$T(2) = 60\%$ of $T(1)$ $= T(0) \times 0.6 \times 0.6$ $= T(0) \times 0.6^2$
Write the difference after 3 minutes.	$T(3) = 60\%$ of $T(2)$ $= T(0) \times 0.6^2 \times 0.6$ $= T(0) \times 0.6^3$
Write the difference after 4 minutes.	$T(4) = 60\%$ of $T(3)$ $= T(0) \times 0.6^3 \times 0.6$ $= T(0) \times 0.6^4$
Write the difference after m minutes.	$T(m) = T(0) \times 0.6^m$
What is the desired difference?	**Desired temperature difference** $= 50° - 20° = 30°$
Write the known function values.	$T(0) = 430,\ T(m) = 30$
Substitute into the model.	$30 = 430 \times 0.6^m$
Divide by 430.	$0.6^m = \frac{30}{430} = 0.06976$
Take logarithms of both sides.	$\log 0.6^m = \log 0.06976\ldots$
Use the law $\log a^n = n \log a$.	$m \log 0.6 = \log 0.06976\ldots$
Divide by log 0.6.	$m = \frac{\log 0.06976\ldots}{\log 0.6}$
Use your calculator to evaluate.	$= 5.2123\ldots$
Add the water temperature to $T(m)$ to model the temperature of the iron.	**Temperature of the iron after m minutes is given by $430 \times 0.6^m + 20$°C after m minutes in the water.**
Write the answer.	**The fork temperature can be modelled as $430 \times 0.6^m + 20$ °C and can be handled after about 5 minutes.**

You can use the kind of modelling in Examples **33**, **34** and **35** in most situations where you can work out a growth (or reduction) factor.

Important!

Exponential growth and decay models

The value of a quantity, Q, after time, t, if the growth factor is g, is given by $Q = Ag^t$, where A is the quantity at time $t = 0$. If the quantity is decreasing, the growth factor is less than 1.

Example 36

There are two groups of humpback whales in Australian waters. The east coast population was estimated to be about 3600 in 1999 and was increasing at a rate of approximately 10.9% a year. The population before whaling commenced was estimated to be 27 000. Whaling of humpbacks did not stop until 1963. Assume that the rate of growth has been the same from 1963 to the present and will continue until the population reaches about 20 000, when it will start to decrease.

a Make a mathematical model for the population from 1963 until it reaches 20 000.
b Predict when the population will reach 20 000.
c Estimate what the population of east coast humpbacks was in 1963.

Solution

a Work out the growth factor.	**Growth factor = 1 + 0.109 = 1.109**
You know about the 1999 population so choose that as the starting point.	**Let $y = 0$ in 1999.**
Write the population for $y = 0$.	**Population at $y = 0$ is $A = 3600$.**
Use the general model $Q = Ag^t$.	**Population = 3600×1.109^y, where y is the number of years after 1999.**
b Substitute population = 20 000.	$20\,000 = 3600 \times 1.109^y$
Divide by 3600.	$1.109^y = \dfrac{20\,000}{3600} = 5.5555\ldots$
Take logarithms of both sides.	$\log 1.109^y = \log 5.555\ldots$
Use the law $\log a^n = n \log a$.	$y \log 1.109 = \log 5.555\ldots$
Divide by log 1.109.	$y = \dfrac{\log 5.5555\ldots}{\log 1.109} = 16.5747\ldots$
Round off and add to 1999.	**The population will reach 20 000 around 2016.**
c Work out the value of y.	$y = 1963 - 1999 = {}^{-}36$
Substitute into the model.	**Population = $3600 \times 1.109^{-36} \approx 84.849\ldots$**
State the answer.	**There would have been about 85 humpback whales left in the east coast population in 1963.**

Exercise 11.4 Exponential equations

See Example 30

Understanding

Extra questions
Exercise 11.4
MAT10NAEQ00063

1 Solve each of the following equations.

a $6^y = 36\sqrt{6}$ **b** $4^p = 64\sqrt{4}$ **c** $3^q = 81$
d $5^b = 625$ **e** $10^z = 1000\sqrt{10}$ **f** $10^r = 100\sqrt{10}$
g $4^n = 256$ **h** $10^q = 10\ 000\sqrt{10}$ **i** $4^x = 256$
j $3^y = 729$

Fluency

See Example 31

2 Solve each of the following equations.

a $2^{3q-1} = 32$ **b** $10^{k-5} = 100\ 000$ **c** $3^{2z+5} = 81\sqrt{3}$
d $6^{2p-3} = 216$ **e** $2^{2c-3} = 32\sqrt{2}$ **f** $5^{x-2} = 25\sqrt{5}$
g $3^{2g-2} = 9$ **h** $6^{2u+2} = 36\sqrt{6}$ **i** $6^{3x-2} = 216$
j $5^{3c+5} = 25$

See Example 32

3 Solve each of the following equations.

a $18 \times 1.149^x = 23$ **b** $15 \times 0.795^w = 25$ **c** $9 \times 1.285^b = 73$
d $19 \times 1.173^u = 188$ **e** $20 \times 1.261^g = 57$ **f** $8 \times 0.536^h = 79$
g $20 \times 1.235^z = 118$ **h** $5 \times 1.447^t = 91$ **i** $6 \times 1.459^z = 184$
j $11 \times 1.134^b = 83$

See Example 33

4 **a** How long would it take \$500 at 5% interest compounding annually to amount to \$3000?
b How long would it take \$5000 at 4.8% interest compounding monthly to amount to \$15 000?
c How long would it take \$4600 at 7.2% interest compounding weekly to amount to \$6000?
d How long would it take \$2100 at 4.8% interest compounding daily to amount to \$5000?
e How long would it take \$4000 at 3.7% interest compounding monthly to amount to \$7600?

Problem solving

See Example 34

5 An algal bloom on a town water lake in the Murray-Darling basin covers an area of only 250 m^2. Fertiliser in run-off into the lake is causing the bloom to increase at the rate of 3.2% each day. If the bloom covers more than a third of the surface, the water will become toxic and will require expensive treatment with copper salts before it can be used for drinking. The lake covers an area of 4.56 square km. A majority of the council think they should do nothing to upset the local farmers as there is a chance that heavy rain will occur and wash out the lake. Use a mathematical model to find how long they have before they must act.

Worked solutions
Exercise 11.4
MAT10NAWS00068

⑥ Under favourable conditions, such as warm, moist spring weather in a grain crop, the population size of field mice can increase at the rate of about 25% a week. 200 mice/ha will cause crop losses of about 12%. About 20 mice/ha survive the winter in a particular crop and the conditions are favourable for breeding. The cost of baiting for mice is about the same as 5% of the amount the farmer can expect to make from his grain crop. Baiting does not kill all the mice, but will reduce the population by about 90%.

- **a** What mouse population size would cause a crop loss of about 5%?
- **b** Use a mathematical model to find how long would it take the mice to reach this level.
- **c** The crop will be harvested about 20 weeks after the end of winter. The farmer baits when the mouse population reaches the 5% damage level. Does the farmer need to do a second baiting?

See Example 35

7 A cup of tea or instant coffee cools by about 15% of the difference between its temperature and the room temperature each minute. Putting a typical amount of milk from the fridge into a cup will cool it by about 10°C. A cup of tea of coffee is drinkable once its temperature drops below 70°C, but most people will not drink the last bit if it is below 35°C. Air-conditioned offices typically have the temperature set at 25°C. For employees in air-conditioned offices, use a mathematical model to work out the following.

- **a** How long it takes a cup of tea or coffee without milk to cool to a drinkable temperature.
- **b** How long it takes for someone who puts their milk in first.
- **c** How long it takes for someone who puts their milk in after about a minute.
- **d** For each of the above cases, how long it is before their cuppa is too cold.

Worked solutions
Exercise 11.4
MAT10NAWS00068

⑧ Carbon 14 is a radioactive isotope of carbon that is produced high above the Earth as a result of cosmic rays from the Sun colliding with nitrogen in the upper atmosphere. As a result, about a trillionth of the carbon dioxide in the air has a carbon 14 atom. Since carbon dioxide is used by plants in photosynthesis, about a trillionth of the carbon in all living things is actually carbon 14. After they die, about 1.2% of the carbon 14 decays back to nitrogen each century. Examination of the carbon 14 in a particular piece of wood reveals that the amount has decreased to 80% of the original trillionth. How old is the piece of wood?

See Example 36

9 Kruger National Park is over a century old and is the largest reserve in Africa. Over the years there has been much controversy about the culling of elephants and it ceased in 1989. If an elephant population is not reduced by natural events such as long droughts or culling, the population increases by about 7% each year. The park can only sustain about 8000 elephants, but by 2004 there were about 11 500 elephants. Use a model without culling or natural disasters to estimate each of the following.

- **a** The number of elephants in 2008.
- **b** The number of elephants in 2010.
- **c** The number of elephants in 1989.

Reasoning

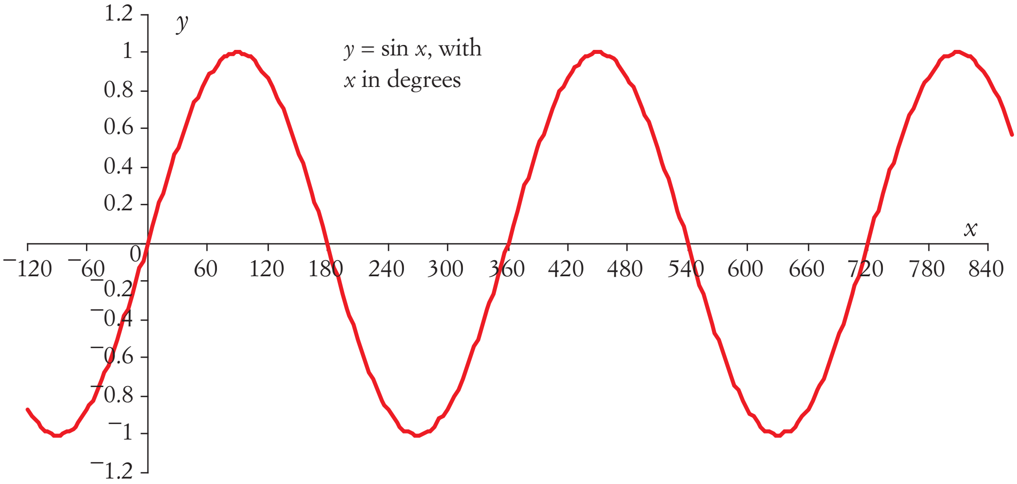
Worked solutions
Exercise 11.4
MAT10NAWS00068

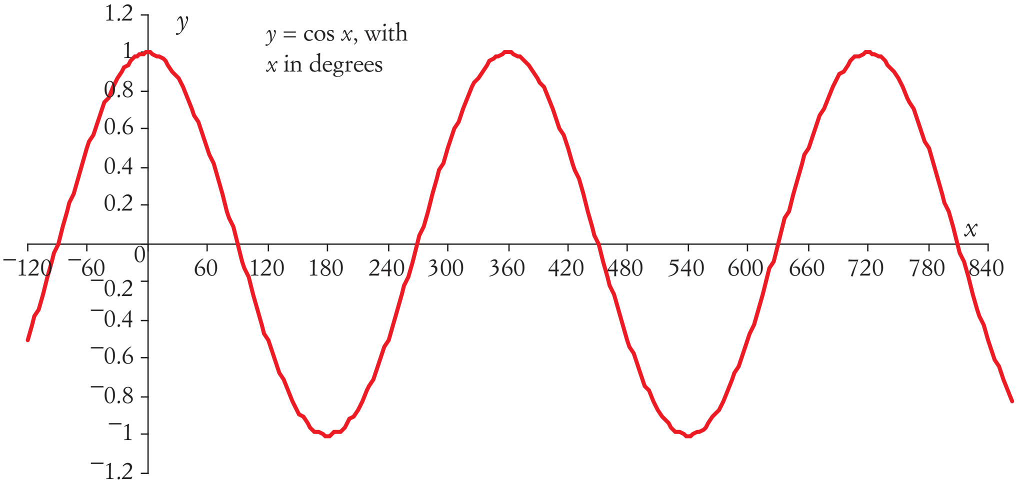
⑩ Derive a general formula for the solution of the equation $a^x = b$, where a and b are positive values.

11.5 Trigonometric equations

In Chapter **10**, you solved some very simple trigonometric equations. To find an angle using one of the trigonometric functions you solved something like $\cos x = 0.857$. This is actually a trigonometric equation with the solution $x \approx 31°$. For some cases, there is an exact answer like $\tan x = \sqrt{3}$.

You also saw that sine and cosine repeat the same values after the angle reaches 360° because the points on the unit circle are the same. The graphs of sine and cosine are drawn again below for extended values of the angle.





You will notice that sine and cosine have the same shape and vary between $^{-}1$ and 1. The graphs are shifted along the x-axis by 90° from each other. The shift of the graphs and the work you did in Chapters **5** and **10** shows you that $\cos x = \sin (x - 90°)$.

Functions or physical phenomena that repeat after a while are called **periodic**. The time between repetitions is called the **period**. Functions that look like the sine function are called **sinusoidal** functions. The heights of tides, the voltage in electric power, radio waves, TV waves, movement of a pendulum bob, sound waves and the motion of a guitar string can all be modelled using sinusoidal functions. In fact any periodic function can be written as a sum of sine functions using *Fourier analysis*.

Up until now, you have used the degree to measure angles. However, for most modelling using sin and cos, the **radian** is used instead.

Important!

Radian angle measure

A **radian** is the angle that is subtended at the centre of a circle by an arc of the circumference that is equal in length to the radius of the circle. The sign for radians is ^{c}, but it is not usually written. There are 2π radians in a revolution, so $2\pi^c = 360°$ and $\pi = 180°$.

If you write 'an angle of 0.9', this really means 0.9 radians. The main reason for using radians is that it simplifies many calculations and formulas in maths and science.

Example 37

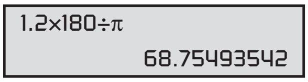
a Change 1.2 to degrees.

b Change 125° to radians.

Solution

a Multiply by the ratio $\frac{180}{\pi}$ to change. $\mathbf{1.2\ radians = 1.2 \times \frac{180}{\pi}}$

Enter as 1.2 × 180 ÷ π.

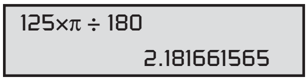


```
1.2×180÷π
          68.75493542
```

Write the answer. $\mathbf{1.2 \approx 68.8° \ (or\ 68°45')}$

b Multiply by the ratio $\frac{\pi}{180}$ to change. $\mathbf{125° = 125 \times \frac{\pi}{180}\ radians}$

Enter as 125 × π ÷ 180.



```
125×π ÷ 180
          2.181661565
```

Write the answer. $\mathbf{125° \approx 2.182}$

You need to be able to solve simple trigonometric equations to use sinusoidal models of phenomena. This includes finding all the solutions in a given domain. You need to know a little about the inverse trigonometric functions: arcsin x and arccos x.

Important!

Arcsine and arccosine

Arcsin x means the values of θ for which $\sin\theta = x$. It is also written as $\mathbf{\sin^{-1}x}$. The **principal value** of arcsin x is the value between $\frac{^-\pi}{2}$ and $\frac{\pi}{2}$ ($^-90°$ and $90°$) inclusive.

Arccos x means the values of θ for which $\cos\theta = x$. It is also written as $\mathbf{\cos^{-1}x}$. The principal value of arccos x is the value between 0 and π (0° and 180°) inclusive.
People often mean only the principal value when they write arcsin x or arccos x. When you use the trigonometric function $\sin^{-1}$ or $\cos^{-1}$ on your calculator, it gives only the principal value.

It is useful to remember that $90° = \frac{\pi}{2} \approx 1.57$, $180° = \pi \approx 3.14$, $270° = \frac{3\pi}{2} \approx 4.71$ and $360° = 2\pi \approx 6.28$. You should set your calculator to radians when solving most trigonometric equations.

Example 38

CAS TI-Nspire exercise
Equations
MAT10NATI00018

CAS ClassPad exercise
Equations
MAT10NACP00018

Solve each of the following for $^-2\pi \le x \le 2\pi$.

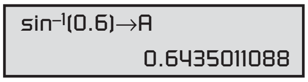
a $\sin x = 0.6$ **b** $\cos x = {}^-0.43$

Solution

a Write the problem.

$\mathbf{\sin x = 0.6}$

Enter as sin⁻¹ 0.6 STO A



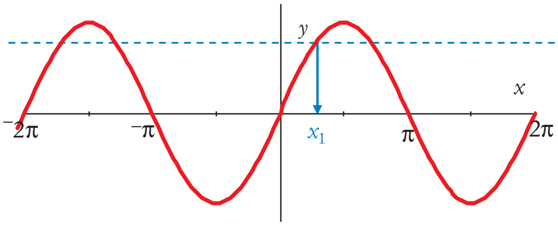
sin⁻¹(0.6)→A
0.6435011088

Write the principal value.

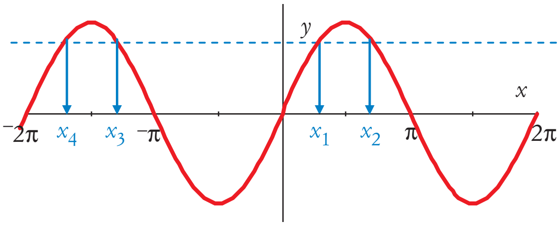
$\mathbf{x_1 = \sin^{-1}0.6 \approx 0.6435}$

Roughly sketch the sine function for $^-2\pi \le x \le 2\pi$ (about $^-6.3 \le x \le 6.3$).

Mark the principal value in its approximate position on the sketch and draw a straight line across through the curve.



Mark the other values of x, using the intersections of the straight line and the curve.

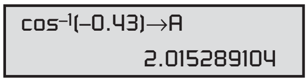
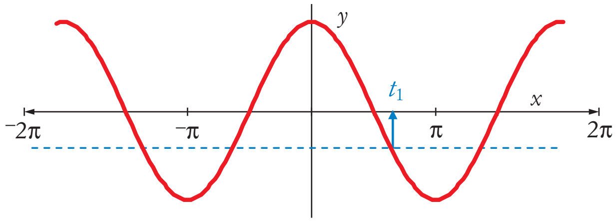
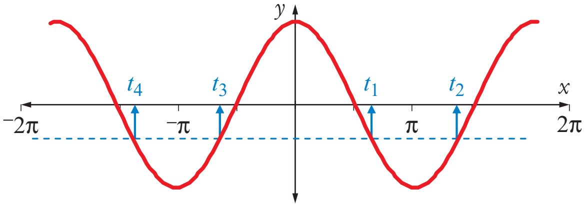


Show how to get the other values.

$\mathbf{x_2 = \pi - x_1,\ x_3 = -\pi - x_1,\ x_4 = {}^-2\pi + x_1}$

Enter as π − A

π−A
2.498091545

Complete all the values.	$x_2 \approx 2.4981$, $x_3 \approx {}^{-}3.7851$, $x_4 \approx {}^{-}5.6397$
Write the answers.	**The solutions are approximately $^{-}5.6397$, $^{-}3.7851$, 0.6435 and 2.4981.**
b Write the problem.	$\cos t = {}^{-}0.43$
Enter cos⁻¹ +/– 0.43 STO A	 cos⁻¹(–0.43)→A 2.015289104
Write the principal value.	$t_1 = \cos^{-1}({}^{-}0.43) \approx 2.0153$
Roughly sketch the cosine function for $^{-}6.3 \le x \le 6.3$ ($^{-}2\pi \le x \le 2\pi$). Mark the principal value in its approximate position on the sketch and draw a straight line across through the curve.	
Mark the other values of t, using the intersections of the straight line and the curve.	
Show how to get the other values.	$t_2 = 2\pi - t_1$, $t_3 = -t_1$, $t_4 = {}^{-}2\pi + t_1$
Enter as 2 × π – A.	 2×π–A 4.267896203
Complete all the values.	$t_2 \approx 4.2679$, $t_3 \approx {}^{-}2.0153$, $t_4 \approx {}^{-}4.2679$
Write the answers.	**The solutions are approximately $^{-}4.2679$, $^{-}2.0153$, 2.0153 and 4.2679.**

Example 39

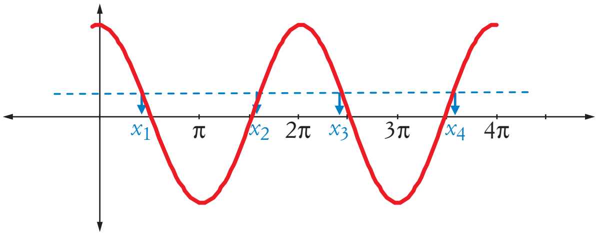
Solve the equation $4 \sin x = 1$ for the first 4 positive values of x.

Solution

Write the equation.	$4 \sin x = 1$
Divide by 4 to simplify.	$\sin x = 0.25$
Find the principal value and store it.	$x_1 = \sin^{-1} 0.25 \approx 0.2527$

Roughly sketch the sine function for $0 \leq x \leq 12.6$ $(0 \leq x \leq 4\pi)$.

Indicate the approximate positions of the solutions.



Show how to get the other values. $\boldsymbol{x_2 = 2\pi - x_1, x_3 = 2\pi + x_1, x_4 = 4\pi - x_1}$

Use the stored value on your calculator. $\boldsymbol{x_2 \approx 6.0305, x_3 \approx 6.5359, x_4 \approx 12.3137}$

Write the answers. **The first 4 positive values are approximately 0.25, 6.03, 6.54 and 12.31.**

In practical examples, you may find it easier to substitute a dummy variable into the trigonometric function to make it easier to solve.

Example 40

A long spring with a small lead ball on the end is compressed and released. The distance of the ball from the centre position is given by $d = 8 \cos 0.5t$ cm, where t is the time in seconds from its release. Negative values of d signify that the spring is stretched.

a Find the first 2 times that the spring is at a compression of 5 cm.

b Find the period of oscillation of the ball and spring.

Solution

a Write the function. $\boldsymbol{d = 8 \cos 0.5t}$

State the value of d. $\boldsymbol{d = 5}$

Substitute the value of d. $\boldsymbol{5 = 8 \cos 0.5t}$

Divide by 8. $\boldsymbol{\cos 0.5t = \frac{5}{8} = 0.625}$

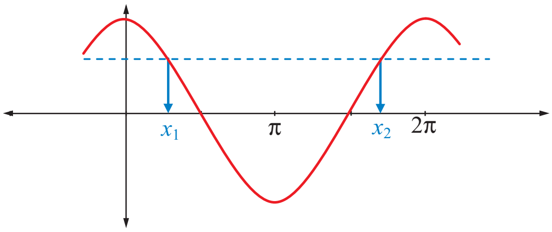
Write a simplified trigonometric equation. $\boldsymbol{\cos x = 0.625, \text{ where } x = 0.5t}$

Find the principal value of x. $\boldsymbol{x_1 = \cos^{-1} 0.625}$

Use your calculator and store the value. $\boldsymbol{\approx 0.8957}$

Roughly sketch the cosine function for $0 \leq x \leq 6.3$ $(0 \leq x \leq 2\pi)$, since you are only interested in positive times.

Show the approximate positions of the principal value and the next value on the sketch.



Find the other value of x.	$x_2 = 2\pi - x_1$
Use the stored value on your calculator.	≈ 5.3875
Now write equations using $x = 0.5t$.	$x_1 = 0.5t_1 \approx 0.8957$ and $x_2 = 0.5t_2 \approx 5.3875$
Find the values of t.	$t_1 \approx 1.791$ and $t_2 \approx 10.775$
Write the answer.	**The spring is at a compression of 5 cm about 1.8 seconds after release, and again about 10.8 seconds after release.**
b Write the value of x for the end of the first oscillation.	$x = 2\pi$
Find the value of t for that value of x.	$0.5t = 2\pi$
Solve for t.	$t = 4\pi \approx 12.566$
Write the answer.	**The period of oscillation is about 12.6 seconds.**

Exercise 11.5 Trigonometric equations

Understanding

1 Change each of the following to degrees.

See Example 37

Extra questions
Exercise 11.5
MAT10NAEQ00064

a 1.5 **b** 2.1 **c** 1 **d** 1.9
e 4.9 **f** 1.4 **g** $^{-}0.1$ **h** 5.7
i 0.7 **j** $^{-}4.7$ **k** 4.5 **l** $^{-}1.4$
m $^{-}0.5$ **n** $^{-}4.4$ **o** 5.8

2 Change each of the following to radians.

a 346° **b** 59° **c** $^{-}74$° **d** 240°
e 77° **f** $^{-}229$° **g** 221° **h** 345°
i $^{-}83$° **j** 288° **k** 356° **l** $^{-}257$°
m 241° **n** 313° **o** 89°

3 Solve each of the following equations for solutions between 0 and 2π inclusive.

See Example 38

a $\sin q = 0.008$ **b** $\cos r = {}^{-}0.409$ **c** $\sin k = 0.212$ **d** $\sin b = {}^{-}0.485$
e $\cos e = {}^{-}0.822$ **f** $\sin q = {}^{-}0.741$ **g** $\sin u = 0.179$ **h** $\cos m = 0.341$
i $\sin c = {}^{-}0.182$ **j** $\sin m = {}^{-}0.551$

4 Solve each of the following equations for solutions between $^{-}2\pi$ and 2π inclusive.

a $\sin n = {}^{-}0.824$ **b** $\sin m = {}^{-}0.542$ **c** $\cos z = {}^{-}0.543$ **d** $\cos y = 0.495$
e $\cos k = {}^{-}0.324$ **f** $\cos x = {}^{-}0.318$ **g** $\sin c = 0.465$ **h** $\sin k = {}^{-}0.304$
i $\sin k = {}^{-}0.118$ **j** $\cos r = 0.758$

5 Solve each of the following equations for solutions between 0 and 4π inclusive.

a $\sin b = 0.146$ **b** $\sin t = 0.046$ **c** $\sin k = 0.579$ **d** $\cos v = {}^{-}0.047$
e $\cos q = {}^{-}0.617$ **f** $\sin m = 0.188$ **g** $\cos a = 0.485$ **h** $\sin h = {}^{-}0.752$
i $\cos h = {}^{-}0.62$ **j** $\cos t = 0.273$

6 Solve each of the following equations for solutions between $^-\pi$ and π inclusive.

- **a** $\sin k = {}^-0.628$
- **b** $\cos q = 0.185$
- **c** $\cos w = {}^-0.804$
- **d** $\cos u = {}^-0.779$
- **e** $\cos e = 0.653$
- **f** $\sin p = {}^-0.692$
- **g** $\sin r = 0.713$
- **h** $\cos w = 0.011$
- **i** $\sin q = 0.891$
- **j** $\sin w = 0.474$

Fluency

See Example 39

7 Find solutions between $^-2\pi$ and 2π inclusive for each of the following equations.

- **a** $4.7 \cos v = {}^-3.3323$
- **b** $0.6 \sin a = 0.1092$
- **c** $5.4 \sin n = 3.2508$
- **d** $7.9 \cos u = 7.7973$
- **e** $4.8 \sin u = {}^-0.2208$
- **f** $6.6 \sin e = 4.9962$
- **g** $0.8 \cos h = {}^-0.7592$
- **h** $1.8 \sin m = {}^-0.0684$
- **i** $9 \sin w = 2.043$
- **j** $6.7 \sin q = {}^-5.8223$
- **k** $5 \cos g = 4.745$
- **l** $6.4 \sin y = {}^-4.9792$
- **m** $0.1 \cos m = {}^-0.0648$
- **n** $7.6 \sin n = {}^-6.2016$
- **o** $4 \sin a = {}^-1.324$
- **p** $1.2 \sin m = 0.3432$
- **q** $8.1 \cos h = 3.5802$
- **r** $1.2 \cos c = {}^-0.132$
- **s** $0.8 \sin k = 0.6312$
- **t** $8.5 \cos x = 1.3855$
- **u** $2 \cos w = 0.682$
- **v** $2.9 \sin w = {}^-2.117$
- **w** $6 \sin a = 4.386$
- **x** $5.4 \cos b = {}^-2.3436$
- **y** $8.6 \sin g = {}^-0.7826$
- **z** $2.7 \cos w = 1.6362$

Worked solutions
Exercise 11.5
MAT10NAWS00069

8 Find the first four positive solutions for each of the following equations.

- **a** $0.8 \sin 6e = {}^-0.0632$
- **b** $7.5 \cos 6y = {}^-6.135$
- **c** $3.4 \cos 17a = 0.9316$
- **d** $7 \sin 8v = {}^-6.027$
- **e** $1.9 \cos 9x = {}^-0.1634$
- **f** $4.2 \sin 6n = {}^-2.6544$
- **g** $2.4 \sin 18p = {}^-0.4584$
- **h** $3.8 \sin 11m = {}^-1.7936$
- **i** $3.5 \sin 7m = {}^-1.8375$
- **j** $1.4 \cos 17b = 0.2856$
- **k** $5.9 \sin 3v = 4.9737$
- **l** $1.7 \sin 13p = {}^-1.428$
- **m** $6.1 \sin 12h = {}^-2.0252$
- **n** $4.6 \cos 0.25v = {}^-4.1124$
- **o** $0.8 \cos 0.24u = 0.5368$
- **p** $4.5 \sin 0.24q = 3.411$
- **q** $2.5 \sin 0.47t = {}^-0.3075$
- **r** $7 \cos 0.99a = 2.912$
- **s** $0.3 \sin 0.45h = 0.1653$
- **t** $3.9 \sin 0.48u = {}^-1.521$
- **u** $4.5 \cos 0.37p = {}^-2.8485$
- **v** $5.1 \sin 0.99c = 0.5661$
- **w** $1.9 \cos 0.67m = {}^-0.6194$
- **x** $0.7 \cos 0.27q = {}^-0.6272$
- **y** $4.2 \sin 0.86p = 2.6502$
- **z** $8.9 \cos 0.34g = {}^-6.4436$

Problem solving

See Example 40

9 A pendulum swings so that the distance of the bob from the centre of the swing is given by $d = 10 \sin 2.09t$ cm, where t is the time in seconds. Negative distances are considered to be towards the left and positive ones to the right.

- **a** Find the first 2 times when the pendulum bob is 5 cm to the left of the centre.
- **b** Find the first 4 times when the pendulum bob is 4 cm from the centre.
- **c** How long does the pendulum take for one complete swing (there and back again)?

Worked solutions
Exercise 11.5
MAT10NAWS00069

10 The maximum temperature in Marble Bar closely follows the rule $T = 34 + 7.5 \cos 0.52m$ °C, where m is the number of months after January (i.e., January is 0, February is 1, and so on).

- **a** What would you expect the maximum temperature to be in December?
- **b** What would you expect the maximum temperature to be in June?
- **c** When would you expect a maximum of 37.8°C?
- **d** When would you expect a maximum of 30°C?

11 The minimum temperatures in Mt Isa closely follow the rule $T = 18 + 7 \cos 0.52m$ °C, where m is the number of months after December (i.e. January is 1, February is 2, etc.).

a When would you expect the minimum to be 12°C?

b When would you expect the minimum to be 24°C?

12 The monthly rainfall in Strahan closely follows the rule $r = 118 + 53 \sin 0.52m$ mm, where m is the number of months after April (i.e. May is 1, June is 2, and so on).

a When would you expect about 170 mm of rain in the month?

b When would you expect about 145 mm of rain?

13 The D string on a guitar moves up and down so that the distance of the centre of the string from the rest position is given by $d = 4 \sin 1841t$ mm, where t is the time in seconds from when it is first strummed.

a Find the first 4 times that the string is 3 mm up from the rest position.

b Find the first 4 times that the string is 2 mm down from the rest position.

c Find how many times the string vibrates in 1 second.

Worked solutions
Exercise 11.5
MAT10NAWS00069

Reasoning

14 On a particular day there is a high tide at Portland of 3.69 m at 0724 h and a low tide of 1.41 m at 1351 h. Use the model $h = a + A \cos kt$, to find when the tidal level is 2 m. Carefully explain your reasoning.

Chapter 11 summary

Quiz
Equations
MAT10NAQZ00011

- An **equation** is a mathematical sentence with an equals sign and expressions on each side.
- A **linear** equation produces a straight line when the expression is plotted for different values of the variable.
- A **solution** of an algebraic equation is a value of the variable that makes the equation true. A solution **satisfies** the equation.
- **Inverse operations** are opposite operations such as addition/subtraction and multiplication/division.
- To solve a linear equation isolate the variable by doing inverse operations in the reverse order to the normal order of operations. Do the same thing to each side of the equation.
- A **linear inequality** is like a linear equation, but instead of having an equals sign, it has one of the signs $>$, $<$, $\geq$ or $\leq$. It is also called an **inequation**.
- For any inequality $x > y$
  - $x + a > y + a$ for any number a
  - $x - a > y - a$ for any number a
  - $ax > ay$ for any *positive* number a
  - $ax < ay$ for any *negative* number a
  - $\frac{x}{a} > \frac{y}{a}$ for any *positive* number a
  - $\frac{x}{a} < \frac{y}{a}$ for any *negative* number a

  The same laws hold for $\geq$, $<$ and $\leq$.
- Equations considered *at the same time* make an equation system called **simultaneous equations**.
- A **simultaneous solution** is a solution that satisfies all the equations at the same time.
- The simultaneous solution of two equations in two unknowns is the intersection of their lines on a graph.
- To solve simultaneous equations by the **substitution method** you use one equation to express one variable in terms of the other. Then you substitute this into the other equation to make a linear equation in one variable. Solve this and then find the other variable.
- To solve simultaneous equations by the **elimination method** you make the coefficients of one variable opposites by multiplying the equations by suitable constants. Then add the equations to make an equation with only one variable. Solve this and then find the other variable.
- A **quadratic equation** is one that can be expressed in the form $ax^2 + bx + c = 0$, where x is the variable and a, b and c are constants.
- You can solve simultaneous equations involving a quadratic by graphing or by using the linear equation for substitution.
- You can use **completing the square** to solve a quadratic equation by:
  1. dividing (if necessary) to make the coefficient of x^2 equal to 1
  2. moving the constant to the RHS
  3. adding half the square of the coefficient of x to both sides
  4. factorising the LHS as a perfect square

Chapter 11 summary

- You can solve quadratic equations by factorisation using the **zero product rule:** if $ab = 0$ then $a = 0$ or $b = 0$.
- The solutions to a quadratic equation $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, provided $a \neq 0$.
- An **exponential equation** is one where the unknown is involved in the exponent of a power. Exponential equations are usually solved by converting the exponent into a coefficient using logarithms.
- You can often model exponential **growth** or **decay** as $Q = Ag^t$, where Q is the quantity after time t, A is the initial quantity at time $t = 0$ and g is the growth factor. For decreasing quantities the growth factor is less than 1.
- **Periodic** functions or phenomena repeat the same values after a time called the **period**.
- Functions that look like the sine function are called **sinusoidal** functions.
- A **radian** is the angle subtended at the centre of a circle by an arc of the circumference that is equal in length to the radius of the circle. The radian sign ^{c} is usually omitted. $\pi = 180°$.
- **Arcsine** is the inverse of the sine function. If $\sin\theta = x$ then $\arcsin x = \theta$. The **principal value** of arcsin is the value between $^-\frac{\pi}{2}$ and $\frac{\pi}{2}$ ($^-90°$ and $90°$) inclusive.
- **Arccosine** is the inverse of the cosine function. If $\cos\theta = x$ then $\arccos x = \theta$. The **principal value** of arccos is the value between 0 and π ($0°$ and $180°$) inclusive.
- Arcsin and arccos are often written as $\mathbf{sin}^{-1}$ and $\mathbf{cos}^{-1}$ respectively, and some people restrict them to the principal values.

Chapter 11 review

Understanding

1 Solve each of the following equations.

a $m + 12 = 7$ **b** $4g = {}^{-}24$ **c** $7 - p = 18$ **d** $\frac{36}{d} = {}^{-}9$

2 Solve the equations $y = 2x - 3$ and $y = 9 - x$ by graphing.

See Example 8

3 For each of the following, solve the inequality and show your solution on a number line.

a ${}^{-}7y \geq 9$ **b** ${}^{-}q + 7 \leq 0$ **c** $\frac{q}{4} < 9$ **d** $n - 3 > {}^{-}7$

See Example 19

4 Solve each of the following equations.

a $y^2 = 9$ **b** $5t^2 = 80$ **c** $25g^2 = 16$

See Example 20

5 Solve the equation $(v + 7)^2 = 144$

See Example 24

6 Solve each of the following equations.

a $x(x + 5) = 0$ **b** $(a - 4)(a + 7) = 0$ **c** $(3m + 2)(2m - 5) = 0$

See Example 30

7 Solve each of the following equations.

a $2^k = 64$ **b** $3^z = 9\sqrt{3}$

See Example 37

8 Change 43° to radians.

See Example 38

9 Find solutions between ${}^{-}2\pi$ and 2π for each of the following equations.

a $\sin x = {}^{-}0.635$ **b** $\cos x = 0.594$

Fluency

See Example 1

10 Solve each of the following equations.

a $\frac{z}{7} + 5 = 9$ **b** $8t - 6 = {}^{-}54$

See Example 13

11 Find the solution of the equations $y + 2x = 10$ and $2y - 3x + 8 = 0$ by graphing.

See Examples 2, 3

12 Find the roots of each of the following equations.

a $5b - 3 = 2b - 12$ **b** $\frac{2c - 3}{5} = 3$ **c** $24 = 18 + 6(5 - b)$

See Example 4

13 Solve each of the following equations.

a $\frac{k}{6} + \frac{k}{4} = \frac{1}{2}$ **b** $\frac{5c}{2} - \frac{c}{9} = \frac{43}{3}$ **c** $\frac{2g + 3}{4} - \frac{g - 5}{3} = 2\frac{5}{6}$

See Example 9

14 For each of the following, solve the inequality and show your solution on a number line.

a ${}^{-}5x + 2 < {}^{-}5$ **b** $6h + 4 \leq 3$

See Example 10

15 For each of the following, solve the inequality and show your solution on a number line.

a $7x - 7 > {}^{-}43 - 2x$ **b** $3(5z - 2) - 6 \leq 4(4z - 3) + 4$

c $\frac{5x - 3}{5} - \frac{6x - 7}{4} < \frac{1}{10}$

See Example 15

16 Find the solution of the simultaneous equations $a + 7b = 19$ and $2a - 3b = {}^{-}13$ by substitution.

See Example 16

17 Find the solution of the simultaneous equations $7u + 4w = 0$ and $9u + 5w = {}^{-}1$ by elimination.

18 Solve each of the following equations by completing the square. *See* Examples 21, 22

a $y^2 + 10y + 21$ **b** $a^2 + 8a - 20$ **c** $15b^2 - 13b - 6$

19 Solve each of the following equations by factorisation. *See* Example 25

a $x^2 - 5x = 0$ **b** $w^2 - 13w + 42 = 0$ **c** $y^2 - 7y - 18 = 0$

d $c^2 + 6c = 16$ **e** $28h^2 - 15h - 25 = 0$ **f** $28g^2 = 9g + 4$

20 Solve the equation $4m^2 + 7m - 2 = 0$ using the quadratic formula. *See* Example 26

21 Find the simultaneous solution of $y = 2x + 4$ and $y = x^2 - 4$ by graphing. *See* Example 14

22 Solve $2x^2 + 7x + 1 = 0$ using the quadratic formula and express your answers both in surd and approximate form.

23 Solve $\frac{3}{2a} + \frac{1}{a} = 5$

24 Solve the following pairs of simultaneous equations. *See* Example 17

a
$$\frac{a}{6} - \frac{b}{2} = 1$$
$$\frac{6a}{5} + \frac{3b}{10} = 3$$

b
$$\frac{x+2y}{3} + \frac{x-3y}{2} = 5$$
$$\frac{2x}{5} - \frac{x-3y}{4} = 1$$

25 Solve $e^2 - 12e + 24 = 0$ by completing the square. *See* Example 23

26 Solve the simultaneous equations $y = x^2 + 4x - 14$ and $y = 2x + 1$ algebraically. *See* Example 29

27 Solve each of the following equations. *See* Examples 31, 32

a $2^{4r-5} = 8\sqrt{2}$ **b** $20 \times 0.935^p = 52$ **c** $12 \times 1.225^c = 114$

28 How long would it take $1200 at 6.6% interest compounding monthly to amount to $4300? *See* Example 33

29 Find solutions between $^-2\pi$ and 2π for the equation $6.2 \sin t = 5.1$. *See* Example 39

30 Find the first four positive solutions of the equation $8 \cos 9x = {}^-3$.

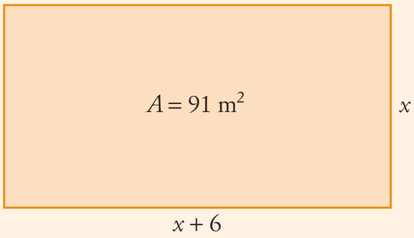
Problem solving

31 A rectangular garden bed is 0.5 m longer than it is wide. Find its dimensions if its perimeter is 9 m. *See* Example 6

32 Liam is twice as old as his son. Together their ages total 78. Write an equation and find their ages. *See* Example 7

See Example 28

33 A rectangle has a length that is 6 m longer than its width. Find its dimensions if its area is 91 m^2.



34 A woman is three times as old as her daughter, but in 10 years she will only be twice as old as her daughter. Write simultaneous equations and find their present ages.

35 Solve the following simultaneous equations:

$$\frac{1}{d} + \frac{3}{g} = 10 \text{ and } \frac{2}{d} + \frac{4}{g} = 16$$

36 Solve $3(x - 5)^2 - 4(x - 5) - 7 = 0$.

See Example 35

37 Feral deer are an emerging problem in Australia. The Sambar species has spread from Western Port and the ranges east of Melbourne through the alpine forests of north-east Victoria, Gippsland and Southern New South Wales. The population was thought to number about 40 000 in 2011 and is increasing at the rate of 10% per year. If not culled, when would their population reach 100 000?

38 Potassium-argon dating is used to work out the age of rocks created by lava flows. Every 1250 million years, half of the potassium isotope ^{40}K changes to the argon isotope ^{40}Ar which is trapped in the minerals. A geologist sent a sample to the lab and was told that 72% of the ^{40}K had changed to ^{40}Ar.

See Examples 35, 36

a What percentage of the potassium-40 changes in 100 million years?

b How old is the rock sample?

39 A bungee rope is tested by putting a large ball of lead on it and letting it drop. From the time 4 seconds after being dropped, the lead ball moves so that its distance above the ground is given by $4 + 1.5 \sin 0.7t$. Find the first 3 times that the ball is 3 m above the ground.

See Example 40

Reasoning

40 A lead sinker weight thrown into water sinks so that its depth is given by $d = 3t^2 - 6t$. Find the time that it takes to sink to a depth of 8 metres and explain your reasoning.

See Example 26

41 Complete the square for the equation $x^2 + 4x + 9 = 0$ and hence show that there are no real answers on the number line.

42 If $x > y$ and $a < 0$, prove that $\frac{x}{a} < \frac{y}{a}$.

See Example 12

43 If a population of n is increasing at p% per year, work out a formula for the population after y years.

Answers

Chapter 1

Exercise 1.1

1 **a** 512 **b** $\frac{1}{1024}$ **c** 1
d $\frac{1}{49}$ **e** $\frac{1}{13}$ **f** $\frac{1}{343}$
g $\frac{1}{144}$ **h** 1 **i** $\frac{1}{625}$

2 **a** $4^3 \times 5^3$ **b** $6^7 \times 4^7$ **c** $7^4 \times 11^4$
d $6^8 \times 7^8$ **e** 1 **f** $2^{12} \times 7^{12}$

3 **a** 5^{12} **b** 7^{10} **c** 3^{28}
d 11^{48} **e** 1 **f** 5^{25}

4 **a** 5 **b** 13 **c** 2
d 6 **e** 3 **f** 4
g 10 **h** 17 **i** 30

5 **a** 0.0008 **b** 7.616 **c** 2.981
d 70.728 **e** 5.453 **f** 6.364
g 0.149 **h** 0.002 **i** 1.0098

6 **a** 18.119 **b** 21.031 **c** 9
d 22.499 **e** 2.398 **f** 204.705

7 **a** 4^{10} **b** 7^6 **c** 15^6
d 6^{19} **e** 10^{26} **f** 9^{16}
g 14^{16} **h** 3^{17} **i** 25^{20}

8 **a** 8^2 **b** 3^4 **c** 7^{15}
d 11^{11} **e** 1 **f** 15
g 14^6 **h** 17^2 **i** 12^4

9 **a** $7^4 \times 5^4$ **b** $6^6 \times 3^6$ **c** $9^4 \times 11^4$
d $6^7 \times 8^7$ **e** $12^8 \times 13^8$ **f** $2^{12} \times 7^{12}$

10 **a** 7^{35} **b** $\frac{1}{6^4}$ **c** 1
d $\frac{1}{4^{45}}$ **e** 8^7 **f** $\frac{5^{18}}{6^9}$
g $\frac{7^5}{8^{20}}$ **h** $\frac{11^6}{7^3}$ **i** 8^5

11 **a** 5 **b** 5 **c** 7
d 3 **e** 8 **f** 2
g 6 **h** 3 **i** 5

12 **a** 7.07 **b** 6.18 **c** 3.04 **d** 3.55
e 9.75 **f** 1.12 **g** 12.13 **h** 7.22
i 4 **j** 13.13 **k** 2.40 **l** 225.70

13 **a** $\frac{1}{a^2}$ **b** $\frac{1}{4^3}$ **c** $\frac{1}{g^7}$ **d** $\frac{1}{5^4}$
e $\frac{1}{y^6}$ **f** $\frac{1}{c^4d^8}$ **g** $\frac{4}{h^4}$ **h** $\frac{1}{256h^4}$
i $\frac{2}{k}$ **j** $\frac{7}{p^4}$ **k** $\frac{8}{q^6}$ **l** $\frac{v^2}{4}$

14 **a** y^{-4} **b** 4^{-3} **c** h^{-9}
d $k^{-6}t^4$ **e** $p^{-6}y^2$ **f** $4i^{-6}j^8$
g $3m^{-5}y^{-2}$ **h** $2m^{-1}z^{-4}$ **i** x^8
j $2x^4y^4$ **k** $3bk^{-3}$ **l** $4r^{-9}w^6$

15 **a** $\sqrt{6}$ **b** $\sqrt[4]{z}$
c $\sqrt[3]{216}$ **d** $\sqrt[5]{y^3}$ or $(\sqrt[5]{y})^3$
e $\sqrt{49^3}$ or $(\sqrt{49})^3$ **f** $\sqrt[3]{v^7}$ or $(\sqrt[3]{v})^7$
g $\sqrt[3]{8^7}$ or $(\sqrt[3]{8})^7$ **h** $\sqrt[4]{16^5}$ or $(\sqrt[4]{16})^5$
i $\sqrt{m^5}$ or $(\sqrt{m})^5$ **j** $\sqrt[4]{u^{-1}}$ or $(\sqrt[4]{u})^{-1}$ or $\sqrt[4]{\frac{1}{u}}$ or $\frac{1}{\sqrt[4]{u}}$
k $\sqrt{\left(\frac{1}{9}\right)^{-1}}$ or $\left(\sqrt{\frac{1}{9}}\right)^{-1}$ or $\sqrt{9}$
l $\sqrt[5]{p^{-3}}$ or $(\sqrt[5]{p})^{-3}$ or $\sqrt[5]{\frac{1}{p^3}}$ or $\frac{1}{\sqrt[5]{p^3}}$ or $\frac{1}{(\sqrt[5]{p})^3}$

16 **a** $k^{\frac{1}{2}}$ **b** $3^{\frac{1}{4}}$ **c** $5^{\frac{1}{2}}$ **d** $4^{\frac{1}{5}}$
e $w^{\frac{1}{3}}$ **f** $c^{\frac{3}{2}}$ **g** $64^{\frac{4}{3}}$ **h** $e^{\frac{3}{4}}$
i $28^{\frac{4}{9}}$ **j** $q^{\frac{6}{5}}$ **k** $8^{\frac{5}{2}}$ **l** $16^{\frac{7}{6}}$

17 **a** 8 **b** 1 **c** 2 **d** 1
e 6 **f** 9 **g** 1 **h** 9
i 343 **j** 216 **k** 16 **l** 64
m $\frac{1}{4}$ **n** $\frac{16}{9}$ **o** 2 **p** 1
q $\frac{64}{27}$ **r** $\frac{1}{36}$ **s** 125 **t** $\frac{2}{5}$
u $\frac{27}{8}$ **v** 4 **w** 100 000 **x** $\frac{1}{10\,000}$

18 **a** $\frac{4^6}{11^6}$ **b** $\frac{7^8}{3^4}$ **c** $\frac{2^{56}}{5^6}$
d $\frac{2^3}{5^{10}}$ **e** $3^3 \times 5^{11}$ **f** $\frac{7^8}{5^{22}}$
g $\frac{1}{2^8 \times 7}$ **h** $7^8 \times 2^6$ **i** 1

19 **a** $\frac{2^4}{3}$ **b** $\frac{3^3}{7^4}$ **c** $\frac{3}{2^2}$
d $\frac{5^{11}}{2^4}$ **e** $3^6 \times 5^2$ **f** $\frac{5^3}{2^8}$

20 **a** $x^{12}y^9$ **b** $\frac{x^4}{y^{20}}$ **c** $x^{36}y^3$
d 1 **e** $\frac{1}{a^7}$ **f** c^8
g $\frac{n^3}{3m}$ **h** $\frac{q^4}{5^2}$ **i** $\frac{2^4rt^2}{s^2}$

21 **a** $\frac{1}{r^2}$ **b** $\frac{12}{y^9}$ **c** $3p^4$
d $\frac{2}{j^6}$ **e** $\frac{1}{6m^{13}}$ **f** $\frac{1}{4k^{14}}$
g $\frac{15g}{h^2}$ **h** $5k^{18}m^{14}$ **i** $16v^{11}$
j $\frac{4q^{17}}{p^{13}}$ **k** $\frac{4e^{12}g^7}{3}$ **l** $\frac{1}{2}c^3n^{19}$

22 **a** $a^{4\frac{1}{2}}$ **b** $b^{3\frac{1}{2}}$ **c** $\frac{6}{n^{\frac{1}{2}}}$
d $12g^{6\frac{1}{3}}$ **e** $12w^{3\frac{2}{3}}p^{2\frac{1}{2}}$ **f** $10m^{3\frac{1}{2}}n^{2\frac{1}{3}}$
g $\frac{21u^3}{s^{3\frac{1}{2}}}$ **h** $\frac{24p^{1\frac{1}{2}}}{q^{\frac{5}{12}}}$

23 **a** 2^4 **b** 5^{k+1} **c** 3×2^3
d 2^m3^{2n+3m} **e** $3^{2i+2} \times 7^3$ **f** 2^3

Exercise 1.2

1 **a** 1 **b** 3 **c** 0
d $^{-}1$ **e** $\frac{1}{2}$ **f** $^{-}3$
g $\frac{1}{3}$ **h** $1\frac{1}{2}$ **i** $2\frac{1}{2}$

2 **a** 1.7634, $10^{1.7634} \approx 58$
b 2.2041, $10^{2.2041} \approx 160$
c 1.2041, $10^{1.2041} \approx 16$

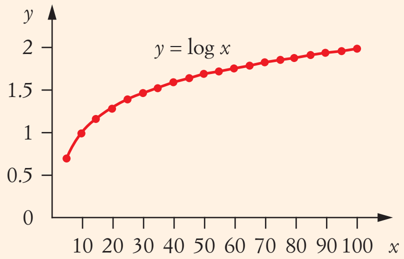
d $^{-}1.3010$, $10^{-1.3010} \approx 0.05$
e 1.1761, $10^{1.1761} \approx 15$
f 2.9494, $10^{2.9494} \approx 890$
g 1.8261, $10^{1.8261} \approx 67$
h 1.7993, $10^{1.7993} \approx 63$
i 1.7959, $10^{1.7959} \approx 62.5$

3 **a** 1.7559 **b** 3.7559 **c** $^{-}1.2441$
4 **a** 1.4533 **b** 3.4533 **c** $^{-}0.5467$
5 **a** 4.9883 **b** $^{-}0.0117$ **c** $^{-}3.0117$
6 **a** 2.0899 **b** 4.0899 **c** $^{-}0.9101$
7 **a** 4 **b** 3 **c** 5
d 3 **e** 3 **f** 4
g 10 **h** 1.5 **i** 1.5
8 **a** 1.893 **b** 0.528 **c** 4.644
d 16.952 **e** 5.422
9 **a** 4 **b** $^{-}2$ **c** 1
d 3.322 **e** 2.5

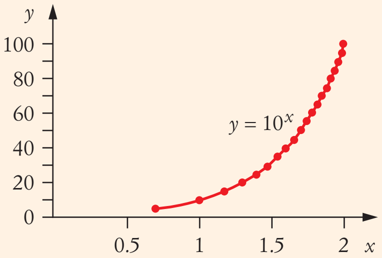
10 **a**

x	5	10	15	20	25
log *x*	0.6990	1	1.1761	1.3010	1.3979
x	30	35	40	45	50
log *x*	1.4771	1.5441	1.6021	1.6532	1.6990
x	55	60	65	70	75
log *x*	1.7404	1.7782	1.8129	1.8451	1.8751
x	80	85	90	95	100
log *x*	1.9031	1.9294	1.9542	1.9777	2

b



c

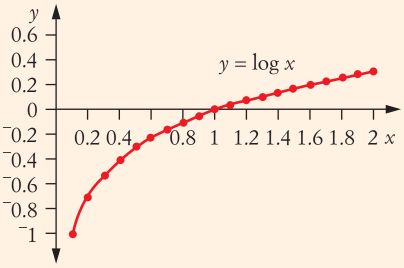


d At the same scale, the graphs mirror images in the line $y = x$.

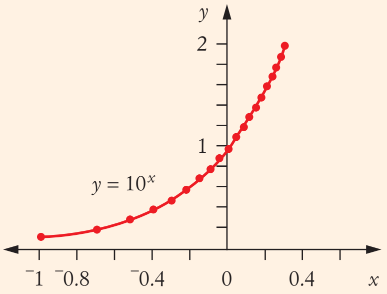
11 **a**

x	0.1	0.2	0.3	0.4	0.5
log *x*	$^{-}1$	$^{-}0.6990$	$^{-}0.5229$	$^{-}0.3979$	$^{-}0.3010$
x	0.6	0.7	0.8	0.9	1.0
log *x*	$^{-}0.2218$	$^{-}0.1549$	$^{-}0.0969$	$^{-}0.0458$	0
x	1.1	1.2	1.3	1.4	1.5
log *x*	0.0414	0.0792	0.1139	0.1461	0.1761
x	1.6	1.7	1.8	1.9	2.0
log *x*	0.2041	0.2304	0.2553	0.2788	0.3010

b



c

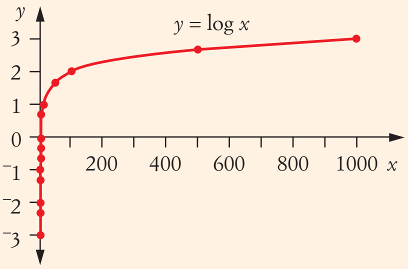


d At the same scale, the graphs mirror images in the line $y = x$.

12 **a**

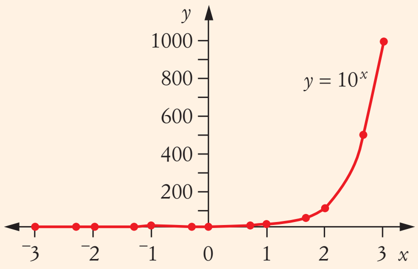
x	0.001	0.005	0.01	0.05	0.1
log *x*	$^{-}3$	$^{-}2.301$	$^{-}2$	$^{-}1.301$	$^{-}1$
x	0.5	1	1	10	50
log *x*	$^{-}0.301$	0	0.699	1	1.699
x	100	500	1000		
log *x*	2	2.699	3		

b



Answers

c



d At the same scale, the graphs mirror images in the line $y = x$.

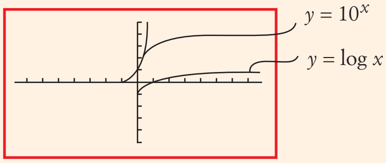
13 **a** $\log 3.5 \approx 0.5441$, $\log 350 \approx 2.5441$

b $\log 350 = \log 3.5 + \log 100$

14 **a** $\log 18 \approx 1.2553$, $\log 3 \approx 0.4771$, $\log 6 \approx 0.7782$

b $\log 18 = \log 3 + \log 6$

15



The graphs mirror images in the line $y = x$

16 **a** Domain = real numbers, range = positive numbers

b Domain = positive numbers, range = real numbers

17 **a** 1.6654 **b** 3.6654 **c** 1.3446 **d** 4.3446

18 **a** 1.5011 **b** $^-3.4989$ **c** 5.5011

d $^-1.4989$ **e** 7.5011

19 **a** 2.2286 **b** 1.2458 **c** 169.3 **d** 17.6

e

Number	Logs
116.72	≈ 2.0671
23.81	≈ 1.3768
Subtract for division.	0.6903
Antilog 0.6903 ≈ 4.9	

f

Number	Logs
16.34	≈ 1.2133
43.67	≈ 1.6402
Add for multiplication.	2.8535
Antilog 2.8535 ≈ 713.7	

20 **a** $x \approx 1.5943$ **b** $x \approx 0.5943$ **c** $x \approx 17.6730$

d $x \approx 9.9031$ **e** $x \approx 3.7310$ **f** $x \approx 2.2295$

g $x \approx 1.7055$ **h** $x \approx 2.7055$ **i** $x \approx {}^-1.0766$

j $x \approx 1.5349$

21–25 Teacher to check proofs.

Exercise 1.3

1 **a** Yes **b** No **c** No **d** Yes

e No **f** Yes **g** Yes **h** No

i No **j** Yes **k** No **l** Yes

2 **a** $2\sqrt{3}$ **b** $2\sqrt{5}$ **c** 6 **d** $4\sqrt{2}$

e $3\sqrt{3}$ **f** $2\sqrt{7}$ **g** $3\sqrt{6}$ **h** $6\sqrt{2}$

i $4\sqrt{6}$ **j** $2\sqrt{30}$ **k** $4\sqrt{7}$ **l** $14\sqrt{2}$

3 **a** $\sqrt{24}$ **b** $\sqrt{45}$ **c** $\sqrt{48}$ **d** $\sqrt{175}$

e $\sqrt{72}$ **f** $\sqrt{192}$ **g** $\sqrt{99}$ **h** $\sqrt{490}$

i $\sqrt{325}$ **j** $\sqrt{504}$ **k** $\sqrt{80}$ **l** $\sqrt{90}$

4 **a** Rational **b** Rational **c** Irrational

d Rational **e** Irrational **f** Rational

g Irrational **h** Rational **i** Rational

j Rational **k** Irrational **l** Irrational

5 **a** $\sqrt{6}$ **b** $2\sqrt{7}$ **c** $\sqrt{30}$ **d** $\sqrt{33}$

e 11 **f** $14\sqrt{3}$ **g** $10\sqrt{2}$ **h** $12\sqrt{3}$

6 **a** $6\sqrt{15}$ **b** $48\sqrt{42}$ **c** $20\sqrt{26}$ **d** 60

e 42 **f** $60\sqrt{2}$ **g** $70\sqrt{2}$ **h** 120

7 **a** $\sqrt{11}$ **b** 2 **c** $\sqrt{3}$ **d** $2\sqrt{3}$

e $\sqrt{10}$ **f** 6 **g** $4\sqrt{2}$ **h** $2\sqrt{3}$

i $1\frac{1}{2}$ **j** $\dfrac{10\sqrt{2}}{3}$

8 **a** $4\sqrt{5}$ **b** $10\sqrt{3}$

c $3\sqrt{13}$ **d** $4\sqrt{5}$

e $6\sqrt{7}$ **f** $4\sqrt{10} + 6\sqrt{5}$

g $4\sqrt{6} + 4\sqrt{5}$ **h** $\sqrt{2} + 10\sqrt{5}$

9–15 Check your answer with your teacher.

16 **a** $3\sqrt{2}$ **b** $2\sqrt{3}$ **c** $6\sqrt{5}$ **d** $3\sqrt{11}$

e $9\sqrt{3}$ **f** $\sqrt{3}$ **g** $3\sqrt{10}$ **h** $15\sqrt{7}$

17 **a** $21\sqrt{2}$ **b** $12\sqrt{3} - 7\sqrt{2}$ **c** $4\sqrt{7} + 7\sqrt{5}$

d $7\sqrt{10} - 6\sqrt{2}$ **e** $5\sqrt{5} + 6\sqrt{2}$ **f** $8\sqrt{7} + 18\sqrt{5}$

18 $\sqrt{15}$ cm

19–26 Check your answer with your teacher.

27 **a** $^-21\sqrt{2} - 11\sqrt{3}$ **b** $^-11\sqrt{3} - 3\sqrt{5} - 8\sqrt{2}$

c $6\sqrt{5} + 3\sqrt{2} + 8\sqrt{3}$ **d** $6\sqrt{5} + 30\sqrt{3}$

28 **a** $25\sqrt{6} + 24\sqrt{5} - 5\sqrt{2}$ **b** $20\sqrt{6} + 17\sqrt{2} + 12\sqrt{3}$

c $79\sqrt{2} + 12\sqrt{3} - 17\sqrt{6}$ **d** $64\sqrt{5} - 82\sqrt{6} + 29\sqrt{2}$

29 **a** $\dfrac{\sqrt{5}}{5}$ **b** $\dfrac{\sqrt{2}}{4}$ **c** $2\sqrt{2}$

d $\dfrac{5\sqrt{3}}{3}$ **e** 2 **f** $\sqrt{3}$

g $\dfrac{\sqrt{15}}{5}$ **h** $\dfrac{5\sqrt{21}}{14}$ **i** $\dfrac{\sqrt{3}}{\sqrt{5}\sqrt{a}}$

30 **a** $16\sqrt{6}$ m **b** $32\sqrt{2}$ m

Chapter review

1 **a** 5^{37} **b** $\frac{1}{6^8}$ **c** 1

2 **a** 9 **b** 25 **c** 100

3 **a** $3^4 \times 5^4$ **b** $4^5 \times 7^5$ **c** $5^3 \times 11^3$

4 **a** 7 **b** 15 **c** 3

5 **a** 7^2 **b** 23^{12} **c** 11^{60}

6 **a** 6.32 **b** 9.03 **c** 2.50

7 **a** $\dfrac{7^6}{5^9}$ **b** 5^{11} **c** $7^{12} \times 11^2 \times 2^4$

8 **a** $\dfrac{1}{2^2 \times 3^2}$ **b** 3×5^9 **c** $2^8 \times 5^4$

9 **a** $\dfrac{y^8}{x^4}$ **b** $\dfrac{1}{a^5}$ **c** $\dfrac{1}{6q}$

10 **a** x^{11} **b** $5k^9$ **c** $27a^{12}b^6$

d $\dfrac{9t^6}{2}$ **e** $\dfrac{2^5}{7}$

11 **a** $\frac{8p^{14}}{3q^6}$ **b** $\frac{1}{2}$ **c** $\frac{1}{27}$ **d** 5

12 **a** Rational **b** Rational **c** Irrational
d Rational **e** Irrational **f** Rational

13 **a** 5 **b** $^-2$ **c** 2.01

14 3 **15** $\sqrt{112}$

16 **a** $2\sqrt{2}$ **b** $3\sqrt{7}$ **c** $6\sqrt{2}$
d $12\sqrt{10}$ **e** $2\sqrt{6}$ **f** $5\sqrt{6} - \sqrt{7}$

17 Check your answer with your teacher.

18 $6\sqrt{2}$

19 **a** 5 **b** $^-6$ **c** 2.5

20 **a** 1.7541 **b** 4.7541 **c** 2.2459 **d** 5.2459

21 **a** 3.6180 **b** $^-3.3820$ **c** 7.6180 **d** $^-9.3820$

22 **a** 2.6792 **b** 0.9290 **c** 477.75 **d** 8.49

e

Number	Logs
125.36	≈ 2.0982
19.87	≈ 1.2982
Subtract for division.	0.8000
Antilog 0.8 ≈ 6.3	

f

Number	Logs
23.55	≈ 1.3720
35.22	≈ 1.5468
Add for multiplication.	2.9188
Antilog 2.9188 ≈ 829.5	

23–25 Teacher to check proofs.

24 Check your answer with your teacher.

25 **a** $^-8\sqrt{2} + 2\sqrt{5}$ **b** $^-2\sqrt{2} - 2\sqrt{5}$
c $22\sqrt{3} - \sqrt{5} + 8\sqrt{6}$ **d** $49\sqrt{2} + 43\sqrt{6} - 10\sqrt{5}$

26 $24\sqrt{3}$ cm

Chapter 2

Exercise 2.1

1 **a** Yellow, 7 **b** Australia, 8 **c** Desert, 6
d Jack, 9 **e** Olivia, 5

2 **a** Median = 4, mode = 4
b Median = C, mode = C
c Median = A, mode = A
d Median = ***, mode = ***, ****+
e Median = M, mode = PG

3 **a** Mean = $6.\overline{2}$, median = 7, modes = 3, 10, range = 10
b Mean ≈ 10.8, median = 9.5, mode = 8, range = 12
c Mean ≈ 10.3, median = 10, mode = 10, range = 17
d Mean ≈ 37.0, median = 36, mode = 35, range = 19
e Mean ≈ 14.6, median = 14, mode = 13, range = 7

4 **a** Median = 26, $Q1$ = 22.5, $Q3$ = 27.5, IQR = 5
b Median = 6, $Q1$ = 4, $Q3$ = 7, IQR = 3
c Median = 34.5, $Q1$ = 32.5, $Q3$ = 35.5, IQR = 3
d Median = 15, $Q1$ = 14, $Q3$ = 16, IQR = 2
e Median = 44.5, $Q1$ = 41, $Q3$ = 47, IQR = 6

5 **a** Median = 7, $Q1$ = 5, $Q3$ = 8, IQR = 3
b Median = 22, $Q1$ = 20, $Q3$ = 23, IQR = 3
c Median = 6, $Q1$ = 5, $Q3$ = 7, IQR = 3
d Median = 12, $Q1$ = 8, $Q3$ = 15, IQR = 7
e Median = 46, $Q1$ = 44, $Q3$ = 51.5, IQR = 7.5

6 **a** Mean = 14.5, SD ≈ 2.57
b Mean = 7.875, SD ≈ 3.30
c Mean = 49.5, SD ≈ 4.95
d Mean ≈ 14.14, SD ≈ 1.55
e Mean ≈ 100.22, SD ≈ 5.57

7 **a** Mean ≈ 9.97, SD ≈ 2.02
b Mean = 29.7, SD ≈ 2.37
c Mean = 17, SD ≈ 2.20
d Mean = 4.25, SD ≈ 2.12
e Mean = 105.16, SD ≈ 2.27

8 **a** 26, 28.5, 30, 32, 41 **b** 2, 6, 12, 15, 20
c 7, 14, 15.5, 21, 35 **d** 1, 9, 13, 14, 16
e 37, 53.5, 58, 60.5, 73

9 **a** 30, 33, 34, 35.5, 39 **b** 3, 6, 8, 10, 16
c 8, 10.5, 13.5, 16.5, 29 **d** 101, 103, 104, 105.5, 109
e 23, 27.5, 30, 32, 36

10 **a** Mode **b** Mode **c** Mean
d Mode **e** Median

11 5, 4 and 125 resp.

12 About 36.2 years.

13 About 25.6 mL.

14 **a** 6.48 kg **b** 32 400 kg
c Only 50 plants were checked and there is no information about the other factors that affect fruit production. However, the pattern is clear so it is probably sufficient.

15 **a** 106 **b** 423
c 407 rallies seems a lot and there is a clear pattern, so it is probably sufficient.

16 **a** Mean **b** Median **c** Mean
d An amount higher than the median, say $Q3$.
e An amount lower than the median, say $Q1$.

17 **a** The amounts increase to \$1364, \$1039.50, \$913 and \$803.
b The amounts increase to \$1340, \$1045, \$930 and \$830.
c About 730, 850, 945, 1500, 3000. The 850, 1500 and 3000 could vary.

18 **a** About \$954.50 **b** About \$157 500

19 ★★★★

20 The median will be lower than the mean, and the mode is probably lower than the median.

21 The mean increases to about 171.8 cm and the median probably increases to about 173 cm.

22 **a** Summary: 5, 7, 9.5, 14.5, 20; probably skewed right.
b Summary: 162, 163, 164.5, 174, 183; probably skewed right.
c Summary: 0, 8, 10, 13, 19; probably symmetrical.
d Summary: 35, 41.5, 45, 47.5, 51, probably symmetrical.
e Summary: 13, 14, 15, 18, 20; probably skewed right.

23 Suburban stores could have younger customers who like different types of music to those that shop in the city.

24 In the Australian federal electoral system, the number of seats won can change dramatically with quite a small change in voting, particularly if results are close. The polls only ask 20 or 30 people in each electorate, so the results in each seat could be quite different to the overall percentages. In three days, the intentions of people could have changed by, say, 0.5% and this could be enough to make this change. The polls could have been correct at the time for the people asked, but still be wrong overall.

25 Employers would definitely supply the gross wages, but employees might give their wages after tax or other deductions, or not want to give their actual earnings.

Exercise 2.2

1 0.1, 0.2 and 24 are non-compliant, so should not be used.

Stopping times (s)

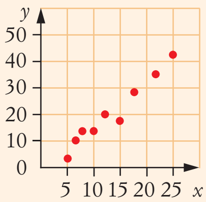
Stem	Leaf
1	8 9 9
2	1 2 3 6 6 6 7 7 8 8
3	0 1 1 4 4 4 5 6 7 7 7 8
4	1 2 3 6
5	0

2 320 is non-compliant, so should not be used.

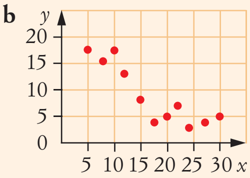
University entrance marks

Stem	Leaf
3	5 6 7 8
4	0 0 2 4 6 6 6
5	0 0 1 2 4 6 6 7 8
6	0 0 2 3 3 4 6 6 9 9
7	2 2 5 8
8	4 4 7 8 8
9	1 2 5

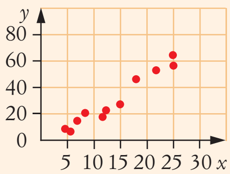
3 a



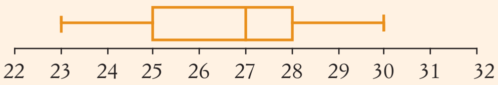
b



c



4 a



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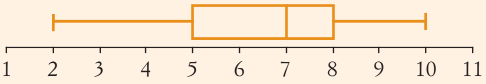
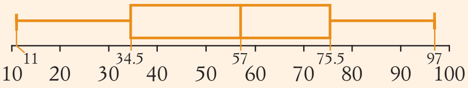
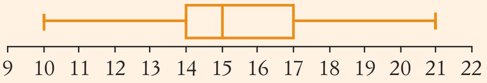
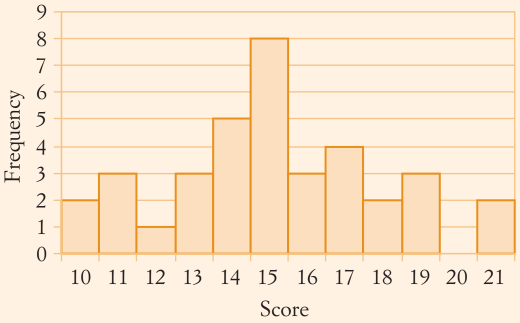


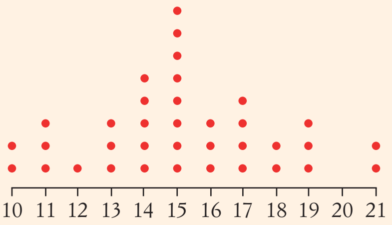
c



5 a

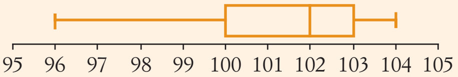

Box-and-whisker

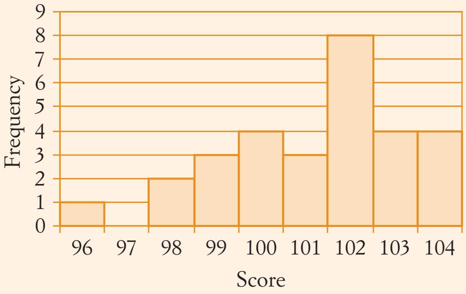

Histogram

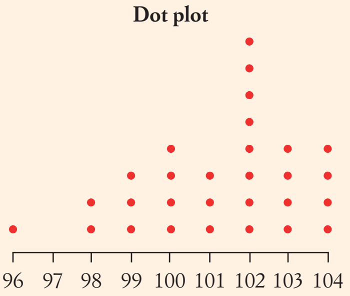

Dot plot

The box-and-whisker plot, histogram and dot plot all show a fairly symmetrical distribution.

b

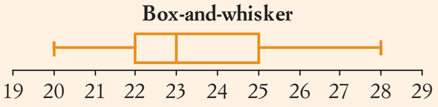

Box-and-whisker

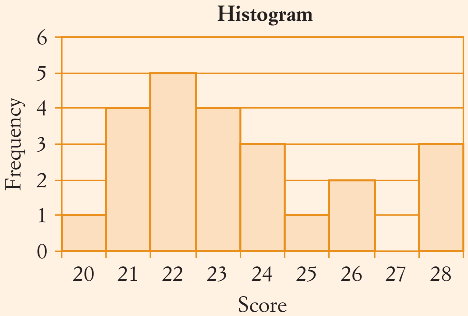

Histogram

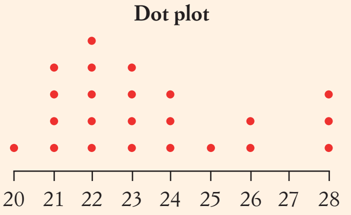


The box-and-whisker plot, histogram and dot plot all show a distribution skewed to the left.

c

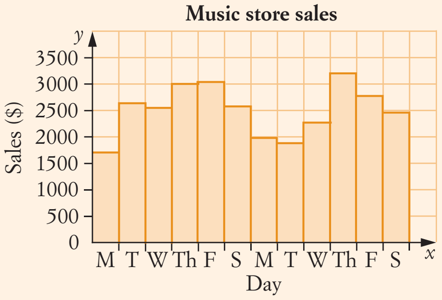






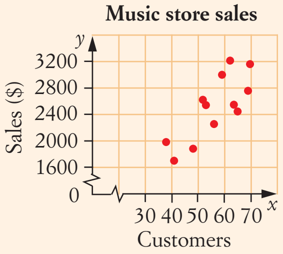
The box-and-whisker plot, histogram and dot plot all show a distribution skewed to the right.

6 a



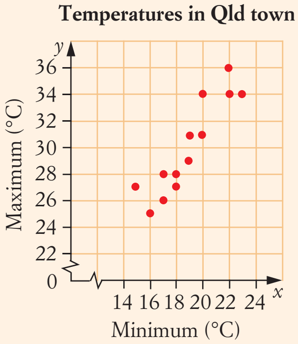
b Their biggest sale days are Thursdays and Fridays, and their smallest are Mondays.

c



d Greater numbers of customers and higher sales go together.

7 a



b Yes, they are higher or lower together.

8 a There does not seem to be any real difference between Victoria and Western Australia.

Time taken by Year 10 students to travel to school (minutes)

Vic	Key: 5|1 = 15 minutes = 1|5	WA
9 5 5 4 1	0	2 3 3 5 5
8 5 5 5 5 0 0 0 0 0 0 0	1	0 0 0 0 0 0 5 5
5 5 2 0 0 0	2	0 0 0 0
7 5	3	0 6
5 0	4	
0	5	0

b Victoria: mean ≈ 18.2, median = 15, range = 49, WA: mean = 15.2, median = 10, range = 48. The means and medians are higher for Victoria, so there is actually a difference, although the ranges are almost the same.

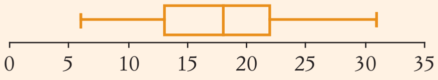
9 a Medium size cars generally cost more to run than small cars.

Car running costs ($/week)

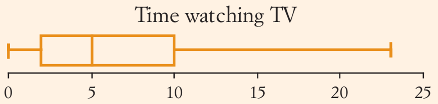
Small	Key: 5|1 = $51	Medium
2	3	
9	3	
4 2 2 2 2	4	0
9 9 9 8 7 6	4	5
1 1	5	1 2 2 4
6	5	5 6 7
1	6	0 0 0 2

b Small: mean ≈ \$46.41, median = \$46, range = 29; Medium: mean ≈ \$54.15, median = \$55, range = 22. There is a real difference in running costs, although the medium cars have a lower range of costs.

10 The times, ranging from 6 to 31 minutes, are fairly symmetrical about the median.



11 The times watching TV are skewed to the right.



12 It would give a good summary of sales for each month, but the manager would probably want more details concerning the kinds of things sold.

13 A scale like this does not provide much discrimination, so a boxplot is unlikely to give much information. It is only 5 points anyway, and these are very likely to be the same as the 5 number summary, so it wouldn't provide any real feedback. Very unhappy customers are mostly not going to 'stay on the line' anyway.

14 **a** That Melbourne house prices are decreasing.

b It is only for 1 year, so the trend could be only temporary.

15 The graph shows the percentage of applicants who applied. It does not show the actual numbers. Western Australia has fewer people than NSW, Vic. and Qld, so the actual numbers could be less.

Exercise 2.3

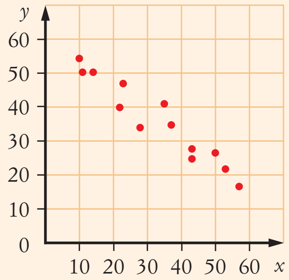
1 **a** Strong positive **b** Weak negative

c Strong negative **d** No relationship

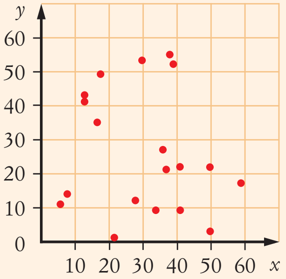
e Weak positive **f** Weak negative

2 **a** 20 **b** 16, 34, 35 **c** None **d** 5 **e** 36, 62

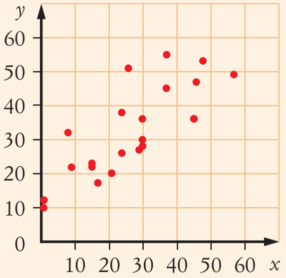
3 **a** Strong negative



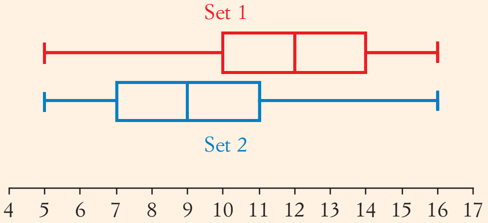
b No relationship



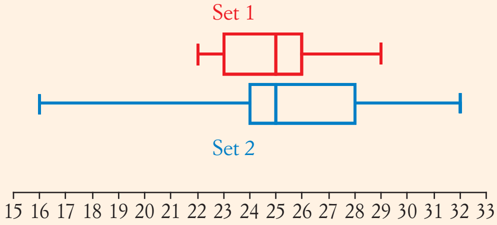
c Weak positive



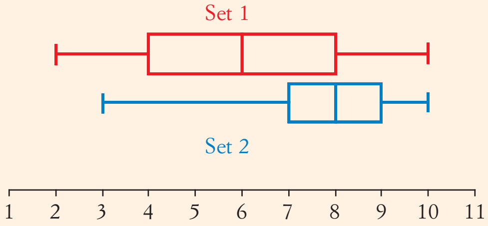
4 **a**



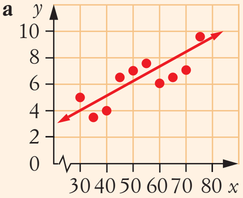
b



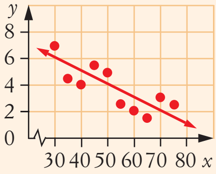
c



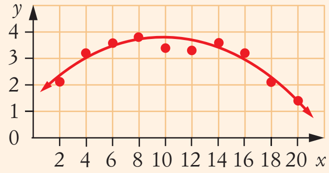
5 **a**



b



c



6 **a** $y \approx 0.09x + 1.4$ **b** $y \approx {}^{-}0.09x + 8.6$

7 **a** $y \approx {}^{-}2.3x + 122$ **b** $y \approx 1.7x + 4.2$

8 **a**

		Accidents in last 2 years					
		1	**2**	**3**	**4**	**5**	**Total**
Age group	**17–21**	16	24	21	14	12	87
	22–26	16	14	12	9	9	60
	27–31	9	7	7	4	2	29
	32–41	4	2	2	0	0	8
	Over 41	4	2	0	0	0	6
	Total	**49**	**49**	**42**	**27**	**23**	**190**

b Younger drivers

c 17–21, 30%; 22–26, 30%; 27–31, 21%; 32–41, 0%, Over 41, 0%

d Younger drivers seem more prone to multiple accidents.

e The numbers for 5-year age groups over 26 are a bit low. However, if they are classified as older (over 26) and younger (26 or less) then the sample is sufficient to get good conclusions.

9 **a**

		Reading age						
		12	**13**	**14**	**15**	**16**	**17**	**Total**
Sex	**Male**	14	17	24	31	24	12	122
	Female	7	10	17	36	29	19	118
	Total	**21**	**27**	**41**	**67**	**53**	**31**	**240**

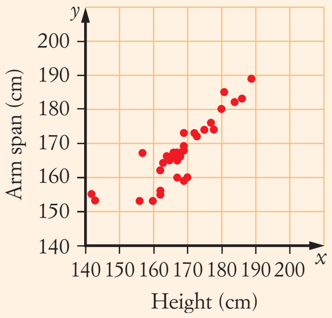
b More girls are good readers (reading age over 15).

c Girls appear better, 41% to 30%.

d All of the frequencies are reasonable, so conclusions are likely to be reliable.

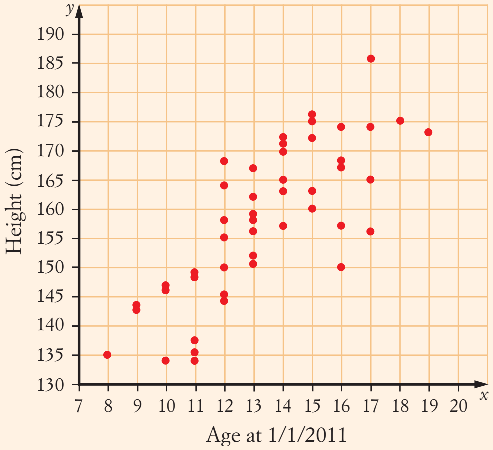
e 35 boys and 48 girls out of the totals is enough to support the conclusions.

10 **a** **Arm span and height**



b There is a strong positive relationship; taller students have larger arm spans.

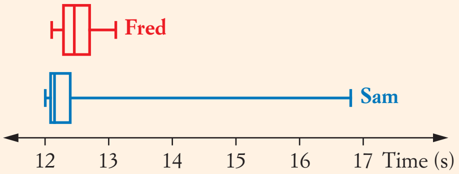
11 **a** **Age and height**



b There is a weak positive relationship. Older students are often taller.

c You would not expect to find any relationship as people have mostly finished growing when they reach university.

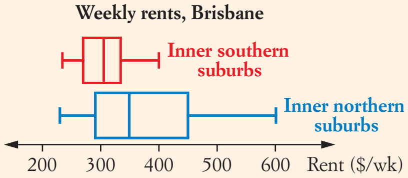
12 **a** **Sprinters' times over 100 m**



b Sam's mean (12.64) is almost the same as Fred's, despite one really bad time that makes his range bigger. His median is much lower, at 12.15 to 12.45.

c Sam, because he is faster, except for one time when he may have been sick.

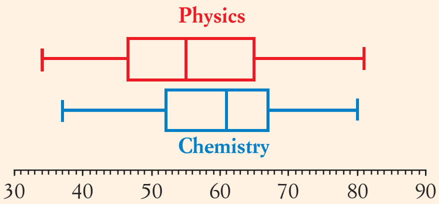
13 a



b Rental costs in the inner northern suburbs are generally higher, but also spread over a much greater range than rental costs in the inner southern suburbs.

c The mean ($370 to $308), the quartiles and the median are all higher for the northern suburbs so it is quite a bit more expensive.

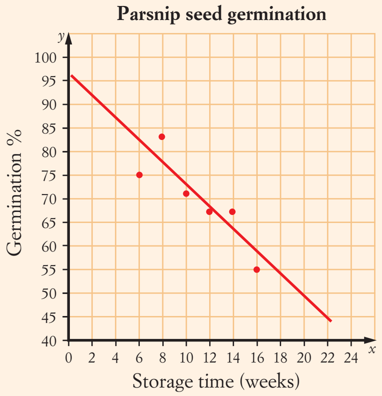
14 a



b Students generally did better in Chemistry than in Physics.

c The mean for Chemistry (59.8) is higher than that for Physics (56.3), so this reinforces the conclusion from the median and quartiles of the boxplot.

15 a

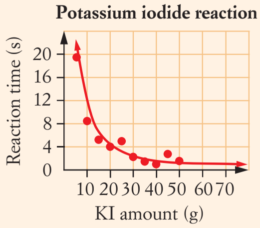


b As the storage time increases, the germination rate decreases.

c Probably about 94%.

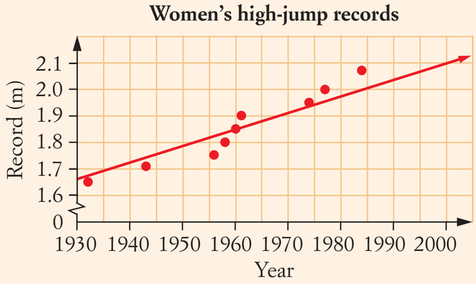
d Probably about 50%.

16 a



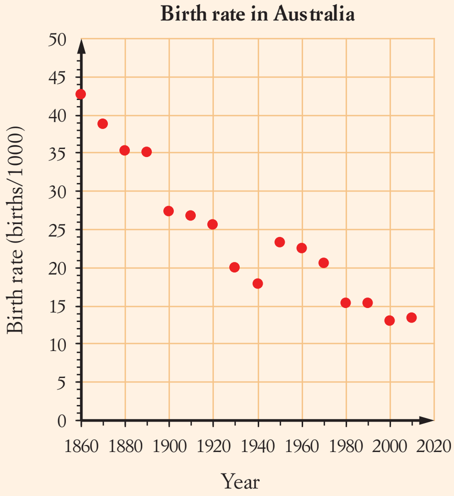
b Less than 1 s.

17 a



b About 2.1 m (2000 result was 2.01 m).

18 a



b $b \approx {}^{-}0.172y + 356.5$

c 3.9 births per 1000 people

d There is certainly a decline, but it is unlikely to continue indefinitely, so it may yet level out.

19

	Mean	Sample SD
Year 8	8.65	1.35
Year 10	8.27	1.48
Year 12	7.93	1.09

Older students say they sleep less than younger ones. Year 12 students are more consistent in how much they say they sleep than Year 8 and 10 students.

20 1950–1959: mean ≈ 2.37, SD ≈ 35s, 2000–2009: mean ≈ 3.37, SD ≈ 33 s. The newer hits are about a minute longer, but vary by about the same amount.

21 The male student is a little more unusual than the female student (1.20 SDs and 1.06 SDs from the mean).

22 You would be more likely to find a girl over 180 cm than one under 145 cm.

23 There is no information about the number of drivers in each group or how far they generally drive.

24 The people who bought the board games are already board game buyers, so the marketing firm will not find out how to appeal to a wider group. It is a good approach if they want to find out about existing buyers.

25 It may be a reasonable approach, but they probably should ask students from lower grades as well.

26 It could be used to set a 'use-by' date, but you would need to decide on an acceptable germination rate.

27 Extrapolation is always difficult, and in this case changes of equipment or methods could make substantial differences.

28 2073, but this assumes that the general trend continues. This is extremely unlikely.

Chapter review

1 Spaghetti, 6

2 Median = Poor, Mode = Very poor

3 **a** Mean ≈ 10.43, Median = 11, Mode = 13, Range = 9

b Mean ≈ 31.7, Median = 32, Mode = 33, Range = 12

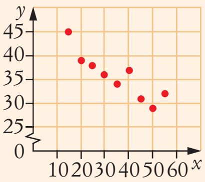
4

Rising times of Year 10 students on Monday

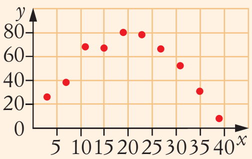
Key: 6|30 = 6:30 a.m.

5	30
6	0 15 15 30 30 30 30 45
7	0 0 0 0 0 0 15 30 30 30 30 30 30 30 45 45
8	0 0 0 0 30

5 **a**



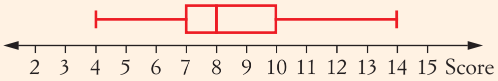
b



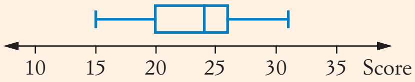
6 **a** Median = 7, $Q1$ = 4.5, $Q3$ = 8, IQR = 3.5

b Median = 12, $Q1$ = 8.5, $Q3$ = 15, IQR = 6.5

7 **a**



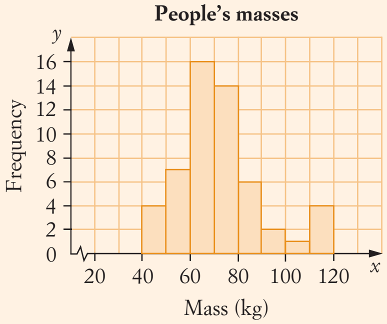
b



8 **a** Weak positive **b** Strong negative **c** None

9 Mean = 9.75, SD ≈ 5.07

10 **a**

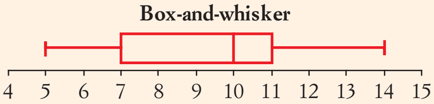


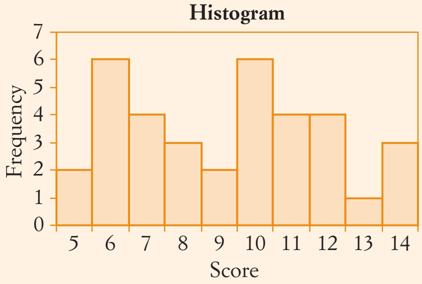
b Most people have masses between 50 and 90 kg.

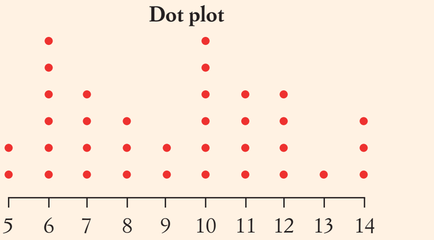
11 **a** 39, 43.5, 49, 53.5, 56 **b** 12, 15, 16, 17, 20

12 **a** Mean price **b** Median income

13



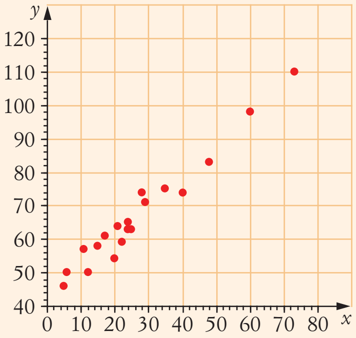




The histogram and dot plot show a bimodal distribution consistent with the box-and-whisker plot.

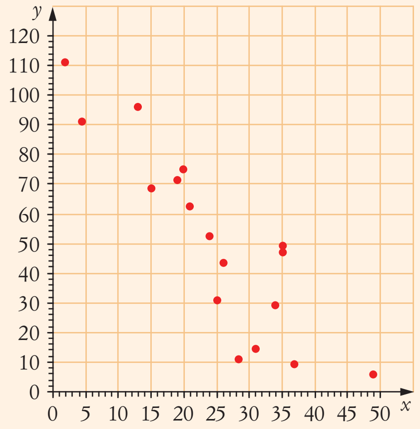
14 **a** 4 **b** No outliers

15 **a**



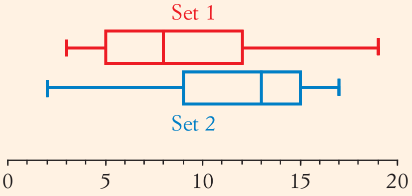
Strong positive

b



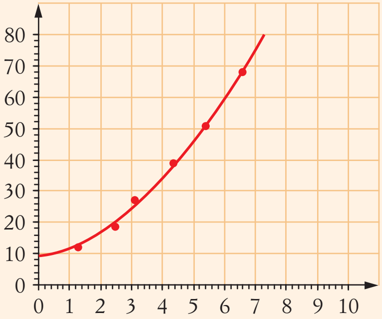
Weak negative

16

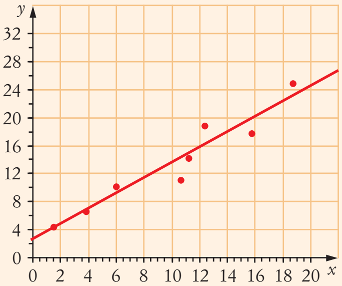


17 72

18 **a**



b



19 **a** 1 apple **b** About 120 apples (using the mean)

20 **a**

Greyhound times (s)

Houndog **Key 3|5 = 53 seconds**		**Chaser** **Key 5|3 = 53 seconds**
7	3	
8 7 4	4	
9 9 8 3 3 1 0	5	3 7 8 9
6 4 1 0 0	6	0 1 4 4 4 6 8 8 8 9
3 3 1 0	7	1 6 6 7
	8	2 3

b Houndog is generally faster.

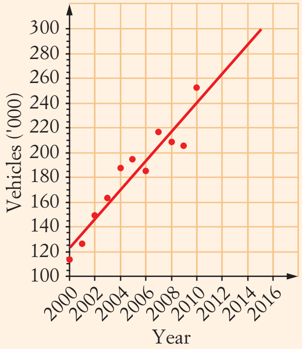
c The means (57.85 to 67.2 s), and medians (59 to 67 s) reinforce the conclusion from the stem-and-leaf plot. The ranges (36 to 30 s) are similar.

21 **a**

		Errors in spelling					
		1	**2**	**3**	**4**	**5**	**Total**
Height (cm)	**150–159**	0	1	3	5	1	10
	160–169	8	6	9	2	9	34
	170–179	11	2	6	7	9	35
	180–189	2	3	2	2	2	11
	Over 189	2	0	0	1	1	4
	Total	**23**	**12**	**20**	**17**	**22**	**94**

b There is no pattern.

22 **a** **New sports utilities sales in Australia**

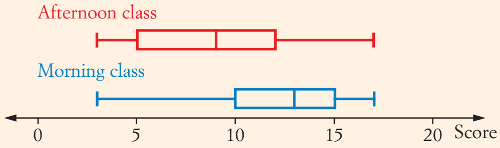


b Yes, strong positive relationship.

c About 290 000

d $S = 11\,800\,Y - 23\,477\,000$

23 **a** **Exam results**



b The morning class appears to have had an advantage.

c The means (12.2 to 9.0) reinforce the conclusion from the medians and quartiles in the boxplots.

24 **a** $S = 10\,900\,Y - 21\,741\,000$ **b** 277 000

25 Geography: Mean ≈ 43.6%, SD ≈ 16.9%, History: Mean ≈ 53.1%, SD ≈ 10.6%, so it appears that Geography is harder. It doesn't settle the argument because this could just be a bad result for Geography, or the students doing Geography might be less interested.

26 The price of bananas is 0.43 SDs from the mean, so it is slightly more unusual than oranges which were 0.41 SDs from the mean.

27 The mean would be lower than the median because the low values would affect it more than the median.

28 The new mean would be (72 × 20 + 3 × 80) ÷ 23 ≈ 73 kg. The median would probably not change because one of the new students must have a lowish weight.

29 The morning class might just be a better class.

30 The distribution is skewed to the left so the mean would be lower than the median. A quarter of the results are in each of the ranges 12–17, 17–19, 19–20, 20–22, so this gives an overall mean of about 18.5.

31 The survey is only of students completing Year 12, so it does not cover all school leavers. The headline is thus misleading.

32 The first company has a much larger sample than the second, but their method is poor. People would have a much better idea of sizes from the second survey, so it is more likely to be accurate. A large sample will not compensate for poor methods.

33 If there were changes in costs, the economic situation or fashion, the prediction could be quite wrong.

Chapter 3

Exercise 3.1

1 **a** $A = 12$ **b** $A = 1$ **c** $A = 35$
d $A = 3$ **e** $A = 13$ **f** $A = 21$
g $A = 75$ **h** $A = 147$ **i** $A = 37$

2 **a** $9v^8, 8$ **b** $^{-}4z^8, 8$
c $7x^3, 3$ **d** Not a polynomial
e Not a polynomial **f** $7n^7, 7$
g $4w^3, 3$ **h** Not a polynomial
i $^{-}9w^8, 8$ **j** $5u^6, 6$

3 **a** $10c^2 - 5r$ **b** $4m^2 - 2g$
c $10x^2 - 8x + 2$ **d** $4ce^2 - 4ac^2 - 4ce$
e $2ab - 3b^2 - 4$ **f** $2a^2 - 3ac$
g $3mn + 6mn^2$ **h** $6m^2 + 5m^3 - 3mn$
i $4g^2 + 5g^3 - 7$ **j** $5a^2b^2 - 3ab^2 + 7ab$
k $24pqr + 8p^2qr$ **l** $pq + qr + pr$
m $9pq^2 - 6pq + 4p^2q$ **o** $24fmr^2 - 20m$
o $31xy^2z - 3xyz^2$ **q** $21a^2 - 6a^3$
q $11ab$ **r** $^{-}4t^3 + 24t^2 - 41t$

4 **a** 20 **b** 23 **c** 55 **d** 72
e 2 **f** $2\frac{2}{3}$ **g** 24 **h** 149

5 **a** $b = 15$ m **b** $t = 12.5$ s
c $b = 5$ m **d** $h \approx 7.6$ cm
e $h \approx 21.05$ m **f** $u \approx 1.97$ m/s
g $m = 2.667 \times 10^{-8}$ kg **h** $R \approx 0.0745$ atm-L/mole/K

6 **a** $t = \dfrac{2u - v}{3au}$ **b** $c = \dfrac{Ik - G}{kt}$
c $b = m(c - a) + a$ **d** $u = sa - at$ or $a(s - t)$
e $d = \dfrac{bc}{a}$ **f** $a = cv^2 + b$
g $b = \dfrac{a^2 - c^2}{2}$ **h** $c = \dfrac{(d + 5)^2}{b}$

7 **a** $36a^5c^4k^6$ **b** $6h^2u^2x^6$ **c** $30k^6p^5r^6$
d $8e^6n^3y$ **e** $20c^2n^3w$ **f** $12a^4g^6n^4$
g $14a^2h^5k^5$ **h** $10a^4h^4v^3$ **i** $63e^4p^3t^3$
j $42h^4r^2u^3$ **k** $8b^4h^4k^2$ **l** $45b^4q^5t^4$
m $\dfrac{7pt^2}{6k}$ **n** $\dfrac{2ev^2}{5u}$ **o** $\dfrac{8}{7gu^3}$
p $\dfrac{1}{9pu^2}$ **q** $\dfrac{3}{2v}$ **r** $\dfrac{6}{5ah^2}$
s $6k$ **t** $\dfrac{6u^2y}{7}$ **u** $\dfrac{3c^2u^2v^2}{8}$
v $\dfrac{6a}{h}$ **w** $\dfrac{9}{8g}$ **x** $\dfrac{5mu^2}{6q^2}$
y $\dfrac{6bc}{n^3}$ **z** $\dfrac{3qv}{5}$

8 **a** $\dfrac{3h^5}{2n^7u^2}$ **b** $\dfrac{4t^8w}{5}$ **c** $\dfrac{3}{5e^7p^4t^3}$
d $2a^{19}c^{18}$ **e** $20b^{14}q^{11}v^3$ **f** $\dfrac{5u^4}{2a^4p^8}$
g $4n^{12}p^{18}v^{19}$ **h** $12a^{15}t^{12}y^{11}$ **i** $2m^{11}q^5r^5$
j $4h^3m^2u^2$ **k** $15b^{20}c^5h^9$ **l** $2a^6b^{14}p^{14}$
m $\dfrac{4k^{11}u^3}{w^2}$ **n** $b^{17}h^{19}r^{13}$ **o** $\dfrac{4kn^2}{5p^4}$
p $\dfrac{5a^3}{4g^6m^2}$ **q** $2a^7c^{13}g^{11}$ **r** $\dfrac{u^3}{5r^6t^8}$
s $\dfrac{3n^8}{be^2}$ **t** $\dfrac{3}{5em^6u}$ **u** $\dfrac{4t}{5b^7m}$
v $\dfrac{2}{3qtw}$ **w** $12b^{20}h^{16}m^7$ **x** $20b^{22}g^{18}v^{12}$
y $5e^{14}p^{16}q^5$ **z** $12g^9n^{18}v^{11}$

9 **a** $\dfrac{^{-}5b^7g^4}{a^2}$ **b** $\dfrac{3x}{4h^4k^7}$ **c** $\dfrac{7c}{a^7m^9}$
d $\dfrac{^{-}4c^6e^2}{9h^2}$ **e** $^{-}12a^2p^3$ **f** $\dfrac{1}{k^4p^8r^2}$
g $\dfrac{-bt}{9r^3}$ **h** $\dfrac{^{-}9b^7q^2}{p^2}$ **i** $\dfrac{^{-}5a^7m}{7c^7}$
j $\dfrac{3n^5}{b^3r}$ **k** $\dfrac{4}{a^3e^4k^3}$ **l** $\dfrac{^{-}2t^4}{5q^4}$
m $\dfrac{^{-}3}{2b^3h^2k^{10}}$ **n** $\dfrac{^{-}b^8k^2}{5p^6}$ **o** $72m^7p^5r^5$
p $\dfrac{2a^6v^5}{c}$ **q** $42c^4$ **r** $\dfrac{^{-}3m^6q^5}{7t^5}$
s $\dfrac{^{-}7b^{10}e^3k}{4}$ **t** $^{-}4b^6h^8v^2$ **u** $\dfrac{^{-}6e^4}{a^3m}$

v $\frac{56e^5n^3}{m}$ **w** $\frac{35g}{k^5t^3}$ **x** $\frac{6}{e^8g^{10}m^8}$

y $^{-}5c^8g^7h^5$ **z** $\frac{30g}{c}$

10 **a** $\frac{^{-}7k^3}{3a^4e}$ **b** $\frac{^{-}12a^5c^{12}}{u^9}$ **c** $\frac{^{-}42}{nr^3u^8}$

d $\frac{7}{3ak^{10}u^2}$ **e** $\frac{u^2}{3c^{10}}$ **f** $\frac{6t^3}{5a^{12}k^5}$

g $\frac{^{-}42c^{10}m^3}{k^{11}}$ **h** $\frac{14p^3q^6}{h^2}$ **i** $\frac{^{-}5c^2g^3}{9m}$

j $\frac{24b^2}{p^7r}$ **k** $\frac{^{-}24a^{17}g^{12}}{k^6}$ **l** $\frac{^{-}5e^3}{2a^9c^{18}}$

11 **a** $\frac{12e^5u^{22}}{k^{16}z^3}$ **b** $\frac{6n^9v^{13}w^{11}}{5m^{11}}$ **c** $\frac{^{-}60a^9v^3}{m^8u^{10}}$

d $\frac{^{-}7w}{6k^{10}n^3z^{14}}$ **e** $\frac{e^{15}}{4a^{21}z^6}$ **f** $\frac{^{-}35g^8m^4}{r^{22}v^{11}}$

g $\frac{3a^6k^{15}z}{5y^6}$ **h** $\frac{30q^9v^{10}}{u^5w^4}$ **i** $\frac{7g^5k^{13}wx^{17}}{10}$

j $\frac{28}{g^2m^{13}n^{11}u^9}$ **k** $\frac{2h^2k^2}{5b^{14}e}$ **l** $\frac{132n^{23}p^2}{r^8x^2}$

m $156a^{10}e^2q^{19}$ **n** $\frac{63}{b^{12}mw^5z^2}$ **o** $\frac{^{-}2t}{13e^4m^{20}}$

p $\frac{^{-}28c^{13}r^8}{q^{12}}$ **q** $\frac{12m^{11}}{a^{10}k^{11}u^{14}}$ **r** $\frac{9qv^{15}}{5h^{17}x^{13}}$

s $\frac{^{-}12t^5}{11n^2q^7u^9}$ **t** $\frac{11n^4v^6}{a^{14}r}$ **u** $\frac{10p^2q^6}{c^{17}m^9}$

v $\frac{13e^{24}u^2}{p^7q^7}$ **w** $\frac{^{-}4a^5z^6}{13n^9y^8}$ **x** $\frac{8g^{14}u^4}{7h^{18}m^5}$

y $\frac{^{-}91e^{11}n^3}{c^4q^{19}}$ **z** $\frac{7n^{11}}{13k^{20}t^{10}y^{16}}$

12 **a** $\frac{4qu}{5}$ **b** $\frac{8}{15w}$ **c** $\frac{4t}{7kn}$

d $\frac{4m}{7a}$ **e** $\frac{5ae^2}{7gw^2}$ **f** $\frac{10kw}{13u}$

13 **a** 10 **b** $^{-}226$ **c** 200 **d** 59
e $^{-}65$ **f** 481 **g** $^{-}701$ **h** $^{-}2561$
i 715 **j** $^{-}125$ **k** 19 **l** 192
m 18 288 **n** 6 **o** 2958 **p** 18 256

14 **a** $^{-}2x^5 - 2x^2 - 3x + 12$
b $4x^7 + 5x^6 - 14x^4 + 14x^2 + 4x - 6$
c $^{-}2x^5 + 9x^4 - 17x^3 + 5$
d $5x^6 - 9x^4 + 8x^3 + 4x^2 + 3x +5$
e $4x^7 + 4x^4 - 9x^3 + 8x^2 + 2x - 8$
f $2x^5 +16x^4 - 2x^2 + x - 2$
g $^{-}5x^5 + 18x^4 - 9x^3 - 6x^2 - 2x - 2$
h $4x^7 + 2x^5 - 5x^4 + 8x^3 + 8x^2 + 2x - 13$
i $^{-}4x^7 + 5x^4 + 8x^3 - 10x^2 - x + 11$
j $^{-}5x^6 - 2x^5 + 9x^4 - 8x^3 - 6x^2 - 6x +7$

15 **a** $u = 6$ **b** $m = {}^{-}4$ **c** $a = 5$
d $E = 17.5$ **e** $h = 45$ **f** $g = 10$
g $t = 3.75$ **h** $c \approx 21.8$ (or $2\sqrt{119}$)
i $w = 349.8$ **j** $r = 6.5$

16 A formula has more than one variable and is an equation that usually has one of the variables on its own on the left-hand side. An equation is just a statement of equality and could have any number of variables, including none. However, most simple equations have one variable.

17 Proof (use a counterexample)

Exercise 3.2

1 **a** $a^2 + 5a$ **b** $m^2 - mp$ **c** $2g^2 + 3fg$
d $4z^2 - 3wz$ **e** $3x^2 + 2xy$ **f** $8c^2 - 7c$
g $^{-}5m^2 + 3mn$ **h** $^{-}2f^2 + 5fh$ **i** $^{-}2d^2 - 3de$
j $^{-}8c^2 + 28c$ **k** $^{-}6n^2 - 18n$ **l** $^{-}15ck + 10k^2$

2 **a** $x^2 + 7x + 12$ **b** $2 + 3b + b^2$
c $a^2 + 9a + 20$ **d** $12 + 8a + a^2$
e $y^2 + 7y + 6$ **f** $a^2 + 3a + 2$
g $m^2+12m + 35$ **h** $a^2 + 8a + 12$
i $n^2 + 6n + 8$

3 **a** $^{-}10(7q + 3)$ **b** $5(15q - 8)$ **c** $7(4y - 5)$
d $4(11a - 14)$ **e** $^{-}9(8v + 9)$ **f** $13(10c - 11)$
g $^{-}15(6c + 5)$ **h** $13(7t - 9)$ **i** $^{-}15(3u + 11)$
j $^{-}2(9n + 13)$ **k** $^{-}14(2w + 15)$ **l** $14(5r + 9)$
m $4(12b + 5)$ **n** $9(10g + 7)$ **o** $^{-}2(11x + 6)$
p $13(6k + 11)$ **q** $^{-}12(5u + 13)$ **r** $5(4p + 1)$
s $^{-}11(6v + 7)$ **t** $40(2h + 3)$ **u** $^{-}11(13x + 5)$
v $3(11e + 13)$ **w** $7(11m - 5)$ **x** $14(10t + 11)$
y $^{-}7(a + 5)$ **z** $5(11k - 2)$

4 **a** $2a^2 + 5a + 3$ **b** $2a^2 + 7a + 3$
c $3x^2 + 14x + 8$ **d** $8y^2 + 22y + 15$
e $35g^2 - 3g - 2$ **f** $18a^2 + 6a - 4$
g $49m^2 - 1$ **h** $15x^2 - 19x + 6$
i $25y^2 + 40y + 16$ **j** $12y^2 + 25y + 12$
k $49n^2 - 25$ **l** $8 + 12c - 8c^2$

5 **a** $(a + b)^2 = a^2 + 2ab + b^2$
b $(x + y)^2 = x^2 + 2xy + y^2$
c $(w - r)^2 = w^2 - 2rw + r^2$
d $(a + 2)^2 = a^2 + 4a + 4$
e $(x - 3)^2 = x^2 - 6x + 9$
f $(m - 5)^2 = m^2 - 10m + 25$
g $(g + 7)^2 = g^2 + 14g + 49$
h $(6 + b)^2 = 36 + 12b + b^2$
i $(3 - n)^2 = 9 - 6n + n^2$
j $(k + 8)^2 = k^2 + 16k + 64$

6 **a** $3a^2 - 9a - 18$ **b** $7x - 21y + 28z$
c $4x^2 + 20x + 28$ **d** $12m - 6n + 18p$
e $3m^2 - 9mn - 6mp$ **f** $8g - 6g^2 + 2ag$
g $x^2 - 25$ **h** $y^2 - 4y - 21$
i $x^2 + 17x + 60$ **j** $k^2 + 2k - 99$
k $x^2 + x - 30$ **l** $18 + 7n - n^2$

7 **a** $13x + 47$ **b** $32a + 15$ **c** $30d + 43$
d $17c$ **e** $5y^2 - 7y$ **f** $^{-}8h$
g $12x^3 - 16x^2 + 10x$ **h** $12a^3 + 6a^2 - 78a$
i $13x^3 - 7x^2 - 2x$ **j** $8a^3 + 3a^2 + 59a$

8 **a** $^{-}12(12b + 13e + 1)$ **b** $12(13p - 3u - 3)$
c $3(14x + 15k + 6)$ **d** $12(12m - 13t + 9)$
e $7(5z + 11m - 4)$ **f** $^{-}9(10m + 7e + 8)$

g ${}^-3(6b + 11g + 2)$
h $2(12q + 13w + 15)$
i $4(12x - 5e + 6)$
j ${}^-7(12t + 5x + 13)$
k $10(13p - 9n - 11)$
l ${}^-4(15w + 11c + 13)$
m $3(4a - 15h + 1)$
n $5(p - 11r + 7)$
o $6(14q - 9e + 5)$
p ${}^-8(13c + p + 2)$
q $15(8x - 12m + 13)$
r $8(5t + 11a - 13)$
s $15(7h + 9y + 9)$
t $10(11k - 12m + 2)$
u $5(y - 15r - 8)$
v $2(13h - 2x - 12)$
w $14(3v - 13n - 8)$
x $5(8r - 12m - 15)$
y $4(10c + 13m + 9)$
z $8(2k - 15x - 6)$

9 a $12h^{-10}m - 54h^{-6}m^5$
b ${}^-18n^{-5}x^{-2} - 15n^{-5}x^3z^3$
c $48t^3w^{-3}x^4 - 30t^{-3}w^{-3}$
d $12a^{-7}c - 14a^{-1}cw^{-6}$
e $15b^{-1}x^{-6} - 21b^{-1}x^{-2}$
f $8b^6n^5 - 6b^7n^{-1}$
g $40a^{-4}k^6 - 25a^{-1}k^4$
h $48a^2b^5 + 56a^2b^6k^7$
i ${}^-4k^6t^4x^{-1} - 3k^6t^4$
j $8b^{-8}c^{-5} + 28b^{-1}c^{-5}z^{-1}$
k $24g^{-2}p^4r^{-1} + 3g^{-2}p^8$
l $42m^{-2}q^{-4} + 49m^3q^{-4}$
m $25k^6m^{-5} + 15k^3m^{-7}$
n $35h^{-6}k^2 + 15h^{-6}k^{-2}p^7$
o ${}^-45k^{-7}r^4v - 25k^{-12}r^4$
p $24eh^{-3} - 40e^6h^{-3}p^5$
q $28h^{-3}r^{-1}z^{-2} + 35h^{-3}r^{-4}$
r ${}^-36a^{-6}y^{-9} - 16a^{-13}y^{-3}$
s $18e^2k^2 + 9k^{-1}$
t $20u^{-9} + 5u^{-3}$
u $30e^2n^{-3}v - 48e^{-1}n^{-3}$
v $35r^8t^6 - 10r^6t^6y^{-5}$
w ${}^-16c^{-5}tz^6 - 72c^{-5}t$
z ${}^-8n^{-3}t^4 - 7n^{-5}t^3$
y $72mw^4 + 32m^{-2}w^6$
z $4g^5t^3 + 18g^5t^{-4}u^2$

10 a $2a^2 -12 - 2a$
b $2x^2 - 11x + 15$
c ${}^-5a^2 - 17a + 12$
d $2k^2 - 13k + 20$
e $2m^2 - 7m + 3$
f ${}^-6x^2 + 19x + 7$
g ${}^-4p^2 - 4pq - q^2$
h $2n^2 + mn - 15m^2$
i $6x^2 + 13xy + 6y^2$
j ${}^-28x^2 + 29xy - 6y^2$
k ${}^-16a^2 + 66ab - 35b^2$
l ${}^-4xy - 3y^2 - x^2$
m $64u^2 - 144e^2$
n $72g^2 + 77gv + 20v^2$
o $80e^2 + 198et + 121t^2$
p $77a^2 + 8ax - 45x^2$
q $165x^2 - 41xr + 2r^2$
r $126c^2 + 41cz + 3z^2$
s $196x^2 - 224xm + 39m^2$
t $56x^2 + 183xp + 135p^2$
u $6v^2 - 5vc - 4c^2$
v $60p^2 + 23pz - 20z^2$
w $105y^2 - 188yr - 13r^2$
x $22m^2 - 189mq + 180q^2$
y $14z^2 - 61zu - 195u^2$
z $70z^2 + 37za - 3a^2$

11 a $a^2 + 2ab + b^2$
b $x^2 - 2xy + y^2$
c $t^2 + 4t + 4$
d $g^2 + 6g + 9$
e $81 - 18h + h^2$
f $a^2 + 12a + 36$
g $m^2 + 22m + 121$
h $100 - 20b + b^2$
i $f^2 - 2fg + g^2$
j $u^2 + 18u + 81$
k $64 + 16v + v^2$
l $y^2 - 26y + 169$

12 a $a^2 - 4$
b $x^2 - 9$
c $e^2 - 25$
d $n^2 - 16$
e $j^2 - 81$
f $q^2 - 16$
g $h^2 - 4$
h $w^2 - 25$
i $a^2 - r^2$
j $t^2 - p^2$
k $y^2 - g^2$
l $a^2 - p^2$

13 a ${}^-42c^2 + 4q^2 + 12x^2 - 36cq - 19cx - 36qx$
b $35n^2 + 64t^2 + 4x^2 + 76nt + 41nx - 49tx$
c $15a^2 - 63e^2 - 6m^2 + 48ae + 19am + 53em$
d $3p^2 + u^2 - 25y^2 - 4pu + 19py - 2uy$
e $8k^2 - 42x^2 - 72y^2 + 12kx + 4ky - 94xy$
f ${}^-14a^2 + 4e^2 + 14p^2 + 25ae - 28ap + 18ep$
g ${}^-18a^2 + 24m^2 - 45t^2 - 32am + 1at + 26mt$
h $72e^2 - 21u^2 - 36y^2 - 4eu + 52ey - 8uy$
i ${}^-45q^2 - 4r^2 - 18u^2 + 58qr + 54qu - 73ru$
j ${}^-56k^2 + 56w^2 + 20z^2 + 15kw + 22kz - 41wz$
k $10p^2 - 9r^2 - 72v^2 - 40pr - 33pv - 45rv$
l $9b^2 - 10m^2 + 36u^2 + 10bm - 33bu - 52mu$
m $27g^2 + 27h^2 - 15x^2 - 3gh - 51gx + 67hx$
n $18e^2 - 30g^2 + 54v^2 + 9eg - 117ev + 3gv$
o ${}^-9c^2 + 45p^2 - 56r^2 + 6cp - 36cr + 37pr$
p $7b^2 + 20u^2 + 6v^2 - 44bu - 21bv - 28uv$
q ${}^-18e^2 - 54h^2 + 2e + 7h - 2m + 15eh - 9em + 18hm$
r $5a^2 - 24e^2 + 48a + 5e + 8z - 4ae + az - 54ez$
s $6e^2 - 14h^2 + 64q^2 - 21eh - 74eq + 38hq$
t ${}^-18b^2 + 24p^2 + 8y^2 - 14bp - 8by - 34py$
u $54q^2 + 18u^2 - 35w^2 + 6qu - 7qw + 36uw$
v ${}^-1a^2 + 35b^2 - 30u^2 + 13ab + 11au + 91bu$
w ${}^-4a^2 + 10c^2 + 28a + 5c + 5h + 43ac + 28ah - 45ch$
x $27h^2 - 7n^2 + 5p^2 + 39hn - 14hp - 89np$
y ${}^-15a^2 + 7c^2 + 16n^2 - 41ac + 32an - 13cn$
z $4q^2 + 24t^2 - 30x^2 - 47qt + 52qx - 55tx$

14 a $85u^2 - 146ut + 55t^2$
b $11y^2 - 16yg - 24g^2$
c $27p^2 - 57pn + 52n^2$
d ${}^-38b^2 + 70bp - 11p^2$
e $11c^2 + 45cg + 41g^2$
f $49t^2 - 92ta + 41a^2$
g $34c^2 - 33cq - 36q^2$
h $75p^2 + 50pv - 35v^2$
i $23q^2 - 105qr + 7r^2$
j $59h^2 + 91hy + 7y^2$
k $48z^2 + 27zh + 47h^2$
l $23r^2 - 15rn - 44n^2$
m $50h^2 + 142hq + 114q^2$
n $4v^2 - 4vt$
o ${}^-11m^2 - 2my - 4y^2$
p $18x^2 - 61xb + 5b^2$
q $25a^2 - 34ab + 4b^2$
r $20c^2 - 10cp + 5p^2$
s $50h^2 - 81ah$
t $21g^2 + 9gh - 108h^2$
u $11n^2 - 26nc - 12c^2$
v $76q^2 - 38qa - 20a^2$
w $7h^2 + 161hp + 36p^2$
x $56e^2 + 90ez + 61z^2$
y $46c^2 + 71cw + 23w^2$
z $6u^2 - 24uh + 14h^2$

15 a $48q^{-10}r^5v^2 + 50q^{-7}r^4v^{-2} - 64q^{-5}r^7v - 54r^5v^3$
b ${}^-24k^5r^0u^{-1} - 15k^7r^2u^{-4} - 42k^6r^4u^3 + 45r^0u^2$
c $64k^0q^6t + 47k^{-3}q^7t^2 - 56k^{-2}q^4t^6 + 9q^6t^7$
d ${}^-97g^{-1}t^8u^{-6} - 7g^{-6}t^7u^6 - 42g^{-12}t^2u^3 + 40t^8u^{-3}$
e $27r^{-6}u^{-5}y^3 - 38ru^{-5}y^{-7} + 63r^4u^2y^6 - 2u^{-5}y^9$
f ${}^-45b^{-2}h^6q + 15b^{-5}h^6q^3 + 6b^{-4}h^3q^2 - 3h^6q^3$
g ${}^-44a^0n^{12}v^{-5} + 84a^{-4}n^{11}v^5 - 2a^{-9}n^7v - 16n^{12}v^{-4}$
h ${}^-20m^{-3}t^{11}z^{-6} + 48m^{-6}t^8z^6 - 15m^{-12}t^5z^5 - 8t^{11}z^{-1}$
i $11g^5k^{-5}u^2 + 11g^2k^1u^{-3} + 4g^4k^{-2}u - 15k^{-5}u^3$
j ${}^-35b^5u^{-2}w^{-1} - 20b^2u^{-4}w^5 + 18b^1u^{-7}w + 72u^{-2}w^0$
k ${}^-21q^{11}t^{12}w^{-4} + 28q^4t^{13}w^6 - 7q^0t^6w^{-2} + 35t^{12}w^{-6}$
l ${}^-21g^{-6}q^{-9}t^{-5} + 58g^{-3}q^{-7}t^{-5} + 18g^{-8}q^{-4}t^5 + 6q^{-9}t^0$
m $g^4n^0p^2 + 28g^2n^4p^{-2} - 6g^4n^2p^4 + 21n^0p^6$
n ${}^-144a^0w^4y + 19a^4w^{-2}y^2 - 64a^5w^2y^4 - 27w^4y^5$
o ${}^-42e^1n^{10}t^{-1} + 117e^5n^0t^6 - 56e^4n^4t^{-6} + 45n^{10}t^{-7}$
p $15h^8n^{-7}t + 4h^3n^1t^{-3} + 10h^4n^{-4}t^3 - 54n^{-7}t^4$

q $41a^4m^5p + 94a^{-1}m^8p^2 - 6a^0m^3p^{-2} + 10m^5p^{-1}$
r $^{-}27g^{-2}p^{-10}y^4 - 54g^{-4}p^{-4}y^{-4} + 27p^{-6}y^3$
$- 63p^{-10}y^7 + 81g^{-4}p^{-3}y^{-4}$
s $45h^{10}z - 21h^6yz^3 - 27h^7y^{-3}z^6 - 3z^7$
t $48k^{-3}n^{14}q^6 - 12k^{-7}n^{11}q^7 + 5k^{-1}n^7q^3 - 35n^{14}q^9$
u $90k^3q^{-6}r - 21k^5q^{-4}r^{-4} - 24k^6q^{-2}r^6 + 54q^{-6}r^7$
v $11a^9p^{12}x^2 + 16a^7p^9x^5 - 4a^9p^7x^7$
w $^{-}54a^{-5}m^{-3}r^2 - 78a^{-7}m^{-4}r^3 - 9a^{-5}m^{-6}r^{-6}$
$+ 42m^{-3}r^{-4}$
x $^{-}35e^7k^{-1}w - 19e^4k^{-2}w^4 + 12e^5k^{-5}w^2 + 28k^{-1}w^3$
y $34e^{11}r^{-9}v - 51e^5r^{-1}v^{-2} + 8e^6r^{-7}v^{-3} + 21r^{-9}v^{-2}$
z $72a^{-2}b^{-11}u^{-1} - 48a^2b^{-10}u^{-5} - 21a^1b^{-6}u^6 + 63b^{-11}u^5$

16 **a** $2a(10a - 9p + 9u)$ **b** $14b(9k + 10y + 15b)$
c $e(3y + 11e + 7h)$ **d** $-c(8c + 15v + 10t)$
e $13b(13y + 14v + 15b)$ **f** $14g(5h + 12g + 10z)$
g $10a(a - v + 15z)$ **h** $^{-}11c(9n + 14w + 10c)$
i $8a(2t - 15a - 5q)$ **j** $7b(12b - u - 7h)$
k $2c(5r - 6p - 6c)$ **l** $^{-}12e(v + 2e + 9w)$
m $5k(k + 9r - 13v)$ **n** $7x(11y - 7z - 8x)$
o $2b(6u + b + 7m)$ **p** $^{-}5n(15n + 4t + 11p)$
q $^{-}11c(9z + 7m + 11c)$ **r** $^{-}3n(12p + 13n + 4q)$
s $8a(6a + 11e + 6y)$ **t** $2a(15h - 11z - 14a)$
u $10e(5v - 8e - 11m)$ **v** $12a(11a - y - 13m)$
w $-k(13w + 3p + 6k)$ **z** $4k(p - 9k - 15x)$
y $14u(7u - 12v - 9x)$ **z** $15g(q - 7u - 8g)$

17 **a** $^{-}14e^2k^2q^3(12ek^4 + 13k^4q^2 + 14e^5q^4)$
b $11w^3yz^5(4y^2z^5 - 13zw + 6yw^3)$
c $^{-}7m^4n^5u^5(5u^2m^2 + 9m^5n + 10u^4)$
d $2g^3r^3y(15g^3y^3 - 14y^2r^3 - 9g^3r^5)$
e $^{-}7c^5y^3z^3(4z^3 + 7y^2c^3 + 5z^3c)$
f $10hm^3u(7m^6h - 11h^3u^2 + 11m^4u^3)$
g $2gp^4q^5(15g^5 + 7pq^2 - 12g^5q)$
h $a^2r^4z(4r^5z - z^2a^5 + 3r^3a^4)$
i $4a^2m^4p^2(5pa + 6a^5m^4 - 7p^4m^4)$
j $11bm^2q^2(10b^4q^2 - 11q^3m^4 + 10b^2m^5)$
k $8an^5x(15x^3 + n^5a^5 - 4xa)$
l $11cp^5v^3(6p^8c^4 - 5c^4v^4 - 7p^5v)$
m $7be^2h^3(11b^5 + 10e^2h^2 - 15b^5h^4)$
n $15e^5r^4z^2(5r^3z^2 + 8z^4e^5 + 10r^4e^3)$
o $11h^4x^4y^4(2y^4h^3 - 5h^5x^2 - 6y)$
p $7a^2hu^2(15a^2u^5 + u^2h^2 - 4a^4h^5)$
q $^{-}7e^2tz^5(5z^5 + 8t^4e^3 + 4z^3e)$
r $2h^2u^5(13q^{10}h^5 + 6hu^4 + 7qu^4)$
s $^{-}10e^4v^2x^3(13e + 14v^3x^2 + 12e^5x^2)$
t $a^5k^5t^3(5k^4t^4 + 12t^3a^3 + 4k^5a^3)$
u $13c^5x^5y(4y^4c - 3c^3x - 10y^5x^2)$
v $15eu^2z^3(7e^3z^4 - 4z^2u + 7e^5u^3)$
w $t^2xz^5(7z^2 - 3xt^2 + 9z^5t^5)$
x $^{-}11g^2tx^4(8t^8g^4 + 9x + 11x^2)$
y $^{-}8g^3xz^4(2 + z^3 + 4g^3z^4)$
z $^{-}13p^3r^3(7p^5r^2 + 8r^3b^5 + 7p^2b^4)$

18 $14x^7 - 2x^6 + 49x^5 - x^4 - 12x^3 + 21x^2 - 42x$

19 $^{-}6x^8 + 3x^7 - 7x^6 - 6x^4 + 21x^3 - 16x^2 + 21x$

20 $63x^8 + 45x^7 - 63x^5 - 36x^4 - 9x$

21 $35x^{10} + 5x^9 - 37x^8 - 57x^7 + 69x^6 + 71x^5 - 58x^4 - 21x^3 +$
$24x - 3$

22 **a** $4a^2 + 12a + 9$ **b** $9x^2 + 6x + 1$
c $25m^2 - 20m + 4$ **d** $9h^2 - 42h + 49$
e $64u^2 - 48u + 9$ **f** $81k^2 - 36k + 4$
g $16d^2 + 40d + 25$ **h** $81k^2 - 36k + 4$
i $4a^2 + 20am + 25m^2$ **j** $16g^2 + 24gs + 9s^2$
k $49h^2 + 28hk + 4k^2$ **l** $64d^2 - 16dq + q^2$

23 **a** $4b^2 - 1$ **b** $9c^2 - 4$ **c** $25m^2 - 9$
d $1 - 4x^2$ **e** $16m^2 - 1$ **f** $64g^2 - 9$
g $4a^2 - p^2$ **h** $c^2 - 9d^2$ **i** $64d^2 - 4$
j $64r^2 - 25y^2$ **k** $49y^2 - 16d^2$ **l** $36 - 9p^2$

24 **a** $a^2 + b^2 + 2ab - 16$
b $x^2 + y^2 + 2xy - 1$
c $c^2 + 4d^2 - 4cd - 9$
d $4x^2 + y^2 - 4xy + 12x - 6y + 9$
e $m^2 + n^2 - 2mn - 4$
f $b^2 - a^2 + 8b + 16$
g $4c^2 - d^2 - 6d - 9$
h $4r^2 - 9t^2 - s^2 + 6st$
i $\frac{4}{x^2} - \frac{1}{y^2} + \frac{2}{y} - 1$
j $\frac{1}{a^2} - \frac{1}{b^2} + \frac{4}{b} - 4$

25 **a** $\frac{g^2t^6\left(3a^7g^2 + 13g^5t^6 + 13at^2\right)}{a^3}$
b $\frac{15z^5\left(11z^3q^3 - 9q^4v + 14z^2v^2\right)}{q^4v^3}$
c $\frac{^{-}7c^6u^3\left(14yc^6 + 3c^4u^6 + 4y^5u^6\right)}{y^4}$
d $\frac{2m^3u(9m^6u - 4u^4q^5 + 8m^6q^4)}{q^6}$
e $\frac{13t^4z^5\left(t^7m^2 + 15mz^6 + 4tz\right)}{m^3}$
f $\frac{3\left(14p^{11}h^5 - 5h^7q^3 + p^4q^4\right)}{h^6p^6q^7}$
g $\frac{2(5hp^5 - 12e^4 - 11e^2p^5)}{e^6h^3p^7}$
h $\frac{9q^7\left(5z^5 + 14z^4h^4 - 4q^5h^3\right)}{h^2z^4}$
i $\frac{^{-}10a^5g^8(2x^3a + 13a^4g^4 + 15x^3g^3)}{x^6}$
j $\frac{15r^3\left(2q^6r^5 - 15n^2 + 7nr^5\right)}{n^7q^6}$

26 **a** $4p^2 - 4p - 15$ **b** $4p^2 + 4p - 15$
c $\frac{\pi(36m^2 + 60m + 25)}{4}$

27 **a** $\left(n+\frac{1}{2}\right)^2 = n^2+n+\frac{1}{4} = n(n+1)+\frac{1}{4}$

b $\left(3\frac{1}{2}\right)^2 = 3\times 4+\frac{1}{4} = 12\frac{1}{4}$

ii $\left(19\frac{1}{2}\right)^2 = 19\times 20+\frac{1}{4} = 380\frac{1}{4}$

28 Area of outside circle $= \pi R^2$ and the area of the inside circle $= \pi r^2$.

Area of annulus $= \pi R^2 - \pi r^2$.

Now $\pi(R-r)(R-r) = \pi[R(R+r) - rR + r)]$

$= \pi(R^2 + Rr - Rr - r^2) = \pi R^2 - \pi r^2$ QED.

Exercise 3.3

1 **a** $\frac{9a+8y}{12}$ **b** $\frac{2g+3m}{9}$ **c** $\frac{14m+n}{10}$

d $\frac{4c-27k}{18}$ **e** $\frac{45c-4r}{40}$ **f** $\frac{15g-28y}{20}$

g $\frac{5bn}{48}$ **h** $\frac{cp}{288}$ **i** $\frac{ek}{56}$

j $\frac{16g}{63u}$ **k** $\frac{21e}{4h}$ **l** $\frac{49m}{72v}$

m $\frac{11p}{6}$ **n** $\frac{3q}{20}$ **o** $\frac{19a}{42}$

p $\frac{25m}{24}$ **q** $\frac{11q}{18}$ **r** $\frac{-c}{4}$

s $\frac{21b^2}{8}$ **t** $\frac{5c^2}{36}$ **u** $\frac{21a^2}{8}$

v $\frac{5}{24}$ **w** $\frac{3}{4}$ **x** $\frac{1}{60}$

y $\frac{^{-}13b}{30}$ **z** $\frac{53b}{18}$

2 **a** $\frac{111h-89}{42}$ **b** $\frac{17g-35}{30}$ **c** $\frac{58k+41}{10}$

d $\frac{60h-35}{28}$ **e** $\frac{80g-95}{18}$ **f** $\frac{16b-3}{12}$

g $\frac{52g+22}{15}$ **h** $\frac{22n-1}{6}$ **i** $\frac{14a-1}{12}$

j $\frac{34a-55}{40}$ **k** $\frac{67b-10g}{40}$ **l** $\frac{6a+11e}{9}$

m $\frac{14h-25y}{6}$ **n** $\frac{38r+41t}{12}$ **o** $\frac{32g+41x}{40}$

p $\frac{43g-34m}{10}$ **q** $\frac{16q-7r}{10}$ **r** $\frac{29h-19t}{12}$

s $\frac{23g-8x}{10}$ **t** $\frac{30c-5e}{12}$

3 **a** $\frac{35-6r}{36}$ **b** $\frac{27r+51}{20}$ **c** $\frac{6e+13}{12}$

d $\frac{8-11k}{6}$ **e** $\frac{5v+13}{8}$ **f** $\frac{n+36}{15}$

g $\frac{38c+29}{24}$ **h** $\frac{37-13h}{30}$ **i** $\frac{17h-55}{10}$

j $\frac{k+2}{12}$ **k** $\frac{v-22e}{18}$ **l** $\frac{15c+37t}{12}$

m $\frac{u+24v}{30}$ **n** $\frac{17t-7k}{6}$ **o** $\frac{2g+r}{2}$

p $\frac{21c+38p}{40}$ **q** $\frac{3c+13p}{5}$ **r** $\frac{71t-62e}{40}$

s $\frac{15q-11p}{12}$ **t** $\frac{6g-31v}{18}$

4 **a** $\frac{21q^2-61q+28}{40}$ **b** $\frac{6e^2+55e+56}{35}$

c $\frac{32a^2-4a-21}{45}$ **d** $\frac{6e^2+5e-21}{48}$

e $\frac{35g^2-44g+12}{6}$ **f** $\frac{35c^2+101c+72}{54}$

g $\frac{4b^2-27b-81}{10}$ **h** $\frac{3a^2-26a-9}{108}$

i $\frac{54e^2+39e-5}{54}$ **j** $\frac{18b^2-45b+25}{40}$

k $\frac{21p^2+50pq+25q^2}{18}$ **l** $\frac{32b^2-68bc+35c^2}{20}$

m $\frac{24n^2-35nq+4q^2}{216}$ **n** $\frac{7c^2+41cm+30m^2}{12}$

o $\frac{5n^2-42np+16p^2}{12}$ **p** $\frac{6a^2+37ae+35e^2}{28}$

q $\frac{5c^2+16cg+12g^2}{24}$ **r** $\frac{45g^2+8gn-21n^2}{32}$

s $\frac{18p^2-33pq+14q^2}{24}$ **t** $\frac{72g^2+25gu+2u^2}{48}$

5 **a** $\frac{35n-63}{12n-18}$ **b** $\frac{63p-49}{12p+10}$ **c** $\frac{72p+8}{72p-63}$

d $\frac{14a+7}{72a-56}$ **e** $\frac{20c+15}{56c+32}$ **f** $\frac{28h+7}{21h-12}$

g $\frac{24u-9}{16u-20}$ **h** $\frac{35e+45}{32e-12}$ **i** $\frac{45a-10}{49a-14}$

j $\frac{10a-35}{18a+30}$ **k** $\frac{35n-42p}{81n+45p}$ **l** $\frac{72c-63h}{18c+8h}$

m $\frac{63e+28p}{15e+9p}$ **n** $\frac{3e+27n}{48e+56n}$ **o** $\frac{72a+56h}{14a-49h}$

p $\frac{27a+24e}{30a+5e}$ **q** $\frac{32g+56v}{18g+3v}$ **r** $\frac{7g-63h}{48g-42h}$

s $\frac{21h+35p}{45h-40p}$ **t** $\frac{5b-15r}{24b+15r}$

6 **a** $\frac{2b^6}{15g^5y^3}$ **b** $\frac{n^4r^8u^5}{2}$

c $\frac{3m^4u^2w^2}{32v^5}$ **d** $\frac{3b^7qw}{8}$

7 **a** $\frac{23a+128k}{42}$ **b** $\frac{3c+18k}{5}$

c $\frac{32a^2+8ak-60k^2}{36}$ **d** $\frac{18h^2-77ph+80p^2}{24}$

e $\frac{6k+9w}{2}$ **f** $\frac{22a-29e}{24}$

g $\dfrac{30a^2 + 21ak - 36k^2 + 28a - 32k}{36}$

h $\dfrac{30p^2 + 93hp - 42h^2 + 32h - 28p}{24}$

8 **a** $\dfrac{8m + 5}{2}$ **b** $\dfrac{10p - 1}{6}$ **c** $\dfrac{22v + 12u}{15}$

9 **a** $\dfrac{4t^2 + 12t + 9}{25}$ **b** $\dfrac{6b^2 - b - 2}{60}$

c $\dfrac{7m^2 + 33m + 20}{20}$

10 **a** A, $\dfrac{57q + 23r}{24}$ **b** B, $\dfrac{11h - 12b}{12}$

c No error **d** C, $\dfrac{27k - 63u}{25k + 5u}$

11 Find the areas of each of the parts (rectangle, triangle and quarter circle) and add to get $\dfrac{(408 + 27\pi)x^2 - (344 + 36\pi)x + 48 + 12\pi}{192}$ cm^2.

Chapter review

1 **a** $7p$ **b** $4t - 2h$ **c** $3xy - 2x^2 - 3y$

2 **a** $m^2 - 3m$ **b** $6k^2 - 8km$

c $x^2 + 7x + 10$ **d** $4e^2 + 7e - 15$

3 **a** $6(5b + 3)$ **b** $^{-}3(3m + 7)$

4 $d = 25$

5 **a** $\dfrac{27q + 64w}{24}$ **b** $\dfrac{5p}{18}$ **c** $\dfrac{91b^2}{54}$ **d** $\dfrac{91r}{18t}$

6 **a** $5m^6, 6$ **b** Not a polynomial **c** $^{-}5x^9, 9$

7 **a** $\dfrac{6n}{7u}$ **b** $42g^4k^3r^3$

8 **a** $r = \dfrac{A}{2\pi s}$ **b** $u = x - Zs$

9 **a** $20t + 13$ **b** $31v - 17v^2$

c $3m^2p^2t^2 - 9m^2p^3t + 6mp^3t^2$ **d** $30k - 9k^2 - 6k^3$

10 **a** $^{-}40b^2c^{-3}g^5 - 112b^{-3}c^{-3}$

b $7a^2 + 7c^2 + 36x^2 + 1ac - 75ax + 4cx$

c $^{-}19g^2 + 31gp + 25p^2$

d $18m^{19}u^{-3} + 12g^{10}m^{-1}u^{10} - 4g^7m^9u^3 - 32m^{19}$

11 **a** $16 - a^2$ **b** $h^2 + 6h + 9$

12 **a** $14g(3n + 14w - 15g)$

b $9ar^5w^4(7r^5w^2 - 15wa^3 + 11r^4a)$

c $\dfrac{8(3z^3n^9y^9 + 11ny^4 + 5zy^9)}{n^6y^6z^3}$

13 **a** $\dfrac{74q + 27r}{42}$ **b** $\dfrac{13p + 41}{20}$

c $\dfrac{40b^2 - 77bv + 36v^2}{16}$ **d** $\dfrac{5p + 5}{48p - 54}$

e $\dfrac{128g^6h^2}{125k^8}$ **f** $\dfrac{x^2 - u^2}{24}$

14 **a** $\dfrac{7t^3}{9b^2u}$ **b** $\dfrac{^{-}40e^3}{h^9}$ **c** $\dfrac{^{-}6u^{14}}{7e^7t^{11}y^5}$

d $\dfrac{20g^4x^7}{p^{11}}$ **e** $\dfrac{^{-}63b^{13}t^3}{g^{10}v^8}$

15 **a** 146 **b** 1210 **c** 452

d 472 **e** 560 **f** 22

16 **a** $6x^5 - 9x^4 - 7x^3 - 8x^2 + 6x - 8$

b $6x^5 + 4x^3 - 14x^2 + 6x - 11$

c $6x^5 - 14x^3 - 4x^2 - 6x - 5$

17 $\dfrac{5e^2v}{3w}$

18 $28x^8 - 32x^7 + 15x^6 + 19x^5 - 30x^4 + 24x^3 - 48x^2 + 54x - 30$

19 **a** $49 - 42q + 9q^2$ **b** $36y^2 - 25$

c $25h^2 + 20hv + 4v^2$ **d** $m^2 + 2mn + n^2 - 36$

20 **a** $n \approx 0.203$ **b** $a = \frac{4}{3}$

21 $\dfrac{34y - 38}{15}$ cm

22 **a** $15n^2 + mn - 6m^2$ **b** $\dfrac{2m^2 - mn - 15n^2}{12}$ cm^2

23 **a** B, $\dfrac{m + 14n}{12}$

b A is unnecessary, also error in C, $\dfrac{12b^2 - 17bc + 6c^2}{12}$

24 $(10n + 5)^2 = 100n^2 + 100n + 25 = 100n(n + 1) + 25$. To find the square of a number ending in 5, multiply the tens digit by one more than the tens digit and put 25 on the end of it. For 65^2, $6 \times 7 = 42$, so $65^2 = 4225$.

25 Proof using examples.

26 Proof.

Chapter 4

Exercise 4.1

1 **a** 73 000 m^2 **b** 0.048 km^2 **c** 0.57 km^2

d 2.5 ha **e** 82.7 m^2 **f** 3420 m^2

2 **a** 144 cm^2 **b** 198 cm^2 **c** 1.35 m^2

d 2.6 m^2 **e** 70 mm^2 **f** 2.38 m^2

g 1144 cm^2

3 **a** 196 mm^2 **b** 4.68 m^2 **c** 3285 cm^2

d 9.12 m^2 **e** 27.36 cm^2 **f** 0.9 km^2

g 2290.2 cm^2 **h** 176.7 mm^2 **i** 3.4 m^2

4 **a** 0.6 m^2 **b** 1963.5 mm^2 **c** 150 cm^2

d 1.2 m^2 **e** 113.1cm^2 **f** 7.1 m^2

g 2185 cm^2 **h** 3.99 m^2

5 **a** 1.375 m^2 **b** 63 cm^2 **c** 1.75 m^2

d 2.94 cm^2 **e** 6.665 m^2 **f** 684 m^2

g 210 cm^2 **h** 621 mm^2 **i** 8.37 m^2

6 **a** 1045.6 mm^2 **b** 113.5 cm^2 **c** 56.7 cm^2

d 4185.4 mm^2 **e** 7238.2 m^2 **f** 1.5 km^2

7 **a** 619 cm^2 **b** 172.5 cm^2 **c** 43.5 m^2

8 **a** 26.46 cm^2 **b** 1088.64 mm^2 **c** 7.75 m^2

9 **a** 370.2 cm^2 **b** 1 878 500 m^2 (1.8785 km^2)

c 179.0 cm^2

10 **a** 654.5 m^2 **b** 641.4 m^2 **c** 256 313.7 mm^2

d 2315.2 cm^2 **e** 141.5 mm^2 **f** 137.7 m^2

11 **a** 372.2 cm^2 **b** 38.79 m^2

12 **a** 11.96 cm^2 **b** 29.4 cm^2 **c** 1710.1 cm^2

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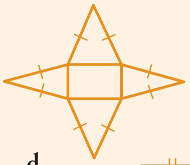
13 **a** 4.5 cm^2 **b** 13.0 cm^2 **c** 8.9 cm^2
14 15.0 cm
15 10.5 cm by 10.5 cm
16 **a** 26.24 m^2 **b** 194.34 m^2 **c** 63.99 m^2
17 160.2 cm^2
18 40 cm^2
19 Circle area $= \pi r^2$
Semicircle area $= \frac{1}{2}\pi(2r)^2$
Shaded area $= \frac{1}{2} \times \pi \times (2r)^2 - \pi r^2 = \pi r^2$ (where r is the radius of the smaller circle)

Exercise 4.2

1 **a** Cube or regular hexahedron
b Triangular prism
c Tetrahedron **d** Slant cone
e Hexagonal prism **f** Hexagonal pyramid
g Dodecahedron **h** Slant pentagonal pyramid
2 **a** Cube or regular hexahedron
b Rectangular prism
c Hexagonal prism **d** Cylinder
e Triangular prism **f** Pentagonal prism
3 Other arrangements of nets are possible.

a

b



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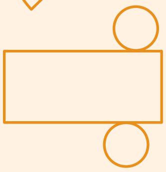

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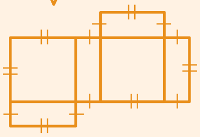

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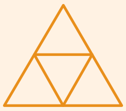

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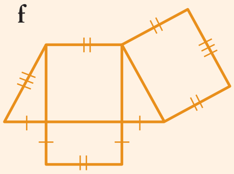

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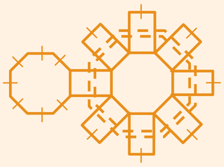
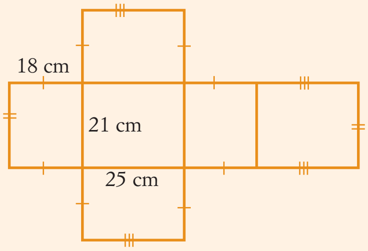


4 **a**



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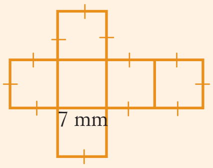

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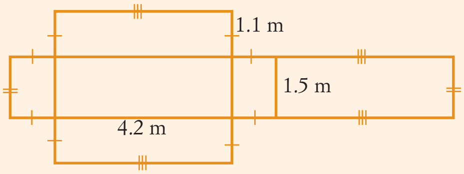

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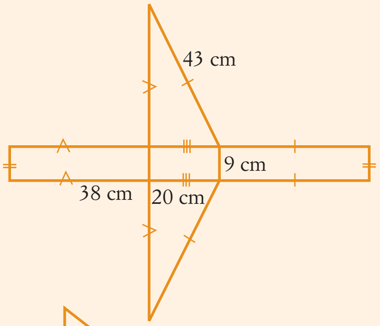

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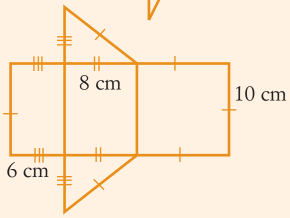

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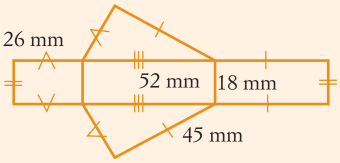

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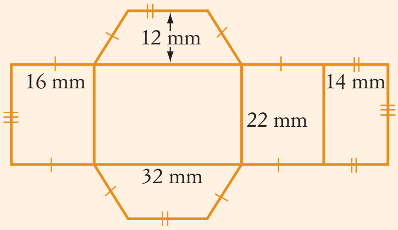

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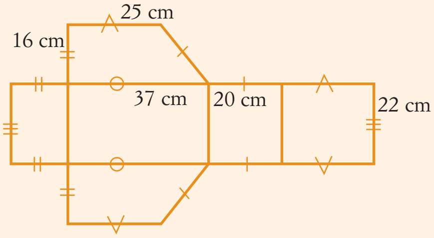
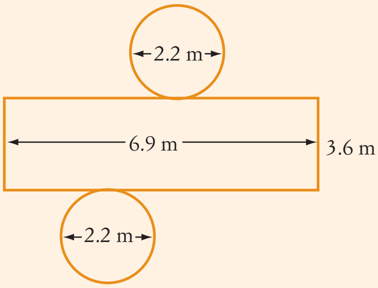


6 **a**



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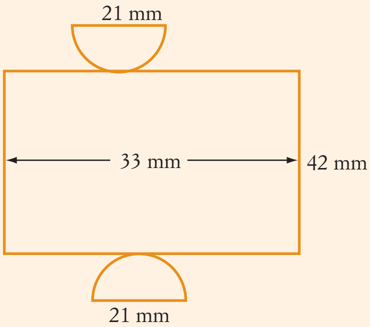

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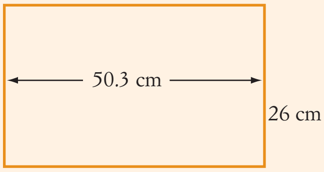
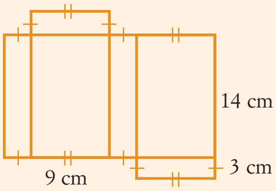


7 **a**



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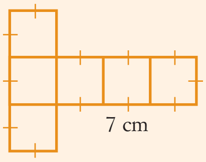

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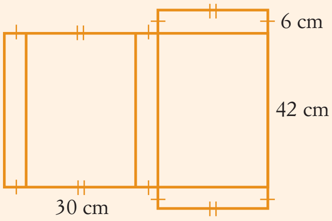
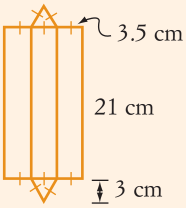


8 **a**



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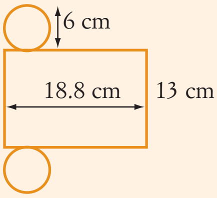

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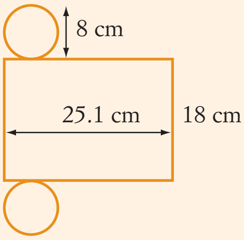
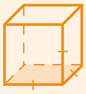
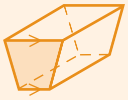


9 **a** 2706 cm^2 **b** 294 mm^2 **c** 25.14 m^2
d 1670.9 cm^2 **e** 288 cm^2 **f** 3358 mm^2
10 **a** 2268 mm^2 **b** 3148 cm^2
11 **a** 32.5 m^2 **b** 1731.8 mm^2 **c** 1306.9 cm^2
12 **a** 390 cm^2 **b** 294 cm^2 **c** 3384 cm^2
13 **a** 231 cm^2 **b** 301.6 cm^2 **c** 552.9 cm^2
14 **a** 872 cm^2 **b** 1858 mm^2 **c** 12.46 m^2
d 5558 cm^2 **e** 7.612 m^2 **f** 1459.2 cm^2
15 **a** 483.8 cm^2 **b** 4222.3 mm^2 **c** 35.2 m^2
d 444.1 cm^2 **e** 45 521.7 cm^2 **f** 1212.3 cm^2
16 **a** 3487.2 cm^2 **b** 103.8 m^2 **c** 22 900.6 mm^2
d 32 716.5 cm^2 **e** 151 449.9 cm^2 **f** 9900.7 mm^2
17 **a** 1528 cm^2 **b** 22.74 m^2 **c** 896 mm^2
d 736 cm^2 **e** 319.74 m^2 **f** 226.44 cm^2
18 **a** 4632 cm^2 **b** 64.45 m^2 **c** 3076 mm^2
19 **a** 3966.3 cm^2 **b** 3732.2 mm^2
c 4.1 m^2 **d** 1558.2 cm^2
20 **a** 1360 cm^2 **b** 2525 mm^2 **c** 27.92 m^2
21 **a** 681.7 cm^2 **b** 233.3 mm^2
c 66.8 m^2 **d** 512.7 cm^2
22 **a** 314.2 cm^2 **b** 2463 mm^2 **c** 64.3 m^2
d 4356.6 cm^2 **e** 5.7 m^2 **f** 9351.7 mm^2
23 **a** 825 cm^2 **b** 429 mm^2 **c** 171.4 m^2
24 **a** 112 cm^2 **b** 2304 mm^2
25 **a** $x = 10$ mm **b** 384 mm^2
26 **a** 3267.3 cm^2 **b** 718.6 m^2 **c** 1545.7 mm^2
27 **a** 9000 mm^2 **b** 873.8 cm^2 **c** 46.46 m^2
28 **a** 40.7 m^2 **b** 166.3 cm^2 **c** 2102.1 mm^2
29 **a** 1256.6 cm^2 **b** 2827.4 mm^2 **c** 1134.1 cm^2
30 **a** 763.4 mm^2 **b** 39.3 m^2 **c** 374.1 cm^2
31 Other shapes are possible.

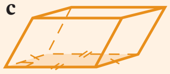
a



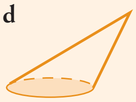
b



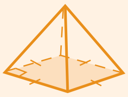
c



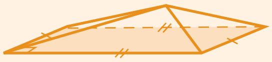
d



e

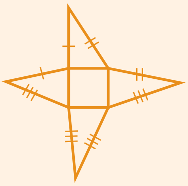


f

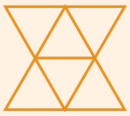


32 Other arrangements of nets are possible.

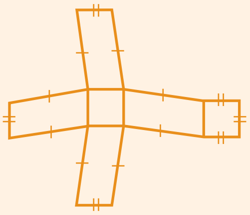
a



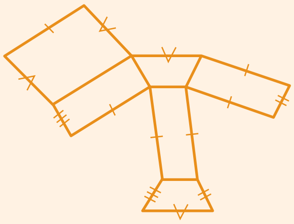
b



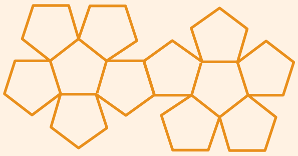
c



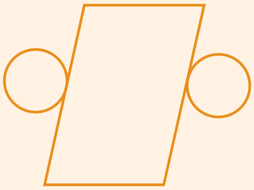
d



e



f



33 **a** Slant cylinder **b** Regular octahedron
c Regular icosahedron **d** Regular dodecahedron
34 **a** 7206.7 cm^2 **b** 54 788.5 mm^2
c 80.5 m^2 **d** 13 609.4 cm^2
35 172 m^2 **36** 49.3 m^2
37 **a** 11.4 m (rounded) **b** 207.4 m^2
c 25.9 L is required so 7 tins are needed at a cost of $875.
38 1979.2 cm^2 **39** 519.9 mm^2
40 **a** 67.48 m^2 **b** $3846.36
41 **a** 22.9 m^2 **b** 2.5 L
42 **a** 3.8 m × 6.5 m **b** $1953.75
43 7 full revolutions (6.4)
44 4397 mm^2 **45** 235 000 mm^2 (2350 cm^2)

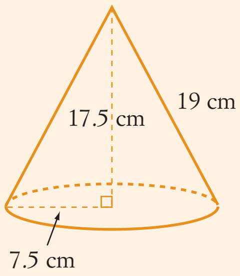
Exercise 4.3

1 **a** 2.4 L **b** 3200 cm^3 **c** 7500 mL
d 0.73 L **e** 41 mL **f** 8200 L
g 432 cm^3 **h** 0.003 24 m^3 **i** 2.32 L
2 **a** 0.0054 kL **b** 27 000 cm^3 **c** 0.0043 kL
d 354 000 m^3 **e** 25 L **f** 840 000 cm^3

3

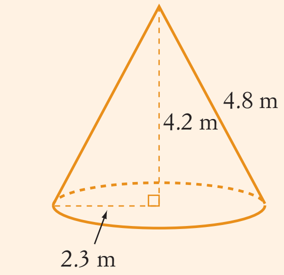
	Length	Width	Height	Volume
a	15 cm	12 cm	8 cm	1440 cm^3
b	4 m	3 m	10 m	120 m^3
c	17 mm	11 mm	22 mm	4114 mm^3
d	26.4 cm	17.5 cm	14.4 cm	6652.8 cm^3
e	6.7 m	3.2 m	1.5 m	32.16 m^3

4 **a** 27 cm^3 **b** 192 cm^3 **c** 0.648 m^3
d 19 800 cm^3 **e** 312 cm^3
5 **a** 19 974.2 cm^3 **b** 4712.4 mm^3 **c** 8.6 m^3
d 1125.4 cm^3 **e** 6.4 m^3 **f** 5695.7 mm^3
6 **a** 82 m^3 **b** 619.7 cm^3 **c** 3.927 m^3
d 2205.3 cm^3 **e** 3234 mm^3
7 **a** 179.6 cm^3 **b** 41 629.8 mm^3 **c** 8580.2 cm^3
d 10.3 m^3 **e** 190.85 cm^3
8 **a** 1026 cm^3 **b** 2.55 m^3 (2.5498 m^3)
c 39.3 cm^3 (39.294 cm^3)
9 **a** 234 cm^3 **b** 1248 cm^3 **c** 0.8528 m^3
d 7290 cm^3 **e** 3.6 m^3 **f** 192 mm^3
10 **a** 110.25 cm^2 **b** 111.4 cm^3 **c** 500.3 cm^3
11 **a** 700 cm^3 **b** 288 m^3 **c** 17.3 cm^3
12 **a** 14 024.1 mm^3 **b** 27.2 m^3 **c** 7125.1 cm^3
d 3.2 m^3 **e** 130 765.7 cm^3 **f** 61.5 mm^3
13 **a** 58.3 cm^3 **b** 57.9 m^3 **c** 131.5 m^3
14 **a** 1.6 m^3 **b** 603.2 cm^3 **c** 7485.9 mm^3
15 **a** 1838.8 cm^3 **b** 25.5 m^3 **c** 14 365.5 mm^3
16 **a** 37 875 cm^3 **b** 7863.1 cm^3
c 867 415.1 cm^3 **d** 147 436.1 cm^3
17 **a** 1206.4 cm^3 **b** 1013.7 cm^3 **c** 86 295.1 cm^3
18 **a** 2805 cm^3 **b** 15 620 cm^3 **c** 8833.3 cm^3
19 **a** 2348.2 cm^3 **b** 14 678.1 cm^3 **c** 6.16 m^3
20 **a** 33 895.7 cm^3 **b** 1.7 m^3 **c** 410 103 mm^3
21 1308.3 cm^3
22 **a** **i**



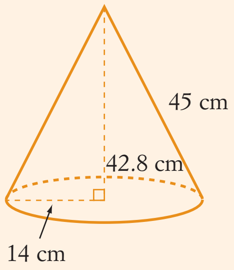
ii 1028.3 cm^3

b **i**



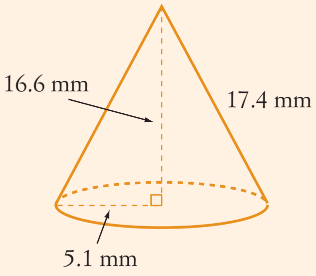
ii 23.3 m^3

c **i**



ii 8784.7 cm^3

d **i**



ii 453.1 mm^3

23 1624.6 cm^3 **24** 72 cm^2 **25** 20 solids

26 **a** 2400 m^3 **b** 2400 kL **c** $16\frac{2}{3}$ hours

27 **a** 50 **b** 500 **c** About 52%.

28 **a** About 1413 times. **b** About 4878 km.

29 **a** 44 m^3 **b** About 21 min.

30 **a** 44.7 m^3 **b** About 1.2 m. **c** $2570.40

31 **a** 193.8 m^3 **b** About 1.6%. **c** 183.7 m^2

32 **a** 146 m **b** 137 m

33 The Aztec pyramid by about 1 900 000 m^3.

34 No, the 'anchor' has a mass of approximately 31.33 t.

35 $\frac{\text{Volume of sphere}}{\text{Volume of cylinder}} = \frac{2}{3}$

Chapter review

1 **a** 0.0567 km^2 **b** 38 700 m^2

c 0.57 km^2 **d** 2.5 ha

2 **a** 289 cm^2 **b** 360 cm^2 **c** 338.12 m^2

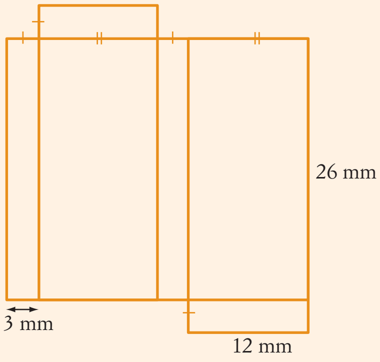
d 224 cm^2 **e** 331.5 mm^2

3 136 mm^2

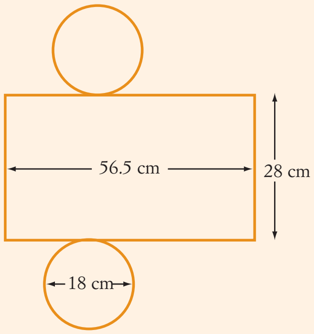
4 **a** Cylinder **b** Square pyramid **c** Pentagonal prism

5 **a** Rectangular pyramid **b** Cone

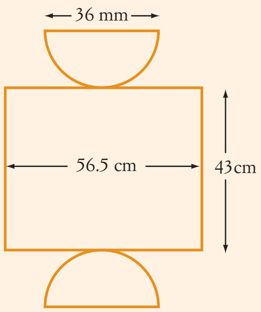
6 **a**



b



c



7 **a** 2750 mL **b** 5430 L

c 271 cm^3 **d** 0.003 600 kL

e 49 000 cm^3 **f** 0.002 800 kL

8 **a** 204.1 cm^2 **b** 52.8 mm^2

9 2830 cm^2 **10** 44.6 cm^2

11 **a** 304.8 cm^2 **b** 10.9 m^2

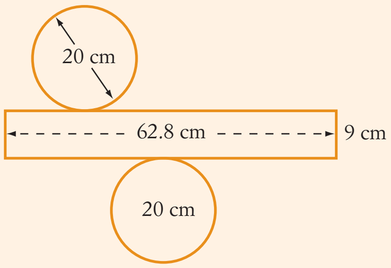
12 **a** $m^2 = p^2 + q^2$ **b** $p = 1.4$

13 **a** 2.16 m^2 **b** 234.9 cm^2

14 852 mm^2 **15** 2092.3 cm^2 **16** 3449.5 mm^2

17 **a** 1298 cm^2 **b** 1140 mm^2 **c** 25.12 m^2

18 **a**



b 1193.8 cm^2

19 **a** 678.6 cm^2 **b** 622 mm^2

20 **a** 523.1 cm^2 **b** 41.2 m^2

21 **a** 16 400.7 cm^2 **b** 24.7 m^2

22 **a** 480 cm^2 **b** 650.3 mm^2

c 36.3 m^2 **d** 1592.8 cm^2

23 **a** 4216 cm^2 **b** 23.3 m^2

c 3631.7 mm^2 **d** 3402.3 cm^2

24 **a** 27 cm^3 **b** 192 cm^3 **c** 19 974.2 mm^3

d 82 m^3 **e** 3.927 m^3 **f** 22.4 cm^3

25 77 cm^3 **26** 540 cm^3 **27** 2.288 m^3

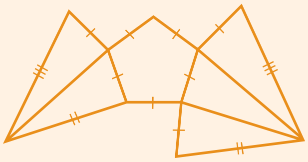
28 **a** 112.2 m^3 **b** 6792.9 cm^3 **c** 15 239.9 mm^3

d 5111.3 m^3 **e** 22 066.6 cm^3 **f** 139.1 m^3

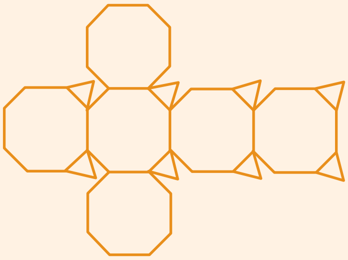
29 **a** Slant rectangular pyramid

b Octahedron (truncated tetrahedron)

30 **a**



b



31 17 mm **32** 24.4 m **33** 96.74 m^2

34 **a** 68 204 cm^3 **b** 21 275 cm^3

35 **a** 122.6 cm^3 **b** 2236.4 cm^3

36 11.4 cm^3

37 No, the water will weigh about 6.9 tonnes.

38 **a** \$1380.63 **b** 127.8 m^3 **c** Almost 9 hours.

39 72 mm

Chapter 5

Exercise 5.1

1 **a**, **d**, **f**, **g** are rising; **b**, **c**, **h**, **i**, are falling; **e** and **j** are neither.

2 **a** and **h** are in the first quadrant; **b, d** and **g** are in the second quadrant; **c** and **e** are in the third quadrant; **f** is in the fourth quadrant.

3 **a** $(2, \frac{1}{2})$ **b** $(^{-}1, 3)$ **c** $(^{-}\frac{1}{2}, ^{-}2)$ **d** $(1, ^{-}1\frac{1}{2})$

e $(1, 1)$ **f** $(^{-}3, 1\frac{1}{2})$ **g** $(^{-}3\frac{1}{2}, \frac{1}{2})$ **h** $(^{-}1, \frac{1}{2})$

4 **a** $\sqrt{8} \approx 2.83$ units **b** $\sqrt{52} \approx 7.21$ units

c 13 units **d** $\sqrt{274} \approx 16.55$ units

e $\sqrt{162} \approx 12.73$ units **f** $\sqrt{162} \approx 12.73$ units

g $\sqrt{61} \approx 7.81$ units **h** $\sqrt{148} \approx 12.17$ units

i $\sqrt{73} \approx 8.54$ units **j** $\sqrt{233} \approx 15.26$ units

5 **a** $(5, 1)$ **b** $\left(^{-}1, \frac{1}{2}\right)$ **c** $(0, 6)$ **d** $\left(\frac{1}{2}, \frac{1}{2}\right)$

e $(^{-}5, 5)$ **f** $\left(\frac{1}{2}, 1\frac{1}{2}\right)$ **g** $\left(^{-}3\frac{1}{2}, 2\frac{1}{2}\right)$ **h** $\left(4\frac{1}{2}, ^{-}2\right)$

i $\left(^{-}4\frac{1}{2}, ^{-}8\right)$ **j** $\left(\frac{1}{2}, ^{-}3\right)$

6 **a** 2 **b** $^{-}1$ **c** $\frac{^{-}2}{3}$ **d** $\frac{1}{2}$ **e** 0

f 3 **g** $\frac{1}{3}$ **h** $\frac{^{-}1}{3}$ **i** $^{-}3$

j Not defined (division by 0)

7 **a** 3 **b** $^{-}2$ **c** 2 **d** 7 **e** $\frac{1}{8}$

f $^{-}2$ **g** $^{-}1$ **h** 1 **i** 0 **j** $\frac{^{-}3}{4}$

k Not defined (division by 0) **l** $\frac{^{-}3}{2}$

8 **a** **i** $\sqrt{41}$ **ii** $(3.5, 5)$ **iii** $\frac{4}{5}$

b **i** $\sqrt{18}$ **ii** $(5.5, 2.5)$ **iii** $^{-}1$

c **i** $\sqrt{29}$ **ii** $(^{-}4, 3.5)$ **iii** $^{-}2\frac{1}{2}$

d **i** $\sqrt{170}$ **ii** $(1.5, 2.5)$ **iii** $\frac{^{-}1}{13}$

e **i** $\sqrt{58}$ **ii** $(^{-}3.5, 3.5)$ **iii** $\frac{^{-}3}{7}$

f **i** $\sqrt{130}$ **ii** $(^{-}3.5, 5.5)$ **iii** $\frac{7}{9}$

g **i** $\sqrt{392}$ **ii** $(2, 1)$ **iii** $^{-}1$

h **i** $\sqrt{50}$ **ii** $(8.5, ^{-}0.5)$ **iii** 1

i **i** $\sqrt{37}$ **ii** $(^{-}4.5, ^{-}4)$ **iii** $^{-}6$

j **i** $\sqrt{34}$ **ii** $(^{-}5.5, 0.5)$ **iii** $\frac{^{-}3}{5}$

k **i** $\sqrt{41}$ **ii** $(^{-}2.5, ^{-}2)$ **iii** $\frac{^{-}4}{5}$

l **i** $\sqrt{148}$ **ii** $(0, 0)$ **iii** $\frac{^{-}1}{6}$

9 **a**, **b**, **d** and **e** are collinear.

10 **a** $a = \pm 4$ **b** $a = 8$ or 0 **c** $a = 0$ or 4

d $a = {^{-}2}\frac{2}{3}$ **e** $a = {^{-}5}$ or 1

11 **a** $b = 3$ **b** $b = 2$

12 B is $(^{-}12, ^{-}3)$. **13** Q is $(11, 4)$. **14** R is $(2, ^{-}13)$.

15 $\sqrt{41}$, $\sqrt{10}$, $\sqrt{29}$, $\sqrt{85}$ and $\sqrt{37}$ units.

16 Sides: $\sqrt{10}$, 3, $\sqrt{34}$ and $\sqrt{5}$ units. Diagonals both 5 units.

17 All four sides are $\sqrt{20}$ units, so it is a rhombus.

18 Sides are $\sqrt{65}$, $\sqrt{18}$ and $\sqrt{65}$ units.

19 **a** $\sqrt{13}$, $\sqrt{13}$ and $\sqrt{26}$ units. **b** Isosceles triangle

20 Midpoints are $(4, 1)$, $\left(4\frac{1}{2}, ^{-}3\frac{1}{2}\right)$, $(1, ^{-}2)$ and $\left(\frac{1}{2}, 2\frac{1}{2}\right)$.
For the quadrilateral formed by the midpoints: midpoints of diagonals are both $\left(2\frac{1}{2}, \frac{^{-}1}{2}\right)$; *or* gradients of sides are $^{-}9, \frac{^{-}3}{7}, {^{-}9}, \frac{^{-}3}{7}$. The midpoints of *MNOP* form a parallelogram.

21 **a** Both $(1, 0)$. **b** Parallelogram.

22 **a** $\sqrt{20}$, $\sqrt{20}$, $\sqrt{20}$ and $\sqrt{20}$ units. **b** Rhombus

23 Gradients of sides are $\frac{1}{2}$, $^{-}3$, $\frac{1}{2}$ and $^{-}3$; *or* midpoints of diagonals are both $\left(\frac{1}{2}, \frac{^{-}1}{2}\right)$.

24 18.75 units^2 **25** $D = (^{-}3, ^{-}4)$

26 *DEFG* is a parallelogram because opposite sides are parallel and equal.

27 $(^{-}5, ^{-}3)$, $(5, ^{-}1)$ or $(3, 3)$.

28 The quadrilateral is a parallelogram because opposite sides are parallel and equal.

29 **a** $D\left(2, \frac{1}{2}\right)$, $E\left(\frac{1}{2}, 2\frac{1}{2}\right)$

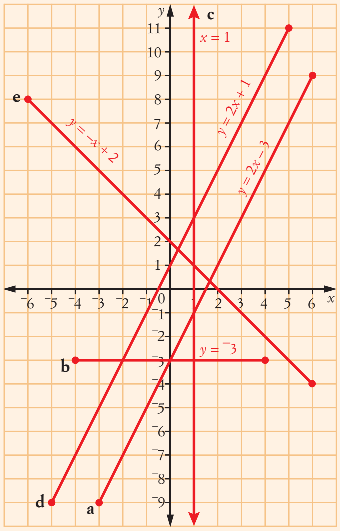
b $m_{DE} = m_{BC} = \frac{^{-}4}{3}$

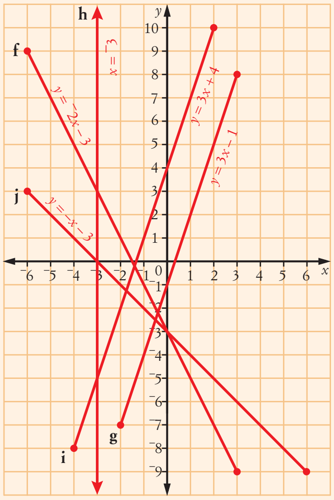
c $DE = \frac{5}{2}$ units, $BC = 5$ units.

Exercise 5.2

1 **a** xi **b** vii **c** viii
d v **e** i **f** ii
g ix **h** xii **i** x

2





Answers are in the order: x-intercept, y-intercept.

a $\left(1\frac{1}{2}, 0\right)$ and $(0, {}^{-}3)$. **b** No x-intercept, $(0, {}^{-}3)$.
c $(1, 0)$, no y-intercept. **d** $\left({}^{-}\frac{1}{2}, 0\right)$ and $(0, 1)$.
e $(2, 0)$ and $(0, 2)$. **f** $\left({}^{-}1\frac{1}{2}, 0\right)$ and $(0, {}^{-}3)$.
g $\left(\frac{1}{3}, 0\right)$ and $(0, {}^{-}1)$. **h** $({}^{-}3, 0)$, no y-intercept.
i $\left({}^{-}1\frac{1}{3}, 0\right)$ and $(0, 4)$. **j** $({}^{-}3, 0)$ and $(0, {}^{-}3)$.

3

a	Yes	No	No	Yes	No
b	No	Yes	Yes	No	Yes
c	No	No	No	Yes	Yes
d	No	No	Yes	No	Yes
e	No	No	Yes	No	Yes

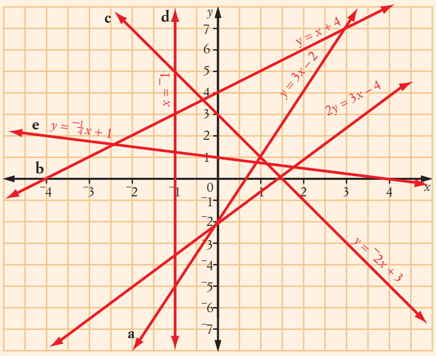
4 Answers are in the order: x-intercept, y-intercept.

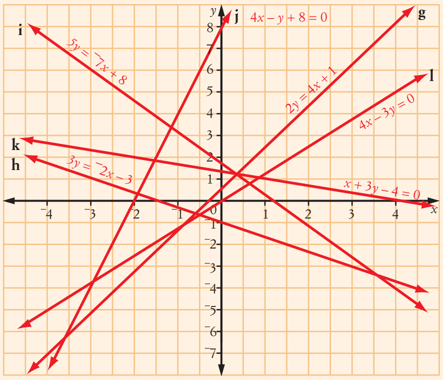
a $({}^{-}2, 0)$ and $(0, 3)$ **b** $(3, 0)$ and $(0, {}^{-}4)$
c $(4, 0)$ and $\left(0, 1\frac{1}{3}\right)$ **d** $\left({}^{-}2\frac{1}{2}, 0\right)$ and $\left(0, {}^{-}2\frac{1}{2}\right)$
e $\left({}^{-}\frac{1}{2}, 0\right)$ and $\left(0, 1\frac{1}{2}\right)$ **f** $(3, 0)$ and $(0, 5)$

5 **a** $y = 4$ **b** $y = 5$ **c** $y = {}^{-}1$
d $y = 5$ **e** $y = 0$ **f** $y = {}^{-}4$
g $y = 1$ **h** $y = 7$ **i** $y = {}^{-}3$
j $y = {}^{-}8$ **k** $y = 3$ **l** $y = 1$

6 **a** $x = {}^{-}2$ **b** $x = 3$ **c** $x = 2$
d $x = 0$ **e** $x = 4$ **f** $x = {}^{-}3$
g $x = 6$ **h** $x = {}^{-}4$ **i** $x = 7$
j $x = {}^{-}8$ **k** $x = 3$ **l** $x = 5$

7

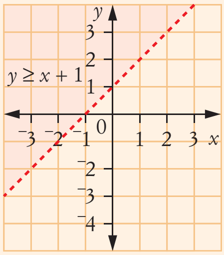




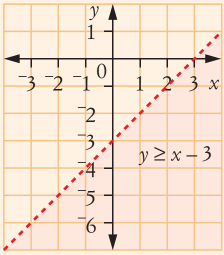
Intercepts are in the order: x-intercept, y-intercept.

a **i** 3 **ii** $\frac{2}{3}, {}^{-}2$
b **i** 1 **ii** ${}^{-}4, 4$
c **i** ${}^{-}2$ **ii** $1\frac{1}{2}, 3$
d **i** Not defined. **ii** ${}^{-}1$, no y-intercept.
e **i** ${}^{-}\frac{1}{4}$ **ii** 4, 1
f **i** $1\frac{1}{2}$ **ii** $1\frac{1}{3}, {}^{-}2$
g **i** 2 **ii** ${}^{-}\frac{1}{4}, \frac{1}{2}$
h **i** ${}^{-}\frac{2}{3}$ **ii** ${}^{-}1\frac{1}{2}, {}^{-}1$
i **i** ${}^{-}1\frac{2}{5}$ **ii** $1\frac{1}{7}, 1\frac{3}{5}$
j **i** 4 **ii** ${}^{-}2, 8$
k **i** ${}^{-}\frac{1}{3}$ **ii** $4, 1\frac{1}{3}$
l **i** $1\frac{1}{3}$ **ii** 0, 0

8 **a**



b



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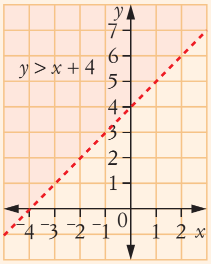
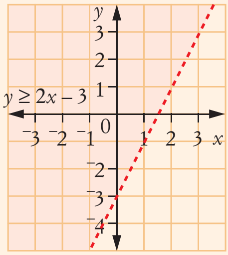
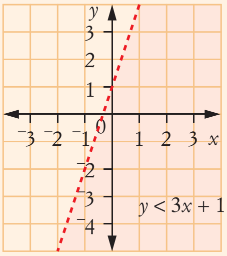


d



e



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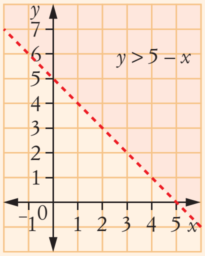
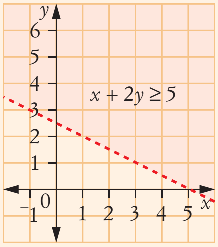
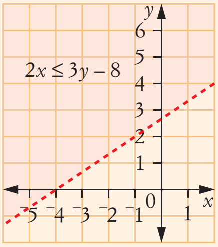


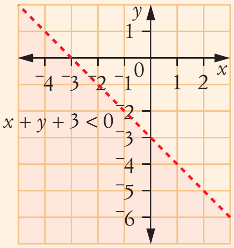
9 **a**



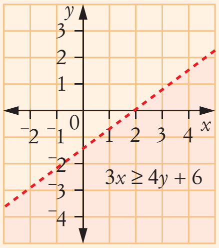
b



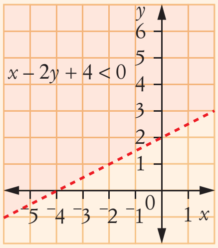
c



d



e



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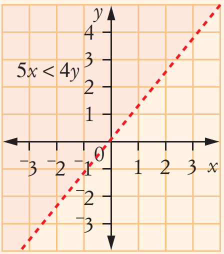


10 **a** $y = 2x + 4$ **b** $y = {}^{-}3x + 1$
c $y = 5x - 6$ **d** $y = 3x - 2$
e $y = {}^{-}3$ **f** $y = {}^{-}4x$

11 **a** $4x - y - 1 = 0$ **b** $2x - 3y - 6 = 0$
c $4x + 3y + 3 = 0$ **d** $5x - 2y + 4 = 0$
e $10x - 3y - 12 = 0$ **f** $7x + 4y + 7 = 0$

12 Answers are in the order: gradient, *y*-intercept.
a 2, 5 **b** ${}^{-}3$, 5 **c** ${}^{-}1$, 1 **d** ${}^{-}7$, 4
e 1, 2 **f** ${}^{-}8$, 0 **g** $\frac{1}{2}, \frac{1}{2}$ **h** ${}^{-}7$, 14
i $\frac{1}{3}$, 2 **j** ${}^{-}\frac{4}{3}, 2\frac{2}{3}$ **k** $\frac{7}{3}$, 0

13 **a** $y = 2x + 1, 2x - y + 1 = 0$
b $y = -x + 3, x + y - 3 = 0$
c $y = x + 1, x - y + 1 = 0$
d $y = {}^{-}4x - 8, 4x + y + 8 = 0$
e $y = {}^{-}\frac{3}{4}x + 6, 3x + 4y - 24 = 0$
f $y = \frac{7}{3}x + 3, 7x - 3y + 9 = 0$

14 $x = 2$ **15** $y = {}^{-}\frac{2}{3}x + 4, 2x + 3y - 12 = 0$

16 **a** $x + 2y - 2 = 0, y = {}^{-}\frac{1}{2}x + 1$
b $4x - 3y + 12 = 0, y = \frac{4}{3}x + 4$
c $x - 2y - 4 = 0, y = \frac{1}{2}x - 2$
d $4x - y + 4 = 0, y = 4x + 4$
e $8x - 3y - 24 = 0, y = \frac{8}{3}x - 8$
f $3x + 5y + 15 = 0, y = {}^{-}\frac{3}{5}x - 3$
g $x - 5y - 5 = 0, y = \frac{1}{5}x - 1$
h $3x - 2y + 12 = 0, y = \frac{3}{2}x + 6$
i $x + y + 4 = 0, y = -x - 4$
j $x + y - 2 = 0, y = -x + 2$

17 **a** 2 **b** (0, 4) **c** $2x - y + 4 = 0$

18 **a** $3x + y - 18 = 0, y = {}^{-}3x + 18$
b $x - y - 2 = 0, y = x - 2$
c $2x - y + 3 = 0, y = 2x + 3$
d $3x + y + 6 = 0, y = {}^{-}3x - 6$
e $4x + y - 16 = 0, y = {}^{-}4x + 16$
f $x + y - 3 = 0, y = -x + 3$
g $x - y - 1 = 0, y = x - 1$
h $4x - y - 14 = 0, y = 4x - 14$
i $5x + 8y + 17 = 0, y = {}^{-}\frac{5}{8}x - 2\frac{1}{8}$
j $2x + y + 10 = 0, y = {}^{-}2x - 10$
k $2x - y = 0, y = 2x$
l $2x - y - 13 = 0, y = 2x - 13$

19 **a** $2x - y - 4 = 0$ **b** $2x + 3y + 1 = 0$
c $2x - 3y - 7 = 0$ **d** $x = 2$
e $4x - y + 2 = 0$ **f** $2x + y - 10 = 0$

20 **a** and **e** are perpendicular, **b** and **f** are parallel, **c** and **d** are neither.

21 **a** $m_2 = {}^{-}\frac{1}{3}$ **b** $m_2 = \frac{3}{4}$ **c** $m_2 = {}^{-}5$
d $m_2 = \frac{10}{31}$ **e** $m_2 = \frac{-b}{a}$ **f** $m_2 = h$

22 **a** $b = 1$ **b** $b = {}^{-}3\frac{1}{4}$

23 **a** Parallel **b** Parallel **c** Not parallel
d Parallel **e** Parallel **f** Not parallel

For question 24, the equation of the parallel line is given first.

24 **a** $4x - y - 24 = 0, x + 4y + 11 = 0$
b $3x - 5y + 31 = 0, 5x + 3y - 5 = 0$
c $3x + 2y - 4 = 0, 2x - 3y + 19 = 0$
d $7x - 4y - 19 = 0, 4x + y + 9 = 0$

25 $m_1 = m_2 = {}^{-}\frac{2}{3}$

26 $m_1 \times m_2 = {}^{-}\frac{4}{5} \times \frac{5}{4} = {}^{-}1$

27 $m_{AB} \times m_{AC} = \frac{5}{2} \times {}^{-}\frac{2}{5} = {}^{-}1$; Area = 29 units^2

28 $m_{LK} = m_{MN} = \frac{3}{2}$; $m_{KN} = m_{LM} = {}^{-}\frac{2}{3}$;
$m_{LK} \times m_{KN} = {}^{-}1$; $LK = MN = \sqrt{13}$ units;
$KN = LM = 3\sqrt{13}$ units. Area = 39 units^2

29 $m_{HI} = m_{KJ} = {}^{-}3$,
$m_{IJ} = m_{HK} = \frac{1}{3}$; $m_{HI} = m_{IJ} = {}^{-}3 \times \frac{1}{3} = {}^{-}1$; and
$HI = IJ = JK = KH = \sqrt{10}$

30 $m_{JK} \times m_{KL} = \frac{1}{2} \times 2 = {}^{-}1$

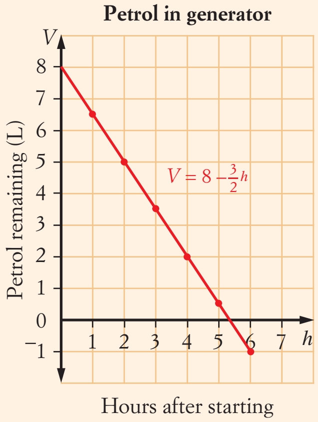
31 $m_{PM} \times m_{PQ} = {}^{-}\frac{2}{3} \times \frac{3}{2} = {}^{-}1$

32 *PQRS* is a kite because two pairs of adjacent sides are equal and its diagonals bisect at right angles.

33 **a**

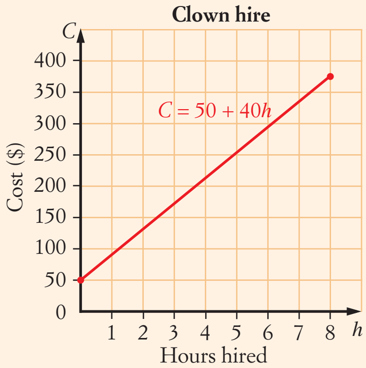
h	0	1	2	3	4	5	6
V	8	6.5	5	3.5	2	0.5	${}^{-}1$

b



c Linear function **d** After $3\frac{1}{3}$ hours.
e 5 L
f The volume of petrol in the generator before it is started.
g The number of hours when the generator runs out of petrol.

34 **a**



b Linear function.

c $170 **d** 3.75 hours

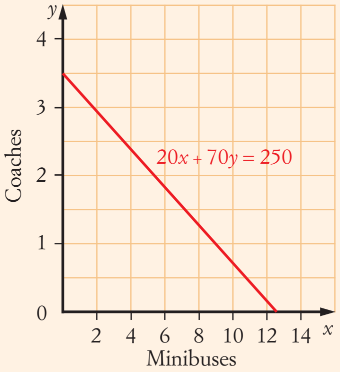
e The cost of hiring the clown regardless of how many hours he or she works. It is sometimes called a service fee or a call-out fee.

f The hourly rate of pay for the clown

35 **a** $20x + 70y$ pupils

b $20x + 70y = 250$

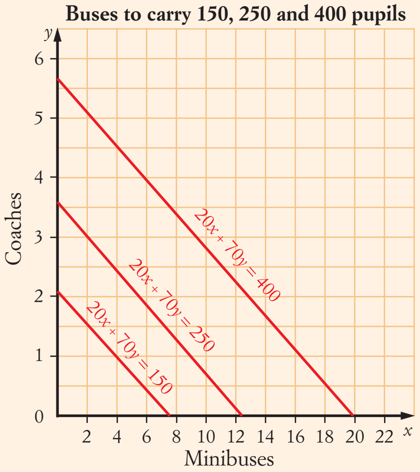
c Buses to carry 250 pupils



d 9 minibuses and 1 coach, or 2 minibuses and 3 coaches.

e There are other possibilities but these would carry approximately 250 pupils.

f

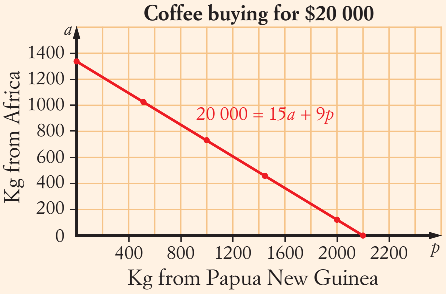

Buses to carry 150, 250 and 400 pupils

g The graphs are parallel.

36 **a** $20\,000 = 15a + 9p$

b

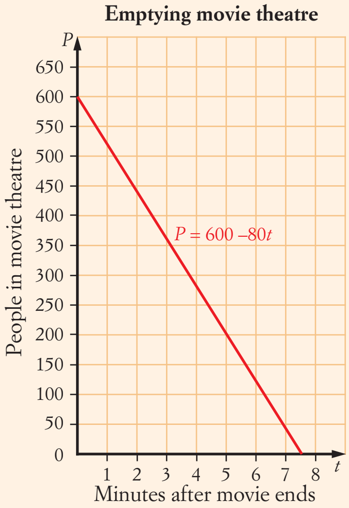
p	0	500	1000	1500	2000
a	1333	1033	733	433	133


Coffee buying for $20 000

c About 850 kg of African coffee.

37 **a** $P = 600 - 80t$

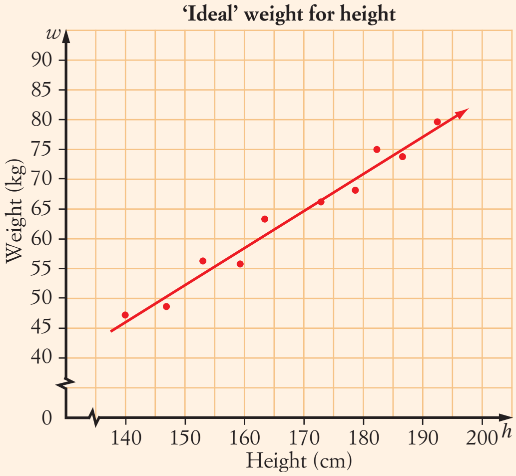
b


Emptying movie theatre

c About 3.8 minutes.

38 $4x - y + 17 = 0$

39 **a**


'Ideal' weight for height

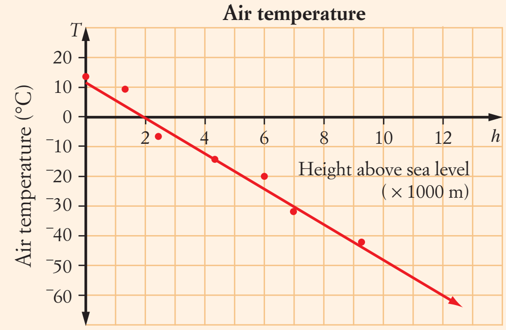
b About 65 kg.

c About 202 cm.

d $w = 0.63h - 43$ (Answers may vary slightly.)

e 64.1 kg (using equation) and 65 kg (from graph); 203 cm (using equation) and 202 cm (from graph).

40 **a**



b About 17°C.

c About 11 100 m.

d $T = {}^{-}0.0063h + 15$ (Answers may vary slightly.)

e $^{-}35.4$°C

f 11 900 m

Exercise 5.3

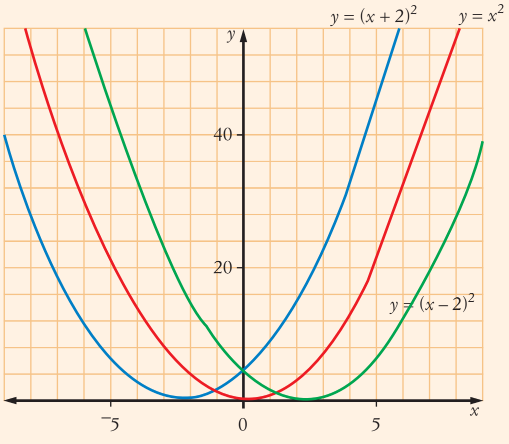
1 Straight lines: **b**, **d**, **m**, **i**, **k**
Parabolas: **e**, **j**, **o**
Hyperbolas: **a**, **f**
Circles: **c**, **h**
Exponentials: **g**, **n**
None: **l**

2 **a** E **b** K **c** J **d** D
e I **f** F **g** B **h** A
i G **j** C **k** L **l** H

3 **a** G **b** J **c** K **d** L
e C **f** E **g** H **h** D
i I **j** B **k** A **l** F

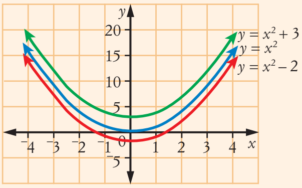
4 **a** $y = x^2$ **b** $y = {}^{-}x^2$ **c** $y = \frac{1}{2}x^2$
d $y = \frac{1}{5}x^2$ **e** $y = \frac{1}{3}x^2$ **f** $y = 0.25x^2$

5 **a**



b If $a > 0$, the graph shifts to the left. If $a < 0$, the graph shifts to the right.

6 **a**



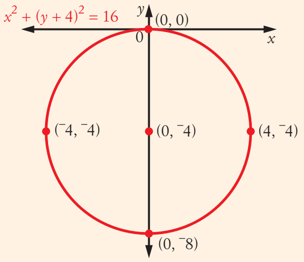
b **i** For all: $x = 0$
For parts **ii** and **iii**, answers are stated in the order: $y = x^2$, $y = x^2 + 3$, $y = x^2 + {}^{-}2$
ii (0, 0), (0, 3), (0, $^{-}2$)
iii 0, 3, $^{-}2$

c When $a > 0$, the graph is shifted up, compared to the graph of $y = x^2$.
When $a < 0$, the graph is shifted down compared to the graph of $y = x^2$.

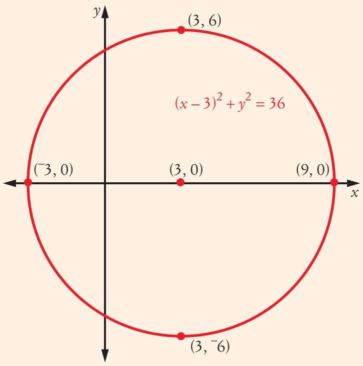
7 **a** $(x - 4)^2 + (y - 5)^2 = 9$
b $(x + 2)^2 + (y - 4)^2 = 64$
c $(x - 3)^2 + (y + 6)^2 = 25$
d $x^2 + y^2 = 81$
e $(x + 3)^2 + (y + 5)^2 = 4$

8 **a** **i** (0, 0) **ii** 4
b **i** (0, 0) **ii** 1
c **i** ($^{-}2$, 7) **ii** 5
d **i** ($^{-}5$, $^{-}3$) **ii** 3
e **i** (6, $^{-}9$) **ii** 11
f **i** (4, 8) **ii** 8
g **i** ($^{-}3$, $^{-}2$) **ii** 10
h **i** ($^{-}10$, 4) **ii** 15

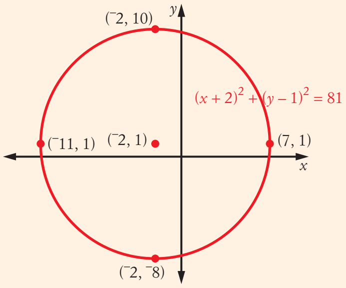
9 **a**



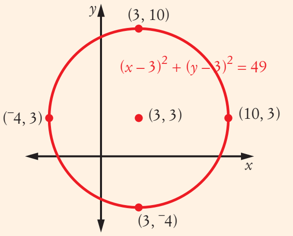
b



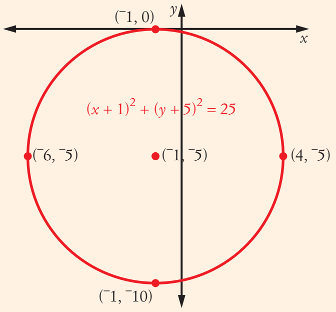
c



d



e



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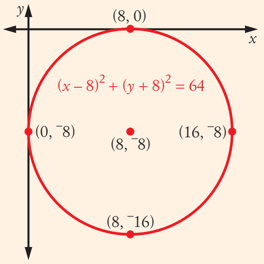
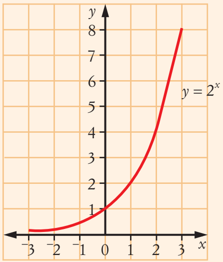


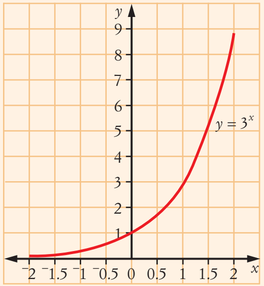
10 a



b **i** 1 **ii** 2 **iii** 1.4 **iv** 3

c **i** 0 **ii** 2 **iii** 2.8 **iv** $^-$0.3

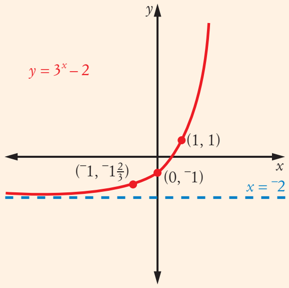
11 a



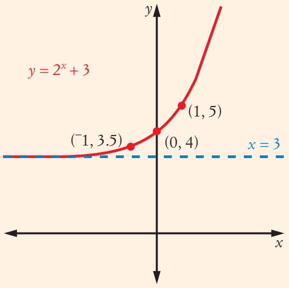
b **i** 1 **ii** 5.2 **iii** 1.9 **iv** 7.2

c **i** 1 **ii** 1.5 **iii** 0.8 **iv** 0.9

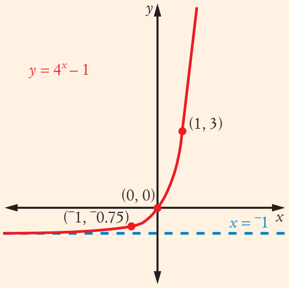
12 a



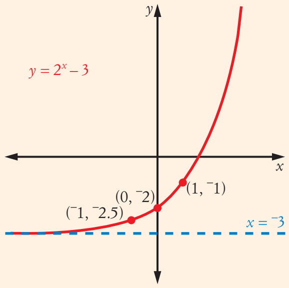
b



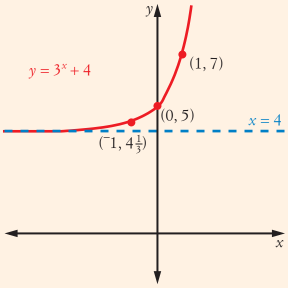
c



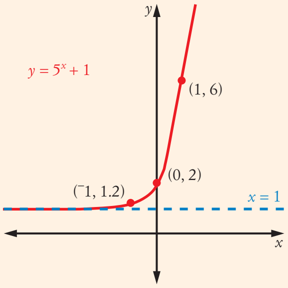
d



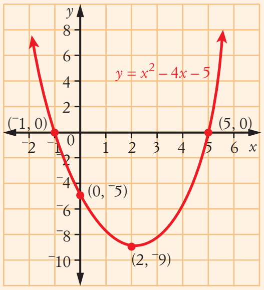
e



f

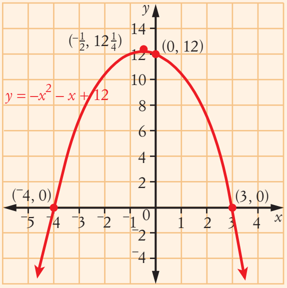


13 a i



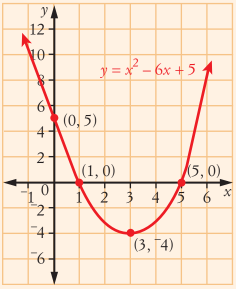
ii Minimum

b i



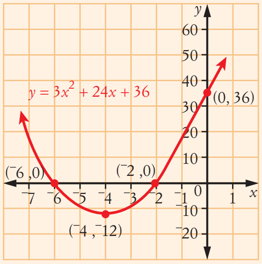
ii Maximum

c i



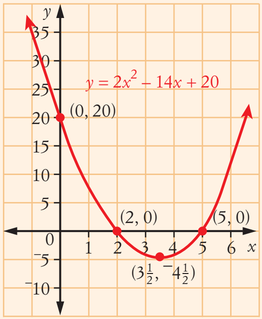
ii Minimum

d i



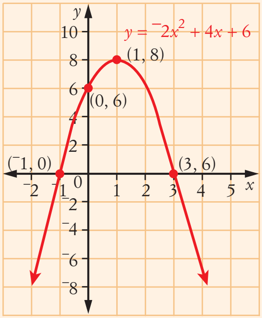
ii Minimum

e i



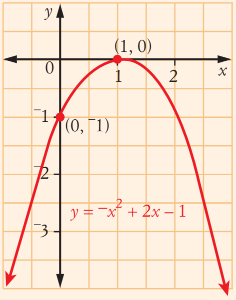
ii Minimum

f i



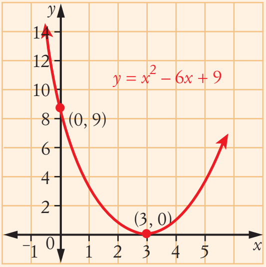
ii Maximum

g i



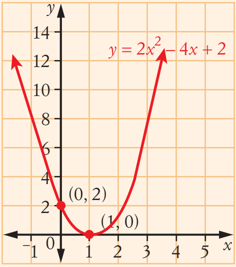
ii Minimum

h **i**



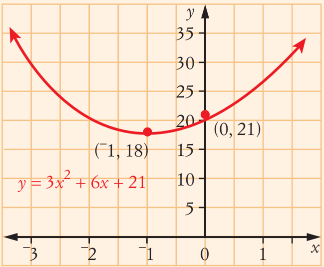
ii Minimum

i **i**



ii Minimum

j **i**



ii Minimum

14 **a** $y = x^2$ **b** $y = -x^2$ **c** $y = x^2 - 2$

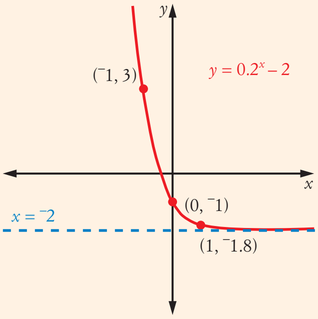
d $y = -x^2 + 3$ **e** $y = x^2 + \frac{1}{2}$ **f** $y = -x^2 - \frac{2}{3}$

g $y = -x^2 - 6$

15 **a** *A* **b** *C* **c** None

d None **e** *E*

16 **a**



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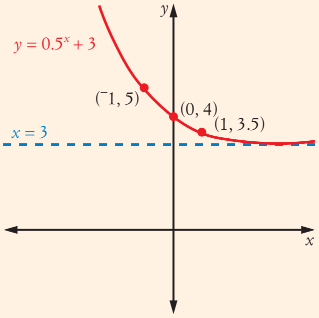

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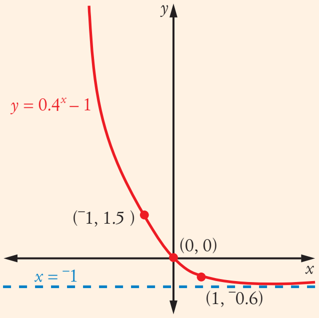

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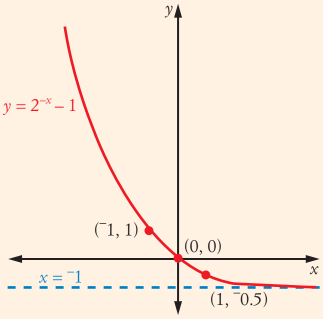
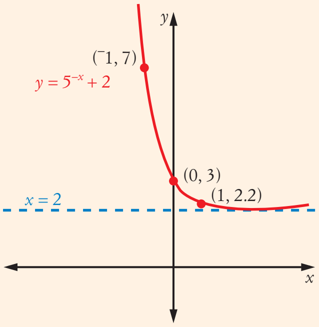
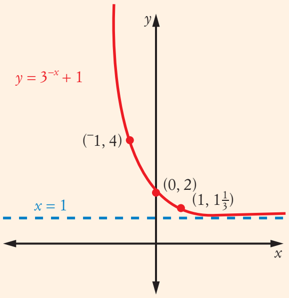


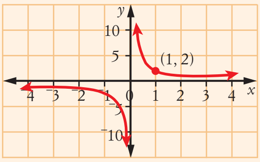
e



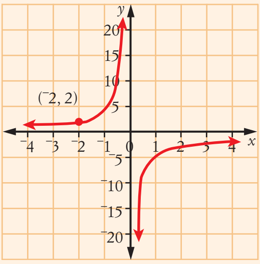
f



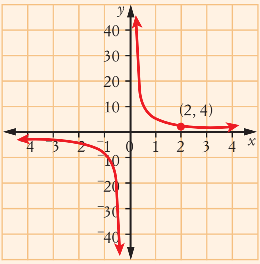
17 **a**



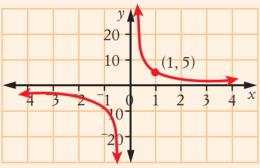
b



c



d



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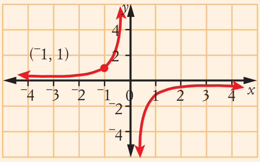

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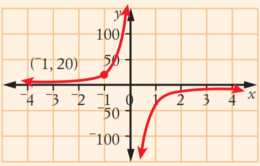

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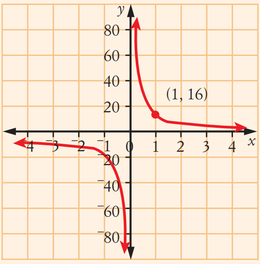

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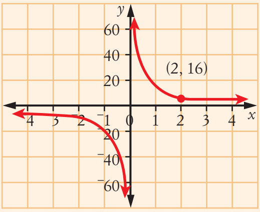


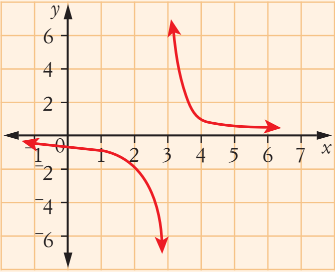
18 **a** On **b** Below **c** Above
d On **e** Above

19 **a** Below **b** On **c** On
d Above **e** Below

20 **a**

x	$^{-}1$	0	1	2	2.8	3.2	4	5	6
y	$^{-}0.25$	$^{-}\frac{1}{3}$	$^{-}0.5$	$^{-}1$	$^{-}5$	5	1	0.5	$^{-}\frac{1}{3}$

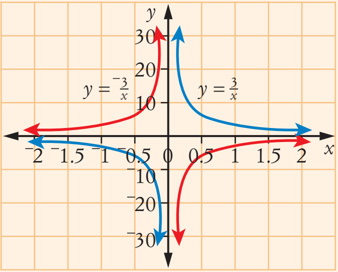
b



c Hyperbola

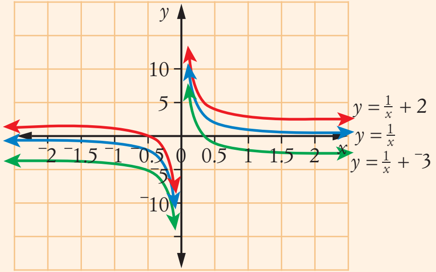
d $y = 0$ and $x = 3$

21 **a**



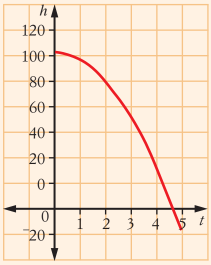
b Each branch of one hyperbola is reflected in the y-axis to give the branches of the other hyperbola.

22 **a**



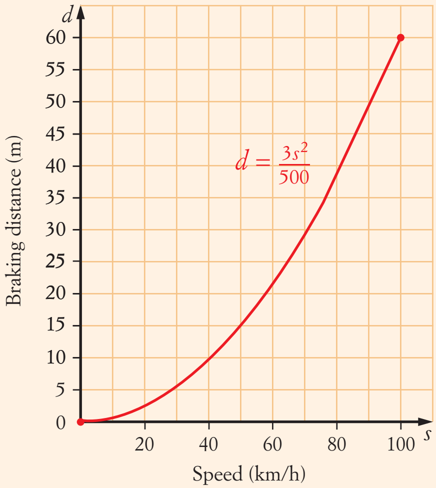
b When $a > 0$, the graph is shifted up compared to the graph of $y = \frac{1}{x}$.
When $a < 0$, the graph is shifted down compared to the graph of $y = \frac{1}{x}$.

23 **a**



b 105 m **c** 19.2 m **d** 4.7 s

24 Car braking



a About 9 m (8.7 m).

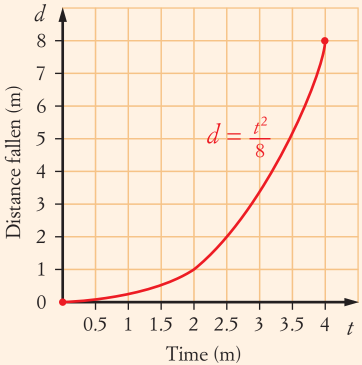
b About 34 m (33.8 m).

c About 71 km/h (70.7 km/h).

25 **a** On **b** Inside **c** Outside

d Find the distance to the centre of the circle and compare it with the radius.

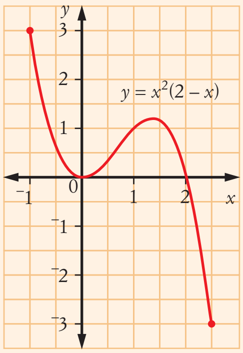
26 Falling ball



a About 1.7 m. **b** About 6.9 m.

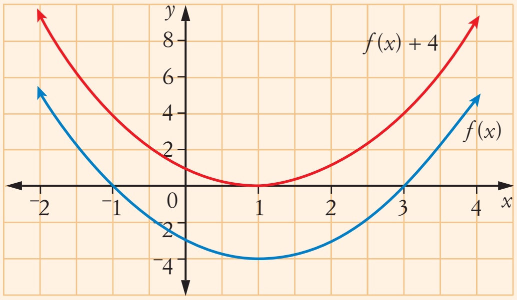
c About 3.6 s after it begins to fall.

27 **a**

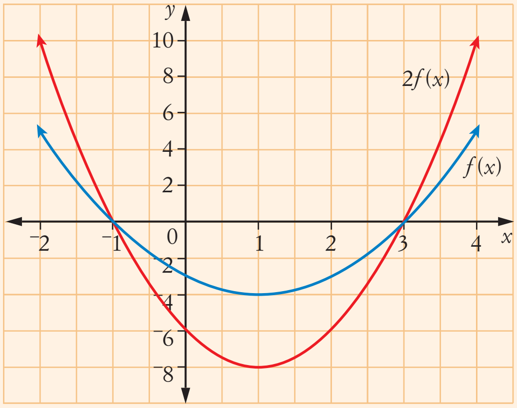


b About 1.2. **c** About 1.3 $(1\frac{1}{3})$.

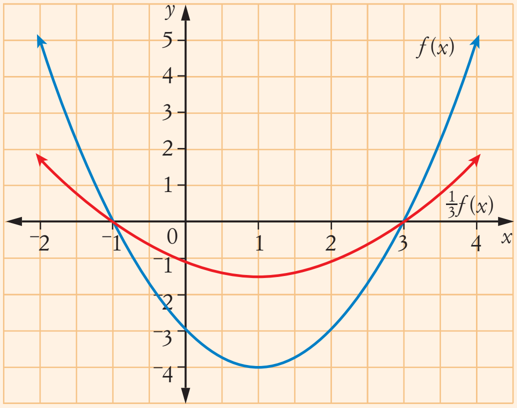
28 **a**



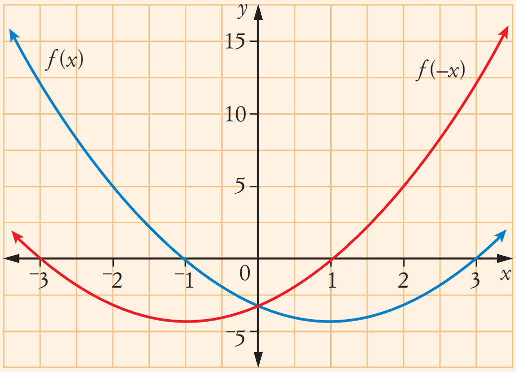
b



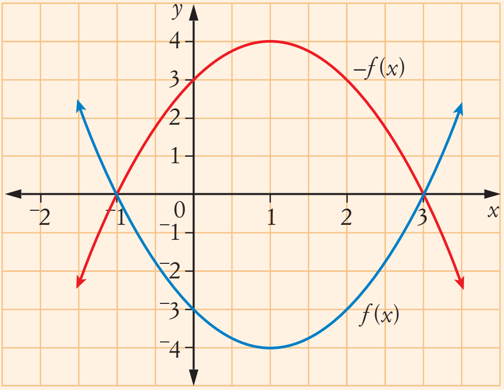
c



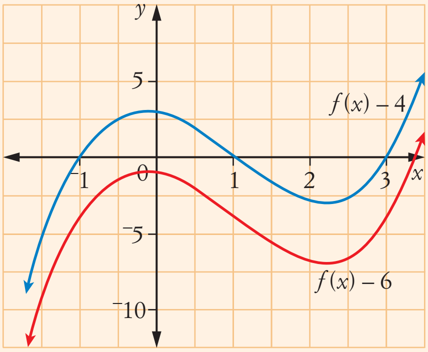
d



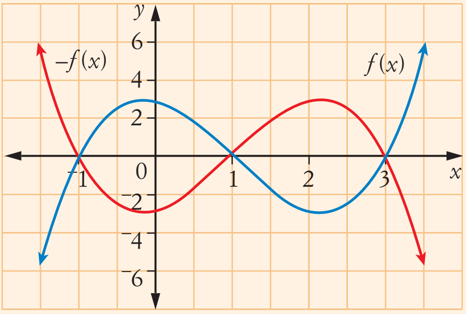
e



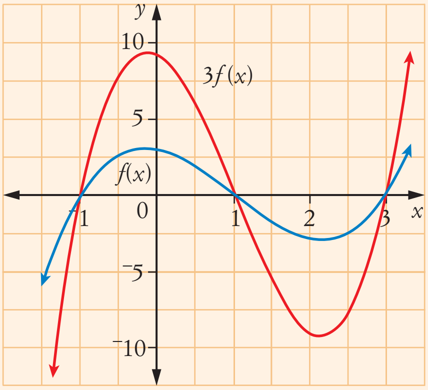
29 **a**



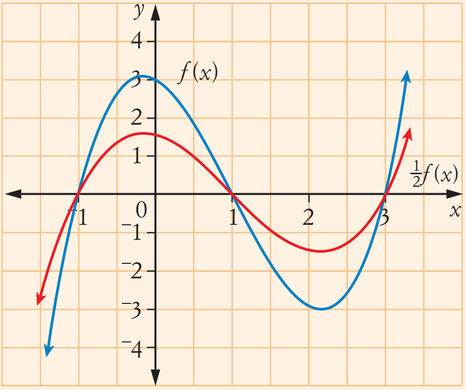
b



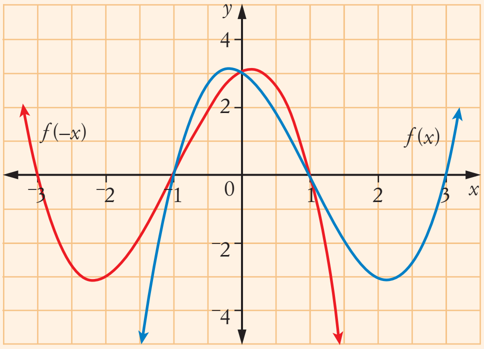
c



d



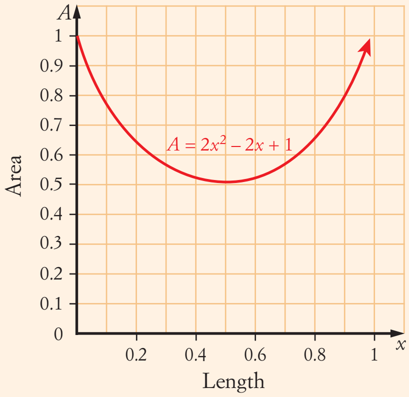
e



30 **a** $ER = 1 - x$ and $ER = FS = GP = DQ$.

b Use Pythagoras' theorem to show that the side of the smaller square measures $\sqrt{2x^2 - 2x + 1}$.

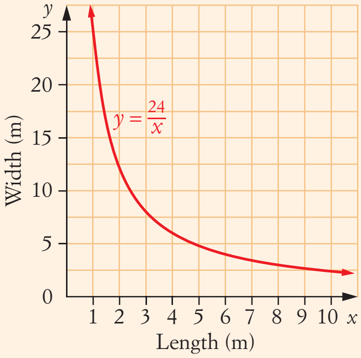
c Area of Smaller square



d $x = 0.5$ units, $A = 0.5$ units^2

31 **a** $24 = xy$ or $y = \frac{24}{x}$

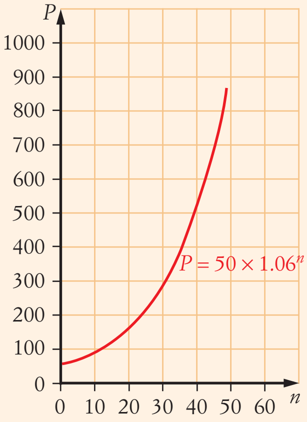
b Area of sheet of fabric



c About 3.7 m.

32 **a** 50 **b** About 101. **c** About 287.

d



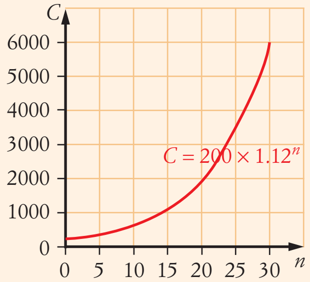
e About 39.5 months.

33 **a** 200

b About 621.

c About 1929.

d

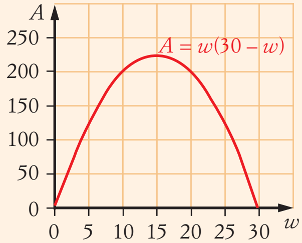


e About 24 weeks.

34 **a** $l = 30 - w$

b $A = w(30 - w)$

c



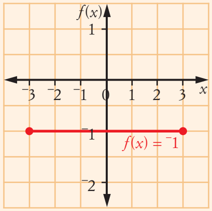
d 15 m by 15 m.

Chapter review

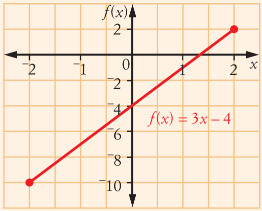
1 **a** $\frac{^{-}3}{2}$ **b** 0 **c** 1 **d** $^{-}4$

2 **a** C **b** D **c** A **d** B

3



4

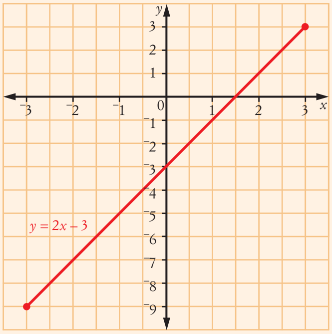


5 **a** Linear **b** Linear

c Not linear **d** Not linear

Answers

6



x-intercept = (1.5, 0), y-intercept = $(0, {}^{-}3)$.

7 Not on the line. **8** $y = 7$

9 **a** **iii** **b** **vi** **c** **ii** **d** **i** **e** **v** **f** **iv**

10 13 units **11** $(\frac{1}{2}, {}^{-}2)$ **12** ${}^{-}2$

13 $y = 3x - 1$ or $3x - y - 1 = 0$

14 Gradient $= \frac{1}{2}$, y-intercept = (0, 2)

15 $y = 3x + 11$, $3x - y + 11 = 0$

16 $2x + 3y - 5 = 0$

17 $y = -x + 1$, $x + y - 1 = 0$ **18** Perpendicular

19 **a** Neither **b** Parallel **c** Perpendicular

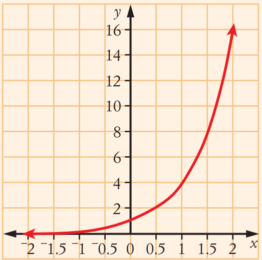
20 **a** $m_1 = \frac{{}^{-}1}{4}$ **b** $m_1 = \frac{{}^{-}2}{5}$ **c** $m_1 = 2\frac{1}{2}$

21 $y = 3x - 2$ or $3x - y - 2 = 0$

22 $3x - y - 2 = 0$ (parallel), $x + 3y + 16 = 0$ (perpendicular).

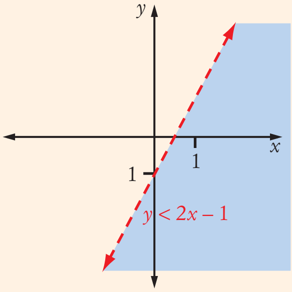
23 $(x + 1) + (y - 3)^2 = 16$

24 **a**

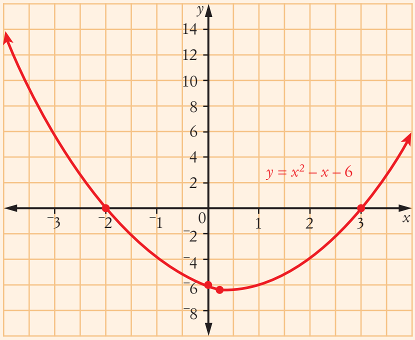


b 3 **c** 1.3

25

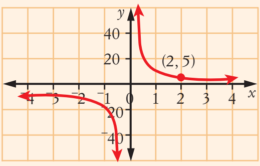


26 **a**



b Minimum

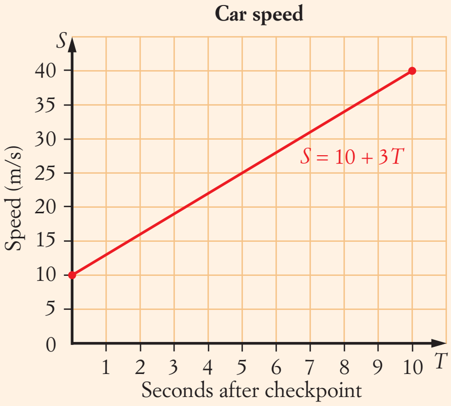
27



28 $B = (2, 1)$ **29** Opposite sides are parallel.

30 Opposite sides are equal and adjacent sides are perpendicular.

31

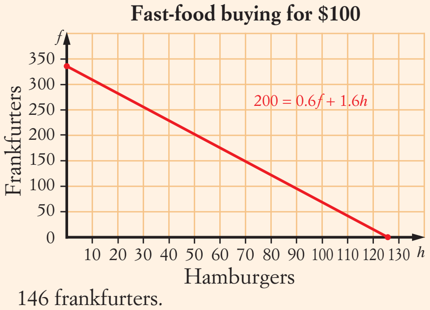


b After about 6.7 seconds. **c** About 22 m/s.

32 **a** $200 = 0.6f + 1.6h$

b

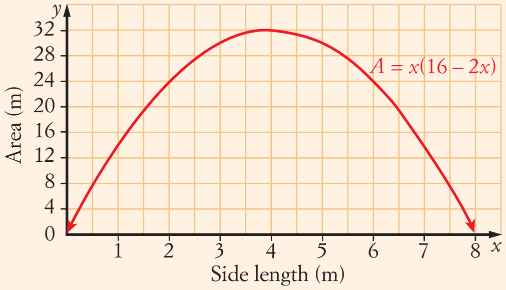
h	0	25	50	75	100	125
f	333	267	200	133	67	0



c 146 frankfurters.

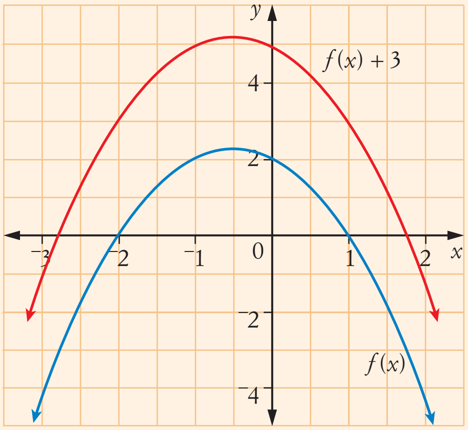
33 **a** $A = x(16 - 2x)$

b Chicken pen

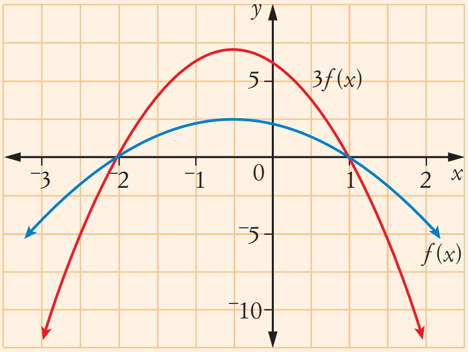


c 32 m^2 **d** 4 m by 8 m

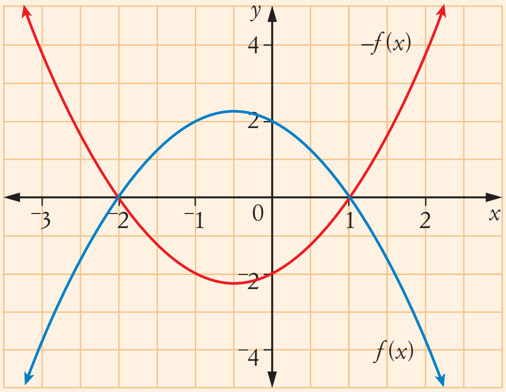
34 **a**



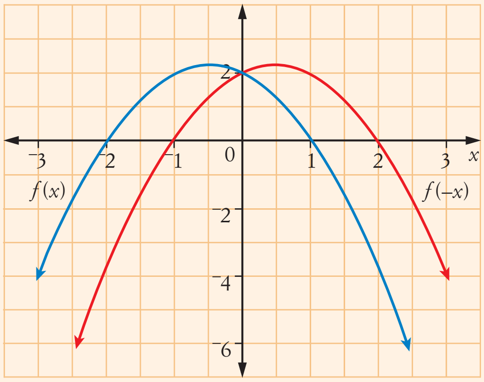
b



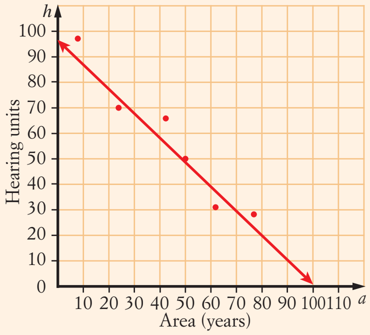
c



d



35 **a** Hearing ability



b $h = -a + 100$

c About 70 hearing units.

d About age 80.

Chapter 6

Exercise 6.1

1 **a** \$90 **b** \$30 **c** \$50 **d** \$52.50 **e** \$90

2 **a** 7 : 3 **b** 9 : 1 **c** 8 : 3 **d** 29 : 7 **e** 3 : 1

3 **a** Brisbane 0.29, Sydney 0.417, Melbourne 0.513, Adelaide 0.497, Perth 0.503, Darwin 0.02, Hobart 0.483.

b About 47. **c** About 65.

4 **a** 0.121 **b** 0.879

c About 48 pearls. **d** About 3314 oysters.

5 **a** $\frac{1}{4} = 0.25$ **b** 4 **c** $\frac{1}{13} \approx 0.077$

d $\frac{3}{13} \approx 0.231$ **e** $\frac{1}{13} \approx 0.077$ **f** $\frac{9}{13} \approx 0.692$

g $13.85 \approx 14$ cards

6 **a** {SuncapZinc: RP, RY, YP, YY, BP. BY}

b $\frac{1}{6} \approx 0.167$(yellow)

7 **a** 18.49% **b** Yes

8 **a** $\frac{4}{5} = 0.8$ **b** $\frac{1}{11} \approx 0.090\ 91$ **c** $\frac{1}{7} \approx 0.1429$

d $\frac{3}{5} = 0.67\ 7$ **e** $\frac{5}{12} \approx 0.4167$

9 **a** 0.000 82 **b** About 71 40-year-old men.

c \$536.67 **d** \$683.33

e It allows about 80% for costs and profit, so it is reasonable.

f It allows only about 48% for costs and profit so it is quite good.

10 a {G1G2, G1G3, G1Y1, G1Y2, G2G3, G2Y1, G2Y2, G3Y1, G3Y2, Y1Y2}

b Unmatched socks **c** $\frac{3}{10} = 0.3$

d $\frac{1}{10} = 0.1$ **e** 3 : 2

11 a About 56.83%. **b** About 3.63%.

c About 0.29%. **d** About 47 : 3.

12 The calculation is done by working out where the centre of the coin lands. Since the table is divided into the squares, this includes all the area of the table. If the squares didn't go right to the edge it would make a difference. If you were allowed another go if your coin fell off, it would increase your chance of winning. For example, if the table was only 5 cm square, your probability would go up to about 33.6%, but for larger tables it would not go up as much.

Exercise 6.2

1 $\frac{1}{4}$

2 $\frac{9}{25} = 0.36$

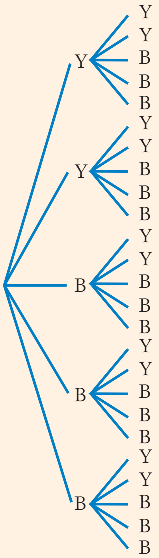
3 $\frac{1}{8}$ **4** $\frac{1}{16}$ **5** $\frac{1}{216}$

6 a $\frac{1}{16}$ **b** $\frac{1}{64}$

7 $\frac{12}{25} = 0.48$

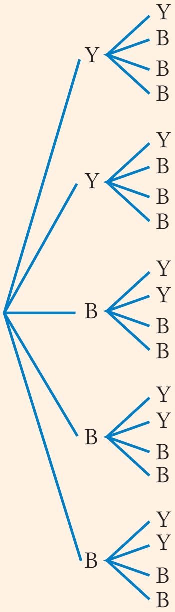
8 $\frac{10}{13}$

9 a



b $\frac{4}{25} = 0.16$ **c** $\frac{9}{25} = 0.36$ **d** $\frac{12}{25} = 0.48$

10 a

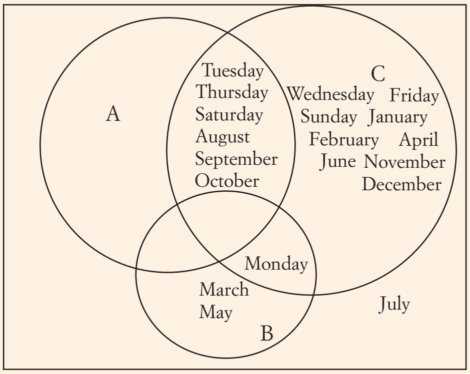


b $\frac{1}{10} = 0.1$ **c** $\frac{3}{10} = 0.3$ **d** $\frac{3}{5} = 0.6$

11 a i $\frac{1}{3} \approx 0.333$ **ii** $\frac{2}{3} \approx 0.667$ **iii** $\frac{5}{9} \approx 0.556$

b i $\frac{1}{5} = 0.2$ **ii** $\frac{4}{5} = 0.8$ **iii** $\frac{3}{5} = 0.6$

12 a

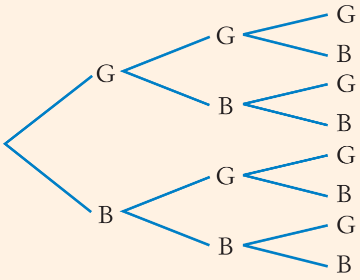


b $\frac{6}{19}$ **c** $\frac{15}{19}$ **d** 0

e $\frac{9}{19}$ **f** $\frac{1}{19}$ **g** $\frac{18}{19}$

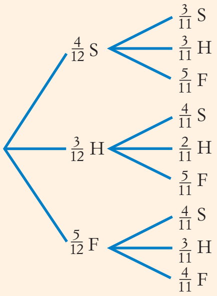
h $\frac{10}{19}$ **i** $\frac{1}{19}$ **j** $\frac{3}{19}$

13 a



b $\frac{3}{8}$ **c** $\frac{1}{2}$ **d** $\frac{1}{8}$

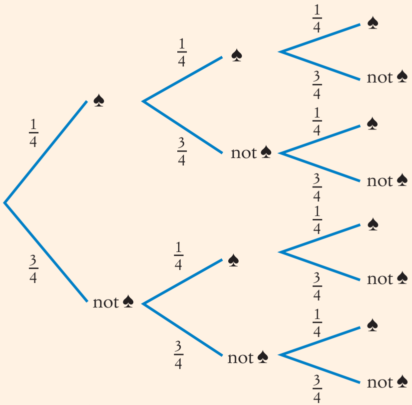
14 a



b $\frac{1}{22}$ **c** $\frac{1}{11}$ **d** $\frac{5}{33}$ **e** $\frac{2}{11}$

f $\frac{5}{22}$ **g** $\frac{10}{33}$ **h** $\frac{19}{66}$ **i** $\frac{47}{66}$

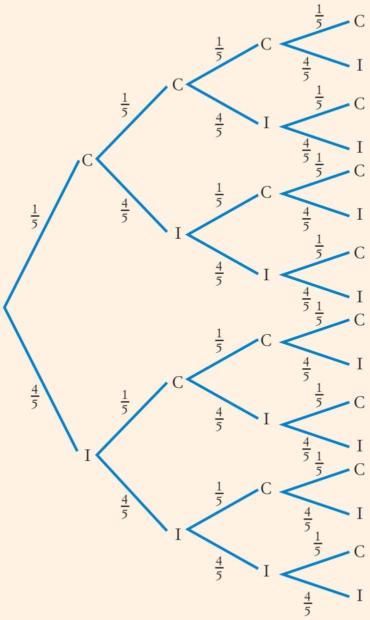
15 a



b $P(3S) = \frac{1}{64} = 0.015625$

c $P(0S \text{ or } 1S) = \frac{27}{32} = 0.84375$

16 a



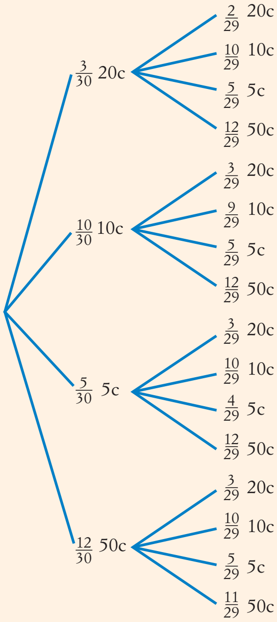
b $P(4C) = \frac{1}{625} = 0.0016$

c $P(2C \text{ or } 3C \text{ or } 4C) = \frac{113}{625} = 0.1808$

d About 9 people.

17 a $\frac{13}{150} \approx 0.08667$ **b** $\frac{19}{50} = 0.38$ **c** $\frac{31}{50} = 0.62$

18 a



b $\frac{124}{435} \approx 0.2851$ **c** $\frac{311}{435} \approx 0.7149$ **d** $\frac{94}{145} \approx 0.6483$

e $\frac{28}{145} \approx 0.1931$ **f** $\frac{12}{145} \approx 0.08276$ **g** $\frac{22}{29} \approx 0.7586$

h $\frac{98}{145} \approx 0.6759$ **i** $\frac{7}{29} \approx 0.2414$ **j** $\frac{21}{29} \approx 0.7241$

19 3H is 7 : 1, 2H is 5 : 3, 2T is 5 : 3 and 3T is 7 : 1.

20 a $\frac{1}{2} = 0.5$ **b** $\frac{5}{27} \approx 0.1852$

c Non-smokers have a 31% lower probability of having lots of colds.

d It is only some Year 10 students, and some of the frequencies are very low, so it is not a good sample size.

e Yes, but the small sample size makes it doubtful.

21 a $\frac{32}{147} \approx 0.2177$ **b** $\frac{17}{43} \approx 0.3953$

c The table doesn't show no accidents, so we cannot say. However, it appears that of those who do have accidents, younger drivers have more.

d Yes

22 a About 36%. **b** About 81%.

c Non-indigenous people are about 45% more likely to live to 65, so it appears that the health of Aboriginal and Torres Strait Islanders is worse than other Australians.

d Yes, and the sample size is very large, so the results are reliable.

23 a $P(0) = \frac{9}{16} = 0.5625$, $P(1) = \frac{3}{8} = 0.375$, $P(2) = \frac{1}{16} = 0.0625$

b About 56 people, 38 people, and 6 people respectively.

24 $\frac{2}{3} \approx 0.667$

25 a $\frac{1}{17} = 0.0588$ **b** $\frac{4}{17} = 0.235$ **c** $\frac{13}{17} = 0.765$

d $\frac{15}{34} = 0.441$ **e** About 94 times.

26 0.352

27 a $\frac{33}{66\,640} \approx 0.000\,495\,2$ **b** $\frac{33}{16\,660} \approx 0.001\,981$

28 a 7 : 2 **b** 8 : 1 **c** 2 : 1

29 **a** $\frac{1}{10} = 0.1$ **b** $\frac{11}{20} = 0.55$ **c** $\frac{1}{5} = 0.2$
d $\frac{3}{5} = 0.6$ **e** $\frac{13}{20} = 0.65$

30 There are 20 planks that are longer than 4.5 m (L) out of 60 planks. On the branch of the simplified tree diagram going L-L-L, the fractions are $\frac{20}{60}$, $\frac{19}{59}$ and $\frac{18}{58}$. The probability of getting 3 lengths that are longer than 4.5 m is thus $P(3 > 4.5) = \frac{20 \times 19 \times 18}{60 \times 59 \times 58} = \frac{57}{1711} \approx 0.03\ 331$.

31 There are 4 leaky bags (L) and 16 good bags (G) out of 20 bags altogether. On the branch of the simplified tree diagram going G-G-G, the fractions are $\frac{16}{20}$, $\frac{15}{19}$ and $\frac{14}{18}$. The probability of getting 3 good bags is thus $P(3\mathrm{G}) = \frac{16 \times 15 \times 14}{20 \times 19 \times 18} = \frac{28}{57} \approx 0.4912$.

32 You can draw a tree diagram with the first stage as the rotor, the second as the bearings and the third as the case. At each stage you can get a bad component (B) or a good component (G). The fitting of one component to another are independent events, so the probabilities are multiplied. Only the G-G-G branch gives a motor that is not a lemon. It has a probability of $0.97 \times 0.98 \times 0.99 \approx 0.9411$. the probability of getting a lemon is thus about 5.89%. You cannot just add the percentages because there will be overlap between the poor fittings.

Exercise 6.3

1 **a** $\frac{1}{3} \approx 0.3333$ **b** $\frac{2}{11} \approx 0.1818$ **c** $\frac{1}{5} = 0.2$
d $\frac{1}{6} \approx 0.1667$ **e** $\frac{1}{6} \approx 0.1667$ **f** $\frac{1}{3} \approx 0.3333$
g $\frac{1}{2} = 0.5$ **h** $\frac{2}{11} \approx 0.1818$

2 **a** $\frac{1}{19} \approx 0.05\ 263$ **b** $\frac{5}{19} \approx 0.2632$ **c** $\frac{3}{19} \approx 0.1579$
d $\frac{2}{19} \approx 0.1053$ **e** $\frac{4}{19} \approx 0.2105$ **f** $\frac{4}{19} \approx 0.2105$
g $\frac{4}{19} \approx 0.2105$ **h** $\frac{3}{19} \approx 0.1579$ **i** $\frac{2}{19} \approx 0.1053$
j $\frac{4}{19} \approx 0.2105$

3 **a** $\frac{2}{9} \approx 0.2222$ **b** $\frac{1}{6} \approx 0.1667$ **c** $\frac{1}{9} \approx 0.1111$
d $\frac{1}{9} \approx 0.1111$ **e** $\frac{1}{6} \approx 0.1667$ **f** $\frac{1}{6} \approx 0.1667$
g $\frac{2}{9} \approx 0.2222$ **h** $\frac{2}{9} \approx 0.2222$ **i** $\frac{1}{171} \approx 0.005\ 848$
j $\frac{2}{57} \approx 0.03\ 509$

4 **a** $\frac{1}{221} \approx 0.004\ 525$ **b** $\frac{1}{17} \approx 0.058\ 82$
c $\frac{11}{221} \approx 0.049\ 77$ **d** $\frac{105}{221} \approx 0.4751$
e $\frac{20}{221} \approx 0.090\ 50$ **f** $\frac{1}{17} \approx 0.058\ 82$
g $\frac{4}{17} \approx 0.2353$

5 **a** $\frac{8}{455} \approx 0.017\ 58$ **b** $\frac{4}{1365} \approx 0.002\ 941$
c $\frac{8}{455} \approx 0.017\ 58$ **d** $\frac{8}{455} \approx 0.017\ 58$
e $\frac{4}{455} \approx 0.008\ 791$ **f** $\frac{4}{455} \approx 0.008\ 791$
g $\frac{4}{455} \approx 0.008\ 791$ **h** $\frac{3}{1365} \approx 0.002\ 197$
i $\frac{8}{455} \approx 0.017\ 58$ **j** $\frac{4}{81} \approx 0.043\ 96$

6 $\frac{414}{455} \approx 0.9099$ **7** $\frac{444}{455} \approx 0.9758$

8 $\frac{86}{91} \approx 0.9451$ **9** $\frac{1}{12} \approx 0.083\ 33$

10 $\frac{1}{6} \approx 0.1667$

11 **a** $\frac{11}{81} \approx 0.1358$ **b** $\frac{11}{24} \approx 0.4583$ **c** $\frac{1}{2} = 0.5$
d $\frac{3}{16} = 0.1875$ **e** $\frac{3}{71} \approx 0.042\ 25$ **f** $\frac{3}{56} \approx 0.053\ 57$
g $\frac{19}{27} \approx 0.7037$ **h** $\frac{19}{82} \approx 0.2317$ **i** $\frac{41}{71} \approx 0.5775$
j $\frac{41}{64} \approx 0.6406$

12 $\frac{114}{169} \approx 0.6746$ **13** 0.8

14 $P(\text{2K and no better}) = 2 \times \frac{3}{49} \times \frac{44}{48} = \frac{11}{98} \approx 0.1122$; $P(\text{3A and no better}) = 6 \times \frac{2}{49} \times \frac{47}{48} \times \frac{44}{47} = \frac{11}{49} \approx 0.2245$; so you would be better off discarding the king.

15 **a** $\frac{17}{27} \approx 0.6296$ **b** $\frac{94}{127} \approx 0.7402$ **c** $\frac{14}{71} \approx 0.1972$

d While most people with black hair have brown eyes and nearly three-quarters of people with blond hair have blue eyes, they could have any eye colour, and only about 1 in 5 people with red hair have green eyes. Green eyes are uncommon anyway.

e $P(\text{black hair/brown eyes}) = \frac{17}{55} \approx 0.3091$, so it is an inaccurate stereotype.

f Unless the site is a reliable source, it is unwise to assume the results are reliable, although the sample size is not particularly small.

16 $\frac{4}{5} = 0.8$

17 $\frac{143}{2235} \approx 0.063\ 98$

18 The correct answer is $\frac{2}{11}$. Instead of using the number of ways of getting a total of 4 from the restricted sample space, the student used the number from the whole sample space.

19 The correct answer is $\frac{37}{138}$. All three students used the number of students with blue eyes in the restricted sample space correctly, but used the wrong sample space in the formula. The first student used 'blue eyes' as the sample space, the second used the whole sample and the third used the number of students with brown hair and brown eyes.

20 The correct answer is $\frac{37}{195} \approx 0.1897$, using conditional probability for pairs of apple, orange, banana, peach and pear cards and then the addition principle. The student did not take the first card into account when working out the probability of pairs, so he/she was assuming independence for each pair, although the student did correctly use the addition principle.

Chapter review

1 $\frac{1}{8} = 0.125$

2 **a** 0.16 **b** 0.36 **c** 0.48

3 **a** $\frac{4}{51} \approx 0.078\ 43$ **b** $\frac{16}{51} \approx 0.3137$ **c** $\frac{4}{25} = 0.16$
d $\frac{3}{5} = 0.6$ **e** $\frac{19}{25} = 0.76$ **f** 14

4 \$36 **5** 5 : 1 **6** $\frac{16}{25} = 0.64$

7 **a** $\frac{1}{24} \approx 0.041\ 67$ **b** $\frac{1}{12} \approx 0.083\ 33$
c $\frac{5}{36} \approx 0.1389$ **d** $\frac{53}{72} \approx 0.7361$

8 **a** $\frac{4}{99} \approx 0.040\ 40$ **b** $\frac{4}{99} \approx 0.040\ 40$
c $\frac{4}{165} \approx 0.024\ 24$ **d** $\frac{8}{99} \approx 0.080\ 81$

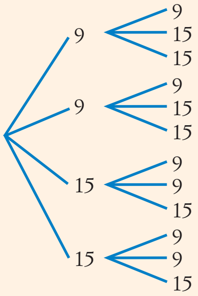
9 $\frac{10}{49} \approx 0.2041$

10 **a** $\frac{22}{27} \approx 0.8148$ **b** $\frac{31}{63} \approx 0.4921$

11 **a** $\frac{1}{3} \approx 0.3333$ **b** $\frac{2}{3} \approx 0.6667$ **c** About 22

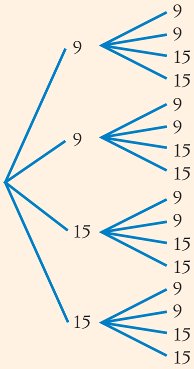
12 $\frac{5}{9} \approx 0.5556$

13 **a**



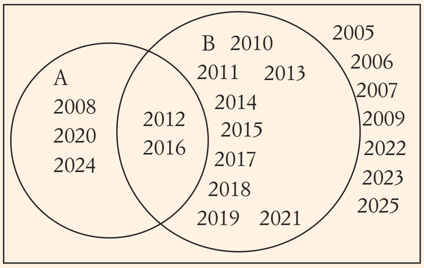
b $\frac{1}{6} \approx 0.1667$ **c** $\frac{2}{3} \approx 0.6667$

14 **a**



b $\frac{1}{4} = 0.25$ **c** $\frac{1}{2} = 0.5$

15 **a**



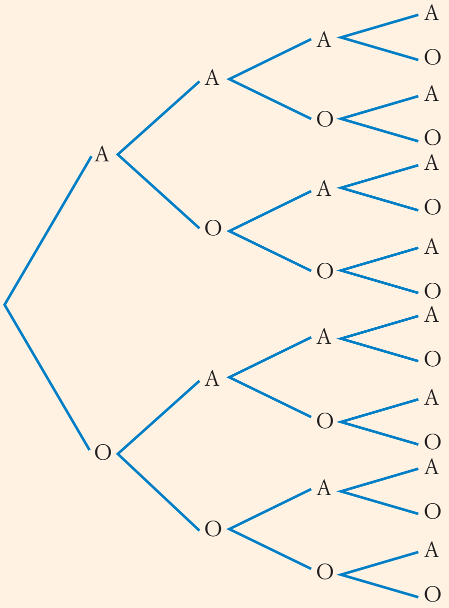
b $\frac{5}{21} \approx 0.2381$ **c** $\frac{11}{21} \approx 0.5238$ **d** $\frac{2}{21} \approx 0.095\ 24$

e $\frac{2}{3} \approx 0.6667$ **f** $\frac{1}{7} \approx 0.1429$

16 **a** {B1B2, B1R1, B1R2, B1B, B2R1, B2R2, B2B, R1R2, R1B, R2B}

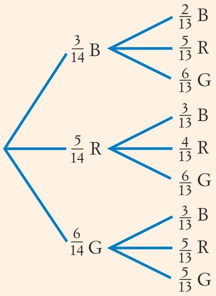
b $\frac{2}{5} = 0.4$

17 **a**



b $\frac{3}{16} = 0.1875$ **c** $\frac{3}{4} = 0.75$ **d** $\frac{1}{16} = 0.0625$

18 **a**



b $\frac{10}{91} \approx 0.1099$ **c** $\frac{15}{182} \approx 0.082\ 42$ **d** $\frac{15}{91} \approx 0.1648$

e $\frac{30}{91} \approx 0.3297$ **f** $\frac{9}{13} \approx 0.6923$

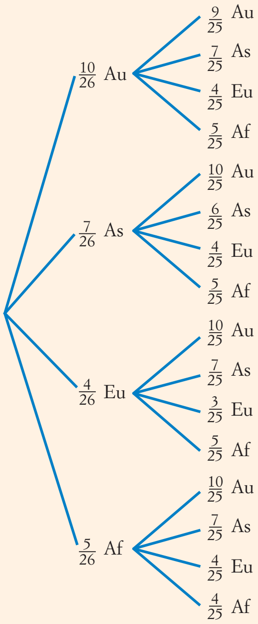
19 **a** $\frac{4}{11} \approx 0.3636$ **b** $\frac{3}{5} = 0.6$

c $\frac{1}{6} \approx 0.1667$ **d** $\frac{3}{22} \approx 0.1364$

20 $\frac{8}{9} \approx 0.8889$

21 **a** $\frac{23}{80} = 0.143\ 75$ **b** $\frac{27}{80} = 0.3375$

22 **a**



b $\frac{21}{325} \approx 0.064\ 62$ **c** $\frac{243}{325} \approx 0.7477$

d $\frac{4}{65} \approx 0.061\ 54$ **e** $\frac{37}{130} \approx 0.2846$

23 $\frac{97}{150} \approx 0.6467$ **24** $\frac{5}{8} = 0.525$

25 **a** $\frac{7}{30} \approx 0.2333$ **b** About 27. **c** About 133.

26 **a** $\frac{29}{44} \approx 0.6591$ **b** $\frac{3}{5} = 0.6$

c Considering the sample size, there is no real difference.

27 **a** {T1T2, T1T3, T1T4, T1F1, T1F2, T2T3, T2T4, T2F1, T2F2, T3T4, T3F1, T3F2, T4F1, T4F2, F1F2}

b $\frac{7}{15} \approx 0.4667$

28 **a** $\frac{1}{200} = 0.005$ **b** $\frac{3}{25} = 0.12$

c $\frac{3}{200} = 0.015$ **d** $\frac{2}{25} = 0.08$

29 $\frac{2}{3} \approx 0.6667$

30 **a** $\frac{36}{67} \approx 0.5373$ **b** $\frac{36}{107} \approx 0.3364$ **c** $\frac{5}{8} = 0.525$

d 0.4627 of students who work make >$100 compared to 0.375 of students who received gifts.

31 $\frac{78}{625} = 0.1248$ **32** $\frac{3}{20} = 0.15$

33 **a** $\frac{1}{8} = 0.125$ **b** $\frac{1}{3} \approx 0.3333$

c $\frac{5}{24} \approx 0.2083$ **d** $\frac{19}{24} \approx 0.7917$

34 **a** $\frac{3}{185} \approx 0.016\ 21$ **b** $\frac{153}{10\ 545} \approx 0.014\ 51$

c $\frac{36}{3515} \approx 0.010\ 24$ **d** $\frac{18}{925} \approx 0.019\ 46$

e $\frac{51}{925} \approx 0.055\ 14$

35 Of the 55 recordings, 14 are netball or basketball. A simplified tree diagram for picking 3 recordings, basketball/netball (BN) or other (O) will have a branch that goes BN-BN-BN that has probabilities $\frac{14}{55}$, $\frac{13}{54}$ and $\frac{12}{53}$. The probability of getting all three basketball or netball is thus $\frac{14 \times 13 \times 12}{55 \times 54 \times 53} = \frac{364}{26\ 235} \approx 0.013\ 87$. It could also be done by conditional probability.

36 The student has not taken into account the order in which the heads and tails appear. When the order is taken into account, there are 16 ways the coins could fall, and only one of them has 4 heads, so the correct probability is $\frac{1}{16} = 0.0625$.

37 There are actually only 11 ways that one of the dice could show 4, because in one case they are both showing 4. Of the 11 ways they could show 4, only 2, 3 and 4 or 4 and 3, have a total of 7. Thus the probability is $\frac{2}{11} \approx 0.1818$ or about 18%.

38 The person probably looked at the total number of people who got $10–19, which was 16 out of the 40 who got pocket money to get 40%. Instead, the person should have used the restricted sample space consisting only of the people who got pocket money. So the correct numerator is actually only 11, giving $\frac{11}{40} = 27.5\%$.

39 The answer to question **20** can be calculated as the part of a single square where the centre of the circle overlaps the lines of the square. This is all but a square 1 cm by 1 cm in the middle of the large square. However, if the piece was not allowed to be partly off the board, there would be part of the outside squares where it would not count. This would be the outside 1 cm of these squares. The board is 24 cm by 24 cm, so the part where the piece was allowed would be 22 × 22 $\text{cm}^2 = 484\ \text{cm}^2$. All 64 squares would have 1 cm^2 where the pieces do not overlap, so the probability would be $\frac{64}{484} = \frac{105}{121}$ $\approx$ 0.8678 which is a little less than in the original case.

40 The class member worked out the probability using the number intending to do Physics in the whole sample space instead of restricting the sample space to the 10 intending to take Chemistry. Of the 10 intending to do Chemistry, there are 5 intending to take Physics, so the probability is actually 50%.

Chapter 7

Exercise 7.1

1 **a** $(x + y)(x - y)$ **b** $(4 + a)(4 - a)$
c $(8 + b)(8 - b)$ **d** $(w + 7)(w - 7)$
e $(p + q)(p - q)$ **f** $(a + 5)(a - 5)$
g $(c + 9)(c - 9)$ **h** $(x + 11)(x - 11)$
i $(6 + d)(6 - d)$ **j** $(10 + m)(10 - m)$

2 **a** $(m + n)^2$ **b** $(x - y)^2$
c $(a + 2)^2$ **d** $(w + 3)^2$
e $(h - k)^2$ **f** $(b - 5)^2$
g $(n + 6)^2$ **h** $(m - 6)^2$
i $(x - 7)^2$ **j** $(k + 9)^2$

3 **a** $(a + m)(a - m)$ **b** $(x + 1)(x - 1)$
c $(c + 3)(c - 3)$ **d** $(y + 4)(y - 4)$
e $(5 + w)(5 - w)$ **f** $(7 + a)(7 - a)$
g $(b + 10)(b - 10)$ **h** $(g + 9)(g - 9)$
i $(13 + n)(13 - n)$ **j** $(x + 8)(x - 8)$

4 **a** $(a+b)^2$ **b** $(m-n)^2$
c $(h-3)^2$ **d** $(x+10)^2$
e $(p+5)^2$ **f** $(g-6)^2$
g $(x-7)^2$ **h** $(h+3)^2$
i $(5-n)^2$ **j** $(p+4)^2$
k $(3-x)^2$ **l** $(y+9)^2$

5 **a** $(b+m)(6+a)$ **b** $(y-g)(7+d)$
c $(b-a)(5+c)$ **d** $(m+q)(m+n)$
e $(b+a^2)(a-3)$ **f** $(m^2-3)(2m+1)$
g $(a+b)(x+y)$ **h** $(x+a)(y-z)$
i $(d-p)(a+b)$

6 **a** $(2a+5b)(2a-5b)$ **b** $(4x+3y)(4x-3y)$
c $(7m+4n)(7m-4n)$ **d** $(3a+8b)(3a-8b)$
e $(2x+11y)(2x-11y)$ **f** $(6x+5y)(6x-5y)$
g $(12m+5n)(12m-5n)$ **h** $(8p+7q)(8p-7q)$
i $(9a+4b)(9a-4b)$

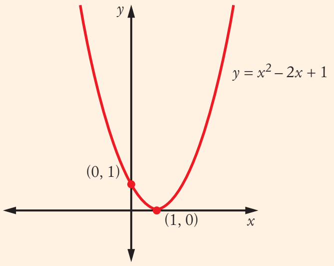
7 **a** $(2x-1)^2$ **b** $(3x+1)^2$
c $(4g-1)^2$ **d** $(2a-b)^2$
e $(3p+q)^2$ **f** $(5x-y)^2$
g $(x+8b)^2$ **h** $(b+11c)^2$
i $(a-9b)^2$ **j** $(a-8b)^2$
k $(3h+k)^2$ **l** $(4g+1)^2$
m $(2m-n)^2$ **n** $(4p+q)^2$
o $(5x-1)^2$ **p** $(7y-4)^2$

8 **a** $(2a+3)^2$ **b** $(3x-5)^2$
c $(4m-3)^2$ **d** $(3n+2)^2$
e $(5k-4)^2$ **f** $(2a-7)^2$
g $4(m-3)^2$ **h** $(9x+2)^2$
i $(4a-5)^2$ **j** $(8m+7)^2$

9 **a** $(y+1)(y-1)(2x-1)$ **b** $(m^2n-1)(n-1)$
c $(x+z)(h+g+j)$ **d** $(d-c)(a-f-g)$
e $(3+a+b)(a-b)$ **f** $(x-4)(4x+1)$
g $(a+4)(c+d)$ **h** $(xy-4)(x+3)$
i $(a+1)(ab-1)$ **j** $2(3a^2-2)(a-1)$

10 **a** $(a-4)(a-2)$ **b** $(a+b-3)(a+b+3)$
c $(x-11)(x-3)$ **d** $(5-2m)(2m-1)$
e $(9-4a)(4a-5)$ **f** $(x-8)(x+2)$
g $(11x+4)(x-16)$ **h** $(11a-1)(5a+17)$
i $(11-8x)(20x-31)$ **j** $-3y(2x+y)$
k $3b(2a-b)$

11 **a**



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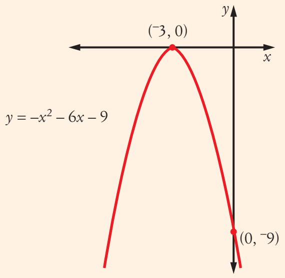

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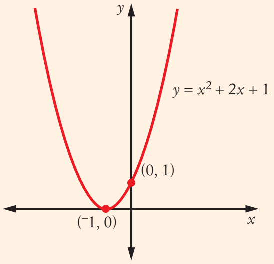

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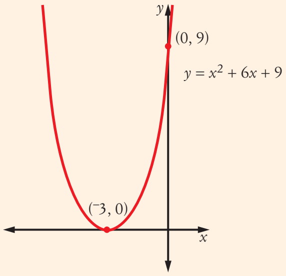

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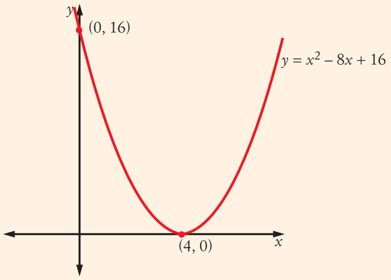

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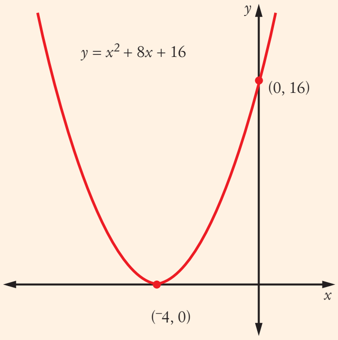

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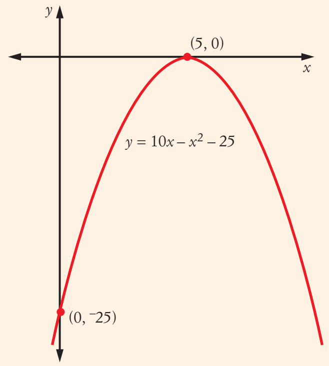

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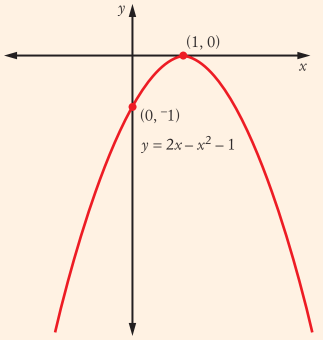


12 $A = \pi r_1^2 - \pi r_2^2 = \pi(r_1 - r_2)(r_1 + r_2) = \pi d_1 d_2$

13 **a** b^2 m^2 **b** a^2 m^2

c $(b^2 - a^2)$ m^2 **d** $(b + a)(b - a)$

e 17 400 m^2; easier to calculate with the factorised expression.

14 Proof

Exercise 7.2

1 **a** $(a + 4)(a + 1)$ **b** $(x + 5)(x + 1)$
c $(y + 2)(y + 5)$ **d** $(m + 1)(m + 9)$
e $(p - 4)(p - 1)$ **f** $(b - 3)(b - 4)$
g $(a + 1)(a - 4)$ **h** $(x + 2)(x - 4)$
i $(y - 3)(y + 4)$ **j** $(a - 2)(a + 10)$

2 **a** $(g + 2)(g + 6)$ **b** $(h + 3)(h + 2)$
c $(k + 7)(k + 6)$ **d** $(x + 2)(x + 3)$
e $(t + 6)(t + 1)$ **f** $(z + 5)(z + 6)$
g $(t + 6)(t + 9)$ **h** $(y + 4)(y + 3)$
i $(a + 7)(a + 4)$ **j** $(w + 5)(w + 7)$

3 **a** C **b** A **c** D
d D **e** C

4 **a** 3 and 5 **b** $^-9$ and 3
c $^-8$ and 5 **d** 8 and 4
e $^-9$ and 4 **f** $^-8$ and $^-8$
g 5 and $^-3$ **h** $^-9$ and 6

5 **a** $(2x + 7)(x - 1)$ **b** $(3a - 2)(2a + 3)$
c $(4x + 3)(3x - 5)$ **d** $(5y + 6)(y - 4)$
e $(7x + 3)(2x + 1)$ **f** $(3x - 1)^2$
g $(5a - 2)(4a + 3)$ **h** $(5x + 1)(2x - 7)$
i $(6a - 1)(a - 3)$ **j** $(4x + 3)(2x - 9)$

6 **a** $(a + 1)(a + 3)$ **b** $(x + 1)(x + 8)$
c $(y + 1)(y + 6)$ **d** $(m + 2)(m + 3)$
e $(x + 4)(x + 2)$ **f** $(p + 4)(p + 5)$
g $(p + 2)(p + 8)$ **h** $(g + 5)(g + 3)$
i $(x + 6)(x + 4)$ **j** $(x + 10)(x + 3)$
k $(a + 10)(a + 4)$ **l** $(m + 14)(m + 3)$
m $(x - 2)(x - 1)$ **n** $(a - 3)(a - 1)$
o $(y - 2)(y - 3)$ **p** $(x - 9)(x - 1)$
q $(y - 2)(y - 5)$ **r** $(a - 2)(a - 8)$
s $(x - 3)(x - 7)$ **t** $(w - 2)(w - 9)$
u $(a - 3)(a - 8)$ **v** $(a - 2)(a - 16)$
w $(y - 15)(y - 1)$ **x** $(b - 7)(b - 5)$

7 **a** $(b - 3)(b + 2)$ **b** $(a + 4)(a - 5)$
c $(x + 3)(x - 4)$ **d** $(y + 6)(y - 7)$
e $(c - 3)(c + 1)$ **f** $(a + 2)(a - 4)$
g $(n - 5)(n + 3)$ **h** $(m - 8)(m + 1)$
i $(a - 5)(a + 2)$ **j** $(x - 5)(x + 2)$
k $(y - 6)(y + 2)$ **l** $(b - 9)(b + 3)$
m $(p + 6)(p - 1)$ **n** $(k - 1)(k + 4)$
o $(a - 2)(a + 3)$ **p** $(c + 5)(c - 4)$
q $(x + 8)(x - 5)$ **r** $(y - 1)(y + 6)$
s $(a - 2)(a + 6)$ **t** $(b - 1)(b + 7)$
u $(k - 2)(k + 8)$ **v** $(p - 4)(p + 8)$
w $(a - 5)(a + 7)$ **x** $(x - 4)(x + 9)$

8 **a** $(e - 5n)(e - 9n)$ **b** $(w - 7t)(w + 15t)$
c $(a - 5g)(a - 3g)$ **d** $(b - 9v)(b + 12v)$
e $(p - e)(p + 4e)$ **f** $(c + 3m)(c - 11m)$
g $(z + m)(z - 11m)$ **h** $(y + 11n)(y - 5n)$
i $(g + 13e)(g - 6e)$ **j** $(g - 9p)(g + 11p)$
k $(h + 7w)(h + 14w)$ **l** $(z + 15q)(z + 14q)$
m $(z - 15u)(z + 4u)$ **n** $(x + 7z)(x + 12z)$
o $(q - 4n)(q + 7n)$ **p** $(p + r)(p - 3r)$
q $(x - 6v)(x - 8v)$ **r** $(k + 3p)(k + p)$
s $(h + 9z)(h - 12z)$ **t** $(c - 10p)(c - 13p)$
u $(z - 10r)(z + 4r)$ **v** $(t + 3p)(t - 9p)$
w $(g - 2q)(g + 12q)$ **x** $(k - 12m)(k - 2m)$

9 **a** $(2x + 3)(x + 3)$ **b** $(2a + 5)(a + 2)$
c $(2x + 3)(x + 5)$ **d** $(3m + 2)(m + 4)$
e $(3x + 2)(x + 6)$ **f** $(3b + 4)(b + 3)$
g $(5x + 2)(x + 4)$ **h** $(5c + 9)(c + 1)$
i $(2x + 3)^2$ **j** $(4b + 1)(b + 3)$
k $(5p + 2)(p + 1)$ **l** $(5h + 1)(h + 2)$
m $(2a + 1)(3a + 2)$ **n** $(6x + 1)(x + 2)$
o $(3y + 2)(y + 2)$ **p** $(3x + 4)(2x + 5)$
q $(3n + 2)(4n + 7)$ **r** $(4x + 3)(2x + 3)$
s $(4y + 5)(3y + 5)$ **t** $(4w + 3)^2$
u $(5x + 6)(3x + 2)$ **v** $(8r + 3)(4r + 5)$
w $2(3a + 4)(2a + 1)$ **x** $(7x + 4)(6x + 7)$

10 **a** $(2x + 1)(x - 5)$ **b** $(5a + 4)(a - 1)$
c $(2x + 3)(x - 5)$ **d** $(2m + 1)(m - 1)$
e $(3x + 1)(x - 1)$ **f** $(6b + 1)(b - 1)$
g $(2x + 1)(x - 3)$ **h** $(2h + 1)(h - 2)$
i $(6y + 1)(y - 2)$ **j** $(5x + 1)(x - 2)$
k $(9n + 1)(n - 2)$ **l** $(10x + 1)(x - 2)$

m $(2a+1)(a-4)$ **n** $(x+2)(7x-1)$
o $(10y-1)(y+2)$ **p** $(4x-1)(x+4)$
q $(5x-1)(x+3)$ **r** $(2b-1)(b+3)$
s $(3x-1)(x+3)$ **t** $(4w-1)(w+3)$
u $(5x-1)(x+4)$ **v** $(6r-1)(r+4)$
w $(2x-1)(x+4)$ **x** $(2a-1)(a+5)$

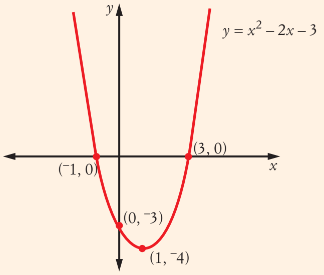
11 **a** $(5x-1)(x+8)$ **b** $(3a+2)(a-4)$
c $(7b+5)(b-3)$ **d** $(x+2)(5x-3)$
e $(3y+1)(y+6)$ **f** $(5x-9)(x-3)$
g $(7f-5)(f-7)$ **h** $(11k-1)(k-8)$
i $(2a+7)(6a-5)$ **j** $(4y+7)(5y-12)$
k $(6x+11)(4x+11)$ **l** $(2d+7)(3d-11)$
m $(2x+5)(x-2)$ **n** $(5m+4)(6m-11)$
o $(3x+1)(2x-5)$ **p** $(2c+5)(3c-4)$
q $(3x+1)^2$ **r** $(3a+4)(6a-1)$
s $(2p+5)(6p-5)$ **t** $(3x+2)(5x-1)$
u $(4a+1)(2a-3)$ **v** $(7x+1)(x-4)$
w $(n+4)(9n-1)$ **x** $(3x+11)(2x-7)$

12 **a** $(9z-8n)(3z+4n)$ **b** $(9b+14g)(3b+2g)$
c $(6p+e)(14p-5e)$ **d** $(5v+11k)(v-9k)$
e $(10b+11n)(4b+5n)$ **f** $(3m-2p)(5m+4p)$
g $(7x+11u)(14x+5u)$ **h** $(6p-11n)(12p-11n)$
i $(8x+7g)(6x-11g)$ **j** $(2v-15n)(15v-2n)$
k $(6m-5h)(4m-15h)$ **l** $(3z-4m)(15z+m)$
m $(8b+9h)(14b+3h)$ **n** $(9y+8a)(7y-10a)$
o $(8m-12r)(3m-2r)$ **p** $(8n+a)(13n-10a)$
q $(13u+15g)(11u-6g)$ **r** $(11e-14u)(10e+9u)$
s $(15x-14p)(12x+p)$ **t** $(13t-7c)(15t+c)$
u $(6r-k)(6r-11k)$ **v** $(3x-b)(x+14b)$
w $(9z-13n)(11z-3n)$ **x** $(8t+3p)(t+3p)$

13 Four values: $^{-}7$, $^{-}5$, 5, 7

14 Six values: $^{-}11$, $^{-}5$, $^{-}1$, 1, 5, 11

15 **a**



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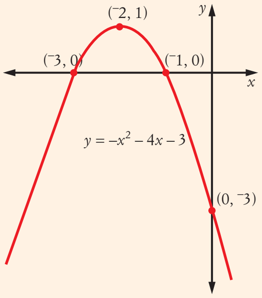

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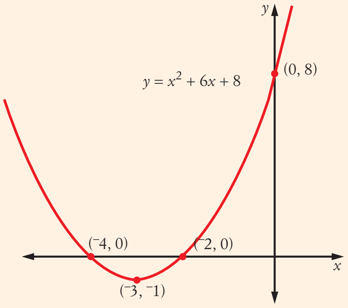

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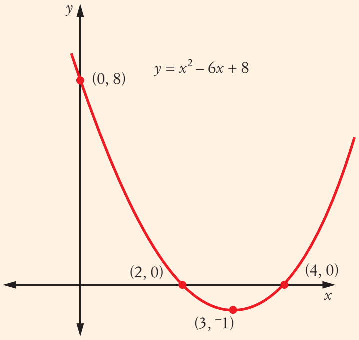

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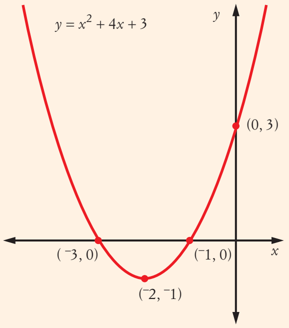

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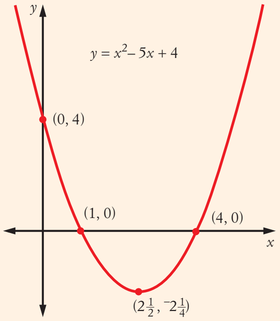

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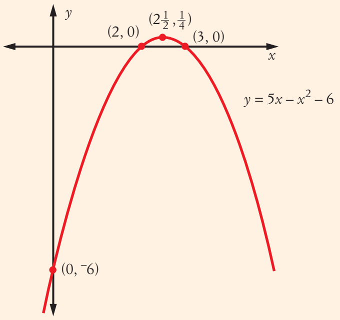

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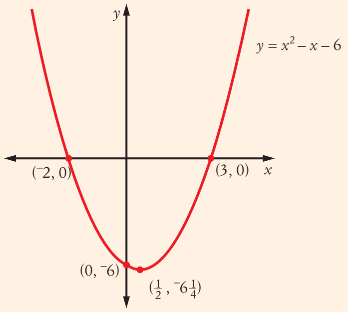


16 Nine values: 2, 6, 12, 20, 30, 42, 56, 72, 90

17 Nine values: 3, 8, 15, 24, 35, 48, 63, 80, 99

18 Six values: 117, 126, 145, 155, 176, 187

19 None, because they always occur in positive and negative pairs.

Exercise 7.3

1 $p(2) = 0$ **2** $q(^-3) = 0$ **3** $r(1) = 0$

4 $m(2) = 0$ and $m(^-2) = 0$

5 $n(3) = 0$ and $n(^-1) = 0$

6 **a** $3(x + 2y)(x - 2y)$
b $5(a + 3b)(a - 3b)$
c $8(m + 2n)(m - 2n)$
d $3ab(a + 3b)(a - 3b)$
e $6xy(x + 2y)(x - 2y)$
f $7pq(p + 3q)(p - 3q)$
g $7n(2n + 1)(2n - 1)$
h $3x(3x + 1)(3x - 1)$
i $4h(2h + 1)(2h - 1)$
j $2(3x + 1)(3x - 1)$

7 **a** $2(n - 4k)(n - 2k)$
b $7(n - 9y)(n - 4y)$
c $5(q - 7t)(q - 6t)$
d $4(u + 4t)(u - 3t)$
e $3(w + 7m)^2$
f $8(n - 4w)(n + 7w)$
g $7(z - 2w)(z - 4w)$
h $6(a - 5c)(a - 8c)$
i $3(t - 4h)(t + 6h)$
j $8(a - 5m)(a + 3m)$

8 **a** $8(2w - 7c)(3w + 7c)$
b $8(2m - 9x)(8m - 3x)$
c $3(4q - 3m)(q - 3m)$
d $2(3c + 7m)(7c - 4m)$
e $2(2p - 9r)(4p - 7r)$
f $7(7a + 6m)(a - 5m)$
g $9(2n + 3q)^2$
h $7(2c + 5v)(4c - 7v)$
i $2(8t + w)(t - 9w)$
j $4(6h + 5q)(9h - 7q)$

9 **a** $x^2 - 3x + 5, R = {}^-12$
b $3x^2 - 8x + 14, R = {}^-29$
c $x^2 - 2x - 4, R = {}^-4$
d $2x^3 - 2x^2 + 4x - 3, R = 0$
e $2x^3 - 4x^2 + 9x - 29, R = 88$
f $x^3 + 2x^2 + 6x + 15, R = 30$
g $x^3 + 4x^2 + 14x + 45, R = 135$
h $2x^4 + 3x^3 + 3x^2 + 7x + 14, R = 26$
i $3x^4 - 2x^3 - x^2 - 2x + 3, R = {}^-1$
j $2x^4 + 3x^3 + 6x^2 + 10x + 22, R = 45$

10 **a** $(a - 4)(a - 2)$
b $(a + b - 3)(a + b + 3)$
c $(x - 11)(x - 3)$
d $8(5 - 2m)(2m - 1)$
e $9(9 - 4a)(4a - 5)$
f $5(x - 8)(x + 2)$
g $(11x + 4)(x - 16)$
h $(11a - 1)(5a + 17)$
i $(11 - 8x)(20x - 31)$
j ${}^-9y(2x + y)$
k $6b(2a - b)$
l $(a^2 + b^2)(a + b)(a - b)$
m $(x^2 + 4)(x + 3)(x - 3)$
n $(a^2 + 7)(a + 3)(a - 3)$
o $5(4a^2 + n^2)(2a + n)(2a - n)$

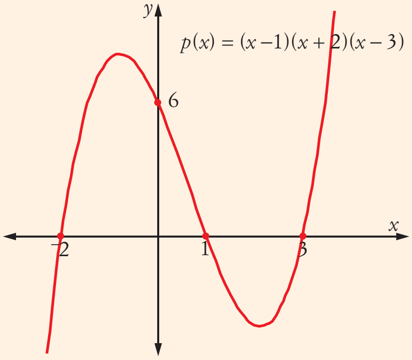
11 **a** $(5a^2 + 4)(a + 1)$
b $(b^2 + 2)(b - 5)$
c $(m^2 + 3)(5m - 1)$
d $(a - 3)(a + b + 3)$
e $(x - 2y)(3x + 6y + 1)$
f $(6 + a^2)(1 + a)$
g $(x + y - a)(x + y + a)$
h $(y - a - 3)(y + a + 3)$
i $(3x - p + 1)(3x + p - 1)$
j $(6k + 1 + 5h)(6k + 1 - 5h)$

12 **a** $(2xy - x + y)(2xy + x - y)$
b $(3ab - x - p)(3ab + x + p)$
c $(x + y - z)(x + y + z)$
d $(m - n - p)(m - n + p)$
e $(z - x - y)(z + x + y)$
f $(a - b - 2)(a - b + 2)$
g $(p + q - 5)(p + q + 5)$
h $(a - b - 2)(a - b + 2)$
i $(m - 3 - 4k)(m - 3 + 4k)$
j $(m - 5 - 4c)(m - 5 + 4c)$
k $(r - s - 4)(r + s + 4)$
l $(a - 2c + 3m)(a + 2c - 3m)$
m $(4x - y - z)(4x - y + z)$
n $(a^2 - 5 - b)(a^2 - 5 + b)$

13 **a** $4(a - 3)(a - 4)$
b $10(x + 3)(x + 2)$
c $5(2m + 5)(2m - 5)$
d $3a^2(2a + 5)(2a - 5)$
e $8(m + 2np)(m - 2np)$
f $(a - b + 4)(a - b + 1)$
g $(a + 3b)(3a + b)$
h $(x^2 + 5)(x + 2)(x - 2)$
i $a(3y + 1)(2y + 1)$
j $16(2 - 3y)(2 + y)$
k $2y^2(3xy + 1)(2xy - 3)$
l $(8 - 3h)(5 - h)$
m $(x^2 - 2)(9x^2 + 16)$

14 **a** $(x + 2)(x - 1)(2x - 3)$
b $(x - 2)(x + 1)(x + 5)$
c $(x - 1)(x + 2)(3x - 2)$
d $(x + 2)(x - 2)(2x + 5)$
e $(x - 2)(x - 1)(2x^2 - 2x + 3)$
f $(x + 3)(x + 2)(3x - 1)(x + 1)$
g $(x + 4)(x - 2)(x + 2)(x + 3)$
h $(x - 3)(x + 1)(x^2 + 4x + 2)$
i $(x - 3)(x - 1)(x + 2)(x + 1)$
j $(x + 3)(x - 2)(2x^2 - 3x - 1)$

15 **a**



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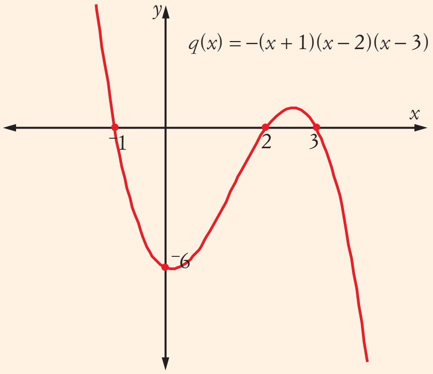

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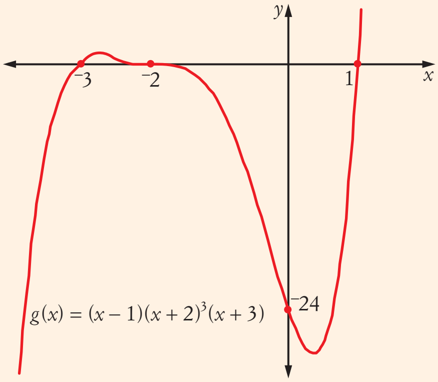

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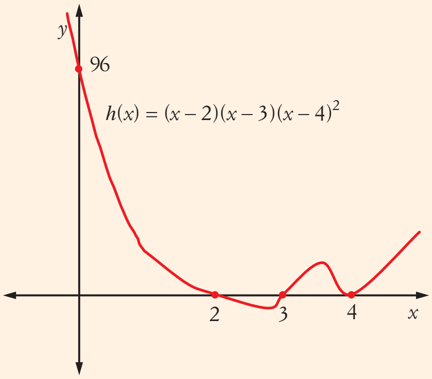

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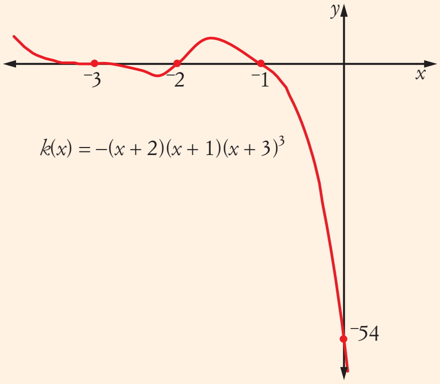

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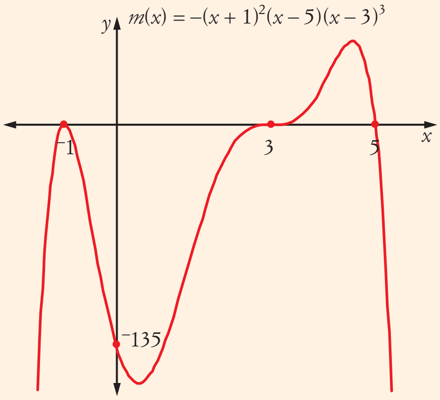
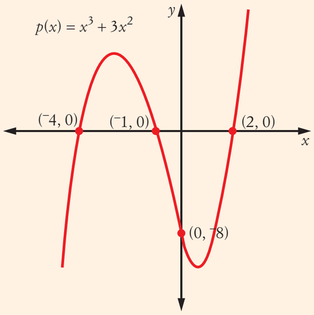
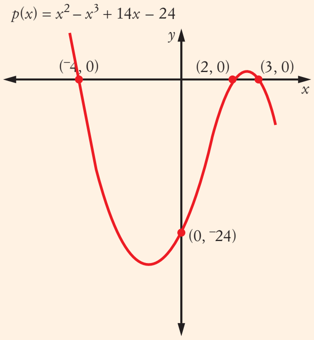


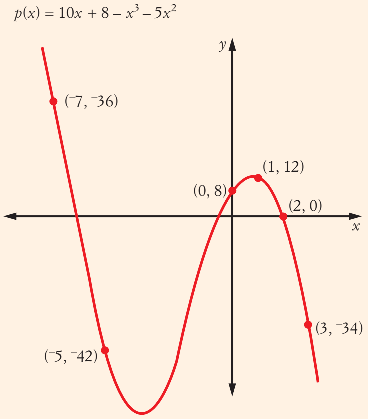
16 **a** $p(x) = x^3 + 3x^2 - 6x - 8 = (x+1)(x-2)(x+4)$



b $p(x) = x^2 - x^3 + 14x - 24 = -(x-3)(x+4)(x-2)$

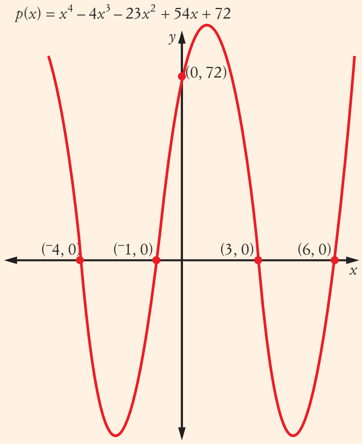


c $p(x) = 10x + 8 - x^3 - 5x^2 = -(x-2)(x^2 + 7x + 4)$



d $p(x) = x^4 - 4x^3 - 23x^2 + 54x + 72$

$= (x - 3)(x + 1)(x - 6)(x + 4)$



17 a 441 **b** 361 **c** 961

d 841 **e** 10 201

Chapter review

1 a $(m + p)(m - p)$ **b** $(x - 3)^2$ **c** $(g + 5)^2$

2 a $(r + 4)(r - 4)$ **b** $(h + 6)^2$ **c** $(k - 7)^2$

3 a $(x + 2y)(3x - y)$ **b** $(b + c)(a + d)$

c $(x - 3y)(x - 6z)$

4 a $(m + 4)(m + 3)$ **b** $(x - 8)(x + 2)$

5 a $(x + 2)(x + 8)$ **b** $(g + 4)(g + 7)$

6 $p({}^{-}2) = 0$ and $p(3) = 0$.

7 a $(5g + 6t)(5g - 6t)$ **b** $(4u - 7)^2$ **c** $(3m + n)^2$

8 a $(m - 1)(7m^2 + 4a)$ **b** $(6 - m + n)(6 + m - n)$

9 a $(x - 6)(x + 7)$ **b** $(p - 5)(p + 3)$

c $(k - 8)(k - 2)$ **d** $(a + 4)(a + 9)$

10 a $(y + 2k)(y + 9k)$ **b** $(m + z)(m + 7z)$

c $(n - 11z)(n + 12z)$

11 a $5x(x - 4y)(x + 4y)$ **b** $2(m + 7a)(m - 4a)$

c $3(q + 8t)(q - 9t)$ **d** $7(g - 6m)(g + 4m)$

12 a $(g - 10p)(g - 3p)$ **b** $(8u - 7g)(6u + 7g)$

c $(3t + 7b)(5t + 9b)$

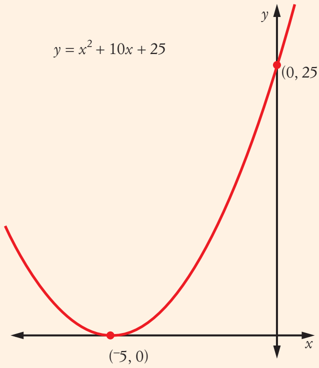
13 a $(2u + 5)(5u + 8)$ **b** $(4b + 3)(7b - 3)$

c $(8a + 3)(7a - 9)$ **d** $(4a - 7)(a - 7)$

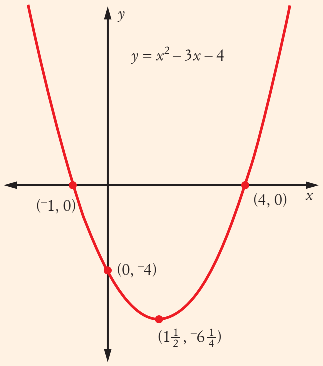
14 a $3x^3 - x^2 + x - 1,\ R = {}^{-}1$

b $2x^4 + x^3 + 4x^2 + 11x + 22,\ R = 38$

15 a



b



16 Six: $^{-}13, {}^{-}8, {}^{-}7, 7, 8, 13$

17 Six: $^{-}19, {}^{-}8, {}^{-}1, 1, 8, 19$

18 a $^{-}7(3a + 2)(3a + 8)$ **b** $(x - 2)(3x + 6 + y)$

c $3x^2y^2(x^2 - 4y + 5xy^2)$ **d** $(4p^2 - 9q)(4p^2 + 9q)$

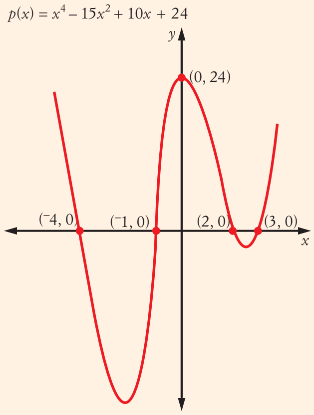
e $(3p - 10)(2p - 13)$

19 a $(x + 3)(x - 1)(2x^2 + x + 2)$

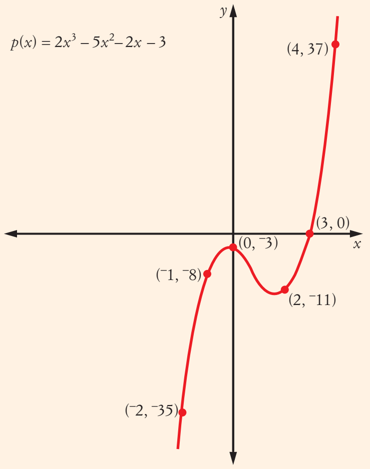
b $(x + 1)(x + 2)(3x + 1)(x - 1)$

20 a $p(x) = x^4 - 15x^2 + 10x + 24$

$= (x - 3)(x - 2)\ (x + 4)(x + 1)$



b $p(x) = 2x^3 - 5x^2 - 2x - 3 = (x - 3)(\ 2x^2 + x + 1)$



21 Proof
22 Eight: 4, 10, 18, 28, 40, 54, 70, 88
23 Two: 144 and 176

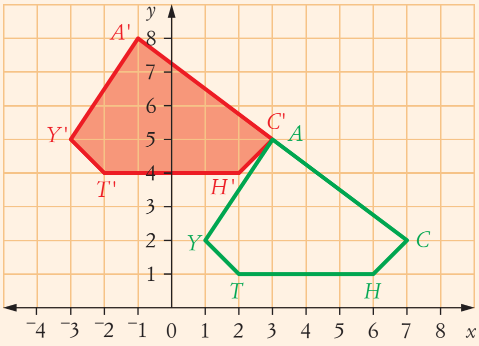
Chapter 8

Exercise 8.1

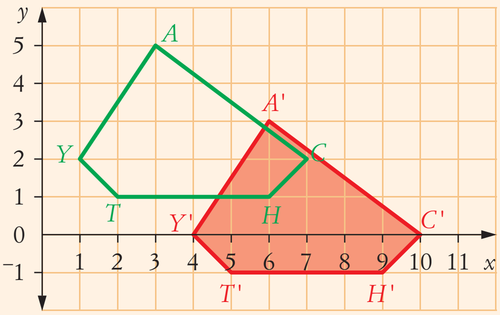
1 **a** $B(1, 2) \longrightarrow B'(0, 1)$, $I(2, 3) \longrightarrow I'(2, 3)$, $R(2, 2) \longrightarrow R'(2, 1)$, $D(6, 3) \longrightarrow D'(10, 3)$, $Y(3, 1) \longrightarrow Y'(4, {}^{-}1)$
b $B(1, 2) \longrightarrow B'(1, 2)$, $I(2, 3) \longrightarrow I'(2, 1)$, $R(2, 2) \longrightarrow R'(1, 1)$, $D(6, 3) \longrightarrow D'(2, {}^{-}3)$, $Y(3, 1) \longrightarrow Y'(0, 0)$
c $B(1, 2) \longrightarrow B'(2, {}^{-}1)$, $I(2, 3) \longrightarrow I'(3, 0)$, $R(2, 2) \longrightarrow R'(3, {}^{-}1)$, $D(6, 3) \longrightarrow D'(7, 0)$, $Y(3, 1) \longrightarrow Y'(4, {}^{-}2)$
d $B(1, 2) \longrightarrow B'(1, 4)$, $I(2, 3) \longrightarrow I'(2, 3)$, $R(2, 2) \longrightarrow R'(2, 4)$, $D(6, 3) \longrightarrow D'(6, 3)$, $Y(3, 1) \longrightarrow Y'(3, 5)$
e $B(1, 2) \longrightarrow B'(7, 2)$, $I(2, 3) \longrightarrow I'(6, 3)$, $R(2, 2) \longrightarrow R'(6, 2)$, $D(6, 3) \longrightarrow D'(2, 3)$, $Y(3, 1) \longrightarrow Y'(5, 1)$

2 **a** *THISWO*, regular hexagon
b *ULDBT*, pentagon
c *HEGLIB*, concave hexagon
d *VNGWEHA*, concave heptagon
e *VPSI*, concave quadrilateral
f *NEDFORTIM*, concave nonagon
g *EANDT*, pentagon
h *DTI*, acute-angled scalene triangle
i *MEA*, acute-angled equilateral triangle
j *GAIN*, rectangle
k *SNCE*, parallelogram
l *VOLDEMNT*, octagon
m *MET*, acute-angled scalene triangle
n *OUR*, right-angled isosceles triangle
o *HER*, obtuse-angled scalene triangle
p *OHAR*, trapezium
q *POTERBYWK*, concave nonagon
r *ROWLING*, heptagon
s *ANDI*, quadrilateral

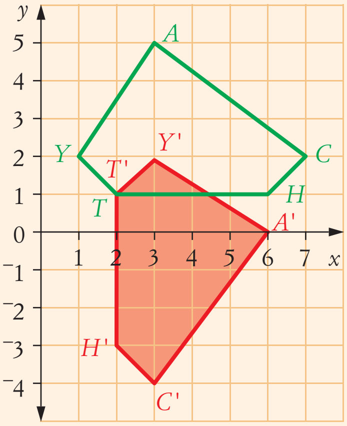
3 **a**



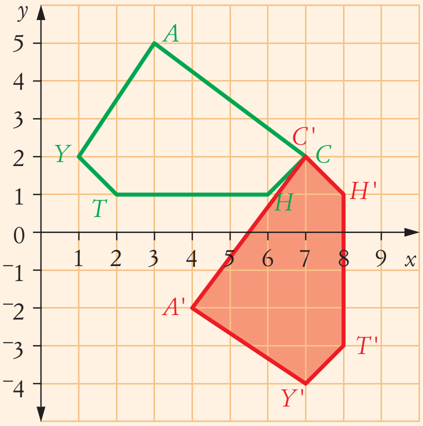
b



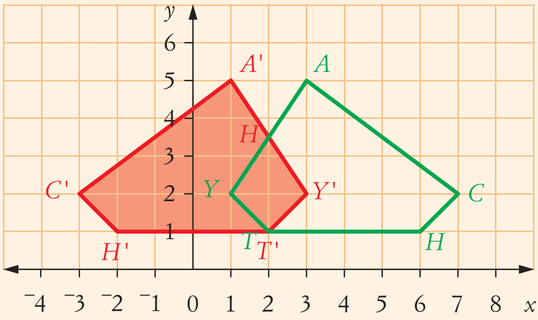
c



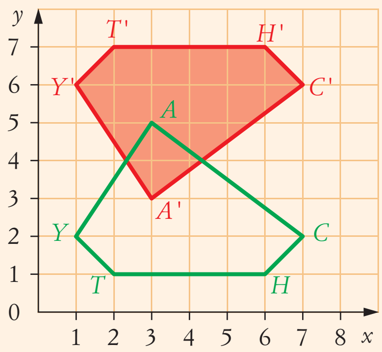
d



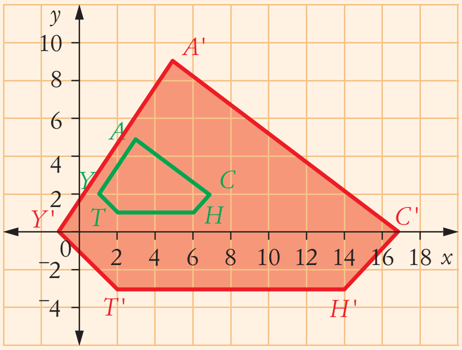
e



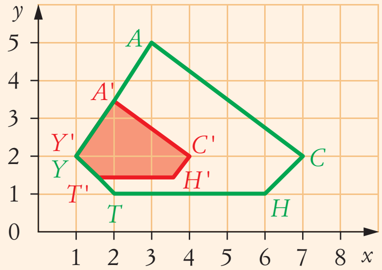
f



g



h



4 **a** Translation 10 units to the right and 3 units up.
b Reflection in the y-axis.
c Translation 8 units to the right and 6 units down.
d Rotation 270° clockwise about $B(^{-}4, 5)$.
e Magnification of 2 with centre $B(^{-}4, 5)$.
f Reflection in the line $y = {}^{-}1$.

5 **a** Magnification with factor 2 and centre (2, 3).
b Rotation 90° clockwise around (1, 2).
c Translation 1 right and 3 down.
d Reflection in line $y = 3$.
e Reflection in line $x = 4$.

6 **a** $P'(0, 1)$, $Q'(3, 4)$, $R'(4, 1)$ and $S'(1, 0)$.
b $P'(3, 6)$, $Q'(6, 9)$, $R'(7, 6)$ and $S'(4, 5)$.
c $P'(5, 3)$, $Q'(2, 6)$, $R'(1, 3)$ and $S'(4, 2)$.
d $P'(^{-}1, {}^{-}1)$, $Q'(4, {}^{-}4)$, $R'(5, {}^{-}1)$ and $S'(2, 0)$.
e $P'(5, 1)$, $Q'(2, {}^{-}2)$, $R'(1, 1)$ and $S'(4, 2)$.
f $P'(1, 3)$, $Q'(^{-}2, 6)$, $R'(1, 7)$ and $S'(2, 4)$.
g $P'(^{-}1, 1)$, $Q'(5, 7)$, $R'(7, 1)$ and $S'(1, {}^{-}1)$.
h $P'\left(1\frac{1}{2}, 3\frac{1}{2}\right)$, $Q'(3, 5)$, $R'\left(3\frac{1}{2}, 3\frac{1}{2}\right)$ and $S'(2, 3)$.

7 **a** equilateral triangle
b parallelogram
c rectangle
d trapezium
e kite
f trapezium
g square
h equilateral triangle
i rhombus
j isosceles triangle
k trapezium
l scalene triangle

8 **a** T **b** T **c** T **d** F **e** T
f F **g** T **h** F **i** T **j** F
k T **l** T **m** F **n** F

9 $A(^{-}3, 1)$, $B(1, {}^{-}4)$, $C(7, {}^{-}3)$, $D(7, 3)$ and $E(3, 4)$

10 **a** Rotation 90° clockwise about (1, 4).
b Translation 8 units to the right and 5 units up.
c Translation 4 units to the left and 8 units down.
d Reflection in the y-axis.
e Reflection in the x-axis.
f Reflection in the line $y = {}^{-}5$.
g Reflection in the line $x = 6$.
h Magnification of 3 with centre $Y(^{-}3, 4)$.

11 **a** A rotation of 90° clockwise about $O(0, 0)$.
b A rotation of 270° clockwise about $O(0, 0)$.

12 **a** A reflection in the x-axis.
b A reflection in the x-axis.

13 $ABCD$: Rotation 270° clockwise about $A(4, 4)$ followed by a reflection in the x-axis.
XYZ: Reflection in the y-axis followed by a translation 8 units to the left and 6 units down.
$PRRS$: Rotation 180° clockwise about $R(^{-}2, {}^{-}9)$ followed by a magnification of 2 with centre $Q(^{-}2, {}^{-}11)$.

14 **a** Reflection in the y-axis.
b Reflection in the y-axis.
c Rotation of 90° clockwise about $O(0, 0)$.
d Rotation of 90° clockwise about $O(0, 0)$.
e Reflection in the line $y = x$.
f Rotation of 180° about $O(0, 0)$.

Exercise 8.2

1 Congruent: **a**, **d** and **f**.
Similar (but not congruent), with scale factors:
b × 2, **e** $\times \frac{1}{2}$, **i** × 1.
Neither (different): **c**, **g**, **h**.

2 **a**, **c**, **d**, **e** and **h** are similar.

3 **a** $\frac{1}{2}$ **b** 2.5 **c** 1.8 **d** $\frac{1}{2}$

4 **a** SSS **b** AAS **c** AAS
d SAS **e** SSS **f** SAS
g AAS **h** RHS **i** AAS

5 **a**, **b**, **c** and **e** are similar.

6 **a** 1.2 **b** Not similar. **c** $\frac{2}{3}$ **d** Not similar.

7 **a** – **b** Teacher to check.
c Yes **d** Yes
e $\triangle ABC$ ||| $\triangle HQP$. The ratio of corresponding sides is equal.

8 **a** – **b** Teacher to check.
c $\triangle DAF$ ||| $\triangle KUT$. The triangles have the same shape but are different sizes.

9 **a** $\triangle ABC\ |||\ \triangle XYZ$ (AAA)
b $\triangle MSQ\ |||\ \triangle AFD$ (AAA)
c No test can be applied.
d $\triangle KLD\ |||\ \triangle DLP\ |||\ \triangle KDP$ (AAA)
e $\triangle MCN\ |||\ \triangle CDN$ (SSS)
f $\triangle XYW\ |||\ \triangle ZUX$ (*SAS*)
g $\triangle PQS\ |||\ \triangle QSR$ (SSS)

10 **a** $\triangle CJQ\ |||\ \triangle MWL$ (AAA)
b $\triangle BIZ$ not similar to $\triangle NPV$
c $\triangle BRT\ |||\ \triangle EAN$ (SSA)
d $\triangle APS\ |||\ \triangle MKR$ (SSS)
e $\triangle CPW\ |||\ \triangle MLE$ (SAS)
f $\triangle AMP\ |||\ \triangle QSD$ (AAA)
g $\triangle PRZ$ not similar to $\triangle DKJ$
h $\triangle AMW\ |||\ \triangle SEL$ (RHS)
i $\triangle WVR\ |||\ \triangle PEN$ (SSS)

11 **a** Congruent to **B** (AAS)
b Congruent to **D** (RHS)
c Congruent to **A** (SSS)
d Congruent to **C** (ASA)

12 Congruent: **a**, $\triangle DYU \equiv \triangle TMP$ (AAS);
b, $\triangle CSG \equiv \triangle LMK$ (RHS);
c, $\triangle BKU \equiv \triangle DPO$ (SSS);
e, $\triangle YED \equiv \triangle NVM$ (SAS);
f, $\triangle HUQ \equiv \triangle JGS$ (SSS);
g, $\triangle OKM \equiv \triangle JNH$ (RHS);
Not necessarily congruent: **d**
Not congruent: **h**, **i**, **j**

13 $A \equiv L$, AAS; $C \equiv I$, SSS; $D \equiv G$, SSS; $E \equiv H$, SAS; $F \equiv K$, RHS. Others are not necessarily congruent.

14 **a** T **b** T **c** F **d** T **e** T
f F **g** F **h** T **i** F **j** F
k F **l** T **m** T

15 **a** 125 cm **b** 146 cm **c** 14 cm

16 **a** 1.25 cm **b** 1.53 cm **17** 35 cm

18 **a** $d = 22.4$ **b** $p = 22.4$ **c** $w = 14$
d $v = 48$ **e** $h = 6$ **f** $k = 10$
g $m = 56$ **h** $n = 2.5$

19 **a** $w = 50°$, $z = 45°$, $x = y = 85°$
b $w = 9$, $x = 4$, $y = 5\frac{1}{3}$
c $m = q = 113°$, $n = 77°$, $p = 80°$
d $m = 8$, $n = 44$, $p = 28$

20 **a** $\angle BAC = \angle EDF$
$\angle ABC = \angle DEF$
$\triangle ABC|||\triangle DEF$(AAA)
$x = 12$ (ratio of corresponding sides is equal)
b $\angle QPS = \angle RTS$
$\angle QSP = \angle RST$
$\triangle QPS|||\triangle RTS$(AAA)
$a = 11.2$ (ratio of corresponding sides is equal)
c $\angle KML = \angle PMO$
$$\frac{KM}{MP} = \frac{LM}{OM}$$
$\triangle KLM\ |||\ \triangle POM$(SAS)
$y = 5$ (ratio of corresponding sides is equal)
d $\angle KIJ = \angle JLA$
$\angle IKJ = \angle AJL$
$\triangle IJK\ |||\ \triangle LAJ$(AAA)
$b = 15$ (ratio of corresponding sides is equal)
e $\angle FBI = \angle RUF$
$\angle FIB = \angle RFU$
$\triangle FBI\ |||\ \triangle RUF$(AAA)
$x = 12$ (ratio of corresponding sides is equal)
f $\angle SDP = \angle PDN$
$$\frac{SD}{PD} = \frac{DP}{DN}$$
$\triangle SDP\ |||\ \triangle PDN$(SAS)
$x = 20$ (ratio of corresponding sides is equal)

21 35.6 **22** 26.7

23 **a** $x = 28°$, $y = 15$ cm
b $n = 112°$, $m = 4.2$ cm
c $a = 21°$, $b = 12$ mm
d $p = 29$ mm, $q = 43$ mm
e $s = 23$ cm, $t = 28$ cm
f $k = 63°$, $j = 3.8$ cm

24 29.5 m **25** 5.6 cm

26 Consider $\triangle WXZ$ and $\triangle XYZ$.

$\angle WXZ = 90° - \angle XWZ$	(angles of a triangle add to 180°)
$\angle XYZ = 90° - \angle XWZ$	(angles of a triangle add to 180°)
$\angle WXZ = \angle XYZ$	(both equal $90° - \angle XWZ$)
$\angle WZX = \angle XZY$	(both equal 90°)
$\triangle WXZ\ |||\ \triangle XYZ$	(All angles are equal)

Use a similar approach to show that $\triangle WXZ\ |||\ \triangle WXY$.
Hence $\triangle WXZ\ |||\ \triangle XYZ\ |||\ \triangle WXY$.

27

$PA = RB$	(given)
$\angle QPA = \angle QRB$	(both equal 90°)
$QP = QR$	(both are sides of a square)
$\triangle QPA \equiv \triangle QRB$	(SAS)
Hence $QA = QB$	(corresponding sides of congruent triangles)

28 170 cm **29** 13.5 m

Exercise 8.3

1 **a** Teacher to check **b** 2
c 3 **d** 540°

2 $a = 90°$, diameter of circle 
$b = 28°$, same arc 
$c = 115°$, exterior angle of cyclic quadrilateral 
$d = 90°$, tangent and radius 
$e = 118°$, angle at centre of circle 
$f = 112°$, cyclic quadrilateral 
$g = 35°$, angle at centre of circle 
$h = 75°$, same arc 
$i = 72°$, exterior angle of cyclic quadrilateral 
$j = 90°$, tangent and radius 

3 $a = 58^\circ$, angle sum of triangle

$b = 58^\circ$, same arc

$c = 38^\circ$, same arc

$d = 43^\circ$, angle at centre of circle

$e = 26^\circ$, same arc

$f = 32^\circ$, same arc

$g = 88^\circ$, angle at centre of circle

$h = 22^\circ$, same arc

$i = 43^\circ$, same arc

$j = 58^\circ$, same arc

$k = 31^\circ$, same arc

$l = 40^\circ$, same arc

4 **a** $\angle DOG = 62^\circ$, opposite angles of parallelogram

So $a = 62^\circ$, vertically opposite

b $\angle SAN = 69^\circ$, supplementary

So $b = 60^\circ$, angle sum of quadrilateral

c $\angle DCA = 42^\circ$, vertically opposite

So $c = 113^\circ$, exterior angle of triangle

d $\angle TSV = 68^\circ$, isosceles triangle

So $d = 112^\circ$, supplementary

e $\angle UER = 139^\circ$, exterior angle of triangle

So $e = 72^\circ$, exterior angle of $\angle UTE$

f $\angle PET = 28^\circ$, isosceles triangle and angle sum of $\angle EPT$

So $\angle SET = 62^\circ$, complementary

So $f = 62^\circ$, isosceles triangle

5 **a** $\angle ASO = 106^\circ$, angles at a point

So $a = 74^\circ$, co-interior

b $\angle ONY = 64^\circ$, complementary

So $b = 64^\circ$, alternate

c $\angle ARL = 76^\circ$, corresponding

So $c = 35^\circ$, angle sum of triangle

d $\angle GEH = 68^\circ$, exterior angle of triangle

So $d = 68^\circ$, alternate

e $\angle LSR = 68^\circ$, isosceles triangle *and* angle sum of triangle

So $e = 68^\circ$, corresponding

f $\angle HEG = 55^\circ$, co-interior

So $f = 55^\circ$, vertically opposite

$\angle OHE = 125^\circ$, opposite angles of parallelogram

So $\angle WHY = 125^\circ$, vertically opposite

So $g = 46^\circ$, angle sum of quadrilateral $YHWS$

6 **a** $\angle MTE = 37^\circ$, isosceles triangle and angle sum of triangle

So $a = 37^\circ$, vertically opposite

b $\angle SGL = 64^\circ$, complementary

So $b = 118^\circ$, angle sum of quadrilateral $SGLA$

c $\angle OYS = 44^\circ$, corresponding

So $c = 95^\circ$, exterior angle of triangle

d $\angle OFT = 78^\circ$, opposite angles of parallelogram

So $d = 168^\circ$, exterior angle of triangle

e $\angle NKS = 70^\circ$, isosceles triangle *and* exterior angle of triangle

So $\angle LKS = 110^\circ$, supplementary

So $e = 70^\circ$, angle sum of quadrilateral $KLES$

7 **a** 900° **b** 1800° **c** 2880°
d 3420° **e** 4140° **f** 5040°

8 **a** 11 **b** 13 **c** 9
d 16 **e** 22 **f** 25

9 **a** $\angle KLM = 90^\circ$, diameter of circle

So $a = 62^\circ$, angle sum of triangle

b $\angle COA = 90^\circ$, tangent and radius

So $\angle OCA = 58^\circ$, angle sum of triangle

So $b = 29^\circ$, angle at centre of circle

c $\angle ADP = 40^\circ$, angle at centre of circle

So $c = 140^\circ$, supplementary

d $\angle DSG = 104^\circ$, cyclic quadrilateral

So $d = 38^\circ$, isosceles triangle *and* angle sum of triangle

e $\angle ASH = 44^\circ$, alternate

So $e = 88^\circ$, angle at centre of circle

f $\angle YLW = 70^\circ$, angle at centre of circle

$\angle LYE = 90^\circ$, tangent and radius

$\angle YEH = 90^\circ$, tangent and radius

So $\angle LHE = 110^\circ$, angle sum of quadrilateral

So $f = 70^\circ$, supplementary

10 $a = 49^\circ$, diameter of circle *and* angle sum of triangle

$b = 54^\circ$, angle sum of triangle *and* isosceles triangle

$c = 72^\circ$, angle at centre of circle

$d = 42^\circ$, angle at centre of circle

$e = 48^\circ$, angle sum of triangle *and* isosceles triangle

$f = 42^\circ$, angle sum of triangle , isosceles triangle *and* angle at centre of circle

$g = 43^\circ$, angle at centre of circle

$h = 47^\circ$, angle sum of triangle *and* isosceles triangle

$i = 47°$, isosceles triangle, diameter of circle *and* angle sum of triangle

$j = 43°$, angle sum of triangle *and* isosceles triangle

$k = 31°$, angle sum of triangle *and* isosceles triangle

$l = 119°$, cyclic quadrilateral

$m = 35°$, angle at centre of circle

$n = 35°$, angle sum of triangle *and* isosceles triangle

$o = 34°$, diameter of circle *and* isosceles triangle

11 $x = 20°$, vertically opposite

$y = 15°$, complementary

$\theta = 51°$, angles at a point

12 $a = 62°$, exterior angle of triangle

$b = 13°$, angle sum of quadrilateral

13 $a = 12°$, alternate

$b = 19°$, co-interior

$c = 11°$, corresponding

14 $a = 41°$, vertically opposite

$b = 18°$, isosceles triangle *and* angle sum of triangle

$c = 17°$, exterior angle of triangle

$d = 15°$, opposite angles of parallelogram

$e = 6°$, alternate

$f = 59°$, angle sum of quadrilateral

15	**a** 167.14°		**b** 135°		**c** 150°		**d** 172°	
16	**a** 9		**b** 12		**c** 72		**d** 24	

17 **a** $6a + 20° = 8a$, exterior angle of cyclic quadrilateral

$2a = 20°$

$a = 10°$

b $4b + 5b = 180° - 90°$, diameter of circle *and* angle sum of triangle

$9b = 90°$

$b = 10°$

c $7c - 11° = 2(3c + 4°)$, angle at centre of circle

$7c - 11° = 6c + 8°$

$c = 19°$

d $6d + 9d = 180°$, cyclic quadrilateral

$15d = 180°$

$c = 12°$

18 $x = 38°$, $y = 102°$

19 $\angle DBC = 63°$

20 $PS = QR$ (given)

$SR = RS$ (common side)

$\angle PSR = \angle QRS$ (both equal 90°)

So $\triangle PSR \equiv \triangle QRS$ (SAS).

So $SQ = RP$ (corresponding sides of congruent triangles).

21 $\angle ZWY = \angle XYW$ (alternate)

$\angle ZYW = \angle XWY$ (alternate)

$WY = YW$ (common side)

So $\triangle ZWY \equiv \triangle XYW$ (AAS).

So $WZ = YX$ (corresponding sides of congruent triangles)

and $WX = YZ$ (corresponding sides of congruent triangles).

22 We know that $\triangle ZWY \equiv \triangle XYW$ (AAS).

$\angle ZWY = \angle XYW$ (corresponding angles of congruent triangles)

$\angle ZYW = \angle XWY$ (corresponding angles of congruent triangles)

So $\angle ZWY + \angle XWY = \angle XYW + \angle ZYW$

So $\angle ZWX = \angle XYZ$

and $\angle WZY = \angle YXW$ (corresponding angles of congruent triangles). QED

23 $AB = CD$ (given)

$BC = DA$ (given)

$DB = BD$ (common side)

So $\triangle ABD \equiv \triangle CDB$ (SSS)

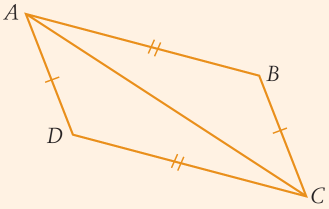
So $\angle ABD = \angle CDB$ (corresponding angles of congruent triangles)

and $\angle CDB = \angle DBC$ (isosceles triangle).

So $\angle ABD = \angle DBC$. QED

Other letters could be used for each of the following proofs.

24



$AB = CD$ (given)

$BC = DA$ (given)

$CA = AC$ (common side)

So $\triangle ABC \equiv \triangle CDA$ (SSS)

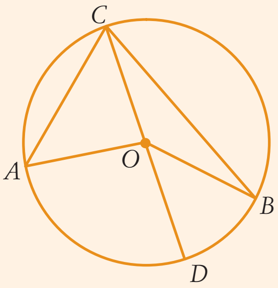
So $\angle CAB = \angle ACD$

So $AB \parallel CD$ (alternate)

Similarly, $\angle BCA = \angle DAC$ and $BC \parallel AD$

So $ABCD$ is a parallelogram. QED

25



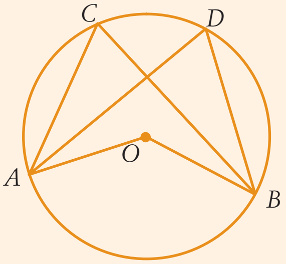
Construct the diameter COD.

Then $AO = BO = CO = r$

Thus $\triangle OAC$ and $\triangle OBC$ are isosceles.

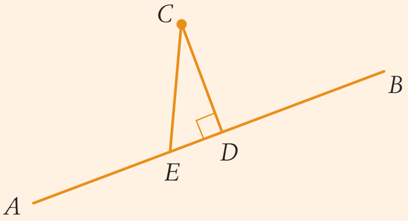
Thus $\angle OAC = \angle OCA$ and $\angle OCB = \angle OBC$.
$\angle BOD = \angle OAC + \angle OCA$, 
So $\angle BOD = 2\angle OCA$
Similarly $\angle AOD = 2\angle OCB$
Now $\angle AOB = \angle BOD + \angle AOD$
$= 2\angle OCA + 2\angle OCB$
$= 2(\angle OCA + \angle OCB)$
$= 2\angle ACB$ **QED**

26



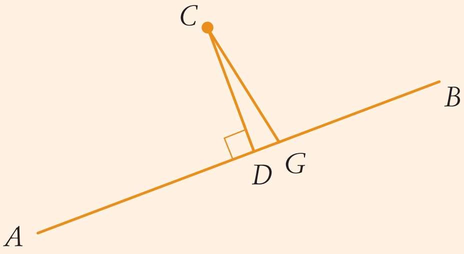
Construct the radii OA and OB.
Then $\angle AOB = 2\angle ACB$ and $\angle AOB = 2\angle ACD$ (by Q25)
Thus $2\angle ACB = 2\angle ACD$
so $\angle ACB = \angle ACD$ **QED**

27



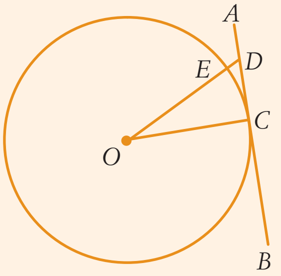
Construct any other line segment CE from the given point C to the line AB.
Then $CE^2 = CD^2 + DE^2$ (Pythagoras).
So $CE = \sqrt{CD^2 + DE^2} > CD$ **QED**

28



Suppose CG, the shortest line segment from the given point G to the line AB is *not perpendicular* to AB.
Construct the perpendicular CD.
Then $CD < CG$ (by Q27).
But CG is the shortest, so this contradicts the assumptions.
Thus CG is perpendicular to AB.

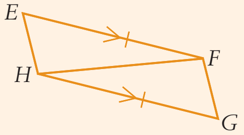
29



Construct the line segment OED from the centre O passing through the circumference at E to the tangent at point D other than C, the intersection of the radius and tangent.
Then $OD = OE + ED$.
But $OE = OC = r$, so $OD > OC$.
Thus the radius OC is the shortest line segment between the point O and the tangent AB.
By Q28 it is perpendicular. **QED**

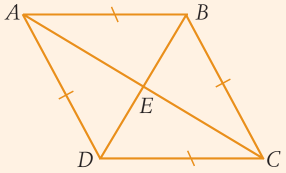
30 $AB = CD$ (given)
$AO = CO$ (both radii)
$BO = DO$ (both radii)
So $\triangle AOB \equiv \triangle COD$ (SSS)
So $\angle AOB = \angle COD$ (corresponding angles in congruent triangles). QED

31



$EF = GH$ (given)
$\angle EFH = \angle GHF$ (alternate )
$FH = HF$ (common side)
So $\triangle EFH \equiv \triangle GHF$ (SAS)
So $\angle EHF = \angle GFH$.
So $EH \parallel GF$ (alternate )
So $EFGH$ is a parallelogram. QED

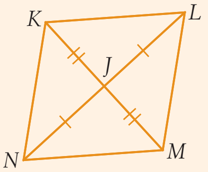
32



$AB = CB$ (given)
$BD = BD$ (common side)
$AD = CD$ (given)
So $\triangle ABD \equiv \triangle CBD$ (SSS)
So $\angle ABE = \angle CBE$.
Now $AB = CB$ (given)
$\angle ABE = \angle CBE$ (proven above)
$BE = BE$ (common side)
So $\triangle ABE \equiv \triangle CEB$ (SAS)
So $\angle AEB = \angle CEB$

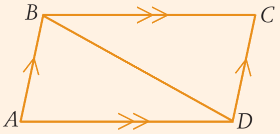
But $\angle AEB + \angle CEB = 180°$ (supplementary)
So $\angle AEB = \angle CEB = 90°$
So $AC \perp BD$. QED

33



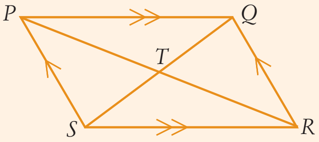
$KJ = MJ$ (given)
$\angle KJL = \angle MJN$ (vertically opposite)
$JL = JN$ (given)
So $\triangle KJL \equiv \triangle MJN$ (SAS)
So $\angle KLJ = \angle MNJ$
So $KL \parallel MN$ (alternate)
Similarly, $\angle KNL = \angle MLN$ and $KN \parallel LN$
So $KLMN$ is a parallelogram. QED

34



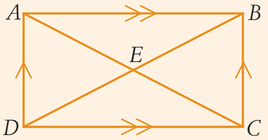
$\angle ABD = \angle CDB$ (alternate)
$BD = DB$ (Common side)
$\angle ADB = \angle CBD$ (alternate)
So $\triangle ABD \equiv \triangle CBD$ (ASA)
So $AD = CB$ and $BA = DC$
So opposite sides are equal. QED

35



$\angle QPT = \angle SRT$ (alternate)
$QP = SR$ (proven in question 4)
$\angle PQT = \angle RST$ (alternate)
So $\triangle PQT \equiv \triangle RST$ (ASA)
So $PT = RT$ and $QT = ST$
So QS and PR bisect each other. QED

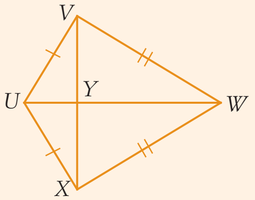
36



$AC = DB$

$AC = DB$ (given)
$AD = DA$ (common side)
$DC = AB$ (proven in question 4)
So $\triangle DAC \equiv \triangle ADB$ (SSS)
So $\angle ADC = \angle DAB$
But $\angle ADC + \angle DAB = 180°$ (cointerior)
So $\angle ADC = \angle DAB = 90°$
So $ABCD$ is a rectangle. QED

37



$UV = UX$ (given)
$VW = XW$ (given)
$WU = WU$ (common side)
So $\triangle UVW \equiv \triangle UXW$ (SSS)
So $\angle VUY = \angle XUY$
Now $UV = UX$ (given)
$\angle VUY = \angle XUY$ (proven above)
$UY = UY$ (common side)
So $\triangle VUY \equiv \triangle XUY$ (SAS)
So $VY = XY$
So UW bisects VX. **QED**

Chapter review

1 a $P(4, 1) \longrightarrow P'(6, 5)$, $G(1, 3) \longrightarrow G'(4, 2)$, $M(5, 5) \longrightarrow M'(2, 6)$, $N(3, 3) \longrightarrow N'(4, 4)$
b $P(4, 1) \longrightarrow P'(^-2, 1)$, $G(1, 3) \longrightarrow G'(1, 3)$, $M(5, 5) \longrightarrow M'(^-3, 5)$, $N(3, 3) \longrightarrow N'(^-1, 3)$

2 a *FATSL*, regular pentagon
b *UGSARE*, concave hexagon
c *APRESTI*, heptagon
d *NTH*, right-angled/scalene triangle
e *EGAR*, kite
f *DENT*, parallelogram
g *ODAY*, trapezium

3 a Neither (different)
b Similar (but not congruent), scale factor 3
c Congruent

4 $a = 90°$, tangent and radius
$b = 36°$, same arc
$c = 95°$, exterior angle of cyclic quadrilateral
$d = 96°$, cyclic quadrilateral
$e = 58°$, angle at centre of circle

5 a Rotation 90° anticlockwise around (3, 4).
b Reflection in the line $x = 1$.

6 a $M'(^-1, 8)$ **b** $M'(4, 2)$

7 b Scale factor 3.

8 B and C are congruent. A and F are similar, with scale factor 2.

9 a $\angle SEA = 46°$, complementary
So $a = 50°$, angle sum of triangle
b $\angle HCE = 87°$, angle sum of quadrilateral
So $b = 93°$, supplementary
c $\angle TPS = 38°$, angle sum of triangle
So $c = 38°$, vertically opposite

d $\angle YBS = 36°$, isosceles triangle *or* angle sum of triangle
So $d = 36°$, corresponding

10 2700°

11 **a** F **b** T **c** F

12 15

13 **a** Similar, AAA **b** Similar, SAS

14 **a** Congruent, SSS **b** Congruent, ASA
c Not necessarily congruent.

15 $A \equiv E$, RHS; $B \equiv F$, ASA; C and D are not necessarily congruent.

16 **a** Congruent to B (SAS) **b** Congruent to C (ASA)

17 $AD = BC$ (given)
$BD = DB$ (common side – hypotenuse)
$\angle ABD = \angle CDB$ (both equal 90°)
$\triangle ABD \equiv \triangle CDB$ (RHS)

18 **a** $\angle DIC = 90°$, diameter of circle
So $a = 33°$, angle sum of triangle
b So $\angle RTA = 90°$, tangent and radius
So $\angle TAR = 27°$, angle sum of $\triangle TAR$
So $\angle CAY = 21°$, vertically opposite
So $b = 43°$, angle sum of $\triangle ACY$

19 **a** 1 : 20 **b** About 47 cm. **c** About 69 cm.

20 **a** About 1.9 mm. **b** About 2.9 mm. **c** About 6.2 mm.

21 **a** $x = 4, y = 12$ **b** $m = 9.6, n = 5.625$

22 2

23 $WX = YZ$ (given)
$WY = YW$ (common side)
$\angle XWY = \angle ZYW$ (given)
$\triangle ABD \equiv \triangle CDB$ (SAS)

24 160° **25** 20

26 $a = 60°$, Exterior angle of triangle
$b = 40°$, corresponding

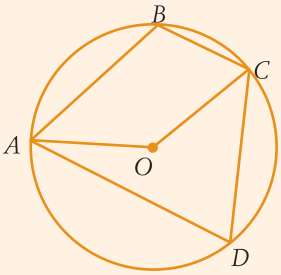
27 **a** Reflection in the line $y = -x$.
b Reflection in the line $y = x$.

28 **a** $(a + 10°) = (3a - 22°)$, same arc
$2a = 32°$
$a = 16°$
b $(3b - 35°) + (3b - 25°) = 180°$, cyclic quadrilateral
$6b = 240°$
$b = 40°$

29 $AB = AD$ (given)
$BC = DC$ (given)
$CA = CA$ (same side)
So $\triangle ABC \equiv \triangle ADC$ (SSS)
Thus $\angle B = \angle D$.
So the angles between the unequal sides in a kite are equal. QED

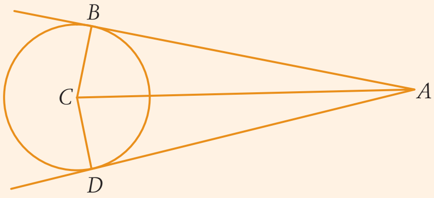
30 21 m

31



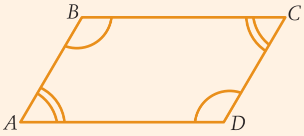
Construct the radii OA and OC.
Then $\angle AOC = 2\angle ADC$ and the reflex angle $\angle AOC = 2\angle ABC$, both .
But $\angle AOC$ + reflex angle $\angle AOC = 360°$, , so $2\angle ADC + 2\angle ABC = 360$.
Thus $\angle ADC + \angle ABC = 180$. **QED**

32



$\angle ABC = \angle ADC$ (tangent and radius)
$AC = AC$ (same)
$BC = DC$ (both radii)
Thus $\angle ABC \equiv \angle ADC$ (RHS)
So $AB = AD$. QED

33



$\angle A + \angle B + \angle C + \angle D = 360°$ (angle sum of quadrilateral)
But $\angle A = \angle C$ and $\angle B = \angle D$ (given)
So $\angle A + \angle B + \angle A + \angle B = 360°$
$\angle A + \angle B = 180°$
So $BC \parallel AD$ (cointerior).
Similarly, $\angle A + \angle D = 180°$ and $AB \parallel DC$.
So $ABCD$ is a parallelogram. QED

Chapter 9

Exercise 9.1

1 **a** \$212.4 **b** \$1176 **c** \$6120
d \$175.87 **e** \$3616.08

2 **a** 1.08 **b** 1.064 **c** 1.038
d 1.124 **e** 1.096

3 **a** 1.006 75 **b** 1.001 25 **c** 1.014 75
d 1.0105 **e** 1.0035

4 **a** 1872% **b** 1115% **c** 2555%
d 390% **e** 2509%
5 **a** 35.01% **b** About 17.9%
6 **a** $4214.78, $1214.78 **b** $6666.4, $1766.4
c $7946.18, $1746.18 **d** $11 058.87, $2658.87
e $18 008.43, $5708.43
7 **a** 4195.88, $195.88 **b** 3911.4, $111.4
c 4897.74, $297.74 **d** 9025.16, $625.16
e 7513.71, $313.71
8 **a** $6211.48, $1211.48 **b** $9736.82, $2736.82
c $19 099.34, $6799.34 **d** $7342.58, $1142.58
e $5679.26, $759.26
9 **a** $12 157.89, $4157.89 **b** $5254.35, $1254.35
c $10 139.88, $3139.88 **d** $12 048.23, $2848.23
e $7119.33, $1379.33
10 **a** $845.6 **b** $148.87 **c** About 37.3%
11 $3153.40, 42.4%
12 $5077.30, 25.2%
13 7%, just over 10 years; 10%, just over 7 years; 14%, just over 5 years; 20%, 4 years, so a bit more than 70 divided by the interest rate.
14 Quarterly and monthly rests make a big difference, but increased frequency makes little difference.

Rate	5	10	15	20
daily	1051.27	1105.2	1161.8	1221.34
weekly	1051.25	1105.1	1161.58	1220.93
fortnightly	1051.22	1105	1161.33	1220.47
monthly	1051.16	1104.7	1160.75	1219.39
quarterly	1050.95	1103.8	1158.65	1215.51
annually	1050	1100	1150	1200

Exercise 9.2

1 $12.02 **2** $44.30 **3** $33.44
4 $8532.40 **5** $12 548.80 **6** $10 980.38
7 $521.64 **8** $912.33 **9** $407.34
10 $589.05 **11** $1248 **12** $8636.13
13 $7935.11 **14** $7751.72 **15** $17 016.34
16 $11 951.83 **17** 5 years
18 6 years 8 months (or 7 years).
19 About 7 years 9 months (or 8 years).
20 About 4 years 11 months (5 years).
21 About 4 years 11.5 months (5 years).
22 About 12 years 4 months (13 years).
23 About 3 years 2.9 months (3 years 3 months).
24 About 4 years 7.8 months (4 years 8 months).
25 About 30 years 9.6 months (30 years 10 months).
26 About $353 000. **27** About $19 300.
28 About $20 900.
29 Calculate the compound interest and the simple rate that gives that interest to get about 13.5%.
30 Choose a figure such as $100, find the growth factor and thus the compound interest and the simple rate that gives that interest to get about 12.6%.
31 Find the ratio of the amount from simple interest to the principal for some amount, say $100. Then find the 12th root for the number of rests to get the growth factor for a quarter. Subtract 1 and multiply by 4 and 100 to change to a nominal percentage of about 12.58%.
32 Find the ratio of the amount from simple interest to the principal for some amount, say $100. Then find the 60th root for the number of rests to get the growth factor for a month. Subtract 1 and multiply by 12 and 100 to work out the nominal percentage: 12.91%.

Exercise 9.3

1 **a** $4485 **b** $17 485 **c** $485.69
2 **a** $7560 **b** $25 060 **c** $240.96
3 **a** $2550.60 **b** $13 450.60 **c** $560.44
4 **a** $4080, $670 **b** $8366.4, $320.0821
c $4400.4, $151.9269 **d** $13 904, $748
e $8295, $308.9103
5 **a** $7.0678 **b** $6.3207 **c** $6.8789
d $7.3899 **e** $7.3376
6 **a** $2348.03 **b** $3547.15 **c** $2162.15
d $3459.56 **e** $5435.80
7 **a** $525 290.80 **b** $584 145.00 **c** $228 916.00
d $725 441.60 **e** $1 096 888.00
8 **a** $508 **b** About 2464%.
9 **a** $645 **b** About 1764%.
c $910 **d** About 2721%.
10 **a** $595 **b** About 840%.
c $715 **d** About 1062%.
11 **a** $412.32 **b** 43.2% **c** $25.88 **d** 132.4%
12 **a** $\frac{\text{Half the total}}{\text{Days in billing period}}$ **b** $1207.82
c 59.1% **d** $3353.60
e $3556.11 **f** 80.5%
13 **a** $6.89 **b** 297.4%
14 $3251.56, $2025.12
15 **a** $352 786.94 **b** $705.24 **c** $164 748
16 **a** $375 000 **b** $417 000 **c** $448 000
17 **a** $285 000 **b** $328 000 **c** $378 000
18 $444 570.90, $3285
19 If you are really strong-willed, choose A because it won't cost anything, otherwise choose B for the low cost.
20 Card A would cost $187.50, Card B $130, Card C $225 and Card D $347.50, so B is best for her.

Chapter review

1 **a** $395.20 **b** $535.30
2 **a** 1.075 **b** 1.097 **c** 1.038
3 **a** 1.004 25 **b** 1.0015
4 **a** 2106% **b** 1174%
5 27% **6** $25.23 **7** $15 394.58
8 **a** $7820 **b** $24 820 **c** $517.08
9 **a** $1732.5, $485.14 **b** $14 508, $327
10 **a** $10 077.70, $2077.70 **b** $10 239.64, $2339.64

11 $12 704.98, $404.98 **12** $448.11
13 $475.20 **14** $24 834.79
15 $7.7182 **16** $2243.85, $807 786
17 $8875.30, $2635.30 **18** $40 342.48, $16 342.48
19 $17 350.32
20 **a** 22.6% **b** 28.1%
21 5 years (4.76)
22 **a** $480 **b** 2433%
c $725 **d** 3084%
23 8 years (7.67)
24 **a** $427.39 **b** 35.7%
c $207.92 **d** 206.5%
25 $2093.98, $2368.46
26 **a** $322 000 **b** $356 000 **c** $379 000
27 9 years (107.8 months). **28** About $23 000.
29 Use constant multiplication of $100 by the growth factor and count how many times it takes to exceed $300 to get 12 years.
30 Work out 12% compound interest on $100 over 4 years, divide by 4 and work out the rate to get 14.34%.
31 Find the ratio of the amount from simple interest to the principal for some amount, say $100. Then find the 24th root for the number of rests to get the growth factor for a month. Subtract 1 and multiply by 12 and 100 to work out the nominal percentage: 19.77%.
32 Assuming that she always paid on time, Card A would cost $408, Card B $318, Card C $369 and Card D $305. Card D would be best for her.

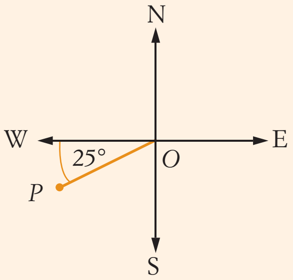
Chapter 10

Exercise 10.1

1 **a** 27.2 m **b** 103.4 cm **c** 17.8 km
d 8.2 cm **e** 18.6 mm **f** 37.7 m
2 **a** $e \approx 35.1$ m **b** $m \approx 24.7$ cm **c** $d \approx 36.2$ m
d $y \approx 30$ mm **e** $n \approx 82.5$ km **f** $r \approx 38.3$ mm
g $k \approx 11.1$ cm **h** $x \approx 15.6$m **i** $q \approx 14.7$ km
3 **a** $w = 300$ m **b** $m = 30$ cm **c** $q = 20$ cm
d $b = 48$ mm **e** $r = 120$ km
4 **a**, **c**, and **f** are right-angled triangles;
b and **d** are acute-angled triangles;
e is an obtuse-angled triangle.
5 **a** 2360.1 mm **b** 3.0 m **c** 55.2 cm
d 644.0 cm **e** 132.9 mm **f** 3.5 km
6 15.2 m **7** 4.2 m **8** 11.4 m
9 5.7 cm **10** 57.8 km **11** 6.8 m

Exercise 10.2

1 **a** $\frac{3}{5}$ **b** $\frac{3}{5}$ **c** $\frac{4}{3}$
d $\frac{4}{5}$ **e** $\frac{3}{4}$ **f** $\frac{4}{5}$
2 **a** $\sin\theta$ **b** $\cos\theta$ **c** $\tan\theta$
3 **a** 0.342 **b** 1 **c** 0.174 **d** 0.934
e 1 **f** 0.158 **g** 0.306 **h** 0.545
i 0.225 **j** 0.017
4 **a** 0.342 **b** 0.743 **c** 0.292 **d** 0.707
e 1 **f** 0.839 **g** 0.985 **h** 0
i 0.707 **j** 0.574
5 **a** 76° **b** 27° **c** 81°
d 44° **e** 9° **f** 26°
g 7° **h** 47° **i** 2°
6 **a** 76° **b** 35° **c** 83°
d 60° **e** 9° **f** 47°
g 15° **h** 24° **i** 2°
7 **a** **i** $\sin\alpha = \frac{4}{5}$, $\cos\alpha = \frac{3}{5}$, $\tan\alpha = \frac{4}{3}$
ii $\sin\beta = \frac{3}{5}$, $\cos\beta = \frac{4}{5}$, $\tan\beta = \frac{3}{4}$
iii $\alpha \approx 53°$, $\beta \approx 37°$
b **i** $\sin\alpha = \frac{15}{17}$, $\cos\alpha = \frac{8}{17}$, $\tan\alpha = \frac{15}{8}$
ii $\sin\beta = \frac{8}{17}$, $\cos\beta = \frac{15}{17}$, $\tan\beta = \frac{8}{15}$
iii $\alpha \approx 62°$, $\beta \approx 28°$
c **i** $\sin\alpha = \frac{12}{13}$, $\cos\alpha = \frac{5}{13}$, $\tan\alpha = \frac{12}{5}$
ii $\sin\beta = \frac{5}{13}$, $\cos\beta = \frac{12}{13}$, $\tan\beta = \frac{5}{12}$
iii $\alpha \approx 67°$, $\beta \approx 23°$
d **i** $\sin\alpha = \frac{7}{\sqrt{58}}$, $\cos\alpha = \frac{3}{\sqrt{58}}$, $\tan\alpha = \frac{7}{3}$
ii $\sin\beta = \frac{3}{\sqrt{58}}$, $\cos\beta = \frac{7}{\sqrt{58}}$, $\tan\beta = \frac{3}{7}$
iii $\alpha \approx 67°$, $\beta \approx 23°$
e **i** $\sin\alpha = \frac{7}{25}$, $\cos\alpha = \frac{24}{25}$, $\tan\alpha = \frac{7}{24}$
ii $\sin\beta = \frac{24}{25}$, $\cos\beta = \frac{7}{25}$, $\tan\beta = \frac{24}{7}$
iii $\alpha \approx 16°$, $\beta \approx 74°$
f **i** $\sin\alpha = \frac{1}{\sqrt{5}}$, $\cos\alpha = \frac{2}{\sqrt{5}}$, $\tan\alpha = \frac{1}{2}$
ii $\sin\beta = \frac{2}{\sqrt{5}}$, $\cos\beta = \frac{1}{\sqrt{5}}$, $\tan\beta = 2$
iii $\alpha \approx 27°$, $\beta \approx 63°$
8 **a** 10 **b** 5.130 **c** 18
d 20.0 **e** 28.868 **f** 45.255
9 **a** $a \approx 23.1$ **b** $b \approx 86.6$ **c** $d \approx 12.7$
d $a \approx 31.4$ **e** $d \approx 77.7$ **f** $m \approx 70.4$
10 **a** $x \approx 8.1$ cm **b** $y \approx 15.9$ m
c $k \approx 16.7$ cm **d** $m \approx 20.6$ mm
e $p \approx 12.3$ km **f** $z \approx 24.1$ mm
11 **a** $a \approx 17.5$ cm **b** $x \approx 11.5$ cm **c** $b = 17.7$ cm
12 **a** 34.99 cm **b** 57.84 mm **c** 139.17 mm
13 **a** 070° **b** 280° **c** 115°
d 248° **e** 207° **f** 143°
14 **a** 250° **b** 100° **c** 295°
d 068° **e** 027° **f** 323°
15 **a**



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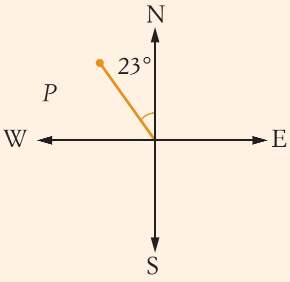

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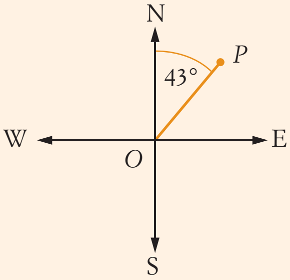

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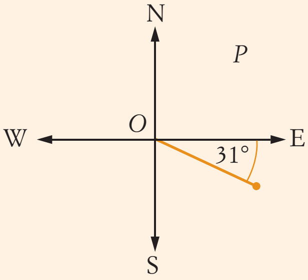

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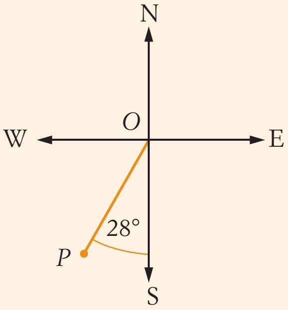

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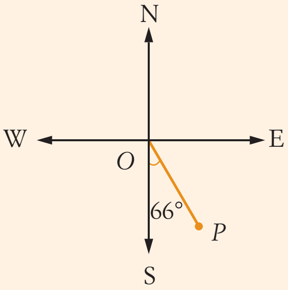

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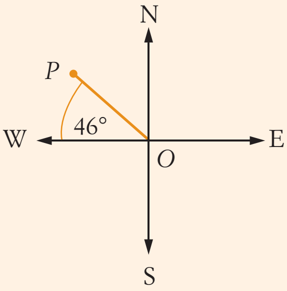
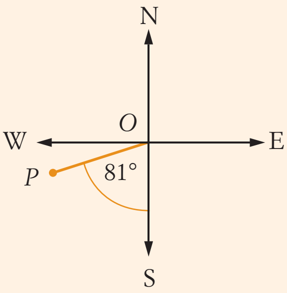


h



16 **a** **i** $x = 9, y = 12, r = 15$

ii $\sin\theta = \tfrac{4}{5}, \cos\theta = \tfrac{3}{5}, \tan\theta = \tfrac{4}{3}$

b **i** $x = {}^{-}12, y = {}^{-}9, r = 15$

ii $\sin\theta = {}^{-}\tfrac{3}{5}, \cos\theta = {}^{-}\tfrac{4}{5}, \tan\theta = \tfrac{3}{4}$

c **i** $x = {}^{-}24, y = 7, r = 25$

ii $\sin\theta = \tfrac{7}{25}, \cos\theta = {}^{-}\tfrac{24}{25}, \tan\theta = {}^{-}\tfrac{7}{24}$

17 **a** **i** $x = 8, y = 15$ **ii** $r = 17$

iii $\sin\theta = \tfrac{15}{17}, \cos\theta = \tfrac{8}{17}, \tan\theta = \tfrac{15}{8}$

b **i** $x = {}^{-}8, y = {}^{-}6$ **ii** $r = 10$

iii $\sin\theta = {}^{-}\tfrac{3}{5}, \cos\theta = {}^{-}\tfrac{4}{5}, \tan\theta = \tfrac{3}{4}$

c **i** $x = 12, y = {}^{-}5$ **ii** $r = 13$

iii $\sin\theta = {}^{-}\tfrac{5}{13}, \cos\theta = \tfrac{12}{15}, \tan\theta = {}^{-}\tfrac{5}{12}$

18 **a** 0 **b** 0 **c** 0 **d** 1

e 0 **f** 0 **g** Not defined **h** $^{-}1$

19 **a** 0.87 **b** 0.5 **c** 0.87 **d** 0.98

e 1 **f** 0.17 **g** 0.17 **h** 0.94

20 **a** $10\sqrt{3}$ **b** $16\sqrt{3}$ **c** $12\sqrt{2}$

21 **a** 60° **b** 45° **c** 30°

22 **a** $\sin\theta = \tfrac{4}{5}, \cos\theta = \tfrac{3}{5}, \tan\theta = \tfrac{4}{3},$

$$\frac{\sin\theta}{\cos\theta} = \tfrac{4}{5} \div \tfrac{3}{5} = \tfrac{4}{3} = \tan\theta$$

b $\sin\theta = \tfrac{24}{25}, \cos\theta = \tfrac{7}{25}, \tan\theta = \tfrac{24}{7},$

$$\frac{\sin\theta}{\cos\theta} = \tfrac{24}{25} \div \tfrac{7}{25} = \tfrac{24}{7} = \tan\theta$$

c $\sin\theta = \dfrac{3}{\sqrt{34}}, \cos\theta = \dfrac{5}{\sqrt{34}}, \tan\theta = \dfrac{3}{5},$

$$\frac{\sin\theta}{\cos\theta} = \frac{3}{\sqrt{34}} \div \frac{5}{\sqrt{34}} = \frac{3}{5} = \tan\theta$$

23 **a** $\tfrac{15}{17}$ **b** $\tfrac{40}{41}$ **c** $\dfrac{5}{\sqrt{29}}$ **d** $\dfrac{9}{\sqrt{106}}$

Answers

24 **a** $w = 52°$ **b** $x = 22°$

c $m = 76.4°$ **d** $p = 8.6°$

e $q = 65°19'$ **f** $k = 32°43'$

25 **a** 0.8 **b** 0.89

c $\cos y = \frac{9}{41}$, $\cos x = \frac{40}{41} = \sin y$

d $\sin b = \frac{12}{13}$, $\sin a = \frac{5}{13} = \cos b$

e $\cos Q = \dfrac{\sqrt{21}}{5} = \sin P$, $\sin Q = \frac{2}{5}$

f $\cos \alpha = \dfrac{3}{\sqrt{58}}$, $\sin \alpha = \dfrac{7}{\sqrt{58}} = \cos \beta$

26 **a** 0 **b** 1 **c** 1

27 **a** 25° **b** 78° **c** 59°

d 13° **e** 82° **f** 46°

28 **a** $^{-}0.84$ **b** 0.93 **c** $^{-}0.76$

d $^{-}0.44$ **e** $^{-}0.29$ **f** 0.13

g 0.62 **h** $^{-}0.44$ **i** $^{-}0.61$

29 **a** $-\cos 50°$ **b** $-\tan 58°$ **c** $\sin 72°$

d $-\tan 17°$ **e** $\sin 38°$ **f** $-\cos 7.2°$

g $\sin 29.4°$ **h** $-\tan 81.8°$ **i** $-\cos 78°$

j $-\tan 45.6°$ **k** $\sin 61.5°$ **l** $\sin 30.3°$

30 **a** $\dfrac{1}{\sqrt{2}}$ **b** 0 **c** 0

d 0.5 **e** $^{-}1$ **f** $^{-}0.5$

g $\dfrac{^{-}1}{\sqrt{2}}$ **h** $\dfrac{\sqrt{3}}{2}$ **i** $^{-}\sqrt{3}$

j $\dfrac{^{-}\sqrt{3}}{2}$

31 **a** 141° **b** 127° **c** 114°

d 113° **e** 172° **f** 135°

g 99° **h** 156° **i** 118°

32 **a** **i** $\frac{4}{5}$ **ii** 53°

b **i** $\frac{12}{13}$ **ii** 67°

c **i** $\frac{24}{25}$ **ii** $16° + 90° = 106°$

d **i** $^{-}\frac{3}{5}$ **ii** $53° + 270° = 323°$

e **i** $\frac{5}{13}$ **ii** 23°

f **i** $^{-}\frac{4}{5}$ **ii** $53° + 180° = 233°$

g **i** $\dfrac{\sqrt{3}}{2}$ **ii** 60°

h **i** $\frac{\sqrt{5}}{3}$ **ii** 48°

33 $\alpha \approx 23°$, $\theta \approx 67°$

34 **a** **i** $\frac{3}{5}$ **ii** $\frac{4}{5}$ **iii** 1

b **i** $\dfrac{\sqrt{3}}{2}$ **ii** $\frac{1}{2}$ **iii** 1

c **i** $\frac{5}{13}$ **ii** $\frac{12}{13}$ **iii** 1

35 **a** 393.3 m **b** 338°

36 **a** 25.1 km **b** 208.6°

37 Proof by demonstration.

38 **a** 283 km **b** 307°

Exercise 10.3

1 **a** 9.2 **b** 19.5 **c** 31.4

d 62.1 **e** 124.3

2 **a** $5.3\ \text{cm}^2$ **b** $113\ \text{mm}^2$

c $208\ \text{m}^2$ **d** $5.38\ \text{m}^2$

3 **a** $b = 6.36$, $c = 12.29$ **b** $e = 11.64$, $d = 11.42$

c $g = 43.55$, $h = 26.60$ **d** $x = 14.37$, $z = 10.06$

e $q = 24.06$, $p = 18.26$ **f** $s = 3.59$, $t = 2.02$

4 **a** 72°18′ or 107°42′ **b** 62°44′ **c** 41°23′

d 20°49′ **e** 50°49′ **f** 110°08′ or 69°52′

5 75.82 mm

6 **a** $C = 68°$, $b = 24.8$ cm, $c = 25.4$ cm

b $M = 61.6°$, $L = 77.1°$, $l \approx 8.86$ cm or
$M = 118.4°$, $L = 20.3°$, $l \approx 3.16$ cm

c $C = 42°55'$, $A = 69°5'$, $a = 6.45$ m

d $P = 64.9°$, $p = 99.6$ mm, $r = 105$ mm

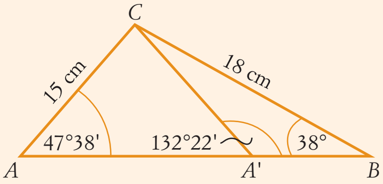
e $E = 47.9°$, $F = 81.7°$, $f = 17.2$ cm

f $Z = 32°34'$, $y = 82.4$ m, $x = 53.9$ m

g $Z = 100°13'$, $Y = 42°4'$, $y = 2.52$ m or
$Z = 79°47'$, $Y = 62°30'$, $y = 3.33$ m

7 $D = 21°8'$ or $158°52'$, $e = 61.27$ cm or 5.30 cm

8 $A = 47°38'$ or $132°22'$



9 **a** 4.36 **b** 23.7 **c** 7.1 **d** 14.4

10 **a** 34.6° **b** 140.1° **c** 103° **d** 19.8°

11 **a** $y = 7$ **b** $r = 2.48$ **c** $a = 18$

d $y = 7.15$ **e** $a = 15.9$

12 **a** $A = 24.15°$, $B = 125.1°$, $C = 30.75°$

b $z = 4.44$, $X = 21.9°$, $Y = 34.07°$

c $E = 26.4°$, $F = 36.3°$, $D = 117.3°$

d $h = 6.78$, $F = 38.1°$, $G = 69.9°$

e $q = 5.1$, $R = 16.1°$, $P = 133.9°$

13 $A = 89.6°$

14 $X = 24.15°$

15 $P = 90°$

16 56.44° or 123.56°

17 Prove by dividing into two equal triangles along the diagonal.

18 **a** $7981\ \text{cm}^3$ **b** $2.17\ \text{m}^3$ **c** $2758\ \text{cm}^3$

19 3.63 m

20 25.48 m

21 **a** Her height is 2992 m above the hill, so her altitude is really about 3397 m.

b The altimeter reading is high.

22 86.05°

23 361.5 m

24 745 m

Exercise 10.4

1 **a** 3.85 m **b** 17 mm **c** 50.25° **d** 41 cm

e 45 mm **f** 32.2° **g** 51.3° or 128.7° **h** 7.3 m

2 **a** 57.5 m **b** 10.8 m **c** 322 m

d 9.2 cm **e** 31.5 cm **f** 16.3 m

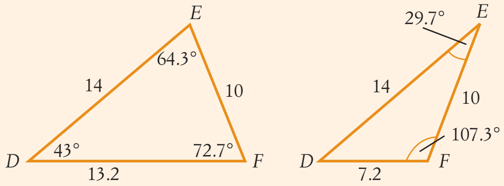
g 625 m **h** 18.3 mm

3 **a** $w \approx 17.26$ m **b** $m \approx 4.08$ m **c** $x \approx 126.59$ m
4 **a** 44.5 m **b** 71.2 m
5 58.5 m
6 **a** 12 800 m **b** 51.2 km/h
7 4.8 m
8 **a** $\theta \approx 59°$ **b** 239°
9 6.9 km **10** 105°
11 **a** 50 km **b** 86.6 km **c** 287.9 km
12 453.2 km
13 **a** $x \approx 2.94$ m **b** $a \approx 3.22$ m **c** $m \approx 1.85$ m
14 202.4 m **15** 68.8 m **16** 24°
17 4.31 km **18** 149.6 m
19 **a** 361 km **b** 94°38′
20 89.8 m **21** 12.75 km
22 45.84 km, 341°28′
23 **a** 422.4 m **b** 512.8 m
24 86 km **25** About 20 m.
26 200.37 . . . ≈ 200 m
27 **a** $x \approx 9.9$ cm, $y \approx 18.8$ cm
b $x \approx 22.8$ cm, $y \approx 21.2$ cm
c $x \approx 6.5$ mm, $y \approx 7.5$ mm
d $x \approx 15.3$ m, $y \approx 30.6$ m
28 41.5 m
29 **a** 31 m **b** 43°18′ **c** 29°8′
30 tower ≈ 7.8 m, building ≈ 12.5 m
31 **a** 12.04 m (south), 9 m (west)
b 44°54′ (south), 53°8′ (west)
c 15.03 m **d** 55°37′
32 1396 m
33 **a** 401 m **b** 233°16′ or S 53°16′ W
34 About 22.9 m high. **35** 076° or N 76° E at 12°47′
36 2.32 m, 139 m^2
37 3.16 m, 161°34′ or S 18°26′ E
38 48.5° **39** 6.9 cm
40 **a** 37.3 m and 57.1 m **b** 43.3 m
41 155 m **42** 77.1 m

Chapter review

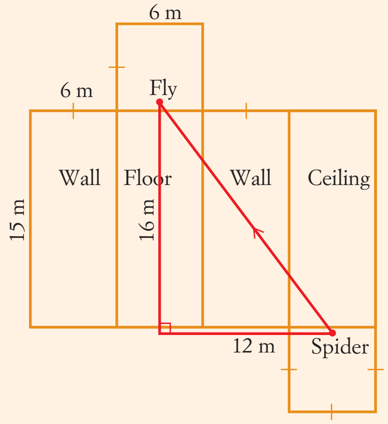
1 $XY^2 = YZ^2 + XZ^2$
2 **a** $\sin\theta = \frac{ZX}{YX}$ **b** $\cos\theta = \frac{ZY}{YX}$ **c** $\tan\theta = \frac{ZX}{ZY}$
3 **a** 0.515 **b** 26°
4 $x = 16$ cm
5 **a** $\tan\alpha = \frac{4}{3}$ **b** $\alpha \approx 53°$
6 $\alpha \approx 28°$, $\beta \approx 62°$
7 **a** $a \approx 21.8$ **b** $b \approx 9.4$
8 **a** $x \approx 16.3$ m **b** $y \approx 35.3°$
c $m \approx 9.3$ km **d** $w \approx 30.2$ mm
9 P 070°, Q 300°; P N 70° E, Q N 60° W
10 **a** $\frac{3}{5}$ **b** $^-\frac{4}{5}$ **c** 143°
11 **a** $\frac{^-2}{\sqrt{8}}$ or $\frac{^-1}{\sqrt{2}}$ **b** 1
c 117° **d** 225°
12 **a** 1 **b** 0 **c** Not defined **d** 0

13 $10\sqrt{3}$
14 **a** 43° **b** 40°
15 $\frac{^-1}{\sqrt{2}}$ **16** 131°
17 35° or 150° **18** 10.6 cm^2
19 **a** 25.7 **b** 37.8°
20



Triangle 1 **Triangle 2**

21 $\theta = 73.4°$ **22** Obtuse-angled
23 **a** 1 **b** $\sqrt{3}$ **c** 30°
d $\frac{^-\sqrt{3}}{2}$ **e** $^-\frac{1}{2}$ **f** $\sqrt{3}$
g $\frac{^-\sqrt{3}}{2}$ **h** $\frac{1}{2}$ **i** $^-\sqrt{3}$
24 About 40 m.
25 20 m

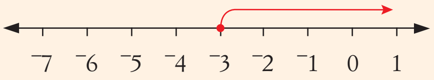


26 86.7 m **27** 50.2 m

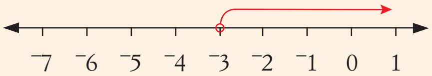
Chapter 11

Exercise 11.1

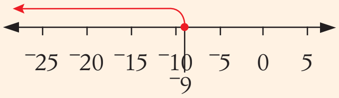
1 **a** $a = {}^-20.1$ **b** $b = {}^-1.5$ **c** $m = {}^-20$
d $h = {}^-20$ **e** $k = {}^-56$ **f** $y = {}^-13.5$
g $x = {}^-120$ **h** $h = 5$ **i** $n = {}^-5$
j $x = {}^-3$ **k** $m = {}^-2$ **l** $n = {}^-2\frac{1}{2}$
m $n = {}^-35$ **n** $d = {}^-2.5$ **o** $a = {}^-11.5$
p $a = {}^-1$ **q** $e = {}^-50$ **r** $r = {}^-1.2$
s $m = {}^-1.9$ **t** $d = {}^-10$
2 **a** $r \geq {}^-3$



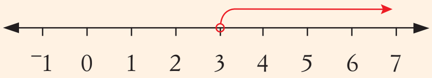
b $r > {}^{-}3$



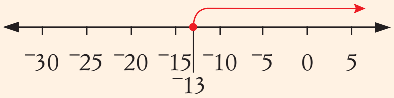
c $b \leq {}^{-}9$



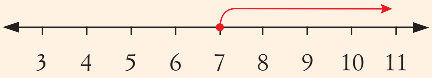
d $v > 3$



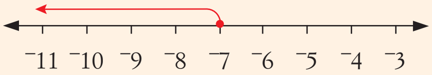
e $g \geq {}^{-}13$



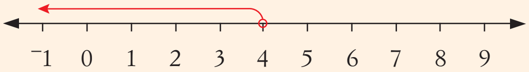
f $h \geq 7$



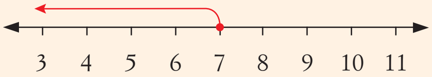
g $w \leq {}^{-}7$



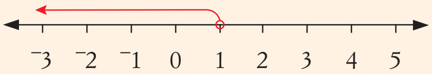
h $m < 4$



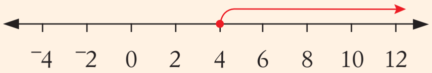
i $x \leq 7$



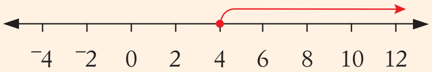
j $e < 1$



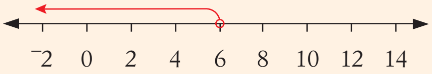
3 a $a \geq 4$



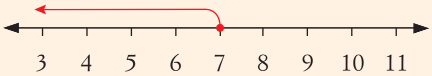
b $m \geq 4$



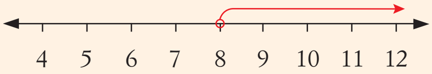
c $w < 6$



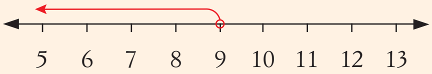
d $m \leq 7$



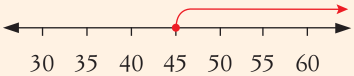
e $q > 8$



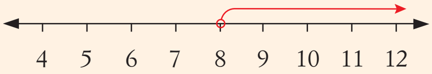
f $e < 9$



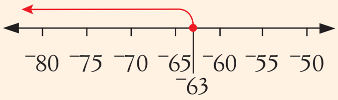
g $r \geq 45$



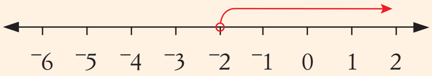
h $h > 8$



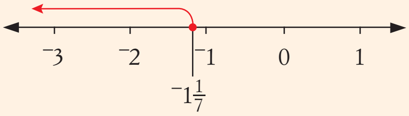
i $v \leq {}^{-}63$



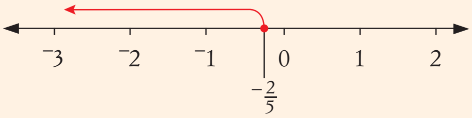
j $e > {}^{-}2$



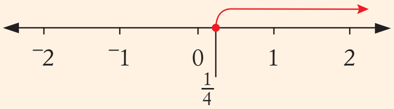
4 a $b \leq {}^{-}1\frac{1}{7}$



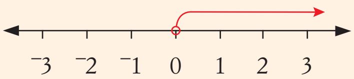
b $x < -\frac{2}{5}$



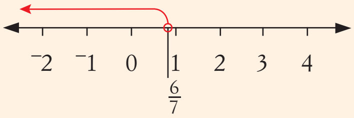
c $e \geq \frac{1}{4}$



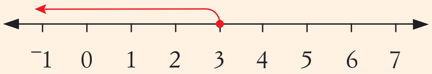
d $q > 0$



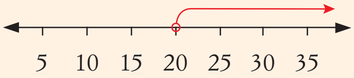
e $z < \frac{6}{7}$



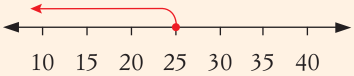
f $h \leq 3$



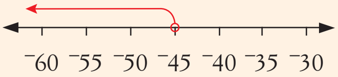
g $e > 20$



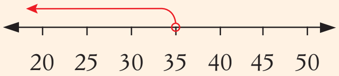
h $g \leq 25$



i $z < {}^{-}45$



j $t < 35$



5 **a** $a = {}^{-}3$ **b** $b = {}^{-}3$ **c** $m = {}^{-}5$
d $n = {}^{-}7$ **e** $k = {}^{-}5$ **f** $m = {}^{-}9$
g $r = {}^{-}12$ **h** $h = {}^{-}7$ **i** $w = {}^{-}40$
j $c = {}^{-}8$ **k** $b = {}^{-}3$ **l** $d = {}^{-}6$

6 **a** $a = {}^{-}9$ **b** $m = {}^{-}5$ **c** $a = {}^{-}2$
d $m = {}^{-}2$ **e** $x = {}^{-}15$ **f** $g = {}^{-}2$
g $k = {}^{-}6$ **h** $w = {}^{-}4$ **i** $x = \frac{1}{2}$
j $a = {}^{-}3$ **k** $m = {}^{-}1\frac{2}{3}$ **l** $x = {}^{-}2$
m $r = {}^{-}3$ **n** $g = {}^{-}10$ **o** $w = {}^{-}2$
p $m = {}^{-}5$ **q** $w = {}^{-}2\frac{2}{3}$ **r** $x = {}^{-}4$
s $d = {}^{-}16$ **t** $b = {}^{-}4$ **u** $n = {}^{-}4$

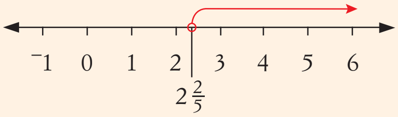
7 **a** $a = {}^{-}13$ **b** $b = {}^{-}4$ **c** $n = {}^{-}7$
d $m = {}^{-}1$ **e** $y = {}^{-}6$ **f** $x = {}^{-}10$
g $d = 5$ **h** $a = {}^{-}2$ **i** $w = {}^{-}3$
j $k = {}^{-}\frac{1}{3}$ **k** $x = {}^{-}1$ **l** $w = 4$
m $x = {}^{-}1$ **n** $a = {}^{-}4$ **o** $k = 3\frac{1}{4}$
p $b = 12$ **q** $y = 8$ **r** $m = {}^{-}6\frac{1}{2}$
s $n = 14$ **t** $x = 4\frac{1}{2}$ **u** $a = 0$

8 **a** $a = 2$ **b** $b = 3$ **c** $m = {}^{-}5$
d $k = 10$ **e** $y = 5$ **f** $x = 2.5$
g $d = {}^{-}2$ **h** $w = {}^{-}2\frac{1}{4}$ **i** $n = 13$
j $x = 3$

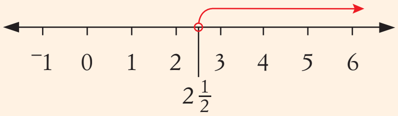
9 **a** $x = 16$ **b** $a = 14$ **c** $b = 12$
d $d = 5$ **e** $a = 12$ **f** $x = {}^{-}20$
g $b = 14$ **h** $m = {}^{-}15$ **i** $n = {}^{-}15$
j $k = 5$ **k** $d = 30$ **l** $x = {}^{-}2\frac{1}{4}$

10 **a** $a = 6$ **b** $b = 19$ **c** $h = {}^{-}1$
d $k = 9$ **e** $n = {}^{-}7\frac{1}{3}$ **f** $b = 9$

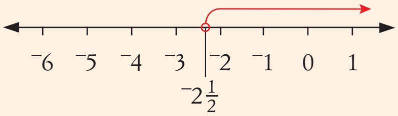
11 **a** $n > 2\frac{2}{5}$



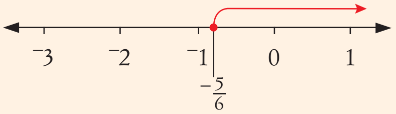
b $p > 2\frac{1}{2}$



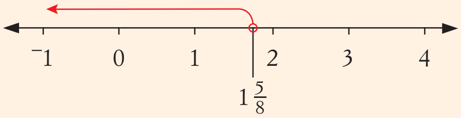
c $c > {}^{-}2\frac{1}{3}$



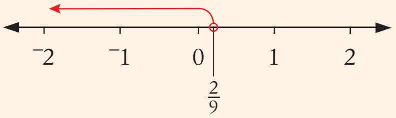
d $e \geq -\frac{5}{6}$



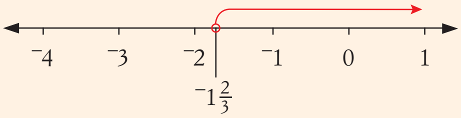
e $k < 1\frac{5}{8}$



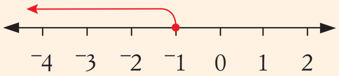
f $y < \frac{2}{9}$



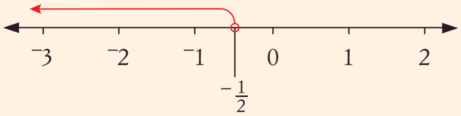
g $c > {}^{-}1\frac{2}{3}$



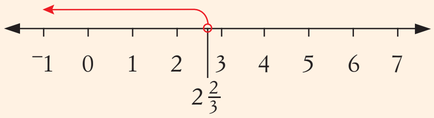
h $k \leq -1$



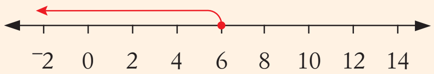
i $z < -\frac{1}{2}$



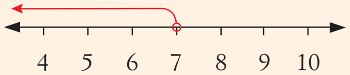
j $u < 2\frac{2}{3}$



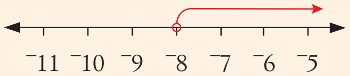
12 a $q \leq 6$



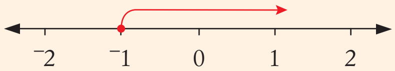
b $b < 7$



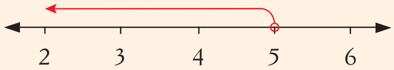
c $u > {}^-8$



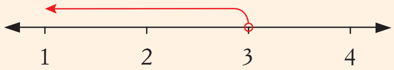
d $e \geq {}^-1$



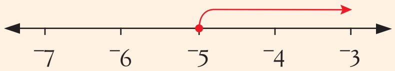
e $p < 5$



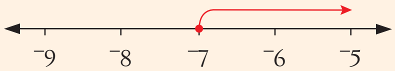
f $k < 3$



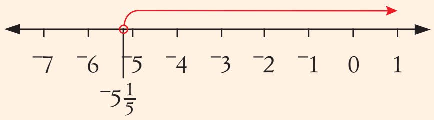
g $b \geq {}^-5$



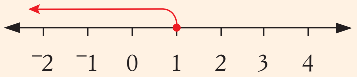
h $w \geq {}^-7$



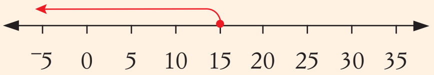
i $c >{}^- 5\frac{1}{5}$



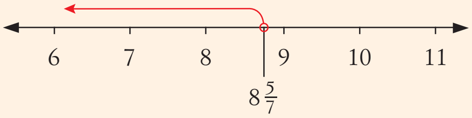
j $a \leq 1$



k $g \leq 15$



l $g < 8\frac{5}{7}$



13 a $a = 1$ **b** $b = 2\frac{2}{3}$ **c** $p = 2$

d $b = 3$ **e** $m = 5\frac{1}{2}$ **f** $c = 4$

g $g = 1$ **h** $n = \frac{2}{9}$ **i** $x = 18\frac{2}{3}$

14 2.5 m **15** 4 km/h

16 100 coins **17** 25 m by 25 m

18 29, 31 and 33

19 a 115.5 °F

b $x = {}^-40$, possible in many places in the US, including California, but not in Australia.

20 9 L to make 10 L of solution.

21 $12 \times \frac{n}{5} + 8 \times \frac{4n}{5} \leq 200$, he can get up to 22 plants.

22 $8.4w \leq 29.5$, about 3.5 kg.

23 $\frac{7A}{3} + 32 \leq 300$, about 114.9 m^2.

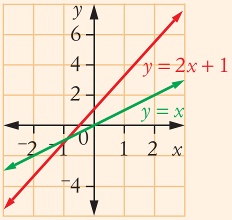
24 $\frac{c - 120}{45} \geq \frac{450}{70}$, \$435 (\$409.29, but there can only be a whole number of blades).

25 Proofs, teacher to check.

Exercise 11.2

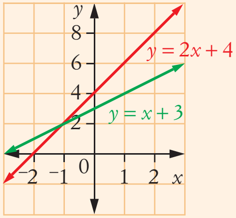
1 a $x = 2$ and $y = 4$ **b** $x = 1$ and $y = 4$

2 a



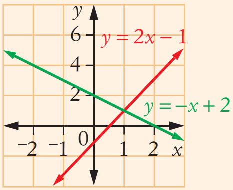
Simultaneous solution = (⁻1, ⁻1).

b



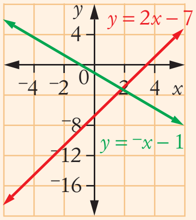
Simultaneous solution = (⁻1, 2).

c



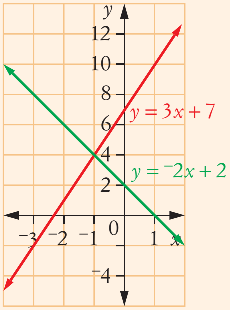
Simultaneous solution = (1, 1).

d



Simultaneous solution = (2, ⁻3).

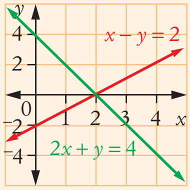
e



Simultaneous solution = (⁻1, 4).

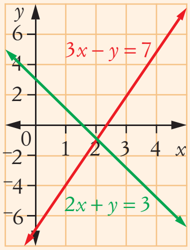
3 **a** $x = {}^-3$ and $y = 9$ or $x = 2$ and $y = 4$.
b $x = 3$ and $y = 9$ or $x = {}^-1$ and $y = 1$.
c $x = {}^-2$ and $y = 4$ or $x = 2$ and $y = 4$.
d $x = 3$ and $y = 9$.

4 **a**



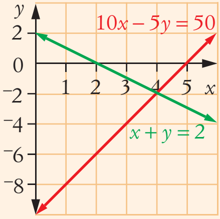
Simultaneous solution = (2, 0).

b



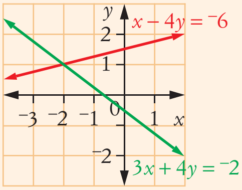
Simultaneous solution = (2, ⁻1).

c



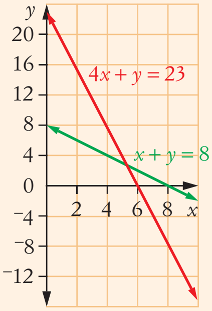
Simultaneous solution = (4, ⁻2).

d



Simultaneous solution = (⁻2, 1).

e



Simultaneous solution = (5, 3).

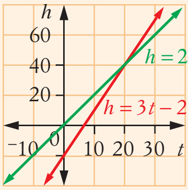
5 **a** $x = 5, y = 2$
b $x = 6, y = 18$
c $a = 2, b = 1$
d $a = 5, b = {}^-7$
e $x = 0, y = 3$
f $c = 8, d = 3$
g $c = 5, d = 4$
h $c = 4, d = 1\frac{1}{2}$
i $x = 2, y = {}^-1$

6 **a** $x = 3, y = 2$
b $x = 5, y = 2$
c $m = 3, n = 1$
d $a = 3, b = 2$
e $m = 1, n = {}^-2$
f $x = 3, y = {}^-3$
g $x = 4, y = {}^-3$
h $a = {}^-2, b = 1$
i $m = {}^-2, n = 1$

7 **a** $x = 0$ and $y = 0$ or $x = 1$ and $y = 1$.
b $x = 0$ and $y = 0$ or $x = 2$ and $y = 4$.
c $x = {}^-1$ and $y = 1$ or $x = 2$ and $y = 4$.
d $x = {}^-2$ and $y = 4$ or $x = 3$ and $y = 9$.
e $x = {}^-3$ and $y = 9$ or $x = 1$ and $y = 1$.
f $x = 1$ and $y = 1$ or $x = 2$ and $y = 4$.

8 **a** $x = 4, y = {}^-2$
b $a = 3, b = 3$
c $m = 2, n = {}^-3$
d $x = 2, y = 4$
e $m = 2, n = {}^-3$
f $c = 4, d = 2$
g $x = 4, y = 1$
h $a = 5, b = {}^-1$
i $x = 4, y = {}^-2$

9 Adult's ticket \$7, child's ticket \$2.

10 

Same height after 20 seconds.

11 18 m **12** Sam 50 bricks, Peta 200 bricks.

13 Strawberries \$2/tray, cherries \$3/tray.

14 $a = 40$ cm, $b = 10$ cm. **15** 2 CDs.

16 Jackie 22/h, Patrick 16/h.

Exercise 11.3

1 **a** $x = {}^{-}2$ or $x = 2$ **b** $a = {}^{-}7$ or $z = 7$

c $b = {}^{-}11$ or $b = 11$ **d** $h = {}^{-}5$ or $h = 5$

e $x = {}^{-}9$ or $x = 9$ **f** $a = {}^{-}2$ or $a = 2$

g $x = {}^{-}3$ or $x = 3$ **h** $b = {}^{-}5$ or $b = 5$

i $x = {}^{-}5$ or $x = 5$ **j** $k = {}^{-}\frac{3}{7}$ or $k = \frac{3}{7}$

k $n = {}^{-}2\frac{1}{4}$ or $n = 2\frac{1}{4}$ **l** $m = {}^{-}\frac{3}{4}$ or $m = \frac{3}{4}$

2 **a** $z = {}^{-}5$ or $z = 1$ **b** $h = 5$ or $h = 15$

c $e = {}^{-}4$ or $e = 14$ **d** $k = {}^{-}3$ or $k = 17$

e $c = {}^{-}2$ or $c = 18$ **f** $q = {}^{-}7$ or $q = 1$

g $q = {}^{-}10$ or $q = {}^{-}4$ **h** $b = {}^{-}3$ or $b = 21$

i $y = {}^{-}1$ or $y = 23$ **j** $m = 0$ or $m = 18$

3 **a** $x = 0$ or $x = {}^{-}1$ **b** $x = 0$ or $x = {}^{-}2$

c $x = 3$ or $x = {}^{-}1$ **d** $f = 4$ or $f = {}^{-}9$

e $x = 10$ or $x = {}^{-}3$ **f** $n = 4$ or $n = {}^{-}3$

g $b = 0$ or $b = {}^{-}8$ **h** $x = 3$

i $m = {}^{-}1\frac{1}{2}$ or $m = 4$ **j** $k = {}^{-}1\frac{1}{2}$ or $k = 3$

k $w = {}^{-}1\frac{2}{5}$ or $w = 1\frac{1}{4}$ **l** $x = {}^{-}\frac{1}{10}$ or $x = 2$

4 **a** $v = 3$ or $v = -1$ **b** $a = 8$ or $a = 9$

c $r = 6$ or $r = {}^{-}5$ **d** $n = {}^{-}5$ or $n = {}^{-}7$

e $w = {}^{-}5$ or $w = 3$ **f** $y = 3$ or $y = -10$

g $k = 13$ or $k = 10$ **h** $g = 4$ or $g = 5$

i $c = {}^{-}11$ or $c = 15$ **j** $n = \frac{{}^{-}5}{6}$ or $n = {}^{-}1\frac{1}{5}$

k $b = 1\frac{1}{5}$ or $b = \frac{{}^{-}1}{3}$ **l** $m = \frac{9}{4}$ or $m = \frac{9}{8}$

m $x = 2\frac{2}{3}$ or $x = \frac{{}^{-}4}{9}$

5 **a** $x = {}^{-}4$ or $x = {}^{-}5$ **b** $a = {}^{-}6$ or $a = {}^{-}2$

c $x = 7$ or $x = 1$ **d** $m = 7$ or $m = {}^{-}4$

e $n = 8$ or $n = {}^{-}5$ **f** $a = {}^{-}9$ or $a = 7$

g $m = {}^{-}8$ or $m = 3$ **h** $k = 9$

i $a = {}^{-}4$ or $a = {}^{-}7$ **j** $x = 0$ or $x = {}^{-}1$

k $x = {}^{-}3$ or $x = 0$ **l** $a = 0$ or $a = 2$

m $d = 0$ or $d = {}^{-}2$ **n** $y = 0$ or $y = 4$

o $x = 0$ or $x = 6$ **p** $x = {}^{-}3$ or $x = {}^{-}2$

q $a = 4$ or $a = {}^{-}5$ **r** $x = 9$ or $x = {}^{-}5$

s $b = 1$ or $b = 7$ **t** $h = 5$ or $h = {}^{-}7$

u $a = 6$ or $a = {}^{-}1$ **v** $u = 2$ or $u = 3$

w $b = 0$ or $b = \frac{1}{4}$ **x** $p = 0$ or $p = \frac{1}{3}$

y $n = 0$ or $n = \frac{2}{5}$

6 **a** $x = 1$ or $x = \frac{1}{2}$ **b** $x = {}^{-}1$ or $x = \frac{1}{3}$

c $x = {}^{-}2$ or $x = \frac{1}{4}$ **d** $x = 3$ or $x = \frac{2}{5}$

e $x = 2$ or $x = {}^{-}1\frac{1}{3}$ **f** $x = 5$ or $x = 1\frac{1}{2}$

g $x = {}^{-}4$ or $x = {}^{-}2\frac{1}{2}$ **h** $x = 4$ or $x = {}^{-}1\frac{1}{2}$

i $x = 5$ or $x = {}^{-}2\frac{1}{3}$ **j** $x = 1$ or $x = {}^{-}\frac{2}{5}$

k $x = {}^{-}1$ or $x = {}^{-}1\frac{3}{4}$ **l** $x = 2\frac{1}{2}$ or $x = {}^{-}4$

7 **a** $g = {}^{-}5 - 2\sqrt{2}$ or $g = {}^{-}5 + 2\sqrt{2}$

b $a = 2 - 2\sqrt{2}$ or $a = 2 + 2\sqrt{2}$

c $t = {}^{-}2 - \sqrt{10}$ or $t = {}^{-}2 + \sqrt{10}$

d $u = {}^{-}7 - 2\sqrt{3}$ or $u = {}^{-}7 + 2\sqrt{3}$

e $n = 3 - \sqrt{11}$ or $n = 3 + \sqrt{11}$

8 **a** $x = {}^{-}1$ or $x = \frac{1}{2}$ **b** $x = {}^{-}1$ or $x = {}^{-}\frac{2}{3}$

c $x = 3$ or $x = 1\frac{2}{3}$ **d** $x = 3$ or $x = \frac{2}{3}$

e $c = {}^{-}3$ or $c = 5\frac{1}{2}$ **f** $a = {}^{-}3$ or $a = 2\frac{1}{3}$

g $a = {}^{-}3$ or $a = {}^{-}\frac{4}{7}$ **h** $h = {}^{-}3\frac{2}{3}$ or $h = {}^{-}\frac{1}{3}$

i $x = 2$ or $x = {}^{-}\frac{3}{5}$ **j** $a = 1$ or $a = \frac{1}{6}$

k $m = \frac{3}{4}$ or $m = 2\frac{1}{2}$ **l** $m = \frac{1}{3}$ or $m = {}^{-}1\frac{1}{2}$

m $h = {}^{-}\frac{5}{6}$ or $h = 1\frac{1}{2}$ **n** $m = \frac{3}{7}$ or $m = 3\frac{1}{2}$

o $x = \frac{2}{3}$ **p** $m = \frac{1}{2}$ or $m = 1\frac{2}{3}$

q $c = 6$ or $c = {}^{-}1\frac{1}{4}$ **r** $b = {}^{-}6$ or $b = \frac{3}{4}$

s $a = {}^{-}\frac{1}{2}$ or $a = 1\frac{1}{2}$ **t** $b = {}^{-}7$ or $b = \frac{1}{7}$

u $m = {}^{-}2$ or $m = 2\frac{1}{2}$ **v** $g = 7$ or $g = {}^{-}1\frac{1}{2}$

w $c = 1\frac{1}{2}$ or $c = 4$

9 **a** $x = {}^{-}2 + \sqrt{3}$ or $x = {}^{-}2 - \sqrt{3}$

b $x = \dfrac{7 - \sqrt{41}}{2}$ or $x = \dfrac{7 + \sqrt{41}}{2}$

c $x = {}^{-}3 - 2\sqrt{2}$ or $x = {}^{-}3 + 2\sqrt{2}$

d $x = 3 - \sqrt{6}$ or $x = 3 + \sqrt{6}$

e $x = \dfrac{2 - \sqrt{14}}{2}$ or $x = \dfrac{2 + \sqrt{14}}{2}$

f $x = \dfrac{{}^{-}3 - \sqrt{7}}{2}$ or $x = \dfrac{{}^{-}3 + \sqrt{7}}{2}$

g $x = \dfrac{{}^{-}3 - \sqrt{3}}{3}$ or $x = \dfrac{{}^{-}3 + \sqrt{3}}{3}$

h $x = \dfrac{1 - \sqrt{17}}{8}$ or $x = \dfrac{1 + \sqrt{17}}{8}$

i $x = \dfrac{5 - \sqrt{10}}{5}$ or $x = \dfrac{5 + \sqrt{10}}{5}$

j $x = \dfrac{3 - \sqrt{41}}{8}$ or $x = \dfrac{3 + \sqrt{41}}{8}$

k $x = \dfrac{{}^{-}2 - \sqrt{6}}{2}$ or $x = \dfrac{{}^{-}2 + \sqrt{6}}{2}$

l $x = \dfrac{{}^{-}1 - \sqrt{6}}{5}$ or $x = \dfrac{{}^{-}1 + \sqrt{6}}{5}$

10 **a** $x \approx {}^{-}0.38$ or ${}^{-}2.62$ **b** $a \approx 6.85$ or 0.15

c $b \approx {}^{-}0.70$ or ${}^{-}4.30$ **d** $w \approx 2.76$ or 0.24

e $n \approx 0.22$ or ${}^{-}2.22$ **f** $x \approx 0.72$ or 2.78

g $x \approx 2.56$ or ${}^{-}1.56$ **h** $x \approx 0.78$ or ${}^{-}1.28$

i $n \approx 1.12$ or ${}^{-}1.79$ **j** $w \approx {}^{-}0.74$ or ${}^{-}2.\ 26$

k $c = 2$ or ${}^{-}\frac{1}{3}$

11 **a** $x = 3$ and $y = 9$ or $x = {}^{-}2$ and $y = 4$

b $x = {}^{-}2$ and $y = {}^{-}5$ or $x = 1$ and $y = {}^{-}2$

c $x = 4$ and $y = 16$ or $x = {}^{-}1$ and $y = 1$
d $x = 4$ and $y = 8$
e $x = 1$ and $y = 3$
f $x = {}^{-}\frac{1}{2}$ and $y = 5\frac{1}{2}$ or $x = 5$ and $y = {}^{-}44$
g $x = \frac{2}{5}$ and $y = 1\frac{4}{5}$ or $x = 1$ and $y = 3$
h $x = {}^{-}5\frac{1}{2}$ and $y = 50$ or $x = 1$ and $y = {}^{-}2$
i $x = 7$ and $y = 40$ or $x = 5$ and $y = 30$
12 **a** $x = 0$ or ${}^{-}1$ **b** $x = {}^{-}3$ or $x = {}^{-}5$
c $y = 5$ or $y = {}^{-}2$ **d** $x = 0$ or $x = 17$
e $b = 2$ or $b = {}^{-}12$ **f** $x = 0$ or $x = 1\frac{2}{3}$
g $w = 1$ or $w = {}^{-}4\frac{1}{4}$ **h** $x = {}^{-}1$ or $x = 8\frac{1}{2}$
i $m = 5\frac{1}{2}$ or $m = 9\frac{1}{2}$ **j** $x = {}^{-}2\frac{1}{4}$ or $x = {}^{-}1\frac{1}{2}$
13 **a** 7 **b** 10 **c** 10 or ${}^{-}9$
14 6 m by 2 m **15** 7 cm
16 $x = 10$ cm, so the dimensions are 10 cm, 24 cm, 26 cm.
17 6 cm by 20 cm by 2 cm
18 **a** 9 integers **b** 12 integers **c** 15 integers
19 $1\frac{3}{5}$ seconds after it was thrown.
20 2.52 cm by 47.57 cm by 10 cm
21 **a** $(x + 3)^2 = {}^{-}1$
b The square root of ${}^{-}1$ cannot be done on the number line.
c $({}^{-}3, {}^{-}1)$

22 $\left(x + \frac{b}{2}\right)^2 = \frac{b^2}{4} - c = \frac{b^2 - 4c}{4}$, which gives

$x = \frac{-b}{2} \pm \sqrt{\frac{b^2 - 4c}{4}} = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$, which is the quadratic formula with $a = 1$.

23 Proof, teacher to check.
24 If $b^2 < 4ac$, then the formula contains the square root of a negative number, which has no place on the real number line.

Exercise 11.4

1 **a** $y = 2\frac{1}{2}$ **b** $p = 3\frac{1}{2}$ **c** $q = 4$
d $b = 4$ **e** $z = 3\frac{1}{2}$ **f** $r = 2\frac{1}{2}$
g $n = 4$ **h** $q = 4\frac{1}{2}$ **i** $x = 4$
j $y = 6$
2 **a** $q = 2$ **b** $k = 10$ **c** $z = {}^{-}\frac{1}{4}$
d $p = 3$ **e** $c = 4\frac{1}{4}$ **f** $x = 4\frac{1}{2}$
g $g = 2$ **h** $u = \frac{1}{4}$ **i** $x = 1\frac{2}{3}$
j $c = {}^{-}1$
3 **a** $x = 1.76$ **b** $w = {}^{-}2.23$ **c** $b = 8.35$
d $u = 14.36$ **e** $g = 4.52$ **f** $h = {}^{-}3.67$
g $z = 8.41$ **h** $t = 7.85$ **i** $z = 9.06$
j $b = 16.07$
4 **a** About 36.7 years.
b About 275.2 months (22 years 11 months).
c About 192 weeks (3 years 36 weeks).
d About 6597.1 days (18 years 27 days).
e About 208.5 months (17 years $4\frac{1}{2}$ months).
5 About 312 days (almost a year).
6 **a** About 83 mice/ha. **b** About 6 weeks (6.396).
c Yes, after about another 10 weeks (10.319).
7 **a** From 100°C, about 3.143 m (3m 9 s).
b About 2.263 m (2 m 16 s).
c About 2.770 m (2 m 46 s).
d 12.398, 11.517 and 12.025 minutes from when it was made.
8 About 1850 years (1848.3).
9 **a** About 15 100 (15 074). **b** About 17 300 (17 258).
c About 4200 (4168).
10 $x = \frac{\log b}{\log a}$

Exercise 11.5

1 **a** 85.9° (85°54′) **b** 120.3° (120°18′)
c 57.3° (57°18′) **d** 108.9° (108°54′)
e 280.7° (280°42′) **f** 80.2° (80°12′)
g ${}^{-}5.7°$ (${}^{-}5°42′$) **h** 326.6° (326°36′)
i 40.1° (40°6′) **j** ${}^{-}269.3°$ (${}^{-}269°18′$)
k 257.8° (257°48′) **l** ${}^{-}80.2°$ (${}^{-}80°12′$)
m ${}^{-}28.6°$ (${}^{-}28°36′$) **n** ${}^{-}252.1°$ (${}^{-}252°6′$)
o 332.3° (332°18′)
2 **a** 6.039 **b** 1.030 **c** ${}^{-}1.290$
d 4.189 **e** 1.344 **f** ${}^{-}4.000$
g 3.857 **h** 6.021 **i** ${}^{-}1.450$
j 5.027 **k** 6.213 **l** ${}^{-}4.490$
m 4.206 **n** 5.463 **o** 1.553
3 **a** 0.008 and 3.1336 **b** 1.9922 and 4.291
c 0.2136 and 2.928 **d** 3.648 and 5.7768
e 2.5357 and 3.7475 **f** 3.9762 and 5.4486
g 0.18 and 2.9616 **h** 1.2228 and 5.0604
i 3.3246 and 6.1002 **j** 3.7252 and 5.6996
4 **a** ${}^{-}2.1732$, ${}^{-}0.9684$, 4.11 and 5.3148
b ${}^{-}2.5688$, ${}^{-}0.5728$, 3.7144 and 5.7104
c ${}^{-}4.1384$, ${}^{-}2.1448$, 2.1448 and 4.1384
d ${}^{-}5.2302$, ${}^{-}1.053$, 1.053 and 5.2302
e ${}^{-}4.3824$, ${}^{-}1.9008$, 1.9008 and 4.3824
f ${}^{-}4.3888$, ${}^{-}1.8944$, 1.8944 and 4.3888
g ${}^{-}5.7996$, ${}^{-}3.6252$, 0.4836 and 2.658
h ${}^{-}2.8327$, ${}^{-}0.3089$, 3.4505 and 5.9743
i ${}^{-}3.0233$, ${}^{-}0.1183$, 3.2599 and 6.1649
j ${}^{-}5.5726$, ${}^{-}0.7106$, 0.7106 and 5.5726
5 **a** 0.1465, 2.9951, 6.4297 and 9.2783
b 0.046, 3.0956, 6.3292 and 9.3788
c 0.6175, 2.5241, 6.9007 and 8.8073
d 1.6178, 4.6654, 7.901 and 10.9486
e 2.2357, 4.0475, 8.5189 and 10.3307
f 0.1891, 2.9525, 6.4723 and 9.2357
g 1.0644, 5.2188, 7.3476 and 11.502
h 3.9927, 5.4321, 10.2759 and 11.7153
i 2.2395, 4.0436, 8.5227 and 10.3268
j 1.2943, 4.9889, 7.5775 and 11.2721
6 **a** ${}^{-}2.4626$ and ${}^{-}0.679$ **b** ${}^{-}1.3847$ and 1.3847
c ${}^{-}2.5048$ and 2.5048 **d** ${}^{-}2.4639$ and 2.4639
e ${}^{-}0.8593$ and 0.8593 **f** ${}^{-}2.3774$ and ${}^{-}0.7643$
g 0.7938 and 2.3478 **h** ${}^{-}1.5598$ and 1.5598
i 1.0995 and 2.042 **j** 0.4938 and 2.6478

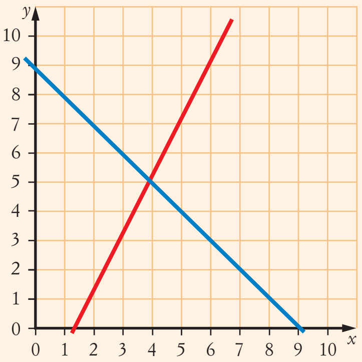
7 a $^{-}3.9243$, $^{-}2.3589$, 2.3589 and 3.9243
b $^{-}6.1002$, $^{-}3.3246$, 0.183 and 2.9586
c $^{-}5.6372$, $^{-}3.7876$, 0.646 and 2.4956
d $^{-}6.1218$, $^{-}0.1614$, 0.1614 and 6.1218
e $^{-}3.0956$, $^{-}0.046$, 3.1876 and 6.2372
f $^{-}5.4245$, $^{-}4.0003$, 0.8587 and 2.2829
g $^{-}3.4624$, $^{-}2.8209$, 2.8208 and 3.4623
h $^{-}3.1036$, $^{-}0.038$, 3.1796 and 6.2452
i $^{-}6.0542$, $^{-}3.3706$, 0.229 and 2.9126
j $^{-}2.0884$, $^{-}1.0532$, 4.1948 and 5.23
k $^{-}5.9625$, $^{-}0.3208$, 0.3207 and 5.9624
l $^{-}2.2501$, $^{-}0.8915$, 4.0331 and 5.3917
m $^{-}4.0074$, $^{-}2.2758$, 2.2758 and 4.0074
n $^{-}2.1872$, $^{-}0.9545$, 4.096 and 5.3287
o $^{-}2.8042$, $^{-}0.3374$, 3.479 and 5.9458
p $^{-}5.9932$, $^{-}3.4317$, 0.29 and 2.8515
q $^{-}5.1702$, $^{-}1.113$, 1.113 and 5.1702
r $^{-}4.6022$, $^{-}1.681$, 1.681 and 4.6022
s $^{-}5.374$, $^{-}4.0508$, 0.9092 and 2.2324
t $^{-}4.8761$, $^{-}1.4071$, 1.4071 and 4.8761
u $^{-}5.0604$, $^{-}1.2228$, 1.2228 and 5.0604
v $^{-}2.3233$, $^{-}0.8183$, 3.9599 and 5.4649
w $^{-}5.4634$, $^{-}3.9614$, 0.8198 and 2.3218
x $^{-}4.2635$, $^{-}2.0197$, 2.0197 and 4.2635
y $^{-}3.0505$, $^{-}0.0911$, 3.2327 and 6.1921
z $^{-}5.3634$, $^{-}0.9198$, 0.9198 and 5.3634
8 a 0.5368, 1.034, 1.584 and 2.0812
b 0.4215, 0.6257, 1.4687 and 1.6729
c 0.0761, 0.2935, 0.4457 and 0.6631
d 0.5224, 0.6557, 1.3078 and 1.4411
e 0.1841, 0.514, 0.8822 and 1.2122
f 0.6376, 0.9332, 1.6848 and 1.9804
g 0.1852, 0.3384, 0.5343 and 0.6875
h 0.3303, 0.5265, 0.9015 and 1.0977
i 0.5278, 0.8186, 1.4254 and 1.7162
j 0.0803, 0.2893, 0.4499 and 0.6589
k 0.3343, 0.7129, 2.4287 and 2.8073
l 0.3184, 0.4066, 0.8017 and 0.8899
m 0.29, 0.4954, 6.5732 and 6.7786
n 10.708, 14.4248, 35.8407 and 39.5575
o 3.4802, 22.6998, 29.6601 and 48.8797
p 3.5843, 9.5056, 29.7643 and 35.6856
q 6.9466, 13.1061, 20.3151 and 26.4746
r 1.1533, 5.1934, 7.4999 and 11.54
s 1.2968, 5.6845, 15.2594 and 19.6471
t 7.3796, 12.2553, 20.4696 and 25.3453
u 6.0979, 10.8837, 23.0795 and 27.8653
v 0.1124, 3.061, 6.459 and 9.4076
w 2.8401, 6.5378, 12.218 and 15.9157
x 9.9314, 13.3397, 33.2024 and 36.6107
y 0.794, 2.859, 8.1 and 10.165
z 7.0011, 11.4788, 25.4811 and 29.9588
9 a About 0.25 and 1.25 seconds.
b About 0.20, 1.31, 1.70 and 2.81 seconds.
c About 3 seconds.

10 a 41.5°C **b** 26.5°C
c February and October **d** April and August
11 a June and August **b** February and December
12 a July **b** May and September
13 a About 0.46, 1.25, 3.87 and 4.66 ms.
b About 1.99, 3.13, 5.40 and 6.54 ms.
14 The time between high tides is 12.9 h so $k = \frac{2\pi}{12.9} \approx 0.487$, the average level is 2.55 m and the variation from slack water is 1.14 m. This gives $d = 2.55 + 1.14 \cos 0.487t$ m, where t is in hours. solving $2 = 2.55 + 1.14 \cos 0.487t$ gives $t = 4.259$ h, so it is at 1140 h.

Chapter review

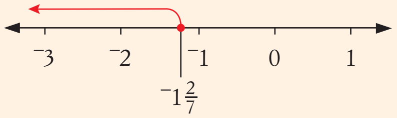
1 a $m = {}^{-}5$ **b** $g = {}^{-}6$ **c** $p = {}^{-}11$ **d** $d = {}^{-}4$

2

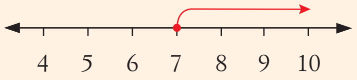


Solution is (4, 5), or $x = 4$, $y = 5$.

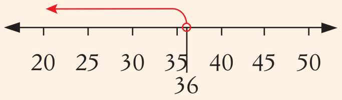
3 a $y \le {}^{-}1\frac{2}{7}$



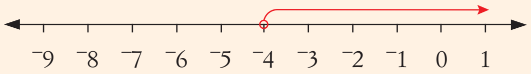
b $q \ge 7$



c $q < 36$



d $n > {}^{-}4$



4 a $y = {}^{-}3$ or $y = 3$ **b** $t = {}^{-}4$ or $t = 4$
c $g = {}^{-}\frac{4}{5}$ or $g = \frac{4}{5}$
5 $v = {}^{-}19$ or $v = 5$

6 **a** $x = 0$ or $x = {}^{-}5$ **b** $a = 4$ or $a = {}^{-}7$

c $m = {}^{-}\frac{2}{3}$ or $m = 2\frac{1}{2}$

7 **a** 6 **b** $2\frac{1}{2}$

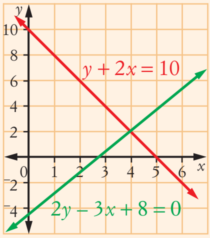
8 0.7505

9 **a** ${}^{-}2.4536$, ${}^{-}0.688$, 3.8296 and 5.5952

b ${}^{-}5.3484$, ${}^{-}0.9348$, 0.9348 and 5.3484

10 **a** $z = 28$ **b** $t = {}^{-}6$

11

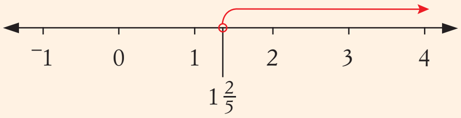


Solution is $x = 4$ and $y = 2$.

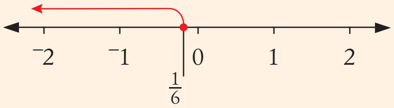
12 **a** $b = {}^{-}3$ **b** $c = 9$ **c** $b = 4$

13 **a** $k = 1\frac{1}{5}$ **b** $c = 6$ **c** $g = 11\frac{1}{2}$

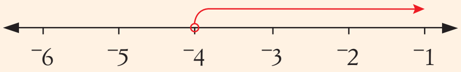
14 **a** $x > 1\frac{2}{5}$



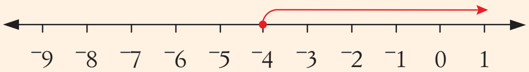
b $h \le {}^{-}\frac{1}{6}$



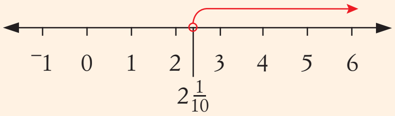
15 **a** $x > {}^{-}4$



b $z \ge {}^{-}4$



c $x > 2\frac{1}{10}$



16 $a = {}^{-}2$ and $b = 3$

17 $u = {}^{-}4$ and $w = 7$

18 **a** $y = {}^{-}7$ or $y = {}^{-}3$ **b** $a = 2$ or $a = {}^{-}10$

c $b = 1\frac{1}{5}$ or $b = {}^{-}\frac{1}{3}$

19 **a** $x = 0$ or $x = 5$ **b** $w = 6$ or $w = 7$

c $y = {}^{-}2$ or $y = 9$ **d** $c = 2$ or $c = {}^{-}8$

e $h = \frac{5}{4}$ or $h = \frac{{}^{-}5}{7}$ **f** $g = \frac{4}{7}$ or $g = \frac{{}^{-}1}{4}$

20 $m = {}^{-}2$ or $m = \frac{1}{4}$

21 $x = {}^{-}2$ and $y = 0$ or $x = 4$ and $y = 12$

22 $x = \dfrac{{}^{-}7 - \sqrt{41}}{4} \approx {}^{-}3.3507$ or $x = \dfrac{{}^{-}7 + \sqrt{41}}{4} \approx {}^{-}0.1492$

23 $a = \frac{1}{2}$

24 **a** $a = 3$ and $b = {}^{-}1$ **b** $x = 6\frac{1}{9}$ and $y = \frac{1}{9}$

25 $e = 6 - 2\sqrt{3}$ or $e = 6 + 2\sqrt{3}$

26 $x = {}^{-}5$ and $y = {}^{-}9$ or $x = 3$ and $y = 7$

27 **a** $r = 2\frac{1}{8}$ **b** $p \approx {}^{-}14.22$ **c** $c \approx 11.09$

28 About 19 years 5 months (232.7 months).

29 0.9659, 2.1757, 7.2491 and 8.4588

30 0.2172, 0.4809, 0.9154 and 1.1790

31 2.5 m by 2 m

32 $2s + s = 78$. Liam is 52 and his son is 26.

33 7 m by 13 m

34 The woman is 30 years old and her daughter is 10 years old.

35 $d = \frac{1}{4}$ and $g = \frac{1}{2}$

36 $x = 4$ or $x = 7\frac{1}{3}$

37 About $9\frac{1}{2}$ (9.61) years.

38 **a** About 5.39%. **b** About 1745 million years.

39 About 9.5, 11.9 and 18.5 seconds after being dropped.

40 4 seconds, the negative 2 seconds is false and represents a time before it was thrown.

41 $(x + 2)^2 = {}^{-}5$, the square root of a negative number has no place on the number line.

42 Proof, teacher to check.

43 $n\left(1 + \frac{p}{100}\right)^y$

Glossary

Word	Definition
addition principle	The rule that the probability of an event consisting of mutually exclusive (separate) events is the sum of their probabilities. $A \cap B = \varnothing \Rightarrow P(A \text{ or } B) = P(A) + P(B)$.
adjacent side	In a right-angled triangle, the side joining the right angle and nominated angle. *See also* hypotenuse, opposite side.
ambiguous case	A triangle with two sides and a non-included acute angle known, where the remaining side and angles have two possibilities (found using the sine rule).
angle of depression	Angle between the horizontal and the line of sight *down* to an object.
angle of elevation	Angle between the horizontal and the line of sight *up* to an object.
appreciation	The increase in value of an item over time, especially if it is a regular increase.
arc	Part of a curve, especially a circle.
arcos	Arccos x means the values of θ for which $\cos\theta = x$. For the principal value, $0 \leq \theta \leq \pi$. *Also* $\cos^{-1}$.
arcsin	Arcsin x means the values of θ for which $\sin\theta = x$. For the principal value, $\frac{^{-}\pi}{2} \leq \theta \leq \frac{\pi}{2}$. *Also* $\sin^{-1}$.
asymptote	A straight line on a graph that is approached by a curve.
bearing	A three-digit number showing the angle in degrees clockwise from north to the direction of an object.
bias	Data or method that is likely to give results generally different to the population. *See also* fair.
bimodal	Having two modes; two scores with equal highest frequency.
binomial products, binomial brackets	Product of two brackets, such as $(2x + 3)(4x - 1)$.
bivariate	Data set where each item has two measurement variables. *See also* univariate.
branch	Part of a curve such as a hyperbola that is not continuous with the rest of the graph.
cardinal number	The numbers 0, 1, 2, 3, . . . etc.; the number of elements in a finite set.
Cartesian plane	A flat surface with horizontal and vertical axes intersecting at 0.
categorical	Data that can be put into categories, such as names of colours.
certain	Will always happen.
chord	A straight line between two points on the circumference of a curve, especially a circle.
class centre	The centre of a true class group in a frequency distribution.

Word	Definition
closure, closed	An operation closed for a set always gives another element of the set when performed on element(s) of the set.
co-interior (angles):	Inside angles on the same side of a transversal over two lines.
common factor	Number or algebraic term that is a factor of each of some given numbers or terms.
common logarithm	The logarithm of a number to the base b. For $10^y = x$, $y = \log_{10} x = \log x$.
commutative (law)	Addition (or multiplication) of two numbers is the same for either number order.
complementary (events)	Two separate events that together make the entire sample space.
completion of the square	Procedure that changes a quadratic expression to the form $a(x + b)^2 + c$.
compound interest	Interest calculated on the current balance of a loan or investment. *See also* reducing interest.
conditional probability	The probability $P(A\|B)$ that A occurs, assuming that B occurs.
congruent (shapes), **congruence**	Shapes that are the same. They may be exactly superimposed, possibly by turning one over.
constant	A number, particularly a term of an expression that is just a number.
continuous	Data that can have any values, such as heights.
converse	For logic or mathematical theorems, the reverse statement.
cosine, cos	The adjacent side divided by the hypotenuse for a nominated angle in a right-angled triangle. *See also* sine, tangent.
cos^{-1}	*See* arccos.
cosine rule	The rule that for a triangle ABC, $a^2 = b^2 + c^2 - 2bc \cos A$ or $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc \cos A}$.
counter example	An example that shows that something is not true.
cross method	Factorisation of a quadratic using a cross to show possible factors.
cross-section	The 2D shape on a flat surface made by cutting through a solid (3D) shape.
cumulative frequency	The progressive total of frequencies of ordered numeric data.
curved surface area	The area of the curved face of a cone, cylinder, or sphere.
cyclic quadrilateral	A quadrilateral whose vertices are on the circumference of a circle.
deductive geometry	Geometry worked out using logical reasoning from assumptions (axioms).
degree (polynomial)	The highest power of a polynomial.
degree (angle)	Angle measure such that a complete revolution is 360 degrees, written as 360°. *See also* minute, second.

Glossary

Word	Definition
dependent (events)	Events where the probability of one is affected by the occurrence of the other. *See also* independent.
dependent (variable)	Variable that is calculated from others; the y-variable of a function.
difference of two squares	Expression of the form $a^2 - b^2 = (a - b)(a + b)$.
discrete	Data that can only have particular values, such as shoe sizes.
distributive (law)	The product of the sum (or difference) of two numbers with another number is the same as the sum (or difference) of the products of that number with each of the two numbers; $a(b \pm c) = ab \pm ac$.
dividend	The number or polynomial that is divided by another to give a quotient and remainder.
division identity	The identity $a = qb + r$, where $0 \leq r < q$ for numbers or the degree of r is less than the degree of q for polynomials.
dummy variable	Variable substituted for an expression to simplify a process.
element	A basic probability outcome; '5' on a die. *See also* sample point.
elimination method	Solution of simultaneous equations by elimination of variables.
empty set	*See* null set.
entire surd	A surdic expression like $\sqrt{20}$ written entirely under the surd sign (instead of $2\sqrt{5}$).
event (probability)	An individual outcome or collection of outcomes.
exponent	The raised number of a power, such as the 4 in 5^4. *See also* index.
exponential	An equation or function with the variable in the exponent such as $y = 5 \times 2^x$ or $3^x = 50$.
expression	Variables and/or numbers connected by operations.
exterior (angle)	Angle on the outside of a polygon made by extending one side.
extrapolation	Estimation of a value outside the range of known values. *See also* interpolation.
factor theorem	If $p(a) = 0$ for a polynomial $p(x)$ then $x - a$ is a factor of $p(x)$.
factorise	Use the distributive law to change an expression into factors, include a bracketed factor.
fair	A sample/question/method that does not favour particular results and will be truly representative of the population. *See also* bias.
five number summary	The lowest, first quartile, median, third quartile and highest values of a frequency distribution.
flat-rate interest	Interest calculated on the initial value of a loan or investment. *See also* simple interest.
formula	An equation that gives one quantity in terms of others. The quantity on its own is called the subject.

Word	Definition
function	A rule that changes a number (input) to a new one (output).
given	For a statement P in mathematical logic, given P means 'assuming that P is true'.
gradient	Property of a straight line obtained by dividing the vertical distance between two points (rise) by the horizontal (run). *See also* slope.
growth factor	Factor that a value is multiplied by to produce the next value, especially when growth is a fixed percentage over time such as compound interest.
grouping	A factorisation method where a common factor is created from separate parts of the expression.
hectare	The area of a square 100 m by 100 m; 10 000 m^2.
Heron's formula	A formula for the area of a triangle: $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $S = \frac{a+b+c}{2}$, from the lengths of the sides a, b and c.
hyperbola	A mathematical relation such as $y = \frac{5}{x}$. Its graph consists of two parts called branches shaped like two right-angled corners with curved bases pointing toward each other.
hypotenuse	The long side of a right-angled triangle (opposite the right angle).
impossible	Will never happen.
independent (events)	Events where the probability of one is not affected by the occurrence of the other.
independent (variable)	Variable that is not calculated from others; the x-variable of a function.
index	The raised number of a power, such as the 4 in 5^4; plural *indices*. *See also* exponent.
inequation, inequality	A statement that one expression is greater than, less than, greater than or equal to or less than or equal to another. It is like an equation with one of the signs $>$, $<$, $\leq$ or $\geq$ instead of $=$.
instalment	Equal payments used to repay a loan at fixed intervals, such as monthly instalments.
intercept	Position where a line intersects an axis.
interest	Charge made for the use of money (obtained as a loan). The amount of the loan is called the principal.
interest rate	Interest stated as a yearly percentage of the amount borrowed.
interpolation	Estimation of a value within the range of known values. *See also* extrapolation.
interquartile range (IQR)	IQR = third quartile − first quartile for a frequency distribution.
intersection (sets)	The set that contains the elements of both of two given sets.

Word	Definition
inverse operations	Operations that do the reverse of each other; × and ÷, etc.
irrational number	A number that cannot be expressed as a ratio of integers such as $\sqrt{2}$.
leading term	The term containing the highest power of a polynomial.
LHS	Left-hand side.
line of best fit	A (usually straight) line drawn as closely as possible to the points on a graph.
linear (rule, equation, function, inequality)	A rule, equation, function or inequality whose graph is a straight line.
logarithm	The logarithm of the number x to the base b, $\log_b x$, is the number y such that $b^y = x$.
mantissa	The value part of a number in scientific notation. The mantissa of 3.52×10^4 is '3.52'.
mean	The arithmetic average, $\bar{x} =$ total ÷ number; for 3, 6, 12, $\bar{x} = 21 \div 3 = 7$.
mean deviation	The average of the absolute value of the differences between data items and the mean of a numerical frequency distribution, given by $\frac{\sum \lvert x - \bar{x} \rvert}{n}$.
median	The central value, by count, of ordered statistical data.
midpoint	The point halfway between two given points.
minute (angle)	$\frac{1}{60}$ of a degree. *See also* second.
mode	The value that occurs most often in statistical data; the score with the highest frequency.
monic quadratic	Quadratic expression or equation where the coefficient of x^2 is 1.
multiplication principle	The rule that the probability of two events both occurring is the product of two probabilities.
mutually exclusive	Sets or events with no common elements.
negative skew	A greater spread of a frequency distribution below the median (to the left) than above the median.
nominal	Categorical data with no natural order, such as the names of countries.
nominal interest rate	Interest rate as an annual rate, especially when rests are not annual.
non-compliant data	Data that is obviously wrong, such as someone's age being 164.
non-monic quadratic	Quadratic expression or equation where the coefficient of x^2 is not 1.
null set	Set with no elements, shown as {} or ø. *See also* empty set.
numeric, numerical	Data that are numbers.

Word	Definition
opposite side	In a right-angled triangle, the short side not joining the right angle and opposite the nominated angle. *See also* adjacent side, hypotenuse.
ordinal	Categorical data having a natural order, such as places in a race.
outlier	A data value well outside the range of most numerical data.
parabola	A mathematical relation such as $y = 3x^2 + 2$. Its graph is a symmetric U-shape with sides curving away from each other.
perfect square	Square of an integer or expression that is the square of a linear expression. *See also* square of a sum, square of a difference.
period, periodic	A periodic function or phenomena repeats the same values after an interval of time called the period.
polygon	Closed 2D figure with straight sides.
polynomial	A (finite) sum of terms of the form ax^k (k a positive integer) such as $3x^5 - 4x^2 + 3$.
population	The entire group that could be relevant to a survey.
positive skew	A greater spread of a frequency distribution above the median (to the right) than below the median.
power	A number multiplied by itself, written as 5^4, say, for $5 \times 5 \times 5 \times 5$.
principal	The initial amount of a loan.
principal value	A single value of arccos or arcsin chosen from a domain that has been restricted to make the relation a function.
prism	Polyhedron where every cross-section parallel to one face is identical to that face.
probability	The chance of an event; experimental or theoretical probability.
proof	A series of mathematical statements that logically show that a statement is true. *See also* QED, RTP, theorem.
pyramid	Polyhedron such that cross-sections parallel to one face are similar and reduce evenly in size to a vertex called the apex opposite this face.
Pythagoras' theorem	Relationship between the sides of a right-angled triangle attributed to Pythagoras (569?–475? BCE).
Pythagorean triple (triad)	Three positive integers a, b, c such that $a^2 + b^2 = c^2$, that could be sides of a right-angled triangle.
QED	Quod erat demonstrandum; Latin phrase at the end of a formal proof to say it is finished. *See also* proof, RTP, theorem.
quadratic	Expression, equation or function with a squared term, of the form $ax^2 + bx + c\ (= 0)$.
quartile	Any of 3 points that divide an ordered data set into 4 equal groups. The highest element may be called the 4th quartile.

Word	Definition
quotient	The result of dividing one number or polynomial by another.
range	The difference between the highest and lowest scores in a set of numeric data.
rationalise	Make rational, such as rationalising the denominator of $\frac{5}{\sqrt{3}}$ to make $\frac{5\sqrt{3}}{3}$.
reducing interest	Interest calculated on the current balance of a loan. *See also* compound interest.
remainder theorem	The remainder when the polynomial $p(x)$ is divided by $x - a$ is $p(a)$.
rest	Time between regular calculations of compound interest.
RHS	Right-hand side.
rise	The vertical distance between two points. *See also* gradient.
root (1)	Reverse of raising to a power. A square root, cube root, fourth root, etc. *See also* surd.
root (2)	The solution of an equation.
RTP	Required to prove; statement of a theorem before its formal proof. *See also* proof, QED, theorem.
run	The horizontal distance between two points. *See also* gradient.
sample	Part of a population (usually used for a survey).
sample point	A basic probability outcome; '5' on a die. *See also* element.
sample space	List of all the simple outcomes in a situation. *See also* event space.
scale factor	The ratio of the sizes of an enlargement or reduction of a shape.
scatter plot	Graph with points representing the values of bivariate data as coordinates.
second (angle)	$\frac{1}{60}$ of a minute $= \frac{1}{3600}$ of a degree.
similar	Shapes that are enlargements or reductions of each other.
simple interest	Interest charged as a percentage of the initial loan or investment, multiplied by the period. *See also* flat-rate interest.
simplification, simplify	Multiply out brackets and/or collect like terms.
simultaneous equations	A system with two (or more) equations considered at the same time.
simultaneous solution	A solution that satisfies all the simultaneous equations.
sine, sin	The opposite side divided by the hypotenuse for a nominated angle in a right-angled triangle. *See also* cosine, tangent.
$\sin^{-1}$	*See* arcsin.
sine rule	The rule that for a triangle ABC, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.
sinusoidal	A graph that looks like that of a sine function.

Word	Definition
skewed (distribution)	An asymmetrical set of scores with an unequal number of scores above and below the mean.
skewed to the left	A greater spread of a frequency distribution to the left of the median (below) than to the right of the median.
skewed to the right	A greater spread of a frequency distribution to the right of the median (above) than to the left of the median.
slant height	The distance from the edge of the base of a cone to the apex.
slope	Property of a straight line obtained by dividing the vertical distance between two points (rise) by the horizontal (run). *See also* gradient.
solution	The answer to a problem, particularly an equation.
spread	Measure of the differences between the values of some data.
square of a difference	An expression of the form $(x - a)^2 = x^2 - 2ax + a^2$. *See also* perfect square.
square of a sum	An expression of the form $(x + a)^2 = x^2 + 2ax + a^2$. *See also* perfect square.
standard deviation	The square root of the average of the squares of the differences between data items and the mean of a numerical frequency distribution given by, $\sigma = \sqrt{\frac{\sum (x - \bar{x})^2}{n}}$ or $\sigma = \sqrt{\frac{\sum x^2}{n} - (\bar{x})^2}$.
standard form	Agreed format of an expression, equation, etc., such as $ax + by + c = 0$ or $ax^2 + bx + c = 0$.
stem-and-leaf plot	Display with high place values arranged vertically (stem) and the remaining place values arranged horizontally (leaves).
subject	The variable on its own (on the left) that is calculated from others in a formula.
subtend	The angle $\angle APB$ is subtended by the object AB at the point P, where A and B are points on the ends of AB, particularly where AB is an arc or chord.
surd (1)	The symbol $\sqrt{\ }$ used to show a root (1). *See also* root (1).
surd (2)	Root of a rational number (especially an integer) that is not rational.
surface area	The area of a surface, particularly the total area of the faces of a solid shape.
synthetic division	A method of dividing a polynomial by $x - a$ that uses only the coefficients.
tangent, tan	The opposite side divided by the adjacent side for a nominated angle in a right-angled triangle. *See also* cosine, sine.
theorem	Precise mathematical statement capable of formal proof consisting of the parts RTP, proof, QED.

Glossary

Word	Definition
transformation	Change in a shape: a rotation, reflection, enlargement, reduction or translation.
transpose	Change a formula to make a particular variable the subject.
tree diagram	Diagram with many branches showing the outcomes in a multi-stage situation.
trigonometric ratio	Commonly, any of cos θ, sin θ or tan θ for an angle θ.
trinomial	Quadratic expression in unfactorised form. *See also* quadratic.
turning point	Point on a curved line such as a parabola where it changes direction.
two means method	Method of finding a line of best fit using the means of the upper and lower halves of data points.
two-way table	A table that lists one set of outcomes horizontally and the other set vertically.
union	The set that contains all the elements in two given sets.
unit circle	Circle of radius 1, especially as used for trigonometric ratios.
unitary method	Method using the value of a unit quantity to calculate a result.
univariate	Data set where each item has only one measurement variable. *See also* bivariate.
universal set	Set of all elements under consideration in a particular context. *See also* Venn diagram.
Venn diagram	A diagram that shows the elements of events (or sets) in overlapping circles within a rectangle representing the universal set.
vertex	The intersection of two sides of a polygon or three or more faces of a solid shape; plural *vertices*.
vertically opposite (angles)	Opposite angles at the intersection of two lines; they are not adjacent.
***x*-intercept**	The point where a line crosses the x-axis, or the distance from this to the origin.
***y*-intercept**	The point where a line crosses the y-axis, or the distance from this to the origin.
zero	Value that makes a function equal to zero. An x-intercept of a line, especially a curved line.
zero product rule	The rule that if the product of two numbers is zero, at least one of them must be zero.

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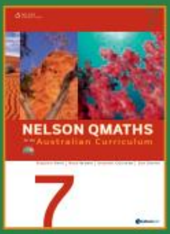

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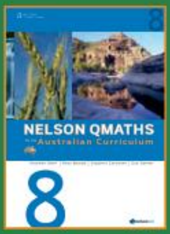

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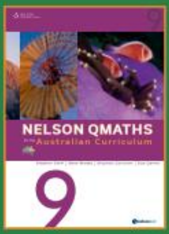

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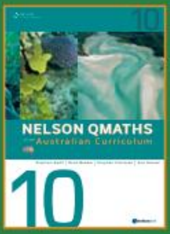

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ISBN: 978-0170194860
9 780170 194860