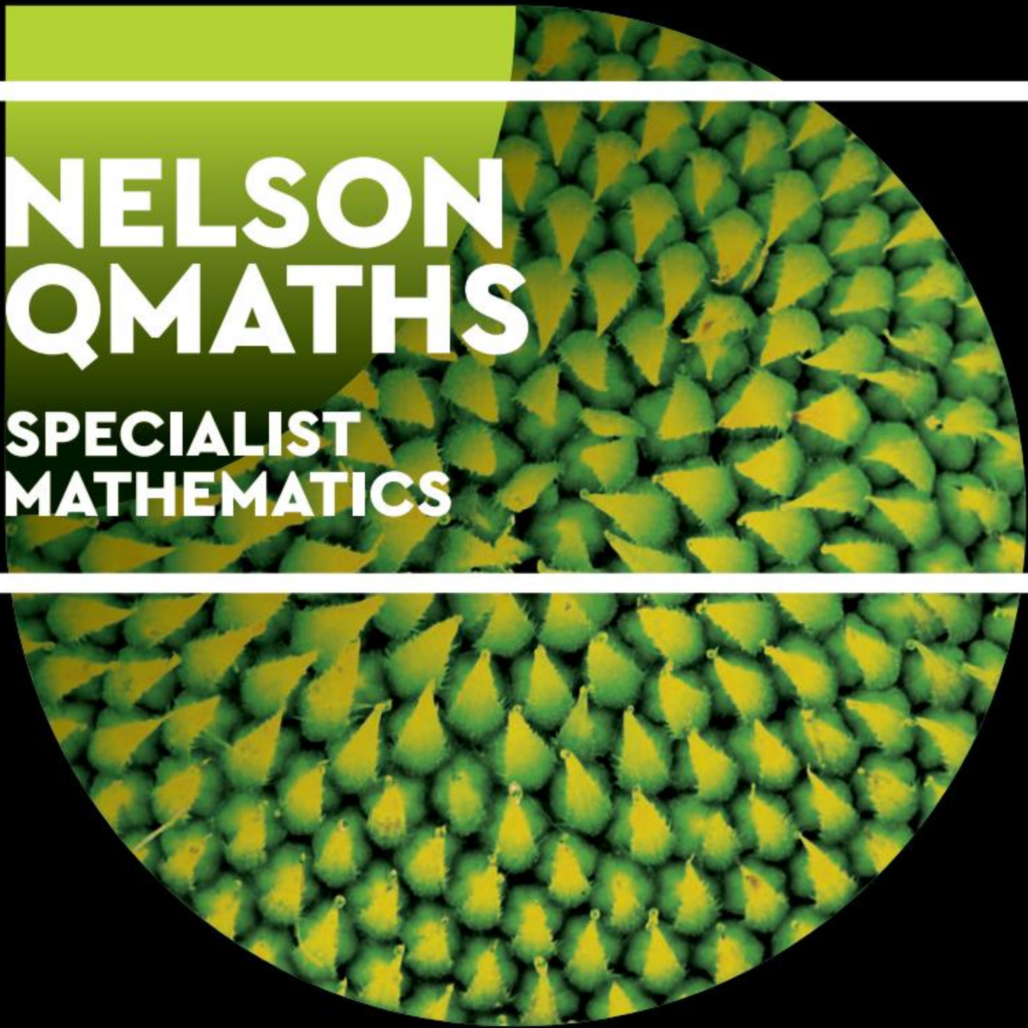
# NELSON QMATHS

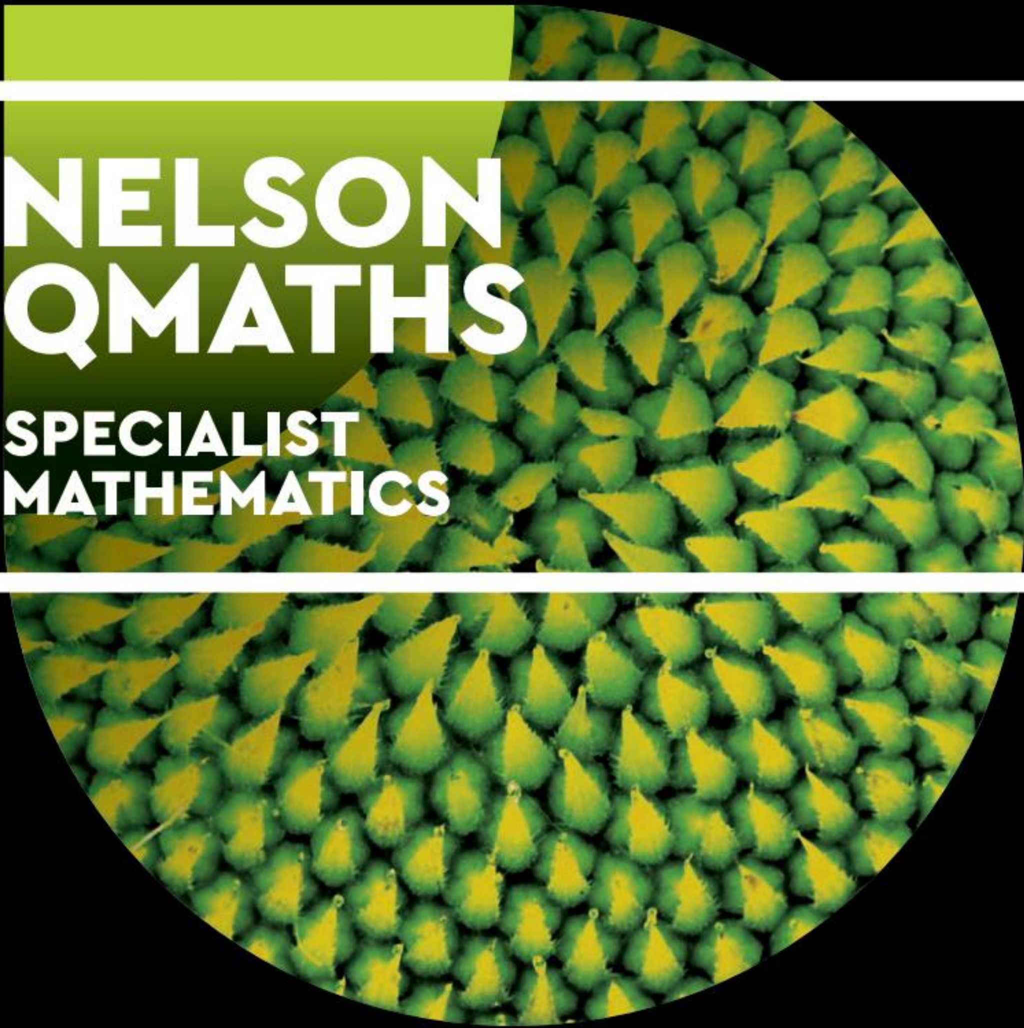
## SPECIALIST MATHEMATICS



YEAR 12

Ross Brodie
Regina Edwards
Stephen Swift

4TH EDITION



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NELSON
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**Nelson QMaths 12 Specialist Mathematics**
**4th Edition**
**Ross Brodie**
**Stephen Swift**
**Regina Edwards**
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# PREFACE

*Nelson QMaths*, Queensland's longest-running senior mathematics series, has been rewritten for the new syllabuses and assessment procedures for implementation from 2019. Based on the Australian Curriculum, four new senior mathematics courses have been introduced into Queensland schools.

- Essential Mathematics
- General Mathematics
- Mathematical Methods
- Specialist Mathematics

With the introduction of external QCE examinations, *Nelson QMaths* will have a renewed focus on assessment, and will include features such as topic tests, practice examinations, *ExamView* question banks and chapter quizzes, video tutorials and worked solutions to all exercise questions. In this book, teachers will find familiar features such as clear worked examples, graded exercises, strong syllabus coverage, Investigations, Technology, chapter summaries, chapter reviews and a glossary/index. We wish all teachers and students using this book every success in embracing the new mathematics courses.

# ABOUT THE AUTHORS

**Ross Brodie** has taught mathematics and been Head of Mathematics at a variety of regional and urban schools in Queensland. He is an experienced teacher and author, having co-written *Nelson QMaths 7–10*, *New QMaths 11–12* and *Nelson Senior Maths 11–12* (Australian Curriculum).

**Regina Edwards** teaches mathematics and science at Cleveland District State High School where she coordinates International Baccalaureate High Level Mathematics and Physics. She was a district panellist for Mathematics C and Physics, and has worked in the Information Technology industry as an analyst.

**Stephen Swift** has taught Mathematics, Science and Computing in urban and regional schools, and was Head of Mathematics at Wellington Point State High School. He is the lead author of *Nelson QMaths 7–10*, *New QMaths 11–12* and *Nelson Senior Maths 11–12* (Australian Curriculum).

## CONTRIBUTING AUTHORS

**Margaret Denham** wrote the topic tests.

**Margaret Denham** and **George Dimitriadis** wrote the worked solutions to all exercise sets.

**Roger Walter** wrote the *ExamView* questions.

**Scott Smith** created the video tutorials.


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# SYLLABUS REFERENCE GRID

| Topics and subtopics | *Nelson QMaths 12 Specialist Mathematics chapter* |
|---|---|
| **UNIT 3: MATHEMATICAL INDUCTION, AND FURTHER VECTORS, MATRICES AND COMPLEX NUMBERS** | |
| **Proof by mathematical induction** | |
| Mathematical induction | 1 Proof by mathematical induction |
| **Vectors and matrices** | |
| The algebra of vectors in three dimensions | 2 3D vectors |
| Vectors and Cartesian equations | 2 3D vectors |
| Systems of linear equations | 4 Vectors and matrices |
| Applications of matrices | 4 Vectors and matrices |
| Vector calculus | 6 Vector calculus |
| **Complex numbers 2** | |
| Cartesian forms | 3 Complex arithmetic |
| Complex arithmetic using polar form | 3 Complex arithmetic |
| The complex plane | 5 Complex polynomials and roots |
| Roots of complex numbers | 5 Complex polynomials and roots |
| Factorisation of polynomials | 5 Complex polynomials and roots |
| **UNIT 4: FURTHER CALCULUS AND STATISTICAL INFERENCE** | |
| **Integration and applications of integration** | |
| Integration techniques | 7 Further applications of integration |
| Applications of integral calculus | 7 Further applications of integration |
| **Rates of change and differential equations** | |
| Rates of change | 9 Differential equations |
| Modelling motion | 9 Differential equations |
| **Statistical inference** | |
| Sample means | 8 Sample means |
| Confidence intervals for means | 10 Confidence intervals for means |

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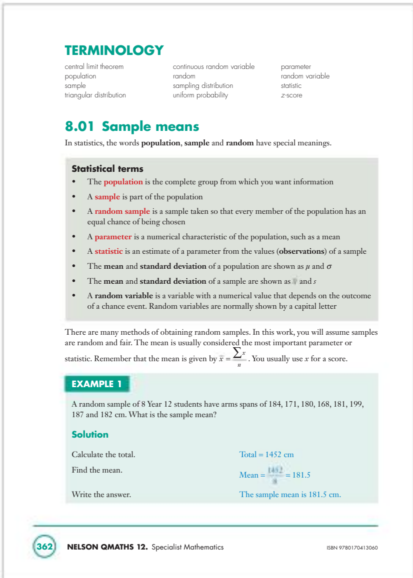
# ABOUT THIS BOOK

## AT THE BEGINNING OF EACH CHAPTER

- Each chapter begins on a double-page spread showing the **Chapter contents** and a list of syllabus subject matter (© Queensland Curriculum and Assessment Authority 2019)


- **Terminology** previews the key words and phrases from within the chapter.



TERMINOLOGY

central limit theorem
population
sample
triangular distribution
continuous random variable
random
sampling distribution
uniform probability
parameter
random variable
statistic
*z*-score

8.01 Sample means

In statistics, the words **population**, **sample** and **random** have special meanings.

**Statistical terms**

- The **population** is the complete group from which you want information
- A **sample** is part of the population
- A **random sample** is a sample taken so that every member of the population has an equal chance of being chosen
- A **parameter** is a numerical characteristic of the population, such as a mean
- A **statistic** is an estimate of a parameter from the values (**observations**) of a sample
- The **mean** and **standard deviation** of a population are shown as $\mu$ and $\sigma$
- The **mean** and **standard deviation** of a sample are shown as $\bar{x}$ and $s$
- A **random variable** is a variable with a numerical value that depends on the outcome of a chance event. Random variables are normally shown by a capital letter

There are many methods of obtaining random samples. In this work, you will assume samples are random and fair. The mean is usually considered the most important parameter or statistic. Remember that the mean is given by $\bar{x} = \frac{\sum x}{n}$. You usually use $x$ for a score.

**EXAMPLE 1**

A random sample of 8 Year 12 students have arms spans of 184, 171, 180, 168, 181, 199, 187 and 182 cm. What is the sample mean?

**Solution**

| | |
|---|---|
| Calculate the total. | Total = 1452 cm |
| Find the mean. | Mean = $\frac{1452}{8}$ = 181.5 |
| Write the answer. | The sample mean is 181.5 cm. |

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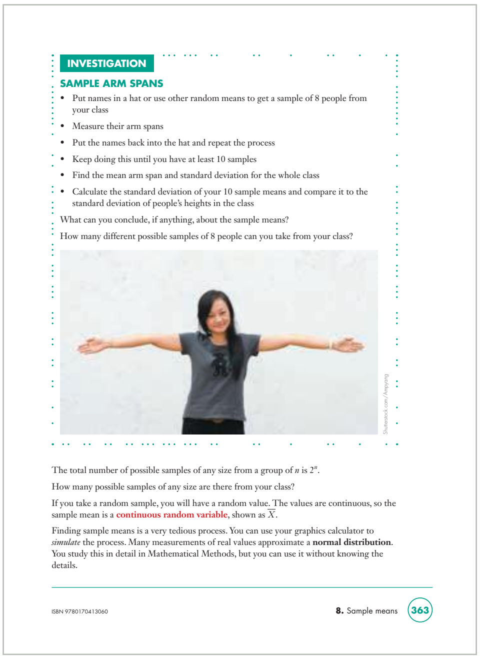
# IN EACH CHAPTER

- Worked examples are explained clearly step-by-step, with the mathematical working shown on the right-hand-side.
- Important facts and formulas are highlighted in a shaded box.
- Important words and phrases are printed in red and listed in the glossary at the back of the book.
- Graded exercises, including **Problem solving** questions, are linked to worked examples and include exam-style problems and realistic applications.
- Worked solutions to all exercise questions are provided on the *NelsonNet* teacher website.
- **Investigations** explore the syllabus in more detail, providing ideas for modelling activities and assessment tasks.
- **Technology** promotes ICT in the classroom, using spreadsheets, graphics calculators and the Internet.
- Instructions are provided for the TI-84 Plus CE, Casio fx-CG20AU and TI-Nspire CX calculators.



INVESTIGATION

SAMPLE ARM SPANS

- Put names in a hat or use other random means to get a sample of 8 people from your class
- Measure their arm spans
- Put the names back into the hat and repeat the process
- Keep doing this until you have at least 10 samples
- Find the mean arm span and standard deviation for the whole class
- Calculate the standard deviation of your 10 sample means and compare it to the standard deviation of people's heights in the class

What can you conclude, if anything, about the sample means?

How many different possible samples of 8 people can you take from your class?

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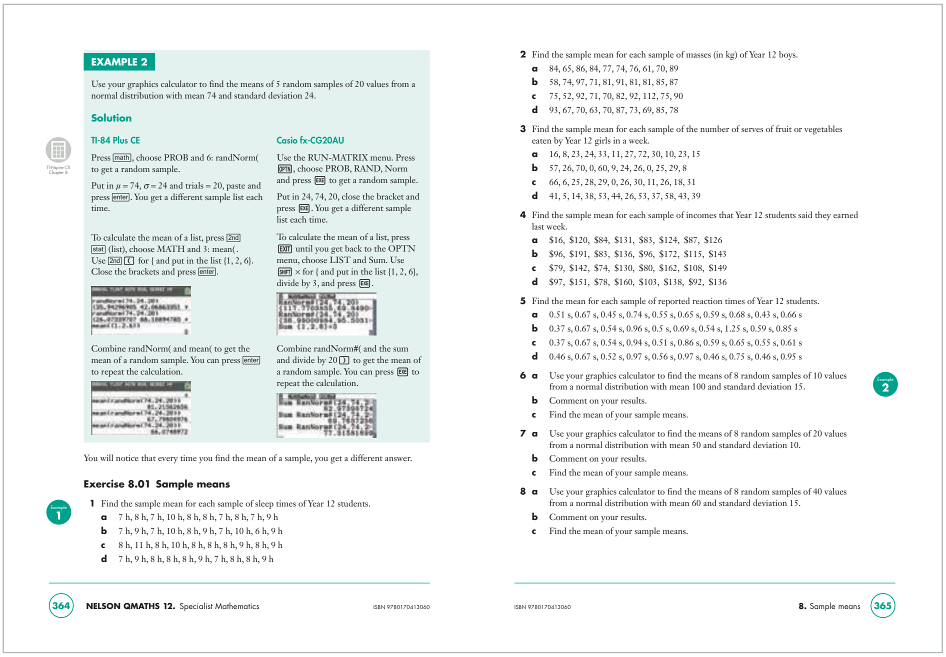
The total number of possible samples of any size from a group of $n$ is $2^n$.

How many possible samples of any size are there from your class?

If you take a random sample, you will have a random value. The values are continuous, so the sample mean is a **continuous random variable**, shown as $\overline{X}$.

Finding sample means is a very tedious process. You can use your graphics calculator to *simulate* the process. Many measurements of real values approximate a **normal distribution**. You study this in detail in Mathematical Methods, but you can use it without knowing the details.

ISBN 9780170413060 **8.** Sample means 363



EXAMPLE 2

Use your graphics calculator to find the means of 5 random samples of 20 values from a normal distribution with mean 74 and standard deviation 24.

Solution

TI-Nspire CX Chapter 8

TI-84 Plus CE

Press [math], choose PROB and 6: randNorm( to get a random sample.

Put in $\mu = 74$, $\sigma = 24$ and trials = 20, paste and press [enter]. You get a different sample list each time.

To calculate the mean of a list, press [2nd] [stat] (list), choose MATH and 3: mean(. Use [2nd] [(] for { and put in the list {1, 2, 6}. Close the brackets and press [enter].

Combine randNorm( and mean( to get the mean of a random sample. You can press [enter] to repeat the calculation.

Casio fx-CG20AU

Use the RUN-MATRIX menu. Press [OPTN], choose PROB, RAND, Norm and press [EXE] to get a random sample.

Put in 24, 74, 20, close the bracket and press [EXE]. You get a different sample list each time.

To calculate the mean of a list, press [EXIT] until you get back to the OPTN menu, choose LIST and Sum. Use [SHIFT] × for { and put in the list {1, 2, 6}, divide by 3, and press [EXE].

Combine randNorm#( and the sum and divide by 20 [)] to get the mean of a random sample. You can press [EXE] to repeat the calculation.

You will notice that every time you find the mean of a sample, you get a different answer.

Exercise 8.01 Sample means

Example 1

1 Find the sample mean for each sample of sleep times of Year 12 students.
a 7 h, 8 h, 7 h, 10 h, 8 h, 8 h, 7 h, 8 h, 7 h, 9 h
b 7 h, 9 h, 7 h, 10 h, 8 h, 9 h, 7 h, 10 h, 6 h, 9 h
c 8 h, 11 h, 8 h, 10 h, 8 h, 8 h, 8 h, 9 h, 8 h, 9 h
d 7 h, 9 h, 8 h, 8 h, 8 h, 9 h, 7 h, 8 h, 8 h, 9 h

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2 Find the sample mean for each sample of masses (in kg) of Year 12 boys.
a 84, 65, 86, 84, 77, 74, 76, 61, 70, 89
b 58, 74, 97, 71, 81, 91, 81, 81, 85, 87
c 75, 52, 92, 71, 70, 82, 92, 112, 75, 90
d 93, 67, 70, 63, 70, 87, 73, 69, 85, 78

3 Find the sample mean for each sample of the number of serves of fruit or vegetables eaten by Year 12 girls in a week.
a 16, 8, 23, 24, 33, 11, 27, 72, 30, 10, 23, 15
b 57, 26, 70, 0, 60, 9, 24, 26, 0, 25, 29, 8
c 66, 6, 25, 28, 29, 0, 26, 30, 11, 26, 18, 31
d 41, 5, 14, 38, 53, 44, 26, 53, 37, 58, 43, 39

4 Find the sample mean for each sample of incomes that Year 12 students said they earned last week.
a \$16, \$120, \$84, \$131, \$83, \$124, \$87, \$126
b \$96, \$191, \$83, \$136, \$96, \$172, \$115, \$143
c \$79, \$142, \$74, \$130, \$80, \$162, \$108, \$149
d \$97, \$151, \$78, \$160, \$103, \$138, \$92, \$136

5 Find the mean for each sample of reported reaction times of Year 12 students.
a 0.51 s, 0.67 s, 0.45 s, 0.74 s, 0.55 s, 0.65 s, 0.59 s, 0.68 s, 0.43 s, 0.66 s
b 0.37 s, 0.67 s, 0.54 s, 0.96 s, 0.5 s, 0.69 s, 0.54 s, 1.25 s, 0.59 s, 0.85 s
c 0.37 s, 0.67 s, 0.54 s, 0.94 s, 0.51 s, 0.86 s, 0.59 s, 0.65 s, 0.55 s, 0.61 s
d 0.46 s, 0.67 s, 0.52 s, 0.97 s, 0.56 s, 0.97 s, 0.46 s, 0.75 s, 0.46 s, 0.95 s

6 a Use your graphics calculator to find the means of 8 random samples of 10 values from a normal distribution with mean 100 and standard deviation 15. (Example 2)
b Comment on your results.
c Find the mean of your sample means.

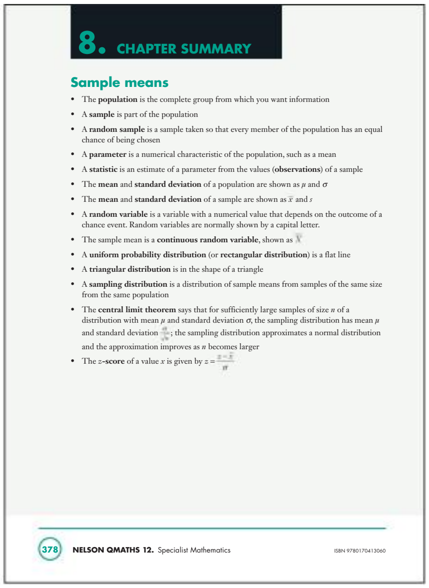
7 a Use your graphics calculator to find the means of 8 random samples of 20 values from a normal distribution with mean 50 and standard deviation 10.
b Comment on your results.
c Find the mean of your sample means.

8 a Use your graphics calculator to find the means of 8 random samples of 40 values from a normal distribution with mean 60 and standard deviation 15.
b Comment on your results.
c Find the mean of your sample means.

ISBN 9780170413060 **8.** Sample means 365

# AT THE END OF EACH CHAPTER

- **Chapter summary** summarises the key concepts, skills and formulas of the chapter.
- **Chapter review** contains revision exercises and are linked to worked examples.
- **Practice examinations** after every 2 to 3 chapters revise the skills and knowledge of those chapters.



8. CHAPTER SUMMARY

Sample means

- The **population** is the complete group from which you want information
- A **sample** is part of the population
- A **random sample** is a sample taken so that every member of the population has an equal chance of being chosen
- A **parameter** is a numerical characteristic of the population, such as a mean
- A **statistic** is an estimate of a parameter from the values (**observations**) of a sample
- The **mean** and **standard deviation** of a population are shown as $\mu$ and $\sigma$
- The **mean** and **standard deviation** of a sample are shown as $\bar{x}$ and $s$
- A **random variable** is a variable with a numerical value that depends on the outcome of a chance event. Random variables are normally shown by a capital letter.
- The sample mean is a **continuous random variable**, shown as $\bar{X}$
- A **uniform probability distribution** (or **rectangular distribution**) is a flat line
- A **triangular distribution** is in the shape of a triangle
- A **sampling distribution** is a distribution of sample means from samples of the same size from the same population
- The **central limit theorem** says that for sufficiently large samples of size $n$ of a distribution with mean $\mu$ and standard deviation $\sigma$, the sampling distribution has mean $\mu$ and standard deviation $\frac{\sigma}{\sqrt{n}}$; the sampling distribution approximates a normal distribution and the approximation improves as $n$ becomes larger
- The $z$-**score** of a value $x$ is given by $z = \frac{x - \bar{x}}{\sigma}$

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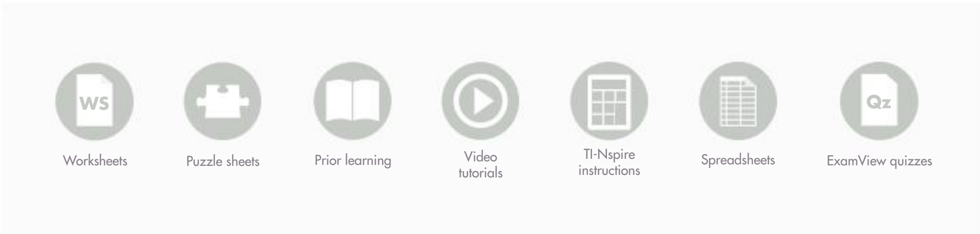
# AT THE END OF THE BOOK

- **Glossary and index** includes a comprehensive dictionary of course terminology.
- **Answers** (with worked solutions on the teacher website).

# NELSONNET STUDENT WEBSITE

Margin icons link to print (PDF) and multimedia resources found on the *NelsonNet* student website, **www.nelsonnet.com.au**. These include:




- **Worksheets** and **puzzle sheets** that are write-in enabled PDFs
- **Prior learning** exercises that revise prerequisite skills for the chapter
- **Video tutorials:** worked examples explained by 'flipped classroom' teachers
- **Graphics calculator instructions** for TI-Nspire CX calculators
- **Spreadsheets:** *Excel* files
- **ExamView quizzes:** interactive and self-marking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# NELSONNET TEACHER WEBSITE

The *NelsonNet* teacher website, also at **www.nelsonnet.com.au**, contains:

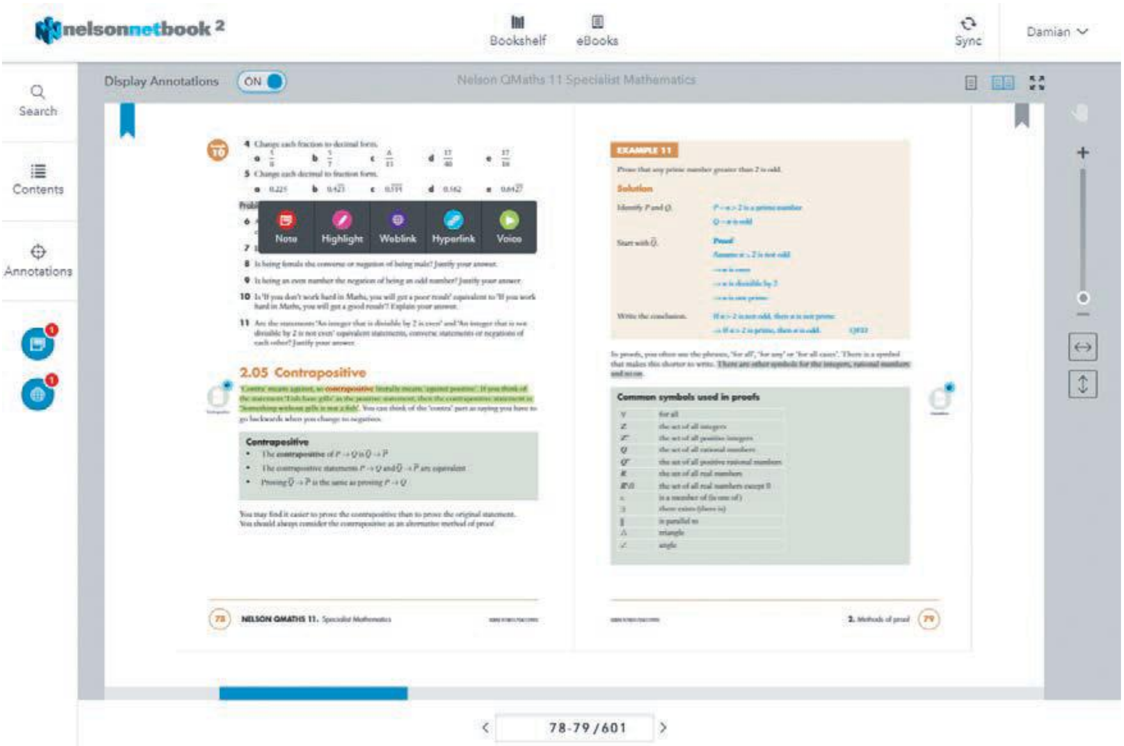
- **Teaching plan**, in Microsoft Word and PDF formats
- **Topic tests**, in Microsoft Word and PDF formats
- **Worked solutions** to each exercise set
- **Chapter PDFs** of the textbook
- **ExamView** exam-writing software and question banks
- **Resource Finder:** search engine for *NelsonNet* resources

Note: Complimentary access to these resources is only available to teachers who use this book as a core educational resource in their classroom. Contact your Cengage Education Consultant for information about access codes and conditions.

# NELSONNETBOOK

**NelsonNetBook** is the web-based interactive version of this book found on *NelsonNet.*

- To each page of NelsonNetBook you can add notes, voice and sound bites, highlighting, weblinks and bookmarks
- **Zoom** and **Search** functions
- Chapters can be customised for different groups of students

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# LIST OF SYMBOLS AND ABBREVIATIONS

| | |
|---|---|
| $=$ | is equal to |
| $\equiv$ | is identically equal to, is congruent to |
| $\neq$ | is not equal to |
| $\approx$ | is approximately equal to |
| $<$ | is less than |
| $>$ | is greater than |
| $\leq$ | is less than or equal to |
| $\geq$ | is greater than or equal to |
| $\vert\vert\vert$ | is similar to |
| $\Vert$ | is parallel to |
| $\in$ | is a member of |
| $\cup$ | union |
| $\cap$ | intersection |
| $\pm$ | plus or minus |
| $\pi$ | pi $\approx$ 3.14 |
| $\subseteq$ | is a subset of |
| $\subset$ | is a proper subset of |
| $\notin$ | is not a member of |
| $\Sigma, \sum_{i=3}^{22} a_i$ | the sum of, the sum of the values of $a_i$ from $a_3$ to $a_{22}$ |
| … | and so on |
| $\{\}, \varnothing$ | the null set |
| $U, \mathbb{E}$ | the universal set |
| $\boldsymbol{C}$ | the complex numbers |
| $\boldsymbol{R}$ | the real numbers |
| $\boldsymbol{R}^+$ | the positive real numbers |
| $\boldsymbol{R}\backslash\{0\}$ | the real numbers excluding 0 |
| $\boldsymbol{Z}, \mathcal{J}$ | the integers |
| $\boldsymbol{Z}^+$ | the positive integers |
| $\boldsymbol{N}$ | the natural numbers |
| $\circ$ | degree |
| $\infty$ | infinity |
| $\Rightarrow$ | implies |
| $\Leftrightarrow$, iff | if and only if |
| $\propto$ | is proportional to |
| $\Delta$ | the discriminant |

| | |
|---|---|
| RTP | required to prove |
| QED | quod erat demonstrandum, statement that a proof is finished |
| $P(k)$ | proposition for the value $k$, like $k^2 + k + 1$ is prime |
| $\sqrt{\ }$ | square root, radical sign |
| $\bar{A}, A'$ | not $A$, the complement of $A$ |
| $\vert A\vert, n(A)$ | the number of elements of set $A$ |
| : | such that |
| LHS | left-hand side |
| RHS | right-hand side |
| $\vert a\vert$ | absolute value of $a$ |
| $m$ | gradient of a straight line |
| $c$ | $y$-intercept of a straight line |
| $(x, y)$ | Cartesian coordinates, ordered pair |
| $^nP_r$ | the number of permutations of $r$ objects from $n$ objects |
| $^nC_r, \binom{n}{r}$ | the number of combinations of $r$ objects from $n$ objects |
| $\log_{10}$ | common logarithm |
| $\ln(x), \log_e(x)$ | the natural logarithm of $x$ |
| $\log_a b$ | the logarithm of $b$ to the base $a$ |
| $e$ | the base of the exponential function, 2.718 281… |
| $e^x$ | the exponential function |
| cos | trigonometric cosine ratio |
| sin | trigonometric sine ratio |
| tan | trigonometric tangent ratio |
| cot | cotangent ratio |
| sec | secant ratio |
| cosec | cosecant ratio |
| $\sin^{-1}(a)$, arcsin $(a)$ | inverse sine, the angle $A$ whose sine is $a$ |
| $\cos^{-1}(a)$, arccos $(a)$ | inverse cosine, the angle $A$ whose cosine is $a$ |
| $\tan^{-1}(a)$, arctan $(a)$ | inverse tangent, the angle $A$ whose tangent is $a$ |
| $\forall$ | for all |


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| | |
|---|---|
| $\exists$ | there exists |
| $\mathbf{p}$, $\underset{\sim}{p}$ | the vector **p** |
| $\lvert\mathbf{p}\rvert$, $p$ | the magnitude (norm) of the vector **p** |
| **AB** | displacement vector from point $A$ to point $B$ |
| **OA**, **a**, $\overline{OA}$ | the position vector of the point $A$ |
| $(r, \theta)$, $\begin{bmatrix} r \\ \theta \end{bmatrix}$ | polar form of the vector with magnitude $r$ and direction $\theta$ |
| $(x, y)$, $\begin{bmatrix} x \\ y \end{bmatrix}$ | component (Cartesian) form of vector |
| **i**, **j**, **k** | unit vectors in the $x$, $y$ and $z$ directions |
| $(x, y, z)$, $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ | component (Cartesian) form of a vector in 3 dimensions, ordered triple |
| $(r, \theta, \phi)$ | polar form of the vector in 3D with magnitude $r$, azimuth $\theta$ and altitude $\phi$ |
| $-\mathbf{a}$ | additive inverse of the vector **a** |
| $\hat{\mathbf{p}}$ | unit vector in the direction of the vector **p** |
| $\Delta\mathbf{v}$ | change in the vector **v** |
| $\mathbf{a} \bullet \mathbf{b}$ | the scalar product (dot product) of the vectors **a** and **b** |
| $\mathbf{a} \times \mathbf{b}$ | the vector product (cross product) of the vectors **a** and **b** |
| $\perp$, ⊾ | perpendicular |
| $\Gamma$ | plane |
| $\angle$ | angle |
| $\vartriangle$ | triangle |
| SSS | side, side, side test of triangle congruency |
| SAS | side, angle, side test of triangle congruency |
| AAS | angle, angle, side test of triangle congruency |
| RHS | right angle, hypotenuse, side test of triangle congruency |
| **A**, $\underset{\sim}{A}$ | the matrix **A** |
| $\mathbf{A} = \left(a_{ij}\right)_{2\times3} = \begin{bmatrix} 3 & -2 & 5 \\ -4 & 1 & 0 \end{bmatrix}$ | the $2 \times 3$ matrix **A** with elements $a_{ij}$ |
| $-\mathbf{A}$ | the additive inverse of matrix **A** |

| | |
|---|---|
| $\mathbf{I}_n$ | the $n \times n$ identity matrix |
| $\delta_{ij}$ | the Kronecker delta function |
| $\mathbf{A}^{-1}$ | the multiplicative inverse of the (square) matrix **A** |
| det **A**, $\lvert\mathbf{A}\rvert$, $A$ | the determinant of the matrix **A** |
| $\mathbf{T}: (x, y) \to (x', y')$ | the transformation that changes the point $(x, y)$ to $(x', y')$ |
| **D** | a dilation |
| $\lambda_1, \lambda_2$ | the dilation factors in the $x$ and $y$ directions for a dilation |
| $\mathbf{R}_\alpha$ | a rotation around the origin through an angle $\alpha$ |
| **M** | a reflection across a line through the origin |
| $\mathbf{S} \circ \mathbf{T}$ | the composition of the transformations **S** and **T** |
| $\mathbf{T}^{-1}$ | the inverse of the transformation **T** |
| $i$ | the imaginary number $\left(\sqrt{-1}\right)$ |
| Re $(z)$ | the real part of the complex number $z$ |
| Im $(z)$ | the imaginary part of the complex number $z$ |
| $\overline{z}$ | the complex conjugate of $z$ |
| $Arg\ (z)$ | the argument of $z$ |
| $\lvert z\rvert$, mod$(z)$ | the modulus of the complex number $z$ |
| cis $(\theta)$ | cos $(\theta)$ + $i$ sin $(\theta)$ |
| $\mathbf{D}^{\mathrm{T}}$, $\mathbf{D}^{\mathrm{t}}$ | the transpose of **D** |
| **r**, $\mathbf{r}_3$ | first order ranking, 3rd order ranking |
| $\mathbf{X}_3$ | Leslie population vector after 3 years |
| $\mathbf{X}_3$ | Markov state vector after 3 transitions |
| $\dot{\mathbf{r}}$ | velocity, derivative of **r** with respect to time |
| $\ddot{\mathbf{r}}$ | acceleration, second derivative of **r** with respect to time |
| $\omega$ | angular velocity |
| $g$ | acceleration of gravity, about 9.8 m/s$^2$ |
| cdf | cumulative distribution function |
| pdf | probability density function |
| $\mu$ | population mean |
| $\overline{x}$ | sample mean |
| $\sigma$, $\sigma_n$ | population standard deviation |
| $s$, $\sigma_{n-1}$ | sample standard deviation |
| $\sigma_{\overline{X}}$ | standard deviation of the sampling distribution $\overline{X}$ |

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# LIST OF MATHEMATICAL VERBS

## A glossary of 'doing words' commonly found in mathematics problems

**analyse:** study and state in detail the relationship of parts of a situation

**apply:** use knowledge or a procedure in a given situation

**calculate:** find a numerical value

**comment:** express an opinion or judgement about a statement or calculation

**communicate:** transmit information to others

**compare:** state similarities and differences and their significance

**consider:** take into account

**construct:** draw an accurate diagram or logically arrange items or ideas

**convert:** change from one form to another

**define:** give the meaning of or identify in exact terms

**demonstrate:** show to be correct

**describe:** state the features of a situation, object, pattern, event, etc.

**determine:** find the answer or make a decision using evidence

**discuss:** give reasons based on evidence for and against a position or proposal

**establish:** introduce and develop a result

**evaluate:** find the value of or state the application, strengths and limitations of a solution

**examine:** state the details and assumptions of a situation

**expand:** remove brackets or change a product to an extended sum of terms

**explain:** state the meaning in logical detail

**explore:** examine or state the details and assumptions of a situation

**factorise:** convert to factors or change a sum of terms to a product of factors

**formulate:** give mathematical expression to an idea or situation

**hence find/prove:** find an answer or prove a result using previous answers or information supplied

**identify:** state the type, name or distinguishing feature of an item or situation

**interpret:** state a conclusion or trend from given information

**investigate:** establish facts, trends or conclusions from collected information

**justify:** give reasons or evidence for an answer or conclusion


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**modify:** change to accommodate different information

**obtain:** find an answer or conclusion

**prove:** use logical steps to establish the truth of

**recall:** remember (and state)

**recognise:** use knowledge to identify features of a situation

**rationalise:** make rational, remove surds

**show that:** in questions where the answer is given, to use mathematical reasoning to prove that the answer is true

**simplify:** reduce the size of numbers in a fraction, or reduce the size of an algebraic expression

**sketch:** draw a diagram that shows the general shape and includes relevant features

**solve:** find the answer or explanation for a problem, particularly the values of variables

**substitute:** replace a variable by a number to calculate an answer

**translate (to mathematical form):** express a situation as mathematical relationships

**verify:** check a solution or result, usually referring to the given situation

**write/state:** give the answer, formula or result without showing any working or explanation (This usually means that the answer can be found mentally, or in one step)

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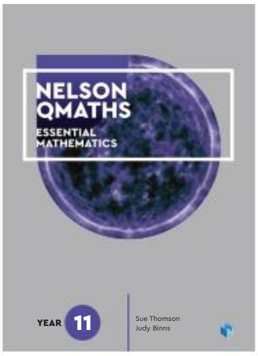
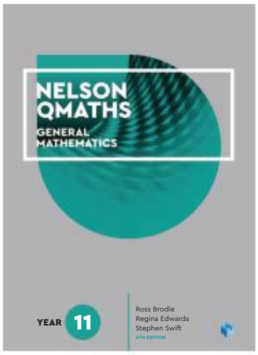
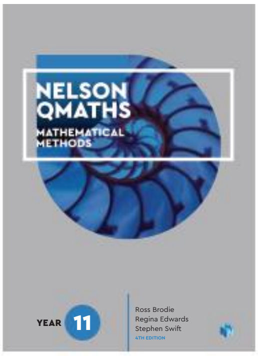
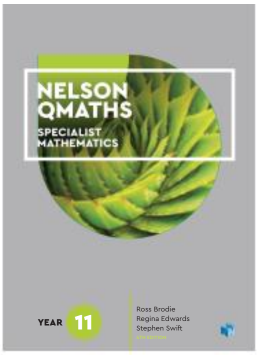
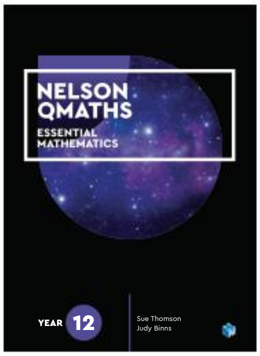
# NELSON QMATHS 11–12 SERIES

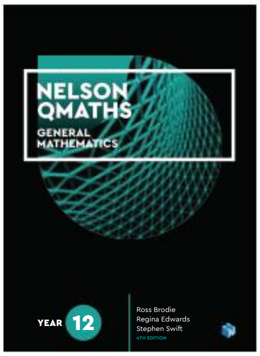


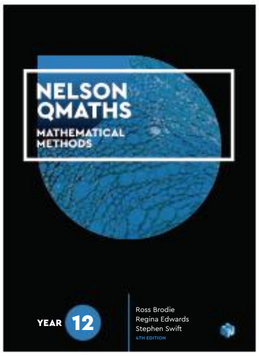



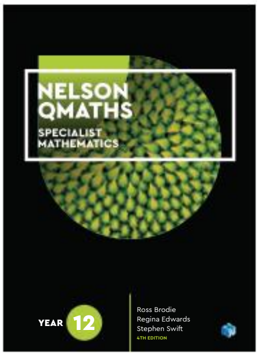





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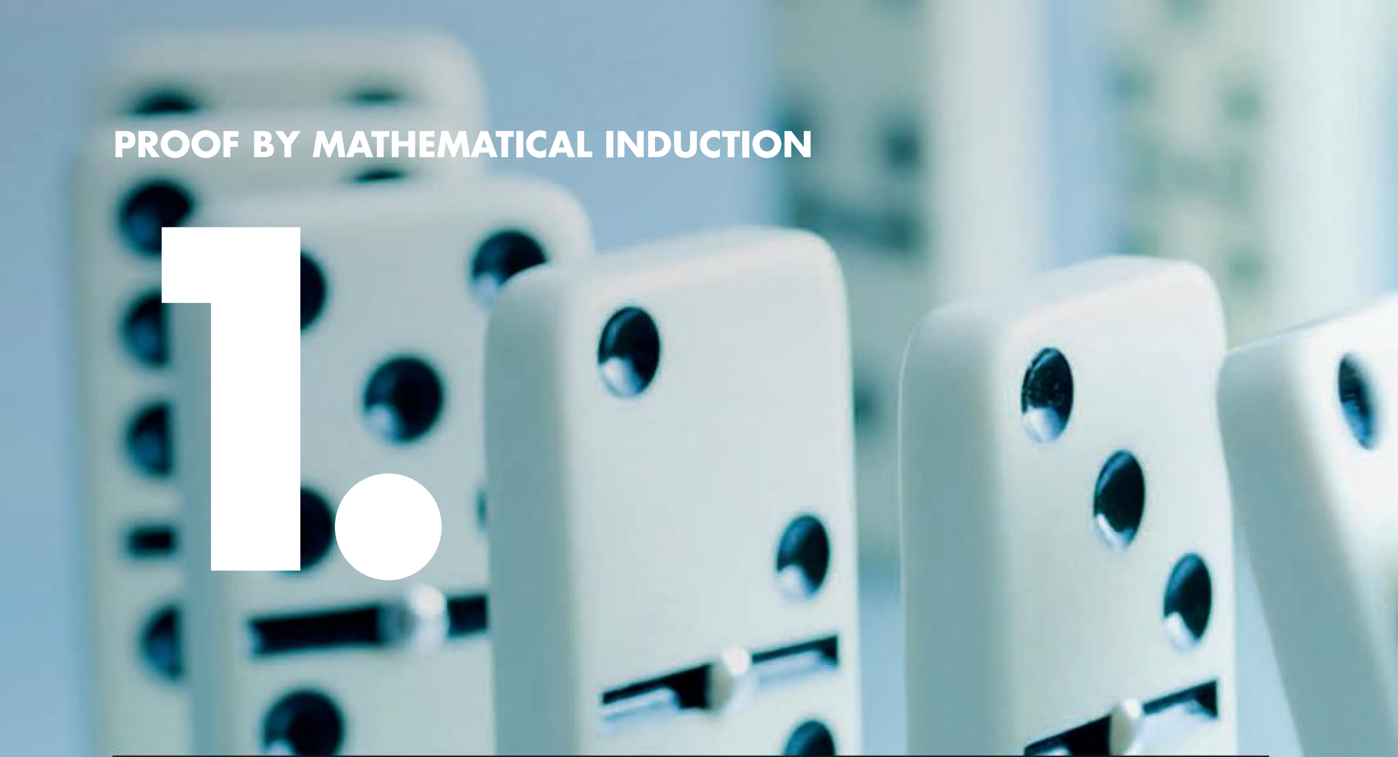


1.

# MATHEMATICAL INDUCTION

Some students tried to create a function that is capable of generating only prime numbers:

$p(n) = n^2 + n + 11$

The results for the first 9 positive integer values of $n$ are shown below.

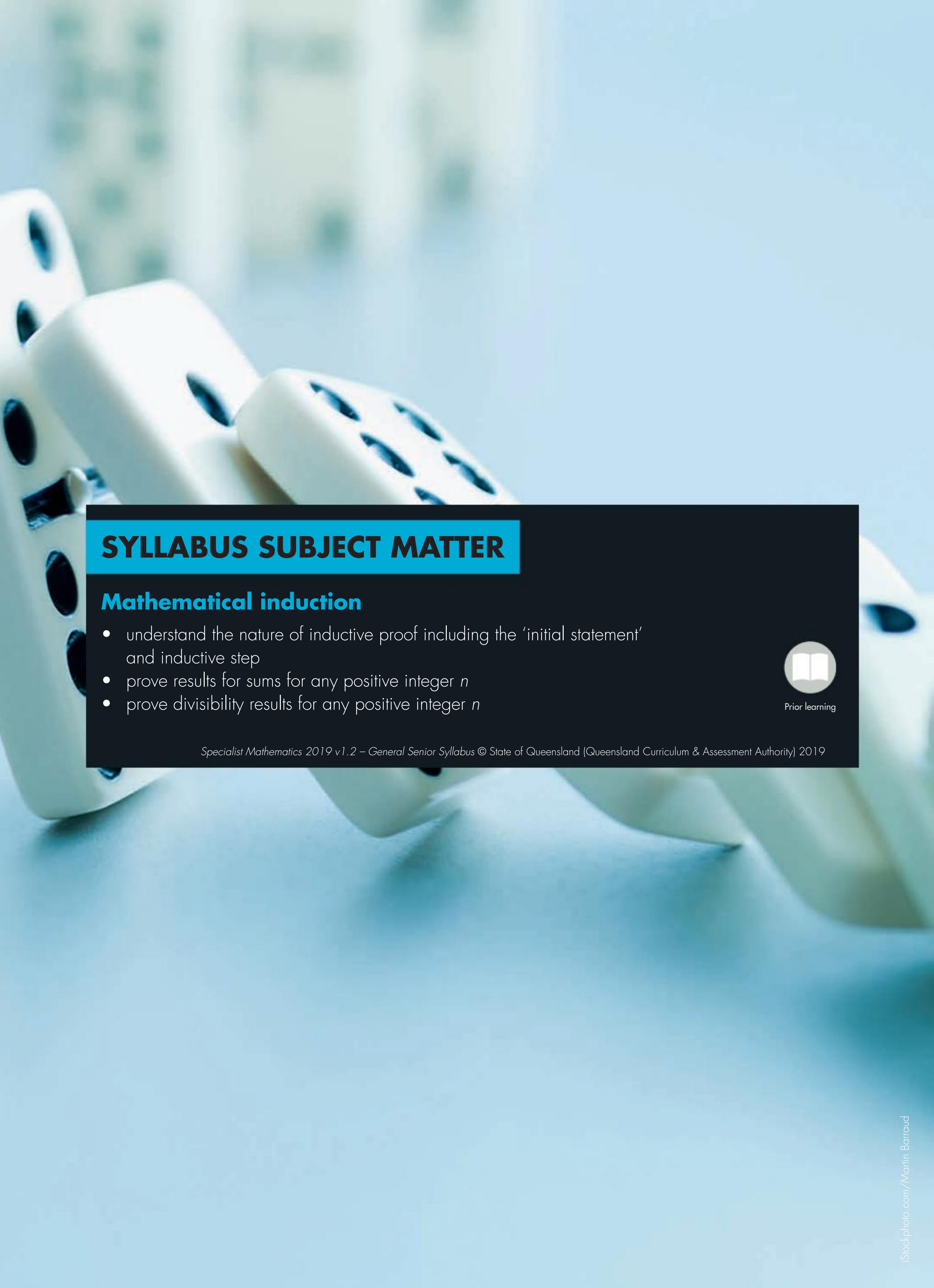
| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| $p(n)$ | 13 | 17 | 23 | 31 | 41 | 53 | 67 | 83 | 101 |

These results look very promising. However, $n = 10$ gives 121, which is $11 \times 11$ and not a prime number.

The mathematicians Euler and Fermat spent a great deal of time trying to develop a prime number formula. They did not succeed either.

Simply substituting values will not prove that a formula is correct for *all* positive integers. The principle of **mathematical induction** is commonly used to prove something is true for all positive integer values of $n$.

1.01 The principle of mathematical induction
1.02 Proofs of integer properties
1.03 Proofs of sums
1.04 Divisibility results
Chapter summary
Chapter review



## SYLLABUS SUBJECT MATTER

### Mathematical induction

- understand the nature of inductive proof including the 'initial statement' and inductive step
- prove results for sums for any positive integer $n$
- prove divisibility results for any positive integer $n$

Prior learning



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## TERMINOLOGY

arithmetic progression
consecutive
even number
geometric progression
inductive hypothesis
inductive method
inductive step
initial step
integer
mathematical induction
natural number
odd number
proof
proposition
QED

Proof by mathematical induction

# 1.01 The principle of mathematical induction

### Mathematical induction

Suppose that a proposition $P(n)$ is defined for every **integer** $n \geq a$, where $a \in Z$.

| | |
|---|---|
| If | $P(a)$ is true |
| and | if $P(k)$ is true then $P(k + 1)$ is also true, |
| then | $P(n)$ is true for all integers $n \geq a$. |

You can think of the principle of **mathematical induction** as being a bit like the *domino effect*. Imagine an infinite set of dominoes lined up as shown below.

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| | |
|---|---|
| If | the first domino topples backward, |
| and | if the $k$th domino topples, then the $(k + 1)$th domino will also topple, |
| then | *all* dominoes will topple. |

The method of mathematical induction is often used in problems where you observe common quantities for $n = 1, 2, 3, \ldots$ and wish to prove a **proposition**, $P(n)$, to describe the general case.

**Proof** by mathematical induction consists of 2 essential parts:

1 the **initial step** showing that the proposition holds for an initial value, such as $n = 1$

2 the **inductive step** showing that if the proposition holds for $n = k$, then the proposition holds when $n = k + 1$.


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The **inductive hypothesis** is the assumption that the proposition is true when $n = k$. To perform the inductive step, you use $P(k)$ to prove the proposition for $n = k + 1$.

## Proof by mathematical induction

**Step 1:** Under RTP (required to prove) state what has to be proved.

**Step 2:** Prove that the proposition is true for the initial value, usually $n = 1$.

**Step 3:** Assume the proposition is true for $n = k$, where $k \in Z^+$.

**Step 4:** Show that it necessarily follows that the proposition is true for $n = k + 1$.

**Step 5:** Make a formal statement such as 'By the principle of mathematical induction $P(n)$ is true for all $n \in Z^+$', followed by **QED**.

The set of **natural numbers** (or positive integers) has the symbol $\mathbf{N}$ or $Z^+$.

Step 2 is the **initial step** while steps 3 and 4 together make up the **inductive step**.

## EXAMPLE 1

Use the method of mathematical induction to prove:

$P(n)$: $1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$ for all $n \Rightarrow Z^+$.

### Solution

State what has to be proved.

**RTP**

$P(n)$: $1 + 3 + 5 + 7 + \ldots + (2n - 1) = n^2$ for all $n \in Z^+$.

The initial step: prove true for $n = 1$.

$\text{LHS} = 2 \times 1 - 1 = 1$ $\quad$ $\text{RHS} = 1^2 = 1$

LHS = RHS

$\therefore P(1)$ is true.

The inductive step: prove $P(k) \Rightarrow P(k + 1)$. First, assume for $n = k$, where $k \in Z^+$.

Assume $P(k)$ is true.

State $P(k)$.

$P(k)$: $1 + 3 + 5 + 7 + \ldots + (2k - 1) = k^2$, $k \in Z^+$

Consider $P(k + 1)$: $1 + 3 + 5 + 7 + \ldots + (2k - 1) + [2(k + 1) - 1] = (k + 1)^2$.

Add the next term to the LHS of $P(k)$.

$$\underbrace{1 + 3 + 5 + 7 + \ldots + (2k - 1)}_{k^2} + [2(k + 1) - 1]$$

$$= k^2 + [2(k + 1) - 1] \quad \text{(using } P(k)\text{)}$$

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| | |
|---|---|
| Expand and express as the RHS of $P(k+1)$. | $= k^2 + 2k + 2 - 1$ |
| Simplify. | $= k^2 + 2k + 1$ |
| Factorise. | $= (k+1)^2$ |
| | $\therefore P(k+1)$ is true. |
| Make a formal statement for $P(n)$. | By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$. **QED** |

## EXAMPLE 2

Use the method of mathematical induction to prove:

$$P(n): 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(2n+1)(n+1)}{6} \text{ for all } n \in \mathbb{Z}^+.$$

### Solution

State what has to be proved.

**RTP**

$$P(n): 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(2n+1)(n+1)}{6}$$

for all $n \in \mathbb{Z}^+$.

Prove true for $n = 1$.

$$\text{LHS} = 1^2 = 1 \qquad \text{RHS} = \frac{1(2\times 1+1)(1+1)}{6} = 1$$

LHS = RHS

$\therefore P(1)$ is true.

Prove $P(k) \Rightarrow P(k+1)$: first assume true for $n = k$, where $k \in \mathbb{Z}^+$.

Assume $P(k)$ is true.

State $P(k)$.

$$P(k): 1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(2k+1)(k+1)}{6}$$

Consider $P(k+1)$: $1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \dfrac{(k+1)[2(k+1)+1][(k+1)+1]}{6}$.

Add the next term to the LHS of $P(k)$.

$$\underbrace{1^2 + 2^2 + 3^2 + \ldots + k^2} + (k+1)^2$$

$$= \frac{k(2k+1)(k+1)}{6} + (k+1)^2 \qquad \text{(using } P(k)\text{)}$$


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Simplify to express as the RHS of $P(k+1)$.

$$= \frac{k(2k+1)(k+1)+6(k+1)^2}{6}$$

Factorise.

$$= \frac{(k+1)[k(2k+1)+6(k+1)]}{6}$$

$$= \frac{(k+1)(2k^2+k+6k+6)}{6}$$

$$= \frac{(k+1)(2k^2+7k+6)}{6}$$

$$= \frac{(k+1)(2k+3)(k+2)}{6}$$

Express in exact form of RHS.

$$= \frac{(k+1)[2(k+1)+1][(k+1)+1]}{6}$$

$\therefore$ $P(k+1)$ is true.

Make a formal statement for $P(n)$.

By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbf{Z}^+$. **QED**

## EXAMPLE 3

Use the method of mathematical induction to prove:

$P(n)$: $3^n > 3 \times 2^n$ for all $n \geq 3$ where $n \in \mathbf{N}$.

### Solution

State what has to be proved.

**RTP**

$P(n)$: $3^n > 3 \times 2^n$ for all $n \in \mathbf{N}$ and $n \geq 3$.

Prove true for $n = 3$.

LHS $= 3^3$
$= 27$

RHS $= 3 \times 2^3$
$= 24$

LHS > RHS

$\therefore$ $P(3)$ is true.

Prove $P(k) \Rightarrow P(k+1)$: first assume true for $n = k$, where $k > 3$.

Assume $P(k)$ is true.

| | |
|---|---|
| State $P(k)$. | $P(k)$: $3^k > 3 \times 2^k$, $k \in N$ |
| Consider $P(k+1)$: $3^{k+1} > 3 \times 2^{k+1}$. | |
| Write the LHS of $P(k+1)$ in terms of $k$. | $3^{k+1} = 3 \times 3^k$<br>$> 3 \times 3 \times 2^k$ (using $P(k)$)<br>$> 3 \times 2 \times 2^k$<br>$= 3 \times 2^{k+1}$<br>$\therefore P(k+1)$ is true. |
| Make a formal statement for $P(n)$. | By the principle of mathematical induction, $P(n)$ is true for all $n \geq 3$, where $n \in N$. **QED** |

## Exercise 1.01 The principle of mathematical induction

### Problem solving

Example 1

**1** Use the method of mathematical induction to prove each formula for $n \in Z^+$.

**a** $1+4+7+10+\ldots+(3n-2)=\dfrac{n(3n-1)}{2}$

**b** $2+4+6+\ldots+2n=n(n+1)$

Example 2

**2** Prove each proposition for $n \in N$.

**a** $1^2+3^2+5^2+\ldots+(2n-1)^2=\dfrac{n(4n^2-1)}{3}$

**b** $2+2^2+2^3+2^4+\ldots+2^n=2^{n+1}-2$

**c** $1^3+3^3+5^3+\ldots+(2n-1)^3=n^2(2n^2-1)$

**3** Show that $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\ldots\left(1-\frac{1}{n}\right)=\frac{1}{n}$ for all integers $n \geq 2$.

Example 3

**4** Prove each inequality by mathematical induction.

**a** $2^n > n$ for all integers $n \geq 1$

**b** $2^n > 2n$ for all integers $n \geq 3$

**c** $3^n > 5 \times 2^n$ for all integers $n \geq 5$

**d** $n^2 \geq 2n+1$ for all integers $n \geq 3$

**5** Prove each inequality by mathematical induction.

**a** $n! > 2^n$ for all integers $n \geq 4$

**b** $n! > 3^n$ for all integers $n \geq 7$

**c** $2^n < 3^n$ for any integer $n \geq 1$


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## 1.02 Proofs of integer properties

### EXAMPLE 4

Use the method of mathematical induction to prove that:

$P(n)$: $3^n$ is **odd** for all $n \in \mathbf{N}$.

### Solution

| | |
|---|---|
| State what has to be proved. | **RTP**<br>$P(n)$: $3^n$ is odd for all $n \in \mathbf{N}$. |
| Prove true for $n = 1$. | $3^1 = 3$<br>3 is odd<br>$\therefore P(1)$ is true. |
| Prove $P(k) \Rightarrow P(k+1)$: first assume true for $n = k$, where $k \in \mathbf{N}$. | Assume $P(k)$ is true. |
| State $P(k)$. | $3^k$ is odd |
| Write $3^k$ as an odd number. | $3^k = 2m + 1$ for some $m \in N$ |
| Write $3^{k+1}$ in terms of $3^k$. | $3^{k+1} = 3 \times 3^k$<br>$= 3 \times (2m + 1)$<br>$= 6m + 3$<br>$= 2(3m + 1) + 1$ |
| State conclusion. | $\therefore P(k + 1)$ is true. |
| Make a formal statement for $P(n)$. | By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbf{N}$. **QED** |

## EXAMPLE 5

Use the method of mathematical induction to prove that:

$P(n)$: $1 \times 1! + 2 \times 2! + 3 \times 3! + 4 \times 4! + \ldots + n \times n! = (n + 1)! - 1$, where $n \in \mathbf{N}$.

### Solution

State what has to be proved.

**RTP**

$P(n)$: $1 \times 1! + 2 \times 2! + 3 \times 3! + 4 \times 4! + \ldots + n \times n! = (n + 1)! - 1$ where $n \in \mathbf{N}$.

Prove true for $n = 1$.

LHS $= 1 \times 1!$
$= 1$

RHS $= (1 + 1)! - 1$
$= 2! - 1$
$= 2 - 1$
$= 1$

LHS = RHS

$\therefore P(1)$ is true.

Prove $P(k) \Rightarrow P(k + 1)$: first assume true for $n = k$.

Assume $P(k)$ is true.

State $P(k)$.

$P(k)$: $1 \times 1! + 2 \times 2! + 3 \times 3! + 4 \times 4! + \ldots + k \times k! = (k + 1)! - 1$

Write the LHS of $P(k + 1)$.

$\underbrace{1 \times 1! + 2 \times 2! + 3 \times 3! + \ldots + k \times k!} + (k + 1)(k + 1)!$

$= (k + 1)! - 1 + (k + 1)(k + 1)!$ (using $P(k)$)

Factorise.

$= (k + 1)!\,[1 + (k + 1)] - 1$

Simplify.

$= (k + 1)!(k + 2) - 1$

Use $(n + 1)! = n! \times (n + 1)$.

$= (k + 2)! - 1$

$\therefore P(k + 1)$ is true.

Make a formal statement for $P(n)$.

By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbf{N}$. **QED**


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## EXAMPLE 6

Given the **recursive function** $t_{n+1} = 10t_n - 9$, where $t_1 = 11$, use the method of mathematical induction to prove $P(n)$: $t_n = 10^n + 1$ for all $n \in \mathbf{N}$.

### Solution

| | |
|---|---|
| State what has to be proved. | **RTP** |
| | If $t_{n+1} = 10t_n - 9$ and $t_1 = 11$, then $P(n)$: $t_n = 10^n + 1$ for all $n \in \mathbf{N}$. |
| Prove true for $n = 1$. | LHS $= t_1 = 11$ (given) |
| | RHS $= 10^1 + 1$ |
| | $= 11$ |
| | LHS = RHS |
| | $\therefore P(1)$ is true. |
| Prove $P(k) \Rightarrow P(k+1)$: first assume true for $n = k$, where $k \in \mathbf{N}$. | Assume $P(k)$ is true. |
| State $P(k)$. | $P(k)$: $t_k = 10^k + 1$, $k \in \mathbf{N}$ |
| Write the LHS of $P(k + 1)$. | $t_{k+1} = 10t_k - 9$ (given) |
| Use the given function. | $= 10 \times (10^k + 1) - 9$ (using $P(k)$) |
| Expand. | $= 10^{k+1} + 10 - 9$ |
| Simplify. | $= 10^{k+1} + 1$ |
| | $\therefore P(k + 1)$ is true. |
| Make a formal statement for $P(n)$. | By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbf{N}$. **QED** |

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## INVESTIGATION

### TOWER OF HANOI

Tower of Hanoi is a classic mathematical puzzle that consists of 3 vertical rods and a number of discs of different sizes that can slide onto the rods. The puzzle begins with the discs all resting on the first rod, in ascending (smallest on top) order. The aim of the puzzle is to move the entire stack to another rod (also in ascending order) in the smallest number of moves. Three rules need to be obeyed:

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- only the top disc can be removed and placed on another stack or an empty rod
- a larger disc cannot be placed on top of a smaller disc
- only one disc can be moved at a time

In this investigation you have $n$ discs. Let $a_n$ be the minimum number of moves needed to solve the puzzle with $n$ discs.

**1** Find a formula for $a_{n+1}$ in terms of $a_n$

**2** Investigate the value of $a_n$ for different values of $n$. Use your findings to formulate a rule for $a_n$ in terms of $n$.

**3** Use the method of mathematical induction to confirm that your rule is correct.

**4** Jayden is sick and has a week off school. His sister Cynthia constructs a Tower of Hanoi puzzle with 20 discs. She dares Jayden to complete the puzzle before he goes back to school. If it takes Jayden half a second to move a disc from one peg to another, can he succeed in this challenge? Justify your answer.


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## Exercise 1.02 Proofs of integer properties

### Problem solving

**1** Prove by mathematical induction that $2^n$ is **even** for every $n \in \mathbf{N}$. Example 4

**2** **a** Prove by mathematical induction that $n^2 - n$ is even for all positive integers.

**b** Find an easier proof for question **2a** by factorising $n^2 - n$.

**3** Use mathematical induction to show that for $n \in \mathbf{Z}^+$: Example 5

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{n}{(n+1)!}=\frac{(n+1)!\ -1}{(n+1)!}$$

**4** Prove that ${}^nC_r = \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{1\times 2\times 3\times \ldots \times r}$, where ${}^nC_0 = 1, {}^nC_n = 1,$ and ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$.

**5** Prove each statement.

**a** $(n + 2)! - n! = (n + 1)(n + 1)! + n \times n!$

**b** $2! \times 4! \times 6! \times \ldots \times (2n)! \geq ((n + 1)!)^n$

**6** Use mathematical induction to prove the derivative of $x^n : \dfrac{d(x^n)}{dx} = nx^{n-1}$

**7** Prove each formula using mathematical induction where $n \in \mathbf{Z}^+$. Example 6

**a** If $a_{n+1} = 5a_n + 4$ and $a_1 = 4$, then $a_n = 5^n - 1$.

**b** If $a_{n+1} = 2a_n - 1$ and $a_1 = 3$, then $a_n = 2^n + 1$.

**8** Show that $(1 + x)^n > 1 + nx + nx^2$ for all $n > 2$, where $n \in \mathbf{Z}^+$ and $x > 0$.

**9** **a** Prove by induction that, if $x > 0$ where $x \in \mathbf{R}$ and $n \geq 2$ where $n \in \mathbf{Z}^+$, then $(1 + x)^n > 1 + nx$.

**b** Let $a_n = \left(1+\dfrac{1}{n}\right)^{n+1}$ for $n \geq 1$.

Show that $\dfrac{a_n}{a_{n+1}} = \dfrac{n+1}{n+2}\times\left(1+\dfrac{1}{n(n+2)}\right)^{n+1}$.

**10** Prove that $t_n = \dfrac{2^n - (-1)^n}{3}$ is odd for $n = 1, 2, 3, \ldots$

## 1.03 Proofs of sums

To prove the formula for a sum of terms, you usually add the next term to the LHS of $P(k)$.

### EXAMPLE 7

Prove by the method of mathematical induction that the formula for the sum of 1, 2, 3, …, $n$ is $\frac{n^2+n}{2}$ for any positive integer $n$.

#### Solution

State what has to be proved.

**RTP**

$P(n)$: $1 + 2 + 3 + \ldots + n = \frac{n^2+n}{2}, n \in \mathbf{N}$.

Prove true for $n = 1$.

LHS = 1

$$\text{RHS} = \frac{1^2+1}{2} = 1$$

LHS = RHS

$\therefore P(1)$ is true.

Prove $P(k) \Rightarrow P(k+1)$: first assume true for $n = k$, where $k \in \mathbf{N}$.

Assume $P(k)$ is true.

State $P(k)$.

$P(k)$: $1 + 2 + 3 + \ldots + k = \frac{k^2+k}{2}, k \in \mathbf{N}$

Write the LHS of $P(k + 1)$.

$$\underbrace{1+2+3+\ldots+k}+(k+1)$$

$$= \frac{k^2+k}{2} + (k+1) \quad \text{(using } P(k)\text{)}$$

Combine fractions.

$$= \frac{k^2+k}{2} + \frac{2k+2}{2}$$

$$= \frac{k^2+3k+2}{2}$$

Obtain $(k + 1)^2$.

$$= \frac{k^2+2k+1+k+1}{2}$$

$$= \frac{(k+1)^2+(k+1)}{2}$$

$\therefore P(k + 1)$ is true.

Make a formal statement for $P(n)$.

By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbf{N}$. **QED**


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## EXAMPLE 8

Use mathematical induction to prove that:

$P(n)$: $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$, for all $n \in \mathbf{Z}^+$.

### Solution

State what has to be proved.

**RTP**

$P(n)$: $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$, $n \in \mathbf{Z}^+$.

Prove true for $n = 1$.

$\text{LHS} = \frac{1}{1\times 2} = \frac{1}{2}$

$\text{RHS} = \frac{1}{1+1} = \frac{1}{2}$

LHS = RHS

$\therefore$ $P(1)$ is true.

Prove $P(k) \Rightarrow P(k+1)$: first assume true for $n = k$, where $k \in \mathbf{Z}^+$.

Assume $P(k)$ is true.

State $P(k)$.

$P(k)$: $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{k(k+1)}=\frac{k}{k+1}$

Write the LHS of $P(k+1)$.

$$\underbrace{\frac{1}{1\times 2}+\frac{1}{2\times 3}+\ldots+\frac{1}{k(k+1)}}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)} \quad \text{(using } P(k)\text{)}$$

Combine fractions.

$$=\frac{k(k+2)+1}{(k+1)(k+2)}$$

Simplify.

$$=\frac{k^2+2k+1}{(k+1)(k+2)}$$

Factorise.

$$=\frac{(k+1)^2}{(k+1)(k+2)}$$

Simplify.

$$=\frac{(k+1)}{(k+2)}$$

$\therefore$ $P(k+1)$ is true.

Make a formal statement for $P(n)$.

By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbf{Z}^+$. **QED**

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## EXAMPLE 9

Use mathematical induction to prove $P(n)$: $\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^n}=\frac{2^n-1}{2^n}$, for all $n \in Z^+$.

### Solution

State what has to be proved.

**RTP**

$P(n)$: $\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^n}=\frac{2^n-1}{2^n}$, $n \in Z^+$.

Prove true for $n = 1$.

$\text{LHS} = \frac{1}{2^1} = \frac{1}{2}$

$\text{RHS} = \frac{2^1-1}{2^1} = \frac{1}{2}$

LHS = RHS

∴ $P(1)$ is true.

Prove $P(k) \Rightarrow P(k+1)$: first assume true for $n = k$.

Assume $P(k)$ is true.

State $P(k)$.

$P(k)$: $\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}=\frac{2^k-1}{2^k}$, $k \in Z^+$

Write the LHS of $P(k+1)$.

$$\underbrace{\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^k}}+\frac{1}{2^{k+1}}$$

$$= \frac{2^k-1}{2^k} + \frac{1}{2^{k+1}} \quad \text{(using } P(k)\text{)}$$

Combine fractions.

$$= \frac{2(2^k-1)+1}{2^{k+1}}$$

Expand.

$$= \frac{2^{k+1}-2+1}{2^{k+1}}$$

Simplify.

$$= \frac{2^{k+1}-1}{2^{k+1}}$$

∴ $P(k+1)$ is true.

Make a formal statement for $P(n)$.

By the principle of mathematical induction, $P(n)$ is true for all $n \in Z^+$. **QED**


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## Exercise 1.03 Proofs of sums

### Problem solving

Example 7

**1** Prove each statement for all $n \in Z^+$.

**a** $1 + 3 + 6 + 10 + \dots$ to $n$ terms $= \frac{n(n+1)(n+2)}{6}$

**b** $5 + 9 + 13 + 17 + \dots + (4n + 1) = n(2n + 3)$

**c** $2 + 10 + 24 + \dots + n(3n - 1) = n^2(n + 1)$

**2** Show that, for every positive integer $n$,

**a** $1 \times 2 + 2 \times 3 + 3 \times 4 + \dots + n(n + 1) = \frac{n(n+1)(n+2)}{3}$

**b** $2 \times 6 + 3 \times 7 + 4 \times 8 \dots$ to $n$ terms $= \frac{n(n+7)(2n+7)}{6}$

**3** Use mathematical induction to prove that:

$$P(n)\text{: } 1^3 + 2^3 + 3^3 + \dots + n^3 = \left(\frac{n(n+1)}{2}\right)^2 \text{ where } n \in N.$$

Example 8

**4** Show for $n \in Z^+$ that:

**a** $\frac{1}{1\times3} + \frac{1}{3\times5} + \frac{1}{5\times7} + \dots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

**b** $\frac{1}{3\times5} + \frac{1}{5\times7} + \frac{1}{7\times9} + \dots$ to $n$ terms $= \frac{n}{3(2n+3)}$

**c** $\frac{5}{1\times2\times3} + \frac{6}{2\times3\times4} + \frac{7}{3\times4\times5} + \dots$ to $n$ terms $= \frac{n(3n+7)}{2(n+1)(n+2)}$

Example 9

**5** Prove that $2^1 + 2^2 + 2^3 + \dots + 2^n = 2^{n+1} - 2$ for all $n \in N$.

**6** Prove each formula, where $n \in Z^+$.

**a** $1^2 - 2^2 + 3^2 - 4^2 + \dots + (-1)^{n-1}n^2 = \frac{(-1)^{n-1}n(n+1)}{2}$

**b** $-2 + 4 - 8 + \dots + (-2)^n = \frac{2}{3}\left[(-2)^n - 1\right]$

**7** Use mathematical induction to prove each formula.

**a** Sum of an **arithmetic progression**:

$$a + (a + d) + (a + 2d) + \dots + [a + (n - 1)d] = \frac{n}{2}[2a + (n-1)d]$$

**b** Sum of an **geometric progression**:

$$a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(1-r^n)}{1-r} \text{ where } r \neq 1.$$

**8** For $n \geq 1$, prove that: $1 + x + x^2 + \dots + x^n = \frac{x^{n+1}-1}{x-1}$, where $x \neq 1$.

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## 1.04 Divisibility results

For any 2 consecutive integers, say $k$ and $k + 1$, one must be even and one must be odd. You can write the even integer as $2p$ and the odd one as $2q + 1$ for some integers $p$ and $q$. Hence their product is given by $k(k + 1) = 2p(2p + 1)$, which is even. Thus it follows that the product of 2 consecutive integers must be even (divisible by 2).

### EXAMPLE 10

Use mathematical induction to prove $P(n)$: $7^n - 4$ is divisible by 3 for all $n \in \mathbf{N}$.

#### Solution

| | |
|---|---|
| State what has to be proved. | **RTP** |
| | $P(n)$: $7^n - 4$ is divisible by 3, $n \in \mathbf{N}$. |
| Prove true for $n = 1$. | $7^1 - 4 = 3$ |
| | 3 is divisible by 3. |
| | $\therefore$ $P(1)$ is true. |
| Prove $P(k) \Rightarrow P(k + 1)$: first assume true for $n = k$, where $k \in \mathbf{N}$. | Assume $P(k)$ is true. |
| Write $7^k - 4$ as a multiple of 3. | $\Rightarrow 7^k - 4 = 3m$, for some $m \in \mathbf{N}$ |
| Write the expression for $(k + 1)$ and work towards it for $k$. | $7^{k+1} - 4 = 7 \times 7^k - 4$ |
| Rearrange and use $P(k)$. | $= 7 \times (3m + 4) - 4$ (using $P(k)$) |
| Expand. | $= 21m + 28 - 4$ |
| Simplify. | $= 21m + 24$ |
| Factorise. | $= 3(7m + 8)$ |
| | $\Rightarrow 7^{k+1} - 4$ is divisible by 3. |
| | $\therefore$ $P(k + 1)$ is true. |
| Make a formal statement for $P(n)$. | By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbf{Z}^+$. **QED** |


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## EXAMPLE 11

Use the method of mathematical induction to prove that for all positive integers $n$, $n(n+1)(n+2)$ is divisible by 6.

### Solution

| | |
|---|---|
| State what has to be proved. | **RTP**<br>$P(n)$: $n(n+1)(n+2)$ is divisible by 6, $n \in \mathbf{Z}^+$. |
| Prove true for $n = 1$. | $1(1+1)(1+2) = 1 \times 2 \times 3$<br>$= 6$<br>6 is divisible by 6.<br>$\therefore P(1)$ is true. |
| Assume that it is true for $n = k$. | Assume $P(k)$ is true. |
| Write as a multiple of 6. | $\Rightarrow k(k+1)(k+2) = 6m$, for some $m \in \mathbf{Z}$. |
| Write the expression for $(k+1)$ and expand. | $(k+1)(k+2)(k+3) = \underbrace{k(k+1)(k+2)} + 3(k+1)(k+2)$<br>$= 6m + 3(k+1)(k+2)$ (using $P(k)$) |
| Use $(k+1)(k+2) = 2p$, where $p$ is an integer. | $= 6m + 3 \times 2p$ for some $p \in \mathbf{Z}$. |
| Simplify. | $= 6m + 6p$ |
| Factorise. | $= 6(m+p)$<br>$\Rightarrow (k+1)(k+2)(k+3)$ is divisible by 6.<br>$\therefore P(k+1)$ is true. |
| Make a formal statement for $P(n)$. | By the principle of mathematical induction,<br>$P(n)$ is true for all $n \in \mathbf{Z}^+$. **QED** |

## EXAMPLE 12

Use the method of mathematical induction to prove $P(n)$.

$P(n)$: $3^{2n+4} - 2^{2n}$ is divisible by 5 for all $n \in \boldsymbol{Z}^+$.

### Solution

| | |
|---|---|
| State what has to be proved. | **RTP** $P(n)$: $3^{2n+4} - 2^{2n}$, $n \in \boldsymbol{Z}^+$. |
| Prove true for $n = 1$. | $3^{2(1)+4} - 2^{2(1)} = 3^6 - 2^2$ |
| | $= 729 - 4$ |
| | $= 725$ |
| | 725 (= 145 × 5) is divisible by 5. |
| | $\therefore P(1)$ is true. |
| Assume true for $n = k$, where $k \in \boldsymbol{Z}^+$. | Assume $P(k)$ is true. |
| Express as a multiple of 5. | $3^{2k+4} - 2^{2k} = 5m$, for some $m \in \boldsymbol{Z}$. |
| Write the expression for $k + 1$ and simplify. | $3^{2(k+1)+4} - 2^{2(k+1)} = 3^{2k+2+4} - 2^{2k+2}$ |
| Write the expression in terms of $3^{2k+4}$. | $= 3^2 \times 3^{2k+4} - 2^2 \times 2^{2k}$ |
| | $= 9 \times (5m + 2^{2k}) - 4 \times 2^{2k}$ (using $P(k)$) |
| Expand. | $= 9 \times 5m + 9 \times 2^{2k} - 4 \times 2^{2k}$ |
| Simplify. | $= 9 \times 5m + 5 \times 2^{2k}$ |
| Factorise. | $= 5(9m + 2^{2k})$ |
| | $\Rightarrow 3^{2(k+1)+4} - 2^{2(k+1)}$ is divisible by 5. |
| | $\therefore P(k + 1)$ is true. |
| Make a formal statement for $P(n)$. | By the principle of mathematical induction, $P(n)$ is true for all $n \in \boldsymbol{Z}^+$. **QED** |


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## Exercise 1.04 Divisibility results

### Problem solving

Example 10

**1** Use mathematical induction to prove each statement for $n \in \boldsymbol{Z}^+$.

**a** $3^n - 1$ is divisible by 2

**b** $3^{2n} - 1$ is divisible by 8

**c** $4^n - 1$ is divisible by 3

**d** $n^7 - n$ is divisible by 7

**2** Prove each statement, where $n \in \boldsymbol{Z}^+$,

**a** $7^n - 3^n$ is divisible by 4

**b** $3^{2n} + 7$ is divisible by 8

**c** $5^n + 6 \times 7^n + 1$ is divisible by 4

Example 11

**3** Prove that for all $n \in \boldsymbol{Z}^+$, the product of any 3 consecutive integers is divisible by 3.

**4** Show that, for any integer $n \geq 1$, $n(n^2 + 2)$ is divisible by 3.

Example 12

**5** Prove for all integers $n \geq 1$:

**a** $5^n + 2(11^n)$ is divisible by 3

**b** $4^n + 6n - 1$ is divisible by 9

**6** **a** Use mathematical induction to prove that $n^3 - n$ is divisible by 6 for all $n \in \boldsymbol{Z}^+$.

**b** Find an easier proof by factorising $n^3 - n$.

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# 1. CHAPTER SUMMARY

## Mathematical induction

- Suppose that a proposition $P(n)$ is defined for every integer $n \geq a$, where $a \in \mathbf{Z}$

  If $P(a)$ is true

  and if $P(k)$ is true then $P(k + 1)$ is also true,

  then $P(n)$ is true for all integers $n \geq a$

- To prove a proposition by mathematical induction, you should follow the steps below.

  **Step 1:** Under RTP (required to prove) state what has to be proved.

  **Step 2:** Prove that the proposition is true for the initial value, usually $n = 1$.

  **Step 3:** Assume the proposition is true for $n = k$, where $k \in \mathbf{Z}^+$.

  **Step 4:** Show that it necessarily follows that the proposition is true for $n = k + 1$.

  **Step 5:** Make a formal statement such as 'By the principle of mathematical induction $P(n)$ is true for all $n \in \mathbf{Z}^+$', followed by QED.


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# 1. CHAPTER REVIEW

## Proof by mathematical induction

Problem solving

**1** Prove by mathematical induction for all $n \in Z^+$ that: Example 1

$$3 + 6 + 9 + \ldots + 3n = \frac{3n(1+n)}{2}$$

**2** Use mathematical induction to prove for all $n \in N$ that: Example 2

$$1^4 + 2^4 + 3^4 + \ldots + = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

**3** Prove each inequality by mathematical induction. Example 3

**a** $4^n > 10 \times 2^n$ for all integers $n \geq 4$

**b** $n! > 2^n$ for all integers $n \geq 4$

**4** Prove that the sum of an odd number of odd numbers is always odd. Example 4

**5** Show by mathematical induction that $(n + 3)! > 2^n$ for all $n \in N$. Example 5

**6** Prove by mathematical induction that: Example 6

If $a_{n+1} = 2a_n - n + 1$ and $a_1 = 3$, then $a_n = 2^n + n$ for all $n \in Z^+$.

**7** Prove each inequality for all $n \in Z^+$ using the method of mathematical induction. Example 7

**a** $\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{n+n} > \frac{13}{24}$ for $n > 1$

**b** $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} \geq \sqrt{n}$ for $n > 1$

Examples 8,9

**8** Prove the following by mathematical induction for all $n \in Z^+$.

$$3 + 7 + 11 + 15 + \ldots \text{ to } n \text{ terms} = 2n^2 + n$$

**9** Prove by induction that for all $n \in Z^+$:

**a** $\frac{1}{3\times4}+\frac{1}{4\times5}+\frac{1}{5\times6}+\ldots+\frac{1}{(n+2)(n+3)}=\frac{n}{3(n+3)}$

**b** $\frac{1}{7\times9}+\frac{1}{9\times11}+\frac{1}{11\times13}+\ldots \text{ to } n \text{ terms}=\frac{n}{7(2n+7)}$

**10** Use mathematical induction to prove that for all $n \in N$:

$$P(n): (1 - x)^n \geq 1 - nx \text{ for } x < 1, n \geq 1$$

**11** Prove by mathematical induction for all $n \in Z^+$ that:

**a** $(1 + 1) + (2 + 1) + (3 + 1) + \ldots + (n + 1) = \frac{n(n+3)}{2}$

**b** $(3 \times 1 - 2) + (3 \times 2 - 2) + (3 \times 3 - 2) + \ldots + (3n - 2) = \frac{n(3n-1)}{2}$

**12** Use mathematical induction to prove that for $n \in N$:

$$\frac{i}{2\times5\times8}+\frac{i}{5\times8\times11}+\frac{i}{8\times11\times14}+\ldots \text{ to } n \text{ terms}=\frac{n(3n+7)i}{20(3n+2)(3n+5)}$$

**13** **a** Prove by induction that:

$\frac{1}{2\times1}+\frac{1}{3\times2}+\frac{1}{4\times3}+\ldots+\frac{1}{n(n-1)}=1-\frac{1}{n}$ for all $n \in Z^+, n > 1$.

**b** Using the above equation, prove that $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}<2$.

Example 10

**14** Use mathematical induction to prove that:

**a** $n^3 + 2n$ is divisible by 3

**b** $3^{4n} - 1$ is divisible by 80


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**15** Prove that $11^n + 4$ is divisible by 5 for all positive integers $n$.

Example 11

**16** Prove that $2^{3n} - 3^n$ is divisible by 5 for all $n \geq 1$.

Example 12

**17** Prove by mathematical induction that all natural numbers (where $n \geq 2$) are divisible by a prime number. Hint: Let $P(n)$ be the statement that every integer $p$, such that $2 \leq p \leq n$, is divisible by a prime number.

**18** If you draw $n$ straight lines on a sheet of paper, they will divide the sheet into regions. Prove by mathematical induction that you can always colour each region either black or white, so that no 2 adjacent regions will be the same colour.

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VECTORS AND MATRICES

# 2. 3D VECTORS

You studied two-dimensional (2D) vectors in Year 11, but motion in the real world is not two-dimensional. In this chapter, you will extend vectors to cover physical space, which is three-dimensional (3D).


2.01 3D coordinates
2.02 3D vectors
2.03 The scalar product
2.04 The vector product
2.05 Vector equations of curves
2.06 Equations of lines in 3D
2.07 Geometric applications of vectors
2.08 Paths of particles
2.09 Equations of planes
2.10 Applications of vectors
Chapter summary
Chapter review

# SYLLABUS SUBJECT MATTER

## The algebra of vectors in three dimensions

- review the concepts of vectors from Unit 1 and extend to three dimensions by introducing the unit vector $\hat{k}$ and the altitude $\varphi$
- prove geometric results (review from the topic Geometric proofs using vectors) in the plane and construct simple proofs in three dimensions.

## Vector and Cartesian equations

- introduce Cartesian coordinates for three-dimensional space, including plotting points and the equations of spheres
- use vector equations of curves in two or three dimensions involving a parameter, and determine a 'corresponding' Cartesian equation in the two-dimensional case
- determine a vector, parametric and Cartesian equation of a straight line and straight-line segment given the position of two points, or equivalent information, in both two and three dimensions
- examine the position of two particles, each described as a vector function of time, and determine if their paths cross or if the particles meet
- define and use the cross product to determine a vector normal to a given plane
- use vector methods in applications, including areas of shapes and determining vector and Cartesian equations of a plane and of regions in a plane

Prior learning

# TERMINOLOGY

altitude
azimuth
circumcentre
component
displacement vector
equation of a plane
inner product
magnitude
normal
parallelepiped
polar coordinates
projection
scalar
vector equation
altitudes
Cartesian equation
circumcircle
cross product
dot product
in-centre
isometric drawing
median
orthocentre
parameter
spherical coordinates
resolving
scalar product
vector product
angular momentum
centroid
collide
determinant
electromagnetic force
in-circle
line segment
norm
orthogonal
parametric vector equation
position vector
right-handed system
unit vectors
work

## 2.01 3D coordinates

You use $x$- and $y$-directions for 2D coordinates; you use the $z$-direction for 3D coordinates. There are 2 possible $z$-directions: they are called left-handed and right-handed.

Think of the $x$-axis and $y$-axis as being clock hands at 9 o'clock or 3 o'clock.

If turning from $x$ to $y$ is clockwise, the $z$-axis points into the clock for a right-handed system.

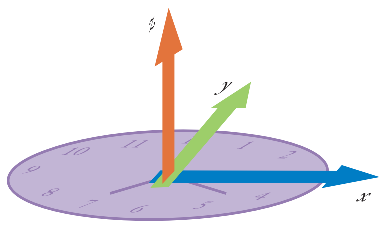
If turning from $x$ to $y$ is anticlockwise, the $z$-axis points out of the clock for a right-handed system.

### 3D Cartesian coordinates

- The $x$, $y$ and $z$ axes are used for 3-dimensional coordinates
- The $z$-direction is chosen to make the system *right-handed*
- The $x$ and $y$ axes are at 90° on a flat plane. The $z$ axis is at 90° to the plane
- If the turn from $x$ to $y$ is clockwise, the $z$-axis is into the plane
- If the turn from $x$ to $y$ is anticlockwise, the $z$-axis is out of the plane

You can use a clock, your right hand, axes on a page, a screw thread or a door to remember the directions.

- *On a clock*:
  - Put $x$-axis towards the 3
  - Put the $y$-axis towards the 12
  - The $z$-axis out of the centre





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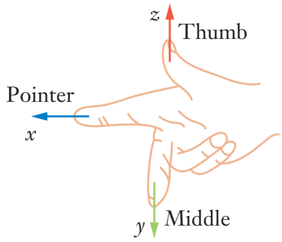
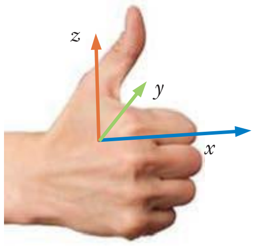
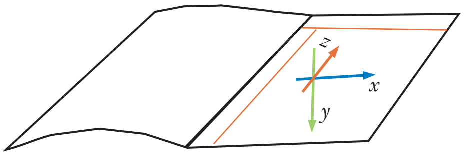
- *On your right hand*:
  - Point along the $x$-axis with your pointer finger
  - Bend your middle finger in the direction of the $y$-axis
  - Put your thumb up in the direction of the $z$-axis



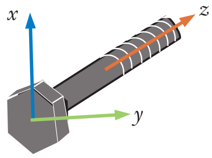

- *On your right hand*:
  - Curl your fingers from the $x$-axis to the $y$-axis
  - Your thumb sticks out in the direction of the $z$-axis



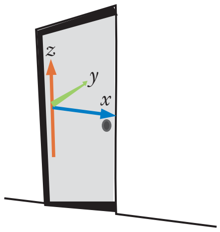

- *On a page*:
  - The $x$-direction is from left to right across the page
  - The $y$-direction is into the page
  - The $z$-direction is up the page




- *On a normal screw thread*:
  - Turn the screw from the $x$-axis to the $y$-axis
  - The screw goes forward in the direction of the $z$-axis




- *On a normal internal door* with the handle on the right-hand side:
  - The $x$-direction is across the door from the hinge to the handle
  - The $y$-direction is through the door
  - The $z$-direction is up the door




When you draw a 3D coordinate system to plot points, you can use boxes to help you place the points in the right positions. You need to foreshorten the axis perpendicular to the page.

You can also use an **isometric** drawing. An **isometric drawing** shows the axes at 120° to each other. Then you mark the scales equally on each axis.

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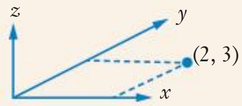
## EXAMPLE 1

**a** Plot $A(2, 3, -1)$ using a box.

**b** Plot $B(-3, 2, 4)$ on an isometric drawing.

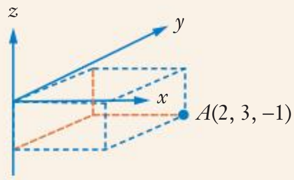
### Solution

**a** Draw the rectangle for the $x$ and $y$ directions on the axes first.



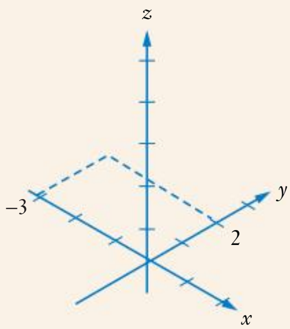

Add the $z$ direction to make a box and name the point.

Make the hidden lines lighter.




**b** Draw the axes and mark the scales equally.

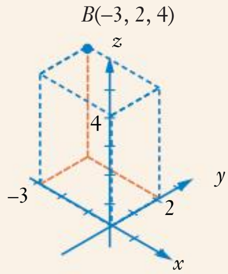
Extend the axes where you need to and draw lines parallel to each axis across from the coordinates on two axes.




Draw a line from the intersection parallel to the third axis.

Draw lines parallel to these three to complete the box and mark the point.

Make the hidden lines lighter.




The first method shown above is generally quicker, but it is not as clear as the second.

In 2D, you used Pythagoras' theorem to show that the distance between $(a_1, a_2)$ and $(b_1, b_2)$ is $d = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}$. The 3D rule is very similar.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## Distance between two points

The distance between $(a_1, a_2, a_3)$ and $(b_1, b_2, b_3)$ in 3D is given by

$$d = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$$

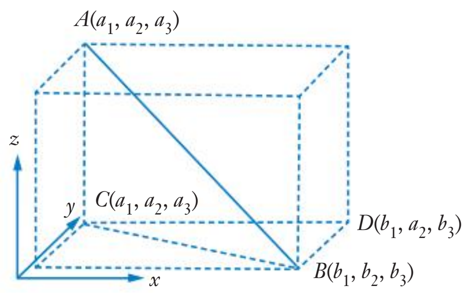
**Proof**

Consider two points $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$.

Put in the points $C(a_1, a_2, a_3)$ and $D(b_1, a_2, b_3)$.

$AC = |b_3 - a_3|$, $CD = |b_1 - a_1|$ and $DB = |b_2 - a_2|$.

$\triangle CDB$ and $\triangle ABC$ are right-angled.




By Pythagoras' theorem,

$CB^2 = CD^2 + DB^2$ and

$$\begin{aligned} AB^2 &= CB^2 + AC^2 \\ &= CD^2 + DB^2 + AC^2 \\ &= |b_3 - a_3|^2 + |b_1 - a_1|^2 + |b_2 - a_2|^2 \\ &= (b_3 - a_3)^2 + (b_1 - a_1)^2 + (b_2 - a_2)^2 \end{aligned}$$

Thus $AB = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$. **QED**

## EXAMPLE 2

Find the distance between:

**a** (3, 5, 3) and (1, –2, 5)

**b** (2, 5, 4) and (–4, 8, 10)

### Solution

**a** Write the formula.

$d = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$

Substitute points and calculate the answer.

It doesn't matter which point you choose as $(a_1, a_2, a_3)$.

$= \sqrt{(1-3)^2 + (-2-5)^2 + (5-3)^2}$

$= \sqrt{57}$

Write the answer.

The distance between (3, 5, 3) and (1, –2, 5) is $\sqrt{57}$.

**b** Write the formula. $d=\sqrt{(b_1-a_1)^2+(b_2-a_2)^2+(b_3-a_3)^2}$

Substitute points and calculate the answer. $=\sqrt{(-4-2)^2+(8-5)^2+(10-4)^2}$

$=9$

Write the answer. It is 9 units from (2, 5, 4) to (–4, 8, 10).

A circle is the 2D figure with all its points the same distance from the centre.

A sphere is the 3D figure with all its points the same distance from the centre.

## Equation of a sphere

The sphere with centre $(a, b, c)$ and radius $r$ has the equation

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2 \qquad \text{or}$$

$$x^2-2ax+y^2-2by+z^2-2cx=r^2-a^2-b^2-c^2$$

You can use the distance formula to derive the equations above.

## EXAMPLE 3

**a** Find the equation of the sphere with centre (1, 6, –4) and radius 7.

**b** Find the position of (–2, 0, –2) in relation to the sphere.

### Solution

**a** Write the general equation. $(x-a)^2+(y-b)^2+(z-c)^2=r^2$

Substitute values. $(x-1)^2+(y-6)^2+(z+4)^2=7^2$

Write the answer. The equation of the sphere with centre (1, 6, –4) and radius 7 is $(x-1)^2+(y-6)^2+(z+4)^2=49$.

**b** Find the distance from the centre. $d=\sqrt{(-2-1)^2+(0-6)^2+(-2+4)^2}$

$=\sqrt{49}$

$=7$

State your conclusion. The distance from the centre is equal to the radius.

Write the answer. The point (–2, 0, –2) lies on the sphere.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**INVESTIGATION**

## POINTS AND SPHERES

Substitute each point into the equation of a sphere with centre (4, –3, –5) and radius 9.

(5, 1, 3), (3, –5, 2), (9, 8, 10)

How are they related to the sphere?

What about the sphere with centre (0, 0, 0) and radius 11 and the points (–2, 6, 9), (6, –6, –7), (3, 5, 4) and (4, 7, 8)?

What general conclusions can you reach about the positions of points in relation to spheres?

## Exercise 2.01 3D coordinates

**1** Plot each point using a box. Example 1

**a** $A(-2, 3, -1)$ **b** $B(3, 2, 4)$ **c** $C(-1, -2, 3)$
**d** $D(3, -2, -3)$ **e** $E(2, -3, 1)$

**2** Plot each point on an isometric drawing.

**a** $A(-2, 4, 1)$ **b** $B(4, 3, -2)$ **c** $C(1, 3, 4)$
**d** $D(-2, 1, -2)$ **e** $E(-3, -2, 2)$

**3** Find the distance between each pair of points. Example 2

**a** (2, 4, 7) and (4, 0, 3) **b** (–4, 6, –2) and (–8, –1, 2)
**c** (2, –3, –4) and (5, 4, –1) **d** (–7, 5, 3) and (1, 2, 3)
**e** (1, 7, 8) and (–4, 4, 5) **f** (3, 8, –4) and (–2, 4, 2)

**4** Find the equation of the sphere with: Example 3

**a** centre (3, –4, 2) and radius 3 **b** centre (2, 3, 6) and radius 7
**c** centre (4, –4, –2) and radius 9 **d** centre (–5, 3, –6) and radius 4
**e** centre (2, 1, 8) and radius 6 **f** centre (–2, –3, –5) and radius 8

**5** Find whether each point is inside, outside or on the sphere given.

**a** $A(2, -1, 5)$, $B(9, 6, 4)$, $C(3, -2, 5)$ and $x^2 - 8x + y^2 - 10y + z^2 - 4z = 4$
**b** $D(2, 3, 5)$, $E(7, 6, 7)$, $F(-3, -4, -2)$ and $x^2 - 6x + y^2 - 10y + z^2 - 2z = 14$
**c** $G(4, 0, 4)$, $H(1, -3, 5)$, $K(1, -3, -4)$ and $x^2 - 2x + y^2 + 6y + z^2 + 8z = 55$
**d** $L(3, 1, 2)$, $M(5, -5, 2)$, $N(4, -3, -3)$ and $x^2 + 4x + y^2 - 8y + z^2 - 6z = 92$
**e** $P(6, 0, 0)$, $Q(6, -4, 0)$, $R(2, 1, 3)$ and $x^2 + 4x + y^2 + 6y + z^2 + 8z = 52$

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## Problem solving

**6** Find the centre and radius of the sphere with equation $x^2 - 12x + y^2 + 10y + z^2 - 8z = 23$.

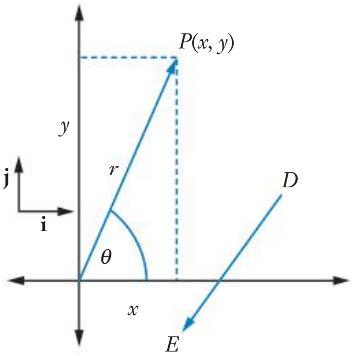
**7** Find whether the spheres with equations $x^2 - 8x + y^2 + 4y + z^2 + 10z + 29 = 0$ and $x^2 + 4x + y^2 - 8y + z^2 - 4z - 25 = 0$ are separate, touch or intersect.

**8** Find the distance between the closest points of the spheres $x^2 + 4x + y^2 + 6y + z^2 + 8z + 13 = 0$ and $x^2 - 4x + y^2 - 10y + z^2 - 4z + 8 = 0$.

WS
Vectors in polar form

# 2.02 3D vectors

You learnt about 2D vectors in Year 11. The **position vector** of $P(x, y)$ is shown as **p** or **OP**, and the **component** form is $(x, y)$ or $x\mathbf{i} + y\mathbf{j}$. The vector from $D$ to $E$ is shown as **DE**. In writing, you can put a wavy line underneath to show it is bold, like $\underset{\sim}{p}$, $\underset{\sim}{OP}$ or $\underset{\sim}{DE}$.




The **polar** form of a vector is shown as $(r, \theta)$, where $r$ is the **magnitude** and $\theta$ is the positive angle between the $x$-axis and the vector. Remember that in Mathematics the positive direction is *anticlockwise*.

The component and polar forms are related by $x = r\cos(\theta)$, $y = r\sin(\theta)$, $r = \sqrt{x^2 + y^2}$ and $\tan(\theta) = \frac{y}{x}$.

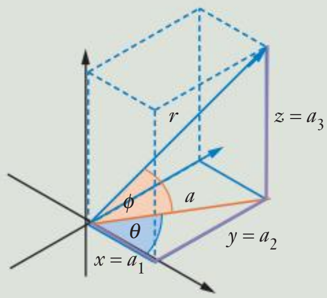
In component form, 3D vectors have 3 components. The polar form has 2 angles.

### Three-dimensional vectors

- The **unit vectors** in the $x$, $y$ and $z$ directions are **i**, **j** and **k** respectively
- The **component** form of a 3D vector is written as

$$\mathbf{a} = (a_1, a_2, a_3),\ \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \text{ or } \mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

- The **polar** (**spherical**) form is $(r, \theta, \phi)$, where $r$ is the **magnitude** (**norm**), $\theta$ is the **azimuth** angle and $\phi$ is the **altitude** angle. $0 \le \theta \le 2\pi$ and $-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$
- The **azimuth** is the angle from the $x$-axis to the **projection** of the vector onto the $xy$ plane. The projection is shown as $a$ in the diagram.
- The **altitude** is the angle between the vector and the $xy$ plane





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From the diagram, $x = a_1 = r\cos(\phi)\cos(\theta)$ $\quad r^2 = x^2 + y^2 + z^2 = a_1^2 + a_2^2 + a_3^2$

$y = a_2 = r\cos(\phi)\sin(\theta)$ $\quad \tan(\theta) = \frac{y}{x} = \frac{a_1}{a_2}$

$z = a_3 = r\sin(\phi)$ $\quad \sin(\phi) = \frac{z}{r} = \frac{a_3}{r}$

- The **norm** (magnitude, size) of **a** is given by $r = a = |\mathbf{a}| = \sqrt{x^2 + y^2 + z^2} = \sqrt{a_1^2 + a_2^2 + a_3^2}$
- The **position vector** of $P(x, y, z)$, **p** or **OP**, is the displacement from the origin to $P$

You need to be familiar with vectors in the form $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$.

## EXAMPLE 4

Express each vector as a linear combination of **i**, **j** and **k** and find its norm.

**a** $(3, 5, -7)$ **b** $\left(10, \frac{\pi}{3}, -\frac{\pi}{6}\right)$ **c** $\begin{bmatrix} -3 \\ 5 \\ -4 \end{bmatrix}$

**d** The displacement from $P(3, -1, 4)$ to $Q(-2, 4, 3)$.

### Solution

**a** Write in terms of **i**, **j** and **k**. $\quad (3, 5, -7) = 3\mathbf{i} + 5\mathbf{j} - 7\mathbf{k}$

Find the norm.

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$
$$= \sqrt{3^2 + 5^2 + (-7)^2}$$
$$= \sqrt{83}$$

**b** Find the $x$-component.

$$a_1 = r\cos(\phi)\cos(\theta)$$
$$= 10\cos\left(-\frac{\pi}{6}\right)\cos\left(\frac{\pi}{3}\right)$$
$$= 10 \times \frac{\sqrt{3}}{2} \times \frac{1}{2}$$
$$= \frac{5\sqrt{3}}{2}$$

Find the $y$-component.

$$a_2 = r\cos(\phi)\sin(\theta)$$
$$= 10\cos\left(-\frac{\pi}{6}\right)\sin\left(\frac{\pi}{3}\right)$$
$$= 10 \times \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2}$$
$$= \frac{15}{2}$$

Find the $z$-component.

$$a_3 = r\sin(\phi)$$
$$= 10\sin\left(-\frac{\pi}{6}\right)$$
$$= 10 \times \left(-\frac{1}{2}\right)$$
$$= -5$$

Write the answer.

$$\left(10, \frac{\pi}{3}, -\frac{\pi}{6}\right) = 2.5\sqrt{3}\mathbf{i} + 7.5\mathbf{j} - 5\mathbf{k}$$

State the norm.

$$|\mathbf{a}| = r = 10$$

**c** Write in terms of **i**, **j** and **k**.

$$\begin{bmatrix} -3 \\ 5 \\ -4 \end{bmatrix} = -3\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}$$

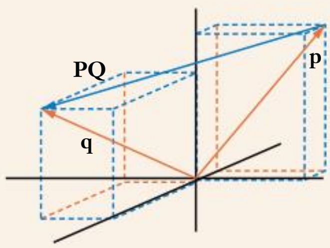
Find the norm.

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$
$$= \sqrt{(-3)^2 + 5^2 + (-4)^2}$$
$$= \sqrt{50} = 5\sqrt{2}$$

**d** Write the position vectors.

$$\mathbf{p} = 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$$
$$\mathbf{q} = -2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$$

Sketch the position and displacement vectors.




Write the relationship.

$$\mathbf{p} + \mathbf{PQ} = \mathbf{q}$$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Rearrange to get **PQ**. $\mathbf{PQ} = \mathbf{q} - \mathbf{p}$

Substitute and calculate the answer. $= (-2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}) - (3\mathbf{i} - \mathbf{j} + 4\mathbf{k})$

$= -5\mathbf{i} + 5\mathbf{j} - \mathbf{k}$

Find the norm.

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$
$$= \sqrt{(-5)^2 + 5^2 + (-1)^2}$$
$$= \sqrt{51}$$

You can see that the relationship between a displacement vector and the position vectors of its endpoints in part **d** above is always true. The displacement from $P$ to $Q$ is always given by **PQ** = **q** − **p**.

You add, subtract and multiply 3D vectors by scalars in the same way as 2D vectors. You also need to be able to find the norm and change between polar and rectangular form.

## EXAMPLE 5

Find:

**a** (5, −2, 4) + (6, 4, −7) **b** (−3, 3, 2) in polar form **c** 4(3, 8, −4) − 2(3, 1, 5)

### Solution

**a** Add the components. $(5, -2, 4) + (6, 4, -7) = (11, 2, -3)$

**b** Find the norm.

$$r = \sqrt{a_1^2 + a_2^2 + a_3^2}$$
$$= \sqrt{(-3)^2 + 3^2 + 2^2}$$
$$= \sqrt{22}$$

Find the azimuth and altitude.

$$\tan(\theta) = \frac{y}{x} = \frac{a_2}{a_1} = \frac{3}{-3} = -1$$

so $\theta = -\frac{\pi}{4}$ or $-45°$

$$\sin(\phi) = \frac{z}{r} = \frac{a_3}{r} = \frac{2}{\sqrt{22}}$$

so $\phi = \sin^{-1}\left(\frac{2}{\sqrt{22}}\right)$

$= 0.4405\ldots$ or $25.239\ldots°$

Write the answer. $(-3, 3, 2) = \left(\sqrt{22}, -\frac{\pi}{4}, \sin^{-1}\left(\frac{2}{\sqrt{22}}\right)\right)$

$\approx (4.69, -0.786, 0.441)$ or $(4.69, -45°, 25°)$

**c** Do the scalar multiplication and add the negative. $4(3, 8, -4) - 2(3, 1, 5) = (12, 32, -16) - (6, 2, 10)$

$= (6, 30, -26)$

It is much easier to add or subtract in component form than in polar form. You should use the formulas above to change a polar vector to component form before adding or subtracting. This is called **resolving** the vector into **orthogonal** (perpendicular) components. Use the $x$, $y$ and $z$ directions unless there is a reason to do otherwise.

## EXAMPLE 6

Find (3, 30°, 60°) + (4, –45°, 150°).

### Solution

Write the problem. $(3, 30°, 60°) + (4, 150°, -45°)$

Change to component form.

$$= (3\cos(30°)\cos(60°), 3\sin(30°)\cos(60°), 3\sin(60°)) + (4\cos(150°)\cos(-45°), 4\sin(150°)\cos(-45°), 4\sin(-45°))$$

Simplify each vector.

$$= \left(3\times\frac{\sqrt{3}}{2}\times\frac{1}{2}, 3\times\frac{1}{2}\times\frac{1}{2}, 3\times\frac{\sqrt{3}}{2}\right) + \left(4\times\left[-\frac{\sqrt{3}}{2}\right]\times\frac{\sqrt{2}}{2}, 4\times\frac{1}{2}\times\frac{\sqrt{2}}{2}, 4\times\left[-\frac{\sqrt{2}}{2}\right]\right)$$

$$= \left(\frac{3\sqrt{3}}{4}, \frac{3}{4}, \frac{3\sqrt{3}}{2}\right) + (-\sqrt{6}, \sqrt{6}, -2\sqrt{6})$$

Add and simplify.

$$= \left(\frac{3\sqrt{3}-4\sqrt{6}}{4}, \frac{3+4\sqrt{2}}{4}, \frac{3\sqrt{3}-2\sqrt{2}}{2}\right)$$


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Find the norm.

$$r=\sqrt{\left(\frac{3\sqrt{3}-4\sqrt{6}}{4}\right)^2+\left(\frac{3+4\sqrt{2}}{4}\right)^2+\left(\frac{3\sqrt{3}-2\sqrt{2}}{2}\right)^2}$$

$$=\sqrt{\frac{27-24\sqrt{18}+96}{16}+\frac{9+24\sqrt{2}+32}{16}+\frac{27-12\sqrt{6}+8}{4}}$$

$$=\frac{\sqrt{27-24\sqrt{18}+96+9+24\sqrt{2}+32+108-48\sqrt{6}+32}}{4}$$

$$=\frac{\sqrt{304-48\sqrt{2}-48\sqrt{6}}}{4}=5.443\ldots$$

Find the azimuth and altitude.

$$\tan(\theta)=\frac{y}{x}=\frac{a_2}{a_1}$$

$$=\frac{3+4\sqrt{2}}{4}\div\frac{3\sqrt{3}-4\sqrt{6}}{4}$$

$$=\frac{3+4\sqrt{2}}{3\sqrt{3}-4\sqrt{6}}$$

$$=-1.881\ldots$$

$$\sin(\phi)=\frac{z}{r}=\frac{a_3}{r}$$

$$=\frac{3\sqrt{3}-2\sqrt{2}}{2}\div 5.443$$

$$=0.434\ldots$$

Write the answer in the original form. That is, in polar form with angles in degrees.

The sum is 5.719, $\tan^{-1}(-1.881\ldots)$, $\sin^{-1}(0.434\ldots)\approx(5.44, 298°, 25.8°)$

If you are asked for an approximate answer, make sure you use the 'exact' answer on your calculator until the last step.

## Exercise 2.02 3D vectors

Example 4

**1** Express each vector as a linear combination of **i**, **j** and **k** and state its norm, correct to 2 decimal places if necessary.

**a** (2, –3, 4) **b** (–3, –2, 4) **c** (3, 1, –5)

**d** (–1, 4, –4) **e** (–3, –1, –2) **f** $\begin{bmatrix} -6 \\ 2 \\ -5 \end{bmatrix}$

**g** $\begin{bmatrix} 4 \\ -7 \\ -2 \end{bmatrix}$ **h** $\begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$ **i** $\begin{bmatrix} -5 \\ -3 \\ 2 \end{bmatrix}$

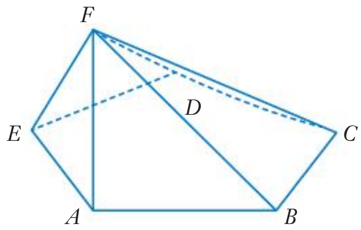
**j** $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ **k** (10, 294°, –27°) **l** (4, 225°, 32°)

**m** (11, 65°, 84°) **n** $\left(16, \frac{5\pi}{3}, \frac{\pi}{6}\right)$ **o** (8, 119°, –70°)

**p** Magnitude 50, azimuth 110°, altitude 25°

**q** Magnitude 80, azimuth $\frac{5\pi}{4}$, altitude $\frac{\pi}{3}$

**2** *ABCDE* is a regular pentagon and *F* is a point not in the plane of the pentagon. The position vectors of *A*, *B*, *C*, *D*, *E* and *F* are **a**, **b**, **c**, **d**, **e** and **f**. Express each of the following vectors in terms of **a**, **b**, **c**, **d**, **e** and **f**.




**a** **AB** **b** **AC** **c** **DF**

**d** **FC** **e** **EB**

Example 5

**3** If **a** = (2, –5, 4), **b** = (–5, 6, 3), **c** = (4, 7, –2) and **d** = (–3, –1, –4), find:

**a** **a** + **b** **b** **a** – **b** **c** 3**a** + 2**b**

**d** 2**a** – 4**b** **e** 5**a** – 2**b** **f** 6**c** + 3**b**

**g** **d** – 4**c** **h** **a** + 2**b** + 3**c** **i** 2**b** – 4**d** – 2**c**

**j** **a** – **b** – **c** + **d**

**4** $\mathbf{p} = \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}, \mathbf{q} = \begin{bmatrix} -3 \\ -2 \\ -6 \end{bmatrix}, \mathbf{r} = \begin{bmatrix} 2 \\ -3 \\ -5 \end{bmatrix}, \mathbf{s} = \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix}, \mathbf{t} = \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$. Find:

**a** **p** + 3**q** **b** 3**r** – **s** + 2**t** **c** **p** + 3**s** – **t**

**d** 4**q** + 3**r** – **p** **e** **t** + **r** + **s** **f** 4**t** – 3**r** + 5**q**

**g** **s** + 3**s** + 5**s** **h** 9**s** **i** –**t** + 4**r** – 2**p**

**j** **r** – **p** + **q** – **t** + **s**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**5** $\mathbf{a} = (12, 30°, -60°)$ and $\mathbf{b} = (15, 150°, 45°)$. Find each expression correct to one decimal place.

**a** $\mathbf{a} + \mathbf{b}$ **b** $\mathbf{a} - \mathbf{b}$ **c** $\mathbf{a} + 2\mathbf{b}$ **d** $2\mathbf{a} - \mathbf{b}$ **e** $3\mathbf{a} + 2\mathbf{b}$

Example 6

**6** $\mathbf{p}$ has norm 30, azimuth $\frac{7\pi}{6}$ and altitude $\frac{\pi}{3}$, and $\mathbf{q}$ has norm 25, azimuth $\frac{7\pi}{4}$ and altitude $\frac{\pi}{6}$. Find each expression, correct to one decimal place.

**a** $\mathbf{p} + \mathbf{q}$ **b** $\mathbf{q} - \mathbf{p}$ **c** $3\mathbf{p} - \mathbf{q}$

**d** $2\mathbf{q} - \mathbf{p}$ **e** $2\mathbf{p} - \mathbf{q}$

## 2.03 The scalar product

WS Scalar product

WS Resultant and unit vectors in 3D space

You defined the scalar product of 2D vectors in Year 11. The scalar product for 3D vectors is defined in exactly the same way.

### 3D scalar product

The **scalar product** (**inner product** or **dot product**) of vectors **a** and **b** is

$$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos(\theta), \text{ where } \theta \text{ is the angle between the vectors}$$

In component form, $(a_1, a_2, a_3) \bullet (b_1, b_2, b_3) = a_1b_1 + a_2b_2 + a_3b_3$

- The scalar product is a real number, *not a vector*
- $\mathbf{i} \bullet \mathbf{i} = \mathbf{j} \bullet \mathbf{j} = \mathbf{k} \bullet \mathbf{k} = 1$ and $\mathbf{i} \bullet \mathbf{j} = \mathbf{j} \bullet \mathbf{k} = \mathbf{k} \bullet \mathbf{i} = 0$
- $(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \bullet (\mathbf{i}b_1 + \mathbf{j}b_2 + \mathbf{k}b_3) = a_1b_1 + a_2b_2 + a_3b_3$
- The angle between two vectors **a** and **b** is given by $\cos(\theta) = \dfrac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$
- The norm is given by $|\mathbf{a}|^2 = \mathbf{a} \bullet \mathbf{a}$
- Non-zero vectors **a** and **b** are perpendicular if and only if $\mathbf{a} \bullet \mathbf{b} = 0$
- Non-zero vectors **a** and **b** are parallel if and only if $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|$
- The projection of **a** on **b** is given by $p = \dfrac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{b}|} = \mathbf{a} \bullet \hat{\mathbf{b}}$ or $\mathbf{p} = \dfrac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{b}|}\hat{\mathbf{b}} = \dfrac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b} = |\mathbf{a}| \cos(\theta)\,\hat{\mathbf{b}} = (\mathbf{a} \bullet \hat{\mathbf{b}})\,\hat{\mathbf{b}}$
- The scalar product is commutative: $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$
- It is distributive over vector addition: $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$

It is tedious to show that the component form of the scalar product above is correct.

However, the other properties are easily shown.

Remember that the projection of **a** on **b** is the component of **a** in the direction of **b**.

## EXAMPLE 7

Find:

**a** the scalar product of vectors of magnitudes 5 and 6 at an angle of 60° to each other

**b** $(2\mathbf{i} - 4\mathbf{j} - 6\mathbf{k}) \bullet (\mathbf{i} + 5\mathbf{j} - 3\mathbf{k})$

**c** the angle between $(2\mathbf{i} - 4\mathbf{j} - 6\mathbf{k})$ and $(\mathbf{i} + 5\mathbf{j} - 3\mathbf{k})$

**d** $(12, 102°, 47°) \bullet (8, 58°, -15°)$

**e** the angle between $(12, 102°, 47°)$ and $(8, 58°, -15°)$

### Solution

**a** Write the formula.

$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos(\theta)$

Substitute the values and calculate the answer.

$= 5 \times 6 \times \cos(60°)$

$= 15$

**b** Use the component formula and calculate the answer.

$(2\mathbf{i} - 4\mathbf{j} - 6\mathbf{k}) \bullet (\mathbf{i} + 5\mathbf{j} - 3\mathbf{k})$

$= 2 \times 1 + (-4) \times 5 + (-6) \times (-3)$

$= 2 - 20 + 18$

$= 0$

**c** Find the angle.

Since $(2\mathbf{i} - 4\mathbf{j} - 6\mathbf{k}) \bullet (\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}) = 0$, the vectors are perpendicular.

Write the answer.

The angle is 90°.

**d** Use $x = r\cos(\theta)\cos(\phi)$, $y = r\sin(\theta)\cos(\phi)$ and $z = r\sin(\phi)$ to change to component form.

$(12, 102°, 47°) \approx (-1.702, 8.005, 8.776)$

$(8, 58°, -15°) \approx (4.095, 6.553, -2.071)$

Calculate the product.

$(-1.702, 8.005, 8.776) \bullet (4.095, 6.553, -2.071)$

$= -1.702 \times 4.095 + 8.005 \times 6.553 + 8.776 \times (-2.071)$

$\approx 27.312$

**e** Find the angle.

$\cos(\theta) = \dfrac{27.312}{12 \times 8} \approx 0.2845$

$\theta = \cos^{-1}(0.2845) \approx 73.5°$

Write the answer.

The angle is about 73.5°.

Finding projections is useful in many applications.


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## EXAMPLE 8

**a** Find the projection of (2, –4, 5) on (–1, 3, 4), correct to 2 decimal places.

**b** Find the vector projection of $\mathbf{F} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ in the direction of $\mathbf{s} = -2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k}$.

### Solution

**a** Find the scalar product.

$$(2, -4, 5) \bullet (-1, 3, 4) = -2 - 12 + 20 = 6$$

Find the magnitude of (–1, 3, 4).

$$|(-1, 3, 4)| = \sqrt{1+9+16} = \sqrt{26}$$

Find the projection.

$$\text{Projection} = \frac{6}{\sqrt{26}} = 1.176\ldots$$

Write the answer.

The projection is about 1.18.

**b** Find the unit vector in the direction of **s**. Remember that the unit vector has magnitude 1, so you divide by the norm.

$$\hat{\mathbf{s}} = \frac{1}{s}\mathbf{s}$$

$$= \frac{1}{\sqrt{4+25+16}}(-2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k})$$

$$= \frac{1}{\sqrt{45}}(-2\mathbf{i} + 5\mathbf{j} - 4\mathbf{k})$$

$$= -\frac{2}{3\sqrt{5}}\mathbf{i} + \frac{5}{3\sqrt{5}}\mathbf{j} - \frac{4}{3\sqrt{5}}\mathbf{k}$$

Find the projection of **F** in the direction of **s**.

$$\text{Projection} = \mathbf{F} \bullet \hat{\mathbf{s}}$$

$$= (\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \bullet \left(-\frac{2}{3\sqrt{5}}\mathbf{i} + \frac{5}{3\sqrt{5}}\mathbf{j} - \frac{4}{3\sqrt{5}}\mathbf{k}\right)$$

$$= -\frac{2}{3\sqrt{5}} - \frac{10}{3\sqrt{5}} - \frac{12}{3\sqrt{5}}$$

$$= -\frac{24}{3\sqrt{5}} = -\frac{8}{\sqrt{5}}$$

Multiply by the unit vector to get the vector projection.

$$\text{Vector projection} = -\frac{8}{\sqrt{5}}\left(-\frac{2}{3\sqrt{5}}\mathbf{i} + \frac{5}{3\sqrt{5}}\mathbf{j} - \frac{4}{3\sqrt{5}}\mathbf{k}\right)$$

$$= 1\tfrac{1}{15}\mathbf{i} - 2\tfrac{2}{3}\mathbf{j} + 2\tfrac{2}{15}\mathbf{k}$$

Write the answer.

The vector projection of **F** on **s** is

$1\tfrac{1}{15}\mathbf{i} - 2\tfrac{2}{3}\mathbf{j} + 2\tfrac{2}{15}\mathbf{k}$.

You can find the answers to Example 8 in other ways.

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## Exercise 2.03 The scalar product

Example 7

**1** Find the scalar product of each pair of vectors with the following norms and angle between them, correct to 2 decimal places if necessary.

**a** 5, 9, 38° **b** $3, 10, \frac{\pi}{4}$ **c** 11, 7, 90° **d** 15, 12, 0°

**e** $7, 2, \frac{\pi}{6}$ **f** $6, 11, \frac{7\pi}{6}$ **g** 9, 9, 150° **h** $8, 12, \frac{5\pi}{3}$

**2** Find each scalar product, correct to 2 decimal places if necessary.

**a** (2, 6, 4) • (5, 10, 1) **b** (3, –4, 5) • (5, –6, –8)

**f** (9, 2, –12) • (6, –3, 4) **d** (8, –3, 5) • (–2, –3, –7)

**h** (–4, 2, –8) • (–5, 1, 3) **f** (1, 0, 5) • (–8, –3, 2)

**3** Find each scalar product.

**a** $\begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}$ **b** $\begin{bmatrix} 2 \\ 9 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \\ 6 \end{bmatrix}$ **c** $\begin{bmatrix} -1 \\ 0 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} -6 \\ 9 \\ 5 \end{bmatrix}$

**4** Find the scalar product of each pair of vectors.

**a** $4\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$ and $4\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$ **b** $6\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$ and $5\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$

**c** $-2\mathbf{i} - 4\mathbf{j} - \mathbf{k}$ and $2\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}$ **d** $-3\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}$ and $2\mathbf{i} - 7\mathbf{j} + 4\mathbf{k}$

**5** Find each scalar product, correct to 2 decimal places if necessary.

**a** (9, 16°, 30°) • (16, 16°, –30°) **b** (3, 45°, 25°) • (16, 60°, –20°)

**c** (19, 127°, 46°) • (6, 250°, 38°) **d** (3, 122°, –42°) • (5, 10°, –28°)

**6** Find the angle between each pair of vectors, correct to one decimal place.

**a** (3, 6, –4) and (0, –2, 7) **b** (0, 3, –2) and (6, 1, 5).

**c** $\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ and $3\mathbf{i} + \mathbf{j} - \mathbf{k}$ **d** $-7\mathbf{i} + 6\mathbf{j} + 9\mathbf{k}$ and $\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$

**e** (6, 35°, 40°) and (8, 187°, 76°) **f** (1, 260°, –14°) and (3, 88°, –82°)

Example 8

**7** Find each projection, correct to 2 decimal places.

**a** (3, –2, 6) on (–2, 1, 5) **b** (–2, 1, 5) on (3, –2, 6)

**c** $-2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ on $5\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ **d** $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ on $-4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

**e** (12, 18°, 24°) on (14, 26°, –40°) **f** (7, 168°, 27°) on (10, 10°, –24°)


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**8** A line passes through the points (1, 3, –4) and (2, –5, –2).

**a** Find a displacement vector parallel to the line.

**b** Find the cosines of the angles between this vector and each of the axes (the *direction cosines* of the line).

**c** Find the *direction angles* $\alpha$, $\beta$ and $\gamma$ of the line—the angles between the line and each of the axes.

## Problem solving

**9** For any vector, prove that $\mathbf{a} \bullet \mathbf{a} = |\mathbf{a}|^2$.

**10** Is the angle between the vectors (9, 42°, 31°) and (8, 12°, 31°) equal to, less than or greater than 30°? Explain your reasoning and prove your answer.

**11** Show that scalar multiplication is commutative.

**12** Show that scalar multiplication is distributive over vector addition.

# 2.04 The vector product

WS Vector product

The vector product is very different to the scalar product. It gives another vector.

### The vector product

The **vector product** $\mathbf{v} = \mathbf{a} \times \mathbf{b}$ is defined by $|\mathbf{v}| = |\mathbf{a}||\mathbf{b}| \sin(\theta)$,

where $\theta$ is the angle between them and **a**, **b** and **v** form a right-handed system.

**v** is perpendicular to both **a** and **b**, so it is also perpendicular to the plane formed by **a** and **b**.

$|\mathbf{i} \times \mathbf{i}| = 1 \times 1 \times \sin(0) = 0$, so $\mathbf{i} \times \mathbf{i} = \mathbf{0}$. Similarly, $\mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$.

$|\mathbf{i} \times \mathbf{j}| = 1 \times 1 \times \sin\left(\frac{\pi}{2}\right) = 1$, and **i**, **j** and **k** form a right-handed system, so $\mathbf{i} \times \mathbf{j} = \mathbf{k}$.

Similarly, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.

Since **i**, **j** and **k** form a right-handed system, **j**, **i** and **k** form a left-handed system.

Hence $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$. Similarly, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$ and $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$.

You can show other properties of the vector product as given on the next page.

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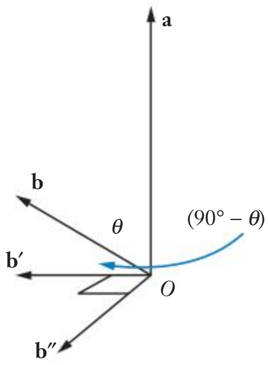
## Vector product properties

- $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$
- $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$
- $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$ and $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$
- The vector product is distributive over vector addition: $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ for any vectors **a**, **b**, **c**
- $\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$, where $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$
- $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any vector **a**
- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ for all vectors **a**, **b**
- The order of multiplication by a **scalar** and a vector product is irrelevant as long as the vectors remain in the same order: $m(\mathbf{a} \times \mathbf{b}) = (m\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (m\mathbf{b})\ \forall\ m \in \mathbf{R}$ and vectors **a**, **b**
- The area of the triangle formed by **a** and **b** is given by $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$
- The area of the parallelogram constructed on **a** and **b** is given by $|\mathbf{a} \times \mathbf{b}|$

**Proof of distributive law**




RTP: $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

Construct **a**, **b** and **c** from the origin point $O$.

Construct the plane through $O$ perpendicular to **a**.

Construct the projection **b′** of **b** on this plane:
$|\mathbf{b}'| = b \cos(90° - \theta)$,

where $\theta$ is the angle between **a** and **b** (see diagram).

Rotate **b′** through 90° anticlockwise in the plane to **b″** as shown.

Now $|\mathbf{b}''| = |\mathbf{b}'| = b\cos(90° - \theta) = b\sin(\theta)$ and **a**, **b** and **b″** form a **right-handed system**.

But $|\mathbf{a} \times \mathbf{b}| = ab\sin(\theta)$ is in the same direction as **b″**, so $\mathbf{a} \times \mathbf{b} = a\mathbf{b}''$.

Construct the projections **c′**, **c″**, $(\mathbf{b} + \mathbf{c})'$ and $(\mathbf{b} + \mathbf{c})''$ in the same way as **b′** and **b″**.

The sum is preserved under the projections and rotations, so $(\mathbf{b} + \mathbf{c})'' = \mathbf{b}'' + \mathbf{c}''$.

Multiplying by the magnitude of **a** gives $a(\mathbf{b} + \mathbf{c})'' = a(\mathbf{b}'' + \mathbf{c}'')$.

Multiplication by a scalar is distributive over vector addition, so $a(\mathbf{b}'' + \mathbf{c}'') = a\mathbf{b}'' + a\mathbf{c}''$.

Thus $a(\mathbf{b} + \mathbf{c})'' = a\mathbf{b}'' + a\mathbf{c}''$.

But $a(\mathbf{b}'' + \mathbf{c}'') = \mathbf{a} \times (\mathbf{b} + \mathbf{c})$, $a\mathbf{b}'' = \mathbf{a} \times \mathbf{b}$ and $a\mathbf{c}'' = \mathbf{a} \times \mathbf{c}$.

Substituting in $a(\mathbf{b} + \mathbf{c})'' = a\mathbf{b}'' + a\mathbf{c}''$ gives $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$. **QED**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You can use the distributive law and the vector products of **i**, **j** and **k** to prove the component form of the vector product. The other properties follow from the basic properties and the formula $A = \frac{1}{2}ab \sin(C)$ for the area of a triangle.

## EXAMPLE 9

Find the vector product of vectors of magnitudes 13 and 18 at an angle of 48° to each other.

### Solution

Find the product.

$$|\mathbf{v}| = |\mathbf{a}|\ |\mathbf{b}|\ \sin(\theta)$$
$$= 13 \times 18 \times \sin(48°)$$
$$\approx 173.9$$

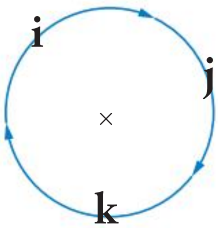
Write the answer.

The product is about 174 perpendicular to both the original vectors.

> You could also say 'perpendicular to the plane of the original vectors'.

The component form formula of the vector product is a bit tedious to remember. You can use a cycle to help. Think of **i**, **j** and **k** as a cycle that keeps going round: **i**, **j**, **k**, **i**, **j**, **k**, **i**, **j**, …

Then:




- **cross products** in the cycle direction give the other unit vector
- cross products in the opposite direction give the negative
- go through each component with the others and put them together

## EXAMPLE 10

$\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{v} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$. Find $\mathbf{u} \times \mathbf{v}$.

### Solution

Write the product.

$$\mathbf{u} \times \mathbf{v} = (\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) \times (-2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$$

Expand the brackets, but leave out $\mathbf{i} \times \mathbf{i}$, $\mathbf{j} \times \mathbf{j}$ and $\mathbf{k} \times \mathbf{k} = \mathbf{0}$.

$$= \mathbf{i} \times 3\mathbf{j} + \mathbf{i} \times (-\mathbf{k}) + (-2\mathbf{j}) \times (-2\mathbf{i}) + (-2\mathbf{j}) \times (-\mathbf{k}) + (4\mathbf{k}) \times (-2\mathbf{i}) + (4\mathbf{k}) \times (3\mathbf{j})$$

Subtract the results for the products that are backwards.

$$= 3\mathbf{k} - (-\mathbf{j}) - (4\mathbf{k}) + (2\mathbf{i}) + (-8\mathbf{j}) - (12\mathbf{i})$$

Collect in the order **i**, **j**, **k**.

$$= -10\mathbf{i} - 7\mathbf{j} - \mathbf{k}$$

You can use the determinant of $2 \times 2$ matrices to find the vector product in component form.

Remember that the **determinant** of the matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by $A = ad - bc$.

Write the vectors out in a $3 \times 3$ array with the unit vectors on top, as shown below for the product $\mathbf{v}_1 \times \mathbf{v}_2$, where $\mathbf{v}_1 = (5, 9, 20)$ and $\mathbf{v}_2 = (3, 5, 11)$.

$$\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 9 & 20 \\ 3 & 5 & 11 \end{matrix}$$

Skip each unit vector column to calculate that component.

Change the sign of the **j** component determinant.

For the **i** component, skip the **i** column and use the determinant $\begin{vmatrix} 9 & 20 \\ 5 & 11 \end{vmatrix}$.

For the **j** component, skip the **j** column and change the sign of $\begin{vmatrix} 5 & 20 \\ 3 & 11 \end{vmatrix}$.

For the **k** component, skip the **k** column and use $\begin{vmatrix} 5 & 9 \\ 3 & 5 \end{vmatrix}$.

*1, 2, miss a few*: $(5, 9, 20) \times (3, 5, 11) = (99 - 100, -(55 - 60), 25 - 27) = (-1, 5, -2)$.

## EXAMPLE 11

Find the area of the triangle formed by the vectors $(-2, 1, -3)$ and $(2, 3, -4)$, correct to 2 decimal places.

### Solution

Write the formula. $A = \frac{1}{2} ab \sin (C)$

Use the vector product. $= \frac{1}{2} |\mathbf{a} \times \mathbf{b}|$

$= \frac{1}{2} |(-2, 1, -3) \times (2, 3, -4)|$

$= \frac{1}{2} |(1 \times (-4) - (-3) \times 3, -(-2 \times (-4) - (-3) \times 2), -2 \times 3 - 1 \times 2)|$

$= \frac{1}{2} |(5, -14, -8)|$

Find the norm of the product. $= \frac{1}{2}\sqrt{5^2 + (-14)^2 + (-8)^2}$

$= \frac{1}{2}\sqrt{285}$

$\approx 8.44$

Write the answer. The area is about 8.44 square units.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## INVESTIGATION

### VECTOR PRODUCTS OF 3 VECTORS

Consider the vectors $\mathbf{a} = (3, -5, -2)$, $\mathbf{b} = (-1, 2, 3)$ and $\mathbf{c} = (4, 5, 6)$.

Find:

- $\mathbf{a} \times \mathbf{b}$
- $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$
- $\mathbf{b} \times \mathbf{c}$
- $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

Compare $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

What can you conclude about the order of the vector product of 3 vectors?

## Exercise 2.04 The vector product

**1** Find the vector product of each pair of vectors with the given norms and angle between them, correct to 2 decimal places if necessary.

Example 9

**a** $3, 8, \frac{3\pi}{2}$ **b** $10, 2, 78°$ **c** $12, 12, 200°$ **d** $5, 2, 0°$

**e** $6, 8, 47°$ **f** $4, 14, 330°$ **g** $7, 11, \frac{\pi}{2}$ **h** $4, 7, \frac{\pi}{3}$

**2** Find each vector product.

Example 10

**a** $(3\mathbf{i} + 4\mathbf{j} + \mathbf{k}) \times (8\mathbf{i} + 8\mathbf{j} - 8\mathbf{k})$ **b** $(-5\mathbf{i} + 6\mathbf{k}) \times (5\mathbf{i} + 9\mathbf{j} + 8\mathbf{k})$

**c** $(-3\mathbf{i} + \mathbf{j} + 7\mathbf{k}) \times (-5\mathbf{i} - 6\mathbf{j} - 3\mathbf{k})$ **d** $(5\mathbf{i} - 5\mathbf{j} - 5\mathbf{k}) \times (-2\mathbf{j} - 6\mathbf{k})$

**e** $(3\mathbf{i} - 7\mathbf{j} - 6\mathbf{k}) \times (5\mathbf{i} + \mathbf{j} - 9\mathbf{k})$ **f** $(\mathbf{i} - 8\mathbf{j} + 7\mathbf{k}) \times (4\mathbf{i} + 4\mathbf{j})$

**g** $(-\mathbf{i} + \mathbf{j} + 9\mathbf{k}) \times (-4\mathbf{i} + 7\mathbf{j} - 8\mathbf{k})$ **h** $(-7\mathbf{i} + 7\mathbf{j} + \mathbf{k}) \times (4\mathbf{i} + 4\mathbf{j} - 9\mathbf{k})$

**3** Find each vector product.

**a** $(8, 9, 2) \times (1, 7, 7)$ **b** $(6, -2, 1) \times (1, -9, -5)$

**f** $(3, 8, -17) \times (9, -6, 2)$ **d** $(5, -6, 1) \times (-8, -6, -4)$

**h** $(-2, 8, -5) \times (-1, 7, 6)$ **f** $(7, 0, 1) \times (-5, -6, 8)$

**4** Find each vector product.

**a** $\begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix} \times \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}$ **b** $\begin{bmatrix} 2 \\ 9 \\ 5 \end{bmatrix} \times \begin{bmatrix} 3 \\ -4 \\ 6 \end{bmatrix}$ **c** $\begin{bmatrix} -1 \\ 0 \\ 7 \end{bmatrix} \times \begin{bmatrix} -6 \\ 9 \\ 5 \end{bmatrix}$

**5** Given the vectors $\mathbf{a} = 3\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + \mathbf{j} - 5\mathbf{k}$, $\mathbf{c} = \mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$, $\mathbf{d} = \mathbf{j} - 2\mathbf{i} + 3\mathbf{k}$ and $\mathbf{e} = 4\mathbf{k} - 3\mathbf{j}$, find:

**a** $\mathbf{a} \cdot \mathbf{b}$ **b** $\mathbf{a} \times \mathbf{b}$ **c** $\mathbf{c} \cdot \mathbf{e}$
**d** $\mathbf{c} \times \mathbf{e}$ **e** $\mathbf{c} \cdot (\mathbf{d} - \mathbf{e})$ **f** $\mathbf{c} \times (\mathbf{d} - \mathbf{e})$
**g** $\mathbf{c} \cdot \mathbf{d} - \mathbf{c} \cdot \mathbf{e}$ **h** $\mathbf{c} \times \mathbf{d} - \mathbf{c} \times \mathbf{e}$ **i** $\mathbf{d} \cdot \mathbf{e}$
**j** $\mathbf{e} \cdot \mathbf{d}$ **k** $\mathbf{e} \times \mathbf{d}$ **l** $\mathbf{d} \times \mathbf{e}$
**m** $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ **n** $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$

Example 11

**6** Find the area of the triangle formed by the vectors (5, 5, 4) and (8, 0, 3).

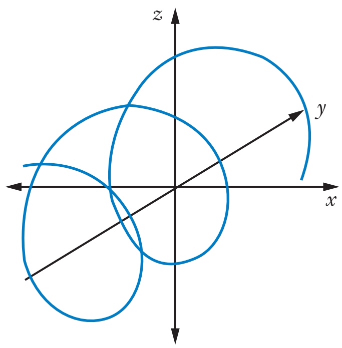
### Problem solving

**7** Find the area of the triangle with vertices (2, 3), (6, 4) and (4, –2).

**8** Find the area of a parallelogram with vertices (4, –2, 3), (–1, 3, 4), (2, 4, –2) and (7, –1, –3).

**9** $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}$. Write a relationship between **a** and **b** and explain your reasoning.

## 2.05 Vector equations of curves




You have studied the equations for two-dimensional lines like straight lines, parabolas, polynomials, exponentials and so on. Most of the equations were in the form $y = f(x)$. You can also use vectors for lines and curves.

Consider the equation $\mathbf{r}(t) = \cos(t)\,\mathbf{i} + 0.3t\,\mathbf{j} + \sin(t)\,\mathbf{k}$ for $-\infty \le t \le \infty$.

The projection of **r** on the *x*-*z* plane has no **j** component, so it is $\mathbf{r}'(t) = \cos(t)\,\mathbf{i} + \sin(t)\,\mathbf{k}$.

The *x* and *z* components of the vector are $x = \cos(t)$ and $z = \sin(t)$. The projection on the *x*-*z* plane is just the unit circle. As *t* increases, the *y* component increases, so the 3D shape is a coil around the *y*-axis.

### Vector equations of curves

- $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where *f*, *g* and *h* are functions is the **vector equation** of a 3D curve.
- The **parameter** *t* may have restrictions on its value.
- Projections on the *y*-*z*, *x*-*z* and *x*-*y* planes have no **i**, **j** and **k** components respectively.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## TECHNOLOGY

### 3D curve sketcher

Download the spreadsheet 3D Sketcher from the NelsonNet website.

Use the spreadsheet to sketch the following curves.

- $\mathbf{r}(t) = 4t\mathbf{i} + 2t\mathbf{j} + 3t\mathbf{k}$
- $\mathbf{r}(t) = t^2\mathbf{i} + 3t\mathbf{j} + 4\mathbf{k}$
- $\mathbf{r}(t) = 4t\mathbf{i} + 2t\mathbf{j} + 3t\mathbf{k}$
- $\mathbf{r}(t) = e^t\mathbf{i} + t\mathbf{j} + 5t\mathbf{k}$

Try making up some curves of your own.

3D Sketcher

You can often use projections of the plane perpendicular to each axis to work out the shape of a curve in 3D.

## EXAMPLE 12

What shape is the curve $\mathbf{r}(t) = 3t\mathbf{i} + \sin(t)\mathbf{j} + t\mathbf{k}$?

### Solution

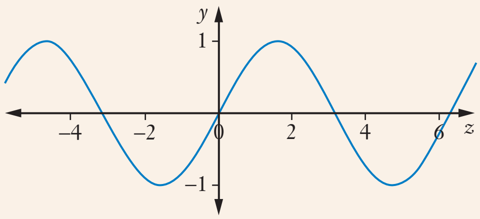
What is the domain of $t$?

As there is no restriction, assume $t \in \boldsymbol{R}$.

Consider the projection on the $y$-$z$ plane, perpendicular to $\mathbf{i}$.

The projection on the $y$-$z$ plane has no $\mathbf{i}$ component, so $\mathbf{r}'(t) = \sin(t)\mathbf{j} + t\mathbf{k}$. Along the $z$-axis, $t$ varies from $-\infty$ to $\infty$ and the function varies as $\sin(t)$.

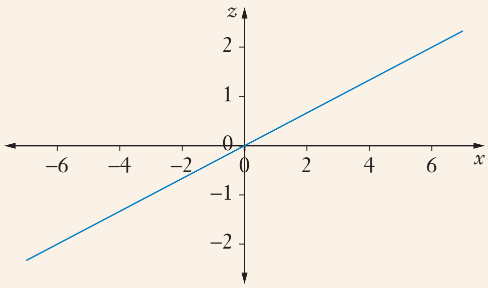
Sketch the projection on the $y$-$z$ plane.




Consider the projection on the $x$-$z$ plane, perpendicular to $y$.

The projection on the $x$-$z$ plane has no $\mathbf{j}$ component, so $\mathbf{r}'(t) = 3t\mathbf{i} + t\mathbf{k}$. Along the $z$-axis, $t$ varies from $-\infty$ to $\infty$ and the function varies as $3t$.

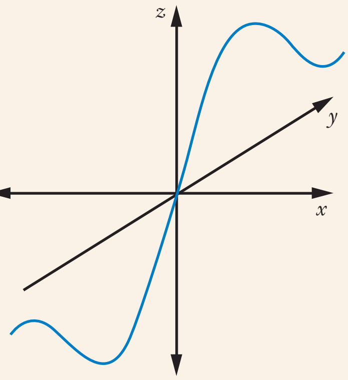
Sketch the projection on the $x$-$z$ plane.




State the shape.

The graph is shaped like sine, but the axis is along the line $x = 3z$.

If you are confident in doing so, you can sketch the graph to illustrate the shape.




You may be asked to write the equation of a 2D vector graph in the form $y = f(x)$. You can also do this for the projection of a 3D curve on one of the planes.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You can change a vector graph in 2D to the **Cartesian** (conventional) form by eliminating the parameter.

## EXAMPLE 13

Change the graph of $\mathbf{r}(t) = 2t\mathbf{i} + 8t^2\mathbf{j}$ to the form $y = f(x)$. Then change it to the Cartesian form and identify the shape.

### Solution

| | |
|---|---|
| Write in the form $x = f(t)$ and $y = g(t)$. | The line is given by $x = 2t$, $y = 8t^2$ |
| Write $t = \ldots$ from the simplest equation. | $t = \frac{x}{2}$ |
| Substitute in the other equation and simplify. | $y = 8t^2$<br>$= 8\left(\frac{x}{2}\right)^2$<br>$= 2x^2$ |
| Write the answer. | The Cartesian form is $y = 2x^2$, which is a parabola with its vertex at the origin. |

## Exercise 2.05 Vector equations of curves

Example 12

**1** What shape is the curve $\mathbf{r}(t) = \cos(t)\mathbf{i} + 0.5t\mathbf{j} + t\mathbf{k}$?

**2** What shape is the curve $\mathbf{r}(t) = t\mathbf{i} + \sin(t)\mathbf{j} + \cos(t)\mathbf{k}$?

**3** What shape is the curve $\mathbf{r}(t) = 3t\mathbf{i} + 4t\mathbf{j} + t\mathbf{k}$?

**4** What shape is the curve $\mathbf{r}(t) = 4t\mathbf{i} - t\mathbf{j} - 2t\mathbf{k}$?

Example 13

**5** Express $\mathbf{r}(t) = t\mathbf{i} + (t-2)^3\,\mathbf{j}$ in Cartesian form and identify the shape.

**6** Write $\mathbf{r}(t) = (3t-2)\mathbf{i} + (4t+1)\mathbf{j}$ in Cartesian form and identify the shape.

**7** Express $\mathbf{r}(t) = \sqrt{t-3}\,\mathbf{i} + 4t\mathbf{j}$ in Cartesian form and identify the shape.

### Problem solving

**8** Express the projection of $\mathbf{r}(t) = \frac{1}{2}t\mathbf{i} + 5(t-3)^2\mathbf{j} + 2t\mathbf{k}$ on the $x$-$y$ plane in Cartesian form and identify the shape.

**9** Express the projection of $\mathbf{r}(t) = 4t\mathbf{i} + (2t-3)\mathbf{j} + t\mathbf{k}$ perpendicular to the $z$-axis in Cartesian form and identify the shape.

**10** Write the projection of $\mathbf{r}(t) = \sqrt{t-3}\,\mathbf{i} + \sqrt{7-t}\,\mathbf{j} + 3t\mathbf{k}$ on the $x$-$y$ plane in Cartesian form and identify the shape.

WS

Equations of lines in space

# 2.06 Equations of lines in 3D

A straight line passing through the points $A(2, -1, 1)$ and $B(4, 1, 5)$ is in the direction of $\mathbf{AB} = 2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$. Any point $P(x, y, z)$ on the line must be in this direction from $A$. The position vector of $P$ will be given by $\mathbf{OP} = \mathbf{OA} + \mathbf{AP}$.

Since $\mathbf{AP}$ is in the same direction as $\mathbf{AB}$, it will be a multiple of $\mathbf{AB}$, say $t\mathbf{AB}$, where $t \in \boldsymbol{R}$.

$\mathbf{OP} = \mathbf{OA} + t\mathbf{AB} = 2\mathbf{i} - \mathbf{j} + \mathbf{k} + t(2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k})$, so $\mathbf{OP} = (2t + 2)\mathbf{i} + (2t - 1)\mathbf{j} + (4t + 1)\mathbf{j}$.

The vector equation of the line is $\mathbf{r}(t) = (2t + 2)\mathbf{i} + (2t - 1)\mathbf{j} + (4t + 1)\mathbf{j}$.

## Equations of lines in 3D

The equation of a straight line through $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ is

- $\mathbf{r}(t) = \mathbf{a} + t\mathbf{AB}$ or $\mathbf{r} = \mathbf{a} + k\mathbf{d}$ as a **vector equation**, where $t, k \in \boldsymbol{R}$, $\mathbf{a}$ is on the line and $\mathbf{AB} = \mathbf{d}$ is in the direction of the line
- $\mathbf{r}(t) = [a_1 + t(b_1 - a_1)]\mathbf{i} + [a_2 + t(b_2 - a_2)]\mathbf{j} + [a_3 + t(b_3 - a_3)]\mathbf{k}$ as a **parametric vector equation**
- $\dfrac{x - a_1}{b_1 - a_1} = \dfrac{y - a_2}{b_2 - a_2} = \dfrac{z - a_3}{b_3 - a_3}$ or $x - \dfrac{a_1}{d_1} = y - \dfrac{a_2}{d_2} = z - \dfrac{a_3}{d_3}$ as a Cartesian equation
- In the vector forms, if $0 \le t \le 1$, the points are on the **line segment** **AB**,

  if $t < 0$, the points are on the line to the left of $A$

  if $t > 1$, the points are on the line to the right of $B$

You can get the vector forms the same way as the equation of the line through $A(2, -1, 1)$ and $B(4, 1, 5)$.

You obtain the Cartesian form by rewriting each of the components as $t = \ldots$ and equating them.

### EXAMPLE 14

**a** Find the vector equations of lines passing through:

  **i** $(-2, 4, 3)$ and $(2, -1, 5)$

  **ii** $(3, 4)$ and $(5, -2)$

**b** Write the equation from part **a** in Cartesian form.


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## Solution

**a** **i** Find the displacement vector.

$\mathbf{d} = \mathbf{AB} = \mathbf{b} - \mathbf{a}$

$= (2, -1, 5) - (-2, 4, 3)$

$= (4, -5, 2)$

Write the formula.

$\mathbf{r}(t) = \mathbf{a} + t\mathbf{AB}$

Substitute values and simplify.

$= -2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k} + t(4\mathbf{i} - 5\mathbf{j} + 2\mathbf{k})$

$= (4t - 2)\mathbf{i} + (4 - 5t)\mathbf{j} + (2t + 3)\mathbf{k}$

**ii** Find the displacement vector.

$\mathbf{AB} = \mathbf{b} - \mathbf{a}$

$= (5, -2) - (3, 4)$

$= (2, -6)$

Write the formula.

$\mathbf{r}(t) = \mathbf{a} + t\mathbf{AB}$

Substitute values and simplify.

$= 3\mathbf{i} + 4\mathbf{j} + t(2\mathbf{i} - 6\mathbf{j})$

$= (2t + 3)\mathbf{i} + (4 - 6t)\mathbf{j}$

> You can substitute $t'$ for $2t$, so $\mathbf{r}(t) = (2t + 3)\mathbf{i} + (4 - 6t)\mathbf{j}$ is the same line as $\mathbf{r}(t) = (t' + 3)\mathbf{i} + (4 - 3t')\,\mathbf{j}$.

**b** **i** Write the Cartesian formula.

$$\frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2} = \frac{z - a_3}{b_3 - a_3}$$

Substitute values and simplify.

$$\frac{x - (-2)}{2 - (-2)} = \frac{y - 4}{(-1) - 4} = \frac{z - 3}{5 - 3}$$

$$\frac{x + 2}{4} = \frac{y - 4}{-5} = \frac{z - 3}{2}$$

**ii** Write the Cartesian formula.

$$\frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2} = \frac{z - a_3}{b_3 - a_3}$$

Substitute values and simplify.

$$\frac{x - 3}{5 - 3} = \frac{y - 4}{(-2) - 4}$$

$$\frac{x - 3}{2} = \frac{y - 4}{-6}$$

$$2 \times \frac{x - 3}{2} = 2 \times \frac{y - 4}{-6}$$

$$x - 3 = \frac{y - 4}{-3} \text{ or } y = -3x + 13$$

## EXAMPLE 15

**a** Find the vector equation of the line segment from $P(1, 4, -1)$ to $Q(5, 1, 1)$.

**b** Is the point $(7, \frac{1}{2}, 2)$ on the line segment $PQ$?

### Solution

**a** Find the displacement vector.

$$\mathbf{PQ} = \mathbf{q} - \mathbf{p}$$
$$= (5, 1, 1) - (1, 4, -1)$$
$$= (4, -3, 2)$$

Write the formula.

$$\mathbf{r}(t) = \mathbf{p} + t\mathbf{PQ}$$

Substitute values and simplify.

$$= \mathbf{i} + 4\mathbf{j} - \mathbf{k} + t(4\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})$$
$$= (4t + 1)\mathbf{i} + (4 - 3t)\mathbf{j} + (2t - 1)\mathbf{k}$$

Write the answer, including the restriction of $t$, for the line segment.

The line segment **PQ** is

$\mathbf{r}(t) = (4t + 1)\mathbf{i} + (4 - 3t)\mathbf{j} + (2t - 1)\mathbf{k}$

for $0 \leq t \leq 1$.

**b** Find the value of $t$ for each coordinate.

$$4t_x + 1 = 7$$
$$t_x = 1.5$$
$$4 - 3t_y = 0.5$$
$$t_y = 1.5$$
$$2t_z - 1 = 2$$
$$t_z = 1.5$$

State the conclusion.

$t = 1.5$ gives the correct coordinates, but $t$ is outside the segment domain, so the point is on the line, but not the segment.


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## Exercise 2.06 Straight lines in 3D

Example 14

**1** Find the vector equation of the line passing through:

**a** (7, –6, –3) and (2, 5, –4)
**b** (–1, –5) and (2, 2)
**c** (2, 1, –6) and (3, 2, 1)
**d** (–2, 6) and (2, –7)
**e** (–1, 4, –5) and (7, –3, –1)

**2** Find the Cartesian equation of the line passing through:

**a** (–4, 3) and (2, 4)
**b** (6, –3, –6) and (–7, 5, 2)
**c** (5, 4, 2) and (7, 3, –5)
**d** (5, –4, 5) and (4, –5, 3)
**e** (1, –2, 5) and (6, 0, –3)

**3** Find the vector equation of the line passing through each point in the direction of the given vector.

**a** (5, 4, –3) and $-7\mathbf{i} + 7\mathbf{j} - \mathbf{k}$
**b** (5, –1) and $4\mathbf{i} - 7\mathbf{j}$
**c** (–5, 6, –3) and $5\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$
**d** (4, –3, –7) and $-6\mathbf{i} + 12\mathbf{k}$
**e** (–7, 7, 4) and $-14\mathbf{i} + 14\mathbf{j}$

**4** Find the vector equation of the line through (1, 4, –4) parallel to the line through (2, 5, –2) and (4, –4, –5).

Example 15

**5** Find the vector equations of the line segments joining each pair of points.

**a** (2, 6, 1) and (2, –6, –8)
**b** (–2, –6, 8) and (–1, –3, 2)
**c** (–4, –5, –6) and (–7, 3, –4)
**d** (–7, –5, 5) and (–6, –2, –7)
**e** (7, –3, –1) and (0, 2, 7)

**6** State whether each point is on the given line or line segment.

**a** (18, –6, –9), $\mathbf{r}(t) = 6\mathbf{i} - 8\mathbf{j} - 3\mathbf{k} + t(6\mathbf{i} + \mathbf{j} - 3\mathbf{k})$
**b** (5, 1, 4), $\mathbf{r}(t) = 7\mathbf{i} - 5\mathbf{j} + 5\mathbf{k} + t(-4\mathbf{i} + 12\mathbf{j} - 3\mathbf{k})$ for $0 \le t \le 1$
**c** (2, –9, –16), $\mathbf{r}(t) = 8\mathbf{i} - 3\mathbf{j} + 6\mathbf{k} + t(3\mathbf{i} + 3\mathbf{j} - 5\mathbf{k})$
**d** (20, –12, –3), $\mathbf{r}(t) = 3\mathbf{i} + 8\mathbf{j} + t(11\mathbf{i} - 13\mathbf{j} - 2\mathbf{k})$ for $0 \le t \le 1$
**e** (–21, 6, –22), $\mathbf{r}(t) = -7\mathbf{i} - 8\mathbf{k} + t(9\mathbf{i} - 4\mathbf{j} + 9\mathbf{k})$

### Problem solving

**7** Explain why the usual Cartesian form of the line through (3, 5, 7) and (4, 9, 7) is invalid and give an alternative.

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WS

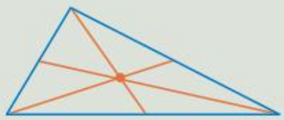
Geometric proofs using vectors

## 2.07 Geometric applications of vectors

You should remember the different triangle 'lines' and 'centres' from Year 11.

### Triangle centres

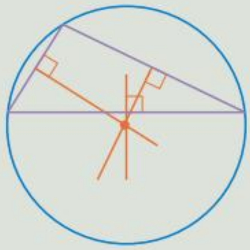
- The **medians** of a triangle join the vertices to the midpoints of the opposite sides
- The **centroid** is the intersection of the medians



Medians and centroid

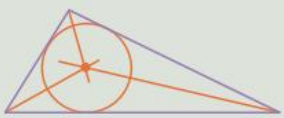
- The **circumcentre** is the intersection of the perpendicular bisectors of each side.
- It is the centre of the **circumcircle** that passes through the vertices

Perpendicular bisectors and circumcentre

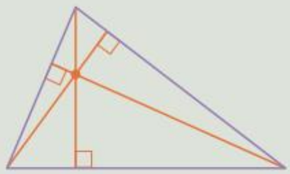


- The **in-centre** is the intersection of the angle bisectors of each vertex. It is the centre of the **in-circle** to which each side is a tangent

Angle bisectors and in-centre



- The **altitudes** of a triangle pass through the vertices and are perpendicular to the opposite sides
- The **orthocentre** is the intersection of the altitudes



Altitudes and orthocentre

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The following vector properties are often used in geometric proofs.

### Vector properties

- The **displacement vector** from $A$ to $B$ is $\mathbf{AB} = \mathbf{b} - \mathbf{a}$
- If $P$ is the midpoint of $AB$, then $\mathbf{p} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$
- The sum of a series of 'end-to-end' displacements is the displacement from the beginning to the end. For example: $\mathbf{AB} + \mathbf{BC} + \mathbf{CD} + \mathbf{DE} = \mathbf{AE}$
- The displacement from $A$ to $A$, $\mathbf{AA} = \mathbf{0}$

You can choose the origin to be anywhere in a geometric proof. Similarly, you can make the $x$ or $y$-axis in any direction that does not lose generality. The most convenient point for the origin is usually at one of the vertices of the shape. It is usually easiest to choose the $x$-axis along one of the sides.

In a proof, remember to clearly state what has to be proven. Show the proof in logical steps and finish with **QED**.

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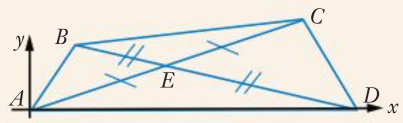
## EXAMPLE 16

Show that if the diagonals of a quadrilateral bisect each other, then it must be a parallelogram.

### Solution

For a quadrilateral *ABCD*, choose the origin at *A*. Put the *x*-axis in the direction of *AD*. Call the intersection of the diagonals *E*.

Sketch a diagram.




Specifically state what has to be proven for the constructed figure.

**RTP**

$BC \parallel AD$ and $AB \parallel DC$

**Proof**

Use the bisection of the diagonals.

$\mathbf{BE} = \frac{1}{2}\mathbf{BD} = \mathbf{ED}$ (*E* is the midpoint of *BD*)

$\mathbf{EC} = \frac{1}{2}\mathbf{AC} = \mathbf{AE}$ (*E* is the midpoint of *AC*)

Use $\triangle BEC$ and $\triangle AED$ to prove that $\mathbf{BC} = \mathbf{AD}$.

$$\begin{aligned}\mathbf{BC} &= \mathbf{BE} + \mathbf{EC} \\ &= \mathbf{ED} + \mathbf{AE} \\ &= \mathbf{AE} + \mathbf{ED} \\ &= \mathbf{AD}\end{aligned}$$

Use the vector equality.

Thus $BC \parallel AD$. (Equal vectors are parallel)

Now use the adjacent triangles for the other sides.

Similarly, $\mathbf{AB} = \mathbf{AE} + \mathbf{EB} = \mathbf{DE} + \mathbf{EC} = \mathbf{DC}$

Thus $AB \parallel DC$. **QED**

You should remember that the sum of the displacements around a polygon is **0**. This can be generalised to the following.

### Displacements in a closed path

Displacements in 2 or 3 dimensions that form a closed path have a sum of **0**.
For example, $\mathbf{PQ} + \mathbf{QR} + \mathbf{RS} + \mathbf{ST} + \mathbf{TP} = \mathbf{0}$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You should be able to see that this is a special case of **PP** = **0**.

The dot product can be used to prove perpendicularity of non-trivial vectors.

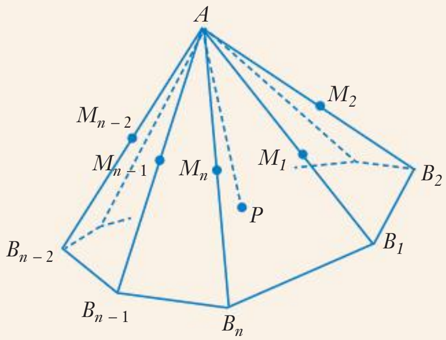
## EXAMPLE 17

Show that the plane formed by the midpoints of the edges from the base to the apex of a pyramid is parallel to the base.

### Solution

Construct a pyramid with apex $A$ and base $B_1B_2B_3\ldots B_n$. Name the midpoints of the edges $B_1A$, $B_2A$, $B_3A$, …, $B_nA$ as $M_1$, $M_2$, $M_3$, …, $M_n$ respectively.

Construct a line $PA$ perpendicular to the base.




State the specific requirement.

**RTP**

$M_1M_2 \perp PA$, $M_2M_3 \perp PA$, …, $M_nM_1 \perp PA$

Use the perpendicularity.

Since $AP$ is perpendicular to the base, then

$B_1B_2 \perp PA$, $B_2B_3 \perp PA$, …, $B_nB_1 \perp PA$

Express $\mathbf{m}_1$ and $\mathbf{m}_2$ in terms of $\mathbf{a}$, $\mathbf{b}_1$ and $\mathbf{b}_2$.

$$\mathbf{m}_1 = \frac{1}{2}(\mathbf{a} + \mathbf{b}_1)$$

$$\mathbf{m}_2 = \frac{1}{2}(\mathbf{a} + \mathbf{b}_2)$$

Express $\mathbf{M_1M_2}$ in terms of $\mathbf{m}_1$ and $\mathbf{m}_2$.

$$\mathbf{M_1M_2} = \mathbf{m}_2 - \mathbf{m}_1$$

Substitute and simplify.

$$= \frac{1}{2}(\mathbf{a} + \mathbf{b}_2) - \frac{1}{2}(\mathbf{a} + \mathbf{b}_1)$$

$$= \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}_2 - \frac{1}{2}\mathbf{a} - \frac{1}{2}\mathbf{b}_1$$

$$= \frac{1}{2}\mathbf{b}_2 - \frac{1}{2}\mathbf{b}_1$$

$$= \frac{1}{2}(\mathbf{b}_2 - \mathbf{b}_1)$$

| | |
|---|---|
| Write the dot product of $\mathbf{M_1M_2}$ and $\mathbf{PA}$ in terms of $\mathbf{m}_1$ and $\mathbf{m}_2$. | $\mathbf{M_1M_2} \bullet \mathbf{PA} = (\mathbf{m}_2 - \mathbf{m}_1) \bullet \mathbf{PA}$ |
| Substitute for $(\mathbf{m}_2 - \mathbf{m}_1)$. | $= \frac{1}{2}(\mathbf{b}_2 - \mathbf{b}_1) \bullet \mathbf{PA}$ |
| Write in terms of $\mathbf{B_1B_2}$. | $= \frac{1}{2}\mathbf{B_1B_2} \bullet \mathbf{PA}$ |
| Use perpendicularity to the base. | $= \frac{1}{2} \times 0 \quad (B_1B_2 \perp PA)$<br>$= 0$ |
| State the implication. | Hence $M_1M_2 \perp PA$. |
| Generalise. | Similarly, $M_2M_3 \perp PA, \ldots, M_nM_1 \perp PA$. **QED** |

The last step in the above generalises the proof of one part to show that the premise holds for all parts.

## Exercise 2.07 Geometric applications of vectors

Example 16

Example 17

**1** Show that the midpoints of the sides of a quadrilateral form a parallelogram.

**2** Show that the plane formed by the midpoints of the edges from the base to the top of a prism is parallel to the base.

### Problem solving

**3** Find a vector expression in terms of **a** and **b** for the position vector of the point $P$ that divides $AB$ in the ratio $k_1 : k_2$.

**4** In the triangle $ABC$, a median is drawn from $A$ to the midpoint $D$ of the side $BC$. The point $M$ divides $AD$ in the ratio 2 : 1.

- **a** Find an expression for **m** in terms of **a**, **b** and **c**.
- **b** Use the result of part **a** to show that the medians are concurrent.
- **c** Write an expression for the position vector of the centroid of $ABC$.

**5** In a triangle $ABC$, the point $D$ divides $BC$ in the ratio $|\mathbf{AB}| : |\mathbf{AC}|$.

- **a** Express **d** in terms of **a**, **b** and **c**.
- **b** Use the scalar product to find $(\cos \angle DAC - \cos \angle DAB)$.
- **c** Use your results to make an inference about the angle bisectors of a triangle.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6 If $G$ is the centroid of a triangle $ABC$, prove that $\mathbf{GA} + \mathbf{GB} + \mathbf{GC} = \mathbf{0}$.

7 $ABCD$ is a tetrahedron. $H$ is the point that divides the line joining $A$ to the centroid of $BCD$ in the ratio 3 : 1.

**a** Find an expression for the position vector of $H$ in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$.

**b** Show that the four lines joining the vertices of the tetrahedron $ABCD$ to the centroids of the opposite faces are concurrent.

8 Find an expression for the midpoint of the line joining the midpoints of $AB$ and $CD$, which are two edges of the tetrahedron $ABCD$.

## 2.08 Paths of particles

You have already studied movement in a straight line in the Mathematical Methods course. You can extend this to movement in the plane or in space using 3D vectors. Vector functions of time are useful for this purpose.

### EXAMPLE 18

The paths of 2 particles moving on a horizontal plane are given by $\mathbf{p}(t) = 2t\mathbf{i} - (4t^2 - 38t - 20)$ and $\mathbf{q}(t) = 3t\mathbf{i} + (105 - 9t)\mathbf{j}$.
If their paths cross, where do they do cross?

#### Solution

| | |
|---|---|
| Write the coordinates of $\mathbf{p}(t)$. | $x = 2t, y = 38t + 20 - 4t^2$ |
| Eliminate $t$. | $t = 0.5x$, so |
| | $y = 38 \times 0.5x + 20 - 4 \times (0.5x)^2$ |
| | $= 19x + 20 - x^2$ |
| Write the path for $\mathbf{p}(t)$. | The path of the first is $y = 19x + 20 - x^2$. |
| Write the coordinates of $\mathbf{q}(t)$. | $x = 3t, y = 105 - 9t$ |
| Eliminate $t$. | $t = \frac{1}{3}x$ |
| | $y = 105 - 9 \times \left(\frac{1}{3}x\right)$ |
| | $= 105 - 3x$ |

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| | |
|---|---|
| Write the path for $\mathbf{q}(t)$. | The other path is $y = 105 - 3x$. |
| Solve the paths. | $19x + 20 - x^2 = 105 - 3x$ |
| | $x^2 - 22x + 85 = 0$ |
| | $(x - 5)(x - 17) = 0$ |
| | $x = 5$ or $x = 17$ |
| | At $x = 5, y = 90$ |
| | At $x = 17, y = 54$ |
| State the intersections. | Their paths cross at (5, 90) and (17, 54). |

Particles **collide** when their coordinates are the same at the same time.

## EXAMPLE 19

Two particles are moving in paths given by the vectors $\mathbf{v}(t) = 6t\mathbf{i} + (3t - 1)\mathbf{j} + (4t + 1)\mathbf{k}$ and $\mathbf{w}(t) = (2t + 20)\mathbf{i} + (5t - 9)\mathbf{j} + (3t + 7)\mathbf{k}$. Find whether they collide, and if they collide, where and when they do.

### Solution

| | |
|---|---|
| Find when the $x$-coordinates are the same. | $6t = 2t + 20$ |
| | $t = 5$ |
| Substitute in the vectors. | $\mathbf{v}(5) = 30\mathbf{i} + 14\mathbf{j} + 21\mathbf{k}$ |
| | $\mathbf{w}(5) = 30\mathbf{i} + 16\mathbf{j} + 22\mathbf{k}$ |
| State if they collide. | The particles do not collide. |

## INVESTIGATION

### MISSING PARTICLES

Two particles do not collide.

- How do you find the distance between them at any time?
- How do you find the minimum distance?
- How do you find by how much they miss each other?
- Find by how much the particles in Example **19** miss each other.


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## Exercise 2.08 Paths of particles

Example 18

1 The paths of 2 particles are given by $\mathbf{m}(t) = (2t + 1)\mathbf{i} + (2t + 3)\mathbf{j}$ and $\mathbf{n}(t) = (4 - t)\mathbf{i} + 2(t + 3)\mathbf{j}$. If their paths cross, where do they do cross?

2 The paths of 2 particles are given by $\mathbf{a}(t) = (t + 2)\mathbf{i} + (t^2 + 9)\mathbf{j}$ and $\mathbf{b}(t) = (9 - t)\mathbf{i} + (3t - 19)\mathbf{j}$. If their paths cross, where do they do so?

3 The paths of 2 particles are given by $\mathbf{d_1}(t) = (2t - 1)\mathbf{i} + (2t + 11)\mathbf{j} + (4t + 3)\mathbf{k}$ and $\mathbf{d_2}(t) = (t + 3)\mathbf{i} + (8t - t^2 + 9)\mathbf{j} + (4t + 9)\mathbf{k}$. If their paths cross, where do they do so?

Example 19

4 Two particles are moving in paths given by the vectors $\mathbf{v}(t) = (5t - 2)\mathbf{i} + (8t - 10)\mathbf{j}$ and $\mathbf{w}(t) = (3t + 12)\mathbf{i} + (7t - 3)\mathbf{j}$. Find whether they collide, and if they collide, where and when they do.

5 The paths of 2 particles are given by $\mathbf{c}(t) = (t^2 - 10t + 28)\mathbf{i} + (3t + 20)\mathbf{j}$ and $\mathbf{d}(t) = (2t - 4)\mathbf{i} + (t^2 - 20)\mathbf{j}$. Find whether they collide, and if they collide, where and when they do.

6 Two particles are moving in paths given by the vectors $\mathbf{s_1}(t) = (3t - 10)\mathbf{i} + (3t + 10)\mathbf{j} + (t^2 - 4t + 6)\mathbf{k}$ and $\mathbf{s_2}(t) = (2t - 4)\mathbf{i} + (t^2 - 8)\mathbf{j} + (24 - t)\mathbf{k}$. Find whether they collide, and if they collide, where and when they do.

### Problem solving

7 A balloon of diameter 20 m is ascending at 3 m/s 200 m away from an archer on horizontal ground. The archer fires an arrow towards the balloon at an angle of 60° to the horizontal and a speed of 50 m/s when the top of the balloon is at a height of 50 m. The arrow is acted upon by gravity, so has a downward acceleration of 9.8 m/s$^2$.

Write vector equations for the positions of the arrow and balloon and hence find whether or not the arrow strikes the balloon.

8 A plane flying from Cairns to Brisbane has a ground speed of 110 m/s and is at a height of 5000 m and 30 km directly south of the airport. It is ascending at 15 m/s and is flying at a bearing of 150°. At the same time, another aircraft is approaching Cairns and is 40 km SE of Cairns at a height of 8000 m. It is flying NW at 115 m/s relative to the ground and is descending at 20 m/s. Write vector equations for the locations relative to the airport and determine whether they should take evasive action to avoid a collision.

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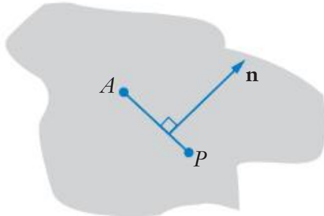
WS
Equations of planes

# 2.09 Equations of planes

Consider the point $A(3, -2, 5)$ in a plane perpendicular to the vector $\mathbf{n} = 5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$.
For any point $P(x, y, z)$ in the plane,

$\mathbf{PA} = \mathbf{p} - \mathbf{a}$ is perpendicular to $\mathbf{n}$.




Thus

$$(x - 3, y + 2, z - 5) \bullet (5, 3, -1) = 0$$
$$5(x - 3) + 3(y + 2) + (-1)(z - 5) = 0$$
$$5x - 15 + 3y + 6 - z + 5 = 0$$
$$5x + 3y - z = 4$$

The equation $5x + 3y - z = 4$ is called the **equation of the plane** because any point on the plane satisfies the equation.

Any other vector perpendicular to the plane will be parallel to $\mathbf{n} = 5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, so will be a multiple of $\mathbf{n}$. You can use the equation to find another point on the plane. You can check that using another point $B$ in the plane and a multiple of $\mathbf{n}$ will give the same equation.

## Equation of a plane

- A perpendicular $\mathbf{n} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$ to a plane $\Gamma$ is called a **normal** of the plane.
- The **equation of a plane** $\Gamma$ through $A(a_1, a_2, a_3)$ with normal $\mathbf{n} = (l, m, n)$ is given by

  $$l(x - a_1) + m(y - a_2) + n(z - a_3) \text{ or } lx + my + nz = k,$$

  where $P(x, y, z)$ is any point in the plane.
- The **vector equation of a plane** $\Gamma$ is given by $\mathbf{n} \bullet (\mathbf{p} - \mathbf{a}) = 0$, $\mathbf{p} \bullet \mathbf{n} = \mathbf{a} \bullet \mathbf{n} = k$ or $\mathbf{n} \bullet \mathbf{r} = \mathbf{n} \bullet \mathbf{a}$, where $P$ is a point in the plane and $\mathbf{r}$ is the position vector of points in the plane

You can substitute $A(a_1, a_2, a_3)$ and $\mathbf{n} = (l, m, n)$ for $A(3, -2, 5)$ and $\mathbf{n} = (5, 3, -1)$ to prove the general rules for the equation of a plane.

### EXAMPLE 20

A plane $\Gamma$ with normal $(2, 3, 5)$ contains the point $A(5, 3, -6)$.

**a** Find the Cartesian equation of the plane.

**b** Write a vector equation of the plane.

**c** Determine whether $(4, -8, 2)$ is on the plane.

**d** Find a relation for the points on $\Gamma$ inside a circle with centre $C(4, 2, -5)$ and radius 4.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## Solution

**a** Write the formula. $lx + my + nz = k$

Substitute the normal. $2x + 3y + 5z = \text{k}$

Substitute $A$ to find $k$. $k = 2 \times 5 + 3 \times 3 + 5 \times (-6)$

$= -11$

Write the answer. The Cartesian equation of the plane is $2x + 3y + 5z = -11$.

**b** Write in vector form. The vector equation of the plane is $\mathbf{n} \bullet \mathbf{p} = -11$, where $\mathbf{n} = (2, 3, 5)$ and $P$ is on the plane.

**c** Substitute the point in the LHS of the Cartesian equation. $\text{LHS} = 2 \times 4 + 3 \times (-8) + 5 \times 2$

$= -6 \neq \text{RHS}$

Write the answer. $(4, -8, 2)$ is not on the plane.

**d** Check that $C$ is on the plane. $\text{LHS} = 2 \times 4 + 3 \times 2 + 5 \times (-5)$

$= -11 = \text{RHS}$

Write the distance of the points inside the circle from $A$. $\sqrt{(x-4)^2 + (y-2)^2 + (z+5)^2} < 4$

Square both sides. $(x-4)^2 + (y-2)^2 + (z+5)^2 < 16$

Rearrange the equation of the plane to isolate one variable.

$2x = -11 - 3y - 5z$

$x = \dfrac{-11 - 3y - 5z}{2}$

Substitute in the relation and simplify.

$\left(\dfrac{-11 - 3y - 5z}{2} - 4\right)^2 + (y-2)^2 + (z+5)^2 < 16$

$(-19 - 3y - 5z)^2 + 4(y-2)^2 + 4(z+5)^2 < 64$

$361 + 9y^2 + 25z^2 + 114y + 190z + 30yz$

$+ 4y^2 - 16y + 16 + 4z^2 + 40z + 100 < 64$

$13y^2 + 29z^2 + 30yz + 98y + 230z < -413$

Write the answer. The points in the circle satisfy the relation

$13y^2 + 29z^2 + 30yz + 98y + 230z < -413$

and $x = \dfrac{-11 - 3y - 5z}{2}$.

There are 3 possible forms of the answer to part **d**, depending on which variable is expressed in terms of the others.

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You can use the cross product to find a normal to a plane containing 3 given points.

## EXAMPLE 21

**a** The plane $\Gamma_1$ contains the points $A(3, 2, -1)$, $B(5, -2, 3)$ and $C(1, 5, 2)$. Find its equation.

**b** Find the equation the plane $\Gamma_2$ through $D(1, 4, 7)$ parallel to $\Gamma_1$.

### Solution

**a** Find the vectors **AB** and **AC**.

$\mathbf{AB} = (5 - 3, -2 - 2, 3 - (-1))$
$= (2, -4, 4)$
$\mathbf{AC} = (1 - 3, 5 - 2, 2 - (-1))$
$= (-2, 3, 3)$

Find the cross product.

$\mathbf{AB} \times \mathbf{AC} = (2, -4, 4) \times (-2, 3, 3)$
$= (-24, -14, -2)$

Any multiple is also perpendicular, so choose a simple one.

$\mathbf{n} = -\frac{1}{2}(-24, -14, -2)$
$= (12, 7, 1)$

Write the equation.

$12x + 7y + z = k$

Substitute a point to fin the value of $k$.

$k = 12 \times 3 + 7 \times 2 + 1 \times (-1) = 49$

Write the answer.

The equation of $\Gamma_1$ is $12x + 7y + z = 49$.

**b** $\Gamma_1$ and $\Gamma_2$ are parallel.

**n** is also a normal for $\Gamma_2$

Write the equation of $\Gamma_2$.

$12x + 7y + z = k_2$

Substitute $D$ to find $k_2$.

$k_2 = 12 \times 1 + 7 \times 4 + 1 \times 7 = 47$

Write the answer.

The equation of $\Gamma_2$ is $12x + 7y + z = 47$.

The equations of parallel planes have the same coefficients, only the constants are different.


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## Exercise 2.09 Equations of planes

Example 20

**1** Find the Cartesian equation of a plane with:

**a** normal (1, –4, 5) and point (–3, 4, –2)

**b** normal $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and point (5, 0, 3)

**c** normal (–3, 0, 6) and point (4, –1, 6)

**2** Find the vector equation of a plane with:

**a** normal $-\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$ and point (4, –4, 5)

**b** normal (1, –6, 3) and point (–2, 4, 4)

**c** normal $5\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$ and point (1, 0, –3)

**3** State whether each point is on the plane $4x - 3y + 4z = 8$.

**a** (1, 6, 4) **b** (–1, 8, 9) **c** (3, 4, 2) **d** (2, 8, 4) **e** (3, –4, –4)

**4** State whether each point is on the plane $(3\mathbf{i} - \mathbf{k}) \bullet \mathbf{p} = -4$.

**a** (2, 5, 10) **b** (0, 3, 2) **c** (5, 6, 2) **d** (1, –4, 7) **e** (1, 0, 7)

**5** Find a relation for points on the plane $2x - 5y + z = 9$ and outside the circle with centre (4, –1, 4) and radius 5.

**6** Find a relation for points on or inside the circle with centre (4, –4, 2), radius 9 and lying on the plane $(2, 3, 5) \bullet \mathbf{p} = 6$

Example 21

**7** Find the Cartesian equation of the plane containing the 3 points:

**a** (1, –1, 3), (4, 6, 2) and (5, 1, 0)

**b** (3, 0, –2), (–4, 2, 3) and (–3, 4, 5)

**8** Find the vector equation of the plane containing the 3 points:

**a** (0, 3, 0), (2, 5, 7) and (–3, 2, 4)

**b** (2, 2, 2), (5, 7, 9) and (–3, 2, –4)

**9** Find the equation of a plane through (1, 6, –8) parallel to $3x - 4y - 2z = 7$.

### Problem solving

**10** Find the equation of a plane containing the point (–2, –3, 5) and the vectors $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $3\mathbf{i} - 4\mathbf{k}$.

**11** Demonstrate that the vector equation of the plane through $A(a_1, a_2, a_3)$ with normal $\mathbf{n} = (l, m, n)$ is given by $\mathbf{p} \bullet \mathbf{n} = k$ and find the value of $k$, where $P$ is a point in the plane.

**12** Explain whether the planes $3x - 2y + z = 4$ and $2y - 3x + 2z = 4$ are parallel, perpendicular or neither.

**13** Find the shortest distance of the plane $3x - y - 4z = 2$ from the origin.

**14** Find the shortest distance of the plane $(3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}) \bullet \mathbf{p} = 10$ from the point (1, 4, 5).

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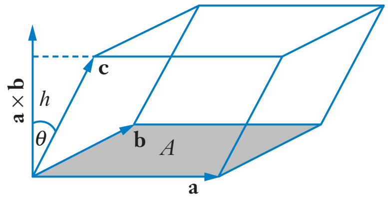
# 2.10 Applications of vectors

The area of a parallelogram formed by **a** and **b** is given by $A = |\mathbf{a} \times \mathbf{b}|$. A third vector **c**, not in the same plane, will make the parallelogram into a 3D shape called a **parallelepiped**. The volume of this will be given by $Ah$, where $h$ is the perpendicular height.




Now $h = c \tan(\theta)$, where $\theta$ is the angle between $\mathbf{a} \times \mathbf{b}$ and **c**.

Thus the volume is $V = |\mathbf{a} \times \mathbf{b}|\ c \tan(\theta) = |\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}|$. The absolute value signs are needed to ensure the volume is positive.

### Volume of a parallelepiped

The volume of the parallelepiped formed by the vectors **a**, **b** and **c** is given by

$$V = |\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}| = |\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}|$$

$\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$ has only one possible order of operations because $\mathbf{b} \times \mathbf{c}$ must be done first to give a vector for the dot product. $|\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}| = |\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}|$ because the volume is fixed.

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## EXAMPLE 22

Find the volume of the parallelepiped formed by the vectors (2, 3, –2), (1, 0, 3) and (–2, 1, –2).

### Solution

Write the rule. $V = |\mathbf{a} \times \mathbf{b} \bullet \mathbf{c}|$

Substitute the vectors and evaluate.

$$= |(2, 3, -2) \times (1, 0, 3) \bullet (-2, 1, -2)|$$
$$= |(3 \times 3 - (-2) \times 0, (-2) \times 1 - 2 \times 3, 2 \times 0 - 3 \times 1) \bullet (-2, 1, -2)|$$
$$= |(9, -8, -3) \bullet (-2, 1, -2)|$$
$$= |-18 - 8 + 6|$$
$$= 20$$

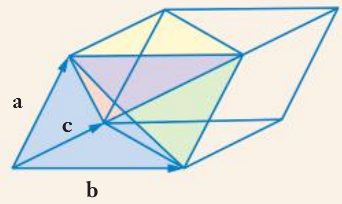
Write the answer. The volume is 20 cubic units.

## EXAMPLE 23

Find the volume of the tetrahedron formed by the vectors **a** = (1, 6, –1), **b** = (6, 0, 1) and **c** = (3, 4, –3).

### Solution

The diagram shows that 3 identical tetrahedrons are needed to make half a parallelepiped. Hence the volume of a tetrahedron is $\frac{1}{6}$th the volume of the corresponding parallelepiped.




Find the volume.

$$V = \frac{1}{6}|\mathbf{a} \times \mathbf{b} \bullet \mathbf{c}|$$
$$= \frac{1}{6}|(1, 6, -1) \times (6, 0, 1) \bullet (3, 4, -3)|$$
$$= \frac{1}{6}|(6, -7, -36) \bullet (3, 4, -3)|$$
$$= \frac{1}{6}|18 - 28 + 108|$$
$$= 16\frac{1}{3} \text{ cubic units}$$

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You have seen many examples of the use of vectors in Science. Displacement, velocity, acceleration and momentum are all vectors. The scalar and vector products are also important. The **work** done (energy transferred) by a force is the scalar product of the force and the displacement while the force is acting. For a force **F** and displacement **s**, $W = \mathbf{F} \cdot \mathbf{s}$.

When you are riding a bicycle, it is harder to keep your balance at low speeds than when you are going faster. This is because the wheels of a moving bike have **angular momentum**. It takes force to change the direction of their spin. This applies to any spinning object and is the principle of the gyroscope. The angular momentum of an object is the cross product of its momentum and radius. The angular momentum is thus either 'spin up' or 'spin down' from the plane of rotation.

You may know that life on Earth is possible because cosmic particles are deflected or trapped by the Earth's magnetic field. Any moving charge in a magnetic field is acted upon by an **electromagnetic force**. The force is equal to $q\mathbf{v} \times \mathbf{B}$, where $q$ is the charge, **v** is the velocity and **B** is the magnetic field strength. Alpha and Beta rays from the Sun are charged particles. When they reach the Earth, a force acts perpendicularly to their motion and the direction of the magnetic field. Some of the particles are deflected back into space, some get through, and some spiral along the magnetic field lines between the Earth's magnetic poles. If they strike the atmosphere at the poles, they cause light to be emitted. This is the shimmering light that can be sometimes seen on dark nights in Tasmania.

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## Exercise 2.10 Applications of vectors

Example 22

**1** Find the volume of the parallelepiped *formed by* $\mathbf{EF} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$, $\mathbf{EI} = 7\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and $\mathbf{EH} = 2\mathbf{j} - 3\mathbf{i} + 2\mathbf{k}$.

Example 23

**2** Find the volume of a tetrahedron formed by the vectors $\mathbf{AB} = (3, 5, 7)$, $\mathbf{AC} = (2, 2, 0)$ and $\mathbf{AD} = (-2, 3, 6)$.


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## Problem solving

**3** Find the volume of the parallelepiped $ABCDEFGH$ with vertices $A(2, -3, 4)$, $B(4, -4, 5)$, $C(3, 0, 0)$, $D(5, -1, 1)$, $E(7, 1, 6)$, $F(9, 0, 7)$, $G(8, 4, 2)$ and $H(10, 3, 3)$.

**4** Find the volume of the tetrahedron with vertices $P(-2, 3, -4)$, $Q(0, 6, -3)$, $R(1, 7, 1)$ and $S(-6, 6, 0)$.

**5** Find an expression in component form for $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

**6** Show that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \bullet \mathbf{c})\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\mathbf{c}$.

**7** Use the result of question **5** to show that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$.

**8** A boat and trailer are winched up an 8 m ramp inclined at 40° to the horizontal. The force of gravity on the boat and trailer is 1500 N. What work is done against gravity?

**9** A yacht is sailing on a bearing of 240° with a force of 6000 N exerted on its sails by a south-easterly wind. What work is done on the yacht when it sails a distance of 1 nautical mile (n.m.)? (1 n.m. = 1852 m)

**10** A downhill skier loses 15% of the energy gained going down a 50 m long slope at an angle of 60° to the horizontal, due to friction and air resistance. What energy is gained (from gravity) if the skier's total mass (with skis, etc.) is 78 kg?

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**11** The Earth has a mass of $6 \times 10^{24}$ kg and orbits the Sun at a radius of 150 000 000 km once a year. What is the angular momentum of the Earth arising from its orbit of the Sun?

**12** Jupiter orbits the Sun every 12 years and is 5 times further from the Sun than the Earth. It has a mass 318 times that of Earth. Compare the angular momentum arising from Jupiter's orbit of the Sun to that of Earth (question **11**).

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# 2. CHAPTER SUMMARY

## 3D vectors

### 3D coordinates

- The $x$, $y$ and $z$ axes are used for 3-dimensional coordinates
- The $z$-direction is chosen to make the system *right-handed*
- Boxes can be use to locate points in a 3D drawing
- A 3D **isometric** drawing shows axes at 120° to each other
- The distance between $(a_1, a_2, a_3)$ and $(b_1, b_2, b_3)$ is given by
$$d = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$$
- The equation of a sphere with centre $(a, b, c)$ and radius $r$ is
$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2, \text{ or}$$
$$x^2 - 2ax + y^2 - 2by + z^2 - 2cx = r^2 - a^2 - b^2 - c^2$$
- In Mathematics, the positive direction is *anticlockwise*

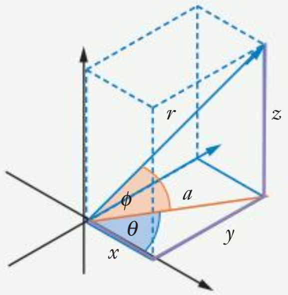
### 3D vectors

- The **unit vectors** in the $x$, $y$ and $z$ directions are **i**, **j** and **k** respectively
- The **component** form of a 3D vector is written as $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ or $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$
- The 3D **polar** (**spherical**) coordinates are $(r, \theta, \phi)$, where $r$ is the **magnitude** (**norm**), $\theta$ is the **azimuth** angle and $\phi$ is the **altitude** angle
- The azimuth is the angle from the $x$-axis to the projection $a$ of the vector onto the $xy$ plane. $0 \le \theta \le 2\pi$
- The **altitude** is the angle between the vector and the $xy$ plane. $-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$

$$a_1 = r\cos(\phi)\cos(\theta) \qquad r^2 = x^2 + y^2 + z^2 = a_1^2 + a_2^2 + a_3^2$$

$$a_2 = r\cos(\phi)\sin(\theta) \qquad \tan(\theta) = \frac{y}{x} = \frac{a_2}{a_1}$$

$$a_3 = r\sin(\phi) \qquad \sin(\phi) = \frac{z}{r} = \frac{a_3}{r}$$




- The **norm** (**magnitude**, **size**) of **v** is given by
$r = a = |\mathbf{a}| = \sqrt{x^2 + y^2 + z^2} = \sqrt{a_1^2 + a_2^2 + a_3^2}$
- The **position vector** of $P(x, y, z)$, **p** or **OP**, is the displacement from the origin to $P$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- Changing a polar vector to perpendicular components is called **resolving** a vector into **orthogonal** (perpendicular) components

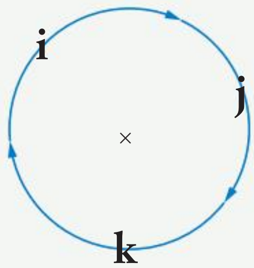
### Scalar product

- The **scalar product** (**inner product** or **dot product**) of vectors **a** and **b** is given by

$$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta), \text{ where } \theta \text{ is the angle between the vectors}$$

- In component form, $(a_1, a_2, a_3) \bullet (b_1, b_2, b_3) = a_1b_1 + a_2b_2 + a_3b_3$
- The scalar product is a real number, *not a vector*
- $\mathbf{i} \bullet \mathbf{i} = \mathbf{j} \bullet \mathbf{j} = \mathbf{k} \bullet \mathbf{k} = 1$ and $\mathbf{i} \bullet \mathbf{j} = \mathbf{j} \bullet \mathbf{k} = \mathbf{k} \bullet \mathbf{i} = 0$
- $(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \bullet (\mathbf{i}b_1 + \mathbf{j}b_2 + \mathbf{k}b_3) = a_1b_1 + a_2b_2 + a_3b_3$
- The angle between two vectors **a** and **b** is given by $\cos(\theta) = \dfrac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$
- The norm is given by $|\mathbf{a}|^2 = \mathbf{a} \bullet \mathbf{a}$
- Non-zero vectors **a** and **b** are perpendicular if and only if $\mathbf{a} \bullet \mathbf{b} = 0$
- Non-zero vectors **a** and **b** are parallel if and only if $\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}|\ |\mathbf{b}|$
- The projection of **a** on **b** is given by $p = \dfrac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{b}|} = \mathbf{a} \bullet \hat{\mathbf{b}}$ or $\mathbf{p} = \dfrac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{b}|}\hat{\mathbf{b}} = \dfrac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b} =$ $|\mathbf{a}| \cos(\theta)\, \hat{\mathbf{b}} = (\mathbf{a} \bullet \hat{\mathbf{b}})\, \hat{\mathbf{b}}$
- The scalar product is commutative: $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a}$
- It is distributive over vector addition: $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$

### Vector product




- The **vector product** $\mathbf{v} = \mathbf{a} \times \mathbf{b}$ is defined as $|\mathbf{v}| = |\mathbf{a}||\mathbf{b}| \sin(\theta)$, where $\theta$ is the angle between them and **a**, **b** and **v** form a right-handed system
- $\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$, where $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$
- Cross products in the cycle direction of components give the other unit vector and in the opposite direction give its negative
- $(a_1, b_1, c_1) \times (a_2, b_2, c_2) = \left( \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}, -\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right)$
- $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$
- $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$
- $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$ and $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$
- The vector product is distributive over vector addition:

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} \text{ for any vectors } \mathbf{a}, \mathbf{b}, \mathbf{c}$$

- $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ for any vector **a**
- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ for all vectors **a**, **b**

- The order of multiplication by a scalar and a vector product is irrelevant as long as the vectors remain in the same order: $m(\mathbf{a} \times \mathbf{b}) = (m\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (m\mathbf{b})\ \forall\ m \in \boldsymbol{R}$ and vectors **a**, **b**
- The area of the triangle formed by **a** and **b** is given by $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$
- The area of the parallelogram constructed on **a** and **b** is given by $|\mathbf{a} \times \mathbf{b}|$

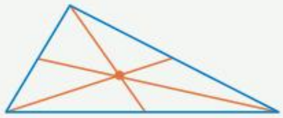
## Curves in 3D

- $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where $f$, $g$ and $h$ are functions, is the **vector equation** of a 3D curve, where $t$ is called the **parameter** $t$
- Projections of a curve on the $y$-$z$, $x$-$z$ and $x$-$y$ planes have no **i**, **j** and **k** components respectively
- The **equation of a straight line** through $A(a_1, a_2, a_3)$ and $B(b_1, b_2, b_3)$ is given by
  - $\mathbf{r}(t) = \mathbf{a} + t\mathbf{AB}$ or $\mathbf{r} = \mathbf{a} + k\mathbf{d}$ as a **vector equation**, where $t, k \in \boldsymbol{R}$, a is on the line and $\mathbf{AB} = \mathbf{d}$ is in the direction of the line
  - $\mathbf{r}(t) = [a_1 + t(b_1 - a_1)]\mathbf{i} + [a_2 + t(b_2 - a_2)]\mathbf{j} + [a_3 + t(b_3 - a_3)]\mathbf{k}$ as a **parametric vector equation**
  - $\frac{x - a_1}{b_1 - a_1} = \frac{y - a_2}{b_2 - a_2} = \frac{z - a_3}{b_3 - a_3}$ or $x - \frac{a_1}{d_1} = y - \frac{a_2}{d_2} = z - \frac{a_3}{d_3}$ as a Cartesian equation

  In the vector forms, if $0 \le t \le 1$, the points are on the line segment **AB**,
  - if $t < 0$, the points are on the line to the left of $A$
  - if $t > 1$, the points are on the line to the right of $B$

## Triangle lines and centres

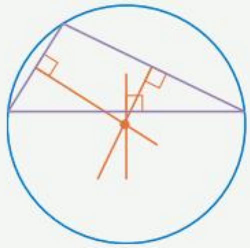
- The **median** of a triangle joins the vertices to the midpoints of the opposite sides
- The **centroid** is the intersection of the medians



Medians and centroid

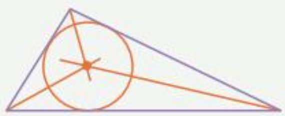
- The **circumcentre** is the intersection of the perpendicular bisectors of each side. It is the centre of the **circumcircle** that passes through the vertices

Perpendicular bisectors and circumcentre



- The **in-centre** is the intersection of the angle bisectors of each vertex. It is the centre of the **in-circle** to which each side is a tangent

Angle bisectors and in-centre

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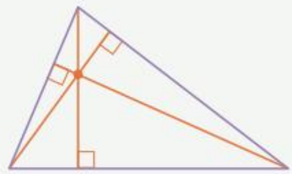
- The **altitudes** of a triangle pass through the vertices and are perpendicular to the opposite sides
- The **orthocentre** is the intersection of the altitudes



Altitudes and orthocentre

## Geometric properties of vectors

- The displacement vector from $A$ to $B$ is $\mathbf{AB} = \mathbf{b} - \mathbf{a}$
- If $P$ is the midpoint of $AB$, then $\mathbf{p} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$
- The sum of a series of 'end-to-end' displacements is the displacement from the beginning to the end. For example: $\mathbf{AB} + \mathbf{BC} + \mathbf{CD} + \mathbf{DE} = \mathbf{AE}$
- The displacement from $A$ to $A$, $\mathbf{AA} = \mathbf{0}$
- Displacements in 2 or 3 dimensions that form a closed path have a sum of **0**. For example, $\mathbf{PQ} + \mathbf{QR} + \mathbf{RS} + \mathbf{ST} + \mathbf{TP} = \mathbf{0}$

## Equations of planes

- A vector $\mathbf{n} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$ perpendicular to a plane $\Gamma$ is called a **normal** of the plane
- The **equation of a plane** $\Gamma$ through $A(a_1, a_2, a_3)$ with normal $\mathbf{n} = (l, m, n)$ is given by

  $$l(x - a_1) + m(y - a_2) + n(z - a_3) \text{ or } lx + my + nz = k,$$

  where $P(x, y, z)$ is any point in the plane
- The **vector equation of a plane** $\Gamma$ is given by

  $$\mathbf{n} \bullet (\mathbf{p} - \mathbf{a}) = 0, \mathbf{p} \bullet \mathbf{n} = \mathbf{a} \bullet \mathbf{n} = k \text{ or } \mathbf{n} \bullet \mathbf{r} = \mathbf{n} \bullet \mathbf{a},$$

  where $P$ is a point in the plane and $\mathbf{r}$ is the position vector of points in the plane

## Vector applications

- Particles will **collide** only if their coordinates are the same at the same time
- The volume of a **parallelepiped** formed by the vectors **a**, **b** and **c** is given by

  $$V = |\mathbf{a} \times \mathbf{b} \bullet \mathbf{c}| = |\mathbf{a} \bullet \mathbf{b} \times \mathbf{c}|$$
- The **work** done (energy transferred) by a force **F** acting through a displacement **s** is given by

  $$W = \mathbf{F} \bullet \mathbf{s}$$
- The **angular momentum** of an object is the cross product of its momentum and radius
- The **electromagnetic force** on a moving charge in a magnetic field is equal to $q\mathbf{v} \times \mathbf{B}$, where $q$ is the charge, **v** is the velocity and **B** is the magnetic field strength

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# 2. CHAPTER REVIEW

## 3D vectors

Example 1

**1** Plot each vector using a box or isometric drawing.

**a** (3, –2, 4) **b** (–1, 3, –2)

Example 2

**2** Find the distance between (1, –5, 3) and (–4, 1, 6).

Example 3

**3** **a** Find the equation of a sphere with centre (4, –3, –2) and radius 6.

**b** Does the point (1, 5, 3) lie inside, outside or on the sphere?

Example 4

**4** **a** Find the norm of (3, 5, –4).

**b** Express (12, –60°, 45°) as a linear combination of **i**, **j** and **k**.

Example 5

**5** **p** = (3, 2, –5) and **q** = (–3, 3, 4). Find 2**p** – 3**q**.

Example 6

**6** $\mathbf{a} = \left(4, \frac{\pi}{3}, \frac{\pi}{6}\right)$ and **b** = (2, –34°, 59°). Find **a** + 4**b** correct to 2 decimal places.

Example 7

**7** Find each scalar product, correct to 2 decimal places where necessary.

**a** Vectors with norms 5 and 9 and an angle of 67° between them

**b** (4, 2, 5) • (–3, 1, 6)

**c** $\left(12, -\frac{\pi}{6}, -\frac{\pi}{6}\right) \bullet \left(9, -\frac{\pi}{6}, \frac{\pi}{6}\right)$

Example 8

**8** Find the projection of 4**i** – 2**j** + 3**k** on **i** + 3**j** – 2**k**.

Example 9

**9** Find, correct to 2 decimal places, the vector product of vectors of magnitudes 6 and 11 with an angle of 108° between them.

Example 10

**10** Find (4**i** + 5**j** – **k**) × (–3**i** + 2**j** + 5**k**).

Example 11

**11** Find the area of the triangle formed by the vectors (3, 9, –5) and (2, –4, 3).

Example 12

**12** What is the shape of $\mathbf{r}(t) = 5t\mathbf{i} - 4t\mathbf{j} + \sin(t)\mathbf{k}$?

Example 13

**13** Express the shape of $\mathbf{r}(t) = (4t + 1)\mathbf{i} - 2t^2\mathbf{j}$ in Cartesian form and identify the shape.


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**14 a** Find the vector and Cartesian equations of the line through (1, 6, –4) and (2, –3, 5).

**b** Is the point (5, –45, –45) on the line?

Example 14

**15** Is the point (7, 4, 8) on the line segment between (4, 1, –1) and (8, 5, 11)?

Example 15

**16** Two particles are moving in the paths $\mathbf{a}(t) = (3t + 2)\mathbf{i} + (7t - 3)\mathbf{j}$ and $\mathbf{b}(t) = (t + 10)\mathbf{i} + (t^2 + 9)\mathbf{j}$. Find if their paths cross and/or if they collide. If they do, where and when does it happen?

Examples 16, 17

**17** Find the Cartesian and vector equations of the plane through (1, –2, 5) with normal (6, 2, –2).

Example 18

**18** Find the equation of the plane containing the points (–3, 4, 1), (2, 5, –4) and (2, 7, 5).

Example 19

**19** Find the volume of the tetrahedron formed by the vectors (3, 5, 7), (–2, –3, 4) and (2, 8, 3).

Examples 20, 21

## Problem solving

**20** Find the overlap or separation of the spheres $x^2 + 12x + y^2 + 10y + z^2 + 8z = 23$ and $x^2 - 4x + y^2 - 8y + z^2 - 6z = 7$.

**21** Find the angle between (4, 7, –2) and (3, –5, 5), correct to the nearest degree.

**22** Find the area of the parallelogram with vertices (2, 8, –4), (6, 2, 7), (5, 1, 3) and (1, 7, –8).

**23** Use vectors to prove that the diagonals of a rhombus are perpendicular.

**24** Find the shortest distance between the plane $3x - 2y + 4z = 6$ and (–1, 5, 7).

**25** Find the volume of the parallelepiped $I(-5, 3, 4)$, $B(2, -4, -3)$, $E(-2, 5, 0)$, $G(-9, 12, 7)$, $Y(-2, 6, 7)$, $U(5, -1, 0)$, $R(1, 8, 3)$, $P(-6, 15, 10)$.

Qz Practice quiz

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# 3. COMPLEX ARITHMETIC


Things that are not real are often called imaginary. As far back as the ancient Greeks, there have been mathematical problems that do not have solutions using the real number system. The Greeks realised that no number existed that satisfies $x^2 = -1$.

As you saw in Year 11, complex numbers can be used to find a solution for equations involving the square root of a negative number. You have already calculated powers of complex numbers using repeated multiplication. The French mathematician Abraham de Moivre (1667–1754) (pronounced *duh mwahv*) developed an alternative method for finding powers of complex numbers.

3.01 Complex number review
3.02 Complex number arithmetic
3.03 Modulus, argument and polar form
3.04 De Moivre's theorem
3.05 Applications of De Moivre's theorem
Chapter summary
Chapter review

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## SYLLABUS SUBJECT MATTER

### Cartesian forms

- review real and imaginary parts Re ($z$) and Im ($z$) of a complex number $z$
- review Cartesian form
- review complex arithmetic using Cartesian form.

### Complex arithmetic using polar form

- prove the identities involving modulus and argument
- prove and use De Moivre's theorem for integral powers

Prior learning

# TERMINOLOGY

Argand diagram
Argand plane
argument
Cartesian form
cis ($\theta$)
complex conjugate
complex number
De Moivre's theorem
imaginary number
modulus
modulus–argument form
polar form
principal argument
realise the denominator
rectangular form
trigonometric form

## 3.01 Complex number review

- A **complex number** $z$ can be written in the form $z = x + iy$, where $x$ and $y$ are real numbers ($x, y \in \boldsymbol{R}$) and $i = \sqrt{-1}$.
- The real part of the complex number $x + iy$ is the real number $x$ and the imaginary part is the real number $y$.

When the imaginary part of the complex number involves a surd, it is usual to write it as $z = 4 - i\sqrt{7}$ so there is no confusion as to whether $i$ is included in the radical sign.

### EXAMPLE 1

Solve the quadratic equation $x^2 - 4x + 13 = 0$.

#### Solution

| | |
|---|---|
| Factorise the quadratic. | $x^2 + 4x + 13$ does not factorise. |
| Use the quadratic formula. | $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ |
| $a = 1$, $b = 4$ and $c = 13$. | $= \dfrac{-(-4) \pm \sqrt{4^2 - 4 \times 1 \times 13}}{2}$ |
| Simplify. | $= \dfrac{4 \pm \sqrt{-36}}{2}$ |
| | $= \dfrac{4 \pm \sqrt{36 \times (-1)}}{2}$ |
| | $= \dfrac{4 \pm \sqrt{36} \times \sqrt{-1}}{2}$ |
| $i = \sqrt{-1}$. | $= \dfrac{4 \pm 6i}{2}$ |
| Simplify. | $= 2 \pm 3i$ |

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Example **1** can also be solved by completing the square:

| | |
|---|---|
| Write the equation. | $x^2 - 4x + 13 = 0$ |
| Complete the square. | $x^2 - 4x + 2^2 + 13 - 2^2 = 0$ |
| Factorise the square. | $(x - 2)^2 + 9 = 0$ |
| Use $i^2 = -1$. | $(x - 2)^2 - 9i^2 = 0$ |
| Factorise the difference of squares. | $(x - 2 - 3i)(x - 2 + 3i) = 0$ |
| Solve. | $x - 2 - 3i = 0$ or $x - 2 + 3i = 0$ |
| | $x = 2 + 3i$ or $x = 2 - 3i$ |

## Complex conjugate

For the complex number $z = x + iy$, its **complex conjugate** is $\bar{z} = x - iy$.

You can see from Example **1** that the roots of the quadratic $x^2 - 4x + 13 = 0$ are complex conjugates. It is possible to show that the product of a complex number and its conjugate is always a real number.

Given the complex number $z = x + iy$ and its complex conjugate $\bar{z} = x - iy$:

$z\bar{z} = (x + iy)(x - iy)$

$= x^2 - i^2y^2$ (Difference of two squares.)

$= x^2 - (-1)y^2$ ($i^2 = -1$)

$= x^2 + y^2$

Since $x, y \in \boldsymbol{R}$, $x^2 + y^2 \in \boldsymbol{R}$.

Hence $z\bar{z}$ is always a real number.

## EXAMPLE 2

If $z = 3 - 5i$, calculate the product of $z$ and its complex conjugate.

### Solution

| | |
|---|---|
| Find the complex conjugate of $z$. | $\bar{z} = 3 + 5i$ |
| Calculate the product. | $z\bar{z} = (3 - 5i)(3 + 5i)$ |
| Expand. | $= 3^2 - (5i)^2$ |
| | $= 9 - 25i^2$ |
| | $= 9 - 25 \times (-1)$ |
| Evaluate. | $= 34$ |

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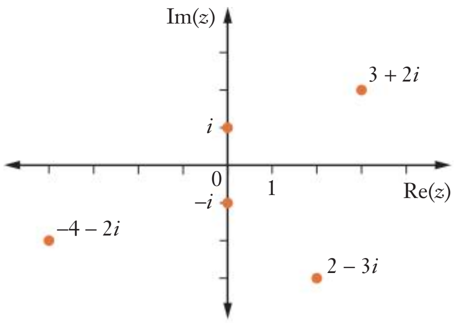
It is possible to represent complex numbers geometrically on an **Argand plane** or **Argand diagram**. In the Argand plane, the horizontal axis is called the real axis, Re($z$), and the vertical axis is called the imaginary axis Im($z$).

The diagram on the right shows the numbers $3 + 2i$, $2 - 3i$, $-4 - 2i$, $i$ and $-i$ represented as points on the Argand plane.

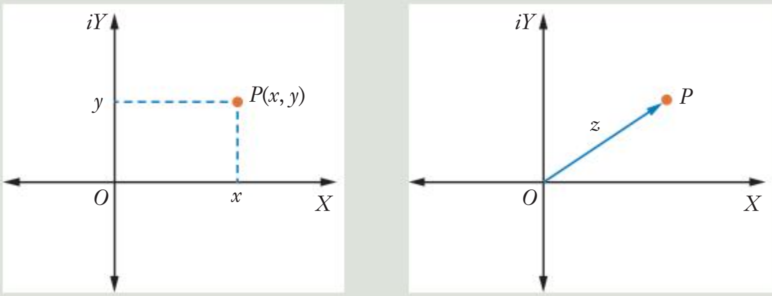



The real axis, Re($z$) is also labelled $X$ while the imaginary axis Im($z$) is also labelled $iY$ to avoid confusion with the Cartesian plane.

Complex numbers written as $x + iy$ are said to be in **Cartesian** or **rectangular form**.

## The Argand plane

The complex number $z = x + iy$ can be represented as the point $P(x, y)$ or by the vector **z** or **OP** on the Argand plane.




## EXAMPLE 3

Given the complex number $z = 4 - 3i$:

**a** state the point $P$ that represents $z$ on the Argand plane

**b** draw the vectors **z** and $\bar{\mathbf{z}}$ on the Argand plane

**c** state the relationship between the vectors **z** and $\bar{\mathbf{z}}$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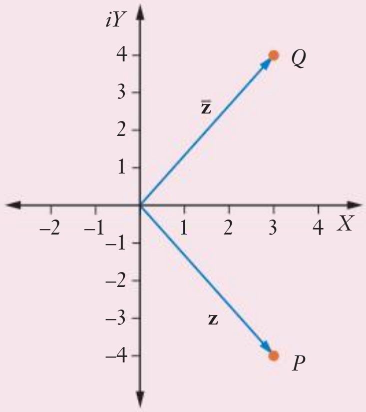
## Solution

**a** Convert the complex number to an ordered pair. $z = 4 - 3i \Rightarrow P(4, -3)$

**b** State the complex conjugate of $z$. $z = 4 + 3i$

Let $Q$ represent $\bar{z}$ on the Argand plane. $\bar{z} = 4 + 3i \Rightarrow Q(4, 3)$

Draw vectors $\mathbf{z}$ and $\bar{\mathbf{z}}$ on the Argand plane.




**c** The vectors $\mathbf{z}$ and $\bar{\mathbf{z}}$ have the same real part but the imaginary parts are opposite in sign.

The vector $\mathbf{z}$ and its complex conjugate $\bar{\mathbf{z}}$ are reflections of each other in the real axis.

## Exercise 3.01 Complex number review

**1** Write each expression in terms of $i$.

**a** $\sqrt{-9}$ **b** $\sqrt{-32}$ **c** $\sqrt{-23}$

**d** $\sqrt{-\frac{1}{4}}$ **e** $\sqrt{-\frac{12}{18}}$ **f** $\sqrt{-\frac{63}{25}}$

**2** Solve each quadratic equation, expressing answers in the form $a \pm ib$. Example 1

**a** $x^2 - 2x + 2 = 0$ **b** $x^2 + x + 1 = 0$

**c** $3x^2 - 4x + 2 = 0$ **d** $x^2 - 6x + 63 = 0$

**e** $2x^2 + x = -5$ **f** $x^2 - 2x + 6 = 0$

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**3** Use the 'completing the square' method to solve each equation.

**a** $x^2 - 4x + 5 = 0$ **b** $x^2 + 6x + 13 = 0$

**c** $x^2 - 5x + 7 = 0$ **d** $x^2 + 6x + 11 = 0$

**4** Evaluate each expression.

**a** $i^3$ **b** $i^4$ **c** $i^5$

**d** $i^{10}$ **e** $i^{31}$ **f** $i + i^2 + i^3 + \ldots + i^8$

**5** Write the complex conjugate of each expression.

**a** $4 - 3i$ **b** $-2 + 5i$ **c** $7 + 3i$

**d** $-2 - 6i$ **e** $3 - i\sqrt{5}$ **f** $3\sqrt{6} + i\sqrt{7}$

Example 2

**6** For each value of $z$, calculate the product $z\bar{z}$.

**a** $z = 3 - 2i$ **b** $z = 5 + 4i$ **c** $z = 5 - 6i$

**d** $z = 4 + 7i$ **e** $z = 1 - 11i$ **f** $z = 8 - 9i$

**g** $z = 4 - 4i$ **h** $z = 9 + 12i$ **i** $z = 6 - 11i$

Example 3

**7** For the complex number $z = 5 + 4i$:

**a** state the point $P$ that represents $z$ on the Argand plane

**b** draw the vectors $\mathbf{z}$ and $\bar{\mathbf{z}}$ on the Argand plane

**c** state the relationship between the vectors $\mathbf{z}$ and $\bar{\mathbf{z}}$

**8** **a** Plot $i^0$, $i^1$, $i^2$ and $i^3$ on the Argand plane.

**b** Calculate $i^0 + i^1 + i^2 + i^3$.

## Problem solving

**9** By examining the coefficients of each quadratic equation, determine if the roots are complex conjugates.

**a** $x^2 - 4x - 9 = 0$ **b** $x^2 + 3x + 11 = 0$

**c** $2x^2 - 3x + 10 = 0$ **d** $4x^2 - 7x - 5 = 0$

**10** State the condition for which the roots of the quadratic $px^2 + qx + r = 0$ are complex conjugates.

**11** Evaluate $\sum_{n=0}^{254} i^n$.


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# 3.02 Complex number arithmetic

WS
Complex number operations

The real part of a complex number is the term not containing $i$ and the imaginary part is the coefficient of $i$. So for $z = 5 + 3i$, $\text{Re}(z) = 5$ and $\text{Im}(z) = 3$.

If the coefficient of $i$ is zero, the number is **real** and if the real part is zero, the number is purely **imaginary**. So $4 + 0i = 4$ is real and $0 + 3i = 3i$ is imaginary.

### The equality of complex numbers

The complex numbers $x_1 + iy_1$ and $x_2 + iy_2$ are equal if both their real and imaginary parts are equal.

$$x_1 + iy_1 = x_2 + iy_2 \text{ iff } x_1 = x_2 \text{ and } y_1 = y_2$$

Addition of complex numbers is defined by separately adding real and imaginary parts. Subtraction is done in a similar way. So if $z = 6 + 11i$ and $w = 8 - 7i$:

$$z + w = (6 + 8) + (11 - 7)i = 14 + 4i$$

and $$z - w = (6 - 8) + [11 - (-7)]i = -2 + 18i$$

### EXAMPLE 4

If $z = 3 + 7i$ and $w = 6 - 3i$, calculate:

**a** $2z + 3w$ **b** $5z - 4w$

#### Solution

**a** Substitute for $z$ and $w$. $2z + 3w = 2(3 + 7i) + 3(6 - 3i)$

Expand. $= 6 + 14i + 18 - 9i$

Add real and imaginary parts. $= (6 + 18) + (14 - 9)i$

Evaluate. $= 24 + 5i$

**b** Substitute for $z$ and $w$. $5z - 4w = 5(3 + 7i) - 4(6 - 3i)$

Expand. $= 15 + 35i - 24 + 12i$

Add real and imaginary parts. $= (15 - 24) + (35 + 12)i$

Evaluate. $= -9 + 47i$

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You can use your calculator to add complex numbers.

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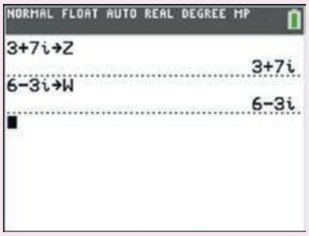
Chapter 3

### TI-84 Plus CE

Assign the value $3 + 7i$ to the variable $z$ by pressing $3 + 7$ $i$ ([2nd] [.]) [sto→] $z$ ([alpha] [2]).

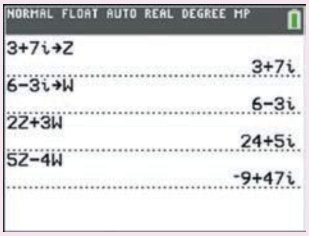
Assign the value $6 - 3i$ to the variable $w$ in a similar way.




To calculate $2z + 3w$ press

$2$ $z$ ([alpha] [2]) $+ 3$ $w$ ([alpha] [−]) [enter].

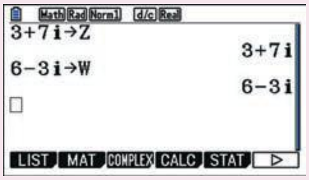
Calculate $5z - 4w$ in a similar way.




### Casio fx-CG 20 AU

Use the Run-Matrix menu.

Press [OPTN] and press [F3] (COMPLEX). Then assign the value $3 + 7i$ to the variable $z$ by pressing $3 + 7$ $i$ ([SHIFT] [0]) [→] $z$ ([ALPHA] [0]).

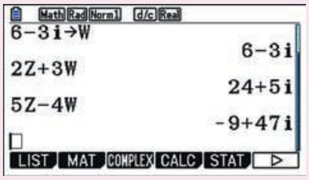



Assign the value $6 - 3i$ to the variable $w$ in a similar way.

To calculate $2z + 3w$ press

$2$ $z$ ([ALPHA] [0]) $+ 3$ $w$ ([ALPHA] [3]) [EXE].

Calculate $5z - 4w$ in a similar way.




You use the distributive law to multiply complex numbers using the fact that $i^2 = -1$.

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then:

$$\begin{aligned} z_1 \cdot z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= x_1x_2 + x_1y_2i + y_1x_2i + iy_1y_2{}^2 \\ &= x_1x_2 - y_1y_2 + x_1y_2i + y_1x_2i \\ &= (x_1x_2 - y_1y_2) + (x_1y_2 + y_1x_2)i \end{aligned}$$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## EXAMPLE 5

If $z = 2 - 5i$ and $w = -5 + 3i$, calculate:

**a** $4zw$ **b** $3z^2$

## Solution

**a** Substitute for $z$ and $w$. $4zw = 4(2 - 5i)(-5 + 3i)$

Expand using the distributive law. $= 4(-10 + 6i + 25i - 15i^2)$

Evaluate. Remember $i^2 = -1$. $= 4(-10 + 31i + 15)$

$= 4(5 + 31i)$

$= 20 + 124i$

**b** Substitute for $z$. $3z^2 = 3(2 - 5i)(2 - 5i)$

Expand using the square of a difference. $= 3(4 - 20i + 25i^2)$

Evaluate. $= 3(4 - 20i - 25)$

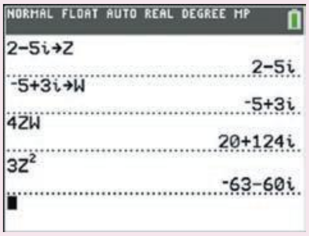
$= 3(-21 - 20i)$

$= -63 - 60i$

You can use your calculator to multiply complex numbers.

### TI-84 Plus CE

Assign the value $2 - 5i$ to the variable $z$ and $-5 + 3i$ to the variable $w$ as previously described.



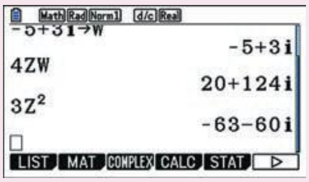

Calculate $4zw$ by pressing
4 $z$ ([alpha] [2]) $w$ ([alpha] [–]) [enter].

Calculate $3z^2$ by pressing
3 $z$ ([alpha] [2]) [x²] [enter].

### Casio fx-CG 20 AU

Use the Run-Matrix menu.

Assign the value $2 - 5i$ to the variable $z$ and $-5 + 3i$ to the variable $w$ as previously described.




Calculate $4zw$ by pressing
4 $z$ ([ALPHA] [0]) $w$ ([ALPHA] [3]) [EXE].

Calculate $3z^2$ by pressing
3 $z$ ([ALPHA] [0]) [$x^2$] [EXE].

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Chapter 3

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## INVESTIGATION

### COMPLEX NUMBER PATTERNS

- Calculate $i^n$ in the form $x + iy$ for each integer from 0 to 10.
- Write a paragraph to describe any patterns you observe.
- Use the pattern you discovered to simplify $i^{201}$.
- Write each of the following expressions in the form $x + iy$:

  $i + 1$

  $i^2 + i + 1$

  $i^3 + i^2 + i + 1$

  $i^4 + i^3 + i^2 + i + 1$

  $i^5 + i^4 + i^3 + i^2 + i + 1$

  $i^6 + i^5 + i^4 + i^3 + i^2 + i + 1$

  $i^7 + i^6 + i^5 + i^4 + i^3 + i^2 + i + 1$

  $i^8 + i^7 + i^6 + i^5 + i^4 + i^3 + i^2 + i + 1$

  $i^9 + i^8 + i^7 + i^6 + i^5 + i^4 + i^3 + i^2 + i + 1$

  $i^{10} + i^9 + i^8 + i^7 + i^6 + i^5 + i^4 + i^3 + i^2 + i + 1$
- Write a paragraph to describe the pattern you observed from the previous step.
- Use this pattern to calculate:

  $$i^{201} + i^{200} + i^{199} + \ldots + i^2 + i + 1$$
- Investigate how to derive the formula:

  $$1 + i + i^2 + \ldots + i^n = \frac{1 - i^{n+1}}{1 - i}$$
- Given that: $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots + \frac{z^n}{n!}$

  $$\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots$$

  $$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots$$

  show that $e^{iz} = \cos(z) + i\sin(z)$.
- Use the previous result to derive Euler's identity, $e^{\pi i} + 1 = 0$.


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You have previously examined the properties of complex conjugates.

**Properties of complex conjugates**

- The product of $z = x + iy$ and its conjugate is $z\bar{z} = x^2 + y^2$.
- The sum of $z = x + iy$ and its conjugate is $z + \bar{z} = 2x$.
- The difference of $z = x + iy$ and its conjugate is $z - \bar{z} = 2yi$.

When dividing by a complex number, you can use the complex conjugate to change it to division by a real denominator. Do this by multiplying the numerator and denominator by the complex conjugate. This is called **realising the denominator**.

To **realise the denominator** of a fraction with the complex denominator $z$, multiply by $\frac{\bar{z}}{\bar{z}}$.

Multiplying by $\frac{\bar{z}}{\bar{z}}$ doesn't change the value as $\frac{\bar{z}}{\bar{z}} = 1$.

## EXAMPLE 6

Calculate:

**a** $(3 - 5i) \div (2 + 3i)$

**b** $\dfrac{-6+8i}{4-7i}$

### Solution

**a** Write as a fraction.

$$(3 - 5i) \div (2 + 3i) = \frac{3-5i}{2+3i}$$

Multiply by $\frac{2-3i}{2-3i}$.

$$= \frac{3-5i}{2+3i} \times \frac{2-3i}{2-3i}$$

Simplify.

$$= \frac{6-9i-10i+15i^2}{2^2-(3i)^2}$$

$$= \frac{-9-19i}{4+9}$$

$$= \frac{-9-19i}{13}$$

Write in Cartesian form.

$$= -\frac{9}{13} - \frac{19}{13}i$$

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**b** Multiply by $\frac{4+7i}{4+7i}$.

$$\frac{-6+8i}{4-7i} = \frac{-6+8i}{4-7i} \times \frac{4+7i}{4+7i}$$

Simplify.

$$= \frac{-24-42i+32i+56i^2}{4^2-(7i)^2}$$

$$= \frac{-80-10i}{16+49}$$

$$= \frac{-80-10i}{65}$$

Cancel and write in Cartesian form.

$$= -\frac{16}{13} - \frac{2}{13}i$$

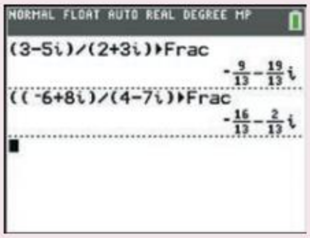
You can use your calculator to multiply complex numbers.

TI-Nspire CX
Chapter 3

### TI-84 Plus CE

Enter the division using [2nd] [.] for $i$.

Use [math], 1:▸Frac to express the answers as fractions.

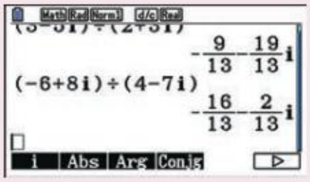



### Casio fx-CG 20 AU

Use the Run-Matrix menu.

Enter the division using [SHIFT] [0] for $i$.

Press [F↔D] to express the answers as fractions.




## Exercise 3.02 Complex number arithmetic

**1** Calculate each expression.

**a** $(5 + 3i) + (2 + 6i)$ **b** $(4 + i) + (-3 + 2i)$ **c** $(5 - 2i) - (3 - 2i)$

**d** $(-3 - 4i) - (12 - 5i)$ **e** $(-5 + 8i) - (9 - 7i)$ **f** $(9 - 13i) + (-11 - 8i)$

Example 4

**2** If $z = 4 + 5i$, $q = 7 - 3i$ and $m = -3 - 4i$, calculate:

**a** $3z + 2q$ **b** $4q - 3m$ **c** $-2z - 5q$

**d** $6m + 4q$ **e** $7q - 6z + 3m$ **f** $8z - 4m - 5q$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**3** Calculate each expression.

**a** $(4+3i)(1-2i)$ **b** $(1-i)(2+i)$ **c** $i(2+i)(2-i)$

**d** $(-4i)2i$ **e** $(5-i)(3-4i)$ **f** $(3+4i)(2+i)$

**g** $(1+i)^2$ **h** $(4-3i)^2$ **i** $(1-2i)^4$

**4** Find the values of $a$ and $b$ ($a, b \in \mathbf{R}$) if:

**a** $a+bi=3(2-5i)+2i(7-i)$

**b** $a+bi=4(-3-6i)-3i(5+2i)$

**5** If $z=2-3i$, $v=-5-7i$ and $w=-8+3i$, calculate:

Example 5

**a** $2zv$ **b** $4wv$ **c** $-3wz$

**d** $zv-vw$ **e** $3z^2+w^2$ **f** $6v^2-4zw$

**6** Evaluate

**a** $(5+i)^2+(5-i)^2$ **b** $(3+i)^4+(3-i)^4$

**7** Write each expression in the form $a+bi$.

Example 6

**a** $\dfrac{1}{3+2i}$ **b** $\dfrac{1}{3+4i}$ **c** $\dfrac{1}{3-4i}$

**d** $\dfrac{1+i}{1-i}$ **e** $\dfrac{3+2i}{5+2i}$ **f** $\dfrac{2-2i}{5i}$

**g** $(1-3i)^{-2}$ **h** $\dfrac{2-3i}{3-4i}$ **i** $\dfrac{1+i\sqrt{3}}{-2\sqrt{3}+2i}$

**j** $\dfrac{1}{4i(3-2i)}$ **k** $\dfrac{(1+2i)(2+3i)}{(4+i)(9+2i)}$ **l** $\dfrac{(3-2i)(1+4i)}{(2+i)(1-2i)}$

**8** If $z=5-2i$, write each expression in the form $a+bi$.

**a** $z\bar{z}$ **b** $z\bar{z}^{-1}$ **c** $z\bar{z}^2$ **d** $\bar{z}(z+\bar{z})$

**e** $(z-\bar{z})^{-1}$ **f** $i(\bar{z}-z)$ **g** $\bar{z}^{-1}$ **h** $\dfrac{z^2}{\bar{z}}$

**9** Show that if $z=a+bi$ and $w=c+di$ (where $a$, $b$, $c$ and $d \in \mathbf{R}$), then:

$$\frac{z}{w}=\frac{ac+bd}{c^2+d^2}+\left(\frac{bc-ad}{c^2+d^2}\right)i$$

## Problem solving

**10** Given that $z = a + bi$ and $z(5 - 7i) = 13 + 41i$, find the values of $a$ and $b$.

**11** Prove that the conjugate of the conjugate of $z$ is $z$ itself.

**12** Prove that:

**a** $\overline{(z^{-1})} = (\overline{z})^{-1}$ **b** $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$

**13** Prove each statement.

**a** $\overline{z + w} = \overline{z} + \overline{w}$ **b** $\overline{zw} = \overline{z}\ \overline{w}$

**14** Find the values of $a$ and $b$ such that $(3a - 4b) + (a + 2b)i = -11 + 3i$.

**15** Prove each statement.

**a** $z\overline{z} = |z|^2$

**b** $|z + w|^2 = |z|^2 + |w|^2 + z\overline{w} + \overline{z}w$

**c** $|zw| = |z||w|$

WS Complex number conversions

WS Modulus and argument

WS Polar complex number operations

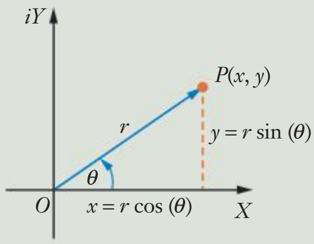
# 3.03 Modulus, argument and polar form

You have seen that any complex number $z = x + yi$ can be represented as a point $(x, y)$ on an Argand diagram. So a complex number expressed in this way is said to be written in Cartesian or rectangular form. It is also possible to write a complex number using the **modulus** (magnitude) of $z$ and the **argument** (angle $z$ makes with the positive direction of the real axis) when $z$ is represented in an Argand diagram. This is known as the **polar form** of a complex number. The polar form is also known as the **modulus–argument** or **trigonometric form**.

### Polar form of a complex number

For the complex number $z = x + yi$:

- the modulus of $z = \text{mod}(z) = |z| = r = \sqrt{x^2 + y^2}$.
- the argument of $z = \arg(z) = \theta$ where $\tan(\theta) = \frac{y}{x}$,
- $x = r\cos(\theta)$ and $y = i\sin(\theta)$.




The polar form of $z$ is:

- $z = (r, \theta)$ or
- $z = r[\cos(\theta) + i\sin(\theta)]$

$\cos(\theta) + i\sin(\theta)$ can also be abbreviated to **cis (θ)**, so the polar form can also be written as $z = r\,\text{cis}(\theta)$.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## EXAMPLE 7

**a** If $\arg(z) = \frac{\pi}{6}$ and $\bmod(z) = 5$, write $z$ in polar form.

**b** Convert $z = 2 + 2i\sqrt{3}$ to polar form.

**c** Convert $z = 2\left[\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right]$ to Cartesian form.

## Solution

**a** Write the polar form of $z$.

$z = r[\cos(\theta) + i\sin(\theta)]$

Use the fact that $\arg(z) = \theta$ and $\bmod(z) = r$.

$= 5\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$

**b** Identify $x$ and $y$.

$z = 2 + 2i\sqrt{3} = x + yi$

So $x = 2$ and $y = 2\sqrt{3}$.

Use the definition to state $r$.

$r = \sqrt{x^2 + y^2}$

Substitute for $x$ and $y$.

$= \sqrt{2^2 + \left(2\sqrt{3}\right)^2} = 4$

Find $\theta$.

$$\tan(\theta) = \frac{y}{x} = \frac{2\sqrt{3}}{2} = \sqrt{3}$$

$$\theta = \frac{\pi}{3}$$

Write the polar from.

$z = (r, \theta) = r[\cos(\theta) + i\sin(\theta)]$

Substitute for known values.

$$z = \left(r, \frac{\pi}{3}\right)$$

$$= 4\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$$

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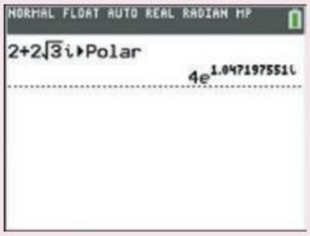
You can also use your graphics calculator to convert to polar form.

TI-Nspire CX
Chapter 3

### TI-84 Plus CE

Make sure the calculator is in radian measure.

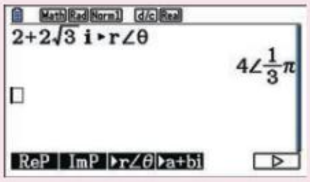
Enter $2 + 2\sqrt{3}i$ and press [math] and select 7: Polar from the CMPLX menu, then press [enter]. The TI-84 expresses cis ($\theta$) as $e\theta^i$. Note $\frac{\pi}{3} \approx 1.047\ 195\ldots$




### Casio fx-CG 20 AU

Use the Run-Matrix menu and make sure the calculator is in radian measure.

Enter $2 + 2\sqrt{3}i$ and press [OPTN] and choose ▸$r\angle\theta$ from the COMPLEX ([F3]) menu, then press [EXE].




**c** Write the polar form.

$$z = 2\left[\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right]$$

Evaluate $\cos\left(\frac{-\pi}{3}\right)$ and $\sin\left(\frac{-\pi}{3}\right)$.

$$= 2\left(\frac{1}{2} + \frac{-\sqrt{3}}{2}i\right)$$

Simplify.

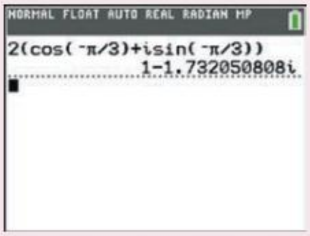
$$= 1 - i\sqrt{3}$$

You can also use your graphics calculator to convert to Cartesian form.

TI-Nspire CX
Chapter 3

### TI-84 Plus CE

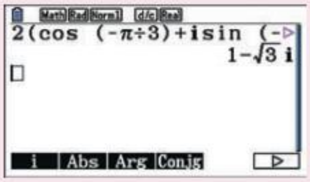
Type the polar form and press [enter].




### Casio fx-CG 20 AU

Use the Run-Matrix menu.

Type the polar form and press [EXE].





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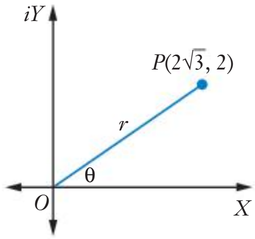
The diagram on the right represents the complex number $z = (r, \theta) = 2\sqrt{3} + 2i$.




The angle that $OP$ makes with the positive $X$-axis is given by

$$\theta = \tan^{-1}\left(\frac{2}{2\sqrt{3}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

So $\theta = \frac{\pi}{6}, 2\pi + \frac{\pi}{6}, 4\pi + \frac{\pi}{6}, \ldots$

There is an infinite number of possible angles.

To avoid confusion, the angle that is normally used is the one in the interval $-\pi < \theta \leq \pi$, and this is called the **principal argument**.

You have seen that adding and subtracting complex numbers is easily done using the Cartesian form. However, when multiplying and dividing complex numbers, the polar form is easier to use than the Cartesian form.

In last year's course you established the following definitions for multiplication and division of complex numbers in polar form.

### Multiplication of complex numbers

You can multiply complex numbers by multiplying their moduli and adding their arguments.

For $z_1 = r_1 \text{ cis } (\theta_1)$ and $z_2 = r_2 \text{ cis } (\theta_2)$:

$$z_1 \times z_2 = r_1 r_2 \text{ cis } (\theta_1 + \theta_2)$$
$$= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$$

and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

$\text{mod}(z_1 z_2) = \text{mod}(z_1) \times \text{mod}(z_2)$ or

$|z_1 z_2| = |z_1| |z_2|$

## Division of complex numbers

You can divide complex numbers by dividing their moduli and subtracting their arguments.

For $z_1 = r_1 \text{ cis } (\theta_1)$ and $z_2 = r_2 \text{ cis } (\theta_2)$:

$$z_1 \div z_2 = \frac{z_1}{z_2} = \frac{r_1}{r_2} \text{ cis } (\theta_1 - \theta_2)$$

$$= \frac{r_1}{r_2} [\cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2)] \qquad z_2 \neq 0$$

and $\quad \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

$$\text{mod}\left(\frac{z_1}{z_2}\right) = \frac{\text{mod}(z_1)}{\text{mod}(z_2)} = \frac{|z_1|}{|z_2|}$$

### EXAMPLE 8

Calculate each expression, writing answers in polar form.

**a** $z \times w$ if $z = 3\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$ and $w = 6\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$

**b** $z \times w$ if $z = 5\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]$ and $w = 7\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right]$

**c** $\frac{z}{w}$ if $z = 8\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$ and $w = 2\left[\cos\left(\frac{3\pi}{5}\right) + i\sin\left(\frac{3\pi}{5}\right)\right]$

### Solution

**a** Use the definition for multiplication. $\quad z_1 z_2 = r_1 r_2 \text{ cis } (\theta_1 + \theta_2)$

Substitute for known values. $\quad z\,w = 3 \times 6\left[\cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right)\right]$

Simplify. $\quad = 18\left[\cos\left(\frac{4\pi + 3\pi}{12}\right) + i\sin\left(\frac{4\pi + 3\pi}{12}\right)\right]$

$$= 18\left[\cos\left(\frac{7\pi}{12}\right) + i\sin\left(\frac{7\pi}{12}\right)\right]$$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**b** Use the definition for multiplication. $z_1 z_2 = r_1 r_2 \text{ cis } (\theta_1 + \theta_2)$

Substitute for known values. $$z\,w = 5 \times 7 = \left[\cos\left(\frac{2\pi}{3} + \frac{5\pi}{6}\right) + i\sin\left(\frac{2\pi}{3} + \frac{5\pi}{6}\right)\right]$$

Simplify. $$= 35\left[\cos\left(\frac{4\pi + 5\pi}{6}\right) + i\sin\left(\frac{4\pi + 5\pi}{6}\right)\right]$$

$$= 35\left[\cos\left(\frac{9\pi}{6}\right) + i\sin\left(\frac{9\pi}{6}\right)\right]$$

$$= 35\left[\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right]$$

Express using the principal argument, i.e. $-\pi < \theta \leq \pi$. $$= 35\left[\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right]$$

**c** Use the definition for division. $$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

Substitute for known values. $$= \frac{8}{2}\left[\cos\left(\frac{\pi}{3} - \frac{3\pi}{5}\right) + i\sin\left(\frac{\pi}{3} - \frac{3\pi}{5}\right)\right]$$

Simplify. Check that the argument is in the interval $(-\pi, \pi]$. $$= 4\left[\cos\left(\frac{5\pi - 9\pi}{15}\right) + i\sin\left(\frac{5\pi - 9\pi}{15}\right)\right]$$

$$= 4\left[\cos\left(\frac{-4\pi}{15}\right) + i\sin\left(\frac{-4\pi}{15}\right)\right]$$

In Year 11, you established a definition for the multiplicative inverse, $z^{-1}$, of the complex number $z$, in polar form.

### Definition of $z^{-1}$

For $z = r \text{ cis } (\theta)$:

$$z^{-1} = r^{-1} \text{ cis } (-\theta) = \frac{1}{r}[\cos(-\theta) + i\sin(-\theta)]$$

and $\arg(z^{-1}) = -\arg(z)$

$$\text{mod}(z^{-1}) = \frac{1}{\text{mod}(z)} = \frac{1}{|z|}$$

The definition for the multiplication of complex numbers can be extended for more than 2 numbers.

### Extended multiplication in polar form

For the complex numbers $z_1 = r_1 \text{ cis } (\theta_1)$, $z_2 = r_2 \text{ cis } (\theta_2)$, $z_3 = r_3 \text{ cis } (\theta_3)$, … , $z_n = r_n \text{ cis } (\theta_n)$,

$$z_1 z_2 z_3 \ldots z_n = r_1 r_2 r_3 \ldots r_n \text{ cis } (\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n)$$
$$= r_1 r_2 r_3 \ldots r_n [\cos(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n) + i \sin(\theta_1 + \theta_2 + \theta_3 + \ldots + \theta_n)]$$

The following results can be derived from this definition:

### Argument and modulus of a product

For the complex numbers $z_1 = r_1 \text{ cis } (\theta_1)$, $z_2 = r_2 \text{ cis } (\theta_2)$, $z_3 = r_3 \text{ cis } (\theta_3)$, …, $z_n = r_n \text{ cis } (\theta_n)$

$$\arg(z_1 z_2 z_3 \ldots z_n) = \arg(z_1) + \arg(z_2) + \arg(z_3) + \ldots + \arg(z_n)$$

and $\quad \text{mod}(z_1 z_2 z_3 \ldots z_n) = |z_1 z_2 z_3 \ldots z_n| = |z_1|\,|z_2|\,|z_3| \ldots |z_n|$

## EXAMPLE 9

**a** If $z = 5\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$, calculate $z^{-1}$.

**b** Convert $z = 2 - 2i$ to polar form and then find $\arg(2 - 2i)^{-1}$ and $|(2 - 2i)^{-1}|$.

**c** If $z = \sqrt{3} \text{ cis}\left(\frac{\pi}{6}\right)$, $w = 2 \text{ cis}\left(\frac{3\pi}{4}\right)$ and $u = 4 \text{ cis}\left(\frac{\pi}{2}\right)$, calculate $\arg(zwu)$, $|zwu|$ and $zwu$.

**d** Calculate $\frac{1}{(2 + 2i\sqrt{3})(1 - i)}$, expressing the answer in polar form.

### Solution

**a** State the definition for $z^{-1}$. $\quad z^{-1} = \frac{1}{r}[\cos(-\theta) + i\sin(-\theta)]$

Substitute for known values. $\quad = \frac{1}{5}\left[\cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)\right]$

**b** Write the number. $\quad z = x + yi = 2 - 2i$

Identify $x$ and $y$. $\quad x = 2$ and $y = -2$


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| | | |
|---|---|---|
| | Write the polar form. | $z = r[\cos(\theta) + i\sin(\theta)]$ |
| | Calculate arg ($z$). | $\tan(\theta) = \frac{y}{x} = \frac{-2}{2} = -1$ |
| | Find $\theta$ where $-\pi < \theta \le \pi$. | $\arg(z) = \theta = \tan^{-1}(-1) = \frac{-\pi}{4}$ |
| | Calculate $\lvert z \rvert$. | $\lvert z \rvert = r = \sqrt{x^2+y^2} = \sqrt{2^2+(-2)^2} = 2\sqrt{2}$ |
| | Write $z$ in polar form. | $z = 2\sqrt{2}\left[\cos\left(\frac{-\pi}{4}\right) + \sin\left(\frac{-\pi}{4}\right)i\right]$ |
| | Use the rule $\arg(z^{-1}) = -\arg(z)$. | $\arg(2-2i)^{-1} = -\left(\frac{-\pi}{4}\right) = \frac{\pi}{4}$ |
| | Use the rule $\lvert z^{-1} \rvert = \frac{1}{\lvert z \rvert}$. | $\lvert (2-2i)^{-1} \rvert = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$ |
| **c** | Use the rule for $\arg(z_1 z_2 z_3 \ldots z_n)$. | $\arg(zwu) = \arg(z) + \arg(w) + \arg(u)$ |
| | Substitute for known values. | $= \frac{\pi}{6} + \frac{3\pi}{4} + \frac{\pi}{2}$ |
| | | $= \frac{17\pi}{12}$ |
| | Use the principal domain, i.e. $-\pi < \theta \le \pi$. | $\arg(zwu) = \frac{-7\pi}{12}$ |
| | Use the rule for $\mathrm{mod}(z_1 z_2 z_3 \ldots z_n)$. | $\lvert zwu \rvert = \lvert z \rvert \lvert w \rvert \lvert u \rvert$ |
| | Substitute for known values. | $= \sqrt{3} \times 2 \times 4$ |
| | | $= 8\sqrt{3}$ |
| | Use the results to write $zwu$. | $zwu = 8\sqrt{3}\left[\cos\left(\frac{-7\pi}{12}\right) + \sin\left(\frac{-7\pi}{12}\right)i\right]$ |
| **d** | Convert $(2 + 2i\sqrt{3})$ to polar form. | Let $z = (2 + 2i\sqrt{3})$ |
| | Find $r$. | $r = \lvert z \rvert = \sqrt{2^2 + (2\sqrt{3})^2} = 4$ |
| | Find arg($z$). | $\tan(\theta) = \frac{2\sqrt{3}}{2} = \sqrt{3}$ |
| | | $\arg(z) = \theta = \tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$ |

| | |
|---|---|
| Write $z$ in polar form. | $z = 4\left[\cos\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{3}\right)i\right]$ |
| Convert $(1 - i)$ to polar form. | Let $w = (1 - i)$ |
| Find $r$. | $r = \lvert w\rvert = \sqrt{1^2 + (-1)^2} = \sqrt{2}$ |
| Find arg($z$). | $\tan(\theta) = \frac{-1}{1} = -1$ <br> $\arg(z) = \theta = \tan^{-1}(-1) = \frac{-\pi}{4}$ |
| Write $w$ in polar form. | $w = \sqrt{2}\left[\cos\left(\frac{-\pi}{4}\right) + \sin\left(\frac{-\pi}{4}\right)i\right]$ |
| Calculate $zw$. | $zw = 4\sqrt{2}\left[\cos\left(\frac{\pi}{3} + \frac{-\pi}{4}\right) + \sin\left(\frac{\pi}{3} + \frac{-\pi}{4}\right)i\right]$ <br> $= 4\sqrt{2}\left[\cos\left(\frac{\pi}{12}\right) + \sin\left(\frac{\pi}{12}\right)i\right]$ |
| Use the rule $z^{-1} = \frac{1}{r}[\cos(-\theta) + i\sin(-\theta)]$ | $(zw)^{-1} = \frac{1}{4\sqrt{2}}\left[\cos\left(\frac{-\pi}{12}\right) + \sin\left(\frac{-\pi}{12}\right)i\right]$ |
| Simplify and write the result. | $\frac{1}{(2+2i\sqrt{3})(1-i)}$ <br> $= \frac{\sqrt{2}}{8}\left[\cos\left(\frac{-\pi}{12}\right) + \sin\left(\frac{-\pi}{12}\right)i\right]$ |

## Exercise 3.03 Modulus, argument and polar form

Example 7

**1** Write each complex number in polar form.

**a** $\arg(z) = \frac{\pi}{3}$ and $\mathrm{mod}(z) = 7$

**b** $\arg(w) = \frac{2\pi}{5}$ and $\mathrm{mod}(w) = 3\sqrt{2}$

**c** $\arg(u) = \frac{-\pi}{4}$ and mod $\lvert u\rvert = 6$

**d** $\arg(v) = \frac{-3\pi}{8}$ and $\mathrm{mod}(v) = 2$

**e** $\arg(w) = \frac{5\pi}{6}$ and mod $\lvert z\rvert = 4\sqrt{3}$

**f** $\arg(z) = \frac{-2\pi}{3}$ and $\lvert z\rvert = 4$

**2** Convert each complex number to polar form.

**a** $1 + i$ **b** $\frac{\sqrt{3}}{2} - \frac{i}{2}$ **c** $-\sqrt{2} + i\sqrt{2}$ **d** $5 - 4i$

**e** $\sqrt{3} + i$ **f** $-7 - 3i$ **g** $-1 - i$ **h** $-1 + i$


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**3** Convert each complex number to Cartesian form.

**a** $5\left[\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right]$

**b** $2\text{ cis }(\pi)$

**c** $4\text{ cis}\left(\frac{5\pi}{6}\right)$

**d** $\frac{2}{\sqrt{2}}\left[\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right]$

**e** $7\left[\cos\left(\frac{-3\pi}{5}\right)+i\sin\left(\frac{-3\pi}{5}\right)\right]$

**f** $4\sqrt{3}\left[\cos\left(\frac{4\pi}{9}\right)+i\sin\left(\frac{4\pi}{9}\right)\right]$

**g** $5\text{ cis}\left(\frac{7\pi}{12}\right)$

**h** $6\left[\cos\left(\frac{-2\pi}{7}\right)+i\sin\left(\frac{-2\pi}{7}\right)\right]$

**4** Calculate each expression, expressing answers in polar form.

Example 8

**a** $z\times w$ if $z=2\left[\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right]$ and $w=\sqrt{2}\left[\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right]$

**b** $u\times v$ if $u=4\left[\cos\left(\frac{5\pi}{6}\right)+i\sin\left(\frac{5\pi}{6}\right)\right]$ and $v=\sqrt{3}\left[\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right]$

**c** $z_1\times z_2$ if $z_1=5\left[\cos\left(\frac{3\pi}{8}\right)+i\sin\left(\frac{3\pi}{8}\right)\right]$ and $z_2=2\left[\cos\left(\frac{4\pi}{7}\right)+i\sin\left(\frac{4\pi}{7}\right)\right]$

**d** $\frac{z}{w}$ if $z=2\left[\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right]$ and $w=4\left[\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right]$

**e** $\frac{u}{v}$ if $u=2\sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right]$ and $v=\sqrt{3}\left[\cos\left(\frac{7\pi}{8}\right)+i\sin\left(\frac{7\pi}{8}\right)\right]$

**f** $\frac{z_1}{z_2}$ if $z_1=8\left[\cos\left(\frac{-\pi}{3}\right)+i\sin\left(\frac{-\pi}{3}\right)\right]$ and $z_2=4\left[\cos\left(\frac{-5\pi}{6}\right)+i\sin\left(\frac{-5\pi}{6}\right)\right]$

**5** Calculate $z^{-1}$ for each expression in polar form.

Example 9

**a** $z=2\left[\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right]$

**b** $z=\sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right]$

**c** $z=2\sqrt{3}\left[\cos\left(\frac{5\pi}{6}\right)+i\sin\left(\frac{5\pi}{6}\right)\right]$

**d** $z=4\left[\cos\left(\frac{3\pi}{8}\right)+i\sin\left(\frac{3\pi}{8}\right)\right]$

**e** $z=6\left[\cos\left(\frac{-2\pi}{3}\right)+i\sin\left(\frac{-2\pi}{3}\right)\right]$

**f** $z=2\sqrt{2}\left[\cos\left(\frac{-3\pi}{4}\right)+i\sin\left(\frac{-3\pi}{4}\right)\right]$

**6** Find $\arg(z)^{-1}$ and $\left|z^{-1}\right|$ for each value of $z$ by first converting $z$ to polar form.

**a** $z=1-i$ **b** $z=2+2i$ **c** $z=1+2i$ **d** $z=2-i$

**e** $z=3-4i$ **f** $z=5+12i$ **g** $z=2-3i$ **h** $z=5-4i$

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**7** If $z = 2\text{ cis}\left(\frac{\pi}{4}\right)$, $w = 3\text{ cis}\left(\frac{2\pi}{3}\right)$ and $u = 4\text{ cis}\left(\frac{5\pi}{6}\right)$, calculate:

**a** $|zw|$ **b** $\arg(zw)$ **c** $zw$ **d** $|wu|$

**e** $\arg(wu)$ **f** $\arg(zwu)$ **g** $|zwu|$ **h** $zwu$

**8** Calculate each expression in polar form.

**a** $\frac{1}{1+i\sqrt{3}}$ **b** $\frac{1}{\frac{3\sqrt{3}}{2}-\frac{3}{2}i}$

**c** $\frac{1}{-3\sqrt{2}+3i\sqrt{2}}$ **d** $\frac{1}{\left(1+i\sqrt{3}\right)\left(-3\sqrt{2}+3i\sqrt{2}\right)}$

**e** $\frac{1}{\left(-2\sqrt{3}+2i\right)\left(\frac{3}{2}+\frac{3\sqrt{3}}{2}i\right)}$ **f** $\frac{1}{\left(\frac{5\sqrt{2}}{2}+\frac{5\sqrt{2}}{2}i\right)\left(-1+i\sqrt{3}\right)}$

**9** Calculate each expression in polar form.

**a** $(1-3i)^{-2}$ **b** $\frac{2-3i}{3-4i}$ **c** $\frac{1+i\sqrt{3}}{-2\sqrt{3}+2i}$

**d** $\frac{1}{4i(3-2i)}$ **e** $\frac{(1+2i)(2+3i)}{(4+i)(9+2i)}$ **f** $\frac{(3-2i)(1+4i)}{(2+i)(1-2i)}$

**10** Calculate each expression in Cartesian form.

**a** $\left[4\text{ cis}\left(\frac{\pi}{3}\right)\right]\left[2\text{ cis}\left(\frac{5\pi}{6}\right)\right]$ **b** $\frac{2\text{ cis}\left(\frac{5\pi}{6}\right)}{4\text{ cis}\left(\frac{\pi}{6}\right)}$ **c** $\frac{6\text{ cis}(170°)}{3\text{ cis}(50°)}$

**d** $\frac{4\text{ cis}(20°)}{8\text{ cis}(80°)}$ **e** $\frac{10\text{ cis}(-60°)}{5\text{ cis}(150°)}$ **f** $[3\text{ cis }(45°)][2\text{ cis }(135°)]$

**11** Find $z_1z_2$ in the form $x + yi$ where:

**a** $z_1 = 2\text{ cis}\left(\frac{\pi}{6}\right)$ and $z_2 = 4\text{ cis}\left(\frac{\pi}{3}\right)$ **b** $z_1 = 10\text{ cis}\left(\frac{\pi}{4}\right)$ and $z_2 = 5\text{ cis}\left(\frac{\pi}{2}\right)$

**c** $z_1 = \sqrt{2}\text{ cis}\left(\frac{3\pi}{4}\right)$ and $z_2 = \sqrt{2}\text{ cis}\left(\frac{\pi}{4}\right)$ **d** $z_1 = 2\text{ cis}\left(\frac{\pi}{3}\right)$ and $z_2 = 2\text{ cis}\left(\frac{\pi}{6}\right)$

**e** $z_1 = \text{cis}\left(\frac{-2\pi}{3}\right)$ and $z_2 = 3\text{ cis}\left(\frac{-2\pi}{3}\right)$ **f** $z_1 = 3\text{ cis}\left(\frac{-\pi}{3}\right)$ and $z_2 = 2\text{ cis}\left(\frac{5\pi}{6}\right)$


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### Problem solving

**12** The alternating current in an electric inductor is given by $I = \frac{V}{Z}$ amperes, where $V$ is the voltage and $Z = R + X_L\,i$ is the impedance. If $V = 6[\cos(40°) + i\sin(40°)]$, $R = 7$ and $X_L = 3$, find the current, giving your answer in Cartesian form.

**13** The current $I$ in an electric circuit with voltage $V$, resistance $R$, capacitive reactance $X_c$ and inductive reactance $XL$ is given by

$$I = \frac{V}{R + (X_L - X_c)i} \text{ amperes.}$$

Find the current if $V = 10[\cos(20°) + i\sin(20°)]$, $R = 7$, $X_L = 4$ and $X_c = 6$, giving your answer in Cartesian form.

**14** Express $\frac{1-i}{1-i\sqrt{3}}$ in modulus argument form and hence find an exact value for $\cos\left(\frac{\pi}{12}\right)$.

**15** Prove that $[r(\cos(\theta) + i\sin(\theta))]^3 = r^3[\cos(3\theta) + i\sin(3\theta)]$.

## 3.04 De Moivre's theorem

WS
Using De Moivre's theorem

It can be demonstrated that the index laws hold for the system of complex numbers ($\boldsymbol{C}$).

### Index laws and complex numbers

If $z \in \boldsymbol{C}$ and $n \in \boldsymbol{Z}^+$, $z^n$ is defined by:

$$z^n = z \times z \times z \times \ldots \text{ to } n \text{ factors.}$$

Furthermore, if $z, z_1, z_2 \in \boldsymbol{C}$ and $m, n \in \boldsymbol{Q}$, then:

$$z^m z^n = z^{m+n}$$

$$(z^m)^n = z^{mn}$$

$$(z_1 z_2)^n = {z_1}^n\, {z_2}^n$$

$$z^0 = 1$$

$$z^{-n} = \frac{1}{z^n}$$

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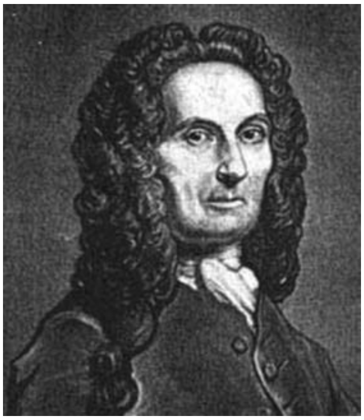
You saw in Year 11 that powers of complex numbers can be calculated when the numbers are expressed in Cartesian form, $z = x + yi$, by repeated multiplication. However, this can prove to be a long and tedious task, especially for large powers.

The method for multiplying complex numbers in trigonometric form developed by the French mathematician Abraham De Moivre (1667–1745) (pronounced *duh mwahv*) provides a relatively simple way to find an integral power of $z$.



Alamy Stock Photo/ART Collection

## De Moivre's theorem

If $z = |z|\ \text{cis}\ (\theta)$ and $n \in \boldsymbol{Q}+$,

then $z^n = |z|^n[\cos (n\theta) + i \sin (n\theta)]$

$z^n = |z|^n\ \text{cis}\ (n\theta)$

**De Moivre's theorem** may be demonstrated to hold for positive integral values of $n$ in the following way.

Let $z_1 = |z_1|\ \text{cis}\ \theta_1$

and $z_2 = |z_2|\ \text{cis}\ \theta_2$

Then $z_1 z_2 = |z_1|\ |z_2|\ \text{cis}\ (\theta_1 + \theta_2)$

Now if $z_1 = z_2 = z = |z|\ \text{cis}\ \theta$

then $z^2 = |z|\ |z|\ \text{cis}\ (\theta + \theta)$

$= |z|^2\ \text{cis}\ (2\theta)$

and $z^3 = z^2 z$

$= [|z|^2\ \text{cis}\ (2\theta)][(|z|\ \text{cis}\ (\theta)]$

$= |z|^3\ \text{cis}\ (2\theta + \theta)$

$= |z|^3\ \text{cis}\ (3\theta)$

and so on.

Hence if follows that, if $n \in \boldsymbol{Q}+$:

$$z^n = |z|^n\ \text{cis}\ (n\theta)$$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## EXAMPLE 10

Use De Moivre's theorem to calculate:

**a** $z^5$ if $z = [\cos(\theta) + i\sin(\theta)]$

**b** $z^{-7}$ if $z = [\cos(2\beta) + i\sin(2\beta)]$

**c** $z^3$ if $z = 4\left[\cos\left(\frac{\pi}{12}\right) - i\sin\left(\frac{\pi}{12}\right)\right]$

**d** $z^8$ if $z = 2\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$

### Solution

**a** Write De Moivre's theorem. $z^n = |z|^n[\cos(n\theta) + i\sin(n\theta)]$

Use $n = 5$ and $|z| = 1$. $z^5 = 1^5[\cos(5\theta) + i\sin(5\theta)]$

Simplify. $= \cos(5\theta) + i\sin(5\theta)$

**b** Use $n = -7$, $\theta = 2\beta$ and $|z| = 1$. $z^{-7} = 1^{-7}[\cos(-14\beta) + i\sin(-14\beta)]$

Simplify. $= \cos(-14\beta) + i\sin(-14\beta)$

**c** Write $z$. $z = 4\left[\cos\left(\frac{\pi}{12}\right) - i\sin\left(\frac{\pi}{12}\right)\right]$

Use $r[\cos(\theta) + i\sin(\theta)] = r\operatorname{cis}(\theta)$. $= 4\operatorname{cis}\left(\frac{-\pi}{12}\right)$

Write $z^3$ in trigonometric form. $z^3 = \left[4\operatorname{cis}\left(\frac{-\pi}{12}\right)\right]^3$

Use De Moivre's theorem. $= 4^3\left[\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right]$

Simplify. $= 64\left[\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right]$

**d** Write $z$. $z = 2\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$

Use $r[\cos(\theta) + i\sin(\theta)] = r\operatorname{cis}(\theta)$. $= 2\operatorname{cis}\left(\frac{3\pi}{4}\right)$

Write $z^8$ in trigonometric form. $z^8 = \left[2\operatorname{cis}\left(\frac{3\pi}{4}\right)\right]^8$

Use De Moivre's theorem. $= 2^8\operatorname{cis}\left(8 \times \frac{3\pi}{4}\right)$

Write in the form $r[\cos(\theta) + i\sin(\theta)]$. $= 256[\cos(6\pi) + i\sin(6\pi)]$

Simplify. $= 256(1 + 0i)$

$= 256$

Complex number expressions need to be converted to polar from before applying De Moivre's theorem.

## EXAMPLE 11

Use De Moivre's theorem to calculate each expression.

**a** $(1+i)^6$ **b** $\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^4$ **c** $\left(1+i\sqrt{3}\right)^4\left(\sqrt{3}+i\right)^2$ **d** $\frac{\left(2+2i\sqrt{3}\right)^5}{\left(1-i\sqrt{3}\right)^3}$

### Solution

**a** Write as $r$ cis $(\theta)$.

$$(1+i)^6=\left[\sqrt{2}\text{ cis}\left(\frac{\pi}{4}\right)\right]^6$$

Use De Moivre's theorem.

$$=\left(\sqrt{2}\right)^6\text{cis}\left(6\times\frac{\pi}{4}\right)$$

Simplify.

$$=8\text{ cis}\left(\frac{3\pi}{2}\right)$$

Write in the form $r[\cos(\theta)+i\sin(\theta)]$.

$$=8\left[\cos\left(\frac{3\pi}{2}\right)+i\sin\left(\frac{3\pi}{2}\right)\right]$$

Simplify.

$$=8(0-i)$$
$$=-8i$$

**b** Write as $r$ cis $(\theta)$.

$$\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^4=\left[1\text{ cis}\left(\frac{\pi}{3}\right)\right]^4$$

Use De Moivre's theorem.

$$=\text{cis}\left(\frac{4\pi}{3}\right)$$

Express in Cartesian form.

$$=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

**c** Write using $r$ cis $(\theta)$.

$$\left(1+i\sqrt{3}\right)^4\left(\sqrt{3}+i\right)^2=\left[2\text{ cis}\left(\frac{\pi}{3}\right)\right]^4\left[2\text{ cis}\left(\frac{\pi}{6}\right)\right]^2$$

Use De Moivre's theorem.

$$=\left[16\text{ cis}\left(\frac{4\pi}{3}\right)\right]\left[4\text{ cis}\left(\frac{\pi}{3}\right)\right]$$

Multiply.

$$=64\text{ cis}\left(\frac{4\pi}{3}+\frac{\pi}{3}\right)$$

Simplify.

$$=64\text{ cis}\left(\frac{5\pi}{3}\right)$$


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$$= 64\left[\cos\left(\frac{5}{3}\pi\right) + i\sin\left(\frac{5}{3}\pi\right)\right]$$

$$= 64\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]$$

$$= 64\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

$$= 32 - 32\sqrt{3}i$$

**d** Write using $r$ cis ($\theta$).

$$\frac{(2+2i\sqrt{3})^5}{(1-i\sqrt{3})^3} = \frac{\left[4\text{ cis}\left(\frac{\pi}{3}\right)\right]^5}{\left[2\text{ cis}\left(\frac{-\pi}{3}\right)\right]^3}$$

Use De Moivre's theorem.

$$= \frac{4^5\text{ cis}\left(\frac{5\pi}{3}\right)}{2^3\text{ cis}(-\pi)}$$

Simplify.

$$= \frac{2^{10}\text{ cis}\left(\frac{5\pi}{3}\right)}{2^3\text{ cis}(-\pi)}$$

Divide.

$$= 2^7\text{ cis}\left(\frac{5\pi}{3} + \pi\right)$$

Simplify.

$$= 128\text{ cis}\left(\frac{8\pi}{3}\right)$$

$$= 128\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$= -64 + 64i\sqrt{3}$$

You have seen how De Moivre's theorem, $z^n = |z|^n\,[\cos(n\theta) + i\sin(n\theta)]$, may be applied for positive integral powers of $n$.

The proof of De Moivre's theorem can be completed using the method of mathematical induction. It is best to prove the theorem in parts.

**Proof of De Moivre's theorem: $z^n = |z|^n[\cos(n\theta) + i\sin n(\theta)]$**

**Part 1: $n = 0$**

Let $n = 0$. $\quad z^0 = 1$

$|z|^0[\cos(0) + i\sin(0)] = 1(1 + 0) = 1$

So the result is true for $n = 0$.

**Part 2: $n$ is a positive integer ($n \in Z^+$)**

Use mathematical induction.

Let $n = 1$. $\quad z^n = z^1 = z$

$$|z|^n[\cos(n\theta) + i\sin(n\theta)] = |z|^1[\cos(\theta) + i\sin(\theta)] = z$$

So the result is true for $n = 1$.

Assume that the result is true for $n = k$.

$$z^k = |z|^k[\cos(k\theta) + i\sin(k\theta)]$$

Let $n = k + 1$. $\quad z^{k+1} = z^k \,.\, z^1$

$$= |z|^k[\cos(k\theta) + i\sin(k\theta)] \bullet |z|^1[\cos(\theta) + i\sin(\theta)]$$

$$= |z|^{k+1}[\cos(k\theta) + i\sin(k\theta)] \bullet [\cos(\theta) + i\sin(\theta)]$$

$$= |z|^{k+1}[\cos(k\theta)\cos(\theta) + i\cos(k\theta)\sin(\theta) + i\sin(k\theta)\cos(\theta) - \sin(k\theta)\sin(\theta)]$$

$$= |z|^{k+1}[\cos(k\theta)\cos(\theta) - \sin(k\theta)\sin(\theta) + i(\sin(k\theta)\cos(\theta) + \cos(k\theta)\sin(\theta))]$$

While this looks complex, the following addition formulas for trigonometry may be used to factorise this expression.

$$\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

$$\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$

Apply the addition formulas for trigonometry.

$$z^{k+1} = |z|^{k+1}[\cos(k\theta + \theta) + i\sin(k\theta + \theta)]$$

$$= |z|[\cos((k + 1)\theta) + i\sin((k + 1)\theta)]$$

Therefore, by mathematical induction, the result is true for $n \in Z^+$.

**Part 3: $n$ is a negative integer ($n \in Z^-$)**

It can be shown that $z^n = |z|^n \operatorname{cis}(n\theta)$ is true for $n = -m$, where $m \in Z^+$.

**Part 4: $n$ is a rational number ($n \in Q$)**

It can be shown that $z^n = |z|^n \operatorname{cis}(n\theta)$ is true for $n =$ where $p, q \in Z$, $q \neq 0$.

Combining the results of parts 1 through 4 establishes $z^n = |z|^n \operatorname{cis}(n\theta)$ for any integral or rational value of $n$. It can actually be proven for all $n \in R$.


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## EXAMPLE 12

Simplify $\dfrac{\left(2[\cos(3\alpha)+i\sin(3\alpha)]\right)^3\times[\cos(5\alpha)+i\sin(5\alpha)]^5}{\left(\sqrt{2}[\cos(2\alpha)+i\sin(2\alpha)]\right)^4}$.

### Solution

Write using $r$ cis ($\theta$).

$$\frac{[2\text{cis}(3\alpha)]^3\times[\text{cis}(5\alpha)]^5}{[\sqrt{2}\text{cis}(2\alpha)]^4}$$

Use De Moivre's theorem.

$$=\frac{8\,\text{cis}(9\alpha)\times\text{cis}(25\alpha)}{2\,\text{cis}(8\alpha)}$$

Use the rules for multiplication and division.

$$= 4\text{ cis }(9\alpha + 25\alpha - 8\alpha)$$

$$= 4\text{ cis }(26\alpha)$$

Write in the form $r[\cos(\theta) + i\sin(\theta)]$.

$$= 4[\cos(26\alpha) + i\sin(26\alpha)]$$

## Exercise 3.04 De Moivre's theorem

**1** Consider the complex number $z = 1 + i$.

- **a** Express $z$ in polar form, that is, $|z|[\cos(\theta) + i\sin(\theta)]$.
- **b** Find $z^4$.
- **c** Find $\dfrac{1}{z^3}$, leaving your answer in polar form.

**2** Consider the complex number $z = 2(1 + i)$.

- **a** Express $z$ in the form $|z|[\cos(\theta) + i\sin(\theta)]$.
- **b** Find $z^5$, leaving your answer in polar form.
- **c** Find $z^{-2}$, leaving your answer in polar form.

**3** Use De Moivre's theorem to calculate each expression in polar form.

Example 10

- **a** $z^7$ if $z = [\cos(\theta) + i\sin(\theta)]$
- **b** $z^{-5}$ if $z = [\cos(3\alpha) + i\sin(3\alpha)]$
- **c** $z^4$ if $z=\sqrt{2}\left[\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right]$
- **d** $z^7$ if $z=\sqrt{3}\left[\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right]$
- **e** $z^{-3}$ if $z=2\left[\cos\left(\frac{\pi}{4}\right)-i\sin\left(\frac{\pi}{4}\right)\right]$
- **f** $z^{-5}$ if $z=\sqrt{2}\left[\cos\left(\frac{-5\pi}{6}\right)+i\sin\left(\frac{-5\pi}{6}\right)\right]$

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**4** Use De Moivre's theorem to find each power given in Cartesian form.

**a** $\left[\sqrt{2}\operatorname{cis}\left(\frac{\pi}{12}\right)\right]^3$

**b** $\left[3\operatorname{cis}\left(\frac{2\pi}{3}\right)\right]^3$

**c** $\left[\operatorname{cis}\left(\frac{4\pi}{5}\right)\right]^{10}$

**d** $\left[\operatorname{cis}\left(\frac{\pi}{4}\right)\right]^8$

**e** $\left[\operatorname{cis}\left(\frac{-\pi}{18}\right)\right]^9$

**f** $\left[\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right]^3$

**g** $\left[\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)\right]^{-6}$

**h** $\left[2\operatorname{cis}\left(\frac{3\pi}{4}\right)\right]^{-4}$

**i** $3[\cos(100°) + i\sin(100°)]^3$

**j** $2[\cos(120°) + i\sin(120°)]^3$

Example 11

**5** Calculate each expression using De Moivre's theorem, expressing your answers in Cartesian form.

**a** $(\sqrt{3} + i)^5$

**b** $(2\sqrt{2} - i2\sqrt{2})^6$

**c** $\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{23}}{2}i\right)^8$

**d** $(2 - 2i\sqrt{3})^4$

**e** $(\sqrt{3} - i)^6$

**f** $(2 + 2i)^3$

**g** $(\sqrt{2} - i\sqrt{2})^7$

**h** $(1 + i\sqrt{3})^4$

**i** $(1 - i\sqrt{3})^4$

**j** $(3 - 4i)^3$

**k** $(1 - i)^{10}$

**l** $(1 + i\sqrt{3})^6$

**6** Use De Moivre's theorem to calculate each expression in Cartesian form.

**a** $(\sqrt{3} - i)^{-3}$

**b** $(1 + i)^6$

**c** $(1 + i)^{-6}$

**d** $(2\sqrt{3} + 2i)^{-3}$

**e** $(\sqrt{3} + i)^{-4}$

**f** $(4 - 4i\sqrt{3})^{-3}$

**g** $(-1 - i)^{10}$

**h** $(-1 + i\sqrt{3})^4$

**i** $(1 + i)^{20}$

**j** $\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2$

**7** Calculate each expression using De Moivre's theorem.

**a** $(1 - i)^3(2 + 2i)^4$

**b** $(-\sqrt{3} + i)^6 + (1 - i\sqrt{3})^4$

**c** $(2 + i)^5(1 - 2i)^3$

**d** $(1 - i)^6(\sqrt{3} + i)^4$

**e** $\frac{(1+i)^3}{(\sqrt{3}+i)^4}$

**f** $\frac{(\sqrt{3}-i)^3}{(1-i)^3}$

**g** $\frac{(3-4i)^5}{(4+3i)^3}$

**h** $\frac{(1+i\sqrt{3})^{-5}}{(\sqrt{3}-i)^{-4}}$

**i** $\frac{(\sqrt{3}+i)^6}{(1-i)^8}$

**j** $\frac{(-1+i\sqrt{3})^5}{\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^{10}}$

**k** $\frac{\left[3\operatorname{cis}\left(\frac{\pi}{12}\right)\right]^5}{\left[3\operatorname{cis}\left(\frac{-\pi}{36}\right)\right]^3}$

**l** $\frac{(1-i)^4(1+i\sqrt{3})^5}{(1+i)^2}$


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**8** Simplify each expression.

Example 12

**a** $[\cos(\theta)+i\sin(\theta)]^4\times[\cos(\theta)+i\sin(\theta)]^5$

**b** $[\cos(\alpha)+i\sin(\alpha)]^3\times[\cos(\alpha)+i\sin(\alpha)]^{-7}$

**c** $[\cos(3\beta)+i\sin(3\beta)]^5\times[\cos(2\beta)+i\sin(2\beta)]^7$

**d** $\left(\sqrt{2}[\cos(\theta)+i\sin(\theta)]\right)^4\times\left(\sqrt{3}[\cos(\theta)+i\sin(\theta)]\right)^{-3}$

**e** $\left(2[\cos(3\alpha)+i\sin(3\alpha)]\right)^5\times\left(\sqrt{2}[\cos(2\alpha)+i\sin(2\alpha)]\right)^5$

**9** If $z=\sqrt{3}\text{ cis}\left(\frac{2\pi}{3}\right)$ and $w=\sqrt{2}\text{ cis}\left(\frac{3\pi}{4}\right)$, calculate:

**a** the modulus and argument of $\frac{z^5}{w^3}$ **b** $z^3\times w^5$

## Problem solving

**10** The binomial theorem shown below can also be proved very neatly using mathematical induction, with methods somewhat similar to De Moivre's theorem.

**a** Show that ${}^nC_r+{}^nC_{r-1}={}^{n+1}C_r$ for $n\geq 1$ and $0\leq r\leq n$.

Remember that ${}^nC_r=\frac{n!}{r!(n-r)!}$

**b** The binomial theorem states that

$$(x+y)^n={}^nC_nx^n+{}^nC_{n-1}x^{n-1}y+{}^nC_{n-2}x^{n-2}y^2+\ldots+{}^nC_1xy^{n-1}+{}^nC_0y^n$$

Prove the binomial theorem by mathematical induction.

# 3.05 Applications of De Moivre's theorem

WS Trigonometric identities using De Moivre's theorem

You can use De Moivre's theorem in conjunction with the binomial theorem to demonstrate certain relationships between the powers of sin ($\theta$) and cos ($\theta$) and the sine and cosine of multiple angles such as $2\theta$, $3\theta$ and so on.

## EXAMPLE 13

**a** Use the binomial theorem to expand $[\cos(\theta)+i\sin(\theta)]^3$.

**b** Use De Moivre's theorem and the result of part **a** to develop a formula for $\cos(3\theta)$.

### Solution

**a** Use the binomial theorem.

$[\cos(\theta)+i\sin(\theta)]^3$

$=\cos^3(\theta)+3\cos^2(\theta)\cdot i\sin(\theta)+3\cos\theta\cdot i^2\sin^2(\theta)+i^3\sin^3(\theta)$

| | | |
|---|---|---|
| | Simplify using $i^2 = -1$. | $= \cos^3(\theta) + i[3\cos^2(\theta)\sin\theta)] - 3\cos(\theta)\sin^2(\theta) - i[\sin^3(\theta)]$ <br> $= [\cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)] + i[3\cos^2(\theta)\sin(\theta) - \sin^3(\theta)]$ |
| **b** | Use De Moivre's theorem. | $[\cos(\theta) + i\sin(\theta)]^3 = \cos(3\theta) + i\sin(3\theta)$ |
| | Equate this result with the result from part **a**. | $\cos(3\theta) + i\sin(3\theta)$ <br> $= \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta) + i[3\cos^2(\theta)\sin(\theta) - \sin^3(\theta)]$ |
| | Equate the real parts. | $\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)$ |
| | Use $\sin^2(\theta) + \cos^2(\theta) = 1$ to eliminate $\sin(\theta)$. | $= \cos^3(\theta) - 3\cos(\theta)[1 - \cos^2(\theta)]$ <br> $= \cos^3(\theta) - 3\cos(\theta) + 3\cos^3(\theta)$ <br> $= 4\cos^3(\theta) - 3\cos(\theta)$ |
| | State the result. | $\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$ |

In the previous example, the trigonometric problem became an algebraic problem when we solved it using De Moivre's theorem. You can also use De Moivre's theorem and the multiplicative inverse of a complex number $\left(z^{-1} = \frac{1}{z}\right)$ to express powers of sin ($\theta$) and cos ($\theta$) in terms of multiple angles.

| | | |
|---|---|---|
| Apply De Moivre's theorem with $n = 1$ and $\lvert z\rvert = 1$. | $z = \cos\theta + \mathrm{i}\sin\theta$ | [1] |
| Take the inverse of both sides. | $\frac{1}{z} = z^{-1} = [\cos(\theta) + i\sin(\theta)]^{-1}$ | |
| Apply De Moivre's theorem with $n = -1$. | $\frac{1}{z} = \cos(-\theta) + i\sin(-\theta)$ | |
| Use $\cos(-\alpha) = \cos(\alpha)$ and $\sin(-\alpha) = -\sin(\alpha)$. | $= \cos(\theta) - i\sin(\theta)$ | [2] |
| Add [1] and [2]. | $z + \frac{1}{z} = 2\cos(\theta)$ | |
| Write in terms of $\cos\theta$. | $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$ | |
| Subtract [2] from [1]. | $z - \frac{1}{z} = 2i\sin(\theta)$ | |
| Write in terms of $\sin\theta$. | $\sin(\theta) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$ | |


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The following results should be learnt for future use.

$$\cos(\theta) = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad \sin(\theta) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

or $$\cos(\theta) = \frac{z + z^{-1}}{2} \qquad \sin(\theta) = \frac{z - z^{-1}}{2i}$$

A similar procedure can be used to establish the general case.

$$\cos(n\theta) = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right) \qquad \sin(n\theta) = \frac{1}{2i}\left(z^n - \frac{1}{z^n}\right)$$

or $$\cos(n\theta) = \frac{z^n + z^{-n}}{2} \qquad \sin(n\theta) = \frac{z^n - z^{-n}}{2i}$$

## EXAMPLE 14

Prove that $\sin(5\theta) + \sin(3\theta) = 2\sin(4\theta)\cos(\theta)$.

### Solution

| | |
|---|---|
| | **RTP** |
| Write the equation. | $\sin(5\theta) + \sin(3\theta) = 2\sin(4\theta)\cos(\theta)$ |
| | **Proof** |
| Write the LHS. | $\text{LHS} = \sin(5\theta) + \sin(3\theta)$ |
| Apply the rules for $\cos(n\theta)$ and $\sin(n\theta)$. | $= \frac{z^5 - z^{-5}}{2i} + \frac{z^3 - z^{-3}}{2i}$ |
| Express as a single fraction. | $= \frac{z^5 - z^{-5} + z^3 - z^{-3}}{2i}$ |
| Write the RHS. | $\text{RHS} = 2\sin(4\theta)\cos(\theta)$ |
| Apply the rules for $\cos(n\theta)$ and $\sin(n\theta)$. | $= 2 \bullet \frac{z^4 - z^{-4}}{2i} \bullet \frac{z + z^{-1}}{2}$ |
| Express as a single fraction. | $= \frac{z^4 z + z^4 z^{-1} - z^{-4} z - z^{-4} z^{-1}}{2i}$ |
| Simplify the numerator. | $= \frac{z^5 + z^3 - z^{-3} - z^{-5}}{2i}$ |

| | |
|---|---|
| Rearrange the numerator as for the LHS. | $=\dfrac{z^5-z^{-5}+z^3-z^{-3}}{2i}$ |
| | $=\text{LHS}$ |
| Write the result. | $\sin(5\theta)+\sin(3\theta)=2\sin(4\theta)\cos(\theta)$ |
| | **QED** |

De Moivre's theorem and the binomial expansion can also be used to find the integrals of the powers of trigonometric functions.

## EXAMPLE 15

**a** Use the binomial expansion of $[\cos(\theta)+i\sin(\theta)]^3$ and De Moivre's theorem to develop a formula for $\sin(3\theta)$ in terms of $\sin(\theta)$.

**b** Use this result to calculate $\int_0^{\frac{\pi}{2}} 4\sin^3(\theta)\,d\theta$.

### Solution

| | | |
|---|---|---|
| **a** | Use the result established in Example **13**. | $\cos(3\theta)+i\sin(3\theta)$ <br> $=\cos^3(\theta)-3\cos(\theta)\sin^2(\theta)+i[3\cos^2(\theta)\sin(\theta)-\sin^3\theta)]$ |
| | Equate the imaginary parts. | $\sin(3\theta)=3\cos^2(\theta)\sin(\theta)-\sin^3(\theta)$ |
| | Use $\sin^2(\theta)+\cos^2(\theta)=1$ to eliminate $\cos(\theta)$. | $=3[1-\sin^2(\theta)]\sin(\theta)-\sin^3(\theta)$ <br> $=3\sin(\theta)-3\sin^3(\theta)-\sin^3(\theta)$ <br> $=3\sin(\theta)-4\sin^3(\theta)$ |
| | State the result. | $\sin(3\theta)=3\sin(\theta)-4\sin^3(\theta)$ |
| **b** | Write an expression in terms of $\sin^3(\theta)$. | $4\sin^3(\theta)=3\sin(\theta)-\sin(3\theta)$ |
| | Write the integral in terms of $\sin(\theta)$. | $\int_0^{\frac{\pi}{2}} 4\sin^3(\theta)\,d\theta=\int_0^{\frac{\pi}{2}}[3\sin(\theta)-\sin(3\theta)]\,d\theta$ |
| | Calculate the definite integral. | $=\left[-3\cos(\theta)+\dfrac{\cos(3\theta)}{3}\right]_0^{\frac{\pi}{2}}$ |


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$$= \left[-3\cos\left(\frac{\pi}{2}\right)+\frac{\cos\left(\frac{3\pi}{2}\right)}{3}\right]-\left[-3\cos(0)+\frac{\cos(0)}{3}\right]$$

$$= 0 + 0 - \left[(-3\times 1)+\frac{1}{3}\right]$$

$$= 2\tfrac{2}{3}$$

## Exercise 3.05 Applications of De Moivre's theorem

Example 13

**1** **a** Use the binomial theorem to expand $[\cos(\theta) + i\sin(\theta)]^3$.

**b** Use De Moivre's theorem and the result of part **a** to develop a formula for $\sin(3\theta)$.

**2** **a** Use the binomial theorem to expand $[\cos(\theta) + i\sin(\theta)]^2$.

**b** Use De Moivre's theorem and the result of part **a** to develop a formula for $\cos(2\theta)$ in terms of $\cos(\theta)$.

**c** Develop a formula for $\cos(2\theta)$ in terms of $\sin(\theta)$.

**3** Let $z = \cos(\theta) + i\sin(\theta)$.

**a** Show that $\frac{1}{z} = \cos(\theta) - i\sin(\theta)$.

**b** Calculate $z^3$.

**c** Show that $\frac{1}{z^3} = \cos(3\theta) - i\sin(3\theta)$.

**4** Use the results of question **3** to answer the following.

**a** Show that $\left(z+\frac{1}{z}\right)^3 = 8\cos^3(\theta)$.

**b** By expanding $\left(z+\frac{1}{z}\right)^3$, prove that $2\cos(3\theta) + 6\cos(\theta) = 8\cos^3(\theta)$.

**c** Express $\cos(3\theta)$ in terms of powers of $\cos(\theta)$.

**5** Use the results of question **3** to answer the following.

**a** Show that $\left(z-\frac{1}{z}\right)^3 = -8i\sin^3(\theta)$.

**b** By expanding $\left(z-\frac{1}{z}\right)^3$, prove that $2i\sin(3\theta) - 6i\sin(\theta) = -8i\sin^3(\theta)$.

**c** Express $\sin(3\theta)$ in terms of powers of $\sin(\theta)$.

**6** Use De Moivre's theorem to show that:

**a** $\cos^2(\theta) - \sin^2(\theta) = \cos(2\theta)$ **b** $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$

Example 14

**7** Using the formulas $\cos(n\theta) = \dfrac{z^n + z^{-n}}{2}$ and $\sin(n\theta) = \dfrac{z^n - z^{-n}}{2i}$, prove each result.

**a** $\sin(12\theta) + \sin(4\theta) = 2\sin(8\theta)\cos(4\theta)$ **b** $\cos(3\theta) + \cos(7\theta) = 2\cos(5\theta)\cos(2\theta)$

**c** $\sin(5\theta) + \sin(\theta) = 2\sin(3\theta)\cos(2\theta)$ **d** $\cos(3\theta) - \cos(9\theta) = 2\sin(6\theta)\sin(3\theta)$

Example 15

**8** **a** Use the binomial expansion of $[\cos(\theta) + i\sin(\theta)]^3$ and De Moivre's theorem to develop a formula for $\cos(3\theta)$ in terms of $\cos(\theta)$.

**b** Use this result to calculate $\int_0^{\frac{\pi}{4}} 4\cos^3(\theta)\,d\theta$.

**9** **a** Use the binomial expansion of $[\cos(\theta) + i\sin(\theta)]^2$ and De Moivre's theorem to develop a formula for $\sin^2(\theta)$ in terms of $\cos(2\theta)$.

**b** Use this result to calculate $\int_0^{\frac{\pi}{2}} \sin^2(\theta)\,d\theta$.

## Problem solving

**10** Using De Moivre's theorem, show that:

**a** $\cos(4\theta) = \cos^4(\theta) - 6\cos^2(\theta)\sin^2(\theta) + \sin^4(\theta)$, and hence show that $\cos(4\theta) = 8\cos^4(\theta) - 8\cos^2(\theta) + 1$

**b** $\sin(4\theta) = 4\sin(\theta)\cos(\theta)[\cos^2(\theta) - \sin^2(\theta)]$

**11** If $z = \cos(\theta) + i\sin(\theta)$, use powers of $(z + z^{-1})$ or $(z - z^{-1})$ to show that

$$\sin^5(\theta) = \frac{1}{16}[\sin(5\theta) - 5\sin(3\theta) + 10\sin(\theta)].$$

**12** Use powers of $(z + z^{-1})$ or $(z - z^{-1})$ to express $\sin^4(\theta)$ as $\frac{1}{8}[3 + \cos(4\theta) - 4\cos(2\theta)]$.

**13** **a** Use the binomial expansion of $[\cos(\theta) + i\sin(\theta)]^4$ and De Moivre's theorem to show:

$$\sin^4(\theta) = \frac{1}{8}\cos(4\theta) - \frac{1}{2}\cos(2\theta) + \frac{3}{8}$$

**b** Hence calculate $\int_0^{\frac{\pi}{2}} \sin^4(\theta)\,d\theta$.

**14** If $z = \cos(\theta) + i\sin(\theta)$, use powers of $(z + z^{-1})$ or $(z - z^{-1})$ to find an expression for $\cos^4(\theta)$ in terms of $\cos(4\theta)$ and $\cos(2\theta)$ and hence calculate $\int_0^{\frac{\pi}{4}} \cos^4(\theta)\,d\theta$.


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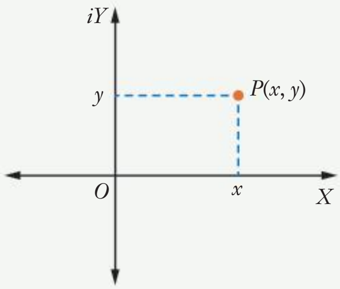
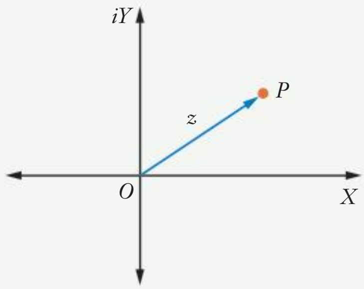
# 3. CHAPTER SUMMARY

## Complex arithmetic

- The **imaginary number** $i$ has the property that $i^2 = -1$, so $i = \sqrt{-1}$
- A **complex number** has the form $z = x + iy$ where $x, y \in \boldsymbol{R}$ and $i = \sqrt{-1}$. The real part of $z = \text{Re}\,(z) = x$ and the imaginary part of $z = \text{Im}\,(z) = y$.
- If $z = x + iy$, the **complex conjugate** is $\bar{z} = x - iy$
- A complex number can be represented geometrically on an **Argand plane** or **Argand diagram**. In the Argand plane, the horizontal axis is called the real axis, Re($z$), and the vertical axis is called the imaginary axis Im($z$). The complex number $z = x + iy$ can be represented as the point $P(x, y)$ or by the vector **z** or **OP** on the Argand plane.



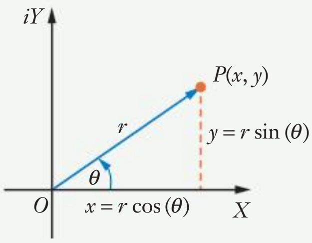

- Complex numbers written as $x + iy$ are said to be in **Cartesian** or **rectangular form**
- The complex numbers $x_1 + iy_1$ and $x_2 + iy_2$ are equal if both their real and imaginary parts are equal, i.e., $x_1 + iy_1 = x_2 + iy_2$ iff $x_1 = x_2$ and $y_1 = y_2$
- To **realise the denominator** of a fraction with denominator $z$, multiply by $\frac{\bar{z}}{\bar{z}}$
- When $z$ is represented in an Argand diagram, the **modulus**, mod ($z$) or $|z|$, is its magnitude and the **argument**, arg ($z$), is the angle that $z$ makes with the positive direction of the real axis.

  For $z = x + yi$:

  mod $(z) = |z| = r = \sqrt{x^2 + y^2}$.

  arg $(z) = \theta$, where $\tan(\theta) = \dfrac{y}{x}$,

  $x = r\cos(\theta)$ and $y = i\sin(\theta)$.




- $\cos(\theta) + i\sin(\theta)$ can be abbreviated to cis ($\theta$)
- The **polar form** of $z$ is $z = (r, \theta)$ or $z = r[\cos(\theta) + i\sin(\theta)]$ or $r$ cis ($\theta$). The polar form is also known as the **modulus–argument** or **trigonometric form**. The value of $\theta$ in the interval $-\pi < \theta \leq \pi$ is called the **principal argument**.

- For $z_1 = r_1 \text{ cis } (\theta_1)$ and $z_2 = r_2 \text{ cis } (\theta_2)$:

$$z_1 \times z_2 = z_1 z_2 = r_1 r_2 \text{ cis } (\theta_1 + \theta_2)$$

where $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ and $\text{mod}(z_1 z_2) = \text{mod}(z_1) \times \text{mod}(z_2)$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \text{ cis } (\theta_1 - \theta_2)$$

where $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$ and $\text{mod}\left(\frac{z_1}{z_2}\right) = \frac{\text{mod}(z_1)}{\text{mod}(z_2)} = \frac{|z_1|}{|z_2|}$

- **De Moivre's theorem** states that if $z = |z| \text{ cis } (\theta)$, then

$$z^n = |z|^n[\cos (n\theta) + i \sin (n\theta)] = |z|^n \text{ cis } (n\theta)$$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# 3. CHAPTER REVIEW

## Complex arithmetic

**1** Solve each quadratic equation, expressing solutions in the form $a \pm ib$. **Example 1**

**a** $x^2 + 2x + 5 = 0$ **b** $4x^2 - 6x + 3 = 0$

**c** $x^2 + x + 2 = 0$ **d** $x^2 - x + 6 = 0$

**2** For each value of $z$, calculate the product $z \cdot \bar{z}$. **Example 2**

**a** $z = 5 - i$ **b** $z = 3 + 2i$ **c** $z = 2 - 5i$

**d** $z = 6 + 3i$ **e** $z = 2 - 9i$ **f** $z = 4 - 13i$

**3** For the complex number $z = -3 + 2i$: **Example 3**

**a** state the point $P$ that represents $z$ on the Argand plane

**b** draw the vectors $\mathbf{z}$ and $\bar{\mathbf{z}}$ on the Argand plane

**c** state the relationship between the vectors $\mathbf{z}$ and $\bar{\mathbf{z}}$

**4** If $a = 6 - i$, $b = -8 + 5i$ and $c = -9 - 7i$, calculate: **Example 4**

**a** $3a + 2c$ **b** $5c - 4b$ **c** $6b - 3a - 7c$

**5** If $u = -4 + 5i$, $v = 6 - 2i$ and $w = -3 - 7i$, calculate: **Example 5**

**a** $3wv$ **b** $uw - vu$ **c** $4v^2 + u^2$

**6** Write each expression in the form $a + bi$. **Example 6**

**a** $\frac{1}{2+5i}$ **b** $\frac{5+3i}{2+3i}$ **c** $\frac{2-i\sqrt{5}}{-3\sqrt{5}+i}$ **d** $\frac{(2-i)(3+2i)}{(3+i)(4+2i)}$

**7** **a** If $\arg(z) = \frac{-2\pi}{3}$ and $\mathrm{mod}(z) = 5\sqrt{2}$, write $z$ in polar form. **Example 7**

**b** Convert $z = 3 + 4i$ to polar form.

**c** Convert $z = 5\left[\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right]$ to Cartesian form.

**8** Calculate each expression in polar form: **Example 8**

**a** $z \times w$ if $z = 2\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$ and $w = \sqrt{2}\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$

**b** $z \times w$ if $z = \left[\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right]$ and $w = \frac{1}{2}\left[\cos\left(\frac{-7\pi}{4}\right) + i\sin\left(\frac{-7\pi}{4}\right)\right]$

**c** $\frac{z}{w}$ if $z = 2\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right]$ and $w = \sqrt{2}\left[\cos\left(\frac{5\pi}{8}\right) + i\sin\left(\frac{5\pi}{8}\right)\right]$

**d** $\frac{z}{w}$ if $z = \sqrt{3}\left[\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right]$ and $w = \frac{1}{\sqrt{3}}\left[\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right]$

Example 9

**9** **a** If $w = 4\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]$, calculate $w^{-1}$.

**b** Convert $z = \frac{-3\sqrt{3}}{2} - \frac{3}{2}i$ to polar form and then find $\arg\left(\frac{-3\sqrt{3}}{2} - \frac{3}{2}i\right)^{-1}$ and $\left|\left(\frac{-3\sqrt{3}}{2} - \frac{3}{2}i\right)^{-1}\right|$.

**c** If $z = 3\text{ cis}\left(\frac{3\pi}{4}\right)$, $w = \sqrt{2}\text{ cis}\left(\frac{-\pi}{3}\right)$ and $u = 2\sqrt{3}\text{ cis}\left(\frac{\pi}{6}\right)$, calculate arg $(zwu)$, $|zwu|$ and $zwu$.

**d** Calculate $\frac{1}{(2-2i)(3-3i\sqrt{3})}$, expressing the answer in polar form.

Example 10

**10** Use De Moivre's theorem to calculate:

**a** $z^8$ if $z = [\cos(\alpha) + i\sin(\alpha)]$

**b** $z^{-3}$ if $z = 3[\cos(2\theta) + i\sin(2\theta)]$

**c** $z^5$ if $z = \sqrt{2}\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5}{6}\right)\right]$

**d** $z^{-6}$ if $z = \sqrt{3}\left[\cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\right]$

Example 11

**11** Use De Moivre's theorem to calculate each expression in Cartesian form.

**a** $(1-i)^5$

**b** $\left(\frac{-\sqrt{3}}{2} + \frac{1}{2}i\right)^3$

**c** $(\sqrt{3}-i)^3(1+i\sqrt{3})^5$

**d** $\frac{(1-i\sqrt{3})^6}{(2+2i\sqrt{3})^4}$

Example 12

**12** Simplify $\frac{\left(\sqrt{2}[\cos(2\theta) + i\sin(2\theta)]\right)^6 \times \left(3[\cos(6\theta) + i\sin(6\theta)]\right)^4}{\left(\sqrt{3}[\cos(3\theta) + i\sin(3\theta)]\right)^5}$.

Example 13

**13** Use the binomial expansion of $[\cos(\theta) + i\sin(\theta)]^4$ and De Moivre's theorem to develop a formula for $\cos(4\theta)$ in terms of $\cos(\theta)$.

Example 14

**14** Use the formulas $\cos(n\theta) = \frac{z^n + z^{-n}}{2}$ and $\sin(n\theta) = \frac{z^n - z^{-n}}{2i}$ to prove that

$2\sin(4\theta)\cos(\theta) = \sin(5\theta) + \sin(3\theta)$

Example 15

**15** **a** Use the binomial expansion of $[\cos(\theta) + i\sin(\theta)]^2$ and De Moivre's theorem to develop a formula for $\sin(\theta)\cos(\theta)$.

**b** Use this result to calculate $\int_0^{\frac{\pi}{2}} [\sin(\theta)\cos(\theta)]\,d\theta$.


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## Problem solving

**16** Given that $z = a + bi$ and $z(-3 - 5i) = -21 - i$, find the values of $a$ and $b$.

**17** Use the $\frac{1}{2}(z^n \pm z^{-n})$ formulas to prove that:

**a** $8\cos^4(\theta) = \cos(4\theta) + 4\cos(2\theta) + 3$

**b** $32\sin^6(\theta) = 10 - 15\cos(2\theta) + 6\cos(4\theta) - \cos(6\theta)$

**c** $2\sin(2\theta)\sin(5\theta) = \cos(3\theta) - \cos(7\theta)$

**18** Prove that $\left[\frac{1+\sin(\theta)+i\sin(\theta)}{1+\sin(\theta)-i\sin(\theta)}\right]^n = \cos\left[n\left(\frac{\pi}{2}-\theta\right)\right] + i\sin\left[n\left(\frac{\pi}{2}-\theta\right)\right]$.

Qz
Practice quiz

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# Practice examination 1 ●○○○

**Time: 90 minutes**
**Perusal time: 5 minutes**
**Marks: 50**

### Instructions

- Students are permitted to bring or use: pens, pencils, highlighters, erasers, sharpeners, rules and an approved graphics calculator.
- Students must show appropriate working and justification to gain full marks.
- A QCAA formula sheet is provided.
- Unless otherwise stated, numerical answers should be exact.
- Unless otherwise indicated, no diagrams in this examination are drawn to scale.
- All written responses must be in English.
- Answer all questions.
- **Students are NOT permitted to bring or use notes of any kind, correction fluid/tape, mobile phones and/or any other unauthorised electronic devices.**

### Question 1 (2 marks)

Proving by mathematical induction that a statement P($n$) is true for all positive integers $n$ involves an initial step and an inductive step. Describe what is proved during:

**a** the initial step

**b** the inductive step

### Question 2 (1 mark)

Prove that $2^n - n \geq 0$ for $n = 1$.

### Question 3 (2 marks)

Prove that if $2^k - k \geq 0$, then $2^{k+1} - k - 1 \geq 0$ for any positive integer $k$.


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**Question 4** (3 marks)

Prove that $3^n \geq n$ for all positive integers.

**Question 5** (1 mark)

Find the distance between the points (3, 5, –4) and (–1, 7, 4).

**Question 6** (2 marks)

**a** Find the equation of the sphere with centre (3, –4, –1) and radius 5.

**b** State whether the point $A(1, 1, 5)$ is inside, on or outside the sphere.

**Question 7** (2 marks)

Express (12, 150°, –45°) as a linear combination of **i**, **j** and **k**.

**Question 8** (2 marks)

$\mathbf{p} = 4\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$ and $\mathbf{q} = 4\mathbf{j} - 2\mathbf{i} + \mathbf{k}$.

**a** Find $\mathbf{p} - 5\mathbf{q}$.

**b** Find $\mathbf{p} \bullet \text{q}$.

**Question 9** (2 marks)

**a** Find the vector product of vectors at an angle of $\frac{\pi}{3}$ with magnitudes 4 and 5.

**b** Find $(-2\mathbf{i} + 3\mathbf{j} + \mathbf{k}) \times (4\mathbf{i} - 7\mathbf{j} + 5\mathbf{k})$.

**Question 10** (2 marks)

Change the equation $\mathbf{r}(t) = 2t\mathbf{i} - (4t + 3)^2\mathbf{j}$ to Cartesian form.

**Question 11** (2 marks)

Find the vector equation of a line passing through (1, 6, –3) and (–2, 2, 4).

**Question 12** (2 marks)

Find the Cartesian equation of a plane with normal (3, 2, –4) that contains the point (1, –2, 3).

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**Question 13** (1 mark)

Express $z = -\sqrt{3} + i$ in trigonometric form.

**Question 14** (1 mark)

Express $z = \text{cis}\,(110.5°)$ in Cartesian form (correct to 4 decimal places).

**Question 15** (1 mark)

For the complex number $z = x + yi$, describe in words the graph of all complex numbers $z$ satisfying the condition that the imaginary part of $z$ is 1.

**Question 16** (1 mark)

Calculate $\frac{5+7i}{2+5i}$.

**Question 17** (1 mark)

Express $-7 + 24i$ in the form $r\,\text{cis}\,(\theta)$.

**Question 18** (1 mark)

Express $\sqrt{3}\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$ in the form $x + yi$.

**Question 19** (1 mark)

Find $\frac{z_1}{z_2}$, where $z_1 = 2\,\text{cis}\,(0.5)$ and $z_2 = 4\,\text{cis}\,(0.8)$.

**Question 20** (2 marks))

Calculate $\left[4\,\text{cis}\left(\frac{\pi}{12}\right)\right]^{-3}$ using De Moivre's theorem, expressing your answer in Cartesian form.

**Question 21** (3 marks)

Prove the formula for the sum of a geometric series,

$S_n = \frac{a(1-r^n)}{1-r}$, by induction.


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### Question 22 (3 marks)

Two particles have paths given by $\mathbf{a}(t) = (6t + 5)\mathbf{i} + (t + 3)^2\mathbf{j} + (3 - t)\mathbf{k}$ and $\mathbf{b}(t) = (3 - 7t^2)\,\mathbf{i} + t^2\mathbf{j} + 4t\mathbf{k}$. Determine whether or not their paths cross and whether or not the particles collide. If either case is true, state where that occurs.

### Question 23 (2 marks)

Find the volume of the parallelepiped formed by the vectors $2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$, $\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$ and $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

### Question 24 (3 marks)

Use the $\frac{1}{2}(z^n \pm z^{-n})$ formulas to prove that:

$$32 \sin^6(\theta) = 10 - 15\cos(2\theta) + 6\cos(4\theta) - \cos(6\theta)$$

### Question 25 (3 marks)

Prove that $\dfrac{2^n - (-1)^n}{3}$ is not divisible by 2 for any positive integer $n$.

### Question 26 (5 marks)

Prove that in three-dimensional space, given any 2 vectors at an angle of 120°, there always exist another 2 vectors at angles of 120° to the given vectors and to each other.

### Question 27 (3 marks)

Prove that $\left[\dfrac{1+\sin(\theta)+i\cos(\theta)}{1+\sin(\theta)-i\cos(\theta)}\right]^n = \cos\left[n\left(\dfrac{\pi}{2}-\theta\right)\right] + i\sin\left[n\left(\dfrac{\pi}{2}-\theta\right)\right]$.

**END OF EXAMINATION**

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# 4.

# VECTORS AND MATRICES

In Year 11, you looked at matrices and their application to transformations of the plane. Matrices are used in Economics and Ecology to model and solve large numbers of linear equations. In this chapter, you will look at matrix solutions of linear equations and a range of simple applications.

4.01 Systems of linear equations
4.02 Linear equations and matrices
4.03 Solving linear equations using matrices
4.04 Applications of linear equations
4.05 Solutions of linear equations
4.06 Dominance matrices
4.07 Leslie matrices
4.08 Further matrix applications
Chapter summary
Chapter review

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## SYLLABUS SUBJECT MATTER

### Systems of linear equations

- recognise the general form of a system of linear equations in several variables and use Gaussian techniques of elimination to solve a system of linear equations
- solve systems of linear equations using matrix algebra
- examine the three cases for solutions of systems of equations — a unique solution, no solution and infinitely many solutions — and the geometric interpretation of a solution of a system of equations with three variables

### Applications of matrices

- model real-life situations using matrices, including Dominance and Leslie
- investigate how matrices have been applied in other real-life situations, e.g. Leontief, Markov, area, cryptology, eigenvectors and eigenvalues.

Note: The external examination may assess only dominance and Leslie matrices.

Prior learning

# TERMINOLOGY

age-specific
consistent equations
dominance matrix
Gaussian elimination
independent system
Markov chain
normal
row echelon form
singular matrix
third-order ranking
transition probability
variable vector
augmented matrix
culling
duopoly
homogeneous system
inverse
non–homogeneous
parallel planes
row reduced echelon form
state vector
total ranking
transpose
coefficient matrix
dependent system
first-order ranking
inconsistent equations
Leslie matrix
non-parallel planes
payoff matrix
second-order ranking
system of linear equations
transition matrix
value vector

## 4.01 Systems of linear equations

WS Systems of equations

You already know how to solve 2 simultaneous equations. This is the simplest case of a **system** of equations. Each equation shows a relationship between variables. Remember that a linear equation has no powers, roots, reciprocals, etc. How many equations and variables are there in each of the following systems of simultaneous linear equations?

**A** $\begin{cases} 3x - 4y = -14 \\ 2x + y = 9 \end{cases}$

**B** $\begin{cases} 3p - 4q + 6v = 5 \\ 4p - 3t + 5v = 7 \\ 6q + 4t - v = 9 \end{cases}$

**C** $\begin{cases} 2a + b + 3c = 4 \\ 3a - 2b - 4c = 4 \\ 2a - 3b - c = -4 \end{cases}$

**A** has 2 equations with 2 variables. **B** has 3 equations with 4 variables. **C** has 3 equations with 3 variables. You can solve **A**: 2 equations with 2 unknowns. You eliminate one variable to get an equation with only one variable. You can generalise this to get the method for more equations.

### Gaussian elimination

**Step 1:** Choose a variable to eliminate

**Step 2:** Use the equation with the simplest coefficient of this variable to eliminate it

**Step 3:** Multiply each of the other equations and the chosen equation by suitable numbers to add them and eliminate the variable

**Step 4:** Repeat the steps above to eliminate as many variables as possible. If you get an equation with only one variable, find its value

**Step 5:** Progressively substitute to find the values of other variables


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Each step of **Gaussian elimination** reduces the number of equations.

## EXAMPLE 1

Solve the equations $\begin{cases} 2a + b + 3c = 4 \\ 3a - 2b - 4c = 4 \\ 2a - 3b - c = -4 \end{cases}$

### Solution

Number the equations and choose which variable to eliminate first.

$$\begin{aligned} 2a + b + 3c &= 4 \quad [1] \\ 3a - 2b - 4c &= 4 \quad [2] \\ 2a - 3b - c &= -4 \quad [3] \end{aligned}$$

$b$ has the simplest coefficient. Multiply [1] by 2 and [2] by 1. $b$ now has the same coefficient, but with opposite signs.

$$\begin{aligned} 4a + 2b + 6c &= 8 \quad [1] \times 2 \\ 3a - 2b - 4c &= 4 \quad [2] \times 1 \\ \hline 7a + 2c &= 12 \quad [4] \end{aligned}$$

Now multiply [1] and [3] by appropriate numbers to eliminate $b$.

$$\begin{aligned} 6a + 3b + 9c &= 12 \quad [1] \times 3 \\ 2a - 3b - c &= -4 \quad [3] \times 1 \\ \hline 8a + 8c &= 8 \end{aligned}$$

Simplify the new equation.

$a + c = 1 \quad [5]$

Now eliminate $c$, as it has the simplest coefficients.

$$\begin{aligned} 7a + 2c &= 12 \quad [4] \times 1 \\ -2a - 2c &= -2 \quad [5] \times -2 \\ \hline 5a &= 10 \end{aligned}$$

Solve to get $a$.

$a = 2$

Substitute in [5].

$2 + c = 1$

$c = -1$

When you substitute back in, use the simplest equations to make the job easier.

Substitute $a$ and $c$ in [1].

$2 \times 2 + b + 3 \times (-1) = 4$

$b = 3$

Check your answers in the *original* equations.

$$\begin{aligned} 2a + b + 3c &= 4 + 3 - 3 = 4 \quad [1] \checkmark \\ 3a - 2b - 4c &= 6 - 6 - (-4) = 4 \quad [2] \checkmark \\ 2a - 3b - c &= 4 - 9 - (-1) = -4 \quad [3] \checkmark \end{aligned}$$

Write the answer.

$a = 2$, $b = 3$ and $c = -1$

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Example **1** has the same number of variables as there are equations. It has only one solution, that is, only one set of answers for $a$, $b$ and $c$. Generally, a system of simultaneous linear equations can have no solution, a unique solution or multiple solutions.
**Systems of linear simultaneous equations** are classified below.

## Systems of linear equations

- **Inconsistent equations** have no solutions
- **Consistent equations** have at least one solution
- A **dependent system** has at least one equation that is a combination of others
- An **independent system** has no equation that is a combination of others.
- A **homogeneous system** of equations has no constant terms: You can write every equation in the form $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + \ldots + a_nx_n = 0$
- A **non-homogeneous system** has at least one equation with a constant term

## EXAMPLE 2

Solve this system of equations.

$$\begin{aligned} 3a - 3b + 2c + 2d &= 7 \\ a - b + 3c + 4d &= 6 \\ a - b - 2c - d &= -4 \\ 3a - 3b + c + 3d &= 3 \\ 2a - 2b + c + 3d &= 2 \end{aligned}$$

### Solution

Number the equations.

$$\begin{aligned} 3a - 3b + 2c + 2d &= 7 \quad [1] \\ a - b + 3c + 4d &= 6 \quad [2] \\ a - b - 2c - d &= -4 \quad [3] \\ 3a - 3b + c + 3d &= 3 \quad [4] \\ 2a - 2b + c + 3d &= 2 \quad [5] \end{aligned}$$

Use [3] to eliminate $a$ from [1].

$$\begin{aligned} 3a - 3b + 2c + 2d &= 7 \quad [1] \\ \underline{-3a + 3b + 6c + 3d} &= \underline{12} \quad [3] \times (-3) \\ 8c + 5d &= 19 \quad [6] \end{aligned}$$

Use [3] to eliminate $a$ from [2].

$$\begin{aligned} a - b + 3c + 4d &= 6 \quad [2] \\ \underline{-a + b + 2c + d} &= \underline{4} \quad [3] \times (-1) \\ 5c + 5d &= 10 \end{aligned}$$

Simplify the new equation.

$$c + d = 2 \quad [7]$$


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| | |
|---|---|
| Use [3] to eliminate $a$ from [4]. | $3a - 3b + c + 3d = 3$ [4]<br>$-3a + 3b + 6c + 3d = 12$ [3] × (–3)<br>$7c + 6d = 15$ [8] |
| Use [3] to eliminate $a$ from [5].<br>Simplify.<br>Don't use a new number for the same one. | $2a - 2b + c + 3d = 2$ [5]<br>$-2a + 2b + 4c + 2d = 8$ [3] × (–2)<br>$5c + 5d = 10$<br>$c + d = 2$ [7] |
| Write the new equations together for clarity. | $8c + 5d = 19$ [6]<br>$c + d = 2$ [7]<br>$7c + 6d = 15$ [8] |
| Use [7] to eliminate $d$. | $8c + 5d = 19$ [6]<br>$-5c - 5d = -10$ [7]<br>$3c = 9$ |
| Solve for $c$. | $c = 3$ |
| Substitute in [7] to find $d$. | $3 + d = 2$ [7]<br>$d = -1$ |
| Substitute in [3]. | $a - b - 6 - (-1) = -4$ [3]<br>$a - b = 1$ |
| You can't eliminate anything else, so substitute in the original equations to check, including $a - b = 1$. | $3a - 3b + 2c + 2d = 3 + 6 - 2 = 7$ [1] ✓<br>$a - b + 3c + 4d = 1 + 9 - 4 = 6$ [2] ✓<br>$a - b - 2c - d = 1 - 6 + 1 = -4$ [3] ✓<br>$3a - 3b + c + 3d = 3 + 3 - 3 = 3$ [4] ✓<br>$2a - 2b + c + 3d = 2 + 3 - 3 = 2$ [5] ✓ |
| Write the answer set. | $a - b = 1$, $c = 3$ and $d = -1$ |

$a = 27$, $b = 26$, $c = 3$ and $d = -1$ is one of the infinite number of solutions.

The equations in Example **2** are consistent, but one of the equations is dependent on others. [2] – [3] and [5] – 2 × [3] both gave $5c + 5d = 10$ [7], so [2] – [3] = [5] – 2 × [3].

This simplifies to [5] = [2] + [3], a dependency. You will see that is now obvious in the original equations. The system is both consistent and dependent.

## EXAMPLE 3

Solve:

$$4h - 3i = 5$$
$$h + 4i = 3$$
$$3h + 2i = -2$$

### Solution

Number the equations.

$$\begin{aligned} 4h - 3i &= 5 \quad [1] \\ h + 4i &= 3 \quad [2] \\ 3h + 2i &= -2 \quad [3] \end{aligned}$$

Use [2] to eliminate $h$ from [1].

$$\begin{aligned} 4h - 3i &= 5 \quad [1] \\ -4h - 16i &= -12 \quad [2] \times (-4) \\ \hline -19i &= -7 \end{aligned}$$

Solve for $i$.

$$i = \frac{7}{19}$$

Substitute in [2] to find $h$.

$$h + 4 \times \frac{7}{19} = 3 \quad [2]$$

$$h = \frac{29}{19}$$

Check the solution in the original equations.

$$4h - 3i = 4 \times \frac{29}{19} - 3 \times \frac{7}{19} = \frac{95}{19} = 5 \quad [1] ✓$$

$$h + 4i = \frac{29}{19} + 4 \times \frac{7}{19} = \frac{57}{19} = 3 \quad [2] ✓$$

$$3h + 2i = 3 \times \frac{29}{19} + 2 \times \frac{7}{19} = \frac{101}{19} \neq -2 \quad [3] ✗$$

The equations do not give a consistent solution.

There is no solution, so the equations are inconsistent.

## Exercise 4.01 Systems of linear equations

Examples 1-3

**1** Solve each system of equations. State if any system is inconsistent, dependent or has no unique solution.

**a**
$$\begin{aligned} 7a + 2b + 4c &= 37 \\ 3a + b + 2c &= 16 \\ 17a + 5b + 11c &= 92 \end{aligned}$$

**b**
$$\begin{aligned} 9a + 9b + 4c - 2d &= 35 \\ 2a + 2b + c - d &= 8 \\ 4a + 4b + 2c - d &= 15 \end{aligned}$$

**c**
$$\begin{aligned} 4e - 8f - 10g + 5h &= 49 \\ 2e - 2f - 3g + 2h &= 17 \\ 2e - 3f - 4g + 2h &= 20 \\ 6e - 10f - 13g + 7h &= 66 \\ 6f - 3e + 8g - 4h &= -38 \end{aligned}$$

**d**
$$\begin{aligned} 3k + 3m + 2n + 3p &= 17 \\ 4k + 4m + 5n + 2p &= 12 \\ 3k + 3m + 2n + 2p &= 14 \\ 2k + 2m + 7n &= -6 \end{aligned}$$


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**e**
$$\begin{aligned} 5b + 3c - 4d + 2e &= -17 \\ 4b + c - 3d + 2e &= -10 \\ -8b - 2c + 8d - 4e &= 24 \\ 6b + c - 6d + 3e &= -17 \end{aligned}$$

**f**
$$\begin{aligned} 3f - 5g + 5h - 5k &= 51 \\ 4g + 4k - 2f - 3h + 37 &= 0 \\ 6f - 11g + 10h - 12k &= 113 \\ 2f - 4g + 3h - 3k &= 32 \end{aligned}$$

**g**
$$\begin{aligned} 2a + 3b + c - d &= 14 \\ d - 2a - 3b - 2c + 11 &= 0 \\ 4a + 6b + 3c - 2d &= 31 \\ 7a + 10b + 4c - 3d &= 44 \end{aligned}$$

**h**
$$\begin{aligned} 2p + q + 2r + 3s + t &= 2 \\ 5p + 2q + 3r + 6s + 2t &= 8 \\ 4p + 2q + 5r + 6s + 2t &= 2 \\ 2p + q + 3r + 4s + t &= -1 \\ 5p + 3q + 9r + 8s + 4t &= 1 \end{aligned}$$

## Problem solving

**2** A theme park has 3 ticket prices—for children, adults and seniors.

- 4 children and their grandmother were charged $420.
- 2 children and their parents paid $360.
- A woman taking her mother for a day out from a retirement home paid $175.
- A family with both parents, 3 children and all 4 grandparents paid a total of $765.

Write equations and solve them to find the ticket prices.


Getty Images/E+/SolStock

**3** To make pre-mixed concrete, dry gravel, sand and cement are mixed with water in a concrete truck. Each ingredient has a different density and different mixtures are used for different purposes. Slump, for example, is a mix of gravel, sand, cement and water in the ratio 5 : 2 : 1 : 2 (by volume). This and other mixes (by volume) for standard (5 $m^3$) loads, and their masses, are shown in the table below.

Use the information to find the density of each material.

| Type | Gravel | Sand | Cement | Water | Mass (tonne) |
|---|---|---|---|---|---|
| **Slump** | 5 | 2 | 1 | 2 | 21.4 |
| **General mix** | 7.2 | 4 | 2 | 2.8 | 21.125 |
| **Foundation** | 10 | 3 | 3 | 4 | 20.35 |
| **Fine finish** | 6 | 5 | 2 | 3 | 20.5 |

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## 4.02 Linear equations and matrices

You can write a system of linear equations as a matrix equation. Rearrange them with variables in the same order on the left and the constant on the right.

### EXAMPLE 4

Write this system of equations as a matrix equation.

$3a - 4b = 2d + 8$

$4a = c - d - 11$

$4b - 7 = 5d + 6c$

#### Solution

Rearrange with the variables in the same order. Write a coefficient for every variable in each equation, even when it is 0 or 1.

$3a - 4b + 0c - 2d = 8$
$4a + 0b - 1c + 1d = -11$
$0a + 4b - 6c - 5d = 7$

Write the corresponding matrix equation.

$$\begin{bmatrix} 3 & -4 & 0 & -2 \\ 4 & 0 & -1 & 1 \\ 0 & 4 & -6 & -5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 8 \\ -11 \\ 7 \end{bmatrix}$$

### Matrix equations

For a system of linear equations written in the matrix form **AX** = **B**,

- **A** is called the **coefficient matrix**
- The column matrix of variables **X** is called the **variable vector**
- **B** is called the **value vector**

You can change a system of variables to a matrix by thinking of the variables as *first*, *second*, *third*, etc. Then you add another column to the coefficient vector for the constants.

### Augmented matrix

For a system of linear equations, the **augmented matrix** consists of the coefficient matrix with the constants in an extra column at the end. This is often separated from the others by a vertical line.


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## EXAMPLE 5

Write this system of equations as an augmented matrix.

$3a - 3b + c - d = 8$
$5d - c + 4a = -3$
$3b - 2a - 3d - 4 = 0$
$3b - 4c + 7d - 2a + 1 = 0$

### Solution

Rearrange with the variables in the same order, including coefficients of 0 or 1.

$$3a - 3b + 1c - 1d = 8$$
$$4a + 0b - 1c + 5d = -3$$
$$-2a + 3b + 0c - 3d = 4$$
$$-2a + 3b - 4c + 7d = -1$$

Write the augmented matrix, including the vertical line.

$$\left[\begin{array}{cccc|c} 3 & -3 & 1 & -1 & 8 \\ 4 & 0 & -1 & 5 & -3 \\ -2 & 3 & 0 & -3 & 4 \\ -2 & 3 & -4 & 7 & -1 \end{array}\right]$$

## Exercise 4.02 Linear equations and matrices

**1** Write each system of linear equations as a matrix equation.

Example 4

**a** $3x + 2y - z = 1$
$x - 2y + 3z = 5$
$3x + 2z = -8$

**b** $2a + 3b - 7c + d = 3$
$a + b + d = 5$
$b + 3c - 2d = -4$
$b - 3a + 3c + 5d = 7$

**c** $3c - a + 2d = 6$
$b - 3a + 2d - c + 8 = 0$
$c + 2d + 7a = b + 8$
$2c - d + 3a + 9 = 0$
$a + b - 3d - 7 = 0$

**d** $3f + 2g + 3h - e = 9$
$2e = 4g - 3h + 2f + 7$
$2e + 3g + 9 = h + f$
$3g - 2e + 4h - f + 6 = 0$
$3e - 2f + 4g - h = 10$

**2** Write each system of linear equations as an augmented matrix.

Example 5

**a** $a + b + c = 10$
$2a - 3b - c = -1$
$b + 2c = 8$

**b** $3x - 2y = 9$
$x + 3y = 9$
$2x + 5y = 3$

**c** $2w - 2y + 4z = 9$
$3x - y - 4z = -8$
$4x = 3w + 5z + 2$
$w - 3x - 4y - 5 = 9$

**d** $k + m + n + p + 5 = 0$
$2k - 3m = 11$
$n - 4k + 2 = 0$
$5m + 2p = 3k + n + 7$

# 4.03 Solving linear equations using matrices

You learned in Year 11 that to solve the matrix equation $\mathbf{AX} = \mathbf{B}$ you multiply by $\mathbf{A}^{-1}$.

Of course, $\mathbf{A}^{-1}$ has to exist for this to work. Remember that only square matrices have inverses.

## Using the inverse

For the equation $\mathbf{AX} = \mathbf{B}$

- the **inverse** of $\mathbf{A}$, $\mathbf{A}^{-1}$, exists if and only if $|\mathbf{A}| \neq 0$
- a matrix that has no inverse is called a **singular matrix**
- the solution is given by $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$

## EXAMPLE 6

Use matrices to solve these equations.

$2w - x - 6y + z = 6$
$6x + 5w + 7y - 3z = -2$
$3w - z + x + y = 4$
$w - 4x - 6y + z = 9$

### Solution

Write the variables in the same order in each equation.

$$\begin{aligned} 2w - x - 6y + z &= 6 \\ 5w + 6x + 7y - 3z &= -2 \\ 3w + x + y - z &= 4 \\ w - 4x - 6y + z &= 9 \end{aligned}$$

The coefficient matrix will be $4 \times 4$, so it could have an inverse. Write as a matrix equation.

$$\begin{bmatrix} 2 & -1 & -6 & 1 \\ 5 & 6 & 7 & -3 \\ 3 & 1 & 1 & -1 \\ 1 & -4 & -6 & 1 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 4 \\ 9 \end{bmatrix}$$


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Use your graphics calculator to solve the matrix equation.

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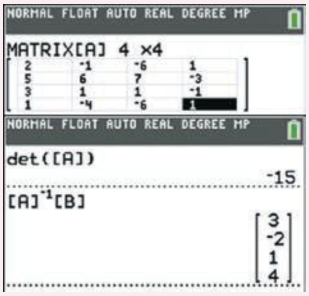
### TI-84 Plus CE

Use [2nd][$x^{-1}$] (matrix) and EDIT to put the coefficient matrix into [A]. Remember to use (–) for negatives.

Then use [2nd][$x^{-1}$] (matrix) MATH and NAMES to find the determinant and check it has an inverse.

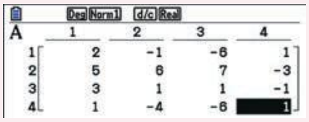
Put the value matrix in [B] and solve the equation using $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$.




### Casio fx-CG20AU

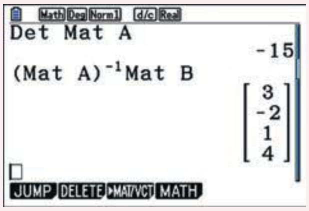
Use the Run-Matrix menu.

Press [F3] (▶MAT/VCT) and put the coefficient matrix in Mat A. Remember to use (–) for negatives.



[EXIT], press [OPTN] and choose det from the MAT/VCT menu. Then press [SHIFT] 2 ) to complete the determinant.

Now put the value matrix as Mat B and solve the equation.




Write the answer. $w = 3, x = -2, y = 1$ and $z = 4$

You cannot use the matrix inverse to solve these equations.

$$5a - 4b + 10c - d = 21$$

$$3a - b + 6c = 11$$

$$6a - 4b + 12c - d = 24$$

There are 4 variables but only 3 equations. The coefficient matrix is singular as it is not square.

Gaussian elimination gives the following:

$6a - 4b + 12c - 2d = 24$

$4b + d = -6$

$d = 2$

Substitution then gives you $a + 2c = 3$, $b = -2$ and $d = 2$.

As augmented matrices this corresponds to:

$$\left[\begin{array}{cccc|c} 5 & -4 & 10 & -1 & 21 \\ 3 & -1 & 6 & 0 & 11 \\ 6 & -4 & 12 & -1 & 24 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 6 & -4 & 12 & -2 & 24 \\ 0 & 4 & 0 & 1 & -6 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 3 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

The last matrix is called the row **reduced echelon form**. In the second row, if you make the leading entry of each row equal to 1, you get the **row echelon form**. You multiply the first row by $\frac{1}{6}$ and the second by $\frac{1}{4}$.

## Row echelon forms

- The leading entry of each non-zero row of the **row echelon form** of a matrix is 1, with zeros below it.
- The **row reduced echelon form** of a matrix is the same, but with zeros below and above the leading 1s.

You can use your graphics calculator to change a matrix to the row reduced echelon form.

You can do this to solve linear equation systems that have singular inverse coefficient matrices.

## EXAMPLE 7

Use your graphics calculator to solve these equations.

$9b - 6a - 2c - 4d = -11$
$4a - 6b + c + 2d = 6$
$2a - 3b - 2d = 3$

### Solution

Write the equations with all the variables in the same (alphabetic) order.

$$\begin{aligned} -6a + 9b - 2c - 4d &= -11 \\ 4a - 6b + 1c + 2d &= 6 \\ 2a - 3b + 0c - 2d &= 3 \end{aligned}$$

Write the augmented matrix.

$$\left[\begin{array}{cccc|c} -6 & 9 & -2 & -4 & -11 \\ 4 & -6 & 1 & 2 & 6 \\ 2 & -3 & 0 & -2 & 3 \end{array}\right]$$

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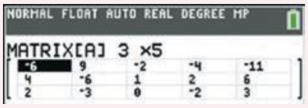
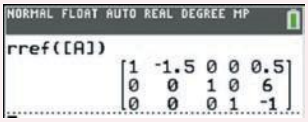
### TI-84 Plus CE

Enter the augmented matrix in [A].



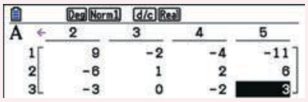

Quit, choose MATH and B: rref (from the matrix menu and complete with [A].



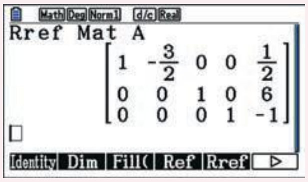

### Casio fx-CG20AU

Use the Run-Matrix menu.

Enter the augmented matrix Mat A.



Press EXIT, OPTN, choose Rref and complete with Mat A.




**Both calculators**

Write the result of the row reduced echelon matrix.

$a - 1.5b = 0.5$, $c = 6$ and $d = -1$

Express the result without fractions.

$2a - 3b = 1$, $c = 6$ and $d = -1$

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You should use the method shown in Example 7 if the coefficient matrix is not square or its determinant is zero.

## Exercise 4.03 Solving linear equations using matrices

Example 6

Example 7

Use matrices and your graphics calculator to solve each system of equations.

**1** 
$4w - 12x + 2y + z = 51$
$3w - 16x + 3y = 66$
$30x - 12w - 5y - 3z = -131$
$13w - 36x + 6y + 3z = 155$

**2** 
$2a + 4b + 4c + 2d = 13$
$4a + 4b + 2c + 2d = 1$
$4a + 2b + 2c + 2d = -5$
$10a + 10b + 6c + 6d = 5$
$2a + b + 3c = 1$

**3** $4e - 7f - 10g + 6h = 91$
$2e - 5f - 6g + 3h = 52$
$6e + f - 8g - 5h = 5$
$8f - 6e + 14g - h = -80$
$f - 2e + 4g + 4h = 5$

**4** $5k - 5m + 6n - 2p = 43$
$4m - 4k - 3n + 2p + 28 = 0$
$5m - 5n + 3p + 41 = 5k$
$6m + 2p + 26 = 6k + n$
$7m - 7k - 3n + 2p + 37 = 0$

**5** $5d - 2c - e + 2f + 2g = 4$
$4c - 9d + 2e - 4f = 4g - 6$
$2f - e + 5d - 3c + 2g = 1$
$3e + 7c - 11d - 5f + 2 = 4g$
$8d - 5c - 2e + 3f + 3g = 2$

**6** $2p + 2q + 2r + 2t + v = 6$
$12p + 5q + 8r + 8t + 4v = 42$
$15p + 8q + 11r + 10t + 5v = 48$
$11p + 4q + 8r + 8t + 4v = 37$
$18p + 9q + 13r + 12t + 6v = 58$

**7** $2a + 4b + 2d + m + 3n = 2$
$a + 2b + 2d + m + 3n = 1$
$2a + 3b + d + m + 3n = 1$
$a + 4b + 4d + 2m + 4n = 4$
$a + 8b + 8d + 4m + 9n = 4$

**8** $5m - 5n + 4p - 8q + 4t = 3$
$3n - m - 2p + 3q = t$
$3m - 3n + 2p - 4q + 2t = 1$
$m - 2n + 2p - 3q + t = 3$
$2m - 2n + 2p - 4q + 2t = 2$

**9** $3a + b + 10c + 9d - 2e = 10$
$4a + 3b + 8c + 6d - 3e = 5$
$4a + 2b + 13c + 11d = 3e + 11$
$2a + b + 8c + 7d - 7 = e$
$4a + 3b + 10c + 8d = 2e + 8$
$3a + 3b + 9c + 7d = e + 5$

**10** $16h - 150g + 80k - 20m = 1$
$35g - 3h - 20k + 5m = -3$
$28g - 4h - 15k + 5m = 4$
$38g - 4h - 20k + 5m = 1$
$3g - 3h - 5k + 5m = 1$
$4h - 14g + 10k - 5m = 3$

## 4.04 Applications of linear equations

You can use simultaneous equations to solve problems with multiple relationships.

### EXAMPLE 8

A trucking company loaded 3 identical trucks with pallets of bananas and pallets of pineapples for delivery in Queensland.

At a weighbridge, the weights of the trucks were measured as 15.2, 14.8 and 14.4 t.

- The first truck had 8 pallets of bananas and 10 pallets of pineapples.
- The second truck had 4 pallets of bananas and 12 pallets of pineapples.
- The third truck had 12 pallets of bananas and 6 pallets of pineapples.

What is the mass of the trucks and pallets of bananas and pineapples?


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## Solution

Choose the variables. Let the mass of the trucks, a pallet of bananas and a pallet of pineapples be $t$, $b$ and $p$ respectively.

Write the equations.

$$t + 8b + 10p = 15.2$$
$$t + 4b + 12p = 14.8$$
$$t + 12b + 6p = 14.4$$

Write as a matrix equation.

$$\begin{bmatrix} 1 & 8 & 10 \\ 1 & 4 & 12 \\ 1 & 12 & 6 \end{bmatrix} \begin{bmatrix} t \\ b \\ p \end{bmatrix} = \begin{bmatrix} 15.2 \\ 14.8 \\ 14.4 \end{bmatrix}$$

Solve using your graphics calculator.

$$\begin{bmatrix} t \\ b \\ p \end{bmatrix} = \begin{bmatrix} 6 \\ 0.4 \\ 0.6 \end{bmatrix}$$

Write the answer. The trucks weigh 6 t, a pallet of bananas 400 kg and a pallet of pineapples 600 kg.

You may have to use augmented matrices for some problems.

## EXAMPLE 9

The amounts paid by some families for snacks at a football game were as follows:

- 2 hamburgers and a steak sandwich $24.00
- 3 hamburgers, 2 steak sandwiches and a chicken burger $50.00
- A hamburger, steak sandwich and chicken burger $26.00
- 2 hamburgers, a steak sandwich and 3 chicken burgers $49.50

What was the price of each snack item?

## Solution

Choose the variables. Let the price be $h$, $k$ and $c$ for one hamburger, steak sandwich and chicken burger respectively.

Write the equations.

$$2h + k = 24$$
$$3h + 2k + c = 50$$
$$h + k + c = 26$$
$$2h + k + 3c = 49.5$$

| | |
|---|---|
| Write as an augmented matrix. | $\left[\begin{array}{ccc\|c} 2 & 1 & 0 & 24 \\ 3 & 2 & 1 & 50 \\ 1 & 1 & 1 & 26 \\ 2 & 1 & 3 & 49.5 \end{array}\right]$ |
| Find the row reduced echelon form using your graphics calculator. | $\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 6.5 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & 8.5 \\ 0 & 0 & 0 & 0 \end{array}\right]$ |
| Write the answer. | The prices were hamburger $6.50, steak sandwich $11 and chicken burger $8.50. |

## Exercise 4.04 Applications of linear equations

Example 8

**1** A mathematician with 3 children said that the sum of their ages is 20, and that the eldest is 5 times as old as the youngest. However, in 6 years, the eldest will only be twice as old as the youngest. What are the present ages of her 3 children?

Example 9

**2** At a school fete, the cost of 3 hamburgers, one steak sandwich and one chicken burger is $12; while 2 steak sandwiches, 2 chicken burgers and 3 sausages in rolls cost $16.50; 3 hamburgers, one steak sandwich, one chicken burger and 3 sausages in rolls cost $16.50; 4 hamburgers, 2 steak sandwiches, 3 chicken burgers and 3 sausages in rolls cost $27; and 2 steak sandwiches, 3 chicken burgers, one sausage in a roll and 2 hamburgers cost $20. What is the price of each food item?


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## Problem solving

**3** For a new play, a theatre group charges different prices for members, subscribers and the general public. The cost for a group made up of a member, 2 subscribers and 3 other people is $151 altogether. Tickets for 2 members and 4 other people cost $164. Three subscribers bring 3 other people, which costs $156. Another group has 3 members, 4 subscribers and 3 others, at a cost of $209. What are the individual prices for members, subscribers and the general public?

**4** A farmer can plant sorghum, canola, sunflowers or a mixture of the 3 crops. He has 25 hectares available, $4000 for costs and 130 hours available for planting, weeding and harvesting during the season. The costs are $162/ha for sorghum, $185/ha for canola and $123/ha for sunflowers. For cultivation, sorghum will take 4 h/ha, canola 7 h/ha and sunflowers 5 h/ha. Ignoring the different returns on the crops, what should the farmer plant to use all his land, all his money and all his time?


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**5** A drinks trolley in a theme park sells soft drinks, thickshakes, milkshakes and juice. The prices of some purchases made by families were as follows:

| | |
|---|---|
| 3 soft drinks, a milkshake and 2 juices | $22.00 |
| 2 soft drinks, 3 thickshakes and a juice | $31.00 |
| 2 thickshakes, 2 milkshakes and 2 juices | $32.00 |
| 3 milkshakes and 4 juices | $31.00 |

Find the individual price of each drink.

**6** A parents' auxiliary has had the following materials donated to make cakes for a stall:

26 eggs
4 kg of self-raising flour
2 kg of plain flour

An orange cake requires 4 eggs and 250 g of each flour. A banana cake requires no eggs, but 500 g of self-raising flour. A chocolate mud cake requires 1 egg and 250 g of each flour. How many of each type of cake should be made to use all of the donated ingredients?

# 4.05 Solutions of linear equations

## INVESTIGATION

### RELATIONSHIPS OF PLANES

#### 2 planes

1. The planes $2x - y + 3z = 5$ and $2x - y + 3z = -2$ have the same normal vector.

   The planes $x + 4y + 2z = -1$ and $2x + y - z = 4$ have different normal vectors.
2. What can you conclude about 2 planes by looking at their equations?
3. What happens when you try to solve the equations $2x - y + 3z = 5$ and $2x - y + 3z = -2$?
4. What happens when you try to solve the equations $x + 4y + 2z = -1$ and $2x + y - z = 4$?
5. What can you conclude about 2 planes by trying to solve their equations?

#### 3 planes

1. The planes $x + 2y - z = 4$, $2x + 4y - 2z = 5$ and $x + 2y - z = 7$ have the same normal vector.

   What can you conclude about these planes by looking at their normal vectors?
2. Two of planes $x + 2y - z = 4$, $2x + 4y - 2z = 5$ and $3x - 2y + z = -1$ have the same normal vector but the other is different.

   What can you conclude about these planes by looking at their normal vectors?
3. The planes $x + 2y - z = 4$, $x - y - 2z = 3$ and $3x - 2y + z = -1$ all have different normal vectors.

   What can you conclude about these planes by looking at their normal vectors?
4. What happens when you try to solve the equations $x + 2y - z = 4$, $2x + 4y - 2z = 5$ and $x + 2y - z = 7$?

   What can you conclude about these planes by trying to solve their equations?
5. What happens when you try to solve the equations $x + 2y - z = 4$, $2x + 4y - 2z = 5$ and $3x - 2y + z = -1$?


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**6** What happens when you try to solve the equations $x + 2y - z = 4$, $x - y - 2z = 3$ and $3x - 2y + z = -1$?

**a** What can you conclude about these planes by trying to solve their equations?

**b** What can you conclude about these planes by trying to solve their equations?

**c** Can you eliminate one variable at a time from the reduced row echelon form?

**d** What does this tell you?

**7** What happens when you try to solve the equations $10x + 16y + 7z = 91$, $4x + 14y - z = 63$ and $2x + 4y + z = 21$?

**a** What can you conclude about these planes by trying to solve their equations?

**b** Can you eliminate one variable at a time from the reduced row echelon form?

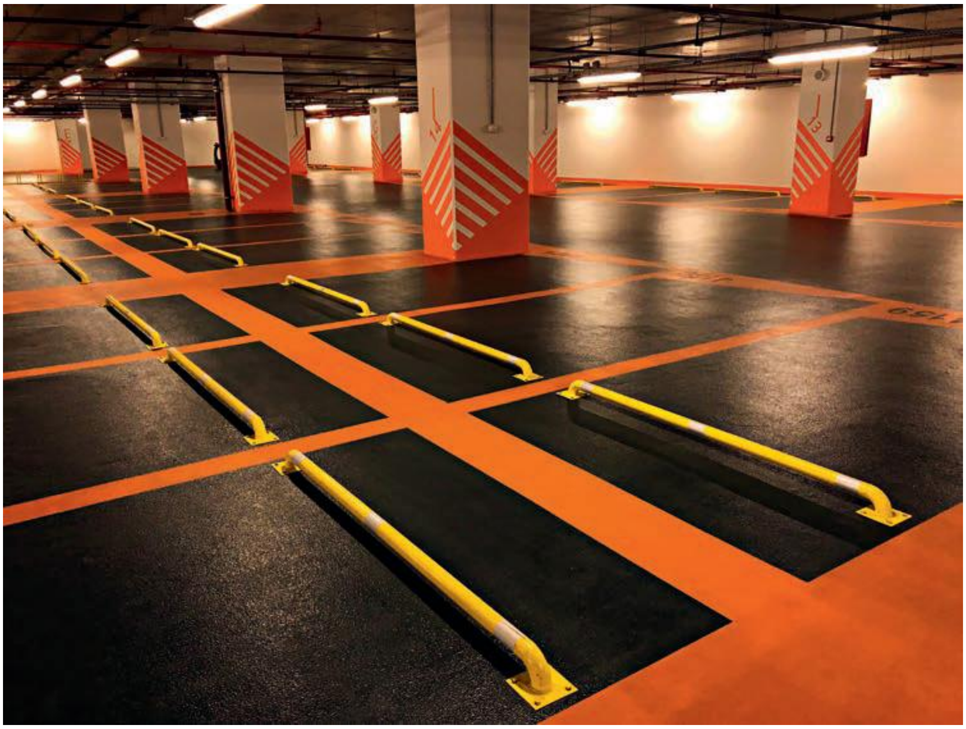
**c** What does this tell you?

**8** What happens when you try to solve the equations $4x - 5y + z = 3$, $x - y = 0$ and $x - 2y + z = 0$?

**a** What can you conclude about these planes by trying to solve their equations?

**b** Can you eliminate one variable at a time from the reduced row echelon form?

**c** What does this tell you?

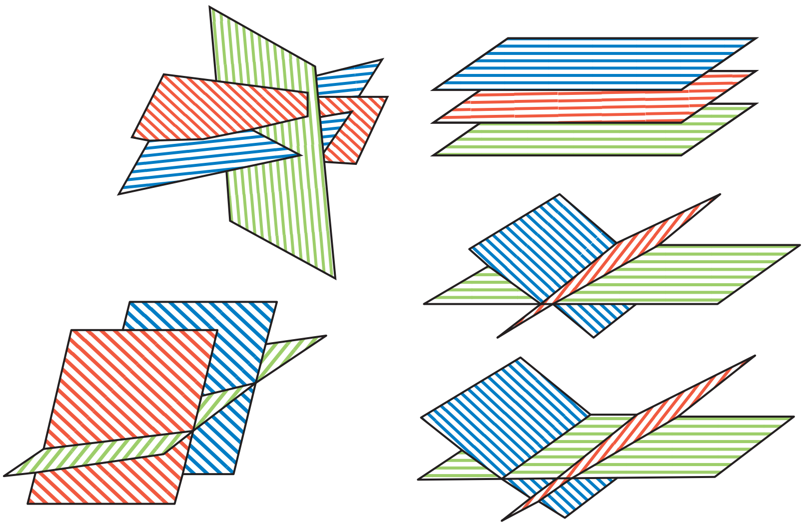


Alamy Stock Photo/Batchelder

From the investigation, you should be able to see the following possibilities.

## Relationships of planes and their equations

- 2 **parallel planes** have equations that are inconsistent
- 2 **non-parallel planes** intersect in a straight line and have equations that are consistent
- 3 parallel planes have the same **normal** and inconsistent equations
- 2 parallel planes with a non-parallel plane have only 2 with the same normal and inconsistent equations
- 3 non-parallel planes that intersect in a point have different normals and equations with a unique solution
- 3 non-parallel planes that intersect in lines have different normals and consistent equations but no unique solution (dependent equations)
- 3 planes can be related in 5 different ways.
  - They can be parallel, corresponding to the normals being the same.
  - 2 can be parallel, with the same normals, but the third non-parallel with a different normal. The third intersects the others in straight lines.
  - They can intersect at a single point, corresponding to a unique solution of the equations.
  - They can intersect in a single line, corresponding to dependent equations.
  - They can intersect in 3 (parallel) lines, corresponding to different normals but inconsistent equations.



 ISBN 9780170413060

2 planes must be either parallel or intersect in a straight line. You can find the intersection of 2 non-parallel planes by eliminating one variable and choosing another as being $t$. The equation of the line will appear to be different, depending on the choice of $t$.

## EXAMPLE 10

Find the relationship of the planes $2x - 3y + z = 4$ and $x + 2y - 2z = -3$.

### Solution

Check the normal.

The normals $(2, -3, 1)$ and $(1, 2, -2)$ are different so the planes are non-parallel.

Eliminate $x$, as it is easiest.

$$\begin{array}{r} 2x - 3y + z = 4 \\ -2x - 4y + 4z = 6 \\ \hline -7y + 5z = 10 \end{array}$$

Let $t = z$, say, and substitute.

$-7y + 5t = 10$

Rearrange.

$$y = \frac{5t - 10}{7} = \frac{5(t-2)}{7} \text{ or } t = \frac{7y + 10}{5}$$

Substitute in one of the planes and isolate $x$.

$$2x - 3 \times \frac{5(t-2)}{7} + t = 4$$

$$2x = \frac{28 + 15(t-2) - 7t}{7} = \frac{8t - 2}{7} \text{ or } t = \frac{2(7x+1)}{8}$$

Write the equation of the line.

$$\mathbf{r}(t) = \frac{2}{7}(4t - 1)\mathbf{i} + \frac{5}{7}(t - 2)\mathbf{j} + t\mathbf{k} \text{ or}$$

$$\frac{7x+1}{4} = \frac{7y+10}{5} = z$$

Write the answer.

The planes intersect in the line

$$\frac{2}{7}(4t - 1)\mathbf{i} + \frac{5}{7}(t - 2)\mathbf{j} + t\mathbf{k} \text{ or}$$

$$\frac{7x+1}{4} = \frac{7y+10}{5} = z$$

You can find the relationship between 3 planes by checking the normals. You should also find the intersection(s) of non-parallel planes.

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## EXAMPLE 11

Find the relationship between the planes $3x - 2y + 4z = 0$, $x - y + 3z = 2$ and $x - 2z = -4$.

### Solution

Consider the normals.

The normals $3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{i} - 0\mathbf{j} - 2\mathbf{k}$ are all different; none of the planes are parallel

Write the matrix equation.

$$\begin{bmatrix} 3 & -2 & 4 \\ 1 & -1 & 3 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -4 \end{bmatrix}$$

Use your graphics calculator to find the determinant of the coefficient matrix.

$$\begin{vmatrix} 3 & -2 & 4 \\ 1 & -1 & 3 \\ 1 & 0 & -2 \end{vmatrix} = 0$$

The coefficient matrix has no inverse, so write the augmented matrix.

$$\text{Augmented matrix} = \left[\begin{array}{ccc|c} 3 & -2 & 4 & 0 \\ 1 & -1 & 3 & 2 \\ 1 & 0 & -2 & -4 \end{array}\right]$$

Use your graphics calculator to find the row reduced echelon form.

$$\text{Row reduced echelon form} = \left[\begin{array}{ccc|c} 1 & 0 & -2 & -4 \\ 0 & 1 & -5 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Use the solution of the equations.

The equations have the solution $x - 2z = -4$, $y - 5z = -6$, so they are dependent and the planes intersect in a single line.

Find the equation of the line, using the easiest substitution for $t$.

Choosing $z = t$:

$x - 2t = -4$, so $x = 2t - 4$ or $t = \dfrac{x+4}{2}$

$y - 5t = -6$, so $y = 5t - 6$ or $t = \dfrac{y+6}{5}$

The equation of the line of intersection is

$\mathbf{r}(t) = (2t - 4)\mathbf{i} + (5t - 6)\mathbf{j} + t\mathbf{k}$

or $\dfrac{x+4}{2} = \dfrac{y+6}{5} = z$

Write the answer.

The 3 planes intersect in the single line

$\mathbf{r}(t) = (2t - 4)\mathbf{i} + (5t - 6)\mathbf{j} + t\mathbf{k}$

or $\dfrac{x+4}{2} = \dfrac{y+6}{5} = z$


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Two parallel planes and a third non-parallel plane have 2 lines of intersection. You can find these using the method in Example **10**.

You can find the intersection lines of planes that intersect in 3 lines by finding the pairwise intersections of the planes. However, since they must be parallel, they must have the same direction vector.

## Exercise 4.05 Solutions of linear equations

**1** Find the relationship of each pair of planes. Example 10

- **a** $x - 2y + z = 7$ and $x - 2y + z = 5$
- **b** $x - y + 2z = -3$ and $2x + 2y - z = 6$
- **c** $3x + y + 2z = 4$ and $x - 3y + 4z = -2$
- **d** $3x - 2y - z = 4$ and $4y + 2z - 6x = 7$
- **e** $5x + y + 2z = -3$ and $3x - 2y - 3z = 4$

**2** Find the relationship of each set of planes. Example 11

- **a** $x + 2y + 2z = -2$, $2x + 7y - 2z = 4$ and $4x + 13y - 2z = 6$.
- **b** $6x + 3y - 2z = 4$, $x + y - z = -3$ and $4x + 2y - z = 4$.
- **c** $x - 2y + 4z = 2$, $2y - 4z - x = 8$ and $3x - 6y + 8z = 4$.
- **d** $2x - 5y + 2z = 9$, $x - 2y + 2z = 7$ and $3x - 3y + 2z = 11$.
- **e** $2x + 3y - 2z = 4$, $x - 3y + z = 2$ and $2x + 3y - 2z = 1$.
- **f** $x + y + 3z = -2$, $2x + 3y + 3z = 0$ and $3z + 5y + 3z = 0$.
- **g** $2x + 4y + 4z = 1$, $4x + 9y + 4z = 4$ and $6x + 13y + 8z = 5$.
- **h** $x - y + 2z = 7$, $x + y + 3z = 5$ and $2x - 2y + 4z = 2$.
- **i** $3x + 3y + 2z = 9$, $2x + y + 2z = 10$ and $5x + 3y + 4z = 22$.

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# 4.06 Dominance matrices

Some tennis players are better than others. You can find it hard to work out the ranking of players because they play many matches. In addition, not all players meet. Players' rankings are often used to decide which players meet in a tournament. This means they should be as accurate as possible.

You can use a matrix to show wins and losses for a group of players.

## Dominance matrices

- A **dominance matrix** is used to compare the wins and losses of a group of 'players'
- Each row is used for one of the players. The entries show the dominance of that player over each of others, in the same player order as the rows
- The entry is 0, 1 or $\frac{1}{2}$:
  - 0 means the player has lost more than they have won against that opponent, or they have not played each other
  - $\frac{1}{2}$ means they have won an equal number of times
  - 1 means the player has won more than lost to that opponent

## EXAMPLE 12

5 players are being ranked. Jane has beaten Mary and Elizabeth, but she lost to Alicia and did not play Consuela. Mary beat Alicia and Consuela, drew with Elizabeth and lost to Jane. Elizabeth lost to Jane and Consuela, drew with Mary but did not play Alicia. Alicia beat Jane and Consuela, lost to Mary and did not play Elizabeth. Naturally, Consuela beat Elizabeth, lost to Alicia and Mary and did not play Jane. Construct the dominance matrix in the order Alicia, Consuela, Elizabeth, Jane and Mary.

### Solution

Use 1 for a win and $\frac{1}{2}$ for a draw. It helps to label the rows and columns so you put the 1s and $\frac{1}{2}$ s in the right places.

The diagonal from the top left to the bottom right has to be 0s because no-one plays themselves.

$$\mathbf{D} = \begin{array}{c} \\ A \\ C \\ E \\ J \\ M \end{array} \begin{array}{c} \begin{array}{ccccc} A & C & E & J & M \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & \frac{1}{2} & 0 & 0 \end{bmatrix} \end{array}$$


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You can see from the last row of Example 12 that Mary, Alice and Jane all beat 2 players.

You would find it harder to work this out for a large number of players.

The product **DU**, where **U** is a column of units, gives the total wins for each player.

For the column of units $\mathbf{U} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, the product $\mathbf{DU} = \begin{bmatrix} 2 \\ 1 \\ \frac{1}{2} \\ 2 \\ 2\frac{1}{2} \end{bmatrix}$ gives the total of the wins for each player in order. What about the total of the losses? From their columns, they all lost to one player. Mary only drew with Elizabeth, who was the lowest scorer.

The **transpose** of a matrix swaps the rows and columns, so $\mathbf{D^T} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & \frac{1}{2} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & 0 \end{bmatrix}$ and

$\mathbf{D^TU} = \begin{bmatrix} 1 \\ 2 \\ 2\frac{1}{2} \\ 1 \\ 1\frac{1}{2} \end{bmatrix}$. The total wins less losses is $\mathbf{DU} - \mathbf{D^TU} = \begin{bmatrix} 2 \\ 1 \\ \frac{1}{2} \\ 2 \\ 2\frac{1}{2} \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 2\frac{1}{2} \\ 1 \\ 1\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 1 \\ 1 \end{bmatrix}$

For this group, the total of wins – losses still doesn't give you a full ranking.

### Dominance matrix rankings

- **First-order ranking** = total wins – total losses. It has the symbol $\mathbf{r}$ or $\mathbf{r_1}$

  For a dominance matrix **D**, $\mathbf{r_1} = \mathbf{DU} - \mathbf{D^TU}$
- **Second-order ranking** = total wins of opponents beaten – total losses of opponents lost to. It has the symbol $\mathbf{r_2}$

  For a dominance matrix **D**, $\mathbf{r_2} = \mathbf{D^2U} - (\mathbf{D^2})^\mathbf{T}\mathbf{U}$
- A **total ranking** adds the first and second-order rankings together.

  The second-order ranking is commonly weighted by 0.5

## EXAMPLE 13

Use the data from Example **12** to find:

**a** the second-order ranking

**b** the unweighted totals and totals weighted by 0.5 for the second order.

### Solution

**a** Write the dominance matrix.

$$\mathbf{D} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

Find the square.

$$\mathbf{D}^2 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & \frac{1}{2} & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 1 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 1 & 1 & \frac{1}{4} \end{bmatrix}$$

Find the transpose.

$$(\mathbf{D}^2)^T = \begin{bmatrix} 0 & 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & 1 \\ 2 & 0 & \frac{1}{4} & \frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$


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| | |
|---|---|
| Find the second-order ranking. | $\mathbf{r_2} = \mathbf{D}^2\mathbf{U} - (\mathbf{D}^2)^T\mathbf{U}$ |

$$= \begin{bmatrix} 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 1 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 1 & 1 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 1 & 1 \\ 2 & 0 & \frac{1}{4} & \frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ \frac{1}{2} \\ 1\frac{1}{4} \\ 3 \\ 3\frac{1}{4} \end{bmatrix} - \begin{bmatrix} 1\frac{1}{2} \\ 2\frac{1}{2} \\ 3\frac{3}{4} \\ 1 \\ 2\frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1\frac{1}{2} \\ -2 \\ -2\frac{1}{2} \\ 2 \\ 1 \end{bmatrix}$$

**b** Find the unweighted total.

$$\mathbf{r_1} + \mathbf{r_2} = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1\frac{1}{2} \\ -2 \\ -2\frac{1}{2} \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\frac{1}{2} \\ -3 \\ -4\frac{1}{2} \\ 3 \\ 2 \end{bmatrix}$$

Find the weighted total.

$$\mathbf{r_1} + 0.5\mathbf{r_2} = \begin{bmatrix} 1 \\ -1 \\ -2 \\ 1 \\ 1 \end{bmatrix} + 0.5\begin{bmatrix} 1\frac{1}{2} \\ -2 \\ -2\frac{1}{2} \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1\frac{3}{4} \\ -2 \\ -3\frac{1}{4} \\ 2 \\ 1\frac{1}{2} \end{bmatrix}$$

Write the answer. The unweighted and weighted totals both rank from best to last as Jane, Alicia, Mary, Consuela, Elizabeth.

Notice that $\mathbf{D}^2$ includes $\frac{1}{4}$ s to count half of the draws of opponents. You can extend dominance matrices to continuous data. You can write the dominance ratio of one player over another as a fraction, decimal or percentage. If a player had won 7 out of 10 encounters with another, their dominance ratio would be 0.7. You could just use the wins and losses instead (7 for the player and 3 for the opponent). You would still get a ranking. However, the results would be different unless each pair of players played the same number of matches. Real-world rankings also place more importance on recent data.

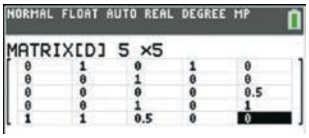
## TECHNOLOGY

## Dominance matrix calculations

You can use your graphics calculator for dominance matrix calculations.

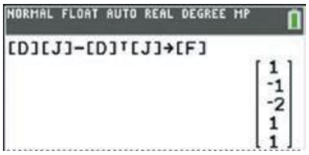
TI-Nspire CX
Chapter 4

### TI-84 Plus CE



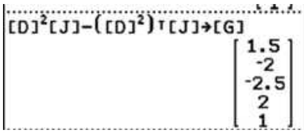

Enter the dominance matrix in [D].

Enter the unit matrix in say, [ J], as [U] is not available on the TI.




The transpose function $^{T}$ in the MATH submenu of the matrix menu is post-fix (comes *after* the matrix).

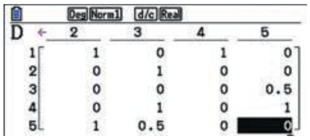
Work out the first-order rankings and store them in [F].




Work out the second-order rankings and store them in [G].


Combine the first- and second-order rankings in the desired weighting.

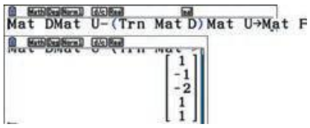
### Casio fx-CG20AU



Use the Run-Matrix menu.

Enter the dominance matrix in Mat D.

Enter the unit matrix in Mat U.




Press EXIT EXIT OPTN and choose the MAT/VCT menu. The transpose function Trn is pre-fix (comes *before* the matrix).

Work out the first-order rankings and store them in [F].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Work out the second-order rankings and store them in [G].

$\leftarrow (\text{Trn Mat } D^2)\text{Mat } U$

$$\begin{bmatrix} \frac{3}{2} \\ -2 \\ -\frac{5}{2} \\ 2 \\ 1 \end{bmatrix}$$

Combine the first- and second-order rankings in the desired weighting.

Mat F+0.5Mat G

$$\begin{bmatrix} 1.75 \\ -2 \\ -3.25 \\ 2 \\ 1.5 \end{bmatrix}$$

**INVESTIGATION**

## SPORTS RANKINGS

Choose a sport for which the season is partway through. Collect information about the performance of each team. Decide how the performance should be coded for a dominance matrix. Then use the matrix to rank the teams in the competition.

Use your rankings to predict the outcomes of the subsequent rounds of the competition. When later rounds have been played, check your predictions against actual events.

Comment on the accuracy of your predictions.

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## Exercise 4.06 Dominance matrices

Examples 12,13

**1** Each head of department in a school gave the principal a list of the top Year 12 students in their subjects in order. The principal constructed the following dominance matrix to work out the school dux.

Use up to 3 orders to find the dux from students 1 to 6.

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

**2** A chess tournament had the following cross-table. Use rankings to as many orders as necessary to rank the players.

| Player | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| **1 Gorenson** | | 1 | | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | | 0 |
| **2 Adams** | 0 | | | $\frac{1}{2}$ | 1 | 0 | 1 | |
| **3 Torre** | | | | 0 | $\frac{1}{2}$ | 0 | 0 | 1 |
| **4 Vukovic** | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 | | | 0 | | $\frac{1}{2}$ |
| **5 Vuipe** | 1 | 0 | $\frac{1}{2}$ | | | | 1 | 0 |
| **6 Murphy** | $\frac{1}{2}$ | 1 | 1 | 1 | | | 1 | |
| **7 Gitani** | | 0 | 1 | | 0 | 0 | | 0 |
| **8 Le Duc** | 1 | | 0 | $\frac{1}{2}$ | 1 | | 1 | |

### Problem solving

**3** A soccer tournament had 5 teams play each other. The final results were:

Dragons vs Cats 1–2
Stars vs Eagles 1–2
Dragons vs Eagles 0–3
Bears vs Dragons 1–0
Eagles vs Cats 2–3
Eagles vs Bears 0–1
Bears vs Cats 0–0
Stars vs Bears 2–2
Stars vs Cats 1–1
Dragons vs Stars 0–3

Rank the teams from 1st to 5th. Explain your method.


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**4** Adam, Beryl, Connie, David, Ewing and Frank have been playing pool against each other for some time. The number of games each has won against the others is shown by the rows of the following table.

| Player | Adam | Beryl | Connie | David | Ewing | Frank |
|---|---|---|---|---|---|---|
| Adam | 0 | 2 | 3 | 1 | 0 | 3 |
| Beryl | 2 | 0 | 0 | 2 | 2 | 4 |
| Connie | 4 | 1 | 0 | 3 | 2 | 1 |
| David | 3 | 2 | 2 | 0 | 4 | 3 |
| Ewing | 4 | 1 | 3 | 3 | 0 | 2 |
| Frank | 1 | 2 | 3 | 4 | 2 | 0 |

Rank the players from 1st to 6th. Explain your method.

**5** A table tennis club records the number of wins that each of the top 6 players had over the others during the season.

| Name | Adams | Feltham | Jones | Sharma | Singh | Smith |
|---|---|---|---|---|---|---|
| Adams | 0 | 8 | 2 | 3 | 3 | 5 |
| Feltham | 1 | 0 | 1 | 1 | 2 | 1 |
| Jones | 3 | 2 | 0 | 2 | 6 | 4 |
| Sharma | 6 | 2 | 3 | 0 | 4 | 2 |
| Singh | 3 | 3 | 1 | 1 | 0 | 3 |
| Smith | 3 | 6 | 4 | 3 | 2 | 0 |

Rank the players from 1st to 6th. Explain your method.


Dreamstime.com/Grosremy

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# 4.07 Leslie matrices

The fish industry needs to know about fish populations to set reasonable catch limits. National Parks and Wildlife needs to know about population trends to protect native wildlife. Both want to make sure that populations are sustainable. Overpopulation of koalas in some parts of Victoria has led to widespread loss of trees and consequent starvation of koalas.

You can use matrices in ecology to model populations over time. In a stable population in a closed area, the birth and death rates are the same. The birth rate for young females is often different to that for older females. The death rate for young animals is often higher than for animals that have survived a few years.

The **Leslie matrix model** is widely used to model the long-term trends of a population of animals.

## Leslie matrix model assumptions

- Females are a fixed proportion of the population, usually half
- The environment, including predation, does not change
- Age-specific birth and death rates do not change

For the birth rates to remain the same, there must be enough space, nesting sites and food in the environment to support breeding.

Consider a population of brushtail possums in a national park that have the following characteristics.

**Age-specific breeding rates of female possums**

| Age (years) | 0–1 | 1–2 | 2–3 | 3–4 | 4–5 |
|---|---|---|---|---|---|
| Breeding rate | 0 | 1.3 | 1.8 | 0.9 | 0.2 |

The **breeding rate** is the average number of live births for a female population in 1 year.

**Age-specific survival rates of female possums**

| Age (years) | 0–1 | 1–2 | 2–3 | 3–4 | 4–5 |
|---|---|---|---|---|---|
| Survival rate | 0.6 | 0.8 | 0.8 | 0.4 | 0 |

The **survival rate** is the average number of possums alive at the beginning of the year who are still alive at the end.

**Base year age-specific female possum population in a national park**

| Age (years) | 0–1 | 1–2 | 2–3 | 3–4 | 4–5 |
|---|---|---|---|---|---|
| Female population | 194 | 82 | 55 | 22 | 6 |


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The population was established by a capture, tag, release and recapture program. Notice that no possums lived longer than 5 years. Possums did not begin breeding until they were at least a year old. You can work out the **age-specific** population in the next year as follows.

The population in the 0–1 age group will be the number born during the year. This will be the total number of possums born to each female age group during the year. For each age group, it will be the product of the number of females and the breeding rate.

The population in other age groups will be product of the survival rate and population of the previous age group.

**Second year age-specific female possum population in a national park**

| Age group | Calculation | Result |
|---|---|---|
| 0–1 | $0 \times 194 + 1.3 \times 82 + 1.8 \times 55 + 0.9 \times 22 + 0.2 \times 6$ | $226.6 \approx 227$ |
| 1–2 | $0.6 \times 194$ | $116.4 \approx 116$ |
| 2–3 | $0.8 \times 82$ | $65.6 \approx 66$ |
| 3–4 | $0.8 \times 55$ | 44 |
| 4–5 | $0.4 \times 22$ | $8.8 \approx 9$ |

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## Leslie matrix model

Population model for a species with $n$ age groups

- The age-specific population is modelled by a vertical vector **X**
- The population in the base year is usually written as $\mathbf{X_0}$, after 1 year as $\mathbf{X_1}$, after 2 years as $\mathbf{X_2}$, and so on
- The **Leslie matrix** is an $n \times n$ square matrix **L** with
  - Age-specific birth rates $B_1, B_2, B_3, \ldots, B_n$ in the top row
  - Age-specific survival rates $S_1, S_2, S_3, \ldots, S_{n-1}$ in the diagonal from $l_{21}$ to $l_{n(n-1)}$
  - All other elements equal to zero
- The population vector in each successive year is given by $\mathbf{X}_n = \mathbf{L}\mathbf{X}_{n-1}$.

## EXAMPLE 14

**a** Write the Leslie matrix for the possum population described on the previous 2 pages.

**b** Find the population vector after 2 years.

**c** Find the population vector after 3 years.

**d** What is the population in the base year and after 1, 2 and 3 years?

### Solution

**a** The population is aged up to 5 years, so **L** is 5 × 5. Write the birth rates in the top row and the survival rates in the diagonal.

$$\begin{bmatrix} 0 & 1.3 & 1.8 & 0.9 & 0.2 \\ 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 \end{bmatrix}$$

**b** Write the rule.

$$\mathbf{X_2} = \mathbf{LX_1}$$

Substitute the values and perform the calculations.

$$= \begin{bmatrix} 0 & 1.3 & 1.8 & 0.9 & 0.2 \\ 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 \end{bmatrix} \begin{bmatrix} 227 \\ 116 \\ 66 \\ 44 \\ 9 \end{bmatrix}$$


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$$= \begin{bmatrix} 311 \\ 136.2 \\ 92.8 \\ 52.8 \\ 17.6 \end{bmatrix} \approx \begin{bmatrix} 311 \\ 136 \\ 93 \\ 53 \\ 18 \end{bmatrix}$$

**c** Write the rule.

Substitute the values and perform the calculations.

$\mathbf{X_3 = LX_2}$

$$= \begin{bmatrix} 0 & 1.3 & 1.8 & 0.9 & 0.2 \\ 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 \end{bmatrix} \begin{bmatrix} 311 \\ 136 \\ 93 \\ 53 \\ 18 \end{bmatrix}$$

$$= \begin{bmatrix} 395.5 \\ 186.6 \\ 108.8 \\ 74.4 \\ 21.2 \end{bmatrix} \approx \begin{bmatrix} 396 \\ 187 \\ 109 \\ 74 \\ 21 \end{bmatrix}$$

**d** The population is twice the female population.

Population each year:

Base year $= 2 \times (194 + 82 + 55 + 22 + 6)$
$= 718$

After 1 year $= 2 \times (227 + 116 + 66 + 44 + 9)$
$= 924$

After 2 years $= 2 \times (311 + 136 + 93 + 53 + 18)$
$= 1222$

After 3 years $= 2 \times (396 + 187 + 109 + 74 + 21)$
$= 1574$

Write the answer.

The populations each successive year are: 718, 924, 1222 and 1574.

The population increases by about 29%, 32% and 29% in the 1st, 2nd and 3rd years respectively.

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You can write $\mathbf{X}_2 = \mathbf{L}\mathbf{X}_1 = \mathbf{L}(\mathbf{L}\mathbf{X}_0) = \mathbf{L}^2\mathbf{X}_0$

$\mathbf{X}_3 = \mathbf{L}\mathbf{X}_2 = \mathbf{L}(\mathbf{L}^2\mathbf{X}_0) = \mathbf{L}^3\mathbf{X}_0$

…

In general, $\mathbf{X}_n = \mathbf{L}^n\mathbf{X}_0$

This will not save time in manual calculation, but it does if you are using a calculator.

## TECHNOLOGY

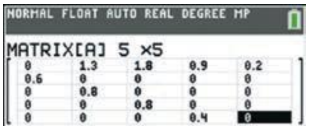
### Leslie calculations on a graphics calculator

You can use your graphics calculator to predict the brushtail possum population and its increases after 20, 21, 30 and 31 years.

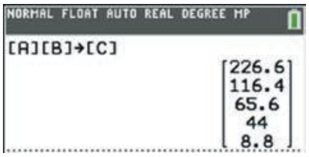
TI-Nspire CX
Chapter 4

#### TI-84 Plus CE

Enter the Leslie matrix and base female population vector in, say, [A] and [B].



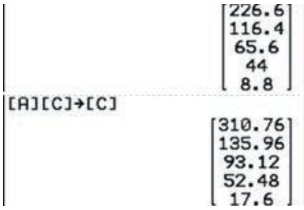

Do the calculation for the first year and store the answer in [C].



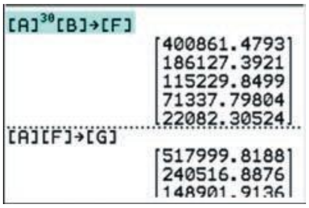

Do the calculation for the second year, but store the answer back in [C].

Press [enter] to repeat for the third year.

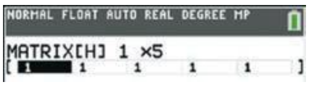
You can do successive years by pressing [enter] again and again.




Type the calculations for 20, 21, 30 and 31 years and store in [D], [E], [F] and [G].




You can use a shortcut to find the population from each vector. Remember the female population is half of the total. If you multiply [1 1 1 1 1] by the vector you get the female total.

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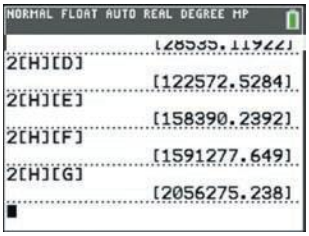
Put this into, say, [H] and do the calculations.

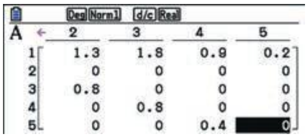
Don't forget to multiply by 2 to get the whole population.



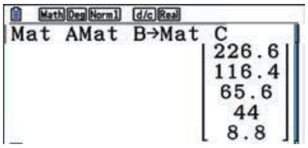

## Casio fx-CG20AU

Use the Run-Matrix menu.

Enter the Leslie matrix and base female population in, say, [A] and [B].



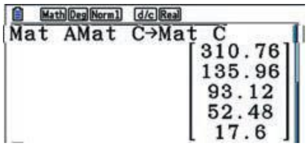

Do the calculation for the first year and store the answer in [C].




Do the calculation for the second year, but store the answer back in [C].

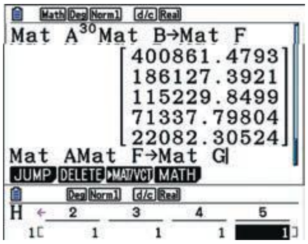
Press enter to repeat for the third year.

You can do successive years by pressing enter again and again.



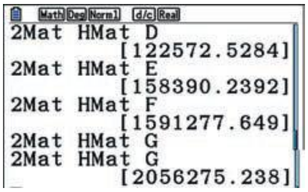

Type the calculations for 20, 21, 30 and 31 years and store in [D], [E], [F] and [G].

You can use a short cut to find the population from each vector. Remember, the female population is half of the total. If you multiply [1 1 1 1 1] by the vector, you get the female total.




Put this into, say, [H] and do the calculations.

Don't forget to multiply by 2 to get the whole population.




The percentage increases from year 20 to 21 and 30 to 31 are both 29.22%. The age distribution of the possum population stabilises so that about 50% is aged 0–1, 23% 1–2, 1% 2–3, 9% 3–4 and 3% 4–5.

The increase in possum population may mean that the rangers in the national park have to limit the population by **culling**. This means that they remove a number or proportion of the possums each year. The culling rate would be set to stabilise the population.

## Leslie model properties

The Leslie population model eventually stabilises so that:

- the age distribution is constant from year to year
- the percentage change in population from one year to the next is constant, so $\mathbf{X}_n = k\mathbf{X}_{n-1}$.
- Simple culling assumes that the same proportion of each age group is removed. For a culling rate $h$, the survival rate and birth rates are multiplied by $(1 - h)$

## EXAMPLE 15

The feral pigeons in the CBD were studied and the following data collected.

| Age (years) | 0–1 | 1–2 | 2–3 | 3–4 |
|---|---|---|---|---|
| Population | 2500 | 1000 | 500 | 150 |
| Birth rate | 0 | 3.5 | 1.5 | 0.5 |
| Death rate | 0.6 | 0.5 | 0.7 | 1 |

Write the Leslie matrix and population vector. Find the population after 21, 30 and 31 years, and hence the annual relative change. Suggest a culling factor to gradually reduce the number of pigeons.

### Solution

The survival rates are the complement of the death rates and the female population is half the population. Use these values in the Leslie matrix and base population vector.

$$\mathbf{L} = \begin{bmatrix} 0 & 3.5 & 1.5 & 0.5 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \end{bmatrix}, \mathbf{X}_0 = \begin{bmatrix} 1250 \\ 500 \\ 250 \\ 75 \end{bmatrix}$$

Use the matrix [1 1 1 1] with your graphics calculator to find the population in the required years.

| Year | 20 | 21 | 30 | 31 |
|---|---|---|---|---|
| Pop$^n$ | 651 177 | 840 243 | 8 017 299 | 10 310 090 |

Find the relative change from 20 to 21 and 30 to 31.

$$\frac{840\,243}{651\,177} = 1.2903\ldots, \frac{10\,310\,090}{8\,017\,299} = 1.2859\ldots$$

Long-term annual relative change is about 1.286.

 ISBN 9780170413060

| | |
|---|---|
| Find the culling factor for a stable population (relative change = 1). | $1.286(1 - h) = 1$<br>$h = 0.2223\ldots$ |
| Write the answer. | The culling factor should be slightly more than 22.2% to gradually reduce the number of pigeons. |

## INVESTIGATION

### FERAL ANIMALS IN AUSTRALIA

There are many feral animals in Australia. Some were introduced in colonial times for hunting, such as foxes and deer. Some are domestic animals that have gone wild, like camels, cats, dogs, pigs, buffalos, donkeys and horses. Cane toads were introduced in an effort to control insects in sugar cane.

Hard-hooved animals like deer, pigs, buffalo and horses are particularly destructive to soils, while feral cats and foxes kill many of our native species. Camels and rabbits out-compete native animals for food.

Shutterstock.com/Stanislav Fosenbauer

Efforts to eradicate or reduce the numbers of feral species have met with little success.

Choose a feral species and find how it affects domestic animals, plants and the environment. Obtain data about the animal and predict its numbers.

Suggest ways that the species could be controlled.

You teacher might want you to write a report.

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## Exercise 4.07 Leslie matrices

Example 14

**1** A population of wallabies is studied and the following data is collected.

| Age (years) | 0–1 | 1–2 | 2–3 | 3–4 | 4–5 | 5–6 | 6–7 | 7–8 |
|---|---|---|---|---|---|---|---|---|
| Population | 1000 | 600 | 500 | 400 | 400 | 300 | 200 | 100 |
| Birth rate | 0 | 0 | 0.8 | 0.9 | 0.7 | 0.5 | 0.3 | 0.1 |
| Survival rate | 0.6 | 0.8 | 0.9 | 0.9 | 0.8 | 0.6 | 0.3 | 0 |

**a** Write the Leslie matrix and initial female population vector.
**b** Find the population vector after 1 year.
**c** Find the population vector after 2 years.
**d** Find the population vector after 3 years.
**e** What is the population in the base year and after 1, 2 and 3 years?

**2** The population of bilbies in an area affected by feral cats is studied and the following data is obtained.

| Age (years) | 0–1 | 1–2 | 2–3 | 3–4 |
|---|---|---|---|---|
| Population | 800 | 500 | 300 | 100 |
| Birth rate | 0.3 | 1.5 | 1.1 | 0.4 |
| Death rate | 0.8 | 0.7 | 0.9 | 1 |

**a** Calculate the survival rates.
**b** Write the Leslie matrix and initial female population vector.
**c** Find the population vector after 1 year.
**d** Find the population vector after 2 years.
**e** Find the population vector after 3 years.
**f** What is the population in the base year and after 1, 2 and 3 years?

**3** A study of a species of frog in a stream that was thought to be slightly polluted resulted in the following data.

| Age (years) | 0–1 | 1–2 | 2–3 | 3–4 | 4–5 |
|---|---|---|---|---|---|
| Population | 25 000 | 1000 | 150 | 80 | 10 |
| Birth rate | 0 | 30 | 40 | 30 | 10 |
| Death rate | 0.98 | 0.9 | 0.85 | 0.95 | 1 |

**a** Write the survival rates.
**b** Write the Leslie matrix and initial female population vector.
**c** Find the population after 1, 2, 3, 10 and 20 years.
**d** What is the population trend?


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**4** Adjust the annual Leslie matrix below to incorporate an annual culling rate of 20%.

Example 15

$$\begin{bmatrix} 0 & 1.5 & 2.1 & 1.2 & 0.5 \\ 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 \end{bmatrix}$$

## Problem solving

**5** A deer park has had problems breeding sufficient animals. A study produced the following data.

| Age (years) | 0–2 | 2–4 | 4–6 | 6–8 | 8–10 | 10–12 |
|---|---|---|---|---|---|---|
| Population | 1000 | 500 | 400 | 200 | 100 | 50 |
| Birth rate | 0 | 0.2 | 0.8 | 0.5 | 0.2 | 0.1 |
| Survival rate | 0.6 | 0.8 | 0.7 | 0.5 | 0.3 | 0 |

Use the Leslie model to find the population trend and suggest what needs to be done to maintain a stable population.

**6** The population of brumbies in a part of Australia was studied. The following data was obtained after a tag-and-release program. An insignificant number of brumbies lived longer than 8 years in the wild. Assume that there is no intervention and conditions remain the same.

| Age (years) | 0–1 | 1–2 | 2–3 | 3–4 | 4–5 | 5–6 | 6–7 | 7–8 |
|---|---|---|---|---|---|---|---|---|
| Population | 600 | 500 | 400 | 340 | 260 | 190 | 100 | 40 |
| Birth rate | 0 | 0.6 | 0.8 | 0.8 | 0.7 | 0.5 | 0.3 | 0 |
| Death rate | 0.3 | 0.2 | 0.2 | 0.3 | 0.4 | 0.6 | 0.8 | 1 |

Find the trend in population. Suggest a culling rate to eradicate the population in about 3 years. Suggest any difficulties in eradication.

**7** A closed saltwater lake supported a population of mud crabs. The crabs were studied and the following data was collected.

| Age (months) | 0–6 | 6–12 | 12–18 | 18–24 | 24–30 | 30–36 |
|---|---|---|---|---|---|---|
| Population | 300 000? | 2500 | 500 | 150 | 100 | 10 |
| Birth rate | 0 | 0 | 600 | 600 | 300 | 100 |
| Death rate | 0.99 | 0.8 | 0.6 | 0.5 | 0.8 | 1 |

Assuming the 300 000 figure is correct, work out the population trend. The crabs cannot be caught in pots until they are big enough to be trapped (over 12 months). By changing the birth and death rates for crabs over 12 months old when they can be caught, find a possible rate for sustainable harvesting. Suggest other ways that sustainable harvesting could be done.

**8** The Devil Facial Tumour Disease was first found in north-eastern Tasmania in 1996. The population of Tasmanian devils in 2000 was studied in a bush area near Hobart before the outbreak of this disease and the following data was obtained.

| Age (years) | 0–1 | 1–2 | 2–3 | 3–4 | 4–5 |
|---|---|---|---|---|---|
| **Population** | 1160 | 464 | 278 | 140 | 20 |
| **Birth rate** | 0 | 0.5 | 2.5 | 1.6 | 0.4 |
| **Death rate** | 0.6 | 0.4 | 0.5 | 0.85 | 1 |

After the outbreak of the disease, the survival rates dropped by 70%.

Find the original general change of the population and the change because of the disease. Suggest possible methods to ensure survival of the species.

iStock/Getty Images Plus/CraigRJD

## 4.08 Further matrix applications

The number of suppliers may be very small for some goods. This can be because the market is small, among for other reasons. For example, in Australia there are only 2 major brewing companies. A situation with only 2 suppliers is called a **duopoly**. True duopolies are rare.

### Small markets

- A market with only 2 suppliers is called a **duopoly**
- If 90% of a market is shared by 2 suppliers, it approximates a duopoly
- A **payoff matrix** shows the profits or market share a supplier in a duopoly will have if the other supplies a particular product from the range they supply


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## EXAMPLE 16

A market for steel fence posts is split between 2 suppliers. They can each make galvanised, painted or untreated posts, but there are setup costs involved in each choice.

Market research shows that the profit for supplying a particular type of post if the competitor makes enough and supplies the whole market with one type is as follows.

| Allsteel's profit $/1000 posts | | | | Blackpost's profit $/1000 posts | | | |
|---|---|---|---|---|---|---|---|
| Allsteel | Blackpost | | | Blackpost | Allsteel | | |
| | U | P | G | | U | P | G |
| Untreated | –$300 | $300 | $500 | Untreated | –$400 | $200 | $600 |
| Painted | $100 | 0 | –$300 | Painted | $200 | –$100 | –$200 |
| Galvanised | $200 | $400 | –$400 | Galvanised | $300 | $400 | –$500 |

**a** Write the payoff matrix for each company.

**b** If Allsteel decides to supply untreated posts, what should Blackpost do?

**c** If Blackpost decides to supply untreated posts, what should Allsteel do?

**d** If each supplier were to supply only one kind, what type would be most likely to be unsupplied?

### Solution

**a** Allsteel's payoff matrix $\mathbf{U_A}$ is constructed with its choices as the rows and Blackpost's choices as the columns. The order of elements is U, P, G for both columns and rows. Blackpost's is done the same way.

$$\mathbf{U_A} = \begin{bmatrix} -300 & 300 & 500 \\ 100 & 0 & -300 \\ 200 & 400 & -400 \end{bmatrix}$$

$$\mathbf{U_B} = \begin{bmatrix} -400 & 200 & 600 \\ 200 & -100 & -200 \\ 300 & 400 & -500 \end{bmatrix}$$

**b** Look at Blackpost's U column.

They should supply galvanised posts.

**c** Look at Allsteel's U column.

They should supply galvanised posts.

**d** Look at the maximum profit for each company.

If one chooses galvanised, the painted is most profitable for the other and vice versa, so untreated posts are most likely to be left unsupplied.

The actual choices will also depend on the sizes of the markets.

In some probability situations, you can use matrices to solve problems. These situations involve a sequence of time intervals of fixed length. In each time interval, there are the same finite number of possibilities called **states**.

The probability of a particular state in any time interval depends only on the previous state.

## Markov chains

- A **Markov chain** exists for a random process with a finite number of different states in a sequence of equal time intervals. The probability of a particular state depends only on the previous state
- The $k$ outcome states of a Markov chain are written as $x_1, x_2, x_3, x_4, \ldots, x_k$
- The probabilities of each state, called **transition probabilities**, are written as $\pi_{ij}$, the probability that the next state is $x_j$, given that the current state is $x_i$
- The matrix $\mathbf{P} = (\pi_{ij})$ is called a **transition matrix**. The total of each row is 1
- The (row) vector of probabilities of each state after $n$ time intervals (or $n$ transitions) is the **state vector** $\mathbf{X}_n$. The initial state is written as $\mathbf{X}_0$

$$\mathbf{X}_n = \mathbf{X}_{n-1}\mathbf{P} = \mathbf{X}_{n-2}\mathbf{P}^2 = \ldots = \mathbf{X}_0\mathbf{P}^n$$

- If any power of $\mathbf{P}$ has no zeros, then the state vector stabilises to the vector $\mathbf{X}$ such that $\mathbf{XP} = \mathbf{X}$
- The probability of the $i$th state after $k$ transitions is written as $P(x_i(k))$. This is the $i$th value of $\mathbf{X}_k$.

You can use a graphics calculator to work out results for a Markov chain.

## EXAMPLE 17

A new company's hire trailers do not need to be returned to the same depot.
It has 4 depots, at Logan Central, Ipswich, New Farm and Aspley.
After a few months of operation, it finds the following pattern of hire and return.

| | | Return depot | | | |
|---|---|---|---|---|---|
| | | Logan | Ipswich | New Farm | Aspley |
| Hire depot | Logan | 50% | 25% | 15% | 10% |
| | Ipswich | 30% | 40% | 25% | 5% |
| | New Farm | 15% | 15% | 35% | 35% |
| | Aspley | 10% | 10% | 20% | 60% |


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The company starts up with 20 trailers at each depot.

**a** What is the probability that a trailer hired at New Farm is returned to Aspley?

**b** Write the transition matrix and initial state vector.

**c** Find the state vectors after 1, 2, 3 and 4 hiring periods.

**d** Find the steady state vector.

**e** What is the significance of the steady state vector?

## Solution

**a** Look up the value in the table.

$P(\text{Aspley} \mid \text{New Farm}) = 0.35$

**b** Use the values in the table for **P**. The trailers are equally distributed at the start, so the probabilities are equal.

$$\mathbf{P} = \begin{bmatrix} 0.5 & 0.25 & 0.15 & 0.1 \\ 0.3 & 0.4 & 0.25 & 0.05 \\ 0.15 & 0.15 & 0.35 & 0.35 \\ 0.1 & 0.1 & 0.2 & 0.6 \end{bmatrix},$$

$$\mathbf{X}_0 = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix}$$

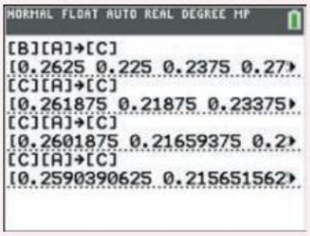
**c** **TI-84 Plus CE**

Enter the transition and state vector matrices in [A] and [B] respectively.

Multiply [B] by [A] and store in [C].

Then multiply [C] by [A] and store in [C].

Press enter to repeat. Do it again.

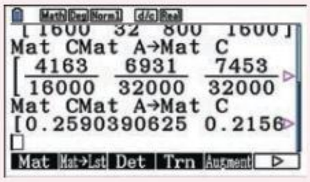



**Casio fx-CG20AU**

Use the Run-matrix menu.

Enter the transition and state vector matrices in Mat A and Mat B respectively.

Multiply Mat B by Mat A and store in Mat C. Then multiply Mat C by Mat A and store back in Mat C. Press EXE to repeat.
Do it again.




Write the vectors.

$\mathbf{X}_1 = [0.2625\ 0.225\ 0.2375\ 0.275]$

$\mathbf{X}_2 = [0.261875\ 0.21875\ 0.23375\ 0.285625]$

$\mathbf{X}_3 = [0.2601875\ 0.21659375\ 0.23290625\ 0.2903125]$

$\mathbf{X}_4 = [0.2590390625\ 0.2156515625\ 0.23275625\ 0.292553125]$

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**d** On your calculator, keep pressing enter/EXE until the result stabilises.

Write the steady state vector.

$\mathbf{X}$ = [0.25762711864417 0.2146892655368 0.23276836158201 0.29491525423739]

**e** Calculate the number of trailers at each depot and write the meaning of the steady state vector.

You could find the steady state vector by solving the matrix equation $\mathbf{XP} = \mathbf{X}$, but this is quite difficult.

$0.2576\ldots \times 80 = 20.610\ldots$

$0.2146\ldots \times 80 = 17.175\ldots$

$0.2327\ldots \times 80 = 18.621\ldots$

$0.2949\ldots \times 80 = 23.593\ldots$

There are usually 20 or 21 trailers at Logan, 17 at Ipswich, 18 or 19 at New Farm and 23 or 24 at Aspley.

Modelling of an economy generally assumes that relationships are linear. This means you can use matrices to model economic problems. Russian-American economist Wassily Leontief divided the US economy into 500 industry sectors. He aimed to forecast the production levels needed to meet consumer demands. Each industry sector of the economy has links with other sectors. These have to be taken into account in the model.

For example, the Australian coal and power industries are linked. If more coal is to be exported, then more power will be used to produce the coal. This means you have to produce more coal to make the extra power. If consumers need more power, then extra power must be produced for the industries that supply materials to power stations. Some power is also used by the power industry, so more is used there as well. This means that the actual increases in production of each industry are difficult to calculate.

## The Leontief input-output model

This model divides the economy into single sectors so that:

- all industry sectors exclusively make a single product; the entire production of an item is grouped together, and nothing else is made by the sector
- the total value of **inputs** equals the total value of **outputs** of each sector
- all outputs are consumed, by other sectors or by consumers, including exports
- the output used by consumers is called the **final demand**
- over the model period, the ratio of inputs to outputs stays the same for each sector. There are no changes in the production methods that affect the amount produced for given inputs
- inputs and outputs are measured in the same units, usually monetary value
- inputs can be outputs of other industry sectors, or **primary inputs**, such as labour, raw materials and land


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Consider a simple example with only 3 sectors: steel, coal and power.

Assume that all other inputs are primary inputs, and all other demand is consumer demand.

**Steel**: Total production = \$300 million

Inputs: Steel = \$20 million, Coal = \$50 million, Power = \$30 million

**Coal**: Total production = \$500 million

Inputs: Steel = \$20 million, Coal = \$40 million, Power = \$50 million

**Power**: Total production = \$400 million

Inputs: Steel = \$10 million, Coal = \$150 million, Power = \$20 million

This is summarised in the table below.

| | | Inputs | | | Final demands | Gross outputs |
|---|---|---|---|---|---|---|
| | | Steel | Coal | Power | | |
| Outputs | Steel | 20 | 20 | 10 | 250 | 300 |
| | Coal | 50 | 40 | 150 | 260 | 500 |
| | Power | 30 | 50 | 20 | 300 | 400 |
| Primary inputs | | 200 | 390 | 220 | | |
| Total inputs | | 300 | 500 | 400 | | |

In each case, total inputs = gross outputs (production). You can calculate the final demands and primary inputs by subtraction.

Steel: Final demand = \$300 – (\$20 + \$20 + \$10) = \$250 million

Primary inputs = \$300 – (\$20 + \$50 + \$30) = \$200 million

Coal: Final demand = \$500 – (\$50 + \$40 + \$150) = \$260 million

Primary inputs = \$500 – (\$20 + \$40 + \$50) = \$390 million

Power: Final demand = \$400 – (\$30 + \$50 + \$20) = \$300 million

Primary inputs = \$400 – (\$10 + \$150 + \$20) = \$220 million

You write each input as a fraction of the total input.

Steel inputs: $\frac{20}{300}, \frac{50}{300}, \frac{30}{300}$ Steel outputs: $\frac{20}{300}, \frac{20}{500}, \frac{10}{400}$

Coal inputs: $\frac{20}{500}, \frac{40}{500}, \frac{50}{500}$ Coal outputs: $\frac{50}{300}, \frac{40}{500}, \frac{150}{400}$

Power inputs: $\frac{10}{400}, \frac{150}{400}, \frac{20}{400}$ Power outputs: $\frac{30}{300}, \frac{50}{500}, \frac{20}{400}$

The final demands, total productions and primary inputs are written as column vectors **D**, **X** and **P** called the **demand matrix**, **production matrix** and **primary inputs matrix** respectively.

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In this case, $\mathbf{D} = \begin{bmatrix} 250 \\ 260 \\ 300 \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} 300 \\ 500 \\ 400 \end{bmatrix}$ and $\mathbf{P} = \begin{bmatrix} 200 \\ 390 \\ 220 \end{bmatrix}$.

Notice that the amount of steel used is given by $\frac{20}{300} \times 300 + \frac{20}{500} \times 500 + \frac{10}{400} \times 400 = 50$

The amounts used by coal and power work the same way. The total production from each sector is equal to the inputs plus consumer demand, so the totals must also be equal.

Total production $= \begin{bmatrix} 300 \\ 500 \\ 400 \end{bmatrix} = \begin{bmatrix} 50 \\ 240 \\ 100 \end{bmatrix} + \begin{bmatrix} 250 \\ 260 \\ 300 \end{bmatrix}$.

$$\text{Now} \begin{bmatrix} 50 \\ 240 \\ 100 \end{bmatrix} = \begin{bmatrix} \frac{20}{300} \times 300 + \frac{20}{500} \times 500 + \frac{10}{400} \times 400 \\ \frac{50}{300} \times 300 + \frac{40}{500} \times 500 + \frac{150}{400} \times 400 \\ \frac{30}{300} \times 300 + \frac{50}{500} \times 500 + \frac{20}{400} \times 400 \end{bmatrix} = \begin{bmatrix} \frac{20}{300} & \frac{20}{500} & \frac{10}{400} \\ \frac{50}{300} & \frac{40}{500} & \frac{150}{400} \\ \frac{30}{300} & \frac{50}{500} & \frac{20}{400} \end{bmatrix} \begin{bmatrix} 300 \\ 500 \\ 400 \end{bmatrix}$$

The fractions are called the **input-output coefficients** and the matrix is called the **input-output matrix**, written as **A**. Altogether, you get the equation $\mathbf{X} = \mathbf{AX} + \mathbf{D}$, called the **input-output equation**.

The equation is usually rearranged to $(\mathbf{I} - \mathbf{A})\mathbf{X} = \mathbf{D}$ or $\mathbf{X} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{D}$.

You can use the equation to find any of the outputs, inputs or consumer demands from the other two.

## EXAMPLE 18

The interaction between 3 industries in a hypothetical economy is shown in this table.

| | | Inputs | | | Final demands |
|---|---|---|---|---|---|
| | | *A* | *B* | *C* | |
| Outputs | *A* | 200 | 100 | 300 | 400 |
| | *B* | 300 | 500 | 100 | 300 |
| | *C* | 400 | 200 | 100 | 400 |

**a** Determine the gross outputs and primary inputs.

**b** Write the input–output matrix **A**.

**c** The demands for *A*, *B* and *C* change to 300, 600 and 800 respectively. Write the new demand matrix.

**d** Find the new production matrix.

**e** Find the new primary inputs matrix.


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## Solution

**a** Add the inputs and final demands horizontally across the table to get the gross outputs.

Gross output ($A$) = 1000
Gross output ($B$) = 1200
Gross output ($C$) = 1100

Subtract the inputs (totals down the table) from the gross outputs.

Primary inputs ($A$) = 1000 – 900 = 100
Primary inputs ($B$) = 1200 – 800 = 400
Primary inputs ($C$) = 1100 – 500 = 600

**b** Divide each input by the total inputs (gross outputs) and put into the matrix.

$$\mathbf{A} = \begin{bmatrix} \frac{200}{1000} & \frac{100}{1200} & \frac{300}{1100} \\ \frac{300}{1000} & \frac{500}{1200} & \frac{100}{1100} \\ \frac{400}{1000} & \frac{200}{1200} & \frac{100}{1100} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{5} & \frac{1}{12} & \frac{3}{11} \\ \frac{3}{10} & \frac{5}{12} & \frac{1}{11} \\ \frac{2}{5} & \frac{1}{6} & \frac{1}{11} \end{bmatrix}$$

**c** Put the values into **D**.

$$\mathbf{D} = \begin{bmatrix} 300 \\ 600 \\ 800 \end{bmatrix}$$

**d** Write the equation for **X** and substitute values.

$$\mathbf{X} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{D}$$

$$= \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{5} & \frac{1}{12} & \frac{3}{11} \\ \frac{3}{10} & \frac{5}{12} & \frac{1}{11} \\ \frac{2}{5} & \frac{1}{6} & \frac{1}{11} \end{pmatrix}\right]^{-1} \begin{bmatrix} 300 \\ 600 \\ 800 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{5} & -\frac{1}{12} & -\frac{3}{11} \\ -\frac{3}{10} & \frac{7}{12} & -\frac{1}{11} \\ -\frac{2}{5} & -\frac{1}{6} & \frac{10}{11} \end{bmatrix}^{-1} \begin{bmatrix} 300 \\ 600 \\ 800 \end{bmatrix}$$

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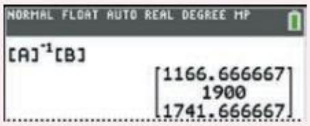
**TI-84 Plus CE**

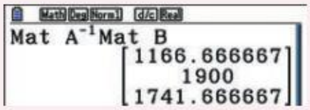
Put in the matrices in [A] and [B] for the calculation.




**Casio fx-CG20AU**

Use the Run-matrix menu.

Put in the matrices in Mat A and Mat B for the calculation.




Write the answer.

$$\mathbf{X} \approx \begin{bmatrix} 1167 \\ 1900 \\ 1742 \end{bmatrix}$$

**e** Find the inputs.

$$A \text{ inputs} = \left(\frac{1}{5} + \frac{3}{10} + \frac{2}{5}\right) \times 1167$$
$$\approx 1050$$
$$B \text{ inputs} = \left(\frac{1}{12} + \frac{5}{12} + \frac{1}{6}\right) \times 1900$$
$$\approx 1267$$
$$C \text{ inputs} = \left(\frac{3}{11} + \frac{1}{11} + \frac{1}{11}\right) \times 1742$$
$$\approx 792$$

Find the primary matrix by subtraction.

$$\mathbf{P} = \begin{bmatrix} 1167 \\ 1900 \\ 1742 \end{bmatrix} - \begin{bmatrix} 1050 \\ 1267 \\ 792 \end{bmatrix} = \begin{bmatrix} 117 \\ 633 \\ 950 \end{bmatrix}$$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## INVESTIGATION

### EIGENVALUES AND EIGENVECTORS

You saw last year that matrices can be used to model transformations. Modelling vectors and their transformations is a very important use of matrices in Calculus, Data analysis, Physics, Engineering, Chemistry and Economics. You use vectors that don't change direction in these applications to simplify calculations.

For a matrix **A**, a vector **v** that doesn't change direction will only change its size.

You can write this as $\mathbf{Av} = \lambda\mathbf{v}$, where $\lambda$ is a scalar. Sometimes $\lambda \in \boldsymbol{R}$ and sometimes $\lambda \in \boldsymbol{C}$.

For example, $\begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ 15 \end{bmatrix} = \begin{bmatrix} 36 \\ 90 \end{bmatrix} = 6\begin{bmatrix} 6 \\ 15 \end{bmatrix}$.

6 is an **eigenvalue** and $\begin{bmatrix} 6 \\ 15 \end{bmatrix}$ is an **eigenvector** for $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$.

**A** actually has 2 non-trivial eigenvalues, 6 and 1.

The eigenvectors are of the form $(2a, 5a)$ for $\lambda = 6$ and $(b, -b)$ for $\lambda = 1$, where $a, b \in \boldsymbol{R}$.

**A** represents a linear transformation of the plane. Eigenvalues and eigenvectors can transform multidimensional spaces. Eigenvectors can change a complex problem to a collection of simpler one-dimensional linear relationships.

You can show that if **A** has an eigenvalue $\lambda$, then $|\mathbf{A} - \lambda\mathbf{I}| = 0$. For a $2 \times 2$ matrix, this is a quadratic equation, so there are at most 2 eigenvalues. For an $n$-dimensional matrix there are at most $n$ eigenvalues. The eigenvectors for an eigenvalue are all in the same direction, so they are linear multiples of each other.

Find eigenvalues for each matrix using $|\mathbf{A} - \lambda\mathbf{I}| = 0$ and write expressions for their eigenvectors.

- $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ has only one eigenvalue
- $\begin{bmatrix} 1 & -2 \\ 8 & 11 \end{bmatrix}$ and $\begin{bmatrix} 3 & 6 \\ 6 & -2 \end{bmatrix}$ have integer eigenvalues
- $\begin{bmatrix} 7 & -2 \\ -4 & 3 \end{bmatrix}$ and $\begin{bmatrix} 6 & -1 \\ -3 & 5 \end{bmatrix}$ have irrational eigenvalues
- $\begin{bmatrix} 4 & -1 \\ 5 & 2 \end{bmatrix}$ and $\begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}$ have complex eigenvalues
- $\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ -1 & -3 & -3 \end{bmatrix}$ has 3 integer eigenvalues, so use your graphics calculator

## Exercise 4.08 Further matrix applications

Example 16

**1** 2 milk distributors, A and B, have bottling plants in the same city. While the market is big enough for standard homogenised milk, it is not big enough for both companies to supply skim, calcium-enriched and super creamy milk. In addition, each of these types requires a special run of the bottling plant, so it is only economical for each company to supply one special type. The following table shows the expected market shares for each type of milk for Company A if Company B supplies each of the other types.

| Company A supplies: | Company B supplies: | | |
|---|---|---|---|
| | Skim | Calcium-enriched | Super creamy |
| **Skim** | 50% | 55% | 35% |
| **Calcium-enriched** | 40% | 55% | 30% |
| **Super creamy** | 60% | 65% | 45% |

**a** Write the market shares of Company A as a payoff matrix.

**b** Write the market shares of Company B as a payoff matrix.

**c** If Company B chooses calcium-enriched, what should Company A supply?

**d** A **dominant strategy** exists for Company A if there is a best choice regardless of Company B's choice. Is there a dominant strategy?

**e** What special types of milk are likely to be unsupplied?

Example 17

**2** The transition matrix for a Markov chain is

$$\mathbf{P} = \begin{bmatrix} 0.2 & 0.3 & 0.15 & 0.35 \\ 0.3 & 0.1 & 0.45 & 0.15 \\ 0.1 & 0.35 & 0.3 & 0.25 \\ 0.4 & 0.1 & 0.2 & 0.3 \end{bmatrix}.$$

**a** What is $\pi_{23}$?

**b** What is the value of $P(x_4 \mid x_3)$?

**c** If $P(x_1(0)) = 0.2$, $P(x_2(0)) = 0.3$, $P(x_3(0)) = 0.4$ and $P(x_3(0)) = 0.1$, find $P(x_4(3))$.


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**3** The following data shows the weather on successive days in Brisbane, coded so that S = sunny, C = cloudy and R = rainy.

| | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| C | S | S | S | C | S | C | R | C | S | C | S | S | S | C | S | C | C | C | C | S | S | S | C | C | C |
| S | S | C | R | S | C | S | S | C | S | S | S | S | R | C | S | S | S | C | C | C | C | C | C | R | R |

Use the data to set up a transition matrix.

**4** The transition matrix for a Markov chain is given by $\boldsymbol{P} = \begin{bmatrix} 0.21 & 0.19 & 0.33 & 0.27 \\ 0.44 & 0.06 & 0.21 & 0.29 \\ 0.1 & 0.45 & 0.27 & 0.18 \\ 0.35 & 0.15 & 0.28 & 0.22 \end{bmatrix}$.

Write down:

**a** $\pi_{22}$ **b** $\pi_{24}$ **c** $\pi_{42}$ **d** $\pi_{23}$

**e** $\pi_{13}$ **f** $P(x_4 \mid x_1)$ **g** $P(x_1 \mid x_3)$ **h** $P(x_3 \mid x_2)$

**i** $P(x_2 \mid x_3)$ **j** $P(x_3 \mid x_1)$.

**5** State whether each vector could be a state vector.

**a** [0.4 0.3 0.4] **b** [0.2 0.1 0.4 0.3] **c** [0.15 0.35 0.5 0.1]

**d** [3 4 3] **e** [0.2 0.2 –0.3 0.3]

**6** State whether each matrix could be a Markov chain transition matrix.

**a** $\begin{bmatrix} 0.1 & 0.5 & 0.4 \\ 0.2 & 0.4 & 0.4 \\ 0.5 & 0.2 & 0.3 \end{bmatrix}$ **b** $\begin{bmatrix} 0.25 & 0.2 & 0.25 & 0.3 \\ 0.25 & 0.25 & 0.25 & 0.3 \\ 0.25 & 0.3 & 0.2 & 0.25 \\ 0.25 & 0.2 & 0.25 & 0.3 \end{bmatrix}$

**c** $\begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.6 & -0.1 & 0.5 \\ 0 & 0.3 & 0.7 \end{bmatrix}$

**7** The transition matrix for a Markov chain is given by $\mathbf{P} = \begin{bmatrix} 0.3 & 0.1 & 0.2 & 0.4 \\ 0.4 & 0 & 0.2 & 0.4 \\ 0.1 & 0.5 & 0 & 0.4 \\ 0.4 & 0 & 0.35 & 0.25 \end{bmatrix}$.

**a** What is the probability of being in state 2 after 3 transitions if the initial state is state 3?

**b** What is the probability of being in state 2 after 3 transitions if the initial state is state 4?

**c** What is the probability of being in state 1 after 5 transitions if the initial state is state 4?

**d** If it exists, find the stable state vector.

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**8** The transition matrix for a Markov chain is given by $\mathbf{P} = \begin{bmatrix} 0.25 & 0.45 & 0.3 \\ 0.15 & 0.35 & 0.5 \\ 0.6 & 0.2 & 0.2 \end{bmatrix}$.

- **a** If $P(x_1(0)) = 0.4$, $P(x_2(0)) = 0.3$, $P(x_3(0)) = 0.3$, find $P(x_2(4))$.
- **b** If $P(x_1(0)) = 0.25$, $P(x_2(0)) = 0.45$, $P(x_3(0)) = 0.3$, find $P(x_3(3))$.
- **c** If $P(x_1(0)) = 0.3$, $P(x_2(0)) = 0.17$, $P(x_3(0)) = 0.53$, find $P(x_3(2))$.
- **d** If $P(x_1(0)) = 0.19$, $P(x_2(0)) = 0.43$, $P(x_3(0)) = 0.38$, find $P(x_1(3))$.
- **e** If $P(x_1(0)) = 0.3$, $P(x_2(0)) = 0.3$, $P(x_3(0)) = 0.4$, find $P(x_2(5))$.

**9** The interaction between 2 industries in a hypothetical economy is shown in the following table.

| | | **Inputs** | | **Final demands** | **Gross Outputs** |
|---|---|---|---|---|---|
| | | ***M*** | ***N*** | | |
| **Outputs** | ***M*** | 20 | 40 | 60 | 120 |
| | ***N*** | 70 | 30 | 30 | 130 |
| | **Primary** | 30 | 60 | | |

- **a** Write the input–output matrix **A**.
- **b** Find the new demand matrix if the demands for $M$ and $N$ change to 50 and 80 respectively.
- **c** Find the new production matrix.
- **d** Find the new primary inputs matrix.

**10** The interaction between 3 industries in a hypothetical economy is shown in the following table.

| | | | **Inputs** | | **Final demands** |
|---|---|---|---|---|---|
| | | ***X*** | ***Y*** | ***Z*** | |
| **Outputs** | ***X*** | 100 | 300 | 400 | 600 |
| | ***Y*** | 200 | 50 | 300 | 400 |
| | ***Z*** | 300 | 200 | 150 | 900 |

- **a** Determine the gross outputs and primary inputs.
- **b** Write the input-output matrix **A**.
- **c** Find the new demand matrix if the demands for $X$, $Y$ and $Z$ change to 400, 600 and 700 respectively.
- **d** Find the new production matrix.
- **e** Find the new primary inputs matrix.


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## Problem solving

**11** Clean Products and Laundry & Kitchen dominate the detergent market in a country. Their profits are related to market share and pricing of detergents. Executives of one company have produced the table below, showing the likely profits of each company under different pricing regimes.

| Clean Products profit × $10^7 | | | | Laundry & Kitchen profit × $10^7 | | | |
|---|---|---|---|---|---|---|---|
| **CP's prices** | **LK's prices** | | | **LK's prices** | **CP's prices** | | |
| | **Low** | **Medium** | **High** | | **Low** | **Medium** | **High** |
| **Low** | –4 | 5 | 12 | Low | –3 | 4 | 11 |
| **Medium** | 2 | 6 | 4 | Medium | 1 | 5 | 4 |
| **High** | –2 | 2 | 10 | High | –1 | 2 | 8 |

Is there a way for both companies to make good profits? Are there any problems with this? Is there a **maximin strategy**—a way to maximise the minimum profit, regardless of what the other company does?

**12** 2 soft drink manufacturers supply the same market with cola, lemonade and/or orange drinks. Since the market in question is very small, they can each afford to supply only one flavour. The market shares that the first manufacturer will obtain are given by the following table.

| First company | Competitor | | |
|---|---|---|---|
| | **Cola** | **Lemonade** | **Orange** |
| **Cola** | 45% | 60% | 65% |
| **Lemonade** | 35% | 50% | 55% |
| **Orange** | 30% | 40% | 55% |

Describe what is likely to happen and state implications for consumers. Explain your methods.

**13** The interaction between the fuel, metal production, transport and manufacturing sectors of an economy is shown in the following table.

| | | Inputs | | | | Gross outputs |
|---|---|---|---|---|---|---|
| | | **Fuel** | **Metal** | **Transport** | **Manufacturing** | |
| **Outputs** | Fuel | 200 | 100 | 400 | 300 | 1500 |
| | Metal | 500 | 100 | 500 | 300 | 1500 |
| | Transport | 400 | 300 | 100 | 400 | 1600 |
| | Manufacturing | 300 | 500 | 200 | 200 | 1700 |

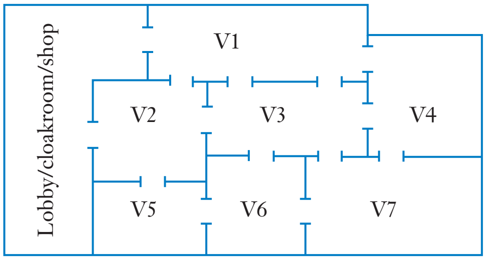
The demands change to 400, 250, 450 and 650 respectively. Find the new production and primary input matrices.

**14** A car hire company provides vehicles from Tullamarine and Avalon airports in Melbourne, and airports in Sydney, Port Macquarie and Brisbane, as well as from city and suburban locations. The percentage of hire cars returned to each airport is shown in the table.

| | | Return depot | | | | |
|---|---|---|---|---|---|---|
| | | **Tullamarine** | **Avalon** | **Sydney** | **Port Macquarie** | **Brisbane** |
| **Hire depot** | **Tullamarine** | 40% | 20% | 25% | 5% | 10% |
| | **Avalon** | 30% | 50% | 10% | 6% | 4% |
| | **Sydney** | 20% | 5% | 45% | 15% | 15% |
| | **Port Macquarie** | 4% | 2% | 30% | 60% | 4% |
| | **Brisbane** | 15% | 1% | 25% | 4% | 55% |

What happens in the long run, and what are the implications?

**15** An art exhibition is partitioned with interconnecting doors in the viewing areas V1 to V7 as shown below. Assuming that people wander randomly through each door, find the proportion of the audience that ends up in each viewing area, ignoring entries and exits from the Lobby area. Comment on the result and the modelling method.

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# 4. CHAPTER SUMMARY

## Vectors and matrices

### Systems of linear equations

- **Gaussian elimination** is the solution of simultaneous linear equations using elimination
- **Inconsistent** equations have no solutions
- **Consistent** equations have at least one solution
- A **dependent** system has at least one equation that is a combination of others
- An **independent** system has no equation that is a combination of others
- A **homogeneous** system of equations has no constant terms: You can write every equation in the form $a_1 x_1 + a_2 x_2 + a_3x_3 + a_4x_4 + \ldots + a_nx_n = 0$
- A **non-homogeneous** system has at least one equation with a constant term
- For a system of linear equations written in the matrix form $\mathbf{AX} = \mathbf{B}$,
  - $\mathbf{A}$ is called the **coefficient matrix**
  - the column matrix of variables $\mathbf{X}$ is called the **variable vector**
  - $\mathbf{B}$ is called the **value vector**
  - the inverse of $\mathbf{A}$, $\mathbf{A}^{-1}$, exists if and only if the determinant $|\mathbf{A}| \neq 0$
  - a matrix that has no inverse is called a **singular matrix**
  - if it exists, the solution is given by $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$
- The **augmented matrix** of a system of linear equations is the coefficient matrix with the constants in an extra column at the end, often separated by a vertical line
- The **row echelon form** of an augmented matrix has the leading entry of each non-zero row of the matrix equal to 1, with zeros below it
- The **row reduced echelon form** of a matrix also has zeros below the leading 1s

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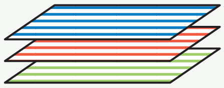
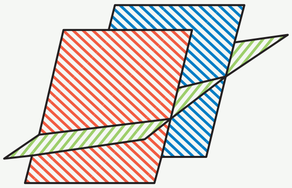
### Relationships of planes and their equations

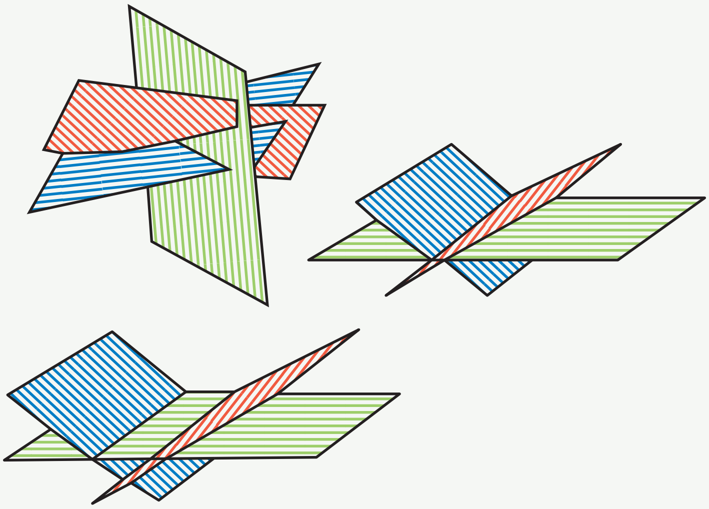
- **2 non-parallel planes** have different normals, consistent equations and intersect in a straight line
- **2 or more parallel planes** have the same normal, inconsistent equations and no intersections



- **3** planes with a third non-parallel plane have one normal different to the others, inconsistent equations and the third intersects the others in 2 straight lines



- **3 non-parallel planes** have different normals and:
  – intersect in a point if the equations are consistent and independent with a unique solution
  – intersect in a single line if the equations are consistent and dependent
  – intersect in 3 parallel lines if the equations are inconsistent

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### Dominance matrices

- A **dominance matrix** is used to compare the wins and losses of a group of 'players'
- Each row is used for one of the players
- The entries in a row show the dominance of the player over the others in the same order as the rows. An entry of 0 is a loss, 0.5 is a draw and 1 is a win
- The **transpose** $\mathbf{D}^{\mathrm{T}}$ of a matrix $\mathbf{D}$ has the rows and columns swapped
- **First-order ranking** is shown as $\mathbf{r}$ or $\mathbf{r}_1$ and is total wins – total losses
- For the vector $\mathbf{U}$ consisting only of 1s, $\mathbf{r}_1 = \mathbf{DU} - \mathbf{D}^{\mathrm{T}}\mathbf{U}$
- **Second-order ranking** is shown as $\mathbf{r}_2$ and is total wins of opponents beaten – total losses of opponents lost to

$$\mathbf{r}_2 = \mathbf{D}^2\mathbf{U} - (\mathbf{D}^2)^{\mathrm{T}}\mathbf{U}$$

- **Total rankings** add the first and second-order rankings. The second-order ranking is commonly weighted by 0.5
- **Third-order ranking**, $\mathbf{r}_3 = \mathbf{D}^3\mathbf{U} - (\mathbf{D}^3)^{\mathrm{T}}\mathbf{U}$

### Leslie matrix models

- Females are taken to be a fixed proportion of the population, usually half
- The environment, including predation, is assumed to stay the same
- Age-specific birth and death rates are assumed to stay the same
- The age-specific population is modelled by a vertical vector $\mathbf{X}$
- The population in the base year is usually written as $\mathbf{X}_0$, after 1 year as $\mathbf{X}_1$, after 2 years as $\mathbf{X}_2$, and so on
- The **Leslie matrix** is an $n \times n$ square matrix $\mathbf{L}$ with
  - age-specific birth rates $B_1, B_2, B_3, \ldots, B_n$ in the top row
  - age-specific survival rates $S_1, S_2, S_3, \ldots, S_{n-1}$ in the diagonal from $l_{21}$ to $l_{n(n-1)}$
  - all other elements equal to zero
- The population vector in each successive year is given by $\mathbf{X}_n = \mathbf{L}\mathbf{X}_{n-1}$
- The Leslie population model eventually stabilises, so the age distribution is constant and the percentage change in population from one year to the next is constant. Thus, $\mathbf{X}_n = k\mathbf{X}_{n-1}$.
- Simple culling assumes that the same proportion of each age group is removed. For a culling rate $h$, the survival and birth rates are multiplied by $(1 - h)$

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## Small markets

- A market with only 2 suppliers is called a **duopoly**
- If 90% of a market is shared by 2 suppliers, it approximates a duopoly
- A **payoff matrix** shows the profits or market share a supplier in a duopoly will make if the other supplies a particular product from the range they supply

## Markov chains

- A **Markov chain** exists for a random process with a finite number of different states in a sequence of equal time intervals. The probability of a particular state depends only on the previous state
- The $k$ outcome states of a Markov chain are written as $x_1, x_2, x_3, x_4, \ldots, x_k$
- The probabilities of each state, called **transition probabilities**, are written as $\pi_{ij}$, the probability that the next state is $x_j$, given that the current state is $x_i$
- The matrix $\mathbf{P} = (\pi_{ij})$ is called a **transition matrix**. The total of each row is 1
- The (row) vector of probabilities of each state after $n$ time intervals (or $n$ transitions) is the **state vector** $\mathbf{X}_n$. The initial state is written as $\mathbf{X}_0$
- $\mathbf{X}_n = \mathbf{X}_{n-1}\mathbf{P} = \mathbf{X}_{n-2}\mathbf{P}^2 = \ldots = \mathbf{X}_0\mathbf{P}^n$
- If any power of $\mathbf{P}$ has no zeros, then the state vector stabilises to the vector $\mathbf{X}$ such that $\mathbf{XP} = \mathbf{X}$


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# 4. CHAPTER REVIEW

## Vectors and matrices

**1** Solve each system of equations by Gaussian elimination.
State if any systems are inconsistent, dependent or has no unique solution.

Examples 1–3

**a** $5x + 21y + 2z = 5$
$4x + 27y + z = 5$
$2x + 12y + z = 4$
$8x + 42y + z = 2$

**b** $a - 2b + c = -6$
$2a - 2b + c = -4$
$6b - 3a - 2c = 13$
$6b - 2a - c = 12$

**c** $4p - 12q - 4r = 0$
$2p - 5q - r = 3$
$8q - 3p + 2r = -3$

**d** $25a + 3b - 6c + 31d = 1$
$17a + 2b - 4c + 21d = 1$
$4c - 22a - 3b - 27d = -2$
$27a + 2b - 8c + 33d = 3$

**2** Write this system of equations as a matrix equation.

Example 4

$3a - 2b + c = 2$

$a + 3b + d = -4$

$3a - 2b + c - 4d = 8$

$3b - a - c - 3d = 2$

**3** Write this system of equations as an augmented matrix.

Example 5

$2k - 5m + p - q = 8$

$k + 2m - 3p + 4q = 0$

$m - 3k + 5p = 2$

$5k + 2m - 3q = 6$

**4** Use matrices to solve each system of equations (where possible).

Examples 6, 7

**a** $3a - 4b + 2c = -10$
$2a - 12b + 7c = -12$
$10b - 2a - 6c = 10$
$5b - 2a - 3c = 7$

**b** $3f + 16g - 5h - 10i = -34$
$2h - f - 6g + 2i = 9$
$2f + 10g - 3h - 7i = -23$
$2f + 18g - 7h - 3i = -21$

**c** $7p + 17q - 3r + 6s = -13$
$2p + 5q - r + 2s = -4$
$2p + 10q - 6r + 5s = 7$
$3p + 11q - 5r + 5s = 2$

**d** $s + 3h - 2i + m = 7$
$3i - 2s - 5h - 2m = -12$
$2s + 7h - 5i + 3m = 19$
$2s + 4h - 2i + 2m = 10$

**5** Find the relationship between each pair of planes.

Example 10

**a** $x - y + 5z = 7$ and $2x + 2y + z = -4$

**b** $3x + 2y - z = 3$ and $2z = 6x + 4y + 3$

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Example 11

**6** Find the relationship between each set of planes.

**a** $3x + 2y - z = 3$, $x + y + 3z = -5$ and $2x - y + z = 4$

**b** $2x + y - z = 5$, $3x - y + 4z = -1$, $4x + 2y - 2z = 7$

**c** $2x + 5y + z = 10$, $17x + 37y + 3z = 67$ and $7x + 16y + 2z = 30$

Examples 12, 13

**7** The dominance matrix for 5 tennis players is shown below.

Use the ranking $\mathbf{r_1} + 0.5\mathbf{r_2}$ to rank the players in order from 1st to 5th.

$$\begin{array}{l} \\ \text{David} \\ \text{Beryl} \\ \text{Toni} \\ \text{Shaan} \\ \text{Jason} \end{array} \begin{array}{c} \begin{matrix} \text{D} & \text{B} & \text{T} & \text{S} & \text{J} \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 1 & 1 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Example 14

**8** $$\begin{bmatrix} 0 & 0 & 6 & 5 & 3 & 2 \\ 0.3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 \end{bmatrix}$$

is the Leslie matrix of a wild species that lives for up to 6 years.

**a** When does the animal start breeding?

**b** What is the survival rate in the first year?

**c** What is the death rate in the third year?

**d** If there are 400 3-year-olds and 300 4-year-olds in a national park in one year, how many 4-year-olds are there the next year?

Example 15

**9** Adjust the Leslie matrix $\begin{bmatrix} 0 & 0 & 6 & 5 & 3 & 2 \\ 0.3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 \end{bmatrix}$ to include annual culling of 15% of all the animals in question **8**.


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Example 16

**10** 2 companies compete in supplying soap in a small market. They each carry many brands, but the soaps can be classified as basic, premium and deluxe brands of soap. Company A's market share with different aggressive marketing strategies are shown below.

| Company A's market share | | | |
|---|---|---|---|
| **Company A pushes** | **Company B pushes** | | |
| | **Basic** | **Premium** | **Deluxe** |
| **Basic** | 40% | 40% | 30% |
| **Premium** | 20% | 40% | 20% |
| **Deluxe** | 40% | 30% | 40% |

Both companies make greater profits from premium and deluxe brands.

Write the payoff matrix for each company.

Example 17

**11** A transition matrix is given by $\begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 \\ 0.2 & 0.2 & 0.3 & 0.3 \\ 0.3 & 0.2 & 0.1 & 0.4 \\ 0.2 & 0.4 & 0.3 & 0.1 \end{bmatrix}$.

**a** Is there a stable-state matrix?

**b** What is the probability of changing from state 3 to state 2?

**c** What is the probability of changing from state 2 to state 1?

**d** Given $\mathbf{X}_0 = [0.25\ 0.25\ 0.25\ 0.25]$, find $\mathbf{X}_1$ and $\mathbf{X}_2$

Example 18

**12** The interaction between the service and transport sectors of an economy is shown in the following table.

| | | **Inputs** | | **Final demands** | **Gross outputs** |
|---|---|---|---|---|---|
| | | **Service** | **Transport** | | |
| **Out puts** | Service | 300 | 600 | 800 | 1700 |
| | Transport | 400 | 200 | 900 | 1500 |

**a** Determine the primary inputs.

**b** Write the input–output matrix **A**.

**c** Find the new demand matrix if the demands for service and transport change to 1000 and 1200 respectively.

**d** Find the new production matrix.

**e** Find the new primary inputs matrix.

## Problem solving

**13** 4 trucks with mixed loads enter a council hard fill site and dump their loads. Each truck can hold 6 $m^3$ of material, and the drivers are able to indicate the amount of each material in their loads. The results are given below.
Find the densities of concrete, gravel and soil (in kg/$m^3$).

| Truck | Entry weight | Exit weight | Concrete ($m^3$) | Gravel ($m^3$) | Soil ($m^3$) |
|---|---|---|---|---|---|
| 1 | 18 400 | 2150 | 2 | 3 | 0.5 |
| 2* | 20 650 | 2350 | 2.5 | 2 | 2 |
| 3 | 17 600 | 3100 | 2 | 2 | 1 |
| 4 | 19 000 | 2750 | 2 | 1.5 | 2.5 |

* Truck 2 was overloaded. The driver was given a second warning.

**14** Different types of solder are alloys of different proportions of tin, copper and lead. The specific gravities (density relative to water) of tin, copper and lead are 8, 6 and 14 respectively, while the specific gravities of three different solders are 9.6, 9.7 and 9.1. Find the percentage composition of each solder given that the first two have 40% lead and the last is 50% copper.

**15** The Beavers, Cats, Hawks, Panthers, Saints and Tigers have played 2 rounds of a football competition. The results of each round are shown in the table below as win, draw or loss.

| | Beavers | Cats | Hawks | Panthers | Saints | Tigers |
|---|---|---|---|---|---|---|
| Beavers | | W W | W D | W D | W W | D W |
| Cats | L L | | D L | D D | W L | D D |
| Hawks | L D | D W | | L L | D W | L L |
| Panthers | L D | D D | W W | | W D | L L |
| Saints | L L | L W | D L | L D | | L L |
| Tigers | D L | D D | W W | W W | W W | |

Rank all teams from 1st to 6th.


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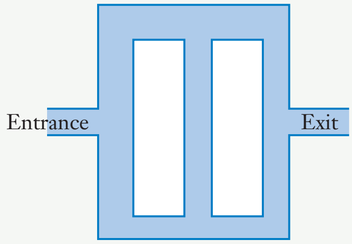
**16** Like many other exotic species, camels have bred in the wild in Australia. Nowadays, they form the only sizeable population of wild camels in the world. The following table gives some estimated population data for the Northern Territory.

| Age (years) | 0–5 | 5–10 | 10–15 | 15–20 | 20–25 | 25–30 |
|---|---|---|---|---|---|---|
| Population | 200 000 | 80 000 | 60 000 | 40 000 | 20 000 | 10 000 |
| Birth rate | 0 | 1.2 | 2.1 | 0.9 | 0.5 | 0 |
| Death rate | 0.6 | 0.3 | 0.2 | 0.4 | 0.6 | 1 |

Suggest suitable strategies to reduce the camel population or sustainably harvest them for export.

**17** An artificial intelligence research robot is programmed to run through the simple maze of 'streets' shown. At T-junctions, the robot never performs a U-turn and there is an equal chance of it going each way. At the corners, it just proceeds around the corner. The robot is put into the entrance. Find the long-term probabilities of the robot emerging from the entrance and exit.




Practice quiz

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COMPLEX NUMBERS 2

# 5. COMPLEX POLYNOMIALS AND ROOTS

While evidence of imaginary numbers dates back to the 1st century AD in the work of Heron of Alexandria, the algebra of complex numbers was not developed until much later.

In the mid 1500s, Gerolamo Cardano documented the solution of third and fourth degree equations that required the existence of $\sqrt{-1}$. However, the acceptance of the legitimate role of complex numbers took many hundreds of years. Over this time, the algebra and calculus of complex numbers developed into extensive fields of study.

In modern times, complex numbers play a vital role in many fields including computer science, electrical engineering, scientific computing, digital electronics and radio astronomy.

5.01 Subsets of the complex plane
5.02 The roots of unity
5.03 Roots of complex numbers
5.04 The remainder theorem
5.05 The factor theorem
5.06 Complex polynomial equations
Chapter summary
Chapter review

## The complex plane (the Argand plane)

- identify subsets of the complex plane determined by straight lines and circles.

## Roots of complex numbers

- determine and examine the $n$th roots of unity and their location on the unit circle
- determine and examine the $n$th roots of complex numbers and their location in the complex plane

## Factorisation of polynomials

- prove and apply the factor theorem and the remainder theorem for polynomials
- consider conjugate roots for polynomials with real coefficients
- solve polynomial equations to order 4

Prior learning



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## TERMINOLOGY

complex polynomial
conjugate root theorem
constant term
degree
dividend
divisor
factor theorem
Fundamental Theorem of Algebra
leading term
locus
multiplicity
quartic
quotient
remainder
remainder theorem
roots of unity
roots of a complex number

# 5.01 Subsets of the complex plane

WS Complex plane graphs

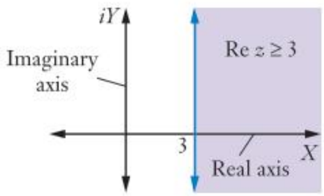
A **locus** (plural **loci**) may be defined as a set of points in a plane that satisfy some given condition. It may help you to think of a locus as a 'path' traced out by a point moving according to some definite law. Often, the locus will be a set of points that is familiar to you. For example, the set of points 2 units from the origin is a circle with radius 2 units.

In order to describe a locus, we could use words or an algebraic equation. In fact, the equation that describes the locus is usually taken to be the locus.

You know that a complex number is any number of the form $z = x + yi$ where $x, y \in \boldsymbol{R}$ and $i = \sqrt{-1}$.

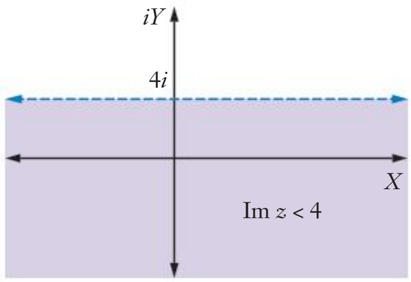
The real part of the complex number is $\text{Re}(z) = x$, while the imaginary part is $\text{Im}(z) = y$.

A relationship such as $\text{Re}(z) \geq 3$ can be represented on an Argand diagram as shown.




This is a locus problem as it involves a set of points that satisfy a given condition.

Consider the sketch of $\text{Im}(z) < 4$ shown at right. Stated more formally as a locus problem, this diagram represents the set of all points $z = x + yi$, such that $\text{Im}(z) < 4$.




When dealing with the graphs of complex numbers, $z = x + yi$ and $z(x, y)$ are used interchangeably.

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Some locus problems deal with the argument of a complex number, as shown in the following example.

## EXAMPLE 1

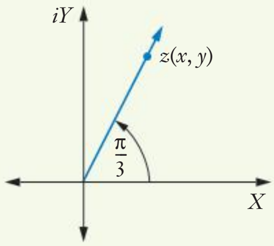
Sketch the graph of the locus of $z(x, y)$ if:

**a** $0 \le \arg(z) \le \frac{\pi}{3}$

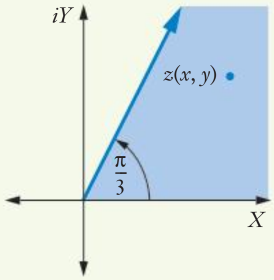
**b** $0 \le \arg(z + 2 - i) \le \frac{\pi}{3}$.

### Solution

**a** $\text{Arg}(z) = \frac{\pi}{3}$ represents all points $z(x, y)$ on the ray that makes an angle of $\frac{\pi}{3}$ with the positive direction of the real axis, as shown in the diagram.



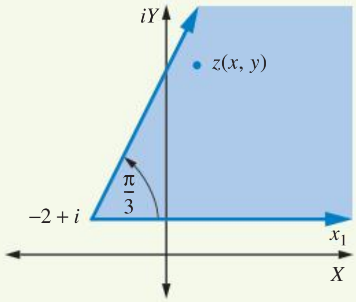

The locus of $z(x, y)$ satisfying $0 \le \arg(z) \le \frac{\pi}{3}$ is the set of all points within the angular region formed by the ray and the real axis, including the boundaries, as shown here.



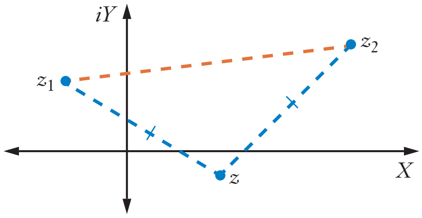

**b** Rearrange $0 \le \arg(z + 2 - i) \le \frac{\pi}{3}$.

$0 \le \arg[z - (-2 + i)] \le \frac{\pi}{3}$

The sketch of the locus in this case is similar to the previous sketch, except that the vertex of the angle has been moved −2 units in the real-axis direction and 1 unit in the imaginary-axis direction.



Consider the complex numbers $z$, $z_1$ and $z_2$ and their representation on an Argand diagram, as shown at right.



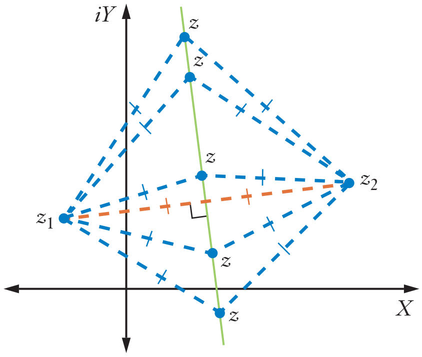

Now consider the locus of $z(x, y)$ such that

$|z - z_1| = |z - z_2|$.

This means that $z$ is equidistant from $z_1$ and $z_2$.

You can see that many such arrangements of $z$, $z_1$ and $z_2$ satisfy the locus condition.

Using basic geometry, it can be shown that the locus of $z$ is the perpendicular bisector of the line drawn through $z_1$ and $z_2$.




## EXAMPLE 2

Describe and sketch the locus of $z(x, y)$ if $|z - 3| > |z + 2 - i|$.

### Solution

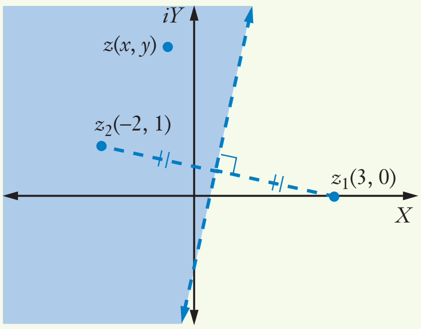
Write the condition. $|z - 3| > |z + 2 - i|$

Write it in the form $|z - z_1| = |z - z_2|$. $|z - 3| > |z - (-2 + i)|$

Identify $z_1$ and $z_2$. $z_1 = 3 + 0i$ and $z_2 = -2 + i$

In this case, $|z - z_1| = |z - z_2|$ represents the perpendicular bisector of the line joining $z_1(3, 0)$ and $z_2(-2, 1)$.

$|z - z_1| > |z - z_2|$ represents the set of all points closer to $z_2(-2, 1)$ than to $z_1(3, 0)$.




Describe the locus. The locus of $z(x, y)$ is the region on the left-hand side of the perpendicular bisector of $z_1z_2$, where $z_1 = 3 + 0i$ and $z_2 = -2 + i$, as shown above.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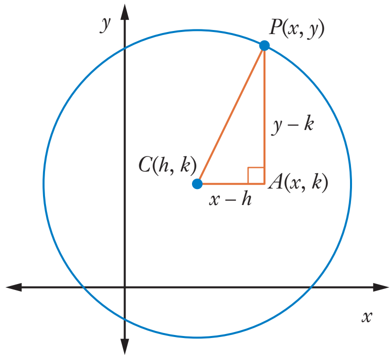
A circle is the set of all points $P$ in a plane that are equidistant from a fixed point $C$ (the centre).




Consider the point $C(h, k)$ and the point $P(x, y)$ equidistant from $C$, as shown at right. You can see that the locus of $P$ is the circumference of a circle.

To find the equation of the locus of $P$, you follow the procedure previously established.

Using Pythagoras' theorem and $\Delta CAP$:

$CA^2 + AP^2 = CP^2$.

Now, using the coordinates of these points:

$(x - h)^2 + (y - k)^2 = CP^2$.

By substituting $r$ for $CP$, you obtain the locus of $P$.

### The Cartesian form of a circle

A circle with centre $(h, k)$ and radius $r$ has the Cartesian equation:

$$(x - h)^2 + (y - k)^2 = r^2$$

It is also possible to represent a circle in the complex plane.

Consider the locus of $z(x, y)$ such that $|z| \leq 3$.
$|z| = 3$ represents all points $z(x, y)$ that are 3 units from the origin, $(0 + 0i)$.

So $\quad |z| = 3 \Rightarrow \sqrt{x^2 + y^2} = 3$

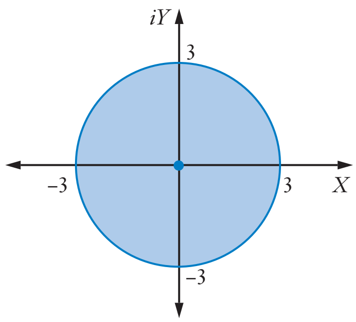
Square both sides. $\quad x^2 + y^2 = 3^2$

This represents a circle with centre the origin and radius 3 units.

For the locus of $z(x, y)$ such that $|z| \leq 3$.

$$|z| \leq 3 \Rightarrow \sqrt{x^2 + y^2} \leq 3$$
$$\Rightarrow x^2 + y^2 \leq 3^2$$

The equation represents all points within and on the boundary of the circle centred at the origin with a radius of 3 units, as shown in the diagram.

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You can state the equation of a circle in the complex plane for centres other than the origin $(0 + 0i)$ by using the fact that $|z_2 - z_1|$ is the distance between $z_1$ and $z_2$. A circle has a constant distance $r$ between a fixed point $z_1$ and the general point $z$ on the circle.

## The complex form of a circle

A circle in the complex plane with centre $z_1 = x_1 + y_1 i$ and radius $r$ has the equation

$$|z - z_1| = r$$

You can also show regions in the complex plane using variations of the equation of a circle to shade inside or outside the circle. For ≤ or ≥, the boundary will be included.

## EXAMPLE 3

Sketch the locus of $z(x, y)$ such that $|z + 1 - 2i| > 5$.

### Solution

Write the inequality. $|z + 1 - 2i| > 5$

Write $z$ in terms of $x$ and $y$. $|x + yi + 1 - 2i| > 5$

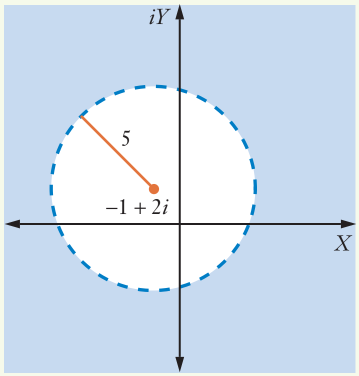
Rearrange. $|(x + 1) + (y - 2)i| > 5$

Remove the modulus signs. $\sqrt{(x+1)^2 + (y-2)^2} > 5$

Square both sides. $(x + 1)^2 + (y - 2)^2 > 5^2$

The equation $(x + 1)^2 + (y - 2)^2 = 5^2$ represents a circle with centre $(-1 + 2i)$ and radius 5 units.

So the locus represented by $(x + 1)^2 + (y - 2)^2 > 5^2$ is the region outside the boundary of the circle with centre $(-1 + 2i)$ and radius 5 units, as shown.





ISBN 9780170413060

## Exercise 5.01 Subsets of the complex plane

Example 1

**1** Sketch the graph of each locus of $z(x, y)$ described.

**a** $\arg(z) = \frac{2\pi}{3}$ **b** $\arg(z) = \frac{-\pi}{4}$ **c** $\arg(z) = \frac{-\pi}{2}$

**d** $\arg(z) = 0$ **e** $0 < \arg(z) \le \frac{\pi}{2}$ **f** $0 \le \arg(z) < \frac{3\pi}{4}$

**g** $\frac{-\pi}{6} \le \arg(z) \le 0$ **h** $\frac{-5\pi}{3} \le \arg(z) < 0$

**2** Sketch the graph of each locus of $z(x, y)$ described.

**a** $\arg(z-1) = \frac{\pi}{3}$ **b** $\arg(z-i) = \frac{\pi}{2}$ **c** $\arg(z-2) = \frac{-2\pi}{3}$

**d** $\arg[z-(1+i)] = \frac{-\pi}{6}$ **e** $\arg[z-(1-2i)] = \frac{2\pi}{3}$ **f** $\arg(z-2i) = \frac{5\pi}{4}$

**g** $\arg[z-(-3)] = \frac{-3\pi}{2}$ **h** $\arg[z-(-1+i)] = \pi$ **i** $0 < \arg(z+i) \le \frac{\pi}{4}$

**j** $\frac{-3\pi}{4} \le \arg(z-i) < 0$ **k** $\frac{-\pi}{4} < \arg(z+3) \le 0$ **l** $0 \le \arg(z-2i) < \frac{\pi}{6}$

**m** $0 < 2\arg(z+2) \le \frac{\pi}{2}$ **n** $\frac{-2\pi}{3} \le \arg(z+2-i) < 0$ **o** $0 < \arg(z-3-i) \le \frac{\pi}{3}$

**p** $\arg(z-2+i) = 0$

Example 2

**3** Describe and sketch each locus of $z = x + yi$ described.

**a** $|z-2| = |z-2i|$ **b** $|z+2i| = |z-1-2i|$ **c** $|z-2+i| = |z+2-i|$

**d** $|z| = |z+1-i|$ **e** $|z+2-3i| < |z+i|$ **f** $|z-1| \ge |z-2i|$

**g** $|z+1| < |z+2-i|$ **h** $|z+1-2i| \le |z+2-2i|$

**i** $|z+2i| > |z-1-2i|$

**4** Sketch the graph of each locus of $z(x, y)$ described.

**a** $|z| > 3$ **b** $|z| > 1$ **c** $|z| = 2$

**d** $|z| < 5$ **e** $|z| \ge 2$ **f** $|z| \le 4$

**5** Find the coordinates of the centre and the radius of the circle for each locus described.

**a** $|z-2| = 5$ **b** $|z-i| = 3$ **c** $\left|z-\left(1+i\sqrt{5}\right)\right| = 2$

**d** $|z-(1-i)| = 6$ **e** $\left|z-\frac{-1+i}{2}\right| \ge \frac{5}{2}$ **f** $|z-2i| \le 2$

Example 3

**6** Sketch the graph of each locus of $z(x, y)$ described.

**a** $|z-1| = 2$ **b** $|z-i| \le 1$ **c** $|z-(1+i)| = 4$

**d** $|z-(1-2i)| < 1$ **e** $|z-(-1+i)| \ge \sqrt{2}$ **f** $|z-5| = 0$

**g** $|z-(-3)| \le 2$ **h** $|z+2| > 5$ **i** $|z+3i| \le 3$

**j** $|z-2+i| < 1$ **k** $|z+3-4i| = 2$ **l** $2|z+2i| = 6$

Problem solving

**7** Sketch the locus of $z(x, y)$ such that $|z| \leq 2$ and $0 \leq \arg(z) < \frac{\pi}{6}$.

**8** Sketch the locus of $z(x, y)$ such that $|z - 1| < 1 < |z|$.

**9** Sketch the locus of $z(x, y)$ such that $2 < |z| \leq 5$ and $-\frac{\pi}{6} \leq \arg(z) < \frac{\pi}{3}$.

**10** If $z \in \boldsymbol{C}$ and $z$ is represented by the point $P(x, y)$, find the Cartesian equation of the locus of $P$ if $|z - 1 - i| = |z - 2|$.

**11** If $z \in \boldsymbol{C}$ and $z$ is represented by the point $P(x, y)$, describe the locus of $P$ where

$$|z - 3i|^2 = 2|z - \bar{z}|.$$

# 5.02 The roots of unity

## INVESTIGATION

### ROOTS OF UNITY

Consider the equation $x^n = 1$ for positive integers $n$.

- Will an equation of this form always have a real solution? Explain your reasoning.
- Could an equation of this form have two real solutions? Explain your reasoning.
- An equation of this form may have complex solutions. How many are there for an equation of this form? Explain your answer.

Consider the equation $x^3 = 1$ for positive integers $n$.

- Draw the graph of $f(x) = x^3 - 1$.
- Use this graph to explain why $x^3 = 1$ has only one real solution.
- Use an algebraic method to find the complex roots of $x^3 = 1$.
- Would the method you developed to solve $x^3 = 1$ be useful in finding the complex roots of, say, $x^7 = 1$? Explain your answer.

If you find the cube root of unity (1), you find the number $x$ such that $x^3 = 1$. In the real number system, this equation has only 1 solution.

$x^3 = 1 \Rightarrow x = \sqrt[3]{1} = 1$

But what about the complex number system?

You know the complex number $z$ can be written as $z = r[\cos(\alpha) + i\sin(\alpha)]$.

You can add an integer multiple of $2\pi$ to the argument of $z$ without changing the number, so:

$$z = r[\cos(\alpha) + i\sin(\alpha)] = r[\cos(\alpha + 2k\pi) + i\sin(\alpha + 2k\pi)], \text{ where } k \in \boldsymbol{Z}.$$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You know that $1 = 1 \text{ cis } (0) = 1[\cos (0) + i \sin (0)]$.

So, to solve $z^3 = 1$, you write

$$z^3 = 1[\cos (2k\pi) + i \sin (2k\pi)] \quad [1]$$

Let the solution be $z = r[\cos (\theta) + i \sin (\theta)]$.

Using De Moivre's theorem,

$$z^3 = r^3[\cos (3\theta) + i \sin (3\theta)] \quad [2]$$

Equating [1] and [2] gives:

$$r^3[\cos (3\theta) + i \sin (3\theta)] = 1[\cos (2k\pi) + i \sin (2k\pi)]$$

This means that $r^3 = 1$ and $3\theta = 2k\pi$.

So $r = 1$ and $\theta = \dfrac{2k\pi}{3}$, where $k \in \mathbf{Z}$.

This means that possible solutions are:

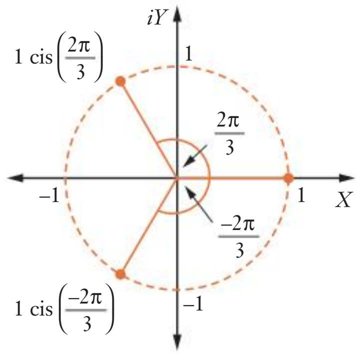
$$\ldots, 1 \text{ cis}\left(\frac{-4\pi}{3}\right), 1 \text{ cis}\left(\frac{-2\pi}{3}\right), 1 \text{ cis } (0\pi), 1 \text{ cis}\left(\frac{2\pi}{3}\right), 1 \text{ cis}\left(\frac{4\pi}{3}\right), \ldots$$

There are an infinite number of solutions as the solutions repeat every $2\pi$, so only answers whose argument is the principal argument, i.e., $-\pi < \dfrac{2k\pi}{3} \leq \pi$ must be chosen.

So the cube roots of unity are:

$$1 \text{ cis}\left(\frac{-2\pi}{3}\right), 1 \text{ cis } (0\pi), 1 \text{ cis}\left(\frac{2\pi}{3}\right)$$

or $\cos\left(\dfrac{-2\pi}{3}\right) + i\sin\left(\dfrac{-2\pi}{3}\right)$, 1, $\cos\left(\dfrac{2\pi}{3}\right) + i\sin\left(\dfrac{2\pi}{3}\right)$




The cube roots of unity may be plotted on an Argand diagram as shown on the right.

You can see that the roots all lie on the unit circle.

The $n$th roots of unity may be defined more generally as follows.

### *n*th roots of unity

An $n$th **root of unity**, where $n \in \mathbf{Z}^+$, is a number $z$ satisfying the equation $z^n = 1$.

The $n$th roots of unity lie on the unit circle and are given by:

$$z = \text{cis}\left(\frac{2k\pi}{n}\right) = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$$

where $k = 0, 1, 2, 3, \ldots, n - 1$ and $-\pi < \dfrac{2k\pi}{n} \leq \pi$.

## EXAMPLE 4

**a** Solve $z^5 = 1$.

**b** Show the roots of $z^5 = 1$ on the unit circle.

**c** Describe the positions of the roots on the unit circle.

### Solution

**a** Write the rule for the 5th root.

$$z = \cos\left(\frac{2k\pi}{5}\right) + i\sin\left(\frac{2k\pi}{5}\right)$$

State the values of $k$.

$$k = 0, 1, 2, 3, 4$$

Let $k = 0$.

$$z_0 = \cos\left(\frac{2\times 0\pi}{5}\right) + i\sin\left(\frac{2\times 0\pi}{5}\right)$$
$$= \cos(0) + i\sin(0)$$

Let $k = 1$.

$$z_1 = \cos\left(\frac{2\times 1\pi}{5}\right) + i\sin\left(\frac{2\times 1\pi}{5}\right)$$
$$= \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)$$

Let $k = 2$.

$$z_2 = \cos\left(\frac{2\times 2\pi}{5}\right) + i\sin\left(\frac{2\times 2\pi}{5}\right)$$
$$= \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right)$$

Let $k = 3$.

$$z_4 = \cos\left(\frac{2\times 3\pi}{5}\right) + i\sin\left(\frac{2\times 3\pi}{5}\right)$$
$$= \cos\left(\frac{6\pi}{5}\right) + i\sin\left(\frac{6\pi}{5}\right)$$

Use the principal argument $-\pi < \frac{2k\pi}{n} \leq \pi$.

$$= \cos\left(-\frac{4\pi}{5}\right) + i\sin\left(-\frac{4\pi}{5}\right)$$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Let $k = 4$.

$$z_5 = \cos\left(\frac{2\times 4\pi}{5}\right) + i\sin\left(\frac{2\times 4\pi}{5}\right)$$

$$= \cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right)$$

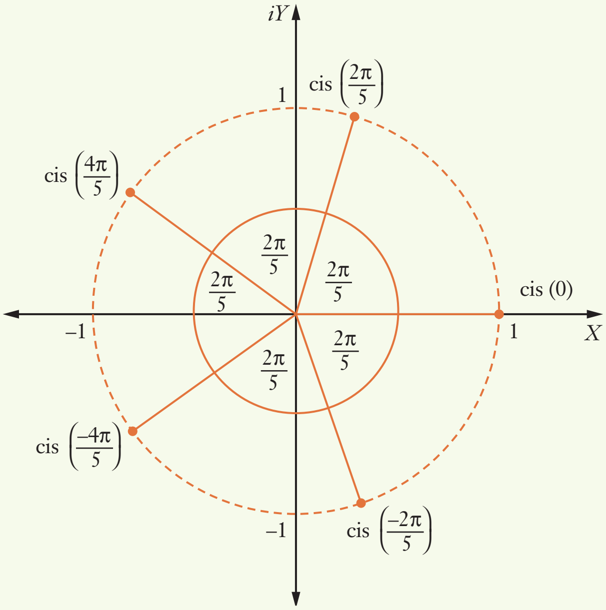
Use the principal argument.

$$= \cos\left(-\frac{2\pi}{5}\right) + i\sin\left(-\frac{2\pi}{5}\right)$$

Write the result.

The roots of $z^5 = 1$ are $\text{cis}\left(\frac{-4\pi}{5}\right)$, $\text{cis}\left(\frac{-2\pi}{5}\right)$, 1, $\text{cis}\left(\frac{2\pi}{5}\right)$, $\text{cis}\left(\frac{4\pi}{5}\right)$.

**b** Represent the roots on the unit circle.




**c** Describe the positions of the roots.

The 5 roots of $z^5 = 1$ are equally spaced on the unit circle. One root is 1 and the remaining adjacent roots are separated by an angle of $\frac{2\pi}{5}$.

You can use a graphics calculator to find the roots of unity.

## EXAMPLE 5

Solve $z^3 = 1$ using a graphics calculator.

### Solution

For both calculators, rearrange the equation $z^3 = 1$ to $z^3 - 1 = 0$.

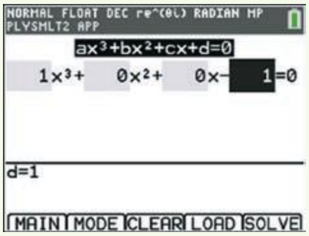
TI-Nspire CX Chapter 5

#### TI-84 Plus CE

Press [apps] and then select 8: PlySmt2.
Select 1: POLYNOMIAL ROOT FINDER.

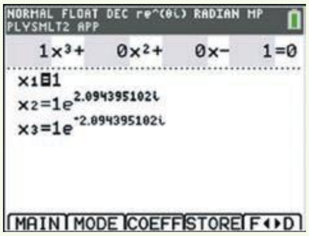
Then select 3 for ORDER as it is a cubic equation, select re^($\theta i$) for the polar form of $z$ and also select RADIAN. Press [graph] (NEXT) to exit the screen.

Now set $a = 1$, accept the '+' sign, set $b = 0$, accept the '+' sign, set $c = 0$, change the sign to '–' and set $d = 1$.




Press [graph] (SOLVE) to display the answers.
Remember that for this calculator $r$ cis $(\theta) = re^{\theta i}$.

Also note that $\frac{2\pi}{3} = 2.094\,395\ldots$

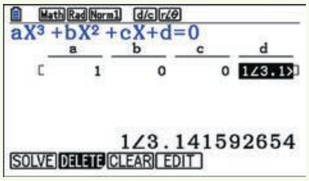



#### Casio fx-CG20AU

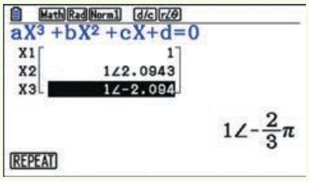
Select the Equation menu and return to the home screen if necessary.

Press [SHIFT] [MENU] (SET UP). Set the Angle to radians (Rad) and the Complex Mode to polar form ($r\angle\ \theta$). Press [EXE].

Press [F2] (Polynomial), [F2] (degree = 3) and then enter $a = 1$, $b = 0$, $c = 0$ and $d = -1$.




Press [F1] (SOLVE) to display the solutions.




**Both calculators**

State the result.

The cube roots of unity are $\text{cis}\left(\frac{-2\pi}{3}\right)$, 1 and $\text{cis}\left(\frac{2\pi}{3}\right)$.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## Exercise 5.02 The roots of unity

Example 4

**1** **a** Solve $z^4 = 1$.

**b** Show the roots of $z^4 = 1$ on the unit circle.

**c** Describe the positions of the roots on the unit circle.

**2** For the fourth roots of unity, prove that:

**a** the sum of all fourth roots of unity is 0

**b** every real fourth root of unity is the additive inverse of the other

**c** each complex fourth root of unity is the conjugate of the other

**d** the product of all fourth roots of unity is –1

**3** **a** Find the seventh roots of unity.

**b** Show the seventh roots of unity on the unit circle.

**c** Describe the positions of the roots on the unit circle.

Example 5

**4** **a** Find the sixth roots of 1 in Cartesian form.

**b** Express the sixth roots of 1 in polar from.

**5** For the eighth roots of unity, $z_0, z_1, z_2, \ldots, z_7$, calculate:

**a** $z_0 + z_1 + z_2 + \ldots + z_7$

**b** $z_0 z_1 z_2 \ldots z_7$

**c** $|z_0| + |z_1| + |z_2| + \ldots + |z_7|$

### Problem solving

**6** For the seventh roots of unity, $z_0, z_1, z_2, \ldots, z_6$, prove that:

**a** $z_0 + z_1 + z_2 + \ldots + z_6 = 0$

**b** the complex roots are conjugate pairs

**c** $z_0 z_1 z_2 \ldots z_6 = 1$

**d** $|z_0| + |z_1| + |z_2| + \ldots + |z_6| = 7$

**7** **a** Given that $z \in \mathbf{C}$, use an algebraic method to find the roots of $z^3 = 27$ in Cartesian form.

**b** Use the previous result to prove that:

**i** the sum of all cube roots of unity is zero

**ii** the product of all cube roots of unity is 1

**iii** each complex cube root of unity is the square of the other

**8** Given that $z \in \mathbf{C}$, use an algebraic method to find the roots of $z^4 = 81$ in Cartesian form.

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**9** **a** Show that, if $w$ denotes either of the cube roots of 1 that is not 1 itself, then the other cube root of 1 (that is not 1) is $w^2$.

**b** Show that a complex number $w$ exists such that the fifth roots may be expressed as $1, w, w^2, w^3$ and $w^4$.

**10** If $\beta$ is a complex root of $z^5 - 1 = 0$, show that $\beta^2, \beta^3$ and $\beta^4$ are the other complex roots, and plot all roots on an Argand diagram.

**11** If $\alpha$ is a complex root of $z^7 - 1 = 0$, show that $\alpha^2, \alpha^3, \alpha^4, \alpha^5$ and $\alpha^6$ are the other complex roots, and plot all roots on an Argand diagram.

WS

Roots of complex numbers

## 5.03 Roots of complex numbers

You can use De Moivre's theorem to find the $n$th roots of a complex number, $z$.

The $n$th **root of a complex number** $z$ is a complex number $w$ such that:

$$w^n = z$$

Write $w$ and $z$ in trigonometric (polar) form.

$$w = s[\cos(\alpha) + i\sin(\alpha)] \text{ and } z = r[\cos(\theta) + i\sin(\theta)]$$

Using De Moivre's theorem, you get

$$w^n = z$$

$$\Rightarrow s^n[\cos(n\alpha) + i\sin(n\alpha)] = r[\cos(\theta) + i\sin(\theta)]$$

The equality of the complex number shows that:

$$s^n = r \Rightarrow s = r^{\frac{1}{n}}$$

and $\cos(n\alpha) = \cos(\theta)$ and $\sin(n\alpha) = \sin(\theta)$

Using the fact that sine and cosine have a period of $2\pi$, you can write:

$$n\alpha = \theta + 2k\pi \Rightarrow \alpha = \frac{\theta + 2k\pi}{n}$$

$$\text{So } w = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

### The *n*th roots of a complex number

Given that $z = r[\cos(\theta) + i\sin(\theta)]$ where $n \in Z^+$ and $w^n = z$, then $z$ has $n$ distinct $n$th roots:

$$w_k = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

where $k = 0, 1, 2, \ldots, n - 1$


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From the definition above you can see that because $|w_k| = r^{\frac{1}{n}}$, the $n$th roots of $z$ lie on the circle with origin (0, 0) and radius $r^{\frac{1}{n}}$ on the complex number plane. It is also clear that the argument of each successive $n$th root is $\frac{2\pi}{n}$ more than the previous root, so that all of the $n$th roots of $z$ are equally spaced on this circle.

## EXAMPLE 6

Find the sixth roots of –27 and show the roots in the complex plane.

### Solution

Write –27 in trigonometric form.

$$z = r[\cos(\theta) + i\sin(\theta)]$$

$$-27 = 27[\cos(\pi) + i\sin(\pi)]$$

Let $w$ be the $n$th root of $z$.

$$w^n = z$$

Use the formula for the $n$th root.

$$w_k = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

Use the values $r = 27$, $n = 6$ and $\theta = \pi$.

$$w_k = \sqrt[6]{27}\left[\cos\left(\frac{\pi + 2k\pi}{6}\right) + i\sin\left(\frac{\pi + 2k\pi}{6}\right)\right]$$

Let $k = 0$.

$$w_0 = \sqrt{3}\left[\cos\left(\frac{\pi + 0}{6}\right) + i\sin\left(\frac{\pi + 0}{6}\right)\right]$$

$$= \sqrt{3}\text{ cis}\left(\frac{\pi}{6}\right)$$

Let $k = 1$.

$$w_1 = \sqrt{3}\left[\cos\left(\frac{\pi + 2\times 1\pi}{6}\right) + i\sin\left(\frac{\pi + 2\times 1\pi}{6}\right)\right]$$

Simplify.

$$= \sqrt{3}\text{ cis}\left(\frac{\pi}{2}\right)$$

$$w_1 = i\sqrt{3}$$

Let $k = 2$.

$$w_2 = \sqrt{3}\left[\cos\left(\frac{\pi + 2\times 2\pi}{6}\right) + i\sin\left(\frac{\pi + 2\times 2\pi}{6}\right)\right]$$

Simplify.

$$= \sqrt{3}\text{ cis}\left(\frac{5\pi}{6}\right)$$

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Let $k = 3$.

$$w_3 = \sqrt{3}\left[\cos\left(\frac{\pi + 2\times 3\pi}{6}\right) + i\sin\left(\frac{\pi + 2\times 3\pi}{6}\right)\right]$$

Simplify.

$$= \sqrt{3}\ \text{cis}\left(\frac{7\pi}{6}\right)$$

Write using the principal argument.

$$= \sqrt{3}\ \text{cis}\left(-\frac{5\pi}{6}\right)$$

Let $k = 4$.

$$w_4 = \sqrt{3}\left[\cos\left(\frac{\pi + 2\times 4\pi}{6}\right) + i\sin\left(\frac{\pi + 2\times 4\pi}{6}\right)\right]$$

Simplify.

$$= \sqrt{3}\ \text{cis}\left(\frac{3\pi}{2}\right)$$

Write using the principal argument.

$$= \sqrt{3}\ \text{cis}\left(-\frac{\pi}{2}\right)$$

$$w_4 = -i\sqrt{3}$$

Let $k = 5$.

$$w_5 = \sqrt{3}\left[\cos\left(\frac{\pi + 2\times 5\pi}{6}\right) + i\sin\left(\frac{\pi + 2\times 5\pi}{6}\right)\right]$$

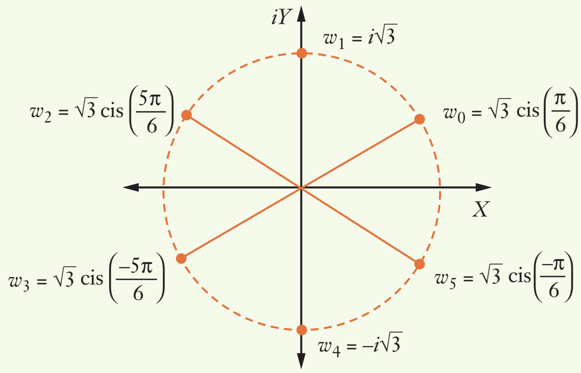
Simplify.

$$= \sqrt{3}\ \text{cis}\left(\frac{11\pi}{6}\right)$$

Write using the principal argument.

$$= \sqrt{3}\ \text{cis}\left(-\frac{\pi}{6}\right)$$

Represent $w_0, w_1, \ldots, w_5$ on an Argand diagram.





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## EXAMPLE 7

Calculate the fifth roots of $32i$ and show the roots on an Argand diagram.

### Solution

Write $32i$ in trigonometric form.

$$z = 32i = 32\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right]$$

Let $w$ be the $n$th root of $z$.

$$w^n = z$$

Use the formula for the $n$th root.

$$w_k = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

Use the values $r = 32$, $n = 5$ and $\theta = \frac{\pi}{2}$.

$$w_k = \sqrt[5]{32}\left[\cos\left(\frac{\frac{\pi}{2} + 2k\pi}{5}\right) + i\sin\left(\frac{\frac{\pi}{2} + 2k\pi}{5}\right)\right]$$

Simplify.

$$= 2\left[\cos\left(\frac{\pi + 4k\pi}{10}\right) + i\sin\left(\frac{\pi + 4k\pi}{10}\right)\right]$$

Let $k = 0$.

$$w_0 = 2\left[\cos\left(\frac{\pi + 0}{10}\right) + i\sin\left(\frac{\pi + 0}{10}\right)\right]$$

$$= 2 \text{ cis}\left(\frac{\pi}{10}\right)$$

Let $k = 1$.

$$w_1 = 2\left[\cos\left(\frac{\pi + 4 \times 1\pi}{10}\right) + i\sin\left(\frac{\pi + 4 \times 1\pi}{10}\right)\right]$$

Simplify.

$$= 2 \text{ cis}\left(\frac{5\pi}{10}\right)$$

$$= 2 \text{ cis}\left(\frac{\pi}{2}\right)$$

$$= 2i$$

Let $k = 2$.

$$w_2 = 2\left[\cos\left(\frac{\pi + 4 \times 2\pi}{10}\right) + i\sin\left(\frac{\pi + 4 \times 2\pi}{10}\right)\right]$$

Simplify.

$$= 2 \text{ cis}\left(\frac{9\pi}{10}\right)$$

Let $k = 3$. $w_3 = 2\left[\cos\left(\frac{\pi + 4 \times 3\pi}{10}\right) + i\sin\left(\frac{\pi + 4 \times 3\pi}{10}\right)\right]$

Simplify. $= 2 \text{ cis}\left(\frac{13\pi}{10}\right)$

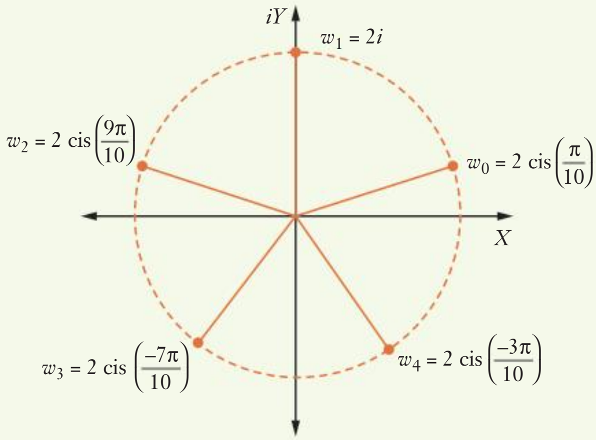
Write using the principal argument. $= 2 \text{ cis}\left(-\frac{7\pi}{10}\right)$

Let $k = 4$. $w_4 = 2\left[\cos\left(\frac{\pi + 4 \times 4\pi}{10}\right) + i\sin\left(\frac{\pi + 4 \times 4\pi}{10}\right)\right]$

Simplify. $= 2 \text{ cis}\left(\frac{17\pi}{10}\right)$

Write using the principal argument. $= 2 \text{ cis}\left(-\frac{3\pi}{10}\right)$

Represent $w_0, w_1, \ldots, w_4$ on an Argand diagram.




## EXAMPLE 8

Solve the equation $w^3 + 2i\sqrt{3} = 2$ and show the roots on the complex plane.

### Solution

Rearrange the equation. $w^3 = 2 - 2i\sqrt{3}$

Convert $2 - 2i\sqrt{3}$ to polar form. $= 4\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]$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Use the formula for the $n$th root.

$$w_k = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

Use the values $r = 4$, $n = 3$ and $\theta = -\dfrac{\pi}{3}$.

$$w_k = \sqrt[3]{4}\left[\cos\left(\frac{-\frac{\pi}{3} + 2k\pi}{3}\right) + i\sin\left(\frac{-\frac{\pi}{3} + 2k\pi}{9}\right)\right]$$

Simplify.

$$= \sqrt[3]{4}\left[\cos\left(\frac{6k\pi - \pi}{9}\right) + i\sin\left(\frac{6k\pi - \pi}{9}\right)\right]$$

Let $k = 0$.

$$w_0 = \sqrt[3]{4}\left[\cos\left(\frac{6\times 0\pi - \pi}{9}\right) + i\sin\left(\frac{6\times 0\pi - \pi}{9}\right)\right]$$

$$= \sqrt[3]{4}\text{ cis}\left(-\frac{\pi}{9}\right)$$

Let $k = 1$.

$$w_1 = 4\left[\cos\left(\frac{6\times 1\pi - \pi}{9}\right) + i\sin\left(\frac{6\times 1\pi - \pi}{9}\right)\right]$$

$$= \sqrt[3]{4}\text{ cis}\left(\frac{5\pi}{9}\right)$$

Let $k = 2$.

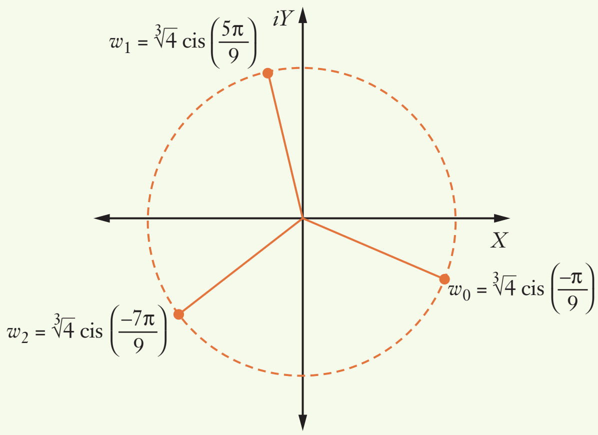
$$w_2 = \sqrt[3]{4}\left[\cos\left(\frac{6\times 2\pi - \pi}{9}\right) + i\sin\left(\frac{6\times 2\pi - \pi}{9}\right)\right]$$

$$= \sqrt[3]{4}\text{ cis}\left(\frac{11\pi}{9}\right)$$

Write using the principal argument.

$$= \sqrt[3]{4}\text{ cis}\left(-\frac{7\pi}{9}\right)$$

Represent the roots on an Argand diagram.

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## Exercise 5.03 Roots of complex numbers

Example 6

**1** Find the fourth roots of 64 and show the roots in the complex plane.

**2** Find the fifth roots of –243 and show the roots in the complex plane.

**3** Find all the complex numbers for which $z^3 = -8$ and show them on an Argand diagram.

Example 7

**4** Calculate the fifth roots of $-32i$ and show the roots on an Argand diagram.

**5** Calculate the sixth roots of $125i$ and show the roots on an Argand diagram.

**6** Solve $z^3 = 27i$ and show the roots using an Argand diagram.

Example 8

**7** Solve the equation $w^3 - i = 1$ and show the roots on the complex plane.

**8** Solve each equation.

**a** $z^3 = -8i$ **b** $z^4 = -1$ **c** $z^5 = 1$ **d** $z^3 = 27i$

**e** $z^3 = -2$ **f** $z^3 = -\frac{8}{27}$ **g** $z^2 = i$ **h** $z^4 = i$

**i** $z^3 = -64i$ **j** $z^3 = 1 - i\sqrt{3}$ **k** $z^4 = 1 + i$ **l** $z^4 = -16$

**9** **a** Find all the fifth roots of 32.

**b** Solve $z^2 = 16i$.

**c** Find all complex numbers $z$ for which $z^4 = \frac{1}{16}$.

**10** Solve each equation.

**a** $z^4 = 1$ **b** $z^4 = 16$ **c** $z^6 = 1$ **d** $z^6 = -64$

**e** $z^5 = 243i$ **f** $z^6 = 27i$ **g** $z^4 = 1 + i$ **h** $z^4 = 2 + 2i\sqrt{3}$

**i** $z^4 = -8 + 8i\sqrt{3}$

**11** **a** Find the square roots of $2 + 2i\sqrt{3}$.

**b** Find all 5 roots of $z^5 - 32 = 0$ and plot them on an Argand diagram.

**c** Solve $z^5 = 16\sqrt{2} + 16i\sqrt{2}$.


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# 5.04 The remainder theorem

WS Polynomial operations

You have previously defined polynomial expressions where the coefficients are real numbers. This definition may be extended to include polynomials with coefficients that are complex.

## Complex polynomials

A **complex polynomial** is an expression of the form

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \ldots + a_1 z^1 + a_0$$

where $a \geq 0$, $a_0, a_1, a_2, \ldots, a_n$ may be complex numbers and $a_0 \neq 0$.

It has **degree** $n$, **leading term** $a_n z^n$ and **constant term** $a_0$.

## EXAMPLE 9

Given the function $f(z) = 4z^3 - 2iz^2 + 3z + 5iz^4 - 7iz + 4i + 3$:

**a** identify the leading term

**b** identify the constant term

**c** identify the degree of $f(z)$

**d** calculate $f(2)$

**e** calculate $f(1 - i)$

### Solution

**a** Rearrange the polynomial with powers in descending order.

$f(z) = 5iz^4 + 4z^3 - 2iz^2 + 3z - 7iz + 4i + 3$

$= 5iz^4 + 4z^3 - 2iz^2 + (3 - 7i)z + (3 + 4i)$

The first term is the leading term.

The leading term $= 5iz^4$

**b** The constant term has no $z$.

The constant term $= 4i + 3$

**c** The degree is the highest power of $z$.

The degree of $f(z) = 4$

**d** Substitute 2 for $z$ and evaluate.

$f(2) = 5i \times (2)^4 + 4 \times 2^3 - 2i \times 2^2 + (3 - 7i) \times 2 + (3 + 4i)$

$= 80i + 32 - 8i + 6 - 14i + 3 + 4i$

$= 41 + 62i$

**e** Substitute $(1 - i)$ and use the fact that $(1 - i)^2 = 1 - 2i + i^2 = -2i$.

$f(2) = 5i \times (1 - i)^4 + 4 \times (1 - i)^3 - 2i \times (1 - i)^2 + (3 - 7i)(1 - i) + (4i + 3)$

$= 5i \times (-2i)^2 + 4 \times [-2i(1 - i)] - 2i \times (-2i) + 3 - 10i - 7 + 4i + 3$

$= 5i \times (-4) - 8i + 8i^2 + 4i^2 - 6i - 1$

$= -13 - 34i$

When you add, subtract, multiply and divide complex polynomials, you need to take care with the coefficients as they have both real and imaginary parts.

## EXAMPLE 10

Given that $p(z) = z^3 + (5 - 2i)z^2 + (7 - 7i)z + 3$

$f(z) = z^2 + (2 - i)z + 4 + 2i$

$d(z) = z + 2 - 3i$

calculate:

**a** $p(z) + f(z)$ **b** $f(z)d(z)$ **c** $p(z) \div d(z)$

### Solution

**a** Write the sum.

$p(z) + f(z) = z^3 + (5 - 2i)\, z^2 + (7 - 7i)z + 3 + z^2 + (2 - i)z + 4 + 2i$

Add the coefficients of powers separately.

$= z^3 + (5 + 1 - 2i)z^2 + (7 + 2 - 7i - i)z + 4 + 3 + 2i$

$= z^3 + (6 - 2i)z^2 + (9 - 8i)z + 7 + 2i$

**b** Write the product.

$f(z)d(z) = [z^2 + (2 - i)z + 4 + 2i][z + 2 - 3i]$

Use the distributive law.

$= z^2(z + 2 - 3i) + (2 - i)z \times z + (2 - i)z \times 2 - (2 - i)z \times 3i + (4 + 2i)z + (4 + 2i) \times 2 - (4 + 2i) \times 3i$

Use the distributive law again. Remember: $i^2 = -1$.

$= z^3 + 2z^2 - 3iz^2 + (2 - i)z^2 + (4 - 2i)z - (6i + 3)z + 4z + 8 - 12i + 2iz + 4i + 6$

Collect like powers and simplify.

$= z^3 + (2 - 3i + 2 - i)z^2 + (4 - 2i - 6i - 3)z + (4 + 2i)z + 14 - 8i$

$= z^3 + (4 - 4i)z^2 + (5 - 6i)z + 14 - 8i$


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**c** Write the quotient.

$$p(z) \div d(z) = \frac{z^3 + (5-2i)z^2 + (7-7i)z + 3}{z + 2 - 3i}$$

Set out as long division.

Divide the $z^3$ term by $z$.

Multiply.

Subtract and bring down.

$$\begin{array}{r} z^2 \phantom{+ (5-2i)z^2 + (7-7i)z + 3} \\ z + 2 - 3i \enclose{longdiv}{z^3 + (5-2i)z^2 + (7-7i)z + 3} \\ \underline{z^3 + (2-3i)z^2} \phantom{+ (7-7i)z + 3} \\ 0 \quad + (3+i)z^2 + (7-7i)z + 3 \end{array}$$

Divide the $z^2$ term by $z$.

Multiply. Note that:

$(3 + i)(2 - 3i) = 9 - 7i$

Subtract and bring down.

$$\begin{array}{r} z^2 + (3+i)z \phantom{+ (7-7i)z + 3} \\ z + 2 - 3i \enclose{longdiv}{z^3 + (5-2i)z^2 + (7-7i)z + 3} \\ \underline{z^3 + (2-3i)z^2} \phantom{+ (7-7i)z + 3} \\ (3+i)z^2 + (7-7i)z + 3 \\ \underline{(3+i)z^2 + (9-7i)z} \phantom{+ 3} \\ 0 \qquad\qquad -2z + 3 \end{array}$$

Divide the $z$ term by $z$.

Multiply.

Subtract.

$$\begin{array}{r} z^2 + (3+i)z - 2 \phantom{+ 3} \\ z + 2 - 3i \enclose{longdiv}{z^3 + (5-2i)z^2 + (7-7i)z + 3} \\ \underline{z^3 + (2-3i)z^2} \phantom{+ (7-7i)z + 3} \\ (3+i)z^2 + (7-7i)z + 3 \\ \underline{(3+i)z^2 + (9-7i)z} \phantom{+ 3} \\ -2z + 3 \\ \underline{-2z - 4 + 6i} \\ 0 + 7 - 6i \end{array}$$

Write the result.

$$p(z) \div d(z) = z^2 + (3 + i)z - 2 + \frac{7 - 6i}{z + 2 - 3i}$$

You can see that the division of $p(z)$ by $d(z)$ results in another polynomial [say $q(z) = z^2 + (3 + i)z - 2$] and a remainder [say $r = 7 - 6i$]. This can be represented symbolically as:

$$p(z) = q(z)d(z) + r$$

Each of these terms has a special name.

- $p(z)$ is called the **dividend** (the polynomial that is being divided)
- $d(z)$ is called the **divisor** (the polynomial by which the dividend is being divided)
- $q(z)$ is called the **quotient** (the polynomial resulting from the division)
- $r$ is called the **remainder** (the polynomial that remains after the division)

You have previously seen this relationship, called the **remainder theorem**, when dealing with real polynomials. The remainder theorem also applies to complex polynomials.

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### The remainder theorem for complex polynomials

When the complex polynomial $P(z)$ is divided by the complex polynomial $D(z)$, the result is a quotient $Q(z)$ and remainder $R$ such that:

$$P(z) = Q(z)D(z) + R$$

If $D(z) = (z - a)$, where $a \in \mathbf{C}$, the remainder, $R = P(a)$.

The last part of the remainder theorem can be confirmed using Example **10 c**. In this example, the dividend is $p(z) = z^3 + (5 - 2i)z^2 + (7 - 7i)z + 3$ and the divisor is $d(z) = z + 2 - 3i$ so $a = -2 + 3i$.

$$\begin{aligned} p(a) &= (-2 + 3i)^3 + (5 - 2i)(-2 + 3i)^2 + (7 - 7i)(-2 + 3i) + 3 \\ &= 46 + 9i + (-49 - 50i) + (7 + 35i) + 3 \\ &= 7 - 6i \end{aligned}$$

and this is the remainder found by long division.

## EXAMPLE 11

Find the remainder when $f(z) = (4 - i)z^3 + (2 - 3i)z^2 - (5 + i)z + 3 - 4i$ is divided by:

**a** $z + 2$ **b** $z - 3i$ **c** $z - 1 + 2i$

### Solution

**a** Calculate $a$.

$z - a = z + 2$, so $a = -2$

Apply the remainder theorem.

$$\begin{aligned} f(-2) &= (4 - i)(-2)^3 + (2 - 3i)(-2)^2 - (5 + i)(-2) + 3 - 4i \\ &= (4 - i)(-8) + (2 - 3i)(4) - (5 + i)(-2) + 3 - 4i \\ &= -32 + 8i + 8 - 12i + 10 + 2i + 3 - 4i \\ &= -11 - 6i \end{aligned}$$

**b** Calculate $a$.

$z - a = z - 3i$, so $a = 3i$

Apply the remainder theorem.

$$\begin{aligned} f(3i) &= (4 - i)(3i)^3 + (2 - 3i)(3i)^2 - (5 + i)(3i) + 3 - 4i \\ &= (4 - i)(-27i) + (2 - 3i)(-9) - 15i + 3 + 3 - 4i \\ &= -108i - 27 - 18 + 27i + 6 - 19i \\ &= -39 - 100i \end{aligned}$$


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| | | |
|---|---|---|
| **c** | Calculate $a$. | $z - a = z - 1 + 2i$, so $a = 1 - 2i$ |
| | Apply the remainder theorem. | $f(1-2i) = (4-i)(1-2i)^3 + (2-3i)(1-2i)^2 - (5+i)(1-2i) + 3 - 4i$ <br> $= (4-i)(-11+2i) + (2-3i)(-3-4i) - (7-9i) + 3 - 4i$ <br> $= -42 + 19i - 18 + i - 4 + 5i$ <br> $= -64 + 25i$ |

## Exercise 5.04 The remainder theorem

Example 9

**1** Given the function $P(z) = 7 + 3iz^2 - 7iz + 5z^2 + 3z + 2z^3$:

**a** identify the leading term

**b** identify the constant term

**c** identify the degree of $P(z)$

**d** calculate $P(-3)$

**e** calculate $P(1+i)$

**2** Identify the leading term and the degree of each polynomial.

**a** $4iz^2 - (3+2i)z + 11 - 7i + (2-i)z^3$

**b** $3z^2 + 7z^3 + (2-3i)z^5 - (6+5i)z - 12i$

**c** $(3+2i)z^3 + 9z - 7z^6 + (5-i)z^4 + 6i$

**d** $(8-3i)z^7 + 6iz^3 - 5z^9 + (3-2i)z$

**e** $2 + 2i - (1-2i)z^6 + 3iz^8 + 3z^2 - 4iz$

**f** $(7-2i)z - (4+2i)z^4 - 4 + (3-i)z^7 + (1-i)z^5$

**3** Identify the constant term of each polynomial shown in question **2**.

**4** Given that $f(z) = z^3 + (3-4i)z^2 + (1+3i)z + 8 - 5i$, calculate:

**a** $f(2)$ **b** $f(-1)$ **c** $f(3i)$ **d** $f(-2i)$ **e** $f(1+i)$

**5** Given that $q(z) = (1-2i)z^3 - (4+i)z^2 + (2-3i)z + 11 + 9i$, calculate:

**a** $q(3)$ **b** $q(-2)$ **c** $q(-2i)$ **d** $q(2+i)$ **e** $q(3-i)$

Example 10

**6** Given that $f(z) = z^3 + (2-3i)z^2 + (5+4i)z + 7 - 2i$, $g(z) = z^2 + (1-2i)z - 5 + 6i$ and $h(z) = z + 3 - 2i$, calculate:

**a** $f(z) + h(z)$

**b** $f(z) + g(z)$

**c** $f(z)h(z)$

**d** $f(z)g(z)$

**e** $g(z) \div h(z)$

**f** $f(z) \div h(z)$

**7** Given that $p(z) = (1-2i)z^3 - (3+i)z^2 + (6-5i)z - 5 + 3i$, $q(z) = (2-3i)z^2 + (2-i)z + 4 + 7i$ and $r(z) = z - 2 + 3i$ calculate:

**a** $p(z) + 2q(z)$

**b** $3q(z) - 4r(z)$

**c** $p(z)r(z)$

**d** $p(z)q(z)$

**e** $q(z) \div r(z)$

**f** $p(z) \div r(z)$

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8 Complete each long division.

a $z-2\overline{)z^3+(1-3i)z^2+(2+4i)z+4}$

b $z+3i\overline{)(1-2i)z^3+(2-6i)z^2+(4-3i)z+2-8i}$

c $z+1-2i\overline{)z^3+(4+3i)z^2+(2-5i)z+9-7i}$

Example 11

9 Given that $g(z) = (-2 + i)z^3 + (3 - 2i)z^2 - (1 - 2i)z + 7 - 4i$, find the remainder when $g(z)$ is divided by:

a $z + 3$ b $z - 2i$ c $z + 2 - i$

10 Find the remainder when $p(z) = z^4 + (3 + 2i)z^3 + 5z^2 - (4 - i)z + 6 + 7i$ is divided by:

a $z - 2$ b $z + 3i$ c $z + 3 - 2i$

### Problem solving

11 When the polynomial $P(z) = 2z^3 - az^2 + bz - 2$ is divided by $z - 1 - i$, the remainder is 0. Find the values of $a$ and $b$.

WS

Remainder and factor theorems

## 5.05 The factor theorem

The remainder theorem states that when $P(z)$ is divided by $z - a$, the remainder is $P(a)$. It follows that if the remainder is 0, then $z - a$ must be a factor of $P(z)$. This is known as the **factor theorem**. You have previously dealt with the factor theorem with real polynomials.

**The factor theorem for complex polynomials**

$(z - a)$, where $a \in C$, is a factor of the complex polynomial $P(z)$ if and only if $P(a) = 0$.

**EXAMPLE 12**

a Show that $z - (1 + 2i)$ is a factor of $P(z) = z^3 - 3z^2 + 7z - 5$.

b Factorise $q(z) = z^3 + 3z^2 + z - 5$.

**Solution**

a Apply the factor theorem. If $z - (1 + 2i)$ is a factor of $P(z)$, then $P(1 + 2i) = 0$.

Substitute for $z$. $z^3 - 3z^2 + 7z - 5 = (1 + 2i)^3 - 3(1 + 2i)^2 + 7(1 + 2i) - 5$

Expand. $= 1 + 6i - 12 - 8i - 3(1 + 4i - 4) + 7 + 14i - 5$

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Simplify. $= (-11 - 2i) - 3(-3 + 4i) + 7 + 14i - 5$

$= -11 - 2i + 9 - 12i + 2 + 14i$

$= 0$

Write the result. $P(1 + 2i) = 0$, so $z - (1 + 2i)$ is a factor of $P(z)$.

**b** Use trial and error to find $q(a) = 0$.

$$q(1) = (1)^3 + 3(1)^2 + 1 - 5$$
$$= 1 + 3 + 1 - 5$$
$$= 0$$

Use the factor theorem. $(z - 1)$ is a factor of $q(z)$.

Use long division.

$$\begin{array}{r} z^2 + 4z + 5 \\ z - 1\overline{)z^3 + 3z^2 + z - 5} \\ \underline{z^3 - z^2} \\ 4z^2 + z \\ \underline{4z^2 - 4z} \\ 5z - 5 \\ \underline{5z - 5} \\ 0 \end{array}$$

State the result. $z^2 + 4z + 5$ is a factor of $q(z)$.

Use the quadratic formula.

$$z = \frac{-4 \pm \sqrt{4^2 - 4 \times 1 \times 5}}{2 \times 1}$$
$$= \frac{-4 \pm \sqrt{-4}}{2}$$
$$= \frac{2(-2 \pm \sqrt{-1})}{2}$$
$$= -2 \pm i$$

Use the roots to state the factors.

$q(-2 + i) = 0$ so $(z + 2 - i)$ is a factor of $q(z)$.

$q(-2 - i) = 0$ so $(z + 2 + i)$ is a factor of $q(z)$.

Use all factors to write the result. $q(z) = (z - 1)(z + 2 - i)(z + 2 + i)$

For the complex polynomial $P(z)$, the factor theorem states that if $P(a) = 0$, then $z - a$ is a factor of $P(z)$, so $z = a$ must be a root of $P(z)$.

It can be shown that an $n$th degree complex polynomial has exactly $n$ roots (including repeated roots). This is known as the **Fundamental Theorem of Algebra**.

It is important to include repeated roots. Consider $f(z) = (z - 1)^2$.

Let $f(z) = 0$: $(z - 1)^2 = 0$

So $(z - 1)(z - 1) = 0$

The roots are 1 and 1.

This is an example of a repeated root. In this case, the root $z = 1$ is said to have a **multiplicity** of 2.

You have previously used the factor theorem to factorise real polynomials, and it can also be used to factorise complex polynomials. When one root of a polynomial is known, the **conjugate root theorem** may be used to find another root.

### Conjugate root theorem

If $P(z)$ is a polynomial with real coefficients and $x + yi$ is a root of $P(z) = 0$, then the complex conjugate, $x - yi$, is also a root of the equation.

If $P(z_1) = 0$, then $P(\overline{z_1}) = 0$.

## EXAMPLE 13

If $z = 1 + i$ is one root of the equation $z^3 + 4z^2 - 10z + 12 = 0$, factorise $P(z) = z^3 + 4z^2 - 10z + 12$.

### Solution

| | |
|---|---|
| Find the conjugate of $1 + i$. | If $z_1 = 1 + i$, then $\overline{z_1} = 1 - i$. |
| Use the conjugate root theorem. | $1 - i$ is another root. |
| Use the factor theorem. | $[z - (1 + i)]$ and $[z - (1 - i)]$ are factors of $z^3 + 4z^2 - 10z + 12$. |
| The product of factors is also a factor. | $[z - (1 + i)] \times [z - (1 - i)]$ must also be a factor. |
| Multiply factors. | $[z - (1 + i)] \times [z - (1 - i)]$ |
| | $= z^2 - z(1 + i) - z(1 - i) + (1 + i)(1 - i)$ |
| Expand. | $= z^2 - z - zi - z + zi + 1^2 - i^2$ |


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| | |
|---|---|
| Simplify. Use $i^2 = -1$. | $= z^2 - 2z + 2$ |
| Use the factor theorem. | $z^2 - 2z + 2$ must be a factor. |

Use long division to find the other factor.

$$\begin{array}{r} z+6 \\ z^2-2z+2\overline{\big)\, z^3+4z^2-10z+12} \\ \underline{z^3-2z^2+2z} \\ 6z^2-12z+12 \\ \underline{6z^2-12z+12} \\ 0 \end{array}$$

| | |
|---|---|
| Write the conclusion. | $(z + 6)$ is the other factor. |
| Write the result. | $P(z) = (z - 1 - i)(z - 1 + i)(z + 6)$ |

**Alternative method**

| | |
|---|---|
| Let $(az + b)$ be the other factor. | $(az + b)(z^2 - 2z + 2) = z^3 + 4z^2 - 10z + 12$ |
| Examine the first term. | $az \times z^2 = z^3$, so $a = 1$ |
| Examine the last term. | $b \times 2 = 12$, so $b = 6$ |
| Write the conclusion. | $(z + 6)$ is the other factor. |
| Write the result. | $P(z) = (z - 1 - i)(z - 1 + i)(z + 6)$ |

## EXAMPLE 14

Write a polynomial function of least degree with integral coefficients that has the zeros $-i$ and 3 (multiplicity 2).

### Solution

| | |
|---|---|
| Assign the polynomial. | Let the polynomial = $P(z)$. |
| Use the factor theorem. | $P(-i) = 0$ so $(z + i)$ is a factor. |
| $P(z)$ has real coefficients, so the conjugate root theorem holds. | Since $-i$ is a root of $P(z)$, its conjugate, $i$, is also a root. |
| Use the factor theorem. | $P(i) = 0$ so $(z - i)$ is a factor. |
| Another root is 3 (multiplicity 2). | $(z - 3)^2$ is a factor of $P(z)$. |

| | |
|---|---|
| Write the polynomial in factored form. | $P(z) = (z + i)(z - i)(z - 3)^2$ |
| Expand. | $= (z^2 + 1)(z^2 - 6z + 9)$ <br> $= z^4 - 6z^3 + 10z^2 - 6z + 9$ |
| State the result. | A polynomial with integral coefficients that has the zeros $-i$ and 3 (multiplicity 2) is $z^4 - 6z^3 + 10z^2 - 6z + 9$. |

## Exercise 5.05 The factor theorem

Example 12

**1** In each case, use the factor theorem to show that $Q(z)$ is a factor of $P(z)$

**a** $Q(z) = z - 2$ and $P(z) = z^3 - 4z^2 + 14z - 20$

**b** $Q(z) = z + 1$ and $P(z) = z^3 - 3z^2 + z + 5$

**c** $Q(z) = z - 3$ and $P(z) = z^3 - 5z^2 + 11z - 15$

**d** $Q(z) = z + 2$ and $P(z) = z^3 - 6z^2 + z + 34$

**2** Factorise each $P(z)$ in question **1**.

**3** Factorise $g(z) = z^3 - 8$.

**4** If $p(z) = z^3 - 2z^2 + 9z - 18$, use the factor theorem to find an integer $a$ such that $p(a) = 0$ and hence factorise $p(z)$.

**5** **a** Show that $(z + 3)$ is a factor of $h(z) = z^3 + (5 - 4i)z^2 + (3 - 16i)z - 9 - 12i$.

**b** Use the result from part **a** to factorise $h(z)$.

**6** **a** Show that $(z - 4)$ is a factor of $q(z) = z^3 - 8z^2 + 29z - 52$.

**b** Factorise $q(z)$.

Example 13

**7** If $z = 1 - i$ is one root of the equation $z^3 + z^2 - 4z + 6 = 0$, factorise $f(z) = z^3 + z^2 - 4z + 6$.

**8** If $z = 2 + 3i$ is one root of the function $z^4 - 4z^3 + 12z^2 + 4z - 13 = 0$, factorise $P(z) = z^4 - 4z^3 + 12z^2 + 4z - 13$.

**9** If $z = 2 - i$ is a root of the function $z^4 - 5z^3 + 7z^2 + 3z - 10 = 0$, factorise $g(z) = z^4 - 5z^3 + 7z^2 + 3z - 10$.

**10** Given $z = 2i$ is a root of the equation $z^4 + 8z^2 + 16 = 0$, completely factorise the polynomial $P(z) = z^4 + 8z^2 + 16$.

Example 14

**11** Write a polynomial function of least degree with integral coefficients that has the zeros $3i$ and $-1$ (multiplicity 2).

**12** Write a polynomial function of least degree with integral coefficients that has the zeros $-4$, $2 - 3i$ and 1.


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**13** Write a polynomial function of least degree with integral coefficients that has the zeros $i$, 2 and $1 - 3i$.

**14** For each polynomial, use the factor theorem to find 2 different integer zeros of $P(z)$ and hence factorise the polynomial.

**a** $P(z) = z^4 - 3z^3 - 5z^2 + 29z - 30$ **b** $P(z) = z^4 + 9z^3 + 45z^2 + 87z + 50$

### Problem solving

**15** State the polynomial $P(z)$ if the leading term is $2z^3$, $z = 2i$ is a zero and $P(0) = -12$.

**16** $R(z)$ is a fifth-degree polynomial with real coefficients. Find $R(z)$ if $R(z)$ has zeros of 0, 2 (multiplicity 2) and $-3i$ and $Q(2) = 26$.

## 5.06 Complex polynomial equations

WS Polynomials with real coefficients

WS Real and imaginary factors

The simplest non-linear complex polynomials to solve are quadratics.

You know that the quadratic equation $az^2 + bz + c = 0$ (where $z \in \boldsymbol{R}$) has roots

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The nature of the roots depends upon the discriminant, $\Delta = b^2 - 4ac$.

If $\Delta > 0$, then there are 2 distinct real roots.

If $\Delta = 0$, then there is one real root.

If $\Delta < 0$, then there is one pair of complex conjugate roots.

The roots of a quadratic are usually denoted as $\alpha$ and $\beta$, so the roots are

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

For real quadratics, you know that the relationship between the sum and product of roots and the coefficients is:

Sum of roots = $\alpha + \beta = \frac{-b}{a}$ and product of roots = $\alpha\beta = \frac{c}{a}$

This relationship also applies to complex quadratics.

Consider $\quad az^2 + bz + c = 0$

Divide through by $a$. $\quad z^2 + \frac{b}{a}z + \frac{c}{a} = 0$

$$z^2 - (\alpha + \beta)\, z + \alpha\beta = 0$$

Or $\quad z^2$ – (sum of roots) $z$ + (product of roots) = 0

This gives the relationship between the sum and product of roots and the coefficients for complex quadratics.

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## EXAMPLE 15

**a** Solve the quadratic $z^2 - 4z + 5 = 0$.

**b** Find the equation of the quadratic $P(z)$ with real coefficients, given that $P(3 - 5i) = 0$.

### Solution

**a** Write the quadratic formula.

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Use $a = 1$, $b = -4$ and $c = 5$ and simplify.

$$= \frac{4 \pm \sqrt{16 - 20}}{2}$$

$$= \frac{4 \pm \sqrt{-4}}{2}$$

$$= \frac{4 \pm 2i}{2}$$

$$= 2 \pm i$$

State the result. The roots are $2 + i$ and $2 - i$.

**b** Use the factor theorem. One root is $3 - 5i$.

Use the conjugate root theorem. The other root is $3 + 5i$.

Find the sum of the roots. $\alpha + \beta = (3 - 5i) + (3 + 5i) = 6$

Find the product of the roots.

$$\alpha\beta = (3 - 5i)(3 + 5i)$$

$$= 9 - 25i^2$$

$$= 34$$

Use the relationship $z^2 - (\alpha + \beta)z + \alpha\beta$. The required quadratic is $z^2 - 6z + 34$.

A cubic polynomial equation has the form $az^3 + bz^2 + cz + d = 0$ with $z \in \boldsymbol{C}$. If all the coefficients are real, the factor theorem and the conjugate root theorem may be used to find the roots of the equation. The Fundamental Theorem of Algebra tells you that that there will be exactly three roots to a cubic polynomial equation. When all the coefficients of the cubic are real, the roots may all be real (with repetitions allowed), or there may be one real root and one pair of complex conjugate roots.

Note that if one of the coefficients is a complex number, then the roots do not occur in conjugate pairs.


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## EXAMPLE 16

**a** Show that $z = 2 - 5i$ is a root of the equation $z^3 - (2 - 5i)z^2 + 16z - 32 + 80i = 0$, and hence find all the roots.

**b** Solve for $z$ if $z^3 - 2iz^2 + 7z - 14i = 0$.

**c** Given that $Q(z) = z^3 + mz^2 + nz + 100$, where $m$ and $n \in \boldsymbol{R}$, and $z = 3 + 4i$ is a zero of $Q(z)$, find the values of $m$ and $n$ and state all the roots of $Q(z)$.

### Solution

**a** Assign the polynomial.

Let $P(z) = z^3 - (2 - 5i)z^2 + 16z - 32 + 80i$

Substitute $2 - 5i$ for $z$.

$$P(2 - 5i) = (2 - 5i)^3 - (2 - 5i)(2 - 5i)^2 + 16(2 - 5i) - 32 + 80i$$

Simplify.

$$= (2 - 5i)^3 - (2 - 5i)^3 + 32 - 80i - 32 + 80i$$

Evaluate.

$$= 0$$

State the conclusion.

$P(2 - 5i) = 0$, so $z = 2 - 5i$ is a root of $P(z)$.

If (2 – 5i) is a root of $P(z)$, then $(z - 2 + 5i)$ is a factor. Use long division to find the other factor.

$$\begin{array}{r} z^2 + 16 \\ z - 2 + 5i\overline{\big) z^3 - (2-5i)z^2 + 16z - 32 + 80i} \\ \underline{z^3 - (2+5i)z^2} \qquad\qquad\qquad \\ 0 + 16z - 32 + 80i \\ \underline{16z - 32 + 80i} \\ 0 \end{array}$$

Factorise $P(z)$.

$$P(z) = (2 - 5i)(z^2 + 16)$$
$$= (2 - 5i)(z + 4i)(z - 4i)$$

State the roots.

The roots of $P(z)$ are $2 - 5i$ and $\pm 4i$.

**b** Group the terms.

$$(z^3 - 2iz^2) + (7z - 14i) = 0$$

Factorise.

$$z^2(z - 2i) + 7(z - 2i) = 0$$

Take out the common factor.

$$(z^2 + 7)(z - 2i) = 0$$

Use $i^2 = -1$.

$$(z^2 - 7i^2)(z - 2i) = 0$$

| | | |
|---|---|---|
| | Use the difference of two squares. | $(z - i\sqrt{7})(z + i\sqrt{7})(z - 2i) = 0$ |
| | State the roots. | Roots $= \pm i\sqrt{7}$ and $2i$. |
| **c** | Apply the conjugate root theorem. | Let $\alpha = 3 + 4i$ and $\beta = 3 - 4i$. |
| | Find the sum of the roots. | $\alpha + \beta = 6$ |
| | Find the product of the roots. | $\alpha\beta = (3 + 4i)(3 - 4i)$<br>$= 9 - 16i^2$<br>$= 25$ |
| | Find the quadratic factor of $Q(z)$. | $z^2 - (\alpha + \beta)z + \alpha\beta = z^2 - 6z + 25$ |
| | Let $(az + b)$ be the other factor. | $(az + b)(z^2 - 6z + 25) = z^3 + mz^2 + nz + 100$ |
| | Examine the first term. | $az \times z^2 = z^3$, so $a = 1$ |
| | Examine the last term. | $b \times 25 = 100$, so $b = 4$. |
| | State the factor. | $(z + 4)$ is the other factor. |
| | Calculate $Q(z)$. | $Q(z) = (z + 4)(z^2 - 6z + 25)$ |
| | Expand | $= z^3 - 6z^2 + 25z + 4z^2 - 24z + 100$ |
| | Simplify. | $= z^3 - 2z^2 + z + 100$ |
| | Identify $m$ and $n$. | $m = -2$ and $n = 1$ |
| | Use the factor theorem. | The roots of $Q(z)$ are $z = 3 \pm 4i$ and $z = -4$. |

You can use the fundamental theorem of algebra to determine the number of roots that higher-order polynomials have. A fourth-degree polynomial is called a **quartic**, so it must have 4 roots.

When finding the roots of higher-order polynomials, a variety of algebraic techniques may need to be used.


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## EXAMPLE 17

**a** Solve for $z$ if $z^4 + z^2 - 90 = 0$.

**b** If $5 + 2i$ is a root of the equation $z^4 - 10z^3 + 22z^2 + mz - 203 = 0$, where $m \in \boldsymbol{R}$, find $m$ and all the roots of the equation.

### Solution

**a** Convert the quartic equation to a quadratic equation using a suitable substitution.

Let $a = z^2$.

$$z^4 + z^2 - 90 = 0$$
$$a^2 + a - 90 = 0$$

Factorise the quadratic.

$$(a + 10)(a - 9) = 0$$

Substitute $z^2$ for $a$.

$$(z^2 - 10)(z^2 - 9) = 0$$

Use $i^2 = -1$.

$$(z^2 - 10i^2)(z^2 - 9) = 0$$

Use the difference of 2 squares.

$z = \pm i\sqrt{10}$ and $z = \pm 3$

**b** Assign the polynomial.

Let $P(z) = z^4 - 10z^3 + 22z^2 + mz - 203$

Apply the conjugate root theorem.

Let $\alpha = 5 + 2i$ and $\beta = 5 - 2i$.

Find the sum and product of the roots. Remember, $i^2 = -1$.

$$\alpha + \beta = 10$$
$$\alpha\beta = 25 - 4i^2 = 29$$

The quadratic formed using the roots must be a factor of $Q(z)$. Write the quadratic factor.

$$z^2 - (\alpha + \beta)z + \alpha\beta = z^2 - 10z + 29$$

So $z^2 - 10z + 29$ is a factor of $Q(z)$.

$Q(z)$ is a quartic, so the other factor must be a quadratic.

$$Q(z) = (z^2 - 10z + 29)(az^2 + bz + c)$$

Compare the first and last terms of $Q(z)$ with the product.

First term $= z^4$, so $a = 1$

Last term $= -203$, so $c = -7$

Write as a product of factors.

$$Q(z) = (z^2 - 10z + 29)(z^2 + bz - 7)$$

Expand and gather terms.

$$= z^4 + (b - 10)z^3 + (22 - 10b)z^2 + (70 + 29b)z - 203$$

| | |
|---|---|
| Compare with the coefficients of $Q(z)$. | $z^3$: $b - 10 = -10$, so $b = 0$<br>$z^2$: $22 - 10b = 22$ (confirms $b = 0$)<br>$z$: $70 + 29b = m$, so $m = 70$ |
| State the result. | $m = 70$, so $Q(z) = z^4 - 10z^3 + 22z^2 + 70z - 203$ |
| Complete the factorisation of $Q(z)$ using completing the square (or the quadratic formula) and the difference of two squares. | $Q(z) = (z^2 - 10z + 29)(z^2 - 7)$<br>$= (z - 5 + 2i)(z - 5 - 2i)(z - i\sqrt{7})(z + i\sqrt{7})$ |
| Identify all roots of $Q(z)$. | The roots of $Q(z)$ are $z = \pm i\sqrt{7}$ and $z = 5 \pm 2i$. |

## Exercise 5.06 Complex polynomial equations

**1** Solve each equation.

**a** $z^2 = 7$ **b** $z^2 = -5$ **c** $z^2 = 2\frac{7}{9}$

**d** $z^2 = -\frac{9}{4}$ **e** $z^2 + 6 = 0$ **f** $z^2 - 9 = 0$

**g** $9z^2 + 1 = 0$ **h** $4z^2 + 1 = 0$ **i** $z^2 + 3z = 3(z - 1)$

**j** $(z - 3)^2 = -4$ **k** $(z + 1)^2 + 5 = 0$ **l** $4(2z + 3)^2 + 1 = 0$

Example 15

**2** Solve each equation.

**a** $z^2 - 2z + 10 = 0$ **b** $z^2 - z + 1 = 0$ **c** $2z^2 + 2z + 3 = 0$

**d** $9z^2 + 6z + 2 = 0$ **e** $z^2 + z + 1 = 0$ **f** $2z^2 - 3z + 2 = 0$

**3** Find the equation of the quadratic $P(z)$ with real coefficients, given that:

**a** $P(1 - 4i) = 0$ **b** $P(2 + 3i) = 0$ **c** $P(-1 - 2i) = 0$

**d** $P(1 - i\sqrt{3}) = 0$ **e** $P(-2 + i\sqrt{5}) = 0$ **f** $P(4 - 7i) = 0$

Example 16

**4** For each equation, show that the given value of $z$ is a root and hence find all the roots of the equation.

**a** $z = 1 - 3i$ and $z^3 - 4z^2 + 14z - 20 = 0$ **b** $z = 2 + 3i$ and $z^3 - 8z^2 + 29z - 52 = 0$

**c** $z = 2 - 4i$ and $z^3 - z^2 + 8z + 60 = 0$ **d** $z = 1 + 5i$ and $z^3 - z^2 + 24z + 26 = 0$

**e** $z = 2 - 5i$ and $z^3 - 6z^2 + 37z - 58 = 0$ **f** $z = 5 + 3i$ and $z^3 - 6z^2 - 6z + 136 = 0$

**5** Solve each equation for $z$.

**a** $z^3 + 2iz^2 + 8z + 16i = 0$ **b** $z^3 - 3iz^2 + 12z - 36i = 0$

**c** $z^3 - 5iz^2 + 10z - 50i = 0$ **d** $z^3 + 6iz^2 + 3z + 18i = 0$

**e** $z^3 - 5iz^2 + 20z - 100i = 0$ **f** $z^3 + 3iz^2 + 24z + 72i = 0$


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**6** Given that $P(z) = z^3 + az^2 + bz + 26$, where $a$ and $b \in \boldsymbol{R}$, and $z = 2 + 3i$ is a zero of $P(z)$, find the values of $a$ and $b$ and state all the roots of $P(z)$.

**7** If $z = -3 + 2i$ is a zero of $R(z) = z^3 + pz^2 + qz - 65$, where $p$ and $q \in \boldsymbol{R}$, find the values of $p$ and $q$ and state all the roots of $R(z)$.

**8** If $z = -4 - 3i$ is a zero of $Q(z) = z^3 + mz^2 + nz + 150$, where $m$ and $n \in \boldsymbol{R}$, find all the roots of $Q(z)$.

**9** Solve each equation for $z$.

Example 17

**a** $z^4 - 16 = 0$ **b** $z^4 + 7z^2 - 144 = 0$ **c** $z^4 - 6z^2 - 216 = 0$

**d** $z^4 + 12z^2 - 160 = 0$ **e** $z^4 - 12z^2 - 108 = 0$ **f** $z^4 + 8z^2 - 768 = 0$

**10** If $4 + 3i$ is a root of the equation $z^4 - 8z^3 + 26z^2 + az + 75 = 0$, where $a \in \boldsymbol{R}$, find $a$ and all the roots of the equation.

**11** If $-3 - 2i$ is a root of the equation $z^4 + 6z^3 + 25z^2 + bz + 156 = 0$, where $b \in \boldsymbol{R}$, find all the roots of the equation.

**12** If $z = di$ is a root of the equation $z^4 - 4z^3 + 22z^2 - 36z + 117 = 0$, find the value of $d$ and all the roots.

**13** **a** Calculate $(z^2 + 1)(2z^2 + 6z + 5)$.

**b** Use the result from part **a** to solve $2z^4 + 6z^3 + 7z^2 + 6z + 5 = 0$.

**14** **a** Calculate $(z^2 + 4)(z^2 + 2z + 2)$.

**b** Use the result from part **a** to solve $z^4 + 2z^3 + 6z^2 + 8z + 8 = 0$.

**15** Solve each equation.

**a** $z^3 + z^2 - 2 = 0$ **b** $z^3 + 5z^2 + 9z + 5 = 0$ **c** $z^3 + 3z^2 + z + 3 = 0$

**d** $z^3 - 3z^2 + 3z - 2 = 0$ **e** $z^3 - 4z^2 + 9z - 10 = 0$ **f** $z^3 - 3z^2 + 7z - 5 = 0$

**16** Solve:

**a** $z^4 + z^2 - 12 = 0$ **b** $z^4 - 2z^3 + 2z^2 - 2z + 1 = 0$

**c** $z^4 - 7z^3 + 11z^2 + 73z - 150 = 0$

## Problem solving

**17** $Q(z)$ is a polynomial of degree 5 with real coefficients. If $Q(z) = 0$ has two roots $2 - 3i$ and $-4i$, how many real roots does it have?

**18** $P(z)$ is a fifth-degree polynomial with real coefficients. If $3i$, $1 - 2i$ and $-2$ are roots of the polynomial, identify $P(z)$.

**19** Solve $z^5 - 2z^4 - 2z^3 + 4z^2 - 3z + 6 = 0$.

**20** Express $Q(z) = z^5 - 3z^4 + 20z^3 + 90z^2 + 64z + 408$ as a product of linear factors.

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# 5. CHAPTER SUMMARY

## Complex polynomials and roots

- A **locus** (plural **loci**) is a set of points in a plane that satisfy some given condition
- The Cartesian form of a circle with centre $(h, k)$ and radius $r$ is:

$$(x - h)^2 + (y - k)^2 = r^2$$

- A circle in the complex plane with centre $z_1 = x_1 + y_1 i$ and radius $r$ has the equation

$$|z - z_1| = r$$

- An ***n*th root of unity**, where $n \in \mathbf{Z}^+$, is a number $z$ satisfying the equation $z^n = 1$. The $n$th roots of unity lie on the unit circle and are given by:

$$z = \text{cis}\left(\frac{2k\pi}{n}\right) = \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)$$

  where $k = 0, 1, 2, 3, \ldots, n - 1$ and $-\pi < \dfrac{2k\pi}{n} \le \pi$.

- The ***n*th roots of a complex number**, $z = r[\cos(\theta) + i\sin(\theta)]$, where $n \in \mathbf{Z}^+$ and $w^n = z$, are:

$$w_k = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

  where $k = 0, 1, 2, \ldots, n - 1$

- A **complex polynomial** is an expression of the form

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \ldots + a_1 z^1 + a_0$$

  where $a \ge 0$, $a_0, a_1, a_2, \ldots, a_n$ may be complex numbers and $a_0 \ne 0$

- For a complex polynomial $P(z)$, the **degree** is the highest power of $z$, the **leading term** is $a_n z^n$ and the **constant term** is $a_0$
- Complex polynomials can be added, subtracted, multiplied and divided in the same way as real polynomials
- If $p(z) = q(z)d(z) + r$, then the **dividend** $= p(z)$, the **divisor** $= d(z)$, the **quotient** $= q(z)$ and the **remainder** $= r$
- The **remainder theorem** for complex polynomials states that when the complex polynomial $P(z)$ is divided by the complex polynomial $D(z)$, the result is a quotient $Q(z)$ and remainder $R$ such that:

$$P(z) = Q(z)D(z) + R$$

- If $P(z) = Q(z)D(z) + R$ and $D(z) = (z - a)$, where $a \in \mathbf{C}$, the remainder, $R = P(a)$


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- The **factor theorem** for complex polynomials states that $(z - a)$, where $a \in \boldsymbol{C}$, is a factor of the complex polynomial $P(z)$ if and only if $P(a) = 0$
- The **Fundamental Theorem of Algebra** states that an $n$th degree complex polynomial has exactly $n$ roots (including repeated roots). A root that is repeated $n$ times has a **multiplicity** of $n$
- The **conjugate root theorem** states that if $P(z)$ is a polynomial with real coefficients and $x + yi$ is a root of $P(z)$, then the complex conjugate, $x - yi$, is also a root of the equation; or, if $P(z_1) = 0$, then $P(\overline{z_1}) = 0$
- A fourth-degree polynomial is called a **quartic** and so it must have four roots

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# 5. CHAPTER REVIEW

## Complex polynomials and roots

Example 1

**1** Sketch the graph of each locus of $z(x, y)$ described.

**a** $0 \le \arg(z) < \frac{\pi}{2}$ **b** $\frac{-\pi}{3} \le \arg(z) \le 0$ **c** $\frac{-7\pi}{4} \le \arg(z) < 0$

**d** $0 \le \arg(z + 2i) \le \frac{\pi}{2}$ **e** $0 < 2\arg(z + 1) \le \frac{\pi}{3}$ **f** $\frac{-3\pi}{4} \le \arg(z + 1 - i) < 0$

Example 2

**2** Describe and sketch each locus of $z = x + yi$ below.

**a** $|z| = |z + 1 + i|$ **b** $|z + 3 - 2i| = |z + i|$

**c** $|z - 1| = |z - i|$ **d** $|z + 1 + 2i| = |z - 2 - 2i|$

Example 3

**3** Sketch the graph of each locus of $z(x, y)$ described.

**a** $|z - (-2)| \le 1$ **b** $|z + 1| > 3$

**c** $|z + 1 - 2i| = 1$ **d** $2|z + i| = 3$

Example 4

**4** **a** Solve $z^9 = 1$.

**b** Show the roots of $z^9 = 1$ on the unit circle.

**c** Describe the positions of the roots on the unit circle.

Example 5

**5** **a** Find the fifth roots of 1 in Cartesian form.

**b** Express the fifth roots of 1 in polar from.

Example 6

**6** Find the cube roots of 64 and show the roots in the complex plane.

Example 7

**7** Calculate the fourth roots of $256i$ and show the roots on an Argand diagram.

Example 8

**8** Solve the equation $w^2 - 2i\sqrt{3} = 2$ and show the roots on the complex plane.

Example 9

**9** Given the function $p(z) = (2 - i)z^2 + 3z^3 - (3 + 2i)z + z^4 - 5i + 8$:

**a** identify the leading term

**b** identify the constant term

**c** identify the degree of $p(z)$

**d** calculate $p(-1)$

**e** calculate $p(2i)$

**f** calculate $p(1 - i)$

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**10** Given that $p(z) = z^3 + (4 - 3i)z^2 + (2 + 2i)z + 5 - i$, $q(z) = (1 + 2i)z^2 + (5 - i)z - 7 + 4i$ and $r(z) = z + 1 + 3i$, calculate:

**a** $p(z) + q(z) + r(z)$ **b** $q(z)r(z)$ **c** $p(z) \div r(z)$

Example 10

**11** Find the remainder when $f(z) = (3 - 2i)z^3 + (1 + 2i)z^2 - (4 + i)z + 2 - 5i$ is divided by:

**a** $z - 2$ **b** $z + 3i$ **c** $z + 2 - 3i$

Example 11

**12** **a** Use the factor theorem to show that $z - (3 + 2i)$ is a factor of $P(z) = z^4 - 4z^3 + z^2 + 26z$.

**b** Use the factor theorem to factorise $q(z) = z^3 + 5z^2 + 4z - 10$.

Example 12

**13** If $(1 + 2i)$ is one root of the function $Q(z) = z^3 - 5z^2 + 11z - 15 = 0$, factorise $P(z)$.

Example 13

**14** Write a polynomial function of least degree with integer coefficients that has the zeros $2i$ and $\frac{3}{5}$.

Example 14

**15** **a** Solve the quadratic $z^2 - 8z + 25 = 0$.

**b** Find the equation of the quadratic $P(z)$ with real coefficients, given that $P(2 - 7i) = 0$.

Example 15

**16** **a** Show that $z = 1 - 3i$ is a root of the equation $z^3 + 2z^2 + 2z + 40 = 0$, and hence find all the roots.

**b** Solve for $z$ if $z^3 - 3iz^2 + 5z - 15i = 0$.

**c** Given that $P(z) = z^3 + az^2 + bz - 30$, where $a$ and $b \in \boldsymbol{R}$, and $z = 1 + 3i$ is a zero of $P(z)$, find the values of $a$ and $b$ and state all the roots of $P(z)$.

Example 16

**17** **a** Solve for $z$ if $z^4 + 6z^2 - 40 = 0$.

**b** If $1 + 2i$ is a root of the equation $z^4 - 2z^3 + 10z^2 + az + 25 = 0$, where $a \in \boldsymbol{R}$, find $a$ and all the roots of the equation.

Example 17

## Problem solving

**18** Sketch the locus of $z(x, y)$ such that $|z| \leq 4$ and $0 \leq \arg(z) \leq -\frac{\pi}{3}$.

**19** State the polynomial $Q(z)$ if the leading term is $3z^3$, $z = -3i$ is a zero and $P(0) = -36$.

**20** $P(z)$ is a polynomial with real number coefficients such that:

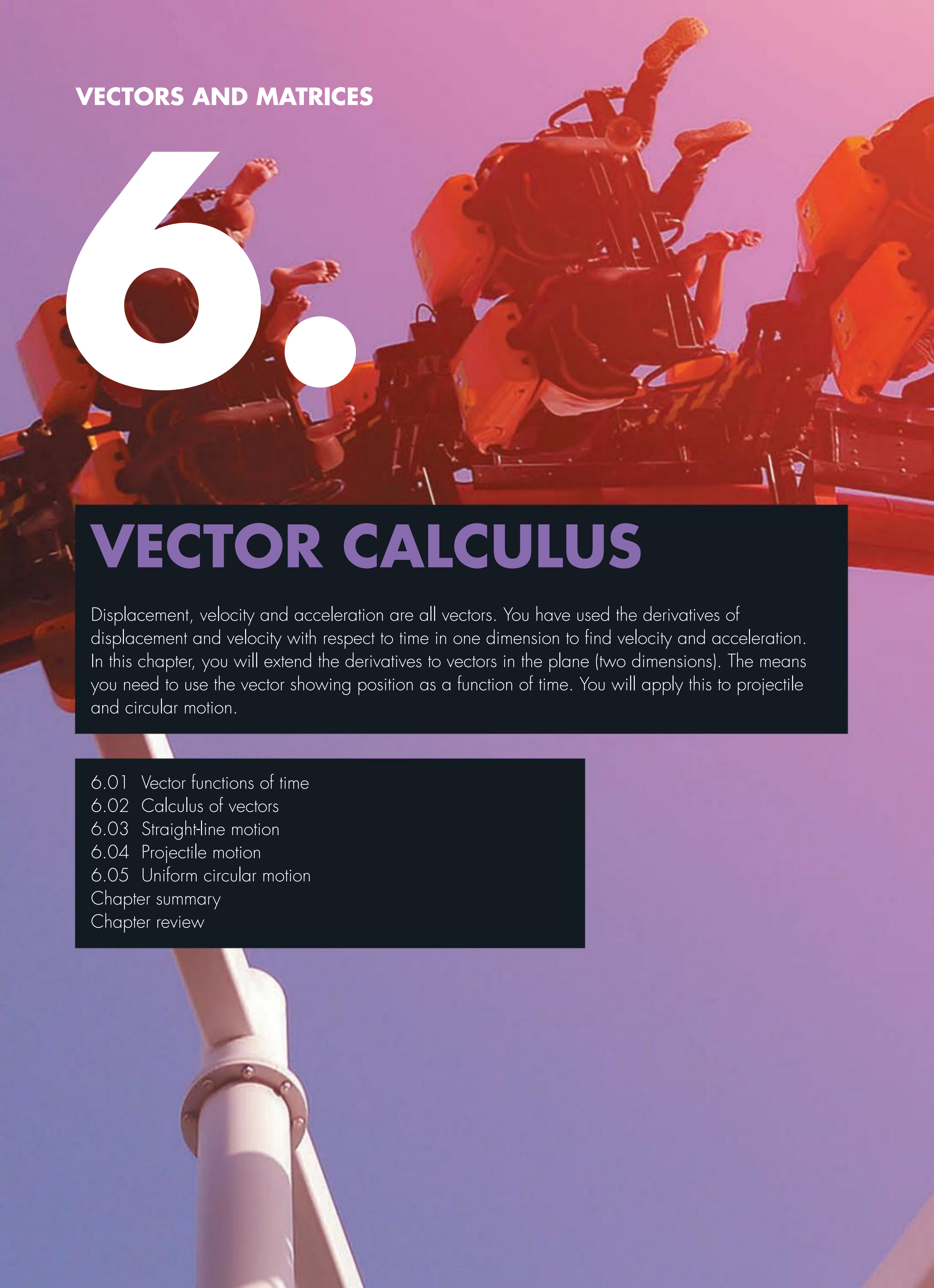
- $P(z)$ is degree 5.
- $z = 3$, $z = 2i$ and $z = 2 - 5i$ are zeros of $P(z)$
- as $z \to -\infty$, $P(z) \to \infty$

Identify the polynomial, leaving your answer in factored form.

**21** $P(z)$ is an $n$th degree polynomial with real number coefficients, where $n \in \boldsymbol{N}$ and $n$ is odd. Explain why at least one of the roots of $P(z)$ must be a real number.

Qz Practice quiz

**22** Solve $z^5 - 4z^4 + 12z^3 + 54z^2 - 13z - 50 = 0$.



# 6. VECTOR CALCULUS

Displacement, velocity and acceleration are all vectors. You have used the derivatives of displacement and velocity with respect to time in one dimension to find velocity and acceleration. In this chapter, you will extend the derivatives to vectors in the plane (two dimensions). The means you need to use the vector showing position as a function of time. You will apply this to projectile and circular motion.

6.01 Vector functions of time
6.02 Calculus of vectors
6.03 Straight-line motion
6.04 Projectile motion
6.05 Uniform circular motion
Chapter summary
Chapter review

## SYLLABUS SUBJECT MATTER

### Vector calculus

- consider position of vectors as a function of time
- derive the Cartesian equation of a path given as a vector equation in two dimensions, including circles, ellipses and hyperbolas
- differentiate and integrate a vector function with respect to time
- determine equations of motion of a particle travelling in a straight line with both constant and variable acceleration
- apply vector calculus to motion in a plane, including projectile and circular motion.

Prior learning

# TERMINOLOGY

acceleration
angular velocity
ellipse
gravity
path
projectile motion
trajectory
velocity
air resistance
Cartesian equation of motion
force
parabola
period
range
uniform circular motion
angle of projection
displacement
frequency
parametric equations
projectile
rectangular hyperbola
vector equation of motion

WS
Parametric and Cartesian equations

## 6.01 Vector functions of time

Consider the motion of a particle with position given by $\mathbf{r}(t) = (5t + 2)\mathbf{i} + (4 - 3t)\mathbf{j}$, where $t$ is in seconds and $\mathbf{r}$ is in metres. You can find the path by eliminating time from the coordinates of its position.

The coordinates are $(5t + 2, 4 - 3t)$, so $x = 5t + 1$ and $y = 4 - 3t$.

You can rearrange the first equation to give $t = \dfrac{x-1}{5} = 0.2(x - 1)$.

Substituting gives $y = 4 - 3 \times 0.2(x - 1)$

$= 4 - 0.6x + 0.6$

or $y = -0.6x + 4.6$

In this case, the motion is a straight line in the direction $\tan^{-1}(-0.6)$ to the horizontal axis.

At $t = 0$, the particle is at (1, 4).

### Vector and Cartesian equations of motion

- The **vector equation of motion** $\mathbf{r}(t) = a(t)\mathbf{i} + b(t)\mathbf{j}$ is equivalent to the **Cartesian** equations $x = a(t)$ and $y(t)$
- The **path** of the motion is derived by eliminating the time $t$

### EXAMPLE 1

A particle moves so that its position at any time $t$ is described by $\mathbf{r}(t) = [5 \cos(2t) + 1]\mathbf{i} + [5 \sin(2t) - 3]\mathbf{j}$, where $t$ is in seconds and $\mathbf{r}$ is in metres.

Express the path in Cartesian form, sketch and describe it.

#### Solution

Write the Cartesian equations.

$x = 5 \cos(2t) + 1$

$y = 5 \sin(2t) - 3$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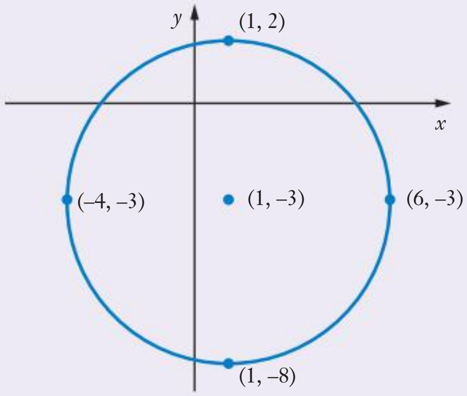
| | |
|---|---|
| Use $\sin^2(\theta) + \cos^2(\theta) = 1$ to eliminate $t$. | $5\cos(2t) = x - 1$ and $5\sin(2t) = y + 3$<br>$[5\cos(2t)]^2 + [5\sin(2t)]^2 = (x-1)^2 + (y+3)^2$<br>$5^2[\cos^2(2t) + \sin^2(2t)] = (x-1)^2 + (y+3)^2$ |
| Use the trigonometric identity. | $(x-1)^2 + (y+3)^2 = 5^2$ |
| State the shape of the path in your answer. | The path is $(x-1)^2 + (y+3)^2 = 5^2$, a circle of radius 5 and centre $(1, -3)$ |
| Sketch the path. |   |

You should recognise the shapes of some curves in **parametric form**. In this work, the parameter is always time, $t$.

## Common curves in parametric form

The parameter in each equation is time, $t$.

- The parametric form of a (horizontal) **parabola** is $x(t) = at^2 + h, y(t) = 2at + k$

  The equivalent Cartesian form is $(y - k)^2 = 4a(x + h)$
- The parametric form of a **circle** is $x(t) = r\cos(\omega t) + h, y(t) = r\sin(\omega t) + k$

  The equivalent Cartesian form is $(x - h)^2 + (y - k)^2 = r^2$
- The parametric form of an **ellipse** is $x(t) = a\cos(\omega t) + h, y(t) = b\sin(\omega t) + k$

  The equivalent Cartesian form is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$
- The parametric form of a **rectangular hyperbola** is $x = t + h, y = \frac{c}{t} + k$

  The equivalent Cartesian form is $y - k = \frac{c}{x - h}$

You can eliminate $t$ from the parametric parabola or rectangular hyperbola by simple substitution.

You can use $\sin^2(\theta) + \cos^2(\theta) = 1$ for the circle and ellipse as in Example **1**.

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## EXAMPLE 2

The position of an object at time $t$ seconds is given by $\mathbf{r}(t) = (3t - 5)\mathbf{j} + \frac{2}{t-1}\mathbf{j}$, where $\mathbf{r}$ is in metres. Express the path in Cartesian form, describe and sketch it.

### Solution

Write the Cartesian equations.

$$x = 3t - 5$$

$$y = \frac{2}{t-1}$$

Use the $x$-coordinate equation to eliminate $t$. Make $t$ the subject, then substitute it into the $y$-coordinate equation.

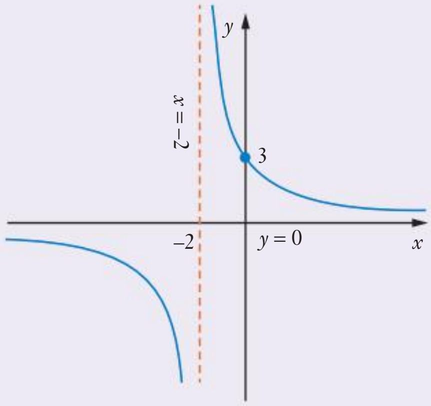
$$t = \frac{1}{3}(x + 5)$$

$$y = \frac{2}{\frac{1}{3}(x+5)-1} = \frac{6}{x+5-3} = \frac{6}{x+2}$$

Describe the path in your answer.

The path is $y = \frac{6}{x+2}$, a rectangular hyperbola, stretched vertically by a factor of 6 with horizontal asymptote $y = 0$, vertical asymptote $x = -2$ and $y$-intercept (0, 3).

Sketch the path.

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## Exercise 6.01 Vector functions of time

Examples 1,2

**1** Find the Cartesian equation for the path of the particle with position vector:

**a** $\mathbf{r}(t) = (4t + 3)\mathbf{i} + (3 - 2t)\mathbf{j}$

**b** $\mathbf{r}(t) = 3t^2\,\mathbf{i} - 4t\mathbf{j}$

**c** $\mathbf{r}(t) = 2\sin(t)\mathbf{i} + 5\cos(t)\mathbf{j}$

**d** $\mathbf{r}(t) = 3\cos(2t)\mathbf{i} + 3\sin(2t)\mathbf{j}$

**e** $\mathbf{r}(t) = 2t\mathbf{i} + \frac{3}{t}\mathbf{j}$

**2** The displacement of a particle at time $t$ is given by $\mathbf{r}(t) = [2 + 4\sin(t)]\mathbf{i} + [4 - 4\cos(t)]\mathbf{j}$. Express the path in Cartesian form, describe and sketch it.

**3** The position of a particle at time $t$ is given by $\mathbf{r}(t) = \frac{1}{t}\mathbf{i} + 9t\mathbf{j}$.

Express the path in Cartesian form, describe and sketch it.

**4** The position of an object at time $t$ is given by $\mathbf{r}(t) = [7\cos(t) + 1]\mathbf{i} + [7\sin(t) - 3]\mathbf{j}$. Express the path in Cartesian form, describe and sketch it.

**5** A particle moves so that its position is given by $\mathbf{r}(t) = 3\cos(t)\mathbf{i} + 5\sin(t)\mathbf{j}$. Express the path in Cartesian form, describe and sketch it.

**6** An object is projected so that its position at any time $t$ is given by $\mathbf{r}(t) = 2t\mathbf{i} + (8t - t^2)\mathbf{j}$, where the components are measured in metres and the time in seconds.

Determine the Cartesian equation for the path of the object, describe and sketch it.

**7** A particle moves so that its position is given by $\mathbf{r}(t) = [12\sin(4t) - 3]\mathbf{i} + [5\cos(4t) + 5]\mathbf{j}$. Find the Cartesian equation of its path, describe and sketch it.

**8** A particle moves along the path $\mathbf{r}(t) = (t + 4)\mathbf{i} + \left(\frac{2}{t} - 3\right)\mathbf{j}$. Find the Cartesian equation of its path, describe and sketch it for $t = 0$ to $10$.

### Problem solving

**9** Show that the Cartesian form of $x(t) = at^2 + h$, $y(t) = 2at + k$ is $(y - k)^2 = 4a(x + h)$.

**10** Show that the Cartesian form of $x(t) = a\cos(\omega t) + h$, $y(t) = b\sin(\omega t) + k$ is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

WS
Derivatives of vectors

WS
Integrals of vector functions

WS
Position, velocity and acceleration

# 6.02 Calculus of vectors

The derivative of a vector is defined in the same way as a normal derivative. The derivatives of constants, sums, products and so on follow the same rules as for scalar derivatives.

The unit vectors **i** and **j** are constants, so their derivatives are 0.

## Vector derivatives

- $\mathbf{r}'(t) = \dfrac{d\mathbf{r}}{dt} = \lim\limits_{h\to 0}\dfrac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$
- Constant vector: $\dfrac{d\mathbf{c}}{dt}$, in particular $\dfrac{d\mathbf{i}}{dt} = \dfrac{d\mathbf{j}}{dt} = \dfrac{d\mathbf{k}}{dt} = 0$
- Vector sum: $(\mathbf{v} + \mathbf{u})' = \mathbf{v}' + \mathbf{u}'$
- Vector difference: $(\mathbf{v} - \mathbf{u})' = \mathbf{v}' - \mathbf{u}'$
- Vector product: $(\mathbf{vu})' = \mathbf{v}'\mathbf{u} + \mathbf{vu}'$
- For $\mathbf{F}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, $\mathbf{F}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$
- **Chain rule**: For $\mathbf{F}(s)$ where $s = s(t)$, $\dfrac{d\mathbf{F}}{dt} = \dfrac{d\mathbf{F}}{ds} \times \dfrac{ds}{dt}$
- For a particle with position $\mathbf{r}(t)$, its **velocity** $\mathbf{v} = \dot{\mathbf{r}} = \dfrac{d\mathbf{r}}{dt}$ and **acceleration** $\mathbf{a} = \ddot{\mathbf{r}} = \dfrac{d\mathbf{v}}{dt}$

## EXAMPLE 3

The position of an object is given by $\mathbf{r}(t) = 10t\mathbf{i} + (20t - 5t^2)\mathbf{j}$.

**a** Find its velocity at time $t$.

**b** Find its acceleration.

**c** Sketch the path of the object and mark the velocity and acceleration vectors at $t = 3$.

### Solution

**a** Find the derivative, treating the components separately.

$\mathbf{v}(t) = \mathbf{r}'(t)$

$= 10\mathbf{i} + (20 - 10t)\mathbf{j}$

Write the answer.

The velocity is $10\mathbf{i} + (20 - 10t)\mathbf{j}$.


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**b** Find the derivative of the velocity.

$\mathbf{a}(t) = \mathbf{v}'(t)$

$= 0\mathbf{i} - 10\mathbf{j}$

$= -10\mathbf{j}$

Write the answer.

The acceleration is $-10\mathbf{j}$.

**c** Write the Cartesian equations of its position.

$x = 10t$

$y = 20t - 5t^2$

Eliminate $t$ and simplify.

$t = 0.1x$

$y = 20 \times 0.1x - 5 \times (0.1x)^2$

$= 2x - 0.05x^2$

Factorise.

$= x(2 - 0.05x)$

State the zeros.

$y = 0$ at $x = 0$ and $x = 40$.

State the maximum.

The maximum is at (20, 20).

Substitute $t = 3$ to find the position, velocity and acceleration.

$\mathbf{r}(3) = 10 \times 3\mathbf{i} + (20 \times 3 - 5 \times 3^2)\mathbf{j}$

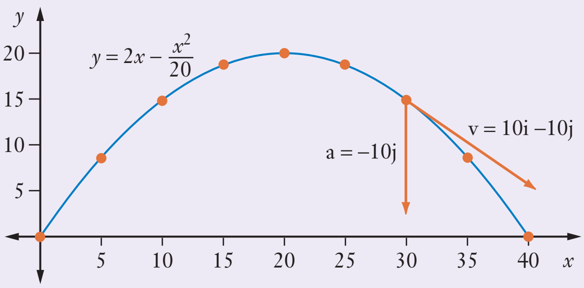
$= 30\mathbf{i} + 15\mathbf{j}$

$\mathbf{v}(3) = 10\mathbf{i} + (20 - 10 \times 3)\mathbf{j}$

$= 10\mathbf{i} - 10\mathbf{j}$

$\mathbf{a}(3) = -10\mathbf{j}$

Sketch the path and mark the velocity and acceleration at $t = 3$.

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You can work backwards to find the position of an object from the velocity.
You need additional information to find the value of the integration constant.

## EXAMPLE 4

The velocity of a particle at any time $t$ is given by $\mathbf{v}(t) = 6\mathbf{i} - 2t\mathbf{j}$, where time is in seconds and distance in metres.

**a** Find its position $\mathbf{r}(t)$, given that it begins from $7\mathbf{i} - 2\mathbf{j}$.

**b** Find its position, velocity and acceleration at $t = 5$ seconds.

### Solution

**a** Write the velocity.

$\mathbf{v}(t) = 6\mathbf{i} - 2t\mathbf{j}$

Find the integral.

$$\mathbf{r}(t) = \int \mathbf{v}\,dt$$

$$= \int (6\mathbf{i} - 2t\mathbf{j})\,dt$$

Separate into components.

$$= \int 6\mathbf{i}\,dt - \int 2t\mathbf{j}\,dt$$

**i** and **j** are constants. The constant of integration will be a vector.

$$= 6t\mathbf{i} - t^2\mathbf{j} + \mathbf{c}$$

Use the position at $t = 0$ to find the value of **c**.

$$\mathbf{r}(0) = 6 \times 0\mathbf{i} - 0^2\,\mathbf{j} + \mathbf{c} = 7\mathbf{i} - 2\mathbf{j}$$

$$\mathbf{c} = 7\mathbf{i} - 2\mathbf{j}$$

Substitute the value of **c** and simplify.

$$\mathbf{r}(t) = 6t\mathbf{i} - t^2\mathbf{j} + 7\mathbf{i} - 2\mathbf{j}$$

$$= (6t + 7)\mathbf{i} - (t^2 + 2)\mathbf{j}$$

**b** Substitute $t = 5$ in the velocity and position equations.

$$\mathbf{v}(5) = 6\mathbf{i} - 2 \times 5\mathbf{j} = 6\mathbf{i} - 10\mathbf{j}$$

$$\mathbf{r}(5) = (6 \times 5 + 7)\mathbf{i} - (5^2 + 2)\mathbf{j}$$

$$= 37\mathbf{i} - 27\mathbf{j}$$

Differentiate the velocity and simplify to find the acceleration.

$$\mathbf{a}(t) = \mathbf{v}'(t)$$

$$= 0\mathbf{i} - 2\mathbf{j}$$

$$= -2\mathbf{j}$$

Substitute $t = 5$ into the acceleration equation.

$$\mathbf{a}(5) = -2\mathbf{j}$$

Write the answer, with the correct units.

At $t = 5$ s, the particle is at $37\mathbf{i} - 27\mathbf{j}$ m with velocity $6\mathbf{i} - 10\mathbf{j}$ m/s and acceleration $-2\mathbf{j}$ m/s$^2$.


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When you integrate acceleration to find position, you need 2 pieces of information. This could be a position and a velocity, or even 2 positions.

## EXAMPLE 5

The acceleration of a particle is given by $\ddot{\mathbf{r}} = 0.6t\mathbf{i} - 0.2\sin(\pi t)\mathbf{j}$, where time is in seconds and acceleration in m/s$^2$. The particle starts from (1, 0) and passes through (3, 0) at $t = 8$ seconds.

**a** Find its position and velocity at any time.

**b** Find its position, velocity and acceleration after 12 seconds.

**c** Describe the motion.

### Solution

**a** Integrate to find the velocity. Remember, the integration constant is a vector.

$$\mathbf{v} = \dot{\mathbf{r}} = \int 0.6t\mathbf{i} - 0.2\sin(\pi t)\mathbf{j}\,dt$$
$$= 0.3t^2\,\mathbf{i} + \frac{0.2}{\pi}\cos(\pi t)\mathbf{j} + \mathbf{c}$$

Integrate again to find the position. Use a different variable for this integration constant.

$$\mathbf{r}(t) = \int 0.3t^2\mathbf{i} + \frac{0.2}{\pi}\cos(\pi t)\mathbf{j} + \mathbf{c}\,dt$$
$$= 0.1t^3\,\mathbf{i} + \frac{0.2}{\pi^2}\sin(\pi t)\mathbf{j} + \mathbf{c}t + \mathbf{d}$$

Use the position given for $t = 0$.

$$\mathbf{r}(0) = 0.1 \times 0^3\,\mathbf{i} + \frac{0.2}{\pi^2}\sin(\pi \times 0)\mathbf{j} + \mathbf{c} \times 0 + \mathbf{d} = \mathbf{i}$$
$$\mathbf{d} = \mathbf{i}$$

Use the position given for $t = 8$.

$$\mathbf{r}(8) = 0.1 \times 8^3\,\mathbf{i} + \frac{0.2}{\pi^2}\sin(\pi \times 8)\mathbf{j} + \mathbf{c} \times 8 + \mathbf{i} = 3\mathbf{i}$$
$$6.4\mathbf{i} + 8\mathbf{c} + \mathbf{i} = 3\mathbf{i}$$
$$\mathbf{c} = -0.55\mathbf{i}$$

Substitute the values of **c** and **d**.

$$\mathbf{v}(t) = 0.3t^2\mathbf{i} + \frac{0.2}{\pi}\cos(\pi t)\mathbf{j} - 0.55\mathbf{i}$$
$$= (0.3t^2 - 0.55)\mathbf{i} + \frac{0.2}{\pi}\cos(\pi t)\mathbf{j}$$
$$\mathbf{r}(t) = (0.1t^3 - 0.55t + 1)\mathbf{i} + \frac{0.2}{\pi^2}\sin(\pi t)\mathbf{j}$$

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| | | |
|---|---|---|
| **b** | Substitute $t = 12$. | $\ddot{\mathbf{r}} = (0.6 \times 12)\mathbf{i} - 0.2 \sin(\pi \times 12)\mathbf{j} = 7.2\mathbf{i}$ <br> $\mathbf{v} = (0.3 \times 12^2 - 0.55)\mathbf{i} + \frac{0.2}{\pi}\cos(\pi \times 12)\mathbf{j}$ <br> $= 42.65\mathbf{i} + \frac{0.2}{\pi}\mathbf{j}$ <br> $\mathbf{r} = (0.1 \times 12^3 - 0.55 \times 12 + 1)\mathbf{i} + \frac{0.2}{\pi^2}\sin(\pi \times 12)\mathbf{j}$ <br> $= 167.2\mathbf{i}$ |
| | Write the answer. | The particle is at $167.2\mathbf{i}$ m along the $x$-axis with velocity $42.65\mathbf{i} + \frac{0.2}{\pi}\mathbf{j}$ m/s and acceleration $7.2\mathbf{i}$ m/s$^2$ 12 seconds after the start. |
| **c** | Describe the motion in the component directions. | The particle is oscillating vertically (along the $y$-axis) while accelerating horizontally (along the $x$-axis). |

## INVESTIGATION

### SIMPLE HARMONIC MOTION

The particle in the above example is oscillating in the $y$ direction. This oscillation is called **simple harmonic motion** (**SHM**), which you will study in Chapter 9, *Differential equations*. The displacement, velocity and acceleration are all modelled by sinusoidal functions.

A particle undergoing SHM has displacement $x = 5(\sin 3t) + 2$ cm, where $t$ is in seconds.

- Find the velocity and acceleration of the particle
- Find the **period** $T$, the time to complete one oscillation
- Find the **frequency** $f$, the number of oscillations in unit time
- Show that the acceleration is given by $\mathbf{a} = -9x$
- Show that the velocity is given by $\mathbf{v} = \pm 3\sqrt{25 - x^2}$

Any object undergoing SHM about the origin has **displacement** of the form $x = A\sin(\omega t + \alpha)$, where $A$ is called the **amplitude**, $\omega$ is the angular velocity and $\alpha$ is the **phase shift**. Show that the following relationships are true.

- The velocity is given by $\mathbf{v} = A\omega\cos(\omega t + \alpha) = \pm\omega\sqrt{A^2 - x^2}$
- The acceleration is given by $\mathbf{a} = -A\omega^2\sin(\omega t + \alpha) = -\omega^2 x$
- The period is given by $T = \frac{2\pi}{\omega} = \frac{1}{f}$ the time to complete one oscillation
- The frequency is given by $f = \frac{\omega}{2\pi} = \frac{1}{T}$

Find some examples that approximate simple harmonic motion. Your teacher might want you to write a report.


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## Exercise 6.02 Calculus of vectors

**1** The position of an object is given by $\mathbf{r}(t) = 3t^2\mathbf{i} + 4\mathbf{j}$.

Example 3

**a** Find its velocity at time $t$.

**b** Find its acceleration.

**c** Find the velocity and acceleration vectors at $t = 2$ and describe the motion.

**2** The position of an object is given by $\mathbf{r} = 2t\mathbf{i} + t^2\mathbf{j}$.

**a** Find its velocity.

**b** Find its acceleration.

**c** Sketch the path of the object for $t = 0$–$10$ and describe its motion.

**3** The velocity of a particle at any time $t$ is given by $\dot{\mathbf{r}} = 5t\mathbf{i} - 2t\mathbf{j}$, where time is in seconds and distance in metres. At $t = 0$, its position is $\mathbf{r} = 3\mathbf{i} - 4\mathbf{j}$.

Example 4

**a** Find its position at time $t$, given that it begins from $7\mathbf{i} - 2\mathbf{j}$.

**b** Find its acceleration at time $t$.

**c** Sketch its path from $t = 0$ to $t = 3$ and describe its motion.

**4** The velocity of a particle at any time $t$ is given by $\mathbf{v}(t) = e^{2t}\mathbf{i} + (3t^2 + 4)\mathbf{j}$. At $t = 0$, it is at (2, 1). Time is in seconds and distance in metres.

**a** Find its position $\mathbf{r}(t)$.

**b** Find its acceleration at time $t$.

**5** The acceleration of a particle is given by $\mathbf{a}(t) = 12t\mathbf{i} - 9\mathbf{j}$, where time is in seconds and acceleration is in m/s$^2$. The particle starts from $3\mathbf{i} + 4\mathbf{j}$ m with velocity $4\mathbf{i} - 2\mathbf{j}$.

Example 5

**a** Find its position and velocity at any time.

**b** Find its position, velocity and acceleration after 5 seconds.

**6** The acceleration of a particle is given by $\ddot{\mathbf{r}} = 9\cos(2\pi t)\mathbf{i} + 15t\mathbf{j}$, where time is in seconds and acceleration is in cm/s$^2$. The particle starts from $3\mathbf{i}$ and has velocity $40\mathbf{j}$ cm/s at 2 seconds.

**a** Find its position and velocity at any time.

**b** Find its position, velocity and acceleration after 4 seconds.

**c** Describe the motion.

### Problem solving

**7** The position of an object is given by $\mathbf{r}(t) = 3\sin(0.5\pi t)\mathbf{i} + 4\cos(0.5\pi t)\mathbf{j}$.

Sketch the path of the object for $t = 0$ to 3 and fully describe its motion.

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**8** The velocity of a particle at any time $t$ is given by $\mathbf{v}(t) = 15\mathbf{j} - \frac{8}{t^2}\mathbf{i}$ m/s, where $t$ is in seconds. At $t = 1$, the particle is at (8, 7), where the distances are in metres.

Sketch the path of the object for $t = 0.5$ to 4 s and fully describe its motion.

**9** The acceleration of a particle is given by $\ddot{\mathbf{r}} = -0.32\pi^2 \sin(0.2\pi t)\mathbf{i} - 0.28\pi^2 \cos(0.2\pi t)\mathbf{j}$, where time is in seconds and acceleration is in m/s$^2$. The particle passes through (11, 2) at 2.5 seconds and (3, –5) at 5 seconds.

Sketch the path of the object for $t = 0$ to 7.5 s and fully describe its motion.

# 6.03 Straight-line motion

You saw last year that an unbalanced force on an object produces acceleration. The relationship is $\mathbf{F} = m\mathbf{a}$, where $\mathbf{F}$ is the force in Newtons, $m$ is the mass in kg and $\mathbf{a}$ is the acceleration in m/s$^2$. You can also derive the following relationships.

### Straight-line motion with constant acceleration

It is sufficient for the vectors $s$, $v$, $u$, and $a$ to be signed to show direction for motion in a straight line. The relationships between these are:

- $v = u + at$
- $v^2 = u^2 + 2as$
- $s = ut + \frac{1}{2}at^2$
- $s = \left(\frac{u+v}{2}\right)t$

where $s$ = displacement, $v$ = velocity, $u$ = initial velocity, $a$ = acceleration and $t$ = time.

Near the Earth, the acceleration of gravity is approximately constant, about 9.8 m/s$^2$.

You can show each of the relationships from $\mathbf{v} = \frac{d\mathbf{s}}{dt}$ and $\mathbf{a} = \frac{d\mathbf{v}}{dt}$.

For example, $v = \int a\,dt = at + c$

At $t = 0$, $v = u$, so $v = at + u = u + at$.

In most cases, you will find it easiest to change quantities to SI units (kg, m, s, etc.) before solving a problem.


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## EXAMPLE 6

Trent is cycling to school and moving at 18 km $h^{-1}$. He passes a parked truck just as it pulls out in the same direction. The truck has a mass of 3 tonnes and the total forward force with which it accelerates for 20 s is 1200 N. It then continues with uniform speed.

**a** How far past the parking spot does the truck reach Trent?

**b** If the truck driver kept accelerating until he passed Trent, how far would it have to travel?

**c** If the truck continued to accelerate until it was 30 m head of Trent, what speed would it reach?

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### Solution

**a** Change Trent's speed to m/s.

$$\text{Trent's speed} = 18 \text{ km/h} = 18 \times \frac{1000}{60 \times 60} \text{ m/s} = 5 \text{ m/s}$$

Calculate the acceleration of the truck.

$$F = ma$$

$$a = \frac{F}{m} = \frac{1200}{3000} = 0.4 \text{ m/s}^2$$

**Method 1**

| | |
|---|---|
| Find the distance they each move in the first 20 seconds. Write the formulas first. | Trent's distance $= ut = 5 \times 20 = 100$ m<br>Truck distance $= ut + \frac{1}{2}at^2$<br>$= 0 + \frac{1}{2} \times 0.4 \times 20^2$<br>$= 80$ m |
| The truck's speed is 3 m/s faster than Trent's. | After the first 20 s, the truck gains 3 m each second on Trent.<br>Distance between them = 20 m |
| Find the time to pass after the first 20 s and the distance the truck travels. | Time to pass $= \frac{20}{3} = 6\frac{1}{3}$ s<br>Distance travelled by truck $= 8 \times 6\frac{1}{3}$<br>$= 53\frac{1}{3}$ m |
| Find the total distance. | Total distance $= 80 + 53\frac{1}{3}$ m<br>$= 133\frac{1}{3}$ m |

**Method 2**

| | |
|---|---|
| Write the distances travelled in time $t$ seconds. | Trent's distance $= 5t$ m<br>Truck distance $= 80 + 8(t - 20)$ m |
| Find when the distances are the same. | $80 + 8(t - 20) = 5t$<br>$80 + 8t - 160 = 5t$<br>$3t = 80$<br>$t = 26\frac{2}{3}$ s |
| Find the distance Trent travels. | Distance of Trent $= 5 \times 26\frac{2}{3}$ m<br>$= 133\frac{1}{3}$ m |
| Write the answer. | The truck passes Trent $133\frac{1}{3}$ m past the parking spot. |


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| | | |
|---|---|---|
| **b** | If the truck continues to accelerate, write the distances travelled at time $t$. | Cyclist distance = $5t$<br>Truck distance = $\frac{1}{2} \times 0.4 \times t^2$ |
| | Find when they are the same. | $5t = 0.2t^2$<br>$0.2t^2 - 5t = 0$<br>$0.2t(t - 25) = 0$<br>$t = 0$ or $t = 25$ |
| | Choose the correct time. | The truck passes Trent when $t = 25$. |
| | Find the distance travelled. | Distance = $5 \times 25 = 125$ m |
| | Write the answer. | If the truck keeps accelerating, it passes Trent 125 m past the parking spot. |
| **c** | Find when the truck's distance is 30 m more than Trent's. | $0.2t^2 - 5t = 30$<br>$t^2 - 25t - 150 = 0$ |
| | Use the quadratic formula. | $t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$<br>$= \frac{-(-25) \pm \sqrt{(-25)^2 - 4 \times 1 \times (-150)}}{2 \times 1}$<br>$= \frac{25 \pm 35}{2} = -5$ s or 30 s |
| | Choose the correct time. | The truck is 30 m ahead at $t = 30$ s. |
| | Find the truck's speed. | Speed = $at = 0.4 \times 30$ m/s<br>$= 12$ m/s |
| | State the answer. | If the truck keeps accelerating until it is 30 m ahead of Trent, it will reach a speed of 12 m/s. |

12 m/s = 12 × 3.6 = 43.2 km/h. The speed limit in a school zone is 40 km/h!

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You need to use integration for variable force and acceleration.

## EXAMPLE 7

A 5 kg object has a force $F = 40t - 6t^2$ acting on it, where $F$ is in newtons and $t$ is seconds. The force acts in a constant direction for 5s. If it starts from rest:

**a** what speed does it reach?

**b** what distance does it cover while the force is acting?

### Solution

**a** Find the acceleration.

$$F = ma$$

$$a = \frac{F}{m}$$

Substitute values.

$$= \frac{40t - 6t^2}{5}$$

$$= 8t - 1.2t^2$$

Use integration to find $v$.

$$\frac{dv}{dt} = 8t - 1.2t^2$$

$$v = \int (8t - 1.2t^2)\,dt$$

$$= 4t^2 - 0.4t^3 + c$$

Find $c$ and write the expression for $v$.

At $t = 0$, $v = 0$, so $c = 0$

$$v = 4t^2 - 0.4t^3$$

Substitute $t = 5$.

$$= 4 \times 5^2 - 0.4 \times 5^3$$

$$= 50 \text{ m/s}$$

Write the answer.

The object reaches a speed of 50 m/s.


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**b** Integrate to find the displacement.

$\frac{ds}{dt} = v = 4t^2 - 0.4t^3$

$s = \int (4t^2 - 0.4t^3)\,dt$

$= \frac{4}{3}t^3 - 0.1t^4 + c$

Find the value of $c$.

At $t = 0$ it hasn't moved, so $s = 0$ and $c = 0$.

Write the expression for $s$.

$s = \frac{4}{3}t^3 - 0.1t^4$

Substitute $t = 5$ s and calculate the result.

$= \frac{4}{3} \times 5^3 - 0.1 \times 5^4$

$= 104.166\ldots$ m

Write the answer.

The object moves about 104.2 m while the force is acting.

## Exercise 6.03 Straight-line motion

Example 6

**1** Find the velocity indicated for each particle moving with constant acceleration in a straight line.

**a** Initial velocity of 4 m/s, acceleration of 3 m/s$^2$ for 5 seconds. Final velocity?

**b** Acceleration of 4 m/s$^2$ for 6 seconds to reach 18 m/s. Original velocity?

**c** Initial velocity of 6 m/s, acceleration of 4 m/s$^2$ over a distance of 8 m. Final velocity?

**d** Acceleration of 3 m/s$^2$ over a distance of 16 m to a velocity of 11 m/s. Original velocity?

**e** Acceleration of 5 m/s$^2$ for 4 s over a distance of 28 m. Original velocity?

**2** Find the displacement for each particle moving in a straight line with a constant acceleration of:

**a** 4 m/s$^2$ for 3 seconds from an initial velocity of 5 m/s

**b** –2 m/s$^2$ for 5 seconds from an initial velocity of 40 m/s

**c** 6 m/s$^2$ from an initial velocity of 14 m/s to a final velocity of 16 m/s

**d** –3 m/s$^2$ from an initial velocity of 18 m/s to rest

**3** Find the constant acceleration in a straight line of each particle.

**a** Initial velocity of 10 m/s and velocity of 4 m/s after 2 seconds.

**b** Changing velocity from 5 m/s to 17 m/s in 3 seconds.

**c** Taking 20 m to change velocity from 10 m/s to 15 m/s.

**d** Taking 48 m to change velocity from 40 m/s to 28 m/s.

**e** Taking 6 seconds from an initial velocity of 4 m/s to cover 150 m.

**4** Find the time taken for constant acceleration in a straight line for each situation.

**a** To change from 40 m/s at 3 m/s$^2$ to 55 m/s.

**b** To cover 18 m from an initial velocity of 3 m/s at 12 m/s$^2$.

**c** To decrease in velocity by 20 m/s from an initial speed of 50 m/s at –4 m/s$^2$.

**d** To go 231 m from an initial speed of 5 m/s at 8 m/s$^2$.

Example 7

**5** An acceleration of $(5t + 3)$ m/s$^2$ acts for 8 seconds on a particle with an initial velocity of 6 m/s in the same direction as the acceleration.

**a** What is the final speed?

**b** What distance does it cover during the time?

**6** A 3 kg object with an initial velocity of 80 m/s has a force of $6 - 3t$ N acting on it for 5 seconds.

**a** What is the final velocity?

**b** What is the displacement?

## Problem solving

**7** A monorail train begins from rest and travels with a uniform acceleration of 1.2 m s$^{-2}$. What will be the speed of the monorail after 30 s, and how far will it have travelled?

**8** A car travelling at 54 km h$^{-1}$ comes to rest with uniform retardation in 5 s. Calculate the acceleration and the distance travelled in the time taken for the car to be brought to rest.

**9** A train moves 48 m in the first 6 seconds of motion and 32 m in the last 2 seconds of 12 seconds of travel at constant acceleration. What is the initial velocity and the total distance travelled?

**10** Debbie is cycling at 16 km/h when she is overtaken by Anne going at a constant 20 km/h. Debbie immediately begins accelerating at a constant rate. At what speed will she pass Anne?

**11** An object of mass 2 kg moves from rest under the action of a force with magnitude $40 - 20t$ N (after $t$ s). When the object reaches a speed of 20 m s$^{-1}$, the force is removed and the object continues with constant speed for 4 s. It is then brought to rest by a constant force of 60 N. Calculate the total distance travelled.

**12** A body of mass 5 kg is acted on by a force in a straight line. The magnitude of the force after $t$ s given by $10(2t - 3t^2)$ N. If the body has an initial velocity of 3 m s$^{-1}$ in the same direction as the force, calculate its velocity after 4 s.


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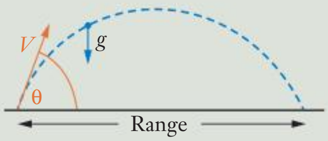
# 6.04 Projectile motion


You apply a force to a ball and then let it go to throw it. After that, the path of the ball is determined by gravity and air resistance. This is called **projectile motion** and applies to shooting arrows, jumping over a puddle and so on. For low speeds and short distances, you can ignore air resistance.

### Projectile motion

- Has an initial **velocity** with a **speed of projection** $V$ and an **angle of projection** $\theta$
- The only forces are those of **gravity** and **air resistance**
- In simple cases, air resistance is small and may be neglected
- The path of the **projectile** is called its **trajectory**
- The final horizontal displacement is called its **range**




You can assume the acceleration of gravity is $g = 9.8$ m/s$^2$ vertically downwards in this work. In most cases, you will find it easiest to separate the horizontal and vertical components of the motion.

## EXAMPLE 8

A projectile is launched from a horizontal plane with a speed of 29.4 m/s at an angle of 30°. Find the:

**a** vector and Cartesian equations of the trajectory

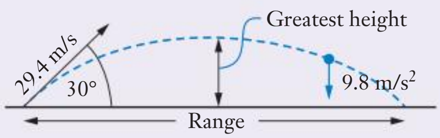
**b** time of flight

**c** range

**d** greatest height

### Solution

**a** Sketch the situation. Remember the acceleration is $g$ m/s$^2$.




Write the directions for the equation. Make **i** vertical and **j** horizontal.

| | | |
|---|---|---|
| | Find the initial velocity in component form. | $\mathbf{v} = 29.4\cos(30°)\mathbf{i} + 29.4\sin(30°)\mathbf{j}$<br>$= 14.7\sqrt{3}\,\mathbf{i} + 14.7\mathbf{j}$ |
| | Write the height and horizontal distance at time $t$. | $\text{Height} = ut + \frac{1}{2}at^2$<br>$= 14.7t - \frac{1}{2} \times 9.8t^2$<br>$= 14.7t - 4.9t^2$<br>$\text{Horizontal distance} = 14.7\sqrt{3}\,t$ |
| | Combine. | $\mathbf{r}(t) = 14.7\sqrt{3}\,t\mathbf{i} + (14.7t - 4.9t^2)\mathbf{j}$ |
| | Eliminate $t$ to get the Cartesian equation. | $x = 14.7\sqrt{3}\,t$, so $t = \frac{x}{14.7\sqrt{3}}$<br>$y = 14.7t - 4.9t^2$<br>$= 14.7\sqrt{3} \times \frac{x}{14.7\sqrt{3}} - 4.9 \times \left(\frac{x}{14.7\sqrt{3}}\right)^2$<br>$= x - \frac{x^2}{132.3}$ |
| | Write the answer. | The vector equation is<br>$\mathbf{r}(t) = 14.7\sqrt{3}\,t\mathbf{i} + (14.7t - 4.9t^2)\mathbf{j}$.<br>The Cartesian equation is $y = x - \frac{x^2}{132.3}$. |
| **b** | The time of flight will be the time when the height returns to zero. | Time of flight is given by height = 0.<br>$14.7t - 4.9t^2 = 0$<br>$4.9t(3 - t) = 0$<br>$t = 0$ or $t = 3$ |
| | Write the answer. | $t = 0$ is the launch, so the time of flight is 3 seconds. |
| **c** | Find the horizontal distance for the time of flight. | Range = horizontal distance at 3 s<br>$= 14.7\sqrt{3} \times 3$<br>$= 44.1\sqrt{3} \approx 76.383\ldots$ m |
| | Write the answer. | The range is $44.1\sqrt{3}$ m, about 76.4 m. |
| **d** | The time taken to reach the greatest height will be half the flight time. | By symmetry, the greatest height is at 1.5 s.<br>$\text{Greatest height} = 14.7 \times 1.5 - 4.9 \times 1.5^2$<br>$= 33.075$ m |
| | Write the answer. | The greatest height reached is 33.075 m. |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You can follow the same steps as in the previous example to find general formulas. The path of a projectile is a parabola.

## Projectile motion formulas

These formulas are for a projectile with initial speed $V$ and angle of projection $\theta$.

- Vector equation of path: $\mathbf{r}(t) = V\cos(\theta)\,t\mathbf{i} + (14.7t - 4.9t^2)\mathbf{j}$
- Cartesian equation of path: $y = x\tan(\theta) - \dfrac{gx^2}{2V^2\cos^2(\theta)}$ or

$$y = x\tan(\theta) - \frac{gx^2}{2V^2}[1 + \tan^2(\theta)]$$

- Time of flight: $T = \dfrac{2V\sin(\theta)}{g}$
- Range: $\dfrac{2V^2\sin(\theta)\cos(\theta)}{g}$ or $\dfrac{V^2\sin(2\theta)}{g}$
- Greatest range: $\dfrac{V^2}{g}$ at $\theta = 45°$
- Greatest height: $\dfrac{V^2\sin^2(\theta)}{2g}$ at $T_{\frac{1}{2}} = \dfrac{V\sin(\theta)}{g}$

The greatest range follows from the fact that the greatest value of $\sin(2\theta)$ is 1 at $\theta = 45°$.

It may be better to learn the methods of working out projectile motion problems than trying to memorise the formulas.


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## EXAMPLE 9

A stone is projected from a platform 12 m above a horizontal field with an initial velocity of 25 m/s at an elevation angle of 55°. A wall of height 3 m is in the path of the stone at a distance of 60 m from the platform. Give your answers correct to 2 decimal places.

**a** What is the greatest height of the stone above the field?

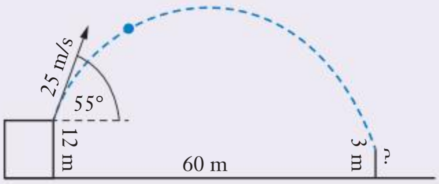
**b** Does the stone clear the wall?

**c** If it clears the wall, how much does it clear it by? If not, where does it hit it?

**d** If the stone clears the wall, where does it land?

### Solution

**a** Sketch the situation.




Write the horizontal and vertical components of the velocity.

$u_x = 25\cos(55°)$ m/s

$u_y = 25\sin(55°)$ m/s

Write the vertical speed at time $t$.

$v_y = 25\sin(55°) - 9.8t$ m/s

At the greatest height, the vertical velocity will be zero. Store the 'exact' answer on your calculator.

At the greatest height, $v_y = 0$.

$25\sin(55°) - 9.8t = 0$

$t = \dfrac{25\sin(55°)}{9.8} = 2.089\ldots$ s

Write an expression for the height.

$h = 12 + 25\sin(55°)\,t - 0.5 \times 9.8t^2$

Substitute the 'exact' value and calculate the answer.

$= 12 + 25\sin(55°) \times 2.089\ldots - 4.9 \times 2.089\ldots^2$

$= 33.397\ldots$ m

Write the answer.

The greatest height is about 33.40 m.

**b** The stone will reach the wall when the horizontal distance is 60 m.

Horizontal distance $= 25\cos(55°)\,t$

Stone reaches wall when $25\cos(55°)\,t = 60$

$t = \dfrac{60}{25\cos(55°)} = 4.184\ldots$ s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| | | |
|---|---|---|
| | Substitute to find the height. | $h = 12 + 25\sin(55°) \times 4.184\ldots - 4.9 \times 4.184\ldots^2$ <br> $= 11.899\ldots$ |
| | Write the answer. | The stone clears the wall. |
| **c** | Find the height above the wall. | It clears the wall by about $11.90 - 3 = 8.90$ m. |
| **d** | Write and solve the equation to find where the stone lands. | Stone lands when $h = 0$. <br> $12 + 25\sin(55°)\,t - 0.5 \times 9.8t^2 = 0$ <br> $4.9t^2 - 25\sin(55°)\,t - 12 = 0$ |
| | Use the quadratic formula. | $t = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ <br> $= \dfrac{-[-25\sin(55°)] \pm \sqrt{[-25\sin(55°)]^2 - 4 \times 4.9 \times (-12)}}{2 \times 4.9}$ |
| | Store the 'exact' answer on your calculator. | $= -0.521\ldots$ or $4.700\ldots$ |
| | Find the horizontal distance. | $x = 25\cos(55°) \times 4.700\ldots$ <br> $= 67.400\ldots$ |
| | Write the answer. | The stone lands about 67.40 m from the platform. |

## TECHNOLOGY

### Projectile motion using a graphics calculator

You could solve the above example more quickly using your graphics calculator.

Make sure your calculator is set to degrees.

The components are $x(t) = 25\cos(55°)\,t$ and

$$y(t) = 12 + 25\sin(55°)\,t - 0.5 \times 9.8t^2$$

Eliminate $t$ by substituting $t = \dfrac{x}{25\cos(55)}$ in $y$ to get

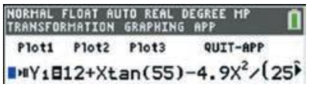
$$y = 12 + \frac{25\sin(55)}{25\cos(55)}x - \frac{4.9}{25^2[\cos(55)]^2}x^2$$

$$= 12 + \tan(55)x - 4.9\,x^2 \div (25^2 \times [\cos(55)]^2)$$

TI-Nspire CX
Chapter 6

### TI-84 Plus CE

Make sure the angle is set to degrees.

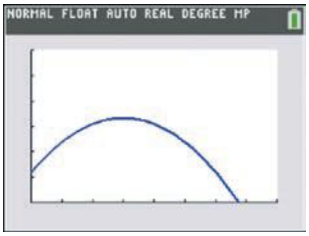



Press [y=] to enter the equation.

Type the function into $Y_1$ and press [enter].

Set the [window] to $0 \le x \le 80$ and $0 \le y \le 60$.

Press [graph] to see the trajectory.

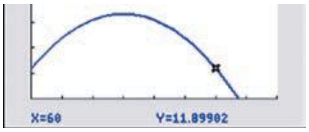



Press [2nd][trace] (calc) and choose 4: maximum.

Follow the prompts to find the maximum.

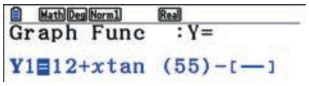

Use calc again and choose 1: value to find the height at the wall ($x = 60$).

Use calc but press 2: zero to find where the stone lands.




### Casio fx-CG20AU

Use the Graph menu.

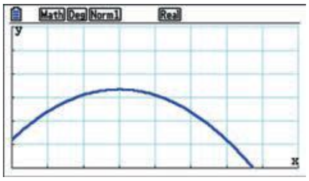



Make sure your calculator is set to degrees.

Type the function into Y1 and press [EXE].

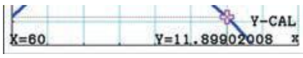
Set the V-Window to $0 \le x \le 80$ and $0 \le y \le 60$.

Press [EXIT] and choose DRAW to see the trajectory.

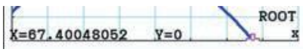



Press [SHIFT] [F5] (G-Solv) and choose MAX to find the maximum.


Use G-Solve again and choose Y-CAL to find the height of the wall ($x = 60$).




Use G-Solve, but choose ROOT to find where the stone lands.

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## Exercise 6.04 Projectile motion

Your teacher will say which problems you should do with your graphics calculator.

**1** For each projectile described, find the horizontal and vertical initial velocities, correct to 2 decimal places. Example 8

**a** Initial velocity = 20 m/s, angle of projection = 35°

**b** Initial velocity = 12 m/s, angle of projection = 62°

**c** Initial velocity = 18 m/s, angle of projection = 84°

**d** Initial velocity = 15 m/s, angle of projection = 9°

**e** Initial velocity = 24 m/s, angle of projection = 42°

**2** Find expressions for the horizontal and vertical positions at time $t$ s of each projectile in question **1**, given the initial heights of 5, 9, 6, 2 and 7 m respectively. Give your answers correct to 2 decimal places.

**3** Find the times taken to reach the greatest height, and the greatest heights reached of each projectile in question **1**, given initial heights of 4, 2, 7, 5 and 9 m respectively. Give your answers correct to 2 decimal places.

**4** Find the times taken to return to the ground and the horizontal distance from the starting point for each projectile described. Assume the ground is level and give your answers correct to 3 decimal places. Example 9

**a** Initial velocity = 12 m/s, angle of projection = 17°, initial height = 6 m

**b** Initial velocity = 8 m/s, angle of projection = 71°, initial height = 4 m

**c** Initial velocity = 16 m/s, angle of projection = 21°, initial height = 3 m

**d** Initial velocity = 32 m/s, angle of projection = 48°, initial height = 10 m

**e** Initial velocity = 25 m/s, angle of projection = 65°, initial height = 5 m

**5** Find the greatest possible height reached above the launching point for projectiles with the following launching speeds. Give your answers correct to one decimal place.

**a** 18 m/s **b** 12 m/s **c** 24 m/s **d** 6 m/s **e** 31 m/s

### Problem solving

**6** James can throw a 120 g stone a maximum distance of 60 m on flat ground. What speed can he throw it at? What about Felecia, who can manage 70 m?

**7** A cannonball is fired over a horizontal plane at an angle of projection of 40° at a speed of 30 m $s^{-1}$. There is an obstacle 8 m high at a distance of 20 m from the firing point. Show that the cannonball clears the object and find the distance it lands from the cannon.

**8** Samson was standing on a mound 1.7 m high. David fired a stone from a slingshot from a position horizontally 120 m away from the bottom of the mound. 2.5 seconds later, the stone struck Samson on the top of the head. Samson was 2.05 m tall. What was the initial speed and angle of projection of the stone?

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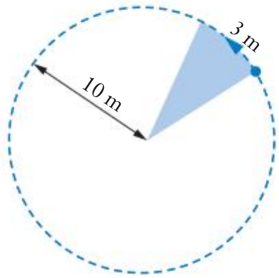
**9** Andrea threw a ball with an initial velocity of 11 m s$^{-1}$ at an angle of 65°. She released the ball 1.8 m above the ground. At what height did the ball hit a vertical wall 10 m away?

**10** A bullet was fired from a high-velocity rifle. The horizontal component of its muzzle velocity was 600 m/s. It hit a target 500 m away at a height of 2 m. What was the angle of elevation of the rifle?

WS Circular motion

## 6.05 Uniform circular motion

Consider an object moving in a circle of radius 10 m at constant speed 3 m/s.




The circumference of the circle is $20\pi$, so it takes $\frac{20\pi}{3}$ seconds for each revolution. The period $T = \frac{2\pi r}{v}$.

A revolution is $2\pi$ radians, so it moves through an angle of $\frac{3}{10}$ radians each second. This is called the **angular velocity** and has the symbol $\omega$. Notice that $\omega = \frac{v}{r}$ or $v = \omega r$.

Put the origin at the centre of the circle, and the $x$-axis in the direction of its starting position. Assume that it is moving anticlockwise.

Then it starts at (10, 0) and $t$ seconds later, it will have moved through an angle of $\frac{3t}{10}$.

You can write its position at time $t$ as $10 \cos\left(\frac{3t}{10}\right)\mathbf{i} + 10 \sin\left(\frac{3t}{10}\right)\mathbf{j}$.

Using $\omega = \frac{3}{10}$ gives you the position as $\mathbf{r} = 10 \cos(\omega t)\mathbf{i} + 10 \sin(\omega t)\mathbf{j}$. You can use $\mathbf{r}$ as the length; it is actually just the radius.

The velocity is $\mathbf{v} = \frac{d\mathbf{r}}{dx} = -10\omega \sin(\omega t)\mathbf{i} + 10\omega \cos(\omega t)\mathbf{j}$

and the acceleration is $\mathbf{a} = \frac{d\mathbf{v}}{dx} = -10\omega^2 \cos(\omega t)\mathbf{i} - 10\omega^2 \sin(\omega t)\mathbf{j}$.

Notice that $\mathbf{a} = -\omega^2\mathbf{r}$, so the acceleration is in the opposite direction to the position (radius).

Now
$$\begin{aligned}\mathbf{v} \cdot \mathbf{a} &= [-10\omega \sin(\omega t)\mathbf{i} + 10\omega \cos(\omega t)\mathbf{j}] \cdot [-10\omega^2 \cos(\omega t)\mathbf{i} - 10\omega^2 \sin(\omega t)\mathbf{j}]\\ &= 100\omega^3 \sin(\omega t)\cos(\omega t) - 100\omega^3 \cos(\omega t)\sin(\omega t)\\ &= 0\end{aligned}$$

Also
$$\begin{aligned}|\mathbf{v}| &= \sqrt{[-10\omega \sin(\omega t)]^2 + [10\omega \cos(\omega t)]^2}\\ &= \sqrt{100\omega^2[\sin^2(\omega t) + \cos^2(\omega t)]}\\ &= 10\omega\end{aligned}$$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Similarly, $|\mathbf{a}| = 10\omega^2 = 10|\mathbf{v}|$

This means that the acceleration is *always perpendicular to the velocity* and $a = vr$.

You can show that the relationships above are true for anything moving in a circle at constant speed (**uniform circular motion**).

## Uniform circular motion

For an object moving in a circle of radius $r$ at constant speed $v$:

- the angular velocity $\frac{d\theta}{dt} = \omega = \frac{v}{r}$, so $v = \omega r$
- the acceleration is given by $\mathbf{a} = -\omega^2\mathbf{r}$, so
- the acceleration is towards the centre of the circle, and $a = \omega^2 r = \frac{v^2}{r}$
- the position at time $t$ is given by $\mathbf{r} = r\cos(\omega t)\mathbf{i} + r\sin(\omega t)\mathbf{j}$
- the period is given by $T = \frac{2\pi r}{v} = \frac{2\pi}{\omega}$

## EXAMPLE 10

A particle of mass 3 kg moves on a smooth horizontal plane in a circle of radius 4 m at a speed of 8 m/s.

**a** What is its angular velocity?

**b** What is its acceleration?

**c** What force must be acting? What is its direction?

### Solution

**a** Angular velocity is the angle in unit time.

$\omega = \frac{8}{4} = 2$ per second

**b** Use the formula.

$a = \omega^2 r = 2^2 \times 4 = 16$ m/s$^2$

You could use $a = \frac{v^2}{r}$ instead of $a = \omega^2 r$.

**c** Use the formula.

$F = ma = 3 \times 16 = 48$ N

Include the direction.

The force is 48 N towards the centre of the circle.

You should already know from whirling things on a string that there must be a force towards the centre keep something going in a circle. On a slope, part of the gravitational force (and acceleration) is exerted down the slope. *Banking* of curves on a road or railway means this component acts towards the centre of the curve.

## EXAMPLE 11

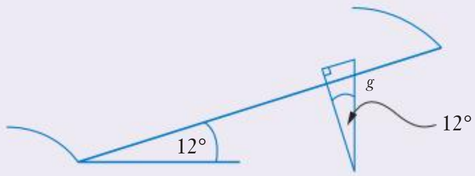
A road curve with a radius of 120 m is banked at an angle of 12°.

**a** What maximum speed should be given for the curve for slippery conditions?

**b** At a speed of 80 km/h, what is the force needed for a car of mass 960 kg to round the curve?

### Solution

**a** Draw the slope on the curve and show the acceleration of gravity in relation to the slope.




Find the component of $g$ down the slope.

$g$ down the slope = 9.8 sin (12°)

Use the formula for acceleration and velocity to find the speed.

$$a = \frac{v^2}{r}$$

$$v^2 = ar$$

$$= 9.8 \sin(12°) \times 120 \text{ m/s}$$

$$= 244.504\ldots \text{ m/s}$$

Find the square root.

$$v = \sqrt{244.504\ldots}$$

$$= 15.636\ldots \text{ m/s}$$

Change to km/h.

$$v = 15.636\ldots \times 3.6 \text{ km/h}$$

$$= 56.291\ldots \text{ km/h}$$

Write the answer.

A maximum speed of 55 km/h should be signposted to allow for slippery conditions.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**b** Change 80 km/h to m/s. $80 \text{ km/h} = \frac{80}{3.6}$

$= 22.222\ldots \text{ m/s}$

Find the acceleration needed. $a = \frac{v^2}{r}$

$= \frac{(22.222\ldots)^2}{120}$

$= 4.115\ldots$

Find the force needed. $F = ma$

$= 960 \times 4.115\ldots$

$= 3950.617\ldots$

Write the answer. A force of about 4000 N is required.

In dry conditions, the tyres of a car exert a sideways force when you are turning, so a higher speed is possible.

## Exercise 6.05 Uniform circular motion

**1** An object is travelling at constant speed in a circular path with a radius of 2.4 m. The period of the motion is 2.8 s. Calculate: Example 10

- **a** the angular velocity
- **b** the speed of the object
- **c** the acceleration of the object.

**2** A weather satellite orbits the Earth in a circular path of radius 7500 km and completes 12 revolutions per day. Calculate its velocity in km $h^{-1}$.

**3** A disc is spinning about its centre at 300 rpm (revolutions per minute). Calculate:

- **a** the angular velocity (in radians/s) of a point on the rim
- **b** the speed of a point 2 cm from the centre.

**4** A particle of mass 4 kg is attached by a string of length 2 m to a fixed point on a smooth flat surface. The particle moves with a speed of 5 m/s. Example 11

- **a** What is the acceleration of the particle?
- **b** Find the tension in the string.
- **c** If the maximum tension that the string can withstand is 150 N, find the greatest number of revolutions per second that the particle can make without breaking the string.

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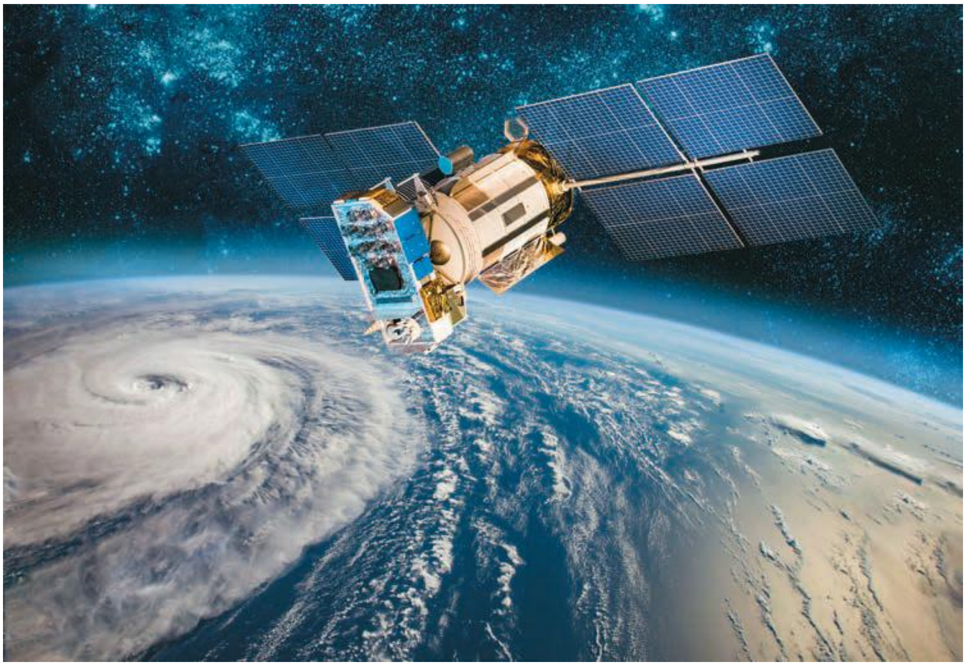
**5** A car has to go around a circular exit ramp for a highway. The ramp has a radius of 50 m and needs to accommodate a speed limit of 45 km/h. At what angle should the curve be banked to avoid using the friction between the tyres and the road?

## Problem solving

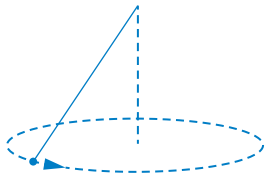
**6** The diameter of a standard adult's bicycle wheel is 28 inches (1 inch ≈ 0.0254 m). The bicycle is moving at a constant speed of 15 km $h^{-1}$. Calculate the acceleration of a piece of chewing gum stuck on the rim.

**7** A telecommunications satellite is placed in a circular orbit of radius 35 000 km. The period of its orbit is 15 h. What is the speed of the satellite in km/h?



Shutterstock.com/Andrey Armyagov

**8** A conical pendulum has a mass of 120 g on the end of a 25 cm long string. The mass goes around in horizontal circle at 80 rpm. What angle does the string make with the vertical?



**9** A body of 1 kg suspended by a fine string from a fixed point describes a circle in a horizontal plane. It makes 3 complete revolutions every 2 s. Find depth of the body below the fixed point, correct to the nearest cm.

**10** A circus performer rides a bicycle around a circular loop-the-loop. If the loop has a radius of 3 m, what is the least speed that the performer must have at the top of the loop in order to remain in contact with it?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# 6. CHAPTER SUMMARY

## Vector calculus

### Vector and Cartesian equations of motion

- The vector equation of motion $\mathbf{r}(t) = a(t)\mathbf{i} + b(t)\mathbf{j}$ is equivalent to the Cartesian equations $x = a(t)$ and $y(t)$
- The **path** of motion is derived by eliminating the time $t$
- The parametric form of a (horizontal) **parabola** is $x(t) = at^2 + h$, $y(t) = 2at + k$
- The equivalent Cartesian form is $(y - k)^2 = 4a(x + h)$
- The parametric form of a **circle** is $x(t) = r\cos(\omega t) + h$, $y(t) = r\sin(\omega t) + k$
- The equivalent Cartesian form is $(x - h)^2 + (y - k)^2 = r^2$
- The parametric form of an **ellipse** is $x(t) = a\cos(\omega t) + h$, $y(t) = b\sin(\omega t) + k$
- The equivalent Cartesian form is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$
- The parametric form of a **rectangular hyperbola** is $x = t + h$, $y = \frac{c}{t} + k$
- The equivalent Cartesian form is $y - k = \frac{c}{x-h}$

### Vector derivatives

- $\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{h\to 0}\frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$
- Constant vector: $\frac{d\mathbf{c}}{dt}$, in particular $\frac{d\mathbf{i}}{dt} = \frac{d\mathbf{j}}{dt} = \frac{d\mathbf{k}}{dt} = 0$
- Vector sum: $(\mathbf{v} + \mathbf{u})' = \mathbf{v}' + \mathbf{u}'$
- Vector difference: $(\mathbf{v} - \mathbf{u})' = \mathbf{v}' - \mathbf{u}'$
- Vector product: $(\mathbf{vu})' = \mathbf{v}'\mathbf{u} + \mathbf{v}\mathbf{u}'$
- For $\mathbf{F}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, $\mathbf{F}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$
- Chain rule: For $\mathbf{F}(s)$ where $s = s(t)$, $\frac{d\mathbf{F}}{dt} = \frac{d\mathbf{F}}{ds} \times \frac{ds}{dt}$
- For a particle with position $\mathbf{r}(t)$, its velocity $\mathbf{v} = \dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt}$ and acceleration $\mathbf{a} = \ddot{\mathbf{r}} = \frac{d\mathbf{v}}{dt}$

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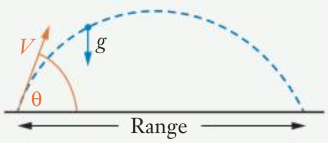
### Straight-line motion with constant acceleration

- $v = u + at$
- $v^2 = u^2 + 2as$
- $s = ut + \frac{1}{2}at^2$
- $s = \left(\frac{u+v}{2}\right)t$
- The acceleration of **gravity** near the Earth is about 9.8 m/s$^2$
- The force and acceleration of an object are related by $F = ma$

### Projectile motion




- The only forces acting in projectile motion are **gravity** and **air resistance** but in simple cases, the air resistance is negligible
- The path of the projectile is called its **trajectory**
- The final horizontal displacement is called its **range**
- The initial velocity is shown as $V$ and the angle of projection as $\theta$
- Vector equation of path: $\mathbf{r}(t) = V\cos(\theta)\, t\mathbf{i} + (14.7t - 4.9t^2)\mathbf{j}$
- Cartesian equation of path: $y = x\tan(\theta) - \frac{gx^2}{2V^2\cos^2(\theta)}$ or

$$y = x\tan(\theta) - \frac{gx^2}{2V^2}[1 + \tan^2(\theta)]$$

- Time of flight: $T = \frac{2V\sin(\theta)}{g}$
- Range: $\frac{2V^2\sin(\theta)\cos(\theta)}{g}$ or $\frac{V^2\sin(2\theta)}{g}$
- Greatest range: $\frac{V^2}{g}$ at $\theta = 45°$
- Greatest height: $\frac{V^2\sin^2(\theta)}{2g}$ at $T_{\frac{1}{2}} = \frac{V\sin(\theta)}{g}$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### Uniform circular motion

- The radius of an object moving in a circle at constant speed is $r$ and the speed is $v$
- The **angular velocity**, $\omega = \dfrac{d\theta}{dt} = \dfrac{v}{r}$, so $v = \omega r$
- The acceleration is given by $\mathbf{a} = -\omega^2\mathbf{r}$
- The acceleration is towards the centre of the circle, and $a = \omega^2 r = \dfrac{v^2}{r}$
- The position at time $t$ is given by $\mathbf{r} = r\cos(\omega t)\mathbf{i} + r\sin(\omega t)\mathbf{j}$
- The period is given by $T = \dfrac{2\pi r}{v} = \dfrac{2\pi}{\omega}$

Alamy Stock Photo/Andrew Holt

# 6. CHAPTER REVIEW

## Vector calculus

Example 1

**1** Find the Cartesian equation for the path of a particle with position vector $\mathbf{r}(t) = (2t - 3)\mathbf{i} + (4t + 9)\mathbf{j}$.

Example 2

**2** Find the Cartesian equation and sketch the graph for the path of a particle with position vector $\mathbf{r}(t) = [6 \sin(t) + 3]\mathbf{i} + [6 \cos(t) - 2]\mathbf{j}$.

Example 3

**3** The position of an object is given by $\mathbf{r}(t) = 2t\,\mathbf{i} + (2t - t^2)\mathbf{j}$.

**a** Find its velocity at time $t$.

**b** Find its acceleration.

**c** Sketch the path of the object and mark the velocity and acceleration vectors at $t = 1.5$.

Example 4

**4** The velocity of a particle at any time $t$ is given by $\mathbf{v}(t) = (2t + 6)\mathbf{i} - 3(t - 4)^2\,\mathbf{j}$, where time is in seconds and distance in metres.

**a** Find its position $\mathbf{r}(t)$, given that it begins from $2\mathbf{i} + \mathbf{j}$.

**b** Find its position, velocity and acceleration at 3 seconds.

Example 5

**5** The acceleration of a particle is given by $\ddot{\mathbf{r}} = (t - 3)\mathbf{i} - 3t\,\mathbf{j}$, where time is in seconds and acceleration in $m/s^2$. The particle passes through (2, 8) at 3 seconds and has velocity $2\mathbf{i} + \mathbf{j}$ at 2 seconds.

**a** Find its velocity at time $t$.

**b** Find its position at time $t$.

Example 6

**6 a** A particle with a velocity of 15 m/s accelerates at 2 m/s for 3 seconds. How far does it travel in this time?

**b** An object travelling at a speed of 10 m/s slows down to 8 m/s over a distance of 6 m. What is its deceleration?

**c** An object of mass 10 kg travelling at 8 m/s has a force of 30 N applied for 5 seconds in the direction of its motion. What is its new velocity?

Example 7

**7** A 5 kg object with an initial velocity of 7 m/s has a force of $6 - 3t$ N acting on it for 4 seconds.

**a** What is the final velocity?

**b** What is the displacement while the force is acting?


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**8** A projectile is fired from a horizontal plane at an angle of 60° and a speed of 24.5 m/s.

Examples 8,9

**a** What is the initial vertical component of the velocity?

**b** What is the initial horizontal component of the velocity?

**c** How long does it take to reach maximum height?

**d** What is the maximum height?

**e** How long does it take to return to the plane?

**f** What is the horizontal distance it travels on the plane?

**9** A particle of mass 4 kg moving in a circle of radius 12 m has a constant speed of 8 m/s.

Example 10

**a** What is its angular velocity?

**b** Find the acceleration.

**c** What force is acting on the particle?

**10** The track in a velodrome is sloped at an angle of 25° on the curve at one end. The curve has a radius of 20 m. A cyclist and his bicycle have a mass of 70 kg.

Example 11

**a** At what speed can the cyclist go round the curve without additional forces, correct to 2 decimal places?

**b** What force is needed to go round the curve at 15 m/s?

## Problem solving

**11** The velocity of a particle is given by $\mathbf{v} = 2\pi \cos(0.5\pi t)\mathbf{j} - 3\pi \sin(0.5\pi t)\mathbf{i}$ m/s, where $t$ is in seconds. It is at (1, 2) at 1 second.

Sketch the path of the particle from 0 to 3 seconds and fully describe its motion.

**12** An express elevator travels 24 floors in 8 s. Each floor is 3 m above the one below. It accelerates for the first 5 floors and decelerates at the same rate for the last 5 floors, travelling at uniform speed in between. Find the speed of the elevator in the middle 14 floors.

**13** A radio controlled car at the start of a straight track of mass 0.5 kg has a force of $10 - 3t$ N applied by its motor for 0.4 seconds to reach maximum speed. It travels at constant speed for another 0.6 seconds before being slowed down gradually in 1.5 s. The car starts and finishes at rest. What total distance does it travel, to the nearest cm?

**14** A ball is thrown from the top of a silo 49 m high. It has an initial speed of 19 m s$^{-1}$ and an angle of projection of 30° upward. At the same time, another ball is thrown downward at the same speed and angle. How far apart do they land, correct to 1 decimal place?

**15** The wheels of a train are 1.5 m apart. Part of the line runs along a circular curve of radius 1.2 km. If the maximum speed of trains on the line is 90 km/h, how high should the outer rail be raised above the inner rail to prevent side-slip?

**16** Show that the Cartesian form of $\mathbf{r}(t) = a \sec(t)\mathbf{i} + b \tan(t)\mathbf{j}$ is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Qz Practice quiz

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# Practice examination 2

**Time: 90 minutes**
**Perusal time: 5 minutes**
**Marks: 50**

## Instructions

- Students are permitted to bring or use: pens, pencils, highlighters, erasers, sharpeners, rules and an approved graphics calculator.
- Students must show appropriate working and justification to gain full marks.
- A QCAA formula sheet is provided.
- Unless otherwise stated, numerical answers should be exact.
- Unless otherwise indicated, no diagrams in this examination are drawn to scale.
- All written responses must be in English.
- Answer all questions.
- **Students are NOT permitted to bring or use notes of any kind, correction fluid/tape, mobile phones and/or any other unauthorised electronic devices.**

### Question 1 (4 marks)

Solve this system of equations by Gaussian elimination.

$$2e + 3f - g - 2h = -2$$
$$e + 2f + 4g + h = 4$$
$$3e - 2f - 5g + 3h = 5$$
$$4e + 3f + g + h = 4$$

### Question 2 (2 marks)

**a** Write this system of equations as a matrix equation.

$$2x - 3y + z = 4$$
$$x + 3y - z = -3$$
$$4x - 2y + z = 8$$

**b** Write this system of equations as an augmented matrix.

$$3a + 2b - d = 0$$
$$5a + 2b + 3c + 4d = -2$$
$$b - 3a + 4c - 5d = 0$$


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## Question 3 (5 marks)

Use matrices to solve each system of equations. State if any system is inconsistent, dependent or has no unique solution.

**a**
$$3x + 7y + 6z = 7$$
$$5x + 11y + 12 = 18$$
$$x + 2y + 3z = 4$$

**b**
$$3a + 2b + 4c - 3d = 17$$
$$9a + 6b + 8c + 2d = 12$$
$$3a + 2b + 2c + d = 2$$
$$6a + 4b + 2c + 5d = -8$$

## Question 4 (3 marks)

Find the relationship between the planes

L1: $2x - 3y + z = 5$

L2: $3x + 4y - z = 4$

L3: $4x - 6y + 2z = -1$

## Question 5 (2 marks)

Sketch the graph of each locus of $z(x, y)$ described.

**a** $\frac{\pi}{3} \le \arg(z) \le 0$

**b** $|z + 2i| > |z - 1 + 2i|$

## Question 6 (3 marks)

**a** Solve $z^6 = 1$.

**b** Show the roots of $z^9 = 1$ on the unit circle.

**c** Describe the positions of the roots on the unit circle.

## Question 7 (1 mark)

Use the factor theorem to factorise $q(z) = z^3 + 3z^2 + 9z - 13$.

## Question 8 (2 marks)

Write a polynomial function of least degree with integer coefficients that has the zeros $3i$ and $\frac{2}{3}$.

## Question 9 (2 marks)

Show that $z = 1 - 2i$ is a root of the equation $2z^3 - 9z^2 + 20z - 25 = 0$, and hence find all the roots.

### Question 10 (1 mark)

Find the Cartesian equation of the path of a particle for which $\mathbf{r}(t) = 5 \cos(2t)\mathbf{i} + 3 \sin(2t)\mathbf{j}$ metres at time $t$ seconds.

### Question 11 (2 marks)

**a** Find the velocity of a particle with position $\mathbf{r}(t) = (t^2 - 2)\mathbf{i} + 5\mathbf{j} - 6t\mathbf{k}$.

**b** A particle with an initial velocity of $3\mathbf{i} + \mathbf{k}$ has acceleration $\mathbf{a}(t) = 5t\mathbf{i} + 6t^2\,\mathbf{j} - 7\mathbf{k}$. Find its velocity at time $t$.

### Question 12 (2 marks)

A particle moving in a straight line begins with velocity 3 m/s and accelerates at 5 m/s for 3 seconds.

**a** Find the distance it travels.

**b** The particle then slows uniformly to a stop over the next 15 m. Find the deceleration.

### Question 13 (1 mark)

A particle moving at 2 m/s accelerates in the same direction as the velocity. The acceleration is given by $\mathbf{a} = 12t$. Find its displacement in 4 seconds.

### Question 14 (3 marks)

3 friends bought music DVDs at a sale from bins at 3 different prices.

- Ami bought 3 from the first bin, 1 from the second and 2 from the third
- Sam bought 2 from the first bin, 3 from the second and 2 from the third
- Toni bought 5 from the first bin, 2 from the second and 1 from the third
- Ami's DVDs cost \$108, Sam's cost \$139, and Toni's cost \$141

Find the cost of one DVD from each bin.

### Question 15 (2 marks)

5 people played tennis against each other.

- Peter lost to Alex and Don, but beat Jim and Terry
- Alex beat Don, Peter and Terry, but did not play Jim
- Don lost to Alex and Terry but beat Peter and Jim
- Jim beat Terry, but lost to Peter and Don

Rank the players up to the second order.


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### Question 16 (3 marks)

**a** Solve $z^5 = 1 + i\sqrt{3}$ and plot the roots on an Argand diagram.

**b** Calculate the area of the figure formed when the roots are joined.

### Question 17 (2 marks)

A ball is thrown at an angle of 60° with speed 24.5 m/s up onto a flat roof 3 m above and over 5 m away from the point of projection. Find how long the ball is in flight, correct to the nearest 0.1 s.

### Question 18 (5 marks)

The reproduction rates of wallaby species native to eastern Australia sometimes change, depending on environmental and other factors.

They begin breeding at about 2 years of age.

The table below shows the birth and death rates of females in good and poor years.

| Age | 1 (Joeys) | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Birth rate (good year) | 0 | 0.5 | 1 | 0.9 | 0.5 | 0.1 |
| Birth rate (poor year) | 0 | 0.4 | 0.6 | 0.4 | 0 | 0 |
| Death rate (good year) | 10% | 5% | 10% | 10% | 30% | 100% |
| Death rate (poor year) | 40% | 30% | 15% | 25% | 100% | |

The population in one area had 500 Joeys and 300, 250, 100, 50 and 20 of the 2-, 3-, 4-, 5- and 6-year-olds respectively.

The subsequent years were good, bad, bad and good. Find the population after these 4 years.

### Question 19 (3 marks)

Sketch the locus of $z(x, y)$ such that $\left|z^2 - \overline{(z^2)}\right| \leq 4$.

### Question 20 (2 marks)

An object being swung in a circle at constant speed moves in a horizontal plane on the end of a string 50 cm long at an angle of 28° to the vertical. Find the time taken to complete one revolution, correct to the nearest 0.01 second.

**END OF EXAMINATION**

INTEGRATION AND APPLICATIONS OF INTEGRATION

# 7. FURTHER APPLICATIONS OF INTEGRATION

In this chapter you will look at various techniques of integration. These techniques are aimed at helping you to find more complicated integrals. You will use the techniques of substitution, integration by parts and partial fractions.

The integration techniques may be new but their application draws on your existing knowledge of algebra, trigonometry and calculus. By studying the techniques in this chapter, you will be able to solve a greater variety of applied calculus problems.


7.01 Integration with trigonometric identities
7.02 Integration by substitution
7.03 Integration of $\frac{1}{x}$
7.04 The inverse trigonometric functions
7.05 Integration using inverse trigonometric functions
7.06 Integration using partial fractions
7.07 Integration by parts
7.08 Areas between curves
7.09 Volumes of solids of revolution
7.10 Simpson's rule
7.11 The exponential probability function
7.12 Applications of the exponential probability function
Chapter summary
Chapter review

# SYLLABUS SUBJECT MATTER

## Integration techniques

- integrate using the trigonometric identities $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$, $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$ and $1 + \tan^2(x) = \sec^2(x)$
- use substitution $u = g(x)$ to integrate expressions of the form $f(g(x))\,g'(x)$
- establish and use the formula $\int \frac{1}{x}\,dx = \ln|x| + c$ for $x \neq 0$
- find and use the inverse trigonometric functions: arcsine, arccosine and arctangent
- find and use the derivative of the inverse trigonometric functions: arcsine, arccosine and arctangent
- integrate expressions of the form $\frac{\pm 1}{\sqrt{a^2 - x^2}}$ and $\frac{a}{a^2 + x^2}$
- use partial fractions where necessary for integration in simple cases
- integrate by parts

## Applications of integral calculus

- calculate areas between curves determined by functions
- determine volumes of solids of revolution about either axis
- use the numerical integration method of Simpson's rule, using technology
- use and apply the probability density function, $f(t) = \lambda e^{-\lambda t}$ for $t > 0$, of the exponential random variable with parameter $\lambda > 0$, and use the exponential random variables and associated probabilities and quantiles to model data and solve practical problems

Prior learning

# TERMINOLOGY

angle sum and difference identities
arctangent (arctan)
exponential distribution
integration by parts
midpoint rule
Pythagorean identity
Simpson's rule
trapezoidal rule
arccosine (arcos)
cumulative distribution function
difference function
integration by substitution
partial fractions
rate of occurrence
solid of revolution
variance
arcsine (arcsin)
expected value
double-angle identities
median
product identities
reliability function
standard deviation
waiting time

## 7.01 Integration with trigonometric identities

WS Trigonometric identities

Trigonometric identities can be used to help change complex integrals into something that can be integrated more easily. You already know a number of trigonometric identities that can be applied to integration.

### Trigonometric identities

**Pythagorean identities**

- $\sin^2(A) + \cos^2(A) = 1$
- $\tan^2(A) + 1 = \sec^2(A)$
- $\cot^2(A) + 1 = \text{cosec}^2(A)$

**Angle sum and difference identities**

- $\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$
- $\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$
- $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$
- $\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$

**Double-angle identities**

- $\sin(2A) = 2\sin(A)\cos(A)$
- $\cos(2A) = \cos^2(A) - \sin^2(A)$
  $= 1 - 2\sin^2(A)$
  $= 2\cos^2(A) - 1$

**Product identities**

- $\sin(A)\sin(B) = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$
- $\cos(A)\cos(B) = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$
- $\sin(A)\cos(B) = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$
- $\cos(A)\sin(B) = \frac{1}{2}[\sin(A + B) - \sin(A - B)]$

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## EXAMPLE 1

**a** Calculate $\int_0^{\frac{\pi}{2}} \sin^2(x)\,dx$

**b** Find $\int \sin(5x)\cos(2x)\,dx$.

### Solution

**a** Use a double-angle identity. $\cos(2A) = 1 - 2\sin^2(A)$

Rearrange. $\sin^2(A) = \frac{1}{2}[1 - \cos(2A)]$

Write the integral and replace $\sin^2(x)$.

$$\int_0^{\frac{\pi}{2}} \sin^2(x)\,dx = \int_0^{\frac{\pi}{2}} \frac{1}{2}[1-\cos(2x)]\,dx$$

Integrate using $\int[1-\cos(2x)]\,dx = \int 1\,dx + \int \cos(2x)\,dx$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} [1-\cos(2x)]\,dx$$

$$= \frac{1}{2}\left[x - \frac{1}{2}\sin(2x)\right]_0^{\frac{\pi}{2}}$$

Evaluate.

$$= \frac{1}{2}\left(\left[\frac{\pi}{2} - \sin(\pi)\right] - [0 - \sin(0)]\right)$$

$$= \frac{\pi}{4}$$

**b** Use a product identity. $\sin(5x)\cos(2x) = \frac{1}{2}[\sin(7x) + \sin(3x)]$

Write the integral and replace the integrand.

$$\int \sin(5x)\cos(2x)\,dx$$

$$= \frac{1}{2}\int[\sin(7x)+\sin(3x)]\,dx$$

Integrate.

$$= \frac{1}{2}\left[-\frac{1}{7}\cos(7x) - \frac{1}{3}\cos(3x)\right]$$

Simplify.

$$= -\frac{1}{14}\cos(7x) - \frac{1}{6}\cos(3x)$$

State the result.

$$\int \sin(5x)\cos(2x)\,dx$$

$$= -\frac{1}{14}\cos(7x) - \frac{1}{6}\cos(3x) + c$$

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## EXAMPLE 2

**a** Find $\int 5\cos^2(4x)\,dx$

**b** Calculate $\int_0^{\pi} \sin^2(x)\,dx$

### Solution

**a** Use a double-angle identity. $\cos(8x) = 2\cos^2(4x) - 1$

Isolate $\cos^2(4x)$. $\cos^2(4x) = \frac{1}{2}[\cos(8x) + 1]$

Write the integral and replace $\cos^2(4x)$. $\int 5\cos^2(4x)\,dx = \int \frac{5}{2}[\cos(8x)+1]\,dx$

Integrate. $= \frac{5}{2}\left[\frac{1}{8}\sin(8x) + x\right]$

Write the result. $\int 5\cos^2(4x)\,dx = \frac{5}{2}\left[\frac{1}{8}\sin(8x) + x\right] + c$

**b** Use a double-angle identity. $\cos(2x) = 1 - 2\sin^2(x)$

Isolate $\sin^2(x)$. $\sin^2(x) = \frac{1}{2}[1 - \cos(2x)]$

Write the integral replacing $\sin^2(x)$. $\int_0^{\pi} \sin^2(x)\,dx = \int_0^{\pi} \frac{1}{2}[1 - \cos(2x)]\,dx$

Integrate. $= \frac{1}{2}\left[x - \frac{1}{2}\sin(2x)\right]_0^{\pi}$

Evaluate. $= \frac{1}{2}\left[\pi - \frac{1}{2}\sin(2\pi) - \left(0 - \frac{1}{2}\sin(0)\right)\right]$

$= \frac{1}{2}\pi$

Recall the derivative of tan $(x)$: $\frac{d}{dx}\tan(x) = 1 + \tan^2(x)$

Using the Pythagorean identity $\frac{d}{dx}\tan(x) = \sec^2(x)$

Similarly, $\frac{d}{dx}\cot(x) = -\mathrm{cosec}^2(x)$

By reversing the differentiation and using these relationships, you can write the integrals of $\sec^2(x)$ and $\mathrm{cosec}^2(x)$.

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**Integrals of sec(x) and cosec(x)**

$$\int \sec^2(x)\,dx = \tan(x) + c \qquad \int \sec^2(kx)\,dx = \frac{1}{k}\tan(kx) + c$$

$$\int \text{cosec}^2(x)\,dx = -\cot(x) + c \qquad \int \text{cosec}^2(kx)\,dx = -\frac{1}{k}\cot(kx) + c$$

## EXAMPLE 3

Find $\int \tan^2(x)\,dx$.

### Solution

| | |
|---|---|
| Use the Pythagorean identity. | $\tan^2(x) + 1 = \sec^2(x) \Rightarrow \tan^2(x) = \sec^2(x) - 1$ |
| Write the integral replacing $\tan^2(x)$. | $\int \tan^2(x)\,dx = \int [\sec^2(x) - 1]\,dx$ |
| Integrate. | $= \tan(x) - x$ |
| Write the result. | $\int \tan^2(x)\,dx = \tan(x) - x + c$ |

## Exercise 7.01 Integration with trigonometric identities

**1** Calculate: Example 1

**a** $\int \sin^2(x)\,dx$ **b** $\int \cos^2(2x)\,dx$ **c** $\int \sin^2(3x)\,dx$

**d** $\int_0^{\frac{\pi}{2}} \cos^2(2x)\,dx$ **e** $\int_0^{\frac{\pi}{2}} \sin^2(5x)\,dx$ **f** $\int_0^{\frac{\pi}{4}} \cos^2(3x)\,dx$

**2** Find:

**a** $\int \sin(3x)\cos(2x)\,dx$ **b** $\int \sin(7x)\cos(4x)\,dx$

**c** $\int \sin(2x)\cos(2x)\,dx$ **d** $\int 2\cos(6x)\cos(2x)\,dx$

**e** $\int 3\sin(6x)\cos(3x)\,dx$ **f** $\int 8\sin(2x)\cos(2x)\,dx$

**3** Find: Example 2

**a** $\int \sin^2(4x)\,dx$ **b** $\int \cos^2(3x)\,dx$ **c** $\int 4\sin^2(2x)\,dx$

**d** $\int 9\cos^2(6x)\,dx$ **e** $\int 8\sin^2(6x)\,dx$ **f** $\int 5\cos^2(5x)\,dx$

**4** Calculate:

**a** $\int_0^{\pi} 2\sin^2(4x)\,dx$ **b** $\int_0^{\frac{\pi}{2}} 4\cos^2(8x)\,dx$ **c** $\int_0^{\frac{\pi}{4}} 2\sin^2(3x)\,dx$

**d** $\int_0^{\frac{\pi}{2}} 3\cos^2(2x)\,dx$ **e** $\int_0^{\frac{\pi}{2}} 4\sin^2(2x)\,dx$ **f** $\int_0^{\frac{\pi}{4}} 4\cos^2(8x)\,dx$

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Example 3

**5** Find:

**a** $\int \cot^2(x)\,dx$ **b** $\int \tan^2(2x)\,dx$ **c** $\int \cot^2(4x)\,dx$

**d** $\int \tan^2(3x)\,dx$ **e** $\int \cot^2\left(\frac{x}{2}\right)dx$ **f** $\int \tan^2\left(\frac{x}{4}\right)dx$

**6** Calculate $\int_0^{\frac{\pi}{12}} \cos(6x)\sin(3x)\,dx$.

## Problem solving

**7** Show that $\int \sin(3x)\cos(3x)\,dx = -\frac{1}{6}\cos^2(3x) + c$

**8** Find $\int \sin^4(x)\,dx$. (*Hint*: You need to use a double-angle identity more than once.)

**9** Find $\int \cos^4(x)\,dx$.

**10** Find $\int \sin^2(x)\cos^2(x)\,dx$.

WS Integration by substitution

Integration by substitution

# 7.02 Integration by substitution

The anti-differentiation formulas do not allow you to evaluate integrals like $\int 2x\sqrt{x^2-3}\,dx$.

To find these types of integrals, a new variable needs to be introduced.

Let $u$ equal the expression in the square root sign in the integrand above.

$$u = x^2 - 3$$

Then $\frac{du}{dx} = 2x$ and so the differential of $u$ is

$$du = 2x\,dx$$

Now look at the integral.

$$\int 2x\sqrt{x^2-3}\,dx = \int \sqrt{x^2-3}\,.2x\,dx$$

Substitute for $u$ and $du$. $= \int \sqrt{u}\,du$

Integrate wrt $u$. $= \frac{2}{3}u^{\frac{3}{2}} + c$

Substitute for $u$. $= \frac{2}{3}(x^2-3)^{\frac{3}{2}} + c$


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You can check this result using the chain rule to calculate the derivative. The chain rule states that for $f(x) = f[u(x)]$, $f'(x) = f'(u) \times u'(x)$, or, for $y = u(x)$, as $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

Differentiate using the chain rule.

$$\frac{d}{dx}\left[\frac{2}{3}(x^2-3)^{\frac{3}{2}} + c\right] = \frac{d}{dx}\left[\frac{2}{3}u^{\frac{3}{2}} + c\right]$$

$$= \frac{3}{2} \times \frac{2}{3}u^{\frac{1}{2}} \times 2x$$

$$= 2x\sqrt{x^2-3}$$

So this procedure works. This method, called **integration by substitution**, works whenever you can write an integral in the form $\int f\left[g(x)\right]g'(x)\,dx$.

**Integration by substitution**

If $u = g(x)$,

$$\int f\left[g(x)\right]g'(x)\,dx = \int f(u)\,du$$

To use the substitution method successfully, you must find a function $u(x)$ so that $u'(x)$ is part of the original and the remainder of the expression is a function of $u$. There is no set method for finding $u(x)$, and it is possible to use different substitutions for the same integral.

## EXAMPLE 4

Find:

**a** $\int\left(4x^2-3x+1\right)^4(8x-3)\,dx$

**b** $\int \sin(x)\cos^4(x)\,dx$

### Solution

**a** Write the integral. $\int\left(4x^2-3x+1\right)^4(8x-3)\,dx$

Let $u = 4x^2 - 3x + 1$. $\frac{du}{dx} = 8x - 3 \Rightarrow du = (8x-3)\,dx$

Write the integral using $u$. $\int\left(4x^2-3x+1\right)^4(8x-3)\,dx = \int u^4\,du$

Integrate. $\int u^4\,du = \frac{1}{5}u^5 + c$

Substitute for $u$. $= \frac{1}{5}(4x^2-3x+1)^5 + c$

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State the result.

$$\int (4x^2 - 3x + 1)^4 (8x - 3)\,dx$$

$$= \frac{1}{5}(4x^2 - 3x + 1)^5 + c$$

**b** Write the integral.

$$\int \sin(x)\cos^4(x)\,dx$$

Let $u = \cos(x)$.

$$\frac{du}{dx} = -\sin(x) \Rightarrow du = -\sin(x)\,dx$$

Write the integral using $u$.

$$\int \sin(x)\cos^4(x)\,dx = -\int u^4\,du$$

Integrate.

$$-\int u^4\,du = -\frac{u^5}{5} + c$$

Substitute for $u$.

$$= -\frac{\cos^5(x)}{5} + c$$

State the result.

$$\int \sin(x)\cos^4(x)\,dx = -\frac{\cos^5(x)}{5} + c$$

When you calculate the definite integral using the method of substitution, you need to change the limits of integration when the variable is changed.

## Substitution rule for definite integrals

If $u = g(x)$, $g'(x)$ is continuous on the interval $[a, b]$,

$$\int_a^b f[g(x)]g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

## EXAMPLE 5

Calculate:

**a** $\int_0^2 (3x+1)^3\,dx$

**b** $\int_0^4 \sqrt{6x+1}\,dx$

### Solution

**a** Write the integral.

$$\int_0^2 (3x+1)^3\,dx$$

Let $u = 3x + 1$

$$\frac{du}{dx} = 3 \Rightarrow du = 3\,dx, \text{ so } dx = \frac{du}{3}$$

Calculate limits for $u$.

When $x = 0$, $u = 1$

When $x = 2$, $u = 7$


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| | |
|---|---|
| Write the integral using $u$. | $\int_0^2 (3x+1)^3\,dx = \int_1^7 u^3 \frac{du}{3}$ |
| Integrate. | $= \frac{1}{3}\left[\frac{1}{4}u^4\right]_1^7$ |
| Evaluate. | $= \frac{1}{12}(7^4 - 1^4)$ |
| | $= \frac{2400}{12}$ |
| State the result. | $\int_0^2 (3x+1)^3\,dx = 200$ |
| **b** Write the integral. | $\int_0^4 \sqrt{6x+1}\,dx$ |
| Let $u = 6x + 1$ | $\frac{du}{dx} = 6 \Rightarrow du = 6\,dx$, so $dx = \frac{du}{6}$ |
| Calculate limits for $u$. | When $x = 0$, $u = 1$ |
| | When $x = 4$, $u = 25$ |
| Write the integral using $u$. | $\int_1^4 \sqrt{6x+1}\,dx = \int_1^{25} \sqrt{u}\,\frac{du}{6}$ |
| Integrate. | $= \frac{1}{6}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^{25}$ |
| Evaluate. | $= \frac{1}{9}\left[25^{\frac{3}{2}} - 1\right]$ |
| State the result. | $\int_1^4 \sqrt{6x+1}\,dx = \frac{124}{9}$ |
| | $= 13\frac{7}{9}$ |

## Exercise 7.02 Integration by substitution

**1** Use integration by substitution to find each integral. (Example 4)

**a** $\int (6x-5)(3x^2 - 5x + 11)\,dx$

**b** $\int (x-1)(4x^2 - 8x + 1)\,dx$

**c** $\int 4x\sqrt{2x^2+5}\,dx$

**d** $\int (x+3)\sqrt{x^2+6x-8}\,dx$

**e** $\int \sin(x)\cos^5(x)\,dx$

**f** $\int 10\cos(x)\sin^4(x)\,dx$

**2** Use integration by substitution to find each integral.

**a** $\int (2x-5)^4\,dx$ **b** $\int (7x+2)^9\,dx$ **c** $\int (3x-7)^5\,dx$

**d** $\int 8(4x-5)^3\,dx$ **e** $\int 12(9x+4)^5\,dx$ **f** $\int \frac{1}{4}(2x+9)^7\,dx$

**3** Find each integral.

**a** $\int \sqrt{3x+8}\,dx$ **b** $\int \sqrt{5x-6}\,dx$ **c** $\int 4\sqrt{2x+7}\,dx$

**d** $\int 4x\sqrt{2x^2-7}\,dx$ **e** $\int \sqrt[3]{8x+7}\,dx$ **f** $\int x\sqrt[3]{6x^2+5}\,dx$

**4** Find each integral.

**a** $\int \frac{4}{\sqrt{8x+7}}\,dx$ **b** $\int \frac{4}{\sqrt{6x-1}}\,dx$ **c** $\int \frac{5}{\sqrt{4x+3}}\,dx$

**d** $\int \frac{x}{\sqrt{x^2+6}}\,dx$ **e** $\int \frac{2x}{\sqrt{3x^2-5}}\,dx$ **f** $\int \frac{5x}{\sqrt{3-4x^2}}\,dx$

**5** Use substitution to find each integral.

**a** $\int \frac{dx}{(2x-5)^2}$ **b** $\int \frac{3}{(x+4)^3}\,dx$ **c** $\int \frac{\sin(x)}{\cos^2(x)}\,dx$

**d** $\int xe^{x^2}\,dx$ **e** $\int 5xe^{3x^2-1}\,dx$ **f** $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

Example 5

**6** Evaluate each definite integral.

**a** $\int_0^2 (x-1)^{12}\,dx$ **b** $\int_0^3 \sqrt{3x+4}\,dx$ **c** $\int_0^2 \frac{dx}{(2x+3)^2}$

**d** $\int_0^{\frac{\pi}{2}} \cos(x)\sin^3(x)\,dx$ **e** $\int_0^{\frac{\pi}{4}} \frac{\sin(x)}{\cos^2(x)}\,dx$ **f** $\int_2^{10} \frac{dx}{\sqrt{5x-1}}$

## Problem solving

**7** Find each integral.

**a** $\int \frac{8e^{x^4}}{x^{-3}}\,dx$ **b** $\int \frac{x}{\sqrt{3x-5}}\,dx$ **c** $\int \frac{4x}{\sqrt{4x+3}}\,dx$

**d** $\int 3x\sqrt{2x+3}\,dx$ **e** $\int 5x\sqrt{x-4}\,dx$

**8** The marginal profit of a surfboard maker is given by

$$P(n) = \frac{n\sqrt{n^2+100}}{10}$$

where $P(n)$ is the profit made in dollars from the $n$th board. The maximum number of boards that can be made in a day is 40. Find the total profit made when only 8 boards are made and the total profit when 30 boards are made.


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# 7.03 Integration of $\frac{1}{x}$

WS Integration of $\frac{1}{x}$

Some of the integration methods you have previously learnt cannot be applied across the complete domain of a function. For example, you know that $\int \frac{1}{x}dx$ is only defined for $x > 0$.

**INVESTIGATION**

## INTEGRATION OF $\frac{1}{x}$ FOR NEGATIVE VALUES

You have already seen that, for positive values of $x$, $\int \frac{1}{x}dx = \ln|x| + c$.

1. Draw a sketch of the function $y = \frac{1}{x}$ for $-10 \le x \le 10$.
2. Compare the area between the curve and $x$-axis from $-5$ to $-2$ with the area from 5 to 2.

   What is the sign of the algebraic area in each case? What is the physical area in each case?
3. What is the value of $y$ at $x = 0$?
4. Can you write some rules about the integration of $\frac{1}{x}$ for positive and negative values of $x$?

In general, it is not possible to integrate functions across points where the function is not defined, as is the case for $y = \frac{1}{x}$ when $x = 0$. So even when the rules for the integration are extended to include negative values of $x$, you still need to exclude integration of regions that include the value $x = 0$.

This means that you can use the rule $\int \frac{1}{x}dx = \ln|x| + c$ to find $\int_{-5}^{-2} \frac{1}{x}dx$ , but not for $\int_{-5}^{5} \frac{1}{x}dx$, because the interval from $-5$ to 5 includes a point where the function is not defined. It is important to remember this restriction.

**The integral of $\frac{1}{x}$**

$\int \frac{1}{x}dx = \ln|x| + c$, provided $x \neq 0$.

## EXAMPLE 6

Use the method of substitution to find $\int \frac{7}{3x-5}dx$.

### Solution

Let $u = 3x - 5$. $\frac{du}{dx} = 3 \Rightarrow du = 3dx$, so $dx = \frac{du}{3}$

Write the integral using $u$. $\int \frac{7}{3x-5}dx = \int \frac{7}{u}\frac{du}{3}$

$= \frac{7}{3}\int \frac{1}{u}du$

Integrate. $= \frac{7}{3}\ln |u| + c$

Substitute for $u$. $= \frac{7}{3}\ln |3x-5| + c$

Write the answer. $\int \frac{7}{3x-5}dx = \frac{7}{3}\ln |3x-5| + c$

The following results may be established using the method shown in the previous example.

### The integral of $\frac{1}{ax+b}$ and $\frac{f'(x)}{f(x)}$

$$\int \frac{1}{ax+b}dx = \frac{1}{a}\ln |ax+b| + c$$

$$\int \frac{f'(x)}{f(x)}dx = \ln |f(x)| + c$$

where $a$ and $b$ are constants, $c$ is the constant of integration and $ax + b, f(x) \neq 0$.

## EXAMPLE 7

Find:

**a** $\int \frac{6x-5}{3x^2-5x+7}dx$ **b** $\int \frac{3x}{8x^2+11}dx$

### Solution

**a** Let $f(x) = 3x^2 - 5x + 7$. $f(x) = 3x^2 - 5x + 7$

Differentiate the denominator. $f'(x) = 6x - 5$


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Since the numerator is the same as the derivative, use the rule for $\int \frac{f'(x)}{f(x)} dx$.

$$\int \frac{6x-5}{3x^2-5x+7} dx = \int \frac{f'(x)}{f(x)} dx$$
$$= \ln|3x^2 - 5x + 7| + c$$

**b** Let $f(x) = 8x^2 + 11$.

$f(x) = 8x^2 + 11$

Differentiate the denominator.

$f'(x) = 16x$

Compare $f'(x)$ with the numerator.

$f'(x) = 16x \Rightarrow \frac{3}{16} \times 16x = 3x$

So $3x = \frac{3}{16} \times f'(x)$

Write the integral in terms of $f(x)$ and $f'(x)$.

$$\int \frac{3x}{8x^2+11} dx = \int \frac{3}{16} \times \frac{f'(x)}{f(x)} dx$$
$$= \frac{3}{16} \int \frac{f'(x)}{f(x)} dx$$

Use the rule for $\int \frac{f'(x)}{f(x)} dx$.

$$= \frac{3}{16} \ln|8x^2 + 11| + c$$

## EXAMPLE 8

Evaluate $\int_0^{\frac{\pi}{4}} \tan(x)\, dx$.

### Solution

Write tan $(x)$ as a fraction.

$\tan(x) = \frac{\sin(x)}{\cos(x)}$

Let $f(x) = \cos(x)$.

$f(x) = \cos(x)$

Differentiate the denominator.

$f'(x) = -\sin(x)$

Write the integral.

$$\int_0^{\frac{\pi}{4}} \tan(x)\, dx = \int_0^{\frac{\pi}{4}} \frac{\sin(x)}{\cos(x)} dx$$

Write the integral in terms of $f(x)$ and $f'(x)$.

$$= \int_0^{\frac{\pi}{4}} -\frac{f'(x)}{f(x)} dx$$

Use the rule for $\int \frac{f'(x)}{f(x)} dx$.

$$= -\left[\ln|\cos(x)|\right]_0^{\frac{\pi}{4}}$$

| | |
|---|---|
| Evaluate. | $= -\left[\ln\left\|\cos\left(\frac{\pi}{4}\right)\right\| - \ln\|\cos(0)\|\right]$ |
| | $= -\ln\left(\frac{1}{\sqrt{2}}\right) + \ln(1)$ |
| | $= \ln\left(\frac{1}{\sqrt{2}}\right)^{-1} + 0$ |
| | $= \ln\left(\sqrt{2}\right)$ |
| Simplify and state the result. | $\int_0^{\frac{\pi}{4}} \tan(x)\,dx = \frac{1}{2}\ln(2)$ |

## Exercise 7.03 Integration of $\frac{1}{x}$

Example 6

**1** Use the method of substitution to find each integral.

**a** $\int \frac{1}{4x+9}\,dx$  **b** $\int \frac{3}{6x-7}\,dx$  **c** $\int \frac{2}{9-4x}\,dx$

**d** $\int \frac{12}{7-4x}\,dx$  **e** $\int \frac{18}{3x-11}\,dx$  **f** $\int \frac{2}{12x+5}\,dx$

Example 7

**2** Find:

**a** $\int \frac{8x-7}{4x^2-7x}\,dx$  **b** $\int \frac{6x+8}{3x^2+8x-4}\,dx$  **c** $\int \frac{x}{x^2+3}\,dx$

**d** $\int \frac{x}{2-3x^2}\,dx$  **e** $\int \frac{x^2+4}{x^3+12x}\,dx$  **f** $\int \frac{dx}{x\ln|x|}$

Example 8

**3** Evaluate:

**a** $\int_0^2 \frac{3}{4+3x}\,dx$  **b** $\int_0^1 \frac{x}{2x^2+3}\,dx$  **c** $\int_0^{\frac{\pi}{4}} \frac{x\cos(x)+\sin(x)}{x\sin(x)}\,dx$

**d** $\int_1^2 \frac{e^{2x}}{e^{2x}+1}\,dx$  **e** $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos(x)\sin(x)}{1-\cos^2(x)}\,dx$  **f** $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos(x)}{2+\sin(x)}\,dx$

**4** For each integral, use the result $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$. Then recalculate the integral by first using algebra to simplify the integrand before integration.

**a** $\int \frac{4x^3}{x^4}\,dx$  **b** $\int \frac{6e^{6x}}{e^{6x}}\,dx$  **c** $\int \frac{7x^{-8}}{x^{-7}}\,dx$

**d** $\int \frac{-\sin(x)e^{\cos(x)}}{e^{\cos(x)}}\,dx$  **e** $\int \frac{x^{-\frac{1}{2}}}{4x^{\frac{1}{2}}}\,dx$  **f** $\int \frac{x^{-\frac{7}{2}}}{x^{-\frac{5}{2}}}\,dx$


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**5** Find:

**a** $\int \frac{3}{x(1+\ln|4x|)}dx$  **b** $\int \frac{20e^{4x}}{4+e^{4x}}dx$  **c** $\int \frac{\cos(x)e^{\sin(x)}}{e^{\sin(x)}}dx$

## Problem solving

**6** Find $\int \frac{3}{1+e^{-3x}}dx$.

**7** Find $\int \frac{2(e^x - e^{-x})}{(e^x + e^{-x})^2}dx$.

**8** The equation $t = \int \frac{dv}{30 - v}$ describes an object moving down an inclined plane subject to a force proportional to the velocity. If the object starts from rest, find the velocity, $v$, as a function of time, $t$.

# 7.04 The inverse trigonometric functions

You have previously studied the calculus of the trigonometric functions sin ($x$), cos ($x$) and tan ($x$). Each of these has an inverse function that is given a special name:

$\sin^{-1}(x)$ = **arcsine** ($x$)  $\cos^{-1}(x)$ = **arccosine** ($x$)  $\tan^{-1}(x)$ = **arctangent** ($x$)

As with the other trig functions, the inverse trigonometric functions are shortened to **arcsin** ($x$), **arcos** ($x$) and **arctan** ($x$).

While the functions sin ($x$), cos ($x$) and tan ($x$) are defined for all values of $x$, the domain of each inverse trigonometric function needs to be restricted so that it satisfies the requirements of a function.

### Inverse trigonometric functions

$\sin^{-1}(x) = y$ if and only if $\sin(y) = x$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

$\cos^{-1}(x) = y$ if and only if $\cos(y) = x$ and $0 \le y \le \pi$.

$\tan^{-1}(x) = y$ if and only if $\tan(y) = x$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

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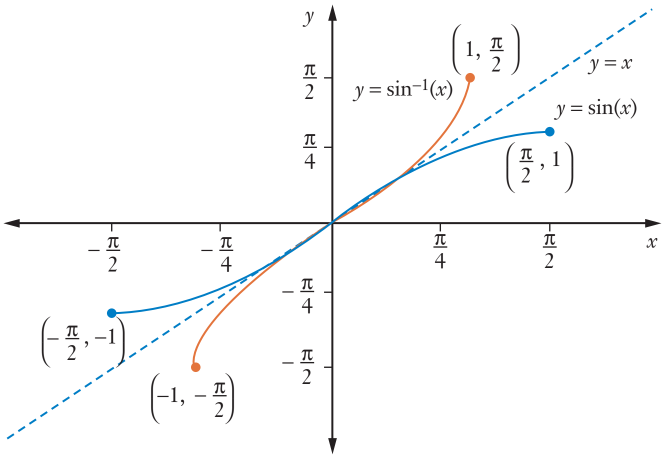
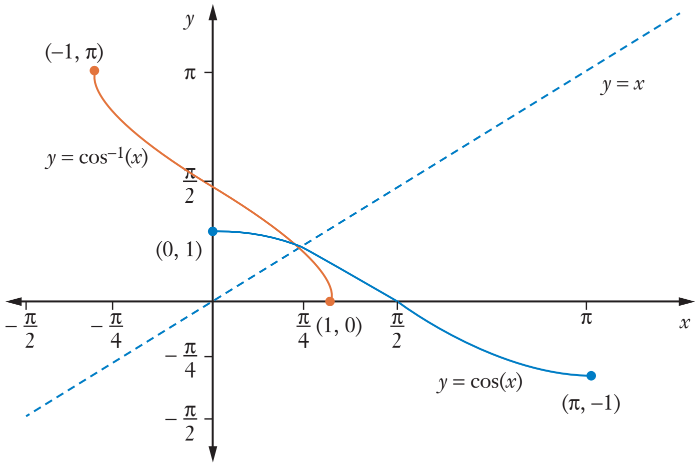
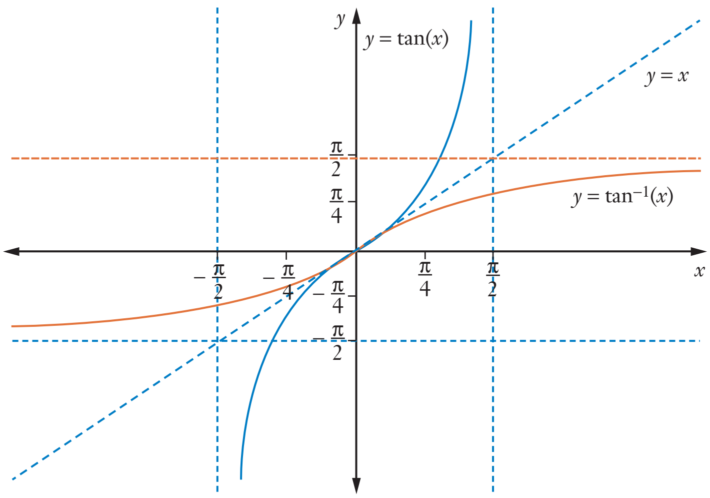
The graph of each inverse trigonometric function is shown below with the corresponding trigonometric function.

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The derivatives of arcsin ($x$), arcos ($x$) and arctan ($x$) are widely used, especially in calculus.

The rule for the derivative of $\sin^{-1}(x)$ is developed below.

$\sin^{-1}(x) = y \Leftrightarrow \sin(y) = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

Start with sin ($y$). $\sin(y) = x$

Use implicit differentiation. $\cos(y)\frac{dy}{dx} = 1$

Rearrange. $\frac{dy}{dx} = \frac{1}{\cos(y)}$

Use the Pythagorean identity $\sin^2(y) + \cos^2(y) = 1 \Rightarrow \cos(y) = \sqrt{1-\sin^2(y)}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2(y)}}$$

Use $x = \sin(y)$. $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$

Use $y = \sin^{-1}(x)$. $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$

The derivative of the other inverse trigonometric functions can be developed in a similar way.

## Derivatives of the inverse trigonometric functions

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$

Derivative of an inverse cosine function

## EXAMPLE 9

Find $\frac{d}{dx}\sin^{-1}(4x)$.

### Solution

Let $u = 4x$.

$$\frac{d}{dx}\sin^{-1}(4x) = \frac{d}{dx}\sin^{-1}(u)$$

Apply the chain rule.

$$= \frac{1}{\sqrt{1-u^2}} \times \frac{du}{dx}$$

Substitute for $u$ and differentiate.

$$= \frac{1}{\sqrt{1-(4x)^2}} \times 4$$

Simplify and state the result.

$$\frac{d}{dx}\sin^{-1}(4x) = \frac{4}{\sqrt{1-(4x)^2}}$$

## EXAMPLE 10

Find $\frac{d}{dx}\tan^{-1}\left(\sqrt{3x}\right)$.

### Solution

Let $u = \sqrt{3x}$.

$$\frac{d}{dx}\tan^{-1}\left(\sqrt{3x}\right) = \frac{d}{dx}\tan^{-1}(u)$$

Apply the chain rule.

$$= \frac{1}{1+u^2} \times \frac{du}{dx}$$

Substitute for $u$ and find $\frac{du}{dx}$.

$$= \frac{1}{1+\left(\sqrt{3x}\right)^2} \times \frac{1}{2\sqrt{3x}} \times 3$$

Simplify and state the result.

$$\frac{d}{dx}\tan^{-1}\left(\sqrt{3x}\right) = \frac{3}{2\sqrt{3x}(1+3x)}$$


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## INVESTIGATION

### TRIGONOMETRIC SUBSTITUTIONS

Functions involving $\sqrt{1-x^2}$ or $\sqrt{a^2-x^2}$ are difficult to integrate without knowing the appropriate substitution. For these functions, the substitutions $x = \sin(u)$ and $x = a\sin(u)$ make the task of integration possible. Even with these substitutions, it is necessary to use the inverse function to $\sin(x)$.

Work in groups of 2 or 3 to find $\int \sqrt{a^2-x^2}\,dx$. You may check your results by referring to a table of standard integrals.

## Exercise 7.04 The inverse trigonometric functions

Example 9

**1** Find each derivative.

**a** $\frac{d}{dx}\cos^{-1}(3x)$ **b** $\frac{d}{dx}\tan^{-1}(-7x)$ **c** $\frac{d}{dx}\sin^{-1}\left(\frac{x}{2}\right)$

**d** $\frac{d}{dx}\arctan\left(\frac{1}{4}x\right)$ **e** $\frac{d}{dx}\arcsin(-6x)$ **f** $\frac{d}{dx}\arccos\left(\frac{x}{3}\right)$

Example 10

**2** Find each derivative.

**a** $\frac{d}{dx}\tan^{-1}(3x^4)$ **b** $\frac{d}{dx}\arcsin(-5x^2)$ **c** $\frac{d}{dx}\cos^{-1}(-2x^3)$

**d** $\frac{d}{dx}\sin^{-1}(3x^2-1)$ **e** $\frac{d}{dx}\arccos(4x^3+5x)$ **f** $\frac{d}{dx}\arctan(-5x^2-6x)$

**3** Determine each derivative.

**a** $\frac{d}{dx}\sin^{-1}(5x^4+1)^3$ **b** $\frac{d}{dx}[\cos^{-1}(5x^2)]^3$ **c** $\frac{d}{dx}[\arcsin(4x^3)]^2$

**4** Determine each derivative.

**a** $\frac{d}{dx}\cos^{-1}\left(\sqrt{x}\right)$ **b** $\frac{d}{dx}\arcsin\left(\sqrt{1-x^4}\right)$ **c** $\frac{d}{dx}\tan^{-1}\left(\sqrt{1-x^2}\right)$

### Problem solving

**5** Calculate $\frac{d}{dx}\left[\frac{x}{\sqrt{1-x^2}}+\sin^{-1}(x)\right]$.

**6** Find $\frac{d}{dx}\tan^{-1}\left(\frac{x}{1-x^2}\right)$.

**7** Show that $\frac{d}{dx}\cot^{-1}(x) = -\frac{1}{1+x^2}$.

**8** Show that $\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}$, where $|x| > 1$.

WS

Inverse trigonometric functions

# 7.05 Integration using inverse trigonometric functions

In the previous section, you used the derivatives of $\sin^{-1}(x)$, $\cos^{-1}(x)$ and $\tan^{-1}(x)$. Reversing these derivative formulas yields the following rules for integrals.

### Integrals involving inverse trigonometric functions

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + c$$

$$\int \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1}(x) + c$$

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + c$$

## EXAMPLE 11

Find $\int \frac{1}{\sqrt{1-9x^2}}\,dx$.

### Solution

Let $u = 3x$.

$$\frac{du}{dx} = 3 \Rightarrow dx = \frac{1}{3}du$$

Write the integral in terms of $u$.

$$\int \frac{1}{\sqrt{1-9x^2}}dx = \int \frac{1}{\sqrt{1-u^2}}\frac{1}{3}du$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{1-u^2}}du$$

Integrate.

$$= \frac{1}{3}\sin^{-1}(u) + c$$

Substitute for $u$ and write the answer.

$$\int \frac{1}{\sqrt{1-9x^2}}dx = \frac{1}{3}\sin^{-1}(3x) + c$$


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Rules for integrals like $\int \frac{1}{\sqrt{a-x^2}}\,dx$, $\int \frac{-1}{\sqrt{a-x^2}}\,dx$ and $\int \frac{1}{a^2+x^2}\,dx$ can be derived from the rules stated above.

Consider $\int \frac{1}{a^2+x^2}dx$

Take out $a^2$ as a common factor. $= \int \frac{1}{a^2\left(1+\frac{x^2}{a^2}\right)}\,dx$

$= \frac{1}{a^2}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}\,dx$

Let $u = \frac{x}{a}$. $\frac{du}{dx} = \frac{1}{a} \Rightarrow dx = a\,du$

Write the integral in terms of $u$. $\frac{1}{a^2}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}dx = \frac{1}{a^2}\int \frac{1}{1+u^2}\,a\,du$

Simplify. $= \frac{1}{a}\int \frac{1}{1+u^2}\,du$

Use the integral rule. $= \frac{1}{a}\tan^{-1}(u) + c$

Substitute for $u$. $= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

The other rules can be developed in a similar way.

**General integrals involving inverse trigonometric functions**

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{-1}{\sqrt{a^2-x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

It should be noted that $\int \frac{-1}{\sqrt{a-x^2}}dx = -\int \frac{1}{\sqrt{a-x^2}}dx = -\sin^{-1}\left(\frac{x}{a}\right) + c$.

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## EXAMPLE 12

Evaluate $\int_{-1}^{1} \frac{1}{\sqrt{16-x^2}}\,dx$, correct to 3 decimal places.

### Solution

Rewrite the integral to match the rule. $\int_{-1}^{1} \frac{1}{\sqrt{16-x^2}}\,dx = \int_{-1}^{1} \frac{1}{\sqrt{4^2-x^2}}\,dx$

Apply the rule. $= \left[\sin^{-1}\left(\frac{x}{4}\right)\right]_{-1}^{1}$

Evaluate. $= \sin^{-1}\left(\frac{1}{4}\right) - \sin^{-1}\left(-\frac{1}{4}\right)$

Use a calculator. $= 0.505\ 360\ldots$

Round off and write the answer. $\int_{-1}^{1} \frac{1}{\sqrt{16-x^2}}\,dx \approx 0.505$

If an exact answer had been required for this example, then it should be stated as
$\sin^{-1}\left(\frac{1}{4}\right) - \sin^{-1}\left(-\frac{1}{4}\right)$.

## Exercise 7.05 Integration using inverse trigonometric functions

Example 11

**1** Find:

**a** $\int \frac{-1}{\sqrt{1-x^2}}\,dx$ **b** $\int \frac{1}{\sqrt{9-x^2}}\,dx$

**c** $\int \frac{1}{1+x^2}\,dx$ **d** $\int \frac{1}{\sqrt{4-x^2}}\,dx$

**e** $\int \frac{1}{16+x^2}\,dx$ **f** $\int \frac{-1}{\sqrt{25-x^2}}\,dx$

**2** Find:

**a** $\int \frac{-1}{\sqrt{8-x^2}}\,dx$ **b** $\int \frac{1}{\sqrt{12-x^2}}\,dx$

**c** $\int \frac{1}{20+x^2}\,dx$ **d** $\int \frac{1}{\sqrt{\frac{1}{9}-x^2}}\,dx$

**e** $\int \frac{1}{\frac{1}{2}+x^2}\,dx$ **f** $\int \frac{-1}{\sqrt{0.25-x^2}}\,dx$


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**3** Evaluate each integral, correct to 3 decimal places where necessary.

Example 12

**a** $\int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{9-x^2}}\,dx$ **b** $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{4-x^2}}\,dx$ **c** $\int_0^{\frac{1}{2}} \frac{1}{4+x^2}\,dx$

**d** $\int_{-2}^{2} \frac{1}{\sqrt{16-x^2}}\,dx$ **e** $\int_0^{\sqrt{2}} \frac{-1}{\sqrt{8-x^2}}\,dx$ **f** $\int_0^{1} \frac{1}{x^2+10}\,dx$

**4** Evaluate each integral, correct to 3 decimal places where necessary.

**a** $\int_0^{\frac{1}{2}} \frac{-5}{\sqrt{25-x^2}}\,dx$ **b** $\int_0^{1} \frac{4}{x^2+1}\,dx$ **c** $\int_0^{1} \frac{7}{\sqrt{4-x^2}}\,dx$

**d** $\int_0^{\frac{1}{2}} \frac{3}{\sqrt{\frac{1}{2}-x^2}}\,dx$ **e** $\int_0^{\frac{1}{4}} \frac{5}{\frac{1}{4}+x^2}\,dx$ **f** $\int_{-\frac{1}{4}}^{\frac{1}{4}} \frac{7}{\sqrt{\frac{1}{8}-x^2}}\,dx$

## Problem solving

**5** Find:

**a** $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$ **b** $\int \frac{e^x}{e^{2x}+1}\,dx$

**6** Evaluate $\int \frac{dx}{x\left(1+[\ln(x)]^2\right)}$.

**7** Find $\int \frac{\arccos(x)}{\sqrt{1-x^2}}\,dx$.

**8** Show that $\int \frac{x}{x^4+16}\,dx = \frac{1}{8}\tan^{-1}\left(\frac{x^2}{4}\right) + c$.

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WS
Partial fractions

Integral calculus

# 7.06 Integration using partial fractions

Finding the integral of a rational function of the form $f(x) = \frac{P(x)}{Q(x)}$ can be tricky. It is possible to express a rational function as a sum of simpler fractions that you can integrate, called **partial fractions**.

Consider the following:

$$\frac{2}{x+3} - \frac{1}{x-2} = \frac{2(x-2)-(x+3)}{(x+3)(x-2)} = \frac{x-7}{x^2+x-6}$$

By reversing this procedure and expressing $\frac{x-7}{x^2+x-6}$ as the partial fractions $\frac{2}{x+3} - \frac{1}{x-2}$, you can now integrate the final expression as follows:

$$\int \frac{x-7}{x^2+x-6}dx = \int\left(\frac{2}{x+3} - \frac{1}{x-2}\right)dx = 2\ \ln|x+3| - \ln|x-2| + c$$

## Partial fractions

Functions of the form $\frac{p}{(x-a)(x-b)}$ or $\frac{px+q}{(x-a)(x-b)}$ can be expressed as partial fractions in the form $\frac{A}{x-a} + \frac{B}{x-b}$.

## EXAMPLE 13

Express $\frac{x+8}{x^2+x-2}$ as partial fractions.

### Solution

Factorise the denominator. $\frac{x+8}{x^2+x-2} = \frac{x+8}{(x-1)(x+2)}$

Write as partial fractions. $\frac{x+8}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$

Write the RHS over a common denominator. $= \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}$

Equate the numerators. $x + 8 = Ax + 2A + Bx - B$

Group like terms. $x + 8 = (A + B)x + (2A - B)$

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| | | |
|---|---|---|
| Equate the variable and constant parts. | $A + B = 1$ | [1] |
| | $2A - B = 8$ | [2] |
| Add [1] and [2] to eliminate $B$. | $3A = 9$ | |
| Solve for $A$. | $A = 3$ | |
| Substitute for $A$ in [1]. | $3 + B = 1 \Rightarrow B = -2$ | |
| Write the partial fractions using $A$ and $B$. | $\frac{x+8}{(x-1)(x+2)} = \frac{3}{x-1} - \frac{2}{x+2}$ | |

Once a rational function is expressed as partial fractions, the integral can be found.

## EXAMPLE 14

Show that $\frac{5x-7}{x^2-4x-5} = \frac{3}{x-5} + \frac{2}{x+1}$ and hence find $\int \frac{5x-7}{x^2-4x-5}\,dx$.

### Solution

| | |
|---|---|
| Write the RHS. | $\text{RHS} = \frac{3}{x-5} + \frac{2}{x+1}$ |
| Write with a common denominator. | $= \frac{3(x+1)+2(x-5)}{(x-5)(x+1)}$ |
| Expand the top and bottom. | $= \frac{3x+3+2x-10}{x^2+x-5x-5}$ |
| Simplify. | $= \frac{5x-7}{x^2-4x-5}$ |
| Write the integral using partial fractions. | $\int \frac{5x-7}{x^2-4x-5}\,dx = \int\left(\frac{3}{x-5} + \frac{2}{x+1}\right)dx$ |
| Use the rules of integration. | $= \int \frac{3}{x-5}\,dx + \int \frac{2}{x+1}\,dx$ |
| | $= 3\int \frac{1}{x-5}\,dx + 2\int \frac{1}{x+1}\,dx$ |
| Integrate. | $= 3 \ln\lvert x-5\rvert + 2 \ln\lvert x+1\rvert + c$ |
| State the result. | $\int \frac{5x-7}{x^2-4x-5}\,dx = 3 \ln\lvert x-5\rvert + 2 \ln\lvert x+1\rvert + c$ |

For some rational functions, the degree of the numerator is greater than the degree of the denominator. In these cases, you can use long division before you find the integral. For example:

$$\frac{x^3+x+1}{x-1}=x^2+x+2+\frac{3}{x-1}$$

so $\displaystyle\int\frac{x^3+x+1}{x-1}=\int\left(x^2+x+2+\frac{3}{x-1}\right)dx=\frac{x^3}{3}+\frac{x^2}{2}+2x+3\ln|x-1|+c$

Other rational functions have repeated factors in the denominator.

## Rational functions with repeated factors in the denominator

Functions of the form $\dfrac{p}{(x-a)^2(x-b)}$ or $\dfrac{px+q}{(x-a)^2(x-b)}$ can be expressed as partial fractions in the form $\dfrac{A}{(x-a)^2}+\dfrac{B}{x-a}+\dfrac{C}{x-b}$.

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## EXAMPLE 15

Find $\int \frac{x^2 - x + 6}{(x-1)^2(x+1)}\,dx$.

### Solution

Write the integrand as partial fractions.

$$\frac{x^2 - x + 6}{(x-1)^2(x+1)} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x+1}$$

Express the RHS with a common denominator.

$$= \frac{A(x+1) + B(x-1)(x+1) + C(x-1)^2}{(x-1)^2(x+1)}$$

Equate the numerators.

$$x^2 - x + 6 = Ax + A + Bx^2 - B + Cx^2 - 2Cx + C$$

$$= (B + C)x^2 + (A - 2C)x + (A - B + C)$$

Equate like terms.

$$B + C = 1 \quad [1]$$

$$A - 2C = -1 \quad [2]$$

$$A - B + C = 6 \quad [3]$$

[3] + [1] $\quad A + 2C = 7 \quad [4]$

[2] + [4] $\quad 2A = 6$

Solve for $A$. $\quad A = 3$

Substitute for $A$ in [4]. $\quad 3 + 2C = 7 \Rightarrow C = 2$

Substitute for $C$ in [1]. $\quad B + 2 = 1 \Rightarrow B = -1$

Write the integral using partial fractions.

$$\int \frac{x^2 - x + 6}{(x-1)^2(x+1)}\,dx = \int \frac{3}{(x-1)^2} - \frac{1}{(x-1)} + \frac{2}{(x+1)}\,dx$$

Use the integral rules.

$$= 3\int \frac{1}{(x-1)^2}\,dx - \int \frac{1}{x-1}\,dx + 2\int \frac{1}{x+1}\,dx$$

Integrate.

$$= -\frac{3}{x-1} - \ln|x-1| + 2\ln|x+1| + c$$

State the result.

$$\int \frac{x^2 - x + 6}{(x-1)^2(x+1)}\,dx = -\frac{3}{x-1} - \ln|x-1| + 2\ln|x+1| + c$$

## Exercise 7.06 Integration using partial fractions

Example 13

**1** Express each rational function as partial fractions.

**a** $\dfrac{5x}{(x+1)(x-4)}$ **b** $\dfrac{8x}{(x+3)(3x+1)}$ **c** $\dfrac{20}{x^2+3x-4}$

**d** $\dfrac{x+34}{(x-6)(x+2)}$ **e** $\dfrac{x+16}{x^2+2x-8}$ **f** $\dfrac{3x+2}{(x-4)(2x+1)}$

Example 14

**2** Show that $\dfrac{2x+5}{(x+1)(x+2)} = \dfrac{3}{x+1} - \dfrac{1}{x+2}$ and hence find $\displaystyle\int \dfrac{2x+5}{(x+1)(x+2)}\,dx$.

**3** Show that $\dfrac{2x+1}{x^2-11x+30} = \dfrac{-11}{x-5} + \dfrac{13}{x-6}$ and hence find $\displaystyle\int \dfrac{2x+1}{x^2-11x+30}\,dx$.

**4** Show that $\dfrac{x+8}{x^2+x-2} = \dfrac{3}{x-1} - \dfrac{2}{x+2}$ and hence find $\displaystyle\int \dfrac{x+8}{x^2+x-2}\,dx$.

**5** Show that $\dfrac{5x-3}{x^2-2x-3} = \dfrac{2}{x+1} + \dfrac{3}{x-3}$ and hence find $\displaystyle\int \dfrac{5x-3}{x^2-2x-3}\,dx$.

Example 15

**6** Find:

**a** $\displaystyle\int \dfrac{15-4x-x^2}{(x+1)(x-2)^2}\,dx$ **b** $\displaystyle\int \dfrac{x-2}{x^2(x-1)}\,dx$ **c** $\displaystyle\int \dfrac{x^2}{(x+2)^2(x-3)}\,dx$

**d** $\displaystyle\int \dfrac{1}{(x+5)^2(x-1)}\,dx$ **e** $\displaystyle\int \dfrac{6x-11}{(x-1)^2}\,dx$ **f** $\displaystyle\int \dfrac{x}{(x+1)(x-1)^2}\,dx$

**7** Change each expression to a polynomial and a proper fraction.

**a** $\dfrac{3x^2+10x-3}{x+4}$ **b** $\dfrac{2x^3-x^2-x-3}{x-1}$ **c** $\dfrac{x^3-2x^2+12x-13}{x+3}$

**8** Use partial fractions to find each indefinite integral.

**a** $\displaystyle\int \dfrac{x-3}{x^2-1}\,dx$ **b** $\displaystyle\int \dfrac{5x^2-2x-6}{(x-1)^2(x+2)}\,dx$ **c** $\displaystyle\int \dfrac{3x^2+3x-12}{(x+1)(x-1)(x-2)}\,dx$

**9** Show that $\dfrac{x-5}{x^2-7x+12} = \dfrac{2}{x-3} - \dfrac{1}{x-4}$ and hence find $\displaystyle\int_{-1}^{1} \dfrac{x-5}{x^2-7x+12}\,dx$.

**10** Show that $\dfrac{5x}{x^2-x-6} = \dfrac{3}{x-3} + \dfrac{2}{x+2}$ and hence find $\displaystyle\int_{-1}^{2} \dfrac{5x}{x^2-x-6}\,dx$.


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### Problem solving

**11** Use long division to express $\dfrac{2x^4 - 3x^3 + 2x^2 + 3x - 5}{x^2 - 2x - 3}$ as a polynomial and proper fraction.

Hence find $\displaystyle\int \frac{2x^4 - 3x^3 + 2x^2 + 3x - 5}{x^2 - 2x - 3}\,dx$.

**12** Express $\dfrac{5x^2 + 2x + 5}{(x-1)(x^2 + x + 2)}$ in the form $\dfrac{A}{x-1} + \dfrac{Bx + C}{x^2 + x + 2}$.

Hence find $\displaystyle\int \frac{5x^2 + 2x + 5}{(x-1)(x^2 + x + 2)}\,dx$.

**13** Find:

**a** $\displaystyle\int \frac{x^2 + 10x + 18}{(x-2)(x^2 + 3x + 4)}\,dx$

**b** $\displaystyle\int \frac{15x^2 - 14x + 14}{(x-1)(3x^2 - 2x + 4)}\,dx$

**c** $\displaystyle\int \frac{10x^2 - 12x + 8}{(x-2)(2x^2 - x + 2)}\,dx$

## 7.07 Integration by parts

It would be useful to have a 'product rule' you could use if you are required to integrate a product of functions, such as $\int x\cos(x)\,dx$. In this case the integrand is the product of the functions $x$ and cos ($x$).

You have already seen that the substitution rule for integration corresponds to the chain rule for differentiation. The product rule for differentiation has a corresponding rule for integration called **integration by parts**.

The product rule states that if $f(x)$ and $g(x)$ are both functions, then

$$\frac{d}{dx}f(x)g(x) = f(x)g'(x) + f'(x)g(x)$$

Integrate: $$\int [f(x)g'(x) + f'(x)g(x)]\,dx = f(x)g(x)$$

$$\int f(x)g'(x)\,dx + \int f'(x)g(x)\,dx = f(x)g(x)$$

Rearrange: $$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

This is the rule for integration by parts.

Some people find the rule is easier to remember using a different notation.

Let $u = f(x)$ and $v = g(x)$.

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Write the product rule. $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

Integrate: $uv = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$

Rearrange: $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

**Integration by parts**

For the functions $f(x)$ and $g(x)$: $\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$

If $u = f(x)$ and $v = g(x)$: $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

To use integration by parts, you need to choose which function is taken as $g(x)$ and which as $f'(x)$. It may take several tries to find the choice that makes the integral easier. It is not necessary to include the constant of integration because it will be part of the new integral at the end.

## EXAMPLE 16

Find $\int x\cos(x)dx$.

### Solution

Choose $f(x)$ and $g'(x)$, where $f'(x)$ will be simpler than $f(x)$. $f(x) = x$ and $g'(x) = \cos(x)$

Work out $f'(x)$ and $g(x)$. $f'(x) = 1$ and $g(x) = \sin(x)$

Write the rule. $\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$

Substitute for known values. $\int x\cos(x)dx = x\sin(x) - \int 1\times\sin(x)dx$

Simplify. $= x\sin(x) - [-\cos(x)] + c$

State the result. $= x\sin(x) + \cos(x) + c$

Integrals involving products with $x$, $x^2$, $x^3$, etc. can often be found in a similar way to the above. It may be necessary to apply integration by parts several times to simplify the integral.


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## EXAMPLE 17

Find $\int e^{3x}x^2\,dx$.

### Solution

| | |
|---|---|
| Choose $f(x)$ and $g'(x)$. | $f(x)=x^2$ and $g'(x)=e^{3x}$ |
| Work out $f'(x)$ and $g(x)$. | $f'(x)=2x$ and $g(x)=\frac{1}{3}e^{3x}$ |
| Write the rule. | $\int f(x)g'(x)\,dx=f(x)g(x)-\int f'(x)g(x)\,dx$ |
| Substitute the functions. | $\int x^2e^{3x}\,dx=x^2\frac{1}{3}e^{3x}-\int 2x\times\frac{1}{3}e^{3x}\,dx$ |
| Simplify. | $=\frac{1}{3}x^2e^{3x}-\frac{2}{3}\int xe^{3x}\,dx$ |

Now apply integration by parts to $xe^{3x}$.

| | |
|---|---|
| Choose $f(x)$ and $g'(x)$. | $f(x)=x$ and $g'(x)=e^{3x}$ |
| Work out $f'(x)$ and $g(x)$. | $f'(x)=1$ and $g(x)=\frac{1}{3}e^{3x}$ |
| Write the rule. | $\int f(x)g'(x)\,dx=f(x)g(x)-\int f'(x)g(x)\,dx$ |
| Substitute for known values. | $\int xe^{3x}\,dx=x\frac{1}{3}e^{3x}-\int 1\times\frac{1}{3}e^{3x}\,dx$ |
| Simplify. | $=\frac{1}{3}xe^{3x}-\frac{1}{3}\int e^{3x}\,dx$ |
| Now do the integral. | $=\frac{1}{3}xe^{3x}-\frac{1}{3}\times\frac{1}{3}e^{3x}$ |
| | $=\frac{1}{3}xe^{3x}-\frac{1}{9}e^{3x}$ |
| Restate the previous result. | $\int e^{3x}x^2\,dx=\frac{1}{3}x^2e^{3x}-\frac{2}{3}\int xe^{3x}\,dx$ |
| Substitute for $\int xe^{3x}\,dx$ and add the constant of integration. | $=\frac{1}{3}x^2e^{3x}-\frac{2}{3}\left(\frac{1}{3}xe^{3x}-\frac{1}{9}e^{3x}\right)+c$ |

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Simplify. $= \frac{x^2e^{3x}}{3} - \frac{2xe^{3x}}{9} + \frac{2e^{3x}}{27} + c$

$= \frac{9x^2e^{3x} - 6xe^{3x} + 2e^{3x}}{27} + c$

Take out the common factor and state the result. $\int e^{3x}x^2dx = \frac{e^{3x}}{27}(9x^2 - 6x + 2) + c$

You will need to use a variety of techniques for products of different expressions.

## EXAMPLE 18

Find:

**a** $\int \sin(x)\cos(x)\,dx$

**b** $\int x^4 \ln|2x|\,dx$

### Solution

**a** Choose the functions. $u = \sin(x)$ and $\frac{dv}{dx} = \cos(x)$

Work out $\frac{du}{dx}$ and $v$. $\frac{du}{dx} = \cos(x)$ and $v = \sin(x)$

Write the rule. $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

Substitute for known values. $\int \sin(x)\cos(x)\,dx = \sin(x)\sin(x) - \int \sin(x)\cos(x)\,dx$

Simplify and rearrange. $2\int \sin(x)\cos(x)\,dx = \sin^2(x)$

Divide by 2 and add the constant. $\int \sin(x)\cos(x)\,dx = \frac{1}{2}\sin^2(x) + c$

**b** If a logarithm function is involved, then choose that factor to be $u$.

Rearrange the integral. $\int x^4 \ln|2x|\,dx = \int \ln|2x|\,x^4\,dx$

Choose the functions. $u = \ln|2x|$ and $\frac{dv}{dx} = x^4$

Work out $\frac{du}{dx}$ and $v$. $\frac{du}{dx} = \frac{1}{x}$ and $v = \frac{x^5}{5}$

Write the rule. $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$


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| | |
|---|---|
| Substitute for known values. | $\int \ln\lvert 2x\rvert\, x^4\, dx = \ln\lvert 2x\rvert \dfrac{x^5}{5} - \int \dfrac{x^5}{5} \cdot \dfrac{1}{x}\, dx$ |
| Simplify and rearrange. | $= \dfrac{1}{5}(x^5 \ln\lvert 2x\rvert) - \dfrac{1}{5}\int x^4\, dx$ |
| Integrate and add the constant. | $= \dfrac{1}{5}(x^5 \ln\lvert 2x\rvert) - \dfrac{1}{25}x^5 + c$ |
| Simplify and state the result. | $\int x^4 \ln\lvert 2x\rvert\, dx = \dfrac{x^5}{25}(5 \ln\lvert 2x\rvert - 1) + c$ |

## Exercise 7.07 Integration by parts

**1** Use integration by parts to find: Example 16

**a** $\int x \sin(x)\, dx$ **b** $\int xe^x\, dx$

**2** Find each integral. Example 17

**a** $\int x^2 e^x\, dx$ **b** $\int x^3 e^{3x}\, dx$

**3** Find each integral. Example 18

**a** $\int x \ln(x)\, dx$ **b** $\int x^2 \ln\lvert x\rvert\, dx$

**4** Find:

**a** $\int \sin^2(x)\, dx$ **b** $\int \ln\lvert 2x\rvert\, dx$ (Let $f(x) = \ln\lvert 2x\rvert$ and $g'(x) = 1$)

**c** $\int \ln\lvert 2x+7\rvert\, dx$ **d** $\int \log_{10}(3x)\, dx$

**e** $\int x^2 \sin(x)\, dx$ **f** $\int x^2 \cos(x)\, dx$

**5** Find:

**a** $\int xe^{2x}\, dx$ **b** $\int x \sec^2(x)\, dx$ **c** $\int x^2 e^{2x}\, dx$

**d** $\int x \cos(3x)\, dx$ **e** $\int xe^{-3x}\, dx$ **f** $\int x \sec(x) \tan(x)\, dx$

**6** Find:

**a** $\int \dfrac{1}{x^2} \ln\lvert x\rvert\, dx$ **b** $\int e^x \sin(x)\, dx$ **c** $\int \sqrt{x} \ln\lvert x\rvert\, dx$

**d** $\int e^x \cos(x)\, dx$ **e** $\int x^3 \ln\lvert x\rvert\, dx$ **f** $\int (\ln\lvert x\rvert)^2\, dx$

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**7** Calculate:

**a** $\int_{\frac{1}{2}}^{1} \ln|x|\, dx$

**b** $\int_{0}^{\frac{1}{2}} \cos^{-1}(x)\, dx$

**c** $\int_{0}^{\frac{\pi}{2}} x\sin(2x)\, dx$

**d** $\int_{0}^{1} xe^{3x}\, dx$

## Problem solving

**8** Use integration by parts with the substitutions $f(x) = \cos^{n-1}(x)$ and $g'(x) = \cos(x)$ to show that:

$$\int \cos^n(x)\, dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\, dx$$

**9** Use integration by parts to show that $\int x^n e^x\, dx = x^n e^x - n\int x^{n-1}e^x\, dx$.

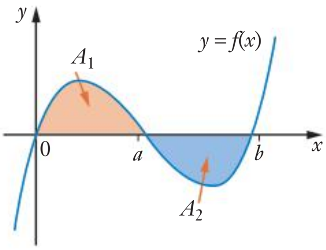
Areas between curves

Areas between curves 2

# 7.08 Areas between curves

You have previously used integration to calculate the areas of regions that lie between the graphs of functions and the $x$-axis.

The diagram shows areas $A_1$ and $A_2$.




$A_1 = \int_0^a f(x)\, dx$ and $A_1 > 0$.

$A_2 = \int_a^b f(x)\, dx$ and $A_2 < 0$.

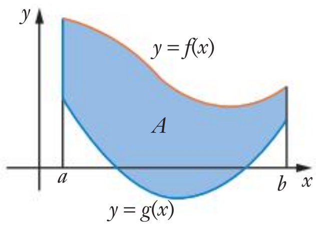
The algebraic area between 0 and $b$ is given by:

$$\int_0^b f(x)\, dx = A_1 - A_2$$

The physical area between 0 and $b$ is given by:

$$\int_0^a f(x)\, dx + \left|\int_a^b f(x)\, dx\right| = A_1 + |A_2|$$

In this section you will use integrals to find the areas of regions that lie between the graphs of 2 functions as shown in the diagram.




The area could be calculated by finding the physical areas under $y = f(x)$ and $y = g(x)$ and subtracting them.

Another way of calculating the area between the graphs of the functions is to calculate the area under the **difference function**, $f(x) - g(x)$.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## Areas between curves

If $y = f(x)$ and $y = g(x)$ are continuous and $f(x) \geq g(x)$ for all $x$ in $[a, b]$, then the area $A$ of the region bounded by the curves $y = f(x)$ and $y = g(x)$, and the lines $x = a$ and $x = b$, is:

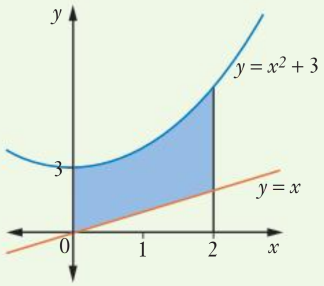
$$A = \int_a^b [f(x) - g(x)]dx$$

## EXAMPLE 19

Find the area of the region bounded by $y = x^2 + 3$ and $y = x$ between $x = 0$ and $x = 2$.

### Solution

Draw a rough diagram. You may want to use a graphics calculator to check the diagram.




Assign the functions. Let $f(x) = x^2 + 3$ and $g(x) = x$

Examine the graphs. $f(x) > g(x)$ between $x = 0$ and $x = 2$, so you can use the area formula.

State the rule. $A = \int_a^b [f(x) - g(x)]dx$

Substitute for known values. $= \int_0^2 [(x^2 + 3) - x]dx$

Simplify. $= \int_0^2 (x^2 - x + 3)dx$

Integrate. $= \left[\frac{x^3}{3} - \frac{x^2}{2} + 3x\right]_0^2$

Evaluate. $= \frac{8}{3} - \frac{4}{2} + 6$

$= 6\frac{2}{3}$

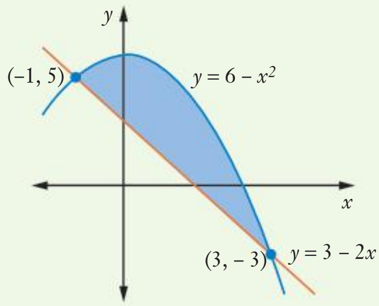
State the result. The area bounded by $y = x^2 + 3$ and $y = x$ between $x = 0$ and $x = 2$ is $6\frac{2}{3}$ square units.

To calculate the area enclosed by 2 functions, say $f(x)$ and $g(x)$, you need to find where they intersect. The zeros of the difference function, $d(x)$, show the points of intersection of the graphs of $f(x)$ and $g(x)$.

## EXAMPLE 20

Find the area enclosed by $y = 6 - x^2$ and $y = 3 - 2x$.

### Solution

| | |
|---|---|
| Assign the functions. | Let $p(x) = 6 - x^2$ and $q(x) = 3 - 2x$ |
| Find the difference function, $d(x)$. | $d(x) = 6 - x^2 - (3 - 2x)$ |
| Rearrange and simplify. | $= -x^2 + 2x + 3$ |
| Factorise. | $= (x + 1)(-x + 3)$ |
| Find the zeros of $d(x)$. | $x = -1$ and $3$ |
| Find the points of intersection. | When $x = -1, y = 5$ |
| | When $x = 3, y = -3$ |
| | Points of intersection = $(-1, 5)$ and $(3, -3)$. |
| Draw a rough diagram and use a graphics calculator to check the it. |  |
| Examine the graphs. | $p(x) > q(x)$ between $x = -1$ and $x = 3$, so you can use the area formula. |
| State the rule. | $A = \int_a^b [p(x) - q(x)]dx$ |
| $p(x) - q(x) = d(x)$ | $= \int_{-1}^{3} (-x^2 + 2x + 3)dx$ |
| Integrate. | $= \left[-\frac{x^3}{3} + x^2 + 3x\right]_{-1}^{3}$ |

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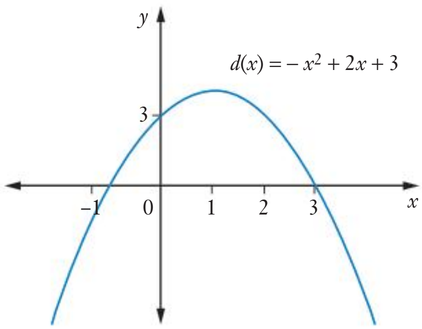
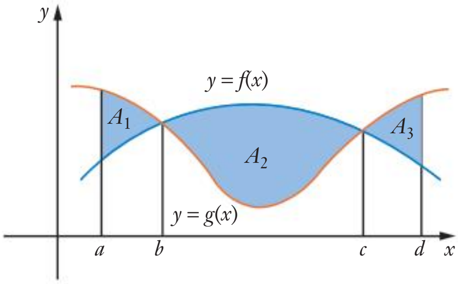
| | |
|---|---|
| Evaluate. | $= \left(-\frac{27}{3}+9+9\right) - \left(\frac{1}{3}+1-3\right)$ |
| | $= 10\frac{2}{3}$ |
| State the result. | The area enclosed by $y = 3 - 2x$ and $y = 6 - x^2$ is $10\frac{2}{3}$ square units. |

In the previous example, if you used $A = \int_a^b [q(x) - p(x)]dx$, the resulting area would have been found to be $-10\frac{2}{3}$ square units.

The graph of $d(x)$ is also useful for calculating the area between 2 functions. The diagram shows the graph of $d(x)$ from the previous example. The graph is positive between $x = -1$ and $x = 3$, and so is the area between $p(x)$ and $q(x)$. Because the graph of $d(x)$ is negative for $x < -1$ and $x > 3$, so would the area between $p(x)$ and $q(x)$.




If you need to find the area between the graphs of two functions $f(x)$ and $g(x)$, where $f(x) > g(x)$ for some values of $x$ but $g(x) > f(x)$ for other values, then the given region needs to be split as shown below.




In this case, the area between $f(x)$ and $g(x)$ in the interval $[a, b]$ is $|A_1| + |A_2| + |A_3|$. The absolute values must be used as the areas will alternate between positive and negative, depending on whether $f(x) > g(x)$ or $g(x) > f(x)$.

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## EXAMPLE 21

Find the area enclosed by $y = x^3 - x$ and $y = 3x$.

### Solution

Assign the functions. $f(x) = x^3 - x$ and $g(x) = 3x$.

Find the difference function, $d(x)$.

$$d(x) = f(x) - g(x)$$
$$= x^3 - x - 3x$$

Simplify.

$$= x^3 - 4x$$

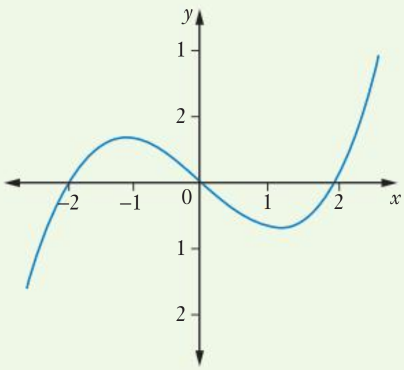
Factorise.

$$= x(x^2 - 4)$$
$$= x(x - 2)(x + 2)$$

Find the zeros of $d(x)$. $x = -2$, 0 and 2

Sketch the difference function.

The graph shows that the area between $f(x)$ and $g(x)$ is positive for $-2 < x < 0$ and negative for $0 < x < 2$.




Find the enclosed area.

$$A = \int_{-2}^{0} (x^3 - 4x)\,dx - \int_{0}^{2} (x^3 - 4x)\,dx$$

Integrate.

$$= \left[\frac{x^4}{4} - 2x^2\right]_{-2}^{0} - \left[\frac{x^4}{4} - 2x^2\right]_{0}^{2}$$

Evaluate.

$$= \left[0 - \left(\frac{16}{4} - 2\times 4\right)\right] - \left[\left(\frac{16}{4} - 2\times 4\right) - 0\right]$$
$$= 4 + 4$$
$$= 8$$

State the result. The area enclosed $y = x^3 - x$ and $y = 3x$ is 8 square units.

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You can use your graphics calculator to check the problem.

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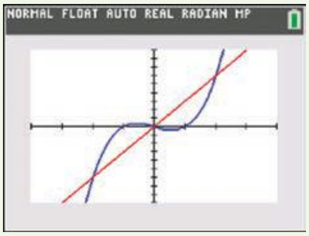


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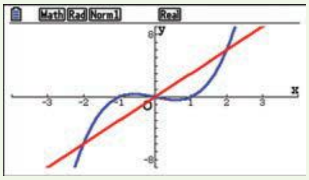


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From the graphs, you can see that $y = x^3 - x$ is above $y = 3x$ for $-2 < x < 0$ and so the area between them is positive, while for $0 < x < 2$, $y = x^3 - x$ is below $y = 3x$ and so the area is negative.

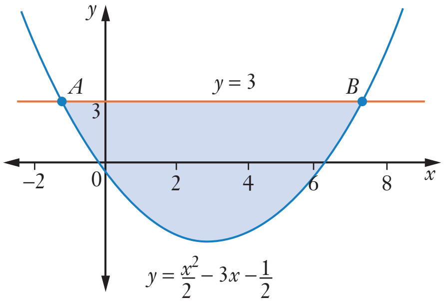
## Exercise 7.08 Areas between curves

**1** For each graph:

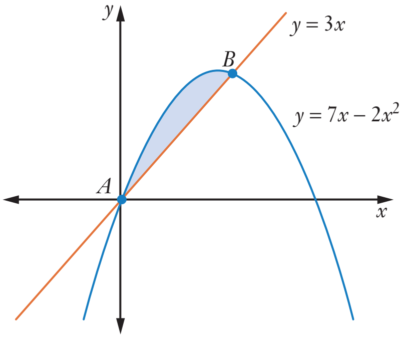
**i** find $A$ and $B$

**ii** calculate the shaded area.

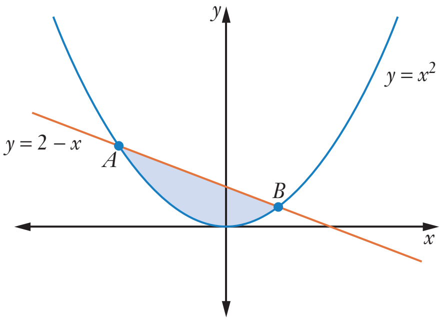
**a**



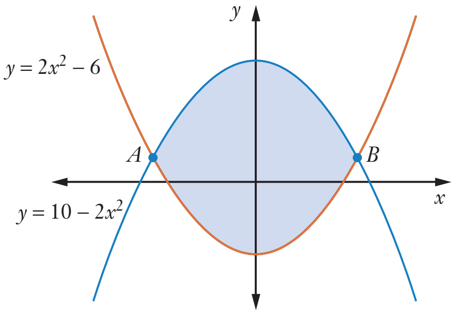

**b**




**c**




**d**




**2** Find the area of the region bounded by $y = -2x^2 - 1$ and $y = x + 3$ between $x = 0$ and $x = 1$.

Example 19

**3** Find the approximate area of the region bounded by $y = e^x$ and $y = x^2 - 1$ between $x = -1$ and $x = 1$.

4 Find the area of the region bounded by $y = 2x^2 - 8x + 10$ and $y = \frac{1}{2}x^2 - 2x - 1$ between $x = 1$ and $x = 2$.

Example 20

5 Find the area enclosed by $y = x^2$ and $y = x^2 - 16$ between $x = 1$ and $x = 4$.

6 Find the area enclosed by $y = x^2$ and $y = \sqrt{x}$.

7 Find the area enclosed by $y = 4x + 16$ and $y = 2x^2 + 10$.

Example 21

8 Find the area enclosed by $y = -\frac{1}{2}x^3 + 2x^2$ and $y = 4x - x^2$.

9 Find the area enclosed by $y = 2x^3 - x^2 - 5x$ and $y = 3x - x^2$.

10 Find the area enclosed by $y = x^3 - x^2 - 7x + 5$ and $y = 2x - 4$.

11 Calculate the area of the region bounded by:

a $y = \sin(x)$ and $y = \cos(x)$ between $x = 0$ and $x = \frac{\pi}{2}$

b $y = \sec^2(x)$ and $y = \sin(x)$ between $x = 0$ and $x = \frac{\pi}{4}$

c $y = x^2$ and $y = x + 6$ between $x = 0$ and $x = 5$

d $y = -2\sec^2(x)$ and $y = 2\cos(x)$ between $x = 0$ and $x = \frac{\pi}{4}$

e $y = x^2$ and $y = -2x^4$ between $x = -1$ and $x = 1$

f $y = 4 - x^2$ and $y = 2 - x$ between $x = -2$ and $x = 3$

g $y = e^x$ and $y = xe^x$ between $x = -1$ and $x = 1$

12 Calculate the area the region bounded by:

a $y = \sqrt[3]{x^2}$ and $y = x$

b $y = 2x^2$ and $y = x^4 - 2x^2$

c $y = x^3 - 3x$ and $y = x$

d $y = x^3 - 5x^2 - 5x + 7$ and $y = 3x - 5$

## Problem solving

13 Calculate the area the region bounded by $y = |x - 2|$ and $y = \sqrt{x}$.

14 Calculate the area the region bounded by $y = x - 1$ and $y^2 = 2x + 6$.

15 Consider the graphs of $f(x) = x(x + 3)$, $g(x) = \frac{4}{x^2}$ and $h(x) = x - \frac{x^2}{4}$ in the first quadrant of the Cartesian plane. If $f(x)$ and $g(x)$ intersect at $(a, 4)$ and $g(x)$ and $h(x)$ intersect at $(b, 1)$, calculate the values of $a$ and $b$ and hence find the area of the region bounded by $f(x)$, $g(x)$ and $h(x)$.

16 A large natural reserve is populated by feral rabbits. It is estimated that their population growth is modelled by the function $P(t) = 60e^{0.02t}$ thousand rabbits per year. Scientists believe that a culling program will alter the growth rate to $Q(t) = -t^2 + 60$ thousand rabbits per year over the next 5 years. If the culling program is successfully implemented, how many fewer rabbits would there be in the reserve after 5 years?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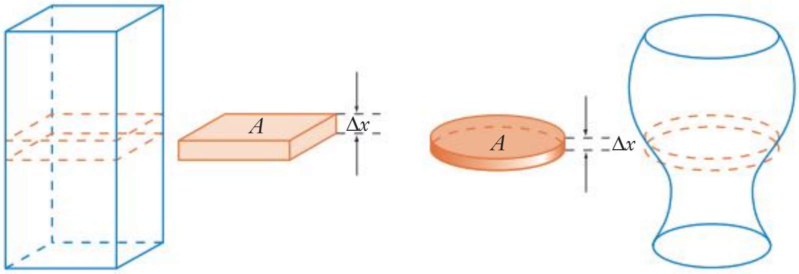
# 7.09 Volumes of solids of revolution

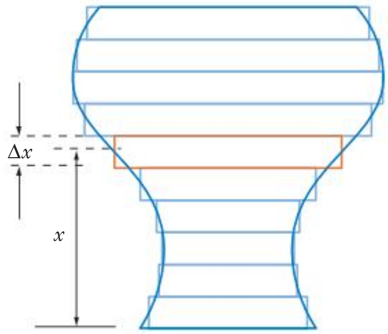
## Volumes by integration



The volume of a prism or cylinder can be calculated by multiplying the area of the base by the height. This really finds the number of cubic units in each layer of the prism. You can approximate the volumes of other solids by slicing them into layers, finding the volume of each layer, and adding the volumes together. The volume of a single slice through a solid is $A\Delta x$, where $A$ is the area of the slice and $\Delta x$ is the thickness of the slice, as shown below.




If successive slices of the same thickness were taken all the way up an object, the total volume would be approximately $V \approx \sum A\Delta x$. The side view of such slices might look like the figure on the right.




The area of each slice actually depends on its position.

It would be more correct to write $A = f(x)$, since the area of the slice is a function of the height, $x$. In order to obtain a more accurate value of $V$, you should make the slices thinner; as $\Delta x \to 0$, the volume will become exact.

$$V \approx \sum A\Delta x$$

But $A = f(x)$, so $\qquad V \approx \sum f(x)\Delta x$

Using the limit $\qquad V = \lim_{\Delta x \to 0} \sum f(x)\Delta x$

But this is the definite integral, so

$$V = \int f(x)\,dx = \int A(x)\,dx$$

The cross-sectional area at $x$ can be written as $A(x)$ to emphasise that it is the area and that it is a function of the 'height' $x$.

**Volume of a solid**

$$V = \int A(x)\,dx$$

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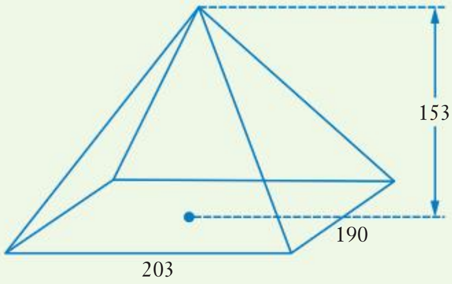
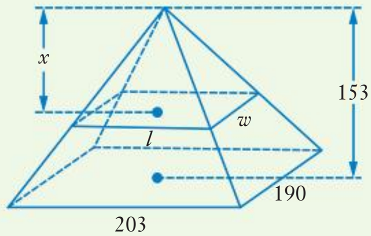
## EXAMPLE 22

Use integration to find the volume of a pyramid of height 153 m with a rectangular base 203 m by 190 m.




### Solution

Re-draw the diagram.




The cross-section of the pyramid is a rectangle with length $l$ and width $w$.

Before you can integrate, you need to express the cross-sectional area as a function of the height, $x$. The direction you choose for the height is arbitrary, so choose the most convenient direction. In this case, it is easiest to measure the 'height' from the top of the pyramid to the base, as shown.

Use ratios to find $l$ and $w$ at height $x$.

$$l = \frac{x}{153} \times 203 \text{ and } w = \frac{x}{153} \times 190$$

Write the area formula at $x$.

$$A = lw$$

Substitute.

$$= \frac{x}{153} \times 203 \times \frac{x}{153} \times 190$$

Simplify.

$$= \frac{38\,570}{23\,409} x^2$$

Write the volume formula.

$$V = \int A(x)\,dx$$

Integrate from $x = 0$ to $x = 153$ to find the volume.

$$= \int_0^{153} \frac{38\,570}{23\,409} x^2\,dx$$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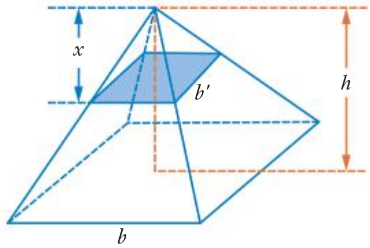
| | |
|---|---|
| Integrate. | $= \frac{38\,570}{23\,409}\left[\frac{x^3}{3}\right]_0^{153}$ |
| Evaluate. | $= \frac{38\,570}{23\,409} \times \frac{153^3}{3}$ |
| | $= 1\,967\,070$ |
| Write the answer. | The volume is 1 967 070 $m^3$. |

The previous example provides a method for proving the formula for the volume of a square-based pyramid with a base length of $b$ and height $h$.




Use ratios to find $b'$ at height $x$.

$$b' = \frac{x}{h} \times b = \frac{bx}{h}$$

So the cross-sectional area at height, $x$ is:

$$A = b'^2 = \frac{b^2x^2}{h^2}$$

Use integrals to find the volume. You need to integrate using the limits $x = 0$ and $x = h$.

$$V = \int_0^h A(x)\,dx = \int_0^h \frac{b^2x^2}{h^2}\,dx = \frac{b^2}{h^2}\int_0^h x^2\,dx$$

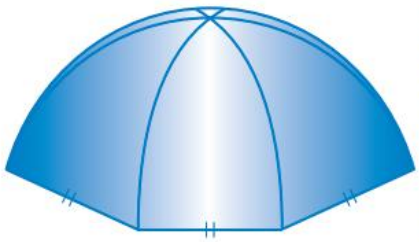
$$= \frac{b^2}{h^2}\left[\frac{x^3}{3}\right]_0^h = \frac{b^2}{h^2} \times \frac{h^3}{3} = \frac{b^2h}{3}$$

So, the volume of a square-based pyramid with base length $b$ and height $h$ is $V = \frac{b^2h}{3}$.

## INVESTIGATION

### VOLUME OF A POP-UP TENT

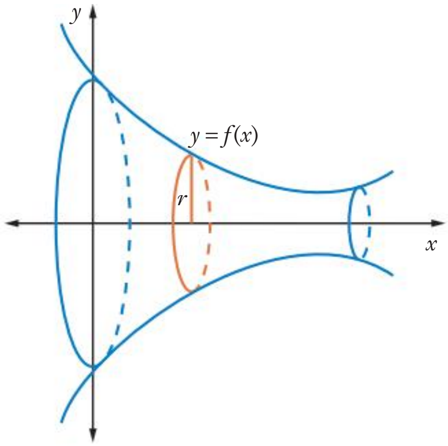
A lightweight ‘pop-up’ tent consists of 6 plastic struts that are inserted into pockets sewn into the joins of the fabric panels. The resulting shape has hexagonal horizontal cross-sections, while vertical cross-sections through the centre are semicircular.



The overall height is 1.3 m. In order to find whether it is safe to use a gas lamp inside, a camper needs to know the volume of air in the tent. Work in groups of 2–3 to find the volume.

## Solids of revolution

Many manufactured objects have a circular cross-section because they can be manufactured by turning on a lathe. Objects of this shape are called **solids of revolution**, and their volumes can be found by using integration. Consider the axis of revolution as being the $x$-axis and write the radius as $y = f(x)$.




Then $y = f(x)$ is the radius of the cross-section, and the cross-section is a circle.

Thus $A = \pi r^2 = \pi y^2$.

The **volume of a solid of revolution** is thus:

$$V = \int A\,dx = \int \pi y^2 dx = \pi \int y^2 dx$$

Of course, you could choose the $y$-axis as the axis of rotation, and in this case you would swap the integration variables.

### Volume of a solid of revolution

The volume of a solid of revolution generated by a curve in the $xy$ plane is given by:

$$V = \pi \int_a^b y^2\,dx \text{ or } V = \pi \int_a^b [f(x)]^2\,dx$$

when $y = f(x)$ between $x = a$ and $x = b$ is rotated about the $x$-axis

or $$V = \pi \int_a^b x^2\,dy \text{ or } V = \pi \int_a^b [f(y)]^2\,dy$$

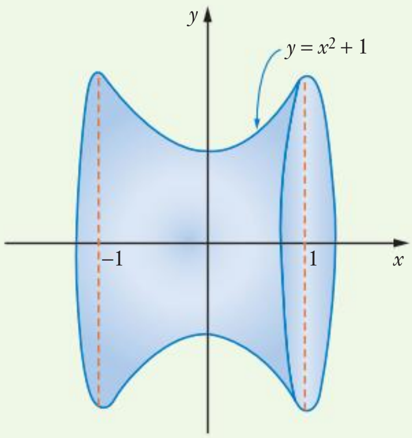
when $x = f(y)$ between $y = a$ and $y = b$ is rotated about the $y$-axis

### EXAMPLE 23

The spool shown is used to store fibre optic cable.

Its shape can be formed by rotating the curve $y = x^2 + 1$ about the $x$-axis from $x = -1$ to $x = 1$ as shown.

Calculate the volume of the spool if the measurements are expressed in metres.





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## Solution

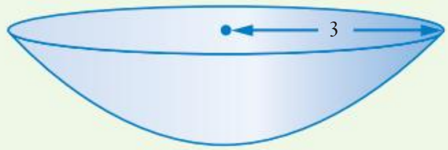
| | |
|---|---|
| Write the formula for a line rotated about the $x$-axis between $x = -1$ and $x = 1$. | $V = \pi \int_{-1}^{1} y^2\,dx$ or $\pi \int_{1}^{-1} [f(x)]^2\,dx$ |
| Substitute for $y$. | $= \pi \int_{-1}^{1} (x^2 + 1)^2\,dx$ |
| Expand the brackets. | $= \pi \int_{-1}^{1} (x^4 + 2x^2 + 1)\,dx$ |
| Integrate. | $= \pi \left[\frac{x^5}{5} + \frac{2x^3}{3} + x\right]_{-1}^{1}$ |
| Evaluate. | $= \pi\left[\left(\frac{1}{5} + \frac{2}{3} + 1\right) - \left(-\frac{1}{5} - \frac{2}{3} - 1\right)\right]$ |
| | $= \frac{56\pi}{15}$ |
| Write the answer. | The volume of the spool is $\frac{56\pi}{15}$ m$^3$, or about 11.7 m$^3$. |

Shutterstock.com/Beata Becla

If you choose the $y$-axis as the axis of rotation, swap the integration variables for $y$ values and find an expression for $x^2$. You will also need to calculate the limits for integration in terms of $y$.

## EXAMPLE 24

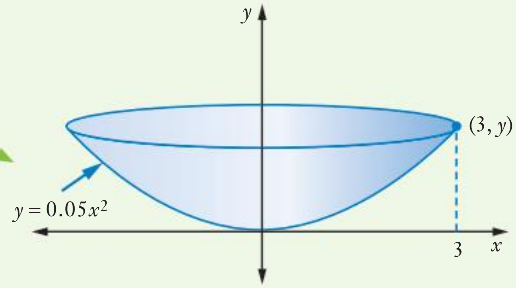
Find the volume of a parabolic radio telescope bowl of radius 3 m formed by rotating the curve $y = 0.05x^2$ about the $y$-axis.




### Solution

Sketch a diagram.

Since the curve is rotated about the $y$-axis, you need to express $x^2$ as a function of $y$.




Rewrite the function.

$y = 0.05x^2 \Rightarrow x^2 = 20y$

The limits of integration will be the values of $y$ that give the correct range of $x$.

When $x = 0$, $y = 0.05x^2$

$= 0.05 \times 0^2$

$= 0$

When $x = 3$, $y = 0.05x^2$

$= 0.05 \times 3^2$

$= 0.45$

Write the formula for volume.

$V = \pi \int x^2 dy$ or $\pi \int [f(y)]^2 dy$

Write in terms of $y$.

$= \pi \int 20y\, dy$

Integrate from $y = 0$ to $y = 0.45$ to find the volume.

$V = \pi \int_0^{0.45} 20y\, dy$

Integrate and evaluate.

$= \pi \left[10y^2\right]_0^{0.45}$

$\approx 6.36$

Write the answer.

The volume is about 6.36 m$^3$.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## Exercise 7.09 Volumes of solids of revolution

**1** Use integration to find the volume of a pyramid of height 15 m with a rectangular base 30 m by 40 m. Example 22

**2** Use integration to find the volume of a cone of height 10 cm with a base of diameter 15 cm.

**3** Calculate the volume generated by revolving the curve $y^2 = 8x$ about the $x$-axis from $x = 0$ to $x = 3$. Example 23

**4** Use a suitable definite integral to find the volumes, in cubic units, that are formed by rotating the following curves about the $x$-axis between the limits shown.

**a** $y = x^2$ from $x = 0$ to $x = 2$

**b** $y = \ln x$ from $x = 1$ to $x = 3$

**c** $y = \dfrac{x^2}{4}$ from $x = 0$ to $x = 4$

**d** $y = \dfrac{1}{2}e^{\frac{x}{2}}$ from $x = 0$ to $x = 5$

**e** $y = 2\sin(x)$ from $x = \dfrac{\pi}{2}$ to $x = \pi$

**f** $y = 3\cos\left(\dfrac{x}{2}\right)$ from $x = 0$ to $x = \dfrac{\pi}{2}$

**5** Calculate the volume generated by revolving the curve $y = 4x^2$ about the $y$-axis from $x = 0$ to $x = 2$. Example 24

**6** Use a suitable definite integral to find the volume, in cubic units, that is formed by rotating each curve about the $y$-axis between the limits shown.

**a** $y = \dfrac{1}{2}x^2$ from $y = 0$ to $y = 5$

**b** $y = 4x^3$ from $y = 0$ to $y = 6$

**c** $y = \ln|x|$ from $y = 0$ to $y = 2$

**d** $y = \dfrac{1}{2}e^{\frac{x}{2}}$ from $x = 1$ to $x = 2$

**e** $y = x^3$ from $x = 0$ to $x = 2$

**f** $x^2 + y^2 = 9$ from $y = 2$ to $y = 3$

**7** Calculate the volume of the solid generated when the region bounded by $y^2 = x^4(1 - x^2)$ is revolved about the $x$-axis.

**8** Calculate the volume of the solid generated when the region bounded by the ellipse $9x^2 - 4y^2 = 36$ is revolved about the $x$-axis.

### Problem solving

**9** A rectangular farm dam is to be built so that all four internal sides slope down from the top at an angle of 30º to the vertical, the top of the dam is 15 m by 25 m and the dam is 2 m deep. Use integration to find the volume of the dam.

**10** A cone with radius $r$ and height $h$ is generated by revolving the line $y = \dfrac{r}{h}$ about the $x$-axis between $x = 0$ and $x = h$. Use a suitable integral to find a formula for the volume of the cone.

**11** A hemisphere of radius 5 cm can be considered to be the solid of revolution of the line $y = \sqrt{25 - x^2}$ around the $x$-axis. Use integration to find the volume of the hemisphere.

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**12** The inside of a pottery urn that a small child could hide in can be modelled as a solid of revolution of the line $y = 0.000\,05x^3 - 0.0045x^2 + 20$ cm around the $x$-axis from $x = 0$ to $x = 80$ cm, with the open end at $x = 0$ and a flat base. Find the maximum volume of oil that could be held by the urn and the volume when it is half-full (i.e. filled halfway to the top).

**13** The interior surface of an hourglass can be considered as having the curve

$$y = \sqrt{1.05 - \frac{1}{1+x^2}}$$

between $x = -5$ and $x = 5$ rotated about the $x$-axis. Find the volume of the hourglass, where all measurements are in centimetres.

**14** The interior surface of a glass *objet d'art* is formed from a molten glass sheet by allowing it to sag through a circular opening. The resulting cross-sectional shape may be modelled by the catenary curve

$$f(x) = 15.43 - 5(e^{0.05x} + e^{-0.05x})$$

between $x = -20$ and $x = 20$, where all measurements are in centimetres. Use rotation about the $y$-axis between $x = 0$ and $x = 20$ to find the volume of the inside of the *objet d'art*.

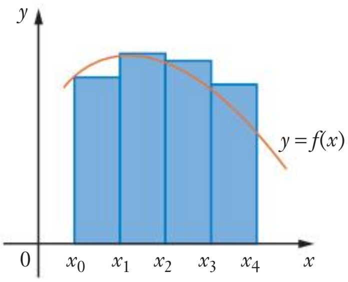
WS Trapezoidal rule

WS Numerical integration

## 7.10 Simpson's rule

There are situations in which it is not possible to find the exact value of a definite integral. In these cases, it is necessary to find an approximate value for the area under the curve.

### Approximations using rectangles



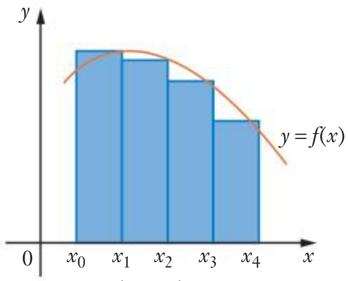
Left Endpoints

The diagram on the right shows the area under $y = f(x)$ for the interval $[a, b]$ divided into a number of rectangular strips.

The area of each strip is given by $A = x_{i-1} \times w$, where $w$ is the width of the strip and $x_{i-1}$ is the left endpoint of the $i$th strip.

The approximate area under the curve for the interval $[a, b]$ using strips drawn with the left endpoints of the strips is:

$$\int_a^b f(x)\,dx \approx A_L = \sum_{i=1}^{n} f(x_{i-1})w$$



Right endpoints

Consider the same area under the curve divided into strips using the right endpoint for each strip.

The area of each strip is now given by $A = x_i \times w$, where $x_i$ is the right endpoint of the $i$th strip.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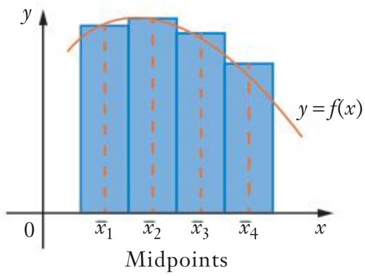
The approximate area under the curve for the interval $[a, b]$ using strips drawn with the right endpoints is:

$$\int_a^b f(x)\,dx \approx A_R = \sum_{i=1}^{n} f(x_i)w$$

A more accurate estimation can be found using the midpoints of rectangular strips as shown on the right.




In this case, the area of each strip is now given by $A = \bar{x}_i \times w$, where $\bar{x}_i$ is the midpoint of the $i$th strip.

The approximate area under the curve for the interval $[a, b]$ using strips drawn with the midpoints is:

$$\int_a^b f(x)\,dx \approx A_M = \sum_{i=1}^{n} f(\bar{x}_i)w$$

This is known as the **midpoint rule** for the estimation of the area under a curve.

## Midpoint rule

$$\int_a^b f(x)\,dx \approx w\sum_{i=1}^{n} f(\bar{x}_i)$$

where the interval $[a, b]$ is divided into $n$ strips of equal width, $w$

$$w = \frac{b-a}{n}$$

$\bar{x}_1, \bar{x}_2, \bar{x}_3, ..., \bar{x}_n$ are the midpoints of the strips

$x_0 = a$ and $x_n = b$

$\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$.

## EXAMPLE 25

Find the approximate area under the curve $y = \frac{1}{x}$ from $x = 1$ to $x = 6$ by calculating the areas of strips 0.5 units wide using the midpoint rule.

### Solution

| | |
|---|---|
| State the width of the rectangles. | $\Delta x = 0.5$ |
| State the midpoints. | Midpoints = 1.25, 1.75, 2.25, …, 5.75 |
| Find a rule for the midpoints using $i$. | Midpoints = $1.25 + 0.5i$ for $i = 0$ to 9 |

Write the midpoint rule. $\int_a^b f(x)\,dx \approx \Delta x \sum_{i=1}^{n} f(\bar{x}_i)$

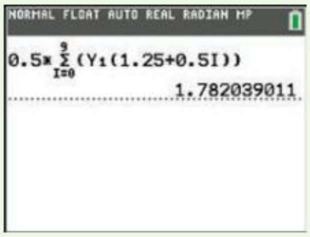
Write the approximation. $\int_1^6 \frac{1}{x}\,dx \approx 0.5 \times \sum_{i=0}^{9} f(1.25+0.5i)$

Use your graphics calculator.

TI-Nspire CX
Chapter 7

### TI-84 Plus CE

First define the function. Press [y=] and enter $1 \div X$ and press [enter]. Then press [2nd][mode][quit].

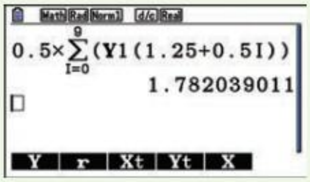



Enter 0.5 [×]. Press [math] and then select 0: summation Σ(. Enter $I$ as the summation variable and the values 0 and 9 for its limits.

Place the cursor inside the summation brackets and press the [vars] button, select Y–VARS and 1:Function. Select 1:$Y_1$. Then type [(]1.25 + 0.5$I$[)] and press [enter] to display the result.

### Casio fx-CG20AU

From the Graph menu, enter $Y1 = 1 \div x$.




Open a Run-Matrix screen and enter 0.5 [×].

Press [F4] (MATH), [F6] (▶) the [F2] (Σ(). Enter $I$ as the summation variable and the values 0 and 9 for its limits.

Place the cursor inside the summation brackets and press the [VARS] button then press [F4] (GRAPH) and select [F1] (Y). Then type 1[(]1.25+0.5$I$[)] and press [EXE] to display the result.

**Both calculators**

Round off and write the answer. $\int_1^6 \frac{1}{x}\,dx \approx 1.782$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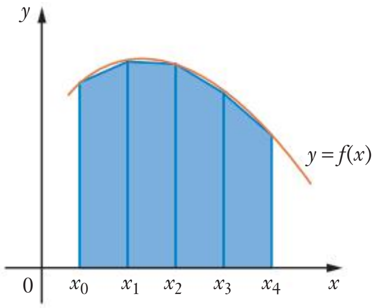
## The trapezoidal rule

A better approximation can be found by replacing the rectangular strips with trapezia. The tops of the trapezia are sloping, so they follow the graph of the function more closely than the rectangles, as shown in the diagram on the right.



Trapezoids

The trapezoidal rule can be formed by averaging the previously stated approximations of the area under the curve using the left and right endpoints.

$$\int_a^b f(x)\,dx \approx \frac{1}{2}\left[\sum_{i=1}^{n} f(x_{i-1})w + \sum_{i=1}^{n} f(x_i)w\right]$$

$$= \frac{w}{2}\left[\sum_{i=1}^{n}[f(x_{i-1}) + f(x_i)]\right]$$

$$= \frac{w}{2}[(f(x_0) + f(x_1)) + (f(x_1) + f(x_2)) + \ldots + (f(x_{n-1}) + f(x_n))]$$

$$= \frac{w}{2}[f(x_0) + 2f(x_1)) + 2f(x_2) + \ldots + 2f(x_{n-1}) + 2f(x_n)]$$

This is called the **trapezoidal rule**.

> ### Trapezoidal rule
>
> $$\int_a^b f(x)\,dx \approx \frac{w}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)]$$
>
> where the interval $[a, b]$ is divided into $n$ trapezia of equal width, $w$
>
> $w = \dfrac{b-a}{n}$
>
> $x_0 = a$ and $x_n = b$
>
> $x_i = a + iw$
>
> An alternate statement of the trapezoidal rule is:
>
> $$\int_a^b f(x)\,dx \approx \frac{w}{2}\left[f(x_0) + 2\sum_{i=1}^{n} f(x_i) + f(x_n)\right]$$
>
> or $\quad \displaystyle\int_a^b f(x)\,dx \approx \frac{w}{2}(E + 2M)$
>
> where $E$ is the sum of the values of $f(x)$ at the ends of the interval and $M$ is the sum of all the values of $f(x)$ at the intervening points $x_1, x_2, x_3, \ldots, x_{n-1}$.

The last form of the trapezoidal rule is the most useful for practical calculations.

## EXAMPLE 26

Use the trapezoidal rule with $n = 5$ to find the approximate value of $\int_1^2 \ln|5x|\,dx$.

### Solution

State the known values. $n = 5, a = 1, b = 2$

Calculate $w$. $w = \frac{b-a}{n} = \frac{2-1}{5} = 0.2$

Write the trapezoidal rule. $\int_a^b f(x)\,dx$

$\approx \frac{w}{2}(E + 2M)$

$= \frac{w}{2}[f(x_0) + f(x_n) + 2(f(x_1) + f(x_2) + \ldots + f(x_{n-1}))]$

Substitute for known values. $= \frac{0.2}{2}[\ln|5| + \ln|10| + 2(\ln|6| + \ln|7| + \ln|8| + \ln|9|)]$

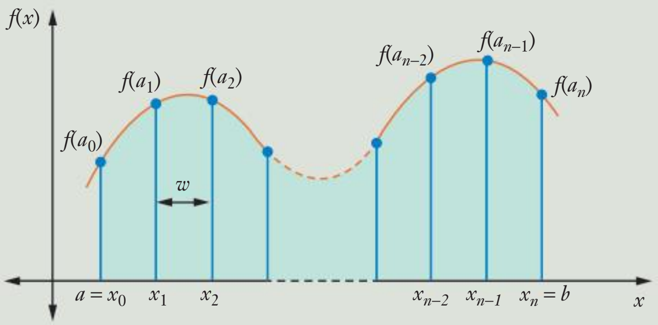
Evaluate. $= 1.994\,069\ldots$

Round and state the result. $\int_1^2 \ln|5x|\,dx \approx 1.994$

## Simpson's rule

More accurate results can be obtained by approximating the tops of the rectangles with parabolas instead of straight lines. Adding the areas of these strips gives **Simpson's rule**.

### Simpson's rule





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The interval from $a$ to $b$ is divided into $n$ strips of equal width $w$ by the points $a = x_0, x_1, x_2, \ldots, x_n = b$, where $n$ is an even number. The area is given by:

$$\int_a^b f(x)\,dx \approx \frac{w}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \ldots + 4f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)]$$

$$= \frac{w}{3}\left[f(x_0) + 4[f(x_1) + f(x_3) + \ldots] + 2[f(x_2) + f(x_4) + \ldots] + f(x_n)\right]$$

$$= \frac{w}{3}(X + 2E + 4O)$$

where $X = f(x_0) + f(x_n)$ is the sum of the end values

$E = f(x_2) + f(x_4) + \ldots + f(x_{n-2})$ is the sum of the even values (apart from the ends)

and $O = f(x_1) + f(x_3) + f(x_5) + \ldots + f(x_{n-1})$ is the sum of the odd values.

The second and third lines of the above formula are more useful in practical calculations.

## EXAMPLE 27

Use Simpson's rule to find an approximate value for $\int_1^5 \frac{8}{x}\,dx$ using strips of width 0.25.

### Solution

State the endpoints of the strips. Points = 1, 1.25, 1.5, …, 4.75, 5

Calculate $n$.

$$w = \frac{b-a}{n}$$

$$\frac{1}{4} = \frac{5-1}{n}$$

$$n = 16$$

Find the end values. $x_0 = 1$ and $x_{16} = 5$

Find a formula for the even values. $x_i = 1.5 + 0.5i$ for $i = 0$ to 6

Find a formula for the odd values. $x_i = 1.25 + 0.5i$ for $i = 0$ to 7

Write Simpson's rule.

$$\int_a^b f(x)\,dx = \frac{w}{3}(X + 2E + 4O)$$

Substitute and use summation notation.

$$\int_1^5 \frac{8}{x}\,dx$$

$$\approx \frac{0.25}{3}\left[f(1) + f(5) + 2\sum_{i=0}^{6} f(1.5 + 0.5i) + 4\sum_{0}^{7} f(1.25 + 0.5i)\right]$$

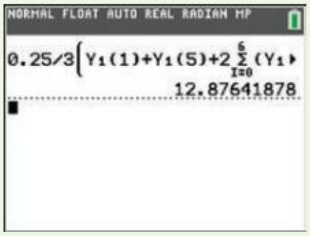
The approximation can now be completed using a graphics calculator as previously described. Manual calculation may also be used but may take a little longer.

TI-Nspire CX Chapter 7

### TI-84 Plus CE

Define the function by entering $Y_1 = 8 \div X$. Then enter the required approximation.

Press enter to display the result.

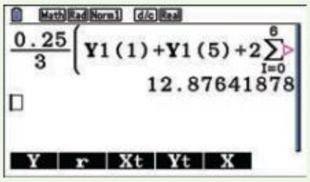



### Casio fx-CG20AU

Define the function by entering $Y1 = 8 \div x$.

Then enter the required approximation.

Press EXE to display the result.




**Both calculators**

Round and state the result. $\int_1^5 \frac{8}{x}\,dx \approx 12.876$

## INVESTIGATION

## AREA OF A CIRCLE

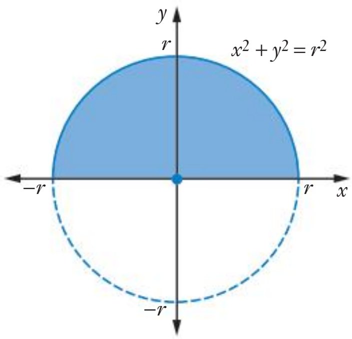
A circle of radius $r$ is drawn with its centre at the origin.

The equation of the circle is given by $x^2 + y^2 = r^2$.

You can then obtain $y = \pm\sqrt{r^2 - x^2}$.

If you consider only the half of the circle above the $x$-axis, you can write $y = \sqrt{r^2 - x^2}$.

So the area of the half-circle can then be written as $\int_{-r}^{r} \sqrt{r^2 - x^2}\,dx$.




1 Use Simpson's rule to find the area of a half-circle of radius 4, using 16 strips.

2 Compare your result from Simpson's rule with the values obtained by other methods.

3 Comment on the accuracy of Simpson's rule.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## Exercise 7.10 Simpson's rule

**1** Find the approximate area under the curve $y = \ln|5x|$ from $x = 2$ to $x = 6$ by calculating the areas of strips 0.5 units wide using the midpoint rule.

Example 25

**2** Use the midpoint rule to find an approximate value for $\int_0^1 x^3\,dx$ with $n = 10$.

**3** Use the midpoint rule to find an approximate area under the curve $y = \frac{1}{x}$ from $x = 1$ to $x = 2$ for $\int_1^2 \frac{1}{x}\,dx$ with 5 strips.

**4** Use the midpoint rule to find an approximate value for $\int_0^1 \sin\left(e^{\frac{x}{2}}\right)dx$ with $n = 4$.

**5** Use the trapezoidal rule to find the approximate area under $y = x^2$ from 0 to 1 with 10 strips.

Example 26

**6** Use the trapezoidal rule to find an approximation for the area under the curve:

**a** $y = x^3 + 8$ from $x = -2$ to $x = 0$ with 4 strips

**b** $y = \sqrt{x^2 + 1}$ from $x = 0$ to $x = 1$ with 5 strips.

**7** Use the trapezoidal rule to find an approximation for the area under the curve:

**a** $y = x^3$ from $x = 0$ to $x = 1$ with 10 strips, correct to 4 decimal places

**b** $y = \frac{1}{x}$ from $x = 1$ to $x = 2$ with 10 strips, correct to 4 decimal places

**c** $y = \frac{1}{\sqrt{1 + x^2}}$ from $x = 0$ to $x = 1$ with 4 strips, correct to 2 decimal places

**d** $y = \frac{1}{4 + x^2}$ from $x = 0$ to $x = 2$ with 10 strips, correct to 2 decimal places.

**8** Use the trapezoidal rule to find an approximation for the area under:

**a** $y = 3x^3 - 2x^2 + x + 3$ from $x = -0.5$ to $x = 1.5$ with 10 strips

**b** $f(x) = e^{x^3 - 2x^2}$ from $x = -2$ to $x = 2$ with 8 strips

**c** $f(x) = \ln|3x^2 + 5|$ from $x = 1$ to $x = 4$ with 12 strips.

**9** Use the trapezoidal rule to find an approximation for:

**a** $\int_2^5 (3x^2 + 5x + 3)\,dx$ with 10 strips

**b** $\int_{-1}^1 \frac{1}{x^2 - x - 5}\,dx$ with 8 strips

**c** $\int_{-3}^3 \sqrt{3x^2 + x^4}\,dx$ with 12 strips.

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Example 27

**10** Use Simpson's rule to find the approximate area under $y = x^2$ from $x = 0$ to $x = 1$ with $n = 4$.

**11** Use Simpson's rule to find an approximation for the area under:

**a** $y = e^{-x^2}$ from $x = 0$ to $x = 6$ with $n = 6$, correct to 4 decimal places

**b** $y = \frac{1}{x}$ from $x = 1$ to $x = 2$ with $n = 6$, correct to 6 decimal places.

**12** Use Simpson's rule to find an approximation for:

**a** $\int_1^e e^x \ln|x|\,dx$ with $n = 8$, correct to 6 decimal places

**b** $\int_1^3 \sqrt{x^2 + 2}dx$ with $n = 8$, correct to 4 decimal places

**c** $\int_{-1}^1 \sqrt{\cos(x)}dx$ with $n = 4$, to correct 6 decimal places.

**13** **a** Calculate $\int_1^6 (x^4 - 3x^2)dx$ using:

**i** the trapezoidal rule with $n = 10$

**ii** Simpson's rule with $n = 10$

**iii** algebraic methods.

**b** Use your answers to calculate the percentage error in the trapezoidal rule and Simpson's rule.

**14** Calculate $\int_0^{\frac{\pi}{2}} \sin(x)dx$ using:

**a** the trapezoidal rule with $n = 6$

**b** Simpson's rule with $n = 6$

**c** algebraic methods.

**15** Calculate $\int_{-1}^1 \frac{dx}{x + 2}$ using:

**a** the trapezoidal rule with $n = 10$

**b** Simpson's rule with $n = 10$.

**16** Calculate $\int_0^3 2^x dx$ using:

**a** the trapezoidal rule with $n = 12$

**b** Simpson's rule with $n = 12$.


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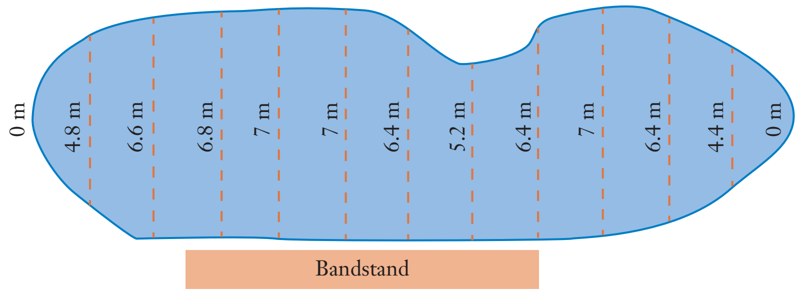
## Problem solving

**17** An area in front of a bandstand is to be covered with an all-weather surface for the audience. The area to be covered is shown below, divided into 2 m strips. Use the trapezoidal rule to find the approximate area, and hence the cost of the surface at $350/m$^2$.




**18** A river, 50 m wide, has the following depths (in metres) from bank to bank. Use Simpson's rule to calculate the cross-sectional area and hence find the volume of water that flows each hour when the rate of flow is a sluggish 3 m/s.

| $W$ | 0 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 | 50 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $D$ | 0 | 1.9 | 3.2 | 4.2 | 4.0 | 3.5 | 3.0 | 2.4 | 1.8 | 0.7 | 0 |

**19** The force needed to crumple the front end of a car at various compressions of the front is shown in the following table. The energy absorption of the crumpling can be calculated as the integral of the force with respect to the amount of compression. Calculate the impact energy absorbed when an identical car undergoes a compression of 20 cm during a collision, using Simpson's rule with 10 divisions. Note that if the distance is in metres and the force is in newtons (N), the energy will be in joules (J).

| **Distance (cm)** | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Force × 1000 (N)** | 50 | 90 | 110 | 120 | 150 | 180 | 200 | 250 | 300 | 400 | 500 | 600 |

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# 7.11 The exponential probability function

## The probability density function

The **probability density function** (PDF) for a continuous random variable is the function, $p(x)$ such that:

$$P(a \le x \le b) = \int_a^b p(x)\,dx$$

This means that $p(x)$ is a probability density function if the probability that the variable lies within an interval is given by the integral of the variable over the same interval.

For any probability density function, the area under the function is 1:

$$P(-\infty \le x \le \infty) = \int_{-\infty}^{\infty} p(x)\,dx = 1$$

You can choose any function that satisfies the conditions to be a probability density function. The choice is governed by how you want to use the distribution you obtain.

You can, for example, apply the particular density function $f(t) = \lambda e^{-\lambda t}$ for $t \ge 0$ using the exponential random variable with parameter $\lambda > 0$. The probability distribution using this function is called the **exponential distribution**. The exponential distribution is important as it can be used to model data and solve problems.

## The exponential distribution

The exponential distribution with parameter $\lambda$ has the probability density function

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & \text{for } t \ge 0 \\ 0 & \text{elsewhere} \end{cases}$$

where $P(a \le t \le b) = \int_a^b f(t)\,dx$

You can see that for the function $f(t) = \lambda e^{-\lambda t}$, $f(t) \ge 0$ for all values of $t$. To show that it is a probability density function, you need only show that the integral is equal to 1.

For the integral $\int_{-\infty}^{\infty} f(t)\,dt$, where $f(t) = \lambda e^{-\lambda t}$, for $t \ge 0$:

$$\int_{-\infty}^{\infty} f(t)\,dt = \int_{-\infty}^{0} f(t)\,dt + \int_{0}^{\infty} f(t)\,dt$$

$$= \int_{-\infty}^{0} 0\,dt + \int_{0}^{\infty} \lambda e^{-\lambda t}\,dt$$

$$= 0 + \left[-e^{-\lambda t}\right]_0^{\infty}$$

$$= 0 + [0 - (-1)]$$

$$= 1$$


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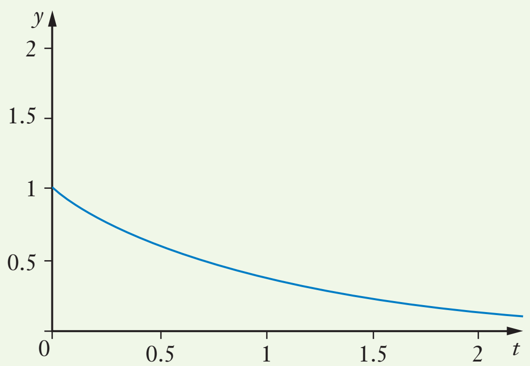
Thus the exponential distribution satisfies the conditions for a probability density function. The distribution is defined for all positive values of $t$. However, since it has a parameter $\lambda$, it is really a family of distributions, with each distribution determined by the value of $\lambda$.

## EXAMPLE 28

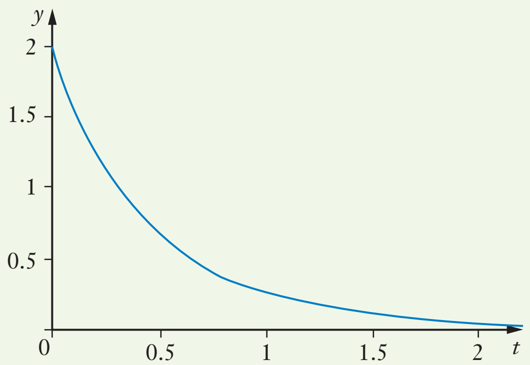
Draw graphs of the exponential distribution $f(t) = \lambda e^{-\lambda t}$ for $\lambda = 1$ and $\lambda = 2$, where $t \geq 0$ and comment on the shapes of the distributions.

### Solution

Draw the graph of $f(t) = \lambda e^{-\lambda t}$ for $\lambda = 1$.




Draw the graph of $f(t) = \lambda e^{-\lambda t}$ for $\lambda = 2$.




Compare the graphs.

For $\lambda = 2$, the distribution is twice as high at $t = 0$ as for $\lambda = 1$ and tapers to zero more quickly. This means that, for $\lambda = 2$, $t$ has a higher probability of being close to zero.

Use the comparison to make a more general statement.

The higher the value of $\lambda$, the more skewed to the left the graph of $y = \lambda e^{-\lambda t}$ is and the higher up the $y$-axis it begins. So for higher values of $\lambda$, there is a higher probability of a lower value of $t$.

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### The effect of $\lambda$ on the exponential probability density function

Use your graphics calculator to plot graphs of $f(x) = \lambda e^{-\lambda x}$ for different values of $\lambda$, where $\lambda > 0$.

Plot the graphs on the same set of axes and summarise your observations regarding how the value of $\lambda$ affects the shape of the probability density function, the area under the curve and hence the probability of $x$ for various ranges of values.

You can use the fact that $P(a \le t \le b) = \int_a^b f(t)\,dx$ with the exponential distribution.

## EXAMPLE 29

A random variable $X$ has an exponential distribution with $\lambda = 0.8$. Find $P(0 \le x \le 2)$.

### Solution

State the exponential distribution. $f(x) = \lambda e^{-\lambda x} = 0.8e^{-0.8x}$

Use the rule $P(a \le t \le b) = \int_a^b f(t)\,dx$ $\quad P(0 \le x \le 2) = \int_0^2 0.8e^{-0.8x}\,dx$

Evaluate the integral.

$$= \left[-\frac{0.8}{0.8}e^{-0.8x}\right]_0^2$$

$$= -e^{-1.6} + e^0$$

$$= 1 - e^{-1.6}$$

$$= 0.798\ 103\ldots$$

Write the answer and round off. $P(0 \le x \le 2) \approx 0.798$

The exponential distribution is particularly important in calculating probabilities of failures. For example, manufacturers are interested in knowing what percentage of items will fail to work after a given time. This can then be compared with the expected life of the item.

You have previously seen that the expected value of a continuous random variable $X$ with probability density function $f(x)$ is given by $E(X) = \int_{-\infty}^{\infty} xf(x)\,dx$. The **expected value** for the exponential distribution is closely related to the parameter $\lambda$.


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## Expected value for the exponential distribution

For an exponential distribution with parameter $\lambda$, $E(X) = \frac{1}{\lambda}$.

This result can be proved using integration by parts, with $f(x) = x$ and $g'(x) = \lambda e^{-\lambda x}$.

Find $f'(x)$ and $g(x)$. $f'(x) = 1$ and $g(x) = -e^{-\lambda x}$

Integrate by parts. $\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$

Use $E(X) = \int_{-\infty}^{\infty} xf(x)\,dx$.

$$E(X) = \int_{-\infty}^{\infty} x\lambda e^{-\lambda x}\,dx$$

$$= \left[-xe^{-\lambda x}\right]_0^{\infty} - \int_0^{\infty} 1 \times \left(-e^{-\lambda x}\right)dx$$

$$= [0-0] + \int_0^{\infty} \left(e^{-\lambda x}\right)dx$$

$$= \left[-\frac{1}{\lambda}e^{-\lambda x}\right]_0^{\infty}$$

$$= \left[0 - \left(-\frac{1}{\lambda}\right)\right]$$

$$= \frac{1}{\lambda}$$

**EXAMPLE 30**

The expected life (in hours) of an incandescent light bulb can be considered to be a random variable with an exponential distribution for which $\lambda = \frac{1}{900}$.

**a** For how many hours would you expect an incandescent bulb to work?

**b** Find the probability that a particular bulb lasts at least 1000 h.

**c** If compact fluorescent bulbs last about 8 times longer than incandescent bulbs, what is the probability that a fluorescent bulb will last at least 1200 hours?

**d** If LED bulbs last about 5 times longer than compact fluorescent bulbs, what is the probability that a LED bulb will last longer than 1200 hours?

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## Solution

**a** Find the expected value. $E(X) = \frac{1}{\lambda} = \frac{1}{\frac{1}{900}} = 900$

State the result. You can expect an incandescent bulb to work for 900 hours.

**b** Write the required probability. $P(x \geq 1000) = P(1000 \leq x < \infty)$

Write the rule for a probability density function $P(a \leq x \leq b) = \int_a^b f(x)\,dx$

Substitute for known values. $P(1000 \leq x < \infty) = \int_{1000}^{\infty} \frac{1}{900} e^{-\frac{x}{900}}\,dx$

Integrate. $= \left[-e^{-\frac{x}{900}}\right]_{1000}^{\infty}$

$= 0 - \left(-e^{-\frac{1000}{900}}\right)$

$= 0.329\ 192\ldots$

Round off and state the result. $P(x \geq 1000) \approx 0.329$

Alamy Stock Photo/Oleksiy Maksymenko Photography

**c** Write $E(X)$ for fluorescent bulbs. $E(X) = 900 \times 8 = 7200$ hours

Find $\lambda$ for fluorescent bulbs. $7200 = \frac{1}{\lambda} \Rightarrow \lambda = \frac{1}{7200}$


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| | |
|---|---|
| Write the required probability. | $P(x \geq 1200) = \int_{1200}^{\infty} \frac{1}{7200} e^{-\frac{x}{7200}} dx$ |
| | $= \left[ -e^{-\frac{x}{7200}} \right]_{1200}^{\infty}$ |
| | $= 0 - \left( -e^{-\frac{1200}{7200}} \right)$ |
| | $= 0.846\,481\ldots$ |
| | $P(x \geq 1200) \approx 0.846$ |
| **d** Write $E(X)$ for LED bulbs. | $E(X) = 7200 \times 5 = 36\,000$ hours |
| Find $\lambda$ for fluorescent bulbs. | $36\,000 = \frac{1}{\lambda} \Rightarrow \lambda = \frac{1}{36\,000}$ |
| Write the required probability. | $P(x \geq 1200) = \int_{1200}^{\infty} \frac{1}{36\,000} e^{-\frac{x}{36\,000}} dx$ |
| | $= \left[ -e^{-\frac{x}{36\,000}} \right]_{1200}^{\infty}$ |
| | $= 0 - \left( -e^{-\frac{1200}{36\,000}} \right)$ |
| | $= 0.967\,216\ldots$ |
| | $P(x \geq 1200) \approx 0.967$ |

## Exercise 7.11 The exponential probability function

**1** Draw a graph of the exponential distribution $f(t) = \lambda e^{-\lambda t}$ (where $t \geq 0$) for each pair of values, and compare the shapes of the distributions. Example 28

**a** $\lambda = 0.25$ and $\lambda = 2$  **b** $\lambda = 0.1$ and $\lambda = 0.2$  **c** $\lambda = 0.7$ and $\lambda = 0.9$

**2** A random variable $X$ has an exponential distribution with $\lambda = 0.6$. Find: Example 29

**a** $P(0 \leq x \leq 3)$  **b** $P(3 \leq x \leq 10)$

**c** $P(9.5 \leq x \leq 10.5)$  **d** $P(x \geq 10)$

**3** A random variable $T$ has an exponential distribution with $\lambda = 3$. Find:

**a** $P(0 \le t \le 0.3)$ **b** $P(0.4 \le t \le 0.6)$

**c** $P(t \ge 0.3)$ **d** $P(t \ge 0.5)$

**4** A random variable $T$ has an exponential distribution with $\lambda = 0.2$. Find:

**a** $P(0 \le t \le 3)$ **b** $P(4 \le t \le 6)$ **c** $P(t \ge 3)$ **d** $P(t \ge 5)$

**5** Find the mean of the exponential distribution with the probability density function

$$f(t) = \begin{cases} me^{-mt} & \text{for } t \ge 0 \\ 0 & \text{elsewhere} \end{cases}$$

**6** An exponential distribution has parameter $\lambda = 1.5$. What is the expected value of the distribution?

**7** What is the parameter $\lambda$ for an exponential distribution with an average value of 16?

Example 30

**8** The length of time (in minutes) an Australia Post worker spends with a customer is known to have an exponential distribution with $\lambda = 0.25$.

**a** What is the average amount of time spent with a customer?

**b** Find the probability that an Australia Post worker spends more than 4 minutes with a customer.

**c** Find the probability that an Australia Post worker spends more than 2 minutes with a customer.

**9** It is known that the average length of time that shoppers spend selecting a Valentine's Day card is 8 minutes. The time spent selecting a Valentine's card can be modelled by an exponential distribution.

**a** State the probability density function for this distribution.

**b** What is the probability that a shopper spends more than 10 minutes looking for a Valentine's Day card?

**c** What is the probability that a shopper spends fewer than 5 minutes looking for a Valentine's Day card?

**d** The day before Valentine's Day, the average time spent by shoppers looking for a card is only 5 minutes. What is the probability that such a shopper spends more than 7 minutes looking for a card?

**10** The manager of a fast-food restaurant knows that the average time customers wait for service is 2.5 minutes. If the time a customer waits to be served is modelled by an exponential distribution, calculate the probability that a customer:

**a** waits more than 4 minutes to be served

**b** waits less than 5 minutes to be served

**c** is served in the first minute

**d** is served within the first 2 minutes


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**11** The number of days ahead of departure that domestic air travellers buy their airline tickets can be modelled by an exponential distribution. Industry research shows that, on average, tickets are purchased 18 days ahead. Find the probability that a traveller will purchase a ticket fewer than 10 days ahead.

**12** On average, a certain machine part lasts 12 years. The length of time that the machine part lasts is exponentially distributed. What is the probability that a machine part lasts:

**a** more than 7 years?

**b** between 9 and 11 years?

## Problem solving

**13** **a** The average response time, in seconds, of students to teacher questions is an exponential distribution with $\lambda \approx 0.07$. Find the probability that the next question will be answered in:

**i** less than 8 s

**ii** 8 s

**iii** more than 8 s.

**b** Comment on the total of these probabilities.

**14** For an exponential distribution with parameter $\lambda$, find the value of $a$ such that $P(0 \leq t \leq a) = 0.8$.

**15** On average, a customer complaint at a particular company takes 10 days before it is resolved. The length of time taken to resolve the complaint is exponentially distributed. After how many days would the complaints supervisor expect 80% of complaints to be resolved?

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WS
Exponential distributions

# 7.12 Applications of the exponential probability function

The exponential probability distribution function is used widely to determine the useful life, reliability and probability of failure of a particular item. While you have already seen that the exponential probability distribution function is defined as $f(t) = \lambda e^{-\lambda t}$ for some parameter $\lambda$ where $\lambda > 0$, the **cumulative distribution function** for an exponential probability density function is also worth knowing.

> The cumulative distribution function for an exponential probability density function with the parameter $\lambda$ is given by $\text{cdf}(t) = 1 - e^{-\lambda t}$.

One of the important properties of the exponential probability distribution is its memoryless property. This refers to the fact that the likelihood of something happening in the future has no relation to whether or not it has happened in the past. In other words, the history (or memory) of the function is irrelevant to the future.

Tossing a fair coin is an example of a probability distribution that is memoryless. The probability of tossing a head is $\frac{1}{2}$ every time you toss the coin. This is true even if, say, the last 10 times you tossed the coin it came tails. The probability of heads in the next throw of a fair coin is always going to be $\frac{1}{2}$.

> For an exponential probability density function with the parameter $\lambda$, the probability of an event occurring in the interval $[p, p + s]$ is the same as the probability that it occurs in the interval $[0, s]$ for any values of $p$ and $s$.

The previous result can be proven using the cumulative density function.

The probability of an event occurring in the interval $[p, p + s]$ is conditional on the fact that the event has not occurred in the interval $[0, p]$.

$$
\begin{aligned}
P[p \le x \le (p+s) \text{ given } x > p] &= \frac{\text{cdf}(p+s) - \text{cdf}(p)}{1 - \text{cdf}(p)} \\
&= \frac{(1 - e^{-\lambda(p+s)}) - (1 - e^{-\lambda p})}{1 - (1 - e^{-\lambda p})} \\
&= \frac{e^{-\lambda p} - e^{-\lambda(p+s)}}{e^{-\lambda p}} \\
&= \frac{e^{-\lambda p}(1 - e^{-\gamma s})}{e^{-\lambda p}} \\
&= 1 - e^{-\lambda s} \\
&= \text{cdf}(s)
\end{aligned}
$$

So $\quad P[p \le x \le (p+s)] = P(0 \le x \le s)$.


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## EXAMPLE 31

At a car park, the exit of cars through a boom gate can be modelled by an exponential distribution with a parameter $\lambda = 5$ and time ($t$) is measured in minutes.

**a** Write the pdf for this distribution.

**b** What is the expected waiting time for a car to exit the car park?

**c** Write the cdf for this distribution.

**d** What is the probability that a car will exit in the next minute?

**e** What is the average rate of cars exiting the car park per minute?

### Solution

**a** State the formula. $f(t) = \lambda e^{-\lambda t}$

Substitute for $\lambda$. $\text{pdf}(t) = 5e^{-5t}$

**b** State the rule for $E(X)$. $E(X) = \frac{1}{\lambda}$

Substitute for $\lambda$. $= \frac{1}{5}$

State the result. The expected waiting time is $\frac{1}{5}$ minutes, or 12 seconds.

**c** State the formula. $\text{cdf}(t) = 1 - e^{-\lambda t}$

Substitute for $\lambda$. $\text{cdf}(t) = 1 - e^{-5t}$

**d** Substitute for $t = 1$. $\text{cdf}(1) = 1 - e^{-5 \times 1}$

Evaluate. $= 0.993\ 262\ldots$

State the result. The probability of a car exiting the car park in the next minute is about 99.33%.

**e** State the rate using the average waiting time. Rate of cars exiting $= \frac{\text{Time period}}{\text{Waiting time}}$

Time period = 1 minute and waiting time $= \frac{1}{5}$ minutes. $\text{Rate/min} = \frac{1\text{ min}}{\frac{1}{5}\text{ min}}$

$= 5$

State the result. The rate of cars exiting the car park is 5 cars per minute.

The previous example shows that for an exponential distribution with $\lambda = 5$, the expected waiting time is also 5. This shows a general result for all exponential distributions involving occurrences over time.

## Properties of the exponential distribution

An exponential distribution for an event that occurs over time ($t$) with parameter $\lambda$ has the following properties.

- The expected **waiting time** $= \frac{1}{\lambda}$
- Average **rate of occurrence** $= \lambda$
- The converse of this result is important and states: If the rate of occurrence of an event is $\lambda$, the occurrence of the event is modelled by an exponential distribution with parameter $\lambda$.

You may have heard the term 'throw-away society'. This refers to the fact that many products are designed to work for long periods, but eventually fail to work due to a number of different reasons. Manufacturers usually provide a limited guarantee. For example, it is common for small electrical appliances to have a 12-month guarantee or warranty period.

An important application of exponential distributions relates to the **reliability** of an item. Reliability is the probability that a product or item will operate properly for a specified period of time without failure. The opposite of reliability is failure and so reliability and failure are complementary events. Reliability can be modelled using an exponential distribution.

## Reliability function

- The **reliability function** for an item is $R(t) = 1 - P(\text{failure})$. For the reliability function, the variable $t$ is the time or the number of uses (cycles) for the item.
- It can be shown that if the average rate of failure is $\lambda$, $R(t) = e^{-\lambda t}$.


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Using this definition, the reliability function $R(t)$ gives the probability that an item will keep working for time $t$ or number of times used, $t$.

## EXAMPLE 32

A computer motherboard component has a constant failure rate of 0.0004 per hour.

**a** What is the probability that the component will work for 3000 hours before it fails?

**b** If 1000 components are tested, how many would be expected to fail within 200 hours?

### Solution

| | |
|---|---|
| **a** State the reliability function. | $R(t) = e^{-\lambda t}$ |
| Substitute for $\lambda$ and $t$. | $R(3000) = e^{-0.0004 \times 3000}$ |
| | $= 0.301\ 194\ldots$ |
| Round off and state the result. | The probability that the component will work for 3000 hours is about 0.301. |
| **b** State the reliability function. | $R(t) = e^{-0.0004t}$ |
| Substitute for $t$. | $R(200) = e^{-0.0004 \times 200}$ |
| | $= 0.923\ 116\ldots$ |
| Calculate the probability of failure in 200 hours. | $P(\text{failure}) = 1 - R(200)$ |
| | $= 1 - 0.923\ 116\ldots$ |
| | $= 0.076\ 883\ldots$ |
| Calculate the expected number of failures. | Expected number $= 1000 \times 0.076\ 883\ldots$ |
| | $= 76.883\ldots$ |
| Round off and state the result. | About 77 of the 1000 components can be expected to fail within 200 hours. |

You can calculate the variance and standard deviation of the exponential distribution using your knowledge of other continuous probability density functions.

You will recall that for a continuous probability density function defined on the interval $[a, b]$,

$$Var(X) = \int_a^b p(x)(x-\mu)^2\,dx, \text{ where } \mu = E(X).$$

For an exponential probability density function with parameter $\lambda$, $E(X) = \frac{1}{\lambda}$, so

$$Var(X) = \int_0^\infty \lambda e^{-\lambda x}\left(x - \frac{1}{\lambda}\right)^2 dx$$

Using integration by parts twice gives $Var(X) = \frac{1}{\lambda^2}$, so the standard deviation, $\sigma = \frac{1}{\lambda}$.

The median is the value below which 50% of values lie, so for the median:

$$\text{cdf}(x) = 0.5 = 1 - e^{-\lambda}$$

So $e^{-\lambda} = 0.5$ and so the median is $\frac{-\ln(0.5)}{\lambda} = \frac{\ln(2)}{\lambda}$.

For an exponential probability density function with parameter $\lambda$:

- **Expected value**, $E(X) = \mu = \frac{1}{\lambda}$,
- **Variance**, $Var(X) = \sigma^2 = \frac{1}{\lambda^2}$
- **Standard deviation**, $\sigma = \frac{1}{\lambda}$.
- **Median** $= \frac{\ln(2)}{\lambda}$.

You may also be required to calculate a quantile for an exponential distribution. You will recall that a quantile $t_\alpha$ is the value of $t$ such that the area below the curve under $t$ is equal to $\alpha$, so that $P(-\infty \le t \le t_\alpha) = \alpha$. You know that for a exponential probability density function, $t \ge 0$, so the expression for $\alpha$ can be written as $P(0 \le t \le t_\alpha) = \alpha$.


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## EXAMPLE 33

The reliability of a particular brand of capacitor is 0.987 for 100 hours of operation. The working life of the capacitor is modelled by an exponential distribution.

**a** What is the median? Explain the result.

**b** For what period is the probability of failure less than 10%?

### Solution

**a** Write the reliability for 100 hours. $R(100) = 0.987$

Write an expression for reliability. $R(t) = e^{-\lambda t}$

Substitute for known values. $0.987 = e^{-100\lambda}$

Solve for $\lambda$. $\ln(0.987) = -100\lambda$

$\lambda = 0.000\ 130\ldots$

State the rule for the median. $\text{Median} = \dfrac{\ln(2)}{\lambda}$

Substitute for $\lambda$. $= \dfrac{\ln(2)}{0.000130\ldots}$

Evaluate. $= 5297.168\ldots$

Round off and state the result. The median is about 5297 hours and so at least 50% of the capacitors will work for at least 5297 hours.

**b** State the condition. $P(\text{failure}) < 0.1$

Reliability and failure are complementary. $R(t) \geq 0.9$

Use the rule for reliability. $e^{-\lambda t} \geq 0.9$

Substitute for $\lambda$. $e^{-0.000\ 130\ldots t} \geq 0.9$

Solve for $t$. $-0.000\ 130\ldots t \geq \ln(0.9)$

$t \geq 805.185\ldots$

Round off and state the result. The probability of failure is less than 10% for a period of less than 805 hours.

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## Exercise 7.12 Applications of the exponential probability function

Example 31

**1** The occurrence of telephone calls into a customer service call centre over time is modelled by an exponential distribution with a parameter of $\lambda = 4$, where time is in minutes.

- **a** Write the pdf for this distribution.
- **b** What is the expected waiting time for a call to come in to the call centre?
- **c** Write the cdf for this distribution.
- **d** What is the probability that a call will come into the centre in the next minute?
- **e** What is the average rate of calls coming into the call centre per minute?

**2** The arrival of cars at a petrol station over time is modelled by an exponential distribution with a parameter of $\lambda = 40$, where time is in hours.

- **a** Write the pdf for this distribution.
- **b** What is the expected waiting time for a car to arrive at the petrol station?
- **c** Write the cdf for this distribution.
- **d** What is the probability that a car will arrive at the petrol station in the next 10 minutes?
- **e** What is the average rate of calls coming into the call centre per hour?

**3** In traffic flow, the term 'time headway' refers to the length of time that elapses between one car passing a given point and another car beginning to pass the same point. For a particular road during heavy traffic, the time headway is modelled by the probability density function $f(t) = 0.5e^{-0.5t}$, where time is in seconds.

- **a** What is the expected time between cars passing a particular point?
- **b** Write the cdf for this distribution.
- **c** What is the probability that a car will pass a given point in the next second?
- **d** What is the average rate of cars passing a given point per minute?

**4** Suppose you usually get 4 phone calls per hour. Calculate the probability that a phone call will arrive within the next hour.

Example 32

**5** A certain brand of resistor has a constant failure rate of 0.002 per hour.

- **a** What is the probability that the resistor will work for 600 hours before it fails?
- **b** If 1000 resistors are tested, how many would be expected to fail within 100 hours?

**6** The number of times that a particular item can be used before it fails is exponentially distributed with an average of 10 000 uses. A particular process requires that the item be used 5000 times. What is the probability that the process will be able to be completed without having to replace the item?

**7** A make and model of ceramic capacitor is known to have constant failure rate of $3 \times 10^{-4}$ failures per hour.

- **a** What is the probability that the capacitor will work for 150 hours before it fails?
- **b** If 500 capacitors are tested, how many would be expected to fail within 1000 hours?

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**8** The reliability of a manufactured item is 0.965 for 200 hours of operation. The working life of the item is modelled by an exponential distribution. Example 33

**a** What is the median? Explain the result.

**b** For what period is the probability of failure less than 5%?

**9** A particular type of light bulb is known to have a reliability of 0.948 for 1000 hours of operation. The life of this type of light bulb can be modelled using an exponential distribution.

**a** What is the median lifetime of this type of light bulb?

**b** For what period is the probability of failure less than 5%?

**10** The reliability of an engine part is 0.998 for 1000 hours of operation. If the life of the part is exponential, for what period is the probability of failure less than 5%?

## Problem solving

**11** The time taken by a reference librarian to answer a query has an average of about 4 minutes. Model the time taken as an exponential distribution, and hence find:

**a** the percentage of queries that take less than 1 minute

**b** the median time

**c** the probability that the next query will take longer than 10 minutes.

**12** **a** 10% of Year 12 students fail to complete a task in 15 minutes. What percentage will fail to complete the task in

**i** 20 min? **ii** 10 min? **iii** 25 min?

**b** How long should be allowed to ensure that 98% of Year 12s complete the task?

**13** The time between successive phone calls at a call centre is known to follow an exponential distribution. If 90% of the intervals between calls are less than 2 minutes, what percentage of the time intervals will be less than 5 minutes?

**14** A computer-controlled machine lathe takes 4 minutes to complete a particular job. It completes this job with a reliability of 0.99. What is its reliability if the time allowed is:

**a** 3 min? **b** 5 min? **c** 10 min?

**15** It is known that 20% of a particular type of battery last longer than 4 years. Use an exponential model to find the guarantee period that a manufacturer should set to ensure that no more than 20% of batteries fail within the guarantee period.

**16** Comment on the value of the median of an exponential distribution with parameter $\lambda$ compared with the mean of the same distribution.

**17** The average time taken to find a spare part for a car at a spare parts distributor is about 6 minutes (if it is in stock). Use the exponential distribution to find:

**a** the proportion of spare parts that take less than 2 minutes to find

**b** the median time

**c** the probability that the next spare part will take longer than 10 minutes to find.

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**18** A study was conducted on 1700 items that have operated for an average of 400 hours. It was found that there had been 150 failures. Assuming an exponential distribution, what percentage of items will be working after 850 hours?

**19** An electronic circuit board consists of:

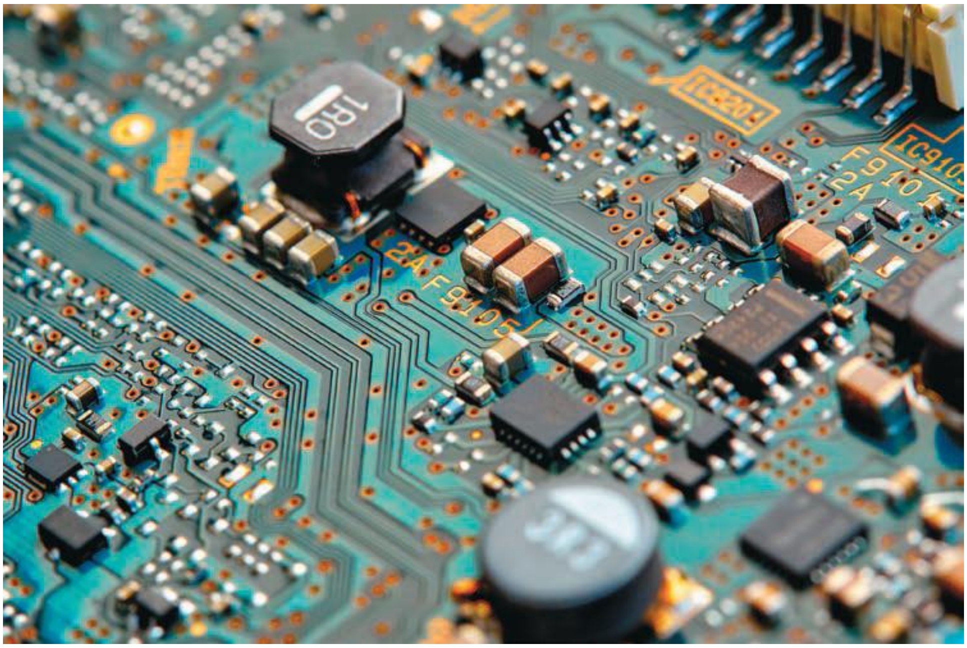
- 6 resistors each having a failure rate of $10^{-6}$/hour
- 4 switches each having a failure rate of $0.5 \times 10^{-6}$/hour
- 3 capacitors each having a failure rate of $0.2 \times 10^{-6}$/hour
- 10 transistors each having a failure rate of $5 \times 10^{-6}$/hour
- 2 diodes each having a failure rate of $2 \times 10^{-6}$/hour

It can be assumed that all connectors and wiring are 100% reliable and all components must operate for the circuit board to work successfully.

**a** Calculate the equivalent failure rate of the circuit board.

**b** What is the probability that the circuit board will still be working after 1000 hours?

**c** What is the probability that the circuit board will still be working after 10 000 hours?

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# 7. CHAPTER SUMMARY

## Further applications of integration

- Pythagorean identities

  $\sin^2(A) + \cos^2(A) = 1$

  $\tan^2(A) + 1 = \sec^2(A)$

  $\cot^2(A) + 1 = \text{cosec}^2(A)$

- Angle sum and difference identities

  $\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)$

  $\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)$

  $\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

  $\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$

- Double-angle identities

  $\sin(2A) = 2\sin(A)\cos(A)$

  $$\begin{aligned}\cos(2A) &= \cos^2(A) - \sin^2(A)\\ &= 1 - 2\sin^2(A)\\ &= 2\cos^2(A) - 1\end{aligned}$$

- Product identities

  $\sin(A)\sin(B) = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$

  $\cos(A)\cos(B) = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$

  $\sin(A)\cos(B) = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$

  $\cos(A)\sin(B) = \frac{1}{2}[\sin(A + B) - \sin(A - B)]$

- The integrals of $\sec^2(x)$ and $\text{cosec}^2(x)$.

  $\int \sec^2(x)\,dx = \tan(x) + c$ and so $\int \sec^2(kx)\,dx = \frac{1}{k}\tan(kx) + c$

  $\int \text{cosec}^2(x)\,dx = -\cot(x) + c$ and so $\int \text{cosec}^2(kx)\,dx = -\frac{1}{k}\cot(kx) + c$

- Integration by substitution, works for integrals in the form $\int_a^b f[g(x)]g'(x)\,dx$.

  If $u = g(x)$, $\int_a^b f[g(x)]g'(x)\,dx = \int f(u)\,du$

  If $u = g(x)$, $g'(x)$ is continuous on the interval $[a, b]$, $\int_a^b f[g(x)]g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$
- $\int \frac{1}{x}\,dx = \ln|x| + c$, provided $x \neq 0$
- $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$
- $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$
- Inverse trigonometric functions

  $\sin^{-1}(x) = y$ if and only if $\sin(y) = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

  $\cos^{-1}(x) = y$ if and only if $\cos(y) = x$ and $0 \leq y \leq \pi$

  $\tan^{-1}(x) = y$ if and only if $\tan(y) = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
- $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}\cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$
- $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + c$
- $\int \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1}(x) + c$
- $\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + c$
- $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$
- $\int \frac{-1}{\sqrt{a^2-x^2}}\,dx = \cos^{-1}\left(\frac{x}{a}\right) + c$
- $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$


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- Functions of the form $\frac{p}{(x-a)(x-b)}$ or $\frac{px+q}{(x-a)(x-b)}$ can be expressed as partial fractions in the form $\frac{A}{x-a}+\frac{B}{x-b}$
- Functions of the form $\frac{p}{(x-a)^2(x-b)}$ or $\frac{px+q}{(x-a)^2(x-b)}$ can be expressed as partial fractions in the form $\frac{A}{(x-a)^2}+\frac{B}{x-a}+\frac{C}{x-b}$
- Integration by parts

  For the functions $f(x)$ and $g(x)$: $\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$

  If $u = f(x)$ and $v = g(x)$: $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$
- For the **functions** $f(x)$ and $g(x)$, the **difference function** is $f(x) - g(x)$
- The area $A$ of the region bounded by the curves $y = f(x)$ and $y = g(x)$, and the lines $x = a$ and $x = b$, is $A = \int_a^b [f(x) - g(x)]dx$
- The volume of a solid, $V$ with cross-sectional area $A$ and height $x$ is $V = \int A(x)\,dx$
- The **volume of a solid of revolution** generated by a curve in the $xy$ plane is given by:

  $$V = \pi\int_a^b y^2\,dx \text{ or } V = \pi\int_a^b [f(x)]^2\,dx$$

  when $y = f(x)$ between $x = a$ and $x = b$ is rotated about the $x$-axis

  or $$V = \pi\int_a^b x^2\,dy \text{ or } V = \pi\int_a^b [f(y)]^2\,dy$$

  when $x = f(y)$ between $y = a$ and $y = b$ is rotated about the $y$-axis
- The **midpoint rule** states that $\int_a^b f(x)\,dx \approx w\sum_{i=1}^{n} f(\bar{x}_i)$, where the interval $[a, b]$ is divided into $n$ strips of equal width, $w = \frac{b-a}{n}$, $\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots, \bar{x}_n$ are the midpoints of the strips, $x_0 = a$ and $x_n = b$ and $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$
- The **trapezoidal rule** states that:

  $$\int_a^b f(x)\,dx \approx \frac{w}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)]$$

  where the interval $[a, b]$ is divided into $n$ trapezia of equal width, $w = \frac{b-a}{n}$, $x_0 = a$ and $x_n = b$ and $x_i = a + iw$

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- The trapezoidal rule may also be written as:

$$\int_a^b f(x)\,dx \approx \frac{w}{2}\left[f(x_0)+2\sum_{i=1}^{n} f(x_i)+f(x_n)\right] \quad \text{or} \quad \int_a^b f(x)\,dx \approx \frac{w}{2}(E+2M)$$

where $E$ is the sum of the values of $f(x)$ at the ends of the interval and $M$ is the sum of all the values of $f(x)$ at the intervening points $x_1, x_2, x_3, \ldots, x_{n-1}$

- **Simpson's rule** states that:

$$\int_a^b f(x)\,dx \approx \frac{w}{3}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\ldots+4f(x_{n-2})+2f(x_{n-1})+f(x_n)]$$

$$=\frac{w}{3}\big[f(x_0)+4[f(x_1)+f(x_3)+\ldots]+2[f(x_2)+f(x_4)+\ldots]+f(x_n)\big]$$

where the interval from $a$ to $b$ is divided into $n$ strips of equal width $w$ by the points

$a=x_0, x_1, x_2, \ldots, x_n=b$, where $n$ is an even number.

- Simpson's rule may also be written as $\int_a^b f(x)\,dx \approx \frac{w}{3}(X+2E+4O)$

where $X=f(a_0)+f(a_n)$ is the sum of the end values

$E=f(a_2)+f(a_4)+\ldots+f(a_{n-2})$ is the sum of the even values (apart from the ends) and

$O=f(a_1)+f(a_3)+f(a_5)+\ldots+f(a_{n-1})$ is the sum of the odd values

- The **exponential distribution** with parameter $\lambda$ has the probability density function

$$f(t)=\begin{cases}\lambda e^{-\lambda t} & \text{for } t\ge 0\\ 0 & \text{elsewhere}\end{cases}$$

where $P(a\le t\le b)=\int_a^b f(t)\,dx$

- The **expected value** for an exponential distribution with parameter $\lambda$, $E(X)=\frac{1}{\lambda}$
- The cumulative distribution function for an exponential probability density function with the parameter $\lambda$ is given by $\text{cdf}(t) = 1 - e^{-\lambda t}$
- For an exponential probability density function with the parameter $\lambda$, the probability of an event occurring in the interval $[p, p+s]$ is the same as the probability that it occurs in the interval $[0, s]$ for any values of $p$ and $s$
- For an exponential distribution for an event that occurs over time ($t$) with parameter $\lambda$, the expected waiting time $=\frac{1}{\lambda}$ and the average rate of occurrence $=\lambda$. Conversely, if the rate of occurrence of an event is $\lambda$, the occurrence of the event is modelled by an exponential distribution with parameter $\lambda$.


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- The **reliability function** for an item is $R(t) = 1 - P(\text{failure})$. It can be shown that if the average rate of failure is $\lambda$, $R(t) = e^{-\lambda t}$.
- For an exponential probability density function with parameter $\lambda$:

  Expected value, $E(X) = \mu = \frac{1}{\lambda}$

  Variance, $Var(X) = \sigma^2 = \frac{1}{\lambda^2}$

  Standard deviation, $\sigma = \frac{1}{\lambda}$

  Median $= \frac{\ln(2)}{\lambda}$

# 7. CHAPTER REVIEW

## Further applications of integration

Example 1

**1** **a** Calculate $\int_0^{\frac{\pi}{2}} \sin^2(x)\,dx$

**b** Find $\int 2\cos(5x)\cos(3x)\,dx$.

Example 2

**2** **a** Find $\int 3\sin^2(2x)\,dx$.

**b** Calculate $\int_0^{\frac{\pi}{2}} 4\sin^2(3x)\,dx$.

Example 3

**3** Find $\int \cot^2\left(\frac{x}{2}\right)dx$.

Examples 4–5

**4** Find:

**a** $\int x\left(3+x^2\right)^9\,dx$

**b** $\int_0^{\frac{\pi}{4}} 2\cos(x)\sin^3(x)\,dx$

Examples 6–7

**5** Find:

**a** $\int \frac{11}{5-7x}dx$

**b** $\int \frac{5x-1}{5x^2-2x-8}dx$

Example 8

**6** Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot(x)\,dx$.

Example 9

**7** Find each derivative.

**a** $\frac{d}{dx}\sin^{-1}(-5x)$

**b** $\frac{d}{dx}\tan^{-1}\left(\frac{x}{2}\right)$

Example 10

**8** Find:

**a** $\frac{d}{dx}\cos^{-1}(-3x^5)$

**b** $\frac{d}{dx}\arcsin\left(\sqrt{2x}\right)$

Example 11

**9** Find $\int \frac{1}{\sqrt{1-16x^2}}\,dx$.

Example 12

**10** Evaluate $\int_{-1}^{1} \frac{1}{\sqrt{9-x^2}}dx$, correct to 3 decimal places.

Example 13

**11** Express each rational function as partial fractions.

**a** $\frac{x-2}{(x+3)(2x-1)}$

**b** $\frac{11x+2}{2x^2-5x-3}$


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**12** Show that $\frac{1}{x^2+3x-4} = \frac{1}{5(x-1)} - \frac{1}{5(x+4)}$ and hence find $\int \frac{1}{x^2+3x-4}\,dx$. Example 13

**13** Find $\int \frac{2x^2-25x-33}{(x-5)(x+1)^2}\,dx$. Example 15

**14** Use integration by parts to find $\int x\sin(x)\,dx$. Example 16

**15** Find $\int x^2\sin(x)\,dx$. Example 17

**16** Find $\int x^4\ln|2x|\,dx$.

**17** Find the area of the region bounded by $y = x^2 + 2$ and $y = -x$ between $x = 2$ and $x = 3$. Example 18

**18** Find the area enclosed by $y = x^2$ and $x + y = 6$.

**19** Find the area enclosed by $y = x^3 - 9x$ and $y = 7x$. Example 19

**20** Use integration to find the volume of a pyramid of height 12 m with a rectangular base 18 m by 26 m. Example 20

**21** Use a suitable definite integral to find the volume, in cubic units, of an object that is formed by rotating the curve $y = \frac{1}{4}x^2$ about the $x$-axis from $x = 0$ to $x = 2$. Example 21

**22** Use a suitable definite integral to find the volume, in cubic units, of an object that is formed by rotating the curve $y = 4 - x^2$ about the $y$-axis from $x = 0$ to $x = 2$. Example 22

**23** Use the midpoint rule to find an approximate value for $\int_1^2 e^{\frac{1}{x}}\,dx$ with $n = 6$. Example 23

**24** **a** Calculate $\int_3^9 (3x^5 - 6x^2 + 8)\,dx$ using: Example 24

- **i** the trapezoidal rule with $n = 12$ Example 25
- **ii** Simpson's rule with $n = 12$
- **iii** algebraic methods. Examples 26–27

**b** Use your answers to calculate the percentage error in the trapezoidal rule and Simpson's rule.

**25** Draw the graph of the exponential distribution $f(t) = \lambda e^{-\lambda t}$ (where $t \geq 0$) for each pair of parameter values, and comment on the shape of the distribution. Example 28

**a** $\lambda = 0.5$ and $\lambda = 1$ **b** $\lambda = 0.3$ and $\lambda = 0.4$ **c** $\lambda = 0.5$ and $\lambda = 0.8$

**26** A random variable $t$ has an exponential distribution with $\lambda = 0.1$. Find: Example 29

**a** $P(0 \leq t \leq 5)$ **b** $P(5 \leq t \leq 15)$ **c** $P(9.5 \leq t \leq 10.5)$ **d** $P(t \geq 10)$

Example 30

**27** The waiting time for a customer's call to be answered by a customer service representative of a telecommunications company can be modelled by an exponential distribution. The average waiting time is known to be 5 minutes.

**a** State the probability density function for this distribution.

**b** Find the probability that a call is answered during the first minute.

**c** Find the probability that a customer waits more than 7 minutes to be answered.

**d** Find the probability that a customer waits less than 5 minutes to be answered.

Example 31

**28** The time taken for students to respond to a teacher's questions is modelled by an exponential distribution with a parameter of $\lambda = \frac{1}{12}$, where time is in seconds.

**a** Write the pdf for this distribution.

**b** What is the expected waiting time for a response to a question?

**c** Write the cdf for this distribution.

**d** What is the probability that a response will occur in the next 8 seconds?

**e** What is the average rate of responses to questions per minute?

Example 32

**29** A new product has a constant failure rate of 0.0025 per hour.

**a** What is the probability that the resistor will work for 200 hours before it fails?

**b** If 1000 resistors are tested, how many would be expected to fail within 50 hours?

Example 33

**30** The reliability of a manufactured item is 0.965 for 200 hours of operation. The working life of the item is modelled by an exponential distribution.

**a** What is the median? Explain the result.

**b** For what period is the probability of failure less than 5%?

## Problem solving

**31** The marginal cost of producing $n$ of pairs of shoes is given by

$$M(n) = \frac{n\sqrt{n^2 + 2500}}{1000}$$

where $n$ is the number of pairs produced, and $M$ is the cost of the last pair in dollars. The fixed costs are $200. Find the average cost of production of:

**a** 20 pairs **b** 50 pairs **c** 200 pairs.

**32** Calculate $\frac{d}{dx}\left[\frac{x}{\sqrt{1-x^2}} - \arccos(x)\right]$.

**33** Show that $\frac{d}{dx}\sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}}$, where $|x| > 1$.

**34** Find $\int \frac{\arcsin(x)}{\sqrt{1-x^2}}\,dx$.


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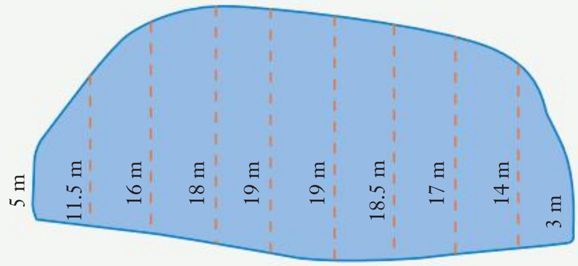
**35** Calculate the area of the region bounded by $x = -y^2 + 10$ and $x = (y - 2)^2$.

**36** A cylinder with radius $r$ and height $h$ is generated by revolving the line $y = r$ around the $x$-axis between $x = 0$ and $x = h$. Use a suitable integral to find a formula for the volume of the cylinder.

**37** A vase may be modelled as a solid of revolution, where the curve $y = \sqrt{x}$ between $x = 1$ and $x = 9$ is rotated about the $x$-axis to form the vase. If the measurements are in centimetres, what is the volume of the vase?

**38** A pointed hat is modelled by rotating the line $y = \sqrt{0.2x}$ from $x = 0$ to $x = 20$ about the $y$-axis. If the measurements are in centimetres, find the volume of the hat.

**39** A lawn is measured for re-turfing as shown, with strips 5 m wide. Use the trapezoidal rule to find the approximate area and hence the cost of turfing at \$120/m$^2$.




**40** The amount of clay wasted by amateur potters when making a particular pot in an adult education class can be considered to be an exponential function for which wasted clay in kilograms is exponentially distributed with $\lambda = 2.1$.

**a** What is the probability that a student will waste less than 1 kg?

**b** What percentage of students will waste more than 2 kg of clay?

**41** When 2 hours is allowed for the completion of a job, 90% of the items produced by a craftsman are of satisfactory quality. If an exponential distribution is assumed, what time would need to be allowed for 98% of the items to be of satisfactory quality?

Qz
Practice quiz

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STATISTICAL INFERENCE

# 8. SAMPLE MEANS

Most countries conduct a population census at regular intervals. There are some very small countries where *everyone* could be included. In other countries, like Australia, some people will be travelling or missed for other reasons. This means it is really a very large sample. You will never know the *exact* mean and standard deviation of a large population.

It is obvious that some samples are better than others. It is likely that larger samples will be better than small samples. How much better? How close will the measurements from a sample be to those of the population? How big a sample do you need? In this chapter, you will start to examine these questions.


8.01 Sample means
8.02 Properties of sample means
8.03 Mean and standard deviation of sample means
Chapter summary
Chapter review

## SYLLABUS SUBJECT MATTER

### Sample means

- examine the concept of the sample mean $\overline{X}$ as a random variable whose value varies between samples, where $X$ is a random variable with mean $\mu$ and the standard deviation $\sigma$
- simulate repeated random sampling from a variety of distributions and a range of sample sizes to illustrate properties of the distribution of $\overline{X}$ across samples of a fixed size $n$, including its mean $\mu$, its standard deviation $\frac{\sigma}{\sqrt{n}}$ (where $\mu$ and $\sigma$ are the mean and standard deviation of $X$) and its approximate normality if $n$ is large
- simulate repeated random sampling from a variety of distributions and a range of sample sizes to illustrate the approximate standard normality of $\frac{\overline{X} - \mu}{(s / \sqrt{n})}$ for large samples ($n \geq 30$), where $s$ is the sample standard deviation

Prior learning



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# TERMINOLOGY

central limit theorem
population
sample
triangular distribution
continuous random variable
random
sampling distribution
uniform probability
parameter
random variable
statistic
$z$-score

## 8.01 Sample means

In statistics, the words **population**, **sample** and **random** have special meanings.

### Statistical terms

- The **population** is the complete group from which you want information
- A **sample** is part of the population
- A **random sample** is a sample taken so that every member of the population has an equal chance of being chosen
- A **parameter** is a numerical characteristic of the population, such as a mean
- A **statistic** is an estimate of a parameter from the values (**observations**) of a sample
- The **mean** and **standard deviation** of a population are shown as $\mu$ and $\sigma$
- The **mean** and **standard deviation** of a sample are shown as $\bar{x}$ and $s$
- A **random variable** is a variable with a numerical value that depends on the outcome of a chance event. Random variables are normally shown by a capital letter

There are many methods of obtaining random samples. In this work, you will assume samples are random and fair. The mean is usually considered the most important parameter or statistic. Remember that the mean is given by $\bar{x} = \frac{\sum x}{n}$. You usually use $x$ for a score.

### EXAMPLE 1

A random sample of 8 Year 12 students have arms spans of 184, 171, 180, 168, 181, 199, 187 and 182 cm. What is the sample mean?

#### Solution

| | |
|---|---|
| Calculate the total. | Total = 1452 cm |
| Find the mean. | Mean $= \frac{1452}{8} = 181.5$ |
| Write the answer. | The sample mean is 181.5 cm. |


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INVESTIGATION

## SAMPLE ARM SPANS

- Put names in a hat or use other random means to get a sample of 8 people from your class
- Measure their arm spans
- Put the names back into the hat and repeat the process
- Keep doing this until you have at least 10 samples
- Find the mean arm span and standard deviation for the whole class
- Calculate the standard deviation of your 10 sample means and compare it to the standard deviation of people's heights in the class

What can you conclude, if anything, about the sample means?

How many different possible samples of 8 people can you take from your class?

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The total number of possible samples of any size from a group of $n$ is $2^n$.

How many possible samples of any size are there from your class?

If you take a random sample, you will have a random value. The values are continuous, so the sample mean is a **continuous random variable**, shown as $\overline{X}$.

Finding sample means is a very tedious process. You can use your graphics calculator to *simulate* the process. Many measurements of real values approximate a **normal distribution**. You study this in detail in Mathematical Methods, but you can use it without knowing the details.

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## EXAMPLE 2

Use your graphics calculator to find the means of 5 random samples of 20 values from a normal distribution with mean 74 and standard deviation 24.

### Solution

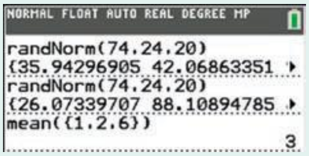
TI-Nspire CX Chapter 8

#### TI-84 Plus CE

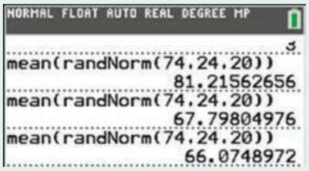
Press [math], choose PROB and 6: randNorm( to get a random sample.

Put in $\mu = 74$, $\sigma = 24$ and trials = 20, paste and press [enter]. You get a different sample list each time.

To calculate the mean of a list, press [2nd] [stat] (list), choose MATH and 3: mean(. Use [2nd] [(] for { and put in the list {1, 2, 6}. Close the brackets and press [enter].




Combine randNorm( and mean( to get the mean of a random sample. You can press [enter] to repeat the calculation.

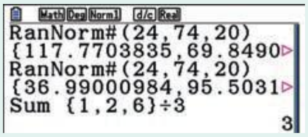



#### Casio fx-CG20AU

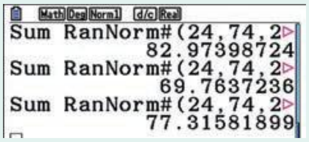
Use the RUN-MATRIX menu. Press [OPTN], choose PROB, RAND, Norm and press [EXE] to get a random sample.

Put in 24, 74, 20, close the bracket and press [EXE]. You get a different sample list each time.

To calculate the mean of a list, press [EXIT] until you get back to the OPTN menu, choose LIST and Sum. Use [SHIFT] × for { and put in the list {1, 2, 6}, divide by 3, and press [EXE].




Combine randNorm#( and the sum and divide by 20 [)] to get the mean of a random sample. You can press [EXE] to repeat the calculation.




You will notice that every time you find the mean of a sample, you get a different answer.

## Exercise 8.01 Sample means

Example 1

**1** Find the sample mean for each sample of sleep times of Year 12 students.

**a** 7 h, 8 h, 7 h, 10 h, 8 h, 8 h, 7 h, 8 h, 7 h, 9 h

**b** 7 h, 9 h, 7 h, 10 h, 8 h, 9 h, 7 h, 10 h, 6 h, 9 h

**c** 8 h, 11 h, 8 h, 10 h, 8 h, 8 h, 8 h, 9 h, 8 h, 9 h

**d** 7 h, 9 h, 8 h, 8 h, 8 h, 9 h, 7 h, 8 h, 8 h, 9 h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**2** Find the sample mean for each sample of masses (in kg) of Year 12 boys.

**a** 84, 65, 86, 84, 77, 74, 76, 61, 70, 89

**b** 58, 74, 97, 71, 81, 91, 81, 81, 85, 87

**c** 75, 52, 92, 71, 70, 82, 92, 112, 75, 90

**d** 93, 67, 70, 63, 70, 87, 73, 69, 85, 78

**3** Find the sample mean for each sample of the number of serves of fruit or vegetables eaten by Year 12 girls in a week.

**a** 16, 8, 23, 24, 33, 11, 27, 72, 30, 10, 23, 15

**b** 57, 26, 70, 0, 60, 9, 24, 26, 0, 25, 29, 8

**c** 66, 6, 25, 28, 29, 0, 26, 30, 11, 26, 18, 31

**d** 41, 5, 14, 38, 53, 44, 26, 53, 37, 58, 43, 39

**4** Find the sample mean for each sample of incomes that Year 12 students said they earned last week.

**a** $16, $120, $84, $131, $83, $124, $87, $126

**b** $96, $191, $83, $136, $96, $172, $115, $143

**c** $79, $142, $74, $130, $80, $162, $108, $149

**d** $97, $151, $78, $160, $103, $138, $92, $136

**5** Find the mean for each sample of reported reaction times of Year 12 students.

**a** 0.51 s, 0.67 s, 0.45 s, 0.74 s, 0.55 s, 0.65 s, 0.59 s, 0.68 s, 0.43 s, 0.66 s

**b** 0.37 s, 0.67 s, 0.54 s, 0.96 s, 0.5 s, 0.69 s, 0.54 s, 1.25 s, 0.59 s, 0.85 s

**c** 0.37 s, 0.67 s, 0.54 s, 0.94 s, 0.51 s, 0.86 s, 0.59 s, 0.65 s, 0.55 s, 0.61 s

**d** 0.46 s, 0.67 s, 0.52 s, 0.97 s, 0.56 s, 0.97 s, 0.46 s, 0.75 s, 0.46 s, 0.95 s

**6** **a** Use your graphics calculator to find the means of 8 random samples of 10 values from a normal distribution with mean 100 and standard deviation 15. Example 2

**b** Comment on your results.

**c** Find the mean of your sample means.

**7** **a** Use your graphics calculator to find the means of 8 random samples of 20 values from a normal distribution with mean 50 and standard deviation 10.

**b** Comment on your results.

**c** Find the mean of your sample means.

**8** **a** Use your graphics calculator to find the means of 8 random samples of 40 values from a normal distribution with mean 60 and standard deviation 15.

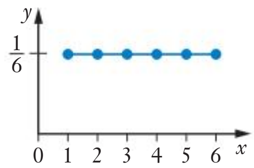
**b** Comment on your results.

**c** Find the mean of your sample means.

# 8.02 Properties of sample means

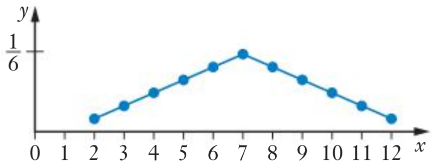
You already know that the probabilities of rolling a 1, 2, 3, 4, 5 or 6 on a normal die are all $\frac{1}{6}$ each.

These make a **uniform probability distribution** (or **rectangular distribution**).




However, you should also realise that the probabilities of rolling totals of 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12 on 2 dice are *not* the same.

They are $\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}$ and $\frac{1}{36}$ respectively.




These make a **triangular distribution**.

What about the distribution of sample means from rolling dice?

## INVESTIGATION

### DICE SAMPLE MEAN DISTRIBUTIONS

You should work in pairs for this investigation. Your teacher will give each pair 2 dice. Make a recording sheet like the one below (or your teacher might supply one).

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| Try | Roll number | | | | | | | | | | Means |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | $\bar{x}_1, \bar{x}_2$ |
| 1 | | | | | | | | | | | |
| 2 | | | | | | | | | | | |
| 3 | | | | | | | | | | | |
| 4 | | | | | | | | | | | |

You need to make about 20 rows.

- One person is to roll the dice
- Record the results of the first 10 rolls across the table in the row for Try 1


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- Write the result for *both* dice in each spot
- Swap rolling and recording after about 6 rows
- For the last column, work out of the mean of the 20 singles and the mean of the 10 pairs for each row
- Work out the means of your sample means
- Is the mean of the sample means for the single die rolls close to 3.5? How close?
- Is the mean of the sample means for the pairs close to 7? How close?

Put all the class results together to make a plot of the sample means.

What shapes are the distributions of the sample means?

From the investigation, it should be obvious that the shape of the distribution of sample means is not the same as the shape of the distribution from which the samples come.

### Sampling distribution

A **sampling distribution** is a distribution of sample means from samples of the same size from the same population.

You can use your graphics calculator to simulate the sampling distributions from different distributions. The total of rolling 2 dice is the sum of 2 results from rolling one die.

## EXAMPLE 3

**a** Use your graphics calculator to simulate the sample means of 50 samples of the totals of 40 rolls of 2 dice.

**b** Draw the sampling distribution.

**c** Comment on your results.

### Solution

**a** Write how to simulate 40 totals of 2 dice with numbers from 1 to 6.

Each total is the sum of 2 random integers from 1 to 6, so 40 totals will be the pairwise sums of 2 sets of random integers from 1 to 6.

The sample mean will be the mean of these 40 totals.

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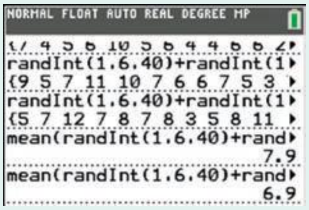
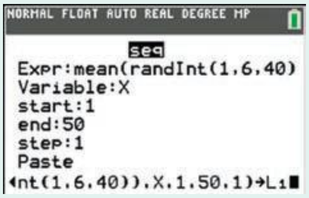
### TI-84 Plus CE

Press [math], choose PROB and 5: randInt( to get a random sample of integers. Put in 1, 6, 40 and paste to get 40 integers from 1 to 6. Then add another set of 40 random integers from 1 to 6 and press [enter]. You get a different sample list each time.

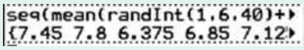
You can simulate the sample mean using 3: mean( from the LIST MATHS menu.




You can generate a sequence of 50 simulated sample means using 5: seq( from the LIST OPS menu. Complete the seq screen and paste, then store to L1, using [2nd] 1 for L1.




After you press [enter] be patient while your calculator processes the commands.

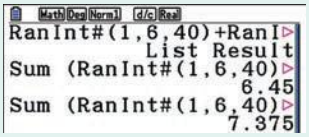



You can examine L1 using 1: Edit in the STAT menu.

### Casio fx-CG20AU

Use the RUN-MATRIX menu. Press [OPTN], choose PROB, RAND, Int to get a random sample of integers. Put in 1, 6, 40 to get 40 integers from 1 to 6 and close the brackets. Then add another set of 40 random integers from 1 to 6 and press [EXE]. Press [EXE] again to List Results. Press [EXIT] and ⓥ to return to the command line.

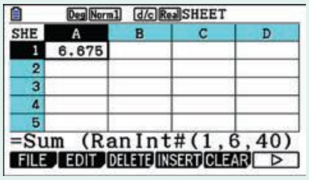
You can simulate the sample mean using Sum from the LIST menu and dividing by 40.




You will use the spreadsheet menu to do the 50 simulated sample means.

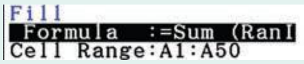
Press menu and choose 4 Spreadsheet.

Press [SHIFT] . (=), enter the formula into cell A1 and press [EXIT] to return to the spreadsheet menu.




Select the cells A1:A50 using [SHIFT] 8 (CLIP). The outline of the cell will change to a double outline. Use the cursor control to arrow down to A50.

Choose EDIT and FILL.




Press [EXE] and be patient while your calculator processes the commands.

You can examine List 1 to see the sample means.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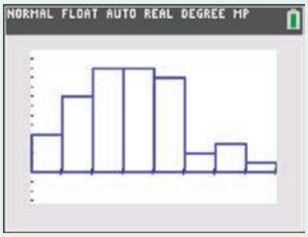
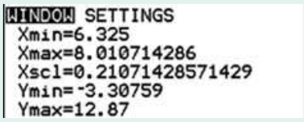
**b** To show a graph, you use statplot. Press 2nd y= and enter to go into Plot 1. Leave it on and set Type to the histogram icon, press zoom and choose 9: ZoomStat.

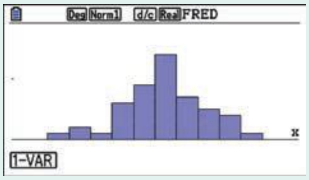
Your calculator will automatically set the scales to show the histogram.




Press window to check the scales.




To see the graph, select the cells A1:A50, and choose GRAPH. Use SET to change StatGraph1 to Hist. It should give XCellRange: A1:A50 and Frequency: 1. Press EXE and choose GRAPH1. Try Start: 6 and Width: 0.2 and press EXE. You can always do the graph again with different settings.




**c** Comment on the results.

Your results will probably be different, but the patterns will be similar.

It is not possible to say whether the result is a triangular distribution or not, but the values in the samples vary from 2 to 12, while those in the sampling distribution are only between 6 and 8.

The rand function on the TI-84 Plus CE and the ran# function on the Casio fx-CG20AU generate a random number from 0 to 1. You can get any random number from, say 5 to 12 using $5 + 7 \times$ rand or $5 + 7 \times$ ran#.

## EXAMPLE 4

**a** Use your graphics calculator to simulate a sample of 50 numbers from a uniform distribution on the interval [10, 16] and draw the graph.

**b** Now simulate 60 sample means of 50 items from a uniform distribution on the interval [10, 16] and draw the graph.

**c** Find the mean and standard deviation of the sampling distribution.

**d** Comment on your results.

### Solution

**a** Write the calculation for a random number $r$.

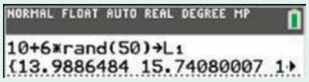
For $r$ between 0 and 1, $10 + 6r$ is from 10 to 16.

TI-Nspire CX
Chapter 8

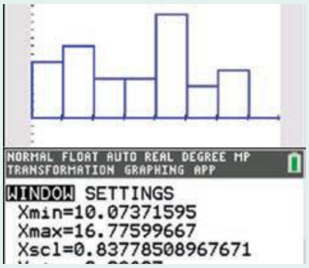
#### TI-84 Plus CE

Press [math] and choose PROB and 1: rand(50) for 50 random numbers.

Create the random numbers and store in L1.



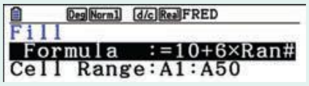

Use statplot as shown in Example **3** to draw a histogram and check the window settings.



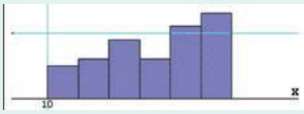

#### Casio fx-CG20AU

Use the Spreadsheet menu (4).

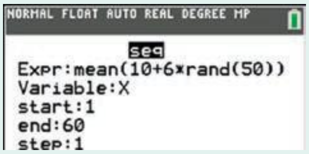
Enter the formula in cell A1 and fill it down to A50. Remember to press [OPTN], choose PROB, RAND and. Ran# for a random number and press [EXIT] to return to the Spreadsheet menu.




Select the cells A1:A50 again and choose GRAPH. Draw a histogram as shown in Example **3**. Use a start of 10 and width of 1.



**b** Use a sequence of 60 means of samples to create the sample means and store in L1.




Press [OPTN], choose PROB, RAND and List to get a list of random numbers.

RanList#(50) gives 50 random numbers. Use the sum and divide by 50 to get the sample mean. Enter the formula in cell A1 and press [EXIT] to return to the Spreadsheet menu.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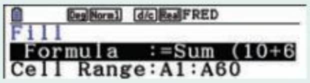
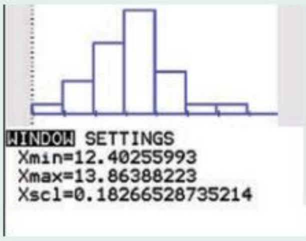
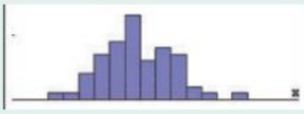
Select the cells A1 to A60 and Fill with the formula.




Draw a histogram of the sample means and check the window settings.

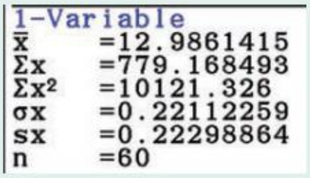



Select the cells A1 to A60 again and draw a histogram. The one shown was for a Start of 12.4 and a Width of 60, but your sample means might need different parameters. Redo the histogram as many times as you need to get the Start and Width to show it properly.



**c** Press [stat] and use CALC 1-Var Stats to find the mean and standard deviation of the sampling distribution in L1.


Use CALC and 1-VAR to find the mean of the sampling distribution in cells A1:A60.




**d** Comment on the results.

Your graphs, means and standard deviations will be different, but the patterns will be similar.

The sampling distribution does not look like the sample distribution. The sampling distribution is more like a normal distribution than a uniform distribution. The values of the sampling distribution are between about 12 and 14 rather than 10 and 16. The mean is very close to 13 and the standard deviation is quite small.

The mean of a uniform continuous variable on the interval [10, 16] is 13 and the standard deviation is about 1.7. The value of $\frac{1.7}{\sqrt{60}}$ is about 0.22, which is close to the standard deviation found for the sampling distribution in the above example.

## Exercise 8.02 Properties of sample means

Examples 3,4

**1 a** Use your graphics calculator to simulate the sample means of 60 samples of the totals of 10 rolls of 2 normal dice.

**b** Draw the sampling distribution and find its mean and standard deviation.

**c** Now simulate the sample means of 60 samples for 30 rolls of 2 dice.

**d** Find the mean and standard deviation.

**e** Now simulate the sample means of 60 samples for 100 rolls of 2 dice.

**f** Graph the sampling distribution and find the mean and standard deviation.

**g** The mean and standard deviation of the total of 2 dice are 7 and about 2.6. Compare the results of **b**, **d** and **f**.

**2 a** Use your graphics calculator to simulate the sample means of 60 samples of 10 rolls of a single normal die.

**b** Draw the sampling distribution and find its mean and standard deviation.

**c** Now simulate the sample means of 60 samples of 30 rolls of one die.

**d** Find the mean and standard deviation.

**e** Now simulate the sample means of 60 samples of 100 rolls of one die.

**f** Graph the sampling distribution and find the mean and standard deviation.

**g** The mean and standard deviation of the results of rolling one die once only are 3.5 and about 1.71. Compare the results of **b**, **d** and **f**.

**3 a** Use your graphics calculator to simulate the sample means of 10 samples of 50 rolls of one die.

**b** Draw the sampling distribution and find its mean and standard deviation.

**c** Now simulate the sample means of 30 samples of 50 rolls of one die.

**d** Find the mean and standard deviation.

**e** Now simulate the sample means of 100 samples of 50 rolls of one die.

**f** Graph the sampling distribution and find the mean and standard deviation.

**g** The mean and standard deviation of the results of rolling one die once only are 3.5 and about 1.71. Compare the results of **b**, **d** and **f**.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**4** **a** Use your graphics calculator to simulate the sample means of 60 samples of 10 numbers from a uniform distribution on the interval 5–25.

**b** Draw the sampling distribution and find its mean and standard deviation.

**c** Now simulate the sample means of 60 samples of 30 numbers from a uniform distribution on the interval 5–25.

**d** Find the mean and standard deviation.

**e** Now simulate the sample means of 60 samples of 100 numbers from a uniform distribution on the interval 5–25.

**f** Graph the sampling distribution and find the mean and standard deviation.

**g** The mean and standard deviation of a uniform distribution on the interval 5–25 are 15 and about 5.8. Compare the results of **b**, **d** and **f**.

## Problem solving

**5** Use your graphics calculator to simulate the sampling distribution of a normal distribution with mean 20 and standard deviation 5 for increasing sample sizes and comment on your results.

**6** Use your graphics calculator to simulate the sampling distribution of a binomial distribution with $n = 20$ and $p = 0.4$ for increasing sample sizes and comment on your results. The mean and standard deviation of the distribution are 8 and about 2.19.

**7** Use your graphics calculator to simulate the sampling distribution of a binomial distribution with $n = 40$ and $p = 0.6$ for increasing sample sizes and comment on your results. The mean and standard deviation of the distribution are 24 and about 3.10.

Getty Images/Evans vestal Ward/NBC/NBCU Photo Bank

# 8.03 Mean and standard deviation of sample means

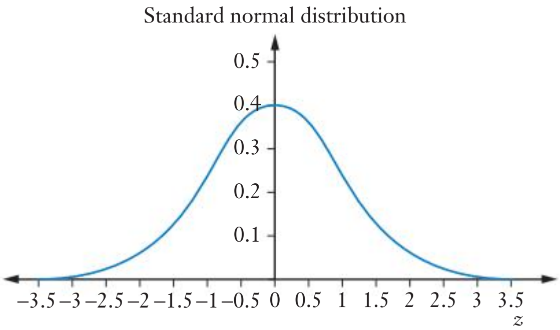
In Exercise 8.02, you saw that as the sample size increases, the shape of the sampling distribution gets closer to a normal distribution. The means get closer to the mean of the underlying distribution. The standard deviation is close to $\frac{\sigma}{\sqrt{n}}$, where $\sigma$ is the standard deviation of the underlying distribution and $n$ is the sample size. As the number of samples increases, the standard deviation gets closer to $\frac{\sigma}{\sqrt{n}}$. This is a consequence of the **central limit theorem**.

## The central limit theorem

For sufficiently large samples of size $n$ of a distribution with mean $\mu$ and standard deviation $\sigma$:

- the sampling distribution has mean $\mu$ and standard deviation $\frac{\sigma}{\sqrt{n}}$
- the sampling distribution approximates a normal distribution
- the approximation improves as $n$ becomes larger

A standard normal distribution has mean 0 and standard deviation 1. Its shape is as shown below. It has the equation $y = \frac{1}{\sqrt{2\pi}}e^{0.5z^2}$.




The variable for a normal distribution is written as $z$. In order to compare another distribution to the standard normal distribution, you calculate $z$-scores.

## Z-scores

The **$z$-score** of a value $x$ is given by $z = \frac{z - \bar{x}}{\sigma}$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## INVESTIGATION

### Z-SCORE VARIATION OF SAMPLING DISTRIBUTIONS

What happens to the distribution of $\frac{\overline{X}-\mu}{(s/\sqrt{n})}$ as the size of a sample increases?

The mean and standard deviation of a uniform distribution on the interval [8, 24] are $\frac{b-a}{2} = 16$ and $\frac{b-a}{2\sqrt{3}} \approx 9.24$ respectively.

Use your graphics calculator to simulate 60 sample means for samples of size 30 from a uniform distribution on the interval [8, 24].

Transform the values of the sample means $\overline{X}$ to $\frac{\overline{X}-\mu}{(s/\sqrt{n})}$ using $\mu = 16$ and $s = 9.24$ and draw a graph of the values.

Do a sampling distribution of samples of:

**a** size 50 **b** size 100 **c** size 200

and repeat the transformation and graph.

What happens to the shape of the distribution of $\frac{\overline{X}-\mu}{(s/\sqrt{n})}$ as the sample size increases?

From the investigation, you should be able to see the validity of the central limit theorem. The proof of the theorem is outside the scope of this course.

## EXAMPLE 5

Samples of 40 are taken from a population with mean 25 and a standard deviation of 6. What are the mean and standard deviation of the sampling distribution?

### Solution

| | |
|---|---|
| Use the rule for the mean. | Mean $= \mu = 25$ |
| Use the formula. | Standard deviation $= \frac{\sigma}{\sqrt{n}}$ |
| | $= \frac{6}{\sqrt{40}}$ |
| | $= 0.9486\ldots$ |
| Write the answer. | The mean and standard deviation are about 25 and 0.95. |

Remember that the mean of a distribution is the **expected value**.

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## EXAMPLE 6

40 samples of bags of tomatoes are weighed and found to have an average weight of 3.2 kg with a standard deviation of 6.8 g. The bags contain an average of 20 tomatoes. What would you expect the average mass and standard deviation of tomatoes to be?

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### Solution

Identify the sampling distribution.

The sampling distribution has 40 bags of tomatoes, each a sample of 20.

40 bags should give reasonably reliable results for the sampling distribution.

The mean mass of a bag = 3.2 kg.

$$\text{Mass of 1 tomato} = \frac{3.2}{20}$$
$$= 0.16 \text{ kg}$$
$$= 160 \text{ g}$$

Use the formula and solve for $\sigma$.

$$\text{Standard deviation} = \frac{\sigma}{\sqrt{n}}$$
$$6.8 = \frac{\sigma}{\sqrt{20}}$$
$$\sigma = 6.8\sqrt{20}$$
$$= 28.6215\ldots$$
$$\approx 29$$

Write the answer.

The average mass of tomatoes would be about 160 g with a standard deviation of about 29 g.


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## Exercise 8.03 Mean and standard deviation of sample means

Example 5

**1** Samples of 25 items were taken from a population with a mean of 48 and a standard deviation of 8. Find the mean and standard deviation of the sample means.

**2** Samples of 50 items were taken from a population with a mean of 100 and a standard deviation of 14. Find the mean and standard deviation of the sample means.

**3** Samples of 100 items were taken from a population with a mean of 67 and a standard deviation of 17. Find the mean and standard deviation of the sampling distribution.

**4** Samples of 70 items were taken from a population with a mean of 85 and a standard deviation of 20. Find the mean and standard deviation of the sample means.

Example 6

**5** The mean and standard deviation of the means of samples of 25 items were 14 and 0.3. What would you expect the mean and standard deviation of the population to be?

**6** The mean and standard deviation of the means of samples of 40 items were 26 and 1.5. What would you expect the mean and standard deviation of the population to be?

**7** The mean and standard deviation of the means of samples of 60 items were 42 and 2.5. What would you expect the mean and standard deviation of the population to be?

**8** The mean and standard deviation of the means of samples of 200 items were 52 and 1.3. What would you expect the mean and standard deviation of the population to be?

### Problem solving

**9** The average mass of large carrots is about 350 g with a standard deviation of about 47 g. What would be the average mass and standard deviation of bags of 6 large carrots?

**10** The average price of renting a house in a Brisbane suburb in 2018 was about $450. This was the average of the averages of samples of 30 from different suburbs. The standard deviation of the average was about $15. What would you expect the average house rental price and the standard deviation to be across Brisbane?

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# 8. CHAPTER SUMMARY

## Sample means

- The **population** is the complete group from which you want information
- A **sample** is part of the population
- A **random sample** is a sample taken so that every member of the population has an equal chance of being chosen
- A **parameter** is a numerical characteristic of the population, such as a mean
- A **statistic** is an estimate of a parameter from the values (**observations**) of a sample
- The **mean** and **standard deviation** of a population are shown as $\mu$ and $\sigma$
- The **mean** and **standard deviation** of a sample are shown as $\bar{x}$ and $s$
- A **random variable** is a variable with a numerical value that depends on the outcome of a chance event. Random variables are normally shown by a capital letter.
- The sample mean is a **continuous random variable**, shown as $\overline{X}$
- A **uniform probability distribution** (or **rectangular distribution**) is a flat line
- A **triangular distribution** is in the shape of a triangle
- A **sampling distribution** is a distribution of sample means from samples of the same size from the same population
- The **central limit theorem** says that for sufficiently large samples of size $n$ of a distribution with mean $\mu$ and standard deviation $\sigma$, the sampling distribution has mean $\mu$ and standard deviation $\frac{\sigma}{\sqrt{n}}$; the sampling distribution approximates a normal distribution and the approximation improves as $n$ becomes larger
- The $z$**-score** of a value $x$ is given by $z = \frac{z - \bar{x}}{\sigma}$


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# 8. CHAPTER REVIEW

## Sample means

**1** Find the mean of this sample of the ages of mothers of Year 12 students.

42, 52, 49, 49, 39, 56, 46, 50, 45, 52, 44, 51, 46, 54, 48

Example 1

**2** **a** Use your graphics calculator to find the means of 8 random samples of 20 values from a normal distribution with mean 30 and standard deviation 6.

**b** Find the mean of your samples.

Example 2

**3** The following procedure is used in this question. A number is drawn from a bag containing 5 marbles numbered 1 to 5 and replaced. A second marble is drawn and the numbers on the marbles are added together.

Examples 3, 4

**a** Use your graphics calculator to simulate the sample means of 50 samples of size 15 for the procedure.

**b** Draw the sampling mean and find its mean and standard deviation.

**c** Repeat **a** and **b** for samples of 75.

**d** Comment on your results.

**4** Samples of 36 are taken from a population with a mean of 40 and a standard deviation of 9. What are the mean and standard deviation of the sampling distribution?

Example 5

**5** The mean and standard deviation of the means of samples of 60 items were 18 and 1.4. What would you expect the mean and standard deviation of the population to be?

Example 6

### Problem solving

**6** Use your graphics calculator to simulate the sampling distribution of a normal distribution with mean 50 and standard deviation 8 for increasing sample sizes. Use 40 samples each time and comment on your results.

**7** The average mass of an onion is 350 g with a standard deviation of 45 g. What would be the average mass and standard deviation of bags of 25 onions?

**8** Samples of 40 boxes of apples are weighed and found to have an average weight of 10.2 kg with a standard deviation of 0.08 kg. The boxes each contained 30 apples. What would you expect the average mass and standard deviation of the apples to be?

Practice quiz

# Practice examination 3 ●●●○

**Time: 90 minutes**
**Perusal time: 5 minutes**
**Marks: 50**

## Instructions

- Students are permitted to bring or use: pens, pencils, highlighters, erasers, sharpeners, rules and an approved graphics calculator.
- Students must show appropriate working and justification to gain full marks.
- A QCAA formula sheet is provided.
- Unless otherwise stated, numerical answers should be exact.
- Unless otherwise indicated, no diagrams in this examination are drawn to scale.
- All written responses must be in English.
- Answer all questions.
- **Students are NOT permitted to bring or use notes of any kind, correction fluid/tape, mobile phones and/or any other unauthorised electronic devices.**

### Question 1 (2 marks)

Find each integral.

**a** $\int \frac{x}{\sqrt{1+5x^2}}\,dx$

**b** $\int \frac{2x-4}{x^2-4x+5}\,dx$

### Question 2 (2 marks)

Calculate each integral.

**a** $\int_2^5 \frac{2x-3}{x^2-3x+5}\,dx$

**b** $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^2(x)-\cos^2(x)}{\sin(x)\cos(x)}\,dx$

### Question 3 (2 marks)

Find each integral by substitution.

**a** $\int 2x\sqrt{x^2-3}\,dx$

**b** $\int \frac{\ln|x|}{x}\,dx$


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## Question 4 (2 marks)

Use integration by parts to find:

**a** $\int x^3 e^x \, dx$

**b** $\int e^{x+\ln|x|} \, dx$

## Question 5 (2 marks)

Consider the region between the curve $y = x^3$, the $x$-axis and the line $x = 3$.
Calculate the volume of the solid generated when the region is revolved about the:

**a** $x$-axis

**b** $y$-axis

## Question 6 (4 marks)

Use the trapezoidal rule to find an approximation for the area under:

**a** $y = \frac{1}{x}$ from $x = 1$ to $x = 4$ with six strips, correct to 2 decimal places

**b** $y = \sqrt{100 - x^2}$ from $x = -10$ to $x = 10$ with 5 strips, correct to 3 decimal places

**c** $y = \frac{\sin(x)}{x}$ from $x = -\pi$ to $x = \pi$ with 9 strips, correct to 3 decimal places

**d** $y = \frac{\sqrt{x}}{5}$ from $x = 0$ to $x = 25$ with 10 strips, correct to 3 decimal places.

## Question 7 (1 mark)

An exponential distribution has parameter $\lambda = 0.2$. What is the expected value of the distribution?

## Question 8 (1 mark)

What is the parameter $\lambda$ for an exponential distribution with an average value of 25?

## Question 9 (1 mark)

How does the shape of the exponential distribution change as the parameter $\lambda$ is increased?

## Question 10 (3 marks)

A random variable $X$ has an exponential distribution with $\lambda = 0.05$. Find:

**a** $P(0 \le x \le 2)$

**b** $P(10 \le x \le 20)$

**c** $P(1.5 \le x \le 2.5)$

## Question 11 (1 mark)

Find the standard deviation of an exponential distribution with parameter $\lambda = 0.6$.

## Question 12 (2 marks)

Find the sample mean of the following sample of times (in hours) spent on the internet.

4, 13, 7, 12, 7, 9, 8, 7, 4, 9, 7, 9, 4, 17, 4, 11, 4, 8, 5, 12, 6, 11, 7, 3

## Question 13 (2 marks)

How would you simulate a sample from a uniform distribution on the interval [6, 20]?

## Question 14 (2 marks)

**a** What is the shape of a uniform distribution on the interval [6, 20]?

**b** What is the shape of the sampling distribution?

## Question 15 (3 marks)

Samples of 49 items are taken from a distribution with mean 60 and standard deviation 8.54. What would be the mean and standard deviation of the sampling distribution?

## Question 16 (2 marks)

A sphere with radius $r$ is generated by revolving the curve $y = \sqrt{r^2 - x^2}$ around the $x$-axis between $x = -r$ and $x = r$. Use an integral to find a formula for the volume of the sphere.

## Question 17 (2 marks)

A developer has a large area of land, but some of it is swampy and must be either filled or excavated for a 'lake' feature. The surveyor measured the width of the low-lying land at 20 m intervals. The measurements are shown in the table. Use Simpson's rule to calculate the area of the swamp in hectares, correct to 3 significant figures.

| Offset (m) | 0 | 20 | 40 | 60 | 80 | 100 | 120 | 140 | 160 | 180 | 200 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Width (m) | 15 | 40 | 80 | 100 | 150 | 175 | 170 | 160 | 120 | 60 | 30 |


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**Question 18** (3 marks)

The time taken by customer service staff at the Australian Taxation Office to deal with telephone queries is exponentially distributed with an average time of 15 minutes. Model the time taken as an exponential distribution, then use the model to find:

**a** the probability that a query will be dealt with in less than 10 minutes

**b** the median time taken for dealing with a query

**c** the percentage of queries that can be expected to take longer than 20 minutes.

**Question 19** (3 marks)

The official mass of a soccer ball is 430 g, but there is an allowance for variation. The standard deviation is about 6 g. A club puts soccer balls in mesh bags of 20 balls for team practice so that everybody in the squad gets a ball, with allowance for some to be damaged or deflated. What would you expect the mean and standard deviation of bags of soccer balls to be?

**Question 20** (3 marks)

Show that $\int \frac{\ln|x|}{x^2}\,dx = -\frac{\ln|x|+1}{x} + c$.

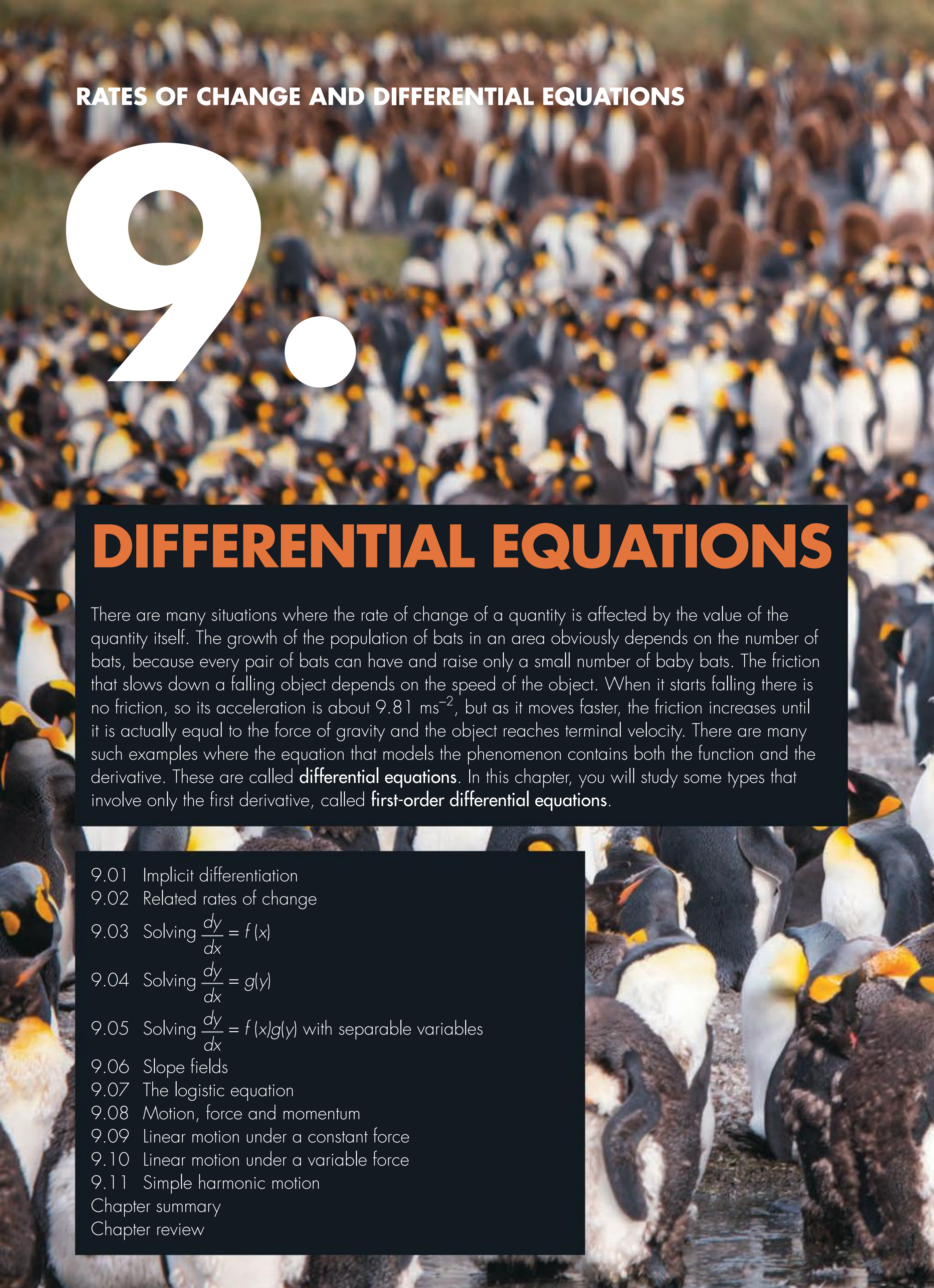
**Question 21** (4 marks)

Show that $\int x\sin^{-1}(x^2)\,dx = \frac{1}{2}x^2\sin^{-1}(x^2) + \frac{1}{2}\sqrt{1-x^4} + c$.

**Question 22** (3 marks)

Trays of Bowen mangoes have a mean mass of 7 kg with a standard deviation of 30 g. The trays have about 15 mangoes on them. The actual tray weighs 100 g. What are the mean mass and standard deviation of Bowen mangoes?

**END OF EXAMINATION**





# 9.

# DIFFERENTIAL EQUATIONS

There are many situations where the rate of change of a quantity is affected by the value of the quantity itself. The growth of the population of bats in an area obviously depends on the number of bats, because every pair of bats can have and raise only a small number of baby bats. The friction that slows down a falling object depends on the speed of the object. When it starts falling there is no friction, so its acceleration is about 9.81 $ms^{-2}$, but as it moves faster, the friction increases until it is actually equal to the force of gravity and the object reaches terminal velocity. There are many such examples where the equation that models the phenomenon contains both the function and the derivative. These are called **differential equations**. In this chapter, you will study some types that involve only the first derivative, called **first-order differential equations**.


9.01 Implicit differentiation

9.02 Related rates of change

9.03 Solving $\frac{dy}{dx} = f(x)$

9.04 Solving $\frac{dy}{dx} = g(y)$

9.05 Solving $\frac{dy}{dx} = f(x)g(y)$ with separable variables

9.06 Slope fields

9.07 The logistic equation

9.08 Motion, force and momentum

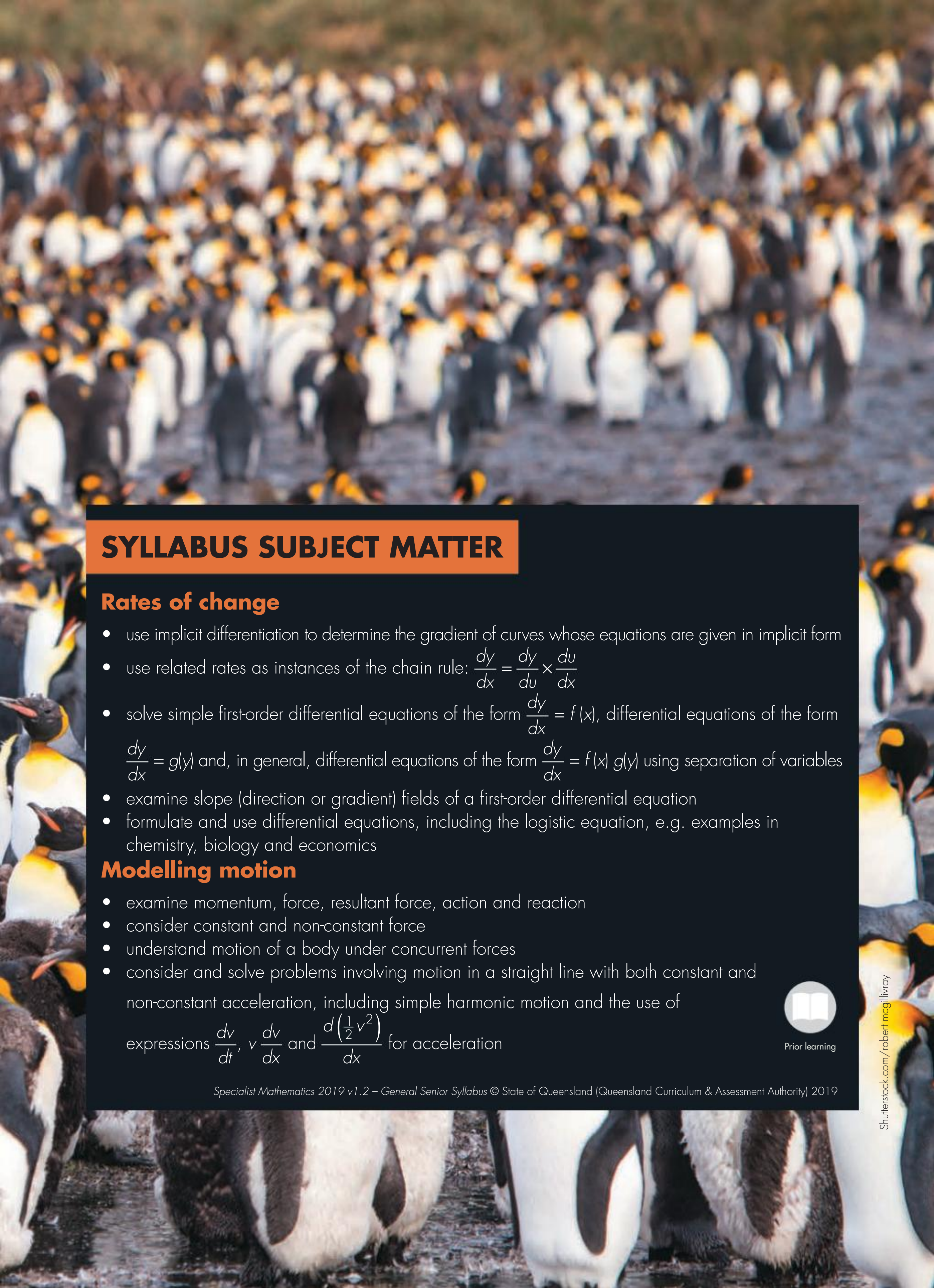
9.09 Linear motion under a constant force

9.10 Linear motion under a variable force

9.11 Simple harmonic motion

Chapter summary

Chapter review



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# SYLLABUS SUBJECT MATTER

## Rates of change

- use implicit differentiation to determine the gradient of curves whose equations are given in implicit form
- use related rates as instances of the chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
- solve simple first-order differential equations of the form $\frac{dy}{dx} = f(x)$, differential equations of the form $\frac{dy}{dx} = g(y)$ and, in general, differential equations of the form $\frac{dy}{dx} = f(x)\, g(y)$ using separation of variables
- examine slope (direction or gradient) fields of a first-order differential equation
- formulate and use differential equations, including the logistic equation, e.g. examples in chemistry, biology and economics

## Modelling motion

- examine momentum, force, resultant force, action and reaction
- consider constant and non-constant force
- understand motion of a body under concurrent forces
- consider and solve problems involving motion in a straight line with both constant and non-constant acceleration, including simple harmonic motion and the use of expressions $\frac{dv}{dt}$, $v\frac{dv}{dx}$ and $\frac{d\left(\frac{1}{2}v^2\right)}{dx}$ for acceleration

Prior learning

# TERMINOLOGY

amplitude
direction field
gradient field
frequency
normal
slope field
terminal velocity
concurrent forces
equilibrium
implicit differentiation
logistic model
point of equilibrium
simple harmonic motion
differential equation
first-order differential equation
initial condition
Newton's laws of motion
related rates of change
tangent

## 9.01 Implicit differentiation

WS
Implicit differentiation

The equation $y = 2x^2 - 3x + 1$ expresses the variable $y$ explicitly in terms of the variable $x$. It is of the form $y = f(x)$. This indicates that the value of $y$ depends on the value of $x$.

However, some functions are defined *implicitly* by a relation between $x$ and $y$ such as $x^2 + y^2 = 9$. This equation represents a circle with centre (0, 0) and radius 3.

To find the derivative of such functions, you use the method of **implicit differentiation**. This is the process of differentiating with respect to the desired variable $x$ while treating the other variables as functions of $x$.

Before looking at implicit differentiation, it is worth revisiting the chain rule.

Consider the equation $z = y^2$

Differentiate wrt $x$. $\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx}$

$= 2y\frac{dy}{dx}$

Substitute for $z$. $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$

This result can be generalised as follows.

**Implicit differentiation**

$$\frac{d}{dx}[f(x)] = \frac{d}{dy}[f(y)] \times \frac{dy}{dx}$$

For an equation in $x$ and $y$, implicit differentiation consists of differentiating both sides of the equation with respect to $x$ and then solving the resulting equation for $y'$.

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**EXAMPLE 1**

Find $\frac{dy}{dx}$ for $x + xy + y^3 = 9$.

**Solution**

Differentiate both sides of the equation. $\frac{d}{dx}(x + xy + y^3) = \frac{d}{dx}(9)$

Differentiate terms separately. $\frac{d}{dx}(x) + \frac{d}{dx}(xy) + \frac{d}{dx}(y^3) = 0$

Use the product rule for $xy$. $1 + \left[\frac{d}{dx}(x) \times y + x\frac{d}{dx}(y)\right] + 3y^2\frac{dy}{dx} = 0$

Simplify. $1 + y + x\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$

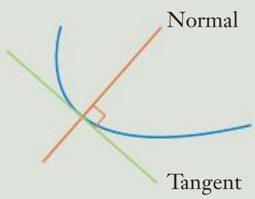
Rearrange and factorise. $\frac{dy}{dx}(x + 3y^2) + 1 + y = 0$

Isolate $\frac{dy}{dx}$. $\frac{dy}{dx} = \frac{1+y}{x+3y^2}$

The equations of conic sections, including circles and ellipses, are not usually explicit as they are not written in the form $y = f(x)$. This means that you can use implicit differentiation to find the gradients of tangents and normals to these and other similar curves.

## Tangents and normals

- A **tangent** to a curve is a line that touches the curve at one point and has the same slope as the curve at that point.
- The **normal** to a curve at a particular point is a line passing through that point that is perpendicular to the curve, and hence perpendicular to the tangent.



## EXAMPLE 2

**a** Find the equation of the tangent to $y^2 = 20x$ at $(5, -10)$.

**b** Find the equation of the normal to $\frac{x^2}{25} + \frac{y^2}{9} = 1$ at $(3, -2.4)$.

### Solution

**a** Write the conic equation. $y^2 = 20x$

Find the derivative.
$$\frac{d}{dx}(y^2) \times \frac{dy}{dx} = \frac{d}{dx}(20x)$$
$$2y \times \frac{dy}{dx} = 20$$

Divide by $2y$ to find the derivative.
$$\frac{dy}{dx} = \frac{10}{y}$$

Substitute $y = -10$ to find the gradient of the tangent, $m$.
$$m = \frac{10}{-10} = -1$$

Write the gradient–point form of a line. $y - y_1 = m(x - x_1)$

Substitute $(x_1, y_1) = (5, -10)$ and $m = -1$. $y - (-10) = -1(x - 5)$

Simplify. $y + 10 = -x + 5$

Write the answer in standard form. $x + y + 5 = 0$

**b** Write the equation of the conic.
$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

Find the derivative.
$$\frac{d}{dx}\left(\frac{x^2}{25}\right) + \frac{d}{dx}\left(\frac{y^2}{9}\right) \times \frac{dy}{dx} = \frac{d}{dx}(1)$$

The derivative of the RHS is 0.
$$\frac{2x}{25} + \frac{2y}{9} \times \frac{dy}{dx} = 0$$

Rearrange to isolate the derivative.
$$\frac{dy}{dx} = -\frac{9x}{25y}$$

Substitute $(3, -2.4)$ to find the gradient of the tangent, $m$.
$$m = \frac{9 \times 3}{25 \times 2.4} = \frac{9}{20}$$

Find the slope of the normal.
$$\text{Slope of normal} = -\frac{20}{9}$$

Write the slope-point form of a line. $y - y_1 = m(x - x_1)$

Substitute values.
$$y - (-2.4) = -\frac{20}{9}(x - 3)$$

Simplify. $9y + 21.6 = -20x + 60$

Write the answer in standard form. $20x + 9y - 38.4 = 0$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## Exercise 9.01 Implicit differentiation

Example 1

**1** Find $\frac{dy}{dx}$ for each equation.

**a** $4x^2 + 9y^2 = 25$ **b** $5x^2 - 3xy + 2y^2 = 6$ **c** $x^2y + y^2x = xy$

**d** $\frac{x^2}{16} + \frac{y^2}{5} = 1$ **e** $3x\sqrt{y} = (2 - y)^2$ **f** $\frac{5}{x} + \frac{9}{y} = 7$

**2** Use implicit differentiation to find $\frac{dy}{dx}$ in terms of $x$ and $y$ for:

**a** $xe^{2y} - 4xy = 5$ **b** $y \sin (y) = x$

**c** $\cos (y^2 + 1) = 3x$ **d** $y \sin (x) - 5x = 0$

**e** $e^{x^2y} = 2x + y$ **f** $y^2 + x^2 - xy + \cos (y) = 0$

Example 2

**3** Find the equation of the tangent to $(x - 4)^2 + (y + 2)^2 = 100$ at (6.8, 7.6).

**4** Find the equation of the tangent to $y^2 = 24x$ at (6, 12).

**5** Find the equation of the tangent to the hyperbola $x^2 - y^2 = 16$ that passes through the point (5, −3).

**6** Find the equation of the tangent to $\frac{(x+1)^2}{36} - \frac{(y-2)^2}{16} = 1$ at (6.5, 5).

**7** Find the equations of the tangents to the ellipse $4x^2 + 9y^2 = 40$ that have a gradient of $-\frac{2}{9}$.

**8** Find the equation of the normal to $\frac{(x-3)^2}{25} + \frac{(y+2)^2}{16} = 1$ at (6, 1.2).

**9** Find the equations of the tangents to the ellipse $\frac{9x^2}{52} + \frac{16y^2}{52} = 1$ that are parallel to the line $y = \frac{9}{8}x - \frac{1}{8}$.

**10** Find the equations of the tangents to the hyperbola $y = \frac{1}{x}$ that pass through point (1, −1).

### Problem solving

**11** Find the intersection of the tangents to $\frac{(x-3)^2}{9} + \frac{(y+2)^2}{4} = 1$ at $x = 4.8$.

**12** Find the intersection of the normals to $\frac{x^2}{16} - \frac{y^2}{9} = 1$ at $y = 3.15$.

**13** For the parabola $y^2 = 4ax$, show that the equation of the tangent at a point $P(x_1, y_1)$ on the parabola is $y_1y = 2a(x + x_1)$.

**14** Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Show that the equation of the tangent to the ellipse with slope $m$ is $y = mx \pm \sqrt{a^2m^2 + b^2}$.

**15** Show that the equation of the tangent, with gradient $m$, to the parabola $y^2 = 4ax$ is $y = mx + \frac{a}{m}$.

**16** Show that the equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at a point $P(x_1, y_1)$ on the ellipse is $b^2x_1x + a^2y_1y - a^2b^2 = 0$.

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WS

Related rates

# 9.02 Related rates of change

It is often possible to relate the change in one quantity to the change in another quantity. For example, if you fill a balloon with air, both the volume and the radius of the balloon increase and their rates of increase are related to each other. The rate of increase of volume and the rate of increase of radius are known as **related rates of change**.

Related rates problems usually involve calculating the rate of change of one quantity in terms of the rate of change of another quantity. This is done by finding an equation that relates the 2 quantities and then using the chain rule to differentiate both sides of the equation with respect to time.

### Related rates of change

The relation between **related rates of change** with respect to time can be found by using the chain rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

### EXAMPLE 3

Given that $y = 4x^3 - 17x^2$ and $\frac{dx}{dt} = 8$, find $\frac{dy}{dt}$ when $x = 3$.

#### Solution

Calculate the rate of change of $y$ wrt $x$, that is, $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 12x^2 - 34x$$

Use related rates to find the rate of change of $y$ wrt $t$, that is, $\frac{dy}{dt}$.

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

Substitute for known values.

$$= (12x^2 - 34x) \times 8$$

Find $\frac{dy}{dt}$ when $x = 3$.

$$= 8(12 \times 3^2 - 34 \times 3)$$
$$= 48$$

State the result.

When $x = 3$, $\frac{dy}{dt} = 48$.

Many related rate problems involve volume and areas of geometric shapes.


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## EXAMPLE 4

Air is pumped into a spherical balloon so that the rate of increase of the radius, $\frac{dr}{dt} = 0.0125$ cm/s. Calculate the rate at which the volume is increasing $\left(\frac{dV}{dt}\right)$ when the radius is 25 cm.

### Solution

Write the rule for the volume of a sphere.

$$V = \frac{4}{3}\pi r^3$$

Calculate the rate of change of $V$ wrt $r$, $\frac{dV}{dr}$.

$$\frac{dV}{dr} = 4\pi r^2$$

Use related rates to find the rate of change of $V$ wrt $t$, $\frac{dV}{dt}$.

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

Substitute for known rates.

$$\frac{dV}{dt} = 4\pi r^2 \times 0.0125$$

Find $\frac{dV}{dt}$ when $r = 25$.

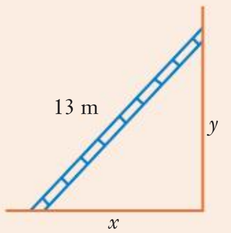
$$\begin{aligned} &= 4\pi \times (25)^2 \times 0.0125 \\ &= 31.25\pi \\ &= 98.174\ldots \end{aligned}$$

Round off and state the result.

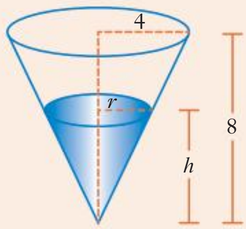
The rate of change of volume of the balloon is $31.25\pi$ cm$^3$/s (about 98 cm$^3$/s) when $r = 25$.

## EXAMPLE 5

**a** A 13 m long ladder rests against a vertical wall, as shown in the diagram on the right. If the base of the ladder is sliding away from the base of the wall at the rate of 0.4 m/s, how fast is the top of the ladder moving down the wall when the bottom of the ladder is 5 m from the base of the wall?




**b** Water is filling a tank that has the shape of an inverted circular cone with top radius 4 m and height 8 m. If the rate at which water is coming in to the tank is 2 m$^3$/min, find the rate at which the water level is rising when the water is 3 m deep.

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## Solution

**a** Use Pythagoras' theorem to write a relationship between $x$ and $y$.

$$x^2 + y^2 = 13^2$$

Differentiate both sides wrt $t$.

$$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = 0$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

Rearrange and isolate $\frac{dy}{dt}$.

$$\frac{dy}{dt} = \frac{dx}{dt} \times \left(-\frac{x}{y}\right)$$

Write the rate of change of $x$ wrt $t$.

$$\frac{dx}{dt} = 0.4$$

Find $y$ when $x = 5$.

$$5^2 + y^2 = 13^2$$
$$y^2 = 144$$
$$y = 12$$

Substitute for $y$ and $\frac{dx}{dt}$.

$$\frac{dy}{dt} = 0.4 \times \left(-\frac{5}{12}\right)$$

Evaluate.

$$= -\frac{5}{3}$$

The negative sign indicates that the top of the ladder is sliding down the wall.

State the result.

The top of the ladder is sliding down the wall at the rate of $\frac{5}{3}$ m/s.

**b** Write the rule for the volume of a cone.

$$V = \frac{1}{3}\pi r^2 h$$

You want to know $\frac{dh}{dt}$, so it is best to eliminate $r$ by expressing $r$ in terms of $h$. Use similar triangles from the diagram.

$$\frac{r}{4} = \frac{h}{8} \Rightarrow r = \frac{h}{2}$$

Write the rule for $V$ in terms of $h$.

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$$

$$V = \frac{\pi}{12}h^3$$

Differentiate wrt $t$.

$$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{\pi}{12}h^3\right)$$

$$= \frac{\pi}{4}h^2\frac{dh}{dt}$$


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Rearrange.

$$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$$

You know $\frac{dV}{dt} = 2$.

$$= \frac{8}{\pi h^2}$$

You want to know the rate when $h = 3$.

$$= \frac{8}{\pi(3)^2}$$

$$= \frac{8}{9\pi}$$

State the result.

The water level is rising at a rate of $\frac{8}{9\pi}$ $m^3$/min (about 0.283 $m^3$/min).

## Steps in solving related rates problems

1 Read the question carefully.

2 Draw a diagram if possible.

3 Assign variables to quantities that are functions of time.

4 Express the given information and the required rate in terms of derivatives.

5 Write an equation that relates the various quantities stated in the question.

6 Differentiate both sides of the equation with respect to time.

7 Substitute the given information into the equation and solve for the unknown rate.

## Exercise 9.02 Related rates of change

Example 3

**1** Find an expression for $\frac{dy}{dt}$ given:

**a** $y = 5x^3 + 4x$ and $\frac{dx}{dt} = 2$

**b** $y = e^{4x}$ and $\frac{dx}{dt} = -1$

**c** $y = x \ln |x|$ and $\frac{dx}{dt} = 4$

**d** $y = \sin (3x)$ and $\frac{dx}{dt} = 8$

**2** Evaluate $\frac{dy}{dt}$ at $x = 2$, given:

**a** $y = 3x^4 - 2x^2$ and $\frac{dx}{dt} = 3$

**b** $y = (4x - 1)^3$ and $\frac{dx}{dt} = -2$

**c** $y = xe^{2x}$ and $\frac{dx}{dt} = 1$

**d** $y = 3x \ln |x|$ and $\frac{dx}{dt} = 4$

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**3** Evaluate $\frac{dy}{dt}$ at $x = \frac{\pi}{4}$, given:

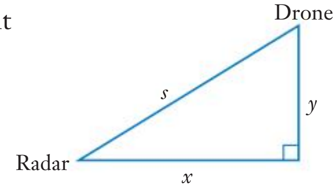
**a** $y = \cos(3x)$ and $\frac{dx}{dt} = -2$ **b** $y = \sin(4x)$ and $\frac{dx}{dt} = 3$

Example 4

**4** Oil is leaking onto a floor and forms a circular pool. The rate of increase of the radius of the pool, $\frac{dr}{dt} = 20$ cm/min. Calculate the rate at which the area of the pool is increasing $\left(\frac{dA}{dt}\right)$ when the radius is 6 cm.

Example 5

**5** A drone is flying towards a radar station at a constant height of 6 km above the ground. The distance between the drone and the radar station ($s$) is decreasing at a constant rate of 320 km/h.




**a** Explain why $\frac{dy}{dt} = 0$.

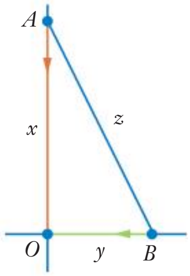
**b** Calculate the horizontal speed of the plane when $s = 10$ km.

## Problem solving

**6** A particle is moving along the curve $y = x^2$ in such a way that when $x = 3$, $\frac{dx}{dt} = 7$ cm/s. Calculate $\frac{dy}{dt}$ at that instant.

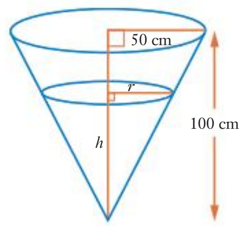
**7** Car $A$ is travelling south at 40 km/h and car $B$ is travelling west at 60 km/h as shown in the diagram.

Both cars are travelling towards the intersection of the 2 roads. At what rate are the cars approaching each other when car $A$ is 1.2 km and car $B$ is 0.5 km from the intersection?




**8** A container has the shape of an inverted cone, a height of 100 cm and a radius at the top of 50 cm.

The container is filled with water at the constant rate of 50 cm$^3$/min. Find the rate at which the level of water is rising in the container at the instant when the volume of water in the basin is $144\pi$ cm$^3$.




**9** A bright light is fixed on the ground 20 m away from a building. A 2 m tall person walks directly away from the light towards the building at 1 m/s. How fast is the length of the shadow cast on the building changing when the person is 12 m from the building?

**10** Oil is leaking from a rusted container. The leaking oil forms a circular slick on the surface of the ocean. The area of the oil slick increases at a rate of $280\pi$ m$^2$/min. Calculate the rate of change of the oil slick's radius when it is 350 m.

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**11** Air is leaking out of a spherical balloon at the rate of 6 $cm^3$/min. What is the rate of change of the surface area when the radius is 1 cm?

**12** The sides of a right-angled triangle are changing, but the perimeter remains at 40 cm. When the sides of the triangle are 8 cm, 15 cm and 17 cm, the hypotenuse is changing at a rate of 1.4 cm/min. Find the rates of change of the other two edges at this instant.

**13** A spherical balloon is being inflated at the rate of 12 $cm^3$/s. Calculate the rate of change of the radius when the radius is 2 cm.

**14** If a circle has area $A$ and circumference $C$, find the rate of change of its area with respect to its circumference.

**15** An observer stands on level ground 500 m away from a launch pad to watch a rocket launch. The rocket blasts off vertically and maintains a velocity of 260 m/sec. Calculate the rate of change of the distance between the observer and the rocket when the rocket is 1200 m above the ground.

**16** A container is shaped like an inverted cone with a radius of 12 cm and a height of 12 cm. The container is being filled with water so that the water level rises at a rate of 1.5 cm/s. At what rate is water being poured into the container when the water level is 6 cm?

**17** A 7 m long ladder is leaning against a vertical wall. The top of the ladder is falling down the wall at the rate of 1.5 m per second. How fast is the bottom of the ladder moving along the ground when the bottom of the ladder is 3 m from the wall?

**18** The width ($w$) of a rectangle is increasing at a rate of 4 cm/s, while the length ($l$) increases at 5 cm/s. At what rate is the area increasing when $w = 6$ cm and $l = 9$ cm?

**19** A spherical balloon is inflated at a rate of 100 $cm^3$/s. How fast is the radius of the balloon changing when the diameter is 50 cm?

Dreamstime.com/Florea Paul Daniel

## 9.03 Solving $\frac{dy}{dx} = f(x)$

WS Simple differential equations

The process of modelling involves developing a mathematical model of a real-world problem using reasoning or evidence from experiments. A mathematical model often involves an equation that contains an unknown function and one or more of its derivatives.
This is known as a **differential equation**.

> The order of a differential equation is the order of the highest derivative involved.
> A **first-order differential equation** involves only the first derivative.

You use integration to find the solution of a differential equation.
Consider the differential equation:

$$\frac{dy}{dx} = 4x^3 - 2x + 3$$

The general solution is $y = x^4 - x^2 + 3x + c$.

More information is required to find a specific solution. If you know, for example, that (1, 8) is a point on the curve $y = x^4 - x^2 + 3x + c$, then the specific solution to the differential equation is $y = x^4 - x^2 + 3x + 5$.

### EXAMPLE 6

Find the general solution to each differential equation.

**a** $\frac{dy}{dx} = (7x - 5)^3$ **b** $\frac{dy}{dt} = 12e^{5t} - 4e^{-3t}$ **c** $\frac{dq}{dm} = \frac{4}{16+m^2}$

#### Solution

**a** Integrate.

$$y = \int (7x-5)^3\, dx$$
$$= \frac{(7x-5)^4}{7\times 4} + c$$

Simplify.

$$= \frac{1}{28}(7x-5)^4 + c$$

**b** Find the integral with respect to $t$.

$$y = \int \left(12e^{5t} - 4e^{-3t}\right) dt$$
$$= \frac{12}{5}e^{5t} + \frac{4}{3}e^{-3t} + c$$

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**c** Find the indefinite integral wrt $m$.

Use the standard integral

$\int \frac{a}{a^2+x^2}dx = \tan^{-1}\left(\frac{x}{a}\right) + c.$

$$q = \int \frac{4}{16+m^2}dm$$
$$= \tan^{-1}\left(\frac{m}{4}\right) + c$$

## EXAMPLE 7

Find the solution of each differential equation.

**a** $\frac{dy}{dx} = \tan^2(x)$ if $y\left(\frac{\pi}{4}\right) = 1$

**b** $\frac{dy}{dx} = \frac{2x+1}{x^2+7x+10}$ if $y(1) = 0$

**c** $\frac{dy}{dx} = \frac{3x-4}{3x^2-8x}$ if $y(2) = 3$

### Solution

**a** Integrate.

$$y = \int \tan^2(x)\,dx$$

Use $1 + \tan^2(x) = \sec^2(x)$.

$$= \int \sec^2(x) - 1\,dx$$
$$= \tan(x) - x + c$$

Substitute for $\left(1, \frac{\pi}{4}\right)$.

$$1 = \tan\left(\frac{\pi}{4}\right) - \frac{\pi}{4} + c$$

Solve for $c$.

$$c = \frac{\pi}{4}$$

State the result.

$$y = \tan(x) - x + \frac{\pi}{4}$$

**b** Factorise the denominator.

$$x^2 + 7x + 10 = (x+5)(x+2)$$

Change $\frac{2x+1}{x^2+7x+10}$ to partial fractions.

$$\frac{2x+1}{x^2+7x+10} = \frac{A}{x+5} + \frac{B}{x+2}$$
$$= \frac{A(x+2)+B(x+5)}{(x+5)(x+2)}$$

Solve for $A$ and $B$.

$A + B = 2$ [1]

$2A + 5B = 1$ [2]

[2] − 2[1] $\Rightarrow 3B = -3 \Rightarrow B = -1$

So $A = 3$.

Write as partial fractions and integrate.

$$\int \frac{2x+1}{x^2+7x+10}dx = \int \frac{3}{x+5} - \frac{1}{x+2}dx$$

Use the rules of integration.

$$= \int \frac{3}{x+5}dx - \int \frac{1}{x+2}dx$$
$$= 3\ln|x+5| - \ln|x+2| + c$$

| | |
|---|---|
| Substitute for (1, 0) | $0 = 3\ln(6) - \ln(3) + c$ |
| Simplify using log laws. | $= \ln\left(\frac{6^3}{3}\right) + c$ |
| Solve for $c$. | $c = -\ln(72)$ |
| Substitute. | $y = 3\ln\lvert x+5\rvert - \ln\lvert x+2\rvert - \ln(72)$ |
| Simplify using log laws. | $= \ln\left\lvert\frac{(x+5)^3}{72(x+2)}\right\rvert$ |
| **c** Integrate. | $y = \int \frac{3x-4}{3x^2-8x}\,dx$ |
| Use $\int \frac{f'(x)}{f(x)}\,dx = \ln\lvert f(x)\rvert + c$. | $= \int \frac{1}{2} \times \frac{6x-8}{3x^2-8x}\,dx$ |
| | $= \frac{1}{2}\ln\lvert 3x^2 - 8x\rvert + c$ |
| Substitute for (2, 3). | $3 = \frac{1}{2}\ln\lvert 3 \times (2)^2 - 8 \times 2\rvert + c$ |
| | $= \frac{1}{2}\ln(4) + c$ |
| | $= \ln(2) + c$ |
| | $c = 3 - \ln(2)$ |
| State the result. | $y = \frac{1}{2}\ln\lvert 3x^2 - 8x\rvert + 3 - \ln(2)$ |

When you apply differential equations to a physical problem, you will usually be interested in finding a particular solution rather than the general solution. For problems involving rates of change, the situation at time $t = 0$ is often used to find particular solutions: $y(t_0) = y_0$ is called the **initial condition**.

## EXAMPLE 8

A rocket is constructed so that its acceleration for a short time after take-off is given by $a = 10t + 2$ ms$^{-2}$, where $t$ is in seconds. It takes off vertically from a height of 100 m above sea level. Use differential equations to find its velocity and height after 4 s.

### Solution

| | |
|---|---|
| Write acceleration as the derivative of velocity. | $\frac{dv}{dt} = 10t + 2$ |
| Integrate. | $v = \int 10t + 2\;dt$ |
| | $= 5t^2 + 2t + c_1$ |

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| | |
|---|---|
| Use the initial condition, i.e. at $t = 0$ $v(0) = 0$. | $0 = 5 \times 0^2 + 2 \times 0 + c_1$ |
| Solve for $c_1$. | $c_1 = 0$<br>$v = 5t^2 + 2t$ |
| Find $v$ when $t = 4$. | $v = 5 \times 4^2 + 2 \times 4$<br>$= 88$ |
| Write velocity as the derivative of the height. | $\frac{dh}{dt} = v = 5t^2 + 2t$ |
| Integrate. | $h = \int 5t^2 + 2t\,dt$<br>$= \frac{5}{3}t^3 + t^2 + c_2$ |
| Use the initial condition, i.e. at $t = 0$ $h(0) = 100$. | $100 = 0 + 0 + c_2$<br>$c_2 = 100$ |
| State the result. | $h = \frac{5}{3}t^3 + t^2 + 100$ |
| Find $h$ when $t = 4$. | $= \frac{5}{3} \times 4^3 + 4^2 + 100$<br>$= 222\frac{2}{3}$ |
| Write the answer. | After 4 s the rocket has a velocity of 88 $ms^{-1}$ and a height of $222\frac{2}{3}$ m above sea level. |


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## Exercise 9.03 Solving $\frac{dy}{dx} = f(x)$

Example 6

**1** Find the general solution to each differential equation.

**a** $\frac{dy}{dx} = 4 + 5x - 9x^2$

**b** $\frac{dy}{dt} = (3t + 5)^3$

**c** $\frac{dp}{dx} = \frac{\cos\sqrt{x}}{\sqrt{x}}$

**d** $\frac{dy}{dm} = \ln|2e^{\sin(m)}|$

**e** $\frac{dm}{dx} = \frac{x+1}{4+x^2}$

**f** $\frac{dv}{dt} = 6e^{3t} - 12e^{-4t}$

**g** $\frac{dq}{dw} = 24w^3 - \frac{12}{w}$

**h** $\frac{dh}{dy} = 8\cos(3y)\sin(3y)$

**i** $\frac{du}{dx} = \frac{4}{x\ln|x|}$

**j** $\frac{dy}{dp} = (p^{-1} + 2)^2$

Example 7

**2** Find the solution to each differential equation.

**a** $\frac{dy}{dx} = (3x^2 - 5x + 4)^4(6x - 5)$ if $y(1) = 3$

**b** $\frac{dy}{dx} = \frac{2x-6}{x^2-3x}$ if $y = -4$ when $x = 1$

**c** $\frac{dy}{dx} = \sin(x)\cos(x)$ if $y\left(\frac{\pi}{4}\right) = 3$

**d** $\frac{dy}{dx} = \frac{5x+2}{x^2}$ if $y(1) = 0$

**e** $\frac{dy}{dx} = \frac{\ln|x|}{x}$ if $y = -5$ when $x = 1$

**f** $\frac{dy}{dx} = xe^{-x^2}$ if $y(0) = 0$

**g** $\frac{dy}{dx} = \sin(x)\cos(2x)$ if $y\left(\frac{\pi}{3}\right) = 1$

**h** $\frac{dy}{dx} = \frac{2x-3}{x^2-x-6}$ if $y(4) = 0$

Example 8

**3** The velocity of a stone thrown up into the air is $v = 20 - 10t$ m s$^{-1}$, where $t$ is in seconds. The stone is thrown from a height of 1.5 m.

**a** Use the fact that $v = \frac{dh}{dt}$ to write a differential equation and find an expression for the height of the stone.

**b** Find the height after 2.5 s.

**4** A projectile moves so that its acceleration is given by $a = 2t + 7$ m s$^{-2}$, where $t$ is in seconds. The projectile is initially at rest at sea level. Use differential equations to find:

**a** the velocity of the projectile after 3 s

**b** the height of the projectile after 4 s.

### Problem solving

**5** The marginal profit of a maker of USB storage devices is given by

$$P(n) = \frac{200n + 4000}{n^2 + 40n + 1}$$

where $n$ is the number of devices made and $P(n)$ is the profit in dollars on the $n$th device. Use differential equations to find the total profit when $n$ devices are made, and hence the profit from 5000 devices.


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**6** The acceleration due to gravity on the surface of the moon is such that the speed of some green cheese thrown upward by an astronaut with a strong arm is given by:

$$v = 25 - 1.6t, \text{ where } v \text{ is in m s}^{-1} \text{ and } t \text{ is in seconds.}$$

Assume that the initial height is 0 m.

**a** Find a formula for the height after $t$ s.

**b** Find the height after 4 s.

**c** Use a quadratic equation to show that the cheese never reaches a height of 200 m.

**7** The rate of energy storage by a spring is given by:

$$\frac{dQ}{dx} = 200x \text{ J m}^{-1}$$

where $Q$ J (joules) is the energy stored and $x$ m is the compression of the spring. No energy is stored when there is no compression. Find a formula for the energy stored for a compression of $x$ m, and thus the energy stored for a compression of 12 cm.

**8** The marginal cost of production of items is the derivative of the cost of production. The marginal cost of producing large tents is given by

$$\frac{dC}{dn} = 0.002n^2 - n + 250$$

where $C$ is the cost in dollars of producing $n$ tents. It also costs \$20 000 to set up the production line before any tents are produced. Find a formula for the cost of producing $n$ tents.

**9** The rate of change of momentum of an object is given by the force acting on the object. An object with a momentum of 50 Ns is acted upon by a force given by $F = 8 - 2t$ N (newtons) for a period of 4 s. Find a formula for the momentum at $t$ s $(0 \le t \le 4)$.

# 9.04 Solving $\frac{dy}{dx} = g(y)$

When dealing with differential equations it is convenient to define the **differential**.

## The differential

For the function $f(x)$, the differential of $f$ is:

$$df = f'(x)\, dx$$

where $dx$ is a very small change in the value of $x$.

Thus, for a value $x_0$,

$$f(x_0 + dx) \approx f(x_0) + df = f(x_0) + f'(x_0)\, dx$$

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## Properties of differentials

The rules for differentials can easily be found using the rules for derivatives.
If $f$ and $g$ are functions and $c$ is a constant:

1 $d(c) = 0$

2 $d(c \times f) = c \times df$

3 $d(f \pm g) = df \pm dg$

4 $d(f \times g) = g \times df + f \times dg$

5 $d\left(\frac{f}{g}\right) = \frac{g \times df + f \times dg}{g^2}$

6 $d(x^n) = nx^{n-1}\,dx$

7 $d(e^x) = e^x\,dx$

8 $d(\ln|x|) = \frac{1}{x}dx$

9 $\lim\limits_{dx \to 0} \frac{df}{dx} = f'(x)$

In the previous section you solved differential equations of the form $\frac{dy}{dx} = f(x)$. In this section you will find solutions to differential equations of the form $\frac{dy}{dx} = g(y)$. The process of finding a solution to an equation like $\frac{dy}{dx} = 2y$ might not be obvious at first, however, rearranging the equation makes it easier to see.

| | |
|---|---|
| Write the equation. | $\frac{dy}{dx} = 2y$ |
| Use the definition of a differential. | $dy = 2y\,dx$ |
| Rearrange. | $\frac{1}{y}\,dy = 2\,dx$ |
| Integrate. | $\int \frac{1}{y}\,dy = \int 2\,dx$ |
| Evaluate. | $\ln\|y\| = 2x + c$ |
| For $y > 0$. | $y = e^{2x+c}$ |

This rule has wide application for problems involving exponential growth and decay.

It is usual to express the rule as follows:

$y = e^{2x+c} = e^{2x} \times e^c = e^c \times e^{2x}$

In this case, the initial condition is $y_0 = e^c$ and so the rule becomes $y = y_0\,e^{2x}$.

This rule can be applied to any equation of the type $\frac{dy}{dx} = g(y)$.

## Solving $\frac{dy}{dx} = ky$

The differential equation $\frac{dy}{dx} = ky$ has the general solution

$y = y_0\,e^{kx}$, where $y_0$ is the initial condition.

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## EXAMPLE 9

Solve each differential equation.

**a** $\frac{dy}{dx} = 5y$ **b** $\frac{dy}{dx} = 8\sqrt{y}$

## Solution

**a** Use the definition of a differential. $dy = 5y\,dx$

Divide both sides by $y$. $\frac{1}{y}dy = 5\,dx$

Integrate. $\int \frac{1}{y}dy = \int 5dx$

$\ln|y| = 5x + c$

Express in exponential form $y = e^{5x+c} = e^c \times e^{5x}$

Let $A = e^c$ and state the result. $y = Ae^{5x}$, where $A = e^c$ is the initial condition.

**b** Invert both sides of the equation. $\frac{dx}{dy} = \frac{1}{8\sqrt{y}}$

Use the definition of differentials. $dx = \frac{1}{8\sqrt{y}}dy$

Integrate. $\int dx = \int \frac{1}{8\sqrt{y}}dy$

$x = \frac{1}{8}\int y^{-\frac{1}{2}}dy$

Evaluate. $= \frac{1}{8} \times \frac{2}{1} \times y^{\frac{1}{2}} + c_1$

Isolate $y$. $\frac{1}{4}y^{\frac{1}{2}} = x - c_1$

The arbitrary constant $c = 4c_1$. $y^{\frac{1}{2}} = 4x - c$

Square both sides. $y = (4x - c)^2$

**Alternative method:** You can also use integration by substitution and the chain rule to solve $\frac{dy}{dx} = g(y)$.

For $\frac{dy}{dx} = 5y$ above, write the equation with the $y$ variable on LHS: $\frac{1}{y}\frac{dy}{dx} = 5$

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Now integrate both sides with respect to $x$.

$$\int \frac{1}{y}\frac{dy}{dx}\,dx = \int 5\,dx$$

Use the rule for integration by substitution.

$$\int \frac{1}{y}\,dy = \int 5\,dx$$

Then continue as shown in Example **9 a**.

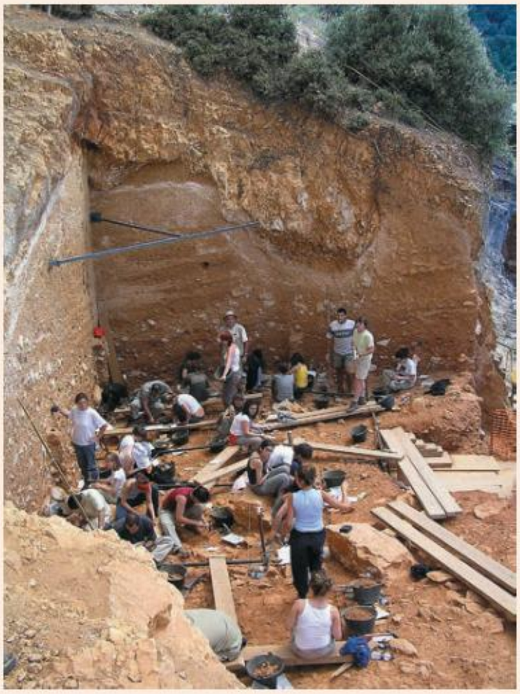
## EXAMPLE 10

The rate of decay of radioactive material depends on the amount of material there is. The half-life is the time taken for half the material present at any time to decay.

The rate of decay of the radioactive material can be written as $\frac{dM}{dt}$, where $M$ is the mass and $t$ is time in years. The half-life of carbon-14 is 5730 years.

In a charcoal sample taken from a midden, 0.05 g of carbon-14 was present. The archaeologist working on the site estimated that 0.27 g of carbon-14 was present at the time of the fire.

Use a differential equation to find the age of the midden.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### Solution

The radioactive material is decaying, so the rate of decay, $k < 0$.

Write the differential equation. $\frac{dM}{dt} = -kM$

Use the rule for the general solution. $M = M_0e^{-kt}$

Use the half-life to calculate $k$. At $t = 5730$, $M = \frac{1}{2}M_0$

Substitute for known values. $\frac{1}{2}M_0 = M_0e^{-5730k}$


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| | |
|---|---|
| Simplify. | $\frac{1}{2} = e^{-5730k}$ |
| Change to log form. | $\ln\left(\frac{1}{2}\right) = -5730k$ |
| Reverse and apply log laws. | $-5730k = -\ln(2)$ |
| Isolate $k$. | $k = \frac{\ln(2)}{5730}$ |
| State the rule for $M$. | $M = M_0e^{-kt}$, where $k = \frac{\ln(2)}{5730}$ |
| 0.27 g of carbon-14 is present initially. | $M = 0.27e^{-kt}$, $k = \frac{\ln(2)}{5730}$ |
| The current value of $M = 0.05$ g. | $0.05 = 0.27e^{-kt}$ |
| Simplify. | $e^{-kt} = \frac{0.05}{0.27}$ |
| Express as a logarithm. | $-kt = \ln\left(\frac{0.05}{0.27}\right)$ |
| Rearrange. | $t = -\ln\left(\frac{0.05}{0.27}\right) \times \frac{1}{k}$ |
| Substitute for $k$ and use the log laws. | $= -\ln\left(\frac{0.05}{0.27}\right) \times \frac{5730}{\ln(2)}$ |
| | $= 13\ 940.857\ldots$ |
| Round off and state the result. | The midden is about 13 940 years old. |

A number of natural situations can be modelled as differential equations, where the rate of change is a linear function of the dependent variable. They can arise in situations where the rate of change of a quantity depends on both the quantity and some other constant. In the case of cooling or heating curves, the constant is derived from the temperature of the surroundings and the characteristics of the object.

## EXAMPLE 11

A piece of very hot iron is dropped into some cold water, which is at 15°C. It is known that the rate of cooling is proportional to the temperature difference between the iron and the water, i.e. $\frac{dT}{dt} = -k(T - 15)$, where $T$ is the temperature of the iron and $t$ is the time in minutes.

After 10 minutes, the iron has cooled to 45°C, and after 13 minutes it is 30°C.
Use a differential equation to find the initial temperature of the iron.

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## Solution

| | |
|---|---|
| Write the differential equation. | $\frac{dT}{dt} = -k(T - 15)$ |
| Use the definition of differentials. | $\frac{1}{T-15}\,dT = -k\,dt$ |
| Integrate. | $\int \frac{1}{T-15}\,dT = \int -k\,dt$ |
| Evaluate. | $\ln\lvert T - 15\rvert = -kt + c$ |
| Change to exponential form. | $T - 15 = e^{-kt + c}$ |
| Isolate $T$. | $T = 15 + Ae^{-kt}$ |
| Substitute for known values. | $45 = 15 + Ae^{-10k} \Rightarrow Ae^{-10k} = 30$ |
| | $30 = 15 + Ae^{-13k} \Rightarrow Ae^{-13k} = 15$ |
| Divide the equations. | $\frac{Ae^{-10k}}{Ae^{-13k}} = \frac{30}{15}$ |
| Simplify. | $e^{3k} = 2$ |
| Write in log form. | $3k = \ln(2)$ |
| Isolate $k$. | $k = \frac{1}{3}\ln(2)$ |
| Substitute for $k$ for $t = 10$. | $Ae^{-\frac{1}{3}\ln(2)\times 10} = 30$ |
| Simplify. | $A\left(e^{\ln(2)}\right)^{-\frac{10}{3}} = 30$ |
| Use the log laws. | $A \times 2^{-\frac{10}{3}} = 30$ |
| Isolate $A$. | $A = 30 \times 2^{\frac{10}{3}}$ |
| Write the model. | $T = 15 + Ae^{-kt}$ |
| Substitute. | $T = 15 + 30 \times 2^{\frac{10}{3}} \times e^{-\frac{1}{3}\ln(2)t}$ |
| Simplify using log laws. | $= 15 + 30 \times 2^{\frac{10-t}{3}}$ |
| Find $T$ when $t = 0$. | $T = 15 + 30 \times 2^{\frac{10}{3}}$ |
| | $= 317.381\ldots$ |
| Round off and state the result. | The initial temperature was about 317.4°C. |


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## INVESTIGATION

### MERCURY POLLUTION

Work in groups of 3 to 4 for this investigation.

A lake of volume 107 $m^3$ has been polluted with approximately 200 tonnes of mercury. River water flows into the lake at the rate of 20 $MLh^{-1}$ and leaves at the same rate, helping to flush the mercury out of the lake (but into the ocean further down). The river contains no mercury and, as it enters, mixes thoroughly with the lake water.


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1 Work out a differential equation for the amount of mercury remaining in the lake, expressed as a concentration of parts per million.

2 Solve the equation to find the time taken for the mercury to decrease to safe levels. You will need to consult the library to find an acceptable level of mercury pollution.

3 Discuss your findings as a class group, focusing on the implications of polluting a largely closed environment. Consider the implications of flushing pollutants into the ocean, given that ocean currents leave the continental shelf largely undisturbed.

## Exercise 9.04 Solving $\frac{dy}{dx} = g(y)$

Example 9

**1** Find a general solution for each differential equation.

**a** $\frac{dy}{dx} = -3y$ **b** $\frac{dx}{dt} = 4x$ **c** $\frac{dy}{dx} = -y$

**d** $\frac{dw}{dm} = 0.01w$ **e** $\frac{dz}{da} = -0.03z$ **f** $\frac{dv}{dt} = -5v$

**2** Find a general solution for each differential equation.

**a** $\frac{dy}{dx} = 2y + 5$ **b** $\frac{dx}{dt} = 7 - 4x$ **c** $\frac{dy}{dx} = 4 - 0.3y$

**d** $\frac{dw}{dm} = 7.1 - 11w$ **e** $\frac{dy}{dx} = 1 - 0.03y$ **f** $\frac{dv}{dt} = -5v - 4$

**3** Find a general solution for each differential equation.

**a** $\frac{dy}{dx} = 6\sqrt{y}$ **b** $\frac{dy}{dt} = 5y + 1$ **c** $\frac{dy}{dx} = 1 + y^2$

**d** $\frac{dy}{dx} = \sqrt{9 - y^2}$ **e** $\frac{dy}{dx} = \frac{3}{y^2}$ **f** $\frac{dy}{dx} = \frac{y^2}{3}$

**4** Show that $y = e^{2t} - \frac{3}{2}$ is a solution to $\frac{dy}{dt} = 2y + 3$.

**5** Solve $\frac{dy}{dt} = 8e^{2t}$ if $y(0) = 4$.

Example 10

**6** The rate of decay of a radioactive substance is proportional to the amount of material there is. The half-life of the substance is known to be 1600 years.

**a** A sample of the substance has a mass of 100 mg. Find a formula for the mass of the sample that remains after $t$ years.

**b** Find the mass after 1800 years.

**c** When will the mass be reduced to 20 mg?

Example 11

**7** The rate at which a hot object cools is approximately proportional to the difference between its temperature and the temperature of the surroundings. A cup of black coffee is made with boiling water. After 2 minutes, its temperature has dropped to 80°C. The room is at 25°C. Use a differential equation to find how much longer it will take to drop to 50°C.

## Problem solving

**8** As the water in a town water supply dam evaporates, the volume of water and the surface area decrease, so the rate of evaporation slows. The rate of evaporation is given by the expression $E = 0.023V$ ML/day. The rate of evaporation is the same as the rate of decrease of the volume. If the initial volume is 18.5 ML and there is no rain for 6 months, what would the volume decrease to, even if no water was used?

**9** Bacteria grow by dividing cells, so the rate of growth of a bacterial colony is proportional to the number of bacreria present. In a deep wound, 20 anaerobic Clostridium bacteria are initially present. After 2 hours, the number has grown to 90. Use a differential equation to determine the number present after time $t$. When the number present reaches 1000, the patient begins to develop symptoms of nerve poisoning from the toxins produced by the bacteria. How long does this take?

**10** The rate of loss of water from a rainwater tank with a leak in the side depends on the pressure at the hole. The pressure depends on the height of water above the leak. A cylindrical tank, 2 m high, has a leak 20 cm from the top, which has not been fixed because it is only a short distance from the top. At 5 h after the tank is filled, the level has dropped by 15 cm. Use a differential equation to model the height of water above the hole as a function of time.


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**11** Antibiotic absorption from the blood depends on its concentration in the blood. An antibiotic, needing to be between 5 ppm and 20 ppm (parts per million), was injected intravenously at 10 a.m. to give a concentration of 20 ppm. After 1 hour, the level had fallen to 16 ppm. Use a differential equation to find the latest time at which a follow-up injection should be given.

**12** The concentration of chlorine present in a swimming pool changes proportionally to the amount present. The concentration at 8 a.m. was measured at 5 ppm. At 9 a.m., it had fallen to 3.8 ppm. Use a differential equation to find when more chlorine needs to be added to ensure that the concentration remains above 1 ppm.

**13** A furnace is switched on at 9 a.m. Heat is supplied at a constant rate; but as the furnace temperature increases, heat is lost at a rate determined by the difference between its temperature and the surrounding temperature. The temperature of both was 20°C at 9 a.m. At 9:30 a.m., the furnace was at 200°C and by 10 a.m., it was at 350°C. The minimum operating temperature is 800°C. Use a differential equation to find the time when the furnace is ready for use and the highest temperature it could be expected to reach.

**14** A dust particle of mass $m$ falls according to the equation $m\frac{dv}{dt} = mg - kv$, where $m$ is the mass, $g$ is the acceleration due to gravity, $v$ is the velocity and $k$ is a constant. Assuming that the particle falls from rest, solve the equation and comment on the solution.

**15** Millikan's oil-drop experiment is a famous experiment to determine the charge on an electron. In order to perform the calculations, it is necessary to solve the equation that describes the speed of small drops of oil falling under gravity. This equation is:

$$\frac{dv}{dt} = 9.8 - \frac{Cv}{D^2} \text{ m s}^{-1}$$

where $C = 3.1 \times 10^{-6}$ m$^2$ s$^{-1}$ and the diameter is $D$ m. The oil drop is assumed to start from rest. Solve the differential equation and express the eventual (terminal) velocity in terms of $D$.

**16** When a capacitor is charged through a resistor, the current $I$, the charge $Q$ on the capacitor and the voltage $V$ across the resistor are related by $Q = CV$ and $V = IR$. This gives

$$E = \frac{dQ}{dt}R + \frac{Q}{C}$$

where $E$ is the voltage of the battery. A capacitor of capacitance 5 μF is charged through a resistance of 8000 Ω from a 12 V supply. Find the charge, voltage and current at time $t$.

**17** Two cups of coffee are made at the same time with boiling water. Milk is immediately added to one cup, but the other is left black. When milk is added, the temperature drops by 10°C, and it drops another 10°C in the next 2 minutes. Assuming that both cups of coffee have the same characteristics for cooling, find the temperatures of both 5 minutes after they are made, given that the room is at 25°C. Can you refine your model to take account of the fact that different people add different amounts of milk? State any assumptions you make in this refinement.

WS

Differential equations

WS

Differential equations and exponentials

Differential equations

# 9.05 Solving $\frac{dy}{dx} = f(x)g(y)$ with separable variables

Differential equations of the form $\frac{dy}{dx} = f(x)g(y)$ have separable variables. You can separate the independent variables ($x$ terms) to one side of the equation and the dependent variables ($y$ terms) to the other.

You can divide both sides by $g(y)$ as $g(y) \neq 0$.

This gives $$\frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

Using differentials $\frac{1}{g(y)}\, dy = f(x)\, dx$

Integrating, we get $\int \frac{1}{g(y)}\, dy = \int f(x)\, dx$

OR

Using integration by substitution,

$$\int \frac{1}{g(y)}\frac{dy}{dx}\, dx = \int \frac{1}{g(y)}\, dy = \int f(x)\, dx$$

### Solving $\frac{dy}{dx} = f(x)\, g(y)$

A differential equation with separable variables $\frac{dy}{dx} = f(x)g(y)$ can be solved using $\int \frac{1}{g(y)}\, dy = \int f(x)\, dx + c$, where $c$ is a constant of integration.

Following the integration described above, a relationship between $x$ and $y$ is obtained. In some cases you can make $y$ the subject.

### EXAMPLE 12

Find all solutions to each differential equation.

**a** $\frac{dy}{dx} = \frac{2x+3}{y^2+5}$

**b** $\frac{dy}{dx} = (y+4)(2x-3)$

**c** $e^y\frac{dy}{dx} - x - 6x^2 = 0$ if $y(1) = 3$

**d** $\frac{dy}{dx} = \frac{\cos(x)}{1-y^2}$ if $y\left(\frac{\pi}{2}\right) = 2$


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## Solution

**a** Write the equation. $\frac{dy}{dx} = \frac{2x+3}{y^2+5}$

Separate the variables. $(y^2 + 5)\ dy = (2x + 3)\ dx$

Integrate. $\int (y^2+5)\,dy = \int (2x+3)\,dx$

$\frac{y^3}{3} + 5y = x^2 + 3x + c$

State as an implicit equation. $\frac{y^3}{3} + 5y - x^2 - 3x = c$

**b** Separate the variables. $\frac{1}{y+4}\frac{dy}{dx} = (2x - 3)$

Integrate. $\int \frac{1}{y+4}\,dy = \int (2x-3)\,dx$

$\ln |y + 4| = x^2 - 3x + c$

You can assume that $y + 4 \geq 0$. $\ln (y + 4) = x^2 - 3x + c$

Express in exponential form. $y + 4 = e^{x^2-3x+c}$

Isolate $y$ and simplify. $y = e^{x(x-3)} \times e^c - 4$

$y = Ae^{x(x-3)} - 4$, where $A = e^c$

**c** Separate the variables. $e^y \frac{dy}{dx} = 6x^2 + x$

Integrate by substitution. $\int e^y\,dy = \int (6x^2 + x)\,dx$

$e^y = 2x^3 + \frac{x^2}{2} + c$

Use the initial condition. $e^3 = 2 \times 1^3 + \frac{1^2}{2} + c$

Solve. $c = e^3 - \frac{5}{2}$

Write the equation. $e^y = 2x^3 + \frac{x^2}{2} + e^3 - \frac{5}{2}$

Use logarithms. $y = \ln\left(2x^3 + \frac{x^2}{2} + e^3 - \frac{5}{2}\right)$

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**d** Separate the variables. $(1 - y^2)\, dy = \cos(x)\, dx$

Integrate. $\int(1 - y^2)dy = \int \cos(x)dx$

$y - \frac{1}{3}y^3 = \sin(x) + c$

Use the initial condition. $2 - \frac{1}{3} \times 2^3 = \sin\left(\frac{\pi}{2}\right) + c$

$-1\frac{2}{3} = c$

State the result. $y - \frac{1}{3}y^3 - \sin(x) + 1\frac{2}{3} = 0$

## EXAMPLE 13

Sales for an energy drink were about 50 000 cans per week. After 4 weeks of a TV advertising campaign the sales increased to 90 000 cans per week. The increase in sales was proportional to both the current weekly sales and the length of the ad campaign. Estimate the sales at the end of the campaign in week 10 and state any problems with the estimate.

### Solution

| | |
|---|---|
| Choose variables. | Let $N$ = weekly sales and $t$ = time in weeks |
| Write an equation for the sales increase. | $\frac{dN}{dt} = kNt$, where $k$ is a constant. |
| Separate the variables. | $\frac{1}{N}\frac{dN}{dt} = kt$ |
| Integrate with respect to $t$. | $\int \frac{1}{N}\frac{dN}{dt}dt = \int ktdt$ |
| Use integration by substitution. | $\int \frac{1}{N}dN = \int ktdt$ |
| Integrate. | $\ln(N) = \frac{1}{2}kt^2 + c$ |
| Write in exponential form. | $N = e^{\frac{1}{2}kt^2 + c} = e^c e^{\frac{1}{2}kt^2} = Ae^{\frac{1}{2}kt^2}$ |
| Substitute $t = 0$ and solve. | $50\,000 = Ae^0$, so $A = 50\,000$ |
| Substitute $t = 4$ and solve. | $90\,000 = 50\,000\, e^{8k}$ |
| Solve for $k$ and keep 'exact' value. | $k = \frac{1}{8}\ln(1.8) = 0.0734\ldots$ |

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| | |
|---|---|
| Substitute $t = 10$. | $N(10) = 50\,000\, e^{50k} = 1\,969\,804.6\ldots$ |
| State your answer. | The sales after 10 weeks should be nearly 2 million. |
| State some likely problems. | The equation can only apply for a limited time because sales are limited. Competitors will probably take action to protect their market share. |

## Exercise 9.05 Solving $\frac{dy}{dx} = f(x)g(y)$ with separable variables

Example 12

**1** Find all solutions to each differential equation.

**a** $\frac{dy}{dx} = 3 - 2y$

**b** $\frac{dy}{dx} = \frac{x^2}{y}$

**c** $\frac{dy}{dx} = (4y - 7)(3x + 1)$

**d** $\frac{dy}{dx} = \frac{5-4x}{3y^2-5}$

**e** $\frac{dy}{dx} = 6x(y-1)^{\frac{2}{3}}$

**f** $\frac{dy}{dx} = x + xy$

**g** $\frac{dy}{dx} = xe^{x+y}$

**h** $\frac{dy}{dx} + e^{3x}y^2 = y^2$

**i** $\frac{dy}{dx} = \frac{x^2}{y^2+1}$

**j** $\frac{dy}{dx} = \frac{(1+x^2)(1+y^2)}{xy}$

**k** $\frac{dy}{dx} = \frac{x-e^{-x}}{y+e^{y}}$

**l** $\frac{dy}{dx} = e^{3x}\, e^{2y}$

**2** Solve each differential equation.

**a** $\frac{dy}{dx} + y = 9x^2y$ if $y(0) = 1$

**b** $\frac{dy}{dx} = \frac{3x^2+4x+3}{2y-2}$ if $y(0) = 1$

**c** $\frac{dy}{dx} = 6x^2(1 + y^2)$ if $y(0) = 0$

**d** $\frac{dy}{dx} = \frac{4x-x^3}{4+y^3}$ if $y(0) = 1$

**e** $\frac{dy}{dx} = \frac{x^2}{y}$ if $y(0) = -4$

**f** $\frac{dy}{dx} = \frac{x}{\sqrt{x^2+16}}$ if $y(3) = 7$

**g** $\frac{dy}{dx} = \frac{4-2x}{3y^2-5}$ if $y(2) = 3$

**h** $\frac{dy}{dx} - \frac{e^{-x}-e^{x}}{3+4y} = 0$ if $y(0) = 1$

**i** $\frac{1}{y^2}\frac{dy}{dx} + x^2 = 0$ if $y(0) = 2$

**j** $\frac{dy}{dx} = \frac{1-2x}{y}$ if $y(1) = -2$

**k** $y\frac{dy}{dx} + xe^{x^2} = 0$ if $y(0) = -2$

**l** $x\frac{dy}{dx} = y^2 - 2y$ if $y(1) = 1$

Example 13

**3** Sales of strawberries after some incidents of sabotage dropped dramatically. After a few weeks, sales at one supermarket started to pick up. The number of punnets sold each week was proportional to both time and the number sold the previous week.
3 weeks after the last incident, 80 punnets were sold. After another 2 weeks, sales had increased to 140 punnets. Estimate the sales in another 2, 4 and 6 weeks.

## Problem solving

**4** In a town of 5000 people, no one has immunity to a new strain of the flu. Anyone who comes in contact with someone who has been infected is certain of becoming infected. If there are initially $N$ people infected with the flu, then the number of contacts between infected and uninfected people in a day is given by $N(5000 - N)$. The rate of change of the number of infected people is given by:

$$\frac{dN}{dt} = 0.0001 \times N(5000 - N), \text{ where } t \text{ is in days.}$$

If one person has the flu to begin with, find how many people have been infected after $t$ days. Use your solution to find out how many people are infected after 14 days.

**5** In the past, victims of some types of crime rarely told police but now, the number reported to police each year has now started to increase. This has been proportional to time and the number reported the previous year. 18 500 such crimes were reported in 2014 across Australia and by 2017 it had increased to 31 300. Predict the number in 2025 and state any problems with the prediction.

**6** 5000 litres of water in a rainwater tank is treated with 50 grams of a chemical to kill Guardia and other bacteria. The house uses about 500 litres a day, but summer rain is adding about 1000 litres each day. The rainwater mixes completely with the water in the tank. Find the amount of chemical left in the tank after 10 days.

**7** Plagues of mice, locusts and other pests depend on the breeding rate and the available space and food. The available space and food will determine the maximum number, $M$, of plague animals that can be supported. The growth of the population will depend jointly on the population and the room for growth measured by $M - N$, where $N$ is the population, so

$$\frac{dN}{dt} = kN(M - N),$$

where $k$ is determined by the breeding rate. If the initial population is $n$, solve the differential equation to show that $N = \dfrac{RMe^{kMt}}{Re^{kPt} + 1}$, where $R = \dfrac{n}{M - n}$.

Alamy Stock Photo/Alan Tunnicliffe

**8** For a mouse plague as described in the previous question, $k = 3 \times 10^{-6}$ and $M = 10\ 000$. Time is measured in days. Determine the population after 1, 2, 3, 5, 10 and 20 weeks if the initial population of mice is 100.

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**9** It is known that the air resistance for the movement of very small objects is proportional to the velocity. However, for larger objects the air resistance is proportional to the square of the velocity, leading to the differential equation

$$m\frac{dv}{dt} = mg - kv^2$$

where $m$ is the mass, $v$ is the velocity, $g$ is the acceleration due to gravity and $k$ is a constant related to the shape and size of the object. Show that the solution to the differential equation is

$$v = \sqrt{\frac{gm}{k}}\left(\frac{e^{ct}-1}{e^{ct}+1}\right), \text{ where } c = 2\sqrt{\frac{kg}{m}}$$

A skydiver has a terminal velocity of 50 $\text{ms}^{-1}$ before opening her parachute. If her mass is 60 kg and $g = 9.8\ \text{ms}^{-2}$, find $k$ and hence calculate her speed after 2, 4 and 6 seconds of free-fall.

## 9.06 Slope fields

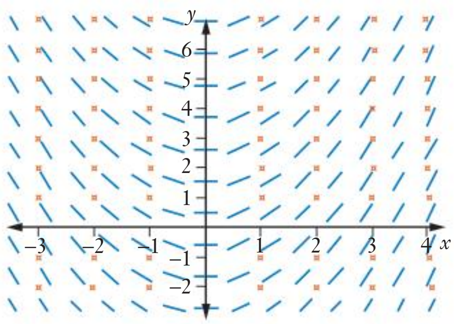
In this section, you will learn how to visualise a differential equation. Consider the equation

$$\frac{dy}{dx} = x$$

This means that $\frac{dy}{dx}$ is the slope of the tangent to the solution of $y$ at point $(x, y)$. Also,

- if $\frac{dy}{dx} = 0$, the tangent line is horizontal
- if $\frac{dy}{dx} > 0$, the tangent line goes up
- if $\frac{dy}{dx} < 0$, the tangent line goes down
- the value of $\left|\frac{dy}{dx}\right|$ determines the steepness of the tangent line

Any solution to the differential equation $\frac{dy}{dx} = x$ has the property that the slope at any point is equal to the $x$-coordinate at that point, as shown in the diagram.




The derivatives are shown as short lines or arrows, with the slope equal to the derivative.

If the solution goes through the point (0, 0), its slope at this point is 0. Likewise if the solution goes through a point with $x = 1$, its slope there is 1 and if it goes through a point where $x = -2$, the slope is also $-2$.

The diagram is called the **slope field**, **gradient field** or **direction field** for the differential equation $\frac{dy}{dx} = x$.

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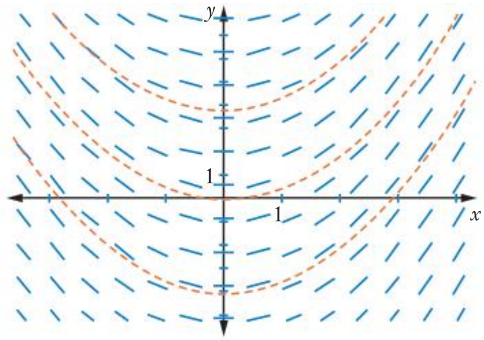
As with all differential equations, there is no unique solution to $\frac{dy}{dx} = x$, but instead a family of solutions. Particular solutions depend on the initial condition. The diagram on the right shows some of the solutions to $\frac{dy}{dx} = x$.




## EXAMPLE 14

For the differential equation $\frac{dy}{dx} = \frac{1}{2}y$:

**a** draw the slope field

**b** indicate the family of curves that are solutions

### Solution

**a** The differential equation relates the slope of the solution and the value of $y$.

$\frac{dy}{dx} = \frac{1}{2}y$

So, for example, when $y = 2$, $\frac{dy}{dx} = 1$

Draw up a table of values of $y$ and $\frac{dy}{dx}$.

| $y$ | –4 | –3 | –2 | –1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| $\frac{dy}{dx}$ | –2 | $-\frac{3}{2}$ | –1 | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 1 | $\frac{3}{2}$ | 2 |

Interpret the table of values.

The table of values shows that when $y = -4$, the slope of the solution must be –2. This is true for all solutions where $y = -4$.

Likewise, when $y = -3$, the slope of the solution must be $-\frac{3}{2}$ for all solutions where $y = -3$.

The same reasoning applies for all other values in the table.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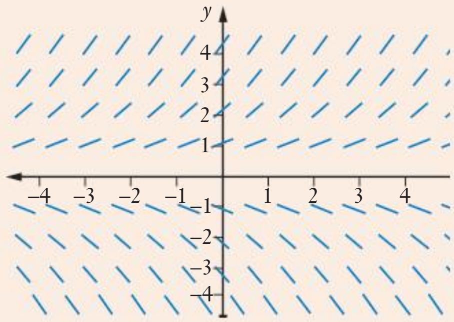
Draw the slope field.

Draw a set of axes for $x$ and $y$.

When $y = -4$, draw a small line with a slope of $-2$.

When $y = -3$, draw a small line with a slope of $-\frac{3}{2}$.

Repeat this procedure for all values in the table.



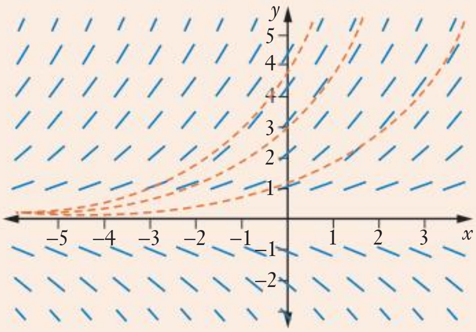

**b** Examine the gradients.

As $y \to 0$, $\frac{dy}{dx} \to 0$, so the $x$-axis is an asymptote.

As $y \to \infty$, $\frac{dy}{dx} \to \infty$.

As $y \to -\infty$, $\frac{dy}{dx} \to -\infty$.

Use the lines drawn for the slopes as a guide to draw in some of the family of curves that are solutions.




Comment on the result.

The family of curves appear to be exponential.

By examining a differential equation, you should be able to identify the key features of its corresponding slope field.

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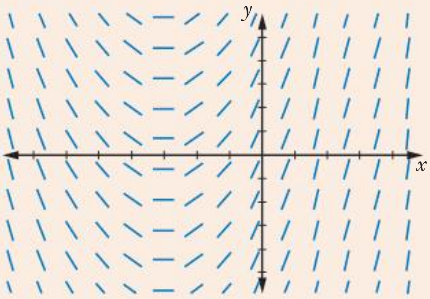
## EXAMPLE 15

Match each differential equation to its slope field.

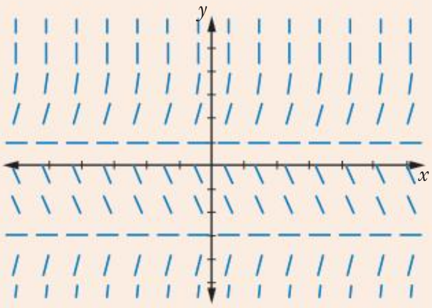
**a** $\frac{dy}{dx} = (y - 1)(y + 3)$

**b** $\frac{dy}{dx} = x + 3$

**A**




**B**




## Solution

**a** Describe the behaviour of the derivative.

When $y = 1$ or $y = -3$, then $\frac{dy}{dx} = 0$, so horizontal asymptotes exist at $y = 1$ and $y = -3$.

When $y < -3$, then $\frac{dy}{dx} > 0$.

When $1 < y < -3$, then $\frac{dy}{dx} < 0$.

When $y > 1$, then $\frac{dy}{dx} > 0$.

Match the behaviour of the derivative with the slope field.

Slope field **B** matches the differential equation $\frac{dy}{dx} = (y - 1)(y + 3)$.

**b** Describe the behaviour of the derivative.

When $y < -3$, then $\frac{dy}{dx} < 0$.

When $y = -3$, then $\frac{dy}{dx} = 0$.

When $y > -3$, then $\frac{dy}{dx} > 0$.

Match the behaviour of the derivative with the slope field.

Slope field **A** matches the differential equation $\frac{dy}{dx} = x + 3$.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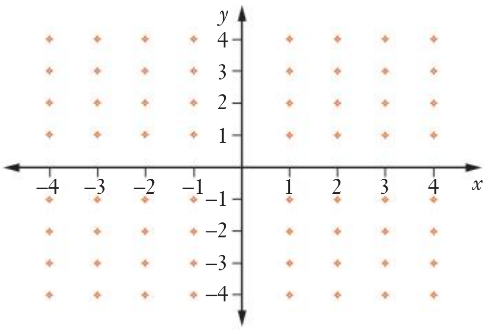
## Exercise 9.06 Slope fields

Copy and complete this grid to draw the slope field for each question.




Example 14

**1** For the differential equation $\frac{dy}{dx} = x - 2$, draw:

**a** the slope field

**b** the solution curve for the initial condition $y = 1$ when $x = 1$

**c** the solution curve for the initial condition $y = 4$ when $x = -2$

**2** For the differential equation $\frac{dy}{dx} = 2y$, draw:

**a** the slope field

**b** the solution curve for the initial condition $y = 1$ when $x = 1$

**c** the solution curve for the initial condition $y = 1$ when $x = -1$

**3** Draw the slope field for the differential equation $\frac{dy}{dx} = x + y$ and draw the solution curves given the initial conditions $y(0) = 0$ and $y(2) = -2$.

**4** Draw the slope field for the differential equation $\frac{dy}{dx} = y - 2$ and draw the solution curves given the initial conditions $y(0) = 0$ and $y(2) = -2$.

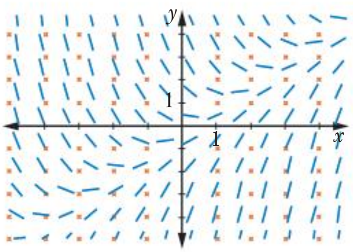
**5** Draw the slope field for the differential equation $\frac{dy}{dx} = y^2 - 4$ and draw the solution curves given the initial conditions $y(0) = 0$ and $y(2) = -2$.

Example 15

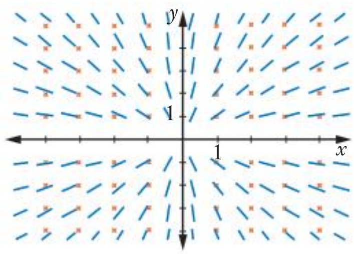
**6** Match each differential equation with its slope field.

**a** $\frac{dy}{dx} = x + 2$ **b** $\frac{dy}{dx} = 3 - y$ **c** $\frac{dy}{dx} = x - y$ **d** $\frac{dy}{dx} = \frac{y}{x}$

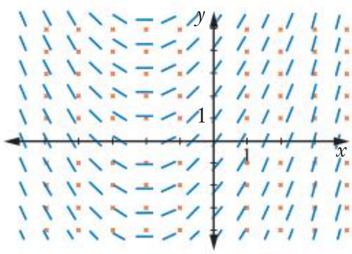
**A**



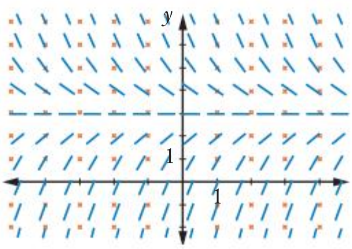

**B**




**C**



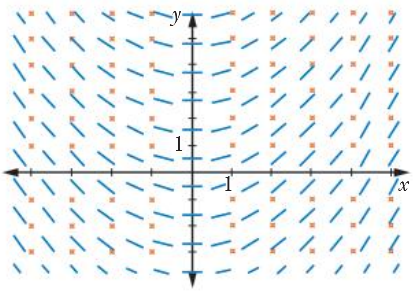

**D**



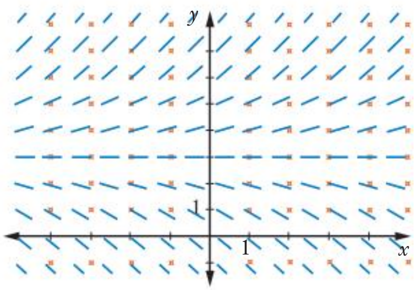
**7** Match each differential equation with its slope field.

**a** $\frac{dy}{dx} = 2x$ **b** $\frac{dy}{dx} = 3y - 1$ **c** $\frac{dy}{dx} = \frac{1}{2}x$ **d** $\frac{dy}{dx} = \frac{y}{3} - 1$

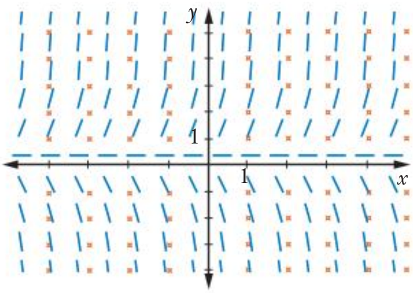
**A**



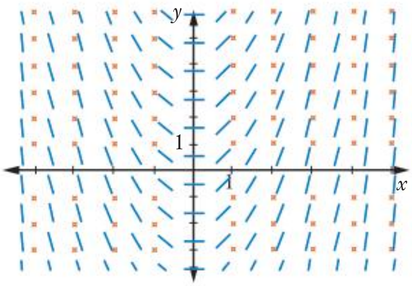

**B**




**C**




**D**

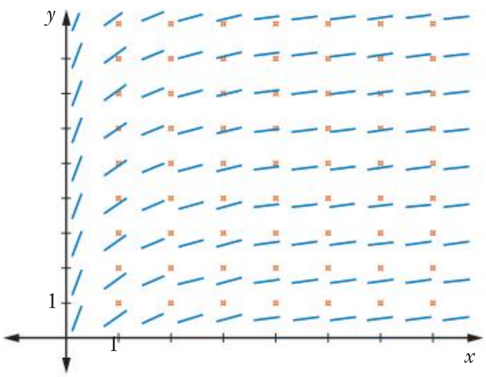



## Problem solving

**8** Draw the slope field for the differential equation $\frac{dy}{dx} = xy$ and draw the solution curves given the initial conditions $y(1) = -2$ and $y(-2) = 4$.

**9** Draw the slope field for the differential equation $\frac{dy}{dx} = \frac{x}{y}$ and draw the solution curves given the initial conditions $y(0) = 0$ and $y(2) = -2$.

**10** The slope field for a particular differential equation is shown.




Which of the following could be the solution to the differential equation?

**A** $y = 2x^2$

**B** $y = \ln|x|$

**C** $y = 2e^{-x}$

**D** $y = \sin(x)$

**E** $y = 3e^x$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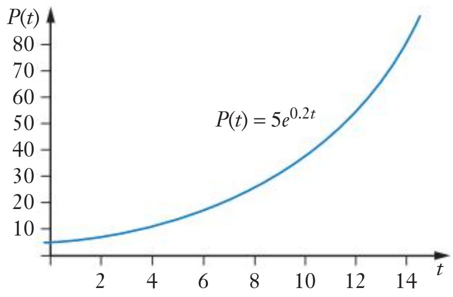
# 9.07 The logistic equation

You have previously examined exponential models of growth and decay. For example, it is possible to use an exponential model for population growth. However, there are limitations to using this type of model.

The graph on the right shows the population modelled by the exponential equation $P(t) = 5e^{0.2t}$. You can see that with this model, the growth of the population increases as time increases and is limitless.




Clearly, this type of model would not be very useful when dealing with the growth of a population of animals or plants as it assumes that the resources required for growth, such as food, water and space, are limitless.

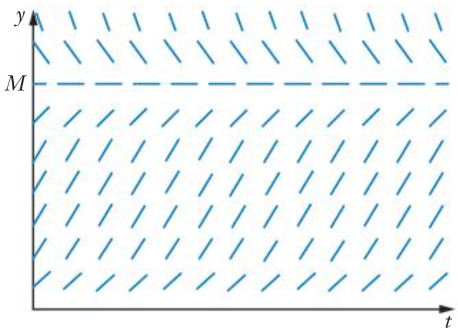
To overcome this issue, an alternative model was developed by the mathematical biologist Pierre-François Verhulst in the 1840s. Verhulst's model used a rate of population growth that is proportional to the rate of breeding $kP$, where $P$ is the current population. As the population grows, the growth is also proportional to $M - P$, where $M$ is the maximum population that can be sustained, also known as the **carrying capacity**.

This gives the following differential equation describing the rate of growth of a biological population:

$$\frac{dP}{dt} = kP(M - P).$$

The maximum population ($M$) is taken to be constant, so replacing $N$ with the variable $y$ gives $\frac{dy}{dt} = ky(M - y)$. This is known as the **logistic model** for population growth.

The slope field for the differential equation $\frac{dy}{dt} = ky(M - y)$ is shown.

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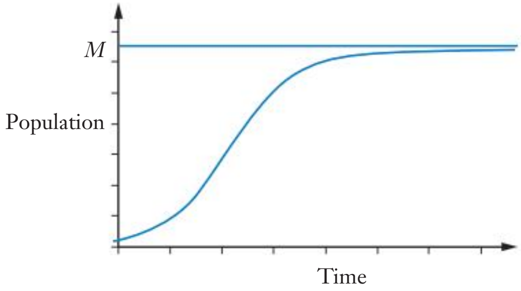
A typical graph that results from the logistic model is shown below. You can see that it is a particular solution curve for the slope field shown on the previous page.




This model shows that initially the population rises almost exponentially, but eventually reaches a limiting value.

## Logistic model of growth

The **logistic model** of growth is given by $\frac{dy}{dt} = ky(M - y)$, where the rate of change of the variable is proportional to both the variable and the difference between the maximum value and the variable.

## EXAMPLE 16

In a town of 5000 people, 10 people have a new strain of the flu. The spread of the flu through the town is given by the differential equation $\frac{dP}{dt} = 0.000\,24P(M - P)$, where $t$ is in weeks and $P$ is the number of people infected.

**a** Find an expression for the number of people infected.

**b** Calculate the number of infected people after 2 weeks and 4 weeks.

### Solution

**a** Write the differential equation.

$$\frac{dP}{dt} = 0.000\,24P(5000 - P)$$

Separate the variables.

$$\frac{1}{P(5000-P)}\frac{dP}{dt} = 0.000\,24$$

Integrate.

$$\int \frac{1}{P(5000-P)}dP = \int 0.000\,24\,dt$$

Write $\frac{1}{P(5000-P)}$ as partial fractions.

$$\frac{1}{P(5000-P)} = \frac{A}{P} + \frac{B}{5000-P}$$

$$= \frac{A(5000-P)+BP}{P(5000-P)}$$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Equate the denominators. $A(5000 - P) + BP = 1$

$$\text{Let } P = 0 \Rightarrow A = \frac{1}{5000}$$

$$\text{Let } P = 5000 \Rightarrow B = \frac{1}{5000}$$

Substitute for $A$ and $B$.

$$\int 0.000\,24\,dt = \int\left[\frac{A}{P} + \frac{B}{5000 - P}\right]dP$$

$$= \int\left[\frac{1}{5000P} + \frac{1}{5000(5000 - P)}\right]dP$$

$$= \frac{1}{5000}\int\left[\frac{1}{P} + \frac{1}{(5000 - P)}\right]dP$$

$$\int\left[\frac{1}{P} + \frac{1}{(5000 - P)}\right]dP = 5000\int 0.000\,24\,dt$$

Find the integrals. $\ln(P) - \ln(5000 - P) = 5000 \times 0.000\,24t + c$

Simplify. $\ln\left(\frac{P}{5000 - P}\right) = 1.2t + c$

Change to exponential form. $\frac{P}{5000 - P} = Ae^{1.2t}$

Isolate $P$. $P = \frac{5000Ae^{1.2t}}{1 + Ae^{1.2t}}$

Multiply top and bottom by $e^{-1.2t}$. $= \frac{5000A}{e^{-1.2t} + A}$

Use the initial conditions, $t = 0$ and $P = 10$. $10 = \frac{5000A}{1 + A}$

Solve for $A$. $A = \frac{10}{5000 - 10} = \frac{1}{499}$

State the logistic model. $P = \frac{5000A}{A + e^{-1.2t}}$, where $A = \frac{1}{499}$

**b** Substitute for $t = 2$. $P(2) = \frac{5000 \times \frac{1}{499}}{\frac{1}{499} + e^{-2.4}}$

Evaluate. $= 108.0654\ldots$

Substitute for $t = 4$. $P(4) = 979.116\ldots$

Round off and state the result. The number of people infected with the flu after 2 weeks is about 108 and about 979 after 4 weeks.

You can use the process shown in the previous example to develop the general solution to the differential logistic equation $\frac{dy}{dt} = ky(M - y)$.

## Solution to the logistic equation

The general solution for the differential logistic equation $\frac{dy}{dt} = ky(M - y)$ is:

$$y = \frac{MA}{A + e^{-kMt}} = \frac{My_0}{y_0 + (M - y_0)e^{-kMt}}$$

where $A = \frac{y_0}{M - y_0}$ and $y_0$ is the initial value.

## EXAMPLE 17

The maximum number of possums that can be supported a national park is estimated to be 3000. When first counted, there were 1200 possums. A second count 4 months later showed that the number had grown to 1300. Use the general solution for the logistic model to predict the number of possums in the park after 1 year.

### Solution

Assign the variable for population. Let the population $= P$

State the parameters. $M = 3000, P_0 = 1200$

State the model using $P$.

$$P = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}$$

Substitute for $M$ and $P_0$.

$$= \frac{3000 \times 1200}{1200 + (3000 - 1200)e^{-3000kt}}$$

$P(4) = 1300$.

$$1300 = \frac{3000 \times 1200}{1200 + 1800e^{-12000k}}$$

Solve for $e^{-3000k}$.

$$1200 + 1800e^{-4 \times 3000k} = \frac{36000}{13}$$

$$e^{-4 \times 3000k} = \frac{\frac{36\,000}{13} - 1200}{1800}$$

$$e^{-3000k} = \sqrt[4]{\frac{\frac{36\,000}{13} - 1200}{1800}}$$

$$= 0.966\,281\ldots$$


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| | |
|---|---|
| Write the logistic model. | $P = \dfrac{3\,600\,000}{1200 + 1800(0.966...)^{12}}$ |
| Evaluate. | $= 1504.603\ldots$ |
| Round off and state the result. | After 12 months, there will be about 1504 possums in the park. |

An alternative statement of the logistic equation is

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

where $P$ is the population, $k$ is the growth factor and $M$ is the maximum population (or carrying capacity). The general solution for a logistic equation in this form is developed in a similar way to that shown previously.

Write the differential equation.
$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

Separate the variables and integrate.
$$\int \frac{1}{P\left(1 - \frac{P}{M}\right)}\, dP = \int k\, dt$$
$$\int \frac{M}{P(M-P)}\, dP = \int k\, dt$$

Use partial fractions.
$$\int \left[\frac{1}{P} + \frac{1}{(M-P)}\right] dP = \int k\, dt$$

Integrate.
$$\ln (P) - \ln (M - P) = kt + C$$
$$\ln \left(\frac{M-P}{P}\right) = -kt - C$$

Express in exponential form.
$$\left(\frac{M-P}{P}\right) = e^{-kt - C} = e^{-C} e^{-kt}$$
$$\frac{M-P}{P} = Ae^{-kt}$$

Solve for $P$.
$$P = \frac{M}{1 + Ae^{-kt}}$$

Let $t = 0$.
$$P_0 = \frac{M}{1 + Ae^{0}}$$

Rearrange.
$$Ae^{0} = A = \frac{M - P_0}{P_0}$$

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### Solution to the logistic equation

The general solution for the differential logistic equation $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$ is:

$$P = \frac{M}{1 + Ae^{-kt}}, \text{ where } A = \frac{M - P_0}{P_0}$$

## EXAMPLE 18

The growth of a population follows the differential equation $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$, where $t$ is in days. The initial population is 100 and the carrying capacity is 1000. The growth rate of the population is 0.06. Calculate:

**a** the population after 30 days

**b** the population after 80 days

**c** the time taken for the population to reach 950

### Solution

**a** State the parameters. $k = 0.06$, $M = 1000$ and $P_0 = 100$

Find $A$. $A = \frac{1000 - 100}{100} = 9$

Write the general solution. $P = \frac{1000}{1 + 9e^{-0.06t}}$

Calculate $P(30)$. $P(30) = \frac{1000}{1 + 9e^{-0.06 \times 30}}$

$= 401.979\ldots$

Round off and state the result. The population is about 402 after 30 days.

**b** Calculate $P(80)$. $P(80) = \frac{1000}{1 + 9e^{-0.06 \times 80}}$

$= 931.039\ldots$

Round off and state the result. The population is about 931 after 80 days.


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**c** Let $P = 950$.

$$950 = \frac{1000}{1+9e^{-0.06t}}$$

Solve for $t$.

$$1 + 9e^{-0.06t} = \frac{20}{19}$$

$$9e^{-0.06t} = \frac{1}{19}$$

$$e^{-0.06t} = \frac{1}{171}$$

Take logarithms.

$$-0.06t = -\ln(171)$$

$$t = \frac{\ln(171)}{0.06}$$

$$= 85.694\ldots$$

Round off and state the result.

The population will reach 950 after about 86 days.

## INVESTIGATION

### POPULATION GROWTH

1 Use an appropriate data source to find the population statistics for a local town or city over the last 20 years. You should choose a place that has not had a mining boom or some other similar phenomenon in that time.

2 Use the first 10 years of your data to set up an exponential model for the population growth.

3 Now use the same data to set up a logistic model for the population growth.

4 Use both models to predict the population each year for the next 10 years that are covered by your data.

5 Compare your predictions for each model with the actual population figures.

6 Discuss the usefulness of each model in predicting human population on a small scale.

## Exercise 9.07 The logistic equation

Example 16

**1** In a boarding school of 2000 students, 5 have a contagious skin infection. The spread of the infection through the school is given by the differential equation $\frac{dP}{dt} = 0.0002P(2000 - P)$, where $t$ is in days and $P$ is the number of students infected.

**a** Prove that $P = \frac{2000A}{A + e^{-0.4t}}$ where $A = \frac{1}{399}$.

**b** Calculate the number of infected students after 2 days.

**c** How many students are infected after a week?

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Example 17

**2** A population of golden perch live in an isolated lake with no natural predators. The maximum number of golden perch that can be supported in the lake is estimated to be 10 000. An initial count of the perch reveals that 400 are present. A second count completed 1 year later showed the number of perch had grown to 1200.

**a** Use the general solution for the logistic model with $t$ measured in years to predict the number of perch in the lake after 2 years.

**b** When will the perch population reach 8000?

**c** How long will it take for the lake to reach half its carrying capacity?

Example 18

**3** The population of a new colony of native bees is given by

$$P = \frac{60000}{1+1000e^{-0.5t}}, \text{ where } t \text{ is in months.}$$

**a** What is the initial population of the colony?

**b** What is the carrying capacity of the colony?

**c** How long will it take the population to reach 50 000?

**4** In an experiment, a population $P$ of bacteria grows in proportion to $P\left(1-\frac{P}{M}\right)$, where $M$ is the maximum population. A large conical flask initially contains 3 mg of bacteria. It is known that the growth medium in the flask can support a maximum population of 100 mg. The scientist supervising the experiment estimates that the population growth rate is 0.2 per hour.

**a** Find the differential equation satisfied by the population ($P$).

**b** Find the general solution for the differential equation.

**c** When will the population reach 25 mg?

**d** When will the population reach 250 mg?

**5** The population of ocean trout in a fish farm follows the logistic model that uses the ratio of the population to the carrying capacity. It is known that the growth rate is $k = 0.3$ per year and the carrying capacity of the farm is 10 000.

**a** Write the differential equation describing the population model for this problem.

**b** If 2500 fish are initially introduced into the lake, find the general solution for the differential equation that models the number of fish in the lake after $t$ years.

**c** Use your model to estimate the number of fish in the lake after 4 years.

**d** Estimate the time it will take for there to be 7000 fish in the lake.

## Problem solving

**6** The spread of a rumour can be modelled with the logistic model. A rumour concerning the timing of the next Federal election begins in the Australian Public Service. When the rumour is first brought to the attention of a reporter, 2% of the public service have heard it. After 20 days, 50% of the public service have heard the rumour.

**a** What is the maximum percentage of public servants who will hear the rumour?

**b** Find the parameters for the logistic model.


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**c** Find the percentage of public servants who have heard the rumour after 10 days.

**d** What percentage of public servants will have heard the rumour after 30 days?

**e** After how many days will 90% have heard the rumour?

**7** At 10 a.m., 10% of stockbrokers have heard about the impending collapse of a company that is listed on the Australian Stock Exchange. By noon, 25% have heard. Use a logistic model to find the time when 75%have heard of the impending collapse.

**8** It has been said that the maximum population that Australia can support is 50 million. The population reached 18 million in 1995 and was 16 million in 1986.Use the logistic model to predict Australia's population in 2030 and in 2050.

**9** The logistic model can be used to predict the death rates from a toxin. The LD-50 quantity is the amount that must be given to kill 50% of the subjects to which a toxin is delivered. A particular poison is tested on laboratory mice for effectiveness as a control for house mice. The LD levels were found to be 2 g for LD-50 and 3 g for LD-70.

**a** What quantity would be needed for:

**i** LD-90? **ii** LD-95? **iii** LD-99?

**b** What percentage would have died with no poison?

**10** The differential equation that describes the spread of information about food supplements among dairy farmers is given by:

$$\frac{dN}{dt} = 0.0002N(4000 - N)$$

where $N$ is the number who know after $t$ months. 30 farmers are given the information at a Department of Primary Industries seminar in March.

**a** Find the number of farmers who know in June.

**b** Find the number who know in October.

**c** Find the number who know in December.

**d** Rewrite the logistic equation in terms of percentages.

**11** A new viral disease is found to spread according to the equation:

$$\frac{dN}{dt} = kN(M - N)$$

where $M$ is the susceptible population and $N$ is the number of infected people at time $t$ months. It is known that $M = 18\ 000\ 000$ and $k = 1.5 \times 10^{-9}$. In March 2018 it was thought that only 100 people were infected. Use the logistic model to find the number of infected people in:

**a** March 2019

**b** June 2020

**c** January 2025

**12** The population of the world was about 6.5 billion in 2005. Birth rates in the same period averaged 37 million per year and death rates averaged 17 million per year. Some experts state that the carrying capacity for world population is 100 billion.

**a** Write the logistic differential equation for these data.

**b** Use the logistic model to estimate the world population in the year 2015 and compare it with the actual population of 7.4 billion.

**c** Use the logistic model to predict the world population in the years 2100 and 2500.

**d** If the carrying capacity for world population is 40 billion, what are your predictions for the population in 2100 and 2500?

**13** A remote island with no feral animals is used to set up a breeding population of bilbies. Initially, 10 bilbies are released on the island, which has an ideal environment and is thought to be able to sustain a population of 600 bilbies. After 2 years, the population has grown to 30 bilbies. Use the logistic model to find:

**a** when the population will reach 200

**b** how many bilbies could be taken back to the mainland in each following year while maintaining the population at 200.


Getty Images/Auscape/UIG

**14** A poison is dissolved in honey and set as ant baits to get rid of a fire ant nest, estimated to have 10 000 ants. When the bait is set, 10 ants are observed to be killed immediately. The nest is observed after 1 week, when the baits are renewed, and the activity is seen to have reduced by 1%. Use the logistic model for the number killed to find how long the baits must be used to ensure that the nest is exterminated. It is considered that the nest will be exterminated when the population reduces to below 50, as this is the number needed to maintain the queen.

**15** The completion of a particular chemical reaction involving a catalyst follows the logistic model. Reactions between molecules of the reactants occur when they are adsorbed onto the catalyst sufficiently close to one another to react. As the reactions proceed, this is less and less likely to occur. When first measured, 5% of the reactions have taken place; and after 5 minutes, another 5% have taken place. How long is it until the reactions proceed to completion (99% reacted)?


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# 9.08 Motion, force and momentum

WS
Resolution of forces

People have always had an intuitive understanding of motion. For example, an object remains at rest unless something changes. To make something move, the object must be subjected to some kind of force, such as a push or a pull.

Sir Isaac Newton (1642–1727) is widely regarded as being responsible for the foundation of the study of the causes of motion. He stated three laws of motion in a work known as the *Principia*, published in 1687. In recognition of his contribution to the study of dynamics, the unit of force is called the newton.

### Newton's 1st law of motion

Unless acted on by a resultant force, a body remains at rest or in uniform motion in a straight line.

What **Newton's first law of motion** means is that particles accelerate because they are being acted on by forces. In the absence of forces, a particle will not accelerate, that is, it will move at constant velocity.

From Newton's first law, you can define a force as something that causes motion or a change in motion. Force is a vector quantity because it has both magnitude and direction. The resultant force is the vector sum of the forces acting on the object. In this course, the terms *body* and *particle* will be used interchangeably to describe an object. As you will be studying the effect of forces acting on an object, you should assume that all forces act through the centre of the object.

Inertia is the natural tendency of an object to remain at rest or in motion at a constant speed along a straight line. The mass of an object is a quantitative measure of inertia. The SI unit of mass is the kilogram.

The SI unit of force is the newton (N), which is the force required to accelerate a mass of 1 kg at 1 m $s^{-2}$.

When a number of forces act on a body at the same time, the motion is determined by the resultant force. **Newton's second law of motion** is a relation between the net force acting on a mass and its acceleration.

### Newton's 2nd law of motion

- The acceleration of a body is proportional to the resultant force that acts on the body and inversely proportional to the mass of the body.
- The acceleration is in the direction of the resultant force. By choosing suitable units, this law becomes $\mathbf{F} = m\mathbf{a}$ where $\mathbf{F}$ and $\mathbf{a}$ are vectors and $m$ is a scalar quantity.
- The direction of the acceleration is the same as the direction of the resultant force.

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An important force is that due to the Earth's gravity. For all objects near the Earth's surface, the Earth exerts a downward gravitational force, which is known as the weight of the object. The magnitude of the weight is given by

$$W = mg, \text{ where } g = -9.8 \text{ m s}^{-2}$$

## EXAMPLE 19

A person is landing a drone of mass 5 kg. Initially, the drone is hovering 25 m above the ground. To commence landing, the thrust provided by the propellers is reduced to 34 N upwards. The force due to gravity is 49 N downwards and a breeze is pushing the drone horizontally with a force of 20 N.

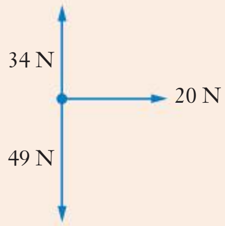
**a** Calculate the acceleration of the drone.

**b** Find the time taken for the drone to reach the ground.

**c** What is the velocity of the drone when it reaches the ground?

### Solution

**a** Draw a diagram.

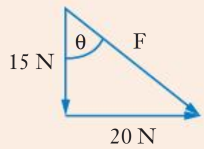



Find the resultant vertical force.

Vertical force = 34 − 49 = −15 N

Use a diagram to find the resultant force, **F**, acting on the drone.

−15 N vertically is 15 N downwards.




Calculate the magnitude of **F**.

$$F = \sqrt{15^2 + 20^2}$$
$$= 25 \text{ N}$$

Calculate the angle that **F** makes with the vertical, $\theta$.

$$\tan(\theta) = \frac{20}{15}$$
$$\theta = 53.13\ldots°$$

Use Newton's second law to find **a**.

$$\mathbf{F} = m\mathbf{a}$$
$$25 = 5\mathbf{a}$$
$$\mathbf{a} = 5 \text{ ms}^{-2}$$

State the answer.

The acceleration of the drone is 5 m s$^{-2}$ downwards at about 53° to the vertical.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| | | |
|---|---|---|
| **b** | Find the vertical component of acceleration. | $a_v = a\cos(\theta)$ |
| | Use calculus to find the vertical velocity, $v_v$. | $\frac{dv_v}{dt} = a_v \Rightarrow \int \frac{dv_v}{dt} dt = \int a_v \, dt$<br>$v_v = a_v t + c$ |
| | Use the initial conditions. | When $t = 0$, $v_v = 0$, so $c = 0$<br>$v_v = a_v t$ |
| | Use calculus to find the vertical distance, $s_v$. | $\frac{ds_v}{dt} = v_v \Rightarrow \int \frac{ds_v}{dt} dt = \int v_v \, dt = \int a_v t \, dt$ |
| | Use the value of $a_v$. | $s_v = \frac{1}{2} a_v t^2 + c$ |
| | Use the initial conditions. | When $t = 0$, $s_v = 25$, so $c = 25$<br>$s_v = \frac{1}{2} a_v t^2 + 25$ |
| | The drone reaches the ground when $s_v = 0$. | $0 = \frac{1}{2} a_v t^2 + 25$ |
| | Solve for $t$. | $t = \sqrt{\frac{-25 \times 2}{a_v}}$ |
| | Substitute for $a_v$. Remember that acceleration acts downwards so it is negative. | $= \sqrt{\frac{-25 \times 2}{-5\cos(53.13...°)}}$<br>$= 4.082...$ |
| | Round off and state the result. | The drone takes about 4 seconds to reach the ground. |
| **c** | Use the formula for velocity. | $v = at$ |
| | Substitute for known values. | $= -5 \times 4.082...$<br>$= -20.412..$ |
| | Round off and state the result. | The drone hits the ground at about $20.4 \text{ m s}^{-1}$. |

The momentum (**p**) of a body is the product of the mass and the velocity:

$$\mathbf{p} = m\mathbf{v}$$

Momentum is a vector quantity since mass is a scalar and velocity is a vector quantity.

Newton's second law of motion can also be stated as:

*The rate of change of momentum of a body is proportional to the magnitude of the resultant force and occurs in the direction of the force.*

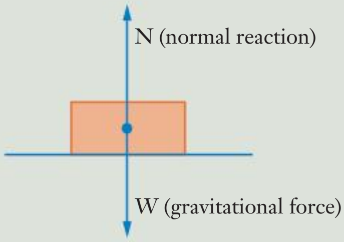
This form of Newton's second law relates to the momentum of a body. The equivalence of the 2 forms of Newton's second law follows directly from the fact that acceleration is the rate of change of velocity.

$$\mathbf{F} = m\mathbf{a} = m\frac{d\mathbf{v}}{dt} = \frac{d(m\mathbf{v})}{dt} = \frac{d\mathbf{p}}{dt}$$

Consider the situation where an ice skater pushes against the handrail enclosing the ice rink. As a result of pushing, the skater starts moving in the opposite direction of the force that was applied to the handrail. **Newton's third law of motion** helps to better understand this situation.

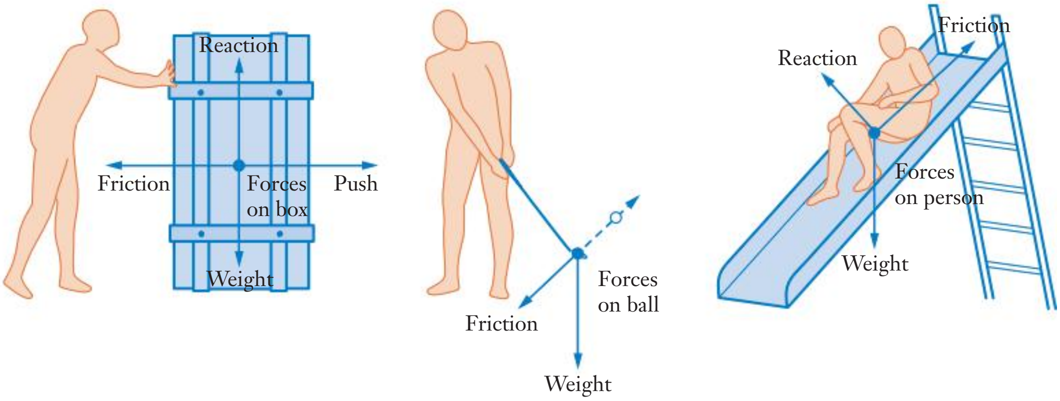
## Newton's 3rd law of motion

For every action there is an equal and opposite reaction.




Newton's third law means that an object resting on a surface has two forces acting on it. One is the Earth's **gravitational force** (**W**), acting towards the centre of the Earth. The other is the **normal reaction**, or simply **reaction** (**N**). The reaction is a force equal in magnitude, but opposite in direction to the gravitational force, as shown at right.

The diagrams will give you an indication of how the forces acting on a body may be represented at a point.




Many situations involve objects joined by a string. A under tension exerts forces on the objects that are attached to either end. The forces are directed inward along the length of the string. You can assume that the string has no mass, and when it passes over any pulley, the pulley's mass can also be ignored. The magnitude of the string's force on either end is the same and is called the string's tension.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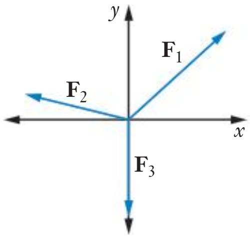
Newton's third law means that forces occur in opposite pairs. It also means that the total momentum of a system remains constant, that is, momentum is conserved.

Many systems have a number of forces that lie in the same plane and pass through a single point.

These are known as **concurrent forces**. If the resultant of the concurrent forces is zero, then the system is said to be in **equilibrium** and the acceleration is also zero.




## EXAMPLE 20

**a** A ball bearing with a mass 200 g rolls along a glass surface at 4 m s$^{-1}$ and bounces off a wall in the opposite direction with a velocity of 3.2 m s$^{-1}$. If the collision takes 1 millisecond, what was the average force exerted by the ball bearing on the wall?

**b** An 64 kg ice skater pushes a large block of ice weighing 250 kg. If the force exerted by the ice skater is 80 N, calculate the acceleration of the skater and the block of ice after the push.

### Solution

**a** Use Newton's second law. $\mathbf{F} = \frac{d\mathbf{p}}{dt}$

Write an expression for the average force. Average force $= \frac{\Delta \mathbf{p}}{\Delta t}$

Calculate the change in momentum. $\Delta \mathbf{p} = mv_1 - mv_2$

Substitute for known values. $= 0.2 \times 4 - 0.2 \times (-3.2)$ kg m s$^{-1}$

$= 1.44$ kg m s$^{-1}$

Use the formula for average force. Average force $= \frac{1.44}{0.001}$

$= 1440$ N

State the answer. The average force exerted on the wall is 1440 N.

**b** State the force for the skater and the block of ice. $\mathbf{F}_{\text{skater}} = 80$ N

$\mathbf{F}_{\text{ice block}} = -80$ N

Use $\mathbf{F} = m\mathbf{a}$. $\mathbf{a} = \frac{\mathbf{F}}{m}$

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| | |
|---|---|
| Substitute for known values. | $\mathbf{a}_{skater} = \frac{80}{64} = 1.25 \text{ m s}^{-2}$ <br> $\mathbf{a}_{ice\ block} = \frac{-80}{250} = -0.32 \text{ m s}^{-2}$ |
| Write the answer. | The skater moves with an acceleration of 1.25 m $s^{-2}$ in the opposite direction to the block of ice, which has an acceleration of 0.32 m $s^{-2}$. |

## Exercise 9.08 Motion, force and momentum

Example 19

**1** A 4 kg mass undergoes an acceleration given by $\mathbf{a} = (2\mathbf{i} + 6\mathbf{j})$ m $s^{-2}$. Find the resultant force acting on the mass.

**2** What resultant force is required to give a mass of 6.5 kg an acceleration of 2.4 m $s^{-2}$ east?

**3** If a resultant force of 42 N produces an acceleration of 1.5 m $s^{-2}$ on an object, what is the mass of the object?

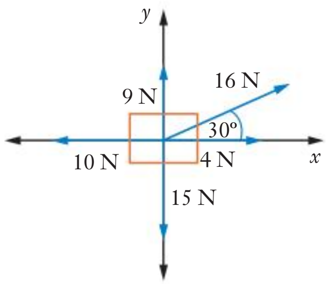
**4** A resultant force of 24 N acts on an object and causes it to reach a velocity of 4.0 m $s^{-1}$ in 2.5 s. What is the mass of the object?

**5** Two concurrent forces act on a particle of mass 4.3 kg. The velocity of the object is $(4\mathbf{i} - 7\mathbf{j})$ m $s^{-1}$. One of the forces is $(3\mathbf{i} - 5\mathbf{j})$ N. What is the other force?

**6** A 5 kg object has a velocity of $3\mathbf{i}$ m $s^{-1}$ at one instant. At 4 seconds later, its velocity is $(10\mathbf{i} + 8\mathbf{j})$ m $s^{-1}$. Find the resultant force acting on the object.

**7** A 950 kg dragster is travelling with an acceleration of 28 m $s^{-2}$. A total force of 2000 N resists the motion of the dragster. What total force is exerted by the engine of the dragster?

**8** Five concurrent forces pull on a 10 kg object, as shown in the diagram on the right. Find the magnitude and direction of the object's acceleration.





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**9** Three concurrent forces $\mathbf{F}_1 = (25.0\text{N}, 42.5°)$, $\mathbf{F}_2 = (15.5\text{N}, 215°)$, and $\mathbf{F}_3 = (20.5\text{N}, 155°)$ accelerate an object with a mass of 8.75 kg.

- **a** What is the resultant force acting on the object?
- **b** What is the magnitude and direction of the object's acceleration?
- **c** An additional force is introduced so that the acceleration of the object is zero. What is the magnitude and direction of the force?

**10** A tennis ball with a mass 100 g strikes the ground at 5 m s$^{-1}$ and bounces up in the opposite direction with a velocity of 3 m s$^{-1}$. The collision takes 5 milliseconds. What was the average force exerted by the tennis ball on the ground?

Example 20

**11** The 85 kg male partner of a figure skating duo pushes his 64 kg female partner with a force of 75 N. Find the acceleration of both partners.


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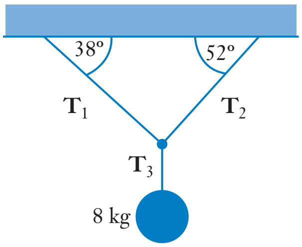
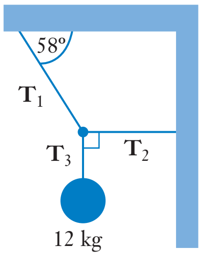
**12** An 8 kg object is suspended by three strings, as shown in the diagram on the right. Find the magnitude of the tension in each string.




**13** A 12 kg object is suspended by 3 strings, as shown in the diagram on the right. Find the magnitude of the tension in each string.

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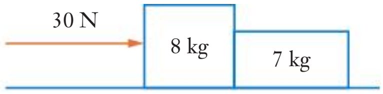
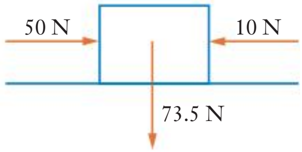
**14** Two blocks are placed next to each other on a frictionless horizontal surface. The masses of the blocks are 8 kg and 7 kg, as shown. A constant horizontal force of 30 N is applied to the 8 kg block.




**a** Calculate the magnitude of the acceleration of the system of blocks.

**b** By drawing separate diagrams for each block, calculate the magnitude of the contact force between the two blocks.

**15** A force of 50 N accelerates a block, as shown on the right. A frictional force of 10 N resists the motion. Calculate:




**a** the normal reaction of the block

**b** the mass of the block

**c** the acceleration of the block

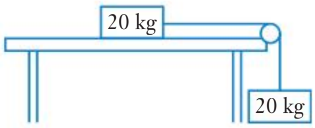
**16** A block of mass 3 kg is acted on by a force $\mathbf{R} = (2\mathbf{i} + 5\mathbf{j})$ N. Calculate the magnitude and direction of the acceleration.

**17** A uniform force acts on a box with a mass of 5 kg and causes the box to accelerate at $(4\mathbf{i} + 8\mathbf{j})$ m s$^{-2}$. Find the magnitude and direction of the force acting on the box.

**18** A body of mass 20 kg is acted on by two forces, $\mathbf{P} = 12\mathbf{i} - 4\mathbf{j}$ and $\mathbf{Q} = 20\mathbf{i} + 32\mathbf{j}$, where both forces are measured in newtons. Calculate the magnitude and direction of the acceleration.

## Problem solving

**19** A 10 kg mass on a rough horizontal surface is joined to a 20 kg mass by a string running over a frictionless pulley, as shown in the diagram on the right. The force due to friction acting on the 10 kg mass is 14 N.




Use separate diagrams for each block to calculate the magnitude of the acceleration of the 10 kg mass.

**20** The apparent weight of a person in an elevator is $\frac{9}{11}$ of her actual weight.

What is the acceleration (including the direction) of the elevator?

**21** A person is standing on a weigh scale (calibrated in newtons) in an elevator. When the elevator is accelerating upward with constant acceleration **a**, the scale reads 757.25 N. When the elevator is accelerating downwards with the same constant acceleration **a**, the scale reads 516.75 N. Determine:

**a** the mass of the person

**b** the magnitude of the acceleration.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# 9.09 Linear motion under a constant force

There are many situations where objects experience linear or straight-line motion. An object falling freely under gravity is one example of this. For linear motion, direction is either positive or negative. The context of the situation will tell you if positive and negative represent up and down, left and right and so on.

There are a number of different situations where a body is subjected to constant, or almost constant, acceleration. Some useful general equations involving the magnitudes of the acceleration, velocity and displacement are presented below.

If acceleration is constant, then $\frac{dv}{dt} = a$

So $v = \int a\,dt \Rightarrow v = at + c$

When $t = 0$, $v = u$, where $u$ = initial velocity, so $u = a \times 0 + c = c$

Thus $v = u + at$

If $s$ is position, you can write $v = \frac{ds}{dt} \Rightarrow \frac{ds}{dt} = u + at$

Integrating gives $s = \int (u + at)\,dt$

$$s = ut + \frac{1}{2}at^2 + c$$

If $s_0 = 0$ is the initial position $s = ut + \frac{1}{2}at^2 + s_0$

If $s$ is the displacement from $s_0$, then $s = ut + \frac{1}{2}at^2$

It is also useful to have an equation of motion that does not contain the variable $t$.

Rewrite $v = u + at$. $a = \frac{v - u}{t}$

Substitute for $a$ into the equation for $s$. $s = ut + \frac{1}{2}\left(\frac{v - u}{t}\right)t^2$

$$s = ut + \frac{1}{2}(v - u)t$$

$$s = \left(\frac{v + u}{2}\right)t$$

Multiply $a$ by $s$. $as = \frac{v - u}{t} \times \left(\frac{v + u}{2}\right)t$

$$2as = v^2 - u^2$$

Rewrite. $v^2 = u^2 + 2as$

These 4 equations of motion are important and should be memorised.

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## Equations for straight-line motion under constant force

For straight-line motion with constant acceleration $a$, velocity $v$, initial velocity $u$, time $t$ and displacement $s$, we have the following equations.

$$v = u + at \qquad s = ut + \frac{1}{2}at^2$$

$$v^2 = u^2 + 2as \qquad s = \left(\frac{v+u}{2}\right)t$$

When the magnitude of velocity is stated as km/h, it is useful to convert to m/s as shown:

$$1 \text{ m s}^{-1} = \frac{1 \text{ m}}{1 \text{ s}} = \frac{\frac{1 \text{ km}}{1000}}{\frac{1 \text{ h}}{3600}} = \frac{3.6 \text{ km}}{1 \text{ h}}, \text{ so } 1 \text{ m/s} = 3.6 \text{ kmh}^{-1}$$

### EXAMPLE 21

A car travelling at 80 km $h^{-1}$ is brought to rest in a distance of 50 m. If the brakes are applied uniformly, calculate:

**a** the deceleration experienced by the car

**b** the time taken for the car to stop.

### Solution

**a** Write the values given.

$v = 0$
$s = 50$
$u = 80 \text{ km h}^{-1}$
$= 80 \div 3.6 \text{ ms}^{-1}$
$= 22.222\ldots \text{ ms}^{-1}$
$a = ?$

Choose an equation containing the desired variables.

$v^2 = u^2 + 2as$

Substitute values.

$0 = (22.222\ldots)^2 + 2a \times 50$

Solve for $a$.

$100a = -(22.222\ldots)^2$
$a = -4.938\ldots$

Write the answer.

The car is decelerating at about 4.94 m $s^{-2}$.


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**b** Choose the most suitable equation. $s = \left(\dfrac{v+u}{2}\right)t$

Substitute values. $50 = \dfrac{22.222\ldots + 0}{2}t$

Solve for $t$. $t = \dfrac{50 \times 2}{22.222\ldots} = 4.5$ s

Write the answer. The car takes 4.5 s to stop.

It is usual to take the downwards direction as negative, so acceleration due to gravity is taken to be $-9.8$ ms$^{-2}$.

## EXAMPLE 22

A stone is thrown vertically upward and is seen passing a point 10 m high on the face of a building 3 s after being thrown. What was the initial velocity of the stone?

### Solution

Write the information.

$s = 10$ m
$t = 3$ s
$a \approx -9.8$ ms$^{-2}$
$u = ?$

Choose the most suitable equation. $s = ut + \frac{1}{2}at^2$

Substitute values. $10 \approx 3u - \frac{1}{2}(-9.8) \times 3^2$

Solve for $u$.

$20 + 88.2 \approx 6u$
$u = 18.033\ldots$

Write the answer. The initial velocity was about 18 ms$^{-1}$.

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There are often multiple ways to solve the same problem.

## EXAMPLE 23

A young man in a wheelchair is moving at a steady 15 km $h^{-1}$. He passes a moped just as it begins to move in the same direction. The moped accelerates uniformly at 0.3 m $s^{-2}$ for 21 s and then continues with uniform speed. How far will the moped travel before reaching the wheelchair?

### Solution

Change the wheelchair's speed to m $s^{-1}$.

$v_W = 15 \div 3.6 = 4\frac{1}{6}$ m $s^{-1}$

Find the wheelchair's position after 20 s.

$s_W = 21 \times 4\frac{1}{6} = 87.5$ m

Find the moped's position after 21 s.

$$\begin{aligned} s_M &= ut + \frac{1}{2}at^2 \\ &= 0 + \frac{1}{2} \times 0.3 \times 21^2 \\ &= 44.1 \text{ m} \end{aligned}$$

Find the distance between them.

Distance between them = 87.5 − 44.1 = 43.4 m

Find the moped's speed after 21 s.

$$\begin{aligned} v_M &= u + at \\ &= 0 + 0.3 \times 21 \\ &= 6.3 \text{ m s}^{-1} \end{aligned}$$

You can use relative speeds.

**Method 1**

Find the speed of the moped relative to the wheelchair.

$v_{M \text{ rel } W} = 8.3 - 4\frac{1}{6} = 2\frac{2}{15}$ m $s^{-1}$

Calculate the time taken to catch up.

$t = \frac{43.4}{2\frac{2}{15}} = 20\frac{11}{32}$ s $\approx 20.34$ s

You can also use distance.

**Method 2**

State the distance of the moped after 21 s.

$s_M = 44.1 + 6.3t$

State the distance of the wheelchair after 21 s.

$s_W = 87.5 + 4\frac{1}{6}t$

The moped and wheelchair meet when the distances are equal.

$$\begin{aligned} s_T &= s_C \\ 44.1 + 6.3t &= 87.5 + 4\tfrac{1}{6}t \end{aligned}$$

Solve for $t$.

$$\begin{aligned} 2\tfrac{2}{15}t &= 43.4 \\ t &= 20\tfrac{11}{32} \text{ s} \end{aligned}$$


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| | |
|---|---|
| Calculate the total distance travelled by the moped. | $s_M = 44.1 + 20\frac{11}{32} \times 6.3$ <br> $= 172\frac{17}{64}$ |
| State the result. | The moped travels $172\frac{17}{64} \approx 172.27$m before reaching the wheelchair. |

## Exercise 9.09 Linear motion under a constant force

**1** A car travelling at 54 km $h^{-1}$ comes to rest with uniform retardation in 5 s. Calculate the acceleration and the distance travelled in the time taken for the car to be brought to rest. Example 21

**2** A rocket car begins from rest and travels with a uniform acceleration of 19.2 m $s^{-2}$. What will be the speed of the rocket car after 6 s, and how far will it have travelled?

**3** A motor vehicle increases its speed from 18 km $h^{-1}$ to 72 km $h^{-1}$ in a distance of 50 m under uniform acceleration. Calculate:

**a** the acceleration

**b** the speed of the vehicle when 25 m has been covered.

**4** An object moves with uniform acceleration for 3 s, in which time it travels 27 m.

It then moves with uniform velocity for the next 5 s, covering 60 m. Calculate:

**a** the initial velocity

**b** the acceleration in the first part of its journey.

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**5** An object begins from rest and after 10 s reaches a speed of 15 m $s^{-1}$. Calculate the acceleration and the distance travelled in the 10 s.

**6** A particle with an initial velocity of 100 m $s^{-1}$ experiences a retardation of 2 m $s^{-2}$. How long will it take to bring the particle to rest, and how far will it travel in this time?

**7** A ball is dropped from a platform that is 120 metres above the ground. Find: Example 22

**a** the time taken for the ball to reach the ground

**b** the ball's speed on impact.

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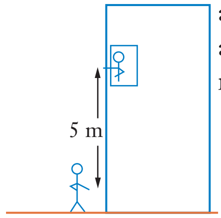
**8** A ball is projected vertically from ground level and rises to a maximum height of 50 m.

**a** Calculate the initial velocity of the ball.

**b** Find the length of time the ball will be in the air.

**9** A stone is thrown directly downward with an initial speed of 6 m $s^{-1}$ from a height of 40 m. When does the stone strike the ground?

**10** A person stands on the ground below a window in a building, as shown in the diagram. The person on the ground throws a ball vertically upward to a person standing in the window 5 m above. After 1.5 s the person in the window catches the ball.




**a** Calculate the initial velocity of the ball.

**b** What was the velocity of the ball just before it was caught?

**11** A ball is projected straight upwards with an initial velocity of 10 m $s^{-1}$ from a 25 m high platform. Find:

**a** the maximum height above the ground reached by the ball

**b** the velocity with which the ball strikes the ground

Example 23

**12** A speeding motorist travelling at 110 km $h^{-1}$ passes a stationary police car. The police car immediately begins to chase the speeding car, accelerating at a constant 2.4 m $s^{-2}$.

**a** How long does it take for the police car to intercept the speeder?

**b** How far did the police car travel before catching up to the speeder?

**c** How fast was the police car travelling when it intercepted the speeder?

**13** A car is initially travelling at a constant velocity of 10 m $s^{-1}$. The car then enters a 110 km $h^{-1}$ zone and begins to accelerate uniformly at 4 m $s^{-2}$. Calculate:

**a** the speed 5 s after acceleration commences

**b** the distance travelled in the 5 s after acceleration commences

**14** A roller-coaster begins from rest and moves with uniform acceleration. It travels 9.5 m in the tenth second after starting its trip. Find:

**a** the acceleration

**b** the total distance covered during the first 5 s of motion.

**15** A cage carrying workers down a mine shaft completes the 675 m descent in 45 s. During the first quarter of the time, the cage is uniformly accelerated, while in the last quarter it is uniformly retarded, the acceleration and retardation being equal in magnitude. Calculate the uniform speed of the cage in the central portion of its descent.

**16** A train carriage moves in a straight line with uniform acceleration.

**a** How far does the carriage travel in 12 s if it moves 48 m in the first 6 s of its motion and 32 m in the last 2 s?

**b** What is the initial velocity?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**17** A force of 19.6 N acts on a brick of mass 2.4 kg for 5 s. Find:

**a** the acceleration

**b** the velocity after 5 s

**c** the distance travelled in this time.

**18** The driver of a car must suddenly apply his brakes to avoid a collision. As a result, the car skids in a straight line for a distance of 15 metres for 3 seconds before stopping.

**a** At what speed was the car travelling as it began to skid?

**b** What was the acceleration of the car during the skid?

**19** An SUV skids to a halt under constant deceleration. The initial speed of the SUV is 26 m $s^{-1}$ and it travels a distance of 40 m before stopping. Find the acceleration of the SUV and the time it takes to stop.

## Problem solving

**20** A particle moves along a straight line with an acceleration of –2 m $s^{-2}$. The particle commences its motion at the origin with a velocity of 10 m $s^{-1}$. Calculate when:

**a** the object returns to the origin

**b** the velocity of the object is zero

**21** Cyclist $X$, riding at 16 km $h^{-1}$ is overtaken and passed by cyclist $Y$, who is travelling at 20 km $h^{-1}$. At this point, $X$ immediately increases speed with uniform acceleration. Show that $X$ will catch $Y$ when she reaches a speed of 24 km $h^{-1}$.

**22** A delivery van driver is travelling at a constant velocity of 10 m/s. The driver realises that she has only 35 seconds to reach her destination, which is 1 km away. The driver then accelerates at 3 m $s^{-2}$ for the rest of the trip. Did she succeed in getting to her destination on time?

**23** A particle moving in a straight line travels 12 m, 18 m and 42 m in successive intervals of 3 s, 2 s and 3 s. Show that these distances are consistent with the proposition that the particle is moving with uniform acceleration.

**24** An express high speed train travelling with constant acceleration passes three stations $A$, $B$ and $C$ without stopping on its way to its final destination. The three stations are on a straight section of the railway and station $B$ is midway between stations $A$ and $C$. If the speeds at which the train passes $A$ and $C$ are 30 m $s^{-1}$ and 50 m $s^{-1}$ respectively, what is the speed at which the train passes station $B$?

**25** A car passes through an intersection at a certain speed and is then subjected to constant acceleration while travelling on a straight road. After travelling for 12 s, the car reaches a bridge that is 300 m from the intersection. It takes the car 6 s to cross the bridge, which is 168 m long. Find the acceleration of the car and its speed when it passes through the intersection.

**26** A ball on the ground is projected vertically upwards with an initial velocity of $u$ m s$^{-1}$. Two seconds later, an identical ball is projected from the same spot with the same initial velocity as the first ball. If $t$ is the time when the two bodies meet and $g$ is the acceleration due to gravity, show that $t = \frac{u+g}{g}$.

WS

Motion with variable forces

## 9.10 Linear motion under a variable force

Because of the relationship $F = ma$, the acceleration is constant when the force is constant. In this section, you will examine the motion of an object subjected to a variable force, and hence variable acceleration. This means that the constant acceleration formulas in the previous section can't be used. In situations where the forces are variable, differential equations must be used.

### EXAMPLE 24

A 5 kg body initially at rest is acted on by a force that has a magnitude that varies with time according to the expression $40t - 6t^2$, where $t$ is in seconds. If the direction of the force remains constant, calculate the velocity of the body after 5 s.

#### Solution

| | |
|---|---|
| Use Newton's second law. | $F = ma$ |
| Substitute for $F$ and $m$. | $40t - 6t^2 = 5a$ |
| Find $a$. | $a = 8t - 1.2t^2$ |
| Use $a = \frac{dv}{dt}$ because you want to find $v$. | $\frac{dv}{dt} = 8t - 1.2t^2$ |
| Integrate to find $v$. | $v = \int (8t - 1.2t^2)\,dt$<br>$= 4t^2 - 0.4t^3 + c$ |
| Use the initial conditions to find $c$. | $v = 0$ when $t = 0$, so $c = 0$ |
| Write the general expression for $v$. | $v = 4t^2 - 0.4t^3$ |
| Substitute $t = 5$. | $= 4 \times 5^2 - 0.4 \times 5^3$<br>$= 50$ m s$^{-1}$ |
| Write the answer. | The velocity after 5 s is 50 m s$^{-1}$. |

In some cases, the force and acceleration may depend on the speed of the object. For this reason, it is useful to have some alternative forms for calculating acceleration using the derivative.

You can use the chain rule to write $\frac{dv}{dx} = \frac{dv}{dt} \times \frac{dt}{dx} = \frac{dv}{dt} \times \frac{1}{\frac{dx}{dt}}$

using the rule for the derivative of the inverse function.

Thus, $\frac{dv}{dx} = a \times \frac{1}{v}$, so $a = v\frac{dv}{dx}$.


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You can use the chain rule with $v = \dfrac{d\left(\frac{1}{2}v^2\right)}{dv}$ to express $a$ in another form.

$$a = v\frac{dv}{dx} = \frac{d\left(\frac{1}{2}v^2\right)}{dv} \times \frac{dv}{dx} = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$$

**Acceleration formulas**

$$a = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$$

**EXAMPLE 25**

A particle moving forward in a straight line is observed to have an acceleration of $5 - 2x$, where $x$ is the displacement at time $t$. Find an expression for the velocity of the particle, given that $v = 4$ at $x = 3$.

**Solution**

Write the appropriate expression for $a$. $\quad a = \dfrac{d\left(\frac{1}{2}v^2\right)}{dx}$

Substitute $a = 5 - 2x$. $\quad \dfrac{d\left(\frac{1}{2}v^2\right)}{dx} = 5 - 2x$

Integrate to find $v^2$.

$$\frac{1}{2}v^2 = \int(5 - 2x)\,dx$$
$$= 5x - x^2 + c$$

Use $v = 4$ at $x = 3$ to find $c$.

$$\frac{1}{2} \times 4^2 = 5 \times 3 - 3^2 + c$$
$$c = 2$$

Write the general expression for $v^2$. $\quad \frac{1}{2}v^2 = 5x - x^2 + 2$

$v$ is positive because the particle is moving forward. $\quad v = \sqrt{10x - 2x^2 + 4}$

Write the answer. $\quad$ The velocity is given by $v = \sqrt{10x - 2x^2 + 4}$.

When a body travels vertically, its weight (force due to gravity) acts vertically downwards. The other force acting on the body is the force of air resistance due to friction between the body and the air. Air resistance acts in the opposite direction to the direction of motion. The air resistance is proportional to some power of the velocity of the body. In the simplest case, air resistance is $kv$, where $k$ is a constant and $v$ is the velocity of the body.

In the case of a body falling under the force of gravity, it is convenient to consider the downwards direction as positive. If the force of air resistance is $kv$, the body's equation of motion is given by $ma = mg - kv$.

As a body falls, eventually the force due to gravity is balanced by the force due to air resistance. When this occurs, the body reaches a limiting velocity called the **terminal velocity** $v_T$, where $v_T = \lim_{t \to \infty} v(t)$. The terminal velocity is constant, so when a body reaches terminal velocity, $a = 0$.

## EXAMPLE 26

An object of mass 9 kg falls from rest from a tall structure during the construction process. The force due to air resistance is $F_A = kv$, where $k$ is a constant and $v$ is the velocity of the object.

**a** Develop an equation for the terminal velocity of the object.

**b** Find the terminal velocity of the object if $k = 4$.

### Solution

**a**

| | |
|---|---|
| Use Newton's 2nd law for gravity. | $F_G = mg$, where $g = 9.8$ m s$^{-2}$ |
| Calculate the resultant force. | $F_R = mg - kv$ |
| Use Newton's 2nd law for downward acceleration. | $F = ma = mg - kv$ |
| Isolate $a$. | $a = g - \frac{kv}{m}$ |
| Write $a$ as a derivative of $v$. | $\frac{dv}{dt} = g - \frac{kv}{m}$ |
| Take out $-\frac{k}{m}$ as a factor to have $v$ by itself. | $\frac{dv}{dt} = -\frac{k}{m}\left(v - \frac{mg}{k}\right)$ |
| Move the expression with $v$ to the LHS. | $\frac{1}{\left(v - \frac{mg}{k}\right)}\frac{dv}{dt} = -\frac{k}{m}$ |
| Find definite integrals using appropriate limits. | $\int \frac{1}{\left(v - \frac{mg}{k}\right)} dv = -\frac{k}{m} dt$ |


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Use integration by substitution, where $m$, $g$ and $k$ are all constants.

$$\ln\left(v - \frac{mg}{k}\right) = -\frac{k}{m}t + c$$

Express in exponential form.

$$v - \frac{mg}{k} = Ae^{-\frac{kt}{m}}$$

$$v = \frac{mg}{k} + Ae^{-\frac{kt}{m}}$$

$$v = \frac{m}{k}\left(g + Be^{-\frac{kt}{m}}\right), \text{ where } B = \frac{kA}{m}$$

Terminal velocity is reached as $t \to \infty$.

As $t \to \infty$, $Be^{-\frac{kt}{m}} \to 0$.

Write an equation for terminal velocity.

$$v_T = \frac{mg}{k}$$

**b** Calculate $v_T$ if $k = 4$.

$$v_T = \frac{9 \times 9.8}{4}$$
$$= 22.05$$

State the result.

The terminal velocity of the object is about 22 m $s^{-1}$.

## Exercise 9.10 Linear motion under a variable force

Example 24

**1** A 1 kg body initially at rest is acted on by a force that has a magnitude that varies with time according to the expression $F = 2t + 5t^2$, where $t$ is in seconds and $F$ is in newtons. If the direction of the force remains constant, find an expression for velocity as a function of time and calculate the velocity of the body after 2 s.

**2** A 3 kg object initially at rest is acted on by a force that has a magnitude that varies with time according to the expression $F = \sin(3t - 2)$, where $t$ is in seconds and $F$ is in newtons. Find an expression for velocity as a function of time and calculate the velocity of the object after 1 s.

**3** An 8 kg object with an initial velocity of 3 m $s^{-1}$ is acted on by a force that has a magnitude that varies with time according to the expression $F = 3e^{0.4t + 5}$, where $t$ is in seconds and $F$ is in newtons. Find an expression for velocity as a function of time and calculate the velocity of the object after 0.1 s.

**4** A 2 kg object with an initial velocity of 7 m $s^{-1}$ is acted on by a force that has a magnitude that varies with time according to the expression $F = 3v$, where $v$ is the velocity of the object and $F$ is in newtons. Find an expression for velocity as a function of time and calculate the velocity of the object after 1 s.

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**5** A particle with a mass of 10 kg is accelerated from rest by a horizontal resultant force with magnitude after $t$ s given by $(3 - 2t)$ N. Find:

**a** an expression for the acceleration of the particle as a function of $t$

**b** the velocity of the particle 6 s after moving from rest.

Example 25

**6** A 2 kg mass moving forward in a straight line is acted on by a force $F = \cos(x)$, where $x$ is the displacement at time $t$ and $F$ is in newtons.

**a** Find an equation for the acceleration of the mass.

**b** Determine an expression for the velocity of the particle, if the initial velocity is 4 m $s^{-1}$.

**7** An object with a mass of 5 kg moving in a straight line is subjected to a force $F = 2x + 7$, where $x$ is the displacement at time $t$ and $F$ is in newtons. If the initial displacement is 0 m and the initial velocity is 2 m $s^{-1}$, find an expression for the velocity of the object.

**8** An object with a mass of 4 kg moving in a straight line is subjected to a force $F = 2v$, where $v$ is the velocity at time $t$ and $F$ is in newtons. If the initial velocity is 3 m $s^{-1}$, find an expression for the velocity of the object as a function of displacement.

**9** A body with a mass of 1 kg moving in a straight line is subjected to a force $F = 4v - v^2$, where $v$ is the velocity at time $t$ and $F$ is in newtons. If the initial velocity is 2 m $s^{-1}$, find an expression for the velocity of the body as a function of displacement.

**10** An object with a mass of 3 kg moving in a straight line is subjected to a force $F = \dfrac{1}{x+1}$, where $v$ is the velocity at time $t$ and $F$ is in newtons. If the initial velocity is $-2$ m $s^{-1}$, find an expression for the velocity of the body.

Example 26

**11** An object with a mass of $m$ kg falls from a height with velocity $v$ m $s^{-1}$. As it falls vertically downwards, the force of resistance to motion is $F = kmv$ N. Find an equation for the velocity as a function of time and hence find the terminal velocity if $k = 0.125$.

**12** An object with a mass of 5 kg falls vertically downwards. The force of resistance to motion is $F = v^2$ N, where $v$ m $s^{-1}$ is the speed of the object at time $t$ s after it fell and the object travels a distance of $x$ m in this time.

**a** Show that the acceleration of the object is given by $a = \dfrac{49 - v^2}{5}$.

**b** Use $a = v\dfrac{dv}{dx}$ to develop an equation for $x$.

## Problem solving

**13** An object of mass 2 kg moves from rest under the action of a force with magnitude after $t$ s given by $40 - 20t$ N. When the object reaches a speed of 20 m $s^{-1}$, the force is removed and the object continues with constant speed for 4 s. It is then brought to rest by a constant force of 60 N. Calculate the total distance travelled.


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**14** A body of mass 5 kg is acted on by a force in a straight line. The magnitude of the force after $t$ s is given by $10(2t - 3t^2)$ N. If the body has an initial velocity of 3 $\text{ms}^{-1}$ in the same direction as the force, calculate its velocity after 4 s.

**15** A 5 kg body with an initial velocity of 14 m $\text{s}^{-1}$ travels in a horizontal plane. The only force acting on the particle is the friction between it and the surface on which it moves. The magnitude of the frictional force is given by $F = 0.2v + 5$ N. Determine:

**a** an expression for velocity in terms of time

**b** the distance travelled by the body before it comes to rest.

**16** A particle with a mass of 2 kg is released from rest and falls downwards through a liquid medium. The resistive force acting on the particle has magnitude $10v$, where $v$ is the velocity of the particle. Find:

**a** $t$ as a function of $v$

**b** $v$ as a function of $t$

**c** the terminal velocity of the particle

**d** the time it takes for the particle to reach a speed equal to half its terminal velocity

**17** A particle of mass $m$ falls under gravity from rest through the air. The magnitude of the resistive force due to air resistance is given by $F = kv$, where $v$ is the speed of the particle.

**a** Find an expression for the velocity of the particle as a function of time.

**b** Use $v = \frac{dx}{dt}$ to find an expression for the displacement as a function of time.

**18** A body of mass $m$ kg is initially at rest before falling under the force of gravity. The force resisting the motion of the body is $F = kv^2$, where $k$ is a constant.

**a** Use the fact that $m\frac{dv}{dt} = mg - kv^2$ to show that $\frac{dv}{dt} = C(b^2 - v^2)$, where $C = \frac{k}{m}$ and $b = \sqrt{\frac{g}{C}}$.

**b** Use techniques of integration and algebra to show that

$$\int \frac{1}{b^2 - v^2} dv = \frac{1}{2b} \ln\left|\frac{b+v}{b-v}\right| + c.$$

**c** Use the results from **a** and **b** above and techniques of integration and algebra to show that

$$v = b\left(\frac{e^{2bCt} - 1}{e^{2bCt} + 1}\right)$$

**d** Find a rule for the terminal velocity of the body.

**19** An object of mass 4 kg is projected vertically upward against air resistance that is proportional to the square of the speed and equal to 20 N when the speed is 10 m $\text{s}^{-1}$. Find the greatest height reached by the object if its initial velocity is 20 m $\text{s}^{-1}$.

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**20** An elastic ball of mass 2 kg is dropped from a height of 50 m and experiences air resistance equal to $kv^2$. The ball reaches the ground with a speed of 30 m s$^{-1}$.

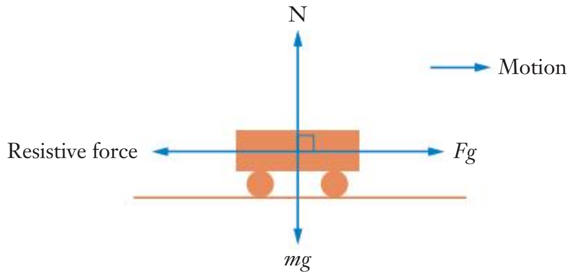
**a** Determine the value of $k$ in the expression for air resistance.

**b** How long after being dropped will the ball first reach the ground?

**c** If the ball rebounds vertically with a speed of 25 m s$^{-1}$, what is the maximum height that it will reach?

**21** A trolley of mass $m$ kg is pulled across a smooth surface under a horizontal force of $F$ kg wt.

While friction between the trolley and the surface is negligible, the trolley is subject to air resistance that is proportional to the speed of the trolley.




If the trolley starts from rest, find an expression for:

**a** the velocity of the trolley as a function of time

**b** the displacement of the trolley as a function of time

**c** the displacement of the trolley as a function of velocity

**d** the terminal velocity. (Hint: Use the expression obtained in part **a**.)

**22** A body of mass $m$ is projected vertically upward with a velocity of $u$. If the air resistance is given by $r = kv$, where $k$ is a constant and $v$ is the speed of the body, find expressions for the greatest height $h$ reached and the time $t_h$ taken to reach that height.

**23** A 1 kg particle falls from rest in a medium in which the resistance to motion is given by $r = kv^2$, where $k$ is a constant and $v$ is the speed.

**a** Show that the distance, $x$, fallen when the velocity is $v$, is given by

$$x = \frac{1}{2k}\ln\left(\frac{g}{g - kv^2}\right).$$

**b** Find the distance fallen when the particle reaches half of its terminal velocity.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# 9.11 Simple harmonic motion

Consider a mass hanging from a spring at rest. If the equilibrium is upset by some vertical interference, the mass will oscillate up and down about the equilibrium position. The mass has an acceleration that is always directed towards the equilibrium position and varies in magnitude directly as the mass' distance from this position.

This type of motion is called **simple harmonic motion** (**SHM**) and is very common in nature.

### Simple harmonic motion

A particle moving with simple harmonic motion moves in a straight line such that its acceleration is always directed to a fixed point and is proportional to its distance from this point. The particle oscillates about the central fixed point or **point of equilibrium** with maximum displacement called the **amplitude**.

The force is given by $\mathbf{F} = -k\mathbf{x}$ where $\mathbf{x}$ is the displacement of the particle from the point of equilibrium.

For the purposes of this section, you can assume that there is no resistance to motion due to friction. Since the motion is in a straight line, the displacement, velocity and acceleration is not normally written in vector notation. You simply use the sign to determine the direction.

Using $F = ma$, you can write $a = -\frac{k}{m}x$. This can be written as $a = -\omega^2 x$ for convenience.

You can use $a = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$ to obtain $\frac{d\left(\frac{1}{2}v^2\right)}{dx} = -\omega^2 x$.

Integrating both sides gives $\frac{1}{2}v^2 = -\frac{1}{2}\omega^2 x^2 + c$, so $v^2 = -\omega^2 x^2 + 2c$.

You can simplify this formula by writing $c$ as $A^2\omega^2$, where $A$ is a constant and is *not* the acceleration.

So $v^2 = A^2\omega^2 - \omega^2 x^2$ and this gives the equation $v = \pm\omega\sqrt{A^2 - x^2}$.

Using $v = \frac{dx}{dt}$, this becomes $\frac{1}{\sqrt{A^2 - x^2}}\frac{dx}{dt} = \pm\omega$.

Integrating gives $\int \frac{1}{\sqrt{A^2 - x^2}}\,dx = \int \pm\omega\,dt$.

So $\int \frac{1}{\sqrt{A^2 - x^2}}\,dx = \pm\omega t + c$.

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Using $\int \frac{1}{\sqrt{A^2 - x^2}}\,dx = \arcsin\left(\frac{x}{A}\right)$ with the previous equation gives

$\arcsin\left(\frac{x}{A}\right) = \pm\omega t + c$, so $x = A \sin(\pm\omega t + c)$.

Using $-\omega$ instead will give $x = A \cos(\pm\omega t + c)$.

It is usual to write the **phase shift** as $\alpha$ rather than $c$ and choose the positive value of $\omega$, giving $x = A \sin(\omega t + \alpha)$, where $A$ is the **amplitude** of the motion, $\omega$ is the **angular velocity** and the time taken for one complete motion (**oscillation**) is given by the **period**, $T = \frac{2\pi}{\omega}$.

The number of cycles in a unit of time is called the **frequency** and is given by $f = \frac{1}{T}$, where time is in seconds, and frequency is measured in hertz (Hz), where 1 Hz = 1 oscillation/second.

In order to avoid confusion between acceleration and amplitude, the 'dot' notation $\ddot{x}$ is used for acceleration and $\dot{x}$ is used for velocity in simple harmonic motion.

## Formulas for simple harmonic motion

The equations for simple harmonic motion with amplitude $A$, phase shift $\alpha$, angular velocity $\omega$, period $T$ and frequency $f$ are:

- Acceleration: $\ddot{x} = -\omega^2 x = -A\omega^2 \sin(\omega t + \alpha)$ or $-A\omega^2 \cos(\omega t + \alpha)$
- Velocity: $v = \dot{x} = \pm\omega\sqrt{A^2 - x^2} = -A\omega \sin(\omega t + \alpha)$ or $A\omega \cos(\omega t + \alpha)$, $v^2 = \omega^2(A^2 - x^2)$
- Displacement: $x = A \cos(\omega t + \alpha)$ or $A \sin(\omega t + \alpha)$
- Period: $T = \frac{2\pi}{\omega} = \frac{1}{f}$, so $\omega = 2\pi f$

## EXAMPLE 27

A particle oscillates 1.2 m on either side of a central position with simple harmonic motion. The period of the motion is 8 s.

**a** What is the maximum speed?

**b** What is the speed when the particle is 0.8 m from the equilibrium position?

### Solution

**a** Write the period formula. $T = \frac{2\pi}{\omega}$

Substitute $T = 8$ and solve for $\omega$. $\omega = \frac{2\pi}{T} = \frac{2\pi}{8} = \frac{\pi}{4}$ radians/s

Write the equation for speed. $\dot{x} = \text{speed} = \pm\omega\sqrt{A^2 - x^2}$


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| | |
|---|---|
| Substitute $a = 1.2$ and $\omega = \frac{\pi}{4}$. | Speed $= \pm\frac{\pi}{4}\sqrt{1.2^2 - x^2}$ |
| Maximum speed occurs when $x = 0$ in either direction. | $= \frac{\pi}{4}\sqrt{1.2^2 - 0}$<br>$= 0.3\pi$ |
| State the result. | The maximum speed is $0.3\pi$ m s$^{-1}$. |
| **b** Use the formula for speed with $x = 0.8$. | Speed $= \pm\frac{\pi}{4}\sqrt{1.2^2 - 0.8^2}$<br>$= 0.702\ldots$ |
| State the result. | The speed when the particle is 0.8 m from the equilibrium position is about 0.70 m s$^{-1}$. |

## EXAMPLE 28

A particle undergoing simple harmonic motion has, at a particular instant, a displacement, velocity and acceleration of 7.5 cm, 7.5 cm s$^{-1}$ and –7.5 cm s$^{-2}$ respectively. Calculate:

**a** the greatest velocity of the particle

**b** the period of the motion.

### Solution

| | |
|---|---|
| **a** Write the acceleration formula. | $\ddot{x} = -\omega^2 x$ |
| Substitute values. | $-0.075 = -\omega^2 \times 0.075$ |
| Solve for $\omega$. | $\omega = 1$ |
| Write the velocity squared formula. | $\dot{x}^2 = \omega^2(A^2 - x^2)$ |
| Substitute values. | $(0.075)^2 = 1^2[A^2 - (0.075)^2]$ |
| Solve for $A$. | $A^2 = 2 \times (0.075)^2$<br>$A \approx 0.106\ldots$ m |
| Write the velocity formula. | $\dot{x} = \pm\omega\sqrt{A^2 - x^2}$ |
| Substitute values, with $x = 0$. Direction doesn't matter. | $\dot{x}_{max} = 1 \times \sqrt{0.106\ldots^2 - 0}$<br>$= 0.106\ldots$ m s$^{-1}$ |
| Write the answer. | The greatest velocity is about 0.106 m s$^{-1}$, in either direction. |

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| | | |
|---|---|---|
| **b** | Write the period formula. | $T = \frac{2\pi}{\omega}$ |
| | Substitute $\omega = 1$. | $= 2\pi$ |
| | Write the answer. | The period is $2\pi$ s. |

The equations of simple harmonic motion can also be used in *reverse*. For example, you can show that a particle is in simple harmonic motion if you can show that the acceleration $\ddot{x} = -\omega^2 x$.

| | |
|---|---|
| Consider a particle that moves in a straight line so that its position at any time $t$ is given by | $x = 3\cos(4t) + 5\sin(4t)$. |
| Differentiate. | $\dot{x} = -12\sin(4t) + 20\cos(4t)$ |
| Differentiate again. | $\ddot{x} = -48\cos(4t) - 80\sin(4t)$ |
| Simplify. | $= -16[3\cos(4t) + 5\sin(4t)]$ |
| | $= -16x$ |

This equates to simple harmonic motion with $\omega = 4$.

## EXAMPLE 29

A particle moves in a straight line so that its acceleration at any time is given by $\ddot{x} = -4x$. If $x = 3$ and $v = -6\sqrt{3}$ when $t = 0$, calculate:

**a** the period **b** the amplitude **c** the displacement at time $t$.

### Solution

| | | |
|---|---|---|
| **a** | Write the acceleration formula for simple harmonic motion. | $\ddot{x} = -\omega^2 x$ |
| | Write the given condition. | $\ddot{x} = -4x$ |
| | Write the conclusion and find $\omega$. | The motion is simple harmonic with $\omega = 2$. |
| | Write the formula for period. | $T = \frac{2\pi}{\omega}$ |
| | Substitute the value of $\omega$. | $= \frac{2\pi}{2}$ |
| | Solve for $T$. | $= \pi$ |
| | Write the answer. | The period is $\pi$. |


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**b** Write the velocity formula. $\dot{x} = \pm\omega\sqrt{A^2 - x^2}$

Substitute values when $t = 0$. $-6\sqrt{3} = -2\sqrt{A^2 - 3^2}$

Solve for $A$.

$$\sqrt{A^2 - 9} = 3\sqrt{3}$$
$$A^2 - 9 = 27$$
$$A^2 = 36$$
$$A = 6$$

Write the answer. The amplitude is 6.

**c** Write the displacement formula. $x = A\cos(\omega t + \alpha)$

Substitute values at $t = 0$. $3 = 6\cos(2 \times 0 + \alpha)$

Solve for $\cos\alpha$.

$$6\cos(\alpha) = 3$$
$$\cos(\alpha) = \frac{1}{2}$$

Write the velocity formula with $\alpha$. $\dot{x} = -A\omega\sin(\omega t + \alpha)$

Substitute values at $t = 0$. $-6\sqrt{3} = -6 \times 2\sin(2 \times 0 + \alpha)$

Solve for $\sin(\alpha)$. $\sin(\alpha) = \frac{\sqrt{3}}{2}$

Use the signs to write the quadrant of $\alpha$. $\alpha$ must be in the 1st quadrant.

Find $\alpha$. $\alpha = \frac{\pi}{3}$

Substitute values to find $x$. $x = 6\cos\left(2t + \frac{\pi}{3}\right)$

Write the answer. The displacement at time $t$ is

$$x = 6\cos\left(2t + \frac{\pi}{3}\right)$$

## Exercise 9.11 Simple harmonic motion

**1** A spring makes 12 vibrations in 3 s. Calculate the period and frequency of the vibration.

**2** A body in simple harmonic motion makes 90 vibrations per minute. Calculate the period and the frequency.

**3** What is the period of a particle undergoing simple harmonic motion with angular velocity $\frac{\pi}{2}$ m s$^{-1}$?

Examples 27, 28

**4** A mass vibrates in simple harmonic motion with an amplitude of 75 cm and a period of 3 s. Calculate:

**a** the frequency

**b** the maximum velocity (the velocity when the displacement $x = 0$)

**c** the velocity when $x = 50$ cm

**d** the maximum acceleration

**e** the acceleration when $x = 50$ cm.

**5** A mass vibrates up and down along a straight line, 20 cm long, in simple harmonic motion with a period of 4 s. Calculate:

**a** the amplitude of the vibration

**b** the speed and acceleration of the body at the midpoint of its path

**c** the speed and acceleration of the body at the upper end of its path

**d** the speed and acceleration of the body when the displacement from the centre is 4 cm.

**6** A particle moves in a straight line such that, at $t$ s, its displacement $x$ m from a fixed point $P$ in its path is given by $x = 8\cos\left(2t + \frac{\pi}{6}\right)$.

**a** Find an expression for acceleration in terms of $x$ only, and hence show that the motion is simple harmonic.

**b** Calculate the period and amplitude of the motion.

**c** Calculate the speed and acceleration when the displacement is $-4$ m.

Example 29

**7** An object undergoes simple harmonic motion so that it oscillates about the equilibrium position with a period of $T = 0.5$ s. The amplitude of the vibration, $a = 2$ cm. Find:

**a** the maximum acceleration

**b** the maximum velocity

**c** the acceleration when the displacement is 1.5 cm

**d** the velocity when the displacement is 1 cm

**e** the equation of motion as a function of time if $x = 0$ at $t = 0$

**8** A particle undergoes motion described by the equation $\ddot{x} = -25x$. When $t = 0$, $x = 0$ and $\dot{x} = 1$.

**a** Find an expression for its displacement at time $t$.

**b** Find an expression for its velocity at time $t$.

**c** Find an expression for its acceleration at time $t$.

**d** Find the period and amplitude of its motion.


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## Problem solving

**9** A particle starts from rest at $x = 10$ and proceeds with simple harmonic motion about $x = 0$ so that after 2 s it reaches $x = 5$. If displacement is in millimetres, find:

**a** an expression for the displacement ($x$) at any time $t$

**b** the speed at $x = 0$

**c** the amplitude and period of motion

**d** the maximum speed

**e** the maximum acceleration.

**10** A particle with simple harmonic motion is moving with a velocity of 3.6 m s$^{-1}$ as it passes through its central position. When the particle is 0.2 m from the central position, it has an acceleration of 4.8 m s$^{-2}$. Calculate:

**a** the amplitude of the oscillation

**b** the period of the oscillation.

**11** The velocity of a particle moving in a straight line is given by $v = 10\sqrt{20 - x^2}$, where $x$ is the displacement from the point $P$.

**a** Show that the motion is simple harmonic.

**b** Calculate the period.

**12** If a particle moves in a straight line such that $(\dot{x})^2 = 8(3 - x^2)$, where $x$ is the displacement at time $t$:

**a** find the acceleration in terms of only $x$

**b** calculate the period and amplitude.

**13** A particle moves in a straight line such that, at $t$ s, its displacement $x$ m from a fixed point $P$ in its path is given by $x = 5\cos\left(\frac{\pi}{2}t + \frac{\pi}{6}\right)$.

**a** Find an expression for acceleration in terms of $x$ only, and hence show that the motion is simple harmonic.

**b** Calculate the period and amplitude of the motion.

**c** Calculate the speed and acceleration when the displacement is −2.5 m.

**14** A particle moves in a straight line with simple harmonic motion. Calculate the time to undergo a complete oscillation if:

**a** the acceleration at a distance of 1.2 m is 2.4 m s$^{-2}$

**b** the acceleration at a distance of 20 cm is 3.2 m s$^{-2}$.

**15** A particle moving with simple harmonic motion has an amplitude of 1.5 m. If the acceleration at a distance of 60 cm from the equilibrium position is 1.2 m s$^{-2}$, find the velocity when the particle is:

**a** in the equilibrium position

**b** 1.2 m from its equilibrium position.

**16** A particle moving with simple harmonic motion passes through two points $P$ and $Q$ that are 56 cm apart with the same velocity. It takes the particle 2 s to move from $P$ to $Q$ and then another 2 s to return to $Q$. Calculate the period and amplitude of the oscillation.

**17** A particle undergoing simple harmonic motion completes one full oscillation every 2 s. If the amplitude of the oscillation is 90 cm, calculate the maximum velocity and the maximum acceleration.

**18** An object in simple harmonic motion performs 150 complete oscillations every minute. If the greatest acceleration achieved is 3 m s$^{-2}$, calculate the greatest velocity and the distance between the extreme points of the motion.

**19** A particle moving with simple harmonic motion performs 45 complete oscillations per minute. The velocity at a point 2.5 cm from the central point is 30 cm s$^{-1}$.

**a** Calculate the greatest distance reached from the central point.

**b** If $P$ and $Q$ are two points positioned 2.5 cm and 4 cm respectively from the centre of motion, find the time taken in moving from $P$ to $Q$.

**20** The speed of a particle is given by $v = 2\sqrt{5+4x-x^2}$ m s$^{-1}$.

**a** Show that the motion is simple harmonic.

**b** Calculate the amplitude and the period.

**21** A component in a machine vibrates with an angular velocity of 120 radians per second and experiences a maximum acceleration of 1000 m s$^{-2}$. Calculate the maximum displacement and the maximum speed of the component if it is moving in simple harmonic motion.

**22** The velocity of a particle moving in a straight line is given by $v = \sqrt{12+8x-4x^2}$, where $x$ is the displacement from a central point.

**a** Show that the motion is simple harmonic.

**b** Calculate the amplitude and the period of oscillation.

**23** It is accepted that the rise and fall of the tide at a particular inlet is simple harmonic motion, with the time difference between successive high tides being 10 h. The entrance to the inlet has a depth of 20 m at high tide and 8 m at low tide. If low tide occurs at 10:00 a.m. on a certain day, find the earliest time at which a vessel requiring a minimum water depth of 15 m can pass through the entrance.

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# 9. CHAPTER SUMMARY

## Differential equations

- **Implicit differentiation** is the process of differentiating an implicit equation with respect to the desired variable $x$ while treating the other variables as functions of $x$
- A **tangent** to a curve is a line that touches the curve at one point and has the same slope as the curve at that point
- The **normal** to a curve at a particular point is a line passing through that point that is perpendicular to the curve, and hence perpendicular to the tangent
- The relationship between **related rates of change** with respect to time can be found using the chain rule $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$
- A mathematical model that involves an equation that contains an unknown function and one or more of its derivatives is known as a differential equation. A **first-order differential equation** involves only the first derivative.
- For problems involving rates of change, the situation at time $t = 0$, $y(t_0) = y_0$ is called the **initial condition**
- For the function $f(x)$, the differential of $f$ is $df = f'(x)\,dx$, where $dx$ is a very small change in the value of $x$. If $f$ and $g$ are functions and $c$ is a constant, the rules for differentials are:

**1** $d(c) = 0$

**2** $d(c \times f) = c \times df$

**3** $d(f \pm g) = df \pm dg$

**4** $d(f \times g) = g \times df + f \times dg$

**5** $d\left(\frac{f}{g}\right) = \frac{g \times df + f \times dg}{g^2}$

**6** $d(x^n) = nx^{n-1}\,dx$

**7** $d(e^x) = e^x\,dx$

**8** $d(\ln|x|) = \frac{1}{x}dx$

**9** $\lim_{dx \to 0} \frac{df}{dx} = f'(x)$

- The differential equation $\frac{dy}{dx} = ky$ has the general solution $y = y_0\,e^{kx}$, where $y_0$ is the initial condition
- A differential equation with separable variables $\frac{dy}{dx} = f(x)g(y)$ can be solved using $\int \frac{1}{g(y)}\,dy = \int f(x)\,dx + c$, where $c$ is a constant of integration
- A **slope field**, **gradient field** or **direction field** is a tool to graphically obtain the solutions to a first-order differential equation

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- The **logistic model** of growth is given by $\frac{dy}{dt} = ky(M - y)$, where the rate of change of the variable is proportional to both the variable and the difference between the maximum value and the variable
- The general solution for the differential logistic equation $\frac{dy}{dt} = ky(M - y)$ is:

$$y = \frac{MA}{A + e^{-kMt}} = \frac{My_0}{y_0 + (M - y_0)e^{-kMt}}, \text{ where } A = \frac{y_0}{M - y_0} \text{ and } y_0 \text{ is the initial value}$$

- The general solution for the differential logistic equation $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$ is:

$$P = \frac{M}{1 + Ae^{-kt}}, \text{ where } A = \frac{M - P_0}{P_0}$$

- **Newton's 1st law of motion** states that unless acted on by a resultant force, a body remains at rest or in uniform motion in a straight line
- **Newton's 2nd law of motion** states that the acceleration of a body is proportional to the resultant force that acts on the body and inversely proportional to the mass of the body. The acceleration is in the direction of the resultant force. Newton's second law can also be written as $\mathbf{F} = m\mathbf{a}$, where $\mathbf{F}$ and $\mathbf{a}$ are vectors and $m$ is a scalar quantity.
- **Newton's 3rd law of motion** states that for every action there is an equal and opposite reaction
- Forces that lie in the same plane and pass through a single point are known as **concurrent** forces. If the resultant of the concurrent forces is zero, then the system is said to be in **equilibrium** and the acceleration is also zero.
- Straight-line motion with constant acceleration $a$, velocity $v$, initial velocity $u$, time $t$ and displacement $s$ obey the following equations

$$v = u + at \qquad s = ut + \frac{1}{2}at^2$$

$$v^2 = u^2 + 2as \qquad s = \left(\frac{v + u}{2}\right)t$$

- In cases where the force and acceleration depend on the speed of the object, the acceleration can be written as $a = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$
- The equation of motion for a body falling under gravity with air resistance force $kv$ can be written as $ma = mg - kv$. As a body falls, eventually the force due to gravity is balanced by the force due to air resistance. As $t \to \infty$, the body reaches a limiting velocity called the **terminal velocity** $v_T$, where $v_T = \lim_{t \to \infty} v(t)$. The terminal velocity is constant and so when a body reaches terminal velocity $a = 0$.


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- A particle moving with **simple harmonic motion** moves in a straight line such that its acceleration is always directed to a fixed point and is proportional to its distance from this point. The particle oscillates about the central fixed point or **point of equilibrium** with maximum displacement (called the **amplitude**). The force is given by $\mathbf{F} = -k\mathbf{x}$, where $\mathbf{x}$ is the displacement of the particle from the point of equilibrium.
- For simple harmonic motion, $\ddot{x}$ is used for acceleration and $\dot{x}$ for velocity. The equations for simple harmonic motion with amplitude $A$, phase shift $\alpha$, angular velocity $\omega$, period $T$ and frequency $f$ are:
  - Acceleration: $\ddot{x} = -\omega^2 x = -a\omega^2 \sin(\omega t + \alpha)$ or $-a\omega^2 \cos(\omega t + \alpha)$
  - Velocity: $v = \dot{x} = \pm\omega\sqrt{A^2 - x^2} = -A\omega \sin(\omega t + \alpha)$ or $A\omega \cos(\omega t + \alpha)$, $v^2 = \omega^2(A^2 - x^2)$
  - Displacement: $x = A\cos(\omega t + \alpha)$ or $A\sin(\omega t + \alpha)$
  - Period: $T = \frac{2\pi}{\omega} = \frac{1}{f}$, so $\omega = 2\pi f$

# 9. CHAPTER REVIEW

## Differential equations

Example 1

**1** Use implicit differentiation to find $\frac{dy}{dx}$ in terms of $x$ and $y$ for each equation.

**a** $8xy + 5x^2 = 3y^2$ **b** $3xe^y - 2xy = 7$ **c** $x^2 \sin(y) = e^y$

Example 2

**2 a** Find the equation of the tangent to $\frac{x^2}{36} + \frac{y^2}{25} = 1$ at $(-3.6, 4)$.

**b** Find the equation of the normal to $\frac{x^2}{25} = 1 + y^2$ at $(13, -2.4)$.

Example 3

**3** Evaluate $\frac{dy}{dt}$ at $x = 2$, given:

**a** $y = (3x^2 - 15)^4$ and $\frac{dx}{dt} = \frac{1}{12}$ **b** $y = 3xe^{4x}$ and $\frac{dx}{dt} = 3$

Example 4

**4** A cubic container has a side of 20 cm. The container is being filled with water so that the rate of increase of the water level $\frac{ds}{dt} = 2$ cm/min. At what rate is the volume of water in the container increasing $\left(\frac{dV}{dt}\right)$ when the water level is 5 cm?

Example 5

**5** Gas is escaping from a spherical balloon at a rate of 2 $m^3$/min.

**a** Find the rate at which the radius is deceasing $\left(\frac{dr}{dt}\right)$ when the radius is 1.2 m.

**b** How fast is the surface area decreasing at the same time?

Example 6

**6** Find the general solution to each differential equation.

**a** $\frac{dy}{dn} = 12\cos(n)$ **b** $\frac{dy}{dt} = 12t^2 + 8$

**c** $\frac{dp}{dx} = \frac{1}{3x^2 + 3}$ **d** $\frac{dx}{dt} = \frac{t}{3t^2 + 8}$

Example 7

**7** Find the solution of each differential equation.

**a** $\frac{dy}{dx} = x\cos(x)$ if $y\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}$

**b** $\frac{dy}{dx} = \frac{1}{\sqrt{1-4x^2}}$ if $y = \pi$ when $x = \frac{1}{2}$

**c** $\frac{dy}{dx} = \frac{1}{4x+3}$ if $y = 3$ when $x = -1$


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**8** An object moves so that its acceleration is given by $a = 6t - 12$ m s$^{-2}$, where $t$ is in seconds. Initially, the object is 5 m above sea level travelling at 9 m s$^{-1}$. Use differential equations to find: Example 8

**a** the velocity of the projectile after 4 s

**b** the height of the projectile after 3 s.

**9** Find the general solution for each differential equation. Example 9

**a** $\frac{dy}{dx} = 0.07y$ **b** $\frac{dt}{dx} = -t$ **c** $\frac{dg}{dh} = 2.8g$

**d** $\frac{dy}{dx} = 0.05y + 0.2$ **e** $\frac{dy}{dq} = -0.8y + 4$ **f** $\frac{dg}{dp} = 1.6g - 4$

**10** The half-life of a radioactive element used in the treatment of cancer is 4 hours. A 12 mg amount is produced at 10 a.m. Example 11

**a** Use a differential equation to find a formula for the amount present at time $t$.

**b** How much of the radioactive element remains at 4 p.m.?

**11** The coffee in a cup is at a temperature of 72°C. The coffee is placed on a table in the sun where the surrounding temperature is 44°C. After 30 minutes the coffee has cooled to 61°C. Example 11

**a** What is the temperature of the coffee after another 20 minutes?

**b** How long does it take for the coffee to cool to 52°C?

**12** Find all solutions to each differential equation. Example 12

**a** $\frac{dy}{dx} = (3y - 1)(2x + 5)$ **b** $\frac{dy}{dx} = y(3y + 2)x$

**c** $\frac{dy}{dx} = \frac{3x^2 - 1}{3 + 2y}$ **d** $y^2 \sin(x) + \frac{dy}{dx} = 0$

**13** Solve each differential equation. Example 12

**a** $\frac{dy}{dx} = \frac{2x}{y + x^2 y}$ if $y(0) = -2$ **b** $\frac{dy}{dx} + \frac{x}{y - 3} = 0$ if $y(0) = 1$

**14** For the differential equation $\frac{dy}{dx} = x - y$: Example 14

**a** draw the slope field

**b** draw the solution curve for the initial condition $y = 0$ when $x = 0$

**c** draw the solution curve for the initial condition $y = 2$ when $x = -2$

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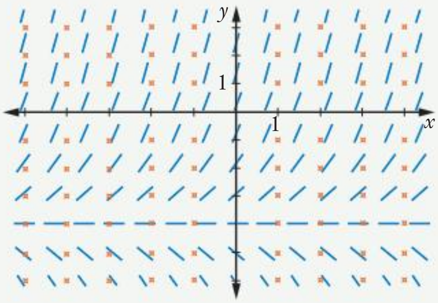
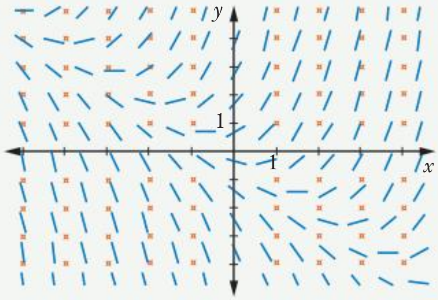
**15** Match each differential equation with its slope field.

**a** $\dfrac{dy}{dx} = \dfrac{x}{y}$ **b** $\dfrac{dy}{dx} = x + y$ **c** $\dfrac{dy}{dx} = 4 + y$ **d** $\dfrac{dy}{dx} = x - 4$

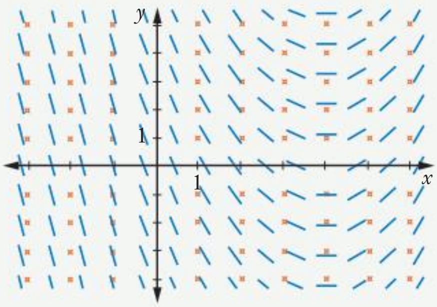
**A**



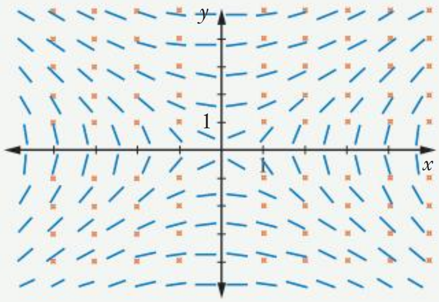

**B**




**C**




**D**




**16** A population of native quolls is housed in a large fenced enclosure with a maximum carrying capacity of 100 quolls. The population growth of the quolls is modelled by the logistic differential equation $\dfrac{dP}{dt} = 0.01P(100 - P)$, where $t$ is measured in years.

There are initially 20 quolls in the enclosure.

**a** Find an expression for the number of quolls in the enclosure.

**b** How many quolls will be present after 2 years?

**c** How long will it take for 80 quolls to be present?

**17** An area in western Queensland can support a maximum of about 30 000 rabbits. Initially there are 3000 rabbits and this population grows to 5000 after 1 month. Use the general solution for the logistic model, with $t$ measured in months, to predict the number of rabbits for months 2 through 12 following the initial count.

**18** A contagious disease spreads through a town with a population of 10 000 people. At the time the outbreak of the disease is discovered, there are 200 infected people. The growth rate of the infected population is estimated to be 1.7 per month. The spread of the disease follows the logistic model $\dfrac{dP}{dt} = kP(1 - \dfrac{P}{M})$, where $P$ is the infected population and $M$ is the total population.

**a** How many infected people are there 20 months after the outbreak?

**b** How long will it take for the number of infected people to reach 7000?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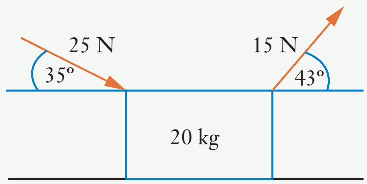
**19** A mass of 20 kg rests on a flat surface. A force of 25 N acts at one end of the mass and a force of 15 N acts at the other, as shown in the diagram.




**a** What is the normal force on the mass?

**b** What is the acceleration of the mass?

Example 20

**20** A tennis ball of mass 58 g strikes a wall horizontally with a velocity of 10 m $s^{-1}$. It rebounds at a velocity of 8 m $s^{-1}$. If the collision lasts for 6 milliseconds, calculate the average force exerted by the tennis ball on the wall.

Example 20

**21** An astronaut pushes on a large piece of space junk with a force of 45 N. If the mass of the piece of space junk craft is 10 000 kg and the mass of the astronaut is 90 kg, what are the accelerations of each after the push?

**22** A car is traveling at 10 m $s^{-1}$, and increases its speed to 30 m $s^{-1}$ with a constant acceleration of 1.5 m $s^{-2}$ along the straight road. Calculate:

Example 21

**a** how long it takes for the car to reach 30 m $s^{-1}$

**b** how far the car travels in this time

**23** A mass is thrown upwards at 15 m/s from a tower that is 60 metres above the ground.

Example 22

**a** Determine the total time that the mass is in the air for before it reaches the ground.

**b** Find the maximum height reached by the mass.

**c** Find the mass' speed when it first strikes the ground.

**24** A car is travelling on a straight road at a constant speed of 54 km $h^{-1}$ and passes stationary motorcycle on an entrance ramp to the road. The motorcycle immediately enters the road and accelerates at a constant 1.4 m $s^{-2}$.

Example 23

**a** How long does it take for the motorcycle to reach the car?

**b** How far did the motorcycle travel before reaching the car?

**c** How fast was the motorcycle travelling when it reached the car?

**25** A 3 kg body is acted on by a force with magnitude at time $t$ s given by $(8t - 5t^2)$ N and constant direction. If the body has an initial velocity of 5 m $s^{-1}$ in the direction of the force, find:

Example 24

**a** an expression for the velocity as a function of time

**b** the velocity at the end of 3 s.

**26** A particle moves in a straight line so that, at time $t$, its displacement from a fixed point (origin) is $x$ and its velocity is $v$. If the acceleration is $(4 - 3x)$ m $s^{-2}$, find $v$ in terms of $x$, given that $v = 5$ when $x = 1$.

Example 25

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Example 26

**27** An object with a mass of $m$ kg falls vertically downwards. The force of resistance to motion is $F = kv$.

**a** Show that $\frac{dv}{dt} = \frac{mg - kv}{m}$.

**b** The time interval for which the object falls is $0 \le t \le v(T)$. Use the definite integral with bounds $t = 0$ and $t = v(T)$ to show that $v(T) = \frac{1}{k}\left(mg - e^{-\frac{kt}{m}}\right)$.

Examples 27, 28

**28** A mass vibrates in simple harmonic motion with an amplitude of 60 cm and a period of 4 s. Calculate:

**a** the frequency

**b** the maximum velocity (the velocity when the displacement $x = 0$)

**c** the velocity when $x = 20$ cm

**d** the maximum acceleration

**e** the acceleration when $x = 20$ cm.

Example 29

**29** An object moves in a straight line so that its acceleration at any time is given by $\ddot{x} = -9x$. If $x = 2\sqrt{2}$ and $v = -6\sqrt{2}$ when $t = 0$, calculate:

**a** the period

**b** the amplitude

**c** the displacement at time $t$.

## Problem solving

**30** A spherical balloon is inflated so that its radius ($r$) increases at a rate of $\frac{3}{r}$ cm/s. How fast is the volume of the balloon increasing when the radius is 12 cm?

**31** Two cars start from the same place at the same time. One travels due east at 80 km/h while the other travels due south at 45 km/h. Calculate the rate of change of the distance between the cars 3 hours later.

**32** Compound interest is paid at a rate such that the increase in the capital at a rest depends on the amount of capital at the time. Instead of expressing interest as being charged over a particular period (monthly, annually, etc.), we can express it as a *continuous interest rate*. This is the rate $r$ such that at any time $\frac{dP}{dt} = rP$, where $P$ is the amount after time $t$. Solve the differential equation and use normal interest calculations to find the continuous interest rate equivalent to 9% compounded monthly.

**33** An oven is heated at a constant rate of 3600 $Js^{-1}$ (watts, W) from its initial temperature of 20°C. At the same time, the oven loses heat to its surroundings at the rate of $7.5(T - 20)$ W, where $T$ is the temperature of the oven. Each 6000 J of energy retained by the oven increases the temperature by 1°C. Use a differential equation to find the temperature of the oven after 5 minutes, and the eventual temperature.


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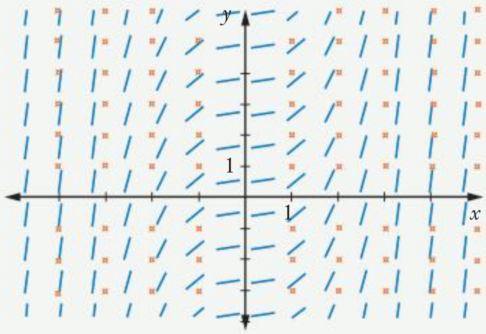
**34** The number of users of a new phone app increases each week. The increase is proportional to both time and the number of users.
3 weeks after it released there are 100 users and after 6 weeks this has grown to 500. Predict the number of users after 10 and 15 weeks and state any problems with the predictions.

**35** The slope field for a particular differential equation is shown. Which of the following could be the solution to the differential equation?




**A** $y = -\frac{y}{x}$

**B** $y = \frac{1}{2}x^2$

**C** $y = 3e^{-x}$

**D** $y = \frac{1}{3}x^3$

**E** $y = \ln|2x|$

**36** An object of mass $m$ kg begins from rest and is pulled across a friction-free surface with a force of $P$ N. The object experiences air resistance of magnitude $kv$, where $v$ is its speed. Find an expression for:

**a** the velocity of the object as a function of time

**b** the terminal velocity

**c** the displacement as a function of time.

**37** A particle moving with simple harmonic motion has velocities of 1.2 m $s^{-1}$ and 0.9 m $s^{-1}$ when it is 0.9 m and 1.2 m respectively from its central position. Find the period of the oscillation and the maximum acceleration of the particle.

Qz
Practice quiz

STATISTICAL INFERENCE

# 10.

# CONFIDENCE INTERVALS FOR MEANS

You have seen that sampling distributions are similar to normal distributions and that the larger the sample size, the more they resemble a normal distribution. To use the normal distribution, you need to know the mean and standard deviation or variance. However, you cannot be certain of a parameter because a population is too big to survey and in most cases, you simply cannot access a whole population anyway. This means you need to know how well a statistic estimates a parameter. Now you can use your previous work in both Mathematical Methods and Specialist Mathematics to use statistics to estimate parameters.

10.01 Confidence intervals of a normal distribution
10.02 Estimating population mean and standard deviation
10.03 Confidence intervals in surveys
Chapter summary
Chapter review

## Confidence intervals for means

- understand the concept of an interval estimate for a parameter associated with a random variable
- examine the approximate confidence interval $\left(\bar{x} - z\frac{s}{\sqrt{n}}, \bar{x} + z\frac{s}{\sqrt{n}}\right)$, as an interval estimate for $\mu$, the population mean, where $z$ is the appropriate quantile for the standard normal distribution
- use simulation to illustrate variations in confidence intervals between samples and to show that most but not all confidence intervals contain $\mu$
- use $\bar{x}$ and $s$ to estimate $\mu$ and $\sigma$, to obtain approximate intervals covering desired proportions of values of a normal random variable and compare with an approximate confidence interval for $\mu$
- collect data and construct an approximate confidence interval to estimate a mean and to report on survey procedures and data quality.

Prior learning



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# TERMINOLOGY

confidence interval
interval estimate
quantile
confidence level
margin of error
estimator
point estimate

WS Normal distributions

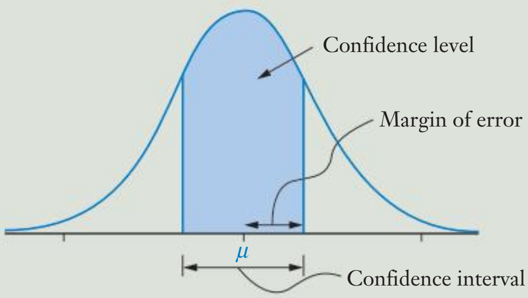
# 10.01 Confidence intervals of a normal distribution

In gardening books, the height of a macadamia tree is given as either 20 m or 15 to 25 m. One is a particular value (point estimate) and the other is an interval that contains the height (interval estimate). Both methods are used in Statistics.

You have already seen that the mean of a sampling distribution is likely to be close to the population mean. It doesn't matter what size samples you use, this will always be the case. However, as you have seen, the larger the sample size, the better the estimate will be. It is better to give an indication of the accuracy of an estimate than the estimate alone. You use a **confidence interval** for this purpose.

## Estimates and confidence intervals

- A **point estimate** of a parameter is a single value obtained from a sample. The statistic used is called an **estimator** of the parameter
- An **interval estimate** of a parameter is an interval that is likely to include the value of the parameter
- A **confidence interval** is a range of values for an estimate of a parameter
- The **confidence level** is the proportion of values of an estimate that are expected to lie within the interval




- For a confidence interval symmetric about the mean, the **margin of error** is half the width of the interval. It is the is the distance of the ends of the interval from the mean

The confidence level is represented by the area under the probability distribution function of a continuous probability distribution.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Remember that the **first quartile** of a statistical distribution is the value below which 25% of the values are found. This is the same as saying the probability of a value being below the first quartile is 0.25. The 40th **percentile** of a statistical distribution is the value below which 40% of the values are found. Again, this is the same as saying that the probability of a value being below the 40th percentile is 0.4.

The general name for values like these is a **quantile**. For a proportion $\alpha$, the quantile $a$ is the value below which the proportion $\alpha$ of the values are found. This is the same as saying the probability of a value being below $a$ is $\alpha$.

You use the inverse function of the cumulative distribution function of a probability distribution to find this value.

## EXAMPLE 1

What is the quantile for $\alpha = 0.87$ for a normal distribution with:

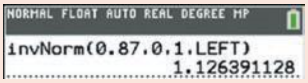
**a** mean 0 and standard deviation 1 (the standard normal distribution)?

**b** mean 42 and standard deviation 5?

### Solution

**a** **TI-84 Plus CE**

Press 2nd vars (distr) and choose 3: invNorm(. Complete the area as 0.87 and leave $\mu$: 0, $\sigma$: 1 and Tail: LEFT. Paste the result as 0.87, and leave the Tail as LEFT. Go down to Paste and press enter.

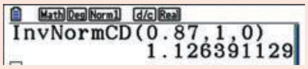



Write the answer.

**Casio fx-CG20AU**

Use the RUN-MATRIX menu.

Press OPTN and choose STAT, DIST, NORM and InvN. Complete the function with 0.87, 1, 0) and press EXE.




The quantile is about 1.1264

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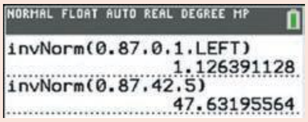
**b** **TI-84 Plus CE**

Complete invNorm( with 0.87, 42 and 5. The default tail is left.




Write the answer.

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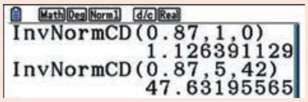
Complete InvNormCD with 0.87, 5 and 42.




The quantile for $\alpha = 0.87$ for a normal distribution with mean 42 and standard deviation 5 is about 47.6.

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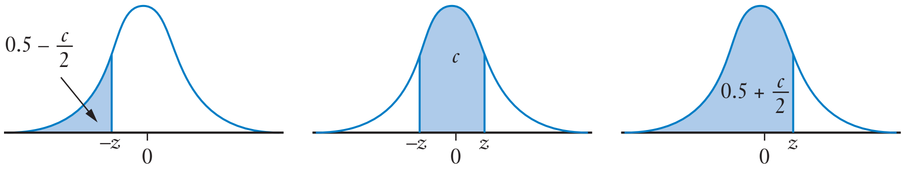
Consider a confidence level for the normal distribution that is symmetrical about the mean. For a confidence level $c$, the value of $\alpha$ for the appropriate quantiles will be $\alpha = 0.5 - \frac{c}{2}$ and $\alpha = 0.5 + \frac{c}{2}$.




The values of the normal distribution for the boundaries of a symmetrical confidence interval will be $-z$ and $z$, where $z$ is the quantile for the confidence level $c$. It is easiest to find the value of $z$ for the upper boundary quantile $\alpha = 0.5 + \frac{c}{2}$.

## EXAMPLE 2

What is the 90% confidence interval for the standard normal variable $Z$?

### Solution

State the mean and standard deviation.

$\mu = 0$ and $\sigma = 1$

Find the value of the upper boundary of $c = 0.9$.

$$\alpha = 0.5 + \frac{0.9}{2}$$
$$= 0.95$$

Use your graphics calculator.

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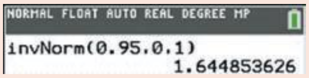
**TI-84 Plus CE**

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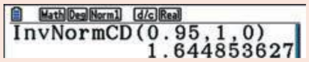


Write the answer.

The 90% confidence interval for the standard normal variable is about [−1.64, 1.64].

It doesn't matter whether you write the interval as [−1.64, 1.64] or (−1.64, 1.64) because the variable is continuous. Of course, you can also write it with order signs as $-1.64 \le z \le 1.64$ or $-1.64 < z < 1.64$.

For other normal variables you need to adjust the procedure. The confidence interval will be $[\mu - z\sigma, \mu + z\sigma]$, where $z$ is the quantile of the standard normal variable, $\mu$ is the mean and $\sigma$ is the standard deviation.


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## EXAMPLE 3

What is the 80% confidence interval for the normal variable $X$ with mean 48 and standard deviation 11.5?

### Solution

Find the value of $\alpha$ for the upper boundary.

$\alpha = 0.5 + \frac{0.8}{2} = 0.9$

**Method 1**

Use your calculator to find $z$ for $\alpha = 0.9$.

$z = 1.28\ldots$

Find the boundaries for the variable $X$ using the number of standard deviations $z$.

The confidence interval $= \mu \pm z\sigma$

$= 48 \pm 1.28\ldots \times 11.5$

$\approx [33.3, 62.7]$

**Method 2**

Use your calculator to find $X$ for the upper boundary. Use $\mu = 48$ and $\sigma = 11.5$.

Upper boundary $= 62.73\ldots$

Find the lower boundary.

Lower boundary $= 48 - (62.73\ldots - 48)$

$= 48 - 14.73\ldots$

$= 33.26\ldots$

Write the answer.

The confidence interval is about [33.3, 62.7].

## Exercise 10.01 Confidence intervals of a normal distribution

**1** Find each quantile for a standard normal distribution, correct to 3 decimal places. Example 1

**a** $\alpha = 0.2$ **b** $\alpha = 0.3$ **c** $\alpha = 0.4$

**d** $\alpha = 0.5$ **e** $\alpha = 0.7$

**2** Find each quantile for a normal distribution with mean 35 and standard deviation 10, correct to 1 decimal place.

**a** $\alpha = 0.1$ **b** $\alpha = 0.6$ **c** $\alpha = 0.8$

**d** $\alpha = 0.9$ **e** $\alpha = 0.95$

**3** Find each quantile for a normal distribution with mean 100 and standard deviation 15, correct to 1 decimal place.

**a** $\alpha = 0.2$ **b** $\alpha = 0.4$ **c** $\alpha = 0.6$ **d** $\alpha = 0.8$ **e** $\alpha = 0.9$

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Example 2

**4** Find each confidence interval for a standard normal distribution.

**a** 80% **b** 95% **c** 97.5% **d** 99% **e** 50%

Example 3

**5** Find each confidence interval for a normal distribution with mean 8 and standard deviation 2.5.

**a** 40% **b** 60% **c** 80% **d** 90% **e** 95%

**6** Find each confidence interval for a normal distribution with mean 72 and standard deviation 16.

**a** 75% **b** 80% **c** 90% **d** 95% **e** 99%

**7** Find each confidence interval for a normal distribution with mean 40 and standard deviation 6.

**a** 60% **b** 80% **c** 94% **d** 99% **e** 99.5%

## Problem solving

**8** A lead-acid accumulator has an average life of 2 years with a standard deviation of 8 months. The manufacturer wants to set the guarantee period so that less than 6% of the batteries are likely to be returned. What should they set it as?


Shutterstock.com/Slavoljub Pantelic

**9** The mean score in a university course is 64 with a standard deviation of 15. What should the mark for a B be set at if it is desired that only 20% of students should get credits or better?

**10** Special tyres used in motor races have an average life of only 30 hours, with a standard deviation of 5 hours. How many tyres would you expect to be changed after 25 to 35 hours of use by a team that raced 6 cars? Assume that all the cars started with 4 new tyres.


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# 10.02 Estimating population mean and standard deviation

WS
Population parameters

You have already seen that the mean of a sample is a good estimator of the mean of a population. The standard deviation of a sampling distribution is given by $\frac{\sigma}{\sqrt{n}}$, where $\sigma$ is the population standard deviation and $n$ is the sample size. That means that the standard deviation of a sample mean is $\frac{\sigma}{\sqrt{n}}$.

You do not usually know the standard deviation of population, so you have to use the standard deviation of a sample as an estimate.

## Estimated mean and standard deviation

- The estimated mean and standard deviation of a population is taken to be the mean and standard deviation of a random sample
- Confidence intervals for the estimated mean are calculated using the estimated mean and the estimated standard deviation of the sampling distribution, $\sigma_{\overline{X}} = \frac{s}{\sqrt{n}}$, where $s$ is the sample standard deviation and $n$ is the sample size, approximately $\left(\bar{x} - z\frac{s}{\sqrt{n}}, \bar{x} + z\frac{s}{\sqrt{n}}\right)$
- The estimated mean and confidence interval will be reasonable estimates if $n \geq 30$

**EXAMPLE 4**

The mean of a sample of 50 items was 28 with a standard deviation of 8. Determine a 95% confidence interval for the population mean $\mu$.

### Solution

Find the estimated sampling distribution standard deviation.

$\sigma_{\overline{X}} = \frac{s}{\sqrt{n}} = \frac{8}{\sqrt{50}} = 1.131\ldots$

Find the upper boundary of the standard normal quantile.

$\alpha = 0.5 + \frac{0.95}{2} = 0.975$

Use your graphics calculator.

$z = 1.9599\ldots$

Find the interval.

Interval $= 28 \pm 1.9599\ldots \times 1.131\ldots$

$= (25.782\ldots, 30.217\ldots)$

Write the answer.

The 95% confidence interval for the population mean is about [25.8, 30.2].

You should memorise the 3 most important confidence intervals.

### Confidence intervals

For the standard normal distribution, the

- 90% confidence interval is [–1.645, 1.645]
- 95% confidence interval is [–1.960, 1.960]
- 99% confidence interval is [–2.576, 2.576]

You have already seen how to use your graphics calculator to find a confidence interval without using the standard normal distribution. When you do this, it is better to find the *lower limit* instead of the upper limit.

## EXAMPLE 5

The mean of a sample of 40 items was 56 with a standard deviation of 9.2. Determine a 90% confidence interval for the population mean.

### Solution

Find the estimated sampling distribution standard deviation.

$$\sigma_{\overline{X}} = \frac{s}{\sqrt{n}} = \frac{9.2}{\sqrt{40}} = 1.45\ldots$$

Find the lower boundary of the standard normal quantile.

$$\alpha = 0.5 - \frac{0.90}{2} = 0.05$$

Use your graphics calculator. Remember you can use Ans (2nd (-) or SHIFT (-)) instead of retyping.

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Chapter 10

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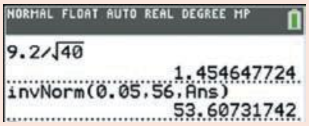


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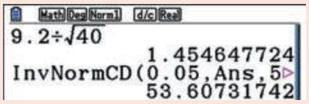


Find the upper boundary.

$$\text{Upper boundary} = 56 + 56 - 53.607\ldots$$
$$= 58.392\ldots$$

Write the answer.

The 90% confidence interval for the population mean is about [53.6, 58.4].

Since the confidence interval is an estimate, it does not make sense to quote it to more than 3 significant figures.


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You may have to calculate the sample mean and standard deviation from data to estimate the sample mean and standard deviation.

## EXAMPLE 6

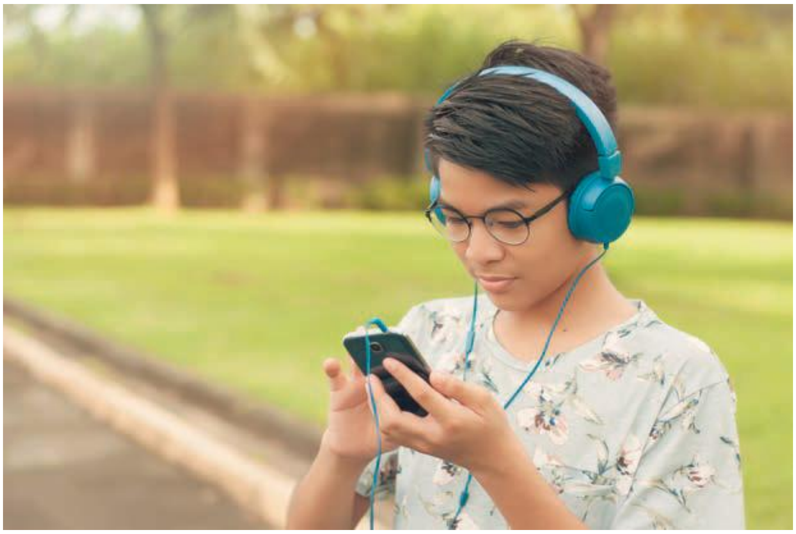
A sample of 20 measurements had values of:

65, 68, 65, 64, 71, 63, 62, 76, 62, 76, 75, 81, 67, 65, 73, 64, 71, 75, 75, 65.

Find the 97% confidence interval for the mean of the population.

### Solution

| | |
|---|---|
| Use your calculator to find the mean and standard deviation of the sample. | $\bar{x} = 69.15, s = 5.597\ldots$ |
| Find the estimated sampling distribution standard deviation. | $\sigma_{\bar{X}} = \frac{s}{\sqrt{n}} = \frac{5.597...}{\sqrt{20}} = 1.251\ldots$ |
| Find the upper boundary of the standard normal quantile. | $\alpha = 0.5 + \frac{0.97}{2} = 0.985$ |
| Use your graphics calculator. | $z = 2.170\ldots$ |
| Find the interval. | Interval $= 69.15 \pm 2.170\ldots \times 1.251\ldots$ $= [66.435\ldots, 71.865\ldots]$ |
| Write the answer. | The 97% confidence interval of the population mean is about [66.4, 71.9]. |



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## INVESTIGATION

### SIMULATED CONFIDENCE INTERVALS

You have already seen how to use your calculator to simulate sample means.

You can also your calculator to find the standard deviations of sample means.

- Use your CAS calculator to set up 30 samples of 25 from a uniform distribution on the interval 0–20.
- Find the mean of each sample.
- Use the standard deviation of the distribution. The mean of this distribution is 10 and the standard deviation is $\frac{20}{2\sqrt{3}}$. What is the (expected) standard deviation of the sampling distribution?
- Determine whether or not $\mu = 10$ is within the 90% confidence interval of the sample mean for each sample.
- Find the class average for the proportion of times the 90% confidence interval contains the mean.
- If you have time, repeat the process for a binomial distribution, a normal distribution, 95% and 80% confidence intervals.
- What can you conclude?

## Exercise 10.02 Estimating population mean and standard deviation

Examples 4,5

**1** The mean of a sample of 40 items was 36 with a standard deviation of 5. Estimate the 95% confidence interval for the population mean.

**2** The mean of a sample of 60 items was 42 with a standard deviation of 10. Estimate the 90% confidence interval for the population mean.

**3** The mean of a sample of 50 items was 107 with a standard deviation of 16. Estimate the 99% confidence interval for the population mean.

**4** The mean of a sample of 30 items was 24 with a standard deviation of 4.8. Estimate the 92% confidence interval for the population mean.

**5** The mean of a sample of 68 items was 45 with a standard deviation of 3.7. Estimate the 85% confidence interval for the population mean.

**6** The mean of a sample of 49 items was 87 with a standard deviation of 4.9. Estimate the 80% confidence interval for the population mean.


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**7** A sample of 30 measurements had values of:

Example 6

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 62 | 33 | 39 | 21 | 39 | 46 | 50 | 46 | 35 | 29 | 39 | 38 | 35 | 45 | 35 |
| 49 | 36 | 50 | 43 | 37 | 38 | 42 | 33 | 46 | 28 | 31 | 55 | 38 | 35 | 42 |

Find the 90% confidence interval for the mean of the population.

**8** A sample of 36 measurements had values of:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 6.2 | 8.3 | 7.2 | 11.1 | 9.4 | 6.2 | 8 | 10.9 | 9.5 | 9.5 | 6.5 | 12.3 |
| 7.1 | 6.7 | 12.1 | 8.8 | 7.4 | 7.9 | 8 | 7.4 | 8 | 8.5 | 11.5 | 10.5 |
| 5.2 | 8.9 | 8.7 | 8.6 | 8.4 | 12.8 | 8.2 | 7.4 | 9.8 | 8.5 | 9.2 | 5.7 |

Find the 95% confidence interval for the mean of the population.

## Problem solving

**9** A sample of 15 Bundaberg farms had these areas (in hectares) planted with tomatoes.

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 12 | 4 | 16 | 7 | 8 | 9 | 20 | 45 | 23 | 32 | 27 | 24 | 18 | 20 | 12 |

Use the data to estimate a 90% confidence interval of the average farm area devoted to tomatoes on Australian farms and state any problems with your estimate.

**10** The monthly sales of new passenger vehicles in Queensland for 2014–2017 were:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 7912 | 7349 | 10 137 | 7327 | 7770 | 11 978 | 8087 | 7710 | 8460 | 7060 | 7161 | 7870 |
| 6561 | 6824 | 8740 | 7000 | 7229 | 10 968 | 7279 | 6523 | 7306 | 6309 | 6875 | 7668 |
| 5996 | 6184 | 7441 | 5563 | 7032 | 9692 | 6571 | 6133 | 6532 | 5906 | 5873 | 6266 |

Estimate a 95% confidence interval of the mean and state any problems with your estimate.

**11** On a Monday morning, 2 classes of Year 12 students were asked how many songs they listened to on their mobile devices the day before. The numbers are shown below.

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 7 | 30 | 2 | 15 | 8 | 15 | 5 | 20 | 0 | 15 | 9 | 30 | 4 | 40 |
| 7 | 20 | 10 | 30 | 3 | 15 | 8 | 20 | 0 | 13 | 9 | 50 | 0 | 20 |
| 9 | 30 | 10 | 20 | 10 | 15 | 12 | 30 | 7 | 30 | 8 | 40 | 1 | 2 |

Estimate the 95% confidence interval for the mean and state any problems with the estimate.

# 10.03 Confidence intervals in surveys

WS Polls and levels of confidence

You often see reports on the news about the results of surveys. When you see or hear the results, they usually quote a point estimate. However, they may add include the sample size and a margin of error in the details.

**EXAMPLE 7**

A poll conducted close to a Federal election was based on 1243 interviews by trained telephone interviewers. Home telephone numbers and the person interviewed within the household were selected at random. According to the poll, the 'two-party preferred' support ALP 51% and Coalition 49% with a stated error margin of 3%.

The headline was 'ALP leads by 2% in latest poll'.

**a** What were the confidence intervals for the ALP and Coalition?

**b** Based on the poll figures, is the headline justified?

**c** Does there appear to be any possible bias?

### Solution

**a** Write the confidence intervals.

Confidence intervals are ALP [48%, 54%] and Coalition [46%, 52%].

**b** Write a conclusion.

Within the unstated level of confidence, the position is between 48 : 52 and 54 : 46, so you can be sure one or the other side is 2% ahead.

**c** State any problems with the survey method.

Home telephone numbers do not include mobiles, and younger people rely more on mobiles. This means there is a bias against younger voters. The interviewers cannot be certain who was interviewed.

State your conclusion.

There is possible bias in the selection of the sample. It is likely to be towards older people.

You cannot really work out the basis of a survey result without knowing the confidence level. Most surveys actually use a confidence level of 95%. This usually gives reasonable results without requiring too large a sample.

### Assumed confidence levels

Unless otherwise stated, the confidence level used in a report is 95%.


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## EXAMPLE 8

The Australian Bureau of Statistics (ABS) conducts monthly interviews of randomly-selected people aged over 15 to compile employment data. They interview 0.32% of this population and change the sample by $\frac{1}{8}$ every month. According to one such survey, the average hours worked by part-time males was 72 hours in the month. There were 1 060 400 males aged over 15 in part-time work out of 19 088 600 in the general population.

**a** Assuming that they are proportionally represented, how many men working part-time were interviewed?

**b** The variance of the hours worked was 369. What was confidence interval?

### Solution

**a** Find 0.32% of the males in part-time employment.

Number interviewed = 0.32% of 1060 400
= 3393

Write the answer.

3393 of those interviewed were part-time.

**b** Use $\text{SD} = \sqrt{\text{Variance}}$

$\text{SD} = \sqrt{369}$

Find the SD of the sampling distribution.

$\sigma_{\bar{x}} = \frac{\sqrt{369}}{\sqrt{3393}}$

$= 0.3297\ldots$

State the assumed confidence level.

Assume the confidence level is 95%

State the value of $z$ to use.

$z \approx 1.96$

Find the confidence interval.

Confidence interval $= 72 \pm 0.3297\ldots \times 1.96$

$\approx [71.4, 72.6]$

Write the answer.

At the 95% level, the confidence interval is [71.4, 72.6].

The ABS uses the international definition of employment, which is 1 hour or more of paid work in a week. The ABS publishes on its website all its methods and the way it calculates results.

## EXAMPLE 9

A political poll states that the results are within a margin of error of 3.5%.

**a** What would the margin of error be at the 90% level of confidence?

**b** Find the margin of error at the 99.9% level of confidence.

### Solution

**a** Write the assumption.

Assume the margin of 3.5% is at the 95% level of confidence.

Use the formula for the margin of error to find the standard deviation of the sampling distribution.

Margin of error $= \sigma_{\bar{x}}\, z$

$3.5\% = 1.96\, \sigma_{\bar{x}}$

$\sigma_{\bar{x}} = \dfrac{3.5\%}{1.96} = 1.7857\ldots\%$

Write the value of $z$ for a 90% confidence interval.

For 90% confidence, $z = 1.645$

Find the margin of error.

Margin of error $= \sigma_{\bar{x}} z$

$= 1.7857\ldots \times 1.645$

$= 2.937\ldots\%$

Write the answer.

At the 90% confidence level, the margin of error would be about 2.9%.

**b** Use your calculator to find the value of $z$ for a 99.9% confidence interval.

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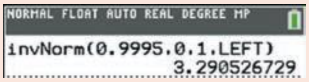


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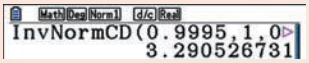


Find the margin of error.

Margin of error $= \sigma_{\bar{x}} z$

$= 1.7857\ldots \times 3.2905\ldots$

$= 5.875\ldots\%$

Write the answer.

At the 99.9% confidence level, the margin of error would be about 5.9%.


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## INVESTIGATION

### SURVEY AT SCHOOL

Choose a topic to do a survey in, such as the number of siblings, cousins, pets, rooms in the house, number of DVDs, music downloads or some other countable amount.
You could survey a person's measurement, a personal best time, personal best jump, length of journey to school, time spent on the internet or some other measurement.

- Choose an aspect for which you can find a mean
- Design a survey, making it as unbiased as you can
- Choose a (random) sample if possible, making sure it is large enough
- Do your survey and calculations, including confidence interval(s)
- Write your conclusions
- Suggest any practical ways your survey could be improved

Your teacher might want you to make a report.

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## Exercise 10.03 Confidence intervals in surveys

Example 7

1 A poll quoted percentages of 30%, 43%, 6% and 21% for the proportions of people with particular views. The margin of error was given as 4%. What were the confidence intervals?

2 A market researcher said the results of a survey showed that mobile phones were replaced after a period of 26 ± 5 months. What was the confidence interval?

Example 8

3 A survey of 47 people gave the average price of a sandwich press as being $49 with a margin of error of $1.50.

**a** What was the confidence interval?

**b** What was the sampling standard deviation?

**c** What was the standard deviation of the amount spent?

4 A newspaper report claimed that Australian adults owned an average of 29 items they used mainly for leisure. The margin of error was given as 2 and the sample size was 40.

**a** What was the confidence interval?

**b** What was the sampling standard deviation?

**c** What was the standard deviation of the number of items?

Example 9

5 A survey had an error margin of 25.

**a** What would be the 90% margin of error?

**b** What would be the 99% margin of error?

6 An error margin of $400 was quoted for the average amount people spent on a entertainment.

**a** What would be the 80% margin of error?

**b** What would be the 98% margin of error?

### Problem solving

7 A market research company conducted interviews to determine the best images to use for baby food containers. Each image was presented to 40 mothers of babies under 6 months old and the mothers rated the attractiveness of the image on a scale from 1 to 10. The most popular image was that of a smiling girl looking directly at the camera with an average rating of 8.5 ± 0.4. The second- and third-ranked images had a crawling baby and a baby being nursed by a young mother on 7.8 ± 0.4 and 7.5 ± 0.4 respectively. Is there any real difference between the ratings? Give detailed reasons for your answer.


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**8** A current affairs TV program asked viewers to state the prison sentence they would give a driver who killed 4 people walking on the footpath while under the influence of drugs. Viewers were also asked for the prison sentence if a driver had lost control of the car after sneezing on the way to the doctor. 300 people responded to the survey with average sentences of 15 years ± 4 months for drug addicts and 2 years ± 0.5 months for sneezing. Would you regard the result 'as powerful support for community anger about weak sentences for drug addicts' as claimed the next night when the poll results were given?

**9** A researcher found that the confidence interval for the increase in scores on a standardised test of reading age after an intensive 4-week course was 10–16 months at a confidence level of 80% for the 35 participants who completed the course. The intensive course cost $2500 for each participant. The fee was non-refundable, but there were still 15 people who did not complete the course. Given the cost of the course, do you think the claim of 'More than a year's improvement in only 4 weeks' is justified?

**10** The average rainfall on King Island in August is 118 ± 5 mm from 36 years of observations. Assume the rainfall is randomly distributed, so the 36 years can count as a random sample. The highest recorded rainfall in August was 214 mm. Does this cast any doubt on the assumption that the rainfall is randomly distributed?

AAP Image/Tourism Tasmania

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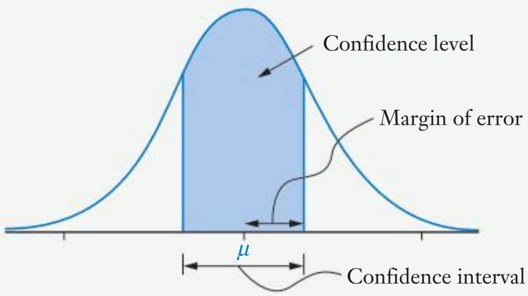
# 10. CHAPTER SUMMARY

## Confidence intervals for means

- A **point estimate** of a parameter is a single value obtained from a sample. The statistic used is called an **estimator** of the parameter
- An **interval estimate** of a parameter is an interval that is likely to include the value of the parameter
- A **confidence interval** is a range of values for an estimate of a parameter
- The **confidence level** is the proportion of values of an estimate that are expected to lie within the interval
- For a confidence interval symmetric about the mean, the **margin of error** is half the width of the interval. It is the distance of the ends of the interval from the mean




- For a proportion $\alpha$, the **quantile** $a$ is the value below which the proportion $\alpha$ of the values are found. This is the same as saying the probability of a value being below $a$ is $\alpha$
- For a confidence level $c$, the value of $\alpha$ for the appropriate quantiles will be $\alpha = 0.5 - \frac{c}{2}$ and $\alpha = 0.5 + \frac{c}{2}$
- The confidence level of a normal variable is given by $[\mu - z\sigma, \mu + z\sigma]$, where $z$ is the quantile of the standard normal variable, $\mu$ is the mean and $\sigma$ is the standard deviation
- The **estimated mean** and **estimated standard deviation** of a population is taken to be the mean and standard deviation of a random sample
- Confidence intervals for the estimated mean are calculated using the estimated mean and the estimated standard deviation of the sampling distribution, $\sigma_{\overline{X}} = \frac{s}{\sqrt{n}}$, where $s$ is the sample standard deviation and $n$ is the sample size, approximately $\left(\bar{x} - z\frac{s}{\sqrt{n}}, \bar{x} + z\frac{s}{\sqrt{n}}\right)$
- The estimated mean and confidence interval will be reasonable estimates if $n \geq 30$
- The **90% confidence interval** of the standard normal distribution is [–1.645, 1.645]
- The **95% confidence interval** of the standard normal distribution is [–1.960, 1.960]
- The **99% confidence interval** of the standard normal distribution is [–2.576, 2.576]
- Unless otherwise stated, the confidence level used in a report is assumed to be 95%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# 10. CHAPTER REVIEW

## Confidence intervals for means

**1** **a** What is the quantile for $\alpha = 0.93$ for a standard normal distribution?

**b** Find the quantile for a proportion of 85% for a normal distribution with mean 15 and standard deviation 4.6.

Example 1

**2** Find the 98% confidence interval for a normal distribution with mean 34 and standard deviation 5.8.

Examples 2,3

**3** The mean of a sample of 80 items was 135 with a standard deviation of 10. Estimate the 99% confidence interval for the population mean.

Examples 4,5

**4** A sample had values of:

Example 6

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 41 | 45 | 36 | 47 | 40 | 43 | 42 | 46 | 36 | 46 | 40 |
| 47 | 30 | 45 | 42 | 50 | 35 | 46 | 36 | 45 | 36 | 46 |
| 34 | 47 | 30 | 45 | 39 | 45 | 37 | 43 | 30 | 44 | 31 |
| 44 | 41 | 44 | 40 | 43 | 41 | 45 | 39 | 45 | 35 | |

Find the 95% confidence interval for the population mean.

**5** The result of a survey of cattle was quoted as being a mean of $574 \pm 42$ kg. What was the confidence interval?

Example 7

**6** A survey of the heights of 80 people gave an average of 176 cm with a margin of error of 2 cm.

Example 8

**a** What was the confidence interval?

**b** What was the sampling standard deviation?

**c** What was the standard deviation of height?

**7** A survey had a margin of error of 6 litres. What would be the 90% margin of error?

Example 9

### Problem solving

**8** A company manufacturing lithium batteries wants to give a guaranteed shelf life. They don't want any more than 0.5% of the batteries returned under warranty. Testing has revealed that the mean shelf life is 24 months with a variance of 6 months. What should their shelf-life guarantee be set at?

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**9** A randomly-selected group of senior high school students said they spent the following amounts of time on homework in a week.

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 9 | 21 | 20 | 8 | 40 | 8 | 7 | 1 | 8 | 20 | 4 | 1 | 10 |
| 3 | 1 | 3 | 1 | 4 | 10 | 35 | 11 | 5 | 1 | 5 | 15 | 12 | 4 |
| 7 | 7 | 0 | 0 | 40 | 21 | 0 | 30 | 1 | 1 | 3 | 40 | | |

Find the 95% confidence interval for the mean and state any problems with the estimate.

**10** The average maximum December temperature in Koyanyama is 35°C ± 0.5°. The average is based on 51 years of temperature records.

Qz
Practice quiz

What is the probability that the average for the December average maximum will exceed 40°C for at least one of the next 3 years? State any problems with your estimate.


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# Practice examination 4 ●●●●

**Time: 90 minutes**
**Perusal time: 5 minutes**
**Marks: 50**

## Instructions

- Students are permitted to bring or use: pens, pencils, highlighters, erasers, sharpeners, rules and an approved graphics calculator.
- Students must show appropriate working and justification to gain full marks.
- A QCAA formula sheet is provided.
- Unless otherwise stated, numerical answers should be exact.
- Unless otherwise indicated, no diagrams in this examination are drawn to scale.
- All written responses must be in English.
- Answer all questions.
- **Students are NOT permitted to bring or use notes of any kind, correction fluid/tape, mobile phones and/or any other unauthorised electronic devices.**

### Question 1 (2 marks)

Find the equations of the tangent and the normal to the hyperbola $\frac{x^2}{7} - \frac{y^2}{7} = 1$ at the point $P(4,-3)$.

### Question 2 (3 marks)

Find the general solutions to each differential equation.

**a** $\frac{dm}{dp} = 7 + 14p$ **b** $\frac{dp}{dx} = 20e^{5x} + 8$ **c** $\frac{dp}{dv} = \frac{v^2 + 4}{v}$

### Question 3 (3 marks)

Find the solution of each differential equation.

**a** $\frac{dy}{dx} = \sin(x)\cos(x)$ if $y = 1$ when $x = \pi$

**b** $\frac{dy}{dx} = \frac{x}{8x^2 - 9}$ if $y = 2$ when $x = 1$

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### Question 4 (2 marks)

Find all solutions to the differential equation $e^{2x+y} - 2e^{x-y}\frac{dy}{dx} = 0$.

### Question 5 (1, 2 marks)

Solve each differential equation.

**a** $\frac{dy}{dx} = xe^{2y}$ if $y(0) = -2$

**b** $\frac{dy}{dx} + y = 9x^2y$ if $y(0) = 1$

### Question 6 (2 marks)

A person is on a small raft with a makeshift sail. The combined mass of the person and the raft is 1200 kg. Paddling the raft causes an average force of 20 N to be applied to the raft in an easterly direction. The wind also exerts a force of 28 N 67° north of east on the raft. What is the acceleration of the raft?

### Question 7 (2 marks)

A car with an initial velocity of 6 m $s^{-1}$ accelerates uniformly at 4 m $s^{-2}$ until it reaches a velocity of 25 m $s^{-1}$.

Calculate the time taken and the distance travelled during this acceleration.

### Question 8 (2 marks)

A particle in simple harmonic motion is moving with a velocity of 1.8 m $s^{-1}$ as it passes through its central position. When the particle is 0.6 m from the central position, it has an acceleration of 2.4 m $s^{-2}$.

Calculate the amplitude and the period of the oscillation.

### Question 9 (2 marks)

An object moving with simple harmonic motion has a period of $\pi$ s and experiences a maximum velocity of 2.4 m $s^{-1}$.

**a** Find the amplitude.

**b** Calculate the velocity when the object is 0.9 m from its central position.

### Question 10 (1 mark)

Find the quantile for a proportion of 95% for a normal distribution with a mean of 48 and a standard deviation of 8.


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**Question 11** (2 marks)

What is the 90% confidence interval for a normal distribution with mean 120 and standard deviation 12?

**Question 12** (3 marks)

What is the 95% confidence interval for the mean of samples of 64 from a population with mean 70 and standard deviation 20?

**Question 13** (3 marks)

Samples of 100 items have a mean of 80 and a standard deviation of 2.4. What are the estimated mean and standard deviation of the population from which they are drawn?

**Question 14** (3 marks)

For the hyperbola $\frac{x^2}{7} - \frac{y^2}{7} = 1$, show that the equation of the tangent to the hyperbola at a point $P(x_1, y_1)$ on the hyperbola is

$$x_1x - y_1y - 7 = 0.$$

**Question 15** (4 marks)

A large capacitor in the power supply of a computer has a capacitance of 3 mF (millifarads). An amateur opened the power supply case and touched the capacitor, initially discharging a voltage of 200 V through the resistance of his body, about 15 000 Ω. A current of 12 mA (milliamps) or more for 1 s can put some people in danger of cardiac arrest or fibrillation.

**a** Use a differential equation from the equations $Q = CV$, $V = IR$ and $I = \frac{dQ}{dt}$ to find whether this accidental discharge could kill.

**b** Refine your model to find the capacitance that could produce a current of 20 mA for 1 s.

**Question 16** (3 marks)

A sample of 30 walnuts have the following masses in grams:

25, 43, 23, 37, 20, 32, 27, 30, 27, 35, 25, 40, 24, 45, 27,
52, 29, 39, 10, 36, 28, 52, 25, 40, 16, 35, 22, 33, 26, 43

What are the estimated mean and standard deviation of the population?

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**Question 17** (3 marks)

A drone is rising vertically at the rate of 5 m/s. A car is travelling under the drone at a constant speed of 20 m/s. The drone is 10 m above the car at the instant that the car is directly below it. What is the rate of change of the distance between the drone and the car 2 seconds after the car is directly below the drone?

**Question 18** (4 marks)

A grey-water system irrigates a garden via a dripper system. The header tank, with a capacity of 100 L, is filled by grey water pumped from a large underground tank that stores the water from the washing machine, shower and hand basin so that it can settle before use. When the pump is on, water enters the header tank at a constant rate of 4 L/min until it is full, and runs out through the dripper system at a rate that depends on the amount of water in the header tank. When the header tank is full, the rate is 2 L/min, and when it is empty there is no water to run out.

**a** How long does the header tank take to fill from empty?

**b** Extend your model to find how long it takes to refill the tank if the pump is restarted when it drops to $p\%$ capacity.

**Question 19** (3 marks)

500 people were selected at random by a market researcher. They were asked how long they had to queue for to get into the last event they went to. 120 people would not answer.

The conclusion was that the average queuing time was 15 minutes ± 23 seconds.

What is the probability that you will have to queue longer than 20 minutes the next time you go to an event?

**END OF EXAMINATION**


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# ANSWERS

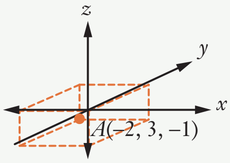
## Chapter 1

Proofs: see worked solutions.

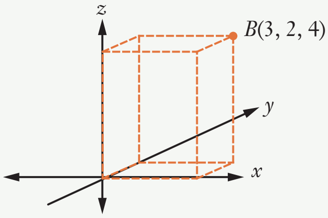
## Chapter 2

### Exercise 2.01

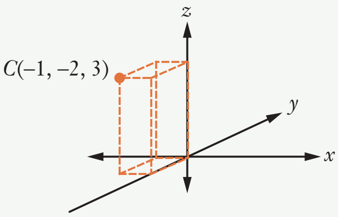
**1** **a**



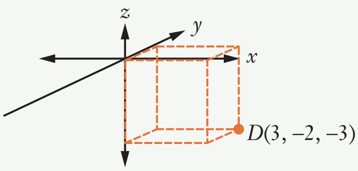

**b**



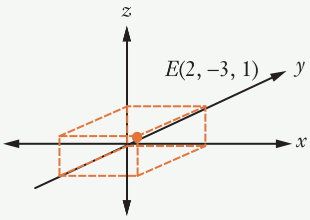

**c**



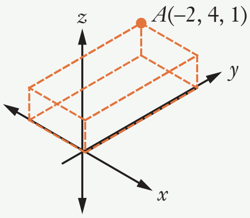

**d**



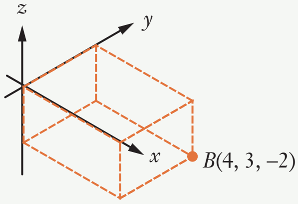

**e**



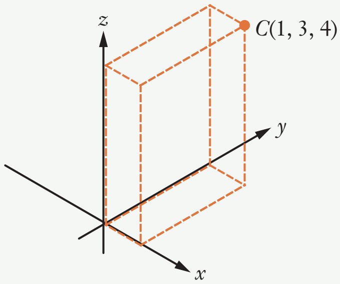

**2** **a**




**b**




**c**

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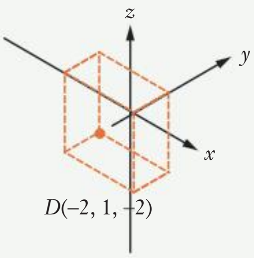
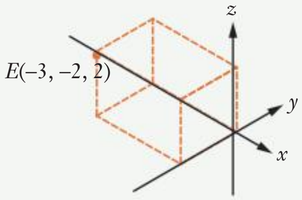
**d**




**e**




**3** **a** 6 **b** 9 **c** $\sqrt{67}$
**d** $\sqrt{73}$ **e** $\sqrt{43}$ **f** $\sqrt{77}$

**4** **a** $9x^2 + 6x + 16y^2 - 8y + 4z^2 + 4z + 20 = 0$
**b** $4x^2 + 4x + 9y^2 + 6y + 36z^2 + 12z = 0$
**c** $16x^2 + 8x + 16y^2 - 8y + 4z^2 - 4z = 45$
**d** $25x^2 - 10x + 9y^2 + 6y + 36z^2 - 12z + 54 = 0$
**e** $4x^2 + 4x + 1y^2 + 2y + 64z^2 + 16z + 33 = 0$
**f** $4x^2 - 4x + 9y^2 - 6y + 25z^2 - 10z = 26$

**5** **a** $A(2, -1, 5)$ is on, $B(9, 6, 4)$ is inside, $C(3, -2, 5)$ is outside
**b** $D(2, 3, 5)$ is inside, $E(7, 6, 7)$ and $F(-3, -4, -2)$ are outside
**c** $G(4, 0, 4)$ is outside, $H(1, -3, 5)$ is on and $K(1, -3, -4)$ is inside
**d** $L(3, 1, 2)$ is inside, $M(5, -5, 2)$ is outside and $N(4, -3, -3)$ is on
**e** $P(6, 0, 0)$ is outside, $Q(6, -4, 0)$ and $R(2, 1, 3)$ are on

**6** (6, −5, 4), 10

**7** Distance between centres = 11, radii 4 and 7, so they touch.

**8** $\sqrt{116} - 9 \approx 1.770$

## Exercise 2.02

**1** **a** $2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$, 5.39 **b** $-3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, 5.39
**c** $3\mathbf{i} + \mathbf{j} - 5\mathbf{k}$, 5.92 **d** $-\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$, 5.74
**e** $-3\mathbf{i} - \mathbf{j} - 2\mathbf{k}$, 3.74 **f** $-6\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$, 8.06
**g** $4\mathbf{i} - 7\mathbf{j} - 2\mathbf{k}$, 8.31 **h** $\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$, 7.28
**i** $-5\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$, 6.16 **j** $\mathbf{i} - \mathbf{k}$, 1.41
**k** $3.624\mathbf{i} - 8.140\mathbf{j} - 4.540\mathbf{k}$, 10
**l** $-2.399\mathbf{i} - 2.399\mathbf{j} + 2.120\mathbf{k}$, 4
**m** $0.486\mathbf{i} + 1.042\mathbf{j} + 10.940\mathbf{k}$, 11
**n** $6.928\mathbf{i} - 12\mathbf{j} + 8\mathbf{k}$, 16
**o** $-1.327\mathbf{i} + 2.393\mathbf{j} - 7.518\mathbf{k}$, 8
**p** $-15.499\mathbf{i} + 42.583\mathbf{j} + 21.131\mathbf{k}$, 50
**q** $-28.284\mathbf{i} - 28.284\mathbf{j} + 69.282\mathbf{k}$, 80

**2** **a** $\mathbf{b} - \mathbf{a}$ **b** $\mathbf{c} - \mathbf{a}$ **c** $\mathbf{f} - \mathbf{d}$
**d** $\mathbf{c} - \mathbf{f}$ **e** $\mathbf{b} - \mathbf{e}$

**3** **a** (−3, 1, 7) **b** (7, −11, 1)
**c** (−4, −3, 18) **d** (24, −34, −4)
**e** (20, −37, 14) **f** (9, 60, −3)
**g** (−19, −29, 4) **h** (4, 28, 4)
**i** (−6, 2, 26) **j** (0, −19, −1)

**4** **a** $\begin{bmatrix} -4 \\ -5 \\ -16 \end{bmatrix}$ **b** $\begin{bmatrix} -4 \\ -7 \\ -14 \end{bmatrix}$ **c** $\begin{bmatrix} 20 \\ 5 \\ -6 \end{bmatrix}$
**d** $\begin{bmatrix} -11 \\ -18 \\ -41 \end{bmatrix}$ **e** $\begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix}$ **f** $\begin{bmatrix} -33 \\ 7 \\ -19 \end{bmatrix}$
**g** $\begin{bmatrix} 36 \\ 18 \\ -27 \end{bmatrix}$ **h** $\begin{bmatrix} 36 \\ 18 \\ -27 \end{bmatrix}$ **i** $\begin{bmatrix} 1 \\ -16 \\ -23 \end{bmatrix}$
**j** $\begin{bmatrix} 1 \\ -6 \\ -15 \end{bmatrix}$

**5** **a** (9.2, 115.7°, 1.3°)
**b** (25.6, 350.9°, −55.3°)
**c** (21.8, 134.1°, 29.7°)
**d** (37.0, 2.0°, −58.0°)
**e** (22.2, 98.1°, −26.7°)

**6** **a** (44.8, 275.8°, 59.2°)
**b** (32.3, 344.6°, −24.7°)
**c** (85.3, 187.5°, 50.1°)
**d** (49.4, 332.1°, −1.1°)
**e** (57.1, 179.6°, 43.7°)

## Exercise 2.03

**1** **a** 35.46 **b** 21.21 **c** 0 **d** 180
**e** 12.12 **f** −57.16 **g** −70.15 **h** 48

**2** **a** 74 **b** −1 **c** 0 **d** −42
**e** −2 **f** 2

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**3** **a** 17 **b** 0 **c** 41

**4** **a** –5 **b** 0 **c** 12 **d** –12

**5** **a** 72 **b** 32.55 **c** 16.50 **d** 1.03

**6** **a** 134.7° **b** 104.3° **c** 102.7°
**d** 158.20° **e** 62.61° **f** 83.92°

**7** **a** 4.02 **b** 3.14
**c** –2.98 (opposite direction)
**d** 0.56 **e** 5.18 **f** –6.58

**8** **a** (1, –8, 2)
**b** $\cos\alpha \approx 0.1204$, $\cos\beta \approx -0.9631$, $\cos\gamma \approx 0.2408$
**c** $\alpha \approx 83.1°$, $\beta \approx 164.4°$, $\gamma \approx 76.1°$

**9** Proof: see worked solutions

**10** It is less than 30° because, as the altitude goes from 0° to 90°, the angle goes from 30° to 0°. Changing both vectors to components and finding the angle using the dot product gives an angle of 25.64°.

**11, 12** Proofs

## Exercise 2.04

**1** All answers are perpendicular to both vectors.
**a** –24 **b** About 19.56
**c** About –49.25 **d** 0
**e** About 35.10 **f** –28
**g** 77 **h** About 24.25

**2** **a** $-40\mathbf{i} + 32\mathbf{j} - 8\mathbf{k}$ **b** $-54\mathbf{i} + 70\mathbf{j} - 45\mathbf{k}$
**c** $39\mathbf{i} - 44\mathbf{j} + 23\mathbf{k}$ **d** $20\mathbf{i} + 30\mathbf{j} - 10\mathbf{k}$
**e** $69\mathbf{i} - 3\mathbf{j} + 38\mathbf{k}$ **f** $-28\mathbf{i} + 28\mathbf{j} + 36\mathbf{k}$
**g** $-71\mathbf{i} - 44\mathbf{j} - 3\mathbf{k}$ **h** $-67\mathbf{i} - 59\mathbf{j} - 56\mathbf{k}$

**3** **a** (49, –54, 47) **b** (19, 31, –52)
**c** (–86, –159, –90) **d** (30, 12, –78)
**e** (83, 17, –6) **f** (6, –61, –42)

**4** **a** $\begin{bmatrix} 13 \\ 29 \\ 26 \end{bmatrix}$ **b** $\begin{bmatrix} 74 \\ 3 \\ -35 \end{bmatrix}$ **c** $\begin{bmatrix} -63 \\ -37 \\ -9 \end{bmatrix}$

**5** **a** –8 **b** $18\mathbf{i} + 19\mathbf{j} + 11\mathbf{k}$
**c** –1 **d** $-32\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$
**e** –18 **f** $21\mathbf{i} + 9\mathbf{j} - 6\mathbf{k}$
**g** –18 **h** $21\mathbf{i} + 9\mathbf{j} - 6\mathbf{k}$
**i** 9 **j** 9
**k** $-13\mathbf{i} - 8\mathbf{j} - 6\mathbf{k}$ **l** $13\mathbf{i} + 8\mathbf{j} + 6\mathbf{k}$
**m** $38\mathbf{i} - 25\mathbf{j} - 107\mathbf{k}$ **n** $-21\mathbf{i} + 83\mathbf{j} - 109\mathbf{k}$

**6** $\frac{1}{2}\sqrt{2114} \approx 22.99$ square units

**7** 11 square units

**8** $\sqrt{2090} \approx 45.72$ square units

**9** $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, so either $\mathbf{a} = \mathbf{0}$, $\mathbf{b} = \mathbf{0}$ or $\mathbf{a}$ and $\mathbf{b}$ are parallel so $\mathbf{b} = k\mathbf{a}$ for some $k \in \boldsymbol{R}$.

## Exercise 2.05

**1** A cosine function with the axis along the line $y = 0.5z$.

**2** A spiral around the $x$-axis.

**3** A straight line

**4** A straight line

**5** $y = (x - 2)^3$, a cubic function translated 2 units to the right.

**6** $y = \frac{4}{3}x + 3\frac{2}{3}$, a straight line

**7** $y = 4x^2 + 12$, a parabola

**8** $y = 5(2x - 3)^2$, a parabola

**9** $y = 0.5x - 3$, a straight line

**10** $x^2 + y^2 = 4$, a circle of radius 2 with centre at the origin.

## Exercise 2.06

**1** **a** $\mathbf{r}(t) = 7\mathbf{i} - 6\mathbf{j} - 3\mathbf{k} + t(-5\mathbf{i} + 11\mathbf{j} - \mathbf{k})$
**b** $\mathbf{r}(t) = -\mathbf{i} - 5\mathbf{j} + t(3\mathbf{i} + 7\mathbf{j})$
**c** $\mathbf{r}(t) = 2\mathbf{i} + 1\mathbf{j} - 6\mathbf{k} + t(\mathbf{i} + \mathbf{j} + 7\mathbf{k})$
**d** $\mathbf{r}(t) = -2\mathbf{i} + 6\mathbf{j} + t(4\mathbf{i} - 13\mathbf{j})$
**e** $\mathbf{r}(t) = -\mathbf{i} + 4\mathbf{j} - 5\mathbf{k} + t(8\mathbf{i} - 7\mathbf{j} + 4\mathbf{k})$

**2** **a** $\frac{x+4}{6} = y - 3$ or $x - 6y + 22 = 0$
**b** $\frac{x-6}{-13} = \frac{y+3}{8} = \frac{z+6}{8}$
**c** $\frac{x-5}{2} = \frac{y-4}{-1} = \frac{z-2}{-7}$
**d** $x - 5 = y + 4 = \frac{z-5}{2}$
**e** $\frac{x-1}{5} = \frac{y+2}{2} = \frac{z-5}{-8}$

**3** **a** $\mathbf{r}(t) = 5\mathbf{i} + 4\mathbf{j} - 3\mathbf{j} + t(-7\mathbf{i} + 7\mathbf{j} - \mathbf{k})$
**b** $\mathbf{r}(t) = 5\mathbf{i} - \mathbf{j} + t(4\mathbf{i} - 7\mathbf{j})$
**c** $\mathbf{r}(t) = -5\mathbf{i} + 6\mathbf{j} - 3\mathbf{k} + t(5\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$
**d** $\mathbf{r}(t) = 4\mathbf{i} - 3\mathbf{j} - 7\mathbf{k} + t(-6\mathbf{i} + 12\mathbf{k})$
**e** $\mathbf{r}(t) = -7\mathbf{i} + 7\mathbf{j} + 4\mathbf{k} + t(-14\mathbf{i} + 14\mathbf{j})$

**4** $\mathbf{r}(t) = \mathbf{i} + 4\mathbf{j} - 4\mathbf{k} + t(2\mathbf{i} - 9\mathbf{j} - 3\mathbf{k})$

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**5** **a** $\mathbf{r}(t) = 2\mathbf{i} + 6\mathbf{j} + \mathbf{k} + t(-12\mathbf{j} - 9\mathbf{k})$ for $0 \le t \le 1$
**b** $\mathbf{r}(t) = -2\mathbf{i} - 6\mathbf{j} + 8\mathbf{k} + t(\mathbf{i} + 3\mathbf{j} - 6\mathbf{k})$ for $0 \le t \le 1$
**c** $\mathbf{r}(t) = -4\mathbf{i} - 5\mathbf{j} - 6\mathbf{k} + t(-3\mathbf{i} + 8\mathbf{j} + 2\mathbf{k})$ for $0 \le t \le 1$
**d** $\mathbf{r}(t) = -7\mathbf{i} - 5\mathbf{j} + 5\mathbf{k} + t(\mathbf{i} + 3\mathbf{j} - 12\mathbf{k})$ for $0 \le t \le 1$
**e** $\mathbf{r}(t) = 7\mathbf{i} - 3\mathbf{j} - \mathbf{k} + t(-7\mathbf{i} + 5\mathbf{j} + 8\mathbf{k})$ for $0 \le t \le 1$

**6** **a** Yes **b** No **c** No
**d** No **e** No

**7** The usual form includes $\frac{z-7}{0}$, which is not defined. The line is in a plane parallel to the $x$-$y$ plane passing through $z = 7$. An alternative form is $x - 3 = \frac{y-5}{4}$ and $z = 7$. You could also write $y = 4x - 7$ and $z = 7$.

## Exercise 2.07

Proofs: see worked solutions

## Exercise 2.08

**1** (4, 6)
**2** They do not cross.
**3** (4, 16, 13) only
**4** (33, 46) at $t = 7$
**5** (12, 44) at $t = 8$
**6** (8, 28, 18) at $t = 6$
**7** Taking the initial position of the arrow as the origin, the position of the arrow is given by $\mathbf{a}(t) = 50t\cos(60°)\mathbf{i} + [50t\sin(60°) - 4.9t^2]\mathbf{j}$. The top of the balloon is at $\mathbf{b}(t) = 200\mathbf{i} + (50 + 3t)\mathbf{j}$ and the bottom is 20 m lower. The arrow passes about 2.8 m below the bottom of the balloon.
**8** Taking the origin as the airport, **i** as South, **j** as East and **k** as upwards, the departing plane is at $\mathbf{d}(t) = [30\,000 + 110t\cos(30°)]\mathbf{i} + 110t\sin(30°)\mathbf{j} + (5000 + 15t)\mathbf{k}$. The arriving plane has position $\mathbf{a}(t) = [40\,000\cos(45°) - 115t\cos(45°)]\mathbf{i} + [40\,000\sin(45°) - 115t\sin(45°)]\mathbf{j} + (8000 - 20t)\mathbf{k}$. They will be the same distance south of the airport about 56.8 s later, but will be separated by 20 538 m in the east-west direction and be at different heights. They should continue on their present courses.

## Exercise 2.09

**1** **a** $x - 4y + 5z = -29$ **b** $3x + 2y - z = 12$
**c** $6z - 3x = 24$
**2** **a** $(-\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}) \bullet \mathbf{p} = -46$
**b** $(\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}) \bullet \mathbf{p} = -14$
**c** $(5\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) \bullet \mathbf{p} = 14$
**3** **a** No **b** Yes **c** Yes **d** No
**e** Yes
**4** **a** Yes **b** No **c** No **d** Yes
**e** Yes
**5** $5x^2 + 26y^2 - 20xy - 28x + 52y > -17$ and $z = 5y - 2x + 9$
**6** $13y^2 + 29z^2 + 30yz - 28y - 76z \le 288$ and $x = \frac{6 - 3y - 5z}{2}$
**7** **a** $19x - 5y + 22z = 90$
**b** $6x - 19y + 16z = -14$
**8** **a** $(15\mathbf{i} - 29\mathbf{j} + 4\mathbf{k}) \bullet \mathbf{p} = -87$
**b** $(30\mathbf{i} + 17\mathbf{j} - 25\mathbf{k}) \bullet \mathbf{p} = 44$
**9** $3x - 4y - 2z = -5$
**10** $4x - 10y + 3z = 37$
**11** Proof: see worked solutions
**12** $\mathbf{n}_1 \bullet \mathbf{n}_2 = -11 \ne |\mathbf{n}_1||\mathbf{n}_2|$, and $\mathbf{n}_1 \bullet \mathbf{n}_2 \ne 0$, so neither.
**13** $\frac{\sqrt{26}}{13}$
**14** $\frac{21\sqrt{29}}{29}$

## Exercise 2.10

**1** 147 c.u.
**2** $7\frac{2}{3}$ c.u.
**3** 55 c.u.
**4** 11.5 c.u.
**5** $[b_1(a_2c_2 + a_3c_3) - c_1(a_2b_2 + a_3b_3)]\mathbf{i} + [b_2(a_3c_3 + a_1c_1) - c_2(a_3b_3 + a_1b_1)]\mathbf{j} + [b_3(a_1c_1 + a_2c_2) - c_3(a_1b_1 + a_2b_2)]\mathbf{k}$
**6, 7** Proofs: see worked solutions
**8** 7713 J
**9** 10.73 MJ
**10** 16.26 kJ
**11** Speed = 29 890 m s$^{-1}$, so $L = 2.690 \times 10^{40}$ kg m$^2$ s$^{-1}$
**12** Angular momentum of Jupiter = 662.5 times that of Earth.


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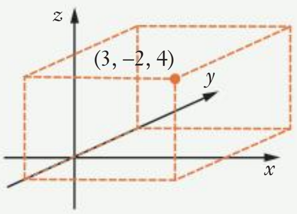
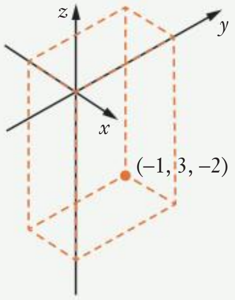
## Chapter review

**1 a**




**b**




**2** $\sqrt{70}$

**3 a** $x^2 + 8x + y^2 - 6y + z^2 - 4z = 7$

**b** Outside

**4 a** $5\sqrt{2}$  **b** $3\sqrt{2}\mathbf{i} - 3\sqrt{6}\mathbf{j} + 6\sqrt{2}\mathbf{k}$

**5** $(15, -5, -22)$

**6** About $5.15\mathbf{i} + 0.70\mathbf{j} + 8.86\mathbf{k}$

**7 a** About 17.58  **b** 20

**c** 108

**8** $\frac{-8}{\sqrt{14}}$

**9** About 62.77 perpendicular to both the original vectors.

**10** $27\mathbf{i} - 17\mathbf{j} + 23\mathbf{k}$

**11** 18.1 square units

**12** A sine wave in the $z$-direction along the line $y = -\frac{4}{5}x$.

**13** $y = -2\left(\frac{x-1}{4}\right)^2$, a parabola with a maximum at $(1, 0)$.

**14 a** $(1, 6, -4) + t(1, -9, 9)$ and $x - 1 = \frac{y-6}{9} = \frac{z+4}{9}$

**b** No

**15** Yes ($t = 0.75$)

**16** They collide at $t = 4$, $x = 14$, $y = 25$.

**17** $\mathbf{p} \bullet (6, 2, -2) = -8$ and $6x + 2y - 2z = -8$

**18** $19x - 45y + 10z = -227$

**19** 20.5 cubic units

**20** Overlap by $16 - \sqrt{194}$

**21** The acute angle is 59°.

**22** $\sqrt{1350} \approx 36.74$  **23** Proof

**24** $\frac{9\sqrt{29}}{29}$  **25** 252

# Chapter 3

## Exercise 3.01

**1 a** $3i$  **b** $4i\sqrt{2}$  **c** $i\sqrt{23}$

**d** $\frac{1}{2}i$  **e** $\frac{\sqrt{6}}{3}i$  **f** $\frac{3\sqrt{7}}{5}i$

**2 a** $1 \pm i$  **b** $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

**c** $\frac{2}{3} \pm \frac{\sqrt{2}}{3}i$  **d** $3 \pm 3i\sqrt{6}$

**e** $-\frac{1}{4} \pm \frac{\sqrt{39}}{4}i$  **f** $1 \pm i\sqrt{5}$

**3 a** $2 \pm i$  **b** $3 \pm 2i$

**c** $\frac{5}{2} \pm \frac{\sqrt{3}}{2}i$  **d** $-3 \pm i\sqrt{3}$

**4 a** $-i$  **b** 1  **c** $i$

**d** $-1$  **e** $-i$  **f** 0

**5 a** $4 + 3i$  **b** $-2 - 5i$  **c** $7 - 3i$

**d** $-2 + 6i$  **e** $3 + i\sqrt{5}$  **f** $3\sqrt{6} - i\sqrt{7}$

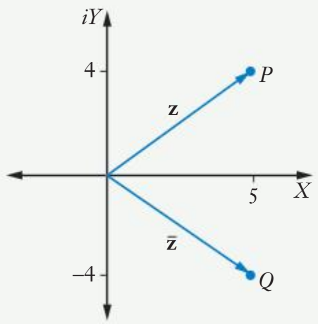
**6 a** 13  **b** 41  **c** 61  **d** 65

**e** 122  **f** 145  **g** 32  **h** 225

**i** 157

**7 a** $P(5, 4)$

**b**




**c** $\mathbf{z}$ and $\overline{\mathbf{z}}$ are reflections of each other in the real axis.

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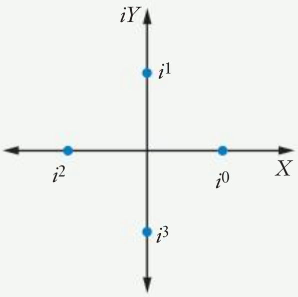
**8 a**




**b** 0

**9** b and c

**10** $4pr > q^2$

**11** $i$ (Hint: $i^0 + i^1 + i^2 + i^3 = 0$. This pattern repeats.)

## Exercise 3.02

**1 a** $7 + 9i$ **b** $1 + 3i$ **c** $2$
**d** $-15 + i$ **e** $4 + 15i$ **f** $-2 - 21i$

**2 a** $26 + 9i$ **b** $37$ **c** $-43 + 5i$
**d** $10 - 36i$ **e** $16 - 63i$ **f** $9 + 71i$

**3 a** $10 - 5i$ **b** $3 - i$ **c** $5i$
**d** $8$ **e** $11 - 23i$ **f** $2 + 11i$
**g** $2i$ **h** $7 - 24i$ **i** $-7 + 24i$

**4 a** $a = 8, b = -1$ **b** $a = -6, b = -39$

**5 a** $-62 + 2i$ **b** $244 + 164i$
**c** $21 - 90i$ **d** $-92 - 40i$
**e** $40 - 84i$ **f** $-116 + 300i$

**6 a** $48$ **b** $56$

**7 a** $\frac{3}{13} - \frac{2}{13}i$ **b** $\frac{3}{25} - \frac{4}{25}i$ **c** $\frac{3}{25} + \frac{4}{25}i$
**d** $i$ **e** $\frac{19}{29} + \frac{4}{29}i$ **f** $-\frac{2}{5} - \frac{2}{5}i$
**g** $-\frac{2}{25} + \frac{3}{50}i$ **h** $\frac{18}{25} - \frac{1}{25}i$ **i** $-\frac{1}{2}i$
**j** $\frac{1}{26} - \frac{3}{52}i$ **k** $-\frac{1}{85} + \frac{18}{85}i$ **l** $\frac{14}{25} + \frac{73}{25}i$

**8 a** $29$ **b** $\frac{21}{29} - \frac{20}{29}i$ **c** $145 + 58i$
**d** $50 + 20i$ **e** $\frac{1}{4}i$ **f** $-4$
**g** $\frac{5}{29} - \frac{2}{29}i$ **h** $\frac{65}{29} - \frac{142}{29}i$

**9** Substitute for $z$ and $w$, simplify and collect real and imaginary parts.

**10** $a = -3, b = 4$

**11** Proof: see worked solutions

**12 a, b** Proofs: see worked solutions

**13 a, b** Proofs – use $z = a + bi$ and $w = c + di$.

**14** $a = -1, b = 2$

**15 a, b, c** Proofs – use $z = a + bi$ and $w = c + di$.

## Exercise 3.03

**1 a** $7\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$

**b** $3\sqrt{2}\left[\cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)\right]$

**c** $6\left[\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right]$

**d** $2\left[\cos\left(\frac{-3\pi}{8}\right) + i\sin\left(\frac{-3\pi}{8}\right)\right]$

**e** $4\sqrt{3}\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right]$

**f** $\sqrt{58}\left[\cos(3.55) + i\sin(3.55)\right]$

**2 a** $\sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$

**b** $\left[\cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)\right]$

**c** $2\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$

**d** $\sqrt{41}\,[\cos(-0.6747) + i\sin(-0.6747)]$

**e** $2\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$

**f** $\sqrt{58}\left[\cos(3.55) + i\sin(3.55)\right]$

**g** $\sqrt{2}\left[\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right]$

**h** $\sqrt{2}\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$

**3 a** $\frac{5}{2} + \frac{5\sqrt{3}}{2}i$ **b** $-2$
**c** $2\sqrt{3} + 2i$ **d** $1 + i$
**e** $-2.163 - 6.657i$ **f** $1.2031 + 6.823i$
**g** $\frac{-5\sqrt{6} + 5\sqrt{2}}{4} + \frac{5\sqrt{6} + 5\sqrt{2}}{4}i$
**h** $3.741 - 4.691i$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**4** **a** $2\sqrt{2}\left[\cos\left(\frac{7\pi}{12}\right)+i\sin\left(\frac{7\pi}{12}\right)\right]$

**b** $4\sqrt{3}\left[\cos\left(\frac{-5\pi}{12}\right)+i\sin\left(\frac{-5\pi}{12}\right)\right]$

**c** $10\left[\cos\left(\frac{53\pi}{56}\right)+i\sin\left(\frac{53\pi}{56}\right)\right]$

**d** $\frac{1}{2}i$

**e** $\frac{2\sqrt{6}}{3}\left[\cos\left(\frac{-\pi}{8}\right)+i\sin\left(\frac{-\pi}{8}\right)\right]$

**f** $2i$

**5** **a** $z=\frac{1}{2}\left[\cos\left(\frac{-\pi}{3}\right)+i\sin\left(\frac{-\pi}{3}\right)\right]$

**b** $z=\frac{\sqrt{2}}{2}\left[\cos\left(\frac{-3\pi}{4}\right)+i\sin\left(\frac{-3\pi}{4}\right)\right]$

**c** $z=\frac{\sqrt{3}}{6}\left[\cos\left(\frac{-5\pi}{6}\right)+i\sin\left(\frac{-5\pi}{6}\right)\right]$

**d** $z=\frac{1}{4}\left[\cos\left(\frac{-3\pi}{8}\right)+i\sin\left(\frac{-3\pi}{8}\right)\right]$

**e** $z=\frac{1}{6}\left[\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right]$

**f** $z=\frac{\sqrt{2}}{4}\left[\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right]$

**6** **a** $\arg(z)^{-1}=\frac{\pi}{4};\ |(z)^{-1}|=\frac{\sqrt{2}}{2}$

**b** $\arg(z)^{-1}=\frac{-\pi}{4};\ |(z)^{-1}|=\frac{\sqrt{2}}{4}$

**c** $\arg(z)^{-1}=-1.107;\ |(z)^{-1}|=\frac{\sqrt{5}}{5}$

**d** $\arg(z)^{-1}=0.4636;\ |(z)^{-1}|=\frac{\sqrt{5}}{5}$

**e** $\arg(z)^{-1}=0.9273;\ |(z)^{-1}|=\frac{1}{5}$

**f** $\arg(z)^{-1}=-1.176;\ |(z)^{-1}|=\frac{1}{13}$

**g** $\arg(z)^{-1}=0.9828;\ |(z)^{-1}|=\frac{\sqrt{13}}{13}$

**h** $\arg(z)^{-1}=0.6747;\ |(z)^{-1}|=\frac{\sqrt{41}}{41}$

**7** **a** 6 **b** $\frac{11\pi}{12}$

**c** $6\left[\cos\left(\frac{11\pi}{12}\right)+i\sin\left(\frac{11\pi}{12}\right)\right]$

**d** 12 **e** $\frac{-\pi}{2}$

**f** $\frac{-\pi}{4}$ **g** 24

**h** $24\left[\cos\left(\frac{-\pi}{4}\right)+i\sin\left(\frac{-\pi}{4}\right)\right]$

**8** **a** $\frac{1}{2}\left[\cos\left(\frac{-\pi}{3}\right)+i\sin\left(\frac{-\pi}{3}\right)\right]$

**b** $\frac{1}{3}\left[\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right]$

**c** $\frac{1}{6}\left[\cos\left(\frac{-3\pi}{4}\right)+i\sin\left(\frac{-3\pi}{4}\right)\right]$

**d** $\frac{1}{12}\left[\cos\left(\frac{11\pi}{12}\right)+i\sin\left(\frac{11\pi}{12}\right)\right]$

**e** $\frac{1}{12}\left[\cos\left(\frac{5\pi}{6}\right)+i\sin\left(\frac{5\pi}{6}\right)\right]$

**f** $\frac{1}{10}\left[\cos\left(\frac{-11\pi}{12}\right)+i\sin\left(\frac{-11\pi}{12}\right)\right]$

**9** **a** $\frac{1}{10}$ cis (2.498) **b** $\frac{\sqrt{13}}{5}$ cis (–0.055 50)

**c** $\frac{1}{2}\text{cis}\left(\frac{-\pi}{2}\right)$ **d** $\frac{\sqrt{13}}{52}$ cis (–0.9828)

**e** $\frac{\sqrt{13}}{17}$ cis (1.626) **f** $\frac{\sqrt{221}}{5}$ cis (1.381)

**10** **a** $-4\sqrt{3}-4i$ **b** $\frac{-1}{4}+\frac{\sqrt{3}}{4}i$

**c** $-1+i\sqrt{3}$ **d** $\frac{1}{4}-\frac{\sqrt{3}}{4}i$

**e** $-\sqrt{3}+i$ **f** –6

**11** **a** $8i$ **b** $25\sqrt{2}(-1+i)$ **c** –2

**d** $4i$ **e** $\frac{-3}{2}+\frac{3\sqrt{3}}{2}i$ **f** $6i$

**12** $0.7542+0.2277i$ A

**13** $1.1120+0.8063i$ A

**14** $\frac{\sqrt{2}}{4}(1+\sqrt{3})$

**15** Proof: see worked solutions

## Exercise 3.04

**1 a** $\sqrt{2}\left[\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right]$ **b** $-4$

**c** $\frac{\sqrt{2}}{4}\left[\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)\right]$

**2 a** $2\sqrt{2}\left[\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right]$

**b** $128\sqrt{2}\left[\cos\left(\frac{5\pi}{4}\right)+i\sin\left(\frac{5\pi}{4}\right)\right]$

**c** $\frac{1}{8}\left[\cos\left(-\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{2}\right)\right]$

**3 a** $\cos(7\theta)+i\sin(7\theta)$

**b** $\cos(-15\alpha)+i\sin(-15\alpha)$

**c** $4\left[\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right]$

**d** $27\sqrt{3}\left[\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right]$

**e** $\frac{1}{8}\left[\cos\left(\frac{-3\pi}{4}\right)+i\sin\left(\frac{-3\pi}{4}\right)\right]$

**f** $\frac{\sqrt{2}}{8}\left[\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right]$

**4 a** $2+2i$ **b** $27$ **c** $1$ **d** $1$

**e** $-i$ **f** $i$ **g** $\frac{1}{8}i$ **h** $\frac{-1}{16}$

**i** $\frac{27}{2}-\frac{27\sqrt{3}}{2}i$ **j** $8$

**5 a** $-16\sqrt{3}+16i$ **b** $4096i$

**c** $1$ **d** $-128+128i\sqrt{3}$

**e** $-64$ **f** $-16+16i$

**g** $64\sqrt{2}+64i\sqrt{2}$ **h** $-8-8i\sqrt{3}$

**i** $-8+8i\sqrt{3}$ **j** $-117-44i$

**k** $-32i$ **l** $64$

**6 a** $\frac{1}{8}i$ **b** $-8i$ **c** $\frac{1}{8}i$

**d** $-\frac{1}{64}i$ **e** $-\frac{1}{32}-\frac{\sqrt{3}}{32}i$ **f** $-\frac{1}{512}$

**g** $32i$ **h** $-8+8i\sqrt{3}$ **i** $-1024$

**j** $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$

**7 a** $128+128i$ **b** $-72+8i\sqrt{3}$

**c** $336-527i$ **d** $-64\sqrt{3}-64i$

**e** $\frac{\sqrt{3}+1}{16}+\frac{\sqrt{3}-1}{16}i$ **f** $2+2i$

**g** $24-7i$ **h** $\frac{1}{4}-\frac{\sqrt{3}}{4}i$

**i** $-4$ **j** $16-16i\sqrt{3}$

**k** $9i$ **l** $32\sqrt{3}+32i$

**8 a** $\text{cis}(9\theta)$ **b** $\text{cis}(-4\alpha)$ **c** $\text{cis}(29\beta)$

**d** $\frac{4\sqrt{3}}{9}\text{cis}(\theta)$ **e** $128\sqrt{2}\text{ cis}(25\alpha)$

**9 a** Modulus $=\frac{9\sqrt{6}}{4}$; argument $=-\frac{11\pi}{12}$

**b** $12\sqrt{6}\text{ cis}\left(\frac{-\pi}{4}\right)$

**10** Proofs: see worked solutions

## Exercise 3.05

**1 a** $[\cos(\theta)+i\sin(\theta)]^3=\cos^3(\theta)-3\cos(\theta)\sin^2(\theta)+i[3\cos^2(\theta)\sin(\theta)-\sin^3(\theta)]$

**b** $\sin(3\theta)=3\sin(\theta)-4\sin^3(\theta)$

**2 a** $[\cos(\theta)+i\sin(\theta)]^2=\cos^2(\theta)-2i\cos(\theta)\sin(\theta)-\sin^2(\theta)$

**b** $\cos(2\theta)=2\cos^2(\theta)-1$

**c** $\cos(2\theta)=1-2\sin^2(\theta)$

**3 a** Proof **b** $\cos(3\theta)+i\cos(3\theta)$ **c** Proof

**4 a, b** Proof: see worked solutions

**c** $\cos(3\theta)=4\cos^3(\theta)-3\cos(\theta)$

**5 a, b** Proof: see worked solutions

**c** $\sin(3\theta)=3\sin(\theta)-4\sin^3(\theta)$

**6–7** Proofs

**8 a** $\cos(3\theta)=4\cos^3(\theta)-3\cos(\theta)$ **b** $\frac{5\sqrt{2}}{3}$

**9 a** $\sin^2(\theta)=\frac{1}{2}[1-\cos(2\theta)]$ **b** $\frac{\pi}{4}$

**10–12** Proofs: see worked solutions

**13 a** Proof **b** $\frac{3\pi}{16}$

**14** $\cos^4(\theta)=\frac{1}{8}\cos(4\theta)+\frac{1}{2}\cos(2\theta)+\frac{3}{8}$ and

$\int_0^{\frac{\pi}{4}}\cos^4(\theta)\,d\theta=\frac{1}{4}+\frac{3\pi}{32}$

## Chapter review

**1 a** $-1\pm 2i$ **b** $\frac{3}{4}\pm\frac{\sqrt{3}}{4}i$

**c** $-\frac{1}{2}\pm\frac{\sqrt{7}}{2}i$ **d** $\frac{1}{2}\pm\frac{\sqrt{23}}{2}i$


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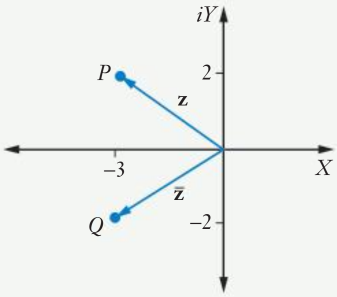
**2** **a** 26 **b** 13 **c** 29
**d** 45 **e** 85 **f** 185

**3** **a** $P(-3, 2)$

**b**




**c** $\mathbf{z}$ and $\overline{\mathbf{z}}$ are reflections of each other in the real axis.

**4** **a** $-17i$ **b** $-13 - 55i$ **c** $-3 + 82i$

**5** **a** $-96 - 108i$ **b** $61 - 25i$ **c** $119 - 136i$

**6** **a** $\frac{2}{29} - \frac{5}{29}i$ **b** $\frac{19}{13} - \frac{9}{13}i$

**c** $-\frac{7\sqrt{5}}{46} + \frac{13}{46}i$ **d** $\frac{9}{20} - \frac{7}{20}i$

**7** **a** $5\sqrt{2}\left[\cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\right]$

**b** $5[\cos(0.9272) + i\sin(0.9272)]$

**c** $\frac{5}{2} - \frac{5\sqrt{3}}{2}i$

**8** **a** $2\sqrt{2}\left[\cos\left(\frac{-11\pi}{12}\right) + i\sin\left(\frac{-11\pi}{12}\right)\right]$

**b** $\frac{1}{2}$

**c** $\sqrt{2}\left[\cos\left(-\frac{\pi}{8}\right) + i\sin\left(-\frac{\pi}{8}\right)\right]$

**d** $3\left[\cos\left(\frac{-5\pi}{6}\right) + i\sin\left(\frac{-5\pi}{6}\right)\right]$

**9** **a** $\frac{1}{4}\left[\cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\right]$

**b** $\frac{1}{3}\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right]$; $\arg(z)^{-1} = \left(\frac{5\pi}{6}\right)$; $|(z)^{-1}| = \frac{1}{3}$.

**c** $\arg(zwu) = \left(\frac{7\pi}{12}\right)$; $|zwu| = 6\sqrt{6}$;
$6\sqrt{6}\left[\cos\left(\frac{7\pi}{12}\right) + i\sin\left(\frac{7\pi}{12}\right)\right]$

**d** $\frac{\sqrt{2}}{24}\left[\cos\left(\frac{7\pi}{12}\right) + i\sin\left(\frac{7\pi}{12}\right)\right]$

**10** **a** $\cos(8\alpha) + i\sin(8\alpha)$

**b** $\frac{1}{27}\cos(-6\theta) + i\sin(-6\theta)$

**c** $4\sqrt{2}\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$

**d** $\frac{1}{27}$

**11** **a** $-4 + 4i$ **b** $i$

**c** $-128\sqrt{3} - 128i$ **d** $-\frac{1}{8} - \frac{\sqrt{3}}{8}i$

**12** $24\sqrt{3}\ \text{cis}(21\theta)$

**13** $\cos(4\theta) = 8\cos^4(\theta) - 8\cos^2(\theta) + 1$

**14** Proof: see worked solutions

**15** **a** $\sin(\theta)\cos(\theta) = \frac{1}{2}[\sin(2\theta)]$ **b** $\frac{1}{2}$

**16** $a = 2, b = -3$

**17** Proofs: see worked solutions

**18** Proof: see worked solutions

## Practice examination 1

**1a** P($n$) is true for $n = 1$

**b** If P($n$) is true for $n = k$, then it is also true for $n = k + 1$

**2–4** Proofs: see worked solutions

**5** $2\sqrt{21}$

**6** **a** $(x-3)^2 + (y+4)^2 + (z+1)^2 = 25$ **b** Outside

**7** $-3\sqrt{6}\mathbf{i} + 3\sqrt{2}\mathbf{j} - 6\sqrt{2}\mathbf{k}$

**8** **a** $-16\mathbf{i} + 15\mathbf{j} - 8\mathbf{k}$ **b** 3

**9** **a** $10\sqrt{3}$ **b** $22\mathbf{i} + 14\mathbf{j} + 2\mathbf{k}$

**10** $y = -(2x+3)^2$

**11** $\mathbf{r}(t) = (3t+1)\mathbf{i} + (6t+4)\mathbf{j} - (7t+3)\mathbf{k}$

**12** $3x + 2y - 4z = -1$

**13** $2\ \text{cis}\left(\frac{5\pi}{6}\right)$

**14** $-0.3502 + 0.9367i$

**15** The line $y = 1$

**16** $\frac{45 - 11i}{29}$

**17** 25 cis (1.8545) or 25 cis (106.3°)

**18** $\frac{\sqrt{3}}{2} + \frac{3}{2}i$

**19** 0.5 cis (−0.3)

20 $\frac{\sqrt{2}}{128} - \frac{\sqrt{2}}{128}i$

21 Proof: see worked solutions

22 Their paths cross at (–25, 4, 8), but they do not collide.

23 32

24–27 Proof: see worked solutions

# Chapter 4

## Exercise 4.01

1 a $a = 5, b = -3, c = 2$

b Consistent, $a + b = 5, c = -3, d = -1$

c $e = 2, f = -2, g = -1, h = 3$

d Consistent, $k + m = 4, n = -2, p = 3$

e $b = -1, c = -2, d = 2, e = 1$

f $f = 2, g = -1, h = 3, k = -5$

g Inconsistent

h $p = 2, q = 3, r = -2, s = -1, t = 2$

2 Adults \$95, Children \$85, Seniors \$80

3 Gravel 6 t/m$^3$, sand 5 t/m$^3$, cement 0.8 t/m$^3$, water 1 t/m$^3$

## Exercise 4.02

1 a $\begin{bmatrix} 3 & 2 & -1 \\ 1 & -2 & 3 \\ 3 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -8 \end{bmatrix}$

b $\begin{bmatrix} 2 & 3 & -7 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ -3 & 1 & 3 & 5 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ -4 \\ 7 \end{bmatrix}$

c $\begin{bmatrix} -1 & 0 & 3 & 2 & 0 \\ -3 & 1 & -1 & 2 & 0 \\ 7 & -1 & 1 & 2 & 0 \\ 3 & 0 & 2 & -1 & 0 \\ 1 & 1 & 0 & -3 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 8 \\ -9 \\ 7 \end{bmatrix}$

d $\begin{bmatrix} -1 & 3 & 2 & 3 \\ 2 & -2 & -4 & 3 \\ 2 & -1 & 3 & -1 \\ -2 & -1 & 3 & 4 \\ 3 & -2 & 4 & -1 \end{bmatrix}\begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ -9 \\ -6 \\ 10 \end{bmatrix}$

The augmented matrices have variables in alphabetical order.

2 a $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 10 \\ 2 & -3 & -1 & -1 \\ 0 & 1 & 2 & 8 \end{array}\right]$

b $\left[\begin{array}{cc|c} 3 & -2 & 9 \\ 1 & 3 & 9 \\ 2 & 5 & 3 \end{array}\right]$

c $\left[\begin{array}{cccc|c} 0 & 2 & -2 & 4 & 9 \\ 3 & 0 & -1 & -4 & -8 \\ 4 & -3 & 0 & -5 & 3 \\ 4 & 1 & -4 & 0 & 14 \end{array}\right]$

d $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & -5 \\ 2 & -3 & 0 & 0 & 11 \\ -4 & 0 & 1 & 0 & -2 \\ -3 & 5 & -1 & 2 & 7 \end{array}\right]$

## Exercise 4.03

1 $w = 2, x = -3, y = 4, z = -1$

2 $a = -4, b = 3, c = 2, d = 0.5$

3 $e = 1.5, f = -2, g = -3.5, h = 6$

4 Consistent, $k - m = 3, n = 4, p = -2$

5 $c = 3, d = 2, e = -2, f = -1, g = 0$

6 Inconsistent

7 $a = 2, b = -0.5, d = 1.5, m = 3, n = -2$

8 Consistent, $m = 2, n = 3, p = q + 5, t = q - 3$

9 $a = 6, b = -6, c = 4, d = -4, e = 3$

10 $g = 2.5, h = 3.5, k = 5.6, m = 6.4$

## Exercise 4.04

1 2, 8 and 10.

2 Hamburger \$2, steak sandwich \$3.50, chicken burger \$2.50, sausage in a roll \$1.50

3 Member \$12, subscriber \$17, public \$35

4 11 ha sorghum, 8 ha canola, 6 ha sunflowers

5 Soft drink \$3, Thickshake \$7, Milkshake \$5, Juice \$4

6 6 orange cakes, 4 banana cakes, 2 chocolate cakes

## Exercise 4.05

1 a Parallel planes

b Planes intersecting in the line $\mathbf{r}(t) = \frac{3}{4}t\mathbf{i} + \frac{1}{4}(5t + 12)\mathbf{j} + t\mathbf{k}$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**c** Planes intersecting in the line $\mathbf{r}(t) = (1 - t)\mathbf{i} + (t + 1)\mathbf{j} + t\mathbf{k}$

**d** Parallel planes

**e** Planes intersecting in the line $\mathbf{r}(t) = t\mathbf{i} + (21t + 1)\mathbf{j} + (-13t - 2)\mathbf{k}$

**2** **a** The planes intersect in the 3 parallel lines

$$\mathbf{r}(t) = \frac{1}{3}(2 - 9t)\mathbf{i} + t\mathbf{j} + \frac{1}{6}(3t - 8)\mathbf{k},$$

$$\mathbf{r}(t) = \frac{1}{5}(4 - 15t)\mathbf{i} + t\mathbf{j} + \frac{1}{10}(5t - 14)\mathbf{k},$$

$$\mathbf{r}(t) = (1 - 3t)\mathbf{i} + t\mathbf{j} + \frac{1}{2}(t - 2)\mathbf{k}$$

**b** The planes intersect in the single point (3, −2, 4).

**c** Two parallel planes intersecting the third in the parallel lines

$$\mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j} + \frac{1}{2}\mathbf{k} \text{ and } \mathbf{r}(t) = (2t + 20)\mathbf{i} + t\mathbf{j} - 7\mathbf{k}$$

**d** The planes intersect in the single point (2, 0, 2.5)

**e** Two parallel planes intersecting the third in the parallel lines

$$\mathbf{r}(t) = t\mathbf{i} + \frac{1}{3}(4t - 5)\mathbf{j} + (3t - 1)\mathbf{k} \text{ and}$$

$$\mathbf{r}(t) = t\mathbf{i} + \frac{1}{3}(4t - 8)\mathbf{j} + (3t - 4)\mathbf{k}$$

**f** The planes intersect in the 3 parallel lines

$$\mathbf{r}(t) = (2 - 2t)\mathbf{i} + t\mathbf{j} + \frac{1}{3}(t - 4)\mathbf{k},$$

$$\mathbf{r}(t) = -2t\mathbf{i} + t\mathbf{j} + \frac{1}{3}t\mathbf{k},$$

$$\mathbf{r}(t) = (1 - 2t)\mathbf{i} + t\mathbf{j} + \frac{1}{3}(t - 3)\mathbf{k}$$

**g** The planes intersect in the single line

$$\mathbf{r}(t) = \frac{1}{2}(3 - 5t)\mathbf{i} + t\mathbf{j} + \frac{1}{4}(t - 2)\mathbf{k}$$

**h** Two parallel planes intersecting the third in the parallel lines

$$\mathbf{r}(t) = \frac{1}{2}(12 - 5t)\mathbf{i} + \frac{1}{2}(-t - 2)\mathbf{j} + t\mathbf{k} \text{ and}$$

$$\mathbf{r}(t) = \frac{1}{2}(6 - 5t)\mathbf{i} + \frac{1}{2}(-t + 4)\mathbf{j} + t\mathbf{k}$$

**i** The planes intersect in the single point (5, −3, 1.5).

## Exercise 4.06

**1** $\mathbf{r}_1 + 0.5\mathbf{r}_2$ gives Student 2 as dux.

**2** $\mathbf{r}_1 + \mathbf{r}_2$ gives −0.5, 0, −5.5, 1.5, −2, 11, −7.5, 4; so the order is Murphy, Le Duc, Vukovic, Adams, Gorenson, Vuipe, Torre, Gitani.

**3** $\mathbf{r}_1 + 0.5\mathbf{r}_2 + 0.25\mathbf{r}_3$ gives −59.75, 11.75, 13.5, 19.75, 14.75; so the order is Bears, Stars, Eagles, Cats, Dragons.

**4** $\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3$ gives −14.5, 8.5, 7, −1, 0.5, −0.5; so the ranking is Beryl, Connie, Ewing, Frank, David and Adam. Other weightings give the same order.

**5** $\mathbf{r}_1 + \mathbf{r}_2$ gives 28, −123, 93, 115, −34.5, 15; so the order is Sharma, Jones, Adams, Smith, Singh, Feltham.

## Exercise 4.07

**1** **a**
$$\begin{bmatrix} 0 & 0 & 0.8 & 0.9 & 0.7 & 0.5 & 0.3 & 0.1 \\ 0.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.9 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0 \end{bmatrix}, \begin{bmatrix} 500 \\ 300 \\ 250 \\ 200 \\ 200 \\ 150 \\ 100 \\ 50 \end{bmatrix}$$

**b**
$$\begin{bmatrix} 630 \\ 300 \\ 240 \\ 225 \\ 180 \\ 160 \\ 90 \\ 30 \end{bmatrix}$$

**c**
$$\begin{bmatrix} 630.5 \\ 378 \\ 240 \\ 216 \\ 202.5 \\ 144 \\ 96 \\ 27 \end{bmatrix}$$

**d**
$$\begin{bmatrix} 631.65 \\ 378.3 \\ 302.4 \\ 216 \\ 194.4 \\ 162 \\ 86.4 \\ 28.8 \end{bmatrix}$$

**e** 3500, 3710, 3868, 4000

**2** **a** 0.2, 0.3, 0.1, 0

**b** $\begin{bmatrix} 0.3 & 1.5 & 1.1 & 0.4 \\ 0.2 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{bmatrix}, \begin{bmatrix} 400 \\ 250 \\ 150 \\ 50 \end{bmatrix}$

**c** $\begin{bmatrix} 680 \\ 80 \\ 75 \\ 15 \end{bmatrix}$ **d** $\begin{bmatrix} 412.5 \\ 136 \\ 24 \\ 7.5 \end{bmatrix}$

**e** $\begin{bmatrix} 357.15 \\ 82.5 \\ 40.8 \\ 2.4 \end{bmatrix}$

**f** 1700, 1700, 1160, 966

**3** **a** 0.02, 0.1, 0.15, 0.05, 0

**b** $\begin{bmatrix} 0 & 30 & 40 & 30 & 10 \\ 0.02 & 0 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0.15 & 0 & 0 \\ 0 & 0 & 0 & 0.05 & 0 \end{bmatrix}, \begin{bmatrix} 12\,500 \\ 500 \\ 75 \\ 40 \\ 5 \end{bmatrix}$

**c** 39 127, 20 551, 26 041, 6217, 1159

**d** Erratic at first, then going down.

**4** $\begin{bmatrix} 0 & 1.2 & 1.68 & 0.96 & 0.4 \\ 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0.64 & 0 & 0 & 0 \\ 0 & 0 & 0.48 & 0 & 0 \\ 0 & 0 & 0 & 0.32 & 0 \end{bmatrix}$

**5** The long-term trend is down by about 10% (10.073…) every 2 years. $1.1(1 + h) = 1$ gives $h \approx 9.1\%$, so they need to bring in about 205 animals every two years (or 105 each year).

**6** The long-term trend is up by about 14% (14.127…) per year. A culling rate of about 12.4% will stabilise the population. To fully eradicate the horses in about 3 years (less than 2 horses), you will need a culling rate of about 92%. Many people will object to killing brumbies because horses are companion animals.

**7** The long-term trend is up by about 19.2% every 6 months. Sustainable harvesting using pots would be about 35.8% (788 crabs). You could also dredge to harvest the whole population, but then the sustainable rate would be about 83%.

**8** The population was originally very stable. It will decrease to about 58 animals in only 10 years. To ensure survival, there should be intensive study of the animals and the disease to try to establish immunity. You could also isolate some healthy animals.

## Exercise 4.08

**1** **a** $\mathbf{U_A} = \begin{bmatrix} 0.5 & 0.55 & 0.35 \\ 0.4 & 0.55 & 0.3 \\ 0.6 & 0.65 & 0.45 \end{bmatrix}$

**b** $\mathbf{U_B} = \begin{bmatrix} 0.5 & 0.6 & 0.4 \\ 0.45 & 0.45 & 0.35 \\ 0.65 & 0.7 & 0.55 \end{bmatrix}$

**c** Super-creamy

**d** Super-creamy

**e** Skim milk

**2** **a** $\pi_{23} = 0.45$ **b** $P(x_4 \mid x_3) = 0.25$

**c** $P(x_4(3)) = 0.265\,3875$

**3** $\mathbf{P} = \begin{bmatrix} 0.541 & 0.417 & 0.042 \\ 0.435 & 0.435 & 0.13 \\ 0.25 & 0.5 & 0.25 \end{bmatrix}$

**4** **a** 0.06 **b** 0.29 **c** 0.15 **d** 0.21
**e** 0.33 **f** 0.27 **g** 0.1 **h** 0.21
**i** 0.45 **j** 0.33

**5** **a** No **b** Yes **c** No **d** No
**e** No

**6** **a** Yes **b** No **c** No

**7** **a** 0.169 **b** 0.1093 **c** 0.3076
**d** About (0.3063 0.1357 0.2101 0.3478)

**8** **a** 0.3335 **b** 0.3324 **c** 0.3320
**d** 0.3315 **e** 0.3334

**9** **a** $\mathbf{A} = \begin{bmatrix} \frac{1}{6} & \frac{4}{13} \\ \frac{7}{12} & \frac{3}{13} \end{bmatrix}$ **b** $\mathbf{D} = \begin{bmatrix} 50 \\ 80 \end{bmatrix}$

**c** $\mathbf{X} \approx \begin{bmatrix} 136.7 \\ 207.6 \end{bmatrix}$ **d** $\mathbf{P} \approx [34.2 \;\; 95.8]$

**10** **a** $\mathbf{X} = \begin{bmatrix} 1400 \\ 950 \\ 1550 \end{bmatrix}$, $\mathbf{P} = [800 \;\; 400 \;\; 700]$


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**b** $\mathbf{A} = \begin{bmatrix} \frac{1}{14} & \frac{6}{19} & \frac{8}{31} \\ \frac{1}{7} & \frac{1}{19} & \frac{6}{31} \\ \frac{3}{14} & \frac{4}{19} & \frac{1}{31} \end{bmatrix}$

**c** $\mathbf{D} = \begin{bmatrix} 400 \\ 600 \\ 700 \end{bmatrix}$ **d** $\mathbf{X} \approx \begin{bmatrix} 1157.0 \\ 1073.3 \\ 1299.7 \end{bmatrix}$

**e** $\mathbf{P} \approx [661.1 \;\; 451.9 \;\; 586.9]$

**11** They could agree to both charge high prices, but this is illegal. Charging medium prices would maximise their minimum returns.

**12** The best strategy for either company is to supply cola, so consumers will only be able to get cola.

**13** (1533.5, 1803.2, 1808.3, 2021.7) and (102.2, 601.1, 452.1, 594.6)

**14** The steady state vector is about (0.220, 0.128, 0.299, 0.177, 0.176), so Tullamarine and Sydney end up with more of the cars. Avalon ends up with the smallest number. If the hiring patterns are not the same as the returns, cars will need to be moved between airports.

**15** V1 has $\frac{1}{6}$, V3 has $\frac{1}{4}$, V5 $\frac{1}{12}$ and the others $\frac{1}{8}$.
This ignores the fact that most people look at everything and go back to items of greater interest. It also assumes that people all change rooms at the same time.

## Chapter review

**1 a** $x = -2, y = \frac{1}{3}, z = 4$

**b** Inconsistent equations, no solution

**c** Dependent, no unique solution $p - 2q = 3$, $q + r = 3$

**d** Dependent, no unique solution, $a = 5 - 2c, b = -2c, d = 2c - 4$

**2** $\begin{bmatrix} 3 & -2 & 1 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & -2 & 1 & -4 \\ -1 & 3 & -1 & -3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 8 \\ 2 \end{bmatrix}$

**3** $\left[\begin{array}{cccc|c} 2 & -5 & 1 & -1 & 8 \\ 1 & 2 & -3 & 4 & 0 \\ -3 & 1 & 5 & 0 & 2 \\ 5 & 2 & 0 & -3 & 6 \end{array}\right]$

**4 a** $a = -2, b = 3, c = 4$

**b** Inconsistent equations, no solution

**c** Consistent equations, dependent, $p + 2q = -1$, $q - r = 4, s = -4$

**d** Consistent equations, dependent, $s + h = 0$, $h - i = 2, m = 3$

**5 a** Planes intersecting in the line $\mathbf{r}(t) = \frac{1}{4}(10 - 11t)\mathbf{i} + \frac{1}{4}(9t - 18)\mathbf{j} + t\mathbf{k}$

**b** Parallel planes

**6 a** Plane intersecting in the single point $(1\frac{4}{5}, -2, -1\frac{3}{5})$.

**b** Two parallel planes intersecting the third in the parallel lines

$t\mathbf{i} + \frac{1}{3}(13 - 11t)\mathbf{j} + \frac{1}{6}(5 - 10t)\mathbf{k}$,

$t\mathbf{i} + \frac{1}{3}(19 - 11t)\mathbf{j} + \frac{1}{3}(4 - 5t)\mathbf{k}$

**c** Planes intersecting in the 3 parallel lines

$\frac{1}{11}(37 - 22t)\mathbf{i} + t\mathbf{j} + \frac{1}{11}(36 - 11t)\mathbf{k}$,

$\frac{1}{3}(10 - 6t)\mathbf{i} + t\mathbf{j} + \frac{1}{3}(10 - 3t)\mathbf{k}$,

$\frac{1}{13}(44 - 26t)\mathbf{i} + t\mathbf{j} + \frac{1}{13}(41 - 13t)\mathbf{k}$

**7** $\mathbf{r}_1 + 0.5\mathbf{r}_2$ gives $\left(\frac{1}{4}, -3\frac{3}{8}, 3\frac{1}{4}, 1\frac{1}{4}, -\frac{1}{2}\right)$ so the order is Toni, Shaan, David, Jason, Beryl.

**8 a** In the 3rd year (i.e., age 2–3)

**b** 30% **c** 40% **d** 240

**9** $\begin{bmatrix} 0 & 0 & 5.1 & 4.25 & 3 & 1.7 \\ 0.255 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.425 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.51 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.34 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.17 & 0 \end{bmatrix}$

**10** $\mathbf{U}_A = \begin{bmatrix} 0.4 & 0.4 & 0.3 \\ 0.2 & 0.4 & 0.2 \\ 0.4 & 0.3 & 0.4 \end{bmatrix}, \mathbf{U}_B = \begin{bmatrix} 0.6 & 0.8 & 0.6 \\ 0.6 & 0.6 & 0.7 \\ 0.7 & 0.8 & 0.6 \end{bmatrix}$

**11 a** Yes, as all entries are non-zero.

**b** 0.2 **c** 0.2

**d** $\mathbf{X}_1 = [0.2 \;\; 0.25 \;\; 0.25 \;\; 0.3]$, $\mathbf{X}_2 = [0.205 \;\; 0.26 \;\; 0.25 \;\; 0.285]$

**12** **a** [1000  700]  **b** $\begin{bmatrix} \frac{3}{17} & \frac{2}{5} \\ \frac{4}{17} & \frac{2}{15} \end{bmatrix}$

**c** $\begin{bmatrix} 1000 \\ 1200 \end{bmatrix}$  **d** $\begin{bmatrix} 2173.4 \\ 1974.7 \end{bmatrix}$

**e** [1278.5  921.5]

**13** Densities of concrete, gravel and soil are thus 3400 kg/m$^3$, 2800 kg/m$^3$ and 2100 kg/m$^3$.

**14** Compositions (in order tin, copper, lead): 20%, 40%, 40%; 25%, 35%, 40%; 15%, 50%, 35%.

**15** $\mathbf{r}_1 + 0.5\mathbf{r}_2 + 0.25\mathbf{r}_3$ gives $(48\frac{3}{4}, -21\frac{1}{8}, -19, -\frac{7}{16}, -40\frac{15}{16}, 32\frac{3}{4})$, so the order is Beavers, Tigers, Panthers, Hawks, Cats and Saints.

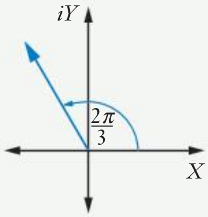
**16** The population increases by about 10.7% each 5 years. Reduce the population cull by more than about 9.7% per 5 years (or 2.0% annually). You can sustainably harvest by that amount.

**17** Entrance 0.4, Exit 0.6

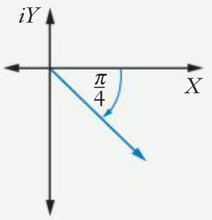
# Chapter 5

## Exercise 5.01

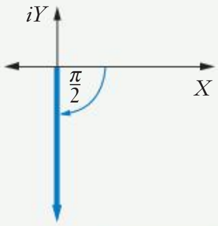
**1** **a**



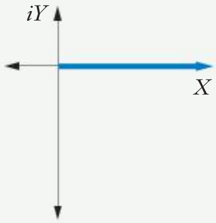

**b**




**c**




**d**

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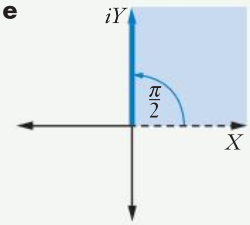


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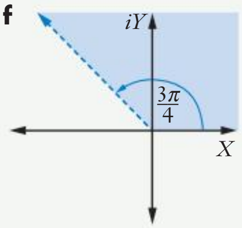


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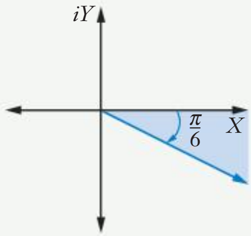


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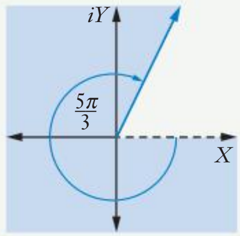


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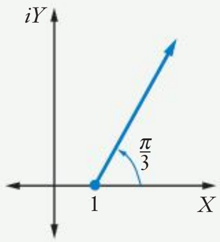


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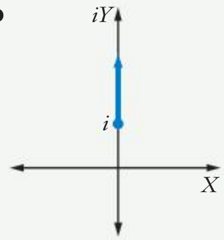


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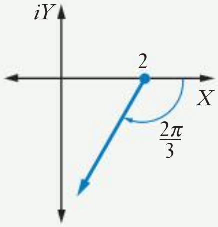


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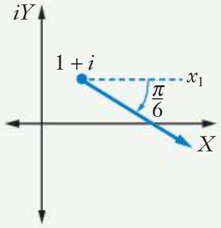


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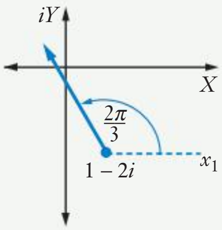


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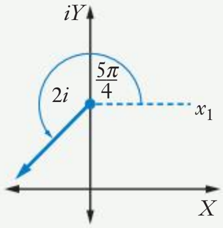


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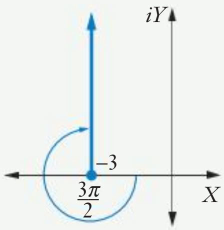


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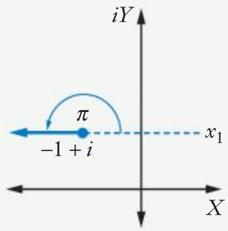



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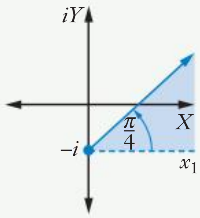


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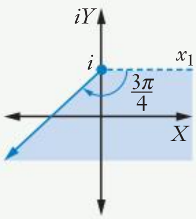


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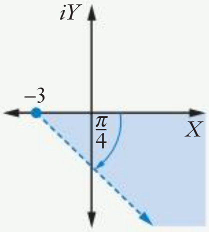


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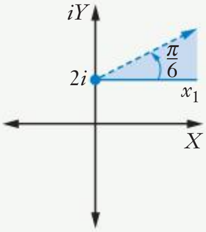


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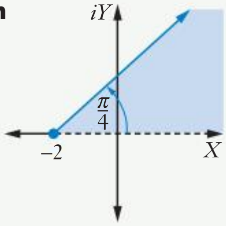


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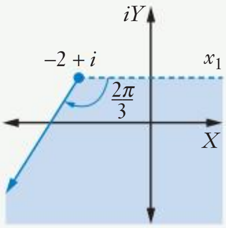


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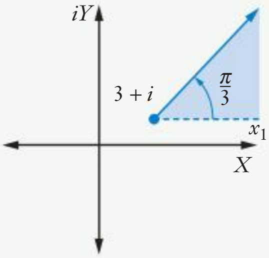


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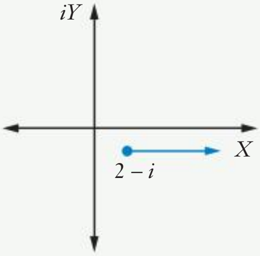
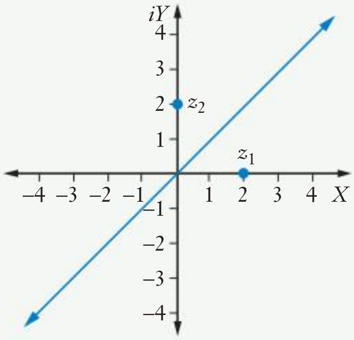
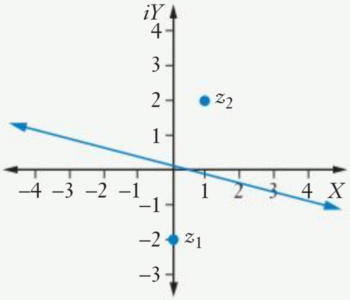


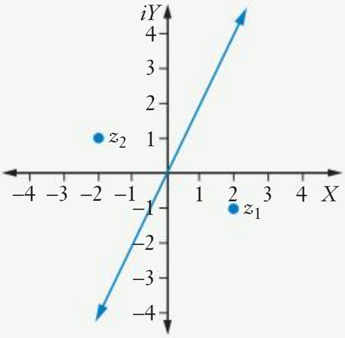

**3** **a** Perpendicular bisector of the line joining $z_1(2, 0)$ and $z_2(0, 2)$.



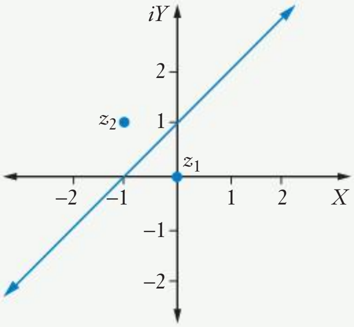

**b** Perpendicular bisector of the line joining $z_1(0, -2)$ and $z_2(1, 2)$.



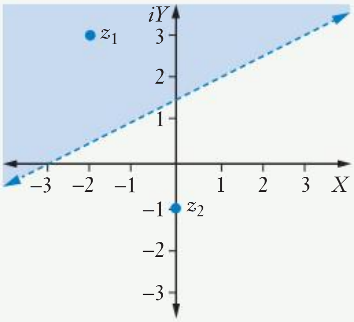

**c** Perpendicular bisector of the line joining $z_1(2, -1)$ and $z_2(-2, 1)$.



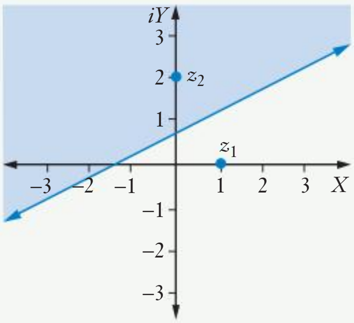
**d** Perpendicular bisector of the line joining $z_1(0, 0)$ and $z_2(-1, 1)$.



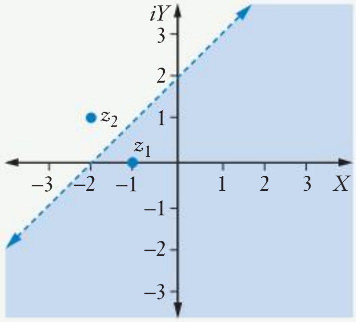

**e** Region left of and above the perpendicular bisector of the line joining $z_1(-2, 3)$ and $z_2(0, -1)$.



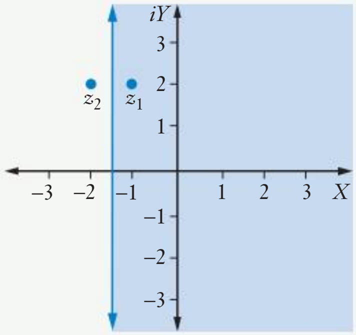

**f** Region left of and above the perpendicular bisector of the line joining $z_1(1, 0)$ and $z_2(0, 2)$, including points on the bisector.



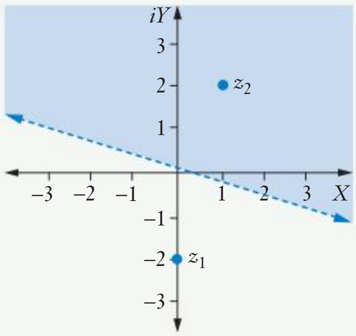

**g** Region right of and below the perpendicular bisector of the line joining $z_1(-1, 0)$ and $z_2(-2, 1)$.




**h** Region right of the perpendicular bisector of the line joining $z_1(-1, 2)$ and $z_2(-2, 2)$, including points on the bisector.




**i** Region right of and above the perpendicular bisector of the line joining $z_1(0, -2)$ and $z_2(1, 2)$.





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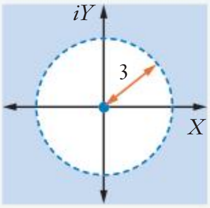
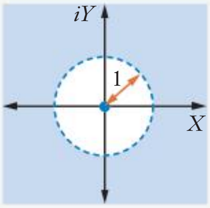
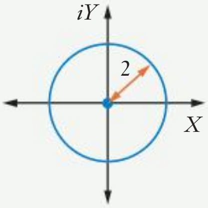
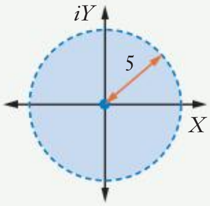
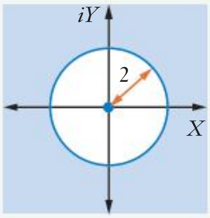
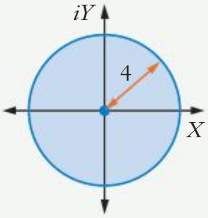
**4** **a** 

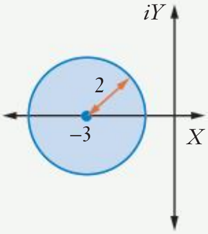

**b** 

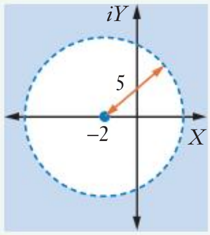

**c** 

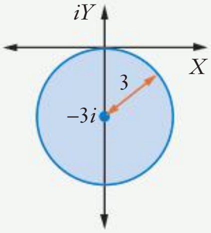

**d** 

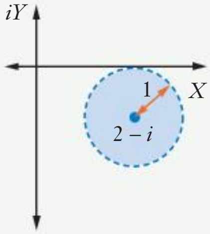

**e** 

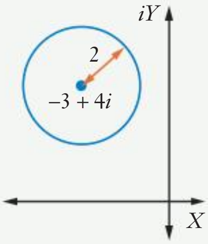

**f** 

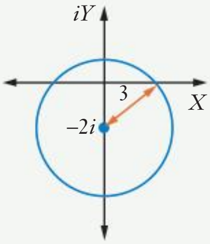

**g** 


**h** 

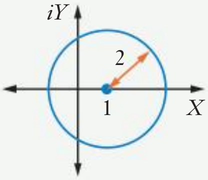

**i** 

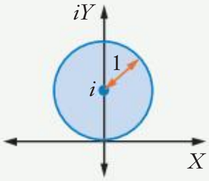

**j** 

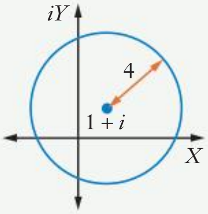

**k** 

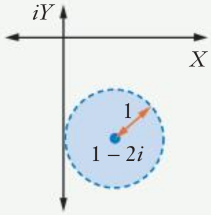

**l** 

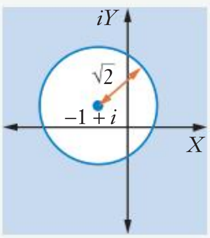

**5** **a** $C = (2, 0), r = 5$ **b** $C = (0, 1), r = 3$
**c** $C = (1 + \sqrt{5}), r = 2$ **d** $C = (0, -1), r = 6$
**e** $C = \left(-\frac{1}{2}, -\frac{1}{2}\right), r = \frac{5}{2}$ **f** $C = (0, -2), r = 2$

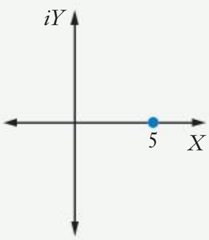
**6** **a** 

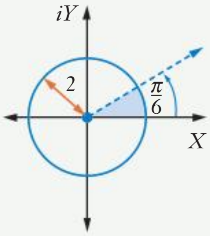

**b** 

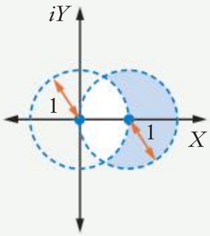

**c** 

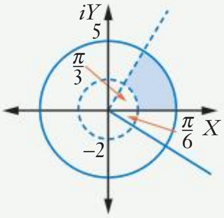

**d** 


**e** 


**f** 


**7** 


**8** 


**9** 

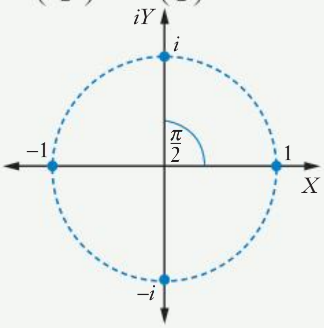

**10** $y = x - 1$

**11** The equation of the locus is $x^2 + (y - 5)^2 = 16$, which is a circle with centre (0, 5) and radius 4.

## Exercise 5.02

**1 a** $\text{cis}\left(\frac{-\pi}{2}\right)$, 1, $\text{cis}\left(\frac{\pi}{2}\right)$, cis ($\pi$) or $-1, -i, 1, i$

**b**

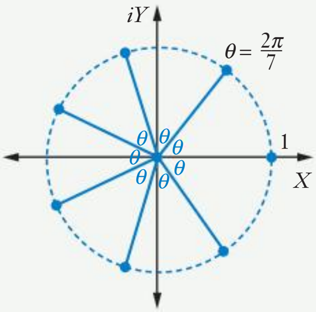



**c** The fourth roots of unity are equally spaced on the unit circle. One root is 1 and the remaining adjacent roots are separated by an angle of $\frac{\pi}{2}$.

**2 a–d** Proofs

**3 a** $\text{cis}\left(\frac{-6\pi}{7}\right)$, $\text{cis}\left(\frac{-4\pi}{7}\right)$, $\text{cis}\left(\frac{-2\pi}{7}\right)$, 1, $\text{cis}\left(\frac{2\pi}{7}\right)$, $\text{cis}\left(\frac{4\pi}{7}\right)$, $\text{cis}\left(\frac{6\pi}{7}\right)$

**b**




**c** The seventh roots of unity are equally spaced on the unit circle. One root is 1 and the remaining adjacent roots are separated by an angle of $\frac{2\pi}{7}$.

**4 a** $-1$, $\text{cis}\left(\frac{-2\pi}{3}\right)$, $\text{cis}\left(\frac{-\pi}{3}\right)$, 1, $\text{cis}\left(\frac{\pi}{3}\right)$, $\text{cis}\left(\frac{2\pi}{3}\right)$

**b** $-1, \frac{-1-i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}, 1, \frac{1+i\sqrt{3}}{2}, \frac{-1+i\sqrt{3}}{2}$

**5 a** 0 **b** $-1$ **c** 8

**6 a–d** Proofs

**7 a** $z^3 = 27 \Rightarrow z^3 - 27 = 0 \Rightarrow (z-3)(z^2+3z+9) = 0$. Factorising the quadratic gives $z = \frac{-3 \pm 3i\sqrt{3}}{2}$.

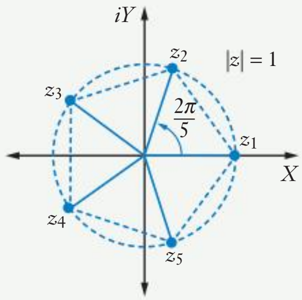
Cube roots are $z = 3$, $z = \frac{-3+3i\sqrt{3}}{2}$ and $z = \frac{-3-3i\sqrt{3}}{2}$.

**b i–iii** Proofs

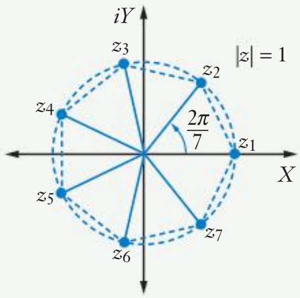
**8** $z^4 = 81 \Rightarrow (z^2-9)(z^2+9) = 0 \Rightarrow z = \pm 3$ or $z = \pm 3i$.

**9** Proofs

**10** Proofs. The roots form a regular pentagon.

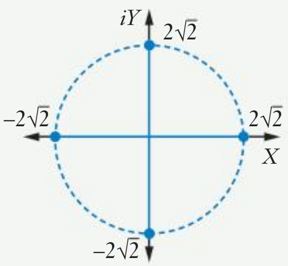



**11** Proofs. The roots form a regular heptagon.




## Exercise 5.03

**1** $2\sqrt{2}, 2i\sqrt{2}, -2\sqrt{2}, -2i\sqrt{2}$





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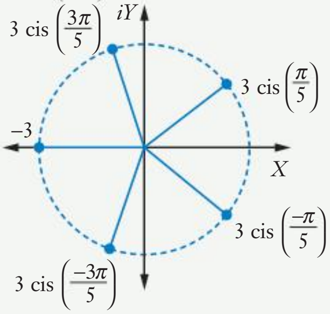
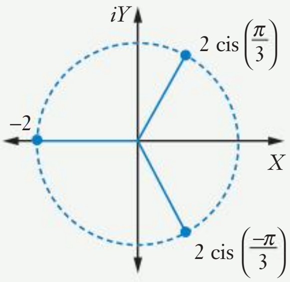
**2** $3 \text{ cis}\left(-\frac{\pi}{5}\right), 3 \text{ cis}\left(-\frac{3\pi}{5}\right), -3, 3 \text{ cis}\left(\frac{\pi}{5}\right),$

$3 \text{ cis}\left(\frac{3\pi}{5}\right)$

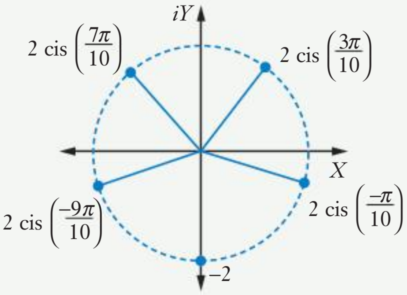



**3** $2 \text{ cis}\left(-\frac{\pi}{3}\right), -2, 2 \text{ cis}\left(\frac{\pi}{3}\right)$



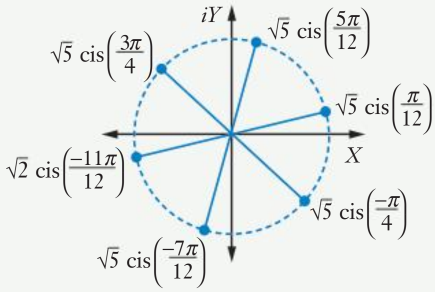

**4** $2 \text{ cis}\left(-\frac{9\pi}{10}\right), 2 \text{ cis}\left(-\frac{\pi}{10}\right), -2i, 2 \text{ cis}\left(\frac{3\pi}{10}\right),$

$2 \text{ cis}\left(\frac{7\pi}{10}\right)$

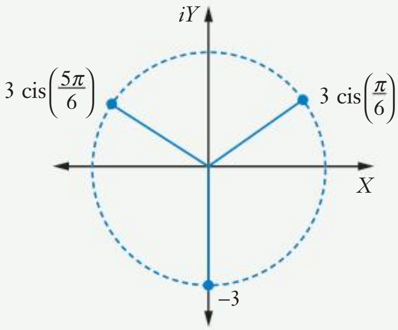



**5** $\sqrt{5} \text{ cis}\left(-\frac{11\pi}{12}\right), \sqrt{5} \text{ cis}\left(-\frac{7\pi}{12}\right), \sqrt{5} \text{ cis}\left(-\frac{\pi}{4}\right),$

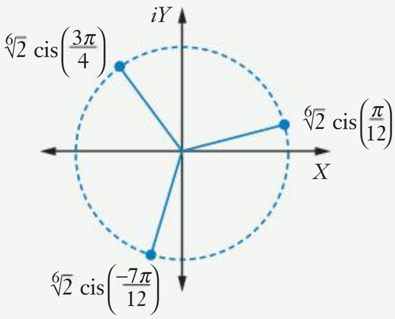
$\sqrt{5} \text{ cis}\left(\frac{\pi}{12}\right), \sqrt{5} \text{ cis}\left(\frac{5\pi}{12}\right), \sqrt{5} \text{ cis}\left(\frac{3\pi}{4}\right)$




**6** $-3i, 3 \text{ cis}\left(\frac{\pi}{6}\right), 3 \text{ cis}\left(\frac{5\pi}{6}\right)$




**7** $\sqrt[6]{2} \text{ cis}\left(-\frac{7\pi}{12}\right), \sqrt[6]{2} \text{ cis}\left(\frac{\pi}{12}\right), \sqrt[6]{2} \text{ cis}\left(\frac{3\pi}{4}\right)$




**8** **a** $\sqrt{3} - i, -\sqrt{3} - i, 2i$

**b** $\frac{1}{2} + \frac{1}{2}i, \frac{1}{2} - \frac{1}{2}i, -\frac{1}{2} + \frac{1}{2}i, -\frac{1}{2} - \frac{1}{2}i$

**c** $1, \text{cis}\left(\frac{2\pi}{5}\right), \text{cis}\left(\frac{4\pi}{5}\right), \text{cis}\left(-\frac{2\pi}{5}\right), \text{cis}\left(-\frac{4\pi}{5}\right)$

**d** $\frac{3\sqrt{3}}{2} + \frac{3}{2}i, -\frac{3\sqrt{3}}{2} + \frac{3}{2}i, -3i$

**e** $-\sqrt[3]{2}, \frac{\sqrt[3]{2}}{2} \pm \frac{\sqrt[3]{2}\sqrt{3}}{2}i$

**f** $-\frac{2}{3}, \frac{1}{3} \pm \frac{\sqrt{3}}{3}i$

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**g** $\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}, -\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}$

**h** $\text{cis}\left(\frac{\pi}{8}\right), \text{cis}\left(\frac{5\pi}{8}\right), \text{cis}\left(-\frac{7\pi}{8}\right), \text{cis}\left(-\frac{3\pi}{8}\right)$

**i** $-2\sqrt{3}-2i, 4i, 2\sqrt{3}-2i$

**j** $\sqrt[3]{2}\ \text{cis}\left(-\frac{\pi}{9}\right), \sqrt[3]{2}\ \text{cis}\left(\frac{5\pi}{9}\right), \sqrt[3]{2}\ \text{cis}\left(-\frac{7\pi}{9}\right)$

**k** $\sqrt[8]{2}\ \text{cis}\left(-\frac{15\pi}{16}\right), \sqrt[8]{2}\ \text{cis}\left(-\frac{7\pi}{16}\right), \sqrt[8]{2}\ \text{cis}\left(\frac{\pi}{16}\right), \sqrt[8]{2}\ \text{cis}\left(\frac{9\pi}{16}\right)$

**l** $\sqrt{2}\pm i\sqrt{2}, -\sqrt{2}\pm i\sqrt{2}$

**9 a** $2, 2\ \text{cis}\left(-\frac{4\pi}{5}\right), 2\ \text{cis}\left(-\frac{2\pi}{5}\right), 2\ \text{cis}\left(\frac{2\pi}{5}\right), 2\ \text{cis}\left(\frac{4\pi}{5}\right)$

**b** $\pm\left(\frac{4}{\sqrt{2}}+\frac{4i}{\sqrt{2}}\right)$ **c** $\pm\frac{1}{2}, \pm\frac{i}{2}$

**10 a** $\pm 1, \pm i$

**b** $\pm 2, \pm 2i$

**c** $\pm 1, \frac{1}{2}\pm\frac{\sqrt{3}}{2}i, -\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

**d** $\pm 2i, \sqrt{3}\pm i, -\sqrt{3}\pm i$

**e** $3i, 3\ \text{cis}\left(-\frac{7\pi}{10}\right), 3\ \text{cis}\left(-\frac{3\pi}{10}\right), 3\ \text{cis}\left(\frac{\pi}{10}\right), 3\ \text{cis}\left(\frac{9\pi}{10}\right)$

**f** $\sqrt{3}\ \text{cis}\left(-\frac{11\pi}{12}\right), \sqrt{3}\ \text{cis}\left(-\frac{7\pi}{12}\right), \sqrt{3}\ \text{cis}\left(-\frac{\pi}{4}\right), \sqrt{3}\ \text{cis}\left(\frac{\pi}{12}\right), \sqrt{3}\ \text{cis}\left(\frac{5\pi}{12}\right), \sqrt{3}\ \text{cis}\left(\frac{3\pi}{4}\right)$

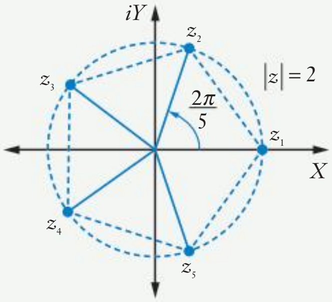
**g** $\sqrt[8]{2}\ \text{cis}\left(-\frac{15\pi}{16}\right), \sqrt[8]{2}\ \text{cis}\left(-\frac{7\pi}{16}\right), \sqrt[8]{2}\ \text{cis}\left(\frac{\pi}{16}\right), \sqrt[8]{2}\ \text{cis}\left(\frac{9\pi}{16}\right)$

**h** $\sqrt{2}\ \text{cis}\left(-\frac{11\pi}{12}\right), \sqrt{2}\ \text{cis}\left(-\frac{5\pi}{12}\right), \sqrt{2}\ \text{cis}\left(\frac{\pi}{12}\right), \sqrt{2}\ \text{cis}\left(\frac{7\pi}{12}\right)$

**i** $\pm\left(\sqrt{3}+i\right), \pm\left(1-i\sqrt{3}\right)$

**11 a** $\pm\left(\sqrt{3}+i\right)$

**b** $2\ \text{cis}\left(-\frac{4\pi}{5}\right), 2\ \text{cis}\left(-\frac{2\pi}{5}\right), 2, 2\ \text{cis}\left(\frac{2\pi}{5}\right), 2\ \text{cis}\left(\frac{4\pi}{5}\right)$




**c** $2\ \text{cis}\left(-\frac{7\pi}{20}\right), -\sqrt{2}-i\sqrt{2}, 2\ \text{cis}\left(\frac{\pi}{20}\right), 2\ \text{cis}\left(\frac{9\pi}{20}\right), 2\ \text{cis}\left(\frac{17\pi}{20}\right)$

## Exercise 5.04

**1 a** $2z^3$ **b** 7 **c** 3
**d** $-11+48i$ **e** $7+10i$

**2 a** $(2-i)z^3, 3$ **b** $(2-3i)z^5, 5$ **c** $-7z^6, 6$
**d** $-5z^9, 9$ **e** $3iz^8, 8$ **f** $(3-i)z^7, 7$

**3 a** $11-7i$ **b** $-12i$ **c** $6i$
**d** 0 **e** $2+2i$ **f** $-4$

**4 a** $30-15i$ **b** $9-12i$ **c** $-28+7i$
**d** $2+17i$ **e** $12+7i$

**5 a** $8-63i$ **b** $-17+27i$ **c** $37+17i$
**d** $34-7i$ **e** $-58-48i$

**6 a** $z^3+(2-3i)z^2+(6+4i)z+10-4i$
**b** $z^3+(3-3i)z^2+(6+2i)z+2+4i$
**c** $z^4+(5-5i)z^3+(5-9i)z^2+30z+17-20i$
**d** $z^5+(3-5i)z^4+(-4+3i)z^3+(28+19i)z^2+(-46-6i)z-23+52i$
**e** $z+2+\frac{1+2i}{z+3-2i}$
**f** $z^2+(-1-i)z+10+5i+\frac{-33+3i}{z+3-2i}$

**7 a** $(1-2i)z^3+(1-7i)z^2+(10-7i)z+3+17i$
**b** $(6-9i)z^2+(2-3i)z+20+9i$
**c** $(1-2i)z^4+(1+6i)z^3+(15-12i)z^2+(-2+31i)z+1-21i$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**d** $(-4-7i)z^5+(-9+2i)z^4+(8-28i)z^3$
$+(1-20i)z^2+(52+33i)z-41-23i$

**e** $(2-3i)z-3-13i+\dfrac{-41-10i}{z-2+3i}$

**f** $(1-2i)z^2-(7+8i)z-32+\dfrac{-69+99i}{z-2+3i}$

**8** **a** $z^2+(3-3i)z+8-2i+\dfrac{20-4i}{z-2}$

**b** $(1-2i)z^2-(4+9i)z-23+9i+\dfrac{29+61i}{z+3i}$

**c** $z^2+(3+5i)z-11-4i+\dfrac{28-25i}{z+1-2i}$

**9** **a** $91-55i$ **b** $-1+18i$ **c** $1-51i$

**10** **a** $58+25i$ **b** $-9+100i$ **c** $-148-28i$

**11** $a=5, b=6$

## Exercise 5.05

**1** **a** $P(2)=0$ **b** $P(-1)=0$ **c** $P(3)=0$
**d** $P(-2)=0$

**2** **a** $(z-1-3i)(z-1+3i)(z-2)$
**b** $(z-2-i)(z-2+i)(z+1)$
**c** $(z-3)(z-1+2i)(z-1-2i)$
**d** $(z-4+i)(z-4-i)(z+2)$

**3** $(z-2)\left(z+1-i\sqrt{3}\right)\left(z+1+i\sqrt{3}\right)$

**4** $(z-2)(z+3i)(z-3i)$

**5** **a** $h(-3)=0$, so $z+3$ is a factor.
**b** $(z+3)(z+1-2i)^2$

**6** **a** $h(4)=0$ so $z-4$ is a factor.
**b** $(z-4)(z-2-3i)(z-2+3i)$

**7** $(z-1-i)(z-1+i)(z+3)$

**8** $(z-2-3i)(z-2+3i)(z+1)(z-1)$

**9** $(z-2-i)(z-2+i)(z-2)(z+1)$

**10** $(z-2i)(z+2i)(z-2i)(2+2i)=(z-2i)^2(z+2i)^2$

**11** $z^4+2z^3+10z^2+18z+9$

**12** $z^4-z^3-3z^2+55z-52$

**13** $z^5-4z^4+15z^3-24z^2+14z-20$

**14** **a** $(z-2)(z+3)(z-2+i)(z-2-i)$
**b** $(z+2)(z+1)(z+3+4i)(z+3-4i)$

**15** $P(z)=2z^3-3z^2+8z-12$

**16** $R(z)=\dfrac{1}{2}z^5-2z^4+\dfrac{13}{2}z^3-18z^2+18z$

## Exercise 5.06

**1** **a** $\pm\sqrt{7}$ **b** $\pm i\sqrt{5}$ **c** $\pm\dfrac{5}{3}$
**d** $\pm\dfrac{3i}{2}$ **e** $\pm i\sqrt{6}$ **f** $\pm3$
**g** $\pm\dfrac{i}{3}$ **h** $\pm\dfrac{i}{2}$ **i** $\pm i\sqrt{3}$
**j** $3\pm2i$ **k** $-1\pm i\sqrt{5}$ **l** $-\dfrac{3}{2}\pm\dfrac{i}{4}$

**2** **a** $1\pm3i$ **b** $\dfrac{1}{2}\pm\dfrac{\sqrt{3}}{2}i$ **c** $-\dfrac{1}{2}\pm\dfrac{\sqrt{5}}{2}i$
**d** $-\dfrac{1}{3}\pm\dfrac{1}{3}i$ **e** $-\dfrac{1}{2}\pm\dfrac{\sqrt{3}}{2}i$ **f** $\dfrac{3}{4}\pm\dfrac{\sqrt{7}}{4}i$

**3** **a** $z^2-2z+17=0$ **b** $z^2-4z+13=0$
**c** $z^2+2z+5=0$ **d** $z^2-2z+4=0$
**e** $z^2+4z+9=0$ **f** $z^2-8z+65=0$

**4** **a** $z=1\pm3i$ and $z=2$ **b** $z=2\pm3i$ and $z=4$
**c** $z=2\pm4i$ and $z=-3$ **d** $z=1\pm5i$ and $z=-1$
**e** $z=2\pm5i$ and $z=2$ **f** $z=5\pm3i$ and $z=-4$

**5** **a** $z=\pm2\sqrt{2}$ and $z=-2i$ **b** $z=\pm2i\sqrt{3}$ and $z=3i$
**c** $z=\pm i\sqrt{10}$ and $z=5i$ **d** $z=\pm i\sqrt{3}$ and $z=-6i$
**e** $z=\pm2i\sqrt{5}$ and $z=5i$
**f** $z=\pm2i\sqrt{6}$ and $z=-3i$

**6** $a=-2, b=5$; roots $=2\pm3i, -2$

**7** $p=-1, q=-17$; roots $=-3\pm2i, 5$

**8** $m=14, n=73$; roots $=-4\pm3i, -6$

**9** **a** $z=\pm2i, \pm2$ **b** $z=\pm4i, \pm3$
**c** $\pm2i\sqrt{3}, \pm3\sqrt{2}$ **d** $\pm2i\sqrt{5}, \pm2\sqrt{2}$
**e** $\pm i\sqrt{6}, \pm3\sqrt{2}$ **f** $\pm4i\sqrt{2}, \pm2\sqrt{6}$

**10** $a=-24$, roots $=4\pm3i$ and $\pm i\sqrt{3}$

**11** $b=72$, roots $=-3\pm2i$ and $\pm2i\sqrt{3}$

**12** $d=\pm3$, roots $=\pm3i$ and $2\pm3i$

**13** **a** $2z^4+6z^3+7z^2+6z+5$
**b** $z=\pm i$ and $-\dfrac{3}{2}\pm\dfrac{1}{2}i$

**14** **a** $z^4+2z^3+6z^2+8z+8$
**b** $z=\pm2i$ and $-1\pm i$

**15** **a** $1, -1\pm i$ **b** $-1, -2\pm i$ **c** $-3, \pm i$
**d** $2, \dfrac{1}{2}\pm\dfrac{\sqrt{3}}{2}i$ **e** $2, 1\pm2i$ **f** $1, 1\pm2i$

**16** **a** $z=\pm2i$ and $\pm\sqrt{3}$ **b** $z=\pm1$ and $\pm i$
**c** $z=2, -3$ and $4\pm3i$

**17** $z=2-3i, 2+3i, 4i$ and $-4i$. As $Q(z)$ has exactly 5 roots, it must have 1 real root.

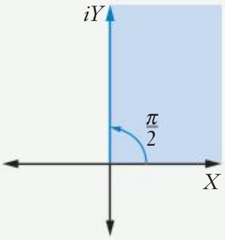
**18** $P(z) = z^5 + 10z^3 + 10z^2 + 9z + 90$

**19** $z = 2, \pm i$ and $\pm\sqrt{3}$

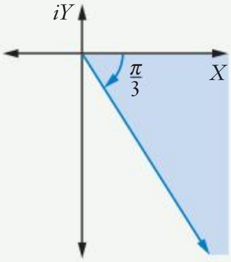
**20** $(z - 3 - 5i)(z - 3 + 5i)(z - 2i)(z + 2i)(z + 3)$

## Chapter review

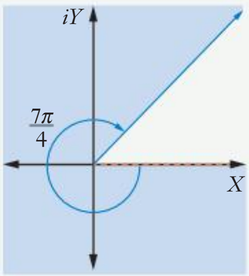
**1 a**



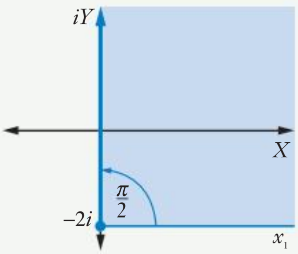

**b**



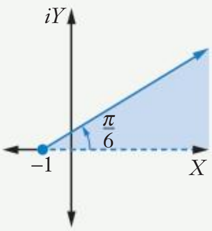

**c**



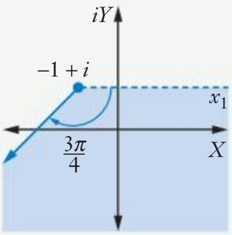

**d**



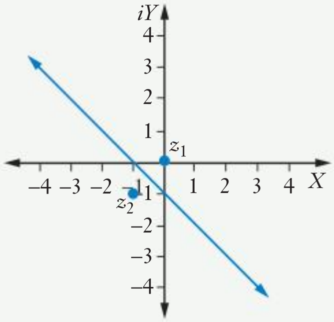

**e**



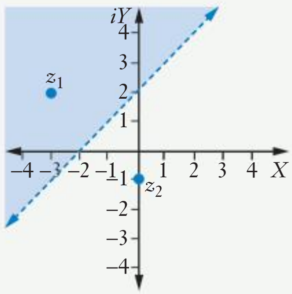

**f**



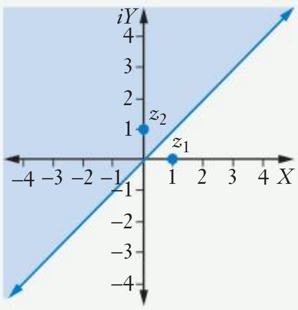

**2 a** Perpendicular bisector of the line joining $z_1(0, 0)$ and $z_2(-1, -1)$.




**b** Region left of and above the perpendicular bisector of the line joining $z_1(-3, 2)$ and $z_2(0, -1)$.




**c** Region left of and above the perpendicular bisector of the line joining $z_1(1, 0)$ and $z_2(0, 1)$, including points on the bisector.





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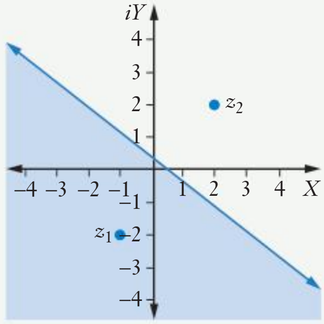
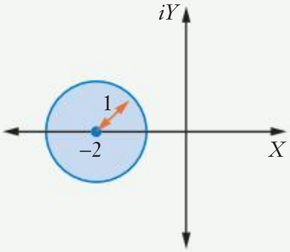
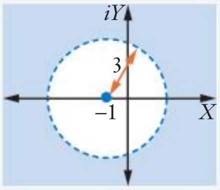
**d** Region left of and below the perpendicular bisector of the line joining $z_1(-1, -2)$ and $z_2(2, 2)$, including points on the bisector.



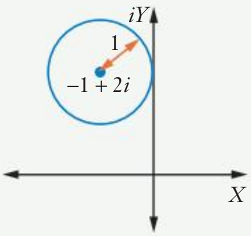

**3 a**



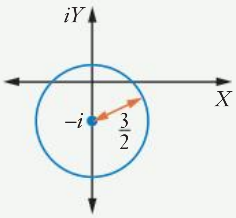

**b**




**c**

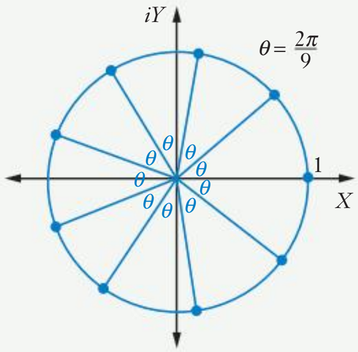



**d**




**4 a** $\text{cis}\left(\frac{-8\pi}{9}\right), \text{cis}\left(\frac{-2\pi}{3}\right), \text{cis}\left(\frac{-4\pi}{9}\right), \text{cis}\left(\frac{-2\pi}{9}\right), 1,$ $\text{cis}\left(\frac{2\pi}{9}\right), \text{cis}\left(\frac{4\pi}{9}\right), \text{cis}\left(\frac{2\pi}{3}\right), \text{cis}\left(\frac{8\pi}{9}\right)$

**b**

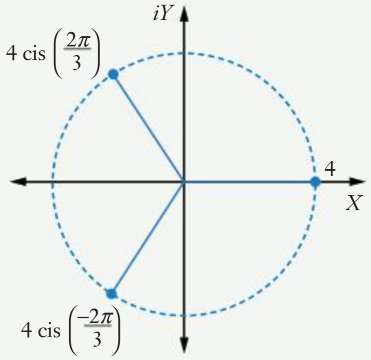



**c** The ninth roots of unity are equally spaced on the unit circle. One root is 1 and the remaining adjacent roots are separated by an angle of $\frac{2\pi}{9}$.

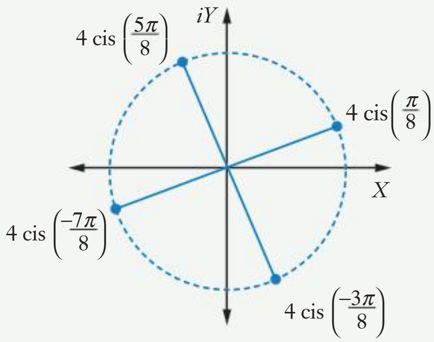
**5 a** $1, 0.3090 + 0.9511i, 0.3090 - 0.9511i, -0.8090 + 0.5878i, -0.8090 - 0.5878i$

**b** $\text{cis}\left(\frac{-4\pi}{5}\right), \text{cis}\left(\frac{-2\pi}{5}\right), 1, \text{cis}\left(\frac{2\pi}{5}\right), \text{cis}\left(\frac{4\pi}{5}\right)$

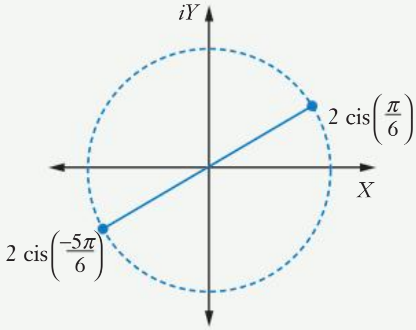
**6** Roots $= 4\text{ cis}\left(-\frac{2\pi}{3}\right)$ or 4 or $4\text{ cis}\left(\frac{2\pi}{3}\right)$




**7** Roots $= 4\text{ cis}\left(-\frac{7\pi}{8}\right)$ or $4\text{ cis}\left(-\frac{3\pi}{8}\right)$ or $4\text{ cis}\left(\frac{\pi}{8}\right)$ or $4\text{ cis}\left(\frac{5\pi}{8}\right)$



**8** Roots $= 2\text{ cis}\left(\frac{\pi}{6}\right)$ or $2\text{ cis}\left(-\frac{5\pi}{6}\right)$




**9** **a** $z^4$ **b** $-5i + 8$ **c** $4$
**d** $11 - 4i$ **e** $20 - 31i$ **f** $-9 - 14i$

**10** **a** $z^3 + (5 - i)z^2 + (8 + i)z - 1 + 6i$
**b** $(1 + 2i)z^3 + 4iz^2 + (1 + 18i)z - 19 - 17i$
**c** $z^2 + (3 - 6i)z - 19 - i + \frac{21 + 57i}{z + 1 + 3i}$

**11** **a** $22 - 15i$ **b** $44 + 70i$ **c** $188 - 102i$

**12** **a** $P(3 + 2i) = 0$
**b** $(z - 1)(z + 3 - i)(z + 3 + i)$

**13** $(z - 1 - 2i)(z - 1 + 2i)(z - 3)$

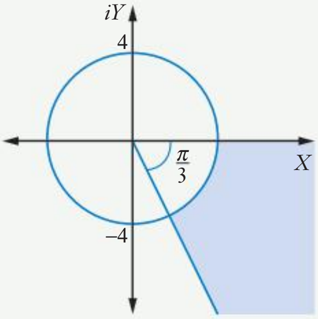
**14** $5z^3 - 3z^2 + 20z - 12$

**15** **a** $z = 4 \pm 3i$ **b** $P(z) = z^2 - 4z + 53$

**16** **a** $z = 1 \pm 3i$ and $z = -4$ **b** $z = \pm i\sqrt{5}$ and $z = 3i$
**c** $a = -5$, $b = 16$; roots $= 1 \pm 3i, 3$

**17** **a** $z = \pm i\sqrt{10}, \pm 2$
**b** $a = -10$, roots $= 1 \pm 2i$ and $\pm i\sqrt{5}$

**18**




**19** $Q(z) = 3z^3 - 4z^2 + 27z - 36$

**20** $P(z) = k(z - 3)(z - 2i)(z + 2i)(z - 2 - 5i)(z - 2 + 5i)$, where $k < 0$.

**21** $P(z)$ must have $n$ roots and $n$ is odd. Any roots of $P(z)$ that are complex must occur as conjugate pairs, so there will always be at least one real root.

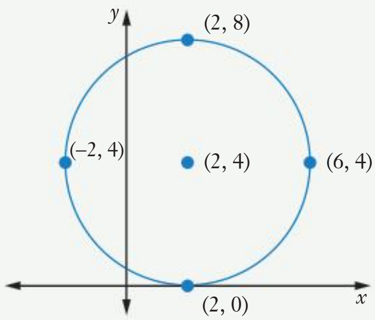
**22** $-2, -1, 1, 3 \pm 4i$

# Chapter 6

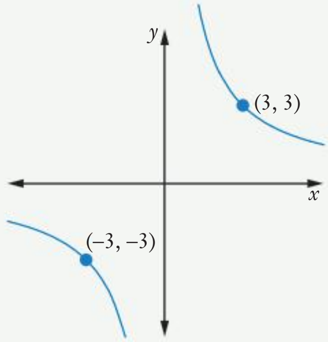
## Exercise 6.01

**1** **a** $x + 2y - 9 = 0$; $y = -\frac{1}{2}x + 4\frac{1}{2}$
**b** $y^2 = \frac{16x}{3}$
**c** $\frac{x^2}{4} + \frac{y^2}{25} = 1$
**d** $x^2 + y^2 = 9$
**e** $y = \frac{6}{x}$

**2** $(x - 2)^2 + (y - 4)^2 = 16$, circle with centre (2, 4) and radius 4.




**3** $y = \frac{9}{x}$, rectangular hyperbola stretched vertically by a factor of 9 with asymptotes at the axes.





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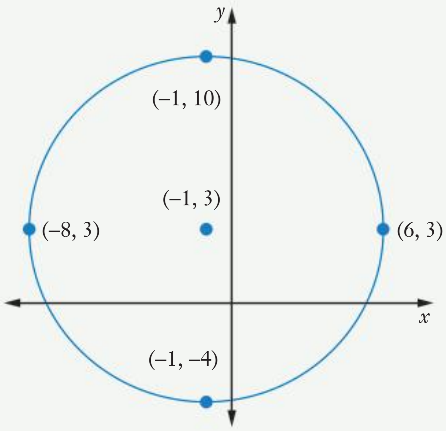
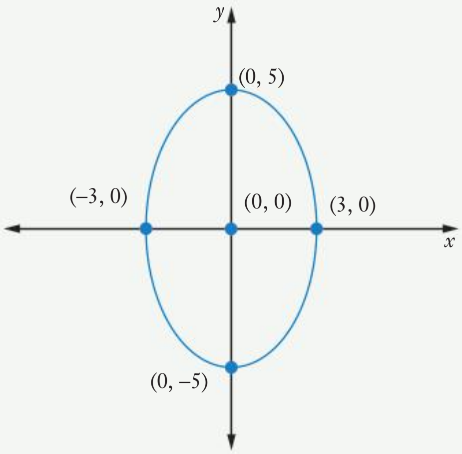
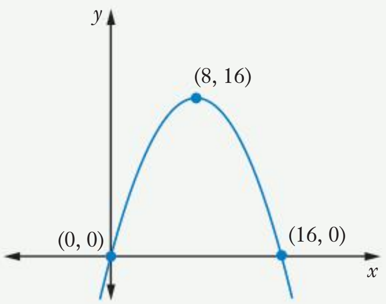
**4** $(x + 1)^2 + (y - 3)^2 = 49$, circle with centre $(-1, 3)$ and radius 7.



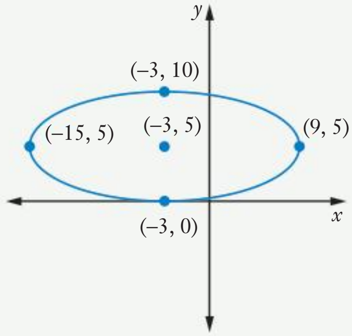

**5** $\frac{x^2}{9} + \frac{y^2}{25} = 1$, ellipse with centre at the origin, major axis of length 10 along the $y$-axis and minor axis of length 3 along the $x$-axis.



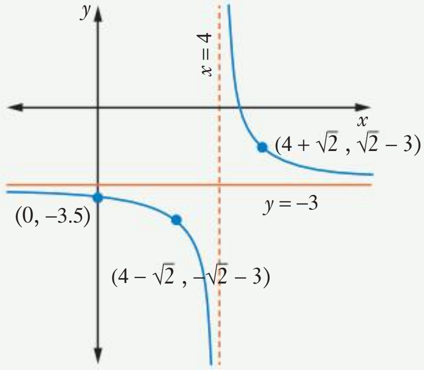

**6** $y = -0.25x(x - 4)$, parabola with maximum at $(8, 16)$, $x$-intercepts 0 and 16, $y$-intercept 0.




**7** $\frac{(x+3)^2}{144} + \frac{(y-5)^2}{25} = 1$, ellipse with centre at $(-3, 5)$, major axis of length 12 along the $x$-axis and minor axis of length 5 along the $y$-axis.




**8** $y = \frac{2}{x-4} - 3$, rectangular hyperbola with centre at $(4, -3)$, stretched vertically by a factor of 2 and asymptotes $x = -3$ and $y = 4$.




**9, 10** Proofs

## Exercise 6.02

**1** **a** $\mathbf{v}(t) = 6t\mathbf{i}$ **b** $\mathbf{a}(t) = 6\mathbf{i}$

**c** $\mathbf{v}(2) = 12\mathbf{i}$, $\mathbf{a}(2) = 6\mathbf{i}$, the object is accelerating along the line $y = 4$

**2** **a** $\mathbf{v}(t) = 2\mathbf{i} + 2t\mathbf{j}$ **b** $\mathbf{a}(t) = 2\mathbf{j}$

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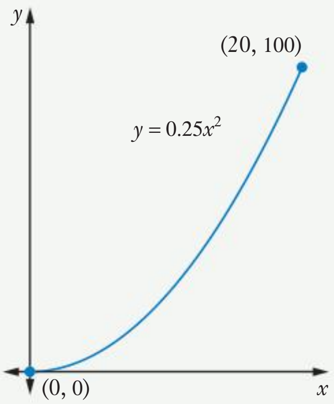
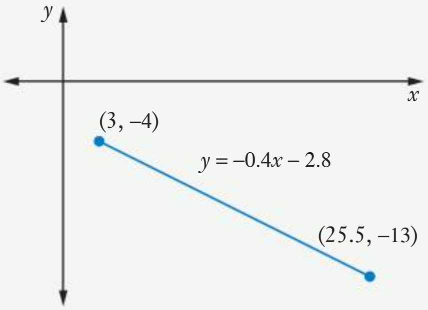
**c** The object follows a parabolic path.




**3 a** $\mathbf{r}(t) = (2.5t^2 + 3)\mathbf{i} - (t^2 + 4)\mathbf{j}$ m

**b** $\mathbf{a}(t) = 5\mathbf{i} - 2\mathbf{j}$ m/s$^2$

**c**




**4 a** $\mathbf{r}(t) = (0.5e^{2t} + 1.5)\mathbf{i} + (t^3 + 4t + 1)\mathbf{j}$ m

**b** $\mathbf{a}(t) = 2e^{2t}\,\mathbf{i} + 6t\mathbf{j}$ m/s$^2$

**5 a** $\mathbf{r}(t) = (2t^3 + 4t + 3)\mathbf{i} - (4.5t^2 + 2t - 4)\mathbf{j}$ m,
$\mathbf{v}(t) = (6t^2 + 4)\mathbf{i} - (9t + 2)\mathbf{j}$ m/s

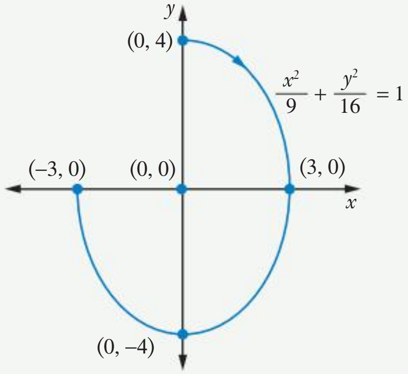
**b** $\mathbf{r}(5) = 273\mathbf{i} - 118.5\mathbf{j}$ m , $\mathbf{v}(5) = 154\mathbf{i} - 47\mathbf{j}$ m/s,
$\mathbf{a}(5) = 60\mathbf{i} - 9\mathbf{j}$ m/s$^2$

**6 a** $\mathbf{r}(t) = \left[3 + \frac{9}{4\pi^2} - \frac{9}{4\pi^2}\cos(2\pi t)\right]\mathbf{i} +$
$(2.5t^3 + 10t)\,\mathbf{j}$ m, $\mathbf{v}(t) = \frac{9}{2\pi}\sin(2\pi t)\mathbf{i} +$
$(7.5t^2 + 10)\mathbf{j}$ cm/s

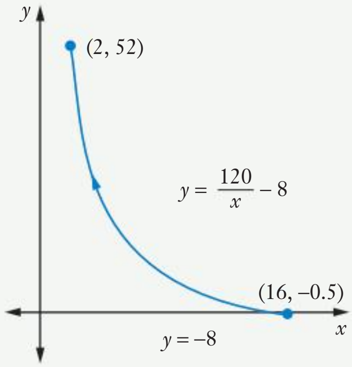
**b** $\mathbf{r}(4) = 3\mathbf{i} + 200\mathbf{j}$ m , $\mathbf{v}(4) = 130\mathbf{j}$ m/s,
$\mathbf{a}(4) = 9\mathbf{i} + 60\mathbf{j}$ m/s$^2$

**c** The particle is oscillating horizontally about the line $x = 3 + \frac{9}{4\pi^2}$ while accelerating vertically.

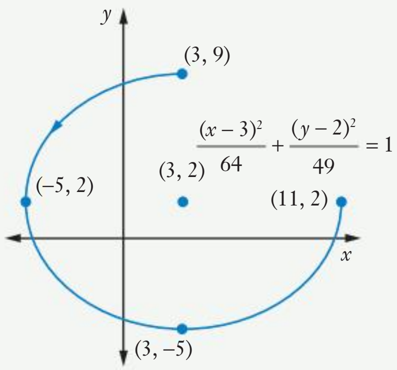
**7** The particle is travelling in a clockwise direction around the ellipse with the equation $\frac{x^2}{9} + \frac{y^2}{16} = 1$. It takes 4 seconds to complete a revolution. In the first 3 seconds, it starts from (0, 4) and goes round to (–3, 0).




**8** The particle is travelling in a hyperbolic path with asymptotes $x = 0$ and $y = -9$. The equation is $y = \frac{120}{x} - 8$. In the time from 0.5 to 4 seconds, it starts from (16, –0.5) and goes to (2, 52).




**9** The particle is travelling in an elliptical path with equation $\frac{(x-3)^2}{64} + \frac{(y-2)^2}{49} = 1$ in an anticlockwise direction. It takes 10 seconds to complete a revolution. In the time from 0 to ??? seconds it travels from (3, 9) to (11, 2).





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## Exercise 6.03

**1** **a** 19 m/s **b** −6 m/s **c** 10 m/s
**d** 5 m/s **e** −3 m/s

**2** **a** 33 m **b** 175 m **c** 5 m **d** 54 m

**3** **a** −3 m/s$^2$ **b** 4 m/s$^2$ **c** $3\frac{1}{8}$ m/s$^2$
**d** −8.5 m/s$^2$ **e** 7 m/s$^2$

**4** **a** 5 s **b** 1.5 s **c** 5 s **d** 7 s

**5** **a** 190 m/s **b** $570\frac{2}{3}$ m

**6** **a** 77.5 m/s **b** $404\frac{1}{6}$ m

**7** 36 m/s, 540 m

**8** −3 m/s$^2$, 37.5 m

**9** 5 m/s and 132 m

**10** 24 m/s

**11** $113\frac{1}{3}$ m

**12** −93 m/s

## Exercise 6.04

**1** **a** 16.38 m/s, 11.47 m/s **b** 5.63 m/s, 10.60 m/s
**c** 1.88 m/s, 17.90 m/s **d** 14.82 m/s, 2.35 m/s
**e** 17.84 m/s, 16.06 m/s

**2** **a** $16.38t$ m, $5 + 11.47t - 4.9t^2$ m
**b** $5.63t$ m, $9 + 10.6t - 4.9t^2$ m
**c** $1.88t$ m, $6 + 17.9t - 4.9t^2$ m
**d** $14.82t$ m, $2 + 2.35t - 4.9t^2$ m
**e** $17.84t$ m, $7 + 16.06t - 4.9t^2$ m

**3** **a** 1.17 s, 10.71 m **b** 1.08 s, 7.73 m
**c** 1.83 s, 23.35 m **d** 0.24 s, 5.28 m
**e** 1.64 s, 22.16 m

**4** **a** 1.521 s, 17.455 m **b** 1.96 s, 5.105 m
**c** 1.562 s, 23.334 m **d** 5.242 s, 112.253 m
**e** 4.835 s, 51.085 m

**5** **a** 16.53 m **b** 7.35 m **c** 29.38 m
**d** 1.84 m **e** 49.03 m

**6** James: about 24.2 m/s; Felecia: about 26.2 m/s

**7** It clears by about 5.07 m and lands about 90.4 m away.

**8** About 50 m/s at an angle of about 16°.

**9** About 57 cm

**10** About 0.62°

## Exercise 6.05

**1** **a** 2.24 rad/s **b** 5.39 m/s **c** 12.1 m/s$^2$

**2** About 23 562 km/h

**3** **a** 31.4 rad/s **b** 0.628 m/s

**4** **a** 12.5 m/s **b** 50 N
**c** $\frac{5\sqrt{3}}{4\pi} \approx 0.689$ rps

**5** $\sin^{-1}\left(\frac{3.125}{9.8}\right) \approx 18.6°$

**6** 48.8 m/s$^2$

**7** 14 661 km/h

**8** About 56°

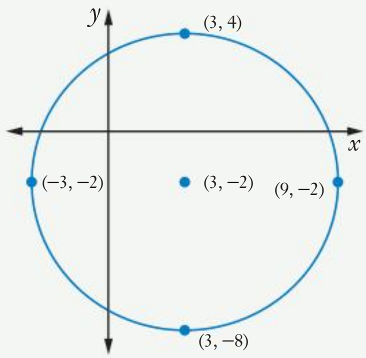
**9** About 11 cm

**10** $\sqrt{29.4} \approx 5.4$ m/s

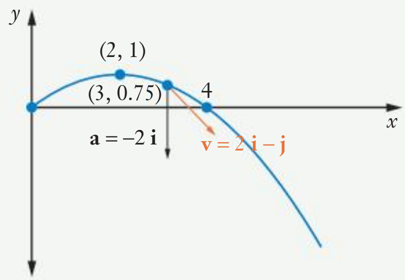
## Chapter review

**1** $y = 2x + 15$

**2** $(x - 3)^2 + (y + 2)^2 = 36$




**3** **a** $\mathbf{v}(t) = 2\mathbf{i} + (2 - 2t)\mathbf{j}$ **b** $\mathbf{a} = -2\mathbf{j}$
**c**




**4** **a** $\mathbf{r}(t) = (t^2 + 6t + 2)\mathbf{i} - [62 + (t - 4)^3]\mathbf{j}$
**b** $\mathbf{r} = 29\mathbf{i} - 61\mathbf{j}$, $\mathbf{v} = 12\mathbf{i} - 3\mathbf{j}$, $\mathbf{a} = 2\mathbf{i} + 6\mathbf{j}$

**5** **a** $\mathbf{v} = (\frac{1}{2}t^2 - 3t + 6)\mathbf{i} + (7 - 1\frac{1}{2}t^2)\mathbf{j}$

**b** $\mathbf{r} = (\frac{1}{6}t^3 - 1\frac{1}{2}t^2 + 6t - 7)\mathbf{i} + (7t - \frac{1}{2}t^3 + \frac{1}{2})\mathbf{j}$

**6** **a** 54 m **b** 3 m/s$^2$ **c** 23 m/s

**7** **a** 7 m/s **b** 31.2 m

**8** **a** $12.25\sqrt{3} \approx 21.2$ m/s **b** 12.25 m/s

**c** $1.25\sqrt{3} \approx 2.17$ s **d** 22.97 m

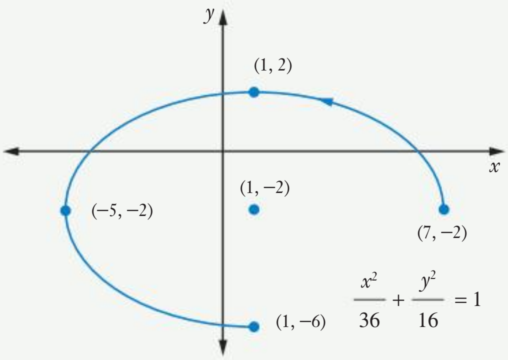
**e** $2.5\sqrt{3} \approx 4.33$ s

**f** $30.625\sqrt{3} \approx 53.04$ m

**9** **a** $\frac{2}{3}$ rad/s **b** $5\frac{1}{3}$ m/s$^2$ **c** $21\frac{1}{3}$ N

**10** **a** 9.10 m/s **b** 787.5 N

**11** The particle is travelling in an anticlockwise direction around the ellipse with the equation $\frac{(x-1)^2}{36} + \frac{(y+2)^2}{16} = 1$. It takes 4 seconds to complete a revolution. In the first 3 seconds it starts from (7, –2) and goes round to (1, –6).




**12** 12.75 m/s

**13** 11.69 m

**14** About 31.9 m

**15** About 8 cm (7.96)

**16** Proof

## Practice examination 2

**1** $e = -2, f = 3, g = -1, h = 4$

**2** **a** $\begin{bmatrix} 2 & -3 & 1 \\ 1 & 3 & -1 \\ 4 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix}$

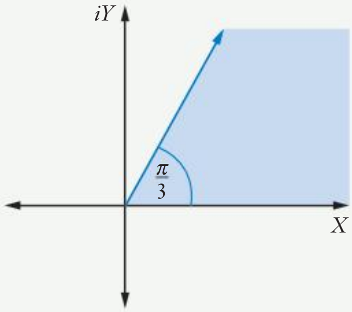
**b** $\left[\begin{array}{cccc|c} 3 & 2 & 0 & -1 & 0 \\ 5 & 2 & 3 & 4 & -2 \\ -3 & 1 & 4 & -5 & 0 \end{array}\right]$

**3** **a** Inconsistent

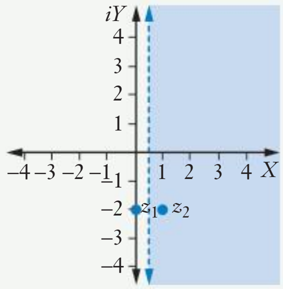
**b** Dependent, $3a + 2b = 2, c = 1.5, d = -3$, no unique solution

**4** L1 and L3 are parallel and intersect L2 and L3 in the lines $9 - 5x = y = \frac{5z-7}{17}$ and $\frac{7-10x}{2} = y = \frac{10z-17}{34}$ respectively.

**5** **a**

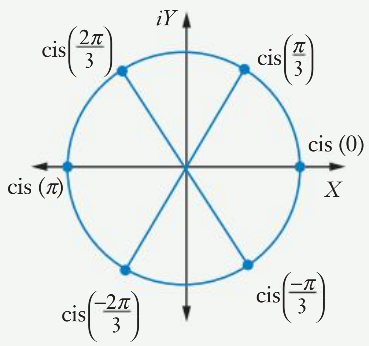



**b** Region right of the perpendicular bisector of the line joining $z_1(0, -2)$ and $z_2(1, -2)$.




**6** **a** $-1 = \text{cis}\,(\pi), \text{cis}\left(\frac{-2\pi}{3}\right), \text{cis}\left(\frac{-\pi}{3}\right), 1 = \text{cis}\,(0), \text{cis}\left(\frac{\pi}{3}\right), \text{cis}\left(\frac{2\pi}{3}\right)$

**b**





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**c** The sixth roots of unity are equally spaced on the unit circle. One root is 1 and the remaining adjacent roots are separated by an angle of $\frac{\pi}{3}$.

**7** $(z + 2 - 3i)(z + 2 + 3i)(z - 1)$

**8** $3z^3 - 2z^2 + 27z - 18$

**9** $1 \pm 2i, \frac{5}{2}$

**10** $\frac{x^2}{25} + \frac{y^2}{9} = 1$

**11** **a** $\mathbf{v}(t) = 2t\mathbf{i} - 6\mathbf{k}$

**b** $\mathbf{v}(t) = (2.5t^2 + 3)\mathbf{i} + 2t^3\mathbf{j} - (7t - 1)\mathbf{k}$

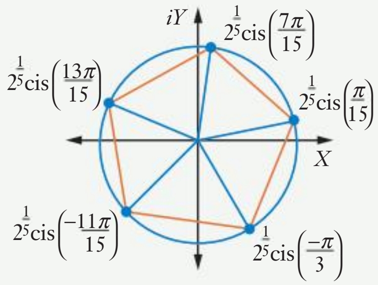
**12** **a** 31.5 m **b** 10.8 m/s$^2$

**13** 136 m

**14** Bin 1 \$15, Bin 2 \$23, Bin 3 \$20

**15** 1st Alex, 2nd Don, 3rd Peter, 4th Terry and last Jim

**16** **a**



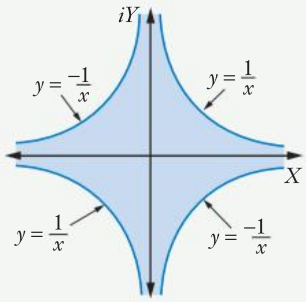

**b** About 3.14 square units.

**17** 4.2 s

**18** About 682 Joeys, 397, 252, 195, 241 and 127 of the 2-, 3-, 4-, 5- and 6-year-olds respectively.

**19** $|z^2 - \overline{(z^2)}| \le 4 \Rightarrow |(z + \bar{z})(z - \bar{z})|^2 \le 4$

So $|2x \times 2iy| \le 4 \Rightarrow |xy| \le 1$




**20** 1.33 s

# Chapter 7

## Exercise 7.01

**1** **a** $\frac{1}{2}\left[x - \frac{1}{2}\sin(2x)\right] + c$

**b** $\frac{1}{2}\left[x + \frac{1}{4}\sin(4x)\right] + c$

**c** $\frac{1}{2}\left[x - \frac{1}{6}\sin(6x)\right] + c$

**d** $\frac{\pi}{4}$ **e** $\frac{\pi}{4}$ **f** $\frac{\pi}{8} - \frac{1}{12}$

**2** **a** $-\frac{1}{10}\cos(5x) - \frac{1}{2}\cos(x) + c$

**b** $-\frac{1}{22}\cos(11x) - \frac{1}{6}\cos(3x) + c$

**c** $-\frac{1}{8}\cos(4x) + c$

**d** $\frac{1}{8}\sin(4x) + \frac{1}{16}\sin(8x) + c$

**e** $-\frac{1}{6}\cos(9x) - \frac{1}{2}\cos(3x) + c$

**f** $-\cos(4x) + c$

**3** **a** $\frac{1}{2}\left[x - \frac{1}{8}\sin(8x)\right] + c$

**b** $\frac{1}{2}\left[x + \frac{1}{6}\sin(6x)\right] + c$

**c** $2\left[x - \frac{1}{4}\sin(4x)\right] + c$

**d** $\frac{9}{2}\left[x + \frac{1}{12}\sin(12x)\right] + c$

**e** $4\left[x - \frac{1}{12}\sin(12x)\right] + c$

**f** $\frac{5}{2}\left[x + \frac{1}{10}\sin(10x)\right] + c$

**4** **a** $\pi$ **b** $\pi$ **c** $\frac{\pi}{4} + \frac{1}{6}$

**d** $\frac{3\pi}{4}$ **e** $\pi$ **f** $\frac{\pi}{2}$

**5** **a** $-x - \cot(x) + c$ **b** $-x + \frac{1}{2}\tan(2x) + c$

**c** $-x - \frac{1}{4}\cot(4x) + c$ **d** $-x + \frac{1}{3}\tan(3x) + c$

**e** $-x - 2\cot\left(\frac{x}{2}\right) + c$ **f** $-x + \tan\left(\frac{x}{4}\right) + c$

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**6** $\frac{\sqrt{2}-1}{9}$

**7** Proof

**8** $\frac{3}{8}x-\frac{1}{4}\sin(2x)+\frac{1}{32}\sin(4x)+c$

**9** $\frac{3}{8}x+\frac{1}{4}\sin(2x)+\frac{1}{32}\sin(4x)+c$

**10** $\frac{1}{8}x-\frac{1}{32}\sin(4x)+c$

## Exercise 7.02

**1** **a** $\frac{1}{2}(3x^2-5x+11)^2+c$

**b** $\frac{1}{16}(4x^2-8x+1)^2+c$

**c** $\frac{2}{3}(2x^2+5)^{\frac{3}{2}}+c$ **d** $\frac{1}{3}(x^2+6x-8)^{\frac{3}{2}}$

**e** $-\frac{1}{6}\cos^6(x)+c$ **f** $2\sin^5(x)+c$

**2** **a** $\frac{1}{10}(2x-5)^5+c$ **b** $\frac{1}{70}(7x+2)^{10}+c$

**c** $\frac{1}{18}(3x-7)^6+c$ **d** $\frac{1}{2}(4x-5)^4+c$

**e** $\frac{2}{9}(9x+4)^6+c$ **f** $\frac{1}{64}(2x+9)^8+c$

**3** **a** $\frac{2}{9}(3x+8)^{\frac{3}{2}}+c$ **b** $\frac{2}{15}(5x-6)^{\frac{3}{2}}+c$

**c** $\frac{4}{3}(2x+7)^{\frac{3}{2}}+c$ **d** $\frac{2}{3}(2x^2-7)^{\frac{3}{2}}+c$

**e** $\frac{3}{32}(8x+7)^{\frac{4}{3}}+c$ **f** $\frac{1}{16}(6x^2+5)^{\frac{4}{3}}+c$

**4** **a** $\sqrt{8x+7}+c$ **b** $\frac{4}{3}\sqrt{6x-1}+c$

**c** $\frac{5}{2}\sqrt{4x+3}+c$ **d** $\sqrt{x^2+6}+c$

**e** $\frac{2}{3}\sqrt{3x^2-5}+c$ **f** $-\frac{5}{4}\sqrt{3-4x^2}+c$

**5** **a** $-\frac{1}{2(2x-5)}+c$ **b** $-\frac{3}{2(x+4)^2}+c$

**c** $\sec(x)+c$ **d** $\frac{1}{2}e^{x^2}+c$

**e** $\frac{5}{6}e^{3x^2-1}+c$ **f** $2\sin\sqrt{x}+c$

**6** **a** $\frac{2}{13}$ **b** $\frac{26\sqrt{13}-16}{9}$ **c** $\frac{2}{21}$

**d** $\frac{1}{4}$ **e** $\sqrt{2}-1$ **f** $\frac{8}{5}$

**7** **a** $2e^{x^4}+c$ **b** $\frac{2}{27}(3x+10)\sqrt{3x-5}+c$

**c** $\frac{1}{3}(2x-3)\sqrt{4x+3}+c$

**d** $\frac{3}{5}(x-1)(2x+3)\sqrt{2x+3}+c$

**e** $\frac{2}{3}(x-4)(3x+8)\sqrt{x-4}+c$

**8** \$36.67, \$1020.76

## Exercise 7.03

**1** **a** $\frac{1}{4}\ln|4x+9|+c$ **b** $\frac{1}{2}\ln|6x-7|+c$

**c** $-\frac{1}{2}\ln|9-4x|+c$ **d** $-3\ln|7-4x|+c$

**e** $6\ln|3x-11|+c$ **f** $\frac{1}{6}\ln|12x+5|+c$

**2** **a** $\ln|4x^2-7x|+c$ **b** $\ln|3x^2+8x-4|+c$

**c** $\frac{1}{2}\ln|x^2+3|+c$ **d** $-\frac{1}{6}\ln|2-3x^2|+c$

**e** $\frac{1}{3}\ln|x^3+12x|+c$ **f** $\ln|\ln(x)|+c$

**3** **a** $\ln\left(\frac{5}{2}\right)$ **b** $\frac{1}{4}\ln\left(\frac{5}{3}\right)$ **c** $\frac{3}{2}\ln(2)$

**d** $\frac{1}{2}\ln\left(\frac{e^4+1}{e^2+1}\right)$ **e** $\frac{1}{2}\ln(2)$

**f** $\ln\left(\frac{2\sqrt{2}+1}{2\sqrt{2}}\right)$

**4** **a** $\ln|x^4|=4\ln|x|+c$

**b** $\ln(e^{6x})=6x+c$

**c** $-\ln|x^{-7}|=7\ln|x|+c$

**d** $\ln(e^{\cos(x)})=\cos(x)+c$

**e** $\frac{1}{2}\ln(x^{\frac{1}{2}})=\frac{1}{4}\ln(x)+c$

**f** $-\frac{2}{5}\ln(x^{\frac{5}{2}})=\ln(x)+c$

**5** **a** $3\ln(1+\ln|4x|)+c$

**b** $5\ln(4+e^{4x})+c$

**c** $\sin(x)+c$

**6** $\ln(e^{3x}+1)+c$


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**7** $-\frac{2}{e^{x}+e^{-x}}+c$

**8** $v = 30(1 - e^{-t})$. Find the integral and then calculate the constant of integration. Apply the log laws and then write in terms of $e$.

## Exercise 7.04

**1** **a** $-\frac{3}{\sqrt{1-9x^2}}$ **b** $-\frac{7}{1+49x^2}$

**c** $\frac{1}{\sqrt{4-x^2}}$ **d** $\frac{4}{16+x^2}$

**e** $-\frac{6}{\sqrt{1-36x^2}}$ **f** $-\frac{1}{\sqrt{9-x^2}}$

**2** **a** $\frac{12x^3}{1+9x^8}$ **b** $-\frac{10x}{\sqrt{1-25x^4}}$

**c** $\frac{6x^2}{\sqrt{1-4x^6}}$ **d** $\frac{6x}{\sqrt{1-(3x^2-1)^2}}$

**e** $-\frac{12x^2+5}{\sqrt{1-(4x^3+5x)^2}}$ **f** $-\frac{10x+6}{1+(-5x^2-6x)}$

**3** **a** $\frac{60x^3(5x^4+1)^2}{\sqrt{1-(5x^4+1)^6}}$ **b** $-\frac{30x\arccos^2(5x^2)}{\sqrt{1-25x^4}}$

**c** $\frac{24x^2\arcsin^2(4x^3)}{\sqrt{1-16x^6}}$

**4** **a** $\frac{1}{2\sqrt{x}\sqrt{1-x}}$ **b** $-\frac{2x}{\sqrt{1-x^4}}$

**c** $\frac{x}{(x^2-2)\sqrt{1-x^2}}$

**5** $\frac{2-x^2}{(1-x^2)^{\frac{3}{2}}}$

**6** $\frac{1+x^2}{x^4-x^2+1}$

**7–8** Proofs

## Exercise 7.05

**1** **a** $\arccos(x)+c$ **b** $\arcsin\left(\frac{x}{3}\right)+c$

**c** $\arctan(x)+c$ **d** $\arcsin\left(\frac{x}{2}\right)+c$

**e** $\arctan\left(\frac{x}{4}\right)+c$ **f** $\arccos\left(\frac{x}{5}\right)+c$

**2** **a** $\arccos\left(\frac{x}{2\sqrt{2}}\right)+c$ **b** $\arcsin\left(\frac{x}{2\sqrt{3}}\right)+c$

**c** $\frac{1}{2\sqrt{5}}\arctan\left(\frac{x}{2\sqrt{5}}\right)+c$ **d** $\arcsin(3x)+c$

**e** $\sqrt{2}\arctan\left(\sqrt{2}x\right)+c$ **f** $\arcos(2x)+c$

**3** **a** 0.238 **b** 0.253 **c** 0.122

**d** $\frac{\pi}{3}$ **e** $-\frac{\pi}{6}$ **f** 0.097

**4** **a** −0.501 **b** $\pi$ **c** $\frac{7\pi}{6}$

**d** $\frac{3\pi}{4}$ **e** 4.64 **f** $\frac{7\pi}{2}$

**5** **a** $\arcsin(e^x)+c$ **b** $\arctan(e^x)+c$

**6** $\arctan(\ln|x|)+c$

**7** $-\frac{1}{2}\arccos^2(x)+c$

**8** Proof. First show that $\int\frac{x}{x^4+16}dx=\frac{1}{2}\int\frac{du}{u^2+16}$.

## Exercise 7.06

**1** **a** $\frac{1}{x+1}+\frac{4}{x-4}$ **b** $\frac{3}{x+3}-\frac{1}{3x+1}$

**c** $\frac{4}{x-1}-\frac{4}{x+4}$ **d** $\frac{5}{x-6}-\frac{4}{x+2}$

**e** $\frac{3}{x-2}-\frac{2}{x+4}$ **f** $\frac{14}{9(x-4)}-\frac{1}{9(2x+1)}$

**2** $3\ln|x+1|-\ln|x+2|+c$

**3** $-11\ln|x-5|+13\ln|x-6|+c$

**4** $3\ln|x-1|-2\ln|x+2|+c$

**5** $2\ln|x+1|+3\ln|x-3|+c$

**6** **a** $2\ln|x+1|-3\ln|x-2|-(x-2)^{-1}+c$

**b** $\ln|x|-\ln|x-1|-2x^{-1}+c$

**c** $\frac{4}{5(x+2)}+\frac{16}{25}\ln|x+2|+\frac{9}{25}\ln|x-3|+c$

**d** $\frac{1}{6(x+5)}-\frac{1}{36}\ln|x+5|+\frac{1}{36}\ln|x-1|+c$

**e** $\frac{5}{x-1}+6\ln|x-1|+c$

**f** $\frac{1}{4}\ln|x-1|-\frac{1}{4}\ln|x+1|-\frac{1}{2(x-1)}+c$

**7** **a** $3x-2+\frac{5}{x+4}$ **b** $2x^2+x-\frac{3}{x-1}$

**c** $x^2-5x+27-\frac{94}{x+3}$

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**8 a** $2\ln|x+1|-\ln|x-1|+c$

**b** $3\ln|x-1|+(x-1)^{-1}+2\ln|x+2|+c$

**c** $3\ln|x-1|-2\ln|x+1|+2\ln|x-2|+c$

**9** $\ln\left(\frac{5}{12}\right)$

**10** $-\ln(4)$

**11** $\frac{2x^4-3x^3+2x^2+3x-5}{x^2-2x-3}$

$=2x^2+x+10+\frac{26x+25}{x^2-2x-3}$ and

$\int\frac{2x^4-3x^3+2x^2+3x-5}{x^2-2x-3}dx$

$=\frac{2x^3}{3}+\frac{x^2}{2}+10x+\frac{1}{4}\ln|x+1|+\frac{103}{4}\ln|x-3|+c$

**12** $A=3, B=2$ and $C=1$;
$3\ln|x-1|+\ln|x^2+x+2|+c$

**13 a** $3\ln|x-2|-\ln|x^2+3x+4|+c$

**b** $3\ln|x-1|+\ln|3x^2-2x+4|+c$

**c** $3\ln|x-2|+\ln|2x^2-x+2|+c$

## Exercise 7.07

**1 a** $\sin(x)-x\cos(x)+c$

**b** $e^x(x-1)+c$

**2 a** $x^2e^x-2xe^x+2e^x+c$

**b** $\frac{1}{3}x^3e^{3x}-\frac{1}{3}x^2e^{3x}+\frac{2}{9}xe^{3x}-\frac{2}{27}e^{3x}+c$

**3 a** $\frac{1}{2}x^2\ln(x)-\frac{1}{4}x^2+c$

**b** $\frac{1}{3}x^3\left(\ln|x|-\frac{1}{3}\right)+c$

**4 a** $\frac{1}{2}[x-\sin(x)\cos(x)]+c$

**b** $x\ln|2x|-x+c$

**c** $\left(x+\frac{7}{2}\right)\ln|2x+7|-x+c$

**d** $x\left[\log_{10}(3x)-\frac{1}{\ln(10)}\right]+c$

**e** $-x^2\cos(x)+2x\sin(x)+2\cos(x)+c$

**f** $x^2\sin(x)+2x\cos(x)-2\sin(x)+c$

**5 a** $\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+c$

**b** $x\tan(x)-\ln|\sec(x)|+c$

**c** $\frac{1}{2}x^2e^{2x}-\frac{e^{2x}}{4}(2x-1)+c$

**d** $\frac{1}{3}x\sin(3x)+\frac{1}{9}\cos(3x)+c$

**e** $-\frac{1}{3}xe^{-3x}-\frac{1}{9}e^{-3x}+c$

**f** $x\sec(x)-\ln|\tan(x)+\sec(x)|+c$

**6 a** $-\frac{1}{x}\ln|x|-\frac{1}{x}+c$

**b** $\frac{1}{2}e^x[\sin(x)-\cos(x)]+c$

**c** $\frac{2x^{\frac{3}{2}}}{3}\ln|x|-\frac{4x^{\frac{3}{2}}}{9}+c$

**d** $\frac{1}{2}e^x[\sin(x)+\cos(x)]+c$

**e** $\frac{x^4}{4}\left(\ln|x|-\frac{1}{4}\right)+c$

**f** $x(\ln|x|)^2-2x\ln|x|+2x+c$

**7 a** $-0.1534$ **b** $0.6576$

**c** $\frac{\pi}{4}$ **d** $4.5746$

**8–9** Proofs

## Exercise 7.08

**1 a i** $A=(-1, 3), B=(7, 3)$

**ii** Shaded area $=\frac{128}{3}$

**b i** $A=(0, 0), B=(2, 6)$

**ii** Shaded area $=\frac{8}{3}$

**c i** $A=(-2, 4), B=(1, 1)$

**ii** Shaded area $=\frac{9}{2}$

**d i** $A=(-2, 2), B=(2, 2)$

**ii** Shaded area $=\frac{128}{3}$

**2** $\frac{31}{6}$ **3** $\frac{3e^2+4e-3}{3e}\approx 3.684$

**4** $\frac{11}{2}$ **5** 48

**6** $\frac{1}{3}$ **7** $\frac{64}{3}$


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**8** 4

**9** 16

**10** $\frac{148}{3}$

**11** **a** $2\sqrt{2}-2$ **b** $\frac{\sqrt{2}}{2}$ **c** $\frac{157}{6}$

**d** $2+\sqrt{2}$ **e** $\frac{22}{15}$ **f** $\frac{49}{6}$

**g** $e-3e^{-1}$

**12** **a** $\frac{1}{10}$ **b** $\frac{128}{15}$ **c** 8

**d** $\frac{863}{6}$

**13** $\frac{13}{6}$ (Find the area from $x=1$ to $x=2$ and the area from $x=2$ to $x=4$.)

**14** 18 (The problem is simplified if you integrate w.r.t. $y$.)

**15** $a=1, b=2$, Area $=\frac{5}{2}$

**16** About 57 200

## Exercise 7.09

**1** 6000 $m^3$

**2** 589.0 $cm^3$

**3** $36\pi$

**4** **a** $\frac{32\pi}{5}$ **b** $\pi[3(\ln(3))^2-6\ln(3)+4]$

**c** $\frac{64\pi}{5}$ **d** $\frac{\pi(e^5-1)}{4}$

**e** $\pi^2$ **f** $\frac{9\pi(\pi+2)}{4}$

**5** $32\pi$

**6** **a** $25\pi$ **b** $\frac{9\pi\sqrt[3]{18}}{5}$

**c** $\frac{\pi(e^4-1)}{2}$

**d** Approximately 4.1304

**e** $\frac{64\pi}{5}$ **f** $\frac{8\pi}{3}$

**7** $\frac{4\pi}{35}$

**8** $24\pi$

**9** 661.2 $m^3$

**10** $V=\frac{1}{3}\pi r^2 h$

**11** 261.8 $cm^3$.

**12** 71 817 $cm^3 \approx$ 71.8 L, 29 078 $cm^3 \approx$ 29.1 L

**13** 24.4 $cm^3$

**14** 3504 $cm^3$

## Exercise 7.10

**1** 11.8054

**2** 0.2488

**3** 0.6919

**4** 0.9470

**5** 0.335

**6** **a** 11.75 **b** 1.150

**7** **a** 0.2525 **b** 0.6938 **c** 0.88 **d** 0.39

**8** **a** 8.45 **b** 1.7326 **c** 9.3482

**9** **a** 178.635 **b** −0.4406 **c** 24.4294

**10** $\frac{1}{3}$

**11** **a** 0.8362 **b** 0.693 170

**12** **a** 8.837 956 **b** 4.9471 **c** 1.827 409

**13** **a** **i** 1357.28 **ii** 1340.04 **iii** 1340

**b** 1.3% and 0.003%

**14** **a** 0.9943 **b** 1.000 026 **c** 1

**15** **a** 1.1016 **b** 1.0987

**16** **a** 10.1241 **b** 10.0989

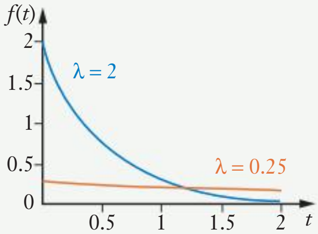
**17** About 136 $m^2$, about \$47 600

**18** About $124\frac{2}{3}$ $m^2$, about $1.35\times10^6$ $m^3$ (flow $\approx$ 374 $m^3\,s^{-1}$)

**19** 32.8 kJ

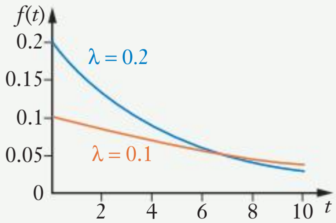
## Exercise 7.11

**1** **a**




The higher value of $\lambda$ starts higher and is skewed more to the left, so it has a higher probability of a lower value of $t$.

**b**




The higher value of $\lambda$ starts higher and is skewed more to the left, so it has a higher probability of a lower value of $t$.

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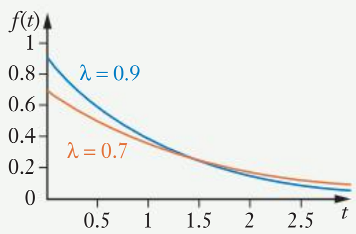
**c** 


The graphs are close to each other, but the higher value of $\lambda$ starts higher and is skewed more to the left, so it has a slightly higher probability of a lower value of $t$.

**2** **a** 0.8347 **b** 0.1628 **c** 0.0015 **d** 0.0025

**3** **a** 0.5934 **b** 0.1359 **c** 0.4066 **d** 0.2231

**4** **a** 0.4512 **b** 0.1481 **c** 0.5488 **d** 0.3679

**5** $\mu = E(T) = \frac{1}{m}$

**6** $\frac{2}{3}$

**7** $\lambda = 0.0625$

**8** **a** 4 min **b** 0.368 **c** 0.607

**9** **a** $f(t) = 0.125e^{-0.125t}$ **b** 0.287 **c** 0.465
**d** 0.247

**10** **a** 0.202 **b** 0.865 **c** 0.330 **d** 0.551

**11** 0.426

**12** **a** 0.558 **b** 0.0725

**13** **a** **i** 0.4288 **ii** 0.0400 **iii** 0.5712
**b** They total more than 1 because they overlap.

**14** $a = -\frac{\ln(0.2)}{\lambda} = \frac{\ln(5)}{\lambda}$

**15** About 16.1 days

## Exercise 7.12

**1** **a** $\text{pdf}(t) = 4e^{-4t}$ **b** $\frac{1}{4}$ min
**c** $\text{cdf}(t) = 1 - e^{-4t}$ **d** 0.982
**e** 4/min

**2** **a** $\text{pdf}(t) = 40e^{-40t}$ **b** $\frac{1}{40}$ hrs
**c** $\text{cdf}(t) = 1 - e^{-40t}$ **d** 0.9987
**e** 40/hr

**3** **a** 2 sec **b** $\text{cdf}(t) = 1 - e^{-0.5t}$
**c** 0.393 **d** 30/min

**4** 0.221

**5** **a** 0.301 **b** 181

**6** 0.6065

**7** **a** 0.956 **b** About 259

**8** **a** Median = 3891, so about 50% of items will continue to work after 3891 hours of operation.
**b** About 288 hours

**9** **a** About 12 980 hrs **b** About 960 hrs

**10** About 25 621 hrs

**11** **a** 22.1% **b** 2.77 min **c** 0.0821

**12** **a** **i** 4.64% **ii** 21.54% **iii** 2.15%
**b** About 25.5 min (25.485)

**13** 99.6%

**14** **a** 0.9925 **b** 0.9875 **c** 0.9752

**15** About 6 months (6.655)

**16** Median = $\frac{\ln(2)}{\lambda}$ and mean = $\frac{1}{\lambda}$; median < mean as ln (2) < 1

**17** **a** 0.2835 **b** About 4 min 10 s
**c** 0.1889

**18** $\lambda = \frac{150}{1700 \times 400}$, $R(850) \approx 0.829$; About 82%

**19** **a** $6.26 \times 10^{-5}$/h **b** 0.939
**c** 0.535

## Chapter review

**1** **a** $\frac{\pi}{4}$ **b** $\frac{1}{8}\sin(8x) + \frac{1}{2}\sin(2x) + c$

**2** **a** $\frac{3}{2}\left[x - \frac{1}{4}\sin(4x)\right] + c$ **b** $\pi$

**3** $-2\cot\left(\frac{x}{2}\right) - x + c$

**4** **a** $\frac{1}{20}(3 + x^2)^{10} + c$ **b** $\frac{1}{8}$

**5** **a** $-\frac{11}{7}\ln|5 - 7x| + c$
**b** $\frac{1}{2}\ln|5x^2 - 2x - 8| + c$

**6** $\frac{1}{2}\ln(2)$

**7** **a** $-\frac{5}{\sqrt{1-25x^2}}$ **b** $\frac{2}{4+x^2}$

**8** **a** $\frac{15x^4}{\sqrt{1-9x^{10}}}$ **b** $\frac{1}{\sqrt{2x}\sqrt{1-2x}}$

**9** $0.25\arcsin(4x) + c$ **10** 0.680

**11** **a** $\frac{5}{(x+3)} - \frac{3}{7(2x-1)}$ **b** $\frac{1}{2x+1} + \frac{5}{x-3}$

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**12** $\frac{1}{5}\ln|x-1| - \frac{1}{5}\ln|x+4| + c$

**13** $5\ln|x+1| - \frac{1}{x+1} - 3\ln|x-5| + c$

**14** $\sin(x) - x\cos(x) + c$

**15** $-x^2\cos(x) + 2x\sin(x) + 2\cos(x) + c$

**16** $\frac{1}{5}x^5\left(\ln|2x| - \frac{1}{5}\right) + c$

**17** $\frac{65}{6}$

**18** $\frac{125}{6}$

**19** 128

**20** $1872\ \text{m}^3$

**21** $\frac{2\pi}{5}$

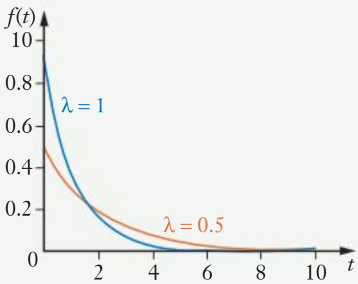
**22** $8\pi$

**23** 2.017

**24** **a** **i** 266 022 **ii** 264 004.5
**iii** 264 000

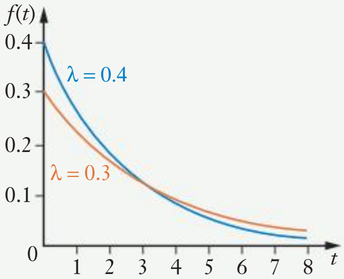
**b** 0.766% and 0.0017%

**25** **a**



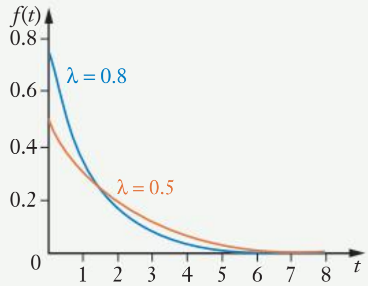

For $\lambda = 1$, the distribution starts twice as high as for $\lambda = 0.5$, and tapers to zero more quickly. This means that, for $\lambda = 1$, $t$ has a higher probability of being close to 0.

**b**




For $\lambda = 0.4$, the distribution starts higher than for $\lambda = 0.3$, and tapers to zero more quickly. This means that, for $\lambda = 0.4$, $t$ has a higher probability of being close to 0.

**c**




For $\lambda = 0.8$, the distribution starts higher than for $\lambda = 0.5$, and tapers to zero more quickly. This means that, for $\lambda = 0.8$, $t$ has a higher probability of being close to 0.

**26** **a** 0.3935 **b** 0.3834
**c** 0.0368 **d** 0.3679

**27** **a** $f(x) = 0.2e^{-\frac{t}{5}}$ **b** 0.181
**c** 0.247 **d** 0.632

**28** **a** $\text{pdf}(t) = \frac{1}{12}e^{-\frac{t}{12}}$ **b** 12 sec
**c** $\text{cdf}(t) = 1 - e^{-\frac{t}{12}}$ **d** 0.487
**e** 5/min

**29** **a** 0.607 **b** 119

**30** **a** Median = 3891, so about 50% of items will continue to work after 3891 hours of operation.
**b** About 288 hours

**31** **a** \$10.52 **b** \$5.52 **c** \$15.39

**32** $\frac{2-x^2}{(1-x^2)^{\frac{3}{2}}}$

**33** Proof

**34** $\frac{1}{2}\arcsin^2(x) + c$

**35** $\frac{64}{3}$ (The problem is simplified if you integrate w.r.t. $y$.)

**36** $V = \pi r^2 h$

**37** $40\pi \approx 126\ \text{cm}^3$

**38** $160\pi \approx 503\ \text{cm}^3$

**39** About 685 $\text{m}^2$, about \$82 200

**40** **a** 0.8775 **b** 1.5%

**41** About 3 h 24 min (3.3979 h)

## Chapter 8

### Exercise 8.01

**1** **a** 7.9 **b** 8.2 **c** 8.7 **d** 8.1

**2** **a** 76.6 **b** 80.6 **c** 81.1 **d** 75.5

**3** **a** 24.3 **b** 27.8 **c** 24.7 **d** 37.6

**4** **a** \$96.38 **b** \$129 **c** \$115.50 **d** \$119.38

**5** **a** 0.593 **b** 0.696 **c** 0.629 **d** 0.677

**6** **a** The means will vary.
**b** The values are all around 100.
**c** The mean is close to 100.

**7** **a** The means will vary.
**b** The values are all around 50.
**c** The mean is fairly close to 50.

**8** **a** The means will vary.
**b** The values are all around 60.
**c** The mean is quite close to 60.

### Exercise 8.02

**1** **a, b** Answers will vary, but will be close to 7 and 0.8.
**c, d** Answers will vary, but will be close to 7 and 0.47.
**e, f** Answers will vary but will be very close to 7 and 0.26.
**g** The means get closer to 7 as the sample size $n$ increases and the standard deviation is close to $\frac{2.6}{\sqrt{n}}$. The distributions look more like a normal distribution as $n$ increases.

**2** **a, b** Answers will vary, but will be close to 3.5 and 0.6.
**c, d** Answers will vary, but will be close to 3.5 and 0.34.
**e, f** Answers will vary but will be very close to 3.5 and 0.19.
**g** The means get closer to 3.5 as the sample size $n$ increases and the standard deviation is close to $\frac{1.71}{\sqrt{n}}$. The distributions look more like a normal distribution as $n$ increases.

**3** **a, b** Answers will vary, but will be close to 3.5 and 0.24.
**c, d** Answers will vary, but will be close to 3.5 and 0.24.
**e, f** Answers will vary but will be very close to 3.5 and 0.24.
**g** The mean and standard deviation get closer to 3.5 and 0.24 as the number of samples increases and the distributions look more like a normal distribution.

**4** **a, b** Answers will vary, but will be close to 15 and 1.8.
**c, d** Answers will vary, but will be close to 15 and 1.1.
**e, f** Answers will vary but will be very close to 15 and 0.58.
**g** The means get closer to 15 as the sample size $n$ increases and the standard deviation is close to $\frac{5.8}{\sqrt{n}}$. The distributions look more like a normal distribution as $n$ increases.

**5** The means get closer to 20, the distributions look more like normal distributions and the standard deviations are close to $\frac{5}{\sqrt{n}}$.

**6** The means get closer to 8, the distributions look more like normal distributions and the standard deviations are close to $\frac{2.19}{\sqrt{n}}$.

**7** The means get closer to 24, the distributions look more like normal distributions and the standard deviations are close to $\frac{3.1}{\sqrt{n}}$.

### Exercise 8.03

**1** 48, 1.6

**2** 100, about 2

**3** 67, 1.7

**4** 85, about 2.4

**5** 14, 1.5

**6** 26, about 9.5

**7** 42, about 19.4

**8** 52, about 18.4

**9** $\bar{x} \approx 2.1$ kg, $s \approx 115$ g

**10** $\mu \approx \$450$, $\sigma \approx \$82$

### Chapter review

**1** 48.2

**2** **a, b** The means vary but will be close to 30.

**3** The means of the sampling means will be close to 6, but will be much closer for the samples of 75. The standard deviations will be about 0.6 and 0.26.

**4** 40 and 1.5

**5** 18 and about 10.8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6 The means will be 50 and the standard deviations will be less than 8, decreasing for larger samples.

7 8.75 kg and 9 g

8 340 g and about 15 g

## Practice examination 3

1 a $\frac{1}{5}\sqrt{1+5x^2}+c$ b $\ln|x^2-4x+5|+c$

2 a $\ln(5)$ b 0

3 a $\frac{2(x^2-3)\sqrt{x^2-3}}{3}+c$ b $\frac{1}{2}[\ln(x)]^2+c$

4 a $e^x(x^3-3x^2+6x-6)+c$
b $e^x(x-1)+c$

5 a $\frac{2187\pi}{7}$ b $\frac{729\pi}{5}$

6 a 1.41 b 142.384 c 3.678 d 16.513

7 5

8 $\lambda = 0.04$

9 As the parameter is increased, the graph starts at a higher point on the $y$-axis and tapers more quickly to the $t$-axis. The greater the value of $\lambda$, the higher the probability is of a value near zero.

10 a 0.0952 b 0.2387 c 0.0452

11 $1\frac{2}{3}$

12 $7\frac{5}{6}$

13 For random numbers $r$ between 0 and 1, calculate $6+14r$.

14 a Rectangular b Normal

15 60, 1.22

16 $V=\frac{4}{3}\pi r^3$

17 About 21 500 m$^2$ = 2.15 ha

18 a 0.4866 b 10.4 min (10 min 24 s)
c 26.4%

19 About 8.6 kg, 1.34 g

20–21 Proof

22 460 g, about 116 g

# Chapter 9

## Exercise 9.01

1 a $-\frac{4x}{9y}$ b $\frac{-10x+3y}{4y-3x}$
c $\frac{y(1-y-2x)}{x(x+2y-1)}$ d $-\frac{5x}{16y}$
e $\frac{6y}{3x+8\sqrt{y}-4y^{\frac{3}{2}}}$ f $-\frac{5y^2}{9x^2}$

2 a $\frac{dy}{dx}=\frac{4y-e^{2y}}{2x(e^{2y}-2)}$
b $\frac{dy}{dx}=\frac{1}{\sin(y)+y\cos(y)}$
c $\frac{dy}{dx}=-\frac{3}{2y\sin(y^2+1)}$
d $\frac{dy}{dx}=\frac{5-y\cos(x)}{\sin(x)}$
e $\frac{dy}{dx}=\frac{2(1-xye^{x^2y})}{x^2e^{x^2y}-1}$
f $\frac{dy}{dx}=\frac{y-2x}{2y-x-\sin(y)}$

3 $7x+24y+230=0$

4 $x-y+6=0$

5 $5x+3y-16=0$

6 $10x-9y-20=0$

7 $2x+9y\pm 20=0$

8 $25x-15y-132=0$

9 $9x-8y\pm 26=0$

10 $(3-2\sqrt{2})x+y-2+2\sqrt{2}=0$ and $(3+2\sqrt{2})x+y-2-2\sqrt{2}=0$

11 (8, –2)

12 (0, 8.75)

13–16 Proofs

## Exercise 9.02

1 a $2(15x^2+4)$ b $-4e^{4x}$
c $4(\ln|x|+1)$ d $24\cos(3x)$

2 a 264 b –1176
c $5e^4$ d $12[\ln(2)+1]$

3 a $3\sqrt{2}$ b –12

4 $240\pi$ cm$^2$/min

5 a The height of the drone does not change, so $\frac{dy}{dt}=0$.
b –400 km/h

6 42 cm/s

7 60 km/h

8 $\frac{25}{18\pi}\approx 0.442$ cm/min

9 $-\frac{5}{8}$ so decreasing by 0.625 m/s

**10** Increasing at 0.4 m/min

**11** $-12$ cm$^2$/min

**12** The shorter side of the triangle is decreasing at 6.4 cm/min and the longer side is increasing at 5 cm/min.

**13** $\frac{3}{4\pi} \approx 0.239$ cm/s

**14** $\frac{dA}{dC} = \frac{C}{2\pi}$

**15** Increasing by 240 m/s

**16** $54\pi \approx 170$ cm$^3$/s

**17** $\sqrt{10} \approx 3.16$ m/s

**18** 66 cm$^2$/s

**19** $\frac{1}{25\pi} \approx 0.0127$ cm/s

## Exercise 9.03

**1** **a** $y = 4x + \frac{5}{2}x^2 - 3x^3 + c$

**b** $y = \frac{1}{12}(3t+5)^4 + c$

**c** $p = 2\sin\sqrt{x} + c$

**d** $y = m\ln(2) - \cos(m) + c$

**e** $m = \frac{1}{2}\left[\ln|4+x^2| + \arctan\left(\frac{x}{2}\right)\right] + c$

**f** $v = 2e^{3t} + 3e^{-4t} + c$

**g** $q = 6w^4 - 12\ln|w| + c$

**h** $h = \frac{4}{3}\sin^2(3y) + c$ or

$h = -\frac{4}{3}\cos^2(3y) + c$ or $h = -\frac{2}{3}\cos(6y) + c$

**i** $u = 4\ln|\ln|x|| + c$

**j** $y = -p^{-1} + 4\ln|p| + 4p + c$

**2** **a** $y = \frac{1}{5}(3x^2 - 5x + 4)^5 - \frac{17}{5}$

**b** $y = 2\ln|x| - 4$ **c** $y = -\frac{1}{4}\cos(2x) + 3$

**d** $y = 5\ln|x| - \frac{2}{x} + 2$ **e** $y = \frac{1}{2}(\ln|x|)^2 - 5$

**f** $y = -\frac{1}{2}e^{-x^2} + \frac{1}{2}$

**g** $y = -\frac{2}{3}\cos^3(x) + \cos(x) + \frac{17}{12}$

**h** $y = \frac{3}{5}\ln|x-3| + \frac{7}{5}\ln|x+2| - \frac{7}{5}\ln(6)$

**3** **a** $h = 20t - 5t^2 + c$ **b** 20.25 m

**4** **a** $v = 30$ m s$^{-1}$ **b** $h = 77\frac{1}{3}$ m

**5** \$1704

**6** **a** $h = 25t - 0.8t^2$ **b** 87.2 m

**c** Proof

**7** $Q = 100x^2$, 1.44 J

**8** $C = \frac{0.002n^3}{3} - \frac{n^2}{2} + 250n + 20\,000$ dollars

**9** Momentum $= 8t - t^2 + 50$ Ns

## Exercise 9.04

**1** **a** $y = Ae^{-3x}$ **b** $x = Ae^{4t}$

**c** $y = Ae^{-x}$ **d** $w = Ae^{0.01m}$

**e** $z = Ae^{-0.03a}$ **f** $v = Ae^{-5t}$

**2** **a** $y = Ae^{2x} - 2.5$ **b** $x = Ae^{-4t} + 1.75$

**c** $y = Ae^{-0.3x} + 13\frac{1}{3}$ **d** $w = Ae^{-11m} + \frac{71}{110}$

**e** $y = Ae^{-0.03x} + 33\frac{1}{3}$ **f** $v = Ae^{-5t} - 0.8$

**3** **a** $y = (3x - A)^2$ **b** $y = Ae^{5t} - \frac{1}{5}$

**c** $y = \tan(x + c)$ **d** $y = 3\sin(x + c)$

**e** $y = \sqrt[3]{9x + c}$ **f** $y = \frac{3}{c - x}$

**4** Proof

**5** $y = 4e^{2t}$

**6** **a** $M = 100e^{-kt}$, where $k = \frac{\ln(2)}{1600}$

**b** About 46 mg

**c** About 3715 years

**7** $T = 25 + 75e^{-0.155t}$ °C ($t$ in min), another 5 min 5 s

**8** About 0.278 ML

**9** $N \approx 20e^{0.752t}$ ($t$ in hours), 5.2 h

**10** $h = 20e^{-0.277t}$ cm ($t$ in hours)

**11** $C \approx 20e^{-0.223t}$ ppm ($t$ in hours), 4:12 p.m.

**12** $C \approx 5e^{-0.274t}$ ppm ($t$ in hours), 1:51 p.m.

**13** $T = 1100 - 1080e^{-0.3646t}$ °C ($t$ in min), 12:31 p.m., 1100°C

**14** $v = \frac{gm}{k}\left(1 - e^{-\frac{k}{m}t}\right)$. It reaches a final velocity of $\frac{gm}{k}$.

**15** $v = \frac{9.8D^2}{C}\left(1 - Ae^{-\frac{C}{D^2}t}\right)$ m s$^{-1}$, $v_1 = \frac{9.8D^2}{C}$ m s$^{-1}$

**16** $Q = (6 \times 10^{-5})(1 - e^{-25t})$ C, $V = 12(1 - e^{-25t})$ V, $I = (1.5 \times 10^{-3})(1 - e^{-25t})$ A ($t$ in s)


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**17** $T = 25 + (T_0 - 25)e^{-0.0835t}$ °C ($t$ in min), 67.8°C (milk) and 74.5°C (black). To refine the model, you could assume that, if the amount added is $p$ times the amount added in the question, the temperature would drop by $10p$ °C; so $T_0 = 100 - 10p$ °C. This would work for values of $p$ less than about 2. For greater values, a more complex procedure involving the volume of the cup would be required.

## Exercise 9.05

**1** **a** $y = \frac{3}{2} + ce^{-2x}$ **b** $3y^2 - 2x^3 = c$

**c** $y = \frac{7}{4} + c\,e^{6x^2+4x}$ **d** $y^3 - 5y - 5x + 2x^2 = c$

**e** $y = 1 + (x^2 + c)^3$ **f** $y = c\,e^{\frac{1}{2}x^2} - 1$

**g** $y = -\ln|e^x(1 - x) + c|$ **h** $y = \dfrac{1}{\frac{1}{3}e^{3x} - x + c}$

**i** $y^3 + 3y - x^3 = c$ **j** $y = \sqrt{cx^2e^{x^2} - 1}$

**k** $y^2 - x^2 + 2(e^y - e^{-x}) = c$ **l** $y = -\frac{1}{2}\ln\left|\frac{2}{3}e^{3x} + c\right|$

**2** **a** $y = e^{3x^3 - x}$

**b** $(y - 1)^2 = x^3 + 2x^2 + 3x$

**c** $y = \tan(2x^3)$

**d** $y^4 + 16y + x^4 - 8x^2 = 17$

**e** $y = -\sqrt{\dfrac{2x^3}{3} + 16}$

**f** $y = \sqrt{x^2 + 16} + 2$

**g** $y^3 - 5y - 4x + x^2 = 8$

**h** $y = -\frac{3}{4} + \frac{1}{4}\sqrt{65 - 8e^x - 8e^{-x}}$

**i** $y = \dfrac{6}{2x^3 + 3}$

**j** $y = -\sqrt{2x - 2x^2 + 4}$

**k** $y = \sqrt{-e^{x^2} + 5}$

**l** $y = \dfrac{2}{3 - x^2}$

**3** Sales at 7, 9 and 11 weeks would be about 324, 993 and 4021 punnets respectively.

**4** $N = \dfrac{1.0002e^{0.5t}}{0.000e^{0.5t} + 1}$, 900

**5** Over 21 million, but it is likely to be too far in the future for the trend to be valid.

**6** 25 g

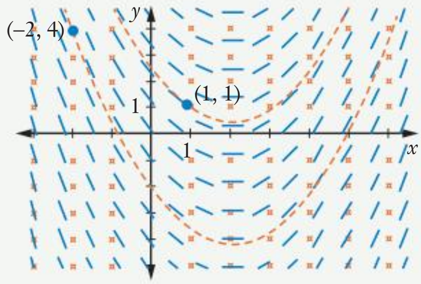
**7** Proof: see worked solutions.

**8** 123, 151, 186, 280, 762 and 4025 mice.

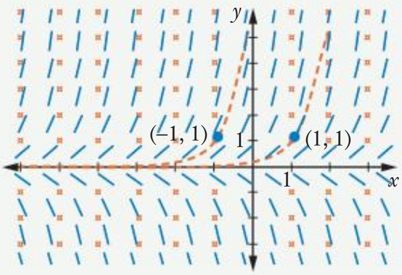
**9** $k \approx 0.2352$. The speeds are approximately 9.68 m s$^{-1}$, 18.65 m s$^{-1}$ and 26.42 m s$^{-1}$.

## Exercise 9.06

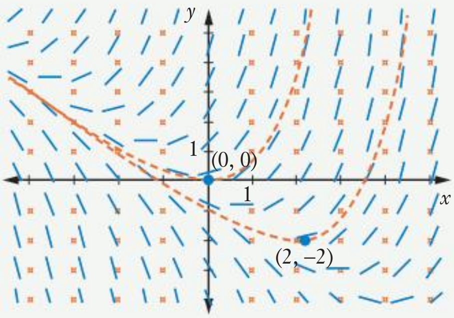
**1** **a–c**



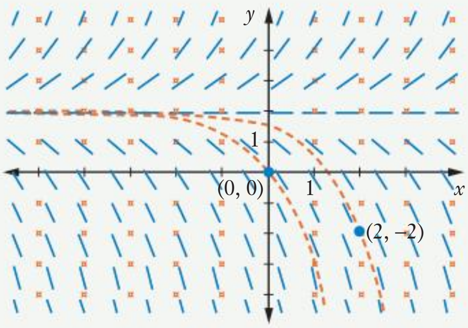

**2** **a–c**




**3**




**4**

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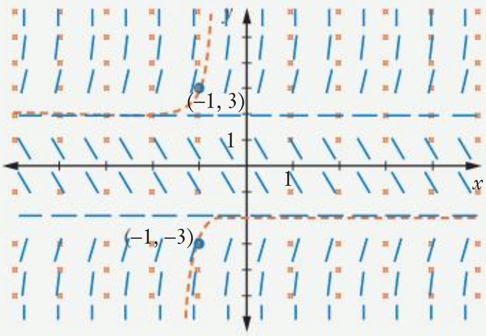
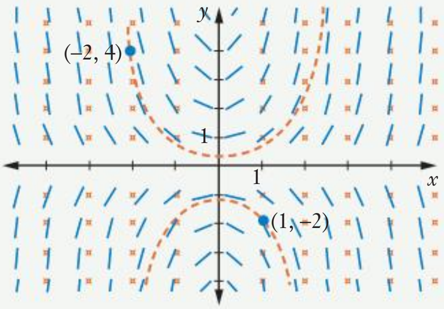
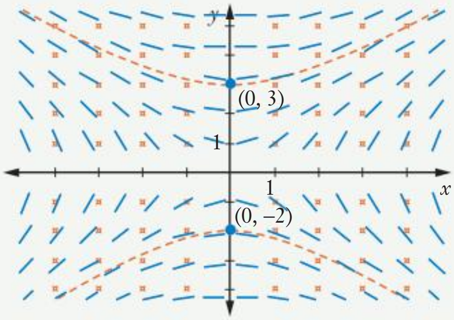
5



6 a – C, b – D, c – A, d – B

7 a – D, b – C, c – A, d – B

8



9



10 B

## Exercise 9.07

1 a Proof: see worked solutions.
b About 11 c About 79

2 a About 3086 perch b About 3.85 years
c About 2.68 years

3 a About 60 b 60 000
c About 17 months

4 a $\frac{dP}{dt} = 0.2P\left(1 - \frac{P}{100}\right)$
b $P = \frac{100}{1 + \frac{97}{3}e^{-0.2t}} = \frac{300}{3 + 97e^{-0.2t}}$
c About 11.9 hours
d $P \le 100$, so it will never reach 250 mg.

5 a $\frac{dP}{dt} = 0.3P\left(1 - \frac{P}{10000}\right)$
b $P = \frac{10000}{1 + 3e^{-0.3t}}$
c About 5253 d About 6.5 years

6 a 100% b $a = 100, y_0 = 2, k \approx 0.001\ 946$
c 12.5% d 87.6% e About 31.3 days

7 4 p.m.

8 26.4 million, 31.2 million

9 a i 4.6 g ii 5.5 g iii 7.4 g b 15.5%

10 a 308 b 2686 c 3640
d Percentage = $\frac{300}{3 + 397e^{-0.8t}}$

11 a 138 b 207 c 915

12 a $\frac{dP}{dt} = \frac{1}{325}\left(1 - \frac{P}{100}\right)$
b 6.69 billion The predicted value is lower than the actual value.
c 8.52 billion, 24.18 billion
d 8.25 billion, 18.83 billion

13 a About 6 years (5.974)
b About 81 each subsequent year

14 5.278 weeks, so they can be removed after 6 weeks.

15 50.5 min

## Exercise 9.08

1 25.3 N at 71.6° above the horizontal.

2 15.6 N east

3 28 kg

4 15 kg

5 $(-3\mathbf{i} + 5\mathbf{j})$ N ($\mathbf{a} = 0$, so the resultant force = 0)

6 13.29 N at 48.8° above the horizontal.

7 28 600 N

8 0.81 N at 14.28° above the horizontal.

9 a 21.04 N at 127.6° b 2.40 m s$^{-2}$ at 127.6°
c 21.04 N at 307.6°

10 160 N

11 Male = 0.88 m s$^{-2}$ and female = 1.17 m s$^{-2}$ in opposite directions.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**12** $T_1 \approx 48.3$ N, $T_2 \approx 61.8$ N, $T_3 = 78.4$ N

**13** $T_1 \approx 138.7$ N, $T_2 \approx 73.48$ N, $T_3 = 117.6$ N

**14** **a** 2 m s$^{-2}$

**b** 8 kg block = 16 N and 7 kg block = 14 N

**15** **a** 73.5 N **b** 7.5 kg **c** 5.33 m s$^{-2}$

**16** 1.8 m s$^{-2}$ at $\tan^{-1}\left(\frac{5}{2}\right) \approx 68.2°$

**17** $20\sqrt{5}$ N at $\tan^{-1}(2) \approx 63.4°$

**18** 2.13 m s$^{-2}$ at $\tan^{-1}\left(-\frac{7}{8}\right) \approx 318.8°$

**19** 6.07 m s$^{-2}$

**20** –1.78 m s$^{-2}$ (1.78 m s$^{-2}$ downwards)

**21** **a** 65 kg **b** 1.85 m s$^{-2}$

## Exercise 9.09

**1** 3 m s$^{-2}$, 37.5 m

**2** 115.2 m s$^{-1}$, 345.6 m

**3** **a** 3.75 m s$^{-2}$ **b** 14.6 m s$^{-1}$

**4** **a** 6 m s$^{-1}$ **b** 2 m s$^{-2}$

**5** 1.5 m s$^{-2}$, 75 m

**6** 50 s, 2500 m

**7** **a** 4.95 s **b** 48.5 m s$^{-1}$

**8** **a** 31.3 m s$^{-1}$ **b** 6.39 s

**9** 2.3 s after being thrown.

**10** **a** 10.68 m s$^{-1}$ **b** –4.02 m s$^{-1}$

**11** **a** 30.1 m **b** 24.3 m s$^{-1}$

**12** **a** 25.5 s **b** 778.4 m **c** 61.1 m s$^{-1}$

**13** **a** 30 m s$^{-1}$ **b** 100 m

**14** **a** 1 m s$^{-2}$ **b** 12.5 m

**15** 20 m s$^{-1}$

**16** **a** 132 m **b** 5 m s$^{-1}$

**17** **a** 8.17 m s$^{-2}$ **b** 40.83 m s$^{-1}$ **c** 102.1 m

**18** **a** 10 m s$^{-1}$ **b** $-3\frac{1}{3}$ m s$^{-2}$

**19** –8.45 m s$^{-2}$ and 3.08 s

**20** **a** 10 s **b** 5 s

**21** Proof

**22** Yes, she reaches her destination in about 22.7 s.

**23** Proof

**24** 41.2 m s$^{-1}$

**25** $a = \frac{1}{3}$ m s$^{-2}$, $u = 23$ m s$^{-1}$. Look at the parts of the journey from the intersection to the bridge and along the bridge separately.

**26** Proof. The second ball will be in motion for $(t - 2)$ seconds.

## Exercise 9.10

**1** $v = t^2 + \frac{5t^3}{3}$, $v(2) = 17\frac{1}{3}$ m s$^{-1}$

**2** $v = -\frac{1}{9}[\cos(3t - 2) - \cos(2)]$, $v(1) = -0.106$ m s$^{-1}$

**3** $v = \frac{15}{16}(e^{0.4t+5} - e^5) + 3$, $v(0.1) = 8.68$ m s$^{-1}$

**4** $v = 7e^{\frac{3}{2}t}$, $v(1) = 31.37$ m s$^{-1}$

**5** **a** $a = \frac{1}{10}(3 - 2t)$ **b** –1.8 m s$^{-1}$

**6** **a** $a = \frac{1}{2}\cos(x)$ **b** $v = \sqrt{\sin(x) + 16}$

**7** $v = \sqrt{\frac{2}{5}x^2 + \frac{14}{5}x + 4}$

**8** $v = \frac{1}{2}x + 3$

**9** $v = 4 - 2e^{-x}$

**10** $v = \sqrt{\frac{2}{3}\ln|x+1| + 4}$

**11** $v = \frac{g}{k}(1 - e^{-kt})$, $v_T = \frac{g}{k}$

**12** **a** Proof **b** $x = \frac{5}{2}\ln\left|\frac{49}{49 - v^2}\right|$

**13** $113\frac{1}{3}$ m

**14** –93 m s$^{-1}$

**15** **a** $v = -25 + 39e^{-\frac{t}{35}}$ **b** 72.1 m

**16** **a** $t = \frac{1}{5}\ln\left|\frac{g}{g - 5v}\right|$

**b** $v = \frac{g}{5}(1 - e^{-5t})$

**c** 1.96 m s$^{-1}$

**d** About 0.139 s

**17** **a** $v = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right)$

**b** $x = \frac{mg}{k}\left[t + \frac{m}{k}\left(e^{-\frac{kt}{m}} - 1\right)\right]$

**18** **a–c** Proofs **d** $v_T = \sqrt{\frac{gm}{k}}$

**19** 11.12 m

**20** **a** $k = 0.003\ 456$ **b** 3.24 s

**c** 30.25 m

**21 a** $v = \frac{Fg}{k}\left(1 - e^{-\frac{k}{m}t}\right)$, where $k$ = resistive force

**b** $x = \frac{Fg}{k}\left(t + \frac{m}{k}e^{-\frac{k}{m}t} - \frac{m}{k}\right)$

**c** $x = \frac{m}{k}\left[\frac{Fg}{k}\ln\left(\frac{Fg}{Fg - kv}\right) - v\right]$

**d** $v_T = \frac{Fg}{k}$

**22** $h = \frac{mu}{k} - \frac{m^2 g}{k^2}\ln\left(1 + \frac{ku}{mg}\right)$, $t_h = \frac{m}{k}\ln\left(1 + \frac{ku}{mg}\right)$

**23 a** Proof **b** $\frac{1}{2k}\ln\left(\frac{4}{3}\right)$

## Exercise 9.11

**1** $\frac{1}{4}$ s, 4 $s^{-1}$

**2** 0.67 s, 1.5 $s^{-1}$

**3** $T = 4$ s

**4 a** $\frac{1}{3}$ $s^{-1}$ **b** $\frac{\pi}{2}$ m $s^{-1}$ **c** $\frac{\pi\sqrt{5}}{6}$ m $s^{-1}$

**d** $-\frac{\pi^2}{3}$ m $s^{-2}$ **e** $\frac{2\pi^2}{9}$ m $s^{-2}$

**5 a** 10 cm

**b** $5\pi$ cm $s^{-1}$, 0 cm $s^{-2}$

**c** 0 cm $s^{-1}$, $-\frac{5\pi^2}{2}$ cm $s^{-2}$

**d** $\pi\sqrt{21}$ cm $s^{-1}$, $-\pi^2$ cm $s^{-2}$

**6 a** $\ddot{x} = -4x = -\omega^2 x$ **b** $\pi$ s, 8

**c** $8\sqrt{3}$ m $s^{-1}$, 16 m $s^{-2}$

**7 a** $32\pi^2$ cm $s^{-2}$ **b** $8\pi$ cm $s^{-1}$

**c** $24\pi^2$ $cm^{-2}$ **d** $4\pi\sqrt{3}$ cm $s^{-1}$

**e** $x = 2\cos\left(4\pi t + \frac{\pi}{2}\right)$

**8 a** $x = 0.2\sin(5t)$ **b** $\dot{x} = \cos(5t)$

**c** $\ddot{x} = -5\sin(5t)$ **d** $T = \frac{2\pi}{5}$, $a = \frac{1}{5}$

**9 a** $x = 10\cos\left(\frac{\pi t}{6}\right)$ **b** $\frac{5\pi}{3}$ mm $s^{-1}$

**c** 12 s, 10 mm **d** $\frac{5\pi}{3}$ mm $s^{-1}$

**e** $\frac{5\pi^2}{3}$ mm $s^{-2}$

**10 a** $\frac{3\sqrt{6}}{10}$ **b** $T = \frac{\pi\sqrt{6}}{6}$ s

**11 a** Proof **b** $\frac{\pi}{5}$

**12 a** $\ddot{x} = -8x$ **b** $T = \frac{\sqrt{2}\pi}{2}$, $a = \sqrt{3}$

**13 a** $\ddot{x} = -\frac{\pi^2 x}{4}$ **b** 4 s, 5

**c** $-\frac{5\pi\sqrt{3}}{4}$ m $s^{-1}$, $\frac{5\pi^2}{8}$ m $s^{-2}$

**14 a** $\sqrt{2}\pi$ s **b** $\frac{\pi}{2}$ s

**15 a** $1.5\sqrt{2}$ m $s^{-1}$ **b** $0.9\sqrt{2}$ m $s^{-1}$

**16** 8 s, $28\sqrt{2}$ cm

**17** $0.9\pi$ m $s^{-1}$, $0.9\pi^2$ m $s^{-2}$

**18** $\frac{3}{5\pi}$ m $s^{-1}$, $\frac{6}{25\pi^2}$ m

**19 a** 6.84 cm **b** 0.053 s

**20 a** Proof **b** 3 m, $\pi$ s

**21** $\frac{5}{72}$, $8\frac{1}{3}$ m $s^{-1}$

**22 a** Proof **b** $a = 2$, $T = \pi$

**23** 12:46 p.m.

## Chapter review

**1 a** $\frac{dy}{dx} = \frac{4y + 5x}{3y - 4x}$ **b** $\frac{dy}{dx} = \frac{2y - 3e^y}{x(3e^y - 2)}$

**c** $\frac{dy}{dx} = \frac{2x\sin(y)}{e^y - x^2\cos(y)}$

**2 a** $5x - 8y + 50 = 0$

**b** $300x - 65y - 4056 = 0$

**3 a** $-108$ **b** $81e^8$

**4** 150 $cm^3$/min

**5 a** $\frac{dr}{dt} = \frac{-1}{2.88\pi}$ **b** $\frac{dSA}{dt} = \frac{10}{3}$

**6 a** $y = 12\sin(n) + c$ **b** $y = 4t^3 + 8t + c$

**c** $p = \frac{1}{3}\arctan(x) + c$ **d** $\frac{1}{6}\ln|3t^2 + 8| + c$

**7 a** $y = x\sin(x) + \cos(x) - \pi$

**b** $y = \frac{1}{2}\arcsin(2x) + \frac{3\pi}{4}$

**c** $y = \frac{1}{4}\ln|4x + 3| + 3$

**8 a** $v = 9$ m $s^{-1}$ **b** $h = 5$ m

**9 a** $y = Ae^{0.07x}$ **b** $t = Ae^{-x}$

**c** $g = Ae^{2.8h}$ **d** $y = Ae^{0.05x} - 4$

**e** $y = Ae^{-0.8q} + 5$ **f** $g = Ae^{1.6p} + 2.5$


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**10** **a** $A \approx 12e^{-0.173t}$ mg ($t$ in hours)

**b** 4.2 mg

**11** **a** 56.2°C

**b** About 1 hour 15 min

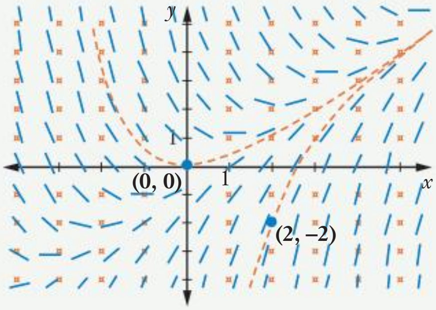
**12** **a** $y = \frac{1}{3} + c\,e^{3x^2+15x}$ **b** $y = \frac{2ce^{x^2}}{3(1-ce^{x^2})}$

**c** $y^2 + 3y - x^3 + x = c$ **d** $y = -\frac{1}{\cos(x)+c}$

**13** **a** $y = -\sqrt{2\ln(1+x^2)+4}$

**b** $y = 3 - \sqrt{4-x^2}$

**14** **a–c**




**15** **a** –D, **b** – B, **c** – A, **d** – C

**16** **a** $P = \frac{100A}{A+e^{-t}}$, where $A = \frac{1}{4}$

**b** About 65 **c** About 2.77 years

**17** 7941, 11 796, 16 152, 20 321, 23 722, 26 155, 27 734, 28 698, 29 262, 29 586, 29 768

**18** **a** About 3795 **b** About 2.8 months

**19** **a** 200.3 N perpendicular to the surface.

**b** 1.572 m $s^{-2}$ parallel to the surface.

**20** 173.3 N

**21** Astronaut = 0.5 m $s^{-2}$ and space junk = 0.0045 m $s^{-2}$ in opposite directions.

**22** **a** $13\frac{1}{3}$ s **b** $266\frac{2}{3}$ m

**23** **a** 5.35 s **b** 71.48 m **c** 37.43 m $s^{-1}$

**24** **a** 21.43 s **b** 321.45 m **c** 30.00 m $s^{-1}$

**25** **a** $v = \frac{4}{3}t^2 - \frac{5}{9}t^3 + 5$ **b** 2 m $s^{-1}$

**26** $v = \sqrt{8x - 3x^2 + 20}$

**27** **a–b** Proofs

**28** **a** $\frac{1}{4}$ $s^{-1}$ **b** $30\pi$ cm $s^{-1}$ **c** $20\pi\sqrt{2}$ cm $s^{-1}$

**d** $15\pi^2$ cm $s^{-2}$ **e** $-5\pi^2$ cm $s^{-2}$

**29** **a** $\frac{2\pi}{3}$ **b** 4 **c** $x = 4\cos\left(3t + \frac{\pi}{4}\right)$

**30** $144\pi$ $cm^3$/s

**31** Increasing by about 91.79 km/h.

**32** $P = P_0e^{rt}$ dollars ($t$ in years), 8.97% p.a.

**33** Temperature after 5 min = 170°C, eventual temperature = 500°C.

**34** $N(10) \approx 22\,688$
$N(15) \approx 39\,062\,500$
10 and 15 weeks are quite long extrapolations from 3 weeks. Other phone apps that supersede or are more popular could be released by other developers.

**35** D

**36** **a** $v = \frac{P}{k}\left(1 - e^{-\frac{k}{m}t}\right)$ **b** $v_T = \frac{P}{k}$

**c** $x = \frac{P}{k}\left(t - \frac{m}{k} + \frac{m}{k}e^{-\frac{k}{m}t}\right)$

**37** $2\pi$ s, 1.5 m $s^{-2}$

# Chapter 10

## Exercise 10.01

**1** **a** –0.842 **b** –0.524 **c** –0.253

**d** 0 **e** 0.524

**2** **a** 22.2 **b** 37.5 **c** 43.4

**d** 47.8 **e** 51.4

**3** **a** 87.4 **b** 96.2 **c** 103.8

**d** 112.6 **e** 119.2

**4** **a** [–1.28, 1.28] **b** [–1.96, 1.96]

**c** [–2.24, 2.24] **d** [–2.58, 2.58]

**e** [–0.67, 0.67]

**5** **a** [6.69, 9.31] **b** [5.90, 10.10]

**c** [4.80, 11.20] **d** [3.89, 12.11]

**e** [3.10, 12.90]

**6** **a** [53.6, 90.4] **b** [51.5, 92.5]

**c** [45.7, 98.3] **d** [40.6, 103.4]

**e** [30.8, 113.2]

**7** **a** [35.0, 45.1] **b** [32.3, 47.7]

**c** [28.7, 51.3] **d** [24.5, 55.5]

**e** [23.2, 56.8]

**8** 1 year (About 11.6 months)

**9** About 77 **10** 16 (16.4)

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## Exercise 10.02

1 [34.5, 37.6]
2 [39.9, 44.1]
3 [101.2, 112.8]
4 [22.5, 25.5]
5 [44.4, 45.6]
6 [86.1, 87.9]
7 [37.3, 42.3]
8 [8.01, 9.23]
9 About 14 to 23 ha, but 15 is too small for this to be reliable and because Bundaberg has a different climate to most of Australia.
10 Between about 6962 and 7887, but the sales seem to be trending down, so it may not be an accurate predictor of future sales.
11 About 11 to 19 [11.3, 18.7], but Sunday may not be typical and the two classes may not be typical of all Year 12s.

## Exercise 10.03

1 [26%, 34%], [39%, 47%], [2%, 10%], [17%, 25%]
2 [21 months, 31 months]
3 **a** [$47.50, $50.50] **b** $0.77
**c** $5.24
4 **a** [27, 31] **b** About 1.02 **c** 6.45
5 **a** About 21 **b** About 33
6 **a** About $262 **b** About $475
7 The top 3 confidence intervals overlap, so the difference could just be sampling variation.
8 The standard deviations are about 3 years and 4 months and the samples are far from random. They are likely to be very unrepresentative because of self-selection bias.
9 The standard deviation was nearly 12 months, so it is not really justified, particularly as you should count everyone.
10 The standard deviation is about 15 mm, so 214 or more has a very low probability (less than $10^{-10}$). This suggests the data is not random, but the records cover only 36 years.

## Chapter review

1 **a** About 1.476 **b** About 19.768
2 About [20.5, 47.5]
3 About [132.1, 137.9]
4 About [39.4, 42.5]
5 [532, 616]
6 **a** [174, 178] **b** About 1.0 **c** About 9.1
7 About 5 (5.04)
8 About 18 months (17.962)
9 About [6.7, 14.1], but there are a lot of 'round' figures that suggest these are not the true amounts of time the students are actually spending on homework.
10 About 1% (0.906%). The records are only for 51 years, so may not be accurate. Also, climate change may alter the weather in Kowanyama, so it may not be accurate.

## Practice examination 4

1 Tangent: $4x + 3y - 7 = 0$; normal: $3x - 4y - 24 = 0$
2 **a** $m = 7p + 7p^2 + c$ **b** $p = 4e^{5x} + 8x + c$
**c** $p = \frac{v^2}{2} + 4\ln|v| + c$
3 **a** $y = -\frac{1}{4}\cos(2x) + \frac{5}{4}$ **b** $\frac{1}{16}\ln|8x^2 - 9| + 2$
4 $y = -\frac{1}{2}\ln|c - e^x|$
5 **a** $y = -\frac{1}{2}\ln|e^4 - x^2|$ **b** $y = e^{3x^3 - x}$
6 0.034 m $s^{-2}$ at 39.8° north of east
7 4.75 s and 73.63 m
8 Amplitude 0.9 m, period $\pi$ s
9 **a** 1.2 m **b** $0.6\sqrt{7}$ m $s^{-1}$
10 61.16
11 About [100, 140], [100.26, 139.74]
12 [65.1, 74.9]
13 80, 0.24
14 Proof
15 **a** $I = \frac{1}{75}e^{-\frac{1}{45}t}$ A ($t$ in s), so a maximum of 13.3 mA would be produced initially, declining to 13 mA after 1 s. It could kill, so it is better to leave large capacitors alone.
**b** $I = \frac{V}{R}e^{-\frac{1}{RC}t}$, so a capacitance of only 0.7 mF at 200 V could kill. To produce 20 mA for 1 s, the voltage would have to be set to at least 307 V with the 3 mF capacitor.
16 About 31.5, 1.8 (1.782)
17 20.12 m/s
18 **a** 34.66 min.
**b** The model can be extended by resetting $t = 0$ to the time when the tank is $p$% full; so $V = 200 - (200 - p)e^{-0.02t}$ L ($t$ in min) and $t = 50\ln|2 - 0.01p)|$ min.
19 About 10% (0.0948)


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# GLOSSARY AND INDEX

**acceleration:** The rate of change of velocity. (p. 242)
**age-specific:** Given in terms of age. (p. 161)
**air resistance:** The retarding force of air friction on an object moving through the air. (p. 255)
**altitude (in geometry):** A perpendicular from one side of a triangle to the opposite vertex. (p. 58)
**altitude (in vector geometry):** The third 3D polar coordinate $\varphi$. The angle from the $x$-$y$ plane to a vector. $-90° \le \varphi \le 90°$. (p. 34)
**amplitude:** The maximum displacement about the point of equilibrium for a particle oscillating during simple harmonic motion. (p. 453)
**angle of projection:** The angle to the horizontal at which a projectile is launched. (p. 255)
**angle sum and difference identities:** Trigonometric identities used to convert trigonometric ratios of sums and differences of angles to products of trigonometric ratios. (p. 278)
**angular momentum:** The cross product of the momentum and radius of an object. (p. 72)
**angular velocity:** The rate of change of angle of an object moving in a circle. (p. 262)
**arccosine ($\cos^{-1}$):** The inverse cosine ratio such that $\cos^{-1}(x) = y$ if and only if $\cos(y) = x$ and $0 \le y \le \pi$. (p. 291)
**arcsine ($\sin^{-1}$):** The inverse sine ratio such that $\sin^{-1}(x) = y$ if and only if $\sin(y) = x$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. (p. 291)
**arctangent ($\tan^{-1}$):** The inverse tangent ratio such that $\tan^{-1}(x) = y$ if and only if $\tan(y) = x$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$. (p. 291)
**Argand diagram:** A diagram (or graph) used to represent complex numbers geometrically. The horizontal axis is called the real axis, Re $(z)$, and the vertical axis is called the imaginary axis Im $(z)$. (p. 84)
**Argand plane:** A plane whose points on which an Argand diagram is drawn. (p. 84)
**argument:** The angle that $z$ makes with the positive direction of the real axis when it is represented in an Argand diagram. Written as arg (z). (p. 94)
**arithmetic progression (AP):** A sequence of numbers in which each term differs from the preceding term by a non-zero constant number $d$. $a$ is the first term and $[a + (n - 1)d]$ is the $n$th term. $a$, $(a + d)$, $(a + 2d)$, …, $[a + (n - 1)d]$. (p. 17)
**augmented matrix:** A matrix from a system of linear equations. The row $a\ b\ c\ \dots\ k$ corresponds to the equation $ax_1 + bx_2 + \dots = k$. (p. 136)
**azimuth:** The second 3D polar coordinate $\theta$. The anticlockwise angle from the $x$-axis to the projection of the vector on the $x$-$y$ plane. $0° \le \theta < 360°$. (p. 34)

**Cartesian equation:** An equation involving the rectangular Cartesian coordinates $x, y$ or $x, y$ and $z$ for a line or plane in 2D or 3D. (p. 53)
**Cartesian equation of motion:** The Cartesian form of the equation of a particle's path. (p. 238)
**Cartesian form (of a complex number):** The form $z = a + bi$. (p. 84)
**central limit theorem:** Sampling distributions approximate the normal distribution and the larger the samples the better the approximation. (p. 374)
**centroid:** The intersection of the medians of a circle. (p. 58)
**circumcentre:** The intersection of the perpendicular bisectors of the side of a triangle. The centre of the circumcircle. (p. 58)
**circumcircle:** The circle through the vertices of a triangle. (p. 58)
**cis ($\theta$):** The abbreviation for $\cos(\theta) + i\sin(\theta)$. (p. 94)

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**coefficient matrix:** A matrix of the coefficients of a system of linear equations. *See also* value vector, variable vector. (p. 136)

**collide:** Two particles collide if they are in the same place at the same time. (p. 64)

**complex conjugate:** For $z = a + ib$, the complex conjugate is $\overline{z} = a - ib$. (p. 83)

**complex number:** A number of the form $z = a + ib$, where $a, b \in \boldsymbol{R}$ and $i = \sqrt{-1}$. (p. 82)

**complex polynomial:** An expression of the form $P(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \ldots + a_1 z^1 + a_0$, where $a \geq 0$, $a_0, a_1, a_2, \ldots, a_n$ may be complex numbers and $a_0 \neq 0$. (p. 215)

**component:** One of the projections of a vector in the direction of the $x$, $y$, or $z$ axes. One of the parts of a vector written in the component form $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. (p. 34)

**concurrent forces:** Forces that lie in the same plane and pass through a single point. (p. 435)

**confidence interval:** A range of values that estimates the value of a parameter to a particular confidence level, particularly the mean. *See also* confidence interval. (p. 472)

**confidence level:** The proportion of values of an estimate that lie inside a confidence interval. (p. 472)

**conjugate root theorem:** A theorem stating that if $P(z)$ is a polynomial with real coefficients and $a + bi$ is a root, then the complex conjugate, $a - bi$, is also a root. (p. 222)

**consistent equations:** A system of linear equations that have at least one solution. (p. 132)

**constant term:** The term $a_0$, where $P(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \ldots + a_1 z^1 + a_0$. (p. 215)

**continuous random variable:** A continuous variable whose value depends on the outcome of a chance event. (p. 363)

**cross product:** The vector $\mathbf{p} \times \mathbf{q}$ of vectors $\mathbf{p}$ and $\mathbf{q}$ with angle $\theta$ between them is perpendicular to both and has magnitude $pq \sin(\theta)$. $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{p} \times \mathbf{q}$ form a right-handed system. (p. 47)

**culling:** Removal of part of a population of animals. (p. 165)

**cumulative distribution function (for an exponential distribution):** For an exponential probability density function with the parameter λ, the cumulative distribution function is $\text{cdf}(t) = 1 - e–\lambda^t$. (p. 342)

**De Moivre's theorem:** A theorem that states if $z = |z| \text{ cis } (\theta)$, then $z^n = |z|^n \text{ cis } (n\theta)$ for $n \in \boldsymbol{Q}+$. (p. 106)

**degree:** The highest power of $z$, where $P(z) = a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \ldots + a_1 z^1 + a_0$. (p. 215)

**dependent system:** a system of linear equations with at least one equation that is a combination of others. (p. 132)

**determinant:** A number determined by a matrix. For the matrix $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det(\mathbf{A}) = |\mathbf{A}| = A = ad - bc$. *See also* singular, non-singular. (p. 48)

**difference function:** For the functions $f(x)$ and $g(x)$, the difference function is $f(x) - g(x)$. (p. 310)

**differential equation:** A mathematical model that involves an equation containing an unknown function and one or more of its derivatives. (p. 396)

**direction field:** A diagram that represents the solutions to a first-order differential equation. (p. 415)

**displacement:** The vector change of position from one point to another, particularly for a moving particle. (p. 246)

**displacement vector:** The vector from one point to another. The change from position $A$ to position $B$ is written as **AB**. (p. 59)

**dividend:** If $p(z) = q(z)d(z) + r$, then the dividend $= p(z)$. (p. 217)

**divisor:** If $p(z) = q(z)d(z) + r$, then the divisor $= d(z)$. (p. 217)

**dominance matrix:** Matrix 0s, $\frac{1}{2}$s and 1s of the dominance of a group of players. Each row shows the dominance of a player over the others. 1 = win, $\frac{1}{2}$ = equal and 0 = loss or unmet. (p. 152)

**dot product:** The value $\mathbf{p} \bullet \mathbf{q} = pq \cos(\theta)$ of vector $\mathbf{p}$ and $\mathbf{q}$ with angle $\theta$ between them. Also given by $\mathbf{p} \bullet \mathbf{q} = p_1 q_1 + p_2 q_2 + p_3 q_3$, where $(p_1, p_2, p_3)$ and $(q_1, q_2, q_3)$ are the component forms of $\mathbf{p}$ and $\mathbf{q}$. (p. 41)

**double-angle identities:** Trigonometric identities used to convert trigonometric ratios of double angles (e.g. $2x$) to trigonometric ratios of single angles (e.g. $x$). (p. 278)

**duopoly:** A market dominated by 2 sellers. (p. 170)

**electromagnetic force**: The force produced on a moving charge by a magnetic field. (p. 72)

**ellipse:** A conic section that looks like a flattened circle. (p. 239)

**equation of a plane:** An equation that specifies the points in a plane. The vector form is $\mathbf{n} \bullet \mathbf{p} = k$ and the Cartesian form is $ax + by + cz = k$, where $\mathbf{n} = (a, b, c)$ is a normal and $\mathbf{p} = (x, y, z)$ is a point in the plane. (p. 66)

**equilibrium:** The state of a system when the resultant of the concurrent forces is zero. (p. 435)

**estimator:** A statistic (that estimates the value of a parameter). (p. 472)


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**even numbers:** Integers that are divisible by 2. They are in the form $n = 2k$, where $k$ is an integer. The even numbers = $\{\ldots, -2, 0, 2, 4, 6, \ldots\} = \{n: n = 2k, k \in \mathbf{Z}\}$. (p. 13)
**expected value:** For an exponential probability density function with parameter λ, the expected value, $E(X) = \mu = \frac{1}{\lambda}$. (pp. 336, 346, 375)
**exponential distribution:** A probability distribution with parameter λ has the probability density function $f(t) = \begin{cases} \lambda e^{-\lambda t} & \text{for } t \geq 0 \\ 0 & \text{elsewhere} \end{cases}$ where $P(a \leq t \leq b) = \int_a^b f(t)\,dx$. (p. 334)

**factor theorem:** For complex polynomials, $(z - a)$ is a factor if and only if $P(a) = 0$. (p. 220)
**first-order differential equation:** A differential equation involving only the first derivative. (p. 396)
**first-order ranking:** Ranking of players based on the total of wins and draws, equal to the difference of the sum of the each row and the corresponding column of the dominance matrix. (p. 153)
**force:** The physical explanation of the acceleration of an object. $\mathbf{F} = m\mathbf{a}$, where $m$ is the mass and $\mathbf{a}$ is the acceleration of the object. (p. 248)
**frequency, $f$ (in simple harmonic motion):** The number of oscillations per second, so $f = \frac{1}{T}$. (pp. 246, 454)
**Fundamental Theorem of Algebra:** An $n$th degree complex polynomial has exactly $n$ roots (including repeated roots). (p. 222)

**Gaussian elimination:** Solution of a system of linear equations by the progressive elimination of variables. (p. 131)
**geometric progression (GP):** A sequence of numbers formed by multiplying the preceding term by a constant $r$ called the common ratio. $a$ is the first term. (p. 17)
**gradient field:** *See* direction field. (p. 415)
**gravity:** The force produced by one mass on another, particularly that produced by the Earth on nearby objects. (p. 248)

**homogeneous system:** System of linear equations whose constant terms are all zero. (p. 132)
**hypothesis:** A general statement of conclusion for which tests are devised to prove or disprove. (p. 5)

**imaginary number:** The number $i$ such that $i^2 = -1$, i.e. $i = \sqrt{-1}$. (p. 119)
**implicit differentiation:** The process of differentiating an implicit equation with respect to the desired variable $x$ while treating the other variables as functions of $x$. (p. 386)
**in-centre:** The intersection of the angle bisector of a triangle. The centre of the in-circle. (p. 58)
**in-circle:** The circle inside a triangle that is tangent to the 3 sides. (p. 58)
**inconsistent equations:** System of linear equations with no solution. (p. 132)
**independent system:** System of linear equations with no equations that are combinations of others. (p. 132)
**initial condition:** The situation at time $t = 0$, i.e. $y(t_0) = y_0$. (p. 398)
**inner product:** *See* **scalar product**. (p. 41)
**integers:** The set of integers, $\mathbf{Z}$, contains the natural numbers, zero and the negative integers $\mathbf{Z}^-$. So the set is $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. (p. 4)
**integration by parts:** An integration technique such that for the functions $f(x)$ and $g(x)$: $\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$. (p. 305)
**integration by substitution:** A technique of integration for integrals in the form $\int f\left[g(x)\right]g'(x)\,dx$. (p. 283)
**interval estimate:** An estimate of a parameter as an interval. (p. 472)
**inverse:** For a non-singular matrix $\mathbf{A}$, the matrix $\mathbf{A}^{-1}$ whose product with $\mathbf{A}$ is the identity matrix. $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$. (p. 138)
**isometric drawing:** A flat drawing of a 3D object with the axes at 120° to each other and equal scales. (p. 29)

**leading term:** The term $a_n z^n$, where $P(z) = a_n z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \ldots + a_1 z^1 + a_0$. (p. 215)
**Leslie matrix:** A matrix that represents the age-specific birth rates and survival rates of an animal. (p. 162)
**line segment:** A finite part of a line. (p. 54)
**locus:** A set of points in a plane that satisfy some given condition. (p. 196)
**logistic model:** A growth model given by $\frac{dy}{dt} = ky(M - y)$, where the rate of change of the variable is proportional to both the variable and the difference between the maximum value $M$, and the variable. (p. 421)

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**magnitude:** size. *See* **norm**. (p. 34)
**margin of error:** Half the width of a symmetric confidence interval. (p. 472)
**Markov chain:** A sequence of a finite system of 'states' at equal time intervals. Each state represents the value of part of the system. Its value depends only on the previous value. (p. 172)
**median (in geometry):** A line from a vertex of a triangle to the midpoint of the opposite side. (p. 58)
**median (in statistics):** The score that divides a statistical or probability distribution into equal halves. (p. 346)
**midpoint rule:** A rule for calculating the approximate area under a curve (definite integral) using rectangular strips. (p. 325)
**modulus:** The magnitude of $z$ when it is represented in an Argand diagram. Written as mod $(z)$ or $|z|$. (p. 94)
**modulus–argument form:** *See* **polar form**. (p. 94)
**multiplicity:** The number of times a root for $P(z) = 0$ is repeated. (p. 222)

**natural numbers:** These are the counting numbers or positive integers. They have the symbol $\boldsymbol{N}$. So the set is $\boldsymbol{N} = \{1, 2, 3, 4, \ldots\}$. (p. 5)
**Newton's first law of motion:** A law that states that unless acted on by a resultant force, a body remains at rest or in uniform motion in a straight line. (p. 431)
**Newton's second law of motion:** A law that states that the acceleration of a body is proportional to the resultant force that acts on the body and inversely proportional to the mass of the body. Can be written as $\mathbf{F} = m\mathbf{a}$, where $\mathbf{F}$ and $\mathbf{a}$ are vectors and $m$ is a scalar quantity. (p. 431)
**Newton's third law of motion:** A law that states that for every action there is an equal and opposite reaction. (p. 434)
**non-homogeneous:** A system of linear equations whose constant terms are not all zero. (p. 132)
**non-parallel planes:** Planes in 3D that are not parallel. (p. 148)
**norm:** The size of a vector, written as $v = |\mathbf{v}|$. (p. 35)
**normal (in vector geometry):** A line that is perpendicular to a plane. (pp. 66, 148)
**normal (in calculus):** The line that is perpendicular to the tangent at a given point on the curve. (p. 387)

**odd numbers:** Integers that are not divisible by 2. Odd numbers are of the form $n = 2k + 1$, where $k$ is an integer. The odd numbers $= \{\ldots, -3, -1, 1, 3, 5, \ldots\} = \{n: n = 2k + 1, k \in \boldsymbol{Z}\}$. (p. 9)
**orthocentre:** The intersection of the altitudes of a triangle. (p. 58)
**orthogonal:** Perpendicular. Orthogonal components are perpendicular to each other. (p. 38)

**parabola:** A conic section whose Cartesian equation is a quadratic. (p. 239)
**parallel planes:** Planes that are parallel to each other. (p. 148)
**parallelepiped:** A 3D shape with 6 parallel faces. The opposite faces are congruent parallelograms. (p. 70)
**parameter (in statistics):** A numerical characteristic of a population. (p. 362)
**parameter (with functions):** A variable common to parts of a geometric equation or related equations such as the equations of lines or planes. (p. 50)
**parametric vector equation:** A vector equation written in terms of a parameter, such as the line $\mathbf{r}(t) = (1 - t)\mathbf{i} + t\mathbf{j} + 4\mathbf{k}$. (p. 54)
**partial fractions:** Functions of the form $\dfrac{p}{(x-a)(x-b)}$ or $\dfrac{px+q}{(x-a)(x-b)}$ can be expressed as partial fractions in the form $\dfrac{A}{x-a} + \dfrac{B}{x-b}$. (p. 300)
**path:** The track of a particle. (p. 238)
**payoff matrix:** A matrix showing the profit or market share of a supplier in a duopoly for one product, with each entry dependent on the supply of each product by the competitor. (p. 170)
**period:** The time for one orbit of an object moving in a circle or the time for one oscillation in simple harmonic motion. (p. 246)
**point estimate:** A single value estimate of the value of a parameter. (p. 472)
**point of equilibrium:** The central fixed point about which a particle oscillates during simple harmonic motion. (p. 453)
**polar coordinates:** Coordinates of a point $(r, \theta, \varphi)$, where $r$ is the norm, $\theta$ the azimuth and $\varphi$ the altitude of the point. Also called spherical coordinates. (p. 34)
**polar form (of a complex number):** The form $z = (r, \theta)$ or $z = r[\cos(\theta) + i\sin(\theta)]$ or $r \operatorname{cis}(\theta)$. (p. 94)
**population:** The complete group from which statistical information is taken. (p. 362)
**position vector:** A vector from the origin of a vector to a point $P$, written as **OP** or **p**. (p. 35)
**principal argument:** For $z = (r, \theta)$, it is the value of $\theta$ in the interval $-\pi < \theta \le \pi$. (p. 97)
**product identities:** Trigonometric identities used to convert products of trigonometric ratios to trigonometric ratios of sums and differences of angles. (p. 278)


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**projectile:** An object launched with an initial velocity but otherwise acted on only by gravity. (p. 255)
**projection:** The component of a vector in a given direction. The vector and its projection form a right-angled triangle with the right angle opposite the vector. (p. 34)
**proof:** An argument that shows something to be true for all cases. (p. 4)
**proof by mathematical induction:** Proof of a statement involving $n$ by showing that it is true for $n = 1$ and that if it is true for $k$, then it is also true for $k + 1$. (p. 5)
**Pythagorean identity:** Trigonometric identity relating $\sin^2(x)$ and $\cos^2(x)$. (p. 278)

**QED:** *Quod erat demonstrandum* (Latin), meaning *demonstrated as required*. (p. 5)
**quantile:** The value of a probability distribution below which a particular proportion of the scores lie. (p. 473)
**quartic:** A fourth degree polynomial. (p. 228)
**quotient:** If $p(z) = q(z)d(z) + r$, then the quotient $= q(z)$. (p. 217)

**random sample:** A selection such that every possibility has an equal probability random variable. (p. 362)
**range:** The horizontal displacement of a projectile when it strikes the ground. (p. 255)
**rate of occurrence:** For an exponential distribution of an event that occurs over time ($t$) with parameter $\lambda$, the average rate of occurrence $= \lambda$. (p. 344)
**realise the denominator:** The process where a fraction with the complex denominator $z$, is multiplied by $\frac{\overline{z}}{\overline{z}}$. (p. 91)
**rectangular form:** *See* **Cartesian form**. (p. 84)
**rectangular hyperbola:** A curve with the Cartesian equation $y = \frac{1}{x}$ or a transformation of this curve. (p. 239)
**recursive function:** Defining a sequence by giving the first term and a method of obtaining each term from the previous term. (p. 11)
**related rates of change:** Rates of change with respect to time found using the chain rule. (p. 390)
**reliability function:** For an exponential distribution, the reliability is the probability that a product or item will operate properly for a specified period given by $R(t) = 1 - P(\text{failure})$. If the average rate of failure is $\lambda$, $R(t) = e^{-}\lambda^{t}$. (p. 344)
**remainder:** If $p(z) = q(z)d(z) + r$, then the remainder $= r$. (p. 217)
**remainder theorem:** When $P(z)$ is divided by $D(z)$, the result is a quotient $Q(z)$ and remainder $R$ such that $P(z) = Q(z)D(z) + R$. (p. 217)
**resolving:** Writing a vector as the sum of components in given directions. (p. 38)
**right-handed system:** Three orthogonal directions related by a right-hand rule. (p. 46)
**root of a complex number:** A number $w$ satisfying the equation $w^n = z$. (p. 208)
**root of unity:** A number $z$ satisfying the equation $z^n = 1$. (p. 203)
**row echelon form:** The leading entry each row of the row echelon form of an augmented matrix is 1, with zeros below it. *See also* augmented matrix, reduced echelon form. (p. 140)
**row reduced echelon form:** The leading entry in each row of the row echelon form of an augmented matrix is 1, with zeros above and below it. *See also* augmented matrix, row echelon form. (p. 140)

**sample:** Part of a population. (p. 362)
**sampling distribution:** The distribution of sample means of random samples of the same size. (p. 367)
**scalar:** A real number. (p. 46)
**scalar product:** *See* **dot product**. (p. 41)
**second-order ranking:** Ranking of players based on the total of wins and draws of opponents, equal to the difference of the sum of the each row and corresponding column of the square of the dominance matrix. (p. 153)
**simple harmonic motion:** An oscillating motion in a straight line motion such that the acceleration of a particle is always directed to a fixed point and is proportional to its distance from this point. (p. 453)
**Simpson's rule:** A rule for calculating the approximate area under a curve (definite integral) using 'rectangular' strips with parabolic arcs as the edge closest to the curve. (p. 328)
**singular matrix:** A matrix that has no inverse. Its determinant is zero. (p. 138)
**slope field:** *See* **direction field**. (p. 415)
**solid of revolution:** A 3D shape generated by rotating a curve around the $x$- or $y$-axis. (p. 320)
**spherical coordinates:** *See* **polar coordinates**. (p. 34)
**state vector:** A vector showing the values of each state of a Markov chain at a particular time. (p. 172)
**statistic:** An approximation of a parameter derived from a sample. (p. 362)
**system of linear equations:** A set of linear equations in the same variables. (p. 132)

**tangent:** The line that touches the curve at one point and has the same slope as the curve at that point. (p. 387)

**terminal velocity:** The limiting velocity reached when the forces causing the motion of a body are equal and opposite the forces resisting motion. (p. 448)

**third-order ranking:** The difference of the sum of the each row and corresponding column of the cube of the dominance matrix. (p. 187)

**total ranking:** The total of first, second, … rankings from a dominance matrix, including possible weightings. (p. 153)

**trajectory:** The path of a projectile. (p. 255)

**transition matrix:** A matrix of the probabilities of each state of a Markov chain. (p. 172)

**transition probability:** Conditional probability of a state of a Markov chain. (p. 172)

**transpose:** A matrix $\mathbf{A}^{\mathrm{T}}$ whose rows and columns are swapped from those of a given matrix **A**. (p. 153)

**trapezoidal rule:** A rule for calculating the approximate area under a curve (definite integral) using trapezoidal strips. (p. 327)

**triangular distribution:** A probability distribution that has a triangular shape. (p. 366)

**trigonometric form:** *See* **polar form**.

**uniform circular motion:** Motion in a circle at constant speed. (p. 263)

**uniform probability distribution:** A probability distribution that is a flat line. (p. 366)

**unit vector:** A vector with unit length 1. (p. 34)

**value vector:** A column matrix whose entries are the constant terms of a system of linear equations. *See also* coefficient vector, variable vector. (p. 136)

**variable vector:** A column matrix whose entries are the variables of a system of linear equations. *See also* coefficient vector, value vector. (p. 136)

**vector equation:** An equation written as a vector or involving a vector or vectors, such as the vector equation of a line. (p. 50)

**vector equation of motion:** The vector form of the position of a particle as a function of time. (p. 238)

**vector product:** *See* **cross product**.

**velocity:** The rate of change of position or displacement. (pp. 242, 255)

**waiting time:** The time spent waiting for an event to occur. (p. 344)

**work:** The dot product of a force and the displacement of a particle while the force is acting. (p. 72)

**z-score:** For a value $x$, the value rescaled to the mean $\bar{x}$ and standard deviation $\sigma$. The formula is $z = \frac{x - \bar{x}}{\sigma}$. (p. 374)


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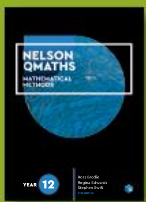


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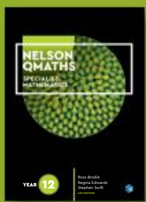


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