PEARSON

GENERAL MATHEMATICS

QUEENSLAND

STUDENT BOOK

12

UNITS 3 & 4

QCE 2019 SYLLABUS

PEARSON

GENERAL MATHEMATICS

QUEENSLAND

STUDENT BOOK

12

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Pearson

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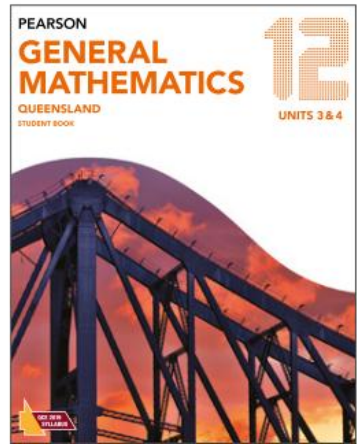

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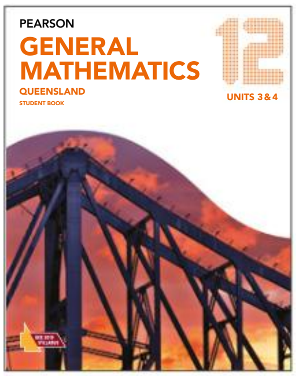


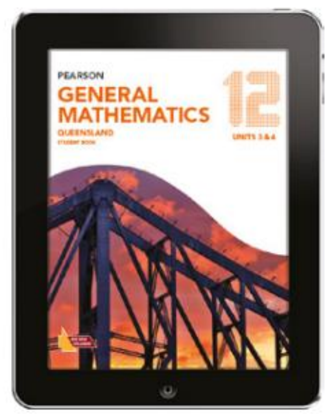
General Mathematics 12
Student book

Student book

The student book has been written by local authors, ensuring quality content and complete curriculum coverage for Queensland, enabling students to prepare with ease and confidence. We have covered the breadth of the content within our exercise questions, from simpler skills-focused questions to those using unfamiliar contexts and application of the theory learnt. The theory, worked examples and question sets are written in line with the assessment objectives, with the aim of familiarising students with QCE cognitive verbs in the process of dependent and guided instruction. Additional interactives that help explain the theory and consolidate concepts have been included throughout all chapters.

Pearson Reader+

Pearson Reader+ is our next-generation eBook. This is an electronic textbook that students can access on any device, online or offline, and is linked to features, interactives and visual media that will help consolidate their understanding of concepts and ideas, as well as other useful content developed specifically for senior mathematics. It supports students with appropriate online resources and tools for every section of the student book, providing access to exemplar worked solutions that demonstrate high levels of mathematical and everyday communication. Students will have the opportunity to learn independently through the Explore further tasks, which have been designed to engage and support conceptual understanding. Additionally, teachers have access to syllabus maps, a teaching program, sample exams, problem-solving and modelling tasks, and additional banks of questions for extra revision.



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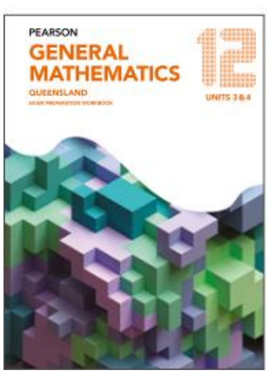


General Mathematics 12
Exam preparation workbook

Exam preparation workbook

Additional component for Year 12 only

The exam preparation workbook provides additional support in preparing students for the external exam. It has been constructed to guide the students through a sequence of preparatory steps and build confidence leading up to the external exam.

How to use this book

Pearson General Mathematics 12 Queensland Units 3 & 4

This Queensland senior mathematics series has been written by a team of experienced Queensland teachers for the QCE 2019 syllabus. It offers complete curriculum coverage, rich content and comprehensive teacher support.

5 Generate the terms by starting with the first term a and adding the difference d to determine the value of each subsequent term.	$t_1 = 23$ $t_2 = 23 + 13 = 36$ $t_3 = 36 + 13 = 49$ $t_4 = 49 + 13 = 62$
6 Interpret the answer.	The first four terms in the sequence are 23, 36, 49, 62.

Arithmetic mean

For the arithmetic sequence: … 5, 11, 17,…

11 is the mean of 5, 11 and 17.

$$\frac{5+11+17}{3} = \frac{33}{3} = 11$$

Also, 11 is the mean of 5 and 17.

$$\frac{5+17}{2} = \frac{22}{2} = 11$$

The arithmetic mean of three terms in an arithmetic sequence is the middle term.

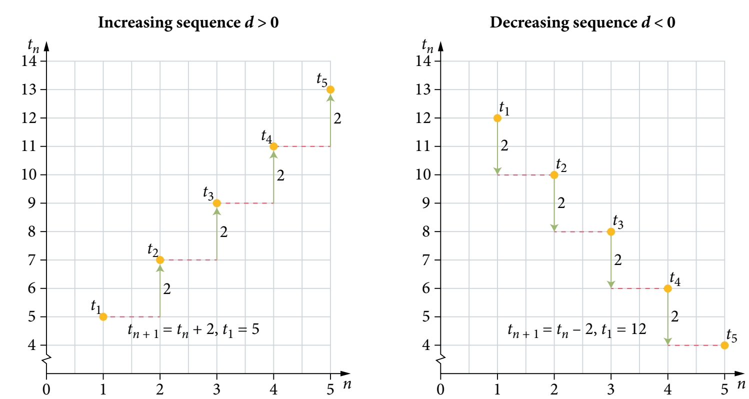
The middle term is also the mean of the terms on either side.

Key information

Key information and rules are highlighted throughout the chapter.

Graphs of arithmetic sequences

Graphs of arithmetic sequences are linear—they rise (or fall) at a steady rate, following a single straight line.



Explore further

Arithmetic sequences and their graphs

Use a spreadsheet to generate and graph arithmetic sequences.

Explore further

This eBook feature provides an opportunity for students to consolidate their understanding of concepts and ideas with the aid of technology, and answer a small number of questions to deepen their understanding and broaden their skills base. These activities should take approximately 5–15 minutes to complete.

6 Determining the value of the nth term

The number of visitors to a park during a fortnight of school holidays begins with 2000 on Monday and increases by 250 each day.

How many people visit the park on the 8th day (the second Monday) of the school holidays?

THINKING	WORKING
1 Recognise the arithmetic sequence and identify the first term a, common difference d and number of terms n.	Arithmetic sequence: $t_1 = 2000$, $d = 250$, $n = 8$
2 Recall the arithmetic term formula. Substitute values for a, d and n to determine the value of the 8th term.	$t_n = t_1 + (n-1)d$ $t_8 = 2000 + (8-1) \times 250 = 3750$
3 Interpret the answer.	On the 8th day, 3750 people visit the park.

Starting an arithmetic sequence with $n = 0$

In many situations, instead of beginning with $n = 1$ and 'term 1', t_1, it is appropriate to begin with $n = 0$, and designate an initial 'term 0', as this makes it easier to match term number to time elapsed after the initial term. For example, t_n could be the value after n weeks. In this case, the general term of the arithmetic sequence is given by $t_n = a + nd$.

If the first term of an arithmetic sequence is $t_0 = a$:
$t_n = a + nd$, and there are $n + 1$ terms in the sequence.

Additional information
Scatter plots
Try the activity, to practise identifying relationships between variables.

Every worked example and question is graded

Every example and question is graded using the three levels of difficulty, as specified in the QCE syllabus:

- simple familiar (1 bar)
- complex familiar (2 bars)
- complex unfamiliar (3 bars).

The visibility of this grading helps ensure all levels of difficulty are well covered.

Meeting the needs of the QCE Syllabus

The authors have integrated both the cognitive verbs and the language of the syllabus objectives throughout the worked examples and questions.

Additional information

These interactives appear in the eBook in two forms, as videos explaining specific concepts or as interactive questions to check students' understanding.

2 Ella bought a car for $14 000. She paid a cash deposit of $2200 and borrowed the balance at an interest rate of 8.48% p.a. compounding every 6 months. Ella intends to pay off the loan and all added interest in 2 or 3 years.

(a) Which recurrence relation models the value of Ella's loan balance at the end of each half year?

A $E_0 = 11800,\ E_{n+1} = 1.0424 \times E_n$ B $E_0 = 14000,\ E_{n+1} = 1.0424 \times E_n$

C $E_0 = 11800,\ E_{n+1} = 1.0848 \times E_n$ D $E_0 = 14000,\ E_{n+1} = 1.0848 \times E_n$

(b) Explain the common error made by a student who had a growth rate of 1.0848 in their solution.

Highlighting common errors

Throughout the exercises, authors have integrated questions designed to highlight common errors frequently made by students. Explanations are given in the worked solutions.

Worked solutions

Fully worked solutions are provided for every question in the student textbook and can be accessed from the accompanying eBook.

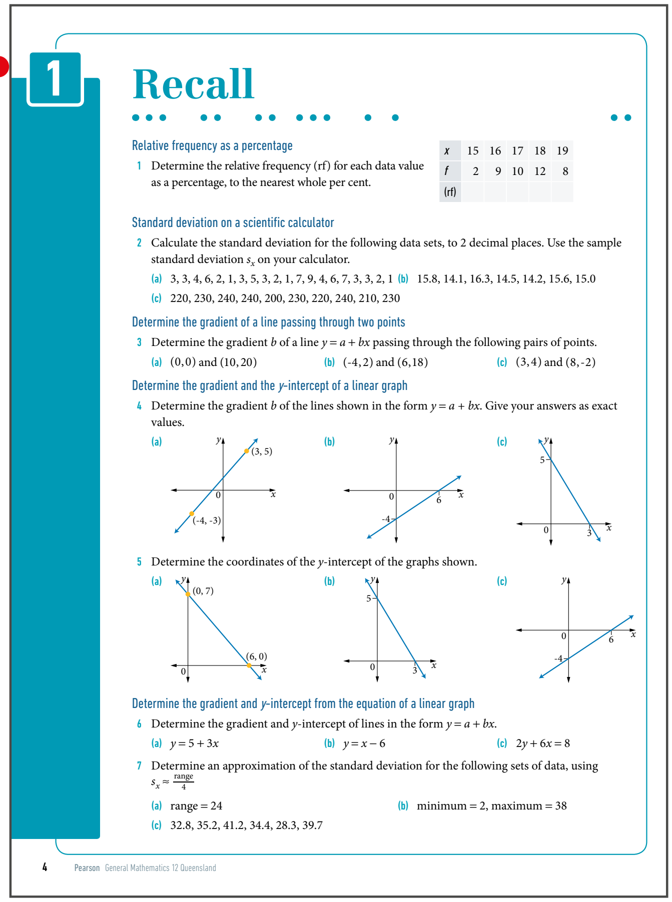
WARNING
If the angular distance is entered using the degrees/minutes/seconds key, the value of D will appear in degrees/minutes/seconds format. To determine the distance in kilometres, convert the value to decimal format at the end.

Warning boxes

Warning boxes are located throughout the chapter to alert students to common errors and misconceptions.

Recall

Each chapter begins with a review of assumed knowledge for the chapter.



1 Recall

Relative frequency as a percentage

1 Determine the relative frequency (rf) for each data value as a percentage, to the nearest whole per cent.

x	15	16	17	18	19
f	2	9	10	12	8
(rf)					

Standard deviation on a scientific calculator

2 Calculate the standard deviation for the following data sets, to 2 decimal places. Use the sample standard deviation s_x on your calculator.

(a) 3, 3, 4, 6, 2, 1, 3, 5, 3, 2, 1, 7, 9, 4, 6, 7, 3, 3, 2, 1 (b) 15.8, 14.1, 16.3, 14.5, 14.2, 15.6, 15.0

(c) 220, 230, 240, 240, 200, 230, 220, 240, 210, 230

Determine the gradient of a line passing through two points

3 Determine the gradient b of a line $y = a + bx$ passing through the following pairs of points.

(a) (0,0) and (10,20) (b) (-4,2) and (6,18) (c) (3,4) and (8,-2)

Determine the gradient and the y-intercept of a linear graph

4 Determine the gradient b of the lines shown in the form $y = a + bx$. Give your answers as exact values.

(a) (b) (c)

5 Determine the coordinates of the y-intercept of the graphs shown.

(a) (b) (c)

Determine the gradient and y-intercept from the equation of a linear graph

6 Determine the gradient and y-intercept of lines in the form $y = a + bx$.

(a) $y = 5 + 3x$ (b) $y = x - 6$ (c) $2y + 6x = 8$

7 Determine an approximation of the standard deviation for the following sets of data, using $s_x \approx \frac{\text{range}}{4}$

(a) range = 24 (b) minimum = 2, maximum = 38

(c) 32.8, 35.2, 41.2, 34.4, 28.3, 39.7

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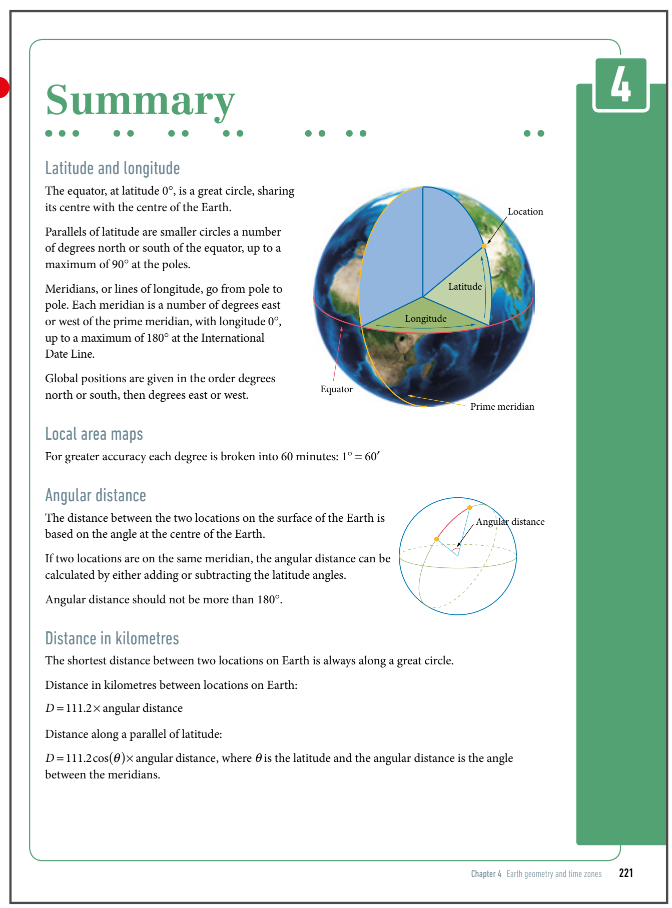
Summary

At the end of each chapter, there is a summary of the key facts and rules discussed in the chapter.



Summary 4

Latitude and longitude

The equator, at latitude 0°, is a great circle, sharing its centre with the centre of the Earth.

Parallels of latitude are smaller circles a number of degrees north or south of the equator, up to a maximum of 90° at the poles.

Meridians, or lines of longitude, go from pole to pole. Each meridian is a number of degrees east or west of the prime meridian, with longitude 0°, up to a maximum of 180° at the International Date Line.

Global positions are given in the order degrees north or south, then degrees east or west.

Local area maps

For greater accuracy each degree is broken into 60 minutes: $1° = 60'$

Angular distance

The distance between the two locations on the surface of the Earth is based on the angle at the centre of the Earth.

If two locations are on the same meridian, the angular distance can be calculated by either adding or subtracting the latitude angles.

Angular distance should not be more than 180°.

Distance in kilometres

The shortest distance between two locations on Earth is always along a great circle.

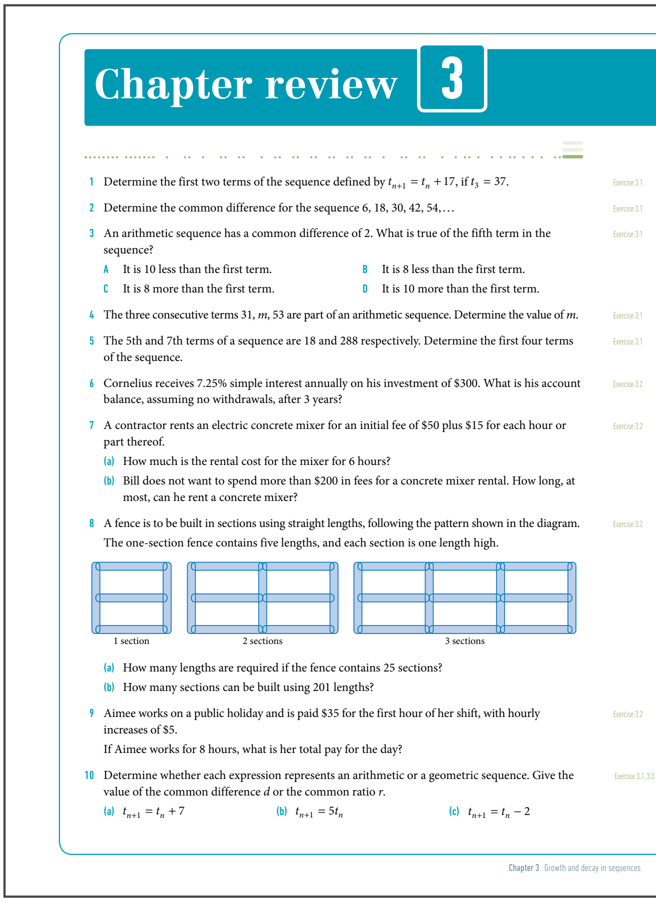
Distance in kilometres between locations on Earth:

$D = 111.2 \times$ angular distance

Distance along a parallel of latitude:

$D = 111.2\cos(\theta) \times$ angular distance, where θ is the latitude and the angular distance is the angle between the meridians.

Chapter 4 Earth geometry and time zones 221



Chapter review 3

1 Determine the first two terms of the sequence defined by $t_{n+1} = t_n + 17$, if $t_3 = 37$. Exercise 3.1

2 Determine the common difference for the sequence 6, 18, 30, 42, 54, … Exercise 3.1

3 An arithmetic sequence has a common difference of 2. What is true of the fifth term in the sequence? Exercise 3.1

A It is 10 less than the first term.
B It is 8 less than the first term.
C It is 8 more than the first term.
D It is 10 more than the first term.

4 The three consecutive terms 31, m, 53 are part of an arithmetic sequence. Determine the value of m. Exercise 3.1

5 The 5th and 7th terms of a sequence are 18 and 288 respectively. Determine the first four terms of the sequence. Exercise 3.1

6 Cornelius receives 7.25% simple interest annually on his investment of $300. What is his account balance, assuming no withdrawals, after 3 years? Exercise 3.2

7 A contractor rents an electric concrete mixer for an initial fee of $50 plus $15 for each hour or part thereof. Exercise 3.2

(a) How much is the rental cost for the mixer for 6 hours?
(b) Bill does not want to spend more than $200 in fees for a concrete mixer rental. How long, at most, can he rent a concrete mixer?

8 A fence is to be built in sections using straight lengths, following the pattern shown in the diagram. The one-section fence contains five lengths, and each section is one length high. Exercise 3.2

1 section 2 sections 3 sections

(a) How many lengths are required if the fence contains 25 sections?
(b) How many sections can be built using 201 lengths?

9 Aimee works on a public holiday and is paid $35 for the first hour of her shift, with hourly increases of $5. Exercise 3.2
If Aimee works for 8 hours, what is her total pay for the day?

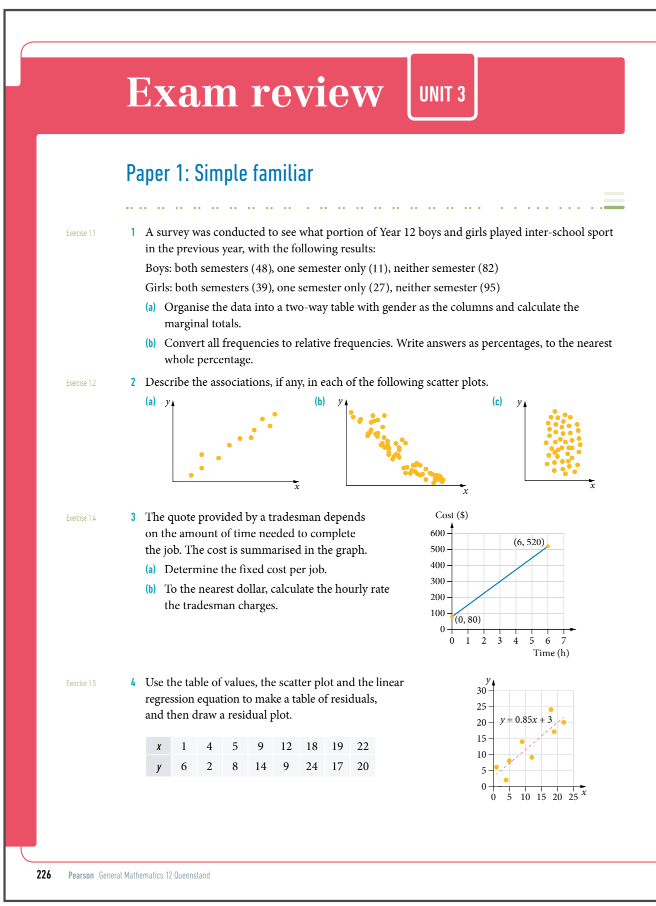
10 Determine whether each expression represents an arithmetic or a geometric sequence. Give the value of the common difference d or the common ratio r. Exercise 3.1, 3.3

(a) $t_{n+1} = t_n + 7$ (b) $t_{n+1} = 5t_n$ (c) $t_{n+1} = t_n - 2$

Chapter 3 Growth and decay in sequences

Chapter review

Every chapter review follows the QCAA examination proportions for levels of difficulty, which is 60% simple familiar, 20% complex familiar and 20% complex unfamiliar.



Exam review UNIT 3

Paper 1: Simple familiar

Exercise 1.1 1 A survey was conducted to see what portion of Year 12 boys and girls played inter-school sport in the previous year, with the following results:
Boys: both semesters (48), one semester only (11), neither semester (82)
Girls: both semesters (39), one semester only (27), neither semester (95)

(a) Organise the data into a two-way table with gender as the columns and calculate the marginal totals.
(b) Convert all frequencies to relative frequencies. Write answers as percentages, to the nearest whole percentage.

Exercise 1.2 2 Describe the associations, if any, in each of the following scatter plots.

(a) (b) (c)

Exercise 1.4 3 The quote provided by a tradesman depends on the amount of time needed to complete the job. The cost is summarised in the graph.

(a) Determine the fixed cost per job.
(b) To the nearest dollar, calculate the hourly rate the tradesman charges.

Exercise 1.5 4 Use the table of values, the scatter plot and the linear regression equation to make a table of residuals, and then draw a residual plot.

x	1	4	5	9	12	18	19	22
y	6	2	8	14	9	24	17	20

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Exam review

Exam reviews provide cumulative practice of content already covered, to prepare students for the end-of-year exam. They have been placed at the end of each unit.

1

Bivariate data analysis

1 Recall

Relative frequency as a percentage

1 Determine the relative frequency (rf) for each data value as a percentage, to the nearest whole per cent.

x	15	16	17	18	19
f	2	9	10	12	8
(rf)					

Standard deviation on a scientific calculator

2 Calculate the standard deviation for the following data sets, to 2 decimal places. Use the sample standard deviation s_x on your calculator.

(a) 3, 3, 4, 6, 2, 1, 3, 5, 3, 2, 1, 7, 9, 4, 6, 7, 3, 3, 2, 1 (b) 15.8, 14.1, 16.3, 14.5, 14.2, 15.6, 15.0

(c) 220, 230, 240, 240, 200, 230, 220, 240, 210, 230

Determine the gradient of a line passing through two points

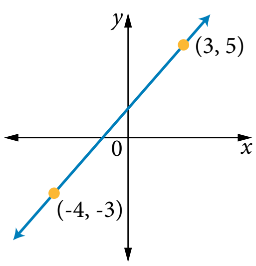
3 Determine the gradient b of a line $y = a + bx$ passing through the following pairs of points.

(a) $(0, 0)$ and $(10, 20)$ (b) $(-4, 2)$ and $(6, 18)$ (c) $(3, 4)$ and $(8, -2)$

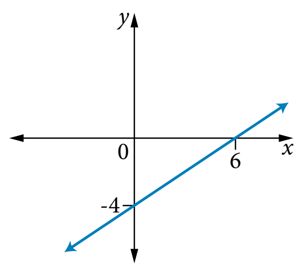
Determine the gradient and the y-intercept of a linear graph

4 Determine the gradient b of the lines shown in the form $y = a + bx$. Give your answers as exact values.

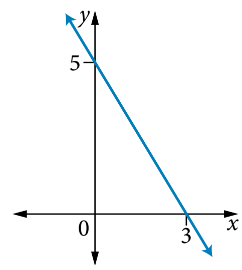
(a)



(b)

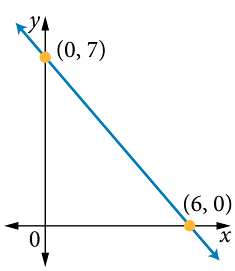


(c)

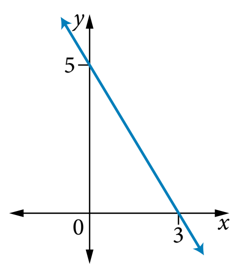


5 Determine the coordinates of the y-intercept of the graphs shown.

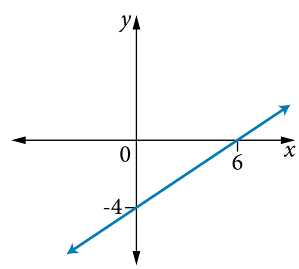
(a)



(b)



(c)



Determine the gradient and y-intercept from the equation of a linear graph

6 Determine the gradient and y-intercept of lines in the form $y = a + bx$.

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(a) range = 24 (b) minimum = 2, maximum = 38

(c) 32.8, 35.2, 41.2, 34.4, 28.3, 39.7

Association between categorical variables 1.1

Bivariate data

Bivariate data is data that comes in pairs. Analysis of bivariate data consists of collecting and pairing data on two variables and then using the data in comparative ways to determine whether a relationship exists between the variables. If a relationship does exist, then the type and strength of the relationship are determined.

Displaying categorical data

For categorical data to be bivariate, the data must be categorised in two ways. A two-way table is a convenient way to display the data. Totals for each row and column, sometimes called *marginal totals*, enable you to calculate the percentage, or proportion, of the whole represented in each subcategory.

1 Constructing a two-way table

A survey is conducted in which young people are asked to choose their favourite film genre from a given list. Each participant's gender is recorded with their response. The results are given below.

Males: Horror (5), Comedy (16), Action (22), Science fiction (16), Romantic comedy (2), Animation (0), Thriller (18)

Females: Horror (18), Comedy (20), Action (4), Science fiction (16), Romantic comedy (48), Animation (30), Thriller (32)

(a) Organise the data into a two-way table with gender as the columns.

THINKING

The categories for one variable go along the top. The categories for the other variable go in a column at the left.

Fill in the numbers.

WORKING

	Males	Females
Horror	5	18
Comedy	16	20
Action	22	4
Science fiction	16	16
Romantic comedy	2	48
Animation	0	30
Thriller	18	32

(b) Calculate the marginal totals.

1 Sum each row and column.

Write in the grand total, by adding either the row totals or the column totals.

	Males	Females	Total
Horror	5	18	23
Comedy	16	20	36
Action	22	4	26
Science fiction	16	16	32
Romantic comedy	2	48	50
Animation	0	30	30
Thriller	18	32	50
Total	79	168	247

2 Interpret the result.

The survey was completed by 247 individuals.

(c) Convert all frequencies to relative frequencies. Write answers as percentages to the nearest whole per cent.

1 Divide each value in the table by the total number of participants surveyed. To convert to a percentage, multiply by 100 and attach the percentage symbol. Round answers as directed.

Replace the numbers with the percentages.

As an example: $\frac{5}{247} \times 100\% = 2\% \,(\text{nearest } \%)$

	Males	Females	Total
Horror	2%	7%	9%
Comedy	6%	8%	~15%
Action	9%	2%	11%
Science fiction	6%	6%	~13%
Romantic comedy	1%	19%	20%
Animation	0%	12%	12%
Thriller	7%	13%	20%
Total	~32%	~68%	100%

2 Interpret the results.

The table shows the percentage of total participants in each category. For example, 9% of respondents prefer horror movies, 15% prefer comedy etc.

32% of respondents are male and 68% are female.

Explore further

Two-way tables

Create a two-way table and then calculate and interpret values using a spreadsheet.

WARNING

The sum of the rows and columns within a percentage frequency table may not always give an accurate reading, due to rounding values throughout the table.

Percentaged two-way tables

The unequal number of males and females in the survey (the total number of entries in each male and female column) makes it difficult to see any difference or similarity that may exist in the answers for the two genders.

If each column is converted to percentages for just that column, so that the total of each column is 100%, a more meaningful analysis of association can be performed.

The differences between the percentages are more clearly seen here. You can see the effect of the variable – gender – on the relative frequencies for each of the movie genre choices. Male participants preferred 'Comedy', 'Action', 'Science fiction' and 'Thriller'. Females predominantly chose 'Romantic comedy', 'Animation' and 'Thriller' films.

	Males (%)	Females (%)
Horror	$\frac{5}{79} \times 100\% \approx 6$	$\frac{18}{168} \times 100\% \approx 11$
Comedy	20	12
Action	28	2
Science fiction	20	9
Romantic comedy	3	29
Animation	0	18
Thriller	23	19
Total	100	100

2 Using a spreadsheet to create a percentaged two-way table

The favourite superheroes of a group of 70 people are shown in the table.

	Spiderman	Batman	Green Lantern	The Hulk	Captain America	Thor	Iron Man
Males	1	7	2	4	2	8	15
Females	6	14	0	3	1	2	5

Using a spreadsheet, produce an appropriately percentaged two-way table so that an analysis of the effect of gender on the choice of superhero can be carried out.

THINKING	WORKING
1 Hypothesise to determine which variable you should sum.	The choice of superhero could be influenced by gender. Determine the total for each gender.
2 Enter the data with more rows than columns, for convenience. Calculate the total number of each column. For example, in cell B9 insert the formula =SUM(B2:B8).	(spreadsheet below)

	A	B	C
1		Males	Females
2	Spiderman	1	6
3	Batman	7	14
4	Green Lantern	2	0
5	The Hulk	4	3
6	Captain America	2	1
7	Thor	8	2
8	Iron Man	15	5
9	Total	39	31

3 In columns D and E, create percentages, to the nearest whole percentage, by dividing each frequency value by the total for that category – here, males and females.

For the spreadsheet method, for the Spiderman, Males (%) cell, insert the formula =B2/B$9. The $ symbol ensures that each value is divided by the same total, B9. Then drag the cross-hair to fill the column.

In cell E2 insert the formula =C2/C$9.

	A	D	E
1		Males (%)	Females (%)
2	Spiderman	3	19
3	Batman	18	45
4	Green Lantern	5	0
5	The Hulk	10	10
6	Captain America	5	3
7	Thor	21	6
8	Iron Man	38	16
9	Total	100	100

4 Interpret the results.

Of the two most popular superheroes, Batman (21 people) and Iron Man (20), 38% of male respondents chose Iron Man as their favourite superhero, while 45% of female respondents chose Batman as a clear favourite superhero.

Describing an association

In any analysis of association:

STEP 1: Decide on the variable that may have influence over the other variable.

STEP 2: Calculate percentages of the variable with influence.

STEP 3: Describe the association in terms of differences in the percentages across the categories.

STEP 4: Interpret the observations in the context of the data.

3 Describing an association from a percentaged two-way table

The following data represents the results obtained on a particular test (marked out of 40), taken by two groups: Group A (students 16 years of age) and Group B (students 18 years of age). The pass mark for the test is 25.

Group A: 37, 26, 31, 23, 34, 38, 29, 17, 33, 26, 23, 21, 29, 30, 33

Group B: 24, 28, 29, 34, 18, 19, 32, 29, 37, 28

Determine whether there is an association between age and the likelihood of passing the test.

THINKING

1 Create a two-way table using the categories described.

Count the number of figures in each category and place the frequency in the relevant column.

Total the frequencies for each of the categories that might influence the outcome.

WORKING

	Pass (≥25 marks)	Fail (<25 marks)	Total
Group A	13	2	15
Group B	7	3	10

2 Convert each frequency to a percentage of the total in each row.

Redraw the table to show the percentages.

	Pass	Fail	Total
Group A	$\frac{13}{15} \times 100\% \approx 87\%$	$\frac{2}{15} \times 100\% \approx 13\%$	100%
Group B	$\frac{7}{10} \times 100\% = 70\%$	$\frac{3}{10} \times 100\% = 30\%$	100%

3 Describe any associations you can identify from the data, giving values and context.

87% of 16-year-olds passed, compared with 70% of 18-year-olds.

It is important to note that this does not lead to the conclusion that younger people are more likely to pass the test. There are likely to be a variety of factors that influence a student's chance of success, including class size, preparation time, test conditions and teacher input.

EXERCISE 1.1 Association between categorical variables

Worked Example

1 An excursion is arranged for students in Year 12 and Year 7 to see a movie together. The students have a choice of popcorn, fries or a choc top to take into the movie with them. The teachers keep a record of the orders, for a later comparison of movie snack choices at different ages: 1

Year 12: popcorn (35), fries (21), choc tops (11)

Year 7: popcorn (40), fries (26), choc tops (50)

(a) Organise the data into a two-way table and calculate the marginal totals.

(b) Convert all frequencies to relative frequencies. Write answers as percentages, to the nearest whole per cent.

2 The manager of a cafe offers a choice of chips, vegetables or salad with each meal. She asks her staff to keep a record of the choices made by different age groups. Here are their findings:

Children: chips (20), vegetables (2), salad (5)

Teenagers: chips (31), vegetables (1) , salad (20)

Adults: chips (10), vegetables (28), salad (63)

(a) Organise the data into a two-way table and calculate the marginal totals.

(b) Calculate each of the following, to the nearest per cent.

(i) the percentage of customers who ordered chips

(ii) the percentage of teenage customers

(iii) the percentage of teenagers who ordered salad

(iv) the difference between the percentage of children who ordered chips and the percentage of adults who ordered chips.

3 For which of the following sets of survey data would you expect very little (or no) association?

A Year 12 and Year 8 students, and amounts of study (small, moderate, large)

B mode of travel to the movies, and age group

C preference for video games or going to the movies, and gender

D ability in Drama, and preference for blue or red

Worked Example 2

4 A group of males and females were asked: 'Which of the following movie villains do you like the most?' Their responses are summarised in the table.

	Lex Luthor	Goldfinger	The Penguin	The Joker	KAOS
Males	46	10	80	40	12
Females	84	22	60	55	28

(a) Produce an appropriately percentaged two-way table.

(b) Which villain is most popular with each gender?

5 A class of Year 12 students spent their Friday Maths lesson breaking open coloured choc-coated peanuts and sultanas to determine whether the outside colour had some influence on whether the chocolate contained a peanut or a sultana. Their findings are shown in the table.

	Blue	Brown	Yellow	White
Peanut	20	24	15	27
Sultana	14	28	26	26

(a) Produce an appropriately percentaged two-way table. Round your answers to the nearest per cent each time.

(b) Which colour was most likely to contain a peanut?

6 An ice-cream vendor wanted to know whether people who chose to eat their ice-cream from a cone were more likely to choose vanilla than those ordering ice-cream in a cup. His findings are shown.

	Vanilla	Other flavour
Cone	165	206
Cup	43	75

(a) Produce an appropriately percentaged two-way table. Round your answers to the nearest per cent each time.

(b) Determine the percentage difference for choosing vanilla in a cone compared to a cup.

(c) The ice-cream vendor wishes to explore proportions among vanilla ice-cream eaters. Determine the proportions of vanilla ice-cream customers eating from cones and from cups.

7 For the data in the tables, assume that categories X and Y are expected to affect the frequencies within the categories A, B and C.

I

	A	B	C
X	50%	30%	20%
Y	40%	40%	20%

II

	A	B	C
X	80%	60%	30%
Y	20%	40%	70%

III

	A	B	C
X	2	15	20
Y	18	15	0

IV

	A	B	C
X	16	10	14
Y	12	19	9

(a) For which of the data can association be explored without further calculation?

A I only B II only C I and II D I and IV

(b) Explain the common error made by a student who chose an incorrect option in part (a).

Worked Example

3

8 The following data about travel time for Year 7 and Year 12 students was collected, to determine whether there is an association between the year level of a student and the time taken to travel to school.

(a) Convert the table to a percentage-based two-way frequency diagram using the travel time categories < 30 minutes and ≥ 30 minutes. Give percentages to the nearest whole per cent.

(b) Use percentage difference to describe the association between year level and travel time to school.

Travel time (minutes)	Number of Year 7 students	Number of Year 12 students
0–<10	25	2
10–<20	40	8
20–<30	24	25
30–<40	12	55
40–<50	4	24
50–<60	2	6

9 The table shows the number of people of various ages with high blood pressure (greater than 140/90 mmHg). The sample size for each age group is 100 males and 100 females.

(a) In which age group does the percentage of women with high blood pressure become higher than the percentage of men with high blood pressure?

(b) Is it reasonable to say that 12% of people aged 18–24 years old have high blood pressure?

Age group	Males	Females
18–24	7	5
25–34	13	4
35–44	19	11
45–54	29	22
55–64	33	27
65–74	37	41
>74	42	52

Source ABS: Australian Health Survey First results

10 Vitamin C is thought by some people to help ward off colds, or at least shorten the length of a cold. In order to test this theory, a Year 12 Biology class conducted a test with 250 Year 11 and 12 students at their school. A controlled experiment was conducted by giving half the students vitamin C tablets to take for the winter and the other half, a placebo, sugar tablets that looked the same as the vitamin C tablets. Students were not told which type of tablet they had been given.

The results of the study are shown in the two-way table.

	Cold	No cold
Sugar tablet	29	96
Vitamin C tablet	24	101

(a) Convert this table to an appropriate percentaged table for investigating whether there is an association between taking vitamin C and the likelihood of getting a cold. Give answers to the nearest whole per cent.

(b) Express any association between taking vitamin C and the likelihood of getting a cold in terms of percentage difference.

11 The table shows the results of a survey of 200 shoppers about a proposal to change the arrangements for childcare at a shopping centre.

(a) Use an appropriately percentaged table to determine whether a person's gender is associated with their likelihood of agreeing with the proposal. Round percentages to the nearest whole per cent.

(b) Express any association between gender and agreement with the proposal in terms of percentage difference.

	Males	Females	Total
Agree	25	88	113
Disagree	45	42	87
Total	70	130	200

12 The data shown in the table represents a group of men and women surveyed and categorised as overweight to obese, or normal to underweight. They were then put into the following subgroups: diabetic, pre-diabetic or within the healthy range (in relation to diabetes).

Data was collected for 800 males and females aged between 24 and 34.

	Overweight to obese		Normal to underweight		Total
	Males	Females	Males	Females	
Diabetic	14	8	20	10	52
Pre-diabetic	9	7	13	5	34
Healthy range	77	105	267	265	714
Total	100	120	300	280	800

(a) Construct a two-way frequency table for the 'Overweight to obese' data by adding a column showing the percentage for males, females and total. Give your answers to the nearest per cent.

(b) Construct a two-way frequency table for the 'Normal to underweight' data by adding a column showing the percentage for males, females and total. Give your answers to the nearest per cent.

(c) What association can you see between the risk of diabetes for the two groups of people?

13 A group of 100 tennis players were observed serving a tennis ball. The table shows the number of unreturnable serves (aces) per set for tennis players with a fast serve, compared to those with a slow serve.

	0 aces	1 ace	2 aces	3 aces	4 aces	Total
Slow serve	7		8	4	0	27
Fast serve	1	8				
Total			22	34		100

(a) Complete the table by using the data provided to determine the missing values.

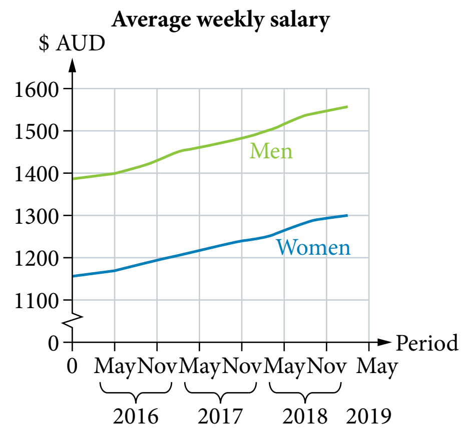
(b) How many players served three aces?

(c) Calculate the percentage of fast and slow servers who served more than two aces. Give your answers to the nearest per cent.

(d) Describe the association observed between the speed of a serve and the number of aces.

14 Information about the average weekly salaries ($AUD) of men and women measured every six months from May 2016 to May 2019 is shown in the graph.

Describe the association between the average weekly salaries of men and women in Australia. Support your conclusions with values, stating the strengths and limitations of your result.



15 The lengths of rivers longer than 100 km in the North and South Islands of New Zealand are listed below.

North Island (km): 172, 290, 425, 193, 105, 175, 241, 161, 158, 154, 182, 137, 143, 132, 119, 137

South Island (km): 209, 169, 288, 209, 138, 121, 177, 161, 322, 145, 121, 108

The approximate areas of the North and South Islands are 114 000 km^2 and 150 000 km^2 respectively.

Investigate whether there is an association between the size of the island and the number of rivers of a particular length.

(a) Organise the data and write your observations for rivers longer than 100 km, 150 km and 200 km.

(b) What do you conclude about there being an association between the area of the island and the number of rivers of particular lengths?

Association between numerical variables 1.2

Comparing numerical data sets

In this section you will examine how to compare two numerical data sets.

It is common for two data sets to have an association where one variable may influence the other. For example, if you were to measure a random group of people's heights and weights you would probably find a positive association between height and weight. The assumed association would be that taller people are more likely to be heavier (as opposed to heavier people being more likely to be tall), because a person's height is independent of how much they weigh.

Scatter plots

A **scatter plot** provides a visual representation of any trend or underlying pattern in the data. A scatter plot is drawn by treating the bivariate data as a series of coordinate pairs and plotting the pairs on a suitable set of axes.

The data used often comes from measurements in real-life situations, so the graph is quite often limited to the positive part of each scale or the first quadrant on the Cartesian plane.

To construct a scatter plot:

- Draw a suitable set of axes with a consistent scale {0, 1, 2, 3…} by noting the highest and lowest value from each set of figures.
- Plot each data pair as a coordinate point.

WARNING

For a scatter plot:

- Do not start at zero on any axis where the values are clustered away from zero.
- Do not join the plotted points.

Explore further

Scatter plots and the line of best fit

Use a spreadsheet to construct a scatter plot, and insert a line of best fit.

4 Constructing a scatter plot

Construct a scatter plot of the following bivariate data set and describe any trend you see.

Person	A	B	C	D	E	F	G	H
Height (cm)	140	160	180	90	100	50	60	120
Weight (kg)	60	75	95	40	50	20	35	65

THINKING

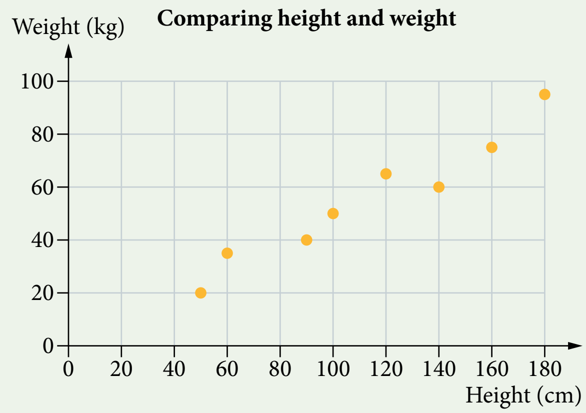
WORKING

1 Draw a suitable set of axes by noting the range of each set of figures.

Plot the points.

In this case, height (cm) is from 50 to 180, so 9 divisions of 20 starting at zero is appropriate.

Weight (kg) varies from 20 to 95, so 5 divisions of 20 starting at zero is appropriate.



2 Comment on any pattern or trend observed.

From the graph, it appears that the taller the person is, the heavier they are.

Explanatory and response variables

Often one of the variables can explain the association, and this is known as the independent or *explanatory variable*. The other variable responds to a change in the explanatory variable and is known as the dependent or *response variable*. In the case of height versus weight, a person's weight is more likely explained by their height than the other way around.

The **explanatory (or independent) variable** is placed on the horizontal axis.

The **response (or dependent) variable** is placed on the vertical axis.

When a saucepan of water is placed on the stove to boil, the water's temperature responds to, or depends upon, the time spent on the hot plate.

When an ice-cream vendor sets up a booth near the beach, the number of ice creams sold will likely respond to the temperature of the day.

Sometimes two variables may both be dependent on other variables. For example, a student's study score in English and Music are related but both depend on other variables such as effort, hours of practice, teacher input and study.

Sometimes variables may have no association at all. For example, a student's study score in English is not related to the length of their hair.

WARNING

Not all bivariate data sets have clear explanatory and response variables.

5 Identifying the explanatory variable

Assuming an association exists, identify the explanatory variable between each of the following pairs:

(i) age and wealth

(ii) age and the number of offspring

(iii) temperature of the day and the number of people at the beach

(iv) number of cigarettes smoked and the chance of getting cancer

(v) volume of petrol remaining in a car's tank and the distance the car has been driven.

THINKING	WORKING
Determine which one of the variables could cause or explain a change in the other.	(i) Age and wealth: The explanatory variable is age. In general, a longer working life is likely to explain an increase in wealth. (ii) Age and the number of offspring: The explanatory variable is age. It takes time to reproduce, so a longer life span is likely to explain an increase in the number of offspring produced by a female. (iii) Temperature of the day and the number of people at the beach: The explanatory variable is temperature. In general, an increase in temperature is likely to explain an increasing beach population. (iv) Number of cigarettes smoked and the chance of getting cancer: The explanatory variable is number of cigarettes smoked. In general, a greater number of cigarettes smoked is likely to explain a higher likelihood of cancer. (v) Volume of petrol remaining in a car's tank and the distance the car has been driven: The explanatory variable is the distance driven. An increasing distance driven explains a decreasing volume of petrol in the tank.

Additional information

Scatter plots

Try the activity, to practise identifying relationships between variables.

Linear trend

A single line that best represents the general pattern of the data is called a *line of best fit*.

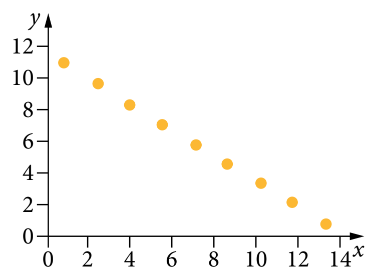
If the pattern is roughly a straight line, a linear association exists.

The trend indicates the general direction of the association. The pattern may show a positive, increasing trend (increasing as you move to the right ↗), meaning that as one variable increases the other also increases. Or it may show a negative, decreasing trend (decreasing as you move to the right ↘), meaning that as one variable increases the other decreases.

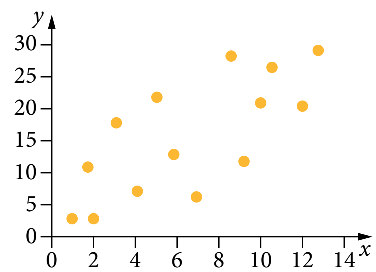
When analysing the scatter plot, look for the following characteristics:

- Is there is an observable pattern?
- Does the pattern show a linear form or non-linear form?
- Give the direction of the slope as positive or negative.
- Identify any outliers – that is, single points that seem to be well outside the general pattern of the rest of the data. If outliers are excluded from the data, the pattern will be easier to see.
- Consider the strength of the association, how the points are spread around and represent the data. A good fit represents a strong trend; a poor fit represents a weak trend.

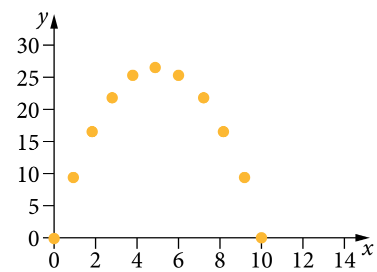
Examples are as follows:



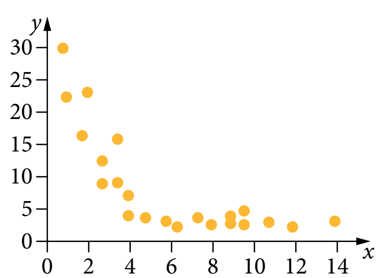
A strong negative linear association



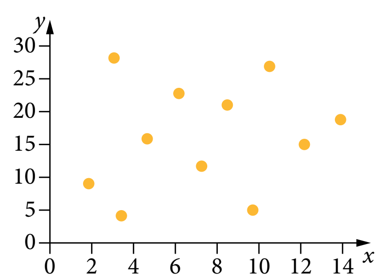
A weak positive linear association



A strong non-linear association



A moderate negative non-linear association

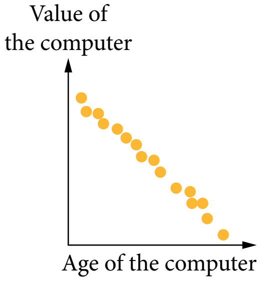


No association

6 Describing associations

Describe the associations, if any, between the variables represented in the following scatter plots.

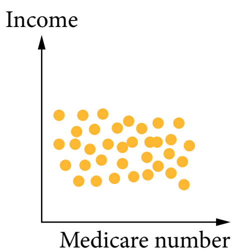
(a)



> **Additional information**
>
> **Identifying correlation**
>
> Try the activity, to practise determining whether scatter plots show a positive or negative correlation.

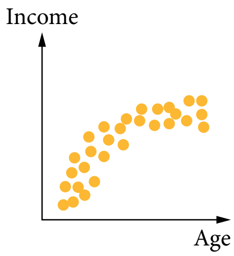
THINKING	WORKING
1 Identify whether a pattern exists. If there is a pattern, describe the strength of the association as strong, moderate or weak.	There is a pattern: the plotted points form a distinct line, so there is a strong association between the two variables.
2 Check for linearity.	The plotted points form a straight line, so the association is linear.
3 Determine the direction of the slope to be positive or negative.	The slope is negative, that is, a decreasing trend; as the explanatory variable increases, the response variable decreases.
4 Interpret the slope.	As a computer ages, the value of the computer decreases.
5 Describe the association.	The association between the age of the computer and the value of the computer is strong, negative and linear.

(b)



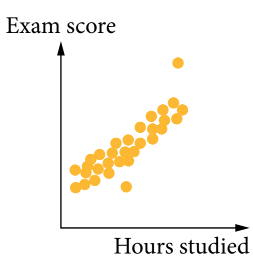
1 Identify whether a pattern exists.	No pattern is evident in the data, suggesting that no relationship exists between the two variables.
2 Describe the association.	There is no apparent connection between income and a person's Medicare number.

(c)



1 Identify whether a pattern exists. If there is a pattern, describe the strength of the association as strong, moderate or weak.	There is a pattern: the data shows a moderate association between the two variables.
2 Check for linearity.	The plotted points do not form a straight line, so the association is non-linear.
3 Determine the direction of the slope to be positive or negative.	The slope is positive and flattens out, an increasing trend. As the explanatory variable increases, the response variable also increases.
4 Interpret the slope.	As people age, personal income increases, plateauing out towards the end of the data.
5 Describe the association.	The association between age and income is moderate, positive and non-linear.

(d)



1 Identify whether a pattern exists. If there is a pattern, describe the strength of the association as strong, moderate or weak.	There is a pattern: there appears to be a moderate association between the two variables.
2 Check for linearity.	The plotted points form a reasonably straight line, so the association is linear.
3 Determine the direction of the slope to be positive or negative.	The slope is positive, an increasing trend: as the explanatory variable increases, the response variable also increases.
4 Interpret the slope.	As the time spent studying increases, the exam score achieved also increases.
5 Describe the association (noting the presence of outliers).	The association between the time spent studying and the exam score is moderate, positive and linear. There are two possible outliers in the data.

More than two variables

When more than two sets of numerical data are given for a group of individuals, you can use bivariate data analysis to explore the association between any pair of variables.

7 Bivariate data analysis when there are three variables

The data set shows the class marks for a Mathematics test, broken into three aspects: exam result out of 50 (ER), average number of hours the student works in a part-time job (PJ) and average time spent studying per week (S).

S	18	11	15	13	19	14	17	12	15	13	9
PJ	14	15	12	10	6	4	8	14	4	16	18
ER	39	35	35	37	43	38	43	36	42	31	27

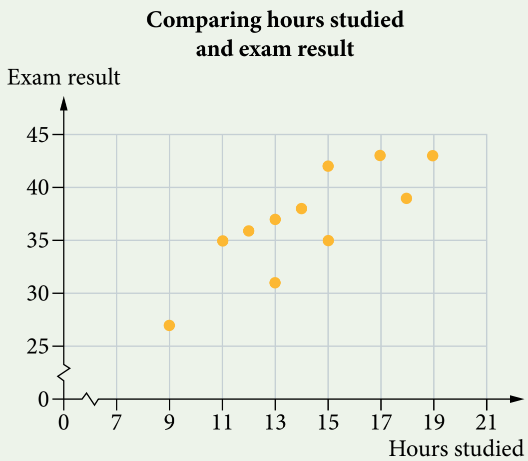
(a) Describe the association between the exam result and time spent studying.

THINKING / WORKING

1 Consider the possible relationship between the exam result and time spent studying.

A positive association may be present, as it is likely that the more a student studies, the better their exam result will be. Exam preparation also relies on other factors, and effective study techniques are very important.

2 Construct a scatter plot with the explanatory variable on the horizontal axis and the response variable on the vertical axis.



3 Describe the trend in terms of strength, direction and form.

There is a moderate positive linear relationship between the number of hours studied and the exam result achieved.

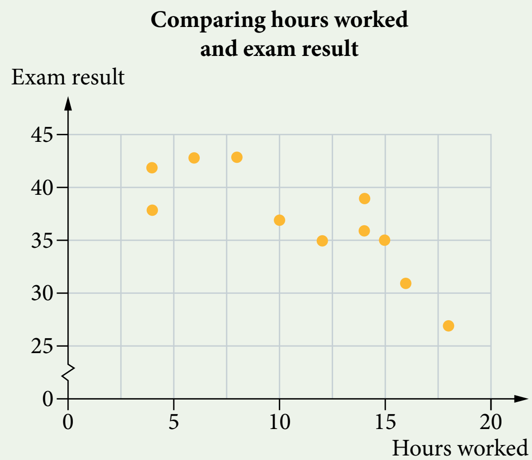
The increasing trend indicates that, as the number of hours of study increases, the exam result also increases.

(b) Describe the association between the exam result and time spent working in a part-time job.

1 Consider the possible relationship between the exam result and time spent working in a part-time job.

A negative association may be present, as it is likely that if students take on too much work they will have less time to study and sleep, which are important aspects in learning and preparing well for assessment such as exams.

2 Construct a scatter plot with the explanatory variable on the horizontal axis and the response variable on the vertical axis.



3 Describe the trend in terms of strength, direction and form.

There is a moderate negative non-linear relationship between the number of hours worked and the exam result achieved.

The decreasing trend indicates that, as the number of hours of part-time work increases beyond approximately 10 hours, the exam result obtained decreases.

(c) Describe the association between the time spent working in a part-time job and the time spent studying.

1 Consider the possible relationship between time spent studying and time spent working in a part-time job.

A negative association may be present, as there are only a set number of hours in a week, so the more time spent working, the less time there is to study. However, there are many reasons for reduced study time including family, sporting or other extra-curricular activities, interest in the subject and the determination of the student. High working hours does not cause low study hours.

2 Construct a scatter plot with the explanatory variable on the horizontal axis and the response variable on the vertical axis.

3 Describe the trend in terms of strength, direction and form.

There is a weak negative non-linear relationship. The negative trend indicates that, as the number of hours of part-time work increases beyond approximately 10 hours, the time spent studying decreases.

EXERCISE 1.2 Association between numerical variables

Worked Example

1 Five friends compete with each other to do the best on their upcoming General Maths Examination. In the final month before the exam they record how much time they spend studying. The final results are given in the table. 4

Construct a scatter plot and describe any pattern you see in terms of strength, direction and form.

Name	Alex	Bel	Cam	Darmi	Echo
Hours of study	30	15	5	10	35
General Maths result (%)	70	45	60	80	95

2 Assuming an association exists, choose the explanatory variable in each of the following pairs. 5

(a) Years of employment; Value of superannuation

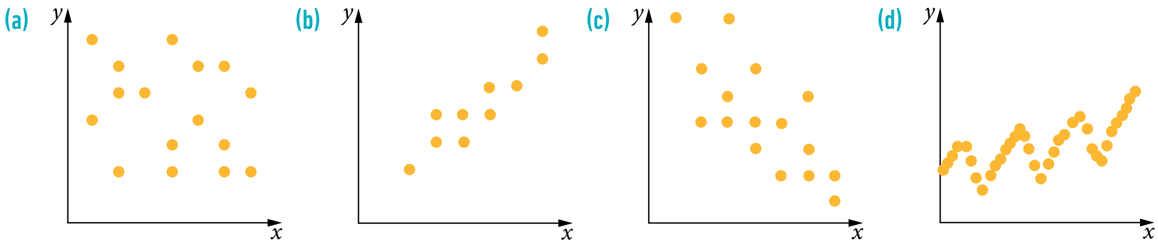
(b) Rainfall; Size of plants

(c) Temperature; Number of ice-creams sold

(d) Waist measurement; Cans of soft drink consumed

Worked Example 6

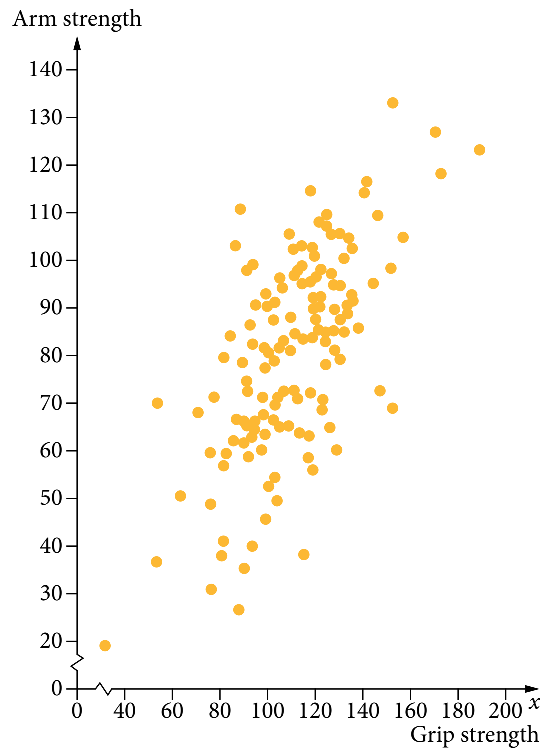
3 Describe the associations, if any, in the following scatter plots.



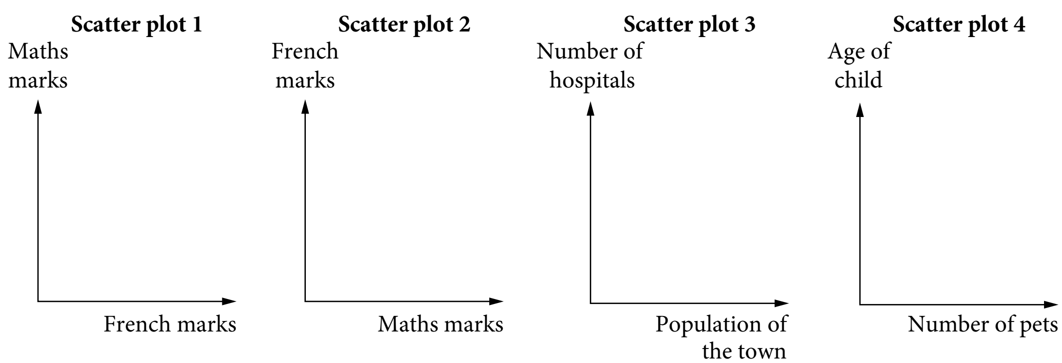
4 Consider the scatter plot comparing arm strength and grip strength.

Which pattern describes the scatter plot?

- **A** strong, positive, linear
- **B** moderate, negative, linear
- **C** no association
- **D** moderate, positive, non-linear



5 A student has started to draw four scatter plots.



(a) On which scatter plot has the student placed the axis titles incorrectly?

A Scatter plot 1 **B** Scatter plot 2 **C** Scatter plot 3 **D** Scatter plot 4

(b) Explain the common error made by a student who chose any of the incorrect options listed in (a).

6 The following data gives the latitude of some Australian cities, along with the average maximum temperature for June for each city.

	Cairns	Townsville	Mackay	Rockhampton	Brisbane	Canberra
Latitude (°S)	16.9	19.3	21.1	23.4	27.5	35.3
Temperature (°C)	26	26	22	24	21	13

	Toowoomba	Sydney	Melbourne	Hobart	Perth	Adelaide
Latitude (°S)	27.6	33.9	37.8	42.9	32.0	34.9
Temperature (°C)	17	18	15	12	19	16

(a) Determine the explanatory variable.

(b) Describe the kind of linear association you would expect to find between the variables, if any.

(c) Graph the scatter plot and describe any association.

7 The following data gives the 1-year-old value and the 5-year-old value for a variety of used cars.

	Mazda 3	Mazda 6	Ford Focus	Subaru Outback	Toyota Camry	Toyota Prado	Kia Cerato
1-year-old value ($'000)	22.0	32.9	19.6	42.4	17.9	51.3	17.5
5-year-old value ($'000)	8.9	16.5	13.9	20.0	12.8	46.0	13.0

(a) Determine the explanatory variable.

(b) Describe the association you would expect to find between the variables, if any.

(c) Graph the scatter plot and describe any association that is revealed.

8 The following data from 2017 gives median house prices for Australian capital cities, along with the population of each city.

	Melbourne	Sydney	Brisbane	Perth	Hobart	Adelaide	Canberra	Darwin
Median house price ($'000)	880	1150	550	550	410	520	720	590
Population ($'000)	4500	5070	2300	2250	220	1700	360	150

(a) Determine the explanatory variable.

(b) Describe the kind of linear association you would expect to find between the variables, if any.

(c) Graph the scatter plot and describe the relationship.

Worked Example 7

9 The following data set shows the final percentage results for 10 students in the three subjects they have in common.

In each case put the first-named variable on the horizontal axis.

Maths	40	60	75	60	80	30	45	75	90	55
English	50	70	70	80	90	50	40	65	75	60
Drama	60	80	50	90	75	40	55	80	80	35

(a) Draw a scatter plot to show any potential association between the students' Maths and English scores.

(b) Draw a scatter plot to show any potential association between the students' Maths and Drama scores.

(c) Draw a scatter plot to show any potential association between the students' Drama and English scores.

(d) Which association appears to be the strongest?

10 The data shown in the table gives the value, age and kilometres travelled, for similar models of Volkswagen Golfs.

Value ($'000)	15.5	18.0	17.0	17.0	20.4	17.5	25.0	21.5	15.6	9.5	14.0	22.5	19.0
Age (years)	7	2	2	3	3	7	5	4	5	10	3	7	3
Km travelled ('000)	63	29	50	53	56	129	54	50	53	184	92	72	35

(a) Which of the three variables would be the response variable in any pairing with the other two variables?

(b) Produce the three possible scatter plots for the data, being mindful of the explanatory variable in each case.

(c) Describe any association that exists between the variables. Compare the strengths and limitations of the associations.

11 The data in the table gives the population, average life expectancy, GDP per capita (a measure of economic wealth) and average adult height, in several countries.

	Population (millions)	Life expectancy (years)	Per capita GDP ($US'000)	Adult height (cm)
Australia	24	82.5	50	168.7
Japan	127	85.0	39	165.0
Spain	49	81.7	27	168.5
New Zealand	4.5	81.2	39	170.5
UK	64	80.7	40	168.6
USA	324	79.8	57	168.8
China	1 474	75.5	8	161.5
Iran	83	71.4	5	163.8
Nigeria	186	53.4	2.2	160.8
Chad	12	50.2	0.7	165.6

(a) Some studies have suggested an association between the general health of a population and height.

(i) Which two variables would you use to investigate this association? Identify the explanatory variable.

(ii) Produce a scatter plot and describe any association it reveals between these two variables.

(b) There is thought to be an association between the general health of a population and its wealth.

(i) Which two variables would you use to investigate this association? Identify the explanatory variable.

(ii) Produce a scatter plot and describe any association it reveals between these two variables.

12 In the previous question, the variable 'population' was not used in any of the scatter plots. Draw three scatter plots to see if the size of a country's population affects any of the other aspects previously investigated. Describe any associations that are revealed.

13 The following experimental data was collected by scientists doing research on climate change. They measured the volume of carbon dioxide, CO_2, produced by vehicles running on petrol and LPG (liquid petroleum gas). (Note that LPG is up to 50% cheaper than petrol per litre, but is less fuel efficient than petrol.)

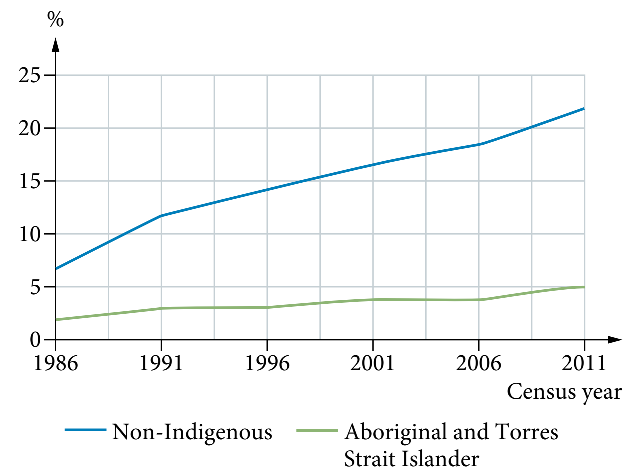
Petrol vehicles								
Petrol burned (L)	10	20	30	40	50	60	70	80
CO_2 produced (L)	13.5	25.6	38	53.5	67	74.2	25	108
LPG vehicles								
LPG burned (L)	10	20	30	40	50	60	70	80
CO_2 produced (L)	11.1	22	32.5	46	57.5	62.8	80.2	90

(a) Draw a scatter plot for each set of data.

(b) Describe the association evident in each scatter plot.

(c) Which fuel produces the least carbon dioxide per litre of fuel burned? Explain your answer.

14 The 2013 Census results published by the Australian Bureau of Statistics (ABS) contained information about the percentage of 15–24 year olds who were in higher education. The results comparing Indigenous and non-Indigenous students are shown in the graph.



(a) Draw a scatter plot of the percentage of Indigenous versus the percentage of non-Indigenous data, using the pair of values for each year.

(b) Describe the association shown by the data in the scatter plot.

(c) By examining the figures for 1986 and 2011, determine the rate of growth for each population and comment upon which population group is increasing its percentage more rapidly.

15 A class of Year 12 students were asked the age of each of their parents. The results are given in the table.

Mother's age	48	47	39	56	59	61	51	53	42	48	50	47	52
Father's age	45	50	42	59	62	61	52	52	44	53	49	53	56

(a) Write a general statement describing the association between the mothers' ages and the fathers' ages. Use a scatter plot to support your description.

(b) One of the students remembers participating in the same activity in Year 8. If the student was able to locate her analysis of the data, explain in what ways it would be the same and in what ways it would be different to the analysis in part (a).

(c) If the students were now asked to submit the age of each parent at the time they were born, explain in what ways the analysis would be the same and in what ways it would be different to the analysis in part (a).

1.3 Correlation and causation

The line of best fit

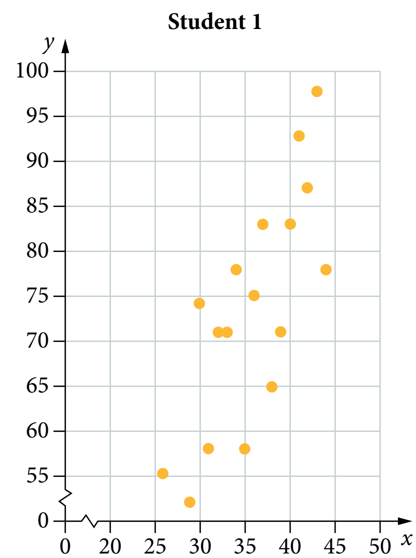
Broad categories such as 'weak', 'moderate' and 'strong' are not always adequate to describe the strength of an association. Several students could have different opinions on the strength of the association, especially if the scales of the axes are shown in different ways.

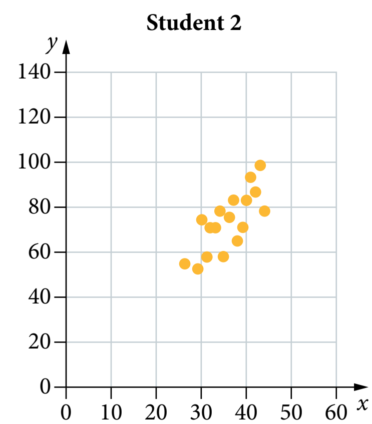
For example, consider the following data displayed by two students:

A	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	26	29
B	74	58	71	71	78	58	75	83	65	71	83	93	87	98	78	55	52

Student 1 has drawn a scatter plot with a broken y-axis to enable a more accurate plot, and has labelled the strength of the association as 'moderate'.

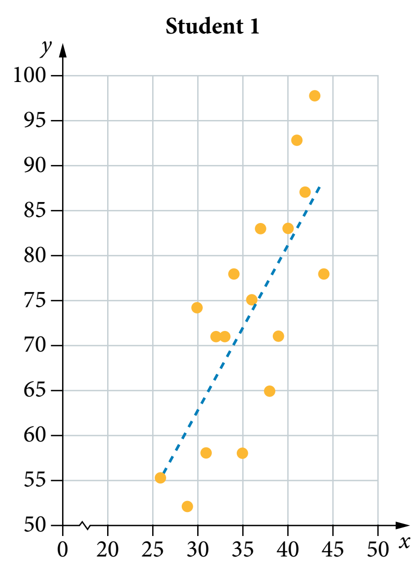
Student 2 has drawn the same scatter plot with the graph starting at $y = 0$, and has labelled the strength of the association as 'strong'.

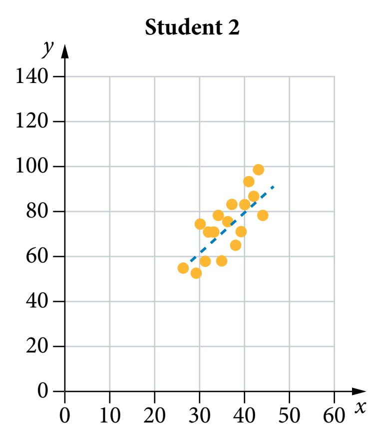




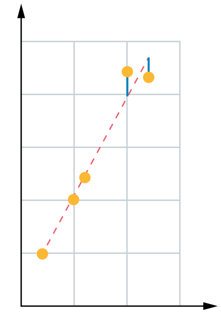
As a result, mathematicians prefer to describe the strength of a linear association between numerical variables using a numerical scale.

Correlation is the numerical measure used to express how closely individual coordinate pairs within a linear association fit a line of best fit, also known as a least-squares line.





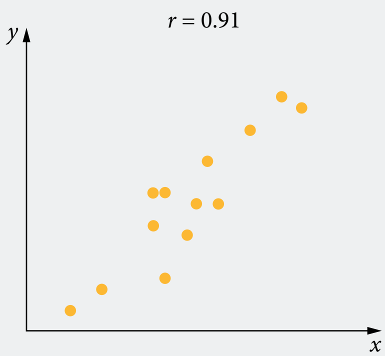
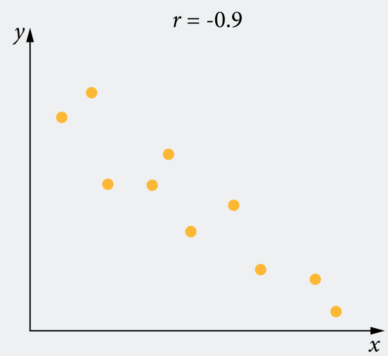
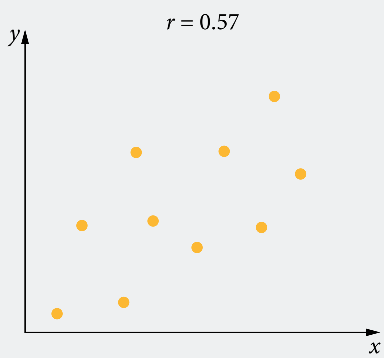
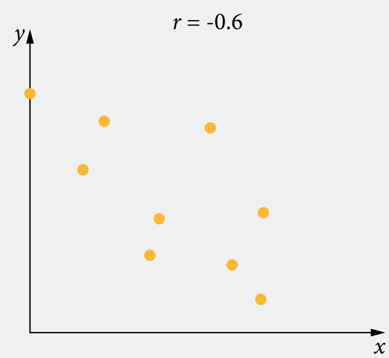
The theoretical line of best fit is likely to be slightly different from any that you would draw by hand, but the aim should be the same: to minimise the vertical distances (residuals) between each point and the line.

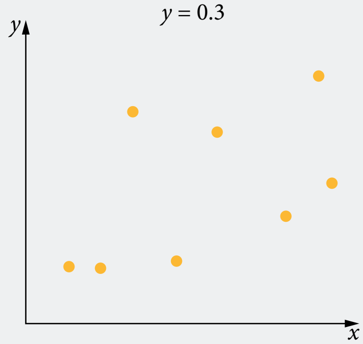
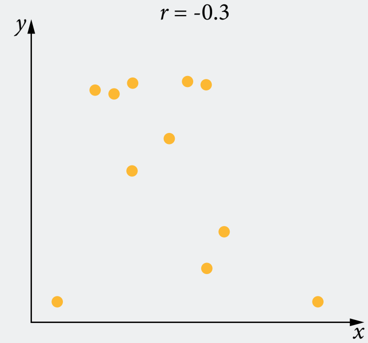
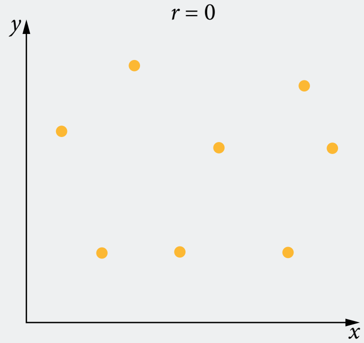


The correlation coefficient (r) is a measure of the strength of a linear relationship between a pair of variables.

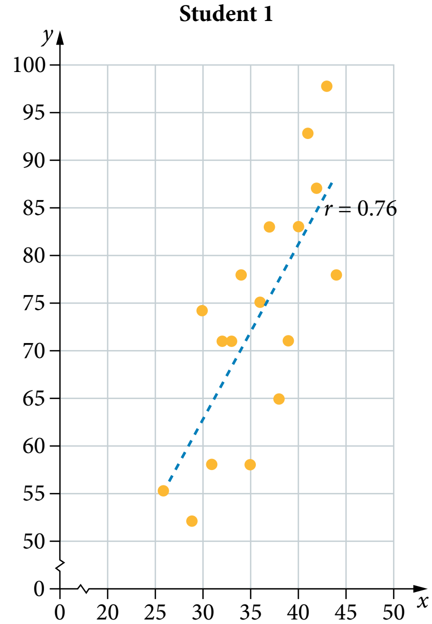
The value of the correlation coefficient *r*

The *correlation coefficient r* is a number between -1 and 1. The value of r is sometimes called *Pearson's correlation coefficient,* after its developer, Karl Pearson.

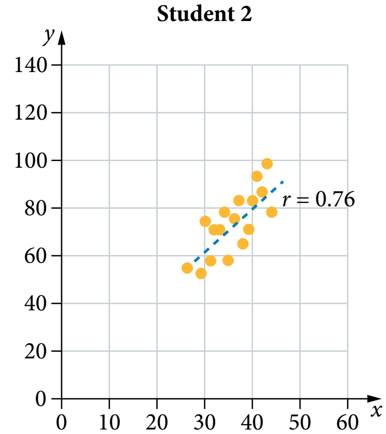
Correlation coefficient	
Positive (increasing trend)	Negative (decreasing trend)
$0.75 \le r \le 1$ Strong positive linear association: all points on or near the line of best fit If $r = 1$, the association represents a perfect positive linear association, with all points lying on the line of best fit. *Example* 	$-1 \le r \le -0.75$ Strong negative linear association: all points on or near the line of best fit If $r = -1$, the association represents a perfect negative linear association, with all points lying on the line of best fit. *Example* 
$0.5 \le r < 0.75$ Moderate positive linear association, with the points spaced on either side of the line of best fit *Example* 	$-0.75 < r \le -0.5$ Moderate negative linear association, with the points spaced on either side of the line of best fit *Example* 

Positive	Negative
$0.25 \le r < 0.5$ Weak positive linear association, with the points spaced on either side of the line of best fit *Example* $y = 0.3$ 	$-0.5 < r \le -0.25$ Weak negative linear association, with the points spaced on either side of the line of best fit *Example* $r = -0.3$ 
$-0.25 < r < 0.25$ No association *Example* $r = 0$ 	

Student 1



Student 2



When the students in the earlier example compare their graphs, the correlation coefficient of the data is $r = 0.76$, indicating a strong positive linear association between the variables.

Calculating *r*

One of the possible formulas for calculating the correlation coefficient, for n values, is:

$r = \frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{x_i - \overline{x}}{s_x} \times \frac{y_i - \overline{y}}{s_y}\right)$, where x_i and y_i are the individual data values, $\overline{x}$ and $\overline{y}$ are the mean values for each variable, and s_x and s_y are the sample standard deviations.

You can use the formula to calculate the correlation coefficient by hand, as shown.

Using technology, calculate the mean and standard deviation for each variable shown in the table below:

x	2	4	6	7	9
y	12	5	9	4	7

$\overline{x} = 5.6 \quad s_x = 2.7018\,(4\text{ d.p.}) \quad \overline{y} = 7.4 \quad s_y = 3.2093\,(4\text{ d.p.})$

x	$\frac{x_i - \overline{x}}{s_x}$	y	$\frac{y_i - \overline{y}}{s_y}$	$\frac{x_i - \overline{x}}{s_x} \times \frac{y_i - \overline{y}}{s_y}$
2	$\frac{2-5.6}{2.7018} \approx -1.3324\ldots$	12	$\frac{12-7.4}{3.2093} \approx 1.4333\ldots$	$-1.3324\ldots \times 1.4333 = -1.9098\ldots$
4	-0.5921	5	-0.7478	0.4428
6	0.1480	9	0.4985	0.0738
7	0.5182	4	-1.0594	-0.5489
9	1.2584	7	-0.1246	-0.1568

Sum the final column:

$$\sum_{i=1}^{n}\left(\frac{x_i - \overline{x}}{s_x} \times \frac{y_i - \overline{y}}{s_y}\right) = -1.9098 + 0.4428 + 0.0738 + -0.5489 + -0.1568$$

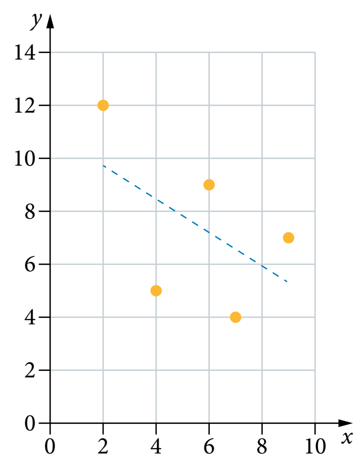
$$= -2.0989$$

And then divide by $n - 1$ or multiply by $\frac{1}{n-1}$:

$$r = \frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{x_i - \overline{x}}{s_x} \times \frac{y_i - \overline{y}}{s_y}\right)$$

$$= \frac{1}{4} \times -2.0989$$

$$= -0.5247$$



A correlation coefficient of $r = -0.5247$ indicates a moderate negative correlation.

> In order to use the value in your analysis, you need to provide a graph to demonstrate that the data is in fact linear.

The formula to calculate the correlation coefficient uses both mean and standard deviation. Both values are affected by outliers (extreme values). Observe the effect if an additional data value $(1, 82)$ is considered:

x	2	4	6	7	9	1
y	12	5	9	4	7	82

Using technology, calculate the mean and standard deviation for each variable:

$\bar{x} = 4.8333\,(4\text{ d.p.})$ $s_x = 3.0605\,(4\text{ d.p.})$ $\bar{y} = 19.8333\,(4\text{ d.p.})$ $s_y = 30.5903\,(4\text{ d.p.})$

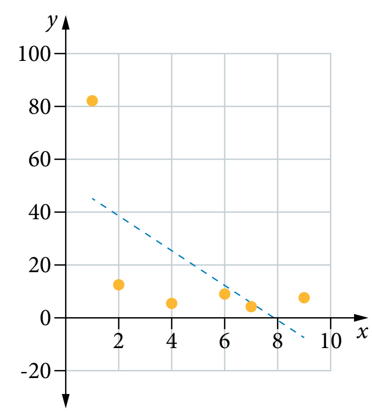
x	$\frac{x_i - \bar{x}}{s_x}$	y	$\frac{y_i - \bar{y}}{s_y}$	$\frac{x_i - \bar{x}}{s_x} \times \frac{y_i - \bar{y}}{s_y}$
2	-0.9258	12	-0.2561	0.2371
4	-0.2723	5	-0.4849	0.1320
6	0.3812	9	-0.3541	-0.135
7	0.7079	4	-0.5176	-0.3664
9	1.3614	7	-0.4195	-0.5712
1	-1.2525	82	2.0322	-2.5454

Outliers have a strong impact on the calculation of r and, if it is believed that the outlier is not a true member of the data set (i.e. it is due to a measuring or recording error), it may need to be removed before any calculation is done.

WARNING

A value of r without a visual assessment is not sufficient to claim a linear association. However, it is assumed that, if the correlation coefficient is being used to measure the strength of an association, the association between two numerical variables is linear, with no outliers.

$$r = \frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{x_i - \bar{x}}{s_x} \times \frac{y_i - \bar{y}}{s_y}\right)$$
$$= \frac{1}{5} \times (0.2371 + 0.1320 + -0.135 + -0.3664 + -0.5712 + -2.5454)$$
$$= -0.65$$



When the outlier is included, the strength of the association seems to increase. Pearson's correlation coefficient does not provide an accurate representation of the data if outliers are present, because the value is calculated using the mean and standard deviation, both of which are sensitive to extreme values.

Although the formula can help you to understand the value of r and how it is affected by outliers, it is quite complex. Your scientific calculator will generate the value of r.

8 Using technology to calculate and interpret the correlation coefficient r

(a) Determine the value of the correlation coefficient r to 2 decimal places, for the following bivariate data set.

Study time (hours per week)	1	6	4	12	3	2	9
Number of hours spent at the gym (per year)	900	490	320	55	650	416	85

METHOD 1: SCIENTIFIC CALCULATOR

THINKING	WORKING
1 Identify the explanatory variable, if there is one.	Hypothesise that the number of hours per year spent at the gym can be explained by, or responds to, the number of hours spent studying. However, either variable could be the explanatory variable.

2 In statistics mode on your calculator, choose linear A+BX.

Enter the explanatory values in the *x* column and the response variable in the *y* column.

In the statistics menu, choose regression and then calculate the correlation coefficient (*r*).

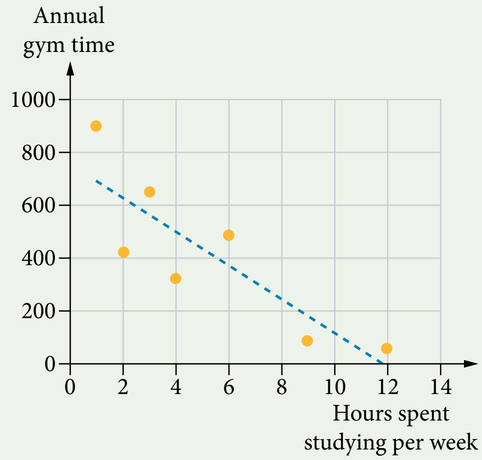
$r = -0.84503\ldots$
$= 0.85$ (2 d.p.)

3 Interpret the value, stating any assumptions or limitations of the result.

A correlation coefficient *r* value within the range $-1 \le r \le -0.75$ indicates a strong negative linear association, with all points on or near the line of best fit. However, this cannot be assumed without graphical verification.

4 Plot the points by hand on a Cartesian plane.

Remove any obvious outliers.



5 Describe the association in terms of strength, direction and form.

The association between the time spent studying and the annual time spent at the gym is a strong negative linear association. The decreasing trend indicates that, as the average number of hours of study increases, the time spent in the gym decreases.

METHOD 2: SPREADSHEET

1 Using an Excel (or similar software) spreadsheet, enter the explanatory values in the *x* column and the response variable in the *y* column.

	A	B
1	n	h
2	1	900
3	6	490
4	4	320
5	12	55
6	3	650
7	2	416
8	9	85

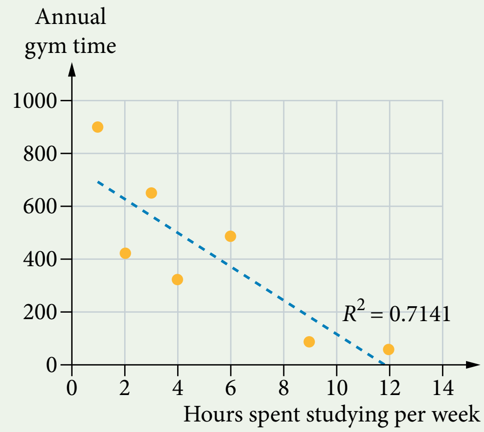
2 Insert a scatter plot.

Right click on any point and add a trendline.

Right click on the trendline, and format the trendline.

Select the linear format.

Display the *R*-squared value on the chart.



3 Interpret the direction of the line to state the sign of the correlation coefficient *r*.

The gradient of the line is negative, indicating that $r < 0$.

4 Write the value of R^2, the coefficient of determination.

$R^2 = 0.7141$

5 Take the square root, acknowledging the possibility of both signs.

$$r = \pm\sqrt{0.7141}$$
$$= \pm 0.84503\ldots$$

6 Determine the correlation coefficient *r* (to 2 decimal places). Choose the sign that matches the sign of the gradient of the trend line.

$\therefore r = -0.85$ (2 d.p.) because the gradient of the line of best fit is negative.

7 Interpret the answer.

A correlation coefficient *r* value in the range $-1 \leq r \leq -0.75$ indicates a strong negative linear association.

8 Describe the association in terms of strength, direction and form.

The association between the time spent studying and the annual time spent at the gym is a strong negative linear association. The decreasing trend indicates that, as the average number of hours of study increases, the time spent in the gym decreases.

Note that just because an association exists, a change in one of the variables does not necessarily *cause* a change in another.

From the previous example, it was evident that as the number of hours of study increased, the annual time spent at the gym decreased. However, that does not necessarily mean that studying less causes you to exercise more and vice versa.

If you had analysed the data where gym attendance was the explanatory variable, decreasing the time spent working out does not necessarily cause an increase in the time spent studying. There are many reasons for people to choose to study or work out, and many variables that can influence the time that an individual can dedicate to each.

Explore further

Correlation and causation

Calculate and interpret the correlation coefficient and the coefficient of determination using technology.

Causation and the coefficient of determination

Consider the following associations.

Variable 1	Variable 2	r	R^2
Hours of structured practice	Piano-playing ability	0.8	0.64
Hours of training	Level of exhaustion after a 5 km run	-0.75	0.56
Hours of productive study	Exam result	0.9	0.81
Maths exam result	English exam result	0.95	0.90
Number of fast food outlets	Number of hospitals	0.9	0.81

Each of these associations has a strong correlation, but can you say that changes in one variable are caused by changes in the other?

Simply knowing that two variables are associated, no matter how strongly, is not sufficient evidence to conclude that the two variables are causally related.

A causal relationship is demonstrated when a change in the explanatory variable *causes* a change in the response variable.

Other reasons for an association:

- Both variables may be responding to a third variable. This is known as a common response, as both variables share a common response to a third variable.
- There may be causation, but the change may also be caused by one or more uncontrolled variables whose effects cannot be disentangled from the effect of the explanatory variable. These are known as confounding variables, and the relationship between the data studied is said to be confounding.
- What is considered to be the response variable may actually be the explanatory variable and be causing the change.
- It may simply be coincidence.

Statisticians calculate a coefficient of determination R^2 to help decide upon the level of causation, but common sense must also be used.

For the first three associations in the table above, there seems to be a clear cause-and-effect situation. A change in variable 1 (the explanatory variable) causes a responsive change in variable 2 (the response variable).

In fact, the value of R^2 can be used to be even more precise, and say:

- 64% of the variation in piano-playing ability is in response to, or can be explained by, a variation in the hours of structured practice; 36% of the variation is due to other factors.
- 56% of the variation in exhaustion after a 5 km run is in response to, or can be explained by, a variation in the number of hours of training; 44% of the variation is due to other factors.
- 81% of the variation in exam results is in response to, or can be explained by, a variation in the hours of productive study; 19% of the variation is due to other factors.

The strong correlation between Maths and English results suggests that it is most likely that both of these results are due to other uncontrolled variables, such as determination, completion of work and hours of study. This would be an example of confounding causation.

The association between the number of hospitals and the number fast food outlets is a good example of a common response. Both variables are responding to the population of the town.

9 Using R^2 to assess causation

Where causation is observed, complete the following statements.

R^2% of the variation in response variable is due to the variation in the explanatory variable.
$(100 - R^2)$% of the variation is due to other factors.

Where causation cannot be verified, discuss the reason.

(a) Distance travelled versus fuel consumed: $r = 0.98$ and $R^2 = 0.96$

THINKING	WORKING
1 Determine whether one variable directly influences the other.	The further you travel, the more fuel is consumed, so a causal relationship is established.
2 Convert the value of R^2 to a percentage.	$R^2 = 0.96$ $= 96\%$
3 R^2% of the variance in the response variable is due to variation in the explanatory variable. $(100 - R^2)$% of the change is due to other factors.	96% of the variation in fuel consumed is due to the variation in distance travelled. 4% of the variation is due to other factors, such as driving conditions and the incline of the terrain.

(b) Number of hours of sunlight versus temperature: $r = 0.5$ and $R^2 = 0.25$

1 Determine whether one variable directly influences the other.	These factors are more likely to respond to a common response – in this case, season or location.
2 Where no direct association is present, decide whether these variables both respond to a single third variable (a common response) or if the situation is more complex.	Hours of sunlight and temperature both respond to the season and/or location, so they both show a common response.

(c) Level of activity and weight gain: $r = -0.7$ and $R^2 = 0.49$

1 Decide whether one variable directly influences the other.	Weight gain would respond to the level of activity undertaken by the participants.
2 Where no direct association is present, explain whether the variables show a common response to a third variable or are affected by confounding variables.	Weight gain and the level of activity are related but most likely confounding. In order for the study to be viable, many confounding variables would need to be controlled. These include age, gender, number of hours of sleep, energy intake and activity type undertaken by participants.

Note that even if, for example, you studied the link between the number of phone calls made by a charity and the amount of donations, and found R^2 to be 0.36, the causation is still there. The number of donations responds to many other confounding variables, such as the persuasiveness of the caller, the financial situation at the time, the areas the calls are made to, and whether there is a current high-profile emergency situation.

Establishing causation is not an exact science. While there are obvious examples of each type of association, in the real world the situation is often somewhere in between, and open to interpretation or further research.

EXERCISE 1.3 Correlation and causation

Worked Example

1 Calculate the correlation coefficient r to 2 decimal places for each of the following bivariate data sets. 8

(a)

Length of trip (km)	0.5	10	40	60	80	100
Number of times parents are asked 'Are we there yet?'	2	5	18	20	25	40

(b)

Computer time (h)	1	6	4	12	3	2
Number of 'Likes' added	24	40	50	120	50	30

(c)

IQ	90	120	85	105	88	103	145	94
Distance of home from the post office (km)	4	2	14	6	48	50	25	32

(d)

Minutes played	30	40	50	60	70	80
Points scored	25	15	30	50	45	65

2 For each data set, draw a scatter plot and calculate r. Then comment on the association (if any). 9

(a)

x	2	5	10	18	22
y	18	12	7	4	2

(b)

Length (m)	1	2	4	7	9	15
Cost ($)	2.80	5.60	11.20	19.60	25.20	42.00

(c)

Number of swimmers	10	25	45	80	100	120	180
Bacteria level (ppm)	0.2	0.5	0.8	0.6	2.3	3.0	2.8

3 The following experimental data was collected.

Rate of reaction (mol/min)	4.2	6.8	10.3	15.6	182	19.5
Temperature (°K)	300	350	400	450	500	550

(a) Calculate the value of the correlation coefficient r to 2 decimal places. Include outliers.

(b) Identify the outlier in the data and exclude it from the data set.

(c) An outlier can exert a large influence on the data. Which statement is true?

A Outliers are generally excluded from the data set so calculations are an accurate representation of the study.

B Outliers must be included because they form a significant part of the data set.

(d) After excluding the outlier, draw a scatter plot of the data and recalculate the value of r.

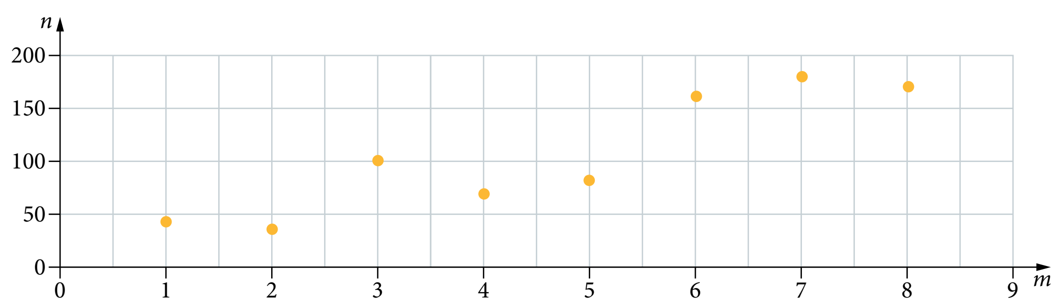
(e) How is it evident in this scenario that excluding the outlier means the points line up with a stronger linear relationship?

(f) Identify a possible error that may have occurred when recording the data.

4 Three bivariate data sets and their scatter plots are shown below.

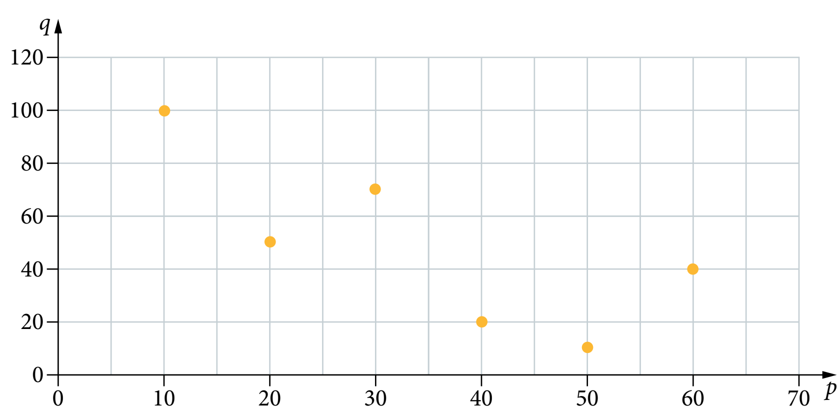
Set A

m	1	2	3	4	5	6	7	8
n	42	34	100	68	80	160	180	170



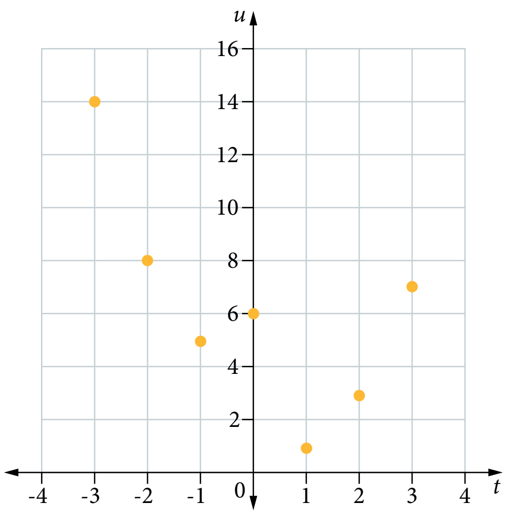
Set B

p	10	20	30	40	50	60
q	100	50	70	20	10	40



Set C

t	-3	-2	-1	0	1	2	3
u	14	8	5	6	1	3	7



Answer the following questions about the bivariate associations.

(a) For which data set is the sample correlation coefficient r closest to -1?

(b) For which data set is the correlation coefficient r closest to 1?

(c) Which data set is non-linear?

(d) Which data set indicates the strongest correlation between its two variables?

5 Assume that the bivariate data sets in this question have a linear association.

(a) Calculate the correlation coefficient r for each set, to 2 decimal places.

(i)

x	1	3	4	6	8
y	11	40	24	70	75

(ii)

x	1	3	4	6	8
y	21	50	34	80	85

(iii)

x	2	6	8	12	16
y	22	80	48	140	150

(b) Compare the results for the three sets of data. Compare the data and explain why this has happened.

6 A bivariate data set is keyed into a scientific calculator, giving the value $r = -0.85$.

(a) Which of the following is a correct assumption?

A The association is negative.

B There is a strong linear association.

C The association is not as strong as data with $r = 0.8$

D An analysis of a trendline in a spreadsheet would have yielded $r = 0.7225$

(b) Explain the common error made by a student who chose the first incorrect option.

7 For the data in the following table use the coefficient of determination to describe the relationship between the explanatory and response variables.

	Variable 1	Variable 2	r	R^2
(a)	Hours of rehearsal	Quality of performance	0.8	0.64
(b)	Number of fast food outlets in a town	Number of hospitals	0.95	0.90
(c)	Number of registered cars	Number of registered motor bikes	0.6	0.36
(d)	Exam result in Physics	Exam result in Maths	0.9	0.81
(e)	Episode number of a reality TV series	Number of contestants remaining	-0.98	0.96

8 Calculate the correlation coefficient r, to 2 decimal places, given each of the following:

(a) $\dfrac{\sum\left(\left(x_i - \bar{x}\right)\times\left(y_i - \bar{y}\right)\right)}{n-1} = 44.23,\ s_x = 10.8,\ s_y = 4.3$

(b) $\dfrac{\sum\left(\left(x_i - \bar{x}\right)\times\left(y_i - \bar{y}\right)\right)}{n-1} = -12.44,\ s_x = 5.6,\ s_y = 2.6$

(c) $\dfrac{\sum\left(\left(x_i - \bar{x}\right)\times\left(y_i - \bar{y}\right)\right)}{n-1} = 850.9,\ s_x = 25.7,\ s_y = 41.2$

(d) $\dfrac{\sum\left(\left(x_i - \bar{x}\right)\times\left(y_i - \bar{y}\right)\right)}{n-1} = -7.5,\ s_x = 2.9,\ s_y = 3.5$

9 The data given in the table has correlation coefficient $r_{xy} = -0.99$.

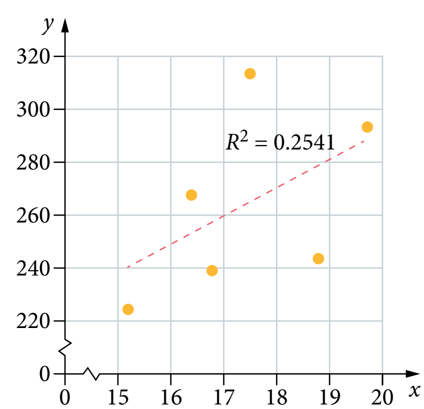
x	1	3	8	10	11
y	25	23	a	5	4

Determine the correlation coefficient, r_{xz} for the data set below.

x	1	3	8	10	11
z	50	46	$2a$	10	8

10 A set of data and a scatter plot are shown, with line of best fit and R^2 value displayed.

	A	B
1	X	Y
2	15.2	225
3	16.4	268
4	16.8	239
5	17.5	314
6	18.8	244
7	19.7	294



Choose an additional point for this data, from within the domain and range of the existing data set, that will increase the value of the correlation coefficient r and the standard deviation of x, s_x and decrease the standard deviation of y, s_y.

1.4 Fitting a linear model

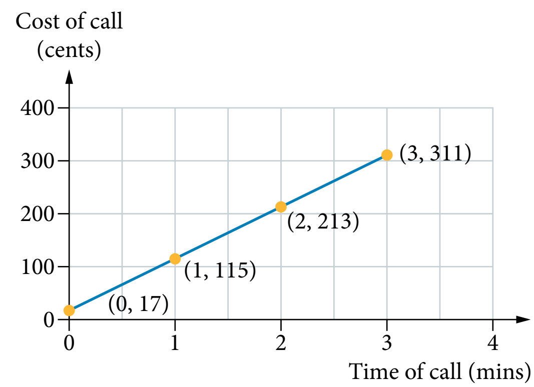
Understanding the gradient and intercept values

In the general equation $y = a + bx$, a represents the y-intercept and b is the gradient or slope of the line. In modelling situations, these values can often be linked to a practical understanding of the information.

Consider this situation. The cost of a phone call with a particular company consists of a 17 cent connection fee and an additional fee of 98 cents per minute.

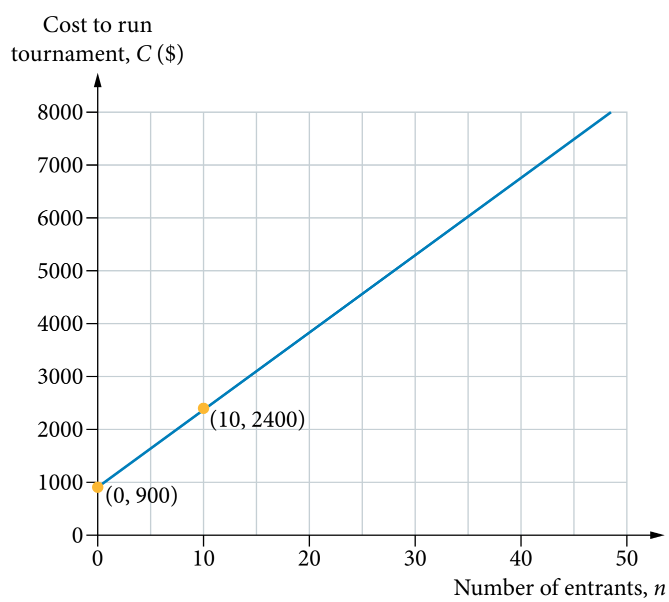
The equation of the graph is:
$$\text{cost of call}(\text{cents}) = 98 \times \text{time of call}(\text{minutes}) + 17.$$

Note that the vertical axis intercept (17) represents the fixed costs and the gradient (98) represents the rate, or cost per minute, of the call.



10 Using a graph to determine the cost per unit

Jayde is in charge of arranging a round robin tournament for her local tennis club. Her fixed costs are the cost of hiring the courts and buying the trophies. The extra costs, which depend upon the number of entrants, are for lunch and tennis balls. Because of the limited number of courts, the maximum number of people who can enter the tournament is 48. The graph shows the total cost of running the event, depending on the number of entrants.



(a) Determine the cost of hiring the courts and buying the trophies.

THINKING	WORKING
1 Read the value of the vertical axis intercept from the graph.	The coordinates of the C-intercept are: $(0, 900)$ $C = 900$
2 Interpret the answer.	The fixed costs of court hire and the purchase of trophies are $900.

(b) Determine the extra cost for every entrant.

1 Determine the gradient.	$\text{slope} = \frac{C_2 - C_1}{n_2 - n_1} = \frac{2400 - 900}{10 - 0} = \frac{1500}{10} = 150$
2 Interpret the answer.	On top of the fixed costs, there is an additional cost of \$150 per entrant to cover the costs of lunch and the purchase of tennis balls.

(c) Write the equation for the cost in terms of the number of entrants.

1 Recall the gradient–intercept form of a linear relationship.	In the form: $y = a + bx$
2 Define the variables.	C is the total cost to run the tournament and n is the number of entrants.
3 Construct a linear equation using the defined variables.	$\text{cost}(\$) = 900 + 150 \times (\text{number of entrants})$ $C = 900 + 150n$

(d) Calculate the cost for 25 entrants, and the corresponding cost per entrant required to cover the cost of the tournament.

1 Substitute the number of entrants into the equation and calculate the cost, the value of C.	Let $n = 25$: $C = 900 + 150n$ $= 900 + 150 \times 25$ $= \$4650$
2 Interpret the value.	If there are 25 entrants, then Jayde's total cost is \$4650.
3 Determine the cost per entrant by dividing the total cost by the number of entrants.	$\frac{\$4650}{25 \text{ entrants}} = \$186/\text{entrant}$ Each entrant would need to pay \$186 to enter the tournament in order for Jayde to break-even.

(e) Determine the number of entrants if Jayde's total cost was \$1950.

1 Substitute the cost into the equation and solve for n.	Let $C = 1950$: $1950 = 900 + 150n$ $1050 = 150n$ $n = 7$
2 Interpret the answer.	If Jayde's total cost was \$1950, there were 7 entrants in the tournament.

A scientific calculator will give the equation of the line of best fit when points are entered in bivariate data mode. If just two pairs of data are entered, the calculator gives the equation to the fixed line through the two points. The same occurs when a spreadsheet is used.

11 Using technology to determine the equation of a line

Use technology to determine the equation of the line passing through the points (1,5) and (3,11).

THINKING

1 In 2-variable statistics mode ($A + BX$), note the algebraic expression your calculator uses to indicate a linear relationship.

Input the values with the x-coordinate for each pair in the first column.

The procedure is the same for an Excel spreadsheet.

WORKING

$y = A + Bx$

	A	B
1	X	Y
2	1	5
3	3	11

THINKING

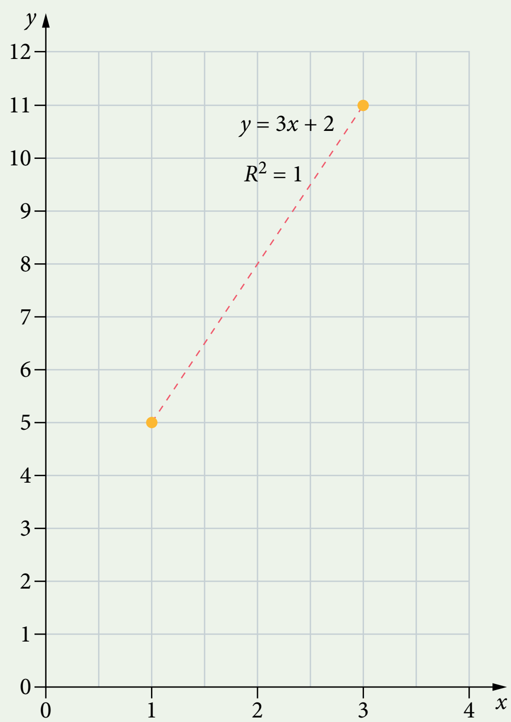
2 On the calculator, use summary statistics to read the values of the parameters (usually A and B).

Check that $r = 1$ (or $R^2 = 1$), because the line is a perfect fit and the gradient is positive. For a negative gradient $r = -1$.

In Excel (or similar software), insert a scatter plot, add a trend line and display its equation.

WORKING

$A = 2$ and $B = 3$



THINKING

3 Substitute the values of A and B into the equation, or read the equation from the graph.

WORKING

The equation of the line passing through the points is $y = 2 + 3x$.

Drawing the line of best fit by eye

With real data you will generally have many more than two points, and often the points will not line up exactly. It is worth determining the equation of the line of best fit manually a few times to understand what is required.

When placing a straight line on a coordinate grid to represent a trend, your aim is to have an equal number of points on both sides of the line, with the total distance of points from the line on each side being the same. *Note*: The line will not necessarily pass through any of the actual points.

Once you have positioned the line of best fit you can determine the gradient, the y-intercept and consequently the equation for the line $y = a + bx$.

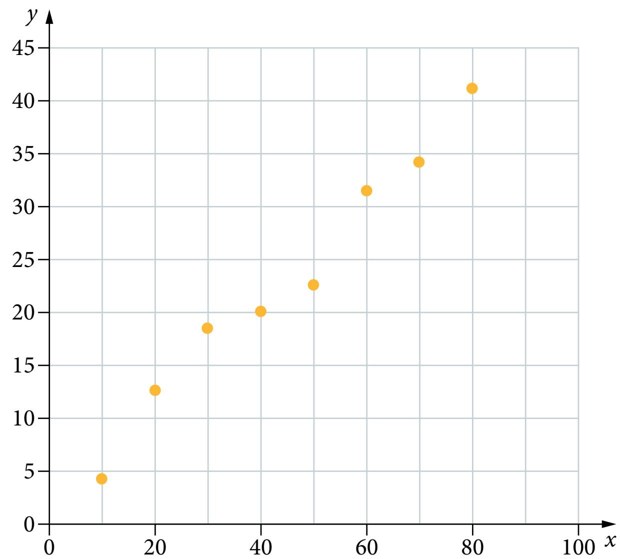
Additional information

Placing a line of best fit

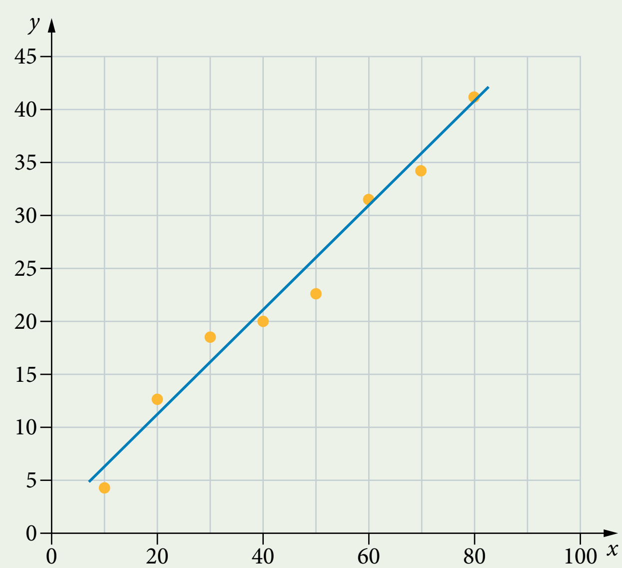
Watch the animation to consolidate your understanding of placing a line of best fit.

12 Placing a line of best fit and determining the equation

Consider the scatter plot below.



(a) Place a line of best fit.

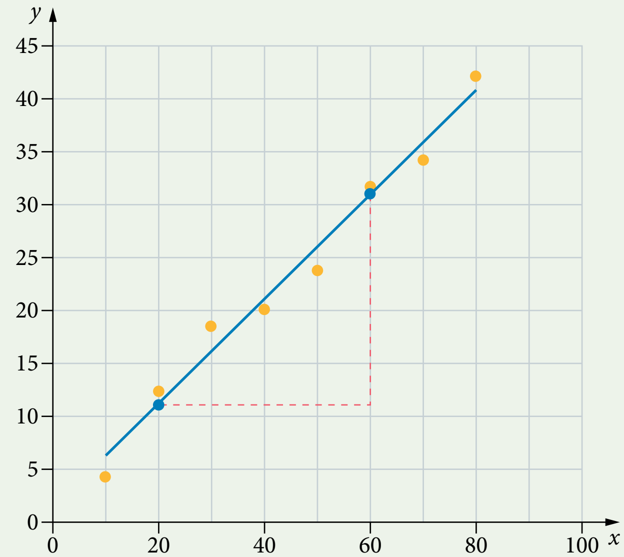
THINKING	WORKING
Use a straight edge to position a line so that there are approximately the same number of points on either side of the line, and the total distance of the points from the line on both sides is roughly the same.	

(b) Calculate the gradient of the line.

Choose two convenient points from the line and calculate the gradient.

$$\text{Gradient} = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}}$$

For this line choose the points (20, 11) and (60, 31).



$$\begin{aligned}\text{Gradient} &= \frac{\text{rise}}{\text{run}}\\ &= \frac{y_2 - y_1}{x_2 - x_1}\\ &= \frac{31 - 11}{60 - 20}\\ &= \frac{20}{40}\\ &= 0.5\end{aligned}$$

(c) Determine the coordinates of the y-intercept.

1 Recall the general equation for a linear graph: $y = a + bx$.

Use $b = 0.5$ from part (b) and substitute either of the points on the line to calculate a.

(20, 11) is used here but (60, 31) would produce the same result.

$y = a + bx$

Substitute $b = 0.5$, $x = 20$ and $y = 11$.

$$\begin{aligned}y &= a + bx\\ 11 &= a + 0.5 \times 20\\ 11 &= a + 10\\ a &= 1\end{aligned}$$

2 Interpret the value.

Graphically this represents a y-intercept with the coordinates (0, 1).

(d) Determine the equation of the line of best fit.

Write the equation by substituting the values for a and b into the gradient–intercept form $y = a + bx$.	The data in the scatter plot has a line of best fit with the equation $y = 0.5x + 1$.

WARNING
The line will be placed in slightly different places by different people, so an answer found in this way can be considered an estimate.

Interpreting the least-squares line

You can use technology to determine the equation of the line of best fit, called the least-squares line, along with the value of the correlation coefficient r. The look of the graph in conjunction with the value of r will allow a sensible interpretation of the linear association between the variables.

If there is a linear association, it is appropriate to interpret the gradient and vertical intercept of the line of best fit in terms of the variables and discuss the strength of the relationship using the correlation coefficient r.

13 Using technology to determine the equation of the least-squares line

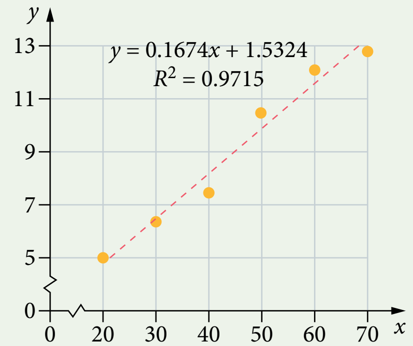
The rate of reaction of two substances at different temperatures is given in the table.

Temperature (°C)	20	30	40	50	60	70
Rate of reaction (mol/min)	5.1	6.4	7.5	10.5	12.1	12.8

Use technology to insert the trend line, and determine the correlation coefficient r and the equation of the least-squares line. Round values to 2 decimal places.

THINKING	WORKING
1 Identify the explanatory variable and define the variables.	The temperature may influence the rate of reaction. Let y represent the rate of reaction in mol/min and x represent the temperature in °C.
2 Enter the data into an Excel (or similar software) spreadsheet with the explanatory variable in the first column.	(spreadsheet below)

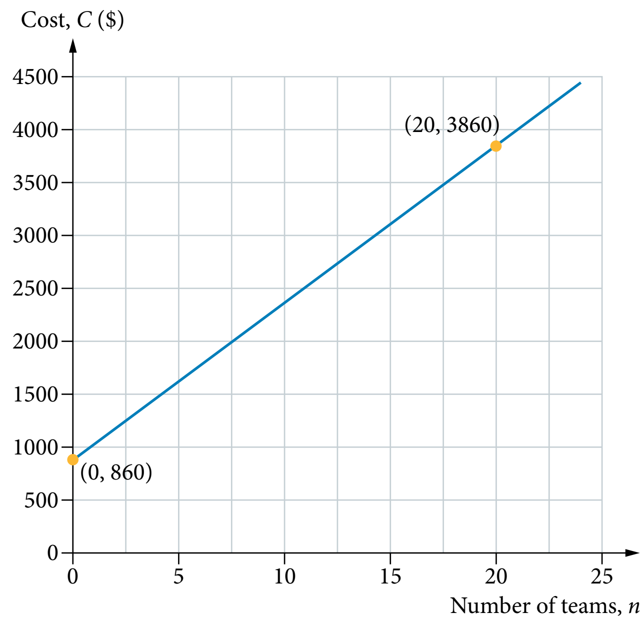
	A	B
1	X	Y
2	20	5.1
3	30	6.4
4	40	7.5
5	50	10.5
6	60	12.1
7	70	12.8

3 Insert a scatter plot. Add a trendline. Display the equation. Display the coefficient of determination R-squared value on the chart.	
4 Interpret the value of the coefficient of determination.	For $R^2 = 0.9715$ $\approx 97\%$ Assuming that the variables are not confounding or responding to a common response, 97% of the variation in reaction rate is in response to, or can be explained by, a variation in temperature. The remaining $(100 - 97)\% = 3\%$ can be attributed to other factors.
5 Determine the correlation coefficient r by taking the square root of R^2, choosing the sign to match the gradient of the graph.	The gradient of the line is positive, therefore $r > 0$. $r = \pm\sqrt{0.9715}$ $r = \pm 0.99\ (2\text{ d.p.})$ $\therefore r = 0.99\ (2\text{ d.p.})$
6 Interpret the value of the correlation coefficient r.	A correlation coefficient $r = 0.99$ indicates a strong (almost perfect) positive linear association. This indicates that as the temperature increases, the rate of reaction also increases.
7 Write the equation in terms of x and y, and then replace these with the given variables, with units.	$y = 1.53 + 0.17x$ $\text{rate of reaction (mol/min)} = 1.53 + 0.17 \times \text{temperature (°C)}$
8 Interpret the gradient and y-intercept.	The equation of the line shows that at 0°C (freezing point), the rate of reaction is 1.53 mol/min and for every degree increase in temperature, the rate of reaction increases by 0.17 mol/min.
9 State any limitations of the model.	Given that the lowest temperature in the data is 20°C, the interpretation of the y-intercept (at freezing point) may not be valid.

EXERCISE 1.4 Fitting a linear model

Worked Example

10 **1** Sonya is arranging a round robin tournament for the local netball association. Her fixed costs are the cost of hiring the courts and buying the trophies. The extra costs, which depend upon the number of teams that enter, are for umpires and catering. The maximum number of teams that can be accepted is 24, due to the limited availability of courts. The graph shows the total cost of running the event, depending on the number of teams entered.



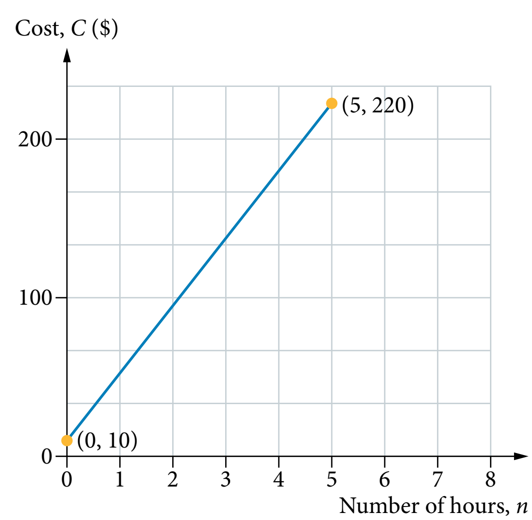
(a) Determine the cost of hiring the courts and buying the trophies.

(b) Determine the extra cost for every team that enters.

(c) Write the equation for the cost in terms of the number of teams.

(d) Calculate the cost for 24 teams.

2 The graph shows the total cost of moving furniture to a nearby suburb. The company charges a fixed price to attend, with an hourly charge for up to 8 hours. Time is rounded up to the nearest whole number of hours.



(a) Determine the fee to attend.

(b) Determine the extra cost for each hour (or part thereof).

(c) Write the equation for the cost in terms of the number of hours.

(d) Calculate the cost for 7 hours.

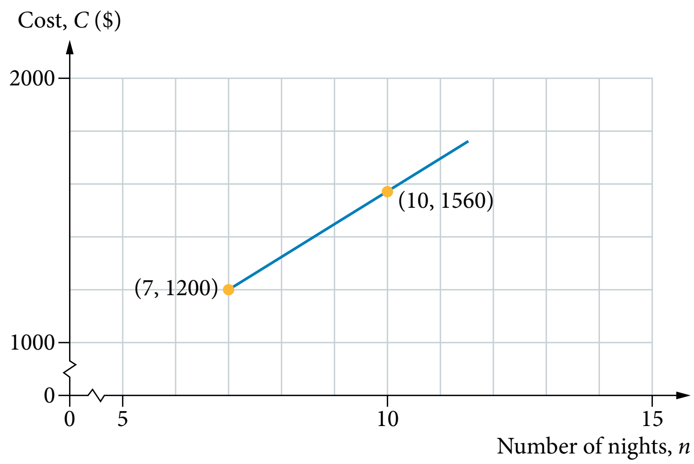
3 The graph shows the cost of accommodation at a Gold Coast resort for a minimum of 7 nights.

(a) Determine the cost for each extra night after the first 7 nights.

(b) If the equation for total cost could give the cost for zero nights, what would this be?

(c) Write the equation for the cost in terms of the number of nights.

(d) Calculate the cost of a 2 week stay.



Worked Example 11

4 Determine the equation of the line through each pair of points.

(a) $(1, 2)$ and $(5, 4)$

(b) $(2, 7)$ and $(5, 1)$

(c) $(12, 4)$ and $(30, 12)$

(d) $(8, 48)$ and $(20, 45)$

5 For the following sets of bivariate data, determine the equation of the least-squares line. Comment on the association between the variables including form, direction and strength. Write all values to 2 decimal places.

(a)

x	14	15	16	17	18
y	27.8	30.0	31.1	33.4	33.9

(b)

x	3	4	5	6	7
y	30.8	23.5	17.5	12.7	3.6

(c)

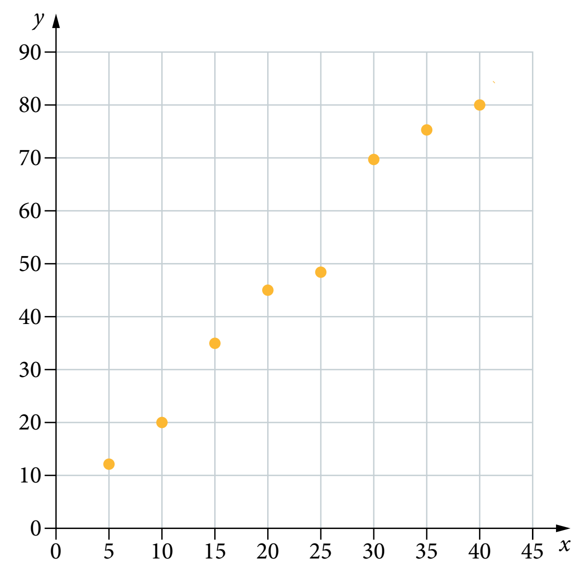
x	20	25	30	35	40
y	6.1	7.2	9.8	10.9	13.4

(d)

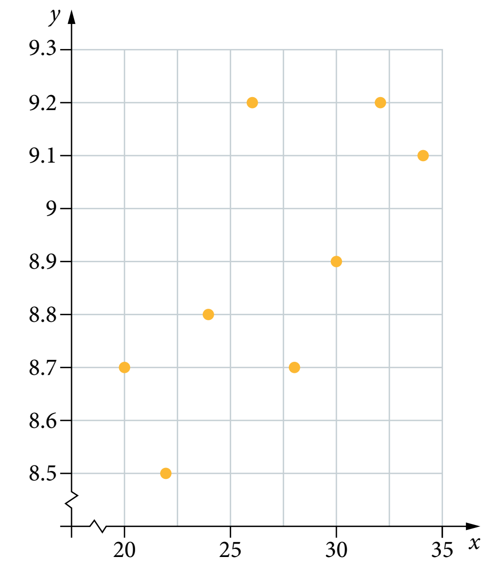
x	6.5	6.7	7.0	8.4	9.6
y	122	141	140	181	185

12

6 Consider the scatter plot shown here. Sketch a line of best fit and determine the equation of the line. Round your answers appropriately.



7 Consider the scatter plot shown here.

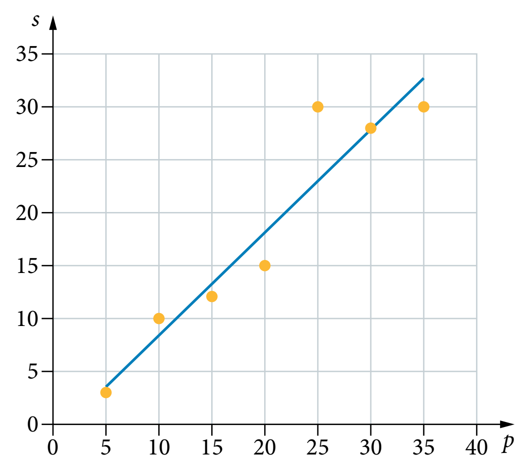


(a) Sketch the line of best fit and determine the equation of the line. Round your answers appropriately.

(b) Explain the common error made by a student who gave an answer in the form $y = 8.6 + bx$.

8 The scatter plot and data shown represent experimental data linking the variables p and s.

p	5	10	15	20	25	30	35
s	3	10	12	15	30	28	30



(a) Without entering the data into your calculator, which of the following is the most likely least-squares equation linking the variables?

A $s = 0.9p - 2$ B $s = 0.9p + 2$

C $s = -0.9p - 2$ D $s = p - 1$

(b) Without calculation, if the point $(35, 30)$ should have been recorded as $(30, 35)$, what would be true of the new gradient?

A It is the same. B It is steeper.

C It is less steep. D It is negative.

9 The scatter plot shown here represents experimental data linking the variables x and y.

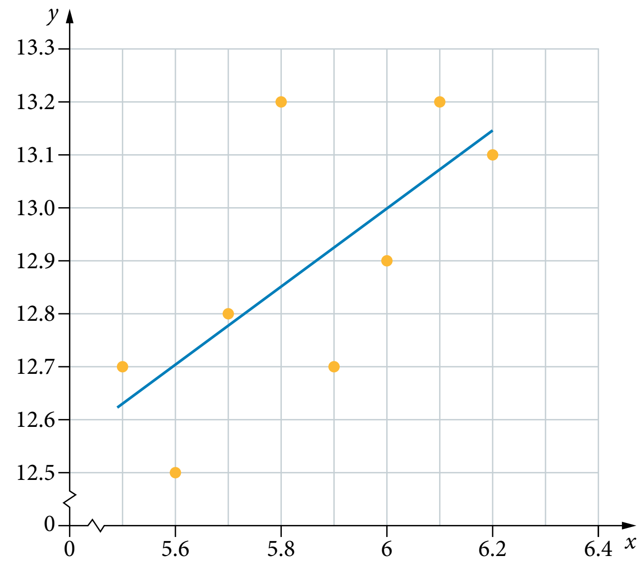
If the line of best fit has the equation $y = a + bx$, determine the value of a.

A 0.75

B 8.6

C 12.5

D 12.57



Worked Example 13

10 The speed and depth of the water flowing at a particular point in a river is recorded at different times.

Depth (m)	Speed (m/s)
0.22	0.47
0.6	0.33
0.78	0.43
0.95	0.42
1.40	0.44
1.75	0.34

(a) Plot the points, fitting the least-squares line, and then determine the correlation coefficient r and the least-squares equation for this line. Round values to 2 decimal places.

(b) Comment on the association using the statistics found.

(c) Interpret the gradient and the y-intercept of the line.

(d) Interpret the coefficient of determination $R^2 = 0.1544$.

11 The results of a weather balloon measuring the air temperature every kilometre as it rises through the atmosphere are shown.

Altitude (km)	0	1	2	3	4	5	6	7
Temperature (°C)	15.0	8.5	2.0	-4.5	-11.0	-17.5	-23.9	-30.5

(a) Determine the correlation coefficient r and the least-squares equation for the line of best fit. Give your answers to 1 decimal place.

(b) Comment on the association in terms of strength, direction and linearity. Interpret the formula.

12 For the following bivariate data sets, calculate to 2 decimal places the correlation coefficient r and the least-squares equation. Let the husband's age be the explanatory variable in each case.

(a) Set A

Husband's age	25	56	80	45	23	32	62	40
Wife's age	23	56	75	46	22	36	57	38

(b) Set B

Husband's age	26	40	70	37	23	32	61	70
Wife's age	22	46	66	41	19	30	57	32

(c) Which set of data fits the least-squares line better? Explain why this is the case.

13 Some people believe that you can predict a person's height by measuring the length of their arm span. The results for 20 students were recorded and are shown in the table.

Person	Height, h (cm)	Arm span, a (cm)	Person	Height, h (cm)	Arm span, a (cm)
1	128	132	11	148	144
2	138	134	12	149	144
3	141	136	13	152	145
4	142	138	14	150	145
5	145	138	15	150	146
6	144	140	16	147	148
7	152	141	17	152	148
8	145	142	18	154	155
9	148	143	19	172	164
10	144	144	20	168	170

(a) To investigate the claim, what would the explanatory variable have to be?

(b) Construct the scatter plot and describe the association.

(c) Calculate the value of the correlation coefficient r, to 2 decimal places.

(d) Determine the least-squares equation, giving values to 2 decimal places.

(e) From this limited data set, do you think arm span is a good way of predicting height? Explain the strengths and limitations of this model.

14 The information in the table relates to each Australian state and territory in June 2013, and is available on the Australian Bureau of Statistics website.

	Area (km^2)	Population ('000)	Motor car thefts	Average weekly earnings of full-time workers ($)
New South Wales	800 642	7 407.7	23 100	1 480
Victoria	227 416	5 737.6	12 100	1 400
Queensland	1 730 648	4 658.6	9 800	1 560
South Australia	983 482	1 670.8	3 800	1 320
Western Australia	2 529 875	2 517.2	7 900	1 800
Tasmania	68 401	513.0	3 000	1 280
Northern Territory	1 349 129	239.5	900	1 580
Australian Capital Territory	2 358	383.4	400	1 820

(a) Complete the following table without doing any calculations.

Association	Explanatory variable	Strength and direction of association	Correlation coefficient estimate
Area versus population			
Area versus motor car thefts			
Area versus average weekly earnings			
Population versus motor car thefts			
Population versus average weekly earnings			
Motor car thefts versus average weekly earnings			

(b) Calculate the value of the correlation coefficient r, to 2 decimal places, for each of the six associations.

(c) For which association(s) would it be appropriate to calculate the least-squares equation? Explain your answer.

(d) Calculate the least-squares equation for the association(s) given in part (c).

15 Four sets of data are being assessed for investigation of linear association.

Set A

x	20	22	24	25	26
y	10	11	14	18	22

Set B

x	20	21	23	27	30
y	50	44	40	38	36

Set C

x	20	24	26	28	30	32	34	36
y	16	15	21	16	19	18	21	17

Set D

x	20	22	24	26	28	30	32	36
y	15	11	15	10	12	9	10	8

(a) Which of the data sets are suitable for linear regression analysis? Support your solution with some initial analysis.

(b) Calculate the least-squares equation for the association(s) given in part (a). Interpret the equation in terms of the gradient and the y-intercept.

(c) Write the equation(s) from (b) in the form $y = m(x - k) + d$, where k is the smallest x-value in the domain. Interpret the other numbers in the equation.

1.5 The least-squares equation and residual analysis

Equation of the least-squares line

When data sets appear to have an association, drawing a scatter plot is a powerful method for observing any trend.

To get a better understanding of the behaviour of data, mathematicians insert a line of best fit.

> The equation of the least-squares line can be determined algebraically using the formula in the form $y = a + bx$, where $b = r\dfrac{s_y}{s_x}$ and $a = \bar{y} - b\bar{x}$.

14 Using the least-squares equation

(a) Determine the equation of the least-squares line, given the following information.

$\bar{x} = 9.5$, $\bar{y} = 17$, $s_x = 1.8$, $s_y = 5.2$ and $r = 0.68$

THINKING	WORKING
1 Substitute the values of the correlation coefficient (r), standard deviation of y (s_y) and standard deviation of x (s_x) to calculate the gradient of the least-squares line, where $b = r\dfrac{s_y}{s_x}$.	$b = r\dfrac{s_y}{s_x}$ $= 0.68 \times \dfrac{5.2}{1.8}$ $= 1.96\dot{4}$
2 Substitute the mean of y ($\bar{y}$), the value of b and the mean of x ($\bar{x}$) in the formula $a = \bar{y} - b\bar{x}$ to calculate the y-intercept of the least-squares line.	$a = \bar{y} - b\bar{x}$ $= 17 - 1.96\dot{4} \times 9.5$ $= -1.66\dot{2}$
3 State the equation of the least-squares line $y = a + bx$, to 1 decimal place.	$y = -1.7 + 2.0x$

(b) For the following set of data, use the formula to calculate the least-squares equation.

x	1	2	3	4	5	6	7	8
y	7	12	29	45	54	57	60	77

1 Use technology to calculate the mean of each variable.	$\bar{x} = 4.5$ $\bar{y} = 42.625$
2 Calculate the standard deviation of each set of data.	$s_x = 2.449$ (3 d.p.) $s_y = 24.547$ (3 d.p.)

3 Calculate the correlation coefficient r.

$r = 0.978\ (3 \text{ d.p.})$

4 Determine the gradient of the least-squares line, where $b = r\dfrac{s_y}{s_x}$.

$$\begin{aligned} b &= r \times \frac{s_y}{s_x} \\ &= 0.978 \times \frac{24.547}{2.449} \\ &= 9.8\ (1 \text{ d.p.}) \end{aligned}$$

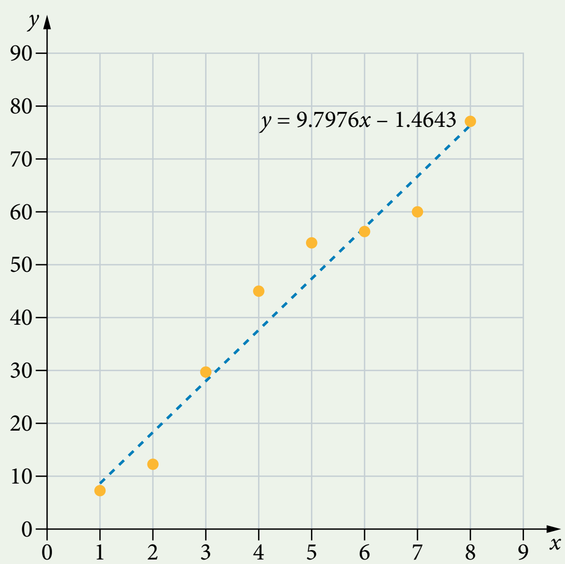
5 Determine the y-intercept of the least-squares line, using the formula $a = \bar{y} - b\bar{x}$.

$$\begin{aligned} a &= \bar{y} - b\bar{x} \\ &= 42.625 - 9.8 \times 4.5 \\ &= \text{-}1.5\ (1 \text{ d.p.}) \end{aligned}$$

6 State the equation of the least-squares line in the form $y = a + bx$.

$y = \text{-}1.5 + 9.8x$

7 Verify using technology.



The least-squares line is the line in a position where the sum of the squares of the vertical difference (residual) between each of the actual points plotted and the predicted points from the equation of that line is minimum.

Explore further

The equation of the least-squares line

Determine and explore the equation of the least-squares line.

Residual plots

When data sets appear to have an association, drawing a scatter plot is a powerful method for observing any trend.

However, the appearance of the scatter plot also depends on the scales used to draw it. Mathematicians use another diagram, called a residual plot, to determine any apparent trend shown by the scatter plot.

A residual plot examines the vertical differences between each point and the line of best fit.

The value of the residual is the (signed) difference between the observed value and the value predicted by the equation of the line of best fit, usually the least-squares equation.

To calculate the value of a residual (residual value, rv), subtract the y-value predicted by the trend line (y_p) from the actual y-value (y_a).

Note: If you have used the least-squares equation, the sum of the residuals will be zero. The values above the line are positive and their total is cancelled out by the total of the negative values below the line.

Explore further

Residual analysis

Use a spreadsheet and the least-squares line to conduct a test of the assumption of linearity with a residual analysis.

Residual value = actual value − predicted value

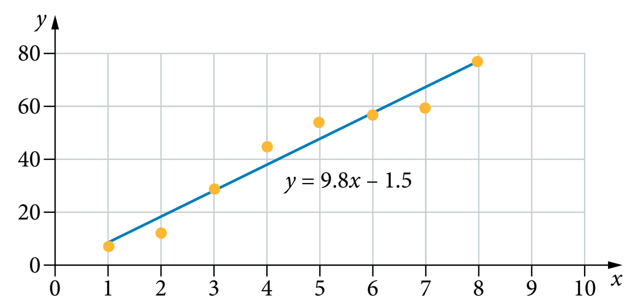
$$\text{rv} = y_a - y_p$$

Where y_a represents the actual y-value and y_p represents the y-value predicted by the least-squares equation, the residual is the vertical difference between the two.

15 Creating a residual plot

Consider the scatter plot and the least-squares equation shown.

x	1	2	3	4	5	6	7	8
y	7	12	29	45	54	57	60	77



(a) Calculate the residuals and create a table.

THINKING

1 Add two rows below the x-values and use the least-squares equation to calculate each residual (by substituting each x value into the equation).

WORKING

x	1	2	3	4	5	6	7	8
y_a	7	12	29	45	54	57	60	77
y_p	8.3	18.1	27.9	37.7	47.5	57.3	67.1	76.9

$y = 9.8x - 1.5$

2 Calculate the residuals using $rv = y_a - y_p$.

x	1	2	3	4	5	6	7	8
y_a	7	12	29	45	54	57	60	77
y_p	8.3	18.1	27.9	37.7	47.5	57.3	67.1	76.9
rv	-1.3	-6.1	1.1	7.3	6.5	-0.3	-7.1	0.1

3 Interpret the residuals.

The residuals show the vertical difference between the actual y-values and the y-values predicted using the least-squares equation.

At $x = 1$, the residual -1.3 indicates that the point $(1, 7)$ sits 1.3 units below the least-squares line, which passes through the point $(1, 8.3)$.

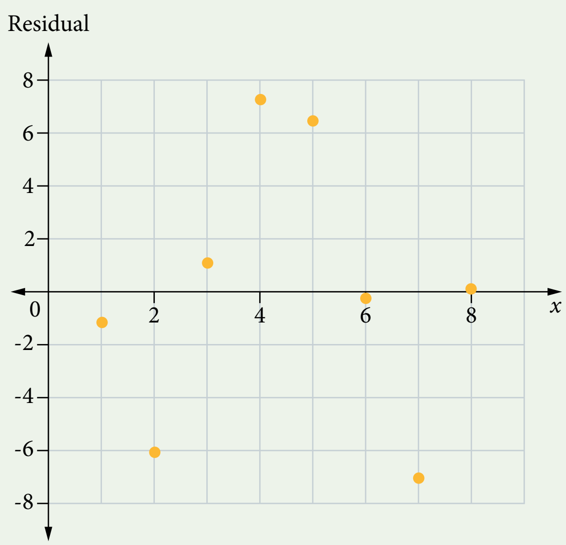
4 Verify the least-squares equation by summing the residuals.

$$\sum rv = -1.3 + (-6.1) + 1.1 + 7.3 + 6.5 + (-0.3) + (-7.1) + 0.1$$
$$= 0.2$$

The sum of the residuals is approximately 0. The line of best fit is most likely given by the least-squares equation but with rounded values, which can cause such a discrepancy.

(b) Construct a residual plot.

1 Use the x-values and the residuals from the table to draw a residual plot.



2 Comment on any patterns found in the residual plot.

The residual plot shows the residual (vertical difference between the actual data value and the predicted data value) of each point.

From the graph, it is evident that the line of best fit cuts the data in half, with two points sitting almost on the line of best fit (i.e. they have a residual of approximately 0), three points sitting above the line (residual value > 0) and three points below the line (residual value < 0).

For data sets with many values, creating a spreadsheet to perform the repetitive tasks will save you time.

16 Creating a residual plot using technology

For the given data, use a spreadsheet to complete the following tasks.

x	1	2	3	4	5	6	7	8
y	7	12	29	45	54	57	60	77

(a) Plot the points, fit the least-squares line and display its equation.

THINKING

Enter the points into a spreadsheet.

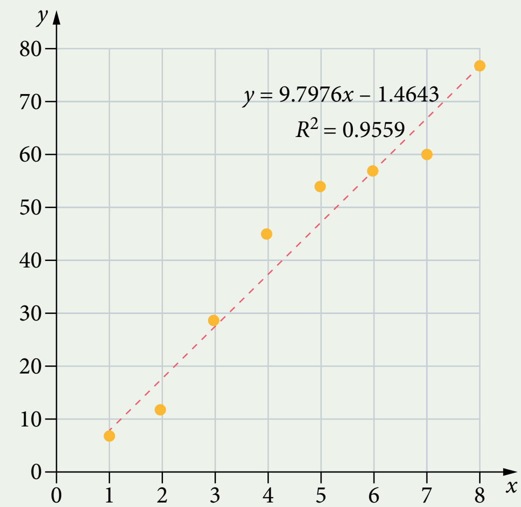
Insert a scatter plot.

Add a trendline.

Display the equation.

The value of the coefficient of determination R^2 is displayed to confirm the strength of the correlation.

WORKING



(b) Use the equation to calculate the predicted y-values and, hence, calculate the residuals.

In column C, add the heading 'y predicted' and in cell C2 input the least-squares formula (=9.80*A2–1.46), then drag the cross-hair to fill the column.

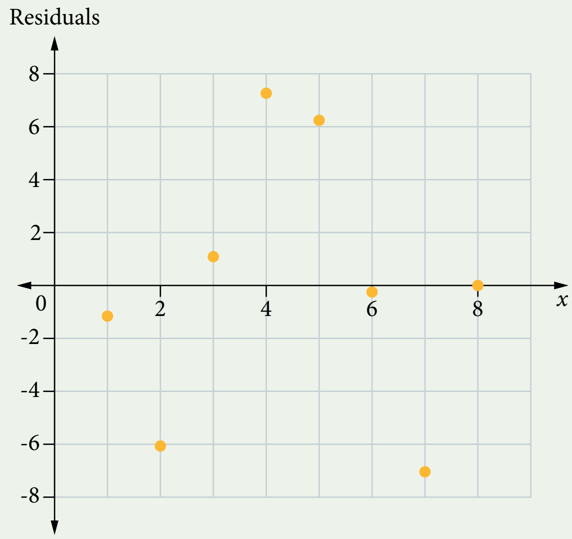
In column D, add the heading 'Residuals'. In cell D2, input the residual formula, rv $= y_a - y_p$ to calculate the residuals (=B2–C2), then drag the cross-hair to fill the column.

	A	B	C	D
1	x	y	y predicted	residuals
2	1	7	8.34	-1.34
3	2	12	18.14	-6.14
4	3	29	27.94	1.06
5	4	45	37.74	7.26
6	5	54	47.54	6.46
7	6	57	57.34	-0.34
8	7	60	67.14	-7.14
9	8	77	76.94	0.06

(c) Construct the residual plot.

Highlight the x column and the residuals column.

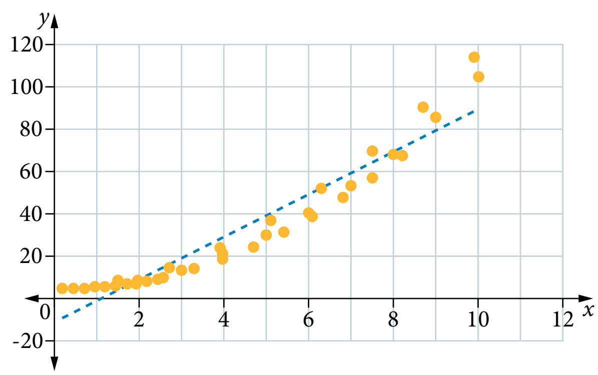
Insert a scatter plot.



In Excel (or other spreadsheet software), values can be made to show fewer decimal places, but the unrounded values are used for any calculation.

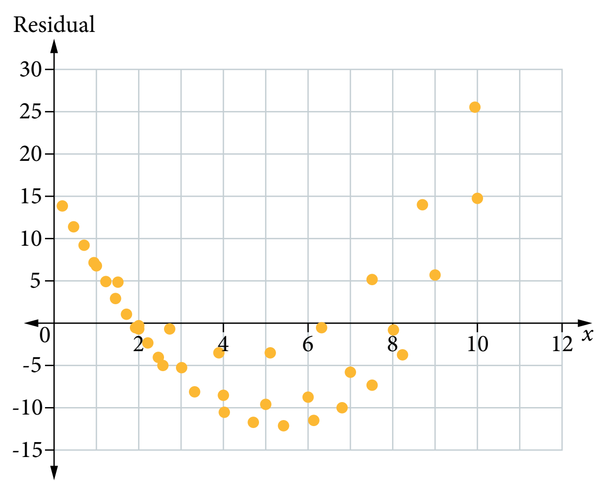
Reading the residual plot

Consider the scatter plot shown at right, with the line of best fit (linear model). The plot shows a non-linear association between variables x and y, with a linear trend line.



The positions of the points relative to the line of best fit show a distinct pattern: for values between 0 and 2, the data points all sit above the line of best fit (residual > 0); between the x-values of 2 to 8 the data points primarily sit below the line of best fit (residual < 0); and for $x > 9$ the data values again sit above the line of best fit (residual > 0).

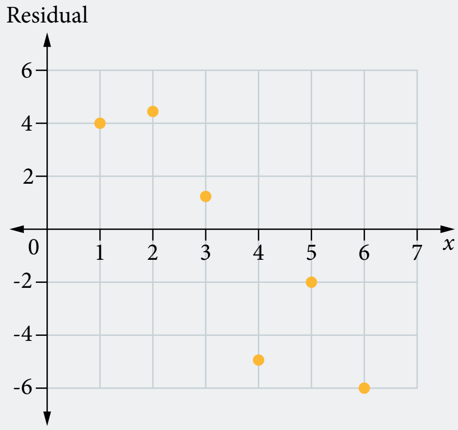
The residual plot, shown at right, can therefore be analysed to state that a non-linear relationship exists between the variables x and y because of the clear pattern of the residuals.



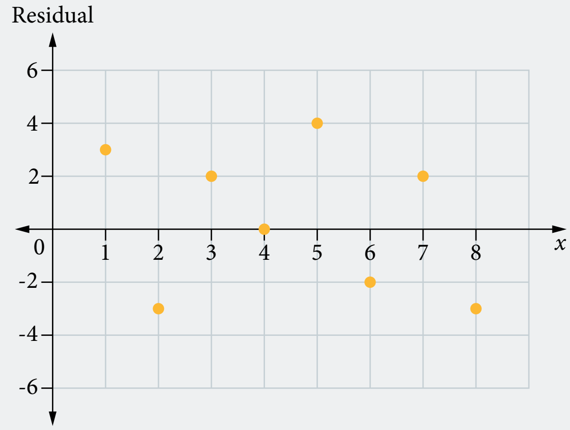
If a linear association exists, then the residual plot will have a random pattern of points.

If the residual plot has a pattern, then the association if any, is not linear.

There appears to be a moderate negative linear trend in the residuals, so linearity is not confirmed.



In this example, the points are random, with no pattern, so linearity is confirmed.

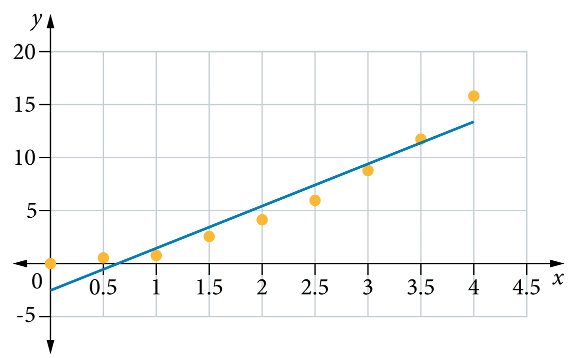


17 Using a residual plot

Consider the scatter plot and data table shown.

x	0	0.5	1	1.5	2	2.5	3	3.5	4
y	0	0.25	1	2.25	4	6.25	9	12.25	16

Without doing any further calculations, create a table showing the sign of each residual value and use this to comment on the assumption of linearity.



THINKING

1 For each point, record whether the residual value will be positive or negative, that is, whether the actual point is above or below the least-squares line.

WORKING

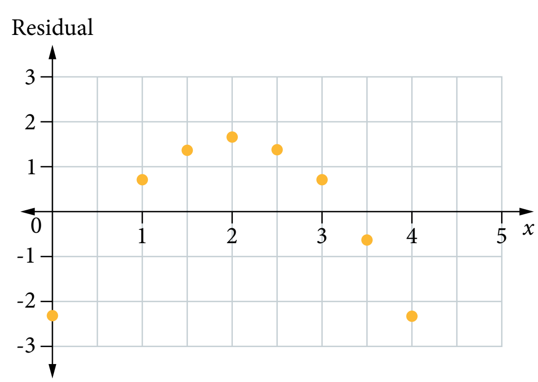
x	0	0.5	1	1.5	2	2.5	3	3.5	4
rv	+	+	−	−	−	−	−	+	+

2 Comment on whether or not the sign changes randomly.

The positive and negative values do not appear randomly, so linearity is not confirmed.

If you calculated the least-squares equation and r from the previous example, you would find that $y = 4x - 2.3$ and $r = 0.96$. It would be easy to assume a strong linear association based on these figures.

The important point to note here is that, despite a strong value of $r = 0.96$, a linear relationship cannot be assumed based on the residual plot shown.



EXERCISE 1.5 The least-squares equation and residual analysis

Worked Example 15

1 For each of the following, calculate the residuals, to 1 decimal place where necessary, and draw a residual plot.

(a) $y = 17.3 + 11x$

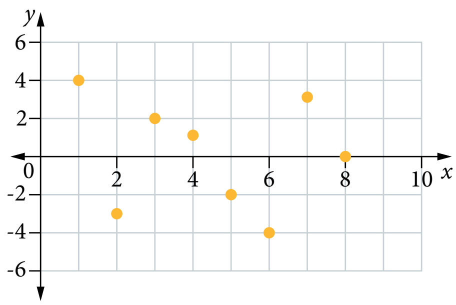
x	1	2	3	4	5	6	7	8
y	25	45	80	30	65	90	70	130

(b) $y = 186 - 2.2x$

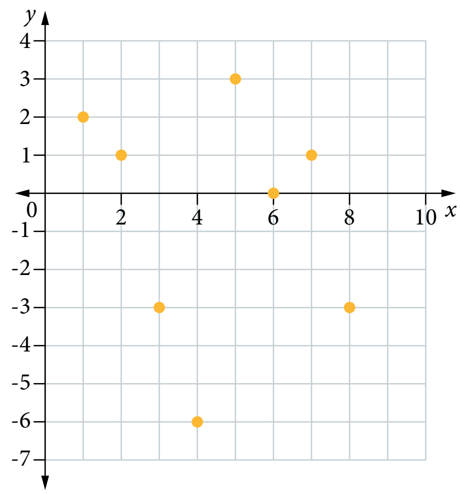
x	10	20	30	40	50	60	70	80
y	180	120	140	70	80	70	30	10

2 Do the following residual plots support a linear association?

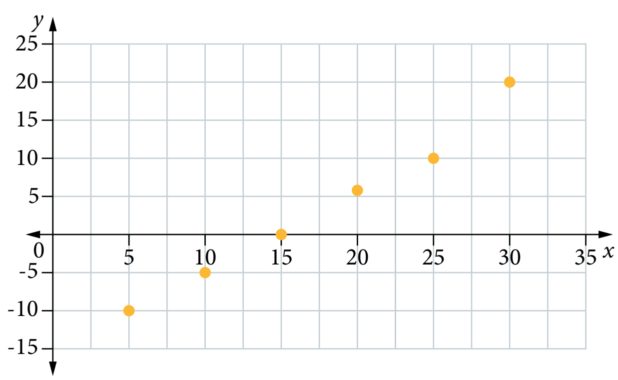
(a)



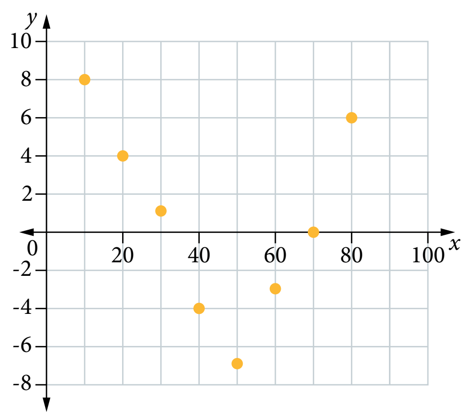
(b)



(c)



(d)

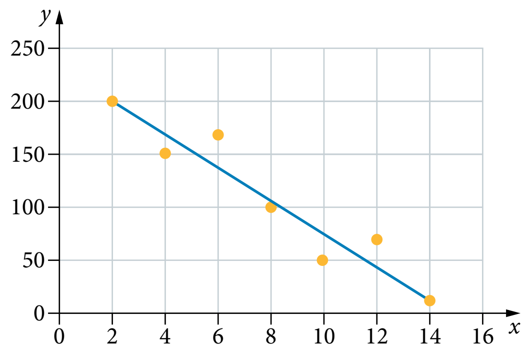


17

3 For the scatter plot and data shown in each case, without doing any further calculations, fill in the table, showing whether the residual value is positive (+), negative (–) or zero (0), and state whether the assumption of linearity is supported.

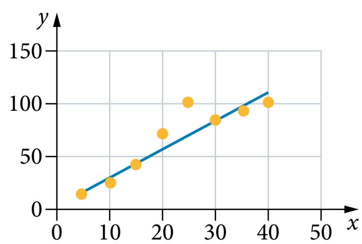
(a)

x	2	4	6	8	10	12	14
y	200	150	160	100	50	60	10
rv							



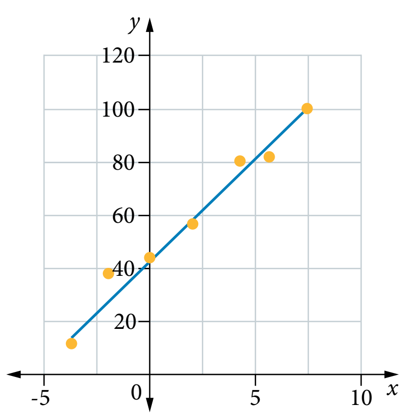
(b)

x	5	10	15	20	25	30	35	40
y	18	22	40	70	100	85	90	100
rv								



(c)

x	-4	-2	0	2	4	6	8
y	12	36	42	55	80	82	100
rv							



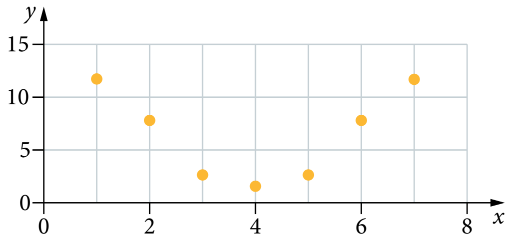
4 Consider the scatter plots and residual plots shown below.

(a) Match each scatter plot (I to IV) with its most likely residual plot (A to D).

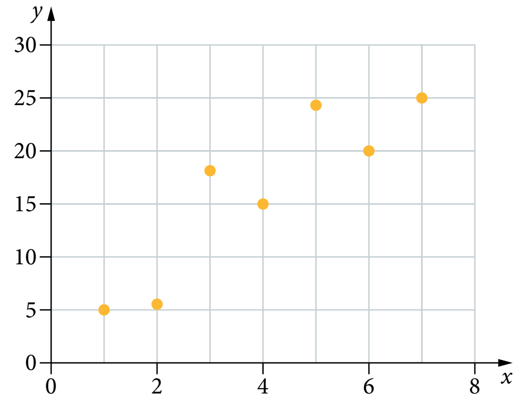
(b) Which residual plots, if any, confirm linearity?

Scatter plots:

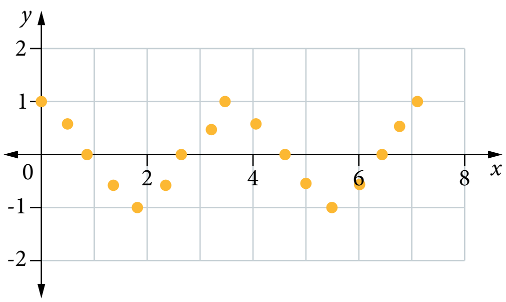
I



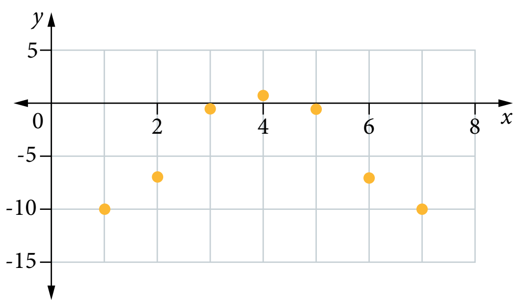
II



III

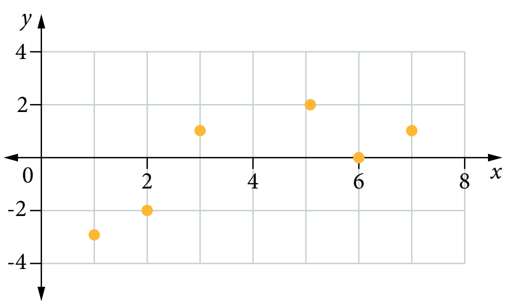


IV

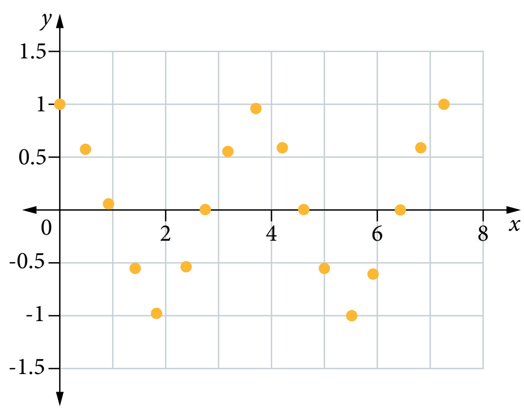


Residual plots:

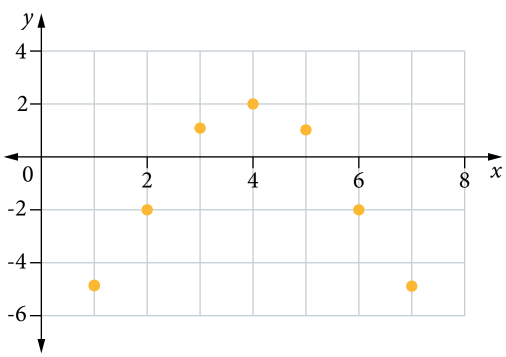
A



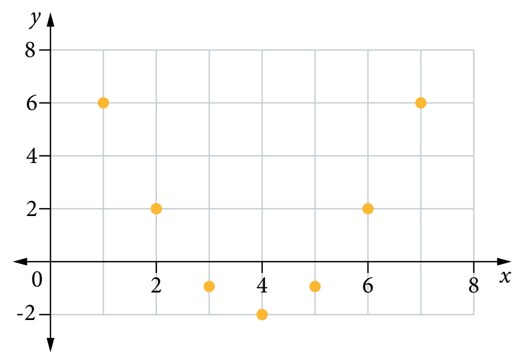
B



C



D



5 The least-squares line for the following data is $y = 11 + 0.5x$.

x	40	50	60	70	80
y	40	25		35	60

(a) The missing y-value is:

A 45 B 41 C 4 D 28

(b) Explain the common error made by a student who calculated the missing y-value to be $11 + 0.5 \times 60$

6 Use spreadsheet software to complete the table of values for each equation.

x	15	20	25	30
y				

(a) $y = 5.4 + 2.3x$

(b) $y = 204.8 - 0.61x$

(c) $y = 49.7x - 26.6$

(d) $y = 428.8 - 4.2x$

7 Complete the table of signed differences, $y_a - y_p$. There is no need to show the positive signs.

(a)

y_a	22	41	40	81	85
y_p	31.2	35.1	41.1	68.8	92.6
$y_a - y_p$					

(b)

y_a	17.8	20.0	21.1	23.4	23.9
y_p	18.1	19.7	21.2	22.8	24.4
$y_a - y_p$					

(c)

y_a	46.1	47.2	49.8	50.9	53.4
y_p	45.9	47.8	49.6	51.5	53.3
$y_a - y_p$					

(d)

y_a	50.8	43.5	37.5	32.7	23.6
y_p	50.7	44.1	37.6	31.1	24.6
$y_a - y_p$					

Worked Example

14 8 Determine the nearest equation of the least-squares line of x on y for the following summarised data sets.

(a) $\bar{x} = 8$, $\bar{y} = 12$, $s_x = 35.6$, $s_y = 41$, $r = 0.85$

(b) $\bar{x} = 11$, $\bar{y} = 22$, $s_x = 6.8$, $s_y = 42$, $r = -0.76$

16 9 Consider the following table of values.

x	15	20	25	30	35	40	45
y	20.1	19.5	20.4	21.6	20.1	20.7	21.6

(a) Determine the equation of the least-squares line.

(b) Use the equation, without rounding, to calculate the predicted values and the residuals, to 1 decimal place.

(c) Construct the residual plot and comment on the linearity of the association.

10 The data gives the average number of days of rain each month in Cairns and the average monthly maximum temperature. Use a residual plot to determine whether there is a linear association between the number of days of rain and the maximum temperatures.

If a linear association is supported, write an equation for the number of days of rain in terms of the temperature.

	J	F	M	A	M	J	J	A	S	O	N	D
Temp (°C)	31	30	30	28	27	25	25	26	27	28	30	31
Days of rain	19	19	19	19	16	12	11	10	9	10	12	16

11 The data in the table gives the average monthly maximum and minimum temperatures in Rockhampton. Use a residual plot to determine whether there is a linear association between maximum and minimum temperatures.

If a linear association is supported, write an equation for the average minimum temperature in terms of the average maximum temperature.

	J	F	M	A	M	J	J	A	S	O	N	D
Max (°C)	32.0	31.3	30.5	28.8	26.0	23.5	23.2	24.8	27.4	29.7	31.3	32.2
Min (°C)	19	19	19	19	16	12	11	10	9	10	12	16

12 The data gives the average number of points scored in a game for some Brisbane Bullets players and the player's height. Use a residual plot to determine whether there is a linear association between points scored and height.

If a linear association is strong enough, write an equation for points scored in terms of height.

Height (cm)	198	208	211	193	195	188	203	206	192
Points	16.8	14.2	10.9	10.2	7.3	7.1	6.8	4.2	2.6

13 The data gives the average head circumference and the average length (height) of baby girls in their first 24 months of life. Use a residual plot to determine whether there is a linear association between head circumference and length.

If a linear association is supported, write an equation for head circumference in terms of length.

Months	0	4	8	12	16	20	24
Length (cm)	49	62.5	69	74	78.7	82.4	86.4
Circumference (cm)	33.9	40.6	43.3	44.9	45.9	46.6	47.2

14 The data below is for 10 Brisbane Roar players who either scored goals or created chances for others to score in the first 10 games of the 2017/2018 season. Use a residual plot to determine whether there is a linear association between the number of shots on goal by a player and the number of created chances (for shots on goal) for other players.

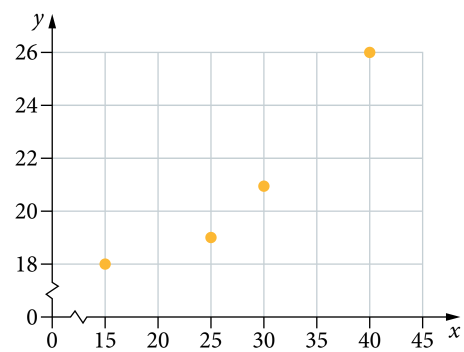
If a linear association is supported, write an equation for the number of created chances for others in terms of the number of shots on goal.

Shots	14	8	8	8	5	4	4	3	2	0	0
Created chances	7	0	3	5	10	4	11	2	3	13	4

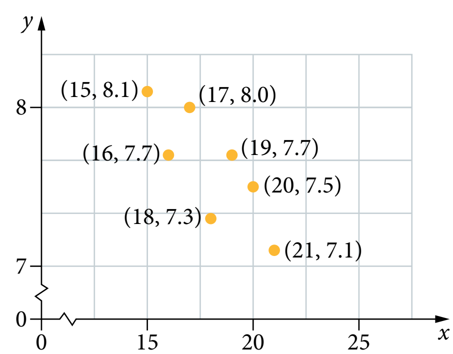
15 Determine the number of data values analysed if $\sum y = 234$, the equation of the least-squares line is $y = 3 + 4x$, $\bar{x} = 2.5$, $s_x = 3$ and $r = 0.8$

16 With small data sets, the inclusion or omission of one or two points can alter the perception of whether the data has a linear association. For this reason, small data sets are considered less reliable than large data sets.

(a) Insert two data points to the scatter plot so that this set of data would be suitable for linear regression analysis. Use a residual plot to support your choice, and calculate the correlation coefficient of the final data set.



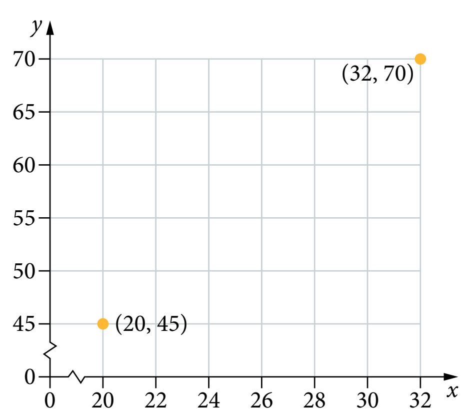
(b) Remove two points from the scatter plot so that this set of data would no longer be suitable for linear regression analysis. Use a residual plot to support your choice.



17 Create a set of data containing the two points shown, as well as three points that lie within the existing domain and range.

(a) The final data set should have a correlation coefficient $r > 0.9$ but should not be suitable for linear regression analysis. Use a residual plot to support your choice.

(b) The final data set should have a correlation coefficient $0.5 < r < 0.75$ but should be suitable for linear regression analysis. Use a residual plot to support your choice.



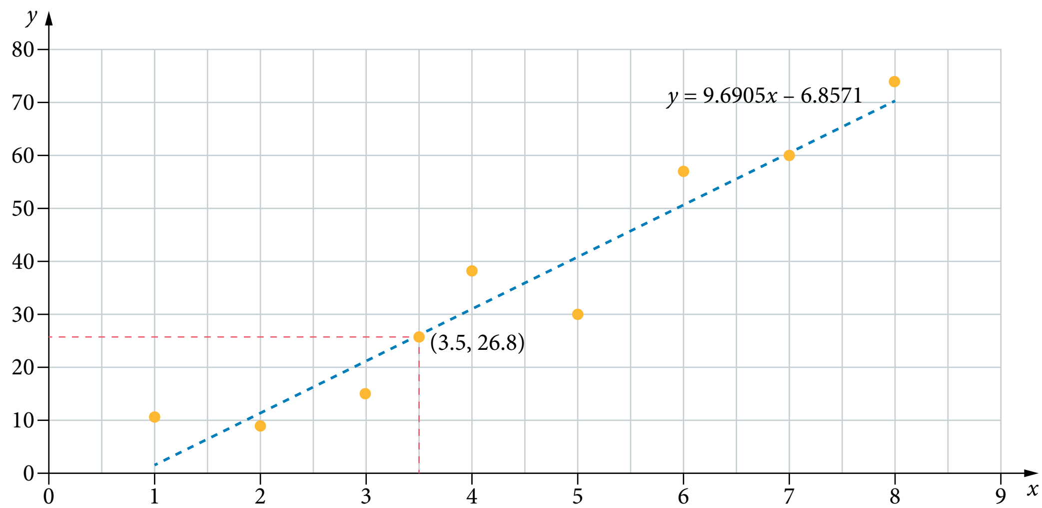
1.6 Making predictions

Reading values from a least-squares line

When data is presented as a scatter plot, you can draw a line of best fit by eye.

To make predictions about the relationship between the variables at points not given on the graph, you can draw a line from one axis to the line of best fit, to predict the corresponding value of the other variable.

If the value used for prediction comes from within the boundaries of the original data, this is called *interpolation* and the predictions are considered to be relatively reliable.



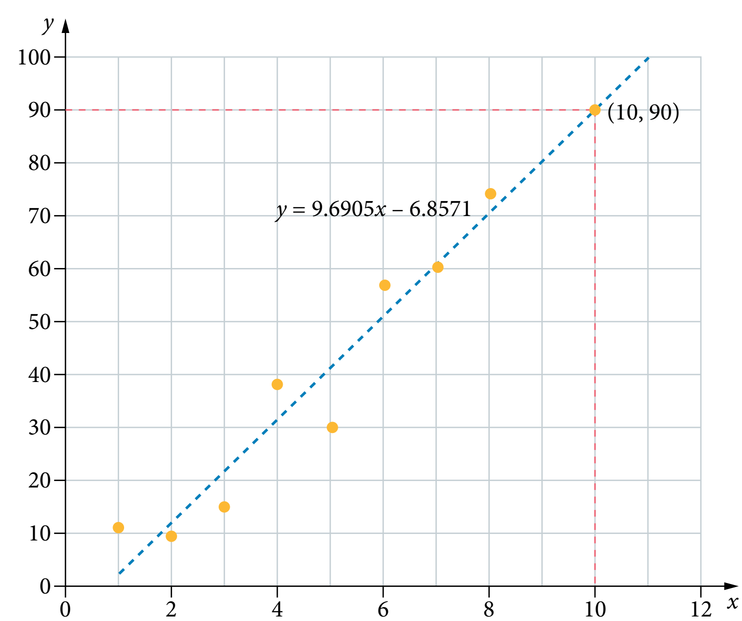
By interpolation, at $x = 3.5$ (the x-value is from within the domain of given data between $x = 1$ and $x = 8$) the predicted y_p value is:

$$y_p = 9.6905 \times 3.5 - 6.8571$$
$$= 26.8\,(1 \text{ d.p.})$$

If the line is extended and used to predict the relationship between the variables outside the boundaries of the original data, this is called *extrapolation* and can be misleading sometimes, as rules do not often extend indefinitely.

By extrapolation, at $x = 10$ (the x-value is from outside the domain of given data between $x = 1$ and $x = 8$) the predicted y_p value is:

$$y_p = 9.6905 \times 10 - 6.8571$$
$$= 90.0\,(1 \text{ d.p.})$$



When extrapolating, the reasonableness of the prediction, including the limitations of the model, needs to be considered.

For example, consider a comparison of the height of volleyball players and their spike height.

You need to determine a reasonable range for which the rule is still valid. For example, if you consider the maximum domain of female heights in volleyball to be 1.5–2 m, then extrapolating to determine the spike height of players who are 0.19 m or 15 m tall is not a reasonable calculation.

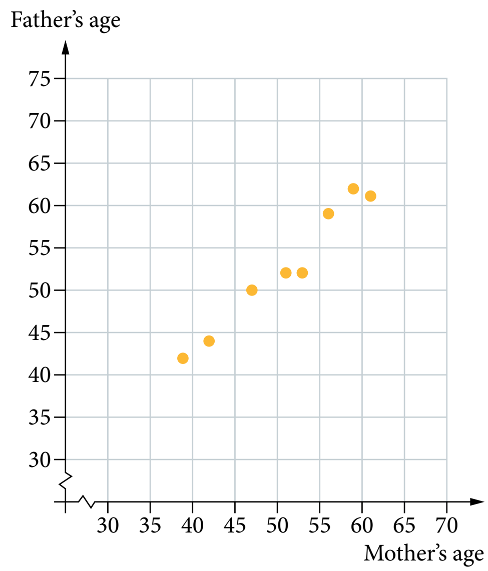
Predicting from within the boundaries of the original data is called **interpolation**.

Predicting from outside the boundaries of the original data is called **extrapolation**.

The less data and the further the least-squares line is from the original data, the less reliable the result.

18 Predicting values from a line of best fit

Consider the scatter plot shown.



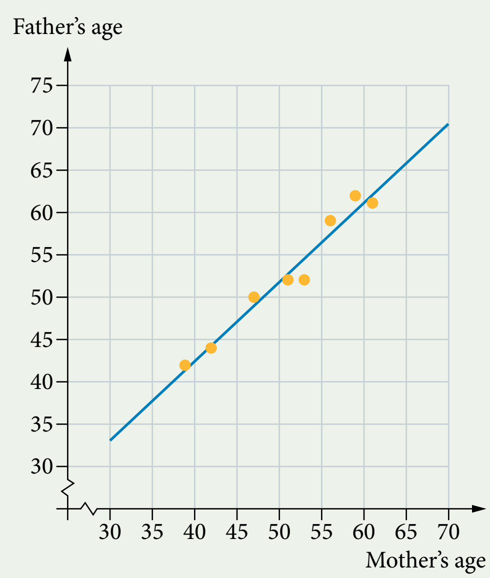
(a) Draw the line of best fit by eye and extend the line to the boundaries of the graph.

THINKING

Place the ruler so that the number of points on each side is the same, then try to minimise the total distance of the points from the line.

Extend the line to the edges of the graph.

WORKING

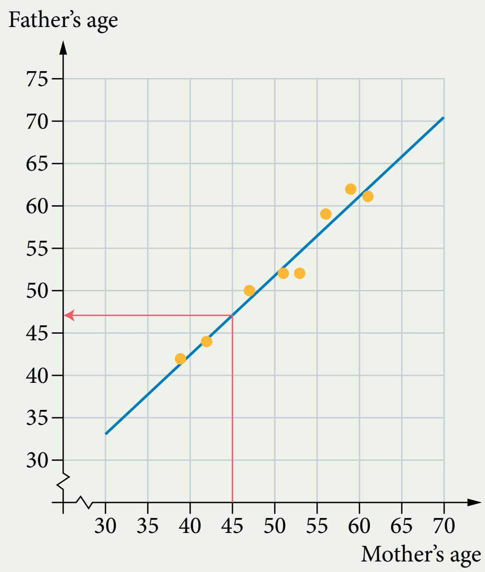


(b) Use your graph to predict each of the following values. Identify whether the process used was interpolation or extrapolation.

(i) Predict the father's age when the mother's age is 45.

1 Draw the line up from the given value and left to the axis.

Read off the value, rounding appropriately.



The coordinates of the point are (45, 47).

2 Interpret the result.

The model predicts the father's age to be 47.

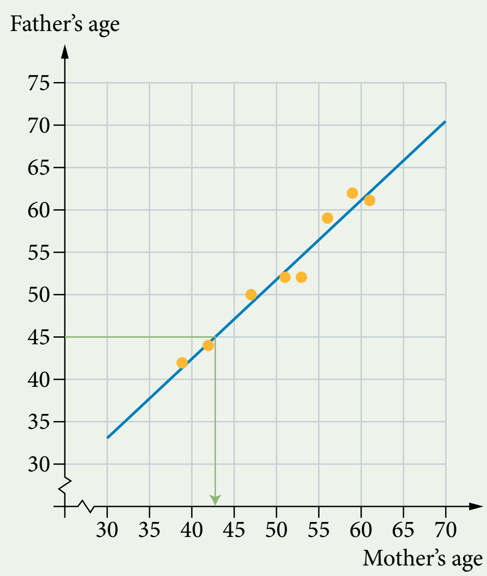
3 Identify the process to be interpolation or extrapolation.

The process involved interpolation, as the given mother's age is within the domain of the data analysed [38, 62].

(ii) Predict the mother's age when the father's age is 45.

1 Draw the line to the right from the given value and down to the axis.

Read off the value, rounding appropriately.



The coordinates of the point are approximately (43, 45).

2 Interpret the answer.

The model predicts the mother's age to be 43.

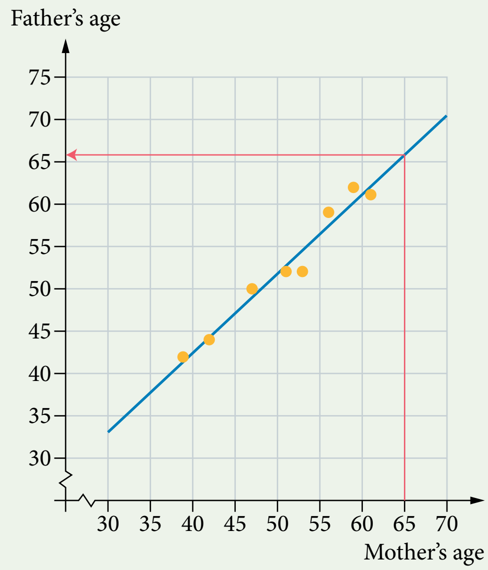
3 Identify the process as interpolation or extrapolation.

The process involved interpolation, as the given father's age is within the range of the data analysed [42, 62].

(iii) Predict the father's age when the mother's age is 65.

1 Draw the line up from the given value and left to the axis.

Read off the value, rounding appropriately.



The coordinates of the point are $(65, 66)$.

2 Interpret the answer.

The model predicts the father's age to be 66.

3 Identify the process to be interpolation or extrapolation.

The process involved extrapolation as the given father's age is outside the range of the data analysed $[42, 62]$.

In calculating the line of best fit, you could have determined the least-squares equation and then solved algebraically. It is a good idea to check your answers in this way.

WARNING

Remember, line of best fit calculations often require rounding, so accuracy is lost.

Using the least-squares equation to make predictions

If the equation of the least-squares line is known, you can predict an unknown variable by solving algebraically.

19 Making predictions from an equation

If water flow speed $= -0.04 \times (\text{depth of water}) + 0.44$, or $s = 0.44 - 0.04 \times d$, where all data values in the analysis were from depths of water between 0.22 m and 1.75 m ($0.22 \leq d \leq 1.75$), use the equation to complete the table.

Depth (m)	Speed (m/s)	Interpolation or extrapolation?
1		
	0.38	
5		

THINKING

1 Identify the given variable in the first row.

Substitute the known s-value into the given equation to determine the corresponding d-value in the first row.

WORKING

$d = 1$ m

$s = -0.04 \times d + 0.44$

$= -0.04 \times 1 + 0.44$

$= 0.40$ m/s

2 Interpret the value.	The predicted water flow at a depth of 1 m is 0.40 m/s.
3 Determine whether the process involved interpolation or extrapolation. Compare the explanatory value with the given domain.	$0.22 \leq 1 \leq 1.75$ This prediction is an example of interpolation.
4 Repeat the steps for the second row of the table.	$s = 0.38$ m/s $s = -0.04 \times d + 0.44$ $0.38 = -0.04 \times d + 0.44$ $0.38 - 0.44 = -0.04 \times d$ $-0.06 = -0.04 \times d$ $\frac{-0.06}{-0.04} = d$ $d = 1.5$ m The predicted water flow at a depth of 1.5 m is 0.38 m/s. $0.22 \leq 1.5 \leq 1.75$ This prediction is an example of interpolation.
5 Repeat the steps for the third row of the table.	$d = 5$ m $s = -0.04 \times d + 0.44$ $= -0.04 \times 5 + 0.44$ $= 0.24$ m/s The predicted water flow at a depth of 5 m is 0.24 m/s. $5 > 1.75$ This prediction lies outside the boundaries of the original data analysed and therefore is an example of extrapolation.
6 Use your answers to complete the table.	(see table below)

Depth (m)	Speed (m/s)	Interpolation or extrapolation?
1	0.40	interpolation
1.5	0.38	interpolation
5	0.24	extrapolation

Making predictions from raw data

To make predictions from raw data you must first create a model, check whether linearity is suitable and determine its strength from the correlation coefficient.

After that you can predict values from the equation with some confidence when interpolating, but with greater reservation when extrapolating, especially for values far from the data set that was used to create the model.

20 Predicting values from raw data

The data gives the latitude (°S) of a number of Australian cities, followed by the average maximum temperature (°C) for February:

Adelaide (34.9, 28.5); Alice Springs (23.7, 35.1); Brisbane (27.5, 28.9); Canberra (35.4, 27.1); Hobart (42.9, 21.6); Melbourne (37.8, 25.8); Perth (32.0, 31.9).

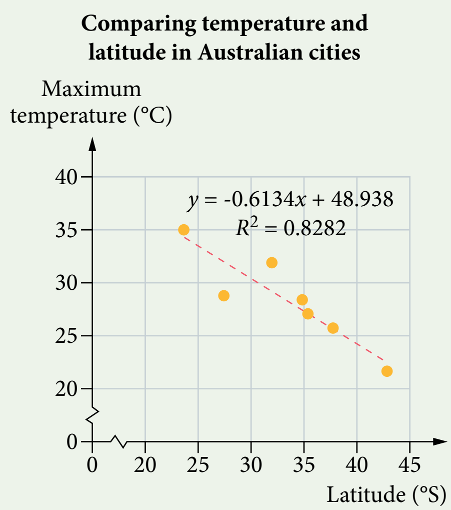
(a) Draw a scatter plot with the line of best fit. Determine the least-squares equation and the correlation coefficient *r*.

THINKING

Use Excel or other spreadsheet software to produce the scatter plot with least-squares line and equation, and display the coefficient of determination R^2 value.

Alternatively, draw the scatter plot by hand to check for linearity, then use a scientific calculator to determine the correlation coefficient *r* and the equation of the least-squares line.

WORKING



$y = -0.6134x + 48.938$

Feb av. max temp(°C) = -0.61 × latitude(°S) + 48.94

Correlation coefficient:

$r = -\sqrt{0.8282}$

$= -0.91$ (2 d.p.)

(b) Comment on the suitability of the model in terms of linearity and strength of association.

1 Describe the scatter plot as linear or not.

The scatter plot suggests a linear association.

If the scatter plot is linear, interpret the value of the correlation coefficient *r*.

There is a strong negative association between the variables: the further you travel from the equator, the lower the maximum average temperatures.

2 Interpret the coefficient of determination.

$R^2 = 0.8282$

$\approx 83\%$

83% of the variation in the average maximum February temperature responds to, or can be explained by, the variation in latitude.

(c) Sydney has a latitude of 33.9°S and Townsville 19.3°S.

(i) For which of the two cities would the model predict the average maximum temperature for February more reliably?

Compare the values with the extreme values of the explanatory data.	For the given data: 23.7°S ≤ latitude ≤ 42.9°S Sydney: 23.7°S ≤ 33.9°S ≤ 42.9°S Townsville: 19.3°S < 23.7°S The prediction for Sydney should be more reliable as interpolation will be used, whereas the Townsville prediction involves extrapolation.

(ii) Predict the average maximum temperature for February for each city.

1 Substitute each value into the least-squares equation.	Sydney: $x = 33.9$ $y = -0.6134x + 48.938$ $= -0.6134 \times 33.9 + 48.938$ $= 28.1\,(1 \text{ d.p.})$ Townsville: $x = 19.3$ $y = -0.6134 \times 19.3 + 48.938$ $= 37.1\,(1 \text{ d.p.})$
2 Interpret the answer.	The predicted average maximum temperatures for February are: Sydney 28.1°C Townsville 37.1°C

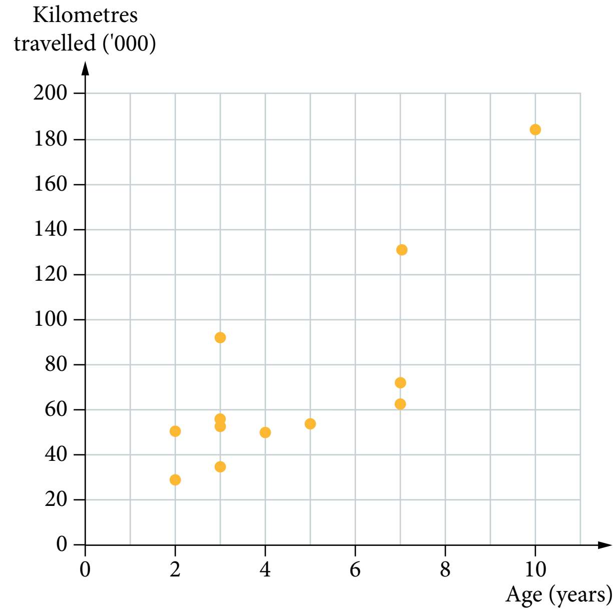
(iii) Given that the average maximum February temperatures are Sydney 26.3°C and Townsville 31.1°C, comment on the accuracy of the residuals.

1 Calculate the residuals using rv = $y_a - y_p$.	Sydney: residual = 26.3 − 28.1 = -1.8°C Townsville: residual = 31.1 − 37.1 = -6.0°C
2 Comment on the accuracy in terms of interpolation or extrapolation.	As expected, the average maximum February temperature for Townsville was predicted with less accuracy, due to extrapolating well beyond the domain of the given data.

EXERCISE 1.6 Making predictions

Worked Example 18

1 The scatter plot shows number of kilometres travelled against age for some used cars.



(a) Draw the line of best fit by eye and extend the line to the boundaries of the graph.

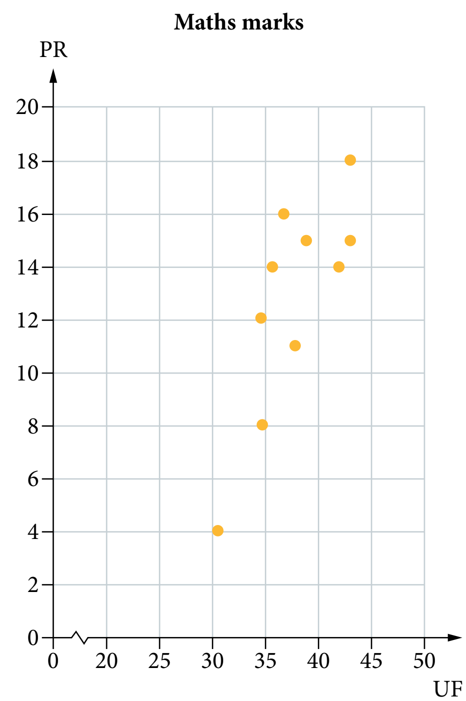
(b) Use your graph to predict each of the following values. Identify the process used as interpolation or extrapolation.

(i) Predict the number of kilometres travelled for a 9-year-old car.

(ii) Predict the age of a car that has travelled 80 000 km.

(iii) Predict the number of kilometres travelled for a 12-year-old car.

2 The scatter plot shows pairs of Maths marks for students in the categories Understanding and fluency (UF), out of 50, and Problem solving and reasoning (PR), out of 20.



(a) Draw the line of best fit by eye and extend the line to the boundaries of the graph.

(b) Use your graph to predict each of the following values. Determine whether the process used was interpolation or extrapolation. Comment on whether any extrapolated values are reasonable.

(i) the PR mark when the UF mark is 40

(ii) the UF mark when the PR mark is 20

(iii) the UF mark when the PR mark is 10

3 The least-squares equation $y = 0.8x - 4.4$ is used to link the value x (\$'000) of a one-year-old car and the value y (\$'000) of a five-year-old car of the same make and model.

(a) Predict the value of a one-year-old Mazda 3 if a five-year-old Mazda 3 just sold for \$12 000.

A \$5200 B \$20 500 C \$9596 D \$15 006

(b) Explain the common error made by a student who calculated the value of the one-year-old Mazda in part (a) to be $y = 0.8 \times 12 - 4.4$ and state how a reasonableness check could have identified the error.

(c) Explain the other kind of common error made by a student who chose either of the other incorrect options in part (a).

4 The least-squares equation
length (m) = 0.2 + 0.7 × diameter (mm)
links the length and diameter of octopus tentacles.
The original data set contains tentacle lengths between 0 m and 4 m. Use the equation to complete the table.

Diameter (mm)	Length (m)	Interpolation or extrapolation?
1		
	2.3	
	10	

5 The least-squares equation
mass (kg) = 0.5 × height (cm) − 1.4
links the height and mass of students in a P–12 school.
The original data set contains student heights from 50 cm to 180 cm. Use the equation to complete the table.

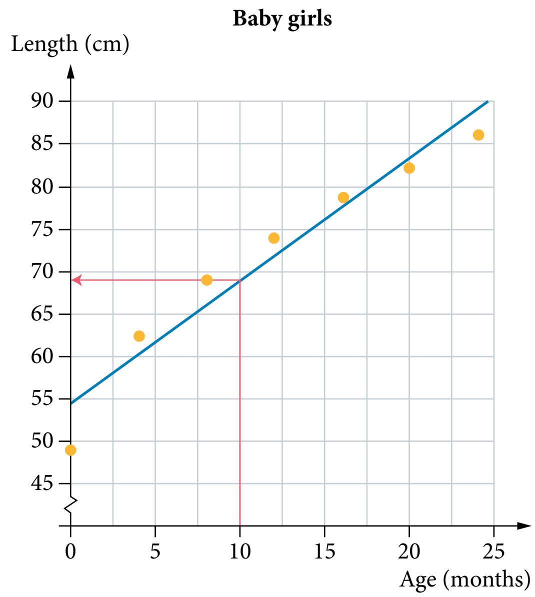
Height (cm)	Mass (kg)	Interpolation or extrapolation?
150		
	70	
	15	

6 The least-squares equation $y = 2.64x - 12.87$ links the population of an Australian state or territory (in thousands) x and the number of motor car thefts in a year y. Predict the number of motor car thefts, to the nearest hundred, expected in each of the following states in a year.

(a) Queensland, population around 5 million

(b) Tasmania, population around 550 000

7 Anika used the line of best fit to estimate the average length of a baby girl of a particular age. Her working is shown here.



Choose the correct statement.

A It is impossible to predict values using Anika's working.

B Fitting a least-squares line is not suitable for this data.

C The linear association is positive and strong.

D The average length of a 10-month-old baby girl is about 69 cm.

8 The least-squares equation $y = 1.37x - 22.22$ links average monthly maximum temperatures x (°C) and average monthly minimum temperatures y (°C) in Rockhampton. In answering the following questions, round values appropriately.

(a) Predict the average minimum temperature in April, given that the April average maximum temperature is 28.8°C.

(b) Predict the average maximum temperature in September, given that the September average minimum temperature is 13.7°C.

Worked Example 20

9 The data gives the latitude (°S) of a number of Australian cities, followed by the average maximum temperature (°C) for August.

	Brisbane	Canberra	Darwin	Melbourne	Perth	Sydney	Townsville
Latitude	27.5	35.4	12.4	37.8	32.0	33.9	19.3
Average max temp	21.7	13.0	31.3	15.0	18.4	18.3	26.0

(a) Draw the scatter plot with a line of best fit. Determine the least-squares equation and the correlation coefficient r. Round all values to 2 decimal places.

(b) Describe the association and causation, where the coefficient of determination $R^2 = 0.95$.

(c) Adelaide is at latitude 34.9°S and Hobart is at 42.9°S.

(i) For which of the two towns would the model predict the average maximum temperature for August more reliably?

(ii) Predict the average maximum temperature for each town.

(iii) Given that the actual temperatures are Adelaide 16.7°C and Hobart 13.0°C, comment on the residuals in terms of whether interpolation or extrapolation was used.

10 The data gives regional population (millions) followed by the latest annual figures for 'unlawful entry' for a number of regions in Queensland.

(a) Draw a scatter plot with line of best fit. Determine the least-squares equation and the correlation coefficient r. Round all values to 2 decimal places.

(b) Describe the association and causation, where the coefficient of determination $R^2 = 0.76$.

(c) Mackay has a regional population of 183 000. Predict the incidence of unlawful entry for the region.

(d) Given that the actual incidence of unlawful entry in the Mackay region is 1343, comment on the residual in terms of crime in the region.

Location	Population (millions)	Unlawful Entry
Brisbane	1.5345	10 274
Central	1.032	6 076
Northern	0.559	7 368
South eastern	0.921	7 422
Southern	0.841	5 681
North Brisbane	0.746	4 602
South Brisbane	0.799	5 672
Capricornia	0.239	1 707
Sunshine Coast	0.352	1 485

11 The data gives youth crime figures as a rate per 100 000 youth in the given state or territory.

	NSW	Vic.	SA	WA	Tas.	ACT
Injury	403.4	308.3	411.7	419.2	362.1	186.5
Theft	1 405.8	447.7	993.6	582.5	395.4	240.2

(a) Draw a scatter plot with line of best fit, using the first row as the explanatory variable. Determine the least-squares equation and the correlation coefficient *r*. Round all values to 2 decimal places.

(b) Describe the association.

(c) Queensland's rate of youth crime involving injury is 330.8. Calculate the rate forecast by the model for Queensland's youth theft rate. Compare this value with the actual value of 588.8. Comment on the result.

(d) Northern Territory has a youth theft rate of 262.7. Calculate the rate forecast by the model for Northern Territory's rate of youth crime involving injury. Compare this value with the actual value of 544.2. Comment on the result.

12 The student results shown below are for two aspects of an oral presentation about poetry. I is receptive modes, including thesis and arguments, and II is productive modes, including use of vocabulary.

I	B^*	A^*	A-	B+	A-	B+	A-	C+	A-	B+	B-	C+
II	B-	A^*	B+	C+	B+	B-	A-	C+	B+	B^*	B^*	B^*

(a) Convert the grades to a numerical scale, with A+ $\leftrightarrow$ 15, A^* $\leftrightarrow$ 14, A- $\leftrightarrow$ 13 and so on down to E- $\leftrightarrow$ 1.

(b) Draw a scatter plot of the numerical data with a line of best fit, using the first row of data as the explanatory variable. Determine the least-squares equation and the correlation coefficient *r*. Round all values to 2 decimal places.

(c) Describe the association.

(d) What score does the model predict for Productive modes when a student scored C- for Receptive modes?

(e) What score does the model predict for Receptive modes when a student scored C- for Productive modes?

13 Two of the many variables that influence temperature and rainfall in Australia are altitude (height above sea level) and latitude (a measure of the distance from the equator), where a larger number represents a greater distance from the equator.

The data below shows the average maximum daily temperature and average yearly rainfall recorded against latitude, all measured at the same altitude.

Latitude (°S)	10	15	20	25	30	35	40	45
Average temperature (°C)	31	28	24	21	16	14	15	10
Average rainfall (mm)	630	550	600	450	250	654	1168	800

(a) Consider the association between latitude and temperature.
  (i) Identify the explanatory variable and draw a scatter plot.
  (ii) Comment on the association between latitude and average maximum temperature in terms of strength, direction and linearity. Describe the trend.
  (iii) Calculate the Pearson correlation coefficient r.
  (iv) Determine the least-squares equation, giving values to 1 decimal place.

(b) Consider the association between latitude and rainfall.
  (i) Identify the explanatory variable and draw a scatter plot.
  (ii) Comment on the association between latitude and average rainfall in terms of strength, direction and linearity. Describe the trend.
  (iii) Calculate the Pearson correlation coefficient.
  (iv) Determine the least-squares equation.

(c) Which of the two associations is stronger? Explain your answer.

(d) For what range of values could interpolation be used to predict the temperature, given the latitude?

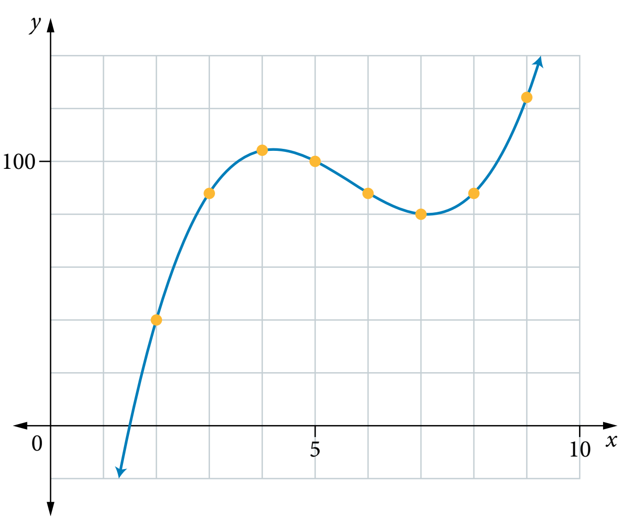
14 There is an association between the heart rate and mass of birds. The smaller the bird, the faster the heart rate. The data for a selection of birds of different sizes is given as mass (g) followed by heart rate (bpm, beats per minute).

canary (20,1000); pigeon(300, 185); crow (341, 378); wild duck (1100, 190); hen (2000, 312); domestic duck (2300, 240); turkey (8750,193)

(a) Select a portion of the data so that a linear association appears to be suitable. Determine the least-squares equation and correlation coefficient for your data.

(b) Use a selection of the data that was not included, to demonstrate the unsuitability of applying a least-squares equation to data whose association is not linear.

15 The graph of $y = 2x^3 - 34x^2 + 180x - 200$ is shown, with points (2, 40), (3, 88), (4, 104), (5, 100), (6, 88), (7, 80), (8, 88) and (9, 124).



(a) Choose any five points from the list that would form a set of data suitable to determine the equation of the least-squares line.

(b) Determine the least-squares equation and correlation coefficient r for the data, rounding to 2 decimal places where appropriate.

(c) Choose one x-value that uses interpolation to predict the y-value from the linear model. Compare the predicted value with the actual value from the curve.

(d) Choose one x-value that uses extrapolation to predict the y-value from the linear model. Compare the predicted value with the actual value from the curve.

(e) Compare the residuals for the interpolated and extrapolated values and comment on the relative suitability of each in terms of the curve given in the question.

Summary

Bivariate data

Bivariate data is data that has two variables.

Where an association exists, the explanatory variable is the one that could explain a change in the response variable The response variable responds to, or can be explained by, a change in the explanatory variable.

Categorical data

A two-way table uses row and column totals to enable calculation of the proportion of the whole for each subcategory.

Appropriately percentaged two-way tables can help identify associations in the data.

Scatter plots

Drawing a scatter plot allows you to see any association that may exist between two numerical data sets.

Where an explanatory variable exists, it is plotted on the horizontal axis.

A positive association means both variables are increasing together. A negative association means one variable increases as the other decreases.

The form of the association is linear or non-linear.

The strength of association

A strong association means the points conform closely to the pattern, e.g. a strong linear association will be represented by points that are close to a straight line.

Where an association exists, the strength can be classified as strong, moderate or weak.

Correlation is a measure of the strength of the linear association between the variables. Only where a linear trend exists, can you calculate the Pearson correlation coefficient r where $-1 \le r \le 1$.

If $r = -1$ the points form a straight line with a negative slope.

If $r = 1$ the points form a straight line with a positive slope.

If $r = 0$ there is a random collection of points with no association.

$0.75 \le r \le 1$ Strong positive linear association
$0.5 \le r < 0.75$ Moderate positive linear association
$0.25 \le r < 0.5$ Weak positive linear association
$-0.25 < r < 0.25$ No association
$-0.5 < r \le -0.25$ Weak negative linear association
$-0.75 < r \le -0.5$ Moderate negative linear association
$-1 \le r \le -0.75$ Strong negative linear association

Causation

The coefficient of determination R^2 can be used to determine how much the variation in the explanatory variable is responsible for the variation in the response variable.

Strong correlation does not imply a causal relationship. Both variables may have a common response to a third value. Causation may exist, but the change may also be caused by one or more uncontrolled, confounding variables whose effects cannot be separated from the effect of the explanatory variable. This is known as confounding.

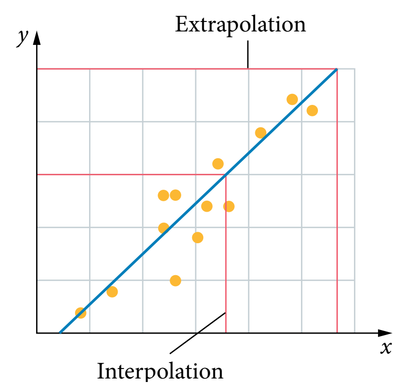
The equation of the least-squares line

Finding the equation of the line of best fit is known as regression analysis. Technology is used to determine the value of the correlation coefficient r, the coefficient of determination R^2 and the equation of the least-squares line in the form $y = a + bx$, where $a = \bar{y} - b\bar{x}$ and $b = r\frac{s_y}{s_x}$.

Making predictions

Interpolation involves predicting a data value from the model graph or equation from within the limits of the original data. Extrapolation involves predicting a data value from outside the limits of the original data.

Interpolation is more likely to give sensible values than extrapolation, especially for values far from the data set that was used to create the model.



Residual analysis

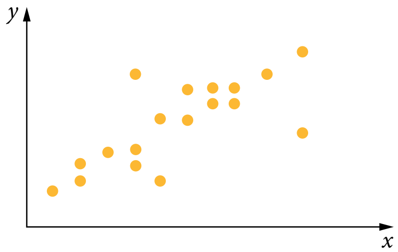
A residual is the signed difference between the observed value y_a and the value predicted by the equation of a line of best fit, y_p, usually the least-squares equation $\text{rv} = y_a - y_p$.

To confirm the assumption of linearity, a residual plot is drawn. If there is a random dispersion of points, the linear assumption is confirmed.

Chapter review 1

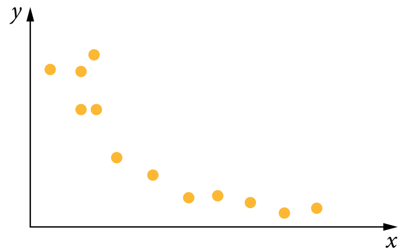
1 Choose the most appropriate description of the association displayed in each graph. Exercise 1.2

(a)



A weak, positive, linear
B moderate, positive, linear
C non-linear
D moderate, negative, linear

(b)



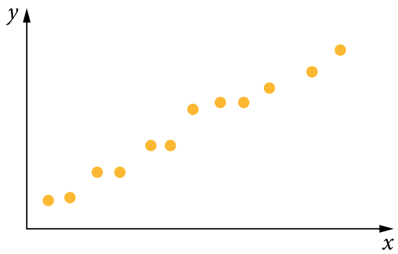
A weak, negative, linear
B moderate, negative, linear
C non-linear
D moderate, positive, linear

2 Assuming an association exists, identify the explanatory variable in each of the following pairs. Exercise 1.2

(a) length of nose and age
(b) hours spent studying and test score
(c) heart rate and intensity of exercise
(d) plant growth and hours of sunlight
(e) tread remaining on a car's tyres and distance travelled

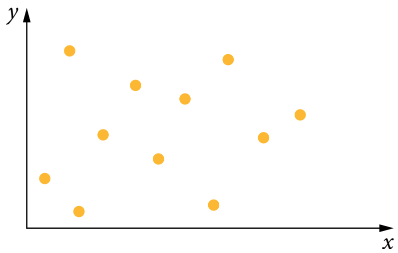
3 For the scatter plot shown, what is the value of the correlation coefficient r? Exercise 1.3

(a)



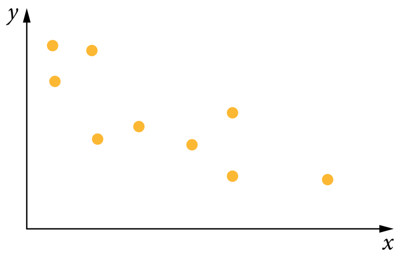
A $-1 < r < -0.7$
B $-0.5 < r < -0.3$
C $0.3 < r < 0.5$
D $0.7 < r < 1$

(b)



A $-1 < r < -0.7$
B $-0.5 < r < -0.3$
C $-0.2 < r < 0.2$
D $0.3 < r < 0.5$

(c)



A $-1 < r < -0.7$
B $-0.5 < r < -0.3$
C $-0.2 < r < 0.2$
D $0.3 < r < 0.5$

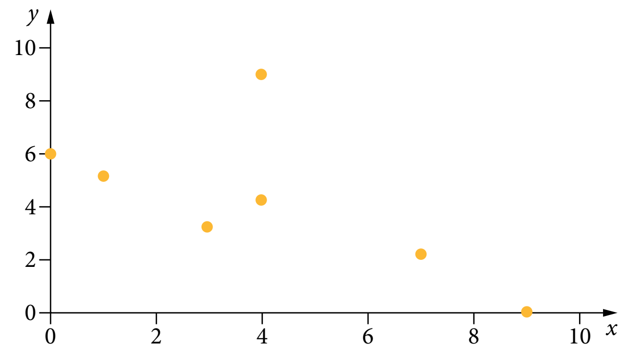
Exercise 1.3

4 Alex and Sam like to note their knowledge of Rugby Union by creating their own statistics. Alex investigates the association between the number of tries and the number of games played. Sam looks at the association between the number of games played and the number of letters in the player's surname. Estimate the values of r that Alex and Sam are likely to find.

Exercise 1.3

5 The correlation coefficient for the scatter plot shown was found to be -0.6.

The point (4,9) was found to be recorded incorrectly and should have been plotted as (4,1). Based on this change, describe the correct correlation coefficient.



Exercise 1.3

6 Consider the following possible causal relationships. Describe the relationship that exists, and identify whether a causal relationship exists, or the common response or confounding variables that may be influencing the data.

(a) The speed travelled and the time taken to complete a journey

(b) The oral health of a pregnant woman and the health of her unborn baby

(c) The number of beach towels sold at a sports store and the number of snowboards sold

(d) The number of grey hairs a person has and the number of heart attacks

(e) The cost of a petrol purchase and the number of litres of petrol purchased

(f) A person's level of education and their level of job satisfaction

Exercise 1.4

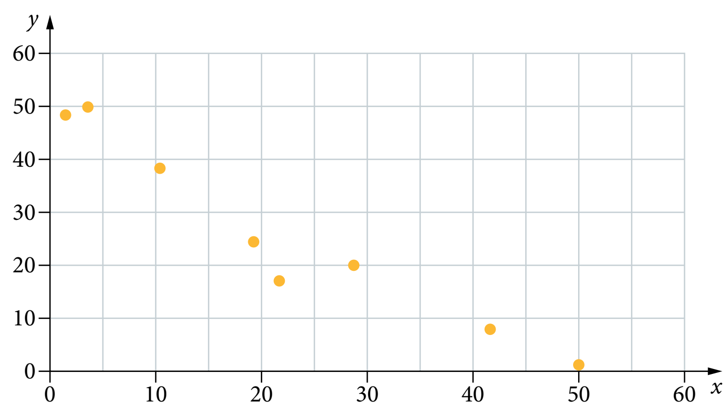
7 Use the scatter plot to answer the following questions.

(a) Place a line of best fit.

(b) Calculate the gradient of the line.

(c) Determine the coordinates of the y-intercept.

(d) Write the equation of the line of best fit.



Exercise 1.5

8 The table shows the number of hours worked and the number of items produced by seven workers.

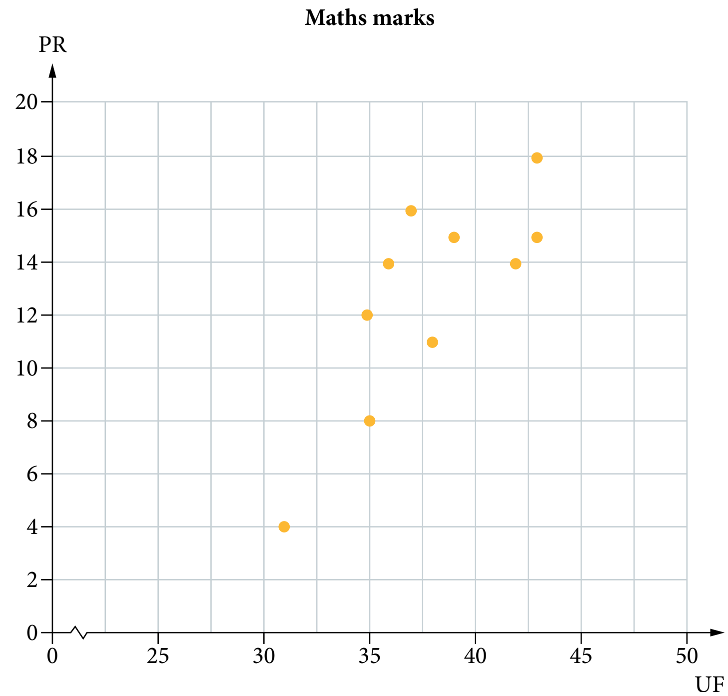
Worker	1	2	3	4	5	6	7
Hours worked	18	38	50	42	24	35	40
Items produced	2 000	2 500	3 200	3 000	1 500	2 400	2 800

The supervisor determines the equation of the least-squares line for the relationship between the number of items produced and the hours worked to be $p = 49h + 763$, where p is the number of items produced and h is the hours worked. When this equation is used to predict the number of items produced after 35 hours of work, what is the residual value?

9 The scatter plot gives pairs of Maths marks for students in the categories Understanding and fluency (UF), out of 50, and Problem solving and reasoning (PR), out of 20.

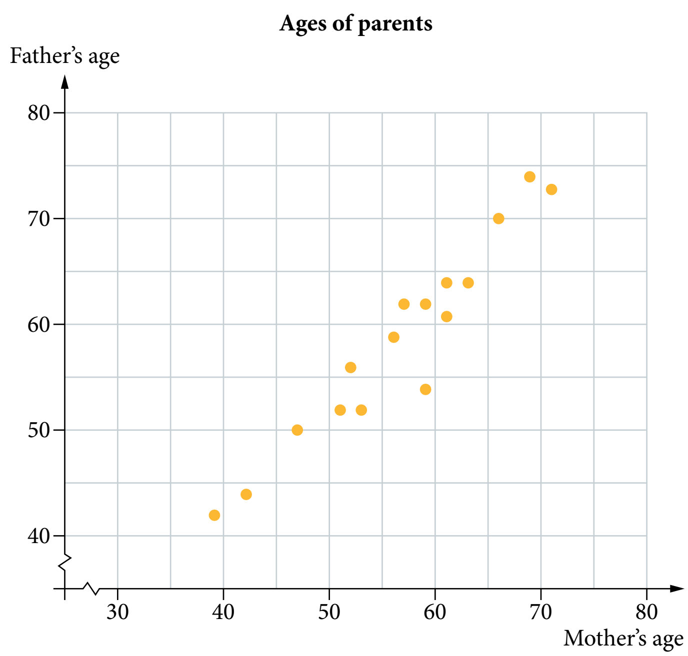
If the data was used to predict marks for students who were not present for one of the tests, what range of values for each variable would allow the process of interpolation to be used?

Exercise 1.6



10 The scatter plot gives ages of parents for a group of young people.

Exercise 1.6



(a) Draw the line of best fit by eye and extend the line to the boundaries of the graph.

(b) Use your graph to predict each of the following ages. Determine whether the process used was interpolation or extrapolation.

(i) the father's age if the mother is 50

(ii) the mother's age if the father is 60

(iii) the mother's age if the father is 80

CHAPTER REVIEW 1

Exercise 1.6

11 The equation of the least-squares line $y = 0.88x - 7.83$ links the average monthly maximum temperatures for Australian cities in November x (°C) and the average monthly minimum temperatures y (°C). Answer the following, rounding values appropriately.

(a) Predict the November average minimum temperature for Townsville, given that the average maximum temperature is 30.7°C.

(b) Predict the November average maximum temperature for Brisbane, given that the average minimum temperature is 18°C.

Exercise 1.3

12 Experimental results indicate that the correlation coefficient r for the least-squares relationship between the mass of food remaining after an ant colony begins eating m and the number of ants at the scene n is -0.4. Describe the relationship.

Exercise 1.2

13 You wish to analyse data involving daily sales of bags of ice and maximum daily temperature.

(a) Which is the explanatory variable?

(b) Will the association be positive or negative?

(c) To what extent does the explanatory variable explain (or cause) change in the response variable? Discuss.

Exercise 1.3

14 The coefficient of determination R^2 between variables x and y is 0.64. Determine the correlation coefficient r.

Exercise 1.3

15 The correlation coefficient for the number of points won by Helena on serve during a tennis match and the number of her first serves in play is 0.6. What percentage of the variance in Helena's success on serve is dependent on her first serve?

Exercise 1.6

16 For the following data, determine the equation of the least-squares line.

(a) $\bar{x} = 37.6$, $\bar{y} = 9$, $s_x = 2.4$, $s_y = 3.2$ and $r = 0.9$

(b) $\bar{x} = 12.4$, $\bar{y} = 1.6$, $s_x = 4.6$, $s_y = 0.4$ and $r = -0.7$

Exercise 1.1

17 A survey of 150 people of various ages was conducted to investigate opinions about hip-hop music. The data collected is shown in the two-way table.

Age	Enjoy	No opinion	Dislike
10–20	4	18	2
20–30	5	6	5
30–40	3	20	27
>40	4	16	40

(a) Convert each value in the table, including the marginal totals, to a percentage of the grand total. Write values to the nearest per cent.

(b) Create an appropriately percentaged table to investigate an association between the variables age and opinion. Round answers to the nearest whole percentage.

(c) Discuss any trends shown in the data.

18 The table represents the results obtained on a Physics test (marked out of 30) taken by male and female students. Exercise 1.1

Females	18	9	17	23	28	22	17	6	21	29
Males	18	24	27	16	15	21	20	19	24	23

Create an appropriately percentaged table to determine whether there is any association between gender and good Physics marks. Classify 'good' as anything above 20.

19 The Mathematics scores for two tests, out of 70 and 20 respectively, for a class of Year 8 students are given in the table. Exercise 1.3

Test 1	46	52	66	65	45	65	59	67	23	69	58	64	18	30
Test 2	6	11	14	12	12	16	14	20	4	20	13	10	6	7

Assuming a linear association, with the explanatory variable as Test 1, calculate and interpret the correlation coefficient.

20 For the data given in the table, assume linear regression analysis is suitable. Exercise 1.4

Name	Abhu	Kiet	Carli	Danni	Ek	Liang	Ramone
Age	1	3	6	9	10	11	12
Shoe size	1	4	7	6	9	9	12

(a) When comparing age and shoe size, the correlation coefficient r is closest to which value?

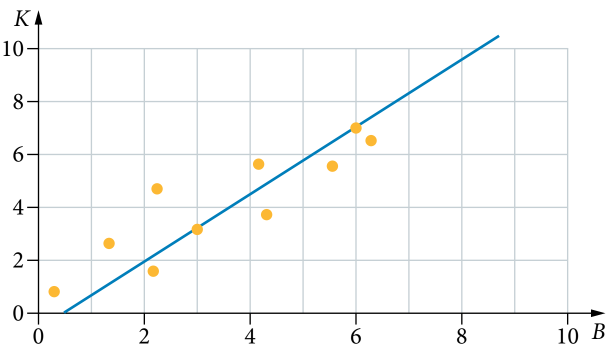
A 0.81 B 0.88 C 0.94 D 8.5

(b) What is the equation of the line of best fit?

A Shoe size = 0.81 × Age − 0.85 B Shoe size = 0.81 × Age + 0.85

C Shoe size = 0.85 × Age + 0.81 D Shoe size = 0.94 × Age + 0.85

21 A scatter plot is shown, along with its line of best fit. Exercise 1.4



Determine the equation of the line of best fit.

A $K = 4B + 3$ B $K = 4B - 3$

C $K = \frac{4}{3}B + 1$ D $K = \frac{4}{3}B - 1$

CHAPTER REVIEW 1

Exercise 1.5

22 The table shows the height (cm) and mass (kg) of players on the roster for the NBL club Perth Wildcats for 2013.

Name	Height (cm)	Mass (kg)	Name	Height (cm)	Mass (kg)
Jermaine Beal	191	92	Mathiang Muo	196	95
Greg Hire	201	99	Jesse Wagstaff	203	100
James Ennis	200	95	Erik Burdon	188	82
Matthew Knight	204	109	Shawn Redhage	202	103
Tom Jervis	211	105	Damian Martin	186	92
Drake U'u	192	97			

Source www.nbl.com.au

(a) Which of the variables, height or mass, is the response variable?

(b) Construct a scatter plot and comment on the association.

(c) Determine the value of the correlation coefficient r, to 2 decimal places.

(d) Determine the least-squares equation to 1 decimal place.

(e) Use the least-squares formula to complete a table of residuals.

(f) Draw the residual plot and comment on the assumption of linearity.

Exercise 1.6

23 The average daily hours of sunshine for July for Australian capital cities are given in the table, along with the average maximum temperatures (°C).

	Adelaide	Brisbane	Canberra	Darwin	Hobart	Melbourne	Perth	Sydney
Sun (h)	4.8	7.3	5.8	10.1	4.4	4.5	6.1	6.6
Temp (°C)	14.9	20.6	11.4	30.5	11.7	13.4	17.9	17

Source: http://www.bom.gov.au/calendar/annual/climate.shtml

(a) Draw a scatter plot by hand to establish a linear association between the variables.

(b) Determine the equation for the least-squares line, with values to 2 decimal places.

(c) Use the equation to predict each of the following, and identify whether each is an example of interpolation or extrapolation:

- **(i)** the hours of sunshine for Alice Springs, given that the average maximum temperature for July is 19.1°C
- **(ii)** the average maximum temperature for Townsville, given that the hours of sunshine for July is 8.4.

24 The least-squares equation for this data set is $y = 1.2x + 39$. Exercise 1.6

x	10	20	30	40	50	60
y	58	70	45	95	102	110

(a) Complete the table of residuals and then draw a residual plot.

x	10	20	30	40	50	60
y	58	70	45	95	102	110
Predicted value						
Residual						

(b) Comment on the assumption of linearity.

(c) The correlation coefficient r for this data is 0.83. To the nearest per cent, what percentage of the variation in the response variable is due to variation in the explanatory variable? Explain any assumptions you need to make about the data.

25 A group of friends challenge each other to do their best on their upcoming exams. For the month before the exams, they record the time they spend studying. Their results for General Maths and English are shown in the table. Exercise 1.6

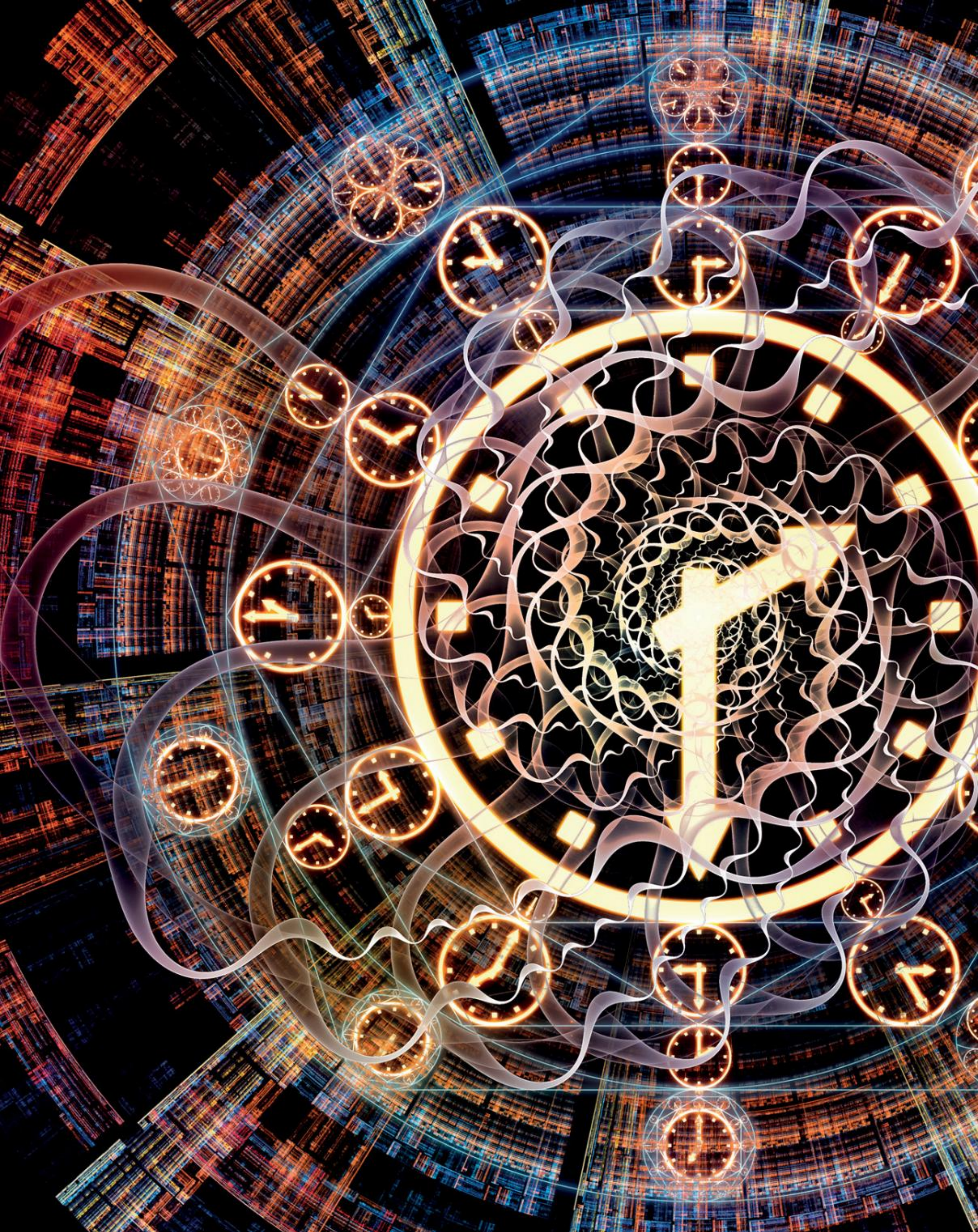
Name	Anika	Brae	Chao	Damon	Ed	Fai
Total hours of study	50	32	10	30	70	85
General Maths (%)	70	50	70	80	95	90
English (%)	85	75	60	45	90	96

(a) Determine the relationship between total hours of study and General Maths result. Support your results with a scatter plot, an equation with values to 1 decimal place, a residual plot and the correlation coefficient r to 2 decimal places.

(b) Determine the relationship between total hours of study and English result. Support your results with a scatter plot, an equation with values to 1 decimal place and the correlation coefficient r to 2 decimal places.

(c) Use your least-squares equation to predict the English result with the following hours of study: 60 hours; 120 hours. Comment on the predictions.

(d) Is there a clear association between hours of study and overall result in English and General Maths? Explain any assumptions you make, and the limitations of your solution.



2

Time series analysis

Recall 88

2 Recall

Identify an outlier

1 An outlier is a data item that does not fit the general pattern of the rest of the data. It may be too large or too small. One definition of when a data value is considered to be an outlier uses the IQR (interquartile range), Q_1 and Q_3 (lower and upper quartiles) to define lower and upper boundaries, called fences. Any data value outside the fences is considered an outlier.

Lower fence: $Q_1 - 1.5 \times \text{IQR}$

Upper fence: $Q_3 + 1.5 \times \text{IQR}$

Identify the outlier(s), if any, in the following data sets.

(a) 2, 4, 6, 8, 12, 12, 14, 15, 42

(b) 0.2, 3.5, 3.8, 4.1, 4.5, 4.9, 7.2

(c) 16, 14, 36, 18, 24, 12, 30, 22

Calculate a mean

2 Use $\bar{x} = \frac{\sum x}{n}$ to calculate the mean of the following sets of data scores.

(a) 12, 14, 16, 18 (b) 0.2, 3.5, 3.8, 4.1, 4.4 (c) 8, 12, 12, 14, 15, 41

Calculate relative frequencies

3 Use relative frequency $= \frac{f}{\sum f}$ to calculate the relative frequency of each x-value as a decimal, to 2 decimal places.

(a)

x	f
12	7
13	20
14	5

(b)

x	83	84	85	86	87
f	12	56	68	42	5

Determine the equation of a line that passes through two points

4 Determine the equation of a line through the given points. Give answers in the form $y = a + bx$, with decimal values where necessary.

(a) $(3, 10)$ and $(8, 50)$ (b) $(40, 75.5)$ and $(60, 12.3)$

Substitute given values into the equation of a line

5 Consider a line with the equation $y = 37.4 - 4.2x$.

(a) Determine the value of y when $x = 3.5$

(b) Determine the value of x, to 1 decimal place, when $y = 21.5$

Interpret the gradient and y-intercept of a line

6 The equation for the distance of a car from home d (km) in terms of time t (hours) is given as $d = 40t + 12$.

(a) Determine the gradient of the graph of the equation, and explain what the value means.

(b) Determine the vertical axis intercept of the graph, and explain what the value means.

Time series

2.1

Graphing time series data

Data that is collected or recorded regularly over time is called *time series data*. This type of data is useful for detecting patterns that may lead to predictions about future trends.

Examples of time series data are:

- unemployment statistics
- the value of the Australian dollar
- new car sales
- maximum and minimum temperatures recorded at the same weather station.

To begin the analysis of time series data, construct a graph. Time is the explanatory variable, plotted on the horizontal axis. The variable of interest, the response variable, is plotted on the vertical axis.

1 Plotting a time series graph

The table represents the sales of Toyota Corolla cars in Australia from 2015 to 2019.

Use a suitable scale and construct a graph of the time series data.

Year	2015	2016	2017	2018	2019
Cars sold	39 013	41 632	36 087	38 799	43 498

THINKING

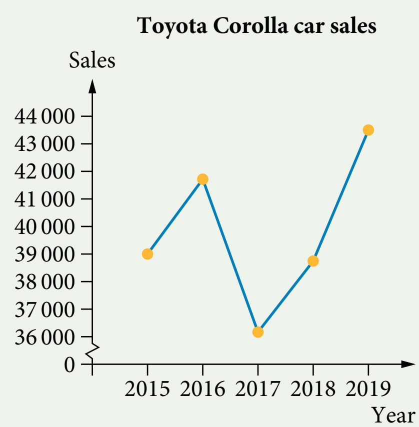
Plot the variable time, given in *years*, on the horizontal axis.

Plot the remaining variable on the vertical axis.

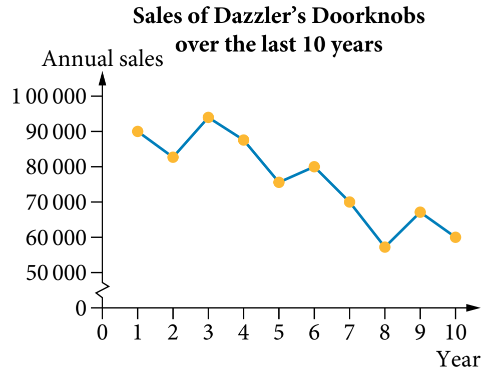
WORKING

The horizontal axis is labelled from 2015 to 2019.

The vertical axis is labelled from 36 000 to 44 000 in increments of 1000.



The following graph shows the annual sales at Dazzler's Doorknobs over 10 years.



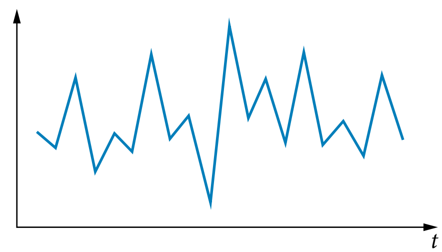
> **Additional information**
> **Graphing a time series**
> Practise graphing a time series.

The graph shows that, over the 10 years, there has been some fluctuation but overall the direction of the graph shows a downward or negative trend for the sales. If the trend were upward, it would be described as a positive trend.

Trends are not always as simple to see as the one in this example. They may be hidden within another type of pattern.

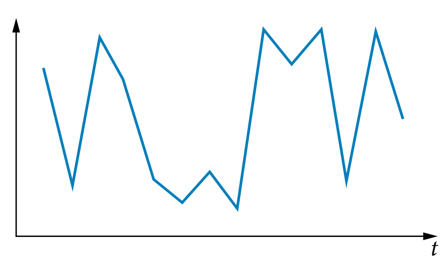
In *cyclical variation* there are repeated patterns, with cycles of variable length. This is often the case with things that are fashionable for a while, fade away and then return to fashion at a later time. An example is sales of skinny ties. Sales are high when the ties are fashionable but poor when they are unfashionable.

In *seasonal variation* there are repeated cyclical fluctuations, with predictable cycles of known and fixed length. Seasonal data can be any type of cyclical data in which the cycles are the same length. *Seasonal factors* are factors such as the time of the day, the day of the week, the month of the year, the quarter of the year, or the season of the year (for example, winter).



The fluctuation may be due to institutional factors such as financial cycles or school years, or may depend on weather patterns. A seasonal pattern can be identified when the peaks and troughs consistently occur after the same time interval. For example, ice-cream sales peak in the warmer months and drop off in the cooler months. In a similar fashion, attendance at a cinema peaks over the weekend and drops away on Mondays and Tuesdays.

Irregular fluctuations show variation but are influenced by unpredictable events and do not fit any of the patterns previously described. There are many possible causes of these variations, such as natural disasters, the economy, war, industrial disputes or, for a small company, turnover of staff. An example is the price of gold, which varies in response to uncertainty in global security.



If a positive or negative trend is not present, the most logical prediction of future values is an average of the values so far.

Trends in irregular graphs

Trendlines can be drawn by hand, but technology removes the need to guess where the line of best fit should go.

2 Using a spreadsheet to fit a trendline

Use the data from Worked example 1 to answer the following questions.

(a) Use technology to create a line graph.

THINKING | WORKING

1 Enter the data into a spreadsheet.

	A	B
1	Year	Cars sold
2	2015	39013
3	2016	41632
4	2017	36087
5	2018	38799
6	2019	43498

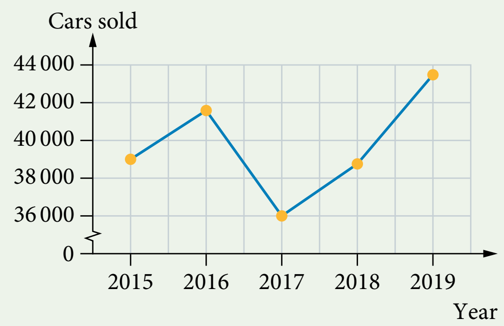
2 Highlight the data and insert a line graph.

Use the data in the first column for the horizontal axis labels.

Format the axes with the same scale as you would use for a hand-drawn graph.

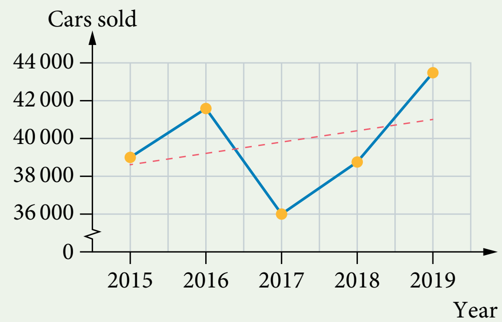
Label the axes.

Note: You can use an *XY* scatter plot with the points joined by lines, if the time values are numbers.



(b) Create a trendline for the graph and comment on its direction.

1 Right-click on any point to create a trendline.



2 Comment on the direction of the trendline.

There is a positive, or increasing trend.

Trendlines and the least-squares line

In your study of bivariate data, you learned that the equation of the line of best fit was calculated using technology as the least-squares equation from the two sets of data. When a line graph is drawn using Excel or other spreadsheet software, the computer treats the time values as categories and uses the values 1, 2, 3, ... as the explanatory variable for the least-squares line.

You will be looking at equations of the trendline later in this chapter.

3 Placing a trendline on a hand-drawn scatter plot

Use the data and time series graph from Worked example 1 to answer the following questions.

(a) Use a scientific calculator to determine the least-squares equation.

THINKING

WORKING

1 In two-variable statistics mode, enter the data into two columns of your scientific calculator.

1	2015	39013
2	2016	41632
3	2017	36087
4	2018	38799
5	2019	43498

2 Use regression analysis to determine the equation of the least-squares line.

A = -1198 027.1, B = 613.7

$y = 613.7x - 1198\,027.1$

(b) Create a trendline for the graph and comment on its direction.

1 Calculate the endpoints of the least-squares line by substituting the first and last values of time into the least-squares equation.

$x = 2015$:

$y = 613.7x - 1198\,027.1$

$= 613.7 \times 2015 - 1198\,027.1$

$= 38\,578.4$

The coordinates of an endpoint are $(2015, 38\,578.4)$

$x = 2019$:

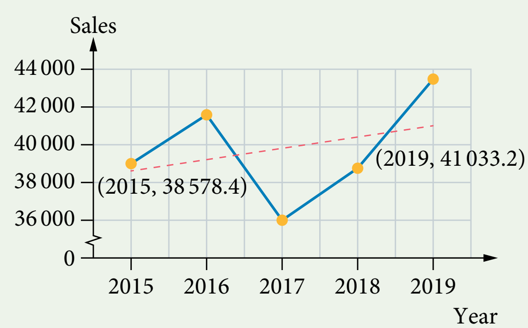
$y = 613.7x - 1198\,027.1$

$= 613.7 \times 2019 - 1198\,027.1$

$= 41\,033.2$

The coordinates of the endpoint are $(2019, 41\,033.2)$

2 On your time series graph, plot the two points and join them to form the trendline.



3 Comment on the direction of the trend.

There is a positive, or increasing trend. That is, over time, sales are increasing.

Outliers

Outliers in time series data are sometimes difficult to recognise, especially if the data is irregular or seasonal. When describing a trend or making predictions about future events, including an outlier might be detrimental to good forecasting.

Additional information

Lines of best fit

Watch the animation to reinforce your understanding of fitting a trendline to a set of data.

If an outlier can be explained by a one-off unanticipated event, so it is known to be truly atypical, a statistician may choose to replace the data point with 'dummy' values for the purpose of showing a trend or making predictions.

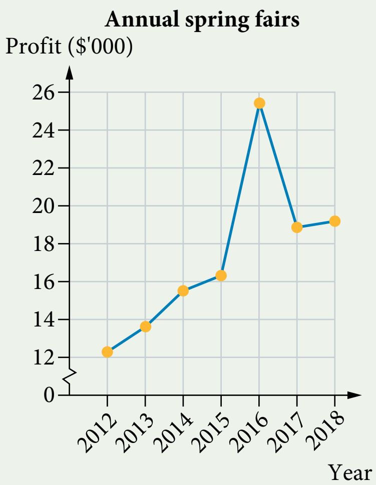
4 Removing the effect of an outlier

The following table shows the profits from a school's annual spring fair from 2012 to 2018.

Year	2012	2013	2014	2015	2016	2017	2018
Profit ($'000)	12.3	13.6	15.5	16.3	25.4	18.9	19.2

(a) Construct a graph of the time series data and predict the year in which the school's centenary was celebrated, with past students returning in great numbers.

THINKING	WORKING
1 Plot the variable time, given in *years*, on the horizontal axis. Plot the remaining variable on the vertical axis.	The horizontal axis is labelled from 2012 to 2018. The vertical axis is labelled from 12 to 26 in increments of 2.



THINKING	WORKING
2 Identify the outlier and determine the year in which this occurred.	It is likely that the centenary year was 2016. However, other reasons for increased profits include participation, items for sale and weather conditions (especially if there are outdoor activities). The value is so high that it should not be used to predict future attendance.

(b) Replace the aberrant point with dummy data. Redraw the graph and comment on the trend.

1 Replace the aberrant value with the average of the values on either side.

2015: 16.3; 2017: 18.9

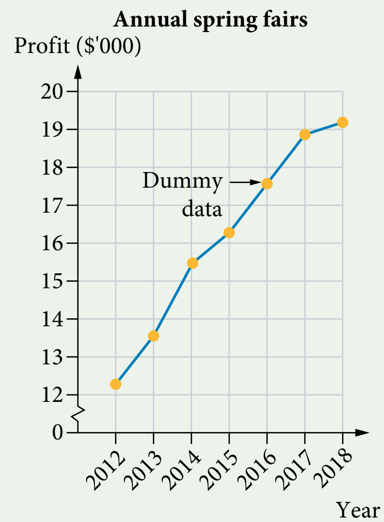
Dummy value for 2016:

$$\frac{16.3 + 18.9}{2} = 17.6$$

2 Redraw the graph with the dummy value replacing the aberrant value.

The horizontal axis is labelled from 2012 to 2018.

The vertical axis is labelled from 12 to 20 in increments of 1.



3 Describe the trend.

Apart from the centenary year, 2016, there is a fairly regular increase in profit each year.

EXERCISE 2.1 Time series

Worked Example 1

1 The table shows wool production in Australia from 2005 to 2013.

Year	2005	2006	2007	2008	2009	2010	2011	2012	2013
Wool production ('000s of tonnes)	519.7	519.9	502.3	458.7	420.3	423.0	429.0	411.0	435.0

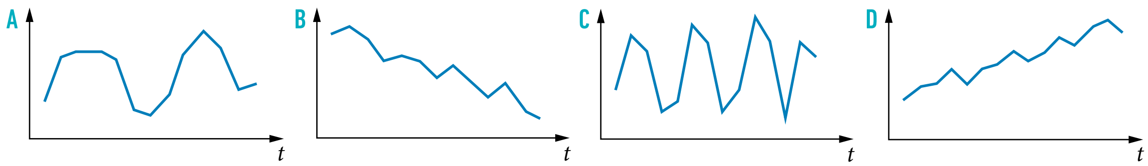
Use a suitable scale and construct a graph of the time series data. Describe any trend or pattern in the data.

2 The numbers of cartons of chocolates manufactured and sold by a factory per day over a 14-day period are given in the table.

Day	Sun	Mon	Tue	Wed	Thu	Fri	Sat	Sun	Mon	Tue	Wed	Thu	Fri	Sat
Cartons	0	42	56	58	54	35	0	0	46	56	60	56	40	0

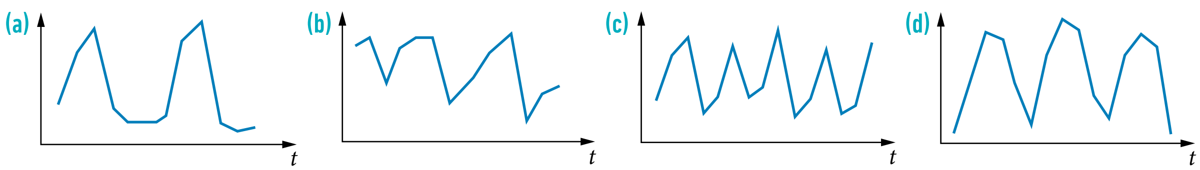
(a) Construct a graph of the time series data to show the pattern of sales.
(b) Comment on the pattern shown in the graph.
(c) Why is it not appropriate to draw a trendline in this situation?

3 Consider the following time series graphs.
(a) Which graph displays seasonal data?

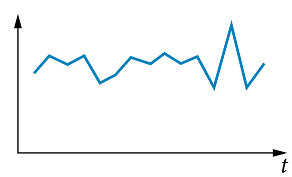


(b) Explain the common error made by a student who chose the option in part (a) that shows cyclical variation.
(c) Describe the pattern shown in each of the other two graphs.

4 For each of the seasonal time series graphs shown, determine the number of points for each cycle.

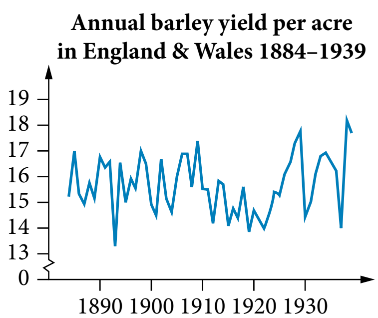


5 Describe the time series graph shown here. Explain how you would predict future values from this data.

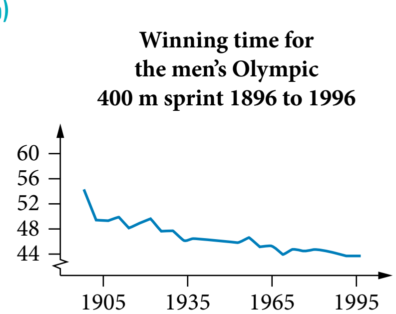


6 Comment on the characteristics (pattern and trend) of each of the following time series graphs.

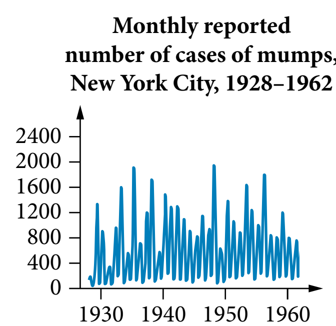
(a)



(b)



(c)



7 Select the type of of pattern that you would expect to see in a time series for each of the following conditions:

A seasonal　B random　C positive trend　D negative trend

(a) sales of tablet computers between 2005 and 2019
(b) monthly rainfall at a particular gauge over a 5 year period
(c) monthly sales of swimsuits over a 5 year period
(d) monthly sales of football cards over a 3 year period
(e) sales of cigarettes between 1960 and 2010

2

8 The table shows beef production in Australia from 2000 to 2013.

Year	2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013
Beef production ('000 tonnes)	1988	2119	2027	2072	2032	2162	2077	2226	2132	2125	2109	2133	2115	2245

Use a spreadsheet to construct a graph of the time series data. Fit the least-squares line and comment on the long-term trend.

4

9 The table shows the attendance at a school's annual fetes from 2012 to 2018.

Year	2012	2013	2014	2015	2016	2017	2018
Attendance (hundreds of people)	22.4	3.8	25.8	29.6	30.1	32.3	33.8

(a) Construct a graph of the time series data, and identify the year in which a weather event occurred that prevented many people from attending.

(b) Replace the aberrant point with dummy data. Redraw the graph and comment on the trend.

3

10 The table shows milk production in Australia from 2003 to 2013.

Year	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012	2013
Milk production ('000 000 L)	10 328	10 076	10 127	10 089	9 583	9 223	9 388	9 023	9 100	9 480	9 200

(a) Construct a graph of the time series data on graph paper. Use a scientific calculator to determine the least-squares equation. Calculate the endpoints of the least-squares line and fit the trendline to your graph.

(b) Comment on the suitability of the trendline to represent the data.

11 The table shows the number of drug offences in Queensland from 2013 to 2017.

Year	2013	2014	2015	2016	2017
Drug offences	54 869	65 434	80 847	88 660	84 259

(a) Construct a graph of the time series data on graph paper. Use a scientific calculator to determine the least-squares equation. Calculate the endpoints of the least-squares line and fit the trendline to your graph.

(b) Comment on the suitability of the trendline to represent the data.

12 The table shows the number of fraud offences in Northern Queensland from 2013 to 2017.

Year	2013	2014	2015	2016	2017
Fraud offences	1 590	1 754	2 308	2 269	3 006

(a) Construct a graph of the time series data on graph paper. Use a scientific calculator to determine the least-squares equation. Calculate the endpoints of the least-squares line and fit the trendline to your graph.

(b) Comment on the suitability of the trendline to represent the data.

13 Milk production at Stan's dairy farm in Victoria per season over 3 years is shown in the table.

Season	Sum	Aut	Win	Spr	Sum	Aut	Win	Spr	Sum	Aut	Win	Spr
Production (ML)	4.8	3.1	1.2	7.8	5.3	2.6	0.8	6.4	4.6	2.8	0.5	6.9

Colin has a smaller farm in Queensland, and the milk production at his farm over the same time period is shown in the table below.

Season	Sum	Aut	Win	Spr	Sum	Aut	Win	Spr	Sum	Aut	Win	Spr
Production (ML)	1.2	0.9	0.8	1.6	1.3	1.1	0.9	1.4	1.2	1.0	0.8	2.0

(a) Construct a graph of the time series data for both farms on the same set of axes.

(b) Compare the seasonality of milk production at the two farms.

14 The quarterly electricity charges for a family home in Springwood, Queensland are given in the table.

Season	Bill ($)	Season	Bill ($)
Spr 2013	415	Spr 2015	201
Sum 2014	419	Sum 2016	389
Aut 2014	348	Aut 2016	356
Win 2014	341	Win 2016	484
Spr 2014	210	Spr 2016	361
Sum 2015	441	Sum 2017	579
Aut 2015	258	Aut 2017	461
Win 2015	369	Win 2017	357

(a) Graph the data and comment on which seasons are the most costly for electricity use for this home. Offer some explanations for the differences.

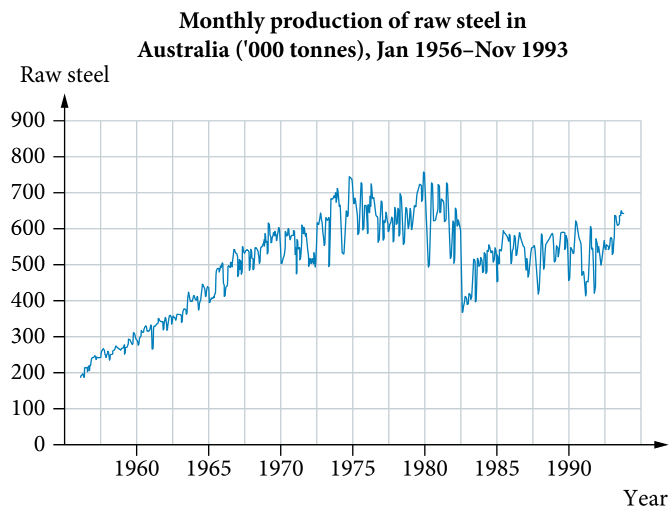
(b) During which year is it most likely that the family travelled in both winter and summer? Estimate dummy data for both points, had the family not travelled. Justify your estimates with calculations.

(c) In the final year of the data shown, the electricity company altered the billing dates by 1.5 months, so one bill was much higher than you would expect. Explain which bill was affected, and estimate the value of the bill if it had been just the amount for the quarter. Justify your estimate with calculations.

(d) Explain the shape of the graph following this aberrant value. Comment on what you would expect the graph of future electricity bills to look like after this change.

(e) Electricity charges increased substantially in the period shown. However, the family had solar panels installed, so each electricity bill was reduced. At one point the number of people living in the home increased. Estimate the time at which each of these events took place.

15 The graph shows monthly production of raw steel in Australia from January 1956 to November 1993.



(a) By breaking the graph into three appropriate sections, describe the changes in production in terms of linear trends.

(b) Write equations for three trendlines to match your descriptions in (a). Determine the values of the endpoints for each line and use them to sketch the lines on the graph.

(c) Calculate the total amount of raw steel produced in each of the following years, according to your model.

(i) 1970 (ii) 1977 (ii) 1991

2.2 Smoothing

Smoothing data

Sometimes the underlying trends in data can be seen more clearly when you remove the effects of random variations in individual data points. This process is known as *smoothing the curve*. A common method of smoothing is to use *moving averages*, which can be done using moving means or moving medians.

Moving means

As you have seen previously, the mean of a data set can be calculated using the formula $\bar{x} = \frac{\sum x}{n}$.

For a time series, the moving mean is calculated for a point by considering the values surrounding it.

When calculating the moving mean, consider the number of points. For smaller data sets, choose a smaller number of points to use in calculating the mean.

The larger the number of points used to calculate the moving mean, the more the variation is removed. However, using larger numbers reduces the number of data points, as more endpoints are lost in the process.

The simplest technique for smoothing time series data is one that smooths data over an odd number of scores, such as 3, 5, …

The 3-point moving mean

In this method, given three consecutive data scores d_1, d_2 and d_3, the middle data score d_2 is replaced with the mean of the three data scores.

$$d_{2\text{ smoothed}} = \frac{d_1 + d_2 + d_3}{3}$$

Note that the 3-point moving mean for the first and last scores cannot be calculated.

The 5-point moving mean

$$d_{3\text{ smoothed}} = \frac{d_1 + d_2 + d_3 + d_4 + d_5}{5}$$

Note that the 5-point moving mean for the first two and last two scores cannot be calculated.

5 The moving mean

The annual migration of German citizens to the USA over the period 1878 to 1888 is given in the table.

Year	1878	1879	1880	1881	1882	1883	1884	1885	1886	1887	1888
Migration ('000 s)	29	35	85	210	251	195	180	124	84	107	110

METHOD 1: BY HAND

(a) Determine the 3-point moving mean, to 1 decimal place.

THINKING

1 Construct a table with the data presented in columns, by adding a third column for the smoothed values. The first and last cells in the Smoothed data column should be left empty.

WORKING

Year	Migration ('000 s)	Smoothed data
1878	29	
1879	35	
⋮	⋮	

2 Calculate the mean of the first three consecutive data scores.

$$1879_{\text{smoothed}} = \frac{29 + 35 + 85}{3}$$
$$= 49.7$$

Year	Migration ('000 s)	Smoothed data
1878	29	
1879	35	49.7
1880	85	
1881	210	
1882	251	
1883	195	
1884	180	
1885	124	
1886	84	
1887	107	
1888	110	

3 Calculate the mean of the next set of three consecutive data scores.

$$1880_{\text{smoothed}} = \frac{35 + 85 + 210}{3}$$
$$= 110$$

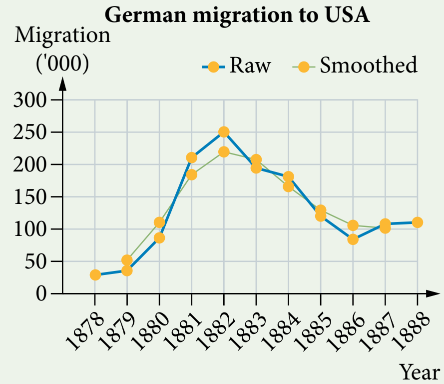
Year	Migration ('000 s)	Smoothed data
1878	29	
1879	35	49.7
1880	85	110.0
⋮	⋮	
1888	110	

4 Continue with this procedure until all moving means have been calculated.

Year	Migration ('000 s)	Smoothed data
1878	29	
1879	35	49.7
1880	85	110.0
1881	210	182.0
1882	251	218.7
1883	195	208.7
1884	180	166.3
1885	124	129.3
1886	84	105.0
1887	107	100.3
1888	110	

(b) Plot both the raw data and the moving mean data on the same set of axes. Comment on the results.

1 Plot the raw data and the smoothed data on the same set of axes. Time is represented by the horizontal axis.



2 Comment on the result.

Only the extreme values have been smoothed out. The overall trends are still clear.

METHOD 2: USING TECHNOLOGY

(a) Determine the 3-point moving mean.

THINKING

WORKING

1 Place the data into two columns.

In the third column, go to the cell midway between the first three data values and enter a formula to calculate the average of the first three data values.

	A	B	C
1	Year	Migration	
2	1878	29	
3	1879	35	=(B2+B3+B4)/3
4	1880	85	
12	1888	110	

2 Click and drag from the cell containing the formula to the lowest valid cell for this calculation.

Always check the formula in one of the other cells to ensure the automatic changes are appropriate.

Note: Even though no decimal places are shown, a spreadsheet retains accuracy to many decimal places.

	A	B	C
1	Year	Migration	
2	1878	29	
3	1879	35	50
4	1880	85	110
5	1881	210	182
6	1882	251	219
7	1883	195	209
8	1884	180	166
9	1885	124	129
10	1886	84	105
11	1887	107	100
12	1888	110	

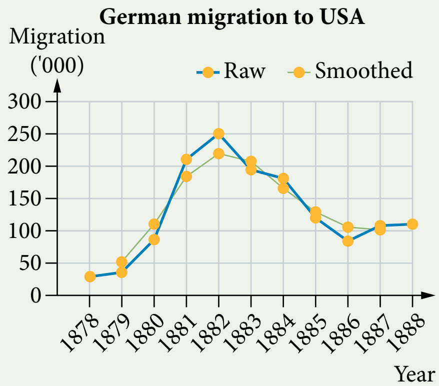
(b) Plot both the raw data and the moving mean data on the same set of axes. Comment on the results.

1 Highlight all the data.

Insert a line graph, choosing one that does not add the values together.

Select the horizontal axis values as the time values.

Ensure the legends are appropriate and add titles where needed.



2 Comment on the result.

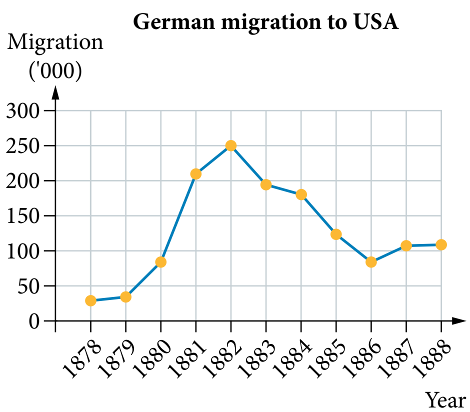
Only the extreme values have been smoothed out. The overall trends are still clear.

Moving medians

Moving medians smooth the graph in the same way as moving means. Because no calculation is needed to determine the median of a set of three values, moving medians can be done directly onto a graph.

6 Smoothing using moving medians

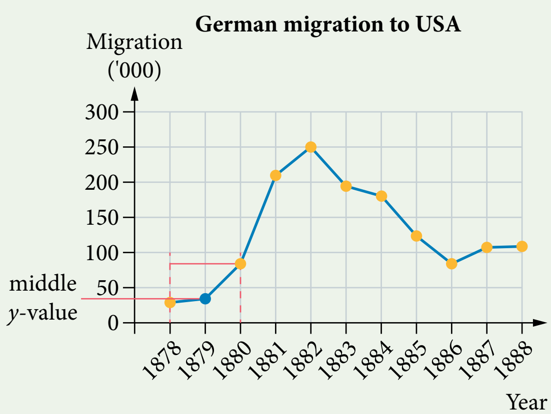
Plot the moving medians on the same set of axes as the original graph. Comment on the result.



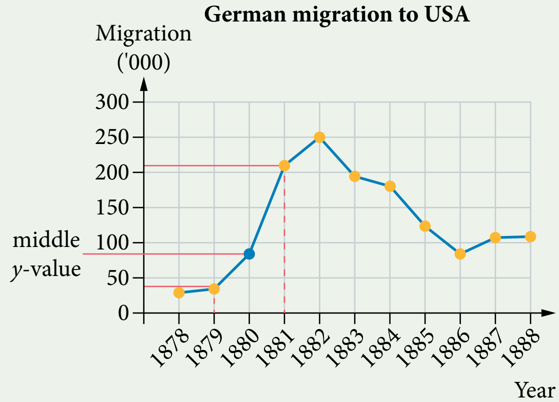
THINKING

WORKING

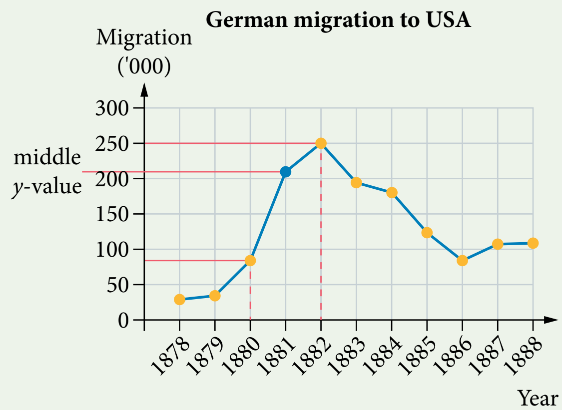
1 Determine the 3-point moving median for the second time period. Do this by taking the median vertical value of the data point and the two either side of it, and plotting the point.



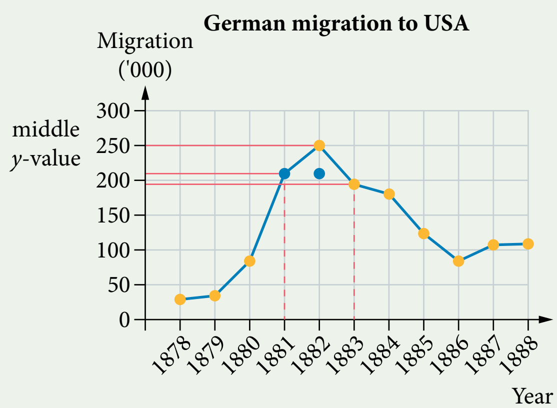
2 Determine the 3-point moving median for the third time period. Do this by taking the median vertical value of the data point and the two either side of it, and plotting the point.



3 Determine the 3-point moving median for the fourth time period. Do this by taking the median vertical value of the data point and the two either side of it, and plotting the point.



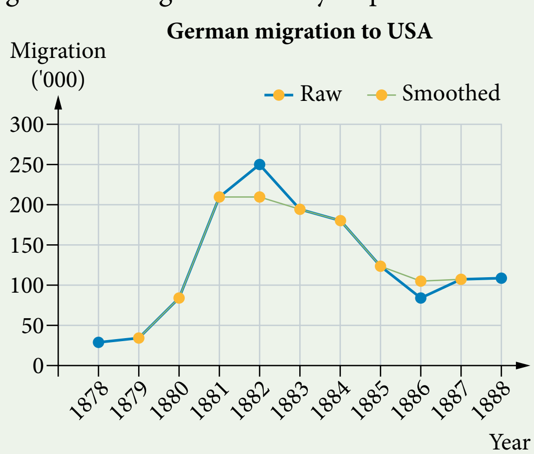
4 Determine the 3-point moving median for the fifth time period. Do this by taking the median vertical value of the data point and the two either side of it, and plotting the point.



For the year 1882, the graph is smoothed because the gradient changes over the 3 year period surrounding it.

5 Repeat the process until no more groups of three are possible. Join the dots with a series of straight lines.

Check the reasonableness of the graph. Where the graph changes from positive to negative (or vice versa) in a 3 year period, the smoothed data will not generally align to an original point.



6 Comment on the result.	The extreme values have been smoothed. Wherever the original graph was rising or falling over the 3 points, the graph remains unchanged. There are two fewer points than in the original graph.

The values could be calculated using a table by choosing the median of each group of values, as shown in the table at right.

Median smoothing shows a smoothing effect on variations where the gradient of the graph of the raw data changes from positive to negative or vice versa within the given time period.

WARNING

For the median, remember to list the data values in ascending order before choosing the middle one.

Year	Migration ('000 s)	Smoothed data 3-point median
1878	29	
1879	35	29, 35, 85
1880	85	35, 85, 210
1881	210	85, 210, 251
1882	251	195, 210, 251
1883	195	180, 195, 251
1884	180	124, 180, 195
1885	124	84, 124, 180
1886	84	84, 107, 124
1887	107	84, 107, 110
1888	110	

Centring

If there are some random events in seasonal data, you may wish to smooth using the cycle length. For example, if quarterly data appears to have an annual cycle, use a 4-point centred moving mean.

The difficulty when smoothing an even number of data points is that the resulting mean lies between adjacent time points in the original data. This can be overcome using a process called *centring*.

The 4-point moving mean

In this method, given four consecutive data scores d_1, d_2, d_3 and d_4, the smoothed value (the mean of the four data scores) is placed between data scores d_2 and d_3.

$$d_{1-4\text{ smoothed}} = \frac{d_1 + d_2 + d_3 + d_4}{4}$$

The next smoothed value (the mean of the next set of four data scores) is placed between data scores d_3 and d_4.

$$d_{2-5\text{ smoothed}} = \frac{d_2 + d_3 + d_4 + d_5}{4}$$

To centre the data, a new mean is calculated for $d_{2-3 \text{ smoothed}}$ and $d_{3-4 \text{ smoothed}}$.

Data	4-point smoothing mean	Centred data
d_1		
d_2		
	$d_{1-4 \text{ smoothed}} = \dfrac{d_1 + d_2 + d_3 + d_4}{4}$	
d_3		$\dfrac{d_{1-4 \text{ smoothed}} + d_{2-5 \text{ smoothed}}}{2}$
	$d_{2-5 \text{ smoothed}} = \dfrac{d_2 + d_3 + d_4 + d_5}{4}$	
d_4		$\dfrac{d_{2-5 \text{ smoothed}} + d_{3-6 \text{ smoothed}}}{2}$
	$d_{3-6 \text{ smoothed}} = \dfrac{d_3 + d_4 + d_5 + d_6}{4}$	
d_5		

7 Smoothing and centring using a 4-point moving mean

The table shows quarterly sales figures over a 3 year period. Smooth and centre the data using a 4-point moving mean.

Year	2018				2019				2020			
Quarter	1	2	3	4	1	2	3	4	1	2	3	4
Sales	400	600	700	300	800	800	1000	600	500	700	1200	600

THINKING

1 Calculate the mean of the first four data points.

Starting with the second data value, calculate the mean of the next set of four points.

Repeat the process until the mean of the last four data points is found.

WORKING

	A	B	C	D
1	Year	Quarter	Sales	4-point moving mean
2	2018	1	400	
3		2	600	$\frac{400 + 600 + 700 + 300}{4} = 500$
4		3	700	$\frac{600 + 700 + 300 + 800}{4} = 600$
5		4	300	$\frac{700 + 300 + 800 + 800}{4} = 650$
6	2019	1	800	$\frac{300 + 800 + 800 + 1000}{4} = 725$
7		2	800	$\frac{800 + 800 + 1000 + 600}{4} = 800$
8		3	1000	$\frac{800 + 1000 + 600 + 500}{4} = 725$
9		4	600	$\frac{1000 + 600 + 500 + 700}{4} = 700$
10	2020	1	500	$\frac{600 + 500 + 700 + 1200}{4} = 750$
11		2	700	$\frac{500 + 700 + 1200 + 600}{4} = 750$
12		3	1200	
13		4	600	

2 Even-valued moving means also need to be centred to align with a time period. To centre, determine the mean of each pair of moving points.

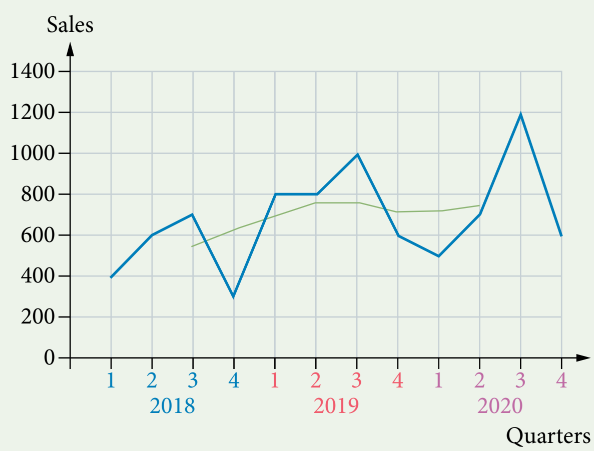
	A	B	C	D	E
1	Year	Quarter	Sales	4-point moving mean	Centered data
2	2018	1	400		
3		2	600	$\frac{400 + 600 + 700 + 300}{4} = 500$	
4		3	700	$\frac{600 + 700 + 300 + 800}{4} = 600$	$\frac{500 + 600}{2} = 550$
5		4	300	$\frac{700 + 300 + 800 + 800}{4} = 650$	$\frac{600 + 650}{2} = 625$
6	2019	1	800	$\frac{300 + 800 + 800 + 1000}{4} = 725$	$\frac{650 + 725}{2} = 687.5$
7		2	800	$\frac{800 + 800 + 1000 + 600}{4} = 800$	$\frac{725 + 800}{2} = 762.5$
8		3	1000	$\frac{800 + 1000 + 600 + 500}{4} = 725$	$\frac{800 + 725}{2} = 762.5$
9		4	600	$\frac{1000 + 600 + 500 + 700}{4} = 700$	$\frac{725 + 700}{2} = 712.5$
10	2020	1	500	$\frac{600 + 500 + 700 + 1200}{4} = 750$	$\frac{700 + 750}{2} = 725$
11		2	700	$\frac{500 + 700 + 1200 + 600}{4} = 750$	$\frac{750 + 750}{2} = 750$
12		3	1200		
13		4	600		

3 Simplify the table.

Note that by using the 4-point moving mean, the total number of data points is reduced from 12 to 8.

	A	B	C	D	E
1	Year	Quarter	Sales		
2	2018	1	400		
3		2	600	500	
4		3	700	600	550
5		4	300	650	625
6	2019	1	800	725	687.5
7		2	800	800	762.5
8		3	1000	725	762.5
9		4	600	700	712.5
10	2020	1	500	750	725
11		2	700	750	750
12		3	1200		
13		4	600		

4 Plot the graph of the raw data and the smoothed and centered data.



5 Interpret the data.	The raw data shows fluctuations, with sales generally increasing in the second and third quarters and dropping in the fourth quarter. From the smoothed data there appears to be an increasing trend in the number of sales per year.

The 4-point moving median

When smoothing data using a 4-point moving median, it is better to use a table rather than working straight from the graph, because centring is required.

Explore further

Smoothing data

Use a spreadsheet to smooth data with 3, 4 and 5 points.

8 Smoothing and centring using a 4-point moving median

The table shows quarterly sales figures over a 3 year period. Smooth and centre the data using a 4-point moving median.

Year	2018				2019				2020			
Quarter	1	2	3	4	1	2	3	4	1	2	3	4
Sales	400	600	700	300	800	800	1000	600	500	700	1200	600

THINKING

WORKING

1 Collect the data in groups of four, starting with the first data value, and organise in ascending order.

	A	B	C	D
1	Year	Quarter	Sales	Data in ascending order
2	2018	1	400	
3		2	600	300, 400, 600, 700
4		3	700	300, 600, 700, 800
5		4	300	300, 700, 800, 800
6	2019	1	800	300, 800, 800, 1000
7		2	800	600, 800, 800, 1000
8		3	1000	500, 600, 800, 1000
9		4	600	500, 600, 700, 1000
10	2020	1	500	500, 600, 700, 1200
11		2	700	500, 600, 700, 1200
12		3	1200	
13		4	600	

2 Calculate the median of each group of data values.

	A	B	C	D	E
1	Year	Quarter	Sales	Data in ascending order	Median
2	2018	1	400		
3		2	600	300, 400, 600, 700	$\frac{400 + 600}{2} = 500$
4		3	700	300, 600, 700, 800	$\frac{600 + 700}{2} = 650$
5		4	300	300, 700, 800, 800	$\frac{700 + 800}{2} = 750$
6	2019	1	800	300, 800, 800, 1000	$\frac{800 + 800}{2} = 800$
7		2	800	600, 800, 800, 1000	$\frac{800 + 800}{2} = 800$
8		3	1000	500, 600, 800, 1000	$\frac{600 + 800}{2} = 700$
9		4	600	500, 600, 700, 1000	$\frac{600 + 700}{2} = 650$
10	2020	1	500	500, 600, 700, 1200	$\frac{600 + 700}{2} = 650$
11		2	700	500, 600, 700, 1200	$\frac{600 + 700}{2} = 650$
12		3	1200		
13		4	600		

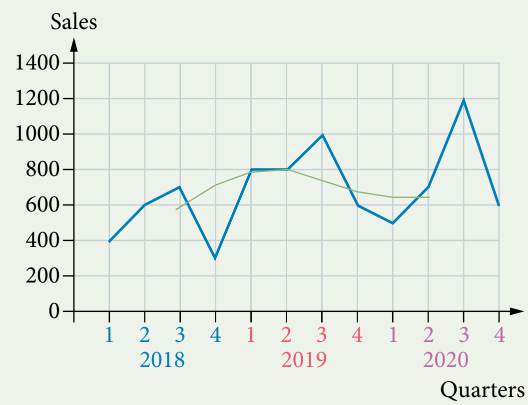
3 Centre the data to align with a time period, by determining the average of sequential medians.

	A	B	C	D	E	F
1	Year	Quarter	Sales	Data in ascending order	Median	Centered data
2	2018	1	400			
3		2	600	300, 400, 600, 700	500	
4		3	700	300, 600, 700, 800	650	$\frac{500 + 650}{2} = 575$
5		4	300	300, 700, 800, 800	750	$\frac{650 + 750}{2} = 700$
6	2019	1	800	300, 800, 800, 1000	800	$\frac{750 + 800}{2} = 775$
7		2	800	600, 800, 800, 1000	800	$\frac{800 + 800}{2} = 800$
8		3	1000	500, 600, 800, 1000	700	$\frac{800 + 700}{2} = 750$
9		4	600	500, 600, 700, 1000	650	$\frac{700 + 650}{2} = 675$
10	2020	1	500	500, 600, 700, 1200	650	$\frac{650 + 650}{2} = 650$
11		2	700	500, 600, 700, 1200	650	$\frac{650 + 650}{2} = 650$
12		3	1200			
13		4	600			

4 Tabulate the results by placing the first calculated centred median alongside the third data point.

	A	B	C	D	E
1	Year	Quarter	Sales		
2	2018	1	400		
3		2	600	500	
4		3	700	650	575
5		4	300	750	700
6	2019	1	800	800	775
7		2	800	800	800
8		3	1000	700	750
9		4	600	650	675
10	2020	1	500	650	650
11		2	700	650	650
12		3	1200		
13		4	600		

5 Plot the graph of the raw data and the smoothed and centered data.



6 Interpret the data.

The raw data shows fluctuations, with sales generally increasing in the second and third quarters and dropping in the fourth quarter. Compared to the data smoothed using a 4-point moving mean, the 4-point moving median shows little increase. The extreme values have little impact on the data smoothed using median smoothing techniques.

EXERCISE 2.2 Smoothing

Worked Example 5

1 The numbers of flamingos migrating to a particular lake annually over the period 1978 to 1988 are given in the table. The figures are in thousands.

(a) Calculate the 3-point moving mean, to 1 decimal place.

(b) Plot the raw data and the moving mean data on the same set of axes.

1978	1979	1980	1981	1982	1983
26	34	40	15	10	14
1984	1984	1985	1986	1987	1988
20	20	35	18	12	20

(c) Which of the following best explains the purpose of using the 3-point moving mean?

A To remove the endpoint data values
B To create another graph to compare values
C To deseasonalise the data
D To reduce extreme irregular fluctations

(d) Explain the common error made by a student who chose the first incorrect option.

2 For data collected monthly, it is best to use a moving mean with how many points?

A 12
B 6
C 4
D 3

3 The table displays the depth of water in a river over a 1 year period.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Depth (m)	10	8	9	12	14	15	16	14	16	12	10	9

Using 3-point moving mean smoothing, what would be the smoothed value for the river depth in July?

4 The table displays the share price for ABBC shares over several days.

Using 5-point moving mean smoothing, determine the smoothed values for the shares on Wednesday and Thursday. Round answers to the nearest cent where necessary.

Day	Mon	Tue	Wed	Thu	Fri	Sat
Price ($)	1.54	1.87	1.63	1.64	1.72	1.66

5 For the sales figures given in the table, construct a 3-point moving mean, correct to 1 decimal place, and then compare the information by plotting both sets of data on the same set of axes. Comment on the overall trend.

Week	1	2	3	4	5	6	7	8	9
Sales	32	58	45	40	65	50	50	80	70

6 For the sales figures given in the table, construct a table of 3-point moving means, to 1 decimal place, and then compare the information by plotting both sets of data on the same set of axes. Comment on the overall trend.

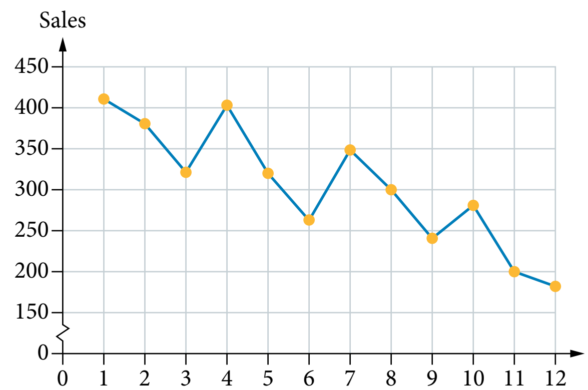
Week	1	2	3	4	5	6	7	8	9	10	11	12
Sales	410	380	320	400	320	260	350	300	240	280	200	180

7 For the sales figures given in the table, construct a table of the 3-point moving medians.

Week	1	2	3	4	5	6	7	8	9	10	11	12
Sales	410	380	320	400	320	260	350	300	240	280	200	180

8 For the graph of sales figures, plot a 3-point moving median on the same set of axes.

Worked Example 6



9 Sales figures for a 2 year period are given in the table. Using 4-point moving mean smoothing, what would be the smoothed and centred value for the third quarter in 2017?

Year	2017				2018			
Quarter	1	2	3	4	1	2	3	4
Sales ('000)	23.4	25.5	38.6	27.9	24.2	27.7	39.4	29.9

10 Rainfall (in mm) is recorded at a farming property in north Queensland over a 2 year period.

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
2017	300	270	290	152	88	20	84	20	26	42	105	201
2018	250	315	270	160	120	80	15	25	21	31	86	183

Using 12-point moving mean smoothing, what would be the smoothed and centred value for September 2017? Give your answer to 1 decimal place.

11 The maximum temperature for Rockhampton for the first 10 days of December 2017 is given in the table.

Day	1	2	3	4	5	6	7	8	9	10
Temperature (°C)	29.1	30.4	31.9	29.1	32.5	34.3	33.2	32.3	30.2	31.1

Using the moving mean smoothing specified, determine the smoothed (and centred, if necessary) value for 8 December. Round your answers to 2 decimal places where necessary.

(a) 2-point (b) 3-point (c) 4-point (d) 5-point

Worked Example 7

12 The table shows quarterly sales figures over a 3 year period. Smooth the data using a 4-point moving mean.

Year	Quarter	Sales	Year	Quarter	Sales	Year	Quarter	Sales
2012	1	600	2013	1	800	2014	1	500
	2	200		2	400		2	300
	3	700		3	900		3	1000
	4	300		4	400		4	600

Worked Example 8

13 The table shows quarterly sales figures over a 3 year period. Smooth the data using a 4-point moving median.

Year	Quarter	Sales	Year	Quarter	Sales	Year	Quarter	Sales
2012	1	600	2013	1	800	2014	1	500
	2	200		2	400		2	300
	3	700		3	900		3	1000
	4	300		4	400		4	600

14 Consider the rainfall data shown in the table.

Season	Sum	Aut	Win	Spr	Sum	Aut	Win	Spr
Rainfall (cm)	25	40	45	60	15	20	50	55

(a) Construct a table of 4-point moving means, centred (to the nearest whole number), then compare the information by plotting both sets of data on the same set of axes.

(b) What was the main reason for smoothing the quarterly rainfall data?

15 The Australian youth prohibited/regulated weapons offence figures over 8 years are given in the table.

2008–09	2009–10	2010–11	2011–12	2012–13	2013–14	2014–15	2015–16
1480	1529	1351	1145	1046	881	878	917

(a) Construct a table of 4-point moving means, centred (to the nearest whole number) and then compare the information by plotting both sets of data on the same set of axes.

(b) Comment on the benefits of each line graph compared with the other.

16 The mean daily sales of chocolate bars at a convenience store over a 4 year period are given in the table. Answer all questions to 1 decimal place where necessary.

Year	Autumn	Winter	Spring	Summer
2015	45	60	50	20
2016	50	75	45	23
2017	60	80	50	25
2018	45	60	70	24

(a) Smooth the data using a 3-point moving mean.

(b) Smooth the data using a 4-point centred moving mean.

(c) Smooth the data using a 5-point moving mean.

(d) Smooth the data using a 12-point centred moving mean.

(e) Explain which moving mean is the most appropriate to use for this set of data.

17 A new cafe is open four days a week. The number of patrons is counted for the first three weeks after the cafe opens.

Week	Fri	Sat	Sun	Wed
1	22	55	37	38
2	56	102	85	83
3	97	124	104	96

(a) Smooth the data with an appropriately numbered moving average. Determine a regression equation from the moving average values. Plot the raw data, with the trendline on the same set of axes.

(b) In what way does the trendline represent the data well, and what important information would be lost if just the trendline was graphed?

18 Each table below gives raw data followed by moving averages for youth offences involving illicit drugs. Calculate the missing values.

(a) New South Wales, 5-point moving averages

2008–09	2009–10	2010–11	2011–12	2012–13	2013–14	2014–15	2015–16
	782	886	847	889			1 005
		822.8	875.2	917.4			

(b) Queensland, 4-point moving averages

2008–09	2009–10	2010–11	2011–12	2012–13	2013–14	2014–15	2015–16
1 324	1 545	1 584	1 751	2 108		2 671	
			1 841.5	2 071.875	2 307.125		

2.3 Seasonal variation

Seasonal index

When a set of data displays a seasonal pattern, the peaks and troughs occur at roughly the same time in a set time period. Removing these seasonal effects can make it easier to observe an underlying trend.

Seasonal variation in time series is measured in terms of an index called a *seasonal index*.

> The seasonal indices are found by dividing the data for a season by the mean of the data for the whole time period.

9 Calculating seasonal indices

Calculate the seasonal indices for the following set of data collected in 3 month intervals over 1 year.

Period	Dec–Feb	Mar–May	Jun–Aug	Sep–Nov
Sales ($)	820	640	430	510

THINKING

1 Calculate the sales made in the 'average season'.

WORKING

$$\text{mean} = \frac{820 + 640 + 430 + 510}{4} = \frac{2400}{4} = 600$$

THINKING

2 Interpret the value.

WORKING

Over the year, the average season shows sales of $600.

THINKING

3 Calculate each seasonal index by dividing the value for each period by the average season.

WORKING

Correct to 2 decimal places:

Period	Dec–Feb	Mar–May	Jun–Aug	Sep–Nov
Seasonal index	$\frac{820}{600} = 1.37$	$\frac{640}{600} = 1.07$	$\frac{430}{600} = 0.72$	$\frac{510}{600} = 0.85$

THINKING

4 Interpret the answer.

WORKING

The seasonal index shows that sales in:

Dec–Feb are 37% higher than the average season

Mar–May are 7% higher than the average season

Jun–Aug are 28% lower than the average season

Sep–Nov are 15% lower than the average season.

> The seasonal indices have a mean of 1. If there are 4 seasons, the sum will be 4. If there are 12 monthly seasonal indices calculated, the sum will be 12.

10 Calculating a missing seasonal index

For each of the following, determine the missing seasonal index.

(a)

Quarter	Spring	Summer	Autumn	Winter
Seasonal index	0.6	1.3	x	0.5

THINKING	WORKING
1 Identify the number of periods and, hence, the sum of the seasonal indices.	There are 4 seasonal indices, so the expected total is 4.
2 Create an equation equating the seasonal indices to the expected total, and solve for the unknown.	$0.6+1.3+x+0.5=4$ $x=4-(0.6+1.3+0.5)$ $x=1.6$
3 Interpret the answer.	The missing seasonal index 1.6 indicates that the data collected in autumn is 60% higher than the average season.

(b)

2 month period	Jan–Feb	Mar–Apr	May–Jun	Jul–Aug	Sep–Oct	Nov–Dec
Seasonal index	$7y$	$2y$	$4y$	$3y$	$2y$	$6y$

1 Identify the number of periods and, hence, the sum of the seasonal indices.	There are 6 seasonal indices, so the expected total is 6.
2 Create an equation equating the seasonal indices to the expected total, and solve for the unknown.	$7y+2y+4y+3y+2y+6y=6$ $24y=6$ $y=0.25$
3 Use the value of the pronumeral to calculate the actual seasonal indices.	(see table below)
4 Evaluate the reasonableness of your answer, by checking that the sum of all the seasonal indices is equal to the number of seasons.	1.75 + 0.5 + 1.0 + 0.75 + 0.5 + 1.5 = 6
5 Interpret the answer.	The seasonal indices show that the data collected in Jan–Feb is 75% higher than the average season, while the periods of Mar–Apr and Sep–Oct are the lowest performing, at 50% lower than the average season.

Step 3 working:

Jan–Feb	Mar–Apr	May–Jun	Jul–Aug	Sep–Oct	Nov–Dec
$7y = 1.75$	$2y = 0.5$	$4y = 1.0$	$3y = 0.75$	$2y = 0.5$	$6y = 1.5$

Averaged seasonal indices

To improve the reliability of the seasonal index, if there are multiple time periods of data, determine the mean seasonal index for each season.

11 Calculating the average seasonal indices

Calculate the seasonal indices for this set of data collected in 3 month intervals over 1 year, then determine the mean index for each season.

	Dec–Feb	Mar–May	Jun–Aug	Sep–Nov	Total
2016 sales ($)	32	60	50	38	180
2017 sales ($)	48	70	60	52	230
2018 sales ($)	45	65	55	45	210

THINKING

WORKING

1 Calculate the average season for each year. The total for each year is given in the table above.

	Dec–Feb	Mar–May	Jun–Aug	Sep–Nov	Mean
2016 sales ($)	32	60	50	38	mean $= \frac{180}{4} = 45$
2017 sales ($)	48	70	60	52	mean $= \frac{230}{4} = 57.5$
2018 sales ($)	45	65	55	45	mean $= \frac{210}{4} = 52.5$

2 Interpret the result.

The average season in 2016 yields $45 in sales, the average season in 2017 yields $57 in sales and the average season in 2018 yields $52 in sales.

3 Determine the seasonal index by dividing the value for each period by the average season for the corresponding year.

	$SI_{Dec-Feb}$	$SI_{Mar-May}$	$SI_{Jun-Aug}$	$SI_{Sep-Nov}$
2016	$\frac{32}{45} = 0.71$	$\frac{60}{45} = 1.33$	$\frac{50}{45} = 1.11$	$\frac{38}{45} = 0.84$
2017	$\frac{48}{57.5} = 0.83$	$\frac{70}{57.5} = 1.22$	$\frac{60}{57.5} = 1.04$	$\frac{52}{57.5} = 0.90$
2018	$\frac{45}{52.5} = 0.86$	$\frac{65}{52.5} = 1.24$	$\frac{55}{52.5} = 1.05$	$\frac{45}{52.5} = 0.86$

4 Identify any trends in the table.

In the periods Mar–May and Jun–Aug, sales in all three years were above average. In the periods Sep–Nov and Dec–Feb, sales were below the average season.

5 Calculate the mean seasonal index for each season over the 3 year period and check the reasonableness of the result.

	$SI_{Dec-Feb}$	$SI_{Mar-May}$
2016–2018 sales	$\frac{0.71 + 0.83 + 0.86}{3} = 0.80$	$\frac{1.33 + 1.22 + 1.24}{3} = 1.26$
	$SI_{Jun-Aug}$	$SI_{Sep-Nov}$
	$\frac{1.11 + 1.04 + 1.05}{3} = 1.07$	$\frac{0.84 + 0.90 + 0.86}{3} = 0.87$

$0.80 + 1.26 + 1.07 + 0.87 = 4$

6	Interpret the result.	On average, over the 3 year period, the sales in: Dec–Feb are 20% below the average season Mar–May are 26% above the average season Jun–Aug are 7% above the average season Sep–Nov are 13% below the average season.

Deseasonalising data

Removing seasonal effects from the data will smooth out fluctuations in the data, and the underlying trend may be more easily identified.

$$\text{Deseasonalised value} = \frac{\text{actual value}}{\text{seasonal index}}$$

From Worked example 11 you know that, on average, sales from December to February are 20% below the average season. It was also stated that, in that period in 2016, sales were made to the value of $32.

Deseasonalising the data value gives you the opportunity to compare all values with the average season, to determine underlying trends in the data. For example:

$$\text{Deseasonalised value} = \frac{32}{0.8} = \$40$$

This value represents the sale equivalent in the average season.

When you graph the raw data against the deseasonalised data, you will see a decrease in the highest-performing periods and an increase in the lowest-performing periods, giving an average long-term trend and making it possible to accurately compare annual performance.

The seasonal indices would normally have been calculated as means from long-term data.

Explore further

Seasonal data

Use a spreadsheet to calculate and interpret seasonal indices and deseasonalise data.

12 Deseasonalising data

Deseasonalise the data, then construct a graph for both the raw data and the deseasonalised data on the same set of axes. Comment on the pattern and trend of the two graphs.

	Dec–Feb	Mar–May	Jun–Aug	Sep–Nov
2016 sales ($)	32	60	50	38
2017 sales ($)	48	70	60	52
2018 sales ($)	45	65	55	45
Seasonal index	0.8	1.26	1.07	0.87

THINKING	WORKING

1 Use the formula

$$\text{Deseasonalised value} = \frac{\text{actual value}}{\text{seasonal index}}$$

to calculate the deseasonalised values.

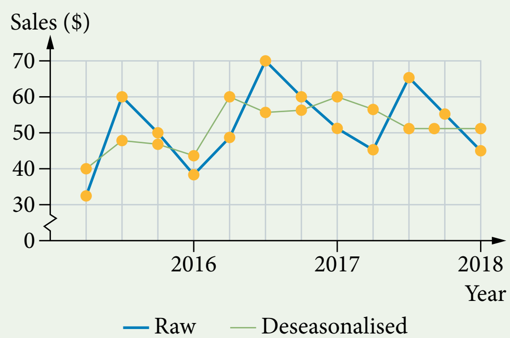
Give answers to 1 decimal place.

	$SI_{Dec–Feb}$	$SI_{Mar–May}$	$SI_{Jun–Aug}$	$SI_{Sep–Nov}$
2016 sales ($)	$\frac{32}{0.8} = 40$	$\frac{60}{1.26} = 47.6$	$\frac{50}{1.07} = 46.7$	$\frac{38}{0.87} = 43.7$
2017 sales ($)	$\frac{48}{0.8} = 60$	$\frac{70}{1.26} = 55.6$	$\frac{60}{1.07} = 56.1$	$\frac{52}{0.87} = 59.8$
2018 sales ($)	$\frac{45}{0.8} = 56.3$	$\frac{65}{1.26} = 51.6$	$\frac{55}{1.07} = 51.4$	$\frac{45}{0.87} = 51.7$

2 Using suitable scales, graph both time series.

Use a different colour for each graph.

The horizontal axis (time) has 12 values and the vertical axis ranges from 30 to 70.



3 Compare the graphs and comment on any trends.

The raw data shows a seasonal pattern, with peak sales occurring in autumn each year.

The deseasonalised data curve is much smoother, with fewer peaks and troughs. There appears to be an overall increasing trend, with better sales in 2017 than in 2016 and 2018.

EXERCISE 2.3 Seasonal variation

Worked Example 9

1 Calculate the seasonal indices for the sales data for four quarters, and interpret the results compared to the average season. Give your answers to 2 decimal places.

Period	Dec–Feb	Mar–May	Jun–Aug	Sep–Nov
Sales ($)	20 000	29 000	31 000	40 000

2 The average rainfall (mm) in Sydney is given in the table.

Month	J	F	M	A	M	J	J	A	S	O	N	D
Rainfall (mm)	102.2	117.6	130.9	128.5	118.6	133.2	96.6	80.7	67.9	76.4	83.6	77.6

(a) Calculate the seasonal indices for each month, to 2 decimal places.

(b) Interpret the seasonal indices for the months with the highest and lowest rainfall.

3 The average rainfall (mm) in Brisbane is given in the table.

Month	J	F	M	A	M	J	J	A	S	O	N	D
Rainfall (mm)	159.6	158.3	140.7	92.5	73.7	67.8	56.5	45.9	45.7	75.4	97.0	133.3

Calculate the seasonal indices, to 2 decimal places:

(a) for each month

(b) for each season, starting with summer (Dec–Feb).

4 The table shows visitor arrivals, in thousands, in Australia over a year. The data has been grouped into periods of 2 months.

Jan–Feb	Mar–Apr	May–Jun	Jul–Aug	Sep–Oct	Nov–Dec	Total
1698	1472	1186	1449	1404	1698	8907

Which expression shows the calculation to determine the seasonal index for May–June?

A $\frac{1186}{1698}$ **B** $\frac{1186}{8907}$ **C** $\frac{1186}{8907 \div 6}$ **D** $\frac{1}{12} \times \frac{1186}{8907}$

Worked Example 10

5 Determine the missing seasonal indices, to 2 decimal places.

(a)

Quarter	Spring	Summer	Autumn	Winter
Seasonal index	1.2	x	0.7	0.9

(b)

2-month period	Jan–Feb	Mar–Apr	May–Jun	Jul–Aug	Sep–Oct	Nov–Dec
Seasonal index	$3y$	y	$2y$	$5y$	$3y$	$4y$

6 Determine and interpret the missing seasonal index, to 2 decimal places.

	Mon	Tue	Wed	Thu	Fri
Seasonal index	0.92	0.88	0.65	1.21	x

7 The table shows the average quarterly rainfall (mm) for Sydney and the seasonal indices.

Complete the table by determining each of the missing values.

	Summer	Autumn	Winter	Spring
Rainfall (mm)				227.9
Seasonal index		1.246	1.023	0.751

8 The table shows the number of sales each day over 3 weeks. Sketch a time series graph for both the raw data and the deseasonalised data on the same set of axes. Comment on whether there was a weekly cycle.

	Monday	Tuesday	Wednesday	Thursday	Friday
Week 1	40	56	80	74	50
Week 2	48	55	75	78	60
Week 3	46	60	82	75	45
Seasonal index	0.724	0.926	1.284	1.228	0.838

9 The table shows quarterly sales of televisions over a 3 year period.

(a) Construct a graph of the time series data.

(b) Comment on the pattern and trend of the graph.

(c) In June 2013, transmission of the analogue TV signal was switched off, leaving only the digital signal. How does this help explain the pattern observed?

(d) Why is it not appropriate to calculate seasonal indices for this data?

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
2011	140	135	140	160
2012	120	130	128	140
2013	180	260	190	130

Worked Example 11

10 Calculate the seasonal indices for the amount of rainfall (cm) over 3 years. Give your answers to 2 decimal places.

	Summer	Autumn	Winter	Spring	Total
2016	120	160	200	180	660
2017	50	200	180	140	570
2018	100	180	150	150	580

11 The table shows the number of visitors, in thousands, to an art gallery over the past three years. The data has been grouped according to season.

	Spring	Summer	Autumn	Winter	Total
2016	11.8	12.4	10.5	8.4	43.1
2017	12.3	12.8	11.8	9.6	46.5
2018	12.6	13.0	12.0	9.8	47.4

(a) Which expression shows the calculation for finding the seasonal index for winter?

A $\frac{8.4}{43.1}$

B $\frac{1}{3}\left(\frac{8.4}{43.1}+\frac{9.6}{43.1}+\frac{9.8}{43.1}\right)$

C $\frac{1}{3}\left(\frac{8.4}{43.1}+\frac{9.6}{46.5}+\frac{9.8}{47.4}\right)$

D $\frac{1}{3}\left(\frac{8.4}{43.1 \div 4}+\frac{9.6}{46.5 \div 4}+\frac{9.8}{47.4 \div 4}\right)$

(b) Explain the common error made by a student who chose the third incorrect option.

(c) Which value is the best approximation for the seasonal index for spring?

A 1.06 B 1.07 C 1.08 D 1.09

12

12 The table shows quarterly sales of DVD players in an electronic goods chain store in 2016–18.

(a) Calculate the seasonal index for each quarter. Give your answers to 3 decimal places.

(b) Deseasonalise the data, giving your answers to the nearest whole number. Then sketch the graphs of the raw and deseasonalised time series data on the same set of axes.

(c) The owner of the stores believes there is a negative trend in sales due to a combination of a move to Blu-ray technology and an increase in online sales. Does the data support the owner's belief?

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
2016	650	620	430	470
2017	610	550	390	450
2018	520	470	280	350

13 The table shows monthly road traffic fatalities in New South Wales for 2011–13.

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
2011	25	30	31	28	27	31	27	35	25	30	37	33
2012	32	25	33	33	31	34	24	36	30	28	35	28
2013	15	33	30	25	24	32	28	32	15	38	36	32

(a) Construct a graph of the time series data.

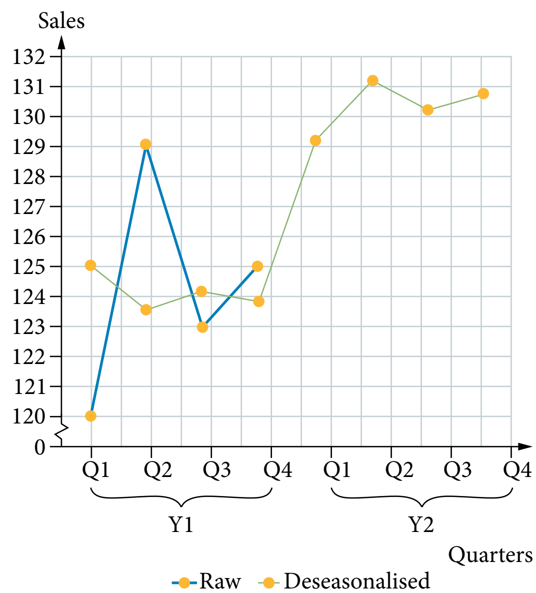
(b) Comment on the pattern and trend of the graph.

(c) Calculate the monthly seasonal indices for January, March, May, August, November and December. Give your answers to 3 decimal places.

(d) Suggest a reason why the indices vary from month to month.

14 The graph shows raw and deseasonalised quarterly sales data over 2 years.

Complete the graph by drawing in the raw data graph line for the second year.



15 The table gives the seasonal indices for monthly rainfall for the Gold Coast Seaway, and the actual rainfall for 2017.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Seasonal index	1.28	1.56	1.14	1.21	1.00	1.13	0.44	0.56	0.40	0.81	1.07	1.25
Actual rainfall (mm)	120	23	394	35	78	132	12	6	0	81	74	72

(a) Given that the average annual rainfall for the Gold Coast Seaway is 1273.5 mm, identify the months in which rainfall was more than 50% below the 2017 average.

(b) For the months that had above-average monthly rainfall, which month had the highest percentage above the mean? What was this percentage, to the nearest per cent?

(c) Determine whether the Gold Coast had above-average or below-average rainfall for 2017 and by how many millimetres.

2.4 Long-term trends

Numerical values of time

When the time values of years, weeks or days are given as numbers, determining the equation of the least-squares line can be straightforward.

13 Determine the least-squares equation of a time series graph

The data below represents the number of parking fines issued by a local council from 2006 to 2014.

Year	2006	2007	2008	2009	2010	2011	2012	2013	2014
Fines	2080	1980	2650	2400	2750	2680	3400	3140	3330

(a) Draw the time series graph.

THINKING

1 Enter the data into a spreadsheet.

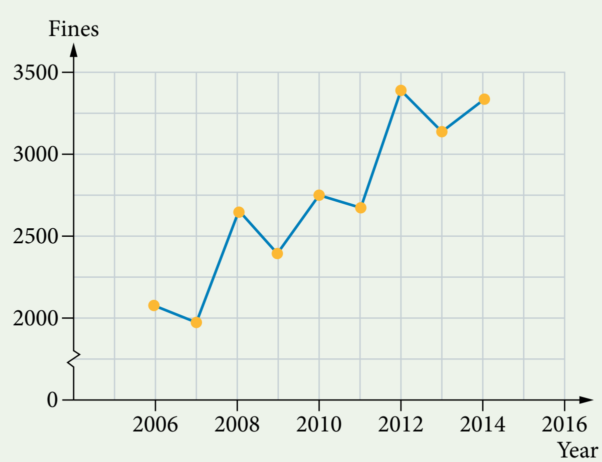
WORKING

	A	B
1	Year	Fines
2	2006	2080
3	2007	1980
4	2008	2650
5	2009	2400
6	2010	2750
7	2011	2680
8	2012	3400
9	2013	3140
10	2014	3330

2 Highlight the data and insert an XY scatterplot.

Ensure that the horizontal axis values are the years, and that the vertical scale matches the values you would choose.

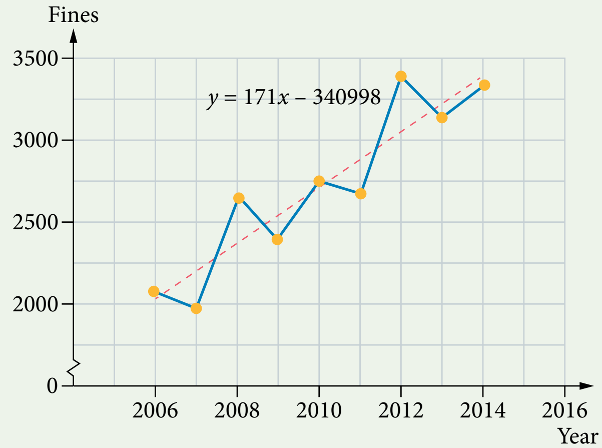
Alternatively, plot the points by hand.



(b) Use technology to determine the least-squares equation.

1 Right-click on any point to add a trendline. Show the equation.

Alternatively, draw a trendline by hand. Enter the values as bivariate data into a scientific calculator with the actual years as the x-values.



2 Read the linear regression equation from the graph or calculator.

The equation of the least-squares line is:
$y = 171x - 340\,998$

3 Interpret the equation in terms of the actual variables.

Number of fines = 171 × year − 340 998

(c) Use the equation of the least-squares line to estimate the number of fines in 2015.

1 Substitute the value of time into your equation.

$x = 2015$
$y = 171x - 340\,998$
$= 171 \times 2015 - 340\,998$
$= 3567$

2 Interpret the answer.

Using extrapolation, the model predicts that the council will issue 3567 fines in 2015.

Time written as a categorical variable

When time is given as days of the week, months of the year, or quarters within a year, a trendline can still be drawn. To use a least-squares equation you must define time in a way that makes it easy to convert between the categorical and numerical scales.

Line graphs in Excel simply use the first value as 1, the second as 2 and so on.

14 Trendlines when time is given as names

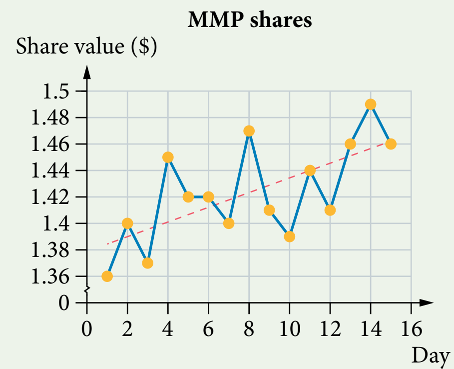
The data below gives the price ($) of MMP shares over a 3 week period.

Week	Mon	Tue	Wed	Thu	Fri
1	1.36	1.40	1.37	1.45	1.42
2	1.42	1.40	1.47	1.41	1.39
3	1.44	1.41	1.46	1.49	1.46

(a) Draw the time series graph and fit a least-squares trendline.

THINKING	WORKING
1 Enter the data into a spreadsheet. Highlight the data and insert a line graph. Add a trendline. Show the equation. Ensure that the horizontal axis values are time, and that the vertical scale matches the values you would choose.	(see spreadsheet and graph below)

	A	B
1	Day	Share value ($)
2	1	1.36
3	2	1.4
4	3	1.37
5	4	1.45
6	5	1.42
7	6	1.42
8	7	1.4
9	8	1.47
10	9	1.41
11	10	1.39
12	11	1.44
13	12	1.41
14	13	1.46
15	14	1.49
16	15	1.46



2 Alternatively, plot the points by hand, then determine the equation of the least-squares line and calculate the values of the endpoints.

Plot the line of best fit on the hand-drawn graph.

$y = 0.0055x + 1.3798$

For $x = 1$:

$y = 0.0055 \times 1 + 1.3798$

$= 1.385$

For $x = 15$:

$y = 0.0055 \times 15 + 1.3798$

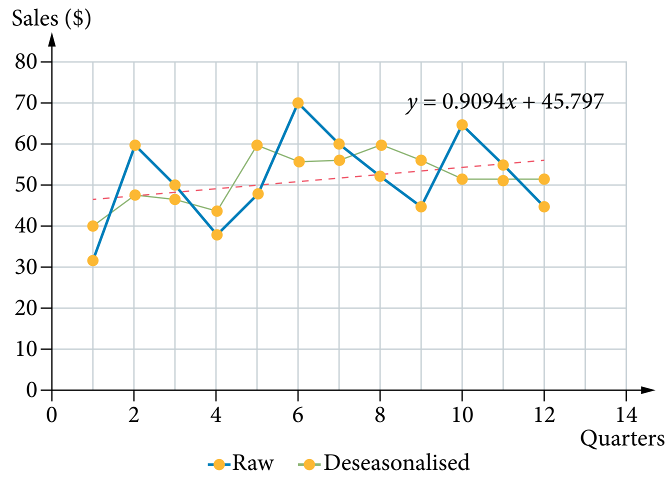
$= 1.462$

Endpoints: $(1, 1.385)$ and $(15, 1.462)$

(b) Estimate the value of the shares on Wednesday of week 4.

THINKING	WORKING
1 Substitute the value of time into the equation and calculate the y-value.	Wednesday of week 4: $x = 18$ $y = 0.0055x + 1.3798$ $= 0.0055 \times 18 + 1.3798$ $= 1.48$ (2 d.p.)
2 Interpret the answer.	The model predicts that the shares will be worth $1.48 by Wednesday of week 4, if the trend continues.

From Worked example 11, the raw and deseasonalised data are shown below.



Time	Raw	Deseasonalised
1	32	40
2	60	47.6
3	50	46.7
4	38	43.7
5	48	60
6	70	55.6
7	60	56.1
8	52	59.8
9	45	56.3
10	65	51.6
11	55	51.4
12	45	51.7
Average seasonal indices		
Dec–Feb	0.8	
March–May	1.26	
June–Aug	1.07	
Sept–Nov	0.87	

To predict future sales (extrapolate) seasonal data, it is more reliable to reduce the fluctuations and determine the equation of the trendline of the deseasonalised values. You can then use the equation of the trendline where the equation

$$y = 0.9094x + 45.797$$

reads as: 'The average seasonal sales in a given quarter is equal to 0.9094 multiplied by the number of quarters after summer in 2016 plus 45.797'.

A long-term forecast or prediction can be made – for example, to determine sales in the March–May quarter in 2020.

March–May 2020 is the 18th quarter.

For $x = 18$

$$\begin{aligned} y &= 0.9094 \times 18 + 45.797 \\ &= 62.17 \ (2 \text{ d.p.}) \end{aligned}$$

The average season's sales in the 2nd quarter of 2020 are expected to be $62.17. However, it is also known that the seasonal index in this quarter is 1.26. That is, sales in the March–May quarter are typically 26% higher than the average season.

To predict sales accurately, the deseasonalised data needs to be reseasonalised using the formula:

$$\begin{aligned} \text{Predicted seasonalised value} &= \text{deseasonalised value} \times \text{seasonal index} \\ &= 62.17 \times 1.26 \\ &= 78.329 \end{aligned}$$

The result shows that the predicted value of sales for the autumn quarter March–May in 2020 is $78.33.

The same principle can be applied to predict sales in spring of 2020 (September–November).

Let $x = 20$

$$\begin{aligned} y &= 0.9094 \times 20 + 45.797 \\ &= 63.99 \ (2 \text{ d.p.}) \end{aligned}$$

As expected with an increasing trend, the expected average sales has increased. You know that sales in spring have a seasonal index of 0.87, meaning expected sales are 13% lower than the average season.

Predicted seasonalised value = deseasonalised value × seasonal index

$= 63.99 \times 0.87$

$= 55.67$

The result shows that the predicted sales for the spring quarter (September–November) in 2020 is $55.67.

EXERCISE 2.4 Long-term trends

Worked Example 13

1 The table shows the percentage of Australian adults who had a smart phone, from 2007 to 2013.

Year	2007	2008	2009	2010	2011	2012	2013
% with a smart phone	19	23	55	62	68	76	84

(a) Draw the time series graph.

(b) Add the trendline and determine the equation of the least-squares line.

(c) Estimate the percentage of adults with a smart phone in 2015. Write your answer to the nearest whole per cent.

(d) How can you explain your answer to part (c)?

2 Sales figures are given over a 7 week period.

Use a linear model to predict the sales for week 20.

Week	1	2	3	4	5	6	7
Sales ($)	452	456	450	459	455	465	463

3 The unemployment rate in Australia for 12 months up to and including November 2017 is given as percentages in the table. Use a linear model to predict the unemployment figures for December 2017 and December 2020.

Dec	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov
5.8	5.7	5.9	5.9	5.7	5.5	5.6	5.6	5.6	5.5	5.4	5.4

4 Some median house prices for Brisbane over several years are given in the table.

Year	1973	1983	1993	2003
Median house price ($'000)	17.5	55.5	136.5	249

(a) Use the data to create a graph and show the least-squares line. Comment on its suitability to predict median house prices by interpolation.

(b) Determine the average annual increase in median price for each of the 10 year periods given by the data. Compare these with the rate suggested by the linear model.

5 A trendline in the form $y = a + bx$ is drawn for data collected each month of 2017 and 2018.

If the model uses $x = 1$ for January 2017 and $x = 2$ for February 2017, what value of x would be used to predict an amount for July 2020?

6 A trendline in the form $y = a + bx$ is drawn for data collected from Monday to Friday each week.

If the model uses $x = 1$ for Monday of week 1 and $x = 2$ for Tuesday of week 1, what day and week is given by $x = 46$?

7 A trendline in the form $y = a + bx$ is drawn for data collected each quarter over several years.

If the model uses $x = 1$ for the first quarter of 1985 and $x = 2$ for the second quarter of 1985, which value of x would be used to predict an amount for the third quarter of 2025?

A 159 B 163 C 167 D 171

8 A trendline in the form $y = a + bx$ is drawn for data collected each quarter over several years.

If the model uses $x = 1$ for the first quarter of 1990 and $x = 2$ for the second quarter of 1990, what value of x would be used to predict an amount for the second quarter of 2019?

9 The equation $y = 14.2 + 0.35x$ models monthly figures for the total amount of money ($'000) in a fund to build a tennis court.

(a) If the model uses $x = 1$ for the beginning of 2015 and $x = 2$ for the beginning of February 2015, what amount is expected to be raised by the beginning of May 2019, to the nearest $10?

A $15 720 B $15 750 C $32 400 D $32 750

(b) Explain the common error made by a student who substituted $x = 52$ into the equation in part (a).

10 The table gives the number of people visiting Queensland over a period of 10 years for the purpose of education.

Year	2007–08	2008–09	2009–10	2010–11	2011–12	2012–13	2013–14	2014–15	2015–16	2016–17
Visitors ('000)	106	103	107	109	112	108	108	125	134	157

Use a linear model to predict the number of visitors in 2020–21. Comment on the suitability of the model and the reliability of the prediction.

11 The ASX 200 is an index that uses the values of 200 shares listed on the Australian sharemarket, while the All Ordinaries index uses 500 shares worldwide.

The table gives the values of both indexes for the first 11 months of 2017.

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov
ASX 200	5621	5712	5865	5924	5725	5722	5721	5715	5682	5909	5970
All Ords	5675	5761	5904	5948	5761	5764	5774	5776	5745	5976	6057

(a) Graph both lines on the same axes to check the suitability of a linear model for each, and to compare the overall trends.

(b) Use the least-squares equations to predict the value of each index for December 2017. Comment on the suitability of the models to predict these values.

(c) Use the least-squares equations to predict the value of each index for December 2018. Comment on the suitability of the models to predict these values.

Worked Example 14

12 The table shows the price ($) of BOM shares over a 3 week period.

Week	Mon	Tue	Wed	Thu	Fri
1	15.24	15.82	15.63	14.12	14.96
2	15.03	14.80	14.76	14.72	14.73
3	14.81	14.31	14.50	14.55	14.48

(a) Create the time series graph and fit a least-squares line.

(b) Estimate the value of the shares on Wednesday of week 10.

13 The table gives ASX 200 values throughout the Global Financial Crisis of 2008–09.

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
2008	5650	5572	5356	5595	5655	5215	4977	5135	4601	4018	3743	3722
2009	3541	3345	3582	3781	3818	3955	4244	4479	4744	4643	4701	4871

(a) Create a graph of the time series data. Model the two distinct trends with separate least-squares lines overlaid onto your graph.

(b) If the first trend had continued, when would the ASX 200 have dropped to less than 1000 for the first time?

(c) If the second trend had continued, in what month would the ASX 200 have first gone above the January 2008 value?

14 The data in the table gives the sales figures ($'000) of a company over a 3 year period.

	1st quarter	2nd quarter	3rd quarter	4th quarter
2016	128	114	119	136
2017	131	118	124	141
2018	138	125	133	149

(a) Deseasonalise the data before graphing the raw data, the deseasonalised data and the least-squares line on the same set of axes. Comment on the suitability of the trendline to predict future sales.

(b) Estimate the sales figures for each quarter of 2019, to the nearest thousand dollars.

15 The data in the table shows the number of yoyos sold at a toy store per month over a 1 year period.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Sales	12	14	9	45	80	36	6	8	10	9	12	4

(a) Create a time series graph. Then determine the equation of the least-squares line in terms of x and y, to 2 decimal places, by labelling January as 1, February as 2, etc.

(b) Using your equation as a model, predict the number of yoyos sold in December of the following year. Comment on the suitability of the model.

(c) Using dummy values, create a new least-squares equation. Using your new equation as a model, predict the number of yoyos sold in December of the following year. Comment on the suitability of the model.

Summary

Data collected over time

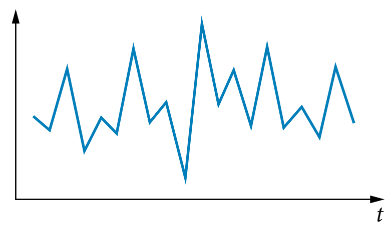
Data that is collected or recorded regularly over time is called time series data. This type of data is useful for detecting patterns that may lead to predictions about future trends.

A time series graph has the observed variable on the vertical axis and time on the horizontal axis.

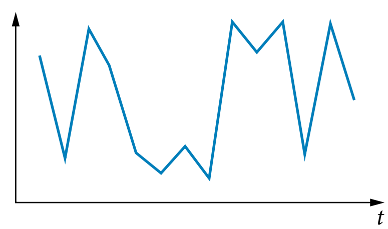
Patterns and trends

Increasing and decreasing trends are not always easy to notice, as they may be hidden within another type of pattern.

Seasonal patterns are repeated *cyclical fluctuations* with predictable cycles of known and fixed length.



Irregular patterns show variation but are influenced by unpredictable events and do not fit any of the patterns described previously.



Smoothing time series: moving means and moving medians

The moving mean for a point is calculated by considering its neighbouring values and calculating the mean.

3-point moving mean: $d_{2\ \text{smoothed}} = \dfrac{d_1 + d_2 + d_3}{3}$

5-point moving mean: $d_{3\ \text{smoothed}} = \dfrac{d_1 + d_2 + d_3 + d_4 + d_5}{5}$

Smoothing with moving medians follows the same process. Three-point moving medians can be found directly from the graph.

When smoothing using an even number of data points, the calculated mean or median will not line up directly with a time period point. The smoothed data requires centring by calculating the mean of two consecutive moving means or medians.

Seasonal variation

When data displays a seasonal pattern, removing the seasonal effects makes the underlying trend easier to see. When given figures for several years, you can improve the reliability of the seasonal index by calculating the mean seasonal index for each season.

$$\text{Seasonal index} = \frac{\text{actual seasonal figure}}{\text{yearly average}}$$

To deseasonalise the figures, use the formula:

$$\text{Deseasonalised value} = \frac{\text{actual value}}{\text{seasonal index}}$$

When you plot the raw figures on the same axis as the deseasonalised figures, you will see that the peaks and troughs have been reduced.

Forecasting

Technology can be used to determine the equation of the least-squares line to show a trend.

You can use the least-squares equation to predict future values by substituting in new values for the time.

Predicting outside the values in the data set is called extrapolation, and the results need to be treated with care.

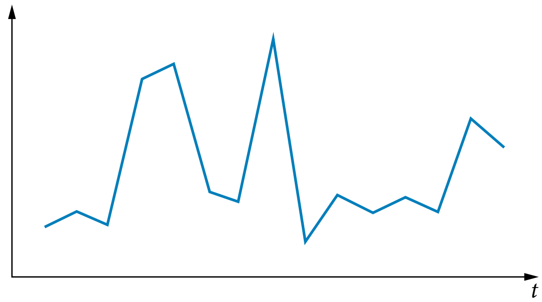
For seasonal data, the most reliable forecast comes from deseasonalising the data, determining the equation of the least-squares line for the deseasonalised data, basing the prediction on this regression line and then converting the deseasonalised figure back to a raw figure (reseasonalising).

Outliers

If an outlier is known to be an aberrant value, it can be replaced by dummy data for the purpose of making meaningful predictions. The simplest dummy data uses the mean of the values on each side of the aberrant value.

Chapter review 2

1 Describe the time series graph shown here. Explain how you would predict future values from this data.



Exercise 2.1

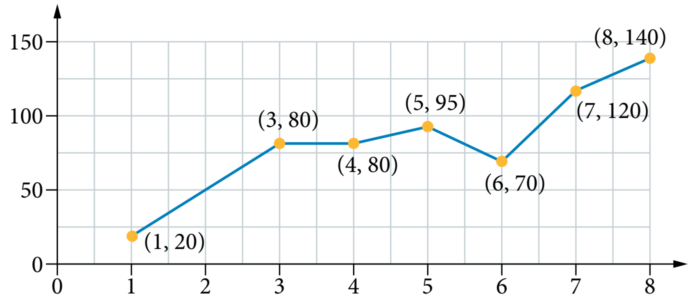
2 What is the best description of the pattern shown in this time series graph?

A cyclical

B negative trend

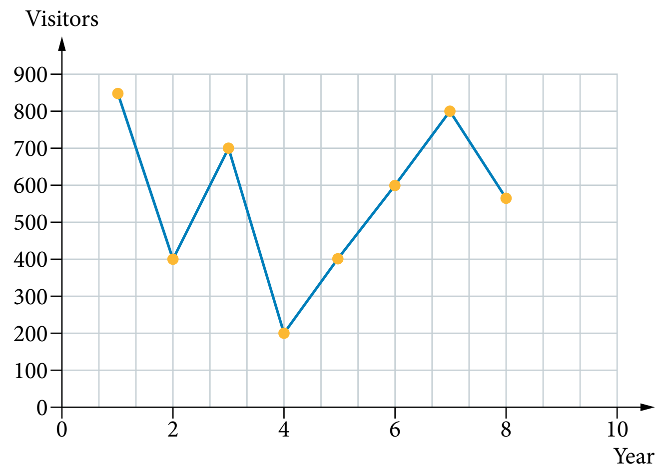
C positive trend

D random



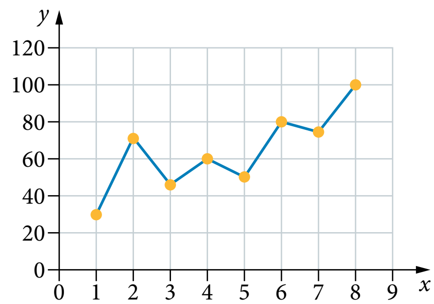
Exercise 2.1

3 What is the best description of the pattern shown in this time series graph?



Exercise 2.1

4 What is the best description of the pattern shown in this time series graph?



Exercise 2.1

Exercise 2.2

5 For the data given in the table, calculate the 3-point moving mean centred on $x = 40$.

x	10	20	30	40	50	60
y	450	580	520	620	660	820

Exercise 2.2

6 Determine the 5-point moving mean for the middle value in the table of data.

t	1	2	3	4	5
y	18	23	30	32	34

Exercise 2.2

7 Use a 3-point moving mean to predict the values of a and b to the nearest whole number.

t	1	2	3	4
y	150	320	530	650
Smoothed		a	b	

Exercise 2.2

8 For the data given in the table, the 3-point moving means have been calculated.

What is the missing original data point, n?

A 8 **B** 9 **C** 10 **D** 11

t	1	2	3	4	5	6
y	5	6	7	n	12	19
		6	7	9	13	

Exercise 2.2

9 Construct a table of the 3-point moving means for the data in the table, and then sketch the graphs of the raw and smoothed data on the same set of axes.

Time	1	2	3	4	5	6	7	8	9	10
Value	12	14	8	7	24	20	15	9	11	12

Exercise 2.2

10 For the values shown here, construct a table of the 7-point moving means, and then sketch graphs of the raw and smoothed data on the same set of axes.

Day	1	2	3	4	5	6	7	8	9	10	11	12
Value	5	5	8	10	12	14	16	19	26	32	35	54

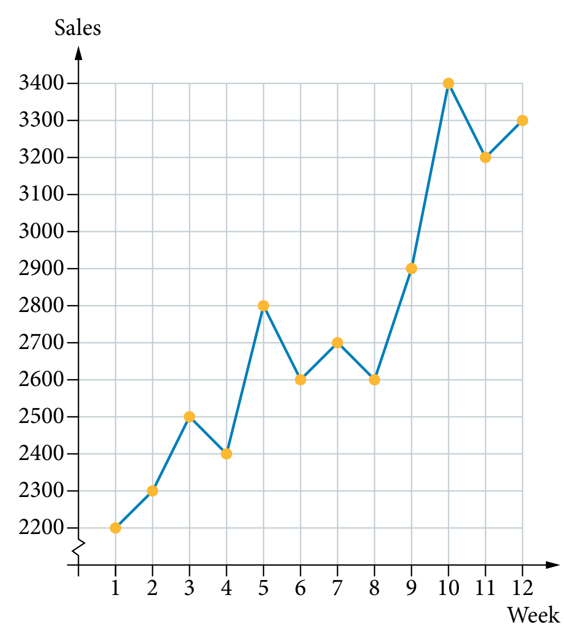
Exercise 2.2

11 For the values given in the table, construct a table of the 3-point moving medians.

Time	1	2	3	4	5	6	7	8	9	10
Value	12	14	8	7	24	20	15	9	11	12

Exercise 2.2

12 For the graph of sales figures shown here, plot the 3-point moving medians on the same set of axes.



13 The quarterly sales figures for Sally's Sleepwear and the associated seasonal indices are given in the table.

Determine the values of m and n, to 2 decimal places.

Quarter	1	2	3	4	Mean
Sales	400	850	700	1200	787.5
Seasonal index	0.51	m	0.89	n	

Exercise 2.3

14 The quarterly sales figures for Mia's Magazines and the associated seasonal indices are given in the table.

Determine the values of x and y, to the nearest whole number.

Quarter	1	2	3	4	Mean
Sales	3 500	2 800	x	y	3 575
Seasonal index	0.98	0.78	1.15	1.09	

Exercise 2.3

15 The seasonal indices for sales of jeans at Juan's Jean Joint are given in the table.

The deseasonalised figure for the second quarter is 845. How many jeans were sold in the second quarter?

Quarter	1	2	3	4
Seasonal index	0.95	1.25	1.10	0.70

Exercise 2.3

16 Sales of T-shirts are seasonal, increasing in warmer weather. Records are kept for over 2 years. The actual sales and seasonal indices are given in the table. One of the seasonal indices is missing.

Season	Winter	Spring	Summer	Autumn
Sales 2017	330	390	540	350
Sales 2018	360	420	530	390
Seasonal index	0.83	0.98	1.29	a

Exercise 2.3

(a) Calculate the missing seasonal index, a.

(b) Deseasonalise the data and then plot the graphs of the actual and deseasonalised data on the same set of axes.

(c) The raw data shows that the highest sales occurred in summer 2017. In which year and season do the deseasonalised figures show the best performances (i.e. more than expected sales)?

17 Determine the seasonal index for each quarter, to 2 decimal places.

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
Sales ($'000)	38.6	49.5	51.2	43.1

Exercise 2.3

18 Given the quarterly data over 3 years and the long-term seasonal indices, seasonalise all the data in the table, writing values to the nearest whole number.

Exercise 2.3

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
2016	320	385	398	384
2017	356	404	420	377
2018	365	421	434	409
SI	0.82	1.04	1.13	1.01

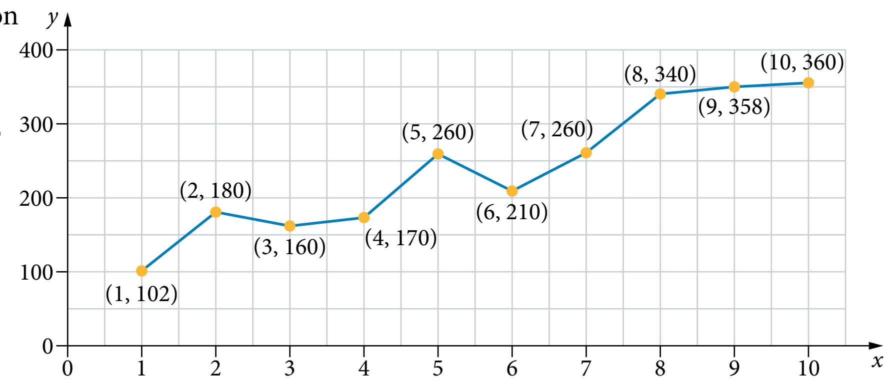
Exercise 2.3

19 Given the deseasonalised daily cafe takings ($) over a week and the seasonal indices for each day, reseasonalise all the data in the table to find the actual takings, to the nearest dollar.

	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Deseasonalised takings ($)	5 250	5 621	5 640	5 594	5 338	5 199	5 303
SI	0.82	0.42	0.85	0.75	1.29	1.65	1.22

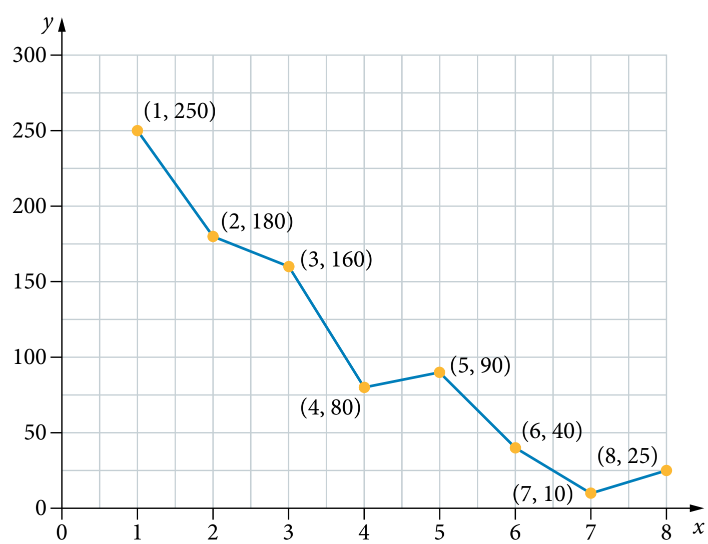
Exercise 2.4

20 Determine the equation of the least-squares line for the time series shown in the graph, to the nearest whole number. Write the equation in context if x is the week number and y is the sales for a new company.



Exercise 2.4

21 Determine the equation of the least-squares line for the time series shown in the graph, to the nearest whole number. Write the equation in context if x is the week number and y is the number of birds sighted in a park as the spring and summer berries have been consumed.



Exercise 2.4

22 The data given in the table shows numbers of visitors at Utehenge over the past 8 years. Utehenge is a replica of Stonehenge made from old utes sprayed grey.

Year	1	2	3	4	5	6	7	8
Visitors	22	32	28	27	35	45	38	49

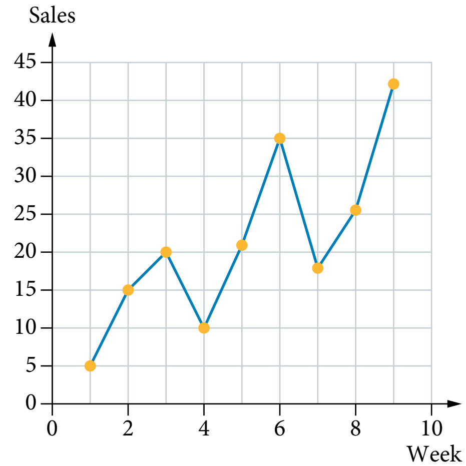
(a) Construct a time series graph for the data, and draw the trendline by eye.

(b) Determine the equation for the least-squares line. Give values in the equation to 2 decimal places.

(c) Predict the number of visitors in 2 years' time.

23 Determine the least-squares equation, to 2 decimal places, for the time series data given by the graph. Determine the endpoints of the trendline and add the line to the existing graph.

Exercise 2.1



24 Construct a table of the centred 4-point moving means, and then sketch the graphs of the raw and smoothed data on the same set of axes.

Exercise 2.2

Time	1	2	3	4	5	6	7	8
Value	20	40	30	50	70	80	70	90

25 Construct a table of the centred 4-point moving medians.

Exercise 2.2

Time	1	2	3	4	5	6	7	8
Value	20	40	30	50	70	80	70	90

26 Which of the following equations best represents the trendline for the time series shown in the graph?

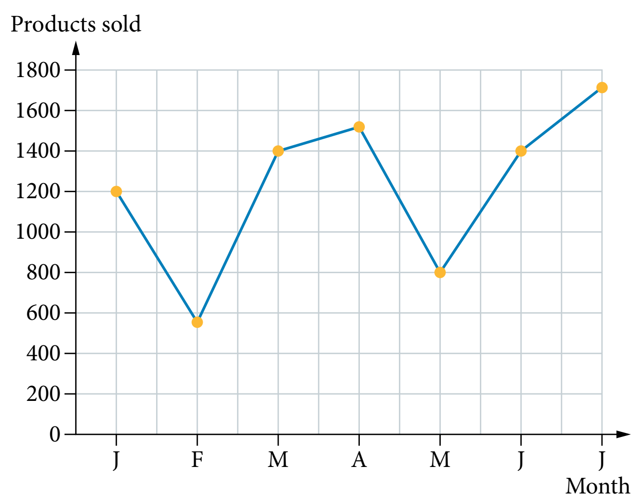
Exercise 2.4

A products sold = number of month + 1000

B products sold = 50 × number of month + 400

C products sold = 91 × number of month + 600

D products sold = 91 × number of month + 860



27 Which of the following equations best represents the trendline for the time series shown in the graph, where q is the quarter, numbering Q1 as $q = 1$?

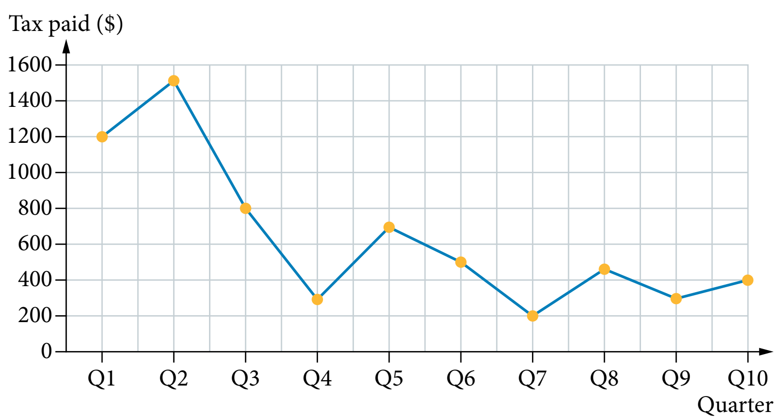
Exercise 2.4

A tax paid (\$) = $1250 - 108q$

B tax paid (\$) = $1250 - 50q$

C tax paid (\$) = $50q - 1250$

D tax paid (\$) = $1250 + 108q$



Exercise 2.3, 2.4

28 The data in the table represents quarterly sales ($'000) at Doug's Dive Shop over 2 years.

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
2017	480	360	210	370
2018	520	350	180	400

(a) Plot the time series graph and add the least-squares line. Determine the least-squares equation, rounded to the nearest whole number.

(b) Calculate the seasonal indices, to 5 decimal places.

(c) Deseasonalise the data, to 2 decimal places. Then sketch the graph of the deseasonalised data and the raw data for the 2 years.

Exercise 2.4

29 The data represents monthly sales over a year. One of the data values comes from a one-off situation that is unlikely to occur again.

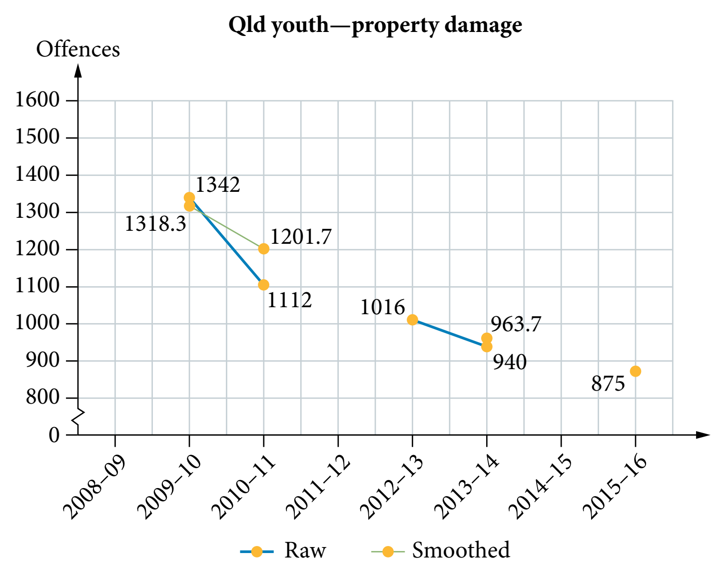
	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Sales	56	66	12	72	87	86	89	94	96	96	112	104

Replace the outlier with a dummy value, then use a least-squares equation to predict sales for May in the following year. Comment on the suitability of the equation to predict this value.

Exercise 2.2

30 The graph gives the raw data and the 3-point moving averages for youth offences with property damage, from 2008–09 to 2015–16.

Calculate the three missing values for each line graph and complete the graph.



31 The table shows the numbers of skis and snowboards sold over 10 years at a ski shop. Exercise 2.4

Year	2003	2004	2005	2006	2007	2008	2009	2010	2011	2012
Skis sold	840	710	730	450	500	200	300	180	100	110
Snowboards sold	220	300	380	340	420	400	580	520	540	600

(a) Draw a time series graph of sales of skis and sales of snowboards on the same set of axes.

(b) Determine separate least-squares equations for the sales of skis and sales of snowboards, to 2 decimal places.

(c) Predict the sales of skis and snowboards in 2014 using the least-squares equations from part **(b)**. Comment on the results.

(d) Calculate the total sales of skis and snowboards together for each year, and draw a time series graph of this data.

(e) Determine the least-squares equation for the total sales. If possible, use equations already written.

(f) At what point does the total sales least-squares equation give unreasonable values when predicting future sales?

(g) At what point would it be invalid to use the total sales least-squares equation to make predictions involving extrapolation into the future?

32 The table shows mean monthly maximum temperatures (°C) at Adelaide airport in 2011 and 2012. Exercise 2.4

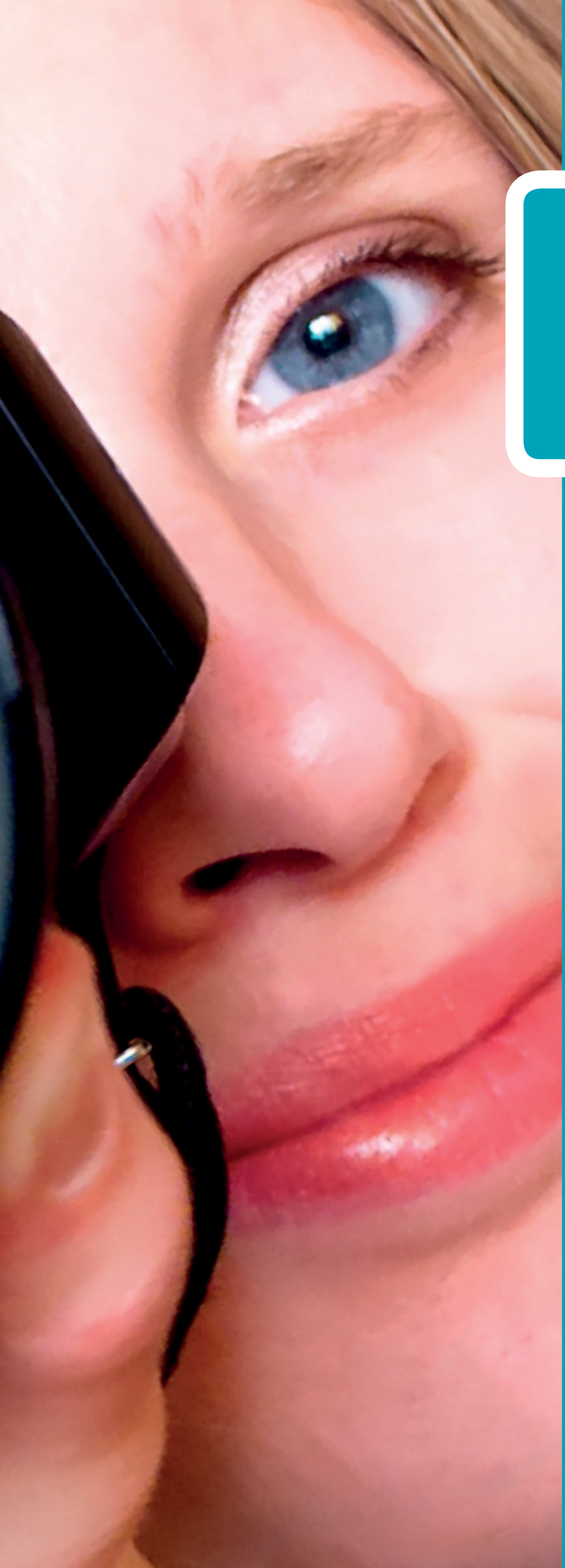
	Jan	Feb	Mar	Apr	May	Jun	July	Aug	Sep	Oct	Nov	Dec
2011	29.1	27.6	24.3	22.3	17.9	16.5	15.5	17.3	20.5	21.3	25.5	26.9
2012	30.0	26.7	25.3	23.6	18.0	15.3	15.1	15.5	19.1	21.9	26.6	27.0

(a) Calculate the monthly seasonal indices for the data, to 2 decimal places.

(b) Determine the values of the deseasonalised data. Give your answers to 1 decimal place.

(c) Plot the deseasonalised data, together with a line of best fit, by eye. Determine the least-squares equation for the line.

(d) Use your analysis to forecast the mean maximum temperatures for the following 3 months.



3

Growth and decay in sequences

3

Recall

Continue a sequence of terms using addition and subtraction

1 Determine the next two terms in each number pattern.

(a) 33, 41, 49, 57, ... (b) 75, 68, 61, 54, ...

Continue a sequence of terms using multiplication and division

2 Determine the next two terms in each number pattern.

(a) 14, 42, 126, 378, ... (b) 243, 81, 27, 9, ...

Substitute values into an expression and simplify

3 Determine the value of each expression after substituting the given values.

(a) $b + (h - 1)e$, given $b = 25$, $h = 21$ and $e = 3$

(b) ar^{n-1}, given $a = 4$, $r = 6$ and $n = 3$

(c) $\frac{u(100 - s^3)}{10 - s}$, given $u = 8$ and $s = 4$

Increase or decrease an amount by a given percentage

4 Increase or decrease each amount by the given percentage using multiplication by a decimal value.

(a) increase 650 by 9% (b) decrease 46 by 7%

(c) decrease 23 000 by 12.5%

Construct graphs by plotting coordinate pairs

5 Construct graphs of the following sets of data. Use the horizontal axis for the independent or explanatory variable (often the first value) in each pair. Use line segments to join successive points.

(a)

Test number	1	2	3	4	5	6
Mark (out of 20)	10	12	14	16	17	20

(b)

Day number	1	2	3	4	5	6	7
Temperature (°C)	16	19	21	18	23	25	22

(c)

Age of company (years)	0	2	4	6	8	10
Net worth ($m)	50	45	40	48	55	75

Solve simultaneous equations

6 Solve the following simultaneous equations.

(a) $y = 41 - 7x$ and $y = 21 - 3x$ (b) $x + 3y = 30$, $x + 13y = -60$

Arithmetic sequences

3.1

Recursion

A sequence is a string of numbers that follow a particular pattern. An arithmetic sequence is one where the difference between consecutive terms (or numbers in the sequence) is constant.

Consider the following sequence of numbers starting with 5.

5, 7, 9, 11, 13, 15, …

The expression t_n represents any term in the sequence, and the expression t_{n+1} represents the next term.

Additional information

Consecutive odd numbers

Watch the video to see an example of an arithmetic sequence.

For example, if t_n is 9, then t_{n+1} is 11. Or t_n could be 13, with t_{n+1} equal to 15.

t_n and t_{n+1} represent successive terms.

The above sequence is now defined using t_n and t_{n+1} as follows, starting with $n = 1$:

$t_{n+1} = t_n + 2$, with $t_1 = 5$, where t_1 represents the first term of the sequence and n represents the term number in the sequence.

When $n = 1$ is substituted into the equation, the result is $t_2 = t_1 + 2$, with $t_1 = 5$, giving $t_2 = 7$, so the first two terms of the sequence are identified.

$t_{n+1} = t_n + 2$ is read as: 'The next term in the sequence is equal to the current term plus two.'

This type of rule, where the next term is defined using the previous term, is called a *recurrence relation*. The process for determining successive terms is called *recursion*.

Using recursion to generate an arithmetic sequence

A sequence like 5, 7, 9, 11, 13, … is an example of an arithmetic sequence, in which there is a common difference between successive terms. In this example, the common difference is 2.

As well as using t_1, a can also be used to denote the first term, and d can be used for the common difference in arithmetic sequences. In the given sequence, $a = 5$ and $d = 2$.

The common difference may be calculated using: $d = t_{n+1} - t_n$

Using this formula with the sequence above:

$d = 7 - 5 = 2$ $\quad$ $d = 9 - 7 = 2$ $\quad$ $d = 11 - 9 = 2$ $\quad$ $d = 13 - 11 = 2$

The general rule for generating an arithmetic sequence using recursion is:

$t_{n+1} = t_n + d$, with $t_1 = a$

where a is the first term and d is the common difference.

1 Using recursion to generate an arithmetic sequence

Generate the first four terms of the following arithmetic sequences.

(a) $t_{n+1} = t_n + 10$, with $t_1 = 7$

THINKING	WORKING
1 Identify the first term $t_1 = a$ and the common difference d.	$t_{n+1} = t_n + 10$, with $t_1 = 7$: $a = 7, d = 10$
2 Generate each subsequent term by substituting $n = 1, n = 2, \ldots$ into the formula until the required number of terms are found.	Second term: $t_2 = t_1 + 10 = 7 + 10 = 17$ Third term: $t_3 = t_2 + 10 = 17 + 10 = 27$ Fourth term: $t_4 = t_3 + 10 = 27 + 10 = 37$
3 Interpret the result.	The first four terms of the sequence are 7, 17, 27, 37.

(b) $t_{n+1} = t_n - 6$, with $t_1 = 100$

THINKING	WORKING
1 Identify the first term $t_1 = a$ and the common difference d.	$t_{n+1} = t_n - 6$, with $t_1 = 100$: $a = 100, d = -6$
2 Generate each subsequent term by substituting $n = 1, n = 2, \ldots$ into the formula until the required number of terms are found.	Second term: $t_2 = t_1 - 6 = 100 - 6 = 94$ Third term: $t_3 = t_2 - 6 = 94 - 6 = 88$ Fourth term: $t_4 = t_3 - 6 = 88 - 6 = 82$
3 Interpret the result.	The first four terms of the sequence are 100, 94, 88, 82.

The general term

The sequence of odd numbers 1, 3, 5, 7, 9, ... is an arithmetic sequence in which each successive term is generated by adding 2 to the previous term.

Steps to generate the terms for this sequence:

STEP 1: a is 1 and the common difference d is 2.

STEP 2: Add one common difference to the first term.

STEP 3: Add two common differences to the first term.

STEP 4: Add three common differences to the first term.

STEP n: Add $(n - 1)$ common differences to the first term.

First term	Second term	Third term	Fourth term	Fifth term	Sixth term	Seventh term	nth term
t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_n
a	$a + 1d$	$a + 2d$	$a + 3d$	$a + 4d$	$a + 5d$	$a + 6d$	$a + (n - 1)d$

So the general term of an arithmetic sequence is: $t_n = a + (n - 1)d$

The nth term in an arithmetic sequence is given by $t_n = t_1 + (n-1)d$ or $t_n = a + (n-1)d$ where a is the first term t_1 and d is the common difference $t_{n+1} - t_n$.

Additional information

Determine the nth term of a sequence

Watch the video to revise determining the nth term in a sequence.

Checking whether a sequence is arithmetic

To check whether a sequence is indeed arithmetic, verify that the difference between successive terms does not vary (i.e. is constant). If given four terms, check that:

$t_4 - t_3 = t_3 - t_2 = t_2 - t_1 = d.$

2 Verifying that a sequence is arithmetic, to determine the value of a term

Determine whether the sequence is arithmetic. If it is arithmetic, calculate the value of the requested term.

(a) 5, 9, 13, 17, … Determine the value of t_{31}.

THINKING	WORKING
1 Determine the difference between successive terms.	$t_4 - t_3 = 17 - 13$ $= 4$ $t_3 - t_2 = 13 - 9$ $= 4$ $t_2 - t_1 = 9 - 5$ $= 4$
2 State whether the sequence is arithmetic. If so, note the first term and common difference.	The difference between terms is constant, therefore the sequence is arithmetic. $t_1 = 5, d = 4$
3 Use the rule for an arithmetic sequence to determine the value of the required term.	$t_n = t_1 + (n-1)d$ $t_1 = 5, d = 4$ and $n = 31.$ $t_{31} = 5 + (31-1) \times 4$ $= 5 + (30) \times 4$ $= 5 + 120$ $= 125$
4 Interpret the answer.	The 31st term in the arithmetic sequence where the first term is 5 and each term increases by 4 is 125.

(b) 6, 13, 27, 55, ... Determine the value of t_{50}.

1 Determine the difference between successive terms.	$t_4 - t_3 = 55 - 27$ $= 28$ $t_3 - t_2 = 27 - 13$ $= 14$
2 State whether the sequence is arithmetic.	The difference between successive terms is not constant, so the sequence is not arithmetic. The requested term need not be found.

(c) 20, 15, 10, 5, ... Determine the value of t_{100}.

1 Determine the difference between successive terms.	$t_4 - t_3 = 5 - 10$ $= -5$ $t_3 - t_2 = 10 - 15$ $= -5$ $t_2 - t_1 = 15 - 20$ $= -5$
2 State whether the sequence is arithmetic. If so, note the first term and common difference.	The difference is constant, therefore this is an arithmetic sequence. $a = 20$, $d = -5$
3 Use the rule for an arithmetic sequence to determine the value of the required term.	$t_n = a + (n-1)d$ $a = 20$, $d = -5$ and $n = 100$. $t_{100} = 20 + (100 - 1) \times (-5)$ $= -475$
4 Interpret the answer.	The 100th term in the arithmetic sequence where the first term is 20 and each term decreases by 5 is -475.

3 Determining a rule for the *n*th term of an arithmetic sequence

For the arithmetic sequence 17, 30, 43, 56, ...

(a) Determine a rule for the *n*th term.

THINKING	WORKING
1 Determine the values of *a* and *d*.	$t_1 = 17$ $d = 30 - 17$ $= 13$ Verify the value of *d*. $d = 43 - 30$ $= 13$

2 Recall the general equation for the nth term of an arithmetic sequence.	$t_n = t_1 + (n-1)d$
3 Substitute values for t_1 and d. Expand and simplify.	$t_n = 17 + (n-1) \times 13$ $= 17 + 13n - 13$ $= 13n + 4$
4 Interpret the result.	The value of the nth term in the arithmetic sequence where the starting term is 17 and subsequent terms increase in value by 13 can be found using the rule $t_n = 13n + 4$.

(b) Use the rule to calculate the value of the 50th term.

1 Substitute for n to find the requested term.	$t_{50} = 13 \times 50 + 4$ $= 654$
2 Interpret the answer.	The 50th term in the arithmetic sequence where the first term is 17 and each term increases in value by 13 is 654.

Using the formula when a term value is known

There are four variables in the formula for the general term: a or t_1, d, n and t_n.

If any three of the variables are known, you can substitute their values into $t_n = a + (n-1)d$ to solve for the unknown.

4 Identifying the position of a term in a sequence

For the arithmetic sequence 20, 27, 34, …

Determine the position in the sequence of the term 300.

THINKING	WORKING
1 Identify the known values.	$t_1 = 20$ $d = 27 - 20$ $= 7$ $t_n = 300$
2 Recall the rule for the nth term.	$t_n = a + (n-1)d$
3 Substitute the known values.	$300 = 20 + (n-1) \times (7)$

4 Solve for n.	$300 = 20 + 7n - 7$ $280 = 7n - 7$ $287 = 7n$ $n = \frac{287}{7}$ $n = 41$
5 Interpret the answer.	300 is the 41st term in the arithmetic sequence where the first term is 20 and each term increases in value by 7.

If two terms of a sequence are known, you can use simultaneous equations to solve for a and d, and hence generate the entire arithmetic sequence.

5 Generating an arithmetic sequence given two terms

Generate the first four terms of the arithmetic sequence whose seventh term is 101 and twentieth term is 270.

THINKING	WORKING
1 Identify the given information.	$t_7 = 101, t_{20} = 270$
2 Create two equations using the general formula.	$t_n = a + (n-1)d$ $a + 6d = 101 \quad [1]$ $a + 19d = 270 \quad [2]$
3 Make a the subject in the first equation. Substitute for a in the second equation and solve for d. *Note*: The elimination method involves subtracting [1] from [2]. Here, this gives $13d = 169$ directly.	From [1]: $a = 101 - 6d$ From [2]: $(101 - 6d) + 19d = 270$ $13d = 270 - 101$ $13d = 169$ $d = \frac{169}{13}$ $d = 13$ The difference between successive terms in the sequence is 13.
4 Substitute for d into one equation to determine the value of a, the first term. Check the result with the other equation.	Substitute $d = 13$ into [1]: $a + 6 \times 13 = 101$ $a + 78 = 101$ $a = 101 - 78$ $a = 23$ Check in [2]: $23 + 19 \times 13 = 270$ The first term in the sequence is 23.

5 Generate the terms by starting with the first term a and adding the difference d to determine the value of each subsequent term.	$t_1 = 23$ $t_2 = 23 + 13$ $= 36$ $t_3 = 36 + 13$ $= 49$ $t_4 = 49 + 13$ $= 62$
6 Interpret the answer.	The first four terms in the sequence are 23, 36, 49, 62.

Arithmetic mean

For the arithmetic sequence: $\ldots 5, 11, 17, \ldots$

11 is the mean of 5, 11 and 17.

$$\frac{5+11+17}{3} = \frac{33}{3} = 11$$

Also, 11 is the mean of 5 and 17.

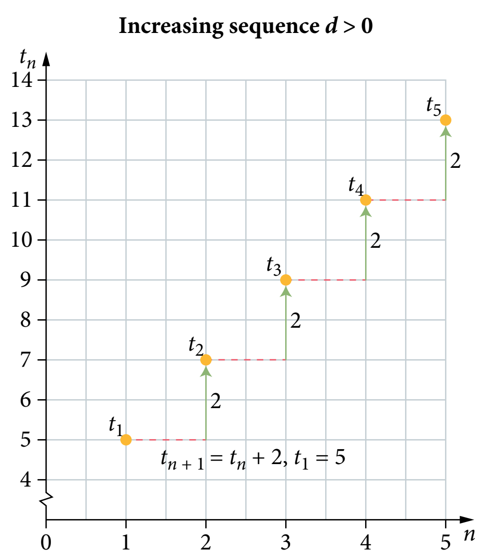
$$\frac{5+17}{2} = \frac{22}{2} = 11$$

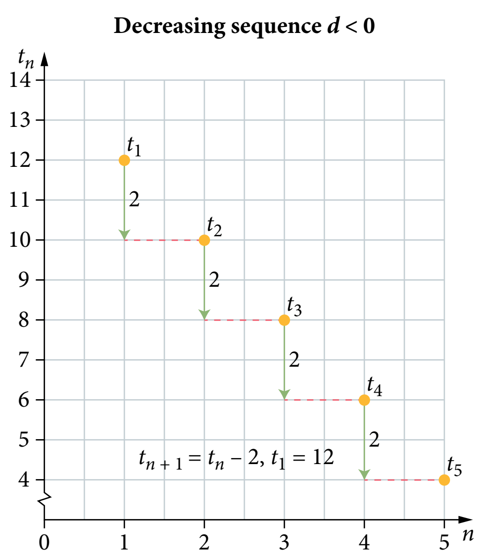
The arithmetic mean of three terms in an arithmetic sequence is the middle term.

The middle term is also the mean of the terms on either side.

Graphs of arithmetic sequences

Graphs of arithmetic sequences are linear—they rise (or fall) at a steady rate, following a single straight line.





Explore further

Arithmetic sequences and their graphs

Use a spreadsheet to generate and graph arithmetic sequences.

EXERCISE 3.1 Arithmetic sequences

Worked Example

1

1 Generate the first four terms of the following arithmetic sequences.

(a) $t_{n+1} = t_n + 2$, with $t_1 = 3$

(b) $t_{n+1} = t_n - 5$, with $t_1 = 45$

(c) $t_{n+1} = t_n + 8$, with $t_1 = 2$

(d) $t_{n+1} = t_n + 11$, with $t_1 = 1$

(e) $t_{n+1} = t_n - 6$, with $t_1 = 4$

(f) $t_{n+1} = t_n - 2$, with $t_1 = 3$

(g) $t_n = t_{n-1} + 15$, with $t_1 = 5$

(h) $t_n = t_{n-1} + 31$, with $t_1 = -10$

(i) $t_n = t_{n-1} - 13$, with $t_1 = 6$

(j) $t_n = t_{n-1} + 0.25$, with $t_1 = 3.5$

2 What is the common difference for the sequence $t_{n+1} = t_n + 36$, with $t_1 = 4$?

3 Generate the first four terms of each arithmetic sequence.

(a) $a = 6, d = 2$

(b) $a = 8, d = 3$

(c) $a = 12, d = -2$

(d) $a = -9, d = 4$

(e) $a = 21, d = -3$

(f) $a = 64, d = 6$

(g) $a = 33, d = -5$

(h) $a = -52, d = 20$

(i) $a = 5000, d = 350$

(j) $a = 1, d = -0.333$

2

4 For each of the following sequences, determine whether the sequence is arithmetic. If it is, calculate the value of the requested term.

(a) 2, 7, 12, 17,… Determine t_{100}.

(b) 6, 8, 10, 12, 14,… Determine t_{30}.

(c) 3, 7, 12, 18,… Determine t_{21}.

(d) 1, 8, 15, 22,… Determine t_{16}.

(e) 9, 10, 11, 12, 13… Determine t_{51}.

(f) 55, 44, 33, 22,… Determine t_{18}.

(g) 12, 9, 6, 3,… Determine t_{26}.

(h) 45, 40, 30, 15,… Determine t_{45}.

(i) 0, 2, 4, 6, 8, 10,… Determine t_{200}.

(j) 72, 64, 56, 48,… Determine t_{33}.

(k) 90, 87.5, 85, 82.5,… Determine t_{73}.

(l) -30, -33, -36, -39,… Determine t_{15}.

(m) 60, 75, 90, 105,… Determine t_{150}.

(n) 4.6, 5, 5.4, 5.8,… Determine t_{66}.

(o) 16.25, 22.5, 28.75, 35,… Determine t_9.

5 For the sequence $t_{n+1} = t_n + 36$, with $t_1 = 4$:

(a) What is the value of the 6th term (t_6)?

A 112 B 184 C 220 D 720

(b) Explain the common error made by a student who chose the second incorrect option.

6 Determine the value of the 20th term of each arithmetic sequence.

(a) $a = -200$ and $d = 26$

(b) $a = 60$ and $d = 7$

(c) $a = -2$ and $d = 6.5$

(d) $a = 11.2$ and $d = -0.6$

7 Determine the first term of the arithmetic sequence with the given conditions.

(a) The common difference is 35 and the tenth term is 814.

(b) The common difference is -15 and the sixth term is -55.

8 Determine the common difference for the arithmetic sequence with the given conditions.

(a) The first term is 45 and the tenth term is 702.

(b) The first term is -51 and the twenty-first term is 229.

(c) The first term is -16 and the fortieth term is -1342.

9 Determine the arithmetic mean of the given terms in each case. Use it to write an arithmetic sequence of three terms with the given values as the first and last terms, respectively.

(a) 8, 20 (b) 4, 44 (c) 19, 37

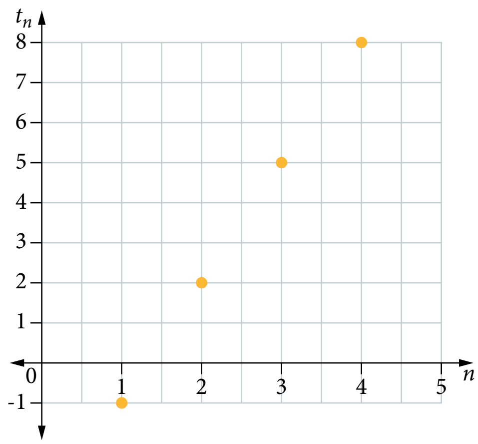
(d) -11, 33 (e) 5, 24 (f) -60, -39

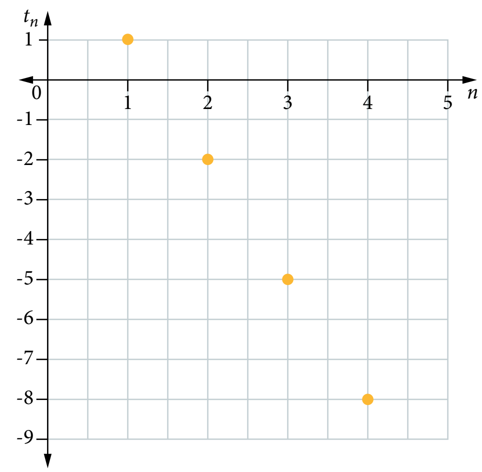
10 Determine the value of the missing term, given that the terms form an arithmetic sequence.

(a) 101, ____, 159 (b) -27, ____, 200

(c) -3000, ____, -560 (d) 52, ____, -58

11 Which of the graphs shows an arithmetic sequence with a first term of -1 and a common difference of -3?

A


B


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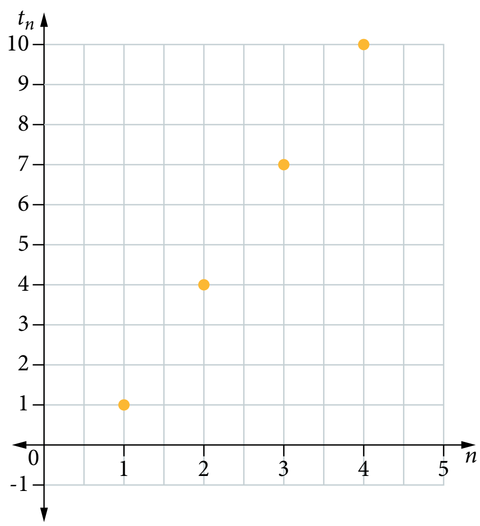
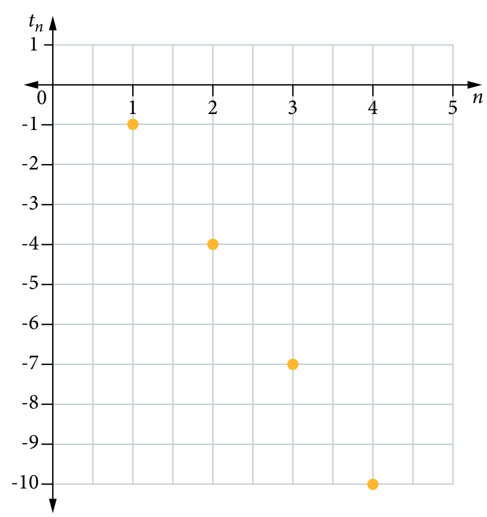


D


Worked Example

3 **12** Determine a rule for the nth term, in terms of n, for each of the following arithmetic sequences. Use your expression to calculate the value of the term required.

(a) 44, 38, 32, 26, ... Determine t_{30}.
(b) -15, 23, 61, 99, ... Determine t_{17}.
(c) 2, -9, -20, -31, ... Determine t_{24}.
(d) 400, 520, 640, 760, ... Determine t_{50}.

13 Determine the position in the arithmetic sequence held by the given term. Write your answer in the form t_n.

(a) $a = 56$ and $d = 2$; $t_n = 88$
(b) $a = -24$ and $d = 6$; $t_n = 288$
(c) $a = 12\,000$ and $d = -440$; $t_n = 4080$
(d) $a = -7.2$ and $d = -0.4$; $t_n = -17.2$

4 **14** Each sequence below is arithmetic. In which position is the final term? Write your answer in the form t_n.

(a) 4, 14, 24, 34, ... , 104
(b) 15, 36, 57, 78, ... , 456
(c) 1000, 994, 988, 982, ... , 100
(d) 67, 113, 159, 205, ... , 10 141

5 **15** List the first four terms of the arithmetic sequence with the given conditions.

(a) The first term is -52 and the 39th term is 442.
(b) The 20th term is 141 and the 30th term is 211.
(c) The 19th term is 58 and the 57th term is 248.
(d) The 4th term is 28 and the 12th term is -28.

16 Graph the given terms of the following sequences against their term number, n. Determine whether the sequence is arithmetic.

(a) 3, 5, 9, 18, ...
(b) 12, 17, 22, 27, ...
(c) -9, 1, 11, 21, ...
(d) 2, 4, 8, 16, ...
(e) -8, -11, -14, -17, ...
(f) 27, 9, 3, 1, ...

17 For the sequence a, $a + 7$, $a + 14$, ..., determine the value of t_{12} if $t_{305} = 2018$.

18 Which term t_n of the arithmetic sequence 15, 45, 75, 105, ... is the first term (i.e. the lowest-numbered term) to exceed the following numbers? Give both the term number and the term value in your answer.

(a) 1000
(b) 5000

19 The following information refers to an arithmetic sequence.

The sum of the first two terms is 47. The sum of the last two terms is -49. $t_8 + t_9 + t_{11} = 0$.

Determine the values of the seven terms referred to, and say how many terms are in the sequence.

20 For the arithmetic sequence where $5t_7 = -4t_{10}$ and $t_8 + t_9 = 3$, determine the rule for the nth term and evaluate the reasonableness of your rule.

Applications of arithmetic sequences 3.2

Problem solving

In situations where a constant increase or decrease is applied, arithmetic sequences can be used to solve problems.

6 Determining the value of the *n*th term

The number of visitors to a park during a fortnight of school holidays begins with 2000 on Monday and increases by 250 each day.

How many people visit the park on the 8th day (the second Monday) of the school holidays?

THINKING	WORKING
1 Recognise the arithmetic sequence and identify the first term a, common difference d and number of terms n.	Arithmetic sequence: $t_1 = 2000,\ d = 250,\ n = 8$
2 Recall the arithmetic term formula. Substitute values for a, d and n to determine the value of the 8th term.	$t_n = t_1 + (n-1)d$ $t_8 = 2000 + (8-1) \times 250$ $= 3750$
3 Interpret the answer.	On the 8th day, 3750 people visit the park.

Starting an arithmetic sequence with $n = 0$

In many situations, instead of beginning with $n = 1$ and 'term 1', t_1, it is appropriate to begin with $n = 0$, and designate an initial 'term 0', as this makes it easier to match term number to time elapsed after the initial term. For example, t_n could be the value after n weeks. In this case, the general term of the arithmetic sequence is given by $t_n = a + nd$.

If the first term of an arithmetic sequence is $t_0 = a$:

$t_n = a + nd$, and there are $n + 1$ terms in the sequence.

7 Using n as time elapsed

Dr Filibuster is a university lecturer who runs an information session. The session begins with a short introduction at noon and then students have the opportunity to speak with professors and collect resources. At noon, 300 students arrive at the lecture theatre. The students attending the session collect the handouts and leave at the rate of four students every minute.

(a) How many students will remain after 30 minutes?

THINKING	WORKING
1 The problem asks for a value after a certain time has elapsed. Recall the version of the arithmetic sequence that starts with $n = 0$.	$t_n = a + nd$, where $a = t_0$
2 Identify the initial population a, common difference d and number of terms n.	$d < 0$ because the student population is decreasing. $a = 300$, $n = 30$ and $d = -4$
3 Substitute the given values and calculate the value of the 30th term.	$t_{30} = 300 + 30(-4)$ $= 300 - 120$ $= 180$
4 Interpret the answer.	There will be 180 students present after 30 minutes.

(b) Assuming the session runs long enough, how long would it take before there were no students left?

1 Use the equation from the first step in part (a).	$t_n = 300 + n \times (-4)$
2 Solve the equation for $t_n = 0$.	$0 = 300 - 4n$ $4n = 300$ $n = 75$
3 Interpret the answer.	There would be no students left after 75 minutes.

Straight line (unit cost) depreciation

Individuals and businesses can claim a tax deduction for the decline in value of equipment purchased for use in earning an income. For example, a builder may purchase a saw bench for $5000, but after a year it may be valued at only $4000. The builder can then claim a tax deduction of the $1000 decline in value. This decline in value is called *depreciation*.

Depreciation can be calculated as a percentage *of the original value*. The reduced value of an item can then be calculated by subtracting this (fixed) amount from the previous year's value. Depreciation calculated this way is called 'straight line' or 'unit cost' or 'flat rate' depreciation.

For example, consider a computer bought for $1000 and depreciated from the original cost at a flat rate of 10%.

The depreciation in the first year is 10% of $1000, $0.10 \times 1000 = \$100$, resulting in a reduced value of $\$1000 - \$100 = \$900$ after one year.

The depreciation in the second year is 10% of $1000, $100, resulting in a reduced value of $\$900 - \$100 = \$800$ after two years.

In this case, the 'unit cost' is $100.

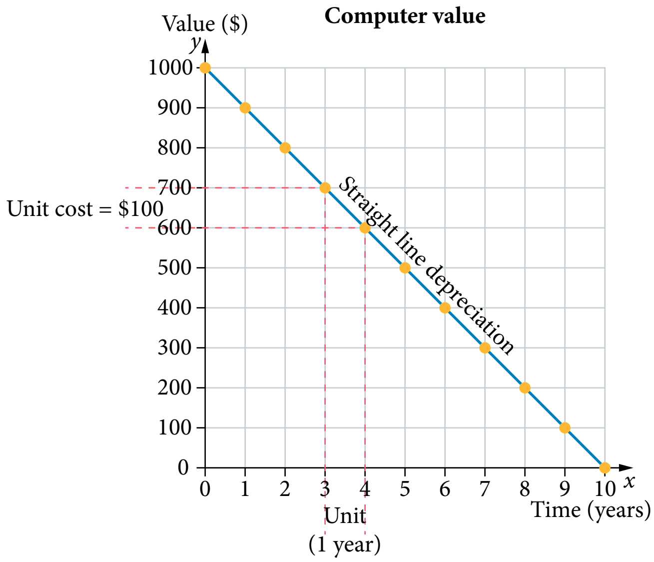
The pattern continues until the 'on paper' value is $0.

When using flat rate depreciation, the value of the item decreases by the same amount each year. A graph of value versus unit produces a straight line, characteristic of an arithmetic sequence.

In general, if the original value of an item is V_0 and the flat rate of depreciation is i%, then the yearly depreciation d is given by $d = i \times V_0$, where i is the decimal equivalent.

Hence the value V_n after n years (or periods) is:

$V_n = V_0 - nd$, where d is the common amount being subtracted. Also, $n = 0, 1, 2, \ldots$



WARNING

In the formula $V_n = V_0 - nd$, d is a positive amount, whereas for the formula $t_n = t_1 + (n-1)d$, d would be negative for decreasing values.

8 Flat rate depreciation using a rule

Donato buys a new cash register for \$2400 and wishes to depreciate it using a flat rate of 9.5%.

(a) By how much does the value of the cash register decrease each year?

THINKING	WORKING
1 Determine the flat annual depreciation amount.	9.5% of \$2400 $= 0.095 \times \$2400$ $= \$228$
2 Interpret the answer.	The cash register will decrease in value by \$228 each year after purchase.

(b) What will be the value of the cash register after 7 years?

1 Identify the initial value V_0 and the number of periods of depreciation n.	The purchase price of the cash register is \$2400, therefore $V_0 = 2400$ and $n = 7$.
2 Use the flat rate depreciation formula. Substitute known values and calculate the value of the 7th term.	$V_n = V_0 - nd$ $V_7 = 2400 - 7 \times 228$ $= \$804$
3 Interpret the answer.	After 7 years, the cash register is valued at \$804.

(c) How long will it take before the value of the cash register is less than \$1000?

1 Use the formula $V_n = V_0 - nd$ and solve $V_n < 1000$ for n.	$V_n = V_0 - nd$ $2400 - n \times 228 < 1000$ $-228n < -1400$ $n > \frac{1400}{228}$ $n > 6.14\ldots$
2 Interpret the answer.	It will take 7 years for the value to be less than \$1000.
3 Check the reasonableness of the solution by substituting the value of n into the equation for V_n.	$V_n = V_0 - nd$ $V_7 = 2400 - 7 \times 228$ $V_7 = 804$ After 7 years the value of the cash register is indeed less than \$1000.

A recurrence relation may also be used to describe the value V_n after n years (or periods):

$V_{n+1} = V_n - d$, with $n = 0, 1, 2, 3, 4, \ldots$, where V_0 indicates the initial value, the value at the *start* of the first year—that is, the value after 0 years.

9 Tabulated flat rate depreciation

A school purchases a new photocopier for \$27 000. The value of the photocopier reduces by a flat rate of 18% of the original cost each year.

Tabulate the value of the photocopier against time, until it is 'written off'—that is, it has no value on paper any more.

THINKING | WORKING

1 Calculate the flat amount reduction in value each year.

$$18\% \text{ of } \$27\,000 = 0.18 \times \$27\,000 = \$4860$$

2 Interpret the result.

The photocopier will decrease in value at a rate of \$4860 per year.

3 Create a table with the initial value next to 0 years.

For each subsequent year, subtract the flat amount from the previous year's value until the value becomes zero or negative.

If using a spreadsheet, enter a subtraction formula for the second value (in cell B3, next to $t = 1$). Here this could be: $= \text{B2} - 4860$. Click and drag down until the value becomes zero or negative.

	A	B
1	Time (years)	Value (\$)
2	0	27000
3	1	22140
4	2	17280
5	3	12420
6	4	7560
7	5	2700
8	6	–2160

4 Adjust the final value to zero, if necessary.

	A	B
1	Time (years)	Value (\$)
2	0	27000
3	1	22140
4	2	17280
5	3	12420
6	4	7560
7	5	2700
8	6	0

5 Interpret the answer.

The photocopier is considered written-off or to have no worth on paper after 6 years.

Simple interest

Simple or 'flat rate' interest is calculated as a percentage of the *original amount* deposited with a bank or other financial institution. The customer receives the same amount of interest at the end of each year (or other time interval).

For example, if a bank offers a 5% simple interest rate, a \$1000 deposit would grow by 5% of \$1000, $0.05 \times 1000 = \$50$ each year if interest is paid annually.

If A_0 is the original amount deposited, and $i\%$ is the flat interest rate, then the interest deposited each year d is:

$$d = \frac{i}{100} \times A_0$$

The total amount A_n accumulated after n years is therefore given by:

$$A_n = A_0 + nd$$

where n is the number of years that have elapsed and d is the common difference.

As with depreciation, $n = 0, 1, 2, \ldots$

Years funds have been in the bank	Interest earned ($)	Calculation of total funds ($)	Total funds in account ($)
0	–	–	1000
1	50	1000 + 50	1050
2	50	1050 + 50	1100
3	50	1100 + 50	1150
4	50	1150 + 50	1200
5	50	1200 + 50	1250

10 Determining the account balance with simple interest

Determine the account balance at the end of the term for the following investment earning simple interest.

Deposit \$9000, interest rate 11.6%, term 5 years

THINKING	WORKING
1 Calculate the interest amount for one year. This is the common difference, d.	$d = 11.6\%$ of \$9000 $= 0.116 \times 9000$ $= \$1044$
2 Interpret the value.	The account increases in value by \$1044 per year in the form of interest earned.
3 Use $A_n = A_0 + nd$ to determine the amount in the account after the given time, n years.	$A_n = A_0 + nd$ $A_5 = 9000 + 5 \times 1044$ $= \$14\,220$
4 Interpret the answer.	The account balance at the end of the 5 year term is \$14 220.

You can calculate the time needed for an account balance to reach a certain amount using the same formula.

11 Calculating time elapsed with simple interest

Mr Bunk deposits \$15 000 with Flat Bank, which pays him a flat (simple) rate of interest of 7.2% of his original deposit per year. Mr Bunk adds no further deposits to this account himself.

How long will it take before Mr Bunk's funds grow to \$35 000?

THINKING	WORKING
1 Calculate the first year's interest. This is the common difference d in the arithmetic sequence of account totals.	$d = 7.2\%$ of \$15 000 $= 0.072 \times 15\,000$ $= \$1080$

2 Interpret the result.	Mr Bunk's account will increase in value by \$1080 per year in the form of interest earned.
3 Use $A_n = A_0 + nd$. Substitute given values and solve for n.	$A_n = A_0 + nd$ $15\,000 + n \times 1080 = 35\,000$ $n \times 1080 = 20\,000$ $n = \frac{20\,000}{1080}$ $= 18.51\ldots$
4 Interpret the answer.	It will take 19 years before the funds grow to \$35 000.
5 Evaluate the reasonableness of the result.	For $n = 18$, $A_{18} = A_0 + nd$ $= 15\,000 + 18 \times 1080$ $= \$34\,440$ For $n = 19$, $A_{19} = A_0 + nd$ $= 15\,000 + 19 \times 1080$ $= \$35\,520$

A recurrence relation may also be used to describe the amount A_n after n years (or periods).
$A_{n+1} = A_n + d$ with $n = 0, 1, 2, 3, 4, \ldots$

In Worked example 11, the question could have been answered by tabulating the values against time. A spreadsheet is particularly useful for the repeated calculations.

EXERCISE 3.2 Applications of arithmetic sequences

Worked Example

1 Lucinda pays her brother 50 cents to clean her room on day 1 of the school holidays. She increases the payment by 20 cents each day over the holiday fortnight. How much does Lucinda pay her brother on day 8 of the holidays? Give your answer in dollars. 6

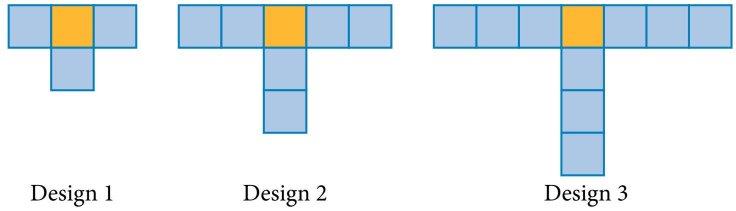
2 As part of a fitness regimen, Michelle runs 2 km per day during the first week, 3 km per day during the second week, 4 km per day during the third week and so on over a 10 week period. Determine the total distance that Michelle runs during week 5.

3 Con decides to do sit-ups each morning before work. He does 40 sit-ups on Monday (day 1), and each day increases the number of sit-ups by 5.

(a) How many sit-ups does Con do on day 30?

(b) On which day does he do 100 sit-ups? Give the name and number of the day.

4 Tia owns her own jewellery company and produces the ‘T’ shapes for a product line of necklaces. Each ‘T’ has a single gold square at its centre, with various numbers of silver squares attached to three sides of the gold square. The first three items in this product line are shown.



(a) How many squares altogether will there be in Design 9?

(b) Which number design contains 49 squares?

5 A stone wall construction starts at 1.12 m and gains 25 cm in height each day.

(a) Determine its height after 6 days.

A 2.37 cm B 151.12 cm C 237 cm D 262 cm

(b) Explain the common error made by a student who used the calculation 5×25 in part (a).

Worked Example 7

6 Mr Di Nero drives a luxury-hire limousine and charges customers $90 flag fall and $4.50 per kilometre travelled.

(a) How much would Mr Di Nero charge for a 20 km trip?

(b) One group wants to drive around, exploring the city for as long as their $500 travel budget will allow. How many kilometres can they travel in Mr Di Nero’s limousine? Give your answer as a whole number.

7 Australia Car Rentals is about to begin trading and is yet to set the flag fall and kilometre charges. The company’s research shows that the average trip will be 10 km, and for this customers will be charged $35.

(a) If the flag fall is $3.50, how much should the company charge per kilometre?

(b) If the company wishes to charge $3 per kilometre, what should be the value of the flag fall?

10

8 Determine the account balance at the end of the term for each of the following, assuming a flat rate of interest.

(a) deposit $93 000, interest rate 5%, term 4 years

(b) deposit $2750, interest rate 9.15%, term 10 years

(c) deposit $12 000 000, interest rate 12.2%, term 3 years

(d) deposit $140 000, interest rate 4.25%, term 20 years

9 Ms Graille is appointed to run a bank and is paid $6.5 million in her first year at the helm. She receives annual increases of $1.5 million. When, after 6 years, the board of directors realise that she has not been performing well at her job, she is released from the bank.

(a) How much was Ms Graille paid in her final year?

(b) What total amount was she paid over the 6 years, including bonus shares worth $10.2 million?

8

10 Zhadan buys a new machine for $29 500 for his business and wishes to depreciate it from the original value using a flat rate of 14%.

(a) By how much does the value of the machine decrease each year?

(b) Determine the value of the machine after 6 years.

(c) How long will it take before the value of the machine is less than $1000?

11 Yarrani purchases a weights bench for $1200 for his fitness centre and depreciates it at 8% per year using the straight line (flat rate) depreciation method.

(a) What is the value of the bench after 5 years?

(b) When does the bench become worthless (on paper)?

Worked Example 9

12 Yuganza buys a new car costing $42 000 for her business, and depreciates it at a flat rate of 15% per year. Construct a table showing the value of the car for the next 5 years.

13 Mira is paying off a $15 000 loan (including interest and other charges) with instalments equal to 18.4% of the initial amount, paid at the end of each year. She uses a spreadsheet to calculate how much she owes over time. She uses a formula with simple subtraction, dragging the formula down the column until the amount owed becomes zero or negative.

(a) What is this first zero or negative value?

A -$1560 B -$1200 C -$280 D $0

(b) How many payments will Mira actually make if the final payment is reduced to what is owed?

A 4 B 5 C 6 D 7

(c) Graph the amount Mira owes against time as a series of dots, starting at $t = 0$.

14 Maurice owns a collection of classic books valued at $3000 at the beginning of 2015. The books are read by his grandchildren and kept on a shelf. He estimates that, in used condition, the collection is depreciating at 8% per year.

(a) Determine the value of Maurice's collection at the beginning of 2020.

(b) What will the value of Maurice's collection be at the end of 2020?

(c) At the beginning of which year will the value of the collection be less than $100 for the first time? Explain any assumptions you need to make.

15 A doughnut machine is valued at $60 000 when new, and depreciates by $0.50 for every batch it produces.

(a) Estimate the value of the machine after it has produced 20 000 batches.

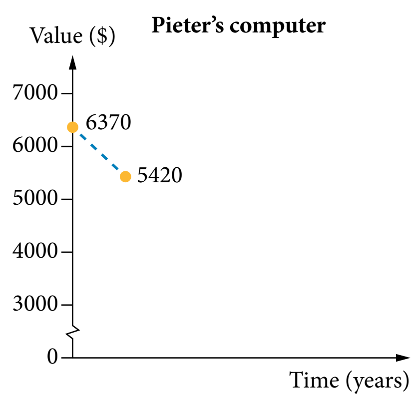
(b) How many batches is the machine expected to produce before being valued at $5000?

16 Cedric deposits $20 000 into a term deposit that earns simple interest of 8% per year. 11

How long will it take before the account grows to at least $32 000?

17 Raul invests $52 000 in a business. He receives a flat rate of interest on his investment. What flat interest rate must he receive if the investment is to grow to $100 000 in 5 years? Answer to 1 decimal place.

18 Pieter uses a spreadsheet and a graph to work out how to depreciate the value of his home computer for tax purposes. His initial attempt is shown.



Pieter decides that the time period for writing off the computer is good, but he doesn't want the deduction in the final year to be different than the rest of the payments.

(a) What flat rate percentage depreciation should Pieter use to make equal depreciation amounts? Give your answer to 2 decimal places.

(b) Graph the paper value of Pieter's computer over time, joining the points with line segments.

3.3 Geometric sequences

Using recursion to generate a geometric sequence

Consider a sequence in which the next term is generated by multiplying the previous term by a constant multiplier. Such a sequence is called a *geometric* sequence.

For example: 2, 6, 18, 54, 162,...

Use t_1 or a to denote the *first term* in a sequence and r for the *common ratio*, which is the number you multiply by to get the next term. So in the sequence above, $a = 2$ and $r = 3$.

The common ratio is defined as: $r = \frac{t_{n+1}}{t_n}$

Using this formula with the sequence above:

$$\begin{aligned} r &= \frac{t_5}{t_4} = \frac{t_4}{t_3} = \frac{t_3}{t_2} = \frac{t_2}{t_1} \\ &= \frac{162}{54} = \frac{54}{18} = \frac{18}{6} = \frac{6}{2} \\ &= 3 \end{aligned}$$

Additional information

A decreasing geometric sequence

View the video for an example of a geometric sequence.

The general rule for generating a geometric sequence using recursion is $t_{n+1} = rt_n$, where $t_1 = a$ and r is the common ratio.

This sequence could be described using recursion as: $t_{n+1} = 3t_n$, with $t_1 = 2$

12 Using recursion to generate a geometric sequence

Generate the first four terms of the following sequences.

(a) $t_{n+1} = 2t_n$, with $t_1 = 5$

THINKING	WORKING
1 Identify the first term and the common ratio.	$t_{n+1} = 2t_n$, with $t_1 = 5$: $a = 5, r = 2$
2 Interpret the result.	The sequence starts with the number 5 and each term in the sequence is generated by multiplying the previous term by 2.
3 Generate each subsequent term by substituting $n = 1, n = 2,\ldots$ into the formula until each of the required terms is found.	First term: $t_1 = 5$ Second term: $t_2 = 2t_1 = 2 \times 5 = 10$ Third term: $t_3 = 2t_2 = 2 \times 10 = 20$ Fourth term: $t_4 = 2t_3 = 2 \times 20 = 40$
4 Interpret the answer.	The first four terms are: 5, 10, 20, 40

(b) $t_{n+1} = \frac{1}{2}t_n$, with $t_1 = 64$

1 Identify the first term and the common ratio.	$t_{n+1} = \frac{1}{2}t_n$, with $t_1 = 64$: $a = 64$, $r = \frac{1}{2}$
2 Interpret the result.	The sequence starts with the number 64 and each term in the sequence is generated by multiplying the previous term by $\frac{1}{2}$.
3 Generate each subsequent term by substituting $n = 1, n = 2, \ldots$ into the formula until each of the required terms is found. *Note*: Multiplying by $\frac{1}{2}$ is the same as dividing by 2.	First term: $t_1 = 64$ Second term: $t_2 = \frac{1}{2} \times t_1 = \frac{1}{2} \times 64 = 32$ Third term: $t_3 = \frac{1}{2} \times t_2 = \frac{1}{2} \times 32 = 16$ Fourth term: $t_4 = \frac{1}{2} \times t_3 = \frac{1}{2} \times 16 = 8$
4 Interpret the answer.	The first four terms are: 64, 32, 16, 8

(c) $t_{n+1} = -3t_n$, with $t_1 = -2$

1 Identify the first term and the common ratio.	$t_{n+1} = -3t_n$, with $t_1 = -2$: $a = -2$, $r = -3$
2 Interpret the result.	The sequence starts with the number -2 and each term in the sequence is generated by multiplying the previous term by -3.
3 Generate each subsequent term by substituting $n = 1, n = 2, \ldots$ into the formula until each of the required terms is found. Notice how a negative r value produces alternating positive and negative terms.	First term: $t_1 = -2$ Second term: $t_2 = -3t_1 = -3 \times (-2) = 6$ Third term: $t_3 = -3t_2 = -3 \times (6) = -18$ Fourth term: $t_4 = -3t_3 = -3 \times (-18) = 54$
4 Interpret the answer.	The first four terms are: -2, 6, -18, 54

The general term

A geometric sequence is generated by multiplying each term by a fixed amount, called the *common ratio*. For positive values of r, when the common ratio is greater than 1, the terms increase in size; when the common ratio is less than 1, the terms decrease in size.

Consider the sequence: 1, 2, 4, 8, 16, 32

This is a geometric sequence in which each successive term is generated by multiplying the previous term by 2.

For this sequence, the *first term* a is 1 and the *common ratio* r is 2.

To determine an equation describing the general term of the sequence:

- To calculate the value of the second term, multiply the first term by the common ratio once.
- To calculate the value of the third term, multiply the first term by the common ratio twice, i.e. the common ratio squared.
- To calculate the value of the fourth term, multiply the first term by the common ratio three times, i.e. the common ratio cubed.
- To calculate the value of the nth term, multiply the first term by the common ratio $n - 1$ times, i.e. multiply the first term by the common ratio to the power of $n - 1$.

First term	Second term	Third term	Fourth term	Fifth term	Sixth term	nth term
t_1	t_2	t_3	t_4	t_5	t_6	t_n
a	$t_1 \times r$ $= a \times r$ $= ar^1$	$t_2 \times r$ $= (ar) \times r$ $= ar^2$	$t_3 \times r$ $= \left(ar^2\right) \times r$ $= ar^3$	$t^4 \times r$ $= \left(ar^3\right) \times r$ $= ar^4$	$t_5 \times r$ $= \left(ar^4\right) \times r$ $= ar^5$	$t_{n-1} \times r$ $= \left(ar^{n-2}\right) \times r$ $= ar^{n-1}$

The general term of an arithmetic sequence is: $t_n = ar^{n-1}$

> The nth term in a geometric sequence is given by $t_n = t_1 r^{n-1}$ or $t_n = ar^{n-1}$
>
> where a is the first term t_1 and r is the common ratio.
>
> If given four terms, verify that the sequence is geometric by:
>
> $$\frac{t_4}{t_3} = \frac{t_3}{t_2} = \frac{t_2}{t_1} = r$$

Verifying that a sequence is geometric

To check whether a sequence is geometric, you need to verify that the ratio between each term and its predecessor does not vary (i.e. is constant).

13 Verifying whether a sequence is geometric and calculating the value of a term

For each of the following sequences, determine whether the sequence is geometric and, if so, calculate the value of the requested term.

(a) Given 2, 6, 18, 54, …, calculate the value of t_{10}.

THINKING	WORKING
1 Determine that the ratio of each term and its predecessor is constant.	$\frac{t_4}{t_3} = \frac{54}{18} = 3$ $\frac{t_3}{t_2} = \frac{18}{6} = 3$ $\frac{t_2}{t_1} = \frac{6}{2} = 3$
2 State whether the sequence is geometric. If so, note the first term and the common ratio.	The ratio between successive terms is constant, therefore the sequence is geometric. $t_1 = 2, r = 3$
3 Use the rule for a geometric sequence to calculate the value of the required term.	$t_n = t_1 r^{n-1}$ where $n = 10$ $t_{10} = 2 \times 3^{10-1}$ $= 39\,366$
4 Interpret the answer.	39 366 is the tenth term in the geometric sequence starting with the number 2 and where each term is 3 times the previous.

(b) Given 360, 180, 90, 45, … , calculate the value of t_8.

THINKING	WORKING
1 Determine that the ratio of each term and its predecessor is constant. *Note*: If the ratio gives a repeating decimal, use fractions instead, to avoid rounding.	$\frac{t_4}{t_3} = \frac{180}{360} = 0.5$ $\frac{t_3}{t_2} = \frac{90}{180} = 0.5$ $\frac{t_2}{t_1} = \frac{45}{90} = 0.5$
2 State whether the sequence is geometric. If so, note the first term and the common ratio.	The ratio between successive terms is constant, therefore the sequence is geometric. $a = 360, r = 0.5$
3 Use the rule for a geometric sequence to find the required term.	$t_n = ar^{n-1}$ where $n = 8$ $t_8 = 360 \times 0.5^{8-1}$ $= 2.8125$
4 Interpret the answer.	2.8125 is the eighth term in the geometric sequence starting with the number 360 and where each term is 0.5 times the previous.

(c) Given 10, 24, 52, 108, …, calculate the value of t_{16}.

1 Check that the ratio of each term and its predecessor is constant.	$\frac{t_4}{t_3} = \frac{108}{52} \approx 2.077$ $\frac{t_3}{t_2} = \frac{52}{24} \approx 2.167$
2 State whether the sequence is geometric.	The ratio of successive terms is not constant, so the sequence is not geometric. The requested term need not be found.

14 Determine the rule for the *n*th term of a geometric sequence

For the geometric sequence 55, 11, $\frac{11}{5}$, $\frac{11}{25}$, …

(a) Determine a rule for the *n*th term.

THINKING	WORKING
1 Determine the values of t_1 and r.	$t_1 = 55$ $r = \frac{t_2}{t_1} = \frac{11}{55} = \frac{1}{5}$
2 Use the general equation for the *n*th term of a geometric sequence. Substitute values for *a* and *r*.	$t_n = t_1 r^{n-1}$ $t_n = 55\left(\frac{1}{5}\right)^{n-1}$
3 Interpret the answer.	The rule to determine the *n*th term of the geometric sequence starting with 55 and where each successive term is $\frac{1}{5}$ of the previous, is $t_n = 55\left(\frac{1}{5}\right)^{n-1}$.

(b) Use the rule to calculate the eighth term.

1 Substitute the required value of n into the rule in the form $t_n = t_1 r^{n-1}$ and calculate t_8.	For $n = 8$ $t_8 = 55\left(\frac{1}{5}\right)^{8-1}$ $= \frac{11}{15625}$
2 Interpret the answer.	The eighth term in the sequence is $\frac{11}{15625}$.

Using the formula when a term value is known

There are four variables in the formula for the general term: a, r, n and t_n.

If any three of the variables are known, you can substitute their values into $t_n = ar^{n-1}$ to solve for the unknown.

15 Identifying the position in a sequence for a given term

For the geometric sequence 10, 30, . . . , what position in the sequence is the term 65 610?

THINKING	WORKING
1 Identify the known values.	$a = t_1 = 10$ $r = \frac{30}{10}$ $= 3$ $t_n = 65610$
2 Recall the rule for the nth term. Substitute the known values and solve for n, using trial and error to evaluate the index.	$t_n = ar^{n-1}$ $65610 = 10(3)^{n-1}$ $(3)^{n-1} = 6561$ Given that: $3^8 = 6561$ $n - 1 = 8$ $n = 9$
3 Interpret the answer.	The ninth term has the value 65 610.

If two terms of a sequence are known, you can use simultaneous equations to solve for a and r and, hence, generate the entire geometric sequence.

16 Generating a geometric sequence given two terms

Generate the first four terms of the geometric sequence given two terms, assuming a positive common ratio if a choice exists.

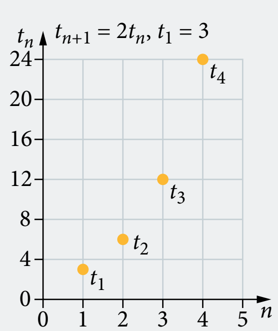
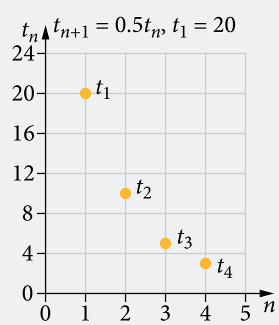
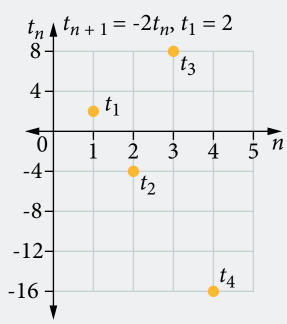
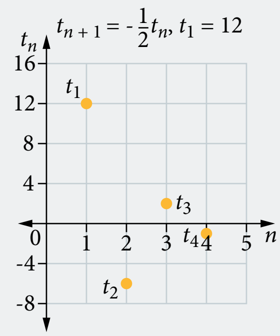
$t_5 = 162$ and $t_9 = 13122$

THINKING	WORKING
1 Create two equations using the general formula.	$t_n = ar^{n-1}$ $t_5 = 162$, so $ar^4 = 162$ [1] $t_9 = 13122$, so $ar^8 = 13122$ [2]
2 Solve the two equations simultaneously using a ratio. In this case r could have been ± 3 if a positive value had not been specified.	$\frac{[2]}{[1]}$ gives: $\frac{ar^8}{ar^4} = \frac{13122}{162}$ $r^4 = 81$ $r = \sqrt[4]{81}$ $r = 3$
3 Substitute and solve for a in one equation. Check the result with the other equation.	In [1]: $a \times 3^4 = 162$ $a = \frac{162}{3^4}$ $= \frac{162}{81}$ $= 2$ Check in [2]: $ar^8 = 2 \times 3^8$ $= 13122$
4 Generate the terms by starting with a and multiplying by r to calculate the value of each subsequent term.	The first four terms of the geometric sequence are: $t_1 = 2$ $\quad t_2 = 2 \times 3 = 6$ $t_3 = 6 \times 3 = 18$ $\quad t_4 = 18 \times 3 = 54$
5 Interpret the answer.	The first four terms in the sequence are: 2, 6, 18, 54.

If n is odd: $r^n = b \Rightarrow r = \sqrt[n]{b}$ and r has the same sign as b.

If n is even: $r^n = b \Rightarrow r = \pm\sqrt[n]{b}$ and b must be positive.

Graphs of geometric sequences

$r > 1$	$0 < r < 1$	$r < -1$	$-1 < r < 0$
As the number of terms increases, the value of the terms increases.	As the number of terms increases, the value of the terms decreases.	The terms alternate between positive and negative. As the number of terms increases, the magnitude of the terms increases.	The terms alternate between positive and negative. As the number of terms increases, the magnitude of the terms decreases.
			

Explore further

Geometric sequences and their graphs

Use a spreadsheet to generate and graph geometric sequences.

EXERCISE 3.3 Geometric sequences

Worked Example 12

1 Generate the first four terms of the following geometric sequences.

(a) $t_{n+1} = 2t_n$, with $t_1 = 9$
(b) $t_{n+1} = 3t_n$, with $t_1 = 4$
(c) $t_{n+1} = 4t_n$, with $t_1 = 2$
(d) $t_{n+1} = 5t_n$, with $t_1 = 1$
(e) $t_{n+1} = -2t_n$, with $t_1 = 3$
(f) $t_{n+1} = -3t_n$, with $t_1 = -5$
(g) $t_{n+1} = 6.5t_n$, with $t_1 = 16$
(h) $t_{n+1} = 1.1t_n$, with $t_1 = 200$
(i) $t_{n+1} = 2.2t_n$, with $t_1 = 500$
(j) $t_{n+1} = 0.9t_n$, with $t_1 = 100$
(k) $t_{n+1} = 0.5t_n$, with $t_1 = 360$
(l) $t_{n+1} = -0.25t_n$, with $t_1 = 4000$

2 Determine the first term a and the common ratio r for the following geometric sequences.

(a) $t_{n+1} = 200t_n$, with $t_1 = 2$
(b) $t_{n+1} = 3.9t_n$, with $t_1 = 50$
(c) $t_{n+1} = 0.88t_n$, with $t_1 = 2.05$
(d) $t_{n+1} = -17t_n$, with $t_1 = 0.003$
(e) $t_{n+1} = 4t_n$, with $t_1 = -4$
(f) $t_{n+1} = 50t_n$, with $t_1 = 25$
(g) $t_{n+1} = -1.08t_n$, with $t_1 = 200$

3 Consider the sequence $t_{n+1} = -4t_n$, with $t_1 = -2$.

(a) What is the second term t_2?

A 8 B -2 C -6 D -8

(b) Explain the common error made by a student who chose the second incorrect option.

(c) What is the sixth term t_6?

A -1024 B -256 C 128 D 2048

4 Generate the first four terms of the geometric sequence, given the following values for a and r.

(a) $a = 6, r = 2$ (b) $a = 1, r = 10$ (c) $a = 20, r = 3$

(d) $a = -2, r = 3$ (e) $a = -5, r = -2$ (f) $a = -6, r = -2.5$

(g) $a = 243, r = \frac{1}{3}$ (h) $a = 40, r = 0.125$ (i) $a = 6000, r = 1.02$

(j) $a = 5000, r = 0.99$ (k) $a = -0.75, r = 2$ (l) $a = 10\,000, r = 1.05$

Worked Example 13

5 Determine whether each sequence is geometric. If it is, determine the value of the requested term.

(a) $5, 15, 45, 135, \ldots, t_{10}$

(b) $1000, 500, 250, 125, \ldots, t_8$

(c) $3, 6, 12, 24, 48, \ldots, t_{21}$

(d) $4, 10, 24, 60, \ldots, t_{19}$

(e) $-1, 2, -4, 8, \ldots, t_{13}$

(f) $60, 12, 2.4, 0.48, \ldots, t_9$ Give your answer to 6 decimal places.

(g) $0.96, -0.48, 0.24, -0.12, \ldots, t_5$

(h) $1, 9, 81, 729, \ldots, t_7$

6 Determine whether each sequence is geometric. If it is, determine the value of the requested term. Express your answers to 3 decimal places.

(a) $0.4, 1, 2.5, 6.25, \ldots, t_{11}$

(b) $18, 19.8, 21.78, 23.958, \ldots, t_{15}$

(c) $1, 4, 9, 16, \ldots, t_{18}$

(d) $20\,000, 19\,000, 18\,050, 17\,145, \ldots, t_5$

(e) $500\,000, 490\,000, 480\,200, 470\,596, \ldots, t_{50}$

(f) $1000, 1050, 1102.5, 1157.625, \ldots, t_{10}$

(g) $1, \frac{10}{3}, \frac{100}{9}, \frac{1000}{27}, \ldots, t_6$

(h) $6, \frac{36}{7}, \frac{216}{49}, \frac{1296}{343}, \ldots, t_{14}$

7 Determine a rule for the nth term, in terms of n, for a geometric sequence given the following values for a and r.

(a) $a = 13$ and $r = 3$ (b) $a = 6$ and $r = -6$ (c) $a = -1000$ and $r = 2$

(d) $a = 552$ and $r = 0.65$ (e) $a = \frac{8}{3}$ and $r = 3.2$

8 Determine the first term a of each geometric sequence, given the following conditions.

(a) $r = 2$ and $t_8 = 9856$

(b) $r = 8$ and $t_5 = 49152$

(c) $r = -2$ and $t_{13} = 430\,080$

(d) $r = 7.5$ and $t_5 = -126\,562.5$

9 Determine the common ratio r for each geometric sequence, given the following values.

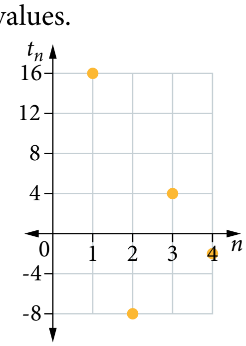
(a) $a = 3$ and $t_4 = 46.875$

(b) $a = 5000$ and $t_5 = 3280.5$

(c) $a = -68\,000$ and $t_3 = -329\,120$

(d) $a = -\frac{3}{7}$ and $t_6 = -7203$

10 Which sequence is represented by the graph?



A $t_{n+1} = 0.5t_n$ with $t_1 = -16$

B $t_{n+1} = -0.5t_n$ with $t_1 = 16$

C $t_{n+1} = -2t_n$, with $t_1 = -16$

D $t_{n+1} = 2t_n$, with $t_1 = 16$

Worked Example

11 Determine a rule for the nth term for each geometric sequence and calculate the value of the specified term. 14

(a) 2, 20, 200, …, t_6
(b) 15, 45, 135, …, t_5
(c) 1.5, 3, 6, …, t_7
(d) 100, 50, 25, …, t_6
(e) 6000, 4800, 3840, …, t_4
(f) -23, 161, -1127, …, t_5
(g) 35, -210, 1260, …, t_8
(h) 0.11, 0.869, 6.8651, …, t_4

12 For each geometric sequence, determine the position of the final term in the sequence. 15

(a) 6, 12, …, 12 288
(b) 11, 33, …, 216 513
(c) 2, 14, …, 33 614
(d) -5, -10, …, -327 680

13 Determine the term number of the given term in the geometric sequence. Write your answer as a value of n.

(a) $a = 14$ and $r = 2$, $t_n = 448$
(b) $a = 69$ and $r = 8$, $t_n = 35\,328$
(c) $a = 9000$ and $r = \frac{2}{3}$, $t_n = 4000$
(d) $a = -61$ and $r = 3.5$, $t_n = -213.5$

14 Generate the first four terms of a geometric sequence (assume a positive common ratio if a choice exists), given the following values. 16

(a) $t_4 = 54$ and $t_9 = 13122$
(b) $t_7 = 16\,384$ and $t_{11} = 4\,194\,304$
(c) $t_3 = -32.5$ and $t_6 = -4062.5$
(d) $t_{11} = 6$ and $t_{14} = -6$

15 Graph the first four terms of each sequence against the term number n.

(a) $t_{n+1} = 1.5t_n$, with $t_1 = 8$
(b) $t_{n+1} = -1.5t_n$, with $t_1 = 8$
(c) $t_{n+1} = 0.5t_n$, with $t_1 = 8$
(d) $t_{n+1} = -0.5t_n$, with $t_1 = 8$

16 For each of the following geometric sequences, give both the term number and the term value in your answer.

(a) Which term of 6, 30, 150, … is the first to exceed a million, 1 000 000?
(b) Which term of 630, 252, 100.8, … is the first to be below a millionth, $\frac{1}{1\,000\,000}$?

17 The following information refers to a geometric sequence.

The product of the first two terms is -1 200 000; the product of the second and third terms is -108 000; the product of the last two terms is -0.637 729 2.

Determine the values of the four terms referred to, and determine the number of terms in the sequence.

3.4 Applications of geometric sequences

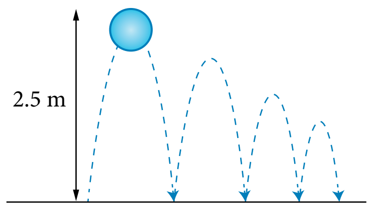
Problem solving

In situations where an amount changes by a constant proportion of the previous amount, geometric sequences can be used to solve problems.

17 Bounce height application

A rubber ball bounces over and over again, reaching 80% of its previous height each time.

If the first bounce reaches a height of 2.5 metres, how high will the 4th bounce be?



THINKING	WORKING
1 Recognise that the problem involves a geometric sequence and identify the first term t_1, common ratio r and term number n.	Geometric sequence: $t_1 = 2.5$, $r = 0.8$, $n = 4$
2 Use the formula for the nth term. Substitute the given values and calculate the value of the nth term t_n.	$t_n = t_1 r^{n-1}$ $t_4 = 2.5(0.8)^{4-1}$ $= 1.28$
3 Interpret the answer.	On the 4th bounce, the maximum height of the ball will be 1.28 m.

Starting a geometric sequence with $n = 0$

There are many applications of geometric sequences that involve time. Examples include population growth, investments and loans. In these situations, instead of beginning with $n = 1$ and term 1, t_1, begin with $n = 0$ and designate an initial term t_0. This makes it easier to match term number to time elapsed after the initial term. For example, t_n could be the value after n weeks. So t_0 represents the initial value or the value at the end of week 0. In this case, the general term of the geometric sequence is given by:

$t_n = ar^n$, with $t_0 = a$

18 Locust numbers application

One summer, a farmer estimates that the number of locusts swarming in a particular region is tripling each week. At the beginning of summer, authorities estimate locust numbers to be 170 000.

(a) Estimate locust numbers 5 weeks later.

THINKING	WORKING
1 Recognise that the problem involves a geometric sequence and t_0 is given.	Geometric sequence: $t_n = ar^n$
Identify the initial population t_0, common ratio r and term number n.	$a = t_0 = 170\,000$, $r = 3$ (tripling), $n = 5$
2 Recall the formula to calculate the nth term. Substitute the given values and calculate the value of the nth term t_n.	$t_n = 170\,000 \times 3^n$ $t_5 = 170\,000 \times 3^5$ $t_5 = 41\,310\,000$
3 Interpret the answer.	After 5 weeks, there will be 41 310 000 locusts.

(b) How many weeks will it take for locust numbers to increase from 170 000 to over a billion (1 000 000 000)?

THINKING	WORKING
1 Recall the formula to calculate the nth term. Substitute the given values and calculate the value of n. Use trial and error to determine the whole number on either side of the required amount.	$1\,000\,000\,000 = 170\,000 \times 3^n$ $3^n = \frac{1\,000\,000\,000}{170\,000}$ $3^n = 5882.3\ldots$ $3^7 = 2187$ and $3^8 = 6561$
2 Interpret the answer.	By the end of the 8th week, locust numbers will be over a billion.

Reducing balance depreciation

The decrease in value of an item as times goes by can also be calculated as a percentage *of the previous year's value*. Because the value decreases over time, so does the amount of yearly depreciation (unlike flat rate depreciation, which remains constant each year).

Depreciation calculated in this way is called 'reducing (or declining) balance depreciation'.

As an example, consider a mobile phone purchased for $1000 and depreciated at 10% per year.

The depreciation in the first year is therefore:

$$10\% \text{ of } \$1000 = 0.01 \times 1000 = \$100$$

resulting in a reduced value of $1000 − $100 = $900.

Year	Depreciation	Reduced balance
1	10% of \$1000 = 0.10×1000 = \$100	\$1000 – \$100 = \$900
2	10% of \$900 = 0.10×900 = \$90	\$900 – \$90 = \$810
3	10% of \$810 = 0.10×810 = \$81	\$810 – \$81 = \$729

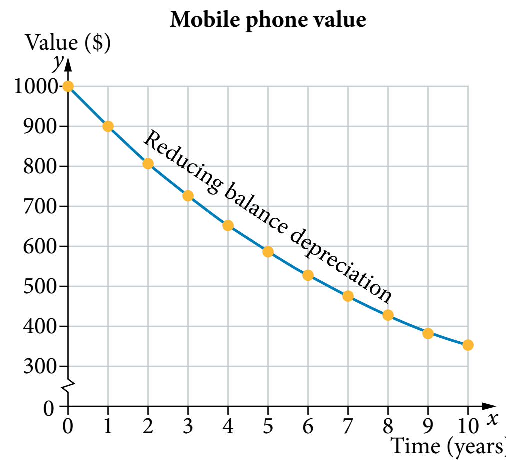
When using reducing balance depreciation, the item value decreases by a smaller amount in each successive year—that is, at a decreasing rate.

In general, if i is the rate of decline expressed as a decimal, the reduction factor or common ratio r is given by:

$r = 1 - i$

So if the original value is V_0, calculate the reduced value V_n after n years (or n reductions) of an item using:

$V_n = V_0(1-i)^n$ or $V_n = V_0 r^n$



This is similar to the geometric sequence formula, except that $n = 0$ is used for the initial term, so n matches the number of years elapsed.

A recurrence relation can also be used to describe the value V_n after n years (or periods): $V_{n+1} = rV_n$

19 Reducing balance depreciation using a rule

Hilda runs a fitness centre and depreciates gym equipment purchased for \$21 000 at a rate of 13% per year using the reducing balance method.

Calculate the value of the equipment after 4 years, rounded to the nearest dollar.

THINKING	WORKING
1 Identify the given information.	$V_0 = 21000$ $i = 0.13$ $n = 4$
2 Use $V_n = V_0(1-i)^n$ to determine the value after the given number of years V_n.	$V_n = V_0(1-i)^n$ $V_4 = 21000(1-0.13)^4$ $= 21000(0.87)^4$ $= \$12031$ (nearest dollar)
3 Interpret the answer.	The equipment is worth \$12 031 after 4 years.

Tabulating depreciated values

Companies often require the 'on paper' value of equipment to be recorded over a number of years, for tax and other administrative purposes. For this reason it is common to see depreciated values recorded from an initial value down to the time at which the equipment is written off. Spreadsheets are ideal for making the repeated calculations.

20 Reducing balance depreciation table

A company establishes a computer server at a cost of \$80 000 in 2015. The company accountant decides to use the reducing balance method and a rate of 25% to calculate annual depreciation (reduction) and, hence, the value of the server each year until its value declines to less than \$5000, at which point it will be 'written off' and scrapped.

(a) Construct a table of values showing the value of the server until it is scrapped. Round values to the nearest dollar.

THINKING

1 Calculate the common ratio.

WORKING

$$\begin{aligned} r &= 1 - i \\ &= 1 - 0.25 \\ &= 0.75 \end{aligned}$$

2 Create a table with Year in the first column and Value in the second. Enter the initial value in the first row. Multiply by the common ratio to calculate each subsequent value. Stop when the value goes below the specified amount.

For the spreadsheet shown, the formula for cell B3 is = B2 * 0.75. Click and drag down until the value is less than 5000.

	A	B
1	Year	Value ($)
2	2015	80000
3	2016	60000
4	2017	45000
5	2018	33750
6	2019	25313
7	2020	18984
8	2021	14238
9	2022	10679
10	2023	8009
11	2024	6007
12	2025	4505

3 Interpret the answer.

The server will be written off and scrapped in 2025 when its value falls below \$5000.

(b) Calculate the depreciation amount in each of the first five years. Round amounts to the nearest dollar.

1 In a new column headed 'Depreciation amount', start next to the second value. Subtract each value from the value above until you have the required number of entries.

For the spreadsheet shown, the formula for cell C3 is = B2 – B3. Click and drag down until there are five entries.

	A	B	C
1	Year	Value ($)	Depreciation amount ($)
2	2015	80000	
3	2016	60000	20000
4	2017	45000	15000
5	2018	33750	11250
6	2019	25313	8438
7	2020	18984	6328

2 Interpret the result.

In the first five years, the server depreciates in value by $20 000, $15 000, $11 250, $8438 and $6328.

WARNING

A spreadsheet uses unrounded values, even when you change the number of decimal places showing. For this reason, manually calculated values may vary slightly from values produced using technology.

Compound interest

Compound interest involves 'interest on interest'. It is calculated on the *total* funds in an account, which grow as interest is paid each year (or other period).

The following table illustrates the compounding effect on a deposit of $1000 earning 5% per year, with interest paid at the end of each year. The funds are compounding yearly.

Years funds have been in the bank	Interest calculation	Interest earned ($)	Total in account ($)
0		–	1000.00
1	0.05 × $1000	50.00	1050.00
2	0.05 × $1050	52.50	1102.50
3	0.05 × $1102.50	55.13	1157.63
4	0.05 × $1157.63	57.88	1215.51
5	0.05 × $1215.51	60.78	1276.29

In general, if i is the interest rate, expressed as a decimal, the growth factor or common ratio r is given by: $r = 1 + i$.

So if the original amount deposited is A_0, the increased amount in the account A_n after n years is: $A_n = A_0(1+i)^n$ or $A_n = A_0 r^n$.

If compounding occurs at intervals other than a year, i is the interest rate, expressed as a decimal, for that period and n is the number of periods over the term of the loan or investment.

The formula is similar to the geometric sequence formula, except that $n = 0$ is used for the initial term, so n matches the number of periods elapsed.

A recurrence relation can also be used to describe the amount A_n after n years (or periods): $A_{n+1} = (1+i)A_n$ or $V_{n+1} = rV_n$

21 Compound interest using a rule

Mrs Goosey deposits \$17 000 in an account that pays 6.9% interest each year, guaranteed. The yearly interest is added to her account automatically.

(a) How much will be in Mrs Goosey's account after 10 years, assuming she makes no more deposits?

THINKING	WORKING
1 Use the compound interest rule for the amount after n periods. Substitute the given values to determine the value of the final amount A_n.	$A_n = A_0(1+i)^n$ $A_{10} = 17\,000(1+0.069)^{10}$ $= 17\,000 \times 1.069^{10}$ $= 33\,130.35$
2 Interpret the answer.	There will be \$33 130.35 in the account after 10 years.

(b) If the same rate of interest is used, but the interest is added monthly, how much will be in the account after 10 years?

1 Calculate the altered values of i and n.	$i = 0.069 \div 12$ $= 0.00575$ $n = 10 \times 12$ months $= 120$ periods.
2 Use the compound interest rule for the amount after n periods. Substitute the given values to determine the value of the final amount A_n.	$A_n = A_0(1+i)^n$ $A_{120} = 17\,000(1+0.00575)^{120}$ $= 17\,000 \times 1.00575^{120}$ $= 33\,826.25$
3 Interpret the answer.	There will be \$33 826.25 in the account after 10 years.

WARNING

Avoid rounding values of i.

Leave the calculated value as an expression (for example, $0.069 \div 12$) if necessary.

EXERCISE 3.4 Applications of geometric sequences

Worked Example

17 **1** A bouncy ball is thrown onto a concrete floor, and bounces to a height of 3.5 metres. The ball continues to bounce, rising to 59% of the height of the previous bounce each time. How high will the eighth bounce be? Write your answer to the nearest millimetre.

2 A gambler first bets \$20 on red at the roulette table and loses. He then doubles his bet each time he loses. How much does the gambler bet on his fifth consecutive loss?

3 Moore's law states that the number of transistors that can be fitted on an integrated circuit (IC) doubles every two years. In the year 2000, it was possible to fit 37.5 million transistors on an IC. How many transistors could fit on an IC in 2010 if Moore's law is correct?

4 Drew notices that the TeraTown reservoir has developed a leak at the bottom of one of its gigantic concrete walls. When he first notices the problem, the leakage rate is 1 drop per minute, but the rate is tripling every minute.

What will the leakage rate be after 8 minutes?

19 **5** Joffa is planning to retire from teaching. In addition to his pension plan, he has set aside a lump sum of \$600 000 for leisure activities. He plans on spending most on leisure in the early years of his retirement, and so he decides to spend his 'leisure funds' in such a way that they will reduce by 60% each year. How much of Joffa's leisure funds will remain after 5 years?

6 Helga buys a new tractor for \$115 000 for her Brussels sprout farm. For tax purposes she depreciates the tractor's value by 22% each year using the reducing balance method. Calculate the value of the tractor after 9 years, rounded to the nearest dollar.

7 \$5000 is invested at 3.5% p.a. compounding annually.

(a) The account balance at the end of 2 years is given by

A $A_2 = 5000(1.35)^2$ **B** $A_2 = 5000(1+3.5)^2$

C $A_2 = 5000(1.035)^2$ **D** $A_2 = 5000(1.07)$

(b) Explain the common error made by a student who used $i = 0.35$ as part of their calculations in part **(a)**.

8 \$20 000 is invested at 4.2% p.a. compounding monthly.

The account balance at the end of 5 years is given by

A $A_5 = 20\,000(1.042)^5$ **B** $A_5 = 20\,000(1.0035)^5$

C $A_{60} = 20\,000(1.042)^{60}$ **D** $A_{60} = 20\,000(1.0035)^{60}$

21 **9** Determine the account balance, to the nearest dollar, at the end of the given term.

(a) Investment: \$78 000 at 7% p.a. compounding annually for 5 years

(b) Investment: \$9 200 000 at 6.35% p.a. compounding annually for 8 years

(c) Investment: \$45 000 at 14.5% p.a. compounding monthly for 10 years

(d) Investment: \$250 000 at 3.15% p.a. compounding quarterly for 3 years

10 Samuel and his partner Rachel invest \$1000 each in superannuation accounts (from which they are unable to withdraw funds). Samuel chooses a risky investment option, and over 5 years his investment decreases in value by 3% of the previous year's balance each year. Rachel chooses a safer investment option, which grows at a rate of 7% per year over the 5 years.

(a) Determine the value of Samuel's superannuation, to the nearest dollar, at the end of the 5 years.

(b) How much is in Rachel's account, to the nearest dollar, after 5 years?

(c) How much more than Samuel does Rachel have after 5 years?

11 Inflation is the rate at which the prices of a particular set of goods and services increase. The inflation rate can be used to estimate what an object may be worth in the future, in the same way that compound interest is calculated.

Assuming an inflation rate of 2.75% per year, estimate the cost of a litre of petrol in 10 years, to the nearest cent, if the current cost is \$2 per litre.

12 Pablo's painting is valued at \$12.5 million. If its value appreciates (increases) by 7% per year, how much will it be valued at in 20 years? Give your answer in millions of dollars, to 1 decimal place.

Worked Example 18

13 An algal bloom is first observed to cover 50 m^2 of a large lake, and the area is quadrupling each day.

(a) Determine the area the algal bloom will cover after a week.

(b) How many days will it take for the bloom to cover more than a billion square metres (1 000 000 000 m^2)?

14 Wilhelmina's worm farm contained 200 worms when she bought it at the end of 2015. The number of worms in the worm farm increases by 12% per year.

(a) How many worms will the farm contain after 4 years?

(b) In which year will the worm population have doubled to 400?

15 The Arachnids Football Club has 15 000 members in 2015, and the CEO estimates that membership will grow by 16% per year. If this growth rate is maintained:

(a) How many members will there be after 5 years?

(b) In which year will the membership first exceed 50 000?

(c) What will the membership of the club increase by in the third year of the plan?

16 Scientists are testing a new antiseptic on a colony of 80 000 bacteria. The number of bacteria appears to be decreasing at the rate of 19% per day. After how many days (at this rate) will the number of bacteria reduce to fewer than 100?

17 The highly rated reality TV show Kitchen Cooking premieres with a national audience of 1200 000 viewers. Ratings then plummet, with viewer numbers decreasing by 15% every episode. The network bosses decide that the show will be axed if its audience falls below 500 000 viewers. How many episodes will there be if ratings continue to decrease at this rate?

18 Brodie drops a rubber ball vertically from a window in his physics lab onto the pavement 45 metres below. He then analyses a video of the bouncing ball, for several bounces. He calculates that the ball always rises to 67% of the height of the previous bounce.

(a) If the first bounce reaches a height of 30 metres, how high will the 5th bounce be? Write your answer to 2 decimal places.

(b) After how many bounces will the ball fail to rise more than 2 metres?

20

19 Ho-Yu values his newly purchased office furniture at $4600 for tax purposes, and depreciates its value by 20% each year using the reducing balance method.

(a) Construct a table of values showing the value of the furniture over five years from the time of purchase.

(b) Ho-Yu claims the depreciation amount (the reduction in value) as a tax deduction each year in his tax return. Calculate how much he claims in each of the first five years, and the total claimed for this period.

20 Skyler buys a carwash and values its equipment at $2 750 000. She depreciates the equipment's value by 16% of the previous year's value each year.

(a) Complete a table showing the value of the equipment and its drop in value after 1, 2 and 3 years.

(b) How many years will it take for the equipment to reduce to less than 10% of its original value?

21 Laabha deposits $500 in an account that pays her 7.5% interest, compounded yearly.

(a) Construct a table showing Laabha's account balance at the end of each of the first 5 years of this investment.

(b) How much interest has Laabha earned after 5 years?

22 Scrooge deposits money in an account that pays him 6% compound interest each year, guaranteed. The yearly interest is added to his account automatically.

How many years after opening the account will it take for Scrooge's investment to double?

23 Best Bank offers customers a compound interest account with an annual interest rate of 7.6%, paid at the end of each year.

(a) How long would it take an initial account balance of $50 000 to grow to over $100 000?

(b) How much would a customer need to invest, to grow their funds to $100 000 in 9 years?

24 The half-life of radioactive material is the time taken for the radioactivity to fall to half of its initial value. A sample of radioactive material used in nuclear medicine causes a count rate meter to make 20 clicks per second or 20 becquerel, 20 Bq. If the half-life of the material is 6.5 days, how long will it take before the count rate falls below 1 Bq? Give your answer to the nearest day.

25 Rupert compares two methods of depreciating the Persian rugs he purchased for his office in 2015 for $45 000. He compares flat rate depreciation at 10% with reducing balance depreciation at 20% over a 10 year period from the year of purchase.

(a) Prepare a table showing valuations using both methods over the 10 year period, and display the values graphically.

(b) Rupert can claim the depreciation amount as a tax deduction. Which depreciation method should Rupert choose, if he wishes to maximise his tax deduction in the first three years?

(c) In which years will the flat rate method produce a greater depreciation than the reducing balance method?

Summary

Sequences

The expression t_n is used to represent any term in a sequence, and t_{n+1} is used to represent the next term. For example, t_n and t_{n+1} represent successive terms.

Arithmetic sequences

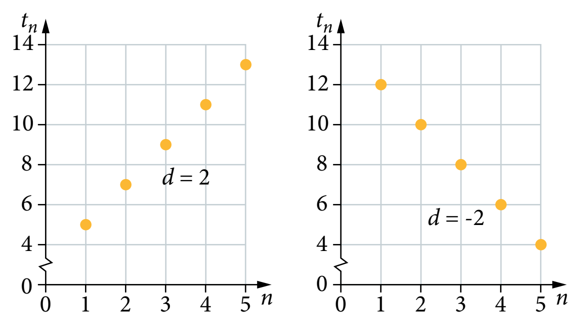
Arithmetic sequences have a constant common difference d given by: $d = t_{n+1} - t_n$.

The general rule for generating an arithmetic sequence using recursion, where a is the first term and d is the common difference, is: $t_{n+1} = t_n + d$, with $t_1 = a$.

The nth term in an arithmetic sequence is given by $t_n = a + (n-1)d$, where a is the first term t_1 and d is the common difference.

The arithmetic mean of three terms in an arithmetic sequence is the middle term. Also, the middle term is the mean of the terms on either side.

Graphs of arithmetic sequences are linear and discrete, with gradient equal to the value of d.



Applications of arithmetic sequences

If starting an arithmetic sequence with $n = 1$, use $t_n = t_1 + (n-1)d$.

If starting an arithmetic sequence with $n = 0$, use $t_n = t_0 + nd$.

Flat rate or unit or straight line depreciation	Simple interest
If the original value of an item is V_0 and the flat rate of depreciation is i, then the yearly depreciation d is given by $d = i \times V_0$. The value V_n after n years (or periods) is: $V_n = V_0 - nd$, where d is the common amount being subtracted.	If A_0 is the original amount invested and i is the flat interest rate, then the interest each year d is: $d = i \times A_0$ The total amount accumulated after n years is given by: $A_n = A_0 + nd$, where n is the number of years that have elapsed and d is the common difference.

Geometric sequences

Geometric sequences have a common ratio r that can be defined by:

$$r = \frac{t_{n+1}}{t_n} = \frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{t_4}{t_3}\ldots$$

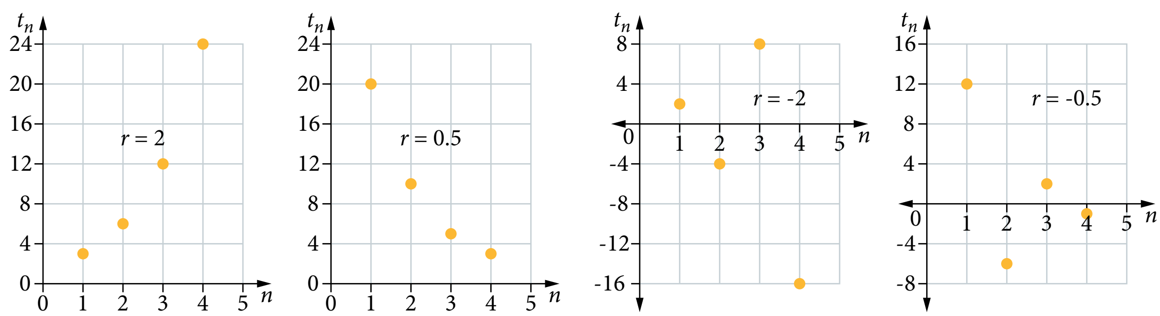
The general rule for generating a geometric sequence using recursion is $t_{n+1} = rt_n$, with $t_1 = a$.

The nth term in a geometric sequence is given by $t_n = ar^{n-1}$,
where a is the first term t_1 and r is the common ratio.

If n is odd: $r^n = b \Rightarrow r = \sqrt[n]{b}$ and r has the same sign as b.

If n is even: $r^n = b \Rightarrow r = \pm\sqrt[n]{b}$ and b must be positive.

Graphs of geometric sequences depend on the value of r for their shape.



Applications of geometric sequences

If starting a geometric sequence with $n = 1$, use $t_n = t_1 r^{n-1}$.

If starting a geometric sequence with $n = 0$, use $t_n = t_0 r^n$.

Reducing balance depreciation	Compound interest
If i is the rate of reduction, expressed as a decimal, the reduction factor is given by $r = 1 - i$. If the original value is V_0, calculate the reduced value after n years of an item using: $V_n = V_0(1-i)^n$ or $V_n = V_0 r^n$	If i is the interest rate, expressed as a decimal, the growth factor is given by: $r = 1 + i$. If the original amount deposited is A_0, the increased amount in the account after n years is: $A_n = A_0(1+i)^n$ or $A_n = A_0 r^n$. If compounding occurs at intervals other than a year, i% is the compound interest rate for that period and n is the number of periods.

Chapter review 3

1 Determine the first two terms of the sequence defined by $t_{n+1} = t_n + 17$, if $t_3 = 37$. Exercise 3.1

2 Determine the common difference for the sequence 6, 18, 30, 42, 54, … Exercise 3.1

3 An arithmetic sequence has a common difference of 2. What is true of the fifth term in the sequence? Exercise 3.1

A It is 10 less than the first term.
B It is 8 less than the first term.
C It is 8 more than the first term.
D It is 10 more than the first term.

4 The three consecutive terms 31, m, 53 are part of an arithmetic sequence. Determine the value of m. Exercise 3.1

5 The 5th and 7th terms of a sequence are 18 and 288 respectively. Determine the first four terms of the sequence. Exercise 3.1

6 Cornelius receives 7.25% simple interest annually on his investment of $300. What is his account balance, assuming no withdrawals, after 3 years? Exercise 3.2

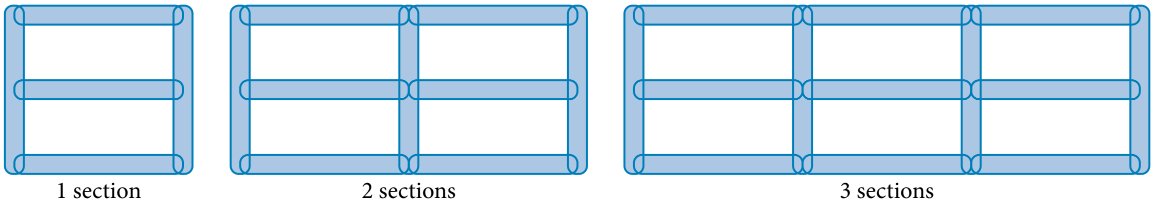
7 A contractor rents an electric concrete mixer for an initial fee of $50 plus $15 for each hour or part thereof. Exercise 3.2

(a) How much is the rental cost for the mixer for 6 hours?

(b) Bill does not want to spend more than $200 in fees for a concrete mixer rental. How long, at most, can he rent a concrete mixer?

8 A fence is to be built in sections using straight lengths, following the pattern shown in the diagram. Exercise 3.2

The one-section fence contains five lengths, and each section is one length high.



(a) How many lengths are required if the fence contains 25 sections?

(b) How many sections can be built using 201 lengths?

9 Aimee works on a public holiday and is paid $35 for the first hour of her shift, with hourly increases of $5. Exercise 3.2

If Aimee works for 8 hours, what is her total pay for the day?

10 Determine whether each expression represents an arithmetic or a geometric sequence. Give the value of the common difference d or the common ratio r. Exercise 3.1, 3.3

(a) $t_{n+1} = t_n + 7$
(b) $t_{n+1} = 5t_n$
(c) $t_{n+1} = t_n - 2$

Exercise 3.1, 3.3

11 Generate the first five terms of each sequence.

(a) $t_{n+1} = 4t_n$, with $t_1 = -3$

(b) $t_{n+1} = t_n + 15$, with $t_1 = -2$

(c) $t_{n+1} = t_n - 23$, with $t_1 = 100$

Exercise 3.1, 3.3

12 Determine the ninth term, and the total of the first 12 terms, of each of the following sequences. Round answers to 2 decimal places where appropriate.

(a) 16, 28, 40, 52, …

(b) 4, 6, 9, 13.5, …

Exercise 3.1, 3.3

13 Determine the first three terms of the following sequences.

(a) an arithmetic sequence with first term 51 and common difference -3

(b) a geometric sequence with first term 1215 and common ratio $\frac{1}{3}$

(c) an arithmetic sequence with 6th term 192 and a common difference of 18

(d) a geometric sequence with common ratio -4 and 5th term 5632

Exercise 3.1, 3.3

14 Determine a rule in terms of n for the general term t_n of each sequence.

(a) 1875, 375, 75, 15, …

(b) 1875, 1784, 1693, 1602, …

Exercise 3.3

15 Describe the geometric sequence 19, 57, 171, 513, … using the recursive form $t_{n+1} = rt_n$ with $t_1 = a$.

Exercise 3.3

16 Determine the sum of the first seven terms of the sequence 6, -12, 24, -48, …

Exercise 3.3

17 Which of the following is a geometric sequence with a common ratio of 5?

A 100, 20, 4, 0.8, …

B 5, 10, 20, 40, …

C 3, 8, 13, 18, …

D 2, 10, 50, 250, …

Exercise 3.3

18 Calculate the sum of the first seven terms of the sequence $\frac{1}{8}, \frac{1}{2}, 2, 8, \ldots$.

Exercise 3.4

19 Professor Paramecium observes a microhabitat in a Petri dish each morning and estimates that the number of microorganisms in the dish increases by 29% each day over a five day period. At the beginning of the fifth day, she estimates that there are 170 microorganisms. Determine the number of microorganisms present on the morning of the first day.

Exercise 3.4

20 Mining production at Mining Boring Co. is 200 tonnes in its first year of operation. If production increases by 17% each year, what will production be in the fifth year? Give your answer to the nearest tonne.

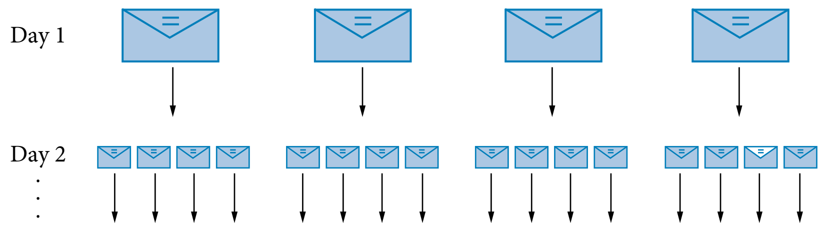
Exercise 3.4

21 A teleconference screen purchased for \$25 200 is depreciated using reducing balance depreciation at a rate of 17% each year. What is the depreciated value of the item, to the nearest dollar, after 3 years?

Exercise 3.4

22 Annika deposits \$5250 in a bank account that earns 5.33% compound interest per year, paid at the end of each year. Determine the total interest earned on Annika's investment after 4 years.

23 A 'chain email' scheme involves the originator of the plan emailing four others, with a request that each person do likewise to four more people, and so on for successive days, as illustrated here. Assume that each email sent is received the next day, and that each recipient sends their emails immediately. Exercise 3.4



(a) How many emails are sent on days 3, 4, 5, 6 and 7?

(b) How many emails will be sent on day 14 if the scheme continues long enough?

24 What is the 50th term in the sequence -20, -17, -14, -11, …? Exercise 3.1

A -977 B 10 C 80 D 127

25 Which number term in the sequence 60, 56, 52, 48, … is equal to -24? Exercise 3.1

A 10 B 22 C 31 D 44

26 Tisha gets a job in which she will be paid \$95 000 per year, with annual pay increases of \$5240 per year. Exercise 3.2

(a) How much will Tisha be paid in the fifth year of her employment?

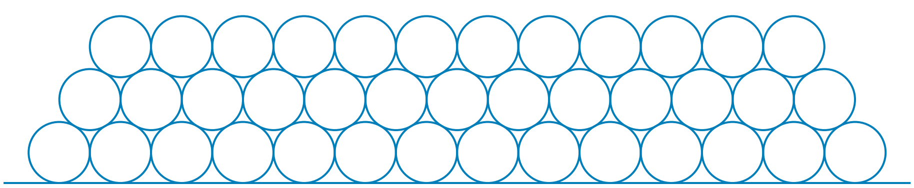
(b) What is the first year in which Tisha will be paid over \$150 000?

27 In 2010 Jill bought office furniture that had an initial value of \$3400. She used straight line depreciation with an annual rate of 12% as a method for valuing the furniture each year thereafter. Exercise 3.2

(a) What was the value of the furniture after 5 years?

(b) Jill decides to 'write off' the furniture when its value falls below \$500. In what year will this occur?

28 A very large pile of identical logs is being stacked in a timber yard, with 14 logs on the bottom (first) layer as shown. Exercise 3.2



Workers are adding logs to the pile. The stacking stops when the top layer contains 4 logs.

(a) How many logs will there be in the 7th layer?

(b) How many layers will be in the stack altogether?

Exercise 3.2

29 Xian plans a fitness program for three weeks. He decides to include running around the local sports oval, which he has measured at 550 m. He plans to run two laps on the first night and then increase the distance run by half a lap each day.

(a) How far does Xian run on the final night of the program?

(b) Show the sequence of metres run each day by marking the points on a graph.

Exercise 3.1, 3.3

30 Determine the first three terms of the following sequences.

(a) an arithmetic sequence with 10th term 1137 and 15th term 1802

(b) a geometric sequence with 7th term 8019 and 11th term 649 539 (assume the common ratio r is positive)

Exercise 3.1, 3.3

31 Determine the values of the first four terms of each of the following sequences. Plot the points on a graph.

(a) $t_{n+1} = t_n + 15$ with $t_1 = -10$

(b) an arithmetic sequence with $a = 88$ and $d = -6$

(c) a geometric sequence with $a = 24$ and $r = 1.5$

Exercise 3.3

32 Determine the first four terms of a geometric sequence with $t_{17} = 589\,824$ and $t_{18} = -1179\,648$.

Exercise 3.4

33 How many years would it take for a car purchased for $30 595 to depreciate to less than $10 000 using a reducing balance annual depreciation rate of 23%?

Exercise 3.4

34 How much compound interest is earned in the fourth year of an investment of $9000 at 16% per year, compounding annually? Give your answer to the nearest dollar.

Exercise 3.4

35 Minke invests $5 million in a property trust that offers a return of 11.3%, compounded yearly.

(a) Determine the value of Minke's investment in 3 years.

(b) For how many years must Minke's funds remain invested in order to grow to over $10 million?

Exercise 3.4

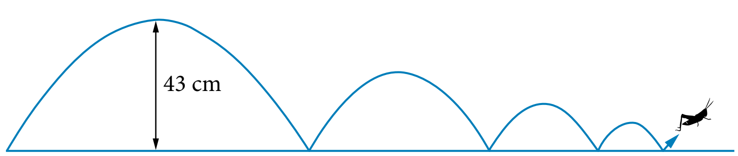
36 Oscar is a professional photographer, and must depreciate the value of his newly purchased digital camera. He paid $3300 for the camera, and depreciates it at 17% per year using reducing balance depreciation.

(a) Determine the value of the camera after 6 years.

(b) How many years does it take before the value of the camera is less than $1000?

Exercise 3.4

37 A grasshopper makes a series of jumps, with each jump being 60% as high as the previous one. The first jump is 43 cm high, as shown in the diagram.



(a) How high is the fourth jump? Give your answer in centimetres, to 1 decimal place.

(b) Which jump in the sequence is the first to fail to reach 5 cm in height?

(c) Determine the total vertical distance the grasshopper travels during the first five jumps. Give your answer to the nearest centimetre.

38 Office equipment worth \$15 000 is to be depreciated for tax purposes using the declining balance method. Determine what annual percentage with this method gives the same depreciated value after 8 years as a flat rate of 7%. Exercise 3.2, 3.4

39 A particular sheet of paper is one-tenth of a millimetre (0.1 mm) thick. Assume there is enough paper of suitable area and sufficient strength to perform each task. Exercise 3.4

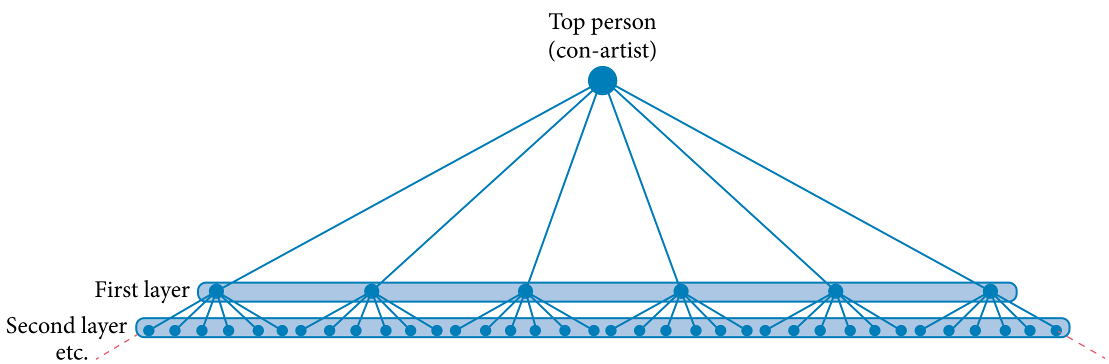
(a) Determine the thickness after the sheet of the paper has been folded in half each of the following number of times. Write your answer in appropriate units, to 1 decimal place.

(i) once **(ii)** 5 times **(iii)** 10 times **(iv)** 20 times

(b) How many times would you have to fold the paper in half to produce at least one kilometre of thickness?

(c) The distance from the Earth to the Moon is approximately 400 000 km. How many times would you need to fold the paper in half to make this distance?

40 An example of a pyramid selling scheme involves a top-level businessperson who recruits six people in the first 'layer' down the pyramid, who must pay him or her a fee of \$100 for the privilege of joining the scheme. These recruits then each recruit six people, collecting a \$100 fee from each. A percentage of the recruitment fees is passed up the pyramid, making the top person wealthy. Exercise 3.4



(a) Write an equation to represent the number of people (N) in the nth layer of the pyramid scheme.

(b) Complete the table, showing the number of people in each layer and the total number of people involved in the scheme once each layer is established, not including the top person.

(c) The world population is approximately 7 billion (7 000 000 000) people. At which layer in the scheme would the number of people involved exceed the world population?

n	N	Total
1		
2		
3		
4		
5		
6		
7		


S
SE
NE
PERVVI
ENSIS
Capricorni
ACIFICVM

4

Earth geometry and time zones

4 Recall

Converting between 24 h time and am/pm time

1 Convert each of the following times:

(a) 12:01 am to 24 h time (b) 2:50 pm to 24 h time (c) 22:10 to am/pm time.

Converting degrees and minutes to decimal degrees

2 Convert each angle to degrees, to 2 decimal places where necessary.

(a) 53°42′ (b) 125°15′ (c) 2°40′

Using a calculator to determine cosine values

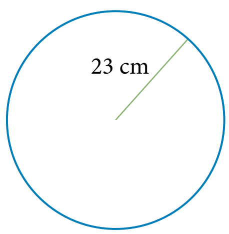
3 Determine each cosine value, to 4 decimal places. Check that your calculator is in degrees mode.

(a) $\cos(30°)$ (b) $\cos(12.6°)$ (c) $\cos(85°)$

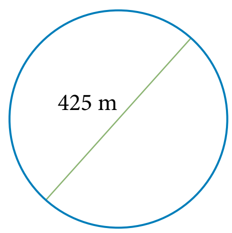
Calculating the circumference of a circle

4 Calculate the circumference of each circle, to the nearest whole number.

(a)



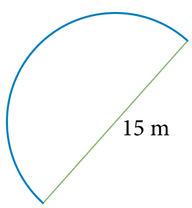
(b)



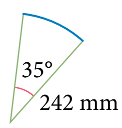
Calculating arc length

5 Calculate the length of each arc, to the nearest whole number.

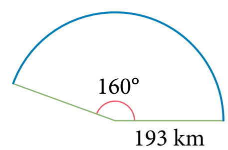
(a)



(b)



(c)



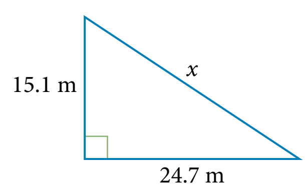
Calculating time elapsed

6 Calculate the time that has elapsed between the pair of given times.

(a) 7:50 am to 4:15 pm (b) 12:40 pm to 6:30 am the next day

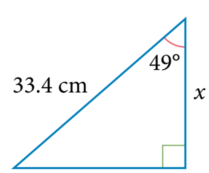
Calculating the length of the hypotenuse

7 Calculate the length of the hypotenuse, to 1 decimal place.



Calculating a length using the cosine ratio

8 Calculate the length of the side marked x, to 1 decimal place.



4.1 Latitude and longitude

The Earth as a sphere

The Earth is approximately spherical in shape. The distance around the equator is slightly greater than the distance from the North Pole to the South Pole and back again, and the surface is not exactly smooth, because of mountain ranges and deep ocean troughs.

For the purposes of this study, think of the Earth as a sphere.

A grid system

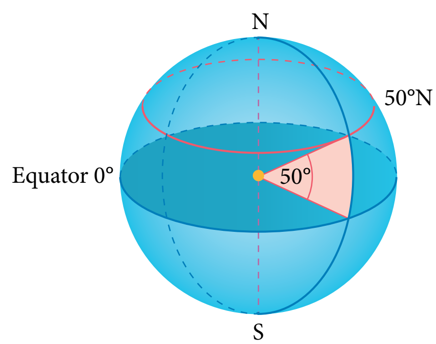
You are familiar with using a coordinate grid system with positive values above the horizontal axis and negative values below the horizontal axis, and positive values to the right of the vertical axis and negative values to the left of the vertical axis.

The grid system for the Earth uses circles (or parallels) of latitude that run east–west around the Earth, and semicircles (called meridians or lines of longitude) that go from pole to pole.

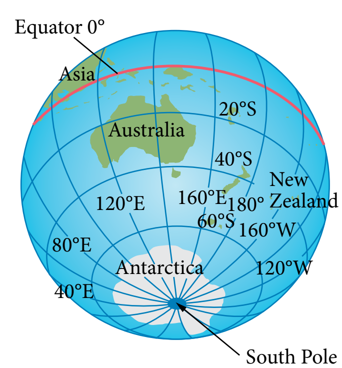
The equator is the circle of latitude midway between the poles. It is called a *great circle*, as it is the largest circle that can be made on the sphere, sharing its centre with the centre of the Earth.

The equator has a latitude of 0° and every other position is given as a number of degrees north or south of the equator, up to a maximum of 90° at the poles. The angle is determined by the angle formed at the centre of the Earth, between the equator and the parallel of latitude.

Every location north of the equator is referred to as the northern hemisphere N, and south of the equator as the southern hemisphere S.

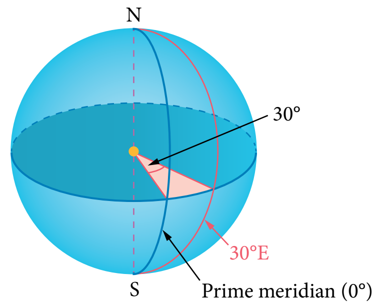


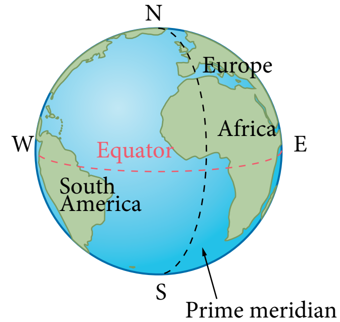
The degrees for the meridians of longitude are based on an arbitrarily chosen prime meridian, given as 0°. The prime meridian goes through the Greenwich Observatory near London, and was chosen at a time when Great Britain was first sending ships to uncharted waters around the globe. Every other meridian has a number of degrees east (to the right) or west (to the left) of the prime meridian, up to a maximum of 180°, where 180°E meets 180°W through, mostly, the Pacific Ocean. Each meridian is half a great circle. So, for example, 20°E and 160°W together form a great circle.



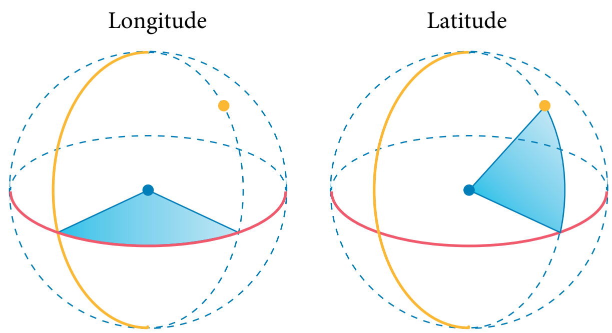
The angle for each meridian is determined by the angle formed at the centre of the Earth between the meridian and the prime meridian.

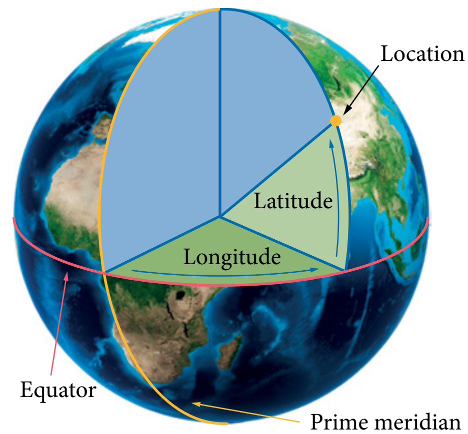
Every location east of the prime meridian is in the eastern hemisphere E. This includes most of Europe and Africa, and all of Asia, Australia and New Zealand. Every location west of the prime meridian, mainly the Americas and a lot of ocean, is in the western hemisphere W.





Together, the two coordinates pinpoint a location on the Earth's surface.

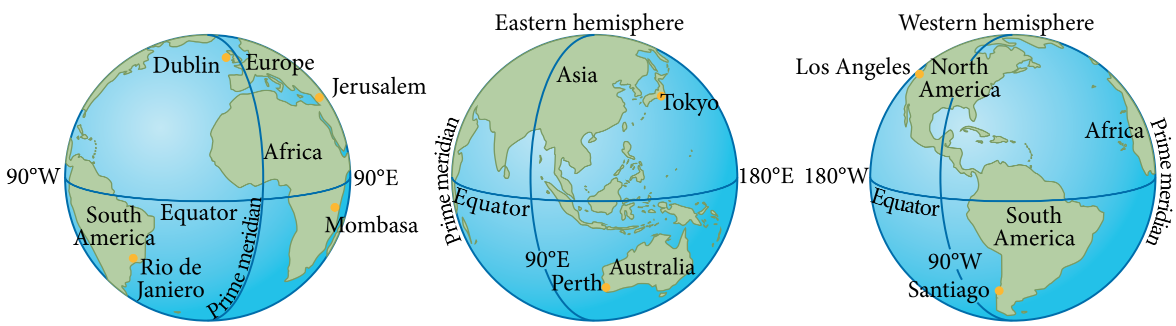




Global positions are given in the order: degrees north or south (latitude), then degrees east or west (longitude).

1 Locating position on a globe

Use the diagrams of globes to locate cities with the given coordinates.



(a) 32°S, 116°E

THINKING	WORKING
1 Use the coordinates to identify which side of the equator (north or south) and which side of the prime meridian (east or west) the city is located in.	32°S indicates that the location is south of the equator. 90–180°E is east (to the right) of the prime meridian, more than half-way into the eastern hemisphere.
2 Use the diagrams to identify a city that is in the correct hemisphere and lies within the correct range of the prime meridian.	32°S, 116°E could be Perth.

(b) 34°N, 118°W

1 Use the coordinates to identify which side of the equator (north or south) and which side of the prime meridian (east or west) the city is located in.	34°N indicates that the location is north of the equator. 90–180°W is west (to the left) of the prime meridian, more than half-way into the western hemisphere.
2 Use the diagrams to identify a city that is in the correct hemisphere and lies within the correct range of the prime meridian.	34°N, 118°W could be Los Angeles.

You can determine coordinates from a globe, a world atlas, a GPS device or any of the location devices available on the internet.

Explore further

Coordinate mapping

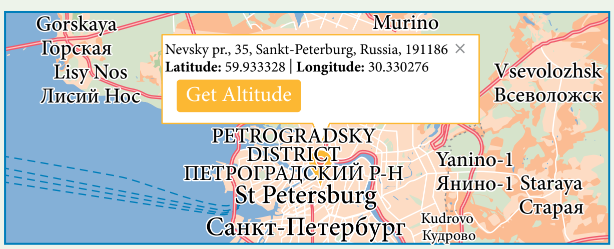
Use a GPS to determine the coordinates of locations anywhere in the world.

2 Locating position from an atlas

Use a world atlas to locate cities with the given coordinates.

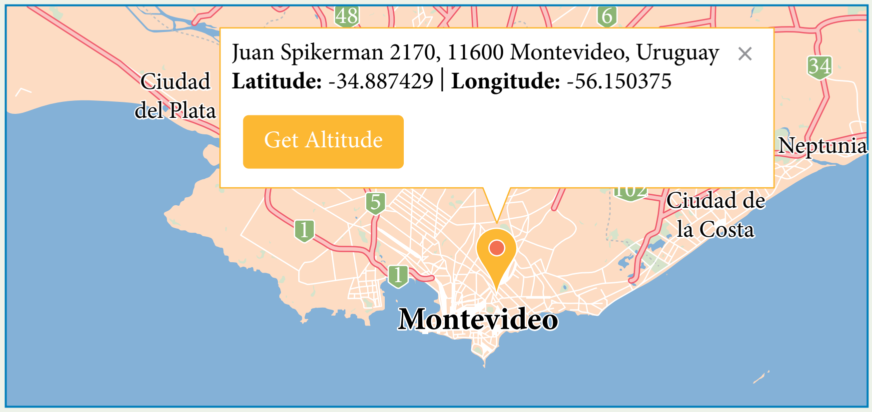
(a) 60°N, 30°E

THINKING	WORKING
1 Locate the section of the sphere from a flat diagram of the whole of Earth.	60°N, 30°E is north of the equator and not far east of the prime meridian, so it is in Europe.
2 Find the individual map that goes with the location on the world map.	You need a map of the western part of Russia. 60°N, 30°E is St Petersburg, Russia.



(b) 35°S, 56°W

1 Locate the section of the sphere from a flat diagram of the whole of Earth.	35°S, 56°W is south of the equator and west of the prime meridian, so it is in South America.
2 Find the individual map that goes with the location on the world map.	You need a map of Uruguay. 35°S, 56°W is Montevideo, Uruguay.



Check all your answers with any electronic device available, including internet location calculators.

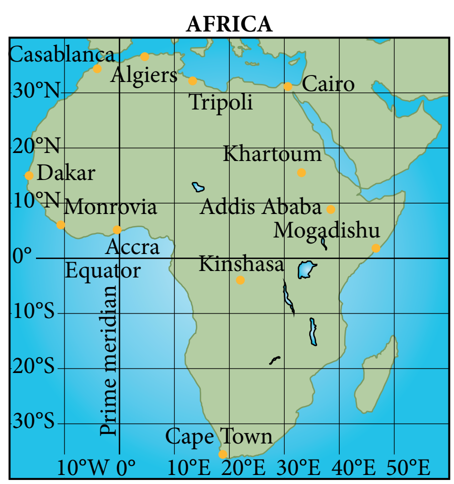
WARNING
The names of cities and countries on printed maps and on globes could be out of date. For example, before 1991 St Petersburg, Russia was Leningrad, USSR.

To estimate the coordinates of places from an atlas, you need to locate the place on a map that has latitude and longitude marked. Because flat maps cannot truly represent the curved surface of the Earth, it is necessary to compromise with straight lines or curves for the meridians. More accurate values can be found in the index of an atlas.

3 Estimating coordinates from an atlas

Use a world atlas to locate each city, then estimate its coordinates.

A map of Africa has been given here for convenience.



(a) Casablanca, Morocco

THINKING	WORKING
1 Locate the city on a flat diagram of the whole of Earth.	Morocco is in the north-western part of Africa.
2 Identify the lines of latitude above and below the city; estimate the degrees north or south. Identify the meridians to the left and right of the city; estimate the degrees east or west.	From this map, Casablanca is 34°N, 7°W. Checking with technology: 33.6°N, 7.6°W

(b) Cape Town, South Africa

1 Locate the city on a flat diagram of the whole of Earth.	Cape Town is in the south-western part of Africa.
2 Identify the lines of latitude above and below the city; estimate the degrees north or south. Identify the meridians to the left and right of the city; estimate the degrees east or west.	From this map, Cape Town is 34°S, 19°E. Checking with technology: 33.9°S, 18.4°E

Local area maps

When giving coordinates in a local area you need greater accuracy. Instead of writing the latitude and longitude with decimal places, each degree is broken into 60 minutes, in the same way as for time in hours.

1 degree = 60 minutes or $1° = 60'$

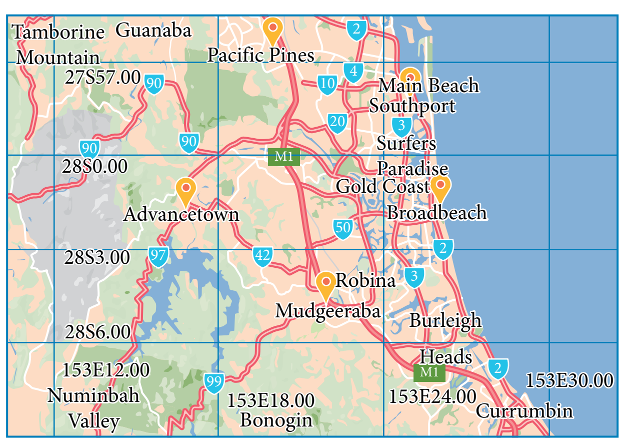
Sometimes few lines of latitude or longitude are shown, with the degrees and minutes not clearly defined. In this case, you need to observe some commonsense rules for Australian locations.

All locations in Australia are south of the equator, so going further south increases the latitude value.

All locations in Australia are east of the prime meridian, so going further east increases the meridian value.

4 Estimating coordinates from a local map

Use the local map to estimate the coordinates of Southport in degrees and minutes.



THINKING	WORKING
1 Identify the lines of latitude above and below the position.	The city is located between 27°57′S and 28°S.
2 Create a scale and use it to estimate the degrees and minutes south.	The city is approximately one-third of the distance from 27°57′S to 28°S, so 27°58′S.
3 Identify the lines of longitude to left and right of the position.	The city is located between 153°24′E and 153°30′E.
4 Create a scale and use it to estimate the degrees and minutes east.	The city is closest to 1 step of 6 to the right of 153°24′E, so 153°25′E.
5 Estimate the coordinates of the city.	Southport is at 27°58′S, 153°25′E.
6 Verify your answer using technology.	27.97°S, 153.42E

EXERCISE 4.1 Latitude and longitude

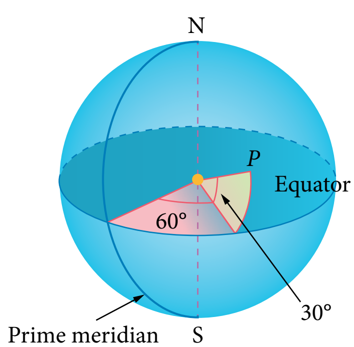
Worked Example

1   1 Use the diagrams of globes from Worked example 1 to locate cities with the given coordinates.

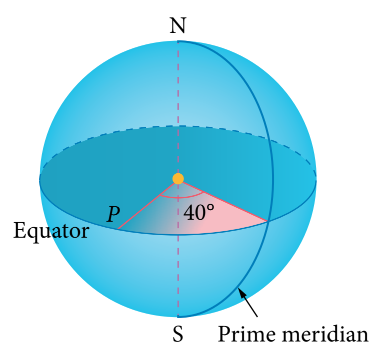
(a) 31°N, 35°E   (b) 23°S, 43°W   (c) 4°S, 40°E   (d) 53°N, 6°W

2 Use the cut-away diagrams of the Earth to determine the coordinates of P.

(a)



(b)



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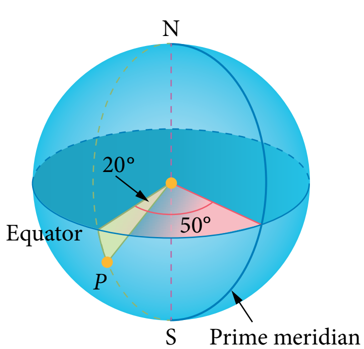
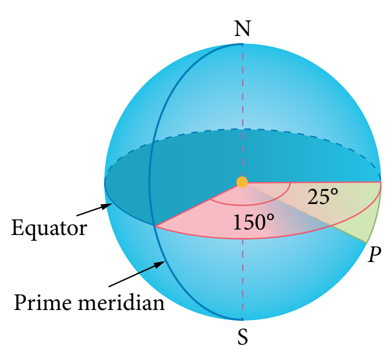


(d)



3 The Noongar community in Western Australia opened an Aboriginal cultural centre at Pinjarra site in 1834.

(a) Which of the following gives the coordinates of the site?

A 116°S, 33°E B 33°S, 116°E C 33°S, 116°W D 33°N, 116°W

(b) Explain the common error made by a student who chose the second incorrect option.

4 Which hemisphere is common to Iran, Northern Ireland, India and Japan?

A northern B eastern C western D both northern and eastern

Worked Example

5 Use a world atlas to locate cities with the given coordinates. 2

(a) 10°N, 67°W (b) 11°N, 107°E (c) 46°S, 171°E (d) 13°S, 39°W

6 Use a map of Australia to locate towns or cities with the given coordinates.

(a) 20°S, 140°E (b) 37°S, 144°E (c) 26°S, 139°E (d) 33°S, 148°E

7 Use maps of Asian countries to locate cities with the given coordinates.

(a) 37°N, 136°E (b) 38°N, 127°E (c) 1°N, 104°E (d) 12°N, 105°E

8 Use a world atlas or the map of Africa provided for Worked example 3 to locate each city, then estimate its coordinates. 3

(a) Algiers, Algeria (b) Addis Ababa, Ethiopia (c) Dakar, Senegal (d) Kinshasa, Congo

9 Convert the coordinates of these Australian capital cities to degrees and minutes, to the nearest minute.

(a) Brisbane: 27.47°S, 153.03°E (b) Sydney: 33.87°S, 151.21°E

(c) Canberra: 35.28°S, 149.13°E (d) Melbourne: 37.81°S, 144.96°E

10 Convert the coordinates of these Australian capital cities to decimal degrees, to 2 decimal places.

(a) Hobart: 42°53′S, 147°20′E (b) Adelaide: 34°56′S, 138°36′E

(c) Perth: 31°57′S, 115°51′E (d) Darwin: 12°28′S, 130°51′E

11 Use the Gold Coast map from Worked example 4 to estimate the coordinates of each location in degrees and minutes. 4

(a) Broadbeach (b) Pacific Pines (c) Advancetown (d) Mudgeeraba

12 The Torres Strait Island region around Cape York has sustained Aboriginal communities for more than 50 000 years.

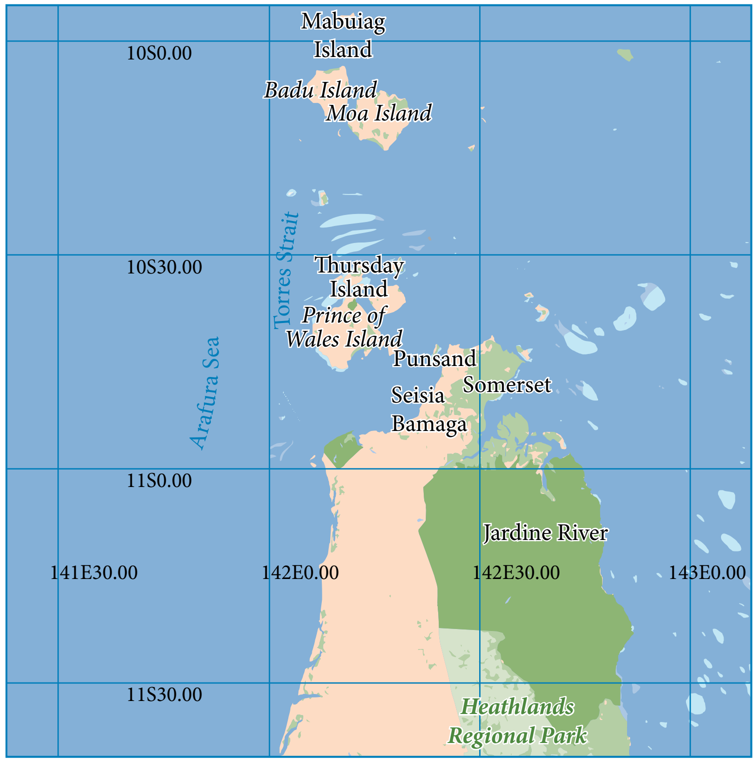
Use the map to estimate the coordinates of each location in degrees and minutes. Assume that the middle of the printed word gives a suitable focus point.

(a) Jardine River

(b) Bamaga

(c) Thursday Island

(d) Somerset



13 The area shown in the map is just north of the Tropic of Capricorn, 23.5°S.

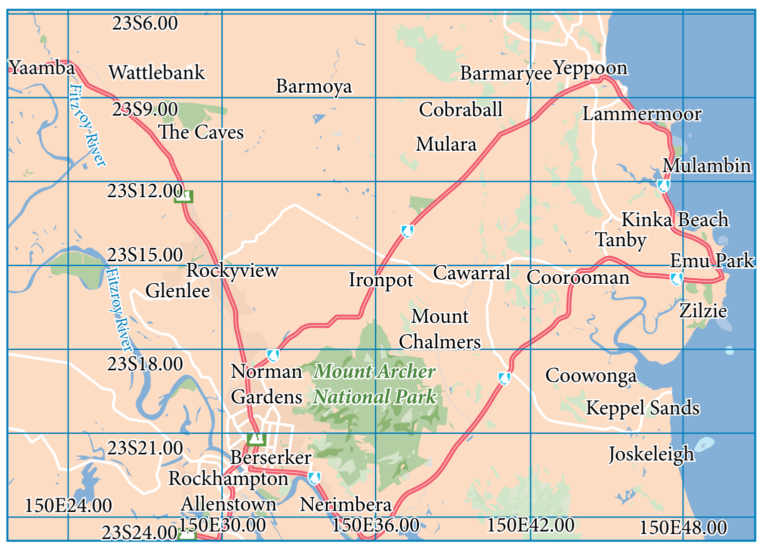
Use the map to estimate the coordinates of each location in degrees and minutes. Assume that the middle of the printed word gives a suitable focus point.

(a) The Caves

(b) Yeppoon

(c) Berserker

(d) Keppel Sands



14 Using a map of the USA, locate and give coordinates for cities that are:

(a) 3°W of Houston, Texas

(b) 2°N and 6°E of Phoenix, Arizona

(c) 3°S and 29°E of Seattle, Washington.

15 Thargomindah, 29°S, 143°49′E, is a long way both south and west of Mackay, 21°9′S, 149°11′E. Mitchell is 68% as far south and 23% as far west of Mackay as Thargomindah.

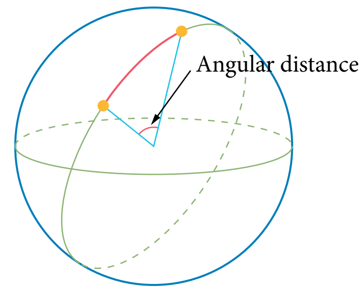
Calculate the coordinates of Mitchell, to the nearest minute.

Distance

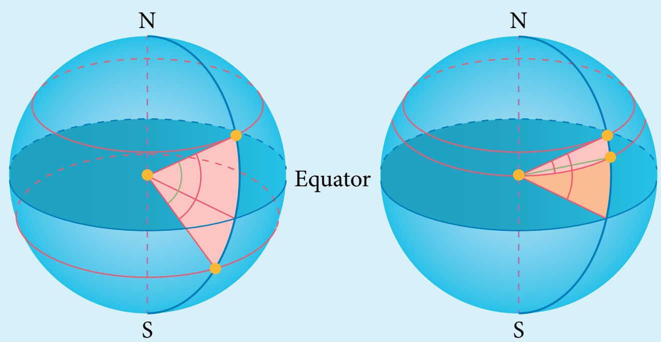
4.2

Angular distance

Most calculations of distance on the surface of the Earth involve the angle made between the two locations, using the centre of the Earth as the vertex. This angle is known as the *angular distance*. Imagine a great circle through the two points slicing the Earth in half, and then imagine the angle at the centre of the Earth.



If two locations are on the same meridian, the angular distance can be calculated by either adding or subtracting the latitude angles.



5 Calculating angular distance between locations on the same meridian

Calculate the angular distance between the two locations.

(a) 14°S, 151°E and 20°N, 151°E

THINKING	WORKING
1 Check that the meridians are the same.	14°S, 151°E and 20°N, 151°E Both locations lie on 151°E of the prime meridian.
2 If the latitudes are on different sides of the equator, add the latitude angles. If the latitudes are on the same side of the equator, subtract the latitude angles.	angular distance $= 14° + 20°$ $= 34°$
3 Interpret the answer.	The angular distance between the two locations is 34°.

(b) 10°S, 5°W and 22°S, 5°W

1 Check that the meridians are the same.	10°S, 5°W and 22°S, 5°W Both locations lie 5°W of the prime meridian.
2 If the latitudes are on different sides of the equator, add the latitude angles. If the latitudes are on the same side of the equator, subtract the latitude angles.	$\text{angular distance} = 22° - 10°$ $= 12°$
3 Interpret the answer.	The angular distance between the two locations is 12°.

(c) 21°57′N, 47°25′E and 27°35′N, 47°25′E

1 Check that the meridians are the same.	21°57′N, 47°25′E and 27°35′N, 47°25′E Both locations lie 47°25′E of the prime meridian.
2 If the latitudes are on different sides of the equator, add the latitude angles. If the latitudes are on the same side of the equator, subtract the latitude angles.	$\text{angular distance} = 27°35' - 21°57'$ $= 5°38'$ If doing this calculation without technology: $\text{angular distance} = 27°35' - 21°57'$ $= 27°35' - (21°35' + 22')$ $= 6° - 22'$ $= 5°60' - 22'$ $= 5°38'$
3 Interpret the answer.	The angular distance between the two locations is 5°38′.

Distance in kilometres

Every great circle on the Earth's surface has the same radius, approximately 6371 km.

To make a direct connection between angular distance and distance in kilometres, use the arc length formula: $l = \frac{\theta}{360} \times 2\pi r$.

Let $r = 6371$ km be the radius of Earth and $\theta = 1$ represent the angular distance between two locations 1° apart. The distance between the locations on the Earth's surface can be calculated as:

$$l = \frac{1}{360} \times 2\pi \times 6371$$
$$= 111.2 \text{ km}$$

Locations on the Earth's surface are separated by a distance of 111.2 km per angular distance of 1°.

> Distance in kilometres between locations on Earth:
> $D = 111.2 \times \text{angular distance}$

6 Calculating distance between places on the same meridian

Calculate the distance between each pair of locations, to the nearest kilometre.

(a) 14°N, 132°W and 2°N, 132°W

THINKING	WORKING
1 Check that the meridians are the same.	14°N, 132°W and 2°N, 132°W Both locations lie 132°W of the prime meridian.
2 Calculate the angular distance.	$\text{angular distance} = 14° - 2° = 12°$
3 Determine the distance by multiplying the angular distance by 111.2 km.	$D = 111.2 \times \text{angular distance} = 111.2 \times 12° = 1334\text{ km (nearest km)}$
4 Interpret the answer.	The locations are both positioned 132°W of the prime meridian, and are separated by an angular distance of 12° and a distance of 1334 km.

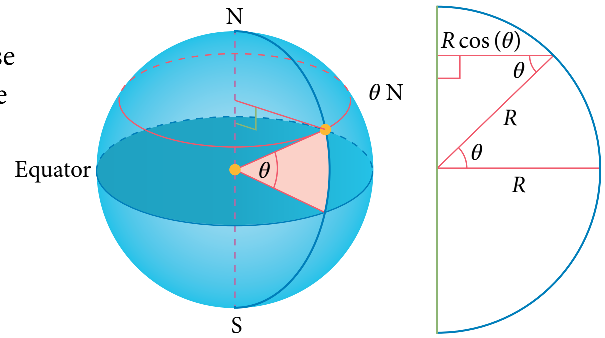
(b) 2°13′N, 143°50′E and 5°23′S, 143°50′E

THINKING	WORKING
1 Check that the meridians are the same.	2°13′N, 143°50′E and 5°23′S, 143°50′E Both locations lie 143°50′E of the prime meridian.
2 Calculate the angular distance.	$\text{angular distance} = 2°13' + 5°23' = 7°36'$
3 Determine the distance by multiplying the angular distance by 111.2 km.	$D = 111.2 \times \text{angular distance} = 111.2 \times \left(7\frac{36}{60}\right)° = 845\text{ km (nearest km)}$
4 Interpret the answer.	The locations are both positioned 143°50′E of the prime meridian, and are separated by an angular distance of 7°36′ and a distance of 845 km.

Distance along a parallel of latitude

The only parallel of latitude that is a great circle is the equator. Distances between locations on the equator use the same formula as distances between locations on the same meridian.

All other parallels of latitude are referred to as small circles, as they have to be downsized from the great circle by a factor of $\cos(\theta)$, where θ (theta) is the angle of the latitude.



Distance along a parallel of latitude:
$D = 111.2\cos(\theta) \times$ angular distance, where θ is the latitude.

$0 \leq \theta \leq 90°$ where θ represents the angle of latitude from the equator. As the angle of latitude increases, the arc length of the small circles decreases.

$\cos(90°) = 0$, which means that the small circles at the poles are reduced to points.

$\cos(0°) = 1$, which indicates that the small circle at the equator is not reduced at all; it is a great circle.

Explore further
Parallels of latitude
Use a spreadsheet to determine distances along parallels of latitude.

7 Calculating distances along a parallel of latitude

Determine the distance along the parallel of latitude between the two locations, to the nearest kilometre.

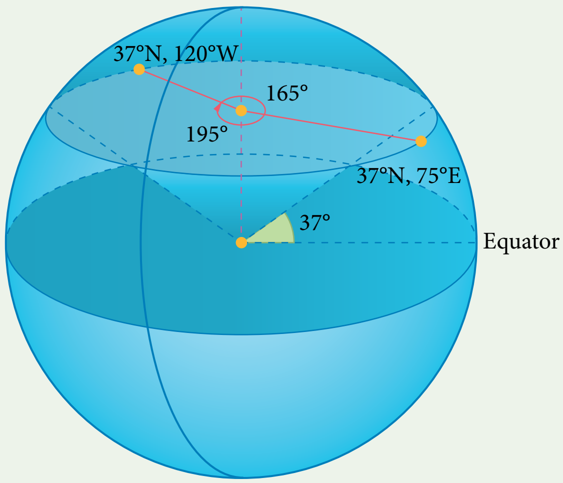
(a) 25°S, 143°E and 25°S, 153°E

THINKING	WORKING
1 Calculate the small circle angular distance.	25°S, 143°E and 25°S, 153°E Both locations have a latitude of 25°S. That is, they are positioned on the small circle 25° south of the equator.
2 If the meridians are on different sides of the prime meridian, add the meridian angles. If the meridians are on the same side of the prime meridian, subtract the meridian angles.	angular distance $= 153° - 143°$ $= 10°$
3 Determine the distance by multiplying the angular distance by 111.2 $\cos(\theta)$, where θ is the angle of latitude.	$D = 111.2\cos(\theta) \times$ angular distance $= 111.2\cos(25°) \times 10°$ $= 1008$ km (nearest km)
4 Interpret the answer.	The locations are both positioned 25° south of the equator, and are separated by an angular distance of 10° on a small circle and a distance of 1008 km.

(b) 31°4′N, 16°33′E and 31°4′N, 5°45′W

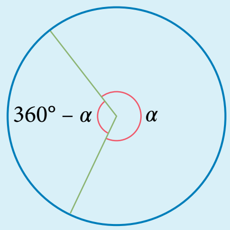
1 Calculate the small circle angular distance.	31°4′N, 16°33′E and 31°4′N, 5°45′W Both locations have a latitude of 31°4′N. That is, they are positioned on the small circle 31°4′ north of the equator.
2 If the meridians are on different sides of the prime meridian, add the meridian angles. If the meridians are on the same side of the prime meridian, subtract the meridian angles.	$\text{angular distance} = 16°33' + 5°45' = 22°18'$
3 Determine the distance by multiplying the angular distance by 111.2 cos (θ), where θ is the latitude.	$D = 111.2\cos(\theta) \times \text{angular distance}$ $= 111.2\cos\left(\left(31\frac{4}{60}\right)^\circ\right) \times \left(22\frac{18}{60}\right)^\circ$ $= 2124\text{ km (nearest km)}$
4 Interpret the answer.	The locations are both positioned 31°4′ north of the equator, and are separated by an angular distance of 22°18′ on a small circle and a distance of 2124 km.

(c) 37°N, 120°W and 37°N, 75°E

1 Calculate the small circle angular distance.	37°N, 120°W and 37°N, 75°E Both locations have a latitude of 37° N. That is, they are positioned on the small circle 37° north of the equator.
2 If the meridians are on different sides of the prime meridian, add the meridian angles. If the meridians are on the same side of the prime meridian, subtract the meridian angles.	$\text{angular distance} = 120° + 75° = 195°$ This gives a major sector angle of 195° between the two locations. The most direct angular distance is the minor sector angle of: 360° − 195° = 165° 

3 Multiply the angular distance by 111.2 cos (θ), where θ is the latitude.	$D = 111.2\cos(\theta) \times$ angular distance $= 111.2\cos(37°) \times 165°$ $= 14\,653$ km (nearest km)
4 Interpret the answer.	The locations are both positioned 37° north of the equator, and are separated by an angular distance (the most direct) of 165° on the small circle and a distance of 14 653 km.

If the angular distance is more than 180°, use the minor sector angle instead.



WARNING
If the angular distance is entered using the degrees/minutes/seconds key, the value of D will appear in degrees/minutes/seconds format. To determine the distance in kilometres, convert the value to decimal format at the end.

Shortest distance between two locations

The shortest distance between two locations on Earth is always along a great circle.

Following a meridian or travelling along the equator both give the shortest distance. However, travelling in an east–west direction along a parallel of latitude does not.

Place a piece of string on a globe with the ends on Paris and Vancouver, and pull the string taut. Although both cities are about 49°N you will find that the shortest path deviates to the north of the parallel of latitude, along a great circle whose centre is the centre of the Earth.

The formula for determining the shortest distance between two locations from their coordinates is very complicated, and you are not expected to be able to use it:
angular distance $= \cos^{-1}\left[\sin(\alpha_1)\sin(\alpha_2) + \cos(\alpha_1)\cos(\alpha_2)\cos(\beta_1 - \beta_2)\right]$, then $D = 111.2 \times$ angular distance to give the distance in kilometres, where the points $A(\alpha_1, \beta_1)$ and $B(\alpha_2, \beta_2)$ have north and east as positive angles, but south and west as negative angles.

To set up a calculation like this in a spreadsheet, all angles would have to be converted to radians by multiplying the angle in degrees by $\frac{\pi}{180}$.

When you use technology to determine distances, it is good to be able to estimate an answer, as this will enable you to identify errors.

Explore further

Great circle calculator

Use a spreadsheet with the angular distance formula to calculate the distance between any locations on Earth, given the coordinates.

The circumference of the Earth is approximately 40 000 km.

The distance from pole to pole is approximately 20 000 km.

The distance from pole to equator is approximately 10 000 km.

The distance from Brisbane to Townsville is approximately 1112 km.

8 Using technology to determine distance from coordinates

Use an online great circle distance calculator or app to determine the distance between positions with the given coordinates.

(a) Auckland, 37°S, 175°E and Los Angeles, 34°N, 118°W

THINKING	WORKING
1 Key in the two sets coordinates. Read the distance in kilometres.	Auckland, 37°S, 175°E to Los Angeles, 34°N, 118°W is 10 500 km.
2 Check that the distance seems reasonable.	This is reasonable, because a quarter of the way around the world is 10 000 km.

(b) Cairns, 27°58′S, 153°25′E and Mount Isa, 27°58′S, 153°25′E

1 Key in the two sets coordinates, leaving a space between degrees and minutes. If a single entry is required, convert the minutes to decimal degrees first. Read the distance in kilometres.	Cairns, 27°58′S, 153°25′E to Mount Isa, 27°58′S, 153°25′E is 783.2 km.
2 Check that the distance seems reasonable.	This is reasonable, because the distance between Brisbane and Townsville is about 1112 km.

If you know the names of the locations, you simply need internet access to determine the distances between them in an instant.

EXERCISE 4.2 Distance

Worked Example 5

1 Calculate the angular distance between the two locations.

(a) 25°S, 142°E and 11°N, 142°E
(b) 18°S, 33°W and 37°S, 33°W
(c) 2°N, 46°E and 61°N, 46°E
(d) 12°S, 100°E and 12°N, 100°E

2 Calculate the angular distance between the two locations.

(a) 65°14′S, 53°20′E and 36°29′N, 53°20′E
(b) 17°40′S, 110°50′W and 10°15′S, 110°50′W
(c) 15°35′N, 130°30′E and 24°28′N, 130°30′E
(d) 28.6°S, 98.3°W and 33.8°N, 98.3°W

3 Two locations on the equator are given as 0°, 106°15′E and 0°, 85°12′W.

(a) The angular distance between the locations is:

A 11°27′ B 21°12′ C 168°33′ D 191°27′

(b) Explain the common error made by a student who gave a reflex angle in part (a).

Worked Example 6

4 Calculate the distance between each pair of locations, to the nearest kilometre.

(a) 51°N, 42°W and 12°S, 42°W
(b) 4.5°N, 106.5°E and 58.6°N, 106.5°E
(c) 14.3°S, 2.2°E and 36.9°S, 2.2°E
(d) 13°S, 55°W and 2°N, 55°W

5 For each of the following, choose the option that gives the distance between the pair of locations.

(a) 13°20′N, 105°15′W and 11°50′N, 105°15′W

A 167 km B 189 km C 2799 km D 278 km

(b) 34°25′S, 75°15′E and 7°12′N, 75°15′E

A 3026 km B 2978 km C 4600 km D 4628 km

(c) 23°30′S, 22°06′E and 31°17′S, 22°06′E

A 875 km B 866 km C 914 km D 6092 km

6 Determine the angular distance around the parallel of latitude between each of the following locations.

(a) 15°4′S, 135°12′E and 15°4′S, 106°24′E
(b) 39°50′N, 65°16′W and 39°50′N, 85°42′E
(c) 33°10′S, 75°45′E and 33°10′S, 110°15′W
(d) 3°N, 6°50′W and 3°N, 1°10′E

7 For each of the angular distances given between two cities, determine the distance between the two locations, to the nearest kilometre.

(a) 10°32′ (b) 43°8′ (c) 2°2′ (d) 163°14′

8 Brisbane is 1112 km from Townsville 'as the crow flies'. What is the angular distance between the two cities?

9 Using a globe of the world of any size, cut a piece of string to match the distance from one of the poles to the equator. Given that this distance is, in reality, about 10 000 km, estimate each of the following distances to the nearest 100 km. For the cities, use an online calculator to enter the names and confirm your estimates.

(a) across Australia from east to west
(b) northern tip of Cape York to southern tip of Tasmania

(c) Brisbane to each of the following:

(i) Cairns
(ii) Sydney
(iii) Adelaide
(iv) Wellington, NZ

Worked Example

10 Use an online great circle distance calculator or app to determine the distance between positions with the given coordinates. 8

(a) 24°N, 105°E and 14°S, 15°W
(b) 2°15′S, 105°50′W and 2°55′S, 105°40′W
(c) 39.4°N, 27.5°E and 28.2°N, 117.8°E
(d) 34°20′N, 25°40′W and 33°10′S, 165°30′W

11 Calculate the distance along the parallel of latitude between the two locations within Australia, to the nearest kilometre. 7

(a) Perth, 31°57′S, 115°51′E and Broken Hill, 31°57′S, 141°28′E
(b) Alice Springs, 23°40′S, 133°53′E and Mount Morgan, 23°40′S, 150°23′E
(c) St George, 28°S, 148°35′E and Boonah, 28°S, 152°41′E
(d) Tamworth, 31°5′S, 150°56′E and Kempsey, 31°5′S, 152°50′E

12 Calculate the distance along the parallel of latitude between the two Pacific Ocean locations, to the nearest kilometre.

(a) Haiphong, Vietnam, 21°N, 106°41′E and Honolulu, Hawaii, 21°N, 157°50′W
(b) Nauru, 0°40′S, 166°56′E and Galapagos Islands, 0°40′S, 91°9′W
(c) Noosa, 26°20′S, 153°5′E and Easter Island, 26°20′S, 109°22′W
(d) New Caledonia, 21°S, 165°37′E and Tonga, 21°S, 175°12′W

13 Calculate the distance along the parallel of latitude between the two European locations, to the nearest kilometre.

(a) Dusseldorf, Germany, 51°10′N, 6°47′E and Dresden, Germany, 51°10′N, 13°44′E
(b) Cork, Ireland, 51°50′N, 8°28′W and Lodz, Poland, 51°50′N, 19°27′E
(c) Amsterdam, Netherlands, 52°23′N, 4°54′E and Hannover, Germany, 52°23′N, 9°44′E
(d) Oporto, Portugal, 41°20′N, 8°38′W and Barcelona, Spain, 41°20′N, 2°10′W

14 Antipodal points lie on opposite ends of any diameter of the Earth.

(a) Determine the angular distance, and distance in kilometres, between any pair of antipodal locations.
(b) Determine the coordinates of locations antipodal to the following:

(i) 25°S, 150°E
(ii) 41°22′N, 23°16′W

15 The distance between locations that are reasonably close can be estimated by thinking of the Earth as flat, with the parallels of latitude forming right angles with the meridians.

Estimate the distance between each pair of towns listed below, to the nearest kilometre.

Explain any assumptions you made, apart from the ones already stated.

(a) Mossman, 16°27′S, 145°22′E and Normanton, 17°40′S, 141°5′E
(b) Maryborough, Queensland, 25°32′S, 152°42′E and Gympie, 26°11′S, 152°40′E

4.3 Time zones

Longitude and time

The Earth revolves on its axis once every 24 hours. During that time, each meridian faces the Sun directly for an instant, experiencing 'local' noon time.

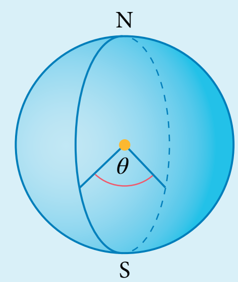
In Queensland our clocks are all set at the same time, yet only the locations exactly on the 150°E meridian experience local noon at midday. Brisbane, at about 153°E, experiences local noon at about a minute before midday, and Mt Isa, at 139°E, experiences local noon at about 12:44 pm.

The Earth revolves at $360° / 24$ hours, or $15° / \text{h}$, so the local time difference between locations can be calculated.

The Earth revolves at $15° / \text{h}$.

Non-coordinated time difference is $\dfrac{\theta}{15}$ hours,

where θ is the angular distance between the meridians.



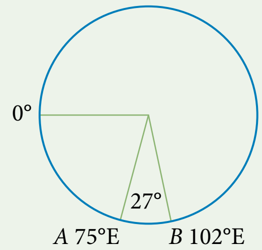
9 Calculating non-coordinated time difference

Calculate the non-coordinated time difference between the two locations and determine which one is ahead.

(a) A: 27°S, 75°E and B: 30°N, 102°E

THINKING	WORKING
1 Ignoring the latitudes, calculate the angular distance between the meridians.	A: 27°S, 75°E and B: 30°N, 102°E For the meridians: $\text{angular distance} = 102° - 75° = 27°$
2 Determine the non-coordinated time difference by dividing the angular distance by 15°.	$\text{time difference} = \dfrac{27}{15} = 1.8\,\text{h} = 1\,\text{h} + (0.8 \times 60)\,\text{min} = 1\,\text{h}\ 48\,\text{min}$

3 Identify the location further east as the one that is further ahead.



B is ahead of A by 1 h 48 min.

(b) A: 37°N, 22°W and B: 24°N, 46°E

1 Ignoring the latitudes, calculate the angular distance between the meridians.

A: 37°N, 22°W and B: 24°N, 46°E

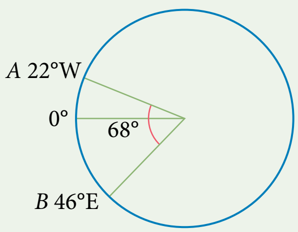
For the meridians:

$$\text{angular distance} = 22° + 46° = 68°$$

2 Determine the non-coordinated time difference by dividing the angular distance by 15°.

$$\begin{aligned}\text{time difference} &= \frac{68}{15}\\ &= 4.533\ldots\text{ h}\\ &= 4\text{ h} + (0.533\ldots \times 60)\text{ min}\\ &= 4\text{ h } 32\text{ min}\end{aligned}$$

3 Identify the location further east as the one that is further ahead.



B is ahead of A by 4 h 32 min.

(c) A: 2°S, 170°E and B: 14°N, 94°W

1 Ignoring the latitudes, calculate the angular distance between the meridians.

A: 2°S, 170°E and B: 14°N, 94°W

For the meridians:

$$\text{angular distance} = 170° + 94° = 264°$$

2 If the angular distance is more than 180°, use the minor sector angle.

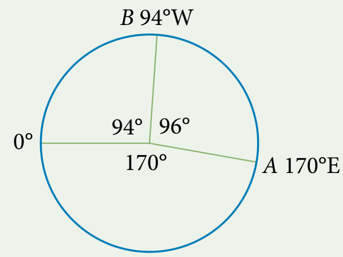
Corrected angular distance:

$360° - 264° = 96°$

3 Determine the non-coordinated time difference by dividing the angular distance by 15°.

$$\text{time difference} = \frac{96}{15}$$
$$= 6.4\,\text{h}$$
$$= 6\,\text{h} + (0.4 \times 60)\,\text{min}$$
$$= 6\,\text{h}\ 24\,\text{min}$$

4 Identify the location further east as the one that is further ahead.



B is ahead of A by 6 h 24 min.

The non-coordinated time differences will approximate the coordinated time differences only roughly.

The most extreme deviation from this approximation occurs when the calculation involves an adjustment, as happened in part (c) of Worked example 9, where the location in the western hemisphere was ahead in hours and minutes. Using a coordinated time system, the time in this location would be a day earlier, so B would be close to 24 h – 6 h 24 min = 17 h 36 min behind A.

Coordinated Universal Time (UTC)

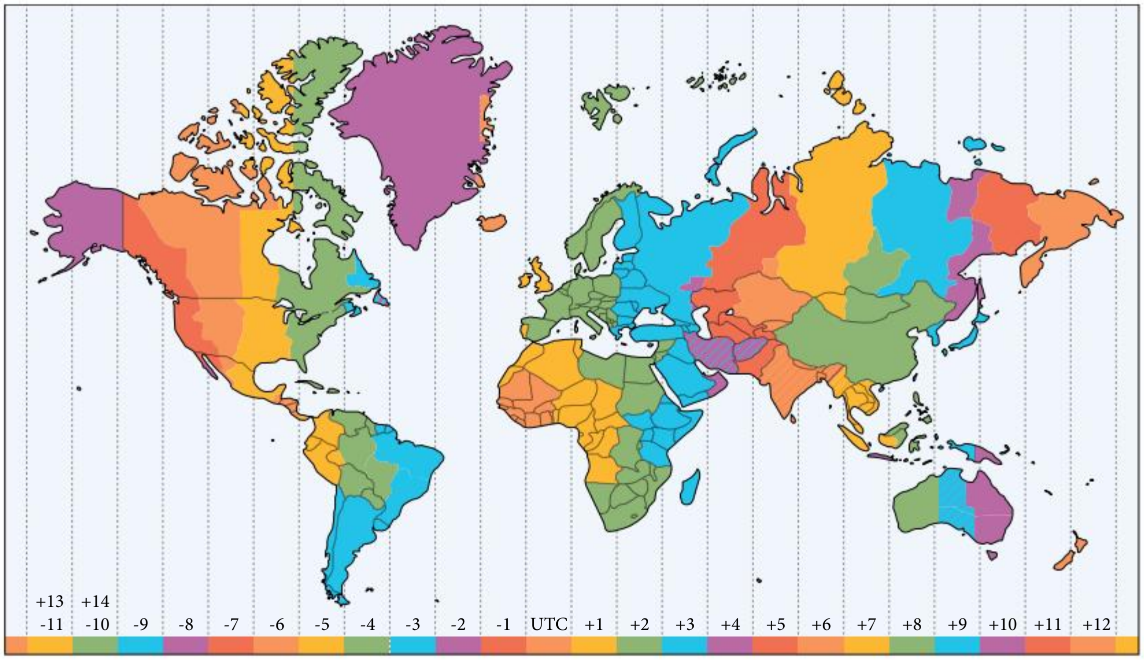
Coordinated Universal Time is a time standard, abbreviated from Universal Time Coordinated (UTC). UTC and its predecessor time zone, Greenwich Mean Time (GMT), are based on the time at the prime meridian. So UTC midday is when the sun is at its zenith over the prime meridian, coinciding with local noon.

To match the 15°/h revolution of the Earth, the time zones UTC + 1, UTC + 2, ⋯ UTC + 12 are centred on the meridians 15°E, 30°E, ⋯ 180°E.

Similarly, the time zones UTC – 1, UTC – 2, ⋯ UTC – 12 are centred on the meridians 15°W, 30°W, ⋯ 180°W.

For convenience, the government of each country or region agrees on a time zone, either for the entire country or for appropriately divided regions within the country.

The time zone map shows the time zones during winter in the southern hemisphere. However, this changes in appearance as places switch between standard time and DST, where individual states and countries, especially those far from the equator, choose to put their clocks forward by an hour for several months.



Queensland, with no daylight saving, experiences a different time zone from the southern states during summer only. Similarly, South Australia has daylight saving time (DST) in summer, but otherwise shares a time zone with the Northern Territory.

The countries/states/regions shaded with stripes use a time zone half an hour ahead of the time indicated by the countries with the same coloured shading.

Australian time zones

There are three main standard time zones within Australia.

WARNING

Eucla in WA uses UTC + 8:45, Australian Central Western Standard Time, ACWST.

Broken Hill in NSW uses ACST.

Lord Howe Island uses UTC + 10:30, Lord Howe Standard Time, LHST.

UTC + 8: Australian Western Standard Time, AWST

UTC + 9:30: Australian Central Standard Time, ACST

UTC + 10: Australian Eastern Standard Time, AEST

Daylight saving in Australia

The Australian Capital Territory, New South Wales, South Australia, Victoria and Tasmania observe daylight saving each summer. Western Australia, Queensland and the Northern Territory do not.

Countries close to the equator generally do not observe daylight saving, as the seasonal contrast in daylight hours does not vary as much as regions with higher latitudes.

Daylight saving time, DST, begins at 2 am on the first Sunday of October, so 2 am becomes 3 am as clocks are moved forward one hour. It ends on the first Sunday of April, when 3 am becomes 2 am as clocks are wound back by an hour.

The states/territories observing DST move to the adjusted time zones.

During DST, South Australia is half an hour ahead of Queensland instead of the usual half an hour behind.

> UTC + 10:30 is Australian Central Daylight Time, ACDT
>
> UTC + 11 is Australian Eastern Daylight Time, AEDT

10 Calculating the time of day within Australia

Calculate the time of day in the given town when it is 2:30 pm in Townsville.

(a) Geraldton, WA

THINKING	WORKING
1 Write the time zones for each location. Check for the possibility of DST.	Townsville: AEST, UTC + 10 Geraldton: AWST, UTC + 8 Neither place observes DST, so ignore this possibility.
2 Calculate the time difference. Adjust the time for the given town.	Geraldton is 2 h behind Townsville. 2:30 pm in Townsville, so: 2:30 pm – 2 h → 12:30 pm in Geraldton.

(b) Port Augusta, SA

(i) in June

THINKING	WORKING
1 Write the time zones for each location. Check for the possibility of DST.	Townsville: AEST, UTC + 10 Port Augusta: ACST, UTC + 9:30
2 Calculate the time difference. Adjust the time for the given town.	Port Augusta is 30 min behind Townsville. 2:30 pm in Townsville, so: 2:30 pm – 0:30 → 2:00 pm in Port Augusta.

(ii) in December

THINKING	WORKING
1 Write the time zones for each location. Check for the possibility of DST.	Townsville: AEST, UTC + 10 Port Augusta: ACDT, UTC + 10:30
2 Calculate the time difference. Adjust the time for the given town.	Port Augusta is 30 min ahead of Townsville. 2:30 pm in Townsville, so: 2:30 pm + 0:30 → 3:00 pm in Port Augusta.

International Date Line (IDL)

For an instant each day, when it is midday at the prime meridian, the whole world is on the same day, and date, with the exception of some islands in the Pacific Ocean, and assuming there is no DST anywhere. New Zealand will have just reached midnight, the eastern states of Australia will be at 10 pm, and Asia, Africa and Europe will all be experiencing afternoon/evening, while all the Americas will still be in the morning of that day.

At any other time of day, the world experiences two different days (and dates) simultaneously. Further west, the time zones are earlier. At the international date line, where 180°E meets 180°W, locations on the eastern side of the line are 24 h ahead of locations on the western side.

> Time zones for locations to the west are generally earlier in the day.
>
> Crossing the IDL going east: go back a day in time, so gain a day.
>
> Crossing the IDL going west: go forward a day in time, so lose a day.

Explore further

Time zones

Use technology to explore Greenwich Mean Time (GMT), the International Date Line (IDL) and Coordinated Universal Time (UTC).

The IDL goes through the Pacific Ocean, following the 180°E/W meridian most of the way. Where it deviates is due mainly to island groups whose governments have chosen to be on the same day as surrounding islands or countries.

11 Calculating the time of day around the world

Using the time zone map, calculate the time of day and date (day/month) in the given town when it is 10:15 am in Sydney on 25/03.

(a) Wellington, NZ

THINKING	WORKING
1 Determine the time zones for each location.	Sydney: AEDT, UTC + 11 Wellington: UTC + 13 Note that NZ observes DST.
2 Calculate the time difference.	Wellington is 2 h ahead of Sydney.
3 Adjust the time for the given town. Adjust the date, if necessary.	10:15 am on 25/03 in Sydney, so: 10:15 am + 2 h → 12.15 pm on 25/03 in Wellington.

(b) Paris, France

1 Determine the time zones for each location.	Sydney: AEDT, UTC + 11 Paris: UTC + 1 Note that DST is the opposite time of year in the northern hemisphere.
2 Calculate the time difference.	Paris is 10 h behind Sydney.
3 Adjust the time for the given town. Adjust the date, if necessary.	10:15 am on 25/03 in Sydney, so: 10:15 am – 10 h → 12:15 am on 25/03 in Paris.

(c) Los Angeles, USA

1 Determine the time zones for each location.	Sydney: AEDT, UTC + 11 Los Angeles: ACDT, UTC − 8
2 Calculate the time difference.	Los Angeles is 19 h behind Sydney.
3 Adjust the time for the given town. Adjust the date, if necessary.	10:15 am on 25 / 03 in Sydney. 10:15 am − 24 h → 10:15 am on 24 / 03 10:15 am + 5 h → 3:15 pm on 24 / 03 The time is 3:15 pm on 24 / 03 in Los Angeles.

Solving problems

An awareness of the time of day in other parts of the world is essential when you have 'live' contact with someone, during a phone or video conversation, for example. You need to be aware of why a British news broadcaster says 'Good morning!' when it is evening. For sports fans who like to watch, say, the Olympic games as they happen, it is much simpler for Queenslanders to watch when the games are held in Asia, with a time difference of, say, 2 hours than when the games are held in Europe.

12 Solving problems involving time zones

A company has offices in Brisbane and London. Standard office hours are 9 am to 5 pm in London, and 8 am to 4 pm in Brisbane. Determine the period of time when a worker in Brisbane can make contact with a co-worker in London, preferably during office hours, during the Australian summer.

THINKING	WORKING
1 Determine the time zones for each location.	Brisbane: AEST, UTC + 10 London: UTC Note that Brisbane does not observe DST.
2 Calculate the time difference. Determine the period of time for the 'away' location in terms of the 'home' location.	London is 10 h behind Brisbane. 9 am to 5 pm in London is 7 pm to 3 am in Brisbane.
3 Determine any overlap in times.	Brisbane's office hours of 8 am to 4 pm do not overlap with London's 7 pm to 3 am.
4 Interpret your solution, exploring alternatives if the preferred times are not possible.	The Brisbane worker must stay at work for 3 hours after 4 pm to talk to their London colleague immediately after that office opens. Alternatively, the London colleague could open the office more than 3 hours early, to talk to their Brisbane colleague as they finish work. Other alternatives involve times near 3 am in Brisbane, which would be unreasonable.

EXERCISE 4.3 Time zones

Worked Example

1 Calculate the non-coordinated time difference between the two locations and state which location is ahead. 9

(a) A: 14°S, 16°E and B: 22°N, 94°E
(b) A: 29°S, 68°W and B: 18°S, 50°W
(c) A: 26°N, 20°E and B: 33°N, 13°W
(d) A: 39°N, 50°W and B: 5°N, 50°E

2 Calculate the local non-coordinated time difference between the two locations and say which one is ahead.

(a) A: 40°S, 110°E and B: 25°N, 150°W
(b) A: 15°N, 95°W and B: 35°N, 99°E
(c) A: 6°S, 142°E and B: 13°S, 80°W
(d) A: 21°N, 135°W and B: 43°N, 132°E

3 Calculate the time of day in the given town when it is 9:45 am in Port Douglas, Qld. 10

(a) Fremantle, WA
(b) Murray Bridge, SA in winter
(c) Melbourne, Vic. in summer
(d) Whyalla, SA in summer

4 Calculate the time of day in Esk, Qld at the time of day and season for the given towns.

(a) 2:20 pm in Sheffield, Tas. in winter
A 2:20 pm B 2:50 pm C 3:20 pm D 4:20 pm

(b) 8:35 am in Canberra in summer
A 9:35 am B 8:05 am C 8:35 am D 7:35 am

(c) 3:30 pm in Alice Springs
A 4:00 pm B 3:30 pm C 4:30 pm D 5:30 pm

(d) 11:15 am in Esperance, WA
A 12:15 pm B 1:15 pm C 11:45 am D 11:15 am

5 The cricket is being broadcast from the MCG (Melbourne Cricket Ground) on Boxing Day.

(a) Determine the local time for viewers in each of the other state and territory capitals when it is 10:25 am in Melbourne.
(b) Determine the time in Melbourne when viewers in each of the other state and territory capitals take their noon break.

6 At what time and on which day will people in Queensland be able to watch live broadcasts of the midnight New Year's Eve fireworks from the following locations?

(a) Auckland, NZ (using the meridian 180°E)
(b) Sydney
(c) London
(d) New York (using the meridian 75°W)

7 For ships at sea, local time is determined from the nearest meridian that is a multiple of 15° from the prime meridian. Determine the UTC time zone for ships in each of the following locations:

(a) 45°45′S, 78°5′E in the Indian Ocean
(b) 43°10′N, 155°30′W in the North Pacific Ocean
(c) 42°50′S, 159°20′E in the Tasman Sea
(d) 23°30′S, 10°15′W in the South Atlantic Ocean.

8 A movie is to be released for download at 6:00 pm from a Hollywood studio in Los Angeles, and simultaneously throughout the US states, excluding Alaska and Hawaii. Use the time zone map and a map of the USA to determine the release time in each of the following states, assuming DST does not apply.

(a) Ohio
(b) Colorado
(c) Texas
(d) Oregon

9 Ships at sea calculate local time coordinated to the closest meridian that is a multiple of 15°.

(a) The time difference between two container vessels at 30°14′N, 35°18′W in the Atlantic Ocean and 25°42′N, 162°50′E in the Pacific Ocean is

A 9 h. B 10 h 47 min. C 11 h. D 13 h.

Worked Example

(b) Explain the common error made by a student who chose the third incorrect option.

11

10 Using the time zone map, calculate the time of day and the date (day/month) in the given town when it is 1:20 pm in Adelaide on 06/02.

(a) Dunedin, NZ (b) Mumbai, India (c) Nice, France (d) Boston, Massachusetts, USA

11 Using the time zone map or online time zone information, calculate the time of day in Bundaberg when it is 6:15 am in each of the following cities. Assume no DST and state 'previous day', 'same day' or 'next day' in each case.

(a) Christchurch, NZ
(b) Ho Chi Minh City, Vietnam
(c) Frankfurt, Germany
(d) Rio di Janeiro, Brazil

12

12 A company has offices in Melbourne and New York. Standard office hours are 8 am to 4 pm in both cities. Determine the period of time during which a worker in Melbourne can make contact with a co-worker in New York, preferably during office hours, during the Australian winter. Note that New York observes DST from early March.

13 Use the time zone map or the internet to determine the time zones for the Asian cities in order to answer this question, and assume there is no DST.

A Townsville company also has offices in Tokyo and Yangon, Myanmar. Normal office hours are 9 am to 5 pm in all three cities. Determine a suitable timeframe for:

(a) a Townsville worker to make contact with a co-worker in Tokyo

(b) a worker in the Tokyo office to make contact with the Townsville office

(c) a worker in the Yangon office to contact the Tokyo office.

14 Three friends about to begin their gap year plan to have a three-way one hour chat once a week on the weekend. Minh is staying at the Sunshine Coast, Hana is off to Los Angeles, and Reese will be based in Beijing. China uses China Standard Time, CST, UTC + 8 all year round. LA uses Pacific Standard Time, PST, UTC – 8 for four months of the year, and Pacific Daylight time, PDT, from early March to early November.

(a) If the girls agree that 'sleep time', 10 pm to 6 am, is out of bounds, what is the period of time available for Minh in Queensland, when they begin in January?

(b) At what local times should Minh tell the others to expect the chat to begin?

(c) What period of time should each of them use if they don't want to have to change their agreed time when PDT begins in early March?

15 Rupert has moved from Manchester, England to Chicago, but still likes to watch Premier League Football games live. Chicago uses Central Standard Time, CST, UTC – 6. Rupert plans to watch the Manchester United versus Newcastle game, which kicks off at St James' Park at 2:15 pm on a Sunday in February. He plans to get up, drive to the gym, about 20 minutes each way, and prepare a snack prior to the game.

Determine the latest time he should leave for the gym if he usually spends 1 h 15 min at the gym.

Time and travel 4.4

Travelling west

Travelling west across time zones from anywhere in Australia could take you to Asia, Europe or Africa. Because you are going in the same direction as the Sun, relative to the Earth, the elapsed time for the journey will seem shorter than it actually is from a casual review of the departure and arrival times.

Departure and arrival times are always 'local' times.

The places may be in different time zones.

Estimated time of arrival is often shortened to ETA.

13 Calculating arrival times, travelling west

A flight departs Brisbane at 9:10 am for Singapore, with an anticipated flight time of 7 h 55 min.

(a) Determine the time in Brisbane when the flight is expected to arrive in Singapore.

THINKING	WORKING
1 Add the time elapsed to the departure time.	9:10 am + 7 h 55 min → 5:05 pm
2 Interpret the answer.	It will be 5:05 pm in Brisbane when the flight is expected in Singapore.

(b) Determine the ETA for the flight in Singapore.

1 Identify the time zones for both locations. Consider DST, if necessary.	Neither location observes DST. Brisbane: AEST, UTC + 10 Singapore: UTC + 8
2 Use the time difference to calculate the unknown time.	Singapore is 2 h behind Brisbane. 5:05 pm − 2 h → 3:05 pm
3 Interpret the answer.	The ETA, Singapore time, is 3:05 pm.

Notice that a departure time of 9:10 am and arrival time of 3:05 pm appears, at a casual glance, to be too short a period for a 7 h 55 min flight.

Travelling east

As long as you do not cross the International Date Line (IDL), places to the east are later in the day, so travelling in the opposite direction to the Sun makes the time between departure and arrival seem longer than it actually is.

14 Calculating arrival times, travelling east

Three flights per week in summer depart the Gold Coast at 06:25 for Queenstown, NZ, with an expected flight time of 3 h 15 min.

(a) Determine the time in Queensland when the flight is expected to arrive in Queenstown.

THINKING	WORKING
1 Add the time elapsed to the departure time.	06:25 + 3 h 15 min → 09:40
2 Interpret the answer.	It will be 9:40 am at the Gold Coast when the flight is expected in Queenstown.

(b) Determine the ETA for the flight in Queenstown.

1 Identify the time zones for both locations. Consider DST, if necessary.	NZ observes DST. Gold Coast: AEST, UTC + 10 Queenstown: UTC + 13
2 Use the time difference to calculate the unknown time.	Queenstown is 3 h ahead of the Gold Coast. 9:40 am + 3 h → 12:40 pm
3 Interpret the answer.	The ETA, Queenstown time, is 12:40 pm.

WARNING

Do not assume that a return flight will have the same duration.

Prevailing winds from the west provide tail winds, making flight times heading east generally shorter.

Travelling across the IDL

If you travel east from Australia to anywhere in the Americas, you will cross the IDL into the western hemisphere, and immediately 'gain a day'. The time on your watch will stay the same, but you will change to the previous day.

If you travel back across the Pacific Ocean, the opposite will happen at the IDL. You will 'lose a day' and advance immediately to the next day.

Travel itinerary

Many trips involve connecting flights, with time spent at airports, or with overnight stays in the city where the airport is located.

A travel itinerary gives the departure and arrival times for each location.

15 Interpreting an itinerary

Nishi is flying from Brisbane to London via Auckland and Los Angeles, with the tail wind making shorter flight times, and stopovers at the airports giving her a chance to stretch her legs. Her itinerary is as follows:

Dep Brisbane 11:15 am, 21/01; arr Auckland 5:25 pm, 21/01; dep Auckland 10:50 pm, 21/01; arr Los Angeles 1:55 pm, 21/01; dep Los Angeles 4:10 pm, 21/01; arr London 10:40 am, 22/01.

(a) Determine the time in Brisbane when Nishi arrives in London.

THINKING	WORKING
1 Determine the time zone for each location. Consider the possibility of DST.	Neither Brisbane nor London have DST in January. Brisbane: AEST, UTC + 10 London: UTC
2 Calculate the time difference.	Brisbane is 10 h ahead of London.
3 Adjust the time for the given town. Adjust the date, if necessary.	10:40 am on 22/01 in London is: 10:40 am + 10 h → 8:40 pm (same day) in Brisbane.
4 Interpret the answer.	The Brisbane time when Nishi arrives in London is 8:40 pm on 22/01.

(b) Calculate the total flying time, in hours and minutes.

1 Identify the departure and arrival times, using times from the same location.	Brisbane times: Initial dep: 11:15 am, 21/01 Final arr: 8:40 pm, 22/01
2 Calculate the total time elapsed.	Time elapsed: 11:15 am (21/01) + 1 day → 11:15 am (22/01) 11:15 am + 25 min → 11:40 am (22/01) 11:40 am + 9 h → 8:40 pm (22/01) 1 day 9 h 25 min = 33 h 25 min
3 Calculate individual stopover times.	Auckland stopover: 5:25 pm to 10:50 pm is 5 h 25 min Los Angeles stopover: 1:55 pm to 4:10 pm is 2 h 15 min Nishi spends a total of 7 h 40 min between flights.
4 Subtract stopover times from total time elapsed to give the total flying time.	Total flying time: 33 h 25 min − 7 h 40 min → 25 h 45 min
5 Interpret the answer.	In total, Nishi spends 25 h 45 min flying.

(c) Determine the individual flight times.

1 Identify the time zone for each location. Consider the possibility of DST.	Brisbane: AEST, UTC + 10 Auckland, in summer: UTC + 13 Los Angeles: UTC – 8 London: UTC
2 Calculate the time differences.	Brisbane is 3 h behind Auckland. Auckland is 21 h ahead of Los Angeles. Los Angeles is 8 h behind London.
3 Adjust the local arrival time to the time zone of the place of departure. Adjust the date, if necessary.	5:25 pm in Auckland is 2:25 pm in Brisbane. 1:55 pm, 21 / 01 in Los Angeles is 10:55 am, 22 / 01 in Auckland. 10:40 am, 22 / 01 in London is 2:40 am, 22 / 01 in Los Angeles
4 Calculate the time elapsed using times from the same location of departure.	Brisbane to Auckland: 11.15 am to 2.25 pm is 3 h 10 min Auckland to Los Angeles: 10.50 pm, 21 / 01 to 10.55 am, 22 / 01 is 12 h 5 min Los Angeles to London: 4.10 pm, 21 / 01 to 2.40 am, 22 / 01 is 10 h 30 min
5 Check that the individual times give the same total as the original calculation.	3 h 10 min + 12 h 5 min + 10 h 30 min = 25 h 45 min

Comparing average speed

Comparing average speeds for flights travelling east with their return flight to the west gives some idea of the tail-wind advantage of flying east.

The average speed of an aircraft from take-off to landing, as calculate from the anticipated departure and arrival times, includes the inevitable time of 'sitting on the tarmac' and taxiing to and from the terminal. The average air speed once the aircraft is in the air will be somewhat higher, and the cruising speed at around 10 000 m will be higher still due to reduced air resistance.

When estimating average air speed, the assumption is made that aircraft follow the path of a great circle, the shortest distance between two points. While this gives a good approximation of the distance, pilots deviate from this path to follow 'highways' in the sky, lining up with a series of navigation points, which ensures substantial separation of planes flying in opposite directions. Smaller aircraft follow paths that hug the coastline as much as possible when flying overseas, so great circle approximations would not be suitable in those cases.

16 Estimating average air speed

The distance between Brisbane and Auckland is 2288 km. Calculate the average speed for each of the following flights, to the nearest whole number.

(a) Brisbane to Auckland, taking 3 h 10 min

THINKING	WORKING
1 Identify the distance travelled in km and the time taken in hours.	Brisbane to Auckland: 2288 km $3\text{ h }10\text{ min} = \left(3 + \frac{10}{60}\right)\text{h}$ $= 3\frac{1}{6}\text{ hours}$
2 Use the formula: speed $= \frac{\text{distance}}{\text{time}}$. Round, if necessary.	$\text{average speed} = \frac{\text{distance}}{\text{time}}$ $= \frac{2288\text{ km}}{3\frac{10}{60}\text{ h}}$ $= 723\text{ km/h (rounded)}$
3 Interpret the answer.	The average speed travelling from Brisbane to Auckland is approximately 723 km/h.

(b) Auckland to Brisbane, taking 3 h 40 min.

THINKING	WORKING
1 Use the formula: speed $= \frac{\text{distance}}{\text{time}}$. Round, if necessary.	$\text{average speed} = \frac{\text{distance}}{\text{time}}$ $= \frac{2288\text{ km}}{3\frac{40}{60}\text{ h}}$ $= 624\text{ km/h}$
2 Interpret the answer.	The average speed travelling from Auckland to Brisbane is approximately 624 km/h.

EXERCISE 4.4 Time and travel

Worked Example

13 **1** A flight departs Brisbane on Monday at 8:45 pm for Dubai, with an anticipated flight time of 14 h 35 min.

(a) Determine the time and day in Brisbane when the flight is expected to arrive in Dubai.

(b) Determine the ETA for the flight in Dubai, if the city is on Gulf Standard Time, GST of UTC + 4, all year.

2 A flight departs Mumbai on a Wednesday in January at 10:40 for Paris, with an anticipated flight time of 8h 40 min.

(a) Determine the time and day in Mumbai when the flight is expected to arrive in Paris.

(b) If Mumbai is UTC + 5:30 all year round, and Paris is UTC + 1 in January, determine the day and the ETA for the flight in Paris.

14 **3** A flight departs Perth, WA on a summer's day at 10:35 am for Sydney, with an anticipated flight time of 4 h 10 min.

(a) Determine the time in Perth when the flight is expected to arrive in Sydney.

(b) Determine the ETA for the flight in Sydney.

4 A flight departs London at 10:15 am for New York, with an anticipated flight time of 8 h 15 min.

(a) Determine the time in London when the flight is expected to arrive in New York.

(b) Determine the ETA for the flight in New York if neither location is observing DST.

5 A flight departs Auckland on a Friday in February at 8:55 pm for Honolulu, with an anticipated flight time of 8h 35 min.

(a) The time and day in Auckland when the flight is expected to arrive in Honolulu is

- A 5:30 pm Friday.
- B 5:30 am Saturday.
- C 5:30 pm Saturday.
- D 5:30 am Sunday.

(b) If Honolulu is UTC – 10 all year round, and Auckland is UTC + 13 in February, determine the day and the ETA for the flight in Honolulu.

- A 7:30 am Saturday
- B 6:30 am Saturday
- C 7:30 am Friday
- D 6:30 am Friday

16 **6** Use an app or the internet to determine the distance between Singapore and London, then calculate the average speed for each of the following flights, to the nearest whole number.

(a) Singapore to London: 14 h 5 min

(b) London to Singapore: 12 h 50 min

7 Use an app or the internet to determine the distance between Townsville and Mount Isa, then calculate the average speed for each of the following flights, to the nearest whole number.

(a) Townsville to Mount Isa: 1 h 40 min

(b) Mount Isa to Townsville: 1 h 35 min

8 Use an app or the internet to determine the distance between New York and Los Angeles, then calculate the average speed for each of the following flights, to the nearest whole number.

(a) New York to Los Angeles: 6 h 30 min

(b) Los Angeles to New York: 5 h 20 min

9 In February, a flight leaves Brisbane at 5:30 pm and its ETA in Perth is 9:05 pm.

(a) If tail winds reduce the flight time by 55 minutes travelling east, what will be the ETA in Brisbane for a flight leaving Perth at 12:50 pm?

A 3:30 pm B 5:20 pm C 7:30 pm D 9:20 pm

(b) Explain the common error made by a student who chose the first incorrect option.

10 Viktor is flying from Sydney to Las Vegas, via Honolulu and Los Angeles. His itinerary is as follows: Worked Example 15

Dep Sydney 9:40 pm on 02/11; arr Honolulu 10:45 am on 02/11; dep Honolulu 10:45 pm on 02/11; arr Los Angeles 6:20 am on 03/11; dep Los Angeles 9:25 am on 03/11; arr Las Vegas 10:49 am on 03/11.

(a) Determine the time and date in Sydney when Viktor arrives in Las Vegas, UTC – 8.

(b) Calculate the total flying time, in hours and minutes.

(c) Determine the individual flight times if Los Angeles and Las Vegas share a time zone and Honolulu is UTC – 10 all year.

11 Sabra is going on holidays to Rome, via Dubai. Her itinerary is as follows:

Dep Brisbane 8:45 pm on 14/02; arr Dubai 5:20 am on 15/02; dep Dubai 9:00 am on 15/02; arr Rome 12:40 pm on 15/02.

(a) Determine the time in Brisbane when Sabra arrives in Rome, UTC + 1.

(b) Calculate the total flying time, in hours and minutes.

(c) Determine the individual flight times, given that Dubai is UTC + 4.

12 Vladimir is flying home to St Petersburg from Edinburgh via Munich. He can choose from the following flights:

Edinburgh to Munich: 10:25 am → 1:40 pm or 12:55 pm → 4:10 pm on Friday; 1:15 pm → 4:30 pm on Sunday; 6:30 pm → 9:45 pm on Monday; Munich to St Petersburg 11:40 am → 4:25 pm, every day except Tuesday.

(a) Choose the Edinburgh to Munich flight that gives the shortest stopover time in Munich. Calculate the length of the stopover.

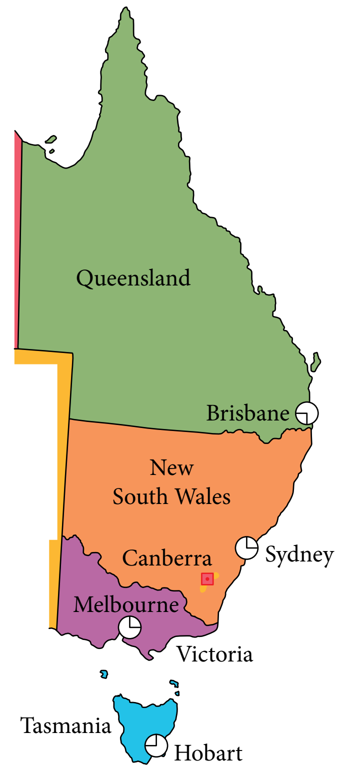
(b) Determine the individual flight times, given that Edinburgh is UTC, Munich is UTC + 1 and St Petersburg is UTC + 3.

13 On Christmas Eve, a flight leaves Brisbane for Melbourne at 11:55 am with an ETA of 3:15 pm.

Use the map to determine the estimated times of arrival, to the nearest five minutes, for each of the following flights. Explain any assumptions you made in answering the question.

(a) Departing Brisbane at 7:10 am for Sydney.

(b) Departing Brisbane at 2:10 pm for Hobart.



14 To return to Brisbane from Ho Chi Minh City, UTC + 7, Nick took the 20:55 flight to Bangkok, also UTC + 7, had a 1 h 40 min stopover, then caught the midnight flight, arriving home on time at 12:05 the next day. The last leg of the journey took him back over Ho Chi Minh City.

In future there may be direct flights from Ho Chi Minh City to Brisbane. If Nick had been able to fly directly home, leaving at the same time, what would his ETA in Brisbane have been? Explain any assumptions you made.

15 Flights from Brisbane to Christchurch, 43°32′S,172°38′E, take just 3 h 30 min for the 2504 km journey, with the return journey taking 3 h 50 min due to head winds.

Flights leave Christchurch in summer for the South Pole.

Even though different aircraft are used for these flights, use the given information to estimate the flight time to the South Pole.

Summary

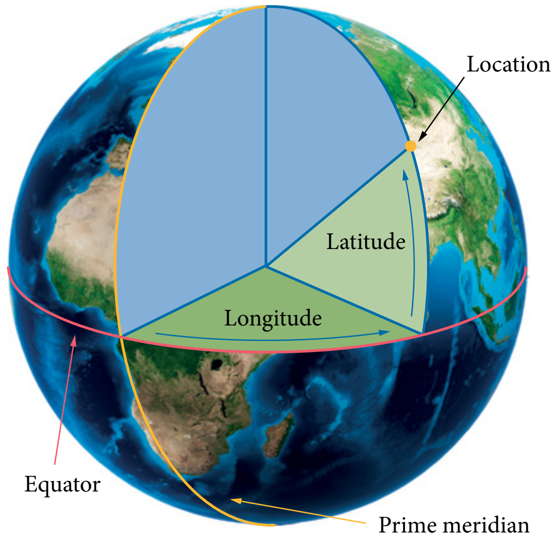
Latitude and longitude

The equator, at latitude 0°, is a great circle, sharing its centre with the centre of the Earth.

Parallels of latitude are smaller circles a number of degrees north or south of the equator, up to a maximum of 90° at the poles.

Meridians, or lines of longitude, go from pole to pole. Each meridian is a number of degrees east or west of the prime meridian, with longitude 0°, up to a maximum of 180° at the International Date Line.

Global positions are given in the order degrees north or south, then degrees east or west.



Local area maps

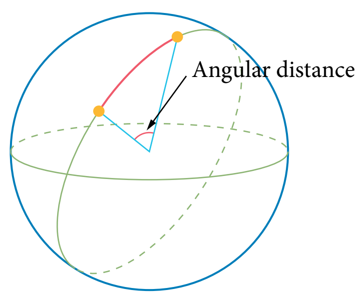
For greater accuracy each degree is broken into 60 minutes: $1° = 60'$

Angular distance

The distance between the two locations on the surface of the Earth is based on the angle at the centre of the Earth.

If two locations are on the same meridian, the angular distance can be calculated by either adding or subtracting the latitude angles.

Angular distance should not be more than 180°.



Distance in kilometres

The shortest distance between two locations on Earth is always along a great circle.

Distance in kilometres between locations on Earth:

$D = 111.2 \times$ angular distance

Distance along a parallel of latitude:

$D = 111.2 \cos(\theta) \times$ angular distance, where θ is the latitude and the angular distance is the angle between the meridians.

Longitude and time

The Earth revolves at $15° / \text{h}$.

Non-coordinated time difference is $\frac{\theta}{15}$ hours, where θ is the angular distance between the meridians.

Coordinated universal time (UTC or GMT)

The time zones UTC + 1, UTC + 2, … are centred on the meridians 15°E, 30°E, ….

The time zones UTC − 1, UTC − 2, … are centred on the meridians 15°W, 30°W, ….

Australian time zones

UTC + 8: Australian Western Standard Time, AWST

UTC + 9:30: Australian Central Standard Time, ACST

UTC + 10: Australian Eastern Standard Time, AEST

Daylight saving in Australia

The Australian Capital Territory, New South Wales, South Australia, Victoria and Tasmania observe daylight saving each summer. Western Australia, Queensland and the Northern Territory do not.

During DST, clocks are advanced by 1 hour.

Travelling east or west

A travel itinerary gives the departure and arrival times for each location.

Departure and arrival times are always 'local' times.

When travelling west, the elapsed time for the journey will seem shorter than it actually is from the departure and arrival times.

When travelling east, the elapsed time for the journey will seem longer than it actually is from the departure and arrival times.

International date line (IDL)

Time zones for locations west of the IDL are generally earlier in the day.

Crossing the IDL going east (to the western hemisphere): go back a day in time.

Crossing the IDL going west (to the eastern hemisphere): go forward a day in time.

Comparing average speed

Aircraft travelling east are generally faster due to the tail-wind advantage.

The average speed of an aircraft from take-off to landing, as calculated from the anticipated departure and arrival times, includes stationary time on the tarmac and time taxiing to and from the terminal.

Chapter review 4

1 Use a world atlas to locate each place, then estimate its coordinates to the nearest degree. Exercise 4.1

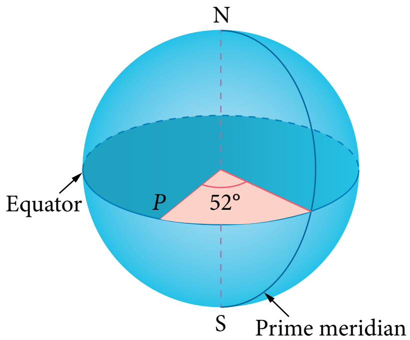
(a) Phnom Penh, Cambodia (b) Cincinnati, Ohio, USA (c) Prague, Czech Republic

2 Determine which continent you would be in at each of the following coordinates. Exercise 4.1

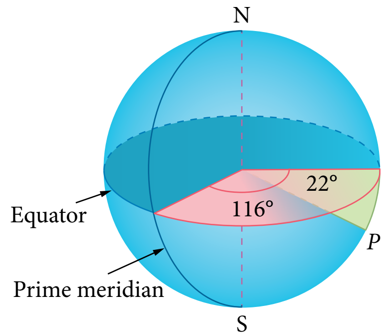
(a) 20°S, 120°E (b) 20°S, 50°W (c) 35°N, 10°E

3 Use the cut-away diagrams of the Earth to determine the coordinates of *P*. Exercise 4.1

(a)



(b)



4 Which hemisphere is common to China, Australia, Malaysia and Italy? Exercise 4.1

A southern B eastern C both southern and eastern D northern

5 Convert the coordinates of Kabul, Afghanistan, 34.56°N, 69.21°E to degrees and minutes, to the nearest minute. Exercise 4.1

6 Calculate the angular distance between the two locations. Exercise 4.2

(a) 5°S, 10°E and 53°N, 10°E (b) 18°S, 100°W and 25°S, 100°W

7 Calculate the angular distance between 14°4′N, 22°16′E and 5°12′N, 22°16′E. Exercise 4.2

8 Two locations on the equator are given as 0°, 99°2′E and 0°, 99°2′W. Exercise 4.2

The angular distance between the locations is

A 9°2′. B 18°4′. C 161°56′. D 198°4′.

9 Determine the angular distance around the parallel of latitude between each of the following locations. Exercise 4.2

(a) 18°S, 149°E and 18°S, 84°E (b) 43°7′N, 18°50′W and 43°7′N, 8°20′E

10 Calculate the distance between 5°N, 80°W and 31°S, 80°W, to the nearest kilometre. Exercise 4.2

11 Use an online great circle distance calculator or app to determine the distance between 16°N, 93°E and 27°S, 28°W. Exercise 4.2

12 Calculate the non-coordinated (non-UTC) time difference between *A*: 25°S, 16°W and *B*: 14°N, 29°E and identify which position is ahead. Exercise 4.3

Exercise 4.3 **13** Calculate the local non-coordinated (non-UTC) time difference between *A*: 38°N, 165°W and *B*: 24°N, 108°E and identify which position is ahead.

Exercise 4.3 **14** Calculate the time of day in the given town when it is 3:40 pm in Bowen in winter.

(a) Port Lincoln, SA **(b)** Canberra, ACT **(c)** Broome, WA

Exercise 4.3 **15** For ships at sea, local time is determined from the nearest meridian that is a multiple of 15° from the prime meridian. Determine the UTC time zone for ships in each of the following locations.

(a) 41.7°N, 49.9°W, the site of the wreck of the *Titanic*

(b) 36°S, 92°E, possible site of the wreck of flight MH370

Exercise 4.3 **16** A movie is to be shown in cinemas at 7:15 pm on Boxing Day throughout the USA. Determine the time and date in Queensland when the movie begins in each of the following US cities.

(a) Los Angeles: UTC – 8 **(b)** New York: UTC – 5

Exercise 4.4 **17** A flight departs Brisbane on Wednesday at 11:40 pm for Bangkok, with an anticipated flight time of 9 h 10 min.

(a) Determine the time and day in Brisbane when the flight is expected to arrive in Bangkok.

(b) Determine the ETA for Bangkok, UTC + 7.

Exercise 4.4 **18** A flight departs Adelaide on a winter evening at 7:45 pm for Hobart, with an anticipated flight time of 1 h 50 min.

(a) Determine the time in Adelaide when the flight is expected to arrive in Hobart.

(b) Determine the ETA for the flight in Hobart.

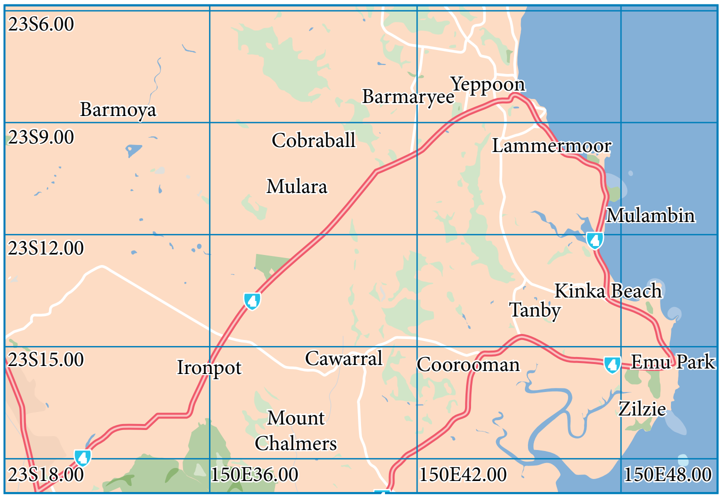
Exercise 4.4 **19** Use an app or the internet to determine the distance between San Diego, California and Atlanta, Georgia, then calculate the average speed for each of the following flights, to the nearest whole number.

(a) San Diego to Atlanta: 4 h 10 min **(b)** Atlanta to San Diego: 4 h 55 min

Exercise 4.1 **20** Use the local Rockhampton district map to estimate the coordinates of each location in degrees and minutes. Assume the middle of the printed word gives a suitable focus point.

(a) Lammermoor

(b) Ironpot



21 Calculate the distance along the parallel of latitude between Münster, Germany, 52°N, 7.6°E and Arnhem, Netherlands, 52°N, 5.9°E, to the nearest kilometre. Exercise 4.2

22 Calculate the distance along the parallel of latitude between Aden, Yemen, 13°N, 45°1′E and San Salvador, El Salvador, 13°N, 89°13′W, to the nearest 100 km. Exercise 4.2

23 Calculate the distance along the parallel of latitude between 25°S, 151°E and 25°S, 161°W, to the nearest kilometre. Exercise 4.2

24 Ships at sea calculate local time coordinated to the closest meridian that is a multiple of 15°. Exercise 4.3

The time difference between two container vessels in the Pacific Ocean at 45°27′N, 139°3′W and 18°5′N, 155°23′E is

A 4 h 22 min. **B** 5 h. **C** 19 h. **D** 19 h 38 min.

25 A flight leaves Los Angeles at 6:00 am with ETA in New York 2:15 pm. Exercise 4.4

If the return flight takes an extra 75 minutes, New York is UTC – 5 and Los Angeles is UTC – 8, what will be the ETA in Los Angeles for a flight leaving New York at 7:00 am?

A 7:30 am **B** 10:30 am **C** 1:30 pm **D** 4:30 pm

26 Benji determines his location outside Oslo to be 60°12′N, 10°5′E. Determine the coordinates of the four locations Benji would be at if he travelled 500 km north, south, east or west. Write each set of coordinates in degrees and minutes, to the nearest minute. Exercise 4.2

27 The distance between locations that are reasonably close can be estimated by thinking of the Earth as flat, with the parallels of latitude forming right angles with the meridians. Exercise 4.2

Estimate the distance between Nouméa, New Caledonia, 22.26°S, 166.45°E and Suva, Fiji, 18.12°S, 178.44°E. Explain any assumptions you made, apart from the ones already stated.

28 Some friends are exploring the possibility of meal-time video call sessions while some of them are travelling. They decide that 6 am, 12 noon and 6 pm will be the start times for any chat. Exercise 4.3

The cities and their time zones are as follows: Brisbane, UTC + 10; Wellington, UTC + 12; Beijing, UTC + 4; London, UTC; New York, UTC – 5; Los Angeles, UTC – 8; Perth, UTC + 8; Tokyo, UTC + 9; Paris, UTC + 1; Tel Aviv, UTC + 2; Honolulu, UTC – 10; Rio de Janeiro, UTC – 3.

Determine all possibilities for meal-time chats between any pair of listed places.

29 Flights from Brisbane to Perth take 5 h 25 min, while Perth to Brisbane takes just 4 h 35 min for the 3606 km journey. Even though the type of aircraft might vary, use the given information to estimate the time saved in flying around the equator in an easterly direction compared with flying west. Explain any assumptions you made in your calculations and the effect they might have on your conclusion. Exercise 4.4

Exam review

UNIT 3

Paper 1: Simple familiar

Exercise 1.1

1 A survey was conducted to see what portion of Year 12 boys and girls played inter-school sport in the previous year, with the following results:

Boys: both semesters (48), one semester only (11), neither semester (82)

Girls: both semesters (39), one semester only (27), neither semester (95)

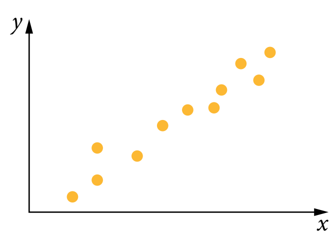
(a) Organise the data into a two-way table with gender as the columns and calculate the marginal totals.

(b) Convert all frequencies to relative frequencies. Write answers as percentages, to the nearest whole percentage.

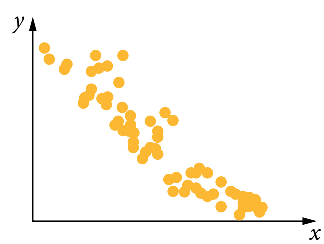
Exercise 1.2

2 Describe the associations, if any, in each of the following scatter plots.

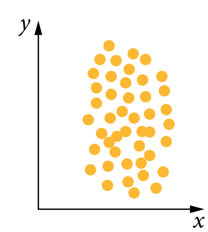
(a)



(b)



(c)

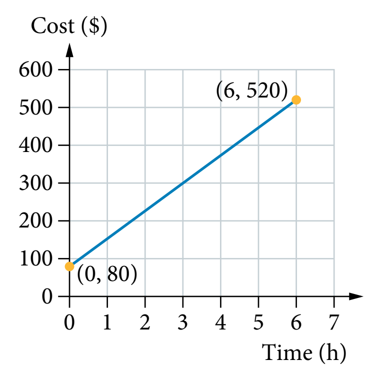


Exercise 1.4

3 The quote provided by a tradesman depends on the amount of time needed to complete the job. The cost is summarised in the graph.

(a) Determine the fixed cost per job.

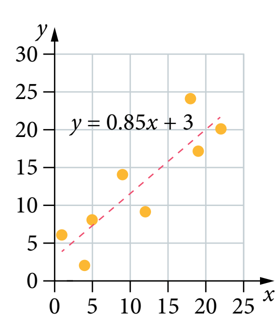
(b) To the nearest dollar, calculate the hourly rate the tradesman charges.



Exercise 1.5

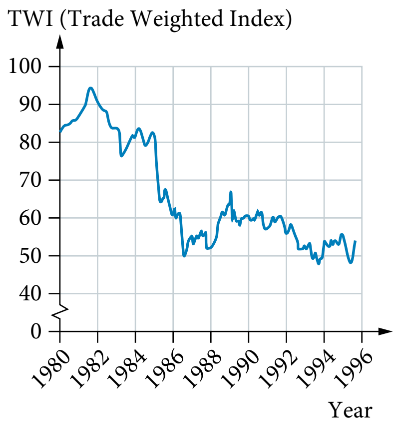
4 Use the table of values, the scatter plot and the linear regression equation to make a table of residuals, and then draw a residual plot.

x	1	4	5	9	12	18	19	22
y	6	2	8	14	9	24	17	20

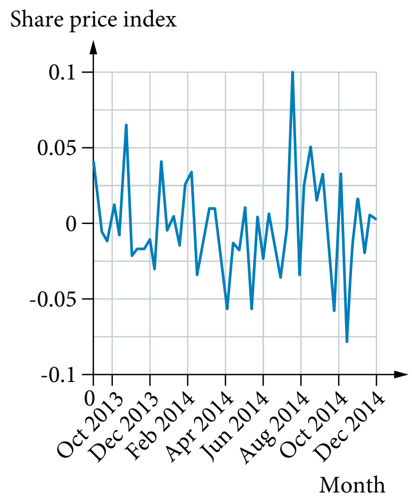


5 Comment on the pattern and trend in each of the following time series graphs. Exercise 2.1

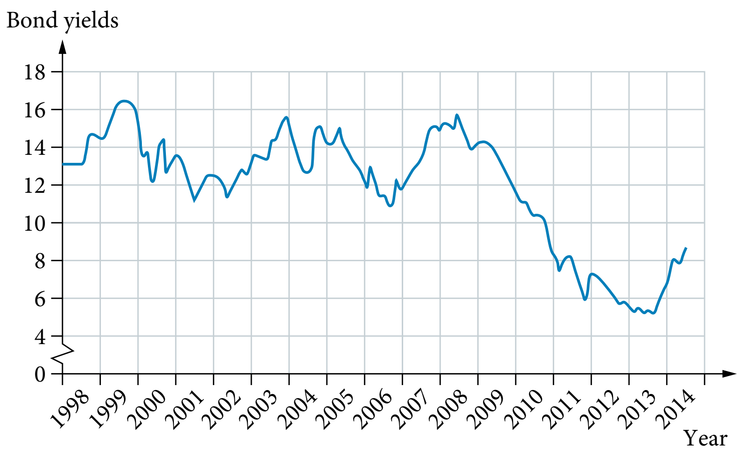
(a) Exchange rate



(b) Monthly stock performance



(c) Government bond yields



6 The table represents the average sale price of a new car over a 40 year period. Exercise 2.1

Year	1950	1960	1970	1980	1990
Sale price ($)	1 510	2 600	3 450	7 200	16 950

(a) Using a suitable scale, construct a graph of the time series data.

(b) Add a trend line to the graph and comment on its direction.

7 Migration numbers to the USA from 1904 to 1911 are given in the table. Exercise 2.2

Year	1904	1905	1906	1907	1908	1909	1910	1911
Migration ('000s)	812.9	26.5	100.5	285.3	782.9	751.8	41.6	878.6

(a) Determine the 3-point moving means, to 1 decimal place.

(b) Plot both the raw data and the moving mean data on the same set of axes. Comment on the results.

Exercise 2.3

8 Calculate the value(s) of each unknown seasonal index.

(a)

Quarter	Spring	Summer	Autumn	Winter
Seasonal index	0.27	1.41	x	0.80

(b)

2 month period	Jan–Feb	Mar–Apr	May–Jun	Jul–Aug	Sep–Oct	Nov–Dec
Seasonal index	x	$3x$	$5x$	x	$\frac{x}{2}$	$2x$

Exercise 3.1, 3.3

9 For each sequence, determine:

(a) the common difference d and the 5th term for the arithmetic sequence: 5, 9, 13, 17, …

(b) the common ratio r and the 5th term for the geometric sequence: -3, 6, -12, 24, …

Exercise 3.1, 3.3

10 Determine the seventh term of the following sequences:

(a) an arithmetic sequence with first term -16 and a common difference of 5

(b) a geometric sequence with first term -8.5 and a common ratio of 4

(c) a geometric sequence with fourth term 34.3 and a common ratio of 7.

Exercise 3.2

11 The number of visitors to a park during a fortnight of school holidays begins with 3000 on Monday and increases by 150 each day.

How many people visit the park on the eighth day (the second Monday) of the school holidays?

Exercise 4.1

12 Use a world atlas to locate the cities with the given coordinates.

(a) 60°N, 24°E **(b)** 12°S, 77°W **(c)** 35°N, 98°W

Exercise 4.2

13 Calculate the distance between each pair of locations, to the nearest kilometre.

(a) 22°N, 118°W and 31°N, 118°W **(b)** 16°22′N, 11°25′E and 9°50′S, 11°25′E

Exercise 4.4

14 A flight departs Melbourne on a Wednesday in November at 8:55 pm for Los Angeles, with an anticipated flight time of 14 h 20 min.

(a) Determine the time and day in Melbourne when the flight is expected to arrive in Los Angeles.

(b) If Los Angeles is UTC – 8, determine the day and the ETA for the flight in Los Angeles.

Paper 2: Complex familiar and complex unfamiliar

Exercise 1.3

1 The prices of two shares seem to have been moving in the same direction for the past 10 months.

HJ shares	3.85	3.9	4	4.05	4.1	4.15	4.4	4.45	4.6	4.65
WZ shares	7.2	7.6	7.4	7.4	8.1	8.1	7.8	8	8.4	8.2

Assuming a linear association, with the explanatory variable appearing first on the table, calculate the correlation coefficient and least-squares equation, to 2 decimal places.

2 The table shows the quarterly sales figures over a 3 year period. Exercise 2.2

Year	2016				2017				2018			
Quarter	1	2	3	4	1	2	3	4	1	2	3	4
Sales	300	250	480	630	280	270	500	650	290	300	520	690

Smooth the data using a 4-point moving mean. Centre where necessary.

3 Calculate the seasonal indices for each 3 month period over the year. Give all answers to 2 decimal places. Exercise 2.3

Year	Sales ($)				
	Dec–Feb	Mar–May	Jun–Aug	Sep–Nov	Total
2016	28	27	21	34	110
2017	31	22	17	35	105
2018	29	26	24	41	120

4 Determine an expression for the nth term of each sequence, and calculate the value of the specified term. Exercise 3.1, 3.3

(a) 1701, 567, 189, 63, …; t_8

(b) 71, 150, 229, 308, …; t_{13}

5 Mrs Gander deposits $15 000 in an account that pays 7.9% compound interest each year. The yearly interest is added to her account automatically. Exercise 3.4

(a) How much will be in Mrs Gander's account after 10 years?

(b) How many years after opening the account would it take for Mrs Gander's investment to triple?

6 Using the time zone map in Chapter 4 or online time zone information, calculate the time in Nambour when it is 2:40 pm in each of the following cities. Assume no DST. Exercise 4.3

(a) Ontario, Canada

(b) Shenzhen, China

(c) Timaru, NZ

7 A set of data was analysed for linear association, and the conclusion was that between 30% and 40% of the variation in y was attributable to variation in x. The data is given here, but the y-value when $x = 22$ was not legible: Exercise 1.6

(15, 45), (16, 55), (18, 50), (19, 60), (20, 75), (21, 55), (22, _), (24, 60)

Determine a possible value for the illegible data point.

8 A young couple are waiting for their investments to increase in value so they can make a deposit of $200 000 on their first home. They have $90 000 in a term deposit that pays a flat rate of 8.5% p.a., with the interest added to the account at the end of each year, but not considered part of the principal. They have $50 000 in an investment account earning 6.8% p.a. interest compounded monthly. Exercise 3.2, 3.4

If the money was invested in mid-June of 2018, when will the couple be able to buy a home?

9 On a map of Western Australia, Esperance, 33°52′S, 121°53′E is 48 mm below and 60 mm to the right of Geraldton, 28°47′S, 114°36′E. Port Hedland is 80 mm above and 35 mm to the right of Geraldton. Calculate the coordinates of Port Hedland, to the nearest minute. Exercise 4.1


2025

5

Compound interest loans and investments

5 Recall

Convert a fraction or decimal to a percentage

1 Convert the following values to percentages.

(a) $\frac{4}{5}$ (b) 1.075 (c) 0.0375

Convert a percentage to a decimal number

2 Convert each of the following percentages to a decimal number.

(a) 9% (b) 15.2% (c) 2.5%

Determine one number as a percentage of another number

3 Determine what percentage the first number is of the second.

(a) 15, 60 (b) 4.5, 45 (c) 24, 12

Calculate a stated percentage of a number

4 Determine the stated percentage for each of the following.

(a) 8% of 400 (b) 35.4% of 2400 (c) 250% of $560

Increase a number by a percentage

5 Increase the number by the given percentage.

(a) 72 by 20% (b) 36.45 by 15% (c) 500 by 6.5%

Solve equations involving indices

6 Solve each equation using trial and error, giving answers in the form required.

(a) $2.5^x = 58$: $a < x < b$, where a and b are consecutive whole numbers

(b) $500 \times 1.05^t = 1000$: $a < t < b$, where a and b are consecutive whole numbers

(c) $240 \times R^7 = 360$, to 2 decimal places

Use recursion to generate a geometric sequence

7 Generate the first four terms of the following sequences.

(a) $t_1 = 200, t_{n+1} = 1.5t_n$ (b) $t_1 = 5000, t_{n+1} = 0.8t_n$

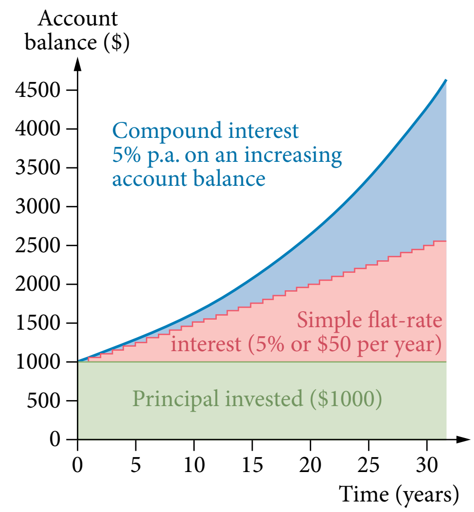
Compound interest and recurrence relations 5.1

Compound interest

Compound interest on a loan or investment is calculated periodically. It is a percentage of the account balance at the time the calculation is made.

Adding interest each period increases the account balance for the next interest calculation. The result is a greater amount of interest for subsequent periods. This is called *compound interest*.

The graph demonstrates the principal amount, with simple interest (or flat rate interest) where a fixed amount of interest (based on the principal invested) is added to the account, and compound interest. The amount of interest earned in the compound interest account increases each compounding period. This is because the interest is paid on the increasing account balance (not the principal invested).



Simple interest is added at the same rate, a fixed-dollar amount each year, once per year, and hence the data is often compared using column graphs or scatter plots.

This is all based on the assumption that the interest rate is fixed (not variable) for the duration of the investment (or loan) and that money is not added to or subtracted from the account.

Tabulating an account

A progressive table of account balances can be helpful in illustrating compounding growth.

The Ersatz Wealth Company is trying to persuade Kim to invest $2000 in one of their accounts. They offer interest at 20% of the increasing investment account balance, calculated annually. The following table outlines growth in an account balance that increases by 20% each year.

Year n	End balance for year n, A_n	Interest calculation for next balance	Interest to increase the balance	Calculation of next balance	Next balance A_{n+1}
0	$A_0 = \$2000$	$0.2 \times \$2000$	$400	$2000 + 400$	$A_1 = \$2400$
1	$A_1 = \$2400$	$0.2 \times \$2400$	$480	$2400 + 480$	$A_2 = \$2880$
2	$A_2 = \$2880$	$0.2 \times \$2880$	$576	$2880 + 576$	$A_3 = \$3456$
3	$A_3 = \$3456$	$0.2 \times \$3456$	$691.20	$3456 + 691.20$	$A_4 = \$4147.20$
4	$A_4 = \$4147.20$	$0.2 \times \$4147.20$	$829.44	$4147.20 + 829.44$	$A_5 = \$4976.64$

The table separates the calculations for the interest earned each year and the account balance at the end of the year. Interest added each year is greater than that added the previous year. The sequence of account balances is a geometric sequence.

A_n: the account balance after the nth calculation of interest.

A_0: the principal, or the initial account balance after zero additions of interest.

Growth factor, *r*

A single growth factor r can be used to obtain the account balance in any year from the account balance in the previous year.

When an account balance A_n increases by a percentage, then $A_{n+1} = r \times A_n$, where r is a growth factor, calculated as follows:

If an account balance increases by i% per period, the growth factor $r = 1 + i$, where i is expressed as a decimal.

When an account balance increases by 20% each compounding period:

$$\begin{aligned} r &= 1 + 0.2 \\ &= 1.2 \end{aligned}$$

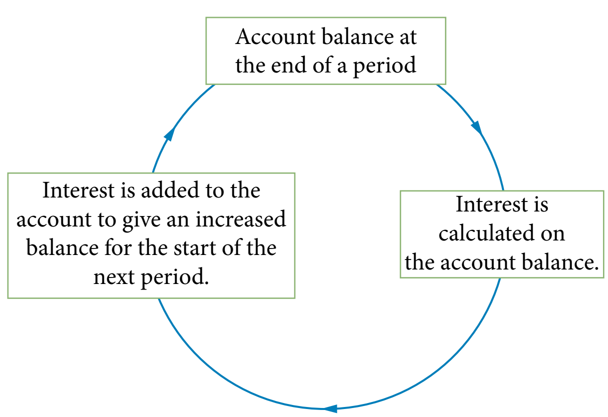
Multiply the balance at the end of any period by 1.2 to determine the balance at the end of the next period.

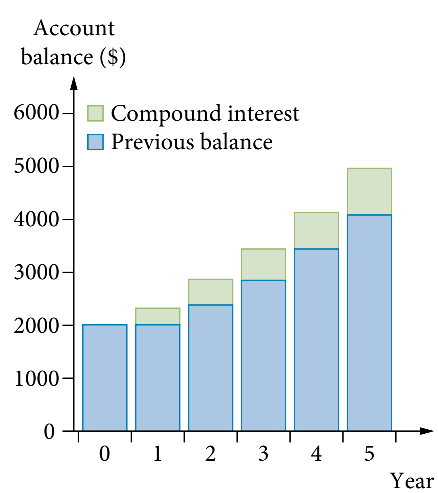
The table shows these calculations of successive account balances from the balance in the previous year, using the growth factor $r = 1.2$.

Year n	Balance at end of year n, A_n	Calculation to find the next balance	Next balance A_{n+1}
0	$A_0 = \$2000$	$1.2 \times 2000 = 2400$	$A_1 = \$2400$
1	$A_1 = \$2400$	$1.2 \times 2400 = 2880$	$A_2 = \$2880$
2	$A_2 = \$2880$	$1.2 \times 2880 = 3456$	$A_3 = \$3456$
3	$A_3 = \$3456$	$1.2 \times 3456 = 4147.2$	$A_4 = \$4147.20$
4	$A_4 = \$4147.20$	$1.2 \times 4147.20 = 4976.64$	$A_5 = \$4976.64$

The graph illustrates the geometric growth of the annual interest and of the account balance value of this compound interest investment.

The diagram shows the recurring cycle of compounding growth.





The cycle means that a recurrence relation can be used to model compound interest growth.

If an investment of $2000 earns interest of 20% each compounding period, the recurrence relation can be written as $A_0 = 2000,\ A_{n+1} = 1.2A_n$.

1 Recurrence relation for a compound interest loan

Write a recurrence relation to model the growth of each loan account balance.

(a) A loan of $8500 with an interest rate of 6% p.a., interest compounded annually.

THINKING	WORKING
1 Identify the parameters. Ensure the period and the rate of interest use the same unit of time.	Principal: $8500 Period: year Rate of interest per period: $i = 6\%$ p.a. $= 0.06$
2 Use the interest rate per period to determine the growth factor.	The balance increases by 6% each year. $r = 1 + i$ $= 1 + 0.06$ $= 1.06$
3 Define the *n*th balance. Write the recurrence relation.	Let L_n be the account balance after n years. $L_0 = 8500,\ L_{n+1} = 1.06 \times L_n$

(b) A loan of $6200 with an interest rate of 7.2% p.a., interest compounded quarterly.

1 Identify the parameters. Ensure the period and the rate of interest use the same unit of time.	Principal: $6200 Period: quarter Rate of interest per period: $i = 7.2\%$ p.a. $= \frac{7.2}{4}\%$ per quarter $= 0.018$
2 Use the interest rate per period to find the growth factor.	$r = 1 + i$ $= 1 + 0.018$ $= 1.018$
3 Define the *n*th balance. Write the recurrence relation.	Let L_n be the account balance after n quarters. $L_0 = 6200,\ L_{n+1} = 1.018 \times L_n$

2 Use a recurrence relation to determine the interest rate

Determine the rate of interest applied to the account balance of a loan modelled by $L_0 = 3300$, $L_{n+1} = 1.048 \times L_n$ with interest charged every six months.

THINKING	WORKING
1 Identify the parameters. Ensure the period and the rate of interest use the same unit of time.	Principal: \$3300 Period: six months Rate of interest: $i = i\%$ per six months $= 2 \times i\%$ p.a.
2 Use the growth factor to calculate the rate of interest for each compounding period.	$L_{n+1} = 1.048 \times L_n$ $= (1 + 0.048) L_n$ Given: $L_{n+1} = rL_n$ and $r = 1 + i$ Hence: $i = 0.048$ $= 4.8\%$ per period The rate of interest per six months is 4.8%.
3 Determine the annual interest rate.	Annual interest rate $= \frac{4.8\%}{6} \times 12$ $= 9.6\%$ p.a. The interest rate is 9.6% p.a.

3 Use a recurrence relation to determine an account balance

The recurrence relation $A_0 = 2000$, $A_{n+1} = 1.025 \times A_n$ models the growth in a bank account balance where compound interest is added each year.

(a) Use the recurrence relation to determine the balance of this account at the end of each year for three years.

THINKING	WORKING
1 Use the recursive formula with successive values of n, where A_0 represents the initial amount.	$A_0 = 2000$, $A_{n+1} = 1.025 \times A_n$ Substitute $n = 0$ where $A_0 = 2000$: $A_1 = 1.025 \times A_0$ $= 1.025 \times 2000$ $= \$2050$ Substitute $n = 1$ where $A_1 = 2050$: $A_2 = 1.025 \times A_1$ $= 1.025 \times 2050$ $= \$2101.25$ Substitute $n = 2$ where $A_2 = 2101.25$: $A_3 = 1.025 \times A_2$ $= 1.025 \times 2101.25$ $= \$2153.78$ (2 d.p.)

2 Interpret the results. | The account balance at the end of each of the first three years is: \$2050, \$2101.25 and \$2153.78 respectively.

(b) Determine the amount of interest earned over the three years.

1 Calculate the difference between the balances at the start and end of the term.

Interest earned:

$$\begin{aligned} A_3 - A_0 &= 2153.78 - 2000 \\ &= \$153.78 \end{aligned}$$

2 Interpret the result.

Over the three years, the account earned a total of \$153.78 in interest.

4 Interpret the values in a table

The table shows the growth of an investment account. Rounding to the nearest cent has been used where appropriate.

Determine the annual rate of interest.

Quarter n	Balance at end of quarter n, A_n (\$)
0	7240.00
1	7326.88
2	7414.80
3	7503.78

THINKING | WORKING

1 Use $A_{n+1} = r \times A_n$ with any pair of values to calculate r.

$$\begin{aligned} r &= \frac{7326.88}{7240} \\ &= 1.012 \end{aligned}$$

Verify a constant or fixed interest rate $\frac{A_{n+1}}{A_n}$:

$$\frac{A_2}{A_1} = \frac{7414.80}{7326.88} = 1.012 \qquad \frac{A_3}{A_2} = \frac{7503.78}{7414.80} = 1.012$$

2 Use $r = 1 + i$ to calculate i.

$$\begin{aligned} 1 + i &= 1.012 \\ 1 + i &= 1 + 0.012 \\ i &= 0.012 \\ i &= 1.2\% \end{aligned}$$

3 Interpret the result.

The quarterly rate of interest is 1.2%.

4 Convert the rate per quarter to an annual rate.

$$\begin{aligned} \text{Annual interest rate} &= 1.2\% \text{ per quarter} \times 4 \text{ quarters} \\ &= 4.8\% \text{ p.a.} \end{aligned}$$

The compound interest rate is 4.8% p.a.

Spreadsheets and recursive sequences

Spreadsheets can generate a recursive sequence of account balances modelled by a recurrence relation.

5 Using a spreadsheet to determine an account balance

\$5000 is invested at 5.4% p.a., compounding monthly. Use a recurrence relation and a spreadsheet to determine the account balance at the end of three years.

THINKING

1 Determine the interest per period and the corresponding growth factor.

WORKING

$$i = \frac{5.4\% \text{ p.a.}}{12} = \frac{0.054}{12} \text{ per month}$$

$$r = 1 + \frac{0.054}{12} = 1.0045$$

2 Determine the recurrence relation.

$A_0 = 5000,\ A_{n+1} = 1.0045 \times A_n$

3 Determine the number of periods of growth.

$$n = 3 \text{ years} = 3 \times 12 \text{ months} = 36 \text{ months}$$

4 Open an Excel spreadsheet.

Enter period values from zero to n.

Here that will be from 0 to 36.

Enter the initial investment value next to $n = 0$ in the Balance column. Here it will be 5000.

Enter a formula in cell B3 to multiply the value in the cell above by r.

Here the formula will be = B2 * 1.0045.

Click and drag down to match the final value of n.

	A	B
1	n	Balance ($)
2	0	5000.00
3	1	5022.50
4	2	5045.10
5	3	5067.80
6	4	5090.61
7	5	5113.52
	…	
35	33	5798.53
36	34	5824.63
37	35	5850.84
38	36	5877.16

5 Interpret the results.

The account balance at the end of three years (36 months) is \$5877.16.

When you use a spreadsheet, if the value of i involves a repeating decimal, key in the formula as a calculation. If the example above had used 5.4% p.a., compounding monthly, the formula in cell B3 would be = B2 * (1 + 0.054 / 12).

WARNING

Even though a spreadsheet shows rounded values, it uses values to many more decimal places.

Hence a spreadsheet's end result may differ slightly from calculations done manually, with rounding. The = ROUND(,) function eliminates this difference.

EXERCISE 5.1 Compound interest and recurrence relations

Worked Example 1

1 Determine a recurrence relation for each of the following.

(a) A loan of \$15 000 has an interest rate of 5.65% p.a. with interest compounded annually. Let L_n be the loan balance at the end of year n.

(b) An investment of \$11 000 has interest added monthly at 8.25% p.a. of the account balance. Let Q_n be the account balance at the end of month n.

(c) Pili borrows \$1000 with compound interest charged monthly at a rate of 2% per month. Let P_n be Pili's loan balance at the end of month n.

(d) Sara opened a savings account with \$467.45. Interest is paid annually on the account balance at a rate of 3.6% p.a. Let S_n be the savings balance at the end of year n.

(e) Ming takes a loan to buy a motorbike. She borrows \$5600 and will pay off the entire loan and any interest added after four years. Interest is charged on the account balance each month at a rate of 8.4% p.a. Let M_n be Ming's loan balance at the end of month n.

(f) Zara has an investment worth \$17 950 and earning an interest rate of 7.2% p.a. compounding monthly. Let L_n be Zara's investment balance at the end of month n.

2 Ella bought a car for \$14 000. She paid a cash deposit of \$2200 and borrowed the balance at an interest rate of 8.48% p.a. compounding every 6 months. Ella intends to pay off the loan and all added interest in 2 or 3 years.

(a) Which recurrence relation models the value of Ella's loan balance at the end of each half year?

A $E_0 = 11\,800,\ E_{n+1} = 1.0424 \times E_n$

B $E_0 = 14\,000,\ E_{n+1} = 1.0424 \times E_n$

C $E_0 = 11\,800,\ E_{n+1} = 1.0848 \times E_n$

D $E_0 = 14\,000,\ E_{n+1} = 1.0848 \times E_n$

(b) Explain the common error made by a student who had a growth rate of 1.0848 in their solution.

3 Determine the rate of interest modelled by each of the following recurrence relations in which interest is compounded annually.

(a) $L_0 = 10\,000,\ L_{n+1} = 1.092 \times L_n$

(b) $L_0 = 8550,\ L_{n+1} = 1.045 \times L_n$

(c) $L_0 = 900,\ L_{n+1} = 1.107 \times L_n$

(d) $L_0 = 1300,\ L_{n+1} = 1.155 \times L_n$

4 Inga invested \$5000 in an account to save for her planned overseas holiday. Interest is added to the account at the end of every month. The table models the growth in the value of her investment for the first 4 months. The values shown have been rounded to the nearest cent.

Month n	Balance at end of month n, A_n
0	\$5000.00
1	\$5000 × 1.008 = \$5040.00
2	\$5040 × 1.008 = \$5080.32
3	\$5080.32 × 1.008 = \$5120.96
4	\$5120.96 × 1.008 = \$5161.93

(a) Determine the monthly rate of interest.

(b) Write a recurrence relation for Inga's investment.

(c) How much will be in the account after 5 months?

5 The recurrence relation $A_0 = 7000,\ A_{n+1} = 1.07 \times A_n$ models the growth in a bank account balance where compound interest is added each year. 3

(a) Use the recurrence relation to determine the balance of this account at the end of each year for 3 years.

(b) Determine the interest earned in 3 years.

6 A loan of \$2000 has interest charged at 6% p.a. compounding every month.

(a) Write a recurrence relation that represents the growth of the balance of this loan every month.

(b) How much interest had been added to the loan account by the end of the second month?

7 An investment of \$12 800 earns compound interest of 5.6% p.a. calculated every 6 months. Which of the following recurrence relations will give the account balance after each 6 month period?

A $A_0 = 12800,\ A_{n+1} = 0.0047 \times A_n$

B $A_0 = 12800,\ A_{n+1} = 1.028 \times A_n$

C $A_0 = 12800,\ A_{n+1} = 0.056 \times A_n$

D $A_0 = 12800,\ A_{n+1} = 1.102 \times A_n$

8 Adele bought a car for which she had to borrow \$17 000 at 6.5% interest compounded annually. She will repay the entire loan with added interest at the end of 3 years. How much interest will she pay?

9 Krishna has invested \$125 000 with interest calculated monthly on the increasing balance at the rate of 7.2% p.a. Determine the amount of interest that is added to the account at each of the following times.

(a) at the end of the first month

(b) at the end of the second month

10 To renovate their home, Chris and Teri borrowed \$60 000 at 6.8% p.a. Interest is added every 6 months on the outstanding loan account balance. They do not make any repayments in the first 2 years.

(a) Construct a table showing, for the first year of this loan, the half-year (n) and the balance at the end of the half-year (A_n).

(b) What is the loan account balance at the end of the first year?

(c) How much total interest has been added to the account in the first year?

(d) How much interest is added in the 6 months after the first year?

11 A new savings account was started with a deposit of \$2834.75 at the end of December. Compound interest, at a rate of 4.4% p.a., will be earned every 3 months.

(a) Write a recurrence relation that represents the growth of the balance of this account.

(b) Determine the balance of the account at the end of March.

(c) Determine the balance of the account at the start of April.

(d) Explain why the account balance at the end of May is the same as the balance at the end of March.

(e) What is the value of the balance in this account after 9 months?

(f) How much interest would this account earn in the first 9 months?

12 Aaron borrowed \$780 at the start of June. Interest is added to the loan account at 7.2% p.a. calculated monthly. Let B_n be the loan account balance at the end of each month.

(a) What does B_1 represent?

(b) Write a recurrence relation that models the growth in the balance of this loan account each month.

(c) If B_k gives the balance at the beginning of October, determine the value of k.

(d) Determine the account balance at the start of August.

13 A \$3500 investment earns 6.25% interest each year, compounded annually. Determine the value of this investment after 5 years.

Worked Example 2

14 Determine the annual rate of interest modelled by each of the following recurrence relations.

(a) $L_0 = 6000,\ L_{n+1} = 1.052 \times L_n$, where interest is charged each 6 months

(b) $L_0 = 3250,\ L_{n+1} = 1.045 \times L_n$, where interest is charged quarterly

(c) $L_0 = 640$, $L_{n+1} = 1.0052 \times L_n$, where interest is charged every month

(d) $L_0 = 107.50$, $L_{n+1} = 1.0025 \times L_n$, where interest is charged every quarter

(e) $S_0 = 442$, $S_{n+1} = 1.005 \times S_n$, where interest is added every month

Worked Example 4

15 The table shows the growth of an investment account n quarters after it started. The values have been rounded to the nearest cent, where appropriate.

Determine the annual rate of interest.

Quarter n	Balance at end of quarter n, A_n ($)
0	5360.00
1	5435.04
2	5511.13
3	5588.29

16 A recurrence relation that models the growth in an investment every month is $A_0 = 6780$, $A_{n+1} = 1.006A_n$.

(a) What is the annual rate of interest applied to this investment?

(b) The table shows the investment account balance at the end of the first 3 months. Identify the error in the table, correct it and explain what the correct value represents.

Month n	Balance at end of month n, A_n ($)	Interest ($)
0	6780.00	
1	6820.68	40.68
2	6861.60	40.92
3	6902.77	41.17
4	6944.19	43.42

5

17 Compound interest at 4.2% p.a. is paid monthly on a savings account that started with $6380.

(a) Write a recurrence relation to represent this situation.

(b) Use a spreadsheet to determine the account balance after 2 years.

18 A loan of $1200 has compound interest charged quarterly at 7.4% p.a. Use a recurrence relation and a spreadsheet to determine the value of this loan after 5 years.

19 An investment of $480 000 earns compound interest at a rate of 7.55% p.a. compounded monthly. Use a spreadsheet to determine the number of months it will take for this investment to grow to $600 000.

20 Alice needs to borrow $45 000. Compound interest will be calculated at 8.4% p.a.

Use a spreadsheet to find the loan account balance after 4 years if interest is compounded:

(a) half-yearly (b) quarterly (c) monthly.

21 To buy a refrigerator, Sandy agrees to a $1200 loan from the Gotcha Home Superstore to be repaid at the end of 2 years. Interest is calculated and added each month, at 4% of the increasing monthly balance. Use a recurrence relation and a spreadsheet to determine when the account balance is first expected to exceed $2000.

22 Grigor is offered interest at 4.25% p.a. compounded monthly if he invests $10 000 for 3 years.

(a) Generate a sequence and determine the value of this investment after 3 years.

(b) After investing his $10 000, Grigor found that he might be able to negotiate a higher interest rate of 4.5% p.a. instead of 4.25% p.a. At what stage must Grigor transfer to the higher rate in order to earn at least $50 more interest than under the previous rate?

23 One account is set up with $5000 earning interest at 6% p.a. compounded monthly. A second account is set up with $5100 earning interest at 6% p.a. compounded annually.

(a) At what point will the first account balance exceed the second account balance for the first time, on a temporary basis?

(b) At what stage will the first account remain ahead of the second on a permanent basis?

5.2 Future value of an investment or a loan

Calculating an account balance directly

Consider a sum of money $\$P$ that grows through interest calculated on the increasing balance at the end of every compounding period.

The growth factor is defined as $r = 1 + i$, where i is the interest rate at which the balance is increasing every compounding period, expressed as a decimal.

Using the definitions above: $A_0 = P$ and $A_{n+1} = rA_n$.

This gives the following sequence of balances after each compounding period:

$$A_0 = P \qquad \begin{aligned} A_1 &= r \times A_0 \\ &= rP \end{aligned} \qquad \begin{aligned} A_2 &= r \times A_1 \\ &= r \times rP \\ &= r^2P \end{aligned} \qquad \begin{aligned} A_3 &= r \times A_2 \\ &= r \times r^2P \\ &= r^3P \end{aligned}$$

> The formula for future value of a compound interest loan or investment is:
>
> $A = P(1+i)^n$ or $A = r^nP$, where $r = 1 + i$
>
> A is the future value or account balance.
>
> P is the principal or initial balance.
>
> n is the number of times interest compounds.
>
> i is the rate of interest for the compounding period.

6 Determine the future value with the compound interest formula

An investment of $12 000 earns compound interest at a rate of 6.5% p.a.

Determine the future value of the investment for each of the following situations.

(a) Interest is compounded annually for 8 years.

THINKING	WORKING
1 Recall the compound interest formula and identify the variables.	$A = P(1+i)^n$ where: $P = 12\,000$, $i = 6.5\%$ p.a. $= 0.065$, $n = 8$
2 Substitute the values to calculate the amount of the investment A.	$A = 12\,000(1+0.065)^8$ $= 12\,000 \times 1.065^8$ $= 19\,859.95$ (2 d.p.)
3 Interpret the result.	Assuming a fixed interest rate, the future value of this investment is $19 859.95.

(b) Interest is compounded daily for 2 years.

1 Recall the compound interest formula and identify the variables.	$A = P(1+i)^n$ where: $P = 12\,000$ $i = \dfrac{0.065}{365}$ (the interest rate per day) $n = 2$ years $= 2 \times 365$ days
2 Substitute the values to calculate the amount of the investment A.	$A = 12\,000\left(1+\dfrac{0.065}{365}\right)^{(2\times 365)}$ $= 13\,665.78\ (2\text{ d.p.})$
3 Interpret the result.	Assuming a fixed interest rate, the future value of this investment is \$13 665.78.

Principal required for a known future value

If you have a specified future value in mind, you can backtrack to determine the amount that needs to be invested, to grow it to the required amount.

7 Determine the required principal when the future value is specified

Determine the amount that must be invested so that an account will grow to at least \$7500 in 6 years. The interest, at 6% p.a., will be calculated quarterly and added to the account. Give your answer to the nearest hundred dollars.

THINKING	WORKING
1 Recall the compound interest formula and identify the variables.	$A = P(1+i)^n$ where: $A = 7500$ $i = \dfrac{6\%\text{ p.a.}}{4}$ $= 1.5\%$ per quarter $= 0.015$ $n = 6$ years $= 6 \times 4$ quarters $= 24$ quarters
2 Substitute the values and solve for P, the principal amount invested.	$7500 = P \times (1.015)^{24}$ $P = \dfrac{7500}{(1.015)^{24}}$ $P = 5246.58\ (2\text{ d.p.})$
3 Round the value as required and interpret the answer.	In order to reach an account balance of \$7500 in 6 years time, you need to invest a principal, to the nearest hundred dollars, of \$5300.

The required period of time

If money is being invested for a specific purpose, you may need to determine how long it will take for the investment to reach a particular target value.

8 Determine the required period of time for a specified future value

Tan invests $14 500 at 4.62% p.a. If compound interest is calculated quarterly, how long will this investment take to be worth at least $20 000?

THINKING	WORKING
1 Recall the compound interest formula and identify the variables.	$A = P(1+i)^n$ $P = 14\,500$ $A = 20\,000$ $i = \frac{4.62\%}{4}$ per quarter $= 1.155\%$ per quarter $= 0.01155$
2 Estimate a time period using simple interest.	Use the simple interest to approximate a term: $I = Pin$ $5500 = 14500 \times 0.0462 \times n$ $n = \frac{5500}{14500 \times 0.0462}$ $= 8.210\ldots$ years $= 8.210\ldots \times 4$ quarters / year $n \approx 32$ quarters
3 Substitute the values and solve for n, the number of quarters.	$20\,000 = 14\,500 \times (1.01155)^n$ $(1.01155)^n = \frac{20\,000}{14\,500}$ $= 1.3793\ldots$
4 Use trial and error to determine the value of n, showing a value above and below the target value.	$(1.01155)^{30} = 1.4113\ldots$ $(1.01155)^{28} = 1.3792\ldots$ $(1.01155)^{29} = 1.3951\ldots$
5 Check that the value of n is reasonable.	$14\,500 \times (1.01155)^{29} = \$20\,230.25$ (2 d.p.)
6 Interpret the value of n as a period of time.	It will take 29 quarters, or 7 years and 3 months, for the investment to be worth at least $20 000.

To estimate the time period required for an investment to grow, use the simple interest formula $I = Pin$.

The 'Rule of 72' can be used to approximate the time required for an investment to double, where $n = \frac{72}{i\%}$

The required rate

You may need the interest rate to be high enough to grow your investment to a particular amount in a specified time.

Explore further

Using the compound interest formula

Generate a sequence and use a formula to calculate the principal, account balance and interest rate of a loan or investment.

9 Determine the required rate for a specified future value

Debra has $30 000 to invest and needs it to grow to at least $35 000 in 3 years. If compound interest is calculated monthly, determine the annual rate of interest required, to 2 decimal places.

THINKING	WORKING
1 Identify the variables.	$P = 30\,000$ $A = 35\,000$ $n = 3$ years $= 3 \times 12$ months $= 36$ months
2 Substitute the values into the compound interest formula and solve for i, the interest rate per period.	$A = P(1+i)^n$ $35\,000 = 30\,000(1+i)^{36}$ $(1+i)^{36} = \frac{35\,000}{30\,000}$ $(1+i)^{36} = 1.166\ldots$ $1+i = \sqrt[36]{1.166\ldots}$ $1+i = 1.004\,291\ldots$ $i = 0.004\,291\ldots$ $i = 0.4291\ldots\%$
3 Convert the value of i to an annual rate by multiplying i by the number of periods per year.	$0.4291\ldots\% \times 12 = 5.15\%$ p.a. (2 d.p.)
4 Interpret the answer.	A rate of 5.15% p.a. is required to grow the investment by the required amount.
5 Evaluate the reasonableness of your answer.	$A = 30\,000\left(1+\frac{0.0515}{12}\right)^{3\times12}$ $A = \$35\,000.66$ (2 d.p.)

EXERCISE 5.2 Future value of an investment or a loan

Worked Example

1 An investment of $36 000 earns compound interest at a rate of 5.02% p.a. Determine the future value when interest is compounded: 6

(a) annually for 7 years (b) monthly for 5 years (c) daily for 4 years

2 A loan of $10 400 earns compound interest every 6 months at a rate of 6% p.a. Determine the future value of this loan in 5 years.

3 Maysha opened a savings account with a deposit of \$2500. Interest is calculated on the increasing balance each month and added to the account. She uses the formula $A = 2500 \times 1.004^{12}$ to calculate the value of her savings account after 1 year. What annual rate of interest is applied to Maysha's account?

A 0.4% B 0.48% C 4% D 4.8%

4 A loan company charges 18.6% per month to customers who need to borrow cash for a short time. Jasmine needed an immediate loan of \$700 to pay for car repairs so she can go to work. She could not repay the loan until after 3 months. How much did Jasmine owe after 3 months?

Worked Example 7

5 Determine the principal amount that must be invested so that an account will grow to at least \$12 500 in 5 years. The interest, at 4.3% p.a., will be calculated quarterly and added to the account. Give your answer to the nearest hundred dollars.

6 Erica wants her savings account balance to be at least \$8000 in 7 years. If the rate of interest is 3.4% p.a. compounded monthly, determine the principal she needs to invest in this account. Give your answer to the nearest 10 dollars.

7 Svetlana expects to receive a tax return of \$2635.74 in 10 days. However, she must borrow money immediately. She will restrict her loan so that the entire tax return, when it arrives, will pay off the loan in full. She borrows from a short-term loan company and they charge 30% per annum, calculated daily. What is the maximum amount of money that Svetlena can borrow? Give your answer to the nearest dollar.

8 An investment of \$8620 earns interest at 4.26% p.a. calculated quarterly on the increasing value. Determine the value of this investment after 8 years.

9 A payday loan, where a person borrows money in the short term until they are paid, has interest added daily at a rate of 0.9% per day. Calculate the amount to be paid back if \$600 is borrowed for a total of 10 days.

10 Two sisters each have \$1000 invested. Carla earns 4.18% p.a. interest calculated quarterly and Teri earns 4.24% p.a. interest calculated every 6 months.

Determine which investment balance is greater after 3 years, and by how much.

11 An investment of \$8000 earns compound interest, calculated quarterly. The value of the account at the end of each of the first two quarters is \$8120.00 and \$8241.80, respectively. Calculate the annual rate of compound interest applied to this investment.

12 An investment of \$15 000 earns interest of 4.1% p.a. for 5 years. Determine the difference in the final account balance after 5 years between interest compounding weekly and annually.

13 A loan of \$12 300 attracts compound interest at a rate of 6.5% p.a. paid half-yearly for 3 years.

(a) The formula calculation that can be used to find the balance due at the end of the term is:

A $A = 12\,300 \times \left(1 + \dfrac{3.25}{100}\right)^3$

B $A = 12\,300 \times \left(1 + \dfrac{6.5}{100}\right)^3$

C $A = 12\,300 \times \left(1 + \dfrac{3.25}{100}\right)^6$

D $A = 12\,300 \times \left(1 + \dfrac{6.5}{100}\right)^6$

(b) Explain the common error made by a student who chose an answer in part (a) with an index of 3.

14 The price of an ounce of gold increased by 2% on each of the past 5 days. Calculate the overall percentage by which the price of gold increased. Give your answer to 2 decimal places.

15 The value of Liam's 1972 Torana LJ has increased by 9% every year. He bought it 6 years ago for \$15 600. Determine its current value, to the nearest hundred dollars.

Worked Example

16 How long will it take for \$2000 to grow to at least \$3000 if compound interest is calculated quarterly at a rate of 10% p.a.? 8

17 Determine the number of months it will take to at least double the value of a \$2500 investment that earns compound interest each month at a rate of 5.4% p.a.

18 William borrows \$25 000 at 4.3% p.a. compounding daily. How many days will it take for the loan account balance to grow by at least \$1000?

19 Keita's loan grew from \$5700.00 to \$7547.70 in 4 years because of the interest that was added to the balance each month. What annual compound rate of interest was applied to this loan? Give your answer to 2 decimal places. 9

20 Grigor invests \$15 000 at a compound interest rate of 6.85% p.a., calculated monthly. How many months will this investment take to double in value?

21 Determine the annual rate of interest needed for an investment of \$10 000 to double in 3 years if the interest is compounded monthly. Give your answer to 2 decimal places.

22 If interest is calculated and added to an account each quarter, what annual rate of interest is needed for an investment of \$14 000 to earn \$2000 interest in 4 years? Give your answer to 2 decimal places.

23 Ken's investment of \$6000 grew to \$6983 in 2 years when interest was calculated and added monthly. Determine the annual rate of interest for this investment. Give your answer to 2 decimal places.

24 Petra is saving for a deposit on a house. Her savings account has a current balance of \$28 407.25. Interest is earned on the account balance at a rate of 4.8% p.a. every quarter.

(a) How much interest will Petra's account earn in the next 2 years?

(b) How long will it take for Petra to have enough in her savings account to pay a deposit of 10% on a property that will cost \$350 000?

25 The price of a house in 2011 was \$625 000. Forty years earlier, it was priced at \$18 500. Calculate the average annual percentage rate by which the price of this house had increased. Give your answer to 1 decimal place.

26 A payment of \$5145.88 is to be invested. Determine the annual rate of interest, calculated monthly, that would give the same account balance after 4 years as when the interest is calculated quarterly at 4.31% p.a. Give your answer to 2 decimal places.

27 Asha earns \$70 000 each year and knows that the median price of a three-bedroom house in her neighbourhood is 650% of her salary. Asha expects that her salary will increase by 4% each year but based on current trends, the median house price is expected to increase by 8% each year.

What percentage of Asha's increased salary will the expected price of a house be in 5 years? Give your answer to the nearest whole per cent.

5.3 Changing parameters

Increasing the rate of interest

How does changing the rate of interest for the same number of compounding periods affect the future value?

Consider the account balances each quarter with interest at 4% p.a. and 8% p.a. charged on a loan of \$1000 for one year, with interest calculated quarterly.

For 4% p.a.	Growth factor
$i = \frac{4\%}{4}$ p.q. $= 1\%$ p.q. $= 0.01$	$r = (1 + i)$ $= 1.01$ $A_0 = 1000$ $A_{n+1} = 1.01A_n$

For 8% p.a.	Growth factor
$i = \frac{8\%}{4}$ p.q. $= 2\%$ p.q. $= 0.02$	$r = (1 + i)$ $= 1.02$ $A_0 = 1000$ $A_{n+1} = 1.02A_n$

Quarter	4% p.a.	8% p.a.
	Balance A_n (\$)	Balance A_n (\$)
0	1000.00	1000.00
1	1010.00	1020.00
2	1020.10	1040.40
3	1030.30	1061.21
4	1040.60	1082.43

- Interest earned for the first quarter for the 4% loan is \$10 and interest earned for the first quarter for the 8% loan is \$20.
- If the rate is doubled, the interest earned in the first period is doubled.
- The total interest for the 4% loan is \$40.60 and the total interest for the 8% loan is \$82.43.
- If the rate is doubled, the interest earned is more than doubled over a term longer than one period. This is due to the compounding effect of a higher rate on a greater balance at the end of each quarter.

If the interest rate is doubled, the interest earned is more than double over any term longer than one period.

Spreadsheets and graphs

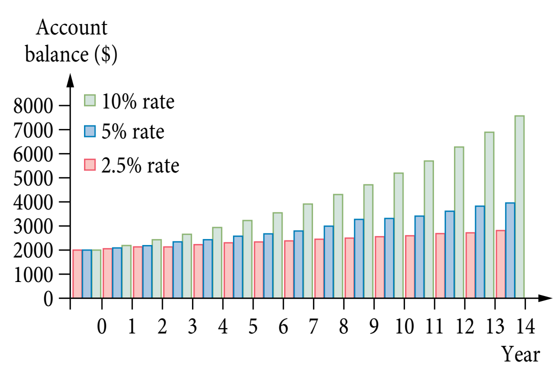
Consider an investment of \$2000 on which compound interest is calculated annually.

The table compares the future value of this investment for different rates of interest and the corresponding recurrence relation for each rate.

Rate	Recurrence relation
2.5% p.a.	$C_0 = 2000,\ C_{n+1} = 1.025 \times C_n$
5% p.a.	$B_0 = 2000,\ B_{n+1} = 1.05 \times B_n$
10% p.a.	$A_0 = 2000,\ A_{n+1} = 1.1 \times A_n$

Bar charts drawn from these tables more clearly illustrate the growth resulting from the three compound interest rates. The graph columns show the account balance for each year.

The compounding, or geometric, growth in the account balance becomes apparent in this comparison, especially with the higher rate of interest. The growth at 10% p.a. increases much faster than the growth at lower rates of interest.



10 Effect of different rates with annual compounding

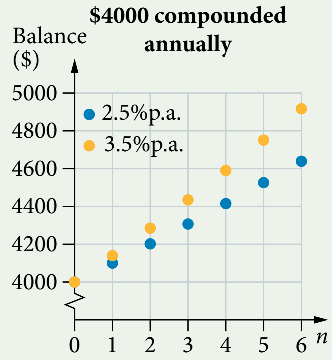
Compare the balance of an investment of \$4000 over 6 years with interest rates of 2.5% p.a. and 3.5% p.a. respectively, compounding annually, in the following ways:

(a) using a spreadsheet and a graph

THINKING	WORKING
1 Determine the recurrence relations.	For 2.5% p.a.: $i = 0.025$, therefore $r = 1.025$; $A_0 = 4000,\ A_{n+1} = 1.025A_n$ For 3.5% p.a.: $i = 0.035$, therefore $r = 1.035$; $A_0 = 4000,\ A_{n+1} = 1.035A_n$
2 Create a spreadsheet with values of n in the first column. In each of the other columns, start with the initial balance in the first row, then create a formula to multiply the value of the cell above by r.	In B3 the formula would be = B2*1.025 and in C3 the formula would be = C2*1.035.

	A	B	C
1	n	2.5% p.a.	3.5% p.a.
2	0	4000.00	4000.00
3	1	4100.00	4140.00
4	2	4202.50	4284.90
5	3	4307.56	4434.87
6	4	4415.25	4590.09
7	5	4525.63	4750.75
8	6	4638.77	4917.02

3 Insert a scatter plot of the balances against n.



(b) by calculating the final balance.

1 Determine the final balance of each investment.	For 2.5% p.a.: $A_6 = 4000(1.025)^6 = \$4638.77$ For 3.5% p.a.: $A_6 = 4000(1.035)^6 = \$4917.02$
2 Calculate the difference in final balances.	Difference: $4917.02 - 4638.77 = 278.25$
3 Interpret the result.	The higher interest rate yielded \$278.25 more over 6 years.

11 Effect of different rates with the same compounding period

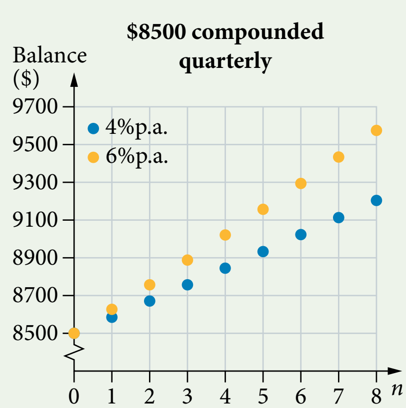
Compare the balance of an investment of $8500 over 2 years with interest rates of 4% p.a. and 6% p.a. respectively, compounding quarterly, in the following ways:

(a) using a spreadsheet and a graph

THINKING	WORKING
1 Determine the recurrence relations.	For 4% p.a.: $i = \frac{4\%}{4}$ per quarter $= 1\%$ per quarter $= 0.01$ And $r = 1.01$ For 6% p.a.: $i = \frac{6\%}{4}$ per quarter $= 1.5\%$ per quarter $= 0.015$ And $r = 1.015$ The recurrence relations are: $A_0 = 8500,\ A_{n+1} = 1.01A_n$ $A_0 = 8500,\ A_{n+1} = 1.015A_n$
2 Determine the number of compounding periods.	$n = 2$ years $= 2 \times 4$ quarters $= 8$ quarters The investment accrues interest for 8 quarters.
3 Create a spreadsheet with values of n in the first column. In each of the other columns, start with the initial balance in the first row, then set up a formula to multiply the value of the cell above by r.	In B3 the formula would be = B2*1.01 and in C3 the formula would be = C2*1.015.

	A	B	C
1	n	4% p.a.	6% p.a.
2	0	8500.00	8500.00
3	1	8585.00	8627.50
4	2	8670.85	8756.91
5	3	8757.56	8888.27
6	4	8845.13	9021.59
7	5	8933.59	9156.91
8	6	9022.92	9294.27
9	7	9113.15	9433.68
10	8	9204.28	9575.19

4 Insert a scatter plot of the balances against n.



(b) by calculating the final balance.

1 Calculate the final balances.	For 4% p.a.: $A_8 = 8500(1.01)^8 = \$9204.28$ For 6% p.a.: $A_8 = 8500(1.015)^8 = \$9575.19$
2 Calculate the difference in final balances.	Difference: 9575.19 – 9204.28 = 370.91
3 Interpret the answer.	The higher interest rate yielded \$370.91 more over 2 years.

Increasing the number of compound periods over the same term

Consider an investment of \$2000, where compound interest is added at a rate of 10% p.a.

The table compares the future value of this investment for interest that compounds annually, six monthly and quarterly.

Let A_n, H_n and Q_n represent the future value of the account balances at the end of n periods.

Period	Annually	Six monthly	Quarterly
Interest rate, each period	$i = 10\%$	$i = \frac{10\%}{2} = 5\%$	$i = \frac{10\%}{4} = 2.5\%$
Growth factor each compounding period	$r = 1 + \frac{10}{100} = 1.1$	$r = 1 + \frac{5}{100} = 1.05$	$r = 1 + \frac{2.5}{100} = 1.025$
Recurrence rule	$A_{n+1} = 1.1 \times A_n$	$H_{n+1} = 1.05 \times H_n$	$Q_{n+1} = 1.025 \times Q_n$
Initial investment	$A_0 = \$2000$	$H_0 = \$2000$	$Q_0 = \$2000$
1 quarter			$Q_1 = \$2050$
Half year (2 quarters)		$H_1 = \$2100$	$Q_2 = \$2101.25$
3 quarters			$Q_3 = \$2153.78$
1 year (4 quarters)	$A_1 = \$2200$	$H_2 = \$2205$	$Q_4 = \$2207.63$
5 quarters			$Q_5 = \$2262.82$
1.5 years (6 quarters)		$H_3 = \$2315.25$	$Q_6 = \$2319.39$
7 quarters			$Q_7 = \$2377.37$
2 years (8 quarters)	$A_2 = \$2420$	$H_4 = \$2431.01$	$Q_8 = \$2436.81$

You should compare balances for the same term:
A_1 with H_2 and Q_4 *or* A_2 with H_4 and Q_8.

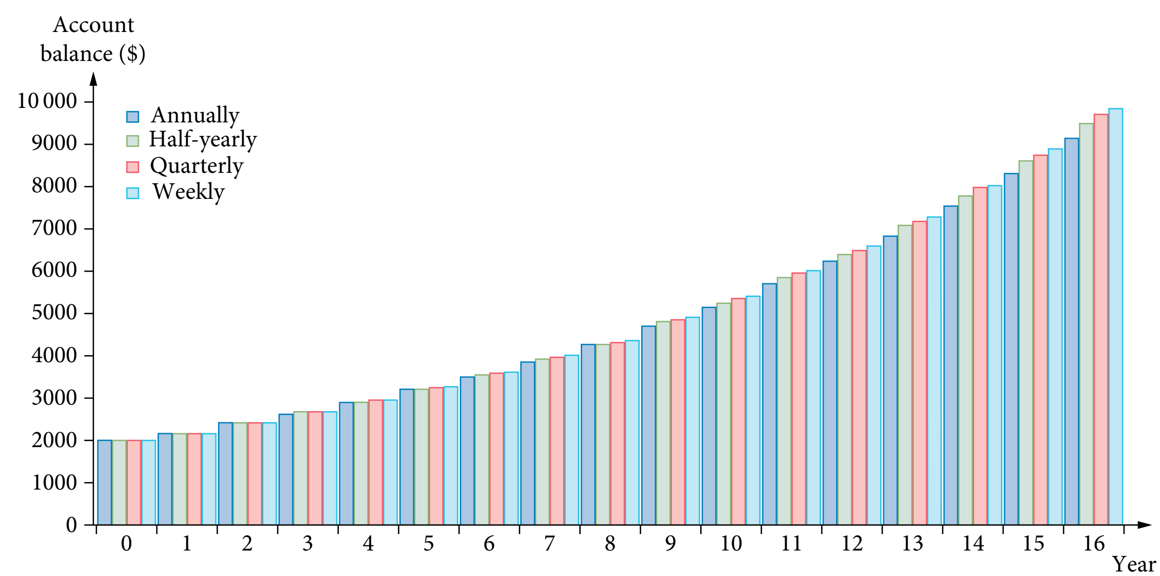
The table shows the correct corresponding balances if this investment continued for five years.

At end of year *n*	Annually	Six monthly	Quarterly
Balance end of year 0	$2000	$2000	$2000
Balance end of year 1	$2200	$2205	$2207.63
Balance end of year 2	$2420	$2431.01	$2436.81
Balance end of year 3	$2662	$2680.19	$2689.78
Balance end of year 4	$2928.20	$2954.91	$2969.01
Balance end of year 5	$3221.02	$3257.79	$3277.23

Interest earned is greater when compounded more frequently in the year.

This is due to the compounding effect of a greater number of interest calculations in the year.

The bar chart includes account balances at the end of each year, where the 10% p.a. compound interest is added weekly, quarterly, half-yearly or annually over a period of 16 years.



Shorter compounding periods have a greater compounding effect on the value of a compound interest loan or investment. Interest is added more frequently and the principal increases earlier and earns more interest.

Interest earned in year 16 of this investment with weekly calculations is almost $105 more than that earned with annual calculations over the same 16 years.

The graph gradually becomes steeper due to compounding, and shows geometric growth in the account balance. The initial $2000 investment has grown to nearly $10 000 after 16 years.

By making the compounding periods shorter and therefore more frequent, greater growth and a larger future value is achieved.

12 Compare an account balance with different compounding periods

Sasha wants to compare two savings account options for investing \$15 000 for four years. Each account pays compound interest at the rate of 5% p.a.

Option 1: Interest is calculated and paid annually.

Option 2: Interest is calculated and paid quarterly.

(a) Determine the account balance after 4 years of Option 1.

THINKING	WORKING
1 Recall the compound interest formula and identify the value of each variable.	$A_n = A_0(1+i)^n$ $A_0 = 15\,000$, $i = 0.05$, $n = 4$
2 Substitute the identified values and calculate the required future value A_n.	$A_4 = 15\,000(1.05)^4$ $= \$18\,232.59$
3 Interpret the result.	Using Option 1, the balance after 4 years will be \$18 232.59.

(b) Determine the account balance after 4 years of Option 2.

1 Recall the compound interest formula and identify the value of each variable.	$A_n = A_0(1+i)^n$ $A_0 = 15\,000$ $i = \frac{5\%}{4}$ per quarter $= 1.25\%$ per quarter $= 0.0125$ $n = 4$ years $= 4 \times 4$ quarters $= 16$ quarters
2 Substitute the identified values and calculate the required future value A_n.	$A_4 = 15\,000(1.0125)^{16}$ $= \$18\,298.34$
3 Interpret the result.	Using Option 2, the balance after 4 years will be \$18 298.34.

(c) After 4 years, which option earns the greater interest and what is the difference?

1 Calculate the difference between the final balances.	Option 1: \$18 232.59 Option 2: \$18 298.34 18 298.34 − 18 232.59 = 65.75
2 Interpret the result.	Option 2 earns \$65.75 more interest over the 2 year investment.

A shorter compounding period will always be the better option for an investment if all other aspects are the same. Sometimes such options come with a fee to be paid to the finance company, in which case the shorter compounding period will only become the better option after enough time has passed for the increased interest to have compensated for the fee.

13 Comparing options involving fees

Peter has \$49 000 in a bank savings account that pays 4.5% p.a. interest every quarter. He discovers that he could transfer to an investment account that also pays 4.5% p.a. interest but compounds weekly. If he does transfer his money, his account will be charged a transfer fee of \$8.00.

Determine whether transferring is the better option for a 6 year term, and calculate the difference in balance with the two options.

THINKING	WORKING
1 Consider the first option. Recall the formula for compound interest, identify the value of each variable and then determine the future value A_n.	Non-transfer option: $A_n = A_0(1+i)^n$ with $A_0 = 49\,000$ $i = \frac{4.5\%}{4}$ per quarter $= 1.125\%$ per quarter $= 0.011\,25$ $n = 6$ years $= 6 \times 4$ quarters $= 24$ quarters $A_4 = 49\,000(1.011\,25)^{24}$ $= \$64\,091.57$
2 Interpret the value.	The balance after 6 years if he does not transfer the account is \$64 091.57.
3 Consider the second option. Recall the formula for compound interest, identify the value of each variable and then determine the future value A_n.	Transfer option: $A_n = A_0(1+i)^n$ with a transfer fee subtracted from the balance transferred: $A_0 = 49\,000 - 8$ $= 48\,992$ $i = 4.5\%$ p.a. $= \frac{4.5\%}{52}$ per week $= \frac{0.045}{52}$ $n = 6$ years $= 6 \times 52$ weeks $= 312$ weeks $A_4 = 48\,992\left(1 + \frac{0.045}{52}\right)^{312}$ $= \$64\,170.29$
4 Interpret the value.	The balance after 6 years if he transfers the account is \$64 170.29.
5 Compare the results and interpret the answer.	Difference: $64\,170.29 - 64\,091.57 = 78.72$ He will be better off by \$78.72 after 6 years if he transfers.

Changing parameters during the term of a loan or investment

For a loan taken over five years, the rate of interest is usually fixed at an agreed percentage. For a loan taken over 25 years, such as a home loan, the rate of interest is usually variable and can be changed if economic conditions change significantly in that time.

Further, if an investor finds an investment that gives a greater rate of interest than the one where they are currently invested, they may change their funds to the higher rate of interest.

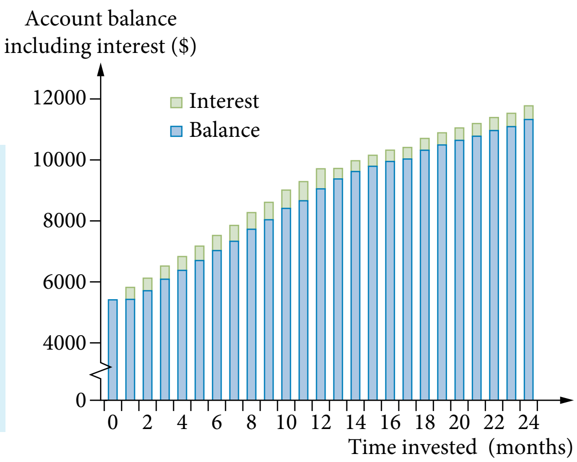
14 Interest rate change during the term of an investment

Arjun invests $5000 in a bank account that paid compound interest each month at a rate of 4.8% p.a. for one year, then reduced to 2.4% p.a. Determine the balance at the end of the two years.

THINKING	WORKING
1 Consider the first year of the investment. Recall the formula for compound interest, identify the value of each variable and then determine the future value A_n.	For the first year: $A_n = A_0(1+i)^n$ with $A_0 = 5000$; $i = \frac{4.8\%}{12}$ per month $= 0.4\%$ per month $= 0.004$; $n = 1$ year $= 12$ months; $A_{12} = 5000(1.004)^{12} = \$5245.35$
2 Interpret the value.	The balance at the end of the first year is $5245.35.
3 Consider the second year of the investment. Recall the formula for compound interest, identify the value of each variable and then determine the future value A_n.	For the second year: $A_n = A_0(1+i)^n$ with $A_0 = 5245.35$; $i = \frac{2.4\%}{12}$ per month $= 0.2\%$ per month $= 0.002$; $n = 1$ year $= 12$ months; $A_{12} = 5245.35(1.002)^{12} = \$5372.63$
4 Interpret the answer.	The balance at the end of the second year is $5372.63.

The graph shows the growth in Arjun's account balance in Worked example 14. The change to the lower compound interest rate of 2.4% p.a. after the first year is clearly evident. A spreadsheet can readily show the growth in this account. The sections of a spreadsheet shown here, with formulas, relate to the worked example above.

If a parameter changes during the term of a loan or investment, a new loan or investment starts at that point.



	A	B	C
1		Year	Balance A_n
2	Rate = 4.8% p.a.	0	5000.00
3		1	5020.00
4		2	5040.08
14		12	5245.35
15	Rate = 2.4% p.a.	13	5255.84
16		14	5266.35
25		23	5361.91
26		24	5372.63

EXERCISE 5.3 Changing parameters

Worked Example

10 **1** Compare the balance of an investment of $12 000 over 5 years with interest rates of 6.2% p.a. and 7.5% p.a. respectively, compounding annually:

(a) using a spreadsheet and a graph (b) by calculating the final balance.

2 Compare the balance of an investment of $20 000 over 10 years with interest rates of 2.9% p.a. and 3.3% p.a. respectively, compounding annually:

(a) using a spreadsheet and a graph (b) by calculating the final balance.

11 **3** Compare the balance of an investment of $7500 over 3 years with interest rates of 2.8% p.a. and 3.6% p.a. respectively, compounding quarterly:

(a) using a spreadsheet and a graph (b) by calculating the final balance.

4 Compare the balance of an investment of $45 000 over 2 years with interest rates of 2.4% p.a. and 4.2% p.a. respectively, compounding monthly:

(a) using a spreadsheet and a graph (b) by calculating the final balance.

12 **5** Anika wants to compare two savings account options for investing $6000 for 5 years. Each account pays compound interest at the rate of 6.25% p.a.

Option 1: Interest is calculated and paid annually.

Option 2: Interest is calculated and paid quarterly.

(a) Determine the account balance after 5 years of Option 1.

(b) Determine the account balance after 5 years of Option 2.

(c) Which option earns the greater interest and what is the difference?

6 A car loan for $18 400 has compound interest charged at 7.2% p.a. for a term of 5 years with no repayments being made until the end of that time. Determine the difference in the total cost between interest being compounded annually versus interest compounded monthly.

7 A loan of $480 000 at 6.45% p.a. is on offer from two financial institutions. One institution calculates compound interest weekly, while the other calculates it monthly. If the loan will be for 25 years, how much more expensive is the loan that compounds weekly if no repayments are made in that time?

8 Andrew has $5000 to put into a savings account that pays 5.25% p.a. interest. Bank A will compound the interest daily. Bank B will compound the interest quarterly.

(a) Which bank would give the greater account balance after one year, and by how much?

A Bank A, $1.78 B Bank A, $2.37 C Bank B, $1.78 D Bank B, $2.37

(b) Explain the common error made by a student who wrote $r = 1.01438\ldots$ for the daily compounding calculation.

9 Con invests $2500 in each of two accounts. Account 1 will pay interest compounding monthly at a rate of 3.75% p.a., while Account 2 will pay interest compounding annually at a rate of 3.77% p.a. Which investment will have a higher balance, and by how much, after 3 years?

10 Consider the following statements about the value of a loan or investment after two years. Identify each statement as true or false, giving a brief explanation.

(a) Compound interest added in the second year will be the same as that added in the first year.

(b) Compound interest at 2.5% added each six months gives the same result as compound interest of 5% added each year.

11 The table shows the balance of a term deposit savings account at the end of the first four years, where interest is added annually.

(a) Determine the rate of interest applied in the first year.

(b) For how many years did this rate of interest apply?

(c) What was the rate of interest applied when it changed?

Year	Balance ($)
0	15 000.00
1	15 675.00
2	16 380.38
3	17 117.49
4	17 802.19

Worked Example

12 Talia has $52 000 in a bank savings account that pays 3.9% p.a. interest compounding every month. She discovers that she could transfer to an investment account that also pays the same rate of interest but compounds weekly. If she transfers her money, her account will be charged a transfer fee of $10.00 at the beginning of the term. 13

Which option gives a better balance, and by how much, in 5 years?

13 A loan of $16 000 can be arranged at the Indigo Dream Finance company. They offer two options, both at an annual interest rate of 6.75%.

Option 1: Interest compounds monthly but there is a $100 application fee added to the loan at the start.

Option 2: Interest compounds 6 monthly and there is no application fee.

Which option is the cheaper over 5 years?

A Option 1 by $3.36

B Option 1 by $103.36

C Option 1 by $143.37

D Option 2 by $243.37

14 An investment account commenced with a deposit of $20 000 and earned compound interest monthly at a rate of 4.2% p.a. At the end of the first year, $8000 was withdrawn from this account. The balance then continued to grow at the same rate as in the first year.

Write a recurrence relation that models the growth of the account balance B_n from the start of the second year.

15 Harry invested $9200 in a bank account. As a welcoming bonus, the account paid compound interest each month at a rate of 4.25% p.a. for one year. At the end of the year, the rate of interest will reduce to 0.5% p.a. 14

(a) Determine the balance at the end of the second year.

(b) How much extra interest could Harry have earned if he had negotiated a rate of 3.75% instead of 0.5% for the second year?

16 In the first four years, the growth in Sayid's investment of $4000 was modelled by $A_0 = 4000$, $A_{n+1} = 1.042A_n$, where n is the number of years.

After this, the interest rate changed and Sayid's investment continued to grow as modelled by $A_4 = x$, $A_{n+1} = 1.0575A_n$.

(a) Determine the value of x.

(b) By how much did the annual rate of interest increase after the first 4 years of Sayid's investment?

(c) Determine the minimum number of years needed for Sayid's initial investment to grow to at least $6000.

17 Kai has put $4500 into a savings account. Interest is added at 3.9% p.a. calculated every quarter on the increasing balance for 5 years. After 5 years, the interest rate is expected to increase to 4.4% p.a.

(a) Determine the account balance at the end of 5 years.

(b) Determine the account balance at the end of 6 years.

(c) By what percentage will Kai's account have grown in 6 years? Give your answer to 2 decimal places.

18 Charlise arranged a loan of $20 000 with compound interest charged each quarter at the rate of 4.5% p.a. for the first year. After the first year, the interest will increase by 2% p.a. The principal, plus all added interest, will be repaid at the end of 2 years. Use recurrence relations and a table to determine the amount that Charlise will have to repay at the end of 2 years.

19 Brad has $15 000 to invest, with interest added every quarter. The options are:

- 8% p.a. with no fee
- 10% p.a. with an initial fee of $500
- 12% p.a. with an initial fee of $2000.

Use a spreadsheet to determine the term for which each option is the best choice. Support your solution with a graph.

20 Simone has borrowed $12 000. Interest is added at 9.6% p.a., calculated every month. She makes a payment of $2000 at the end of each year. Use a spreadsheet to determine how long it will take Simone to pay off the loan. Calculate the amount of the final payment. Verify your solution with a graph.

Effective annual rate of interest

5.4

Comparing investments

As loans can be calculated using compound interest over different periods, there is the need for a standard method to compare loans. The same applies to investments.

The *effective annual rate of interest* is a comparable interest rate that gives the same account balance as would be achieved if the interest on the investment were compounded annually.

Two investments, each of \$1000, earn interest at a nominal rate of 12% p.a.

The *effective* annual rate of interest can be found using the formula:

$$\text{effective rate} = \frac{\text{annual interest earned}}{\text{principal invested}}$$
$$= \frac{\text{end of year balance}}{\text{start of year balance}} - 1$$
$$= \frac{A_n - A_0}{A_0} - 1$$

For investment A, the interest is calculated annually.	For investment B, the interest is calculated each month.
At the end of the first year, the value of investment A is: $1000 \times 1.12 = \$1120$ This is the same as simple interest at 12% p.a. calculated for one year. The percentage change in investment A in the year is: $\text{effective rate} = \frac{\text{end of year balance}}{\text{start of year balance}} - 1$ $= \frac{A_1 - A_0}{A_0} - 1$ $= \frac{1120}{1000} - 1$ $= 1.12 - 1$ $= 0.12$ The effective rate of interest is 12% p.a. Note that this is the same as the nominal, or stated, rate of interest.	$\text{monthly rate of interest} = \frac{12\%}{12}$ $= 1\%$ For this investment, the growth factor each period is $r = 1 + 0.01$ $= 1.01$ At the end of the first year, the value of investment B is: $A = Pr^n$ $= 1000 \times (1.01)^{12}$ $= \$1126.83\ (2\text{ d.p.})$ The percentage change in investment B in the first year is: $\text{effective rate} = \frac{\text{end of year balance}}{\text{start of year balance}} - 1$ $= \frac{A_{12} - A_0}{A_0} - 1$ $= \frac{1126.83}{1000} - 1$ $= 1.12683 - 1$ $= 0.12683$ The effective interest is 12.683% p.a. Note that this is greater than the nominal rate of interest of 12% p.a.

The compounding effect in investment B created a higher balance at the end of the first year. If the annual rate of interest for investment A had been 12.68% p.a. compounding annually it would have grown to \$1126.83 as well.

The rate of 12.68% for investment B is the *effective annual rate of interest*. It represents the annual rate that would be needed to achieve the same account balance at the end of the first year if interest was paid annually rather than monthly.

The *nominal* annual rate of interest is the stated or advertised rate.

The *effective* annual rate of interest takes into account the effect of compounding periods during a year.

When interest compounds only once a year, the effective rate of interest is the same as the nominal rate of interest.

15 Determine the effective annual rate of interest

An investment of $1000 has compound interest of 10% p.a. calculated and added daily.

(a) Determine the investment account balance after one year.

THINKING	WORKING
1 Recall the formula for compound interest, identify the value of each variable and then determine the future value A_n.	$A_n = A_0(1+i)^n$ $A_0 = 1000$ $i = \frac{10\%}{365}$ per day $\quad n = 365$ days $= \frac{0.1}{365}$ $A_{365} = 1000\left(1+\frac{0.1}{365}\right)^{365}$ $= \$1105.16\,(2\text{ d.p.})$
2 Interpret the result.	The balance at the end of the year is \$1105.16. $I = A_{365} - A_0$ $= 1105.16 - 1000$ $= 105.16$ The account has earned \$105.16 in interest.

(b) Use the account balance from part **(a)** to determine the effective annual rate of interest, to 2 decimal places.

THINKING	WORKING
1 Determine the effective growth rate by dividing the end of year (EOY) balance by the start of year (SOY) balance, then subtracting 1.	effective rate $= \frac{\text{end of year balance}}{\text{start of year balance}} - 1$ $= \frac{A_{365} - A_0}{A_0} - 1$ $= \frac{1105.16}{1000} - 1$ $= 1.10516 - 1$ $= 0.10516$
2 Interpret the result.	The effective annual rate of interest is 10.52%, to 2 decimal places.

Developing a formula

The effective rate of interest can be calculated without knowing any of the amounts of money involved.

The effective annual rate of interest is calculated by:

$$\text{effective rate (as a decimal)} = \frac{\text{end of year balance}}{\text{start of year balance}} - 1$$
$$= \frac{A_n - A_0}{A_0} - 1$$
$$= r^n - 1$$
$$= (1+i)^n - 1$$

where n represents the number of compounding periods in the 1 year period.

The rule for calculating the effective annual rate of interest *as a decimal value* is:

$i_{\text{effective}} = r^n - 1$, where $r = 1 + i$

i is the interest rate for each compounding period, expressed as a decimal.

It is important to note that effective interest is calculated over the period of 1 year, so:

n is the number of compounding periods per year.

An alternative form of the rule for the effective annual rate of interest, given on the formula page for this course, is for the decimal value as well. However, the value of i has a different meaning than the way it has been used so far.

$$i_{\text{effective}} = \left(1 + \frac{i}{n}\right)^n - 1$$

Nominal annual rate as a decimal (i)

Number of compounding periods in one year (n)

Explore further

Using the effective interest formula

Determine the effective interest on an investment when comparing annual and monthly interest rates.

The effective annual rate of interest is always greater than the nominal annual rate when interest is added during the year. This is due to the compounding growth that comes from interest being calculated on an increasing account balance at the end of each period through the year. If the compounding period is annual, then the effective annual rate of interest is the same as the nominal rate of interest.

16 Determining the effective annual rate of interest from the nominal rate

A loan of $4800 for 3 years has interest added monthly at the rate of 12.5% p.a. Determine the effective annual rate of interest from the nominal rate.

(a) Use the formula $i_{\text{effective}} = r^n - 1$ to calculate the effective annual rate of interest, to 2 decimal places.

THINKING	WORKING
1 Determine the growth rate per period and the number of compounding periods.	$i = \frac{12.5\%}{12}$ per month $= \frac{0.125}{12}$; $n = 1$ year $= 12$ months Growth factor: $r = 1 + i$ $= 1 + \frac{0.125}{12}$

2 Substitute the values into the effective rate of interest formula.

$$i_{\text{effective}} = r^n - 1$$
$$= \left(1 + \frac{0.125}{12}\right)^{12} - 1$$
$$= 0.13241\ldots$$

3 Interpret the result.

The effective rate is 13.24% p.a.

(b) Use the formula $i_{\text{effective}} = \left(1 + \frac{i}{n}\right)^n - 1$ to determine the effective annual rate of interest, to 2 decimal places.

1 Identify the parameters.

12.5% p.a. compounding monthly, so

$i = 0.125$, $n = 12$

2 Substitute the values into the effective rate of interest formula.

$$i_{\text{effective}} = \left(1 + \frac{i}{n}\right)^n - 1$$
$$= \left(1 + \frac{0.125}{12}\right)^{12} - 1$$
$$= 0.13241\ldots$$

3 Interpret the result.

The effective rate is 13.24% p.a.

If the effective rate is specified, you can use the same formula and backtrack to determine the nominal rate of interest.

17 Determining the nominal rate of interest to give a specified effective rate

A bank wants to offer a savings account with interest added each month, such that the effective annual interest rate is 4.40% p.a. Determine the nominal annual rate of interest for the account.

THINKING

WORKING

1 Recall the formula and identify the parameters.

$$i_{\text{effective}} = \left(1 + \frac{i}{n}\right)^n - 1$$

where $i_{\text{effective}} = 4.40\% = 0.044$, $n = 12$

2 Substitute the values and solve for i, where i represents the nominal annual rate of interest.

$$0.044 = \left(1 + \frac{i}{12}\right)^{12} - 1$$
$$\left(1 + \frac{i}{12}\right)^{12} = 1.044$$
$$1 + \frac{i}{12} = \sqrt[12]{1.044}$$
$$1 + \frac{i}{12} = 1.00359\ldots$$
$$\frac{i}{12} = 0.00359\ldots$$
$$i = 0.0431\ (4\text{ d.p.})$$

3 Interpret the result.

The nominal annual rate is 4.31% p.a.

Calculating rates from the annual growth

If a single year's growth is known, the effective rate can be calculated directly, and the nominal rate can be calculated subsequently.

18 Determining the nominal rate of interest from growth in one year

An investment of $5400 had interest compounded monthly. In the first year, $202.50 interest was added.

Determine the nominal annual rate of interest, to 2 decimal places.

THINKING	WORKING
1 Use the growth in one year to calculate the effective rate.	$i_{\text{effective}} = \dfrac{\text{interest earned}}{\text{start of year balance}} \times 100\%$ $= \dfrac{202.50}{5400} \times 100\%$ $= 3.75\%$ The account balance increased by 3.75%.
2 Recall the effective rate of interest and solve for i, where i represents the nominal annual interest rate expressed as a decimal.	$i_{\text{effective}} = \left(1 + \dfrac{i}{n}\right)^n - 1$ $0.0375 = \left(1 + \dfrac{i}{12}\right)^{12} - 1$ $\left(1 + \dfrac{i}{12}\right)^{12} = 1.0375$ $1 + \dfrac{i}{12} = \sqrt[12]{1.0375}$ $1 + \dfrac{i}{12} = 1.00307\ldots$ $\dfrac{i}{12} = 0.00307\ldots$ $i = 0.0369\ (4\text{ d.p.})$
3 Interpret the result.	The nominal rate is 3.69% p.a.
4 Evaluate the reasonableness of the result.	The effective rate of interest is higher than the nominal or stated interest rate. This is expected, because interest is earned on a higher account balance each period.

If the amounts of money and the periods of time are known, you can use the formula for future value to calculate the effective annual rate of interest.

19 Determining the effective annual rate of interest from a given balance

A loan of \$12 500 has grown to \$16 119.70 in 2 years after compound interest was added each quarter. Determine the annual effective rate of interest, to 2 decimal places.

THINKING	WORKING
1 Use the future value formula $A_n = A_0 r^n$ to determine the growth factor per period.	Compounding quarterly for 2 years: $n = 8$ $A_8 = A_0 r^8$ $16\,119.70 = 12\,500 r^8$ $r^8 = \frac{16\,119.70}{12\,500}$ $r^8 = 1.289\ldots$ $r = \sqrt[8]{1.289\ldots}$ $r = 1.0322\ldots$
2 Interpret the result.	The nominal (stated) annual interest rate for the investment is 3.22% p.a.
3 Use the growth factor per period to determine the growth factor per year and, hence, the interest rate.	Number of compounding periods per year: $n = 4$ $r^4 = (1.0322\ldots)^4$ $= 1.13559\ldots (4\text{ d.p.})$ Therefore: $r = 1 + i$ $= 1 + 0.1356$
4 Interpret the result.	The effective annual rate of interest is 13.56% p.a.
5 Evaluate the reasonableness of your result.	$A_1 = Pin + P$ $= 12\,500 \times 0.1356 \times 1 + 12\,500$ $= 1695 + 12\,500$ $= \$14\,195$ $A_2 = A_1 in + A_1$ $= 14\,195 \times 0.1356 \times 1 + 14\,195$ $= \$16\,119.84$ or $A = P(1 + i)^n$ $= 12\,500 \times (1 + 0.1356)^2$ $= 16\,119.84$ Accounting for rounding errors, the effective annual rate of interest (the rate of interest that would produce the same rate if interest was compounded annually) is 13.56% p.a.

EXERCISE 5.4 Effective annual rate of interest

Worked Example

1 An investment of $4000 has compound interest of 8% p.a. calculated and added daily. 15

(a) Determine the investment account balance after 1 year.

(b) Use the account balance from part (a) to determine the effective annual rate of interest for this investment, to 2 decimal places.

2 A loan of $5000 for 3 years has interest added monthly at the rate of 9.45% p.a. 16

Use the formula to determine the effective annual rate of interest, to 2 decimal places.

3 An investment opportunity is advertised at 12.6% p.a. compounding half-yearly.

(a) The effective annual rate of interest is closest to:

A 6.3% B 13.0% C 25.2% D 26.8%

(b) Explain the common error made by a student who chose the third incorrect option.

(c) Explain why all the incorrect options could be eliminated without calculation being essential in this case.

4 The Francis family wants a loan of $18 000 for 2 years. Company A will charge 9.5% p.a. compounding quarterly and Company B will charge 0.1% lower than company A, but interest will compound monthly.

(a) State the nominal rate of interest charged by each company.

(b) Calculate the effective annual rate of interest charged by each company, to 2 decimal places.

(c) Which company gives the better rate of interest for the Francis family?

(d) Instead of a loan of $18 000, what impact would a loan of $30 000 for 3 years have on the effective annual rate of interest for Company B?

5 A loan of $7800 is being offered at a rate of 7.7% p.a. compounding monthly, or at 7.92% p.a. compounding annually.

(a) Determine the effective annual rate of interest for both options, to 2 decimal places.

(b) Compare the options over the first year.

6 Keely must decide between putting some money into one of two accounts for a year. Account A pays interest on the account balance each quarter at a rate of 3.75% p.a. Account B pays 3.81% at the end of the year. Which account should Keely use?

7 After six months of interest compounding each week, it costs $1359.09 to fully repay a loan of $1200 from the Dodgy Loan company. The annual effective rate of interest charged on this loan is closest to which value?

A 6.6% B 13.3% C 26.5% D 28.3%

8 A bank wants to offer a savings account with interest added each month, such that the effective annual interest rate is 5.05% p.a. Determine the nominal rate of interest for the account, to 2 decimal places. 17

9 Determine the nominal rate of interest that should be advertised for a loan that compounds weekly and costs the same each year as a loan that charges 17.5% p.a. compounding annually.

Worked Example

18 **10** An investment of $7300 had interest compounded monthly. In the first year, the investment earned $244.55 in interest. Determine the nominal rate of interest, to 2 decimal places.

11 An investment of $6600 earned $607.20 in 1 year when interest was compounded each quarter. Determine the nominal rate of interest, to 2 decimal places.

12 An investment grew from $21 400 to $22 900 in 1 year due to the interest being compounded weekly. Determine the nominal rate of interest, to 2 decimal places.

19 **13** A loan of $7300 has grown to $8929.11 in 2 years due to compound interest being added to the loan amount each quarter. Determine the effective annual rate of interest, to 2 decimal places.

14 An investment of $9000 grew by $4426.33 after compounding quarterly for 10 years.

(a) Determine the annual nominal rate of interest. Give your answer to 2 decimal places.

(b) Determine the effective annual rate of interest, to 2 decimal places.

15 Denzel began to create a spreadsheet some time ago, showing the balance for three different accounts after one year. He knows that one was compounded daily, one was compounded monthly and one was compounded quarterly. His notes show that the effective annual rates for the three accounts were 5.2%, 6.4% and 7.9% in no particular order.

	A	B	C	D
1	n	Account A	Account B	Account C
2	0	2000.00	12000.00	500.00
3	1	2012.71	12002.04	506.38

Determine the compounding period and effective annual interest rate for each account.

16 A savings account left untouched by the client for 3 years accrued interest monthly at 3.54% for the first year and 4.83% after that.

(a) Determine the single nominal rate of interest for the entire 3 years that would give the same final balance, to 2 decimal places.

(b) Determine the effective annual rate of interest for the 3 years, to 2 decimal places.

Summary

Compound interest

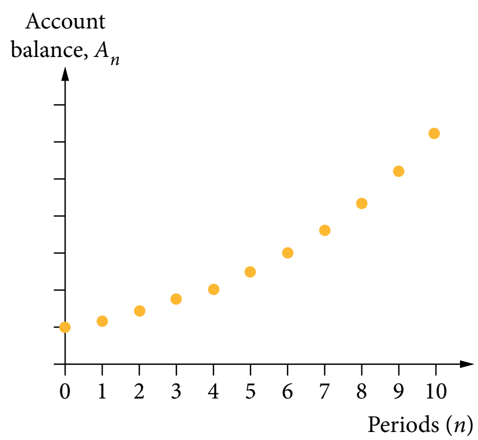
Interest is calculated each period and added to the account balance, and becomes part of the principal for the next compounding period. At each interest calculation, the account balance and the amount of interest added is greater than in the previous compounding period. Such growth is called geometric, or exponential, growth.

Graph of compound interest growth

The graph shows the exponential, or geometric, growth in the value of an asset in a compound interest situation.

Compound interest is calculated on the increasing account balance at the end of each period.

The amount of compound interest increases in each compounding period.



The recurrence relation for a compound interest account:

$A_0 = P$ $A_{n+1} = rA_n$	• A_n is the value of the account after n compounding periods of adding interest • P is principal sum borrowed or invested in dollars • $r = 1 + i$ is the growth factor • i is the compound interest rate each period expressed as a decimal (e.g. 1.6% = 0.016 per quarter)

Future value

The compound interest formula can be used to determine the future value of an investment, loan or asset after n compounding periods:

$A = P(1+i)^n$	• A is the future value or balance, in dollars • P is the principal value or sum borrowed or invested, in dollars • i is the rate of interest per period, expressed as a decimal • n is the number of compounding periods

If interest is compounded annually, then n is the number of years and i is the interest rate per annum expressed as a decimal.

Varying parameters

The future value of an investment over a given time period can be increased by either increasing the compound frequency or increasing the rate of interest earned.

Effective annual rate of interest

An effective annual rate of interest:

- can be used to compare loans or investments with different compounding periods
- takes into account the compounding effect during a year
- represents the nominal rate that would give the same effect if the interest was compounded annually.

When interest compounds *annually*, then:

effective rate of interest = annual nominal rate of interest

When interest compounds *during the year*, then:

effective annual rate of interest > annual nominal rate

Calculating the effective annual rate of interest, $i_{\text{effective}}$:

$$i_{\text{effective}} = \frac{\text{end of year balance}}{\text{start of year balance}} - 1$$

Another form of the formula used to calculate effective annual rate of interest is:

$$i_{\text{effective}} = r^n - 1 \text{ or } i_{\text{effective}} = \left(1 + \frac{i}{n}\right)^n - 1$$

where:

i is the nominal annual rate of interest, as a decimal

n is the number of compounding periods in one year.

Nominal rate of interest

The nominal rate of interest is the annual rate of interest quoted for a loan or investment. If the loan or investment compounds during the year, then the relevant fraction of the annual nominal rate is applied at each interest calculation—for example, one-quarter of the annual nominal rate at each quarterly calculation.

Chapter review 5

1 The annual increase in the annual wage of a worker is modelled by the recurrence relation $W_0 = 68\,400$, $W_{n+1} = 1.035A_n$. *Exercise 5.1*

Each year, this worker's annual wage increases by which percentage?

A 0.035% **B** 0.35% **C** 1.035% **D** 3.5%

2 The annual growth in Patrick's salary is modelled by: $P_0 = 59\,120$, $P_{n+1} = 1.0375P_n$. *Exercise 5.1*

Determine Patrick's salary in 4 years, to the nearest dollar.

3 The rule for a recurrence relation that models the quarterly appreciation in the value of a diamond ring is $V_{n+1} = 1.025V_n$. *Exercise 5.1*

If $V_6 = \$4150$, what is V_7, to the nearest dollar?

4 The price of 1000 shares in the Solar Windmill Company has increased every month. The increase is modelled by the rule $P_{n+1} = 1.05P_n$, where P_n is the price of 1000 shares at the end of month n. *Exercise 5.1*

If the price at the end of May is $P_2 = \$9525.60$, what was the price of 1000 shares at the end of March in the same year?

5 Caleb's investment account earns compound interest calculated annually. A year after opening the account, the balance was \$17 808.00. One year later, the account balance was \$18 876.48. Which is the recurrence relation that models the growth in the account balance each year? *Exercise 5.1*

A $A_0 = 16\,800$, $A_{n+1} = 0.06A_n$ **B** $A_0 = 16\,800$, $A_{n+1} = 1.006A_n$

C $A_0 = 16\,800$, $A_{n+1} = 1.06A_n$ **D** $A_0 = 17\,808$, $A_{n+1} = 1.06A_n$

6 A stamp collection is valued at \$25 000 and is expected to increase in value by 8% every year. Determine the expected increase in value in the third year, to the nearest dollar. *Exercise 5.1*

7 A loan of \$7900 has compound interest charged half-yearly at a rate of 3.8% p.a. Write a recurrence relation that models the growth in this loan account. Let L_n be the value of the loan after n half-years. *Exercise 5.1*

8 The growth in a bank account each month can be modelled by $A_0 = 13\,000$, $A_{n+1} = 1.008A_n$. *Exercise 5.1*

Calculate the nominal annual rate of interest earned by the bank account.

9 Compound interest of 2% p.a. is added each month to a loan of \$8000. Determine the value of this loan after one year. *Exercise 5.2*

10 A statue by a well-known sculptor was bought for \$1250. Every year its value increases by 4%. Determine its value after 5 years, to the nearest dollar. *Exercise 5.2*

11 A rural house was valued at \$480 000 at the start of 2014 and at \$542 000 at the start of 2016. Determine the annual percentage rate of increase in the value of this house over this time, to 2 decimal places. *Exercise 5.2*

Exercise 5.2

12 Terri's annual salary increased by 3.5% p.a. in each of the past 5 years. Her annual salary is now \$80 218. What was Terri's salary 5 years ago, to the nearest dollar?

Exercise 5.1, 5.2

13 Jess invested \$1450 at the end of July 2018. Interest was added to the investment account at 5.4% p.a. calculated monthly. Let J_n be the investment account balance at the end of each month.

(a) What is represented by J_0?

(b) Write a recurrence relation that models the growth in the balance of this loan account each month.

(c) To which account balance does J_1 refer?

(d) What will be the account balance at the end of July 2019?

(e) If the account balance at the end of January 2019 is given by J_k, what does J_{k-1} represent?

(f) What is the account balance at the start of January 2019?

Exercise 5.2

14 Jay and Kono bought a sculpture, created by a widely respected artist, for \$9600. Its value appreciates (increases in value by) 9% every year. To buy the sculpture, they borrowed \$9600 at 7.75% p.a. with interest added monthly on the account balance. After 7 years, they hope to sell the sculpture at the expected increased value and repay the loan plus interest. Calculate the overall profit they should have, to the nearest dollar.

Exercise 5.2

15 A couple decide to put an amount of money into an investment that offers 7% p.a. interest, compounded monthly, for 18 months. How much will they need to invest, to the nearest dollar, if they want a final return of at least \$25 000?

Exercise 5.2

16 After 4 years of earning 4.92% p.a. interest compounded monthly, Darcy's bank balance was \$1045.67. What was the balance for the account 4 years ago?

Exercise 5.2

17 On 1 September 2017, Madison invested her savings of \$8000 with the Skyways Building Society. The interest rate was 4.8% p.a. on investments for fixed term deposits. Interest was paid into the account twice each year. What was the value of Madison's investment on the following dates?

(a) 1 September 2018

(b) 1 September 2019

Exercise 5.2

18 Jaman and Ann decide to put aside an amount of money as an investment for their children's education. They invest for 12 years in a fund that offers 7% p.a. interest compounded monthly. What amount will they need to invest, to the nearest dollar, if they want a final return of at least \$25 000?

Exercise 5.4

19 An amount of \$7250 is deposited into a savings account. Interest is paid at 0.4% of the account balance each month. Calculate the effective annual rate of interest, to 2 decimal places.

Exercise 5.4

20 Consider the following statements about an account that earns 9% p.a. compounding annually. Determine whether each statement is true or false.

(a) The nominal annual rate of interest is 9%.

(b) The effective annual rate of interest is 9%.

21 An amount of \$10 000 is invested for 15 years at 6.75% p.a. Determine the interest earned, to the nearest dollar, for the following compounding periods. Exercise 5.3

(a) annually

(b) monthly

(c) daily

22 A financial adviser offers a client two options for an investment of \$12 500. Exercise 5.3

Option 1: 4.5% p.a. compounded daily

Option 2: 4.6% p.a. compounded half-yearly

(a) Calculate the balance for each investment option after 2 years.

(b) Calculate the balance for each investment option after 10 years.

23 Tess has \$1400 to invest. A credit union account offers interest at 3.95% p.a., compounding quarterly. A bank account offers 3.8% p.a., compounding monthly. After 2 years, by how much would the amount in the credit union account exceed the amount in the bank account? Exercise 5.3

24 The Feelgood Bank offers loans at 5.6% p.a., compounding quarterly for 3 years. Determine the effective annual rate of interest for this loan, to 2 decimal places. Exercise 5.4

25 The table shows the balance in an investment account for the first three quarters, for a loan of \$9700. Exercise 5.4

Year	Balance at end of quarter n, A_n	Calculation to find the next balance	Next balance A_{n+1}
0	$A_0 = \$9700$	$9700 \times \left(1 + \frac{0.085}{4}\right) = 9906.13$	$A_1 = \$9906.13$
1	$A_1 = \$9906.13$	$9906.13 \times \left(1 + \frac{0.085}{4}\right) = 10116.64\ldots$	$A_2 = \$10116.64$
2	$A_2 = \$10116.64$	$10116.64\ldots \times \left(1 + \frac{0.085}{4}\right) = 10\,331.61\ldots$	$A_3 = \$10\,331.61$

What is the effective annual rate of interest for this investment?

26 Ming thinks he has found a strategy to make easy money. He will borrow \$20 000 at an interest rate of 5.4% p.a. compounding quarterly. Then he will invest the money with another company that will pay interest at 5.4% p.a. compounding monthly. At the end of 10 years, he will use his investment balance to pay back the loan plus added interest. Exercise 5.4

(a) Write a recurrence relation for the loan and for the investment.

(b) By calculating the effective annual rate of interest for each account, explain why Ming's strategy works.

5 CHAPTER REVIEW

Exercise 5.2, 5.4

27 The Ibrahim family invested \$240 000 in an online savings account that paid interest each month on the account balance. Initially, the interest rate was 2.85% p.a., but after 12 months it increased to 3.15% p.a.

(a) Calculate the effective annual rate of interest for the first year. Give your answer to 2 decimal places.

(b) Calculate the account balance 3 years after the investment began.

(c) By what percentage, to 2 decimal places, did the investment grow in the first 3 years?

(d) After the first 3 years, the Ibrahim family added another \$40 000 to the investment account at the previous 3.15% p.a. compounding monthly. Two years later, they closed the account and used all the funds to buy a new home for \$863 180. They did not have enough money for the entire amount, so they took out a loan. How much did they need to borrow? Give your answer rounded up to the nearest thousand dollars.

(e) The loan the Ibrahim family took out had interest calculated quarterly at a rate of 5.45% p.a. If no repayments were made on this loan, how much would the Ibrahim family owe on the loan after 15 months? Give your answer to the nearest dollar.

Exercise 5.2

28 Each year, Liang's investment account grows by 6%. How long will it take to at least double the investment?

Exercise 5.2

29 Sharon invested \$2000 at 4.4% p.a., compounding quarterly. After 5 years, she closed that investment account and put all the money into another account that pays 4.7% p.a., compounding monthly. In another 4 years, the value of this new account will be closest to which value?

A \$2200 **B** \$2500 **C** \$2800 **D** \$3000

Exercise 5.2

30 A loan of \$5000 had a rate of interest of 6.3% p.a. compounded quarterly. It was paid when the balance first exceeded \$7500. How long did it take for the loan to reach that balance?

Exercise 5.2

31 An investment of \$4250 earned \$360 interest from 1 July 2017 to 1 July 2018. If the interest was compounded daily, determine the annual rate of interest, to 2 decimal places.

Exercise 5.2

32 Vanita decides to invest her money to save for an extended trip around Australia, planned to start in 36 months. She finds that she will need \$25 000 to buy a second-hand four-wheel-drive. She currently has \$19 500 and is looking for an investment that compounds monthly. What annual rate of interest, to 2 decimal places, will this investment have to earn to eventually pay for her car?

Exercise 5.2

33 When Shanara invested \$1200 at 8% p.a. compounded monthly, the total interest earned was \$324.28. For how long was the original amount invested?

Exercise 5.2

34 How long will it take for an investment of \$2000 to increase by at least \$500 if the rate of interest is 7.5% p.a. compounded weekly? Round your answer to the nearest week. Assume there are 52 weeks in a year.

35 A loan of \$13 800 can be taken at 6.8% p.a., compounding each year, or at 6.55% p.a., compounding monthly. The 6.8% p.a. option is the better choice for the loan for which term? Exercise 5.3

A up to and including 12 months
B up to and including 13 months
C 12 months
D 12 months and 13 months

36 The rates of interest of two investment options were compared. Option 1 started with \$7000 and grew to \$8743.90 in 4 years after earning interest each quarter. Option 2 started with \$9200 and grew to \$10 002.74 after compounding monthly for 18 months. Exercise 5.4

In which investment option was the effective rate of interest greater?

37 Lachlan owes his credit card company \$5500. Interest is added daily at the rate of 21.3% p.a. He could clear the credit card debt by taking out a personal loan of \$5500 for 3 months. He would then be charged interest at 4% p.a. for the first month and then 8.5% p.a., compounding monthly for the remainder of the term of the loan. Exercise 5.2

Determine the amount that Lachlan will save over 3 months by taking the personal loan. Assume that there are 90 days in 3 months.

38 A company invested \$25 000 at 5.5% p.a. compounded monthly. They intend to use the funds to buy a new company car for \$32 000. After 18 months, the rate of interest dropped to 5% p.a. What is the impact of this change of rate on the time the company will need to wait before making the purchase? Exercise 5.2

39 Indiana compares two options for an investment of \$30 000. Exercise 5.3

Option 1: Compound interest 4.25% p.a. calculated and added monthly

Option 2: Compound interest 4.4% p.a. calculated and added annually

(a) After 12 months, by how much would the Option 2 balance exceed the Option 1 balance?

(b) How many months after the end of the first year will the balance for Option 1 be greater than the balance for Option 2?

(c) Indiana decided to invest in Option 2, at 4.4% p.a. After 2 years, she closed this account and reinvested the entire balance at a higher rate of 5% p.a., compounding annually. Determine the balance of the new account after a further 2 years.

(d) If Indiana had invested in Option 1 for the entire 4 years, how would this account balance compare to the account balance found in part **(c)**?

(e) What interest rate, compounding annually, would result in an account having the same balance after 2 years that Option 1 would have after 2 years? Give your answer to 2 decimal places.

6

Reducing balance loans, annuities and perpetuities

6 Recall

Multiply and divide by 100

1 Calculate the following:

(a) 0.086×100 (b) $6.4 \div 100$ (c) $0.53 \div 100$

Convert a percentage to a decimal

2 Convert the following percentages to decimals.

(a) 25% (b) 4.2% (c) 0.8%

Use a recurrence relation

3 Determine the first two terms that follow the initial term A_0 from the mathematical models.

(a) $A_0 = 287, A_{n+1} = A_n - 23$ (b) $A_0 = 350, A_{n+1} = 1.12A_n$

Calculate a compounding growth factor

4 Given that $V_{n+1} = r \times V_n$, where $r = (1+i)$ is the growth factor, calculate the value of r as a decimal number given the following conditions.

(a) V_n increases by 20% as n increases by 1

(b) V_n increases by 6% as n increases by 1

(c) V_n increases by 0.25% as n increases by 1

Use the compound interest formula

5 Use the compound interest formula $A = P(1+i)^n$ or $A_n = A_0 r^n$ to answer each of the following questions.

(a) Determine the value of \$800 after one year if interest at 4% p.a. is calculated on the increasing balance every quarter.

(b) How many months will it take for \$1200 to earn \$200 in compound interest at 6% p.a. calculated each month?

(c) Luke's wage increased by 3.6% for each of the past 4 years. If his weekly wage is now \$925.48, what was his weekly wage 4 years ago?

(d) Wayung bought 200 shares of stock at \$28.11 each. Exactly 4 years later, the value of these shares had increased to \$32.76 each. What was the annual rate of increase in the value of these shares? Give your answer to 2 decimal places.

Use a spreadsheet

6 Use a spreadsheet to answer the following questions.

(a) An investment account begins with \$8285.40. Interest is added monthly at 6.02% p.a. of the increasing balance. How long will it take for this account to reach at least \$9000?

(b) An \$8000 loan at 8.75% p.a. calculated monthly has its interest rate increased to 9.25% p.a. after one year. What is the loan account balance at the end of the second year?

(c) Rogan invested \$780 in an account at 5.08% p.a. calculated monthly. After 2 years, he added another \$500 to the account. If the interest rate continued at 5.08% p.a., determine the value of the investment 6 years after the original investment.

Repaying a loan

6.1

Reducing balance loan

The focus of the previous chapter was on compound interest loans where interest accumulated each period. The value of the loan increased geometrically until the term of the loan was reached. Without periodic repayments during the term of the loan, the amount owing can rapidly build to an overwhelming debt.

Most loans have an agreed term, or duration, and require periodic repayments. A portion of each repayment will pay the interest for the period. The remaining portion will reduce the outstanding loan account balance. This new balance, or principal, is then used to calculate the interest for the next period.

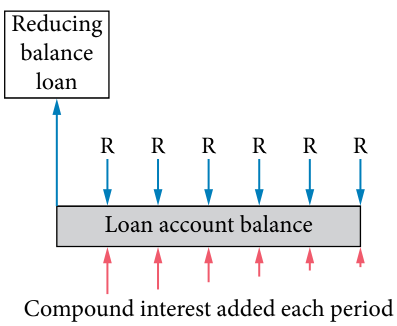
This type of loan is called a reducing balance loan.

> **Additional information**
> **Repaying a loan**
> This activity explores loan repayments.

A reducing balance loan begins with borrowing money, the initial principal, to start a loan account.

The account balance, the principal at the start of each new period, is increased when compound interest is added at regular compounding periods (pink arrows).

The principal is decreased with each periodic repayment (blue arrows, R), made immediately after the interest calculation for that compounding period.



Recurrence relation for a loan with periodic repayments:

L_0 = amount borrowed

L_n = loan account balance after n compounding periods

$L_{n+1} = rL_n - R$ where r is the growth factor due to periodic interest and R is the periodic repayment.

Each repayment is apportioned to paying interest, and the remainder to reducing the principal.

1 A recurrence relation for a loan with periodic repayments

A loan of $10 400 is to be repaid with quarterly repayments of $420. Interest is charged on the outstanding loan balance each quarter at a rate of 6.4% p.a.

(a) Write a recurrence relation that models this loan.

THINKING	WORKING
1 Identify the variables. Ensure the period and the rate of interest use the same unit of time.	Principal: $L_0 = \$10\,400$ Growth rate: $i = \frac{6.4}{4}$ $= 1.6$ so $r = 1.016$ Repayment: $R = \$420$

2 Write a recurrence relation to model this loan and repayment.

$L_{n+1} = r \times L_n - R$

$L_0 = 10\,400,\ L_{n+1} = 1.016 \times L_n - 420$

(b) What was the interest charged for the first quarter?

Determine the interest charged on L_0 for the first quarter.

$\text{Interest} = \frac{i}{100} \times L_0$

$= 0.016 \times 10\,400$

$= \$166.40$

(c) How much of the first repayment was used to reduce the outstanding balance?

Subtract the interest portion from the repayment.

Repayment: \$420

Interest portion: \$166.40

Portion reducing the principal:

$420 - 166.40 = \$253.60$

The account balance after the first repayment is calculated by:

first reduced balance = borrowed principal + interest − repayment

This is the same as:

first reduced balance = borrowed principal − principal portion

2 Analyse the first repayment to determine the next balance

Simon repaid a loan with interest calculated each year at 6.48% p.a. and an annual repayment of \$1240. The interest portion of the first repayment was \$672.30.

(a) What was the principal portion of this repayment?

THINKING	WORKING
1 Subtract the interest portion from the repayment.	Repayment: \$1240 Interest portion: \$672.30 Portion reducing the principal: $1240 - 672.30 = \$567.70$
2 Interpret the result.	Simon paid \$672.30 in interest and \$567.70 off the principal.

(b) How much had Simon borrowed?

1 Use the simple interest formula $I = Pin$ where annual repayment gives $n = 1$.	$i = 6.48\%$ p.a. $= 0.0648$ Interest: \$672.30 $I = Pin$ $672.30 = P \times 0.0648 \times 1$ $P = \frac{672.30}{0.0648}$ $P = 10\,375$
2 Interpret the answer.	Simon had borrowed \$10 375.

(c) What was the loan account balance after the first repayment?

1 Add the interest to the previous balance and subtract the repayment.	$L_1 = L_0 + \text{interest} - \text{repayment}$ $L_1 = 10\,375 + 672.30 - 1240.00$ $= 9807.30$
2 Interpret the result.	The loan account balance after the first repayment was \$9807.30.

3 Analyse the first repayment to determine the nominal rate

Tonia's loan of \$4000 reduced to \$3521.73 after her first monthly repayment of \$500.

(a) How much interest was charged for the first month?

THINKING	WORKING
1 Subtract to calculate the amount of reduction on the principal.	Repayment: \$500 Portion reducing the principal: 4000 – 3521.73 = \$478.27
2 Subtract the principal portion from the repayment.	Interest portion for the first month: 500 – 478.27 = \$21.73

(b) What is the annual rate of compound interest charged for Tonia's loan, to 2 decimal places?

1 Use the simple interest formula $I = Pin$ where $n = 1$.	$P = 4000$, interest: \$21.73 $I = Pin$ $21.73 = 4000 \times i \times 1$ $i = \frac{21.73}{4000}$ $i = 0.005\,4325$
2 Interpret the value.	The monthly rate is 0.543 25%

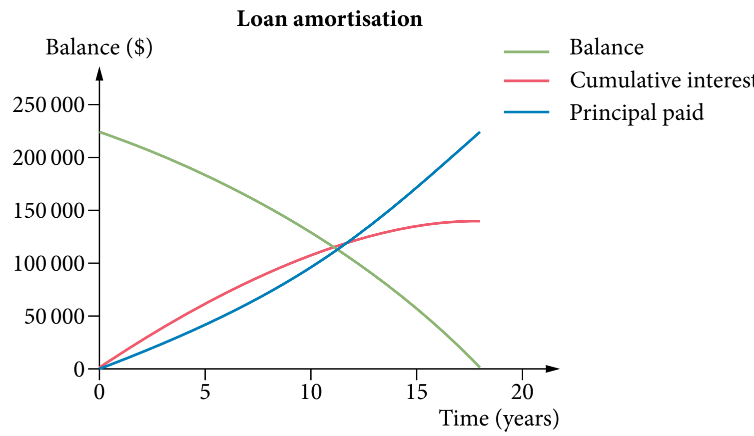
3 Convert to an annual rate and round as required.	Annual rate of interest $= 0.54325\%$ per month $\times 12$ months $= 6.519\%$ p.a.
4 Interpret the result.	The annual or nominal interest rate is 6.52%.

Amortising a reducing balance loan

Repayments for a loan are usually spread over a fixed period of time, called the term of the loan, with equal, periodic repayments.

To amortise a loan means to calculate the equal periodic repayment needed to fully repay the loan plus interest in a given period of time. This distributes the cost of repaying the loan evenly throughout the term of the loan.

When fully amortised, the periodic payments will fully repay the loan by the end of the fixed term.



4 Calculating the term of a loan

Gemma needs $22 350 to pay for home repairs. A bank has offered a loan with interest calculated at 6.6% p.a., charged monthly on the reducing account balance. She will make repayments of $500 per month until the loan is completely repaid.

(a) Write a recurrence relation to model the periodic change in the loan account balance.

THINKING	WORKING
1 Identify the variables. Ensure the period and the rate of interest use the same unit of time.	Principal: $L_0 = \$22\,350$ Growth rate: $i = \frac{6.6\%}{12}$ per month $= 0.55\%$ per month $= 0.0055$ So $r = 1.0055$ Repayment: $R = \$500$
2 Write a recurrence relation to model this loan and repayment.	$L_{n+1} = r \times L_n - R$ $L_0 = 22\,350,\ L_{n+1} = 1.0055 \times L_n - 500$

(b) Use a spreadsheet to determine how long it will take Gemma to repay this loan.

1 Enter values of n in the first column.

Enter the amount borrowed L_0 as the first balance.

Set up a formula for the second balance L_1 to multiply the first balance by the growth factor, then subtract the repayment amount.

Here the formula for B3 is:
=B2 * 1.0055 − 500.

Click and drag down until the balance first becomes negative.

	A	B
1	n	Balance
2	0	22350.00
3	1	21972.93
4	2	21593.78
53	51	220.87
54	52	−277.91

2 Interpret the result.

At the end of month 51, the balance is still \$220.87.

It will take Gemma 52 months to repay this loan.

(c) Determine the amount of the final repayment.

Reduce the usual payment by the amount indicated as the final (negative) balance, altering the final balance to zero.

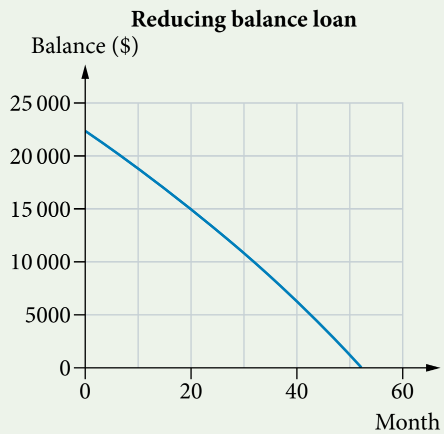
500 – 277.91 = 222.09

The final repayment will be \$222.09.

(d) Graph the balance over the period of the loan. Interpret the shape of the graph.

1 Ensure the final balance is zero.

Sketch a graph showing the balance over the period of the loan.



2 Interpret the shape.

The graph curves down to the right as the interest portion of the repayments decrease, and so the balance decreases by larger amounts each period.

(e) How much interest was Gemma charged for the loan?

1 Determine the total amount repaid by adding together all repayments.	Total repaid $= 51 \times 500 + 222.09$ $= \$25722.09$
2 Subtract the amount borrowed from the total repayments.	Interest = total repaid − total borrowed $= 25722.09 - 22350$ $= \$3372.09$

(f) How much of the final repayment went to paying interest?

Determine the interest charged for a month's loan of the balance in the second-last month.	$L_{51} = \$220.87$ Interest in the final month: $\frac{i}{100} \times L_{51} = 0.0055 \times 220.87$ $= \$1.21\ (2\ \text{d.p.})$

When a loan account balance in a table changes sign from positive to negative, the final, reduced repayment is made at this point, adjusting the balance to zero.

Fixed term loans and balloons

If a reducing balance loan is for a fixed period of time, periodic repayments may not repay the entire loan within the term of the loan.

Any remaining principal after the regular final repayment in a fixed term loan is called a balloon. The final repayment must be increased to include the balloon, to fully repay the loan.

5 Calculating a balloon payment

Tamara borrowed \$10 000 for a four year fixed term. Interest is added annually at a rate of 15% p.a. She will repay \$2000 at the end of each year.

(a) Tabulate the balance due with regular payments over the term of the loan.

THINKING	WORKING
1 Determine the recurrence relation.	$L_0 = 10\,000,\ r = 1.15,\ R = 2000$ $L_{n+1} = 1.15 \times L_n - 2000$
2 Calculate the reducing balances. Once the first term is 'calculated' in the scientific calculator, use the formula: $\times\ 1.15 - 2000 =$	$L_1 = 1.15 \times 10000 - 2000$ $= 9500$ $L_2 = 1.15 \times L_1 - 2000$ $= 1.15 \times 9500 - 2000$ $= 8925$

Then continue to press = for each subsequent term.

$$L_3 = 8925 \times 1.15 - 2000 = 8263.75$$

$$L_4 = 8263.75 \times 1.15 - 2000 = 7503.31\ (2\ \text{d.p.})$$

3 Complete the table.

Year, n	Balance ($)
0	10 000.00
1	9 500.00
2	8 925.00
3	8 263.75
4	7 503.31

(b) Calculate the balloon payment.

1 Add the balloon, the final balance, to the usual payment.

Balloon payment = outstanding balance + usual payment
= 7503.31 + 2000
= $9503.31

2 Interpret the result.

Tamara will pay $9503.31 in place of the regular payment.

The balloon payment, or final repayment that includes a balloon, may be a substantial sum of money. This can mean another loan is needed to find the funds to pay it. Borrowing money to repay borrowed money is sometimes called a debt cycle.

EXERCISE 6.1 Repaying a loan

Worked Example 1

1 A loan of $6000 is to be repaid with quarterly payments of $300. Interest is charged on the reducing balance each quarter at a rate of 5.2% p.a.

(a) Write a recurrence relation that models this loan.

(b) What was the interest charged for the first quarter? Give your answer to the nearest dollar.

(c) How much of the first repayment of $300 was used to reduce the outstanding balance? Give your answer to the nearest dollar.

2 State the interest component and the principal component of the first repayment for each of the loans modelled by the following recurrence relations.

(a) $L_0 = 350,\ L_{n+1} = 1.1 \times L_n - 400$

(b) $L_0 = 800,\ L_{n+1} = 1.05 \times L_n - 100$

(c) $L_0 = 3600,\ L_{n+1} = 1.045 \times L_n - 200$

3 Determine the value of L_1 for each of the following recurrence relations.

(a) $L_0 = 7900$, $L_{n+1} = 1.024 \times L_n - 400$

(b) $L_0 = 22\,000$, $L_{n+1} = 1.076 \times L_n - 540$

(c) $L_0 = 80\,000$, $L_{n+1} = \left(1 + \frac{0.145}{12}\right) \times L_n - 5000$

4 Write a recurrence relation for each of the following reducing balance loans.

(a) \$15 000 borrowed at 9.02% p.a. interest calculated every 6 months with repayments of \$1200 twice a year

(b) \$7684.50 borrowed at 9% p.a. interest calculated monthly with monthly repayments of \$320

(c) \$23 000 borrowed at 12.6% p.a. calculated quarterly with repayments of \$1000 per quarter

5 Natsuo wants to buy a \$9000 second-hand car and borrows the money with interest at 8.46% p.a. calculated monthly. She will fully repay the loan with equal monthly repayments of \$300.

If L_n = the loan account balance after n months, consider the following recurrence rules:

I	$L_{n+2} = 1.00705 \times L_{n+1} - 300$
II	$L_{n+1} = 1.00705 \times L_{n+2} - 300$
III	$L_{n+4} = 1.00705 \times L_{n+3} - 300$
IV	$L_{n+1} = 1.00705 \times L_n - 300$

With appropriate values for n, which of the recurrence rules could be used to calculate L_4?

A I, III and IV only

B II, III and IV only

C IV only

D all of I to IV

Worked Example 2

6 Khaled repaid a loan with interest calculated each year at 7.60% p.a. and an annual repayment of \$1730. The interest portion of the first repayment was \$981.92.

(a) What was the principal portion of this repayment?

(b) How much had Khaled borrowed?

(c) What was the loan account balance after the first repayment?

7 Interest is to be added monthly to a \$22 500 loan at the rate of 6.0% p.a. The loan will be repaid with equal monthly repayments of \$997.20. Complete the table to show the account balance for the first 3 months and the apportioning of each repayment to interest and to reducing the balance. Give your answers to 2 decimal places.

Period, n	Value of loan, L_n (\$)	Equal monthly repayments, R (\$)	Interest (\$)	Reduction in principal (\$)	Next principal, L_{n+1} (\$)
0	22 500.00	997.20		884.70	21 615.30
1	21 615.30		108.08		20 726.18
2	20 726.18	997.20		893.57	

8 A loan of $1380 has interest added each month at 27.6% p.a. The loan will be repaid with three equal monthly repayments of $481.32. Complete the table to show the account balance for the three months and the apportioning of each repayment to interest and to reducing the balance.

Period, n	Value of loan, L_n ($)	Equal monthly repayments, R ($)	Interest ($)	Reduction in principal ($)	Next principal, L_{n+1} ($)
0		481.32	31.74		930.42
1	930.42			459.92	
2	470.50	481.32		470.50	0.00
3	0.00	Loan has been repaid			

9 Periodic repayments of $750 per quarter are made to pay off an $11 000 loan. Interest is charged at 3.05% per quarter.

(a) Write a recurrence relation to model this loan.

(b) A spreadsheet, not yet adjusted for the final payment, shows $L_{19} = 538.97$ and $L_{20} = -194.59$. How much interest was charged in the final quarter of this loan?

(c) How much total interest was charged for this loan?

10 The first four months of an amortisation table for a $13 700 loan are shown.

Calculate the total interest charged throughout this loan.

Period, n	Value of loan, L_n ($)	Equal monthly repayments, R ($)	Reduction in principal ($)	Next principal, L_{n+1} ($)
0	13 700.00	3 526.63	3 364.97	10 335.03
1	10 355.03	3 526.63	3 404.68	6 930.35
2	6 930.35	3 526.63	3 444.85	3 485.50
3	3 485.50	3 526.63	3 485.50	0.00

11 Justin has taken out a loan of $12 000. Interest will be charged each month at a rate of 23.5% p.a. He has agreed to make monthly repayments of $600 until the loan has been fully repaid.

The amortisation table for the first four months of the loan is shown.

Period, n	Value of loan, L_n ($)	Interest added at the end of the next period ($)	Repayment ($)
0	12 000.00	705.00	600.00
1	12 105.00	711.17	600.00
2	12 216.17	717.70	600.00
3	12 333.87	724.61	600.00

(a) Based on the evidence in the table, which one of the following statements is true?

A The interest component is about 85% of the first repayment.

B The outstanding loan account balance is reducing every month.

C The interest charged each month is more than the repayment.

D The loan account balance is reduced by $117.70 at the third repayment.

(b) Explain the common error made by a student who chose the first incorrect option.

12 The table shows the first five periods of amortising a loan of $6750 over three years with interest added every six months. Equal repayments are made every half-year immediately after interest has been calculated.

Complete the final row, making any necessary adjustment to the final repayment.

Period, n	Value of loan, L_n ($)	Interest added at the end of the next period ($)	Balance after interest added ($)	Repayment ($)	Account balance reduced by ($)	L_{n+1} ($)
0	6750.00	270.00	7020.00	1287.64	1017.64	5732.36
1	5732.36	229.29	5961.65	1287.64	1058.35	4674.01
2	4674.01	186.96	4860.97	1287.64	1100.68	3573.33
3	3573.33	142.93	3716.27	1287.64	1144.71	2428.63
4	2428.63	97.15	2525.77	1287.64	1190.49	1238.13
5	1238.13				1238.13	0.00

13 Kane borrowed $5000 at 6% p.a. interest, charged on the outstanding balance every month. He repaid $25 each month. Determine the outstanding balance at the end of two years. Explain what has happened.

14 Aaron keeps track of his reducing balance loan each month with a series of calculations. Some of these calculations are shown in the table.

Month, n	Calculation of balance at the end of month n	Balance at end of month n ($)
1	$L_1 = 1.006 \times 2685 - 225$	2476.11
2	$L_2 = 1.006 \times 2476.11 - 225$	2265.97
3	$L_3 = 1.006 \times 2265.97 - 225$	
4		1841.90

(a) Determine the annual rate of interest for this loan.

(b) How much does Aaron repay each month?

(c) How much was the initial borrowed principal?

(d) Determine the loan account balance for the end of month 3.

(e) Write the calculation Aaron would use to determine the loan account balance at the end of month 4.

(f) Determine the account balance at the end of month 5.

15 Tasha borrowed $3500 at 8% p.a. to be repaid with equal monthly repayments of $597.02.

The amortisation table shows how Tasha intended to repay the loan.

Period, n	Value of loan, L_n ($)	Interest added at the end of the next period ($)	Balance after interest added ($)	Repayment ($)	Account balance reduced by ($)	L_{n+1} ($)
0	3500.00	23.33	3523.33	597.02	573.69	2926.31
1	2926.31	19.51	2945.82	597.02	577.51	2348.80
2	2348.80	15.66	2364.46	597.02	581.36	1767.44
3	1767.44	11.78	1779.22	597.02	585.24	1182.20
4	1182.20	7.88	1190.09	597.02	589.14	593.07
5	593.07	3.95	597.02	597.02	593.07	0.00

(a) How long will it take Tasha to repay this loan?

(b) Determine the total amount of interest that will be charged.

(c) To 2 decimal places, what percentage of the fourth repayment is the interest component?

(d) During the six months, Tasha amended the amortisation table to account for one significant change to the amortisation of the loan. The amended table is shown here.

Period, n	Value of loan, L_n ($)	Interest added at the end of the next period ($)	Balance after interest added ($)	Repayment ($)	Account balance reduced by ($)	L_{n+1} ($)
0	3500.00	23.33	3523.33	597.02	573.69	2926.31
1	2926.31	19.51	2945.82	597.02	577.51	2348.80
2	2348.80	15.66	2364.46	1297.02	1281.36	1064.44
3	1067.44	7.12	1074.56	597.02	589.90	477.54
4	477.54	3.18	480.72	480.72	477.54	0.00
5	0.00	Loan fully repaid				

Determine the change that occurred. Explain how this affected the length of the loan and how much interest Tasha saved.

Worked Example 3

16 Ivana's loan of $5000 reduced to $4431.67 after her first monthly repayment of $600.

(a) How much interest was charged for the first month?

(b) What is the annual rate of compound interest charged for Ivana's loan, to 2 decimal places?

17 Ayu borrowed $14 600 to buy a car, to be repaid at $250 each month. Interest at 6.35% p.a. is charged each month on the reducing balance. Use a spreadsheet to determine how much Ayu still owes after four years.

18 Rainie pays $850 every quarter to repay a loan of $3875. Interest is charged at 7.8% p.a. calculated quarterly.

(a) Write a recurrence relation to model this loan.

(b) Use tabulated values to determine how long it will take to fully repay this loan.

(c) By how much does her fourth repayment of $850 reduce the principal?

(d) How much is her last payment?

19 A loan of $6440 has interest added each quarter at 5.4% p.a. The loan will be repaid in one year with equal quarterly repayments of $1664.70.

Write a recurrence relation to model this loan and construct an amortisation table using the following headings:

Period, n	Value of loan, L_n ($)	Payment at the end of the next period ($)	Interest portion of the payment ($)	Principal reduced by ($)	Next principal, L_{n+1} ($)

4

20 Hilary damages her own car and another car, to the total cost of \$31 600. She is charged with dangerous driving, so the insurance company cannot pay for the repairs and she now needs to borrow the money. Her bank offers a loan with interest calculated at 4.68% p.a., charged monthly on the reducing account balance. She will make a payment of \$900 each month until the loan of \$31 600 is completely repaid.

(a) Write a recurrence relation to model this loan.

(b) Use a spreadsheet to determine how long it will take Hilary to repay this loan.

(c) Calculate the value of the final repayment.

(d) Graph the balance of the loan against time. Explain why the graph is not linear.

(e) How much interest will Hilary pay over the life of her loan?

(f) How much of the final repayment will go to paying interest?

21 A loan of \$7820 has interest added monthly at a rate of 18.6% p.a. and repayments of \$520 per month.

(a) Write a recurrence relation to model this loan.

(b) Determine the first time at which the outstanding loan is reduced to less than half the borrowed amount.

(c) When the balance falls below the repayment amount, the final repayment is increased to include any remaining loan balance. Determine the final repayment.

(d) Calculate the total interest paid.

5

22 Kira has borrowed \$9340 for four years with interest charged at 6.59% p.a. calculated every six months. She can afford to repay only \$1000 every six months. Including the usual repayment, how much will she have to pay at the end of four years, to pay off this loan?

23 A loan of \$7040 for three years has interest charged at 5.47% p.a., calculated monthly on the reducing balance. Repayments of \$200 per month will be made for three years to pay off this loan.

(a) Generate a sequence from a recurrence relation and calculate the loan account balance after five months.

(b) How much of the first repayment of \$200 actually went to paying interest for the first period?

(c) A spreadsheet from the recurrence relation shows $L_{36} = 487.19$. Determine the final repayment that will ensure that the loan is repaid on time.

24 A company borrows \$30 000 for a new computer system at 7.14% p.a. compounding quarterly. It will repay the loan at \$3000 per quarter.

(a) Determine the remaining balance at the end of the first year.

(b) When will the outstanding balance first be less than half the original loan?

(c) What is the term of this loan?

(d) How much should the final repayment be?

(e) What percentage of the first repayment is used to pay interest? Give your answer to 2 decimal places.

25 In 2016, Anita took out a loan to pay for some shares. Compound interest was charged annually and she repaid \$3000 each year towards the loan. One year later she still owed \$18 303.75 on the loan, and another year later she owed \$16 996.85.

Calculate the initial amount borrowed.

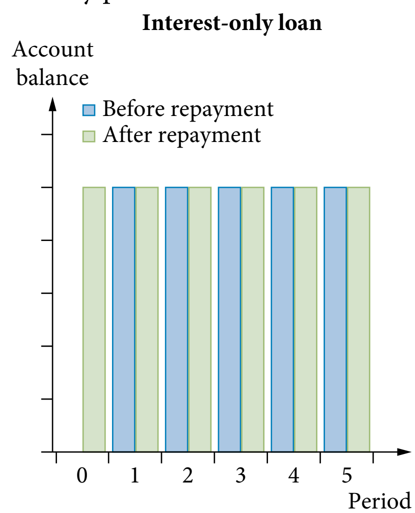
26 Of Yusuf's first repayment on a \$40 200 loan, 85% was directed to paying interest of \$580.48. Calculate the loan account balance after the first repayment.

6.2 Term of a loan

Interest-only loan

If the periodic repayment exactly equals the interest due for each period, the principal (initial amount of money) will remain the same indefinitely. Each repayment will pay only the interest for this constant outstanding balance, equal to the initial borrowed principal, at the end of every period.

An interest-only loan is popular with some investors. Repayments are the minimum needed to avoid an increasing principal. Meanwhile, the value of the asset that was purchased with the loan may increase in value due to inflation or a number of other reasons.



Interest-only loan

Each periodic repayment is exactly 100% of the interest for each period.

The interest due is the same for every period.

The term of the loan is indefinite.

6 Interest-only loan

An interest-only loan of \$9000 has interest charged each quarter at 6.4% p.a.

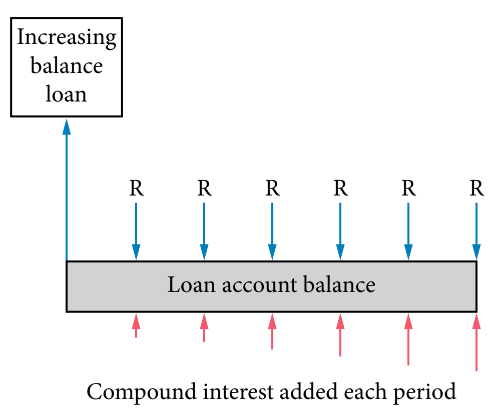
Determine the quarterly payment needed.

THINKING	WORKING
1 Determine the interest for the first period.	Original principal: \$9000 6.4% p.a. quarterly, so interest for one period is: $\frac{0.064}{4} \times 9000 = \$144$
2 Interpret the result.	Repayments of \$144 will pay the interest charged each quarter.

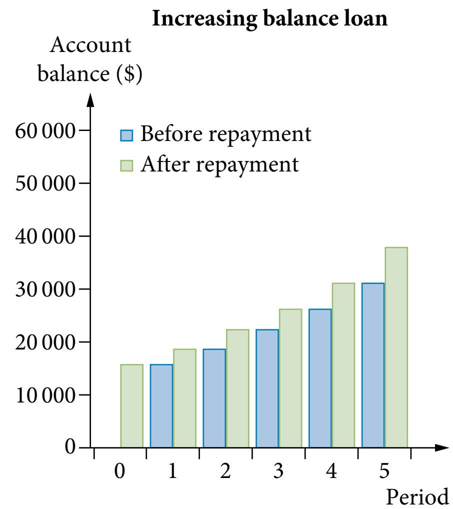
Increasing balance loans

If the periodic repayment is not enough to pay the entire interest for the first period, then the outstanding balance will increase at the end of every period. The rate of increase will depend on how much the first interest calculation exceeds the periodic repayment.

At each repayment, the excess interest beyond the repayment amount also increases and the outstanding loan account balance continues to increase geometrically.



An increasing balance loan should only be considered as a short-term solution, such as when there is a possibility of paying a lump sum, or increasing the repayments at a later date.



Increasing balance loan

Interest for the first period is greater than the repayment amount.

The principal increases after each repayment.

The interest each period increases as the principal increases.

The term of the loan is indefinite.

Types of loans

The change in balance after the first repayment determines the type of loan.

The balance decreases after the first repayment.	Reducing balance loan
The balance increases after the first repayment.	Increasing balance loan
The balance remains the same after the first repayment.	Interest-only loan

7 Interest at the first periodic repayment

Periodic repayments of \$150 are made each month on a loan of \$35 000 with interest calculated on the outstanding loan balance every month at a rate of 6% p.a.

(a) Determine the interest charged for the first period.

THINKING	WORKING
1 Determine the interest rate for one period.	For this loan, a period is one month. $i = \frac{6\%}{12}$ per month $= 0.5\%$ per month $= 0.005$
2 Calculate the interest for the first period.	0.5% of \$35 000 $= 0.05 \times 35\,000$ $=$ \$175

(b) Explain how a repayment of \$150 per month affects this loan.

1 Determine the difference in the principal.	Difference in principal = interest − repayment $= 175 - 150$ $= 25$
2 Interpret the answer.	The principal will increase by \$25 after the first repayment. This is an increasing balance loan.

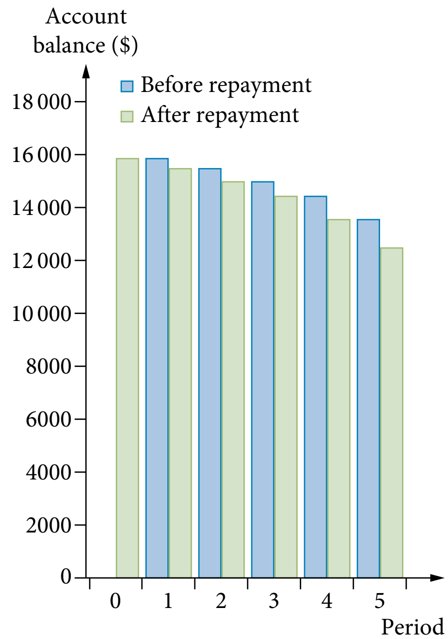
(c) To the nearest $10, what is the least repayment required to make this a reducing balance loan?

Repayments must exceed the interest for the first period.	Interest for the first period is $175. To the nearest $10, in order for this to be a reducing balance loan, the smallest repayment must be $180 per month.

Reducing balance loans

As long as the repayment amount exceeds the interest for the first period, the principal will reduce each period and the loan will eventually terminate when the principal reduces to zero.

For a loan with a fixed repayment, any change in interest rate is likely to lengthen or shorten the loan.



A higher rate of interest will (generally) lengthen the term of a loan.

A lower rate of interest will (generally) shorten the term of a loan.

Explore further

Types of loans

Explore reducing balance, increasing balance and interest-only loans using a spreadsheet.

8 The effect of interest rate on the term of a loan

A car dealership offers a loan of $21 400 to buy a car at 6% p.a. calculated monthly. Repayments of $1200 each month will pay out this loan.

(a) Determine the term of this loan.

THINKING	WORKING
1 Write the recurrence relation.	$L_0 = 21\,400, R = 1200$ $r = 1 + \frac{0.06}{12}$ $= 1 + 0.005$ $= 1.005$ $L_{n+1} = 1.005 \times L_n - 1200$

2 Use a spreadsheet to determine the length of the loan.

A scientific calculator can be used, but you need to carefully count, or write, each balance.

For L_1 in B3 the formula would be:

=B2 * 1.005–1200

	A	B
1	n	Balance
2	0	21400.00
3	1	20307.00
4	2	19208.54
19	17	2056.85
20	18	867.13
21	19	–328.53

3 Interpret the result.

The term of this loan is 19 months.

(b) How is the term of the loan affected if the interest rate is doubled to 12% p.a.?

1 Rewrite the recurrence relation with the new interest rate.

Interest rate

$$r = 1 + \frac{0.12}{12}$$
$$= 1 + 0.01$$
$$= 1.01$$

$L_0 = 21\,400$, $L_{n+1} = 1.01 \times L_n - 1200$

2 Alter your spreadsheet to determine the length of this loan.

For L_1 in B3 the formula would be:
=B2 * 1.01–1200

	A	B
1	n	Balance
2	0	21400.00
3	1	20414.00
4	2	19418.14
20	18	2059.86
21	19	880.46
22	20	–310.74

3 Interpret the result.

The term of the loan has increased from 19 months to 20 months.

For a loan with a fixed interest rate, any change in the size of the repayment is likely to lengthen or shorten the loan.

A higher repayment will (generally) shorten the term of a loan.

WARNING

A lower repayment may lengthen the term of a loan, or change it into a never-ending increasing balance loan.

9 The effect of repayment amount on the term of a loan

The monthly repayment on a \$4000 loan is \$350. Interest at 5.4% p.a. is calculated on the reducing balance each month.

If monthly repayments are doubled from \$350 to \$700, how does this affect the term of the loan?

THINKING	WORKING
1 Determine the growth factor.	$r = 1 + \frac{0.054}{12}$ $= 1.0045$
2 Write the recurrence relations.	Repayment A: $L_0 = 4000,\ L_{n+1} = 1.0045 \times L_n - 350$ Repayment B: $L_0 = 4000,\ L_{n+1} = 1.0045 \times L_n - 700$
3 Set up spreadsheets for each repayment amount. A scientific calculator could be used, with careful counting or recording of the balances.	Repayment A: For L_1 in B3 the formula is: =B2 * 1.0045–350 Repayment B: For L_1 in B3 the formula is: =B2 * 1.0045–700

	A	B
1	n	A
2	0	4000.00
3	1	3668.00
4	2	3334.51
12	10	611.96
13	11	264.71
14	12	–84.10

	A	B
1	n	B
2	0	4000.00
3	1	3318.00
4	2	2632.93
5	3	1944.78
6	4	1253.53
7	5	559.17
8	6	–138.31

4 Interpret the result.	The term is reduced from 12 months to 6 months when repayments are doubled from \$350 to \$700.

Determine a suitable repayment amount

Reducing balance loans are generally set up with a fixed term in mind.

The value of the repayment for a loan of a particular amount and fixed interest rate depends on the term of the loan.

10 Using a spreadsheet to find a suitable repayment

A loan of \$6870 is charged interest each month at 9.5% p.a.

Determine the minimum monthly repayment, to the nearest \$10, needed to repay this loan in less than two years.

THINKING	WORKING
1 Determine the rate of interest per period, expressed as a decimal.	$i = \frac{9.5\%}{12}$ per month $= \frac{0.095}{12}$
2 Determine the interest amount for the first period.	$\frac{9.5\%}{12}$ of $6870 = \frac{0.095}{12} \times 6870$ $= \$54.39$
3 Determine the maximum number of repayments.	$n = 2$ years $= 2 \times 12$ months per year $= 24$ months
4 Write a recurrence relation to model this loan.	$L_0 = 6870$ $r = 1 + \frac{0.095}{12}$ $L_{n+1} = \left(1 + \frac{0.095}{12}\right)L_n - R$
5 Estimate an approximate amount if no interest is charged.	If no interest was charged, the repayment amount would be $\frac{6870}{24} = \$286.25$ per month
6 Create a spreadsheet using repayment amounts of greater than the first interest amount, and multiples of \$10. Fill down to the required value of n. Choose the first value that gives a negative balance for the last term.	Try $R = 300, 310, \ldots$ and fill down to $n = 24$. For L_1 in B3 the first trial would have the formula: =B2 * (1+0.095/12)–300 The successful trial has: =B2 * (1+0.095/12)–320

	A	B
1	n	Balance
2	0	6870.00
3	1	6604.39
24	22	514.16
25	23	198.23
26	24	–120.20

7 Interpret the result.	The monthly repayment, to the nearest \$10, is \$320.

EXERCISE 6.2 Term of a loan

Worked Example

1 An interest-only loan of \$12 880 has interest charged each quarter at 14.68% p.a. Determine the amount of each quarterly payment. 6

2 Calculate the interest charged in each scenario.

(a) A loan of \$500 has interest charged at 2% per month. How much interest is charged in the first month?

(b) A loan of \$1000 has interest charged quarterly at a rate of 6% p.a. How much interest is charged in the first quarter?

(c) A loan of \$4000 has monthly interest charged at a rate of 18% p.a. How much interest is charged in the first month?

(d) A loan of \$12 000 has monthly interest charged at a rate of 9% p.a. How much interest is charged in the first month?

3 Periodic repayments of \$400 are made each month on a loan of \$80 000 with interest calculated on the outstanding loan balance every month at a rate of 9% p.a. 7

(a) Determine the interest charged for the first period.

(b) Explain how a periodic repayment of \$400 per month affects this loan.

(c) Determine the smallest periodic repayment amount required to make this a reducing balance loan. Give your answer to the nearest \$10.

4 Periodic repayments of \$450 are made each quarter on a loan of \$12 000. Interest is 3% p.a. compounding quarterly.

(a) Determine the interest charged for the first period.

(b) How does the answer to part (a) indicate that this is a reducing balance loan?

(c) Determine the account balance after the first repayment.

5 A loan of \$5700 has interest calculated monthly on the outstanding balance at a rate of 6.4% p.a.

(a) Determine the periodic repayment if this is to be an interest-only loan.

(b) What is the outstanding balance after two years?

6 Siobhan owes a finance company \$2063.30. Interest is added to the debt each month at 36% p.a. Siobhan has been trying to repay the loan, making payments of \$50 per month. How long will it take to repay this debt? Select the most appropriate answer.

A This is an interest-only loan. She will never repay the debt, only the interest, unless she increases her periodic repayment.

B This is an increasing balance loan. She will never repay this debt unless she increases her periodic repayment.

C She will repay her debt in approximately four months.

D She will repay her debt in approximately four years.

7 Dillon wants to get an interest-only loan of $2000 to buy some shares. He will be charged interest at the rate of 10.8% p.a. calculated monthly on the account balance.

(a) How much will Dillon have to repay each month?

(b) Dillon plans to sell his shares after three years. Taking into account the interest on the loan, for what price will he need to sell the shares if he wants an overall profit of $750?

8 Vijay and Bhavna plan to borrow $540 000 to buy a house. Initially they plan to make quarterly repayments of $8000. Interest on the loan is 6.55% p.a. compounding quarterly.

(a) Calculate the interest due for the first quarter.

(b) Determine the percentage of their planned quarterly repayment that will be required to pay the interest for the first quarter.

(c) Vijay and Bhavna continue with their plans to pay $8000 per quarter. Determine the balance of their loan after two years.

(d) The bank denies their plan to pay $8000 per quarter and instead insists that the repayments be at least $9000.

Calculate the balance of the loan, with $9000 repayments, after two years.

9 The Dodgy Loan Company has had official complaints about its repayment schedule for loans. It has adjusted contracts so that monthly repayments on a loan of $4000 for one year at 24% p.a. compounding monthly are $200. The company considers that its advertisement 'Repay only $200 per month' is reasonable.

(a) Write the recurrence relation for the loan.

(b) If a non-adjusted spreadsheet shows $L_{12} = \$2390.55$, how much must be paid at the end of the year for this loan arrangement?

(c) The company offers a different loan of $2500 at a rate of 30% p.a. compounding monthly, with repayments of $200 until the loan is repaid. Write a recurrence relation to model this loan.

(d) Given the spreadsheet values $L_{15} = 34.36$ and $L_{16} = -164.78$, after how many months will this loan be repaid, and what will the final repayment be?

(e) Calculate the total amount to be paid for the $2500 loan.

10 Thomas wants to borrow money for a landscaping project. Interest on a loan will be charged at 7.6% p.a. calculated each quarter. Thomas has budgeted for maximum quarterly repayments of $400 for an interest-only loan for three years.

(a) The maximum loan his budget will allow can be calculated by solving which of the following equations?

A $L_0 = 0.076 \times 400$

B $L_0 = \dfrac{0.076}{4} \times 400$

C $L_0 \times \left(\dfrac{0.076}{4}\right) = 400$

D $L_0 \times (1 + 0.076) = 400$

(b) Explain the common error made by a student who chose the third incorrect option.

11 To pay his mobile phone account, Jay had to borrow \$480. He plans to repay the loan at \$60 each month. The table shows the loan account balance L_n at the start of month n, the interest added to the loan account, the monthly repayment, and the loan account balance L_{n+1} at the end of month n.

Month n	Balance at end of month n, L_n (\$)	Interest added at the end of next month (\$)	Repayment at the end of next month (\$)	Balance at end of next month, L_{n+1} (\$)
0	480.00	480.00 × 1.008	60	423.84
1	423.84	423.84 × 1.008	60	367.23
2	367.23	367.23 × 1.008	60	310.17
3	310.17	310.17 × 1.008	60	252.65
4	252.65	252.65 × 1.008	60	194.67

(a) Determine the monthly rate of interest.

(b) Determine the annual rate of interest.

(c) Write a recurrence relation to model the account balance after n months.

(d) Determine Jay's loan account balance after six months.

(e) How long will it take Jay to repay this loan?

(f) How much is his final repayment?

(g) How much total interest will Jay have paid for the loan?

(h) If the rate of interest charged each month had been half that shown in the table but Jay still repaid \$60 each month, what would the total interest charged for the loan have been?

A unchanged

B half as much

C less than half as much

D more than half as much

12 A car dealership offers a loan of \$22 800 to buy a car at 5.4% p.a. calculated monthly. Repayments of \$690 will pay out this loan. Worked Example 8

(a) Use a spreadsheet to determine the term of this loan.

(b) How is the term of the loan affected if the interest rate is halved to 2.70% p.a.?

13 A block of land is advertised for \$380 000. It can be purchased with 10% deposit and the remainder as a loan to be repaid at 6.85% p.a. charged monthly on the outstanding account balance. Repayments are set at \$1300 per month for five years. Describe what will happen to the loan balance over the five years.

14 The monthly repayment on a \$12 000 loan is \$350. Interest at 6.6% p.a. is calculated on the reducing balance each month. If monthly repayments are doubled from \$350 to \$700, use a spreadsheet to determine how this affects the term of the loan. 9

15 A store will arrange the provision and laying of carpet for Jasmine's house. They can lend her the cost of $3670 with 10% p.a. interest calculated quarterly on the reducing balance. She would repay this loan with quarterly payments of $1400.

(a) When will this loan be fully repaid?

(b) How much interest will Jasmine pay for this loan?

(c) Jasmine took the loan of $3670 at 10% p.a. but she could only repay $700 each quarter. When will this loan now be fully repaid?

(d) How much extra interest will this reduced periodic repayment cost her?

16 The Dodgy Loan Company is always willing to help out people who are in a financial crisis. They will lend $4000 for one year to anyone who commits to their loan terms. Interest is charged at a rate of 24% p.a. compounding monthly and repayments are only $65 per month.

(a) Write a recurrence relation to model this loan.

(b) Determine the loan account balance after the first repayment. Explain why the loan balance after the first month would be more than the sum borrowed.

(c) An unadjusted spreadsheet shows $L_{12} = 4201.18$. Determine the final repayment at the end of the year.

(d) How much interest would be charged for this one year loan?

17 Tumi borrowed $14 300 at an interest rate of 7.5% p.a., calculated on the balance every six months. She makes periodic repayments of $500 every six months.

(a) Explain why Tumi's periodic repayments will never repay this loan.

(b) After her fourth repayment, at the end of two years, Tumi realises that she must increase her periodic repayment to stop the outstanding balance from growing. Explain why increasing the repayments to $536.25, starting at her fifth repayment, will still mean she has an increasing balance loan.

(c) Starting with the fifth repayment, how much should Tumi pay every six months for the loan to become an interest-only loan?

Worked Example 10

18 A loan of $10 480 is charged interest each month at 8.8% p.a. Use a spreadsheet to determine the monthly repayment, to the nearest $10, needed to repay this loan in three years.

19 Kieran and Taylah borrowed $300 000 to buy a house at 4.15% p.a. with interest calculated quarterly. They will repay the loan over 25 years.

(a) Use a spreadsheet to determine the quarterly repayment, to the nearest $100.

(b) If they were able to afford repayments of $6000 each quarter, how long would it take to repay the loan?

20 Toby cut up his credit card when he realised it was really a loan card with a large monthly rate of interest for any outstanding account balance. His debt on the credit card is $4400. Toby believes he can only afford to make $75 monthly repayments. His friend has developed a table to show Toby that he must find a way to repay more each month.

Period, n	Balance at end of period, L_n ($)	Payment at the end of the next period ($)	Interest portion of the payment (increases the principal) ($)	Rest of payment, to reduce the principal ($)	Next principal, L_{n+1} ($)
0	4400.00	75.00	81.40	-6.40	4406.40
1	4406.40	75.00	81.52	-6.52	4412.92
2	4412.92	75.00	81.64	-6.64	4419.56
3	4419.56	75.00	81.76	-6.76	4426.32
4	4426.32	75.00	81.89	-6.89	4433.21

(a) Explain what the negative sign in the second-last column indicates.

(b) What is the monthly rate of interest charged for this debt?

(c) What is the minimum amount each month that Toby should repay if he does not want the account balance to increase?

(d) If Toby is able to repay $150 each month, use a spreadsheet to determine how long it would take to repay the debt.

(e) If Toby did pay off the credit card debt with repayments of $150 each month, how much total interest would he have paid in the first 12 months?

(f) Toby's friend has offered him a personal loan of $4400 to repay the credit card debt immediately. Toby will be charged 6% p.a. interest calculated monthly, and he agrees to make monthly repayments of $150. How long will it take Toby to repay this personal loan?

(g) How much total interest will Toby pay his friend in the first 12 months of the personal loan?

21 Lauren wants take out a loan with a term of 30 months at 7.95% p.a. compounding monthly, to invest in the stock market. She can afford to repay no more than $640 each month. Use a spreadsheet to determine the maximum amount that Lauren can borrow, to the nearest $10.

22 To build his new shed, Oliver borrowed $8700 at 6.56% p.a. Interest is calculated quarterly on the reducing balance and Oliver repays $600 every quarter.

(a) Use a spreadsheet to determine how much is owed on the loan after one year.

(b) How much interest will be charged for the loan in the first quarter of the second year?

(c) What is the term of this loan?

(d) What is the final repayment?

(e) To the nearest dollar, how much should Oliver repay each quarter if he wants to pay off this loan in three years?

6.3 Changing parameters

Parameters

During the course of a loan, changes are sometimes made to one of the parameters.

With home loans over a long term, it is usual for the interest rate to change many times.

Because higher repayments reduce the interest paid on loans, borrowers will increase the periodic repayment whenever their budget makes this possible.

11 Rate of interest changes during a loan

Azaria borrowed \$10 000 to pay for her wedding. Interest on the loan was charged quarterly at 20% p.a. with quarterly repayments of \$700. After the first year, Azaria was able to arrange a reduction in the interest rate to 8.0% p.a.

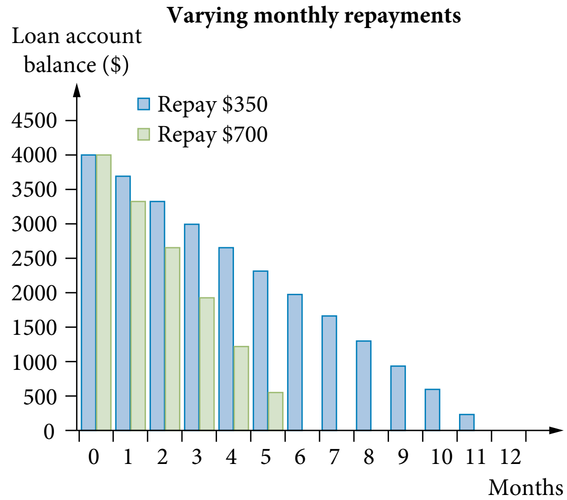
Determine the account balance at the end of the second year.

THINKING	WORKING
1 Write a recurrence relation for the first year.	Repayment: $R = 700$ Growth factor per quarter: $r = 1 + \frac{0.2}{4}$ $= 1.05$ $L_0 = 10\,000,\ L_{n+1} = 1.05 \times L_n - 700$
2 Generate terms for the balance through the first year.	$L_1 = 1.05 \times 10\,000 - 700 = 9800$ $L_2 = 1.05 \times 9800 - 700 = 9590$ $L_3 = 1.05 \times 9590 - 700 = 9369.50$ $L_4 = 1.05 \times 9369.50 - 700 = 9137.98$
3 Write a recurrence relation for the second year.	Repayment: $R = 700$ Growth factor per quarter: $r = 1 + \frac{0.08}{4}$ $= 1.02$ Initial balance: $L_4,\ L_{n+1} = 1.02 \times L_n - 700$
4 Generate terms for the balance through the second year.	$L_5 = 1.02 \times 9137.98 - 700 = 8620.74$ $L_6 = 1.02 \times 8620.74 - 700 = 8093.15$ $L_7 = 1.02 \times 8093.15 - 700 = 7555.01$ $L_8 = 1.02 \times 7555.01 - 700 = 7006.11$
5 Interpret the result.	The amount of the loan still owing after the second year is \$7006.11.

The periodic repayment may be increased or decreased during the term of a loan. This could be due to a wage increase that allows increased repayments or a circumstance where money is needed elsewhere and repayments must decrease.

The graph compares the monthly account balances of a $4000 loan with repayments of $350 and $700.

A greater periodic repayment will usually reduce the term of a loan.



12 Changing the periodic repayment

A loan of $42 000 at 6.4% p.a. interest calculated quarterly has quarterly repayments of $2400. After a year, the repayments will be increased to $4000 per quarter.

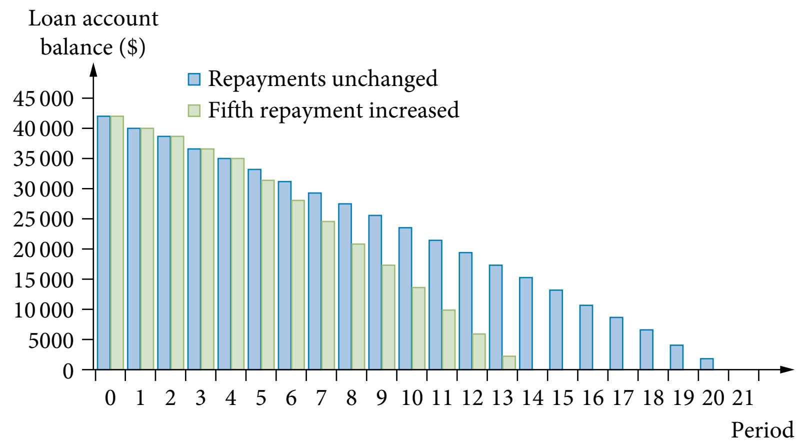
Determine the account balance at the end of 18 months.

THINKING	WORKING
1 Write a recurrence relation for the first year.	Repayment: $R = 2400$ Growth factor per quarter: $r = 1 + \frac{0.064}{4}$ $= 1.016$ $L_0 = 42\,000$, $L_{n+1} = 1.016 \times L_n - 2400$
2 Generate terms for the balance through the first year.	$L_1 = 1.016 \times 42\,000 - 2400 = 40\,272$ $L_2 = 1.016 \times 40\,272 - 2400 = 38\,516.35$ $L_3 = 1.016 \times 38\,516.35 - 2400 = 36\,732.61$ $L_4 = 1.016 \times 36\,732.61 - 2400 = 34\,920.33$
3 Write a recurrence relation for the second part of the loan.	Repayment: $R = 4000$ Initial balance: L_4, $L_{n+1} = 1.016 \times L_n - 4000$
4 Generate terms for the balance through the second part.	$L_5 = 1.016 \times 34\,920.33 - 4000 = 31\,479.06$ $L_6 = 1.016 \times 31\,479.06 - 4000 = 27\,982.72$
5 Interpret the result.	The account balance is $27 982.72 after 18 months.

The graph compares the projected account of a $42 000 loan when:

(a) periodic repayments of $2400 are made for the term of the loan

(b) periodic repayments of $2400 are made for the first four periods and then the repayments are increased to $4000 for the remainder of the term.



The increased repayment from the fifth quarter has significantly decreased the term of the loan.

Missing a periodic repayment

When a repayment is missed, the interest for that period compounds and the account balance increases for the next period.

13 Missing a periodic repayment

A loan of $18 000 attracts 7.2% p.a. interest calculated quarterly. Periodic repayments are $1500 per quarter but the third repayment was missed. All other repayments were made on time.

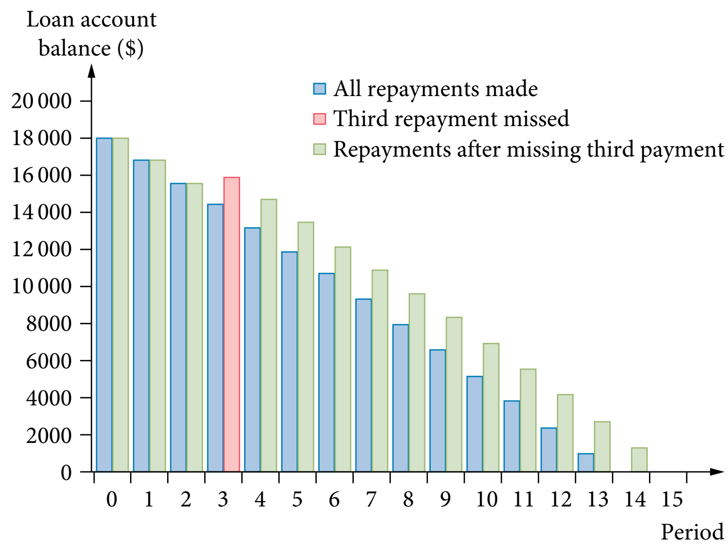
Calculate the loan account balance at the end of the first year.

THINKING	WORKING
1 Write a recurrence relation for the intended situation.	Repayment: $R = 1500$ Growth factor per quarter: $r = 1 + \frac{0.072}{4} = 1.018$ $L_0 = 18\,000,\ L_{n+1} = 1.018 \times L_n - 1500$
2 Generate terms for the first two balance changes.	$L_1 = 1.018 \times 18\,000 - 1500 = 16\,824$ $L_2 = 1.018 \times 16\,824 - 1500 = 15\,626.83$
3 Generate terms for the third balance change, omitting the repayment.	$L_3 = 1.018 \times 15\,626.83 = 15\,908.11$
4 Generate terms for the fourth balance change, reinstating the repayment.	$L_4 = 1.018 \times 15\,908.11 - 1500 = 14\,694.46$
5 Interpret the result.	The loan balance at the end of the first year is $14 694.46.

The graph compares the quarterly account balance of the $18 000 loan from the previous example when:

(a) all repayments are made

(b) the third repayment is missed. The third balance increases by the $281.28 interest charged per quarter.



The account balance decreased for the first two quarters. At the end of the third quarter, the account balance increased as interest was added but the principal was not reduced by a repayment. After this third quarter, the account balance again decreased with each repayment at the same rate as before the missed repayment, but the term of the loan had increased from 14 to 15 quarters.

Making an additional payment

If you can shorten the term of a loan, you will save interest. Making an additional payment if money becomes available is a common strategy, if the loan contract allows it.

14 Making an additional payment

A loan of $50 000 attracts 6.8% p.a. interest calculated quarterly with periodic repayments of $4000 per quarter. An additional $10 000 is paid once only, at the time the third repayment is made.

Use a spreadsheet to determine how much the term of the loan decreased and how much interest was saved by making the additional payment.

THINKING	WORKING
1 Write a recurrence relation for the loan, without the additional payment.	Repayment: $R = 4000$ Growth factor per quarter: $r = 1 + \frac{0.068}{4}$ $= 1.017$ $L_0 = 50\,000,\ L_{n+1} = 1.017 \times L_n - 4000$

2 Create a spreadsheet to show the account balances until the first negative value.

A scientific calculator could be used, with careful counting and recording of terms.

	A	B
1	n	Balance
2	0	50000.00
3	1	46850.00
4	2	43646.45
5	3	40388.44
16	14	679.55
17	15	–3308.90

3 Alter the spreadsheet by changing the formula in one cell in the balance column to account for the additional payment.

Alternatively, use a scientific calculator, recording the balances as you go.

If the formula for L_3 is in B5, the formula in just that cell would change from

=B4 * 1.017–4000

to

=B4 * 1.017–4000–10000

	A	B
1	n	Balance
2	0	50000.00
3	1	46850.00
4	2	43646.45
5	3	30388.44
13	11	805.58
14	12	–3180.72

4 Calculate the difference in the term for the loan.

The extra payment has reduced the term from 15 quarters to 12 quarters, a difference of 9 months.

5 Calculate the difference in total payments, and therefore the saving in interest.

Without the additional payment:

Total paid:

$4000 \times 15 - 3308.90 = \$55\,691.10$

With the additional payment:

Total paid:

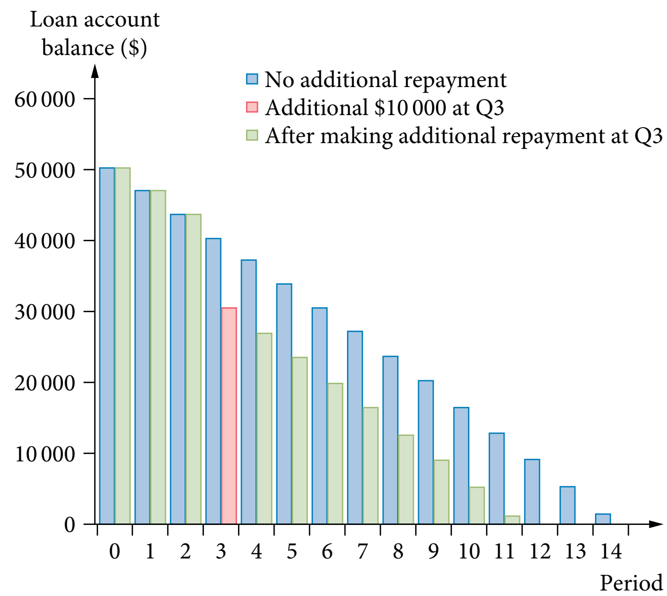
$4000 \times 12 - 3180.72 + 10\,000 = \$54\,819.28$

Saving in interest payments:

$55\,691.10 - 54\,819.28 = \$871.82$

The graph compares the quarterly account balance with regular repayments of the $50 000 loan in the previous example, with the balance if an additional $10 000 payment was made with the third periodic repayment.

Periodic repayments reduce the account balance. At the third quarter, an additional $10 000 payment was made and an immediate extra decrease is evident. Periodic repayments then continued to reduce the account balance at the same rate as before the additional payment.



EXERCISE 6.3 Changing parameters

Worked Example 11

1 Kate borrowed $10 000 for her wedding. Interest on the loan was charged quarterly at 19.6% p.a. with quarterly repayments of $550. After a year, Kate was able to arrange a reduction in the interest rate to 7.6% p.a. Determine the account balance at the end of the second year.

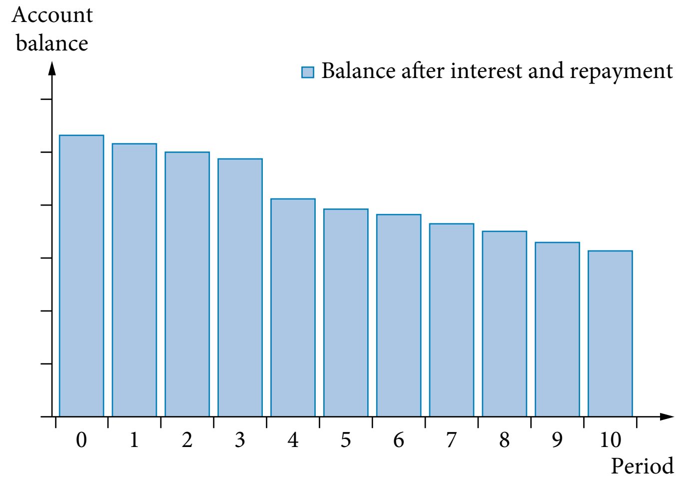
2 Grady borrowed $4000 at 8% p.a. interest calculated quarterly. He began to repay the loan at $500 each quarter. Starting with the third repayment, he increased his periodic repayments to $1000 each quarter.
(a) Write a recurrence relation that models the loan account balance for the first two repayments.
(b) Determine the account balance after the first repayment.
(c) Write a recurrence relation that models the loan account balance from the third repayment.
(d) Determine the account balance one year after Grady borrowed $4000.

3 The table shows the step-by-step calculation of the account balance in the first five months of a $2000 loan.

Month, n	Balance at end of month n, L_n ($)	Interest added at the end of next month ($)	Repayment at the end of next month ($)	Next balance, L_{n+1} ($)
0	2000.00	2000.00×1.01	100	1920.00
1	1920.00	1920.00×1.01	100	1839.20
2	1839.20	1839.20×1.012	100	1761.27
3	1761.27	1761.27×1.012	250	1532.41
4	1532.41	1532.41×1.012	250	1300.80

(a) How can you tell that this loan is a reducing balance loan?
(b) Determine the interest rate per month when the loan first started.
(c) By how much did the first repayment reduce the principal?
(d) Explain the change that applied to this loan in the third month.
(e) What other change to this loan occurred in the first four months and when did this happen?

4 The graph shows the reducing loan account balance when interest is calculated and repayments are made each month.



(a) Which statement is supported by the evidence in the graph?

A The third repayment was increased.

B The fourth repayment was increased.

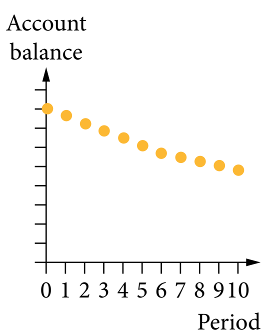
C The fifth repayment was increased.

D One of the repayments was missed.

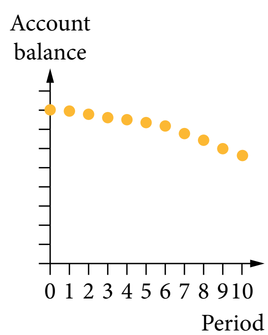
(b) Explain the common error made by a student who was unable to correctly identify the period in which the change to the repayment amount occurred.

5 Which of the following graphs shows a reducing balance loan, where the interest rate decreases at some point while the periodic repayments remain the same?

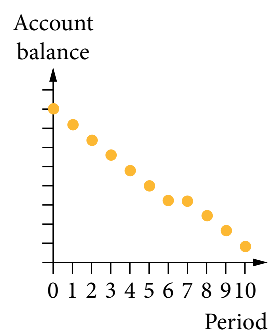
A



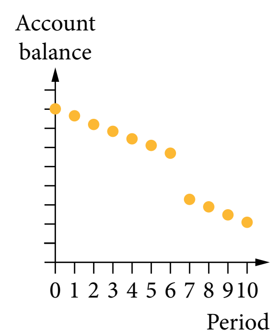
B



C



D



6 Sean missed either the second or third monthly repayment for a loan fully amortised over 24 months with interest calculated monthly. He could not remember which repayment he missed. Can the timing of a missed repayment affect the amortisation of a loan? Explain your answer.

7 Ayush borrows \$295 000 to pay the balance of what is owed on his new house. Interest at 6.5% p.a. is calculated monthly and the monthly repayment is \$3350.

(a) Use a spreadsheet to determine the term of this loan.

(b) The interest rate increases to 6.75% after one year. If Ayush does not change the monthly repayment, by how many months will the loan be extended?

8 To renovate their house, the Chan family borrowed \$45 000 with interest charged monthly at 6.8% p.a. They made periodic repayments of \$300 each month. At the start of the fifth year of this loan, the interest rate was increased to 7.8% p.a. compounding monthly. Use a spreadsheet to determine the number of months it will take to repay this loan.

9 A loan company charges interest each month at only 3% p.a. for the first six months of a loan. After six months, the interest each month changes to 6.95% p.a. Alex can afford a maximum of \$300 per month to repay a \$7400 loan. Use a spreadsheet to determine the minimum term for this loan.

Worked Example 12

10 A loan of \$15 000 attracts 7.2% p.a. interest calculated quarterly. Quarterly repayments are \$1100. After one year, the quarterly repayment will be increased to \$2000. Determine the account balance at the end of 18 months.

11 A loan of \$16 500 has interest charged each quarter at 7.2% p.a. and repayments are \$270 each quarter.

(a) What is the interest portion of the first repayment? Give your answer to the nearest dollar.

(b) Based on your answer to part (a), what type of loan is this?

(c) What is the account balance at the end of the first year?

(d) One additional payment of \$1610.95 is made along with the fourth regular repayment. What effect does this have on the type of loan?

12 A loan of \$30 000 has interest charged every six months at 10% p.a. At the end of each year, a special \$8000 payment is made in addition to the regular six monthly repayment of \$4000.

(a) How long will it take to repay this loan?

(b) How much is the final repayment?

Worked Example

13 A loan of \$13 000 attracts 6.0% p.a. interest calculated quarterly. Periodic repayments are \$1200 per quarter but the third repayment was missed. All other repayments were made on time. 13

Calculate the loan account balance at the end of the first year.

14 Stace borrows \$15 800 for a vacation. Interest is charged at 7.014% p.a. calculated monthly. She begins repaying this loan at \$600 per month but misses the third repayment. She resumes all monthly repayments at the fourth repayment. Calculate the final repayment if Stace pays all that is owed six months after borrowing the money.

15 A loan of \$36 000 attracts 7.2% p.a. interest calculated quarterly with periodic repayments of \$3000 per quarter. An additional \$7000 is paid once only at the time the third repayment is made. 14

Determine how much the term of the loan was reduced by, and how much interest was saved by making the additional payment.

16 Reuben's loan of \$26 540 has monthly interest calculated on the reducing balance at a rate of 6.6% p.a. Monthly repayments of \$900 have been made for the past 11 months. Reuben has just inherited \$5000, which he will add to the 12th repayment.

(a) Use a spreadsheet to determine the account balance at the end of the first two years of this loan.

(b) How many months will it take for Reuben to repay the loan?

17 A loan of \$30 500 is being repaid at \$400 each month. Interest is charged monthly at 6.2% p.a.

(a) Use a spreadsheet to determine the term of this loan.

(b) After 15 months, the interest rate on the loan increased to 6.7%. The term of the loan is to remain the same. What is the amount of the repayment, to the nearest dollar, when the new interest rate takes effect?

18 Compound interest is charged monthly on a loan at 7.25% p.a. After the periodic repayment of \$337.59 that was paid today, the outstanding loan balance is \$14 029.77.

(a) Use a spreadsheet to determine how much time will be saved by a single additional payment of \$5000 two years from today.

(b) How much is the final repayment?

(c) How much interest would be saved by making the extra payment?

6.4 Ordinary annuities and perpetuities

Annuities

An annuity is a compound interest investment with periodic fixed payments, or withdrawals, taken by the owner of the annuity.

The investor purchases an annuity for a principal amount. Interest is added periodically and withdrawals are made periodically as payment to the investor.

Types of annuities

Periodic payments taken from an annuity can mean the account balance will behave in one of three ways after every payment:

- The annuity account decreases after each payment – this is called an ordinary annuity.
- The annuity account increases after each payment – think of this as an increasing value annuity.
- The annuity account stays the same after each payment – this is called a perpetuity.

Perpetuities

A perpetuity can be set up with a fixed rate of interest, with the periodic interest paid to the client as soon as it is earned. Such payments can be used for annual donations, an income in retirement, or any other situation where a regular payment is required from an investment.

The effect of inflation means that regular payments such as these will lose their buying power in the long term.

> A perpetuity is a special type of annuity that pays out the periodic interest earned.
>
> It has the opposite cash flow to an interest-only loan.

15 Determine the principal needed for a perpetuity

A perpetuity earns interest monthly at 4.88% p.a. and pays $1010.77 each month.

Determine the value of the perpetuity.

THINKING	WORKING
1 Convert the annual rate of interest to a monthly rate.	4.88% p.a. $= \frac{4.88\%}{12}$ per month

2 A perpetuity pays the interest earned.	Let P be the value of the perpetuity.
Write an equation for finding the percentage of the investment.	$\frac{4.88\%}{12}$ of $P = 1010.77$
Solve for P, the value of the investment.	$\frac{0.0488}{12} \times P = 1010.77$ $P = 1010.77 \times \frac{12}{0.0488}$ $P = \$248\,550$
3 Interpret the result.	The perpetuity is worth \$248 550.

Relationship between an annuity and a loan

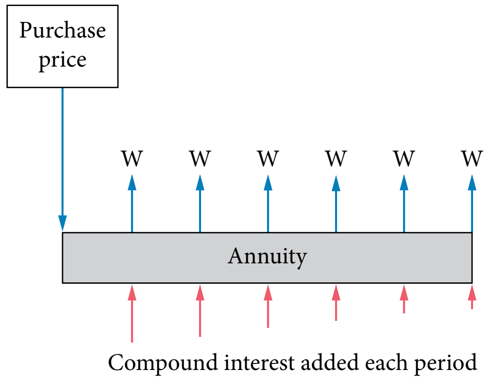
The main difference in the mechanics of an annuity and a loan is the direction of cash flow, represented by the blue arrows (W) in the cash flow diagram below. In loans and annuities, interest is added periodically to increase the account balance before a payment reduces it.

An annuity has the opposite cash flow to that of a loan.

An initial principal amount is used to purchase an annuity.

The principal is increased when interest is added periodically (pink arrows).

The principal is decreased by each fixed periodic payment to the owner of the annuity at the same compounding periods. These payments, or withdrawals, are shown as blue arrows (marked W).



Reducing balance loan:

A principal amount is borrowed at the start, and smaller repayments are paid by the owner periodically.

Annuity:

A principal amount is invested at the start, and smaller payments (withdrawals) are paid to the owner periodically.

Owner of an annuity	Borrower of a loan
Invests an amount, A_0.	Borrows an amount, L_0.
Earns interest each period.	Pays interest each period.
Withdraws periodically.	Makes periodic repayments.
Increasing balance is desirable.	Decreasing balance is desirable.

The recurrence relation that models a loan account after each repayment might be:

$L_0 = 20\,000,\ L_{n+1} = 1.04 \times L_n - 1200$

The loan account balance increases by 4% interest and decreases by a repayment of \$1200 each period.

The recurrence relation that models an ordinary annuity account after each payment to the owner is similar:

$A_0 = 20\,000, A_{n+1} = 1.04 \times A_n - 1200$

The annuity account balance increases by 4% interest and decreases by a withdrawal of $1200 each period.

Eventually, both the loan and the annuity accounts will reduce to zero.

An ordinary annuity is an investment that is bought with a lump sum of money to earn compounding interest and which makes a series of fixed periodic payments (withdrawals) to the owner of the annuity.

Source of each payment

Each periodic payment to the owner of an ordinary annuity has two sources:

- the entire interest earned for the period
- the annuity account balance – this is reduced with each periodic payment.

Each *payment from the annuity* can be calculated using:

Payment = interest earned in the period + reduction in the account balance

The *reduction in the annuity account balance* after each payment can be calculated using:

Reduction in account balance = payment – interest for the period

16 Determine the reduction in account balance after an annuity payment

A $160 000 annuity earns 6% p.a. interest compounding monthly and pays $1200 each month.

How much of the first payment comes from interest and how much comes from reducing the balance?

THINKING	WORKING
1 Determine the interest earned in the first month.	Interest: $\frac{0.06}{12} \times 160\,000 = \$800$
2 Determine the balance of payment required from reducing the principal.	Balance = payment – interest = 1200 – 800 = $400
3 Interpret the result.	The first payment is made up of $800 interest and $400 from the principal.

Amortising an annuity

As with reducing balance loans, an annuity can be amortised. This means that equal payments are calculated and spread through the term of the annuity to reduce the account to zero within a fixed term.

An annuity account balance will eventually reduce to zero if interest earned in the first period is less than the periodic payment.

17 Using a recurrence relation for an ordinary annuity

An annuity is purchased for $50 000 and earns interest at 6% p.a. calculated monthly. It pays $1500 to the owner of the annuity each month.

Use a recurrence relation to determine the balance of this annuity after each of the first three months.

THINKING	WORKING
1 Calculate the growth factor.	$r = 1 + \frac{0.06}{12}$ $= 1.005$
2 Write the recurrence relation.	$A_0 = 50\,000,\ A_{n+1} = 1.005 \times A_n - 1500$
3 Calculate the required balances.	$A_1 = 1.005 \times 50\,000 - 1500$ $= \$48\,750$ $A_2 = 1.005 \times 48\,750 - 1500$ $= \$47\,493.75$ $A_3 = 1.005 \times 47\,493.75 - 1500$ $= \$46\,231.22$

A spreadsheet can be used for annuity calculations, especially for determining the duration, or term.

18 Calculate the final payment

Sean buys an annuity for $120 000 to pay him $10 000 every year. Interest is earned at 3% of the annuity account balance each year.

(a) How long can Sean expect to be paid $10 000 from this annuity?

THINKING / WORKING

1 Calculate the growth factor.

$r = 1 + 0.03$
$= 1.03$

Write the recurrence relation.

$A_0 = 120\,000,\ A_{n+1} = 1.03 \times A_n - 10\,000$

2 Set up a spreadsheet and drag down to the first negative balance.

Alternatively, use a scientific calculator, recording the balance each time.

	A	B
1	n	Balance
2	0	120000.00
3	1	113600.00
4	2	107008.00
17	15	966.95
18	16	–9004.04

3 Interpret this result.

The annuity will pay $10 000 for 15 years with some money left in the account. This could be added to the 15th payment or taken, after interest, at the end of the following year.

(b) Determine the total amount paid by this annuity, assuming the final payment is reduced.

1 Interpret the negative value in the spreadsheet.	The assumed payment of \$10 000 at the end of year 16 is \$9004.04 too much.
2 Add the total of the full payments to the reduced final payment.	Total paid: $10\,000 \times 15 + (10\,000 - 9004.04) = \$150\,995.96$

EXERCISE 6.4 Ordinary annuities and perpetuities

Worked Example

15 **1** A perpetuity earns interest monthly at 3.84% p.a. and pays \$916.32 each month. How much did this perpetuity cost?

2 Consider the following five investments.

Investment I: an annuity earns \$230 interest in a month and pays \$230 per month.

Investment II: an investment modelled by $A_0 = 40\,000$, $A_{n+1} = 1.04 \times A_n - 200$.

Investment III: an investment where each payment exactly matches the interest earned.

Investment IV: an annuity of \$45 000 and the account balance increases slightly after every payment of \$600.

Investment V: an investment of \$45 000 that periodically pays \$600. After five years, the annuity is closed and the investor is paid \$45 600, including the final payment.

Of these five investments, which are perpetuities?

A I, III and V only **B** II and III only **C** II and IV only **D** II, IV and V only

3 Franca wants to invest \$42 000 in a perpetuity that will earn 6.15% p.a. calculated quarterly. Periodic payments are made quarterly from this investment. How much will Franca receive from this perpetuity each quarter?

4 A perpetuity bought for \$140 000 pays \$1960 per quarter. Determine the annual interest rate applied to this perpetuity.

5 An elderly couple would like to purchase an annuity that will pay them \$2000 each month. They also do not want their original investment to decrease. They find an annuity that will pay them 5.4% p.a. calculated monthly. Determine the minimum amount they should invest in this annuity. Give your answer to the nearest dollar.

16 **6** The ordinary annuities listed below have interest compounding monthly, and monthly payments. For each, calculate how much of the first payment comes from interest and how much comes from reducing the principal.

(a) initial principal \$250 000, interest rate 5.4% p.a., payments \$1500

(b) initial principal \$45 000, interest rate 7.5% p.a., payments \$500

(c) initial principal \$125 000, interest rate 4.8% p.a., payments \$600

(d) initial principal \$24 000, interest rate 3.9% p.a., payments \$150

7 An annuity is bought for $40 000 and earns interest at 6.6% p.a. calculated monthly. It pays $2500 to the owner of the annuity each month. 17

Use a recurrence relation to determine the balance of this annuity after each of the first three months.

8 Yvonne purchased an annuity with $230 000 on which she earned interest each month at 4.8% p.a. She had been receiving payments of $1200 each month from this annuity but this is no longer enough. She will end this annuity after receiving the fifteenth $1200 payment, due tomorrow.

(a) How much will Yvonne have to start the new annuity, given that $A_{13} = 226\,271.35$?

(b) Yvonne will use the balance that she withdraws to start a new annuity that will earn 4.7% p.a. and pay her $1500 each month. When she invests this amount in the new annuity, what will be the account balance after the first payment?

9 The table shows the amortisation of an annuity over four years.

Period, n	Balance at end of period n, A_n ($)	Payment withdrawn at the end of the next period ($)	Interest portion of payment ($)	Rest of payment ($)	Balance at end of period $n+1$, A_{n+1} ($)
0	15 000.00	4 500.00	870.00	3 630.00	11 370.00
1	11 370.00	4 500.00	659.46	3 840.54	7 529.46
2	7 529.46	4 500.00	436.71	4 063.29	3 466.17
3	3 466.17	3 667.21	201.04	3 466.17	0.00

(a) Determine the principal used to purchase this annuity.

(b) Determine the periodic payment taken from this annuity.

(c) By how much was the annuity account balance reduced at the end of the second year?

(d) Determine the annual rate of interest earned by this annuity.

(e) Write a recurrence relation that models the annuity account balance of the first three years.

10 Werner purchases an annuity for $60 000 to give him a monthly income of $900. The interest rate is 7.8% p.a. calculated monthly on the reducing principal.

(a) Write a recurrence relation that models the change in the annuity account balance each month.

(b) Complete the table for the first two months of this annuity.

Period, n	Balance at end of period n, A_n ($)	Payment withdrawn at the end of the next period ($)	Interest portion of payment ($)	Rest of payment ($)	Balance at end of period $n+1$, A_{n+1} ($)
0					
1					
2	58 976.69				

(c) The $900 payment for the final payment in the first year will reduce the principal by $547.67. Determine the interest charged for this period.

11 An annuity of $36 750 decreased by $1392.90 after an annual payment of $5000 was taken out by the owner. Determine the annual rate of interest applied to this annuity. Write your answer to 2 decimal places.

12 A quarterly payment of \$5850 from an annuity included the \$3527.19 interest calculated on the reduced account balance that quarter. After this payment, the annuity account balance was reduced to \$218 126.42.

(a) By how much did this payment reduce the account balance?

(b) What was the account balance at the end of the previous quarter?

(c) What was the quarterly rate of interest applied to this annuity?

(d) By how much will the account balance be reduced in the next quarter?

Worked Example 18

13 An annuity is bought for \$110 000 and earns interest at 4.5% p.a. calculated monthly. It pays \$600 per month.

(a) Use a spreadsheet to determine the balance of this annuity after 8 years.

(b) If the withdrawal is doubled to \$1200 per month, determine the balance after 8 years.

(c) Determine the length of this annuity if the withdrawal each month was \$1600, and determine the value of the final payment.

14 The sum of \$80 000 is invested in an annuity on which interest is paid at a rate of 0.4% per month. The annuity pays \$750 each month.

(a) Given that $A_{136} = 2490.77$, what is the term of this annuity?

(b) How much interest will this annuity have earned by the time it ends?

15 Kumar is going to invest \$100 000 in an annuity that earns 6% p.a. interest compounding monthly. He is comparing the term of the annuity between taking periodic monthly payments of \$600 or \$700.

(a) Use a spreadsheet to compare the terms of the annuity in each case, and check the reasonableness of your result.

(b) Which annuity paid more interest, and by how much?

16 Elias purchases an annuity for \$240 000 to pay \$15 000 every year. He wants to compare the effect of earning 6.5% p.a. to 3.25% p.a. (half the rate), compounding annually on the annuity. From Elias' calculations, the balances at 20 years are $A_{20} = 41\,537.44$ and $A_{20} = 263\,295.19$ in no particular order.
Interpret the results in terms of duration of the annuity and total payments, giving actual values where possible.

17 An annuity can be purchased for \$120 000 and pays 4.6% p.a. interest compounding quarterly, with a quarterly payment of \$2000. An alternative is to invest \$120 000 in an annuity that pays 4.6% p.a. interest compounding annually, with an annual payment of \$8000.

(a) What is the account balance after one year for each option?

(b) Explain why the annual option has a higher balance, and in what way the other option is advantageous.

(c) How much extra interest will the annual option earn over its life, given $A_{25} = 7959.72$ for the annual option and $A_{100} = 4758.93$ for the quarterly option?

Increasing-value annuities and annuity investments

6.5

Increasing-value annuities

With ordinary annuities, the withdrawal amount each period takes all the interest earned in the period as well as some of the principal, decreasing the balance.

If the amount of interest each period is higher than the withdrawal, the principal, or capital, is untouched, and the balance actually grows by the portion of the interest not used in the withdrawn amount.

The mechanics of this kind of increasing-value annuity are the same as for an ordinary annuity.

19 Calculating the increased balance

An annuity is purchased for $250 000 and earns interest at 5.7% p.a. calculated monthly. It pays $1000 to the owner of the annuity each month.

Use a recurrence relation to determine the balance of this annuity after each of the first three months.

THINKING	WORKING
1 Calculate the growth factor.	$r = 1 + \frac{0.057}{12}$ $= 1.004\,75$
2 Write the recurrence relation.	$A_0 = 250\,000,\ A_{n+1} = 1.004\,75 \times A_n - 1000$
3 Calculate the required balances.	$A_1 = 1.004\,75 \times 250\,000 - 1000$ $= \$250\,187.50$ $A_2 = 1.004\,75 \times 250\,187.50 - 1000$ $= \$250\,375.89$ $A_3 = 1.004\,75 \times 250\,375.89 - 1000$ $= \$250\,565.18$

Increasing-balance annuities and inflation

When an annuity is set up to provide an income indefinitely, it must be either a perpetuity or an increasing-value annuity. With inflation, the value of both the repayments and the initial principal will lose their buying power over time.

20 Comparing the future value of the original principal with increased balance

An annuity is purchased for $200 000 and earns interest at 7.2% p.a. calculated quarterly. The owner of the annuity is paid $3000 each quarter.

If inflation was 2.5% in the past year, determine whether the investment capital has kept up with inflation.

THINKING	WORKING
1 Calculate the growth factor.	Growth factor for the investment: $r = 1 + \frac{0.072}{4}$ $= 1.018$
2 Write the recurrence relation.	$A_0 = 200\,000,\ A_{n+1} = 1.018 \times A_n - 3000$
3 Calculate the balance at the end of the term.	$A_1 = 1.018 \times 200\,000 - 3000$ $= \$200\,600$ $A_2 = 1.018 \times 200\,600 - 3000$ $= \$201\,210.80$ $A_3 = 1.018 \times 201\,210.80 - 3000$ $= \$201\,832.59$ $A_4 = 1.018 \times 201\,832.59 - 3000$ $= \$202\,465.58$ The balance at the end of the year is $202 465.58.
4 Calculate the future value of the original principal after a year of inflation.	Growth factor for inflation: $r = 1 + \frac{2.5}{100}$ $= 1.025$ Future value of original principal: $20\,000 \times 1.025 = \$205\,000$
5 Compare the future value of the original with the current balance. Write a conclusion.	$\$205\,000 > \$202\,465.58$ The investment capital has not kept up with inflation.

Even when the capital does keep up with inflation, it makes sense that withdrawals from an annuity may have to increase to keep pace with inflation, especially in the long term.

Annuity investments

Investments generally don't start out as lump sums. Before relying on an annuity to provide an income, people may spend years contributing regularly to an investment account, allowing compound interest to grow the balance periodically.

An annuity investment is a compound interest investment with periodic and equal additions to the principal.

A common example of an annuity investment is a savings account that earns interest and to which a fixed additional contribution is added each month.

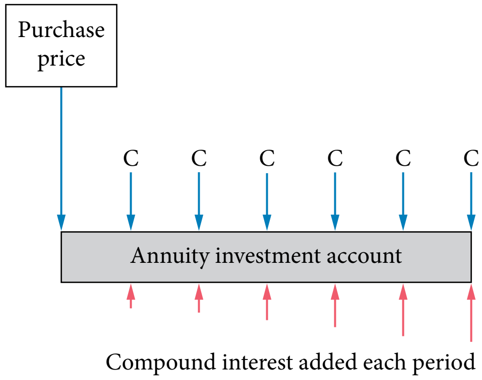
An annuity investment exhibits geometric growth due to the interest calculated on an increasing account balance. The increase is amplified each period by the fixed periodic additional contribution.

Cash flow for an annuity investment is shown in the diagram.

An initial principal amount is used to purchase an annuity investment.

The annuity account is increased when interest is added periodically, shown as red arrows.

The annuity account is further increased by each fixed contribution in the same compounding periods. These contributions are shown as blue arrows, C.



A recurrence relation for an annuity investment would be:

A_0 = purchase price, $A_{n+1} = rA_{n+1} + C$, where \$C is the amount of the regular contribution.

Note the addition symbol in this recurrence relation.

21 Annuity investment

Jessica uses her tax return of \$1337.47 to open a savings account that pays interest of 3.6% p.a. calculated monthly on the account balance at the end of each period. Jessica plans to add to this account by contributing an additional \$80 each month to save for a vacation.

Determine the account balance for the first three months of Jessica's savings account.

THINKING	WORKING
1 Write the recurrence relation.	Growth factor: $r = 1 + \frac{0.036}{12}$ $= 1.003$ $A_0 = 1337.47,\ A_{n+1} = 1.003 \times A_n + 80$
2 Calculate the balance at the end of each period.	$A_1 = 1.003 \times 1337.47 + 80$ $= \$1421.48$ $A_2 = 1.003 \times 1421.48 + 80$ $= \$1505.74$ $A_3 = 1.003 \times 1505.74 + 80$ $= \$1590.26$

22 Annuity investment using technology

Jessica uses her tax return of $1337.47 to open a savings account that pays interest of 3.6% p.a. calculated monthly on the outstanding account balance. Jessica plans to add to this account by depositing $80 each month to save $2500 for a vacation. How long will it take for Jessica's account to reach $2500?

THINKING	WORKING
1 Write the recurrence relation.	Growth factor: $r = 1 + \frac{0.036}{12}$ $= 1.003$ $A_0 = 1337.47$, $A_{n+1} = 1.003 \times A_n + 80$
2 Set up the spreadsheet with the month number and the balance for the end of each month. Click and drag the balance cell with the recurrence rule until the balance is higher than the required amount for the first time. Alternatively, use a scientific calculator, recording the balances as you go.	(see spreadsheet below)
3 Interpret the result.	Jessica will reach her target of $2500 after 14 months.

	A	B
1	n	Balance
2	0	1337.47
3	1	1421.48
4	2	1505.75
…	…	…
15	13	2449.51
16	14	2536.86

Superannuation accounts are an example of annuity investment accounts that become ordinary, interest-only or increasing annuities after retirement. Because of the long-term effect of variation in wages and interest rates, such accounts are not generally set up with fixed parameters. It is not usual to 'purchase' a superannuation account, so the first contribution acts as the purchase price.

EXERCISE 6.5 Increasing-value annuities and annuity investments

Worked Example 19

1 Use a recurrence relation to find the balance of the following increasing-value annuities after each of the first three months.

(a) An annuity is purchased for $120 000 and earns interest at 4.5% p.a. calculated monthly. It pays $300 to the owner of the annuity each month.

(b) An annuity is purchased for $300 000 and earns interest at 3.6% p.a. calculated monthly. It pays $500 to the owner of the annuity each month.

(c) An annuity is purchased for $80 000 and earns interest at 7.5% p.a. calculated monthly. It pays $400 to the owner of the annuity each month.

(d) An annuity is purchased for $225 000 and earns interest at 5.4% p.a. calculated monthly. It pays $800 to the owner of the annuity each month.

2 Answer the following questions about annuity investment accounts.

(a) Which of the following statements is always true?

A The slope of a graph showing the account balance over time is negative.

B The recurrence relation $A_0 = 5000$, $A_{n+1} = 1.055A_n - 250$ is an example that models the growth in the account balance of such an increasing-value annuity.

C The value of A_6 is less than the value of A_5.

D The account balance is increased periodically by added interest and also by a periodic contribution added to the account.

(b) Which one of the following recurrence relations could model the account balance of an increasing-value annuity?

A $A_0 = 15\,000$, $A_{n+1} = 0.98A_n + 240$

B $A_0 = 15\,000$, $A_{n+1} = 1.025A_n + 240$

C $A_0 = 15\,000$, $A_{n+1} = 0.98A_n - 240$

D $A_0 = 15\,000$, $A_{n+1} = 1.025A_n - 240$

3 An annuity investment commences with a deposit of \$20 000 and earns interest at 3% p.a., calculated quarterly. An additional contribution of \$150 is added to the annuity each quarter.

(a) Which recurrence relation models the growth in this annuity?

A $A_0 = 20\,000$, $A_{n+1} = 1.3A_n - 150$

B $A_0 = 20\,000$, $A_{n+1} = 1.0075A_n + 150$

C $A_0 = 20\,000$, $A_{n+1} = 1.025A_n - 150$

D $A_0 = 20\,000$, $A_{n+1} = 1.03A_n + 150$

(b) Explain the common error made by a student who calculated a growth rate of 1.03.

4 Complete the following tables for annuity investments.

(a) A quarterly contribution of \$100 is added to an annuity that earns 2% each quarter.

Annuity balance at end of period n, A_n (\$)	Add interest (\$)	Add periodic contribution (\$)	Amount added to annuity (\$)	Next annuity balance, A_{n+1} (\$)
4800.00				

(b) A monthly contribution of \$400 is added to an annuity that earns interest each month at the rate of 6% p.a.

Annuity balance at end of period n, A_n (\$)	Add interest (\$)	Add periodic contribution (\$)	Amount added to annuity (\$)	Next annuity balance, A_{n+1} (\$)
12 000.00		400.00		

(c) A monthly contribution is added to an annuity that earns 0.48% each month.

Annuity balance at end of period n, A_n (\$)	Add interest (\$)	Add periodic contribution (\$)	Amount added to annuity (\$)	Next annuity balance, A_{n+1} (\$)
6800.00			272.00	

(d) A contribution is added to an annuity at the same time that interest is added.

Annuity balance at end of period n, A_n (\$)	Add interest (\$)	Add periodic contribution (\$)	Amount added to annuity (\$)	Next annuity balance, A_{n+1} (\$)
	148.60	345.00		20 307.03

5 An annuity investment has interest and an additional contribution made monthly. The tables show details for an annuity account for one period.

(a) What is the annual rate of interest?

Period, n	Balance at end of period n, A_n (\$)	Add interest at the end of the next period (\$)	Add fixed contribution (\$)	Amount added to the annuity (\$)	Balance at end of period $n+1$, A_{n+1} (\$)
0	5000	15.00	400.00	415.00	5415.00

(b) If the interest is added quarterly at the rate of 7.2% p.a. what is the balance at the end of the next quarter?

Period, n	Balance at end of period n, A_n (\$)	Add interest at the end of the next period (\$)	Add fixed contribution (\$)	Amount added to the annuity (\$)	Balance at end of period $n+1$, A_{n+1} (\$)
	5590.00		500.00		

(c) What is the additional annual contribution?

Period, n	Balance at end of period n, A_n (\$)	Add interest at the end of the next period (\$)	Add fixed contribution (\$)	Amount added to the annuity (\$)	Balance at end of period $n+1$, A_{n+1} (\$)
	12 580.45	56.09			13 136.54

(d) What was the annuity account balance before the interest and additional contribution were paid this month?

Period, n	Balance at end of period n, A_n (\$)	Add interest at the end of the next period (\$)	Add fixed contribution (\$)	Amount added to the annuity (\$)	Balance at end of period $n+1$, A_{n+1} (\$)
		107.46		387.46	8494.74

6 Determine the account balance for the first three months of the following annuity investment accounts. 21

(a) Callum uses his tax return of $3825.12 to open a savings account that pays interest of 4.2% p.a. calculated monthly on the account balance. He plans to add to this account by contributing an additional $200 each month to save for some multimedia equipment.

(b) Leila makes a profit of $12 000 when she sells her car. She opens an investment account that pays interest of 5.1% p.a. calculated monthly on the account balance. She transfers an additional $300 from her pay to this account each month to save for a working holiday.

(c) Sam sets up an investment account with $18 200 and plans to add $150 each month. The account pays interest of 4.4% p.a. calculated monthly on the account balance.

(d) Mira transfers $8000 to open an investment account to which she contributes an additional $120 each month. The account pays interest of 4.9% p.a. calculated monthly on the account balance.

7 Jhanvi invested an inheritance of $68 700 in an account that paid interest of 1.74% each quarter. She added $750 to this investment every quarter in saving to buy a unit.

(a) Given that $A_4 = 76\,686.98$, how much was in her investment account at the end of seven quarters?

(b) How much interest was added to the account at the end of the eighth quarter?

(c) At the end of two years, instead of paying the periodic $750 into the account, Jhanvi withdrew $30 000. Determine the account balance immediately after this withdrawal.

8 An annuity investment commenced with $43 840 and earns 6.08% p.a. calculated quarterly on the increasing account balance. An additional $2000 contribution is made at the end of every quarter.

(a) Determine the interest earned in the first quarter.

(b) Determine the account balance after the first additional $2000 contribution.

(c) Given that the balance at the end of four years is $91 728.01, how many quarters does it take, from the start, for this account to first exceed $100 000?

9 Fatima has two options for making additional contributions to an annuity investment. Interest on the investment will be calculated on the monthly account balance.

Option 1: Contribute $500 each quarter.

Option 2: Contribute $1000 each half-year.

Which option will make her annuity investment grow more quickly?

10 If inflation was 2.3% in the past year, determine whether each of the following increasing-balance annuities have kept up with inflation. 20

(a) purchased for $150 000, interest 6.6% p.a. calculated quarterly, pays $1500 each quarter

(b) purchased for $250 000, interest 5.4% p.a. calculated quarterly, pays $2000 each quarter

(c) purchased for $60 000, interest 6.8%. p.a. calculated quarterly, pays $100 each quarter

11 Ben used his tax return of $743.51 to open a savings account that paid interest of 3.84% p.a. calculated monthly on the outstanding account balance. He planned to add to this account by contributing an additional $100 each month to save $3500 for a vacation. 22

(a) Write a recurrence relation to model the growth in Ben's savings account.

(b) Use a spreadsheet to determine how long it will take for Ben's account to reach $3500.

12 Flynn is adding $300 each quarter to an investment of $6700. The investment attracts interest of 5.18% p.a. calculated and added to the investment each quarter.

(a) Determine the balance of Flynn's account after one year.

(b) How long will it take for this account to reach $10 000?

(c) At the end of five years, how much has Flynn paid into this account, including the original investment?

(d) How much interest has been added to this account in five years?

13 An investment of $22 000 earns interest at 4.9% p.a. compounding monthly. Additional deposits of $210 per month are made for the first year. Use a spreadsheet to answer the following questions.

(a) What is the balance at the end of the first year?

(b) At the end of the first year, the interest rate will decrease to 4.2% p.a. compounding monthly, and the periodic deposit will increase to $240 each month. Write a recurrence relation to model the growth of this investment, to reflect the changes from the end of the first year.

(c) How long after the beginning of the original investment will the value of this account exceed $35 000?

14 Giaan's superannuation account grows each year at an average rate of 5.2% p.a., to which she makes additional contributions of $5000 each year. After the recalculation of her account at the end of 2015, her superannuation balance was $37 780.29.

(a) Write a recurrence relation to model the growth in Giaan's superannuation account using A_0 to represent the account balance at the end of 2015.

(b) Determine her account balance at the end of 2016.

(c) Use a spreadsheet to determine when her account will first reach at least $100 000.

(d) If Giaan continued to make the same additional contributions and the interest rate remained at an average of 5.2%, how much would be in her superannuation fund by the end of 2040? Give your answer to the nearest thousand dollars.

15 Luke starts an annuity investment with $45 000 that earns interest each month at 6.1% p.a. He is able to add $2000 to this account each month for five years until he retires. When he retires, he reduces his monthly contributions to $100 each month. Use a spreadsheet to determine the value of the annuity investment five years after he retires.

16 Lee wants an annuity investment to grow to $1 000 000 in 35 years. He will invest $45 000 and make additional contributions every month. He finds an annuity that will give him 6.5% p.a. interest calculated monthly. How much will Lee have to contribute every month? Use a spreadsheet and give your answer to the nearest five dollars.

17 Ryan wins a lottery and uses the money to start a savings account. Interest is calculated monthly at 4.88% p.a. Every month he contributes $400 to the account. After two years, the account balance is $151 020.16. Use a spreadsheet to determine how much money he won in the lottery, to the nearest five dollars.

Financial formulas

6.6

Recurrence relations are a useful tool to explore the periodic behaviour of loans and investments, enabling you to understand the pattern of increasing or decreasing value and the compounding effect on interest earned. However, when using recurrence relations to determine values over n periods, you need to perform n calculations. For example, to determine the final value of an investment where weekly deposits are made for 30 years, $30 \times 52 = 1560$ calculations would be required.

Future value of an annuity investment

The future value of an annuity is the value of an investment when a series of n deposits, in dollars, are made periodically, earning a constant rate of interest i% per period.

Consider the following 5 year savings plan. A student has committed to depositing \$85 per week in a savings account paying 4.2% p.a. Determine the amount in the account at the end of the 5 year period.

Creating a recurrence relation where the compounding period corresponds to the payment period:

$A_{n+1} = rA_n + d$ where $d = \$M = \$85$, $A_0 = \$85$ and $r = (1+i) = \left(1 + \frac{0.042}{52}\right)$

Number of deposits, n	Recurrence relation, $A_{n+1} = rA_n + d$	General rule
1	$A_1 = 85$	$A_1 = M$
2	$A_2 = rA_1 + d$ $= \left(1+\frac{0.042}{52}\right) \times 85 + 85$ ≈ 170.07	$A_2 = rA_1 + M$ $= rM + M$
3	$A_3 = rA_2 + d$ $= \left(1+\frac{0.042}{52}\right) \times 170.07 + 85$ ≈ 255.21	$A_3 = rA_2 + M$ $= r(rM + M) + M$ $= r^2M + rM + M$
4	$A_4 = rA_3 + d$ $= \left(1+\frac{0.042}{52}\right) \times 255.21 + 85$ ≈ 340.41	$A_4 = rA_3 + M$ $= r(r^2M + rM + M) + M$ $= r^3M + r^2M + rM + M$
5	$A_5 = rA_4 + d$ $= \left(1+\frac{0.042}{52}\right) \times 340.41 + 85$ ≈ 425.69	$A_5 = rA_4 + M$ $= r(r^3M + r^2M + rM + M) + M$ $= r^4M + r^3M + r^2M + rM + M$
	⋮	⋮

259	$A_{259} = rA_{258} + d$ $= \left(1 + \frac{0.042}{52}\right) \times 24371.37 + 85$ ≈ 24476.06	$A_{259} = rA_{258} + M$ $= r\left(r^{257}M + r^{256}M + \ldots + r^2M + rM + M\right) + M$ $= r^{258}M + r^{257}M + \ldots + r^2M + rM + M$
260	$A_{260} = rA_{259} + d$ $= \left(1 + \frac{0.042}{52}\right) \times 24476.06 + 85$ ≈ 24580.83	$A_{260} = rA_{259} + M$ $= r\left(r^{258}M + r^{257}M + \ldots + r^2M + rM + M\right) + M$ $= r^{259}M + r^{258}M + \ldots + r^2M + rM + M$

In general, assuming the amount is deposited into the account immediately after the interest is deposited, the recurrence relation gives the account balance $\$A_n$ after n deposits of $\$M$, earning $(n-1)$ lots of interest:

$$A_n = M + Mr + Mr^2 + Mr^3 + \ldots + Mr^{n-2} + Mr^{n-1}.$$

To determine the sum of the sequence, consider:

$$rA_n = Mr + Mr^2 + Mr^3 + Mr^4 + \ldots + Mr^{n-1} + Mr^n$$

Therefore:

$$rA_n - A_n = -M + (Mr - Mr) + (Mr^2 - Mr^2) + \ldots + \left(Mr^{n-1} - Mr^{n-1}\right) + Mr^n$$

$$A_n(r-1) = Mr^n - M$$

$$A_n = \frac{Mr^n - M}{r-1}$$

$$A_n = \frac{M\left(r^n - 1\right)}{r-1}$$

For $r = (1+i)$:

$$A_n = \frac{M\left((1+i)^n - 1\right)}{(1+i)-1}$$

$$= M\left[\frac{(1+i)^n - 1}{i}\right]$$

The future value $\$A_n$ of an annuity investment, when n periodic deposits of $\$M$ are made, earning interest at i% per period can be calculated using:

$$A_n = M\left[\frac{(1+i)^n - 1}{i}\right]$$

For the annuity investment above, assuming the student maintained the deposits and the interest rate remained constant, after 5 years:

$i = \frac{0.042}{52}$ per week

$n = 5$ years

$= 5 \times 52$ weeks

$= 260$ payments

$M = \$85$

$$A_{260} = 85 \times \left[\frac{\left(1 + \frac{0.042}{52}\right)^{260} - 1}{\frac{0.042}{52}}\right]$$

$$= 24\,580.83$$

In 5 years the student would have an account balance of $24 580.83.

The student deposited $260 \times 85 = \$22\,100$ and earned $\$24\,580.83 - \$22\,100 = \$2480.83$ interest.

23 Calculating future value

Determine the future value of a retirement fund, and the interest earned, where $120 is deposited at the end of each month for 35 years at a fixed rate of 5% p.a. compounding monthly.

THINKING	WORKING
1 Identify the given information.	$i = \frac{0.05}{12}$ per month $n = 35$ years $= 35 \times 12$ months $= 420$ deposits $M = \$120$
2 Recall the formula for future value.	$A_n = M\left[\frac{(1+i)^n - 1}{i}\right]$
3 Substitute the given information to calculate the future value A_n.	$A_n = 120\left[\frac{\left(1+\frac{0.05}{12}\right)^{420} - 1}{\frac{0.05}{12}}\right]$ $= \$136\,331.09$
4 Interpret the answer.	The future value of the retirement fund in 25 years time is $136 331.09. 420 deposits of $120 are made, totalling $420 \times 120 = \$50\,400$. The account has earned $136\,331.09 - 50\,400 = \$85\,931.09$ in interest.
5 Verify your solution using the recurrence relation in a spreadsheet.	(see spreadsheet below)

	A	B
1	n	A_n
2	1	120
3	2	240.5
4	3	361.5021
420	419	135645.9
421	420	136331.1

24 Calculating the amount to be deposited from a given future value

A couple are devising a 2.5 years savings plan for a holiday with a budget of at least \$30000. Determine the amount of money, to the nearest dollar, that needs to be deposited at the end of each week into a savings account earning a rate of 4.8% p.a. compounding weekly.

THINKING	WORKING
1 Identify the given information.	$A_n = \$30000$ $i = \frac{0.048}{52}$ $n = 2.5$ years $= 2.5 \times 52$ weeks $= 130$ deposits
2 Recall the formula for future value.	$A_n = M\left[\frac{(1+i)^n - 1}{i}\right]$
3 Substitute the given information and solve for M, the amount of each weekly deposit.	$A_n = M\left[\frac{(1+i)^n - 1}{i}\right]$ $30000 = M\left[\frac{\left(1+\frac{0.048}{52}\right)^{130} - 1}{\frac{0.048}{52}}\right]$ $M = \frac{30000}{\left[\frac{\left(1+\frac{0.048}{52}\right)^{130} - 1}{\frac{0.048}{52}}\right]}$ $M = 217.31\ (2\text{ d.p.})$
4 Interpret the answer.	In order to reach their goal of at least \$30000 in 2.5 years, to the nearest dollar, the couple need to deposit at least \$218 weekly.
5 Evaluate the reasonableness of your solution.	Determine the contribution made by the couple: \$218/week × 52 weeks/year × 2.5 years = \$28340 The saving goal (with interest) of \$30000 seems reasonable.

Often when you open an account, you will deposit an initial lump sum. This could be, for example, money received for your birthday, or from selling an item such as a house or a car, or savings transferred from another account. If you then set up an additional savings plan, you need to remember to account for the interest earned on the initial lump sum deposit. This needs to be calculated using the compound interest formula, separately from the future value of the annuity.

25 Future value of an investment with a lump sum and regular deposits

Determine the amount saved when a tax refund of $2875 is deposited into an investment annuity earning 3.98% p.a. compounding monthly and then additional deposits of $160 per month are added at the end of each month for 5 years.

THINKING	WORKING
1 Identify the given information.	The principal invested, $P = \$2875$ $i = \frac{0.0398}{12}$ per month $n = 5$ years $= 5 \times 12$ months $= 60$ deposits
2 Use the compound interest formula to calculate the amount of the invested tax refund.	$A = P(1+i)^n$ $= 2875\left(1+\frac{0.0398}{12}\right)^{60}$ $= \$3506.87$
3 Interpret the value.	Investing the principal alone will give an investment balance of $3506.87 at the end of 5 years.
4 Use the future value formula to determine the amount of the annuity investment based solely on the periodic deposits.	$A_n = M\left[\frac{(1+i)^n - 1}{i}\right]$ $A_{60} = 160\left[\frac{\left(1+\frac{0.0398}{12}\right)^{60} - 1}{\frac{0.0398}{12}}\right]$ $= \$10\,602.46$
5 Interpret the value.	The regular deposits of $160 per month for 5 years have a final balance of $10 602.46.
6 Add the amount earned from the deposited lump sum and the periodic deposits together to determine the balance of the account at the end of the period.	The annuity investment has a balance of $10\,602.46 + 3506.87 = \$14\,109.33$ at the end of the 5 year period.
7 Evaluate the reasonableness of your solution.	The final amount of the annuity investment is greater than the $2875 + 60 \times 160 = \$12\,475$ deposited, and less than the amount that would have been earned if the amount deposited had been done so in a lump sum: $A = P(1+i)^n$ $= 12\,475\left(1+\frac{0.0398}{12}\right)^{60}$ $= \$15\,216.76$

The present value of an annuity investment

The present value of an annuity is the amount of money that needs to be invested in an annuity earning a constant rate of i% per period to receive a series of n equal payments of $\$M$.

For example, if you need to earn \$1000 per month for 3 years, how much needs to be invested in an annuity earning 3.9% p.a. compounding monthly?

Using the compound interest formula:

$A = P(1+i)^n$

The amount to be paid $\$A$ is a constant \$1000 and n gives the length of time in the account for the payment to accrue interest.		
The first payment will have been invested for 1 month.	$A = P(1+i)^n$ $1000 = P\left(1+\frac{0.039}{12}\right)^1$ $P = \frac{1000}{\left(1+\frac{0.039}{12}\right)^1}$ $P = \$996.76$	In order to receive a \$1000 payment (principal + 1 month of interest), you need to initially invest \$996.76
The second payment will have been invested for 2 months.	$A = P(1+i)^n$ $1000 = P\left(1+\frac{0.039}{12}\right)^2$ $P = \frac{1000}{\left(1+\frac{0.039}{12}\right)^2}$ $P = \$993.53$	In order to receive a \$1000 payment (principal + 2 months of interest), you need to initially invest \$993.53

In order to receive two monthly payments of M = \$1000, the amount $\$A$ that you need to invest now is:	$A = P_1 + P_2$ $= \$996.76 + \$993.53$ $= \$1990.29$	In order to receive a \$1000 payment each month for 2 months, you need to initially invest \$1990.29

Number of payments n	Principal investment required:	General rule
1	$A_1 = P_1$ $= \frac{1000}{\left(1+\frac{0.039}{12}\right)^1}$ $= \$996.76$	$A_1 = P_1$ $= \frac{M}{r^1}$
2	$A_2 = \frac{1000}{\left(1+\frac{0.039}{12}\right)^2} + A_1$ $= 993.53 + 996.76$ $= \$1990.29$	$A_2 = P_2 + A_1$ $= \frac{M}{r^2} + \frac{M}{r^1}$
3	$A_3 = \frac{1000}{\left(1+\frac{0.039}{12}\right)^3} + A_2$ $= 990.31 + 1990.29$ $= \$2980.60$	$A_3 = P_3 + A_2$ $= P_3 + (P_2 + P_1)$ $= \frac{M}{r^3} + \frac{M}{r^2} + \frac{M}{r^1}$
4	$A_4 = \frac{1000}{\left(1+\frac{0.039}{12}\right)^4} + A_3$ $= 987.10 + 2980.60$ $= \$3967.70$	$A_4 = P_4 + A_3$ $= P_4 + (P_3 + P_2 + P_1)$ $= \frac{M}{r^4} + \frac{M}{r^3} + \frac{M}{r^2} + \frac{M}{r^1}$
	$\vdots$	$\vdots$
35	$A_{35} = \frac{1000}{\left(1+\frac{0.039}{12}\right)^{35}} + A_{34}$ $= 892.65 + 32139.45$ $= \$33\,032.10$	$A_{35} = P_{35} + A_{34}$ $= P_{35} + (P_{34} + P_{33} + \ldots + P_2 + P_1)$ $= \frac{M}{r^{35}} + \frac{M}{r^{34}} + \ldots + \frac{M}{r^2} + \frac{M}{r^1}$
36	$A_{36} = \frac{1000}{\left(1+\frac{0.039}{12}\right)^{36}} + A_{35}$ $= 889.75 + 33\,032.10$ $= \$33\,921.85$	$A_{36} = P_{36} + A_{35}$ $= P_{36} + (P_{35} + P_{34} + \ldots + P_2 + P_1)$ $= \frac{M}{r^{36}} + \frac{M}{r^{35}} + \ldots + \frac{M}{r^2} + \frac{M}{r^1}$

In general, the recurrence relation gives the account balance $\$A_n$ after n deposits of $\$M$ to be:

$$A_n = \frac{M}{r} + \frac{M}{r^2} + \frac{M}{r^3} + \frac{M}{r^4} + \ldots + \frac{M}{r^{n-2}} + \frac{M}{r^{n-1}} + \frac{M}{r^n}.$$

To determine the sum of the sequence, consider:

$$rA_n = M + \frac{M}{r} + \frac{M}{r^2} + \frac{M}{r^3} + \ldots + \frac{M}{r^{n-3}} + \frac{M}{r^{n-2}} + \frac{M}{r^{n-1}}$$

Therefore:

$$rA_n - A_n = M + \left(\frac{M}{r} - \frac{M}{r}\right) + \left(\frac{M}{r^2} - \frac{M}{r^2}\right) + \ldots + \left(\frac{M}{r^{n-1}} - \frac{M}{r^{m-1}}\right) - \frac{M}{r^n}$$

$$A_n(r-1) = M - \frac{M}{r^n}$$

$$A_n(r-1) = M - Mr^{-n}$$

$$A_n(r-1) = M\left(1 - r^{-n}\right)$$

$$A_n = \frac{M\left(1 - r^{-n}\right)}{r-1}$$

For $r = (1+i)$:

$$\begin{aligned} A_n &= \frac{M\left(1 - r^{-n}\right)}{r-1} \\ &= \frac{M\left(1-(1+i)^{-n}\right)}{(1+i)-1} \\ &= \frac{M\left(1-(1+i)^{-n}\right)}{i} \\ &= M\left[\frac{1-(1+i)^{-n}}{i}\right] \end{aligned}$$

The amount that needs to be invested into an annuity $\$A_n$ earning interest at i% per period, in order to receive n periodic payments of $\$M$ can be calculated using:

$$A_n = M\left[\frac{1-(1+i)^{-n}}{i}\right]$$

For the investment amount above, if you require monthly payments of \$1000 for 3 years, how much do you need to invest in an annuity earning 3.9% p.a. compounding monthly?

$M = \$1000$, $i = \frac{0.039}{12}$ per month and $n = 36$ payments

$$\begin{aligned} A_{36} &= 1000 \times \left[\frac{1-\left(1+\frac{0.039}{12}\right)^{-36}}{\frac{0.039}{12}}\right] \\ &= \$33\,921.85 \end{aligned}$$

In order to receive 36 monthly payments of \$1000, a total of \$36 000, the present value you need to invest in the annuity is \$33 921.85.

26 The present value formula

Determine the present value to be invested, if payments of $125 at the end of each week are to be made from an annuity, for 7 years at a fixed rate of 4.1% p.a. compounding weekly.

THINKING	WORKING
1 Identify the given information.	$i = \frac{0.041}{52}$ per week $n = 7$ years $= 7 \times 52$ weeks $= 364$ payments $M = \$125$
2 Recall the formula for present value.	$A_n = M\left[\frac{1-(1+i)^{-n}}{i}\right]$
3 Substitute the given information to calculate the present value A_n.	$A_{364} = 125\left[\frac{1-\left(1+\frac{0.041}{52}\right)^{-364}}{\frac{0.041}{52}}\right]$ $= \$39\,539.56$
4 Interpret the answer.	In order to receive 364 weekly payments of $125, a total of $364 \times 125 = \$45500$, the present value you need to invest is $39539.56.
5 Your solution can be verified using the recurrence relation in a spreadsheet.	(spreadsheet below)

	A	B	C
1	n	P	A_n
2	1	124.90	124.9015
3	2	124.80	249.7046
4	3	124.70	374.4094
364	363	93.90	39445.74
365	364	93.82	39539.56

27 Calculating the periodic payments from the present value invested

Determine the amount of each monthly payment made over 35 years, if $1.2 million is invested in an annuity earning 4.27% p.a. compounding monthly.

THINKING	WORKING
1 Identify the given information.	$i = \frac{0.0427}{12}$ per month $n = 35$ years $= 35 \times 12$ months $= 420$ payments $A_{420} = \$1\,200\,000$

2 Recall the formula for present value.	$A_n = M\left[\dfrac{1-(1+i)^{-n}}{i}\right]$
3 Substitute the given information to calculate the present value A_n.	$1\,200\,000 = M\left[\dfrac{1-\left(1+\frac{0.0427}{12}\right)^{-420}}{\frac{0.0427}{12}}\right]$ $M = \dfrac{1\,200\,000}{\left[\dfrac{1-\left(1+\frac{0.0427}{12}\right)^{-420}}{\frac{0.0427}{12}}\right]}$ $M = \$5509.37$
4 Interpret the answer.	The investment pays \$5509.37 per month for 35 years.
5 Evaluate the reasonableness of your solution.	If \$1200000 was distributed equally into the 420 payments, each payment (having earned no interest) would be: $\dfrac{1200000}{420} = \$2857.14$ Given that the annuity earns interest, the payment should be higher than this.

EXERCISE 6.6 Financial formulas

Worked Example 23

1 Determine the future value of the following investment annuities.

(a) Deposits of \$50 are made at the end of each month to an account earning 2.85% p.a. compounding monthly, for 10 years.

(b) Deposits of \$20 are made at the end of each week to an account earning 3.4% p.a. compounding weekly, for 6 years.

(c) Deposits of \$285 are made at the end of each quarter to an account earning 4% p.a. compounding quarterly, for 3.5 years.

(d) Deposits of \$105 are made at the end of each week to an account earning 3.85% p.a. compounding weekly, for 5 years.

24

2 Determine how much money needs to be deposited at the end of each period for the following investment annuities.

(a) Planning to save \$20000 in 4 years by depositing money each week into an annuity earning 4.25% p.a. compounding weekly

(b) Planning to save \$850000 for retirement over 45 years by depositing money each week into an annuity earning 3.9% p.a. compounding weekly

(c) Planning to save \$12000 to purchase a car in 3 years by depositing money each month into an annuity earning 3.5% p.a. compounding monthly

(d) Planning to save a \$60000 home deposit in 10 years by depositing money each month into an annuity earning 4.01% p.a. compounding weekly

3 Determine the present value of an annuity investment for each of the following conditions. 26

(a) While travelling around Australia for 2 years, you are to receive weekly payments of $400 from an annuity earning 2.95% p.a. compounding weekly.

(b) During your retirement, you wish to supplement your superannuation by $250 per month from an annuity earning 3.2% p.a. compounding monthly for an anticipated 35 years.

(c) To undertake additional study, you invest in an annuity earning 5.1% p.a. in order to pay $100 per week for 4 years.

(d) You wish to invest in an annuity earning 4.82% p.a. compounding monthly, in order to give monthly payments of $650 for 2.5 years.

4 Determine the value of each of the periodic payments made from the following annuity investments. 27

(a) $25 000 is deposited into an annuity earning 2.93% p.a. compounded weekly. Determine the weekly payments made to the owner if the annuity is fully amortised in 5 years.

(b) After selling an investment property, a couple invest $412 000 in an annuity earning 3.14% p.a. compounded monthly, in order to make an additional contribution to their retirement for 30 years. Determine the monthly payment they will receive.

(c) $5000 is invested in an annuity earning 3.63% p.a. compounded quarterly, to help pay the costs of the first 2 years of further education.

(d) $100 000 is invested in an annuity earning 3.94% p.a. compounded monthly. Determine the value of the monthly payments if the annuity is fully amortised in 50 years.

5 Russell invested $175 000 in an annuity to top up his superannuation income every year. Interest is earned at 3.8% p.a.

(a) Russell anticipates that this contribution will last for 20 years. How much will he receive each year?

(b) Russell actually needs $14 000 per year from this annuity.

(i) Use a spreadsheet to determine how long the annuity will be able to pay this amount.

(ii) What is the total amount paid by this annuity, assuming a reduced payment is taken in the final year?

6 When Jamani's savings account reached $8902.40 at the end of December 2019, he decided to start adding $100 each month into the account, starting at the end of January 2020. The account earns monthly interest at the rate of 5.02% p.a.
Determine Jamani's account balance when he finishes his further education at the end of 2024.

7 Pierre used his tax return of $1259.08 to open a savings account that paid interest of 4.15% p.a. calculated monthly on the account balance. He plans to add to this account by depositing $100 each month to save for a holiday at the end of his university course in 4 years. 25

(a) Determine Pierre's account balance at the end of the 4 year period.

(b) Determine Pierre's account balance at the end of the 4 year period if he is able to deposit an additional $1000 from his tax return at the end of each year.

8 An annuity was purchased for $180 000. Payments of $4200 are to be made each month until the annuity is fully amortised over four years.

(a) The rate of interest is nearest to:

A 0.4723% p.a. B 1.0047% p.a. C 5.66% p.a. D 5.76% p.a.

(b) Explain the common error made by a student who chose the first incorrect option.

9 Kalinda added \$150 every month to her savings account, which started with \$9400 and earned monthly interest at 5.4% p.a. Determine the percentage increase in her account balance, to the nearest whole number, over the first year.

10 Inza is going to open an investment annuity earning 3.94% p.a. compounding weekly, using a deposit of \$14 000 from the sale of her car.

(a) Determine how much Inza needs to contribute each week to save at least \$120 000 for a deposit on a home in 8 years.

(b) Calculate how much interest Inza will earn over the 8 years.

11 Alec and Simone invest \$61 000 in an account that earns interest at 4.2% p.a., calculated monthly for 3 years. They make additional contributions of \$250 each month. At the end of 3 years, the account is closed, with the entire balance reinvested at 4.65% p.a. for another 3 years. Assuming the additional contribution to the investment remains the same, determine the amount of interest Alec and Simone have earned over the 6 year period.

12 Zhi and Mei-Lin saved \$440 000 for their retirement. They bought an annuity that earned interest of 1.05% per quarter and paid \$10 000 each quarter.
Use formulas to estimate how many quarters it takes for their annuity account balance to be approximately half of their original investment.

13 Akeno meets with a financial planner when she is 20 years old and decides to start an investment earning a fixed amount of 3.5% p.a. compounding weekly. She will deposit money weekly throughout her working life, in order to have enough money in the account to retire by the age of 65 and live comfortably on an additional income of at least \$25 000 (plus super) per year paid weekly. Determine an appropriate weekly investment that Akeno should be contributing if she is investing for a 30 year retirement.

14 A popular budgeting rule is the 50/30/20 rule, which states that 50% of your net income should pay for essential items such as housing, bills and food, 30% of your budget should be allocated to discretionary items and 20% of your wage should be saved or used to achieve financial goals.
Dakari starts a casual job in Year 8. He earns an average of \$23 per hour in his casual job, which he plans to continue until he finishes his education in 9 years. He plans to contribute 20% of his weekly wage into an annuity earning 4.5% p.a. compounding weekly.

(a) Use Dakari's circumstances and wage to create a savings plan, and discuss the importance of compound interest. Base your calculations on 15 hours of work per week.

(b) Dakari discusses his saving plan with his parents and decides that, because he is living at home for the duration of his education and has limited living expenses, he is going to aim to save 35% of his wage. How does this compare with his original budget and investment?

15 Olivia is 35 and owns an investment property. After rates and maintenance, she makes a profit of \$320 rent per week. She knows that she can comfortably save an additional \$60 per week, and plans to start an annuity investment earning 3.25% p.a. compounding weekly. If she sells the property, she will make a profit of \$370 000. Olivia's goal is to maximise her investment by the age of 50, to supplement working part-time and eventual retirement. Should Olivia sell or keep her investment property? Justify your solution with mathematical reasoning.

Summary

6

Loans

A sum of money is borrowed at an agreed rate of interest. Interest is charged periodically for the agreed term of the loan.

Periodic repayments are made to the account. Each repayment first pays for any accumulated interest. Any excess will then reduce the principal.

The loan ends when all the principal and accumulated interest has been fully paid.

Recurrence relation for a loan with periodic repayments

L_0 = amount borrowed

L_n = loan account balance after n compounding periods: $L_{n+1} = rL_n - R$ where r is the growth factor due to periodic interest and R is the periodic payment.

Each repayment is apportioned to paying interest, and the remainder to reducing the principal.

The type of loan can be determined by comparing the periodic repayment with the interest calculation for the first period.

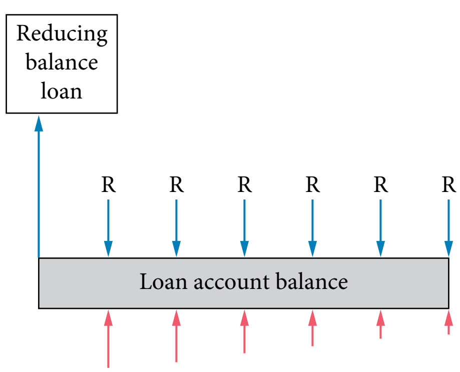
Amount of periodic payment	Effect on account balance	Type of loan
Repayment > first interest calculation	Balance reduces	Reducing balance loan
Repayment = first interest calculation	Balance stays constant	Interest-only loan
Repayment < first interest calculation	Balance increases	Insufficient loan repayments*

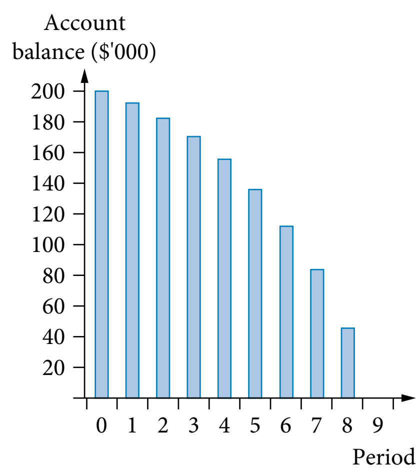
*This is not specifically a type of loan.

Reducing balance loan

This is a compound interest loan with periodic and equal repayments. As the balance decreases, the amount of interest charged also decreases and a greater portion of each repayment goes towards reducing the principal.

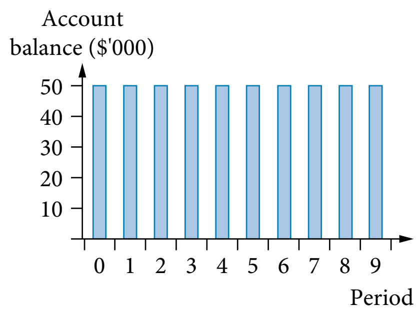
Cash flow for a reducing balance loan:





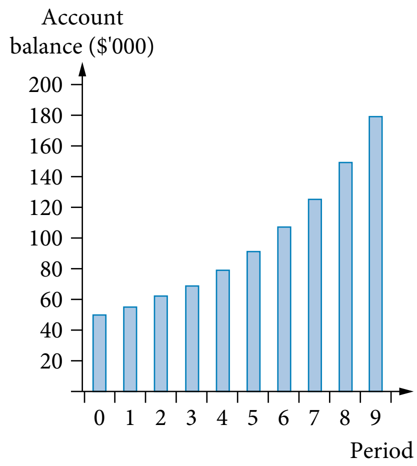
Interest-only loan

Each repayment exactly matches the interest charged for the period. The principal is neither reduced nor increased.



Insufficient loan repayments

If periodic repayments are less than the interest charged for the period, the loan account balance will increase exponentially due to the excess interest being compounded.



Investment

Investments are the reverse of loans. Money is lent by the investor at an agreed rate of interest. Interest is added periodically to the investment account for the agreed term.

Changing a parameter

If a parameter changes during a loan or investment, begin a new calculation that commences with a principal equal to the loan or investment balance just before the change.

Increasing the interest rate

Increasing the interest rate will increase the interest added in a period. Because this increase is compounded, the graph of the account balance will climb more steeply. The term of the loan or investment will generally increase if all other parameters are unchanged.

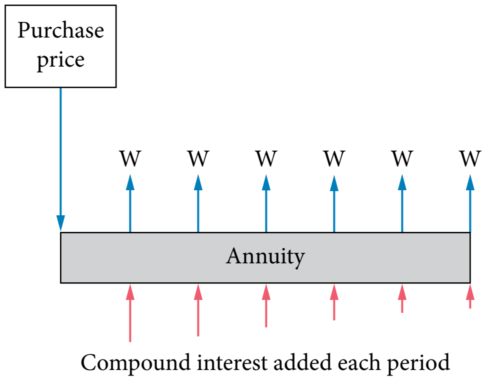
Increasing the payment amount

For a reducing balance loan and an ordinary annuity, increasing the payment amount will reduce the outstanding balance more quickly. The term of the loan or investment will generally decrease if all other parameters are unchanged.

Annuity

An annuity is a compound interest investment with periodic and equal payments taken from the investment. Think of these payments as withdrawals to the owner of the investment.

Cash flow for an annuity:



Recurrence relation for an annuity

A_0 = original investment, $A_{n+1} = rA_n$ − periodic payment

Sources of each payment

The payment taken from an annuity comes from the interest earned in that period. If this is not enough to cover the payment taken, the remainder will come from the account and the principal will be reduced.

What type of annuity?

Amount of periodic payment	Effect on account balance	Type of annuity
Payment > first interest calculation	Balance reduces	Ordinary annuity*
Payment = first interest calculation	Balance remains constant	Perpetuity
Payment < first interest calculation	Balance increases	Increasing annuity

*An ordinary annuity ends when all the principal and added interest has been fully withdrawn.

Perpetuity

A special type of annuity where periodic payments exactly match the interest added each period. This is the reverse of an interest-only loan. The account balance will neither increase nor decrease. A perpetuity ends only when closed by the investor.

Annuity investment

A compound interest investment with periodic and equal contributions made to the investment will always increase in value.

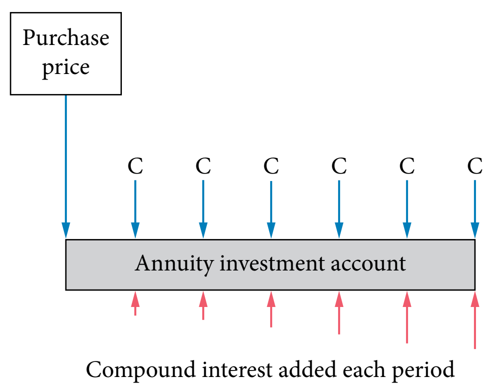
Recurrence relation for an annuity investment:

A_0 = original investment, $A_{n+1} = rA_n$ + periodic contribution

Note the addition symbol in this recurrence relation.

For an increasing annuity, this would increase the account balance more rapidly and is good for the investor.

Cash flow for an annuity investment:



Annuity formulas

The future value $\$A_n$ of an annuity investment, when n periodic deposits of $\$M$ are made, earning interest at i% per period	$A_n = M\left[\dfrac{(1+i)^n - 1}{i}\right]$
The present value $\$A_n$ that needs to be invested in an annuity earning interest at i% per period, in order to receive n periodic payments of $\$M$	$A_n = M\left[\dfrac{1-(1+i)^{-1}}{i}\right]$

Chapter review 6

Exercise 6.1 **1** The recurrence relation for a reducing balance loan is $L_0 = 2400$, $L_{n+1} = 1.01 \times L_n - 100$. Determine the monthly repayments and annual rate of interest.

Exercise 6.1 **2** For a loan of \$5000 at an interest rate of 18% p.a. compounded half-yearly, what amount is owed after a first half-yearly payment of \$820?

Exercise 6.1 **3** A loan of \$10 000 is taken out at 14% p.a. compounded quarterly, with a repayment of \$750 each quarter. Determine the amount owed after three years.

Exercise 6.1 **4** The table shows a mixture of values and expressions in an incomplete amortisation table for a loan to be repaid in three years. Calculate the missing values and match them to the correct positions.

Period, n	Balance at end of period n, L_n (\$)	Interest added at the end of the next period (\$)	Balance after interest added (\$)	Repay (\$)	Account balance reduced by (\$)	Balance at end of period $n+1$, L_{n+1} (\$)
0		450.00			3187.73	
1	6812.27	$4.5\% \times 6812.27$	7118.82	3637.73		$6812.27 - 3331.18$
2	3481.09	156.65				

Exercise 6.1 **5** A university student borrows \$3500 to cover expenses during her course. If the interest rate on a student loan is 5.6% p.a. compounded monthly and she repays \$85 per month, determine the amount owed:

(a) after 6 months

(b) after 1 year.

Exercise 6.1 **6** Jenni takes out a loan of \$39 500 to buy a new car. She must pay interest of 11% p.a. compounded monthly and monthly repayments of \$850. If after 5 years she decides to trade in the car for the latest model, how much does she still owe on her loan?

Exercise 6.2 **7** The recurrence relation $L_0 = 10\,000$, $L_{n+1} = 1.04 \times L_n - 250$ models the balance of a loan. Determine whether the loan is reducing balance, interest-only or increasing balance.

Exercise 6.2 **8** A loan of \$1000 at 9% p.a. is compounded monthly. How many monthly repayments of \$200 will it take to pay off the loan?

Exercise 6.2 **9** Tegan wants to borrow money for two years at 7.9% p.a. compounding monthly to invest in the stock market. She can afford to repay no more than \$500 each month. Determine the maximum amount that Tegan could borrow to ensure that she can afford to repay the interest component of the loan. Give your answer to the nearest hundred dollars.

10 A loan of $19 850 will have interest at 1.02% of the account balance added every month. The loan will be repaid at $775 each month. Use a spreadsheet to determine the term of the loan. Exercise 6.2

11 Repayments on an interest-only loan of $62 000 are $930 each quarter. Calculate the annual rate of interest charged for this loan. Exercise 6.2

12 Sheree makes monthly payments of $313.40 on a $3000 loan. Interest on the loan is compounded every month. It will take Sheree 10 months to repay the loan. The monthly rate of interest charged on this loan is closest to which of the following? Exercise 6.2

A 0.8% B 1% C 4% D 10%

13 Hannah borrows $45 000 to renovate her home. Interest is charged at 6.74% p.a., compounding monthly. To balance her monthly budget, Hannah decides that her monthly repayments will be the minimum required to prevent the loan account balance from increasing beyond $45 000. Exercise 6.2

(a) What is the name of this type of loan?

(b) What are Hannah's monthly repayments?

(c) Two years after the start of the loan, Hannah sold her caravan for $25 000. She added this amount to the periodic 24th monthly payment and continued to pay only the interest for another year. Determine her new monthly payment.

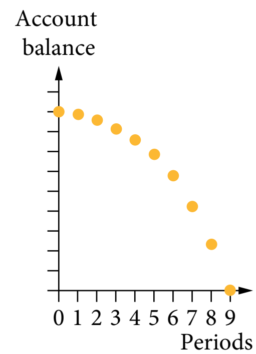
14 To finance his holiday to Canberra, Jeremy takes out a personal loan of $1500 at an interest rate of 6% p.a. compounded quarterly. He makes quarterly repayments of $200. After six months, he decides to increase the repayments to $400 per quarter in order to pay off the loan faster. Exercise 6.3

(a) Complete the amortisation table illustrating Jeremy's repayments and the balance of his loan. Adjust the values in the last row so that the final balance is zero.

Period, n	Principal at end of period n, L_n ($)	Repay ($)	Interest added at the end of the next period ($)	Principal reduced by ($)	Principal at end of period $n+1$, L_{n+1} ($)
0		200.00	22.50	177.50	1322.50
1	1322.50	200.00	19.84	180.16	
2	1142.34		17.14		759.48
3	759.48	400.00		388.61	370.87
4	370.87	400.00	5.56	394.44	-23.57

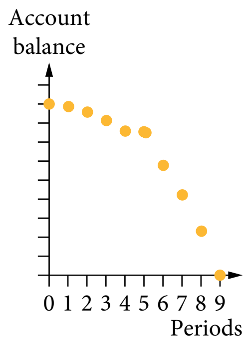
(b) How long will it take for Jeremy to repay the loan?

15 A loan has interest charged monthly at 4.8% p.a. Equal periodic repayments will fully repay this loan in nine months. The graph for this loan is shown here. Exercise 6.3

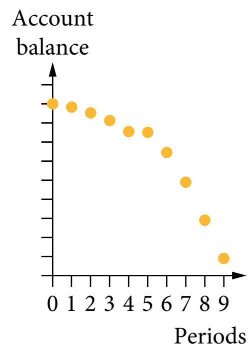


(a) All repayments except the fifth repayment are made during the nine months. If the fifth repayment is missed but all other payments are made, what would the graph look like?

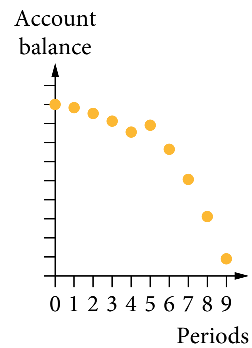
A



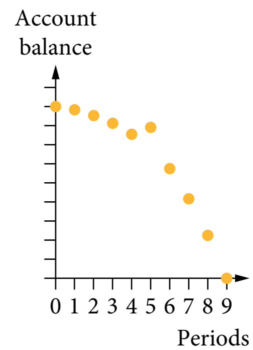
B



C

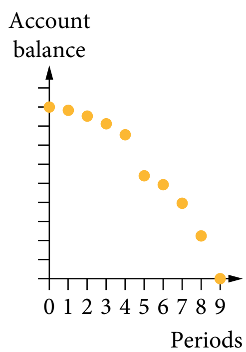


D

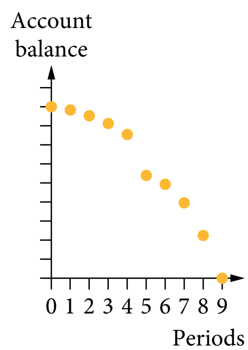


(b) Which of the following graphs shows a reducing balance loan, where all repayments are made on time but an additional payment is made with the fifth repayment?

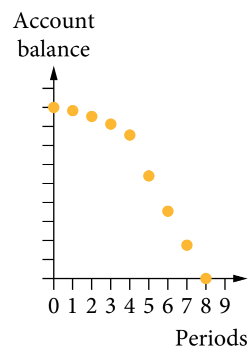
A



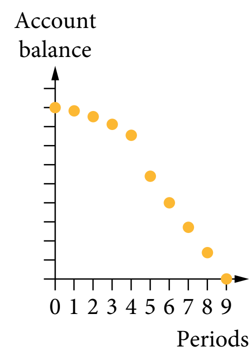
B



C

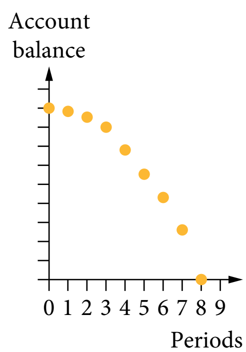


D

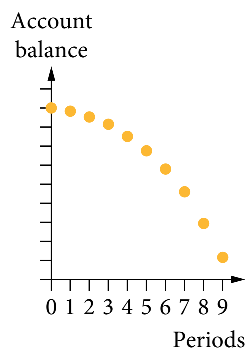


(c) If the repayment each period is 90% of the interest calculated for the first period, the graph would look like which of the following?

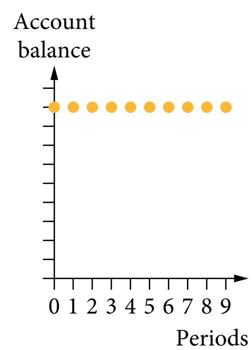
A



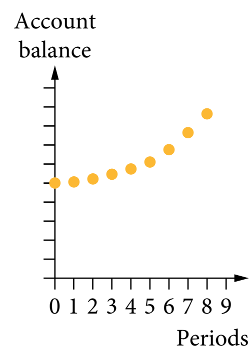
B



C



D



16 Skye has borrowed $6400 for one year at 18.25% p.a. calculated quarterly on the reducing balance. The amortisation table for this loan is shown. *Exercise 6.3*

Period, n	Balance at end of period n, L_n ($)	Interest added at the end of the next period ($)	Balance after interest added ($)	Repay ($)	Account balance reduced by ($)	Balance at end of period $n+1$, L_{n+1} ($)
0	6400.00	292.00	6692.00	1786.57	1494.57	4905.43
1	4905.43	223.81	5129.24	1786.57	1562.76	3342.67
2	3342.67	152.51	3495.18	1786.57	1634.06	1708.61
3	1708.61	77.96	1786.56	1786.56	1708.61	0.00
4	0.00					

Skye missed the first repayment but made a double payment at the end of the second quarter. The amended amortisation table is shown below.

Period, n	Balance at end of period n, L_n ($)	Interest added at the end of the next period ($)	Balance after interest added ($)	Repay ($)	Account balance reduced by ($)	Balance at end of period $n+1$, L_{n+1} ($)
0	6400.00	292.00	6692.00	$0.00	-292.00	6692.00
1	6692.00	305.32	6997.32	3573.14	3267.82	3424.18
2	3424.18	156.23	3580.41	1786.57	1630.34	1793.84
3	1793.84	81.84	1875.68	1875.68	1793.84	0.00
4	0.00					

(a) Explain the amount shown for the principal component of the first repayment.

(b) How much extra interest must Skye pay for this loan due to missing a repayment?

17 An annuity is purchased for $40 000 and attracts interest every six months at the rate of 6.04% p.a. Payment of $2400 is taken by the investor every six months. Determine the recurrence relation for this annuity. *Exercise 6.4*

18 Chang invests $68 000 in an annuity that pays $1316.53 every month, gradually reducing the balance of the annuity to zero. What is the most likely monthly rate of compounding interest earned by this annuity? *Exercise 6.4*

A 1% **B** 2% **C** 3% **D** 4%

19 A perpetuity initially purchased for $210 000 pays a regular monthly income of $850.50. What is the annual rate of interest applied to this perpetuity? *Exercise 6.4*

20 A loan of $4885 is charged interest each month at 8.64% p.a. Determine the monthly repayment required to repay this loan in three years. *Exercise 6.2*

A $145 **B** $155 **C** $165 **D** $175

21 Steph wants an annuity that will pay $6500 each quarter. She purchases one for $210 000 that earns interest of 4.32% p.a. compounding quarterly. *Exercise 6.4*

(a) Write a recurrence relation that models the change in the remaining balance of the annuity.

(b) How long will this annuity be able to make periodic payments?

(c) Calculate the amount of the final, reduced payment.

Exercise 6.4

22 Determine the purchase price of an annuity for which the principal remains unchanged while a payment of $1000 is taken from the annuity each year. Interest is added annually to the annuity at 3.8% p.a.

Exercise 6.4

23 A perpetuity earns 6% p.a. compounding monthly and pays $1240 each month. What will be the account balance of this perpetuity after two years?

Exercise 6.5

24 Minako is saving for a new car and opens a savings account with a deposit of $8000. She adds $400 to this account at the time interest is calculated every three months. The recurrence relation that models the growth in the savings account is:

$M_0 = 8000, M_{n+1} = a \times M_n + b$

After two years, the balance in the account is $12 476.62. Which of the following is closest to the value of a?

A $a = 0.161$ **B** $a = 1.0161$ **C** $a = 1.61$ **D** $a = 1.0644$

Exercise 6.6

25 An investment company states that a $5000 investment will grow to $100 000 in eight years with additional monthly deposits of $600. Determine which of the following annual interest rates, compounding monthly, is being used by the company.

A 0.85% **B** 8.54% **C** 10.20% **D** 10.25%

Exercise 6.5

26 A savings account commences with a deposit of $1000. Each month another $1000 is added to this account. Interest of 5% p.a. compounds every month. What is the account balance after one year?

Exercise 6.1

27 Iren borrowed $9340 to buy a motor scooter, to be repaid at $210 each month for three years. Interest at 6.04% p.a. was charged each month on the reducing balance. Determine the value of the balloon payment needed at the end of three years.

Exercise 6.1

28 Aiden has found a company that will lend him $15 000 to start up his new business. Interest will be charged monthly at 12.65% p.a. The loan must be fully paid out at the end of two years. Aiden will be required to repay the company $450 each month. Calculate the final repayment that will be required.

Exercise 6.2

29 A farmer borrowed $68 000 to buy a new tractor on an interest-only loan. Interest is charged at 5.2% p.a. compounding monthly. The farmer expects to be able to pay a lump sum of $15 000 at the end of every year when the annual payment for leasing some land is paid to her. Her final lump sum will be less than $6000.

(a) How much is each monthly repayment in the first year?

(b) What is the balance at the end of the first year after the lump sum annual payment of $15 000?

(c) How much is each monthly payment in the second year?

(d) What is the final additional payment and when will this be made?

(e) How much interest will the farmer have paid during the term of this loan?

Exercise 6.3

30 A loan was taken for $23 000 at 6.8% p.a., compounding monthly. Repayments of $500 per month had been made for 36 months when the interest rate decreased to 6.05%. Use a spreadsheet to calculate the following amounts.

(a) the outstanding balance after 36 months

(b) the term of the loan, allowing for the decreased rate of interest after 36 months

(c) the amount of the final repayment

(d) the interest charged for the first 36 months

(e) the interest charged for the entire term of the loan

31 Taylah plans to invest in an annuity that earns interest at 7% p.a. for 15 years, paying no less than $600 at the end of every year. Use a spreadsheet to determine the initial cost of the annuity, to the nearest hundred dollars. Exercise 6.4

32 Sam opened a savings account with $7720. The account earned interest of 3.28% p.a. compounding monthly. For six months, he added $150 to the account at the end of each month. At the start of the seventh month, he found another savings account that would earn interest of 3.59% p.a. compounding monthly. He decided to close the first account and transfer all the money to the new savings account. He continued to add the extra $150 each month to the new account. Exercise 6.5

(a) Determine the value of Sam's investment two years after starting the new account.

(b) How much extra interest did Sam earn by moving his money to the other account?

33 Matthew wants to purchase an apartment that will cost him a total of $422 000. He has saved $52 000 for his deposit and he plans to borrow the rest of the purchase price. He has found a company that will lend him $350 000 at 5.9% p.a. compounding monthly, and the rest of the money at 11.8% p.a. compounding monthly with equal monthly repayments. Exercise 6.3

(a) If Matthew purchased the apartment and used his savings for the deposit, how much money would he have to borrow at 5.9% p.a. and how much at 11.8% p.a.?

(b) If Matthew used 30% of his take-home pay of $5760 each month ($1728) towards repaying the smaller loan, how long would it take to repay this loan? Determine the balance of the larger loan when this smaller loan is fully repaid.

(c) The unused portion of the repayment money at the final repayment will be paid to the larger loan for that month. After this payment has been made, what is the outstanding balance of the larger loan?

(d) Once the smaller loan has been repaid, the repayments will be directed to paying off the larger loan. Explain why that will not be enough to reduce the balance of the larger loan.

34 A loan of $12 700 was repaid at $740 per month. Interest at 8.2% p.a. was calculated monthly on the reducing balance. All repayments were made on time, except the last one. The last repayment was missed but the loan was fully repaid the following month. Exercise 6.3

(a) Determine the final repayment, given that the balance after 16 months was $1695.43.

(b) Calculate the total interest paid on the loan.

35 Jan borrowed $380 000 to buy a unit. His interest rate was 5.04% p.a. calculated each quarter. He will repay the loan over 20 years. Exercise 6.3

(a) Use a spreadsheet to determine the quarterly repayment for this loan, to the nearest $10.

(b) The interest rate increased to 5.6% exactly five years after this loan started. If Jan still wanted to repay the initial loan in 20 years, determine his new quarterly repayment.

36 Milly wishes to determine the effect of different interest rates on her investment. She has an initial deposit of $5000 and plans to deposit $350 each month for 15 years. Exercise 6.6

(a) Determine the total amount accrued at the following interest rates, compounded monthly.

(i) 5% p.a. (ii) 3.5% p.a.

(b) What time period would she need if the amount achieved at 5% interest was to be obtained with the current interest rate of 3.5% p.a.?

7

Graphs and networks

7 Recall

Substitute a value into a fractional expression and simplify

1 Determine the value of each expression for the given value of the pronumeral.

(a) $\frac{x(x-3)}{4}$ for $x = 8$ (b) $\frac{n(n+2)}{3}$ for $n = 6$

(c) $\frac{2m(m-5)}{4}$ for $m = 6$

Substitute a value into an equation and then solve for the unknown

2 Substitute the given values and then solve for the unknown value.

(a) For $a + b - c = 5$, solve for b when $a = 4$ and $c = 2$.

(b) For $p + q - r = 1$, solve for p when $q = 8$ and $r = 15$.

(c) For $f + v - e = 2$, solve for e when $f = 3$ and $v = 9$.

Solve equations involving fractions

3 Solve the following equations for x.

(a) $\frac{2(x-3)}{5} = 6$ (b) $\frac{5x}{2} = 20$ (c) $\frac{4x+3}{6} = 1$

Consecutive numbers

4 Determine the missing consecutive whole numbers, given ascending order.

(a) 7, _, 9 (b) 3, _ (c) _, 10

Solving by trial and error

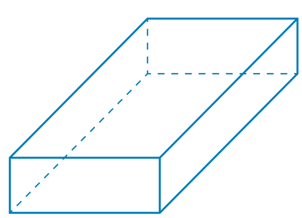
5 Solve each of the following equations for n, given that $n > 0$.

(a) $n(n-1) = 56$ (b) $n(n-1) = 90$ (c) $n(n-1) = 20$

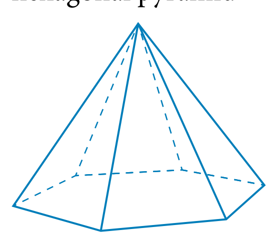
Faces, edges and vertices

6 For each shape, count the number of faces, edges and vertices.

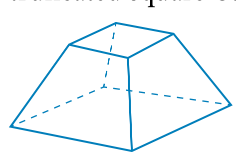
(a) rectangular prism



(b) hexagonal pyramid



(c) truncated square-based pyramid

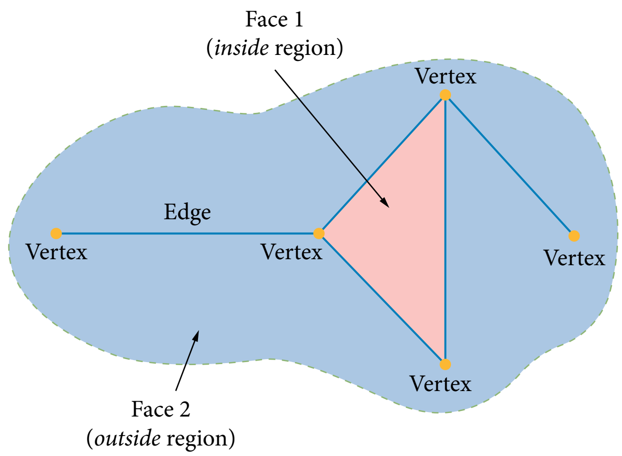


Describing graphs

7.1

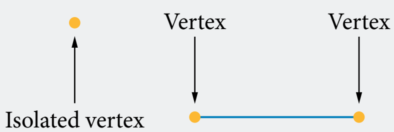
Components of a graph

In this chapter, the term *graph* refers to a *network diagram*, which is a collection of one or more points called vertices (singular vertex) and the line segments that connect them, called edges.



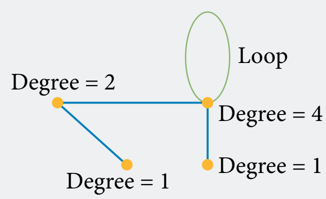
Vertex (v)

A vertex is a point. The plural of vertex is vertices. An isolated vertex is one that is not connected by an edge to another vertex. Vertices are also referred to as nodes.



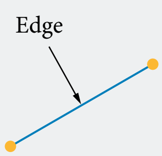
Loop

A loop is an edge that connects a vertex to itself. A single loop adds 2 to the degree of a vertex.



Edge (e)

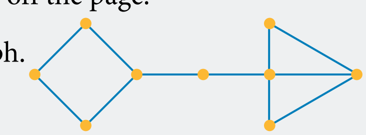
An edge is a line that connects two vertices.



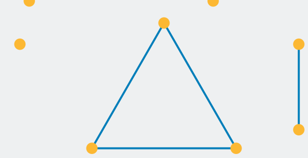
Connected graph

In a connected graph, each vertex is connected directly or indirectly to all other vertices and can be traced without lifting your pen off the page.

This is a connected graph.



This graph is not connected.

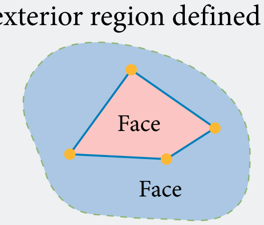


Degree

The degree of a vertex is the number of edges extending from that vertex. An isolated vertex has a degree of 0.

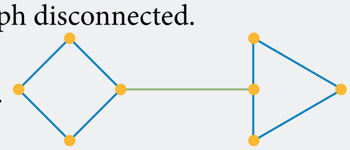
Face (f)

A face is an interior or exterior region defined by edges and vertices. In the figure, the exterior face is shown in blue. This diagram has two faces.



Bridge

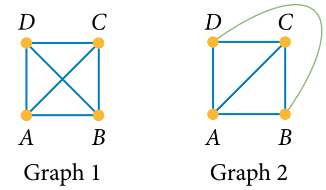
A bridge is an edge in a connected graph that, if removed, leaves the graph disconnected. In the graph shown here, the green edge is a bridge.



Isomorphic graphs

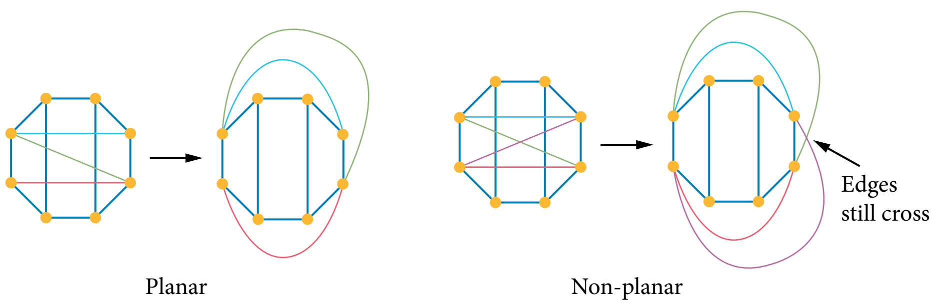
Isomorphic graphs are equivalent. That is, they contain corresponding vertices and edges. This may not always be obvious, because the edges of a graph may have to be stretched or its vertices rearranged to make equivalent connections clearer.

In the following diagram, Graph 1 and Graph 2 are isomorphic. Either one may be morphed into the other, while maintaining the same connections.



Planar graph

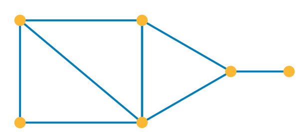
A graph is planar if it can be drawn so that no edges cross.



Euler's formula

Euler's formula expresses the relationship between the numbers of vertices (v), edges (e) and faces (f) in a connected planar graph: $v + f - e = 2$

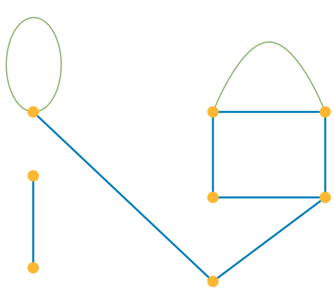
In the connected planar graph shown here, you can see that the equation $6 + 4 - 8 = 2$ is true for $v = 6$, $f = 4$ and $e = 8$.

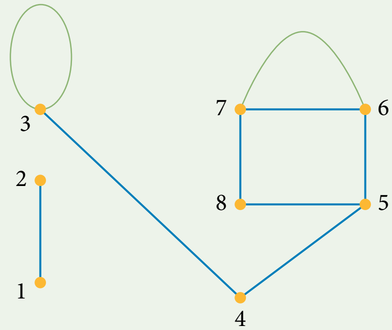
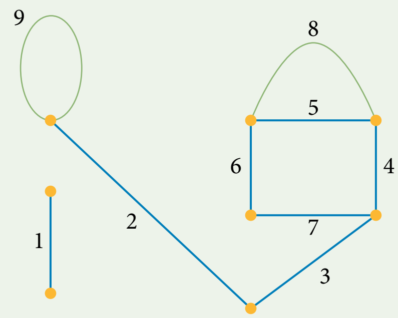
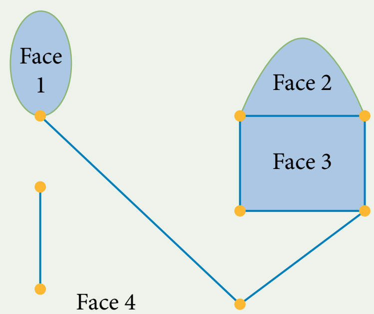


1 The components of a graph

Identify the number of vertices (v), edges (e) and faces (f) in the following graphs, and state whether or not they are connected, planar graphs. Confirm using Euler's formula.

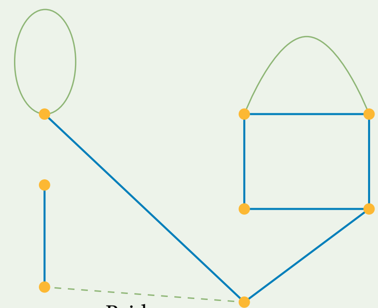
(a)



THINKING	WORKING
1 Identify the number of vertices, by counting the number of points.	 $v = 8$ The graph contains 8 vertices.
2 Identify the number of edges, by counting the number of line segments and loops, if any, between vertices.	 $e = 9$ The graph contains 9 edges.
3 Determine the number of faces by counting the number of interior regions enclosed by edges and the exterior region outside the edges.	 $f = 3 + 1 = 4$ The graph contains 3 interior faces and 1 exterior face.

4 Determine whether the graph is connected or not.

The graph is not connected, because you cannot trace the edges and reach every vertex without lifting your pen off the page. For the graph to be connected, a bridge (an edge connecting the two subgraphs) would need to be inserted, as shown.



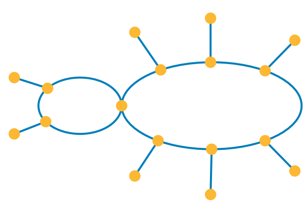
5 Verify the result by substitution into the LHS of Euler's formula $v + f - e = 2$.

$$\begin{aligned} v + f - e &= 8 + 4 - 9 \\ &= 3 \end{aligned}$$

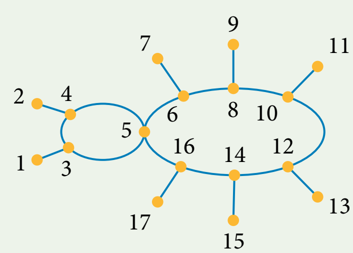
As expected, $v + f - e \neq 2$

because the graph is not connected.

(b)



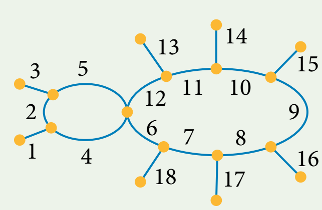
1 Identify the number of vertices, by counting the number of points.



$v = 17$

The graph contains 17 vertices.

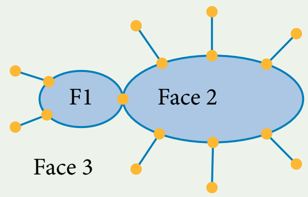
2 Identify the number of edges, by counting the number of line segments and loops, if any, between vertices.



$e = 18$

The graph contains 18 edges.

3 Determine the number of faces by counting the number of interior regions enclosed by edges and the exterior region outside the edges.



$f = 2 + 1 = 3$

The graph contains 2 interior faces and 1 exterior face.

4 Determine whether the graph is connected and planar or not.

The graph is connected, because you can trace the edges and reach every vertex without lifting your pen off the page. The graph is planar because no edges are crossing.

5 Verify the result by substitution into the LHS of Euler's formula $v + f - e = 2$.

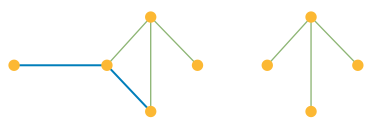
$$\begin{aligned} v + f - e &= 17 + 3 - 18 \\ &= 2 \end{aligned}$$

As expected, $v + f - e = 2$

because the graph is connected and planar.

Subgraphs

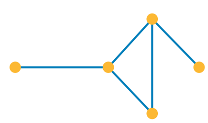
A subgraph contains parts of another graph, but may be missing some vertices or edges. Any edge remaining in a subgraph must join the same vertices as in the original graph.



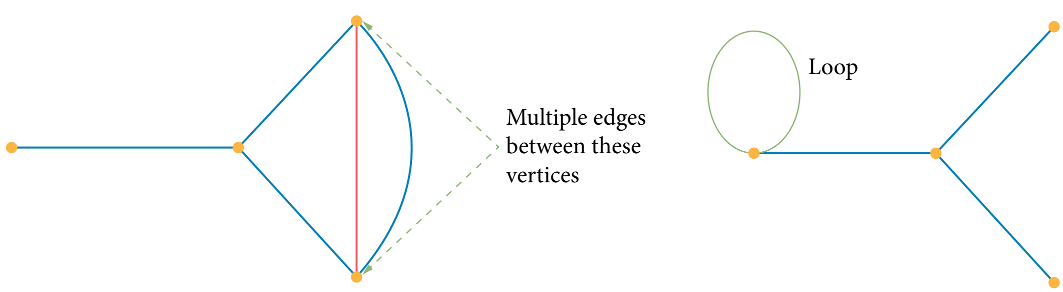
Graph Subgraph

Simple graph

A simple graph contains no loops or multiple edges between vertices. A simple graph may either be connected or disconnected.



Simple graph

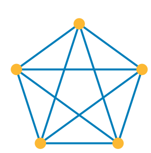


These are NOT simple graphs.

Complete graph

In a complete graph, every vertex is joined once to every other vertex.

The complete graph shown here could be used to represent the different handshakes possible among five people. Each of the five people shakes hands with four others: $5 \times 4 = 20$.



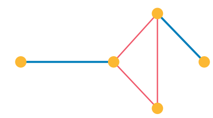
In fact there are $\frac{20}{2} = 10$ handshakes, because each handshake is 'counted' by two people.

The number of edges (e) in a complete graph that has v vertices is given by the equation: $e = \frac{v(v-1)}{2}$.

For the example above, $v = 5$, so $e = \frac{5 \times 4}{2} = 10$.

Cycle

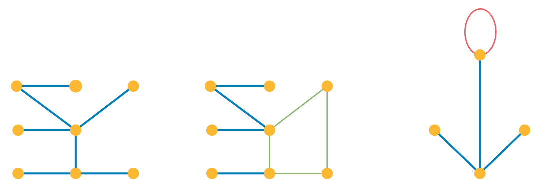
A cycle is a closed path that does not repeat vertices, except for its starting and ending vertex.



Tree

A tree is a connected graph that contains no loops or cycles. Trees have only one exterior face. Trees connect v vertices with $(v-1)$ edges.

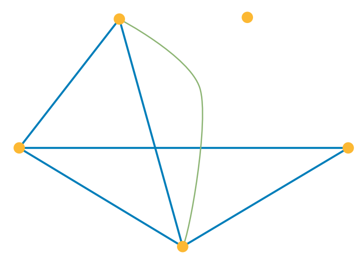
The number of edges (e) in a tree is given by $e = v - 1$, where v is the number of vertices in the tree.



A tree | NOT a tree (contains a cycle) | NOT a tree (contains a loop)

2 Describing a graph

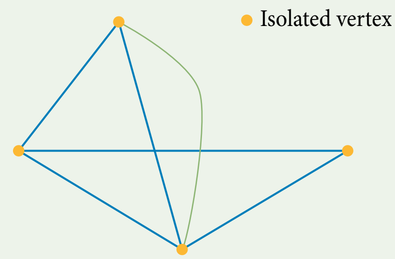
Which of the following terms describes this graph?
tree, complete graph, connected graph, planar graph, simple graph



THINKING

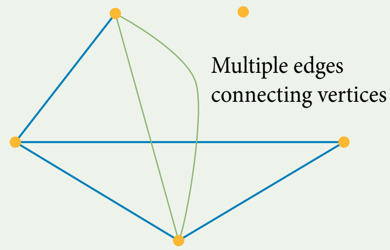
1 Consider the isolated vertex of the graph.

WORKING



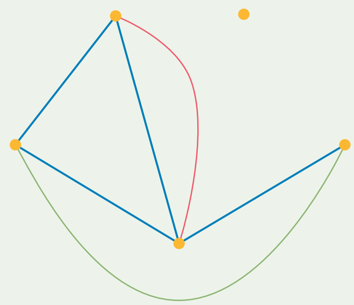
The graph contains an isolated vertex, and therefore is not a tree (where all vertices are connected by $v - 1$ edges), a connected graph (where all vertices can be reached by tracing the edges without lifting your pen off the page) or a complete graph (a graph where each vertex is connected directly to all other vertices).

2 Consider the multiple edges of the graph.



The multiple edges connecting the two vertices shown here mean that the graph does not represent a simple graph (a graph with no multiple edges or loops).

3 Determine whether the graph is planar – if it can be drawn so that no edges cross.



The graph can be redrawn in the plane so that no edges cross. Therefore, the graph is planar.

4 Interpret the result.

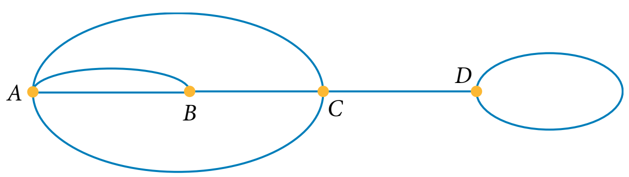
The graph is planar because it can be redrawn with no edges crossing.

The isolated vertex means that the graph is not connected and is therefore not a connected graph, a tree or a complete graph.

The presence of multiple edges connecting two vertices means that the graph is not simple.

3 Types of graphs

Consider the graph shown here, and create the graphs specified in questions (a) to (d).



(a) A subgraph containing:

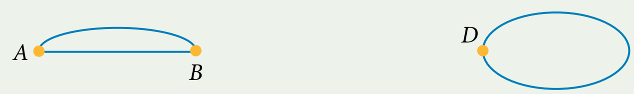
(i) 3 vertices

THINKING

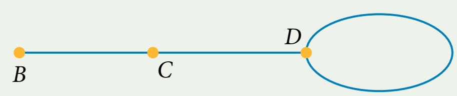
To create a subgraph with a limit on the number of vertices, start by removing the vertex not required, along with any edges connected to that vertex.

WORKING

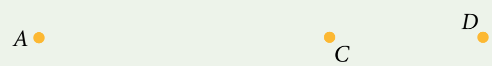
When you remove a vertex, you must also remove all edges attached to that vertex. For example, the subgraph created by removing vertex C and all edges connected to C is:



The subgraph created by removing vertex A and all edges connected to A is:



The subgraph created by removing vertex B and all the edges in the graph is called a degenerate subgraph, where all vertices are isolated and have a degree of 0.



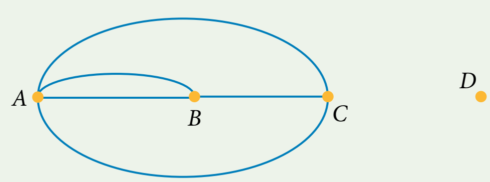
However, there are many subgraphs that contain 3 vertices. Those shown above are just a couple of possibilities.

(ii) 5 edges

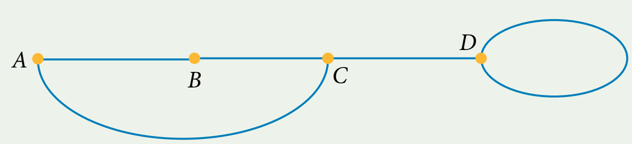
To create a subgraph with a limit on the number of edges, determine whether there are any other conditions, then delete the given number of edges.

The graph has 7 vertices. A subgraph can be connected or disconnected.

When the bridge and the loop at D are removed, the resulting graph is not connected—it has an isolated vertex D.

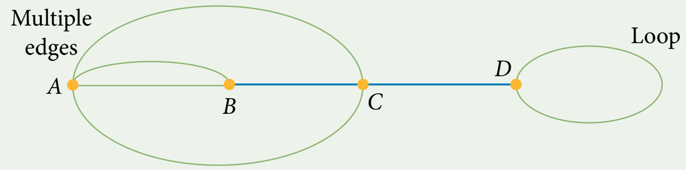


However, any two edges can be removed. For example, if one of the multiple edges between A-B and A-C is removed, the resulting graph is connected.

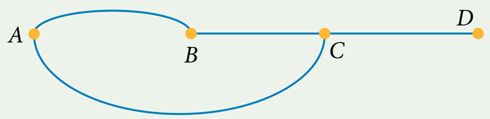


(b) A simple subgraph

1 Identify loops and multiple edges.	Multiple edges connect *A*-*C* and *A*-*B*. There is a loop at vertex *D*.

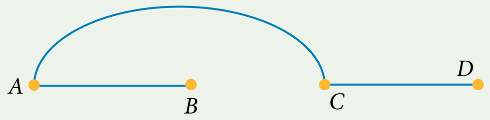


2 Create a simple subgraph by removing loops and multiple edges from the original graph.	Choose a single edge to connect *A* to *B* and a single edge connecting *A* to *C*. The simple graph shown contains a cycle connecting *A*, *B* and *C*.



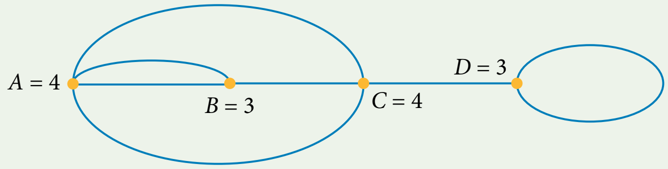
(c) a tree

1 A tree has a single face and connects v vertices with $(v-1)$ edges. Determine the number of edges required to create a tree.	The original graph has 4 vertices (*A*, *B*, *C* and *D*). Therefore a tree will join all 4 vertices using 3 edges.
2 Draw a tree.	Many variations are possible. The subgraph must show all 4 vertices connected by only 3 edges.



(d) an isomorphic graph with vertices arranged in a square

1 An isomorphic graph is equivalent. Determine the degree of each vertex to check the reasonableness of your isomorphic graph.



2 Identify the features of the graph.

The original graph has 4 vertices, connected by 7 edges.

Vertex *A* is connected to *B* with 2 edges and *C* with 2 edges.

Vertex *B* is also connected to *C* with 1 edge.

Vertex *C* is then connected to *D* with 1 edge.

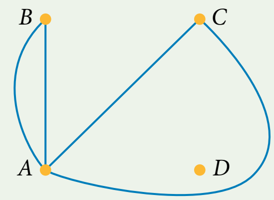
Vertex *D* connects to itself with a loop.

3 Create the template for the isomorphic graph by arranging the vertices.

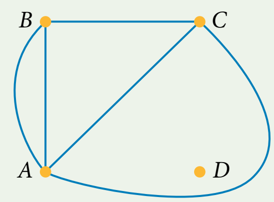
The isomorphic graph is to be the shape of a square. The order of the vertices will vary.

4 Connect the edges to match the description based on the features of the graph.

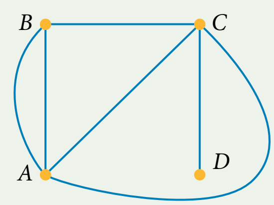
Vertex A is connected to B with 2 edges and C with 2 edges.



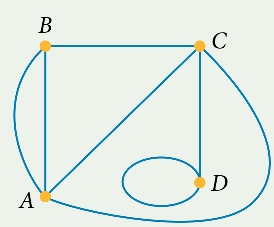
Vertex B is also connected to C with 1 edge.



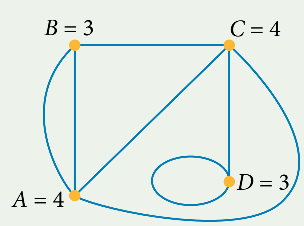
Vertex C is then connected to D with 1 edge.



Vertex D connects to itself with a loop.

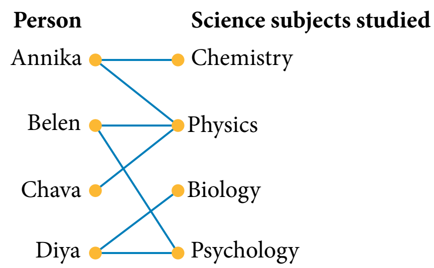


5 Evaluate the reasonableness of the graph by determining the degree of each vertex.



Bipartite graph

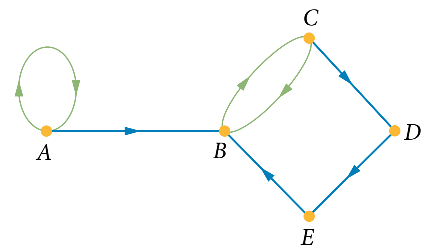
A bipartite graph is one whose vertices may be split into two distinct groups, with each of the graph's edges joining a vertex in one group with a vertex (or vertices) in the other group.



Directed graph (digraph)

A directed graph contains vertices joined by directed edges or *arcs*.

For example, an arrow may indicate a one-way road between *A* and *B*, the result '*A* beats *B*' in a competition, or the relationship '*A* is consumed by *B*' in a food web.

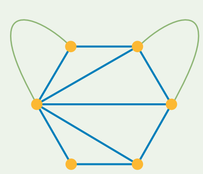


Using the formulas

You can use Euler's formula to calculate the value of v, f or e if you know the values of two of the three pronumerals.

4 Euler's formula

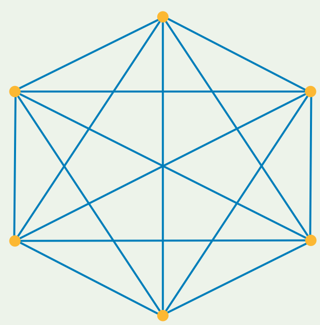
A connected planar graph has 6 vertices and 11 edges. Determine the number of faces (regions) for the graph and draw an example.

THINKING	WORKING
1 Recall Euler's formula. Substitute the given values and solve for f.	$v+f-e=2$ where $v=6$ and $e=11$ $6+f-11=2$ $f-5=2$ $f=7$
2 Interpret the result.	A connected planar graph with 6 vertices and 11 edges has 7 faces.
3 Draw an example. Start with the correct number of vertices. Add edges until all vertices are connected with 11 edges. Count the faces to check the reasonableness of the result.	

If you know a graph is complete and you are given the number of edges, you can use the equation $e=\frac{v(v-1)}{2}$, with trial and error, to determine the number of vertices.

5 Complete graphs

(a) A complete graph has 6 vertices. Determine the number of edges in the graph and the degree of each vertex.

THINKING	WORKING
1 Recall the definition of a complete graph.	A complete graph is one where each vertex is directly connected to all other vertices.
2 Determine the degree of each vertex.	If there are 6 vertices and each vertex is connected to the other 5 vertices, then all vertices have a degree of 5.
3 Recall Euler's formula. Substitute the given values. Simplify the equation.	$e=\frac{v(v-1)}{2}$ where $v=6$ $e=\frac{6(6-1)}{2}$ $=\frac{6\times 5}{2}$ $=\frac{30}{2}$ $=15$
4 Verify your answer graphically.	
5 Interpret the result.	A complete graph with 6 vertices has 15 edges and each vertex has a degree of 5.

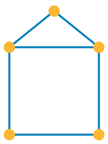
(b) A complete graph has 55 edges. Determine the number of vertices for the graph.

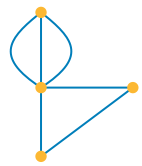
THINKING	WORKING
1 Recall Euler's formula. Substitute the given values. Simplify the equation.	$e=\frac{v(v-1)}{2}$ where $e=55$ $55=\frac{v(v-1)}{2}$ $110=v(v-1)$
2 Use trial and error to determine two consecutive numbers with the required product. Write the algebraic solution.	$11\times 10=110$, so $v=11$
3 Interpret the result.	A complete graph with 55 edges has 11 vertices.

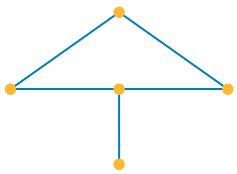
EXERCISE 7.1 Describing graphs

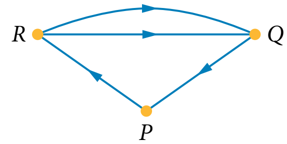
Worked Example

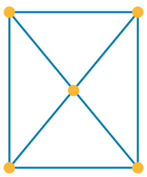
1 Identify the number of vertices (v), edges (e) and faces (f) in each of the following planar graphs, and identify whether or not the graph is connected. Use Euler's rule $v + f - e = 2$ to verify your answer. 1

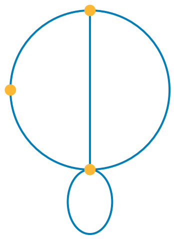
(a)


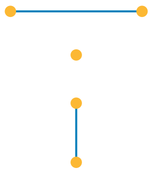
(b)


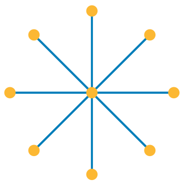
(c)


(d)


(e)


(f)


(g)


(h)


2 Answer the following questions about connected planar graphs.

(a) What is the number of edges in a graph with 16 vertices and 10 faces?

(b) What is the number of vertices in a graph with 10 faces and 18 edges?

(c) What is the number of faces in a graph with 7 vertices and 10 edges?

(d) If the number of faces plus the number of vertices in a planar graph is equal to 20, how many edges connect the graph?

(e) What is the number of edges in a planar graph if it has 9 vertices and 20 faces?

(f) What is the number of vertices in a graph if it has 100 edges and 10 faces?

(g) What is the number of faces in a graph if it has 16 vertices and 50 edges?

(h) What is the sum of the number of faces and number of vertices in a planar graph that has 48 edges?

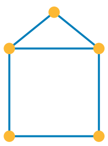
3 Given the number of vertices, faces and edges, decide whether each network is connected and planar.

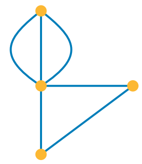
(a) $v = 13, f = 8, e = 19$ (b) $v = 7, f = 12, e = 17$ (c) $v = 29, f = 21, e = 20$

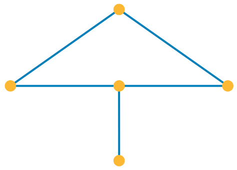
(d) $v = 100, f = 2, e = 98$ (e) $v = 15, f = 70, e = 73$ (f) $v = 255, f = 331, e = 584$

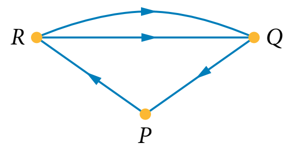
4 From the list of terms below, select all the terms that apply to each of the graphs. 2

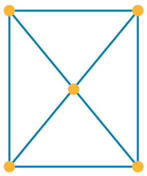
connected, planar, simple, complete, bipartite, directed

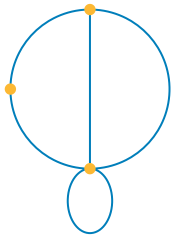
(a)


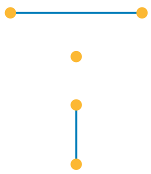
(b)


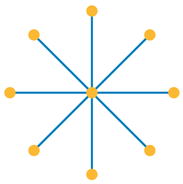
(c)


(d)


(e)


(f)


(g)


(h)


Worked Example

3

5 Consider the graph shown here.

(a) Demonstrate that the graph is planar.

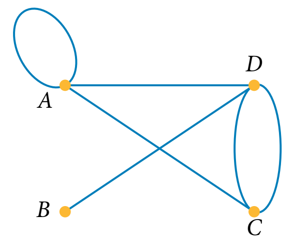
(b) Create a simple subgraph.

(c) Show a tree.

(d) Create an isomorphic graph:

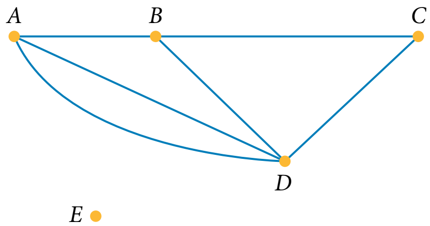
(i) with vertices in the same position

(ii) with vertices arranged in a row.

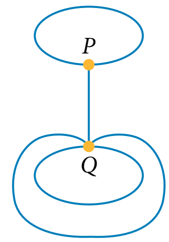


6 Determine the degree of each vertex in the following graphs.

(a)



(b)



7 Determine whether the following statements are true or false.

(a) The sum of the degrees of the vertices is 16.

(b) The graph is connected.

(c) The graph is planar.

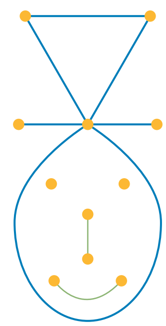
(d) Euler's formula applies to this graph.

(e) The graph is a tree.

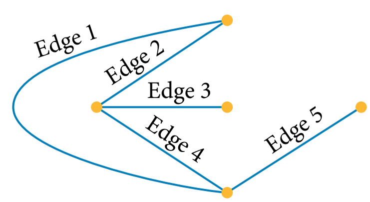
(f) The graph has 3 faces.

(g) The graph is a directed graph.

(h) The graph is simple.

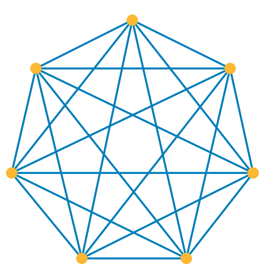


8 Which edges in the graph shown here are bridges?

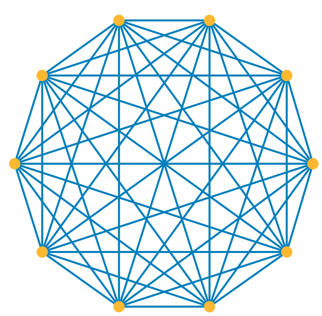


9 Use the equation $e = \frac{v(v-1)}{2}$ to determine how many edges are in each of the complete graphs shown.

(a)



(b)



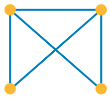
10 How many edges are in a complete graph containing the following number of vertices?

(a) 12 vertices

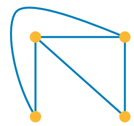
(b) 20 vertices

11 Determine whether each of the pairs of graphs is isomorphic.

(a)

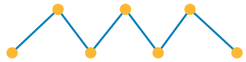


Graph 1

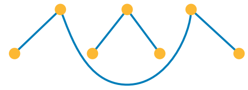


Graph 2

(b)

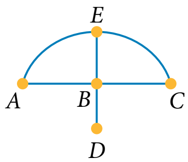


Graph 1

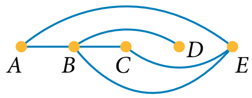


Graph 2

(c)

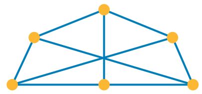


Graph 1

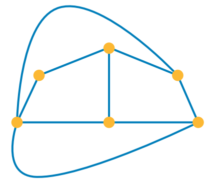


Graph 2

(d)

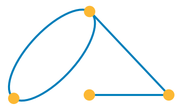


Graph 1

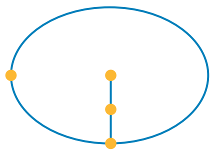


Graph 2

(e)

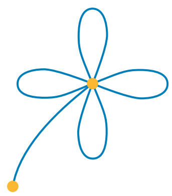


Graph 1

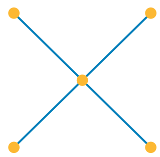


Graph 2

(f)



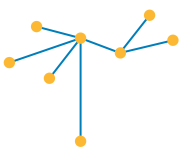
Graph 1



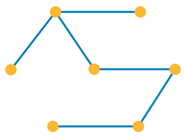
Graph 2

12 Determine whether each of the following graphs is a tree.

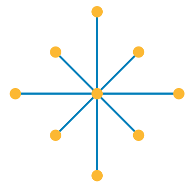
(a)



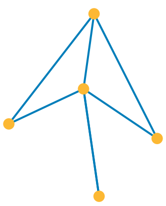
(b)



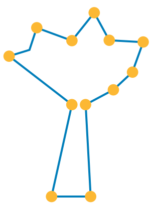
(c)



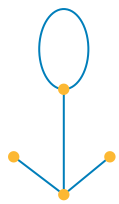
(d)



(e)



(f)



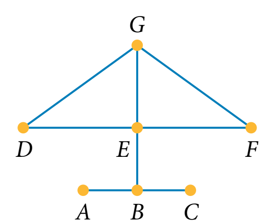
13 Which of the options below would make this network a tree?

A Remove edge *D*-*G*.

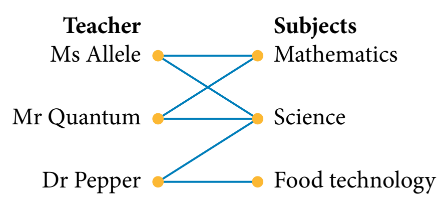
B Remove edge *E*-*B*.

C Remove edges *D*-*E* and *E*-*F*.

D Add an edge between *A* and *D*.



14 The following bipartite graph shows the subjects that three teachers in a small school are qualified to teach at Year 10 level. Determine whether each statement below is true or false.



(a) One of the teachers is qualified to teach all three subjects.

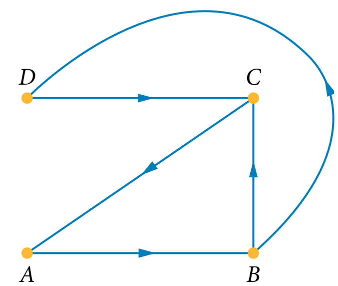
(b) Only one of the teachers is qualified to teach Food technology.

(c) Two of the teachers are qualified to teach Mathematics.

(d) Each of the teachers is qualified to teach Science.

15 Three friends state whether or not they like three types of food. Henry likes pizza and souvlaki. Jean-Luc likes hamburgers and souvlaki. Irina likes all three. Draw a bipartite graph to represent this information.

16 The vertices in the digraph represent the available flights between four cities. State whether each of the statements about this digraph is true or false.



(a) There is a direct flight from A to D.

(b) There is a direct flight from B to D.

(c) The fewest flights required to get from A to D is 2.

(d) There are two ways of getting from A to C.

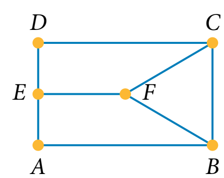
(e) The fewest flights required to get from C to D is 3.

(f) There are three flights that arrive at C.

17 Draw a directed graph to show the following information for four one-way roads. Represent each road as an edge and the intersections as vertices labelled A, B or C.

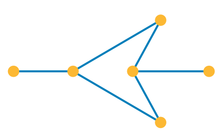
- One road joins points A and B.
- One road joins points B and C.
- Two roads join points A and C.
- Travel is permitted from point A to point B.
- Travel is permitted from point B to point C along one road.
- Travel is permitted from point C to point A along one road.
- Travel is permitted from point A to point C along one road.

18 Observe the graph and answer the following questions.

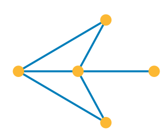


(a) Which of the following is an isomorphic subgraph of the given graph?

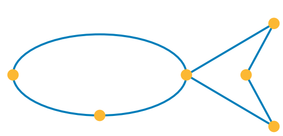
A



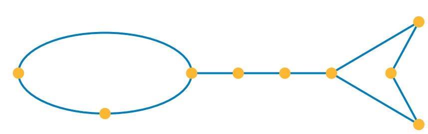
B



C

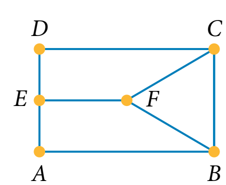


D



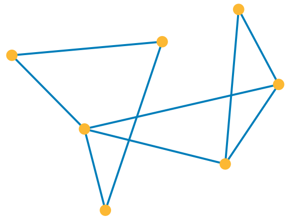
(b) Label the second incorrect option with letters to demonstrate that it is not a subgraph of the original graph.

19 Determine the number of edges that must be removed from the graph to leave a tree. Explain what will happen if more than this number is removed.

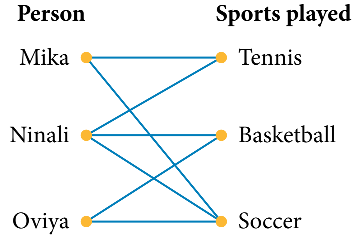


20 For each of the following graphs, draw an isomorphic graph to make it clear that each is planar. Identify the number of vertices (v), edges (e) and faces (f) to confirm that each graph conforms to Euler's rule.

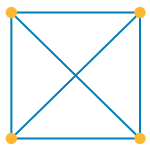
(a)



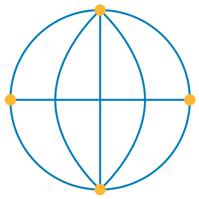
(b)



(c)



(d)



Worked Example 4

21 Determine whether a graph with each of the following sets of properties is planar and connected, then draw a possible version of the graph.

(a) 3 vertices, 4 edges, 3 faces
(b) 4 vertices, 3 edges, 1 face
(c) 2 vertices, 3 edges, 3 faces
(d) 2 vertices, 4 edges, 4 faces
(e) 5 vertices, 4 edges, 1 face
(f) 5 vertices, 6 edges, 3 faces
(g) 8 vertices, 4 edges, 2 faces
(h) 3 vertices, 6 edges, 5 faces

22 Draw a graph that has the following properties.

(a) 4 vertices and connected and planar
(b) 3 vertices, including one with degree 1, one with degree 3 and one with degree 4
(c) 5 vertices, 4 edges and not connected
(d) 4 vertices, 2 loops and 4 edges
(e) 3 vertices and complete
(f) 7 vertices and a tree

23 How many vertices are in a complete graph that has the following number of edges? 5

(a) 15 edges
(b) 36 edges
(c) 6 edges
(d) 28 edges

24 Three committee members volunteer for the following jobs at a fundraising fete.
Aaron is prepared to help with the drinks machine and sausage sizzle.
Bruce is willing to assist on the jumping castle and the drinks machine.
Candice offers to help with the sausage sizzle and the drinks machine.

(a) Draw a bipartite graph to represent this information.
(b) If each member must perform one job, assign each person a job. Write all possible solutions.
(c) Which job is most popular and which is least popular among the committee members?
(d) Which committee member does not wish to cook for the fete?

25 Construct a connected, planar graph that has 8 edges, 2 faces and 5 bridges.

7.2 Practical networks

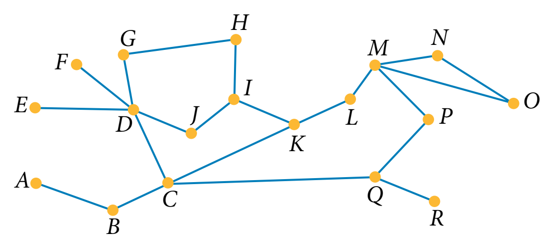
Practical applications

You can use graphs to describe many practical situations, such as bushwalking, social relationships, transportation grids, food webs and round robin sporting events.

A network must show the connections between vertices but does not necessarily show the actual positions of the vertices or the distances between them.

6 Interpreting a graph

A graph representing an underground sewer network is shown here. The vertices labelled A to R represent access points in the network, and the edges represent pipes connecting the points.



(a) Can this network be described as a tree? Justify your answer.

THINKING	WORKING
1 Determine whether the graph contains any cycles or loops in the network.	*G-H-I-J-D-G*, *C-Q-P-M-L-K-C* and *M-N-O-M* are examples of cycles. The network is not a tree because it contains a cycle.
2 Verify your answer by counting the number of vertices and edges and using your understanding of the relationship between the number of vertices and number of edges in a tree.	A tree connects all vertices v with $(v-1)$ edges. There are 18 vertices and 20 edges. For the graph to represent a tree, the vertices should be connected by 17 edges/pipes.

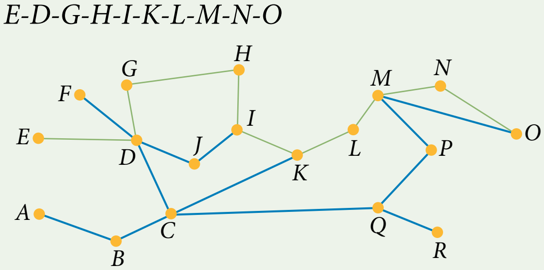
(b) How many access points are there in the network?

Count the number of vertices.	There are 18 access points in the network.

(c) Describe a path from access point E to access point O that comprises a total of 10 access points.

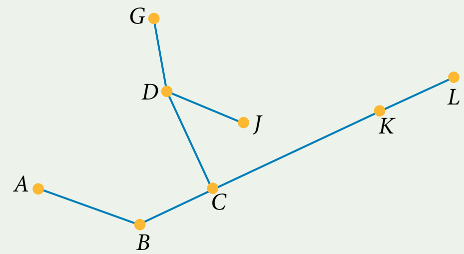
Use trial and error to find a path between the access points, containing the required number of vertices.

E-D-G-H-I-K-L-M-N-O



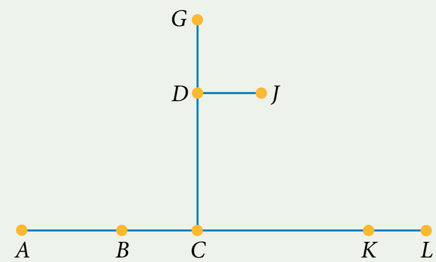
(d) Using only vertical and horizontal line segments, draw a subnetwork of the sewer system that contains only the access points A, B, C, D, G, J, K and L and any connections between them.

1 Draw the subnetwork identified above using the vertices listed and the edges between them.



2 Maintain the same connections as above and draw the network using only vertical or horizontal edges.

There are many possible diagrams.



7 Constructing an undirected network

Construct a graph showing the following pathways between key attractions at a zoo:

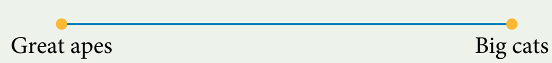
- a path connecting the great apes to the big cats
- a path connecting the reptile house to the nocturnal animals
- paths connecting the elephant enclosure to the great apes, the big cats and the giraffe enclosure
- a path connecting the reptile house to the aquarium
- a path connecting the aquarium to the great apes
- a path connecting the giraffe enclosure to the nocturnal animals.

THINKING

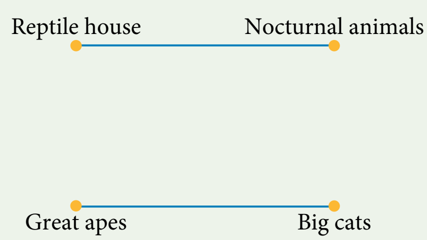
WORKING

1 Use vertices to represent the various animal areas and use edges to represent the paths between them.

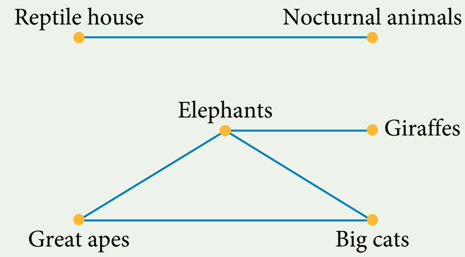
Begin with the first path listed and construct a partial graph with two vertices and an edge between them. Label the vertices.



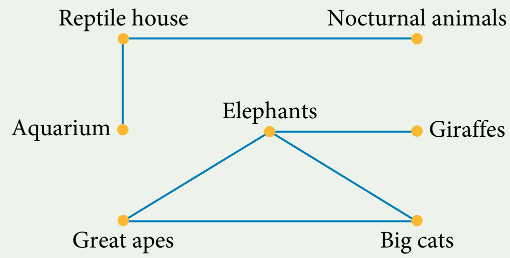
2 Represent the information in the second path listed. This may be done in a number of ways, but be sure to draw the correct connections. Remember, you are constructing a graph, not a map, so exact locations are not important.



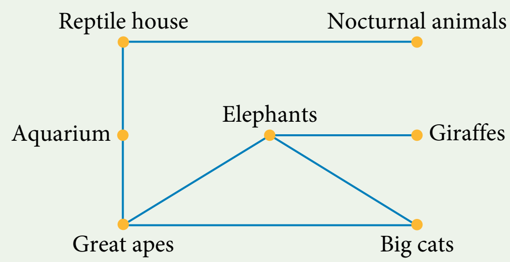
3 Represent the information in the third path listed.



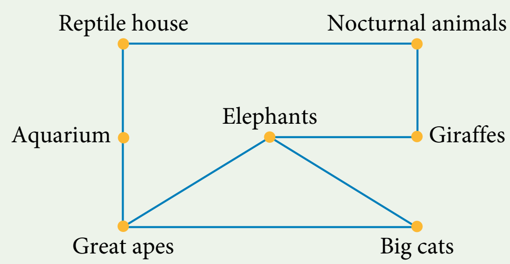
4 Represent the information in the fourth path.



5 Represent the information in the fifth path.



6 Complete the network by representing the information in the last path.



Digraphs

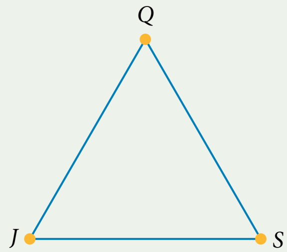
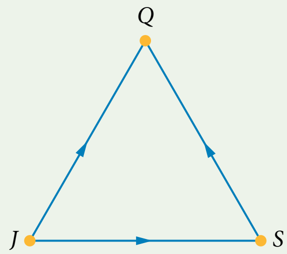
You can use a directed graph (digraph) to represent one-way relationships by including arrows along the edges of the graph that point to a particular vertex.

8 Constructing a digraph

The results of a round robin tournament involving three teams, the Jets (J), the Solar Flares (S) and the Quantum Mechanics (Q), are as follows:

- The Jets defeated the Solar Flares and the Quantum Mechanics.
- The Solar Flares defeated the Quantum Mechanics.

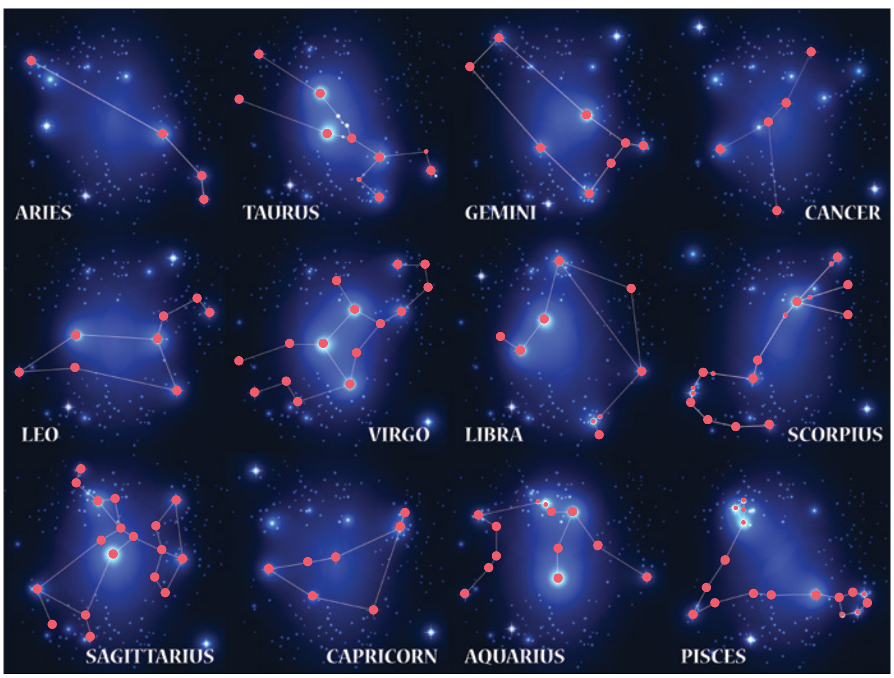
Show this information on a directed graph, where each arrow points towards the losing team and is read as 'defeated'.

THINKING	WORKING
1 Draw three vertices in a triangular pattern. Label each vertex with the first initial of the name of a team.	
2 Along each edge, draw an arrow that points towards the defeated team.	

EXERCISE 7.2 Practical networks

Worked Example 6

1 The figures below show an example of the networks and names of the twelve constellations, including the major stars in each.

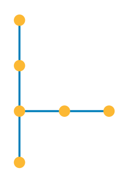


(a) In which constellations do the graphs contain no loops or cycles?

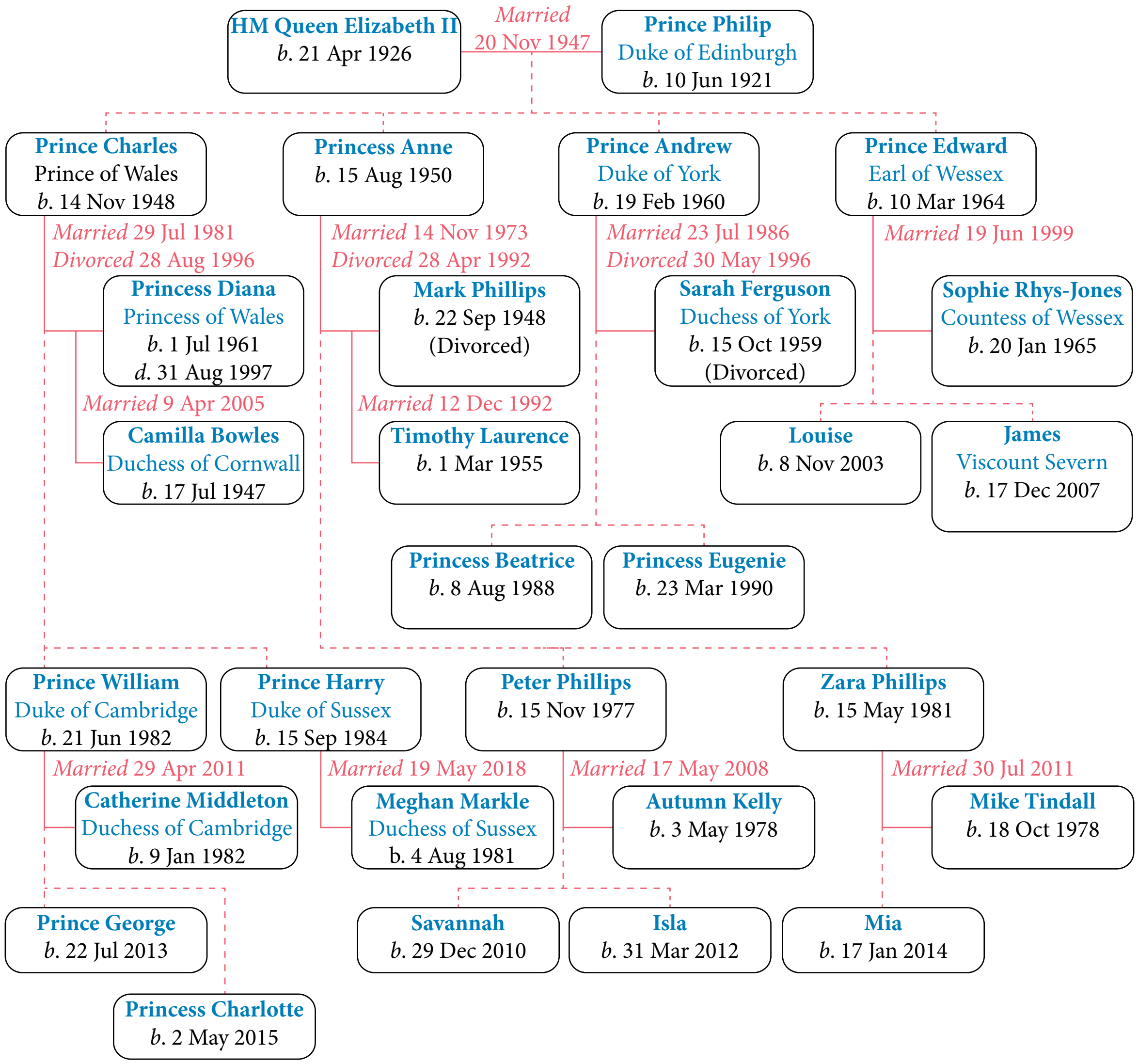
(b) How many stars are in the Virgo constellation?

(c) Which constellations have networks that contain exactly 10 stars?

(d) Which constellation can be represented by the network shown at right?



2 A section of the British Royal Family tree from 1921 to 2018 is shown here.



(a) What are the names of Prince Edward's children?

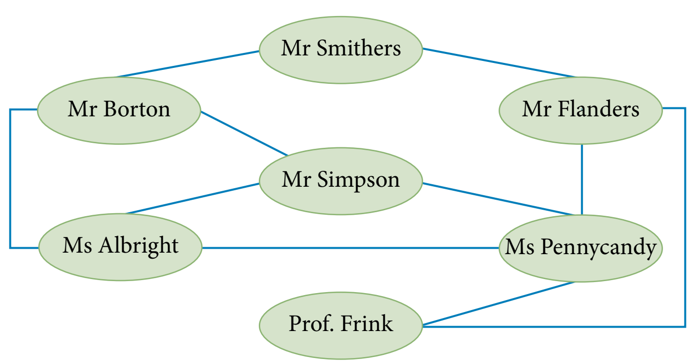
(b) Who are the parents of Zara Phillips?

(c) Was Timothy Laurence born into the Royal family or did he marry into it?

(d) How many members of the British Royal Family were divorced at some point?

(e) The 'line of succession' to the British throne is the list of each first-born child in order from oldest to youngest. Including Queen Elizabeth II, what is the current line of succession to the British throne?

3 A graph showing links to people with common hobbies is shown here. Determine whether each statement is true or false.



(a) Mr Smithers has common hobbies with three others.

(b) Ms Albright does not share any common hobbies with Prof. Frink.

(c) Ms Pennycandy has more hobbies in common with others in the network than anyone else.

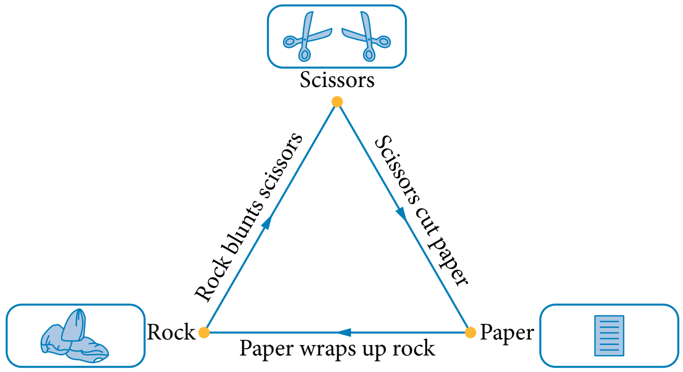
(d) Mr Borton has no hobbies in common with Mr Flanders.

(e) Mr Simpson has hobbies in common with Mr Smithers but not with Prof. Frink.

(f) Mr Flanders has no hobbies in common with Mr Simpson or Prof. Frink.

Worked Example 8

4 The game Rock–Paper–Scissors involves two people simultaneously forming their hands into one of three shapes that represent a rock, paper or scissors. There are three possible outcomes in a game: rock blunts scissors, paper wraps up rock and scissors cut paper. A draw results if two hands form the same shape at the same time. The directed network represents the possible outcomes in a game.



Keira and Lanna play 10 games. Their choices in each game are listed in the table.

Keira	R	R	S	P	P	S	R	S	S	P
Lanna	P	S	R	R	P	R	S	S	P	R

(a) Who is the overall winner and by how many games?

A Lanna wins by 5 games.

B Keira wins by 2 games.

C Lanna wins by 3 games.

D Lanna wins by 2 games.

(b) Explain the common error made by a student who chose the third incorrect option.

5 The map shows an area in New York City that has a number of one-way streets. Arrows along the streets indicate the direction that traffic can flow. North is at the top of the map.

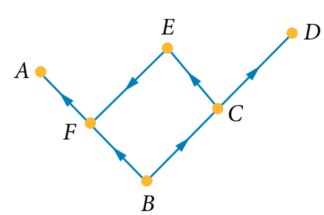
(a) Is it possible to drive in a NE direction along Church Street?

(b) In which general direction will you travel if you drive along Warren Street?

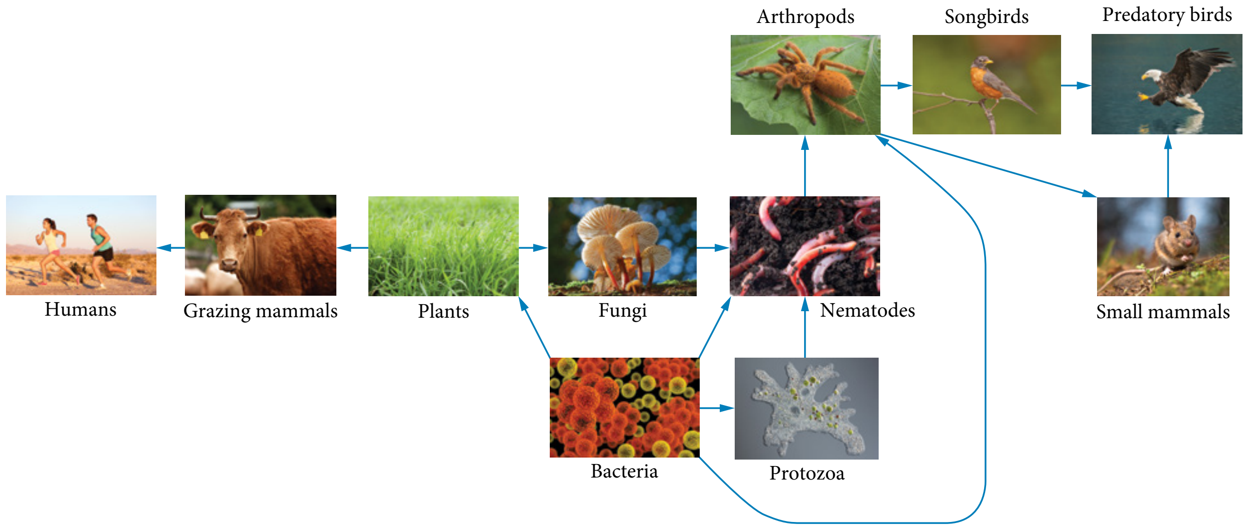
(c) Is it possible to drive in a cycle from the intersection of Leonard Street and West Broadway to Warren Street, along Warren Street then Church Street to Leonard Street and then return to your starting point?

(d) Fred is looking for a parking place and decides to drive around one block, starting at the corner of Duane Street and West Broadway. List the streets he travels along if he begins by driving SE. Remember to include the full street/road type after the name of the street in your answer.

(e) The diagram at right represents intersections and one-way street connections from the map. Label the points (vertices) with the names of the intersections.



6 A food web of selected organisms is shown. The direction of each arrow represents the flow of energy from a producer to a consumer and is read as 'is consumed by'. Answer the following questions based on this food web.



(a) According to the graph, which organisms are:
  (i) consumed by arthropods
  (ii) consumed by nematodes
  (iii) not consumed by other organisms?

(b) Determine whether the following statements are true or false.
  (i) Nematodes are eaten by protozoa.
  (ii) Humans and predatory birds are not eaten by other organisms.
  (iii) Bacteria consume only one type of organism.
  (iv) Small mammals eat predatory birds.
  (v) Songbirds consume arthropods.

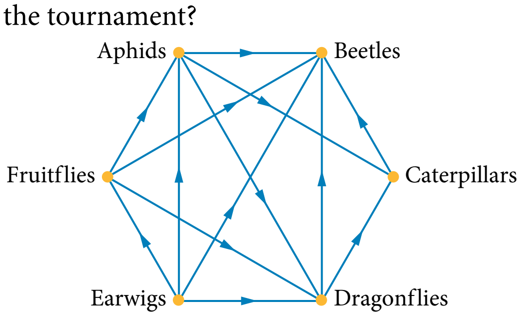
(c) Write a sequence of one organism consuming another to explain how a spider could end up inside an eagle.

7 A mixed netball round robin competition involves six teams, as shown in the network below. Each edge indicates a match that has been played, and each arrow points towards the losing team. The tournament is not yet complete.

(a) Complete the table.

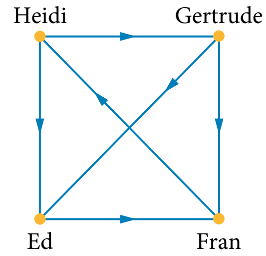
(b) Which is the most successful team in the tournament so far?

(c) Which teams still have to play against each other to complete the tournament?



Team	Number of wins
Aphids	
Beetles	
Caterpillars	
Dragonflies	
Earwigs	
Fruit flies	

8 The digraph shows the results of a four-person round robin tennis competition.
Use the digraph to determine whether each statement is true or false.

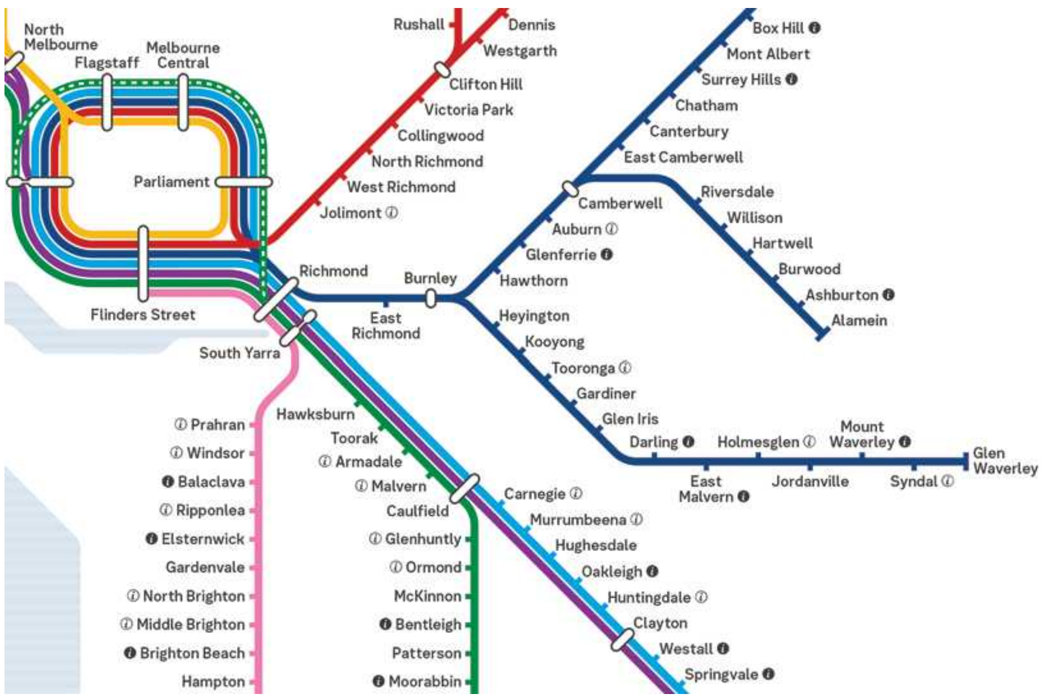


(a) Ed beats Fran, and Gertrude beats Ed.
(b) Gertrude beats Heidi, and Heidi beats Ed.
(c) Heidi beats Gertrude and Ed.
(d) Fran beats Gertrude, and Ed beats Fran.
(e) Ed beats Fran, Gertrude and Heidi.
(f) Fran beats Heidi, and Gertrude beats Fran.

9 Construct a social network showing the following connections.
Jed and Keira are friends. Jed and Liam are friends. Liam and Keira are friends.
Keira and Manuella are friends. Liam and Manuella are friends.

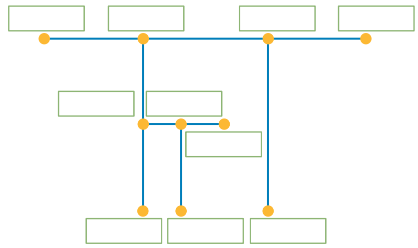
10 A bus travels along the following routes involving stops P, Q, R, S and T.
The route is one-way from P to R. The route is one-way from T to Q.
The route is one-way from R to T. The route is one-way from Q to P.
The route is two-way between R and S (shown without arrows).
Show this information on a directed graph.

11 A map of part of Melbourne's rail network is shown below. (The colours show different zones.)



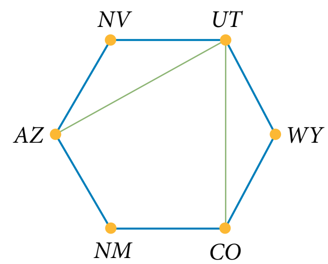
(a) A 'line' is named after the station at the end of the line. How many stations, after Burnley, are on the Glen Waverley line?
(b) On what line is Patterson station?
(c) How many stations are between Box Hill and Flinders Street stations?
(d) At which station should passengers travelling from Elsternwick to Oakleigh change trains?

(e) Copy this simplified version of the Melbourne metro map, and label it using the list below.



Box Hill, Burnley, Carnegie, Caulfield, Flinders Street, McKinnon, Richmond, Ripponlea, South Yarra, Syndal

12 The edges between the vertices in the network shown here represent six US states that share common borders. Each vertex is labelled with its state's two-letter abbreviation. If two states meet at a corner, they are not considered to share a border. The network does not represent the actual geographical locations of the states.



Which of the following maps shows the correct locations of the states?

A

Map A

B

Map B

C

Map C

D

Map D

Worked Example 7

13 The characters in the figure at right represent friends.

(a) Draw the social network among these friends using the information below.

Aaron is friends with Bruce, Helia, Corinne and Freya.

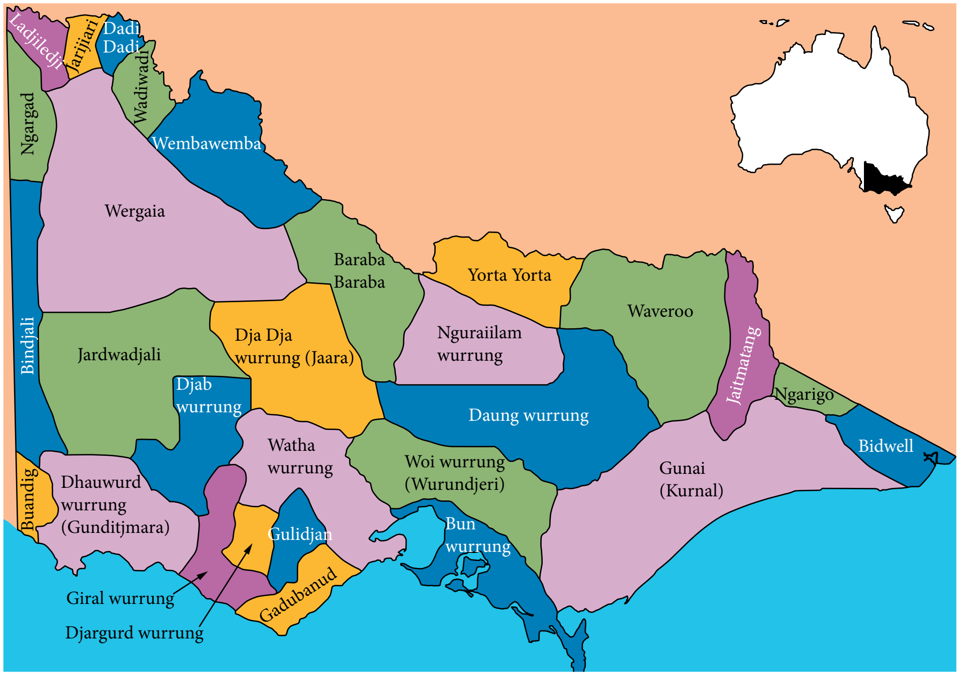
Bruce is friends with Aaron, Corinne, Emillio, Giaan and Helia.

Giaan is friends with Bruce, Desmond and Freya.

Emillio is friends with Bruce, Helia and Corinne.

(b) People in this network use email to contact their friends. Desmond wants to email a party invitation to Freya via one of the other people in their networks. Who is that person?

14 Regions for different Indigenous groups in Victoria are shown on the map.



Construct a network using labelled vertices for the given regions listed below. Connect vertices that share a boundary. (A shared corner is *not* considered a shared boundary.)

Regions: Woi wurrung (WW), Gunai (G), Daung wurrung (DW), Nguraiilam wurrung (NW), Baraba Baraba (BB), Dja Dja wurrung (DD), Jardwadjali (J), Yorta Yorta (YY)

15 Construct a directed graph showing the predatory relationships among the five imaginary organisms listed below. Each arrow along an edge of the graph points away from the organism that is eaten. You can draw an image of each type of organism at each vertex instead of using its name or symbol.

Razor bugs (R) eat the other four organisms.

Electro bugs (E) eat slime worms (S).

Dodecapedes (D) eat brontosaurus beetles (B).

Dodecapedes (D) eat slime worms (S).

Slime worms (S) eat brontosaurus beetles (B).

16 A new airline plans round-trip flights between the major cities, as shown on the map.

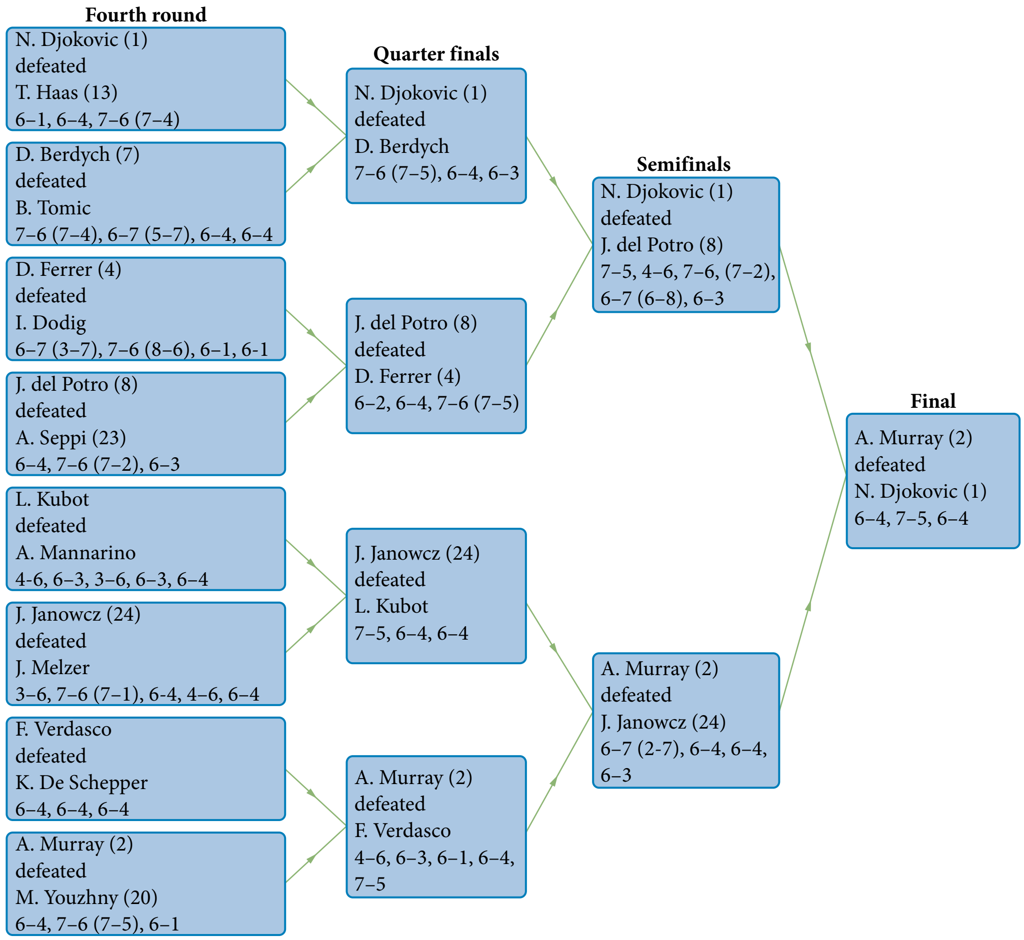
(a) Determine the minimum number of flights required to get from Perth to Alice Springs using this airline.

(b) Select the three longest flights that could be removed but still leave all cities connected by this airline's flights.

(c) Is it possible to add direct flights between Perth and Sydney and between Perth and Brisbane so that the network remains planar?

(d) The managing director of the airline wishes to travel on each flight in a multi-stage trip to check the performance of airline staff. Starting and ending in Melbourne, list the cities in order so that the managing director will minimise the number of flights.

17 The results of the fourth round, quarter finals, semifinals and final of the 2013 Wimbledon men's tennis championship are shown. The graph shows the score for each set, and tie-breaker scores are in parentheses.



(a) Who won the fourth round match between A. Seppi and J. del Potro?

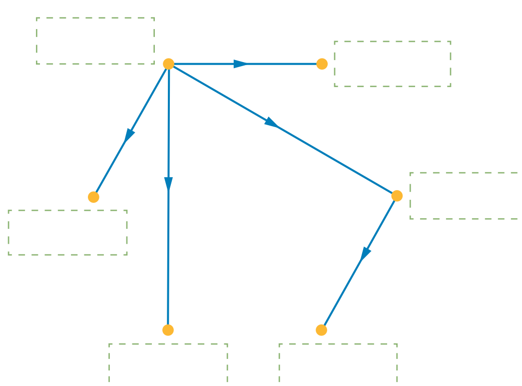
(b) How many sets did J. del Potro win in the fourth round?

(c) How many five-set matches were there in the quarter finals?

(d) Which player lost the fewest games during the semi-finals? (Do not count tie-breakers.)

(e) Who were the players that A. Murray beat, starting with the fourth round?

(f) Copy the following subnetwork, and add a name at each vertex to show the results of the competition. Choose the names from the list below.



Djokovic, Janowcz, Kubot, Murray, Verdasco, Youzhny

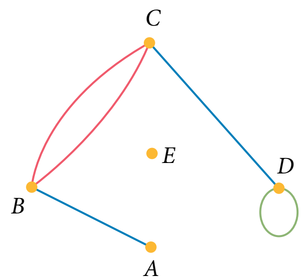
Adjacency matrices

7.3

Undirected graphs

You can use an *adjacency matrix* to show the connections within a graph or network. Each element, or entry, in an adjacency matrix indicates how many connections there are between pairs of vertices.

For example, the value 2 in row *B*, column *C* in the adjacency matrix shown here indicates that there are two connections between vertex *B* and vertex *C*. The value 1 in row *D*, column *D* indicates that there is a loop from vertex *D* to itself.

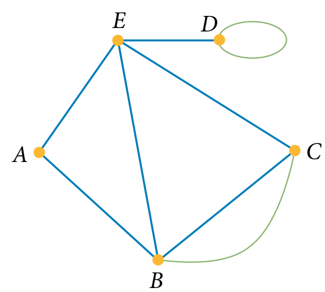


	A	B	C	D	E
A	0	1	0	0	0
B	1	0	2	0	0
C	0	2	0	1	0
D	0	0	1	1	0
E	0	0	0	0	0

An adjacency matrix is symmetric about its main diagonal because the connection between a pair of like vertices is repeated (e.g. *A* is connected to *B* and *B* is connected to *A*).

9 Constructing the adjacency matrix for an undirected graph

Construct an adjacency matrix for the graph.



THINKING

WORKING

1 State the order of the adjacency matrix.

There are 5 vertices in the digraph.

The order of the adjacency matrix is 5×5.

2 State the number of paths (edges) between vertices.

The number of paths between:

A and *B* is 1. *A* and *E* is 1. *B* and *C* is 2.

B and *E* is 1. *C* and *D* is 0. *C* and *E* is 1.

D and *E* is 1.

The loop at *D* is counted as 1.

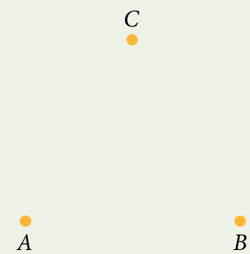
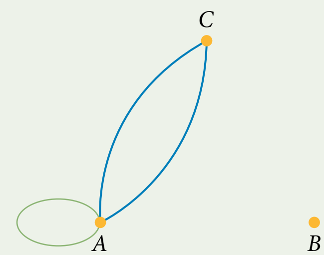
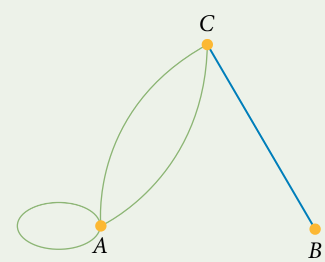
3 Construct the adjacency matrix. Remember to use zeros to show where there are no connections.

	A	B	C	D	E
A	0	1	0	0	1
B	1	0	2	0	1
C	0	2	0	0	1
D	0	0	0	1	1
E	1	1	1	1	0

10 Constructing an undirected graph from its adjacency matrix

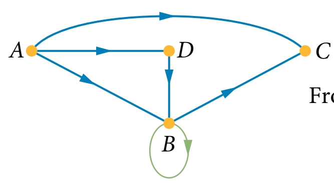
Construct an undirected graph for the adjacency matrix.

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

THINKING	WORKING
1 Label the rows and columns of a 3×3 square matrix using letters.	$\begin{matrix} & A & B & C \\ A & 1 & 0 & 2 \\ B & 0 & 0 & 1 \\ C & 2 & 1 & 0 \end{matrix}$
2 Draw the three vertices in a triangular arrangement.	
3 Look at the first row to identify connections with *A*. Show this information on the graph.	*A* connects to itself. *A* does not connect to *B*. *A* connects to *C* twice. 
4 Look at the second row to identify any additional connections with *B*. Show this information on the graph. Entries in the third row indicate that *C* connects to *A* twice (already drawn), *C* connects to *B* once (already drawn) and *C* does not connect to itself. None of this is new information, so you have completed the graph.	The entry in row B in the adjacency matrix indicates that *B* connects only to *C*. 

Directed graphs

In a directed graph (digraph), an adjacency matrix lists the number of ways to get from one vertex to another by following an arrow along a single edge. In the digraph, there is an arc (a directed edge) from D to B, so the number 1 is in row D, column B of the matrix. There is no arc from B to D, so there is a 0 in row B, column D of the matrix.

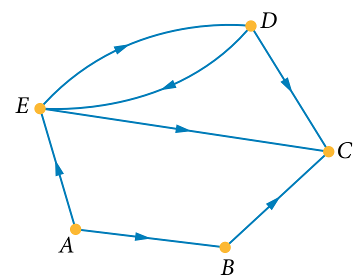


To

From	A	B	C	D
A	0	1	1	1
B	0	1	1	0
C	0	0	0	0
D	0	1	0	0

11 Constructing the adjacency matrix for a directed graph

Construct the adjacency matrix for the digraph.



THINKING

WORKING

1 Determine the order of the adjacency matrix.

There are 5 vertices in the digraph.

The order of the adjacency matrix is 5×5.

2 State the number of arcs from A to each of the other vertices.

Fill in the numbers in the first row with zeros where there is no arc.

A arcs to E and B.

	A	B	C	D	E
A	0	1	0	0	1
B					
C					
D					
E					

3 Construct the adjacency matrix, filling each row for arrows coming from a particular vertex.

B arcs to C.

C is not directed to any other vertices.

D arcs to C and E.

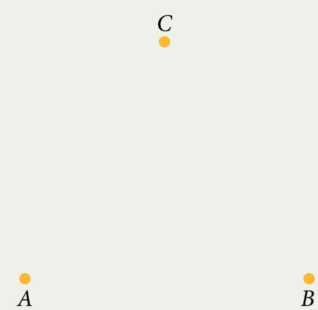
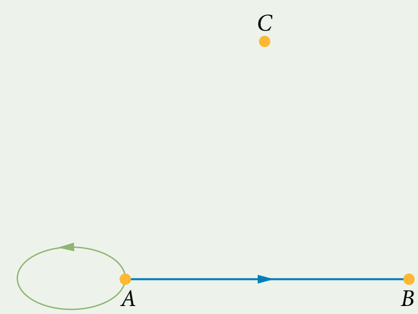
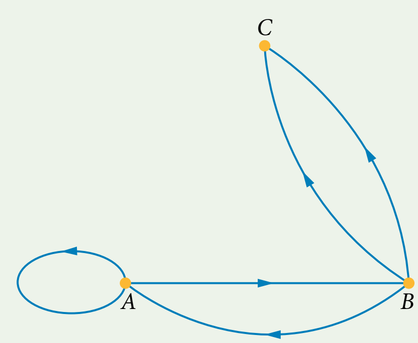
E arcs to C and D.

	A	B	C	D	E
A	0	1	0	0	1
B	0	0	1	0	0
C	0	0	0	0	0
D	0	0	1	0	1
E	0	0	1	1	0

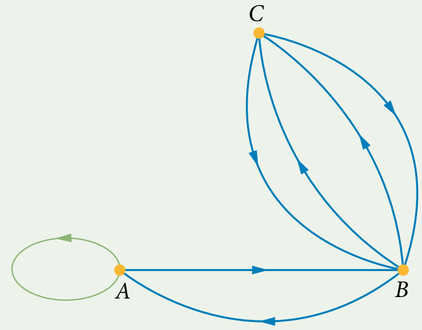
12 Constructing a digraph from an adjacency matrix

Draw a digraph for the adjacency matrix.

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

THINKING	WORKING
1 Label the rows and columns of a square matrix using the letters A, B and C.	$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{matrix} A & B & C \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \end{array}$
2 Draw the three vertices in a triangular arrangement.	
3 Look at row A to identify the connections. Show this information on the graph.	 There is a loop at A and an arc from A to B.
4 Look at row B to identify connections. Show this information on the graph.	 There is an arc from B to A and two arcs from B to C.

5 Look at row C to identify connections. Show this information on the graph.

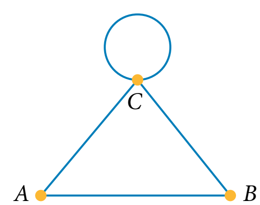


There are two arcs from C to B.

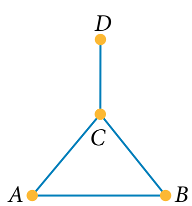
EXERCISE 7.3 Adjacency matrices

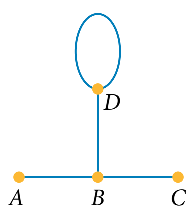
Worked Example 9

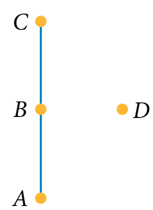
1 Construct the adjacency matrix for the graph shown at right.

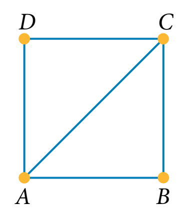


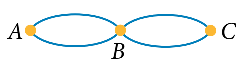
2 Construct the adjacency matrix for each graph below.

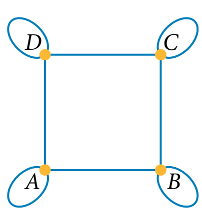
(a)


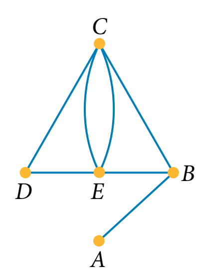
(b)


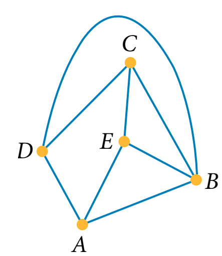
(c)


(d)


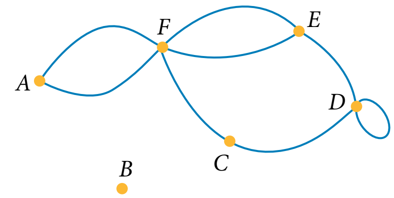
(e)


(f)


(g)


(h)


3 The graph shows connections between six vertices.



(a) Which of the following adjacency matrices correctly represents the graph?

A

	A	B	C	D	E	F
A	0	0	0	0	0	2
B	0	0	0	0	0	0
C	0	0	0	1	0	1
D	0	0	1	2	1	0
E	0	0	0	1	0	2
F	2	0	1	0	2	0

B

	A	B	C	D	E	F
A	1	0	0	0	0	2
B	0	1	0	0	0	0
C	0	0	1	1	0	1
D	0	0	1	2	1	0
E	0	0	0	1	1	2
F	2	0	1	0	2	1

C

	A	B	C	D	E	F
A	0	0	0	0	0	2
B	0	0	0	0	0	0
C	0	0	0	1	0	1
D	0	0	1	1	1	0
E	0	0	0	1	0	2
F	2	0	1	0	2	0

D

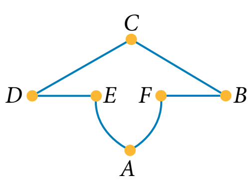
	A	B	C	D	E	F
A	0	0	0	0	0	0
B	0	0	0	0	0	2
C	0	0	0	1	0	1
D	0	0	1	1	1	0
E	0	0	0	1	0	2
F	0	2	1	0	2	0

(b) Explain the common error made by a student who chose the first incorrect option.

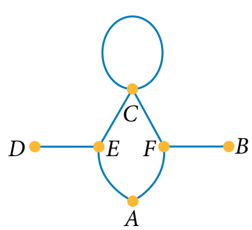
4 Which of the graphs below is represented by the adjacency matrix?

	A	B	C	D	E	F
A	0	0	0	0	1	1
B	0	0	0	0	0	2
C	0	0	0	0	1	1
D	0	0	0	0	2	0
E	1	0	1	2	0	0
F	1	2	1	0	0	0

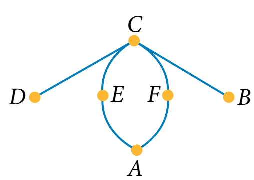
A



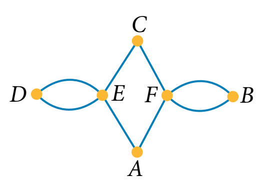
B



C



D



Worked Example 10

5 Draw the undirected graph represented by each adjacency matrix.

(a)

	A	B	C
A	0	1	1
B	1	0	2
C	1	2	1

(b)

	A	B	C	D
A	0	1	1	1
B	1	0	0	1
C	1	0	0	1
D	1	1	1	0

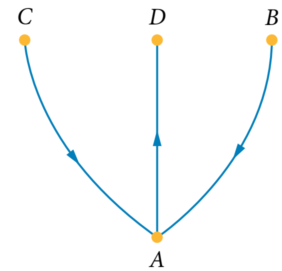
(c)

	A	B	C	D
A	0	2	0	0
B	2	0	1	0
C	0	1	0	1
D	0	0	1	0

(d)

	A	B	C	D	E
A	0	1	2	0	0
B	1	0	0	0	0
C	2	0	0	1	0
D	0	0	1	0	1
E	0	0	0	1	1

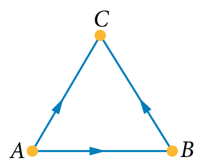
6 Construct the adjacency matrix for the graph.



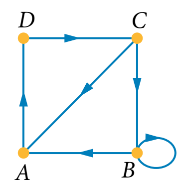
7 Construct the adjacency matrix for each digraph.

(a)

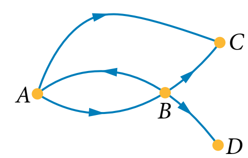
(b)



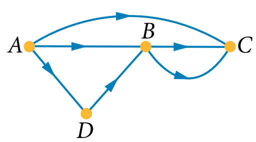
(c)



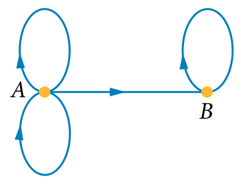
(d)



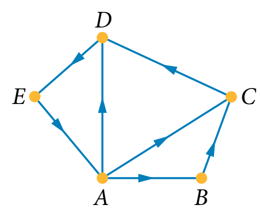
(e)



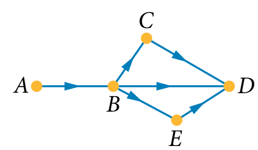
(f)



(g)



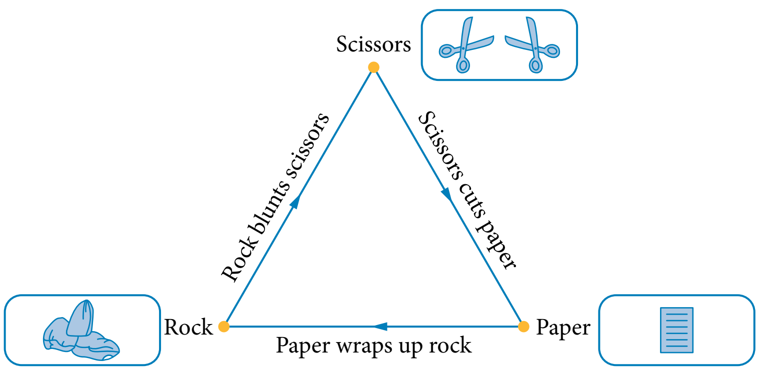
(h)



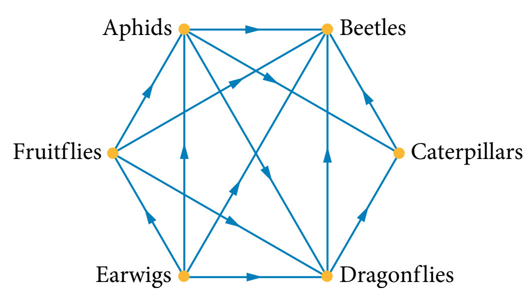
8 Draw a graph to represent the adjacency matrix.

$$\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ \begin{array}{c} A \\ B \end{array} & \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \end{array}$$

9 Construct the adjacency matrix for the rock, paper scissors game.



10 Construct the adjacency matrix for the round robin tournament.



11 Draw a digraph represented by the adjacency matrix.

$$\begin{array}{cc} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

12 Construct the adjacency matrix for a new aeronautical company with flights connecting Brisbane with Rockhampton, Rockhampton with Mackay and Mount Isa, Mackay with Townsville, and Townsville with Cairns. Use I for Mount Isa, and the first letters of the cities for the other vertices.

Worked Example 12

13 Draw a digraph represented by the adjacency matrix in each case.

(a)
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{matrix} A & B & C & D \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

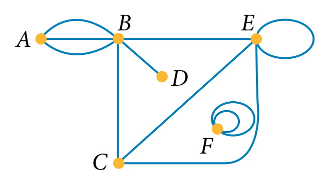
(b)
$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{matrix} A & B & C & D \end{matrix} \\ \begin{bmatrix} 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

14 Draw a digraph represented by the adjacency matrix in each case.

(a)
$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

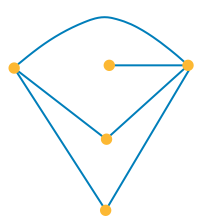
(b)
$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

15 Construct the adjacency matrix to represent the graph.

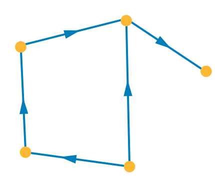


16 For each of the following graphs, construct two possible adjacency matrices.

(a)

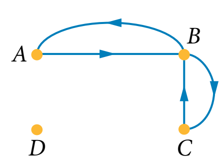


(b)

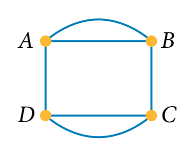


17 For each of the following graphs, determine how the adjacency matrix would change to match the described changes in the graph.

(a) An arc is added from every other vertex to D.



(b) All connections to B are severed (removed).



18 For an undirected graph, describe the changes to the graph given by the changes to the adjacency matrix.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{matrix} A & B & C & D \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 1 & 2 \end{bmatrix} \end{array} \rightarrow \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{matrix} A & B & C & D \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

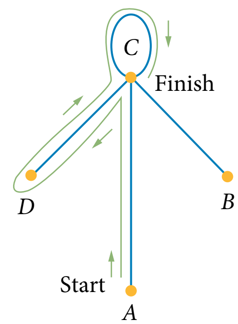
7.4 Walks, trails, paths and cycles

Walks

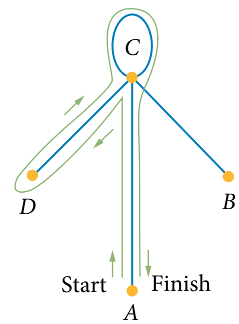
A *walk* is a sequence of connected edges within a graph. A walk may include repeated edges and vertices and may or may not include all edges or vertices.

Walks may be open (end at a different vertex) or closed (start and end at the same vertex).

The walks shown here, in green, are offset slightly from the network, for clarity.



An open walk: *A-C-D-C-C*

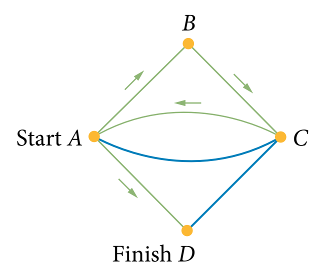


A closed walk: *A-C-D-C-C-A*

Trails

A *trail* is a walk in which no edge is repeated. A vertex, however, may be repeated.

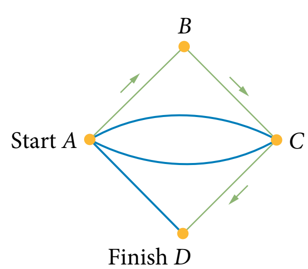
Trails may be open (end at a different vertex) or closed (start and end at the same vertex).



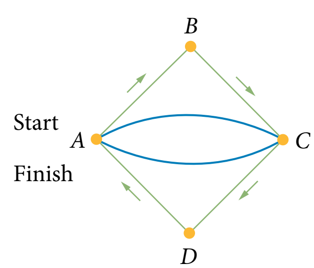
A trail *A-B-C-A-D*

Paths

A *path* is a walk that does not repeat edges or vertices, except possibly the starting vertex. Paths may be open or closed. A closed path is called a cycle.



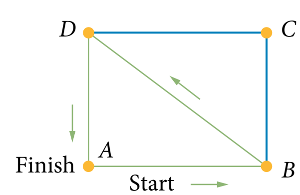
An open path: *A-B-C-D*



A closed path, or cycle: *A-B-C-D-A*

Cycles

A *cycle* is a closed walk that does not repeat vertices, except for its starting and ending vertex. The cycle *A-B-D-A* is shown in green in the graph.



A *walk* is any sequence of vertices joined by edges.

A *trail* is a walk with no repeated edges.

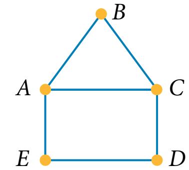
A *path* is a walk with no repeated edges or vertices.

A walk, a trail and a path can be *open* (start and finish at different vertices) or *closed* (start and finish at the same vertex).

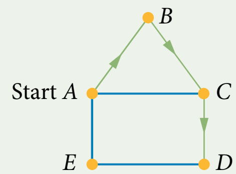
A closed path is called a *cycle*.

13 Identifying trails, paths and cycles

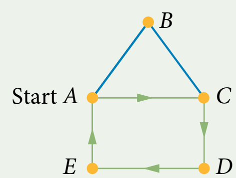
For the graph shown here, describe each of the following as a walk, a trail, an open path or a cycle.



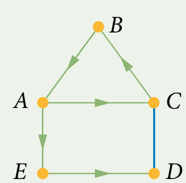
(a) *A-B-C-D*

THINKING	WORKING
1 Trace the walk onto the graph.	
2 Identify any repeated edges or vertices.	There are no repeated edges or vertices, making the walk a path.
3 Compare the position of the start and end points.	The path starts at vertex *A* and ends at vertex *D*. The start and end are different, and therefore it is called 'open'.
4 Interpret the result.	The walk *A-B-C-D* is an open path.

(b) *A-C-D-E-A*

1 Trace the walk onto the graph.	
2 Identify any repeated edges or vertices.	There are no repeated edges or vertices (except the start and finish vertex *A*), making the walk a path.
3 Compare the position of the start and end points.	The path starts at vertex *A* and ends at vertex *A*. The start and end vertex are the same, and therefore it is called 'closed'.
4 Interpret the result.	The walk *A-C-D-E-A* is a closed path, or a cycle.

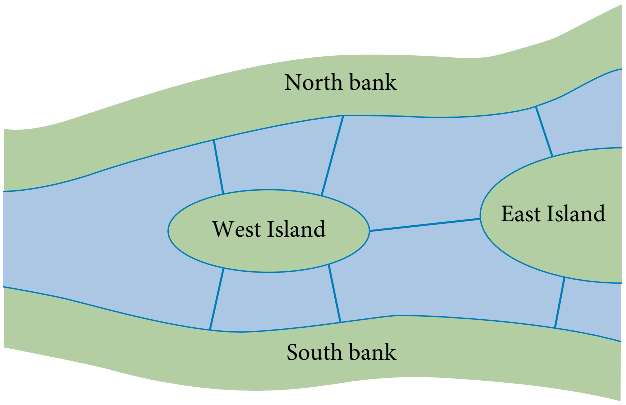
(c) *A-C-B-A-E-D*

1 Trace the walk onto the graph.	
2 Identify any repeated edges or vertices.	There are no repeated edges. There is a repeated vertex *A*, making the walk a trail.

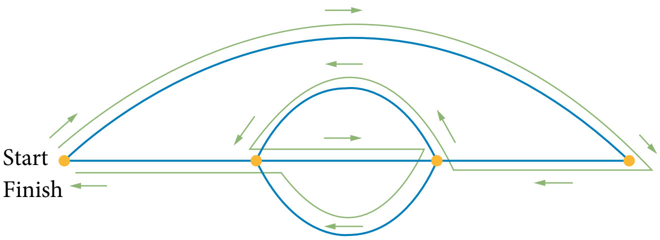
3 Compare the position of the start and end points.	The trail starts at vertex *A* and ends at vertex *D*. The start and end vertex are different, and therefore it is called 'open'.
4 Interpret the result.	The walk *A-C-B-A-E-D* is an open trail.

Eulerian graph or Eulerian trail

Seven bridges in the Prussian city of Königsberg, now the city of Kaliningrad, Russia, connect the northern and southern parts of the city and two islands in the Pregel River. A question that arose hundreds of years ago about these seven bridges was: 'Is there a path a person can take to get through all parts of the city and walk across each bridge exactly once?' In 1735 the mathematician Leonard Euler answered this question in terms of graph theory.

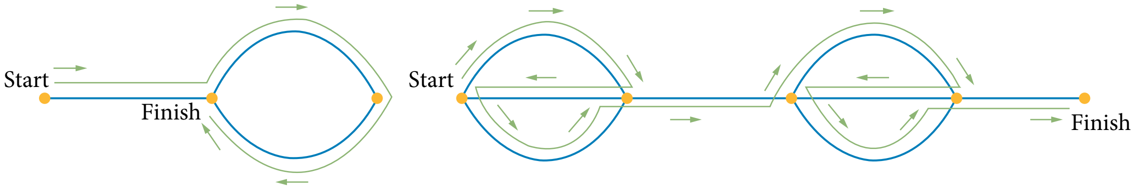


An *Eulerian graph* is one in which a closed trail or cycle uses every *edge of a graph* exactly once, and each vertex is of even degree. This is referred to as an *Eulerian trail*.



Semi-Eulerian graph

A *semi-Eulerian graph* contains an open trail. That is, it uses every *edge* and has exactly two vertices of odd degree.



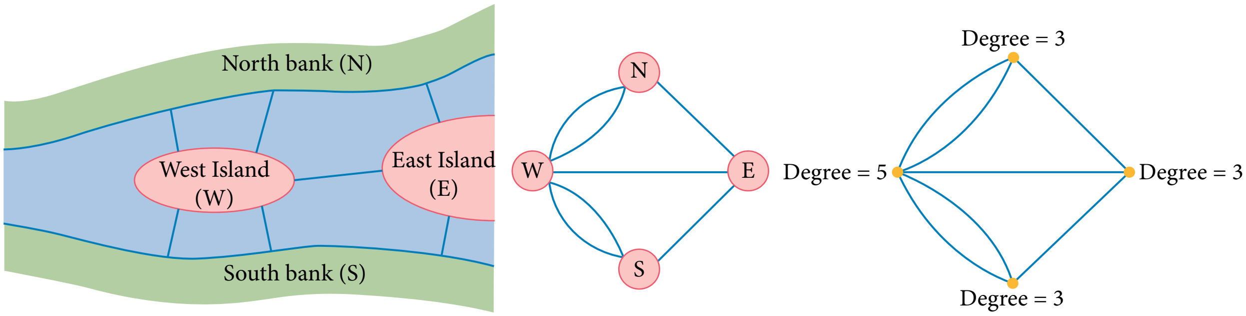
Semi-Eulerian graphs

A semi-Eulerian trail starts at one odd-degree vertex and ends at the other odd-degree vertex.

It is important to notice that when traversing a graph, you must leave an intermediate vertex (a vertex other than the start or finish) after entering it, sometimes more than once. The result is that the degree of that vertex must be an even number.

Every intermediate vertex in an Eulerian graph or a semi-Eulerian graph must have a degree that is even.

In the famous Bridges of Königsberg problem, you can represent the land areas and the bridges between them as shown below. Then you can more easily determine the degree of each vertex.



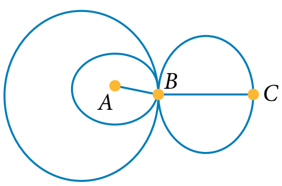
A graph of the Königsberg bridges contains more than two vertices of odd degree. Therefore, neither an Eulerian or semi-Eulerian trail exists, meaning you cannot tour the city by travelling across each bridge exactly once.

Eulerian graphs and semi-Eulerian graphs are traceable; you can trace the whole graph without lifting your pencil, and without drawing the same line twice.

14 Traceable trails

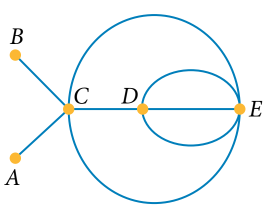
For each graph, determine whether it is possible to trace a trail (no repeated edges) and, hence, describe the graph as an Eulerian graph, a semi-Eulerian graph or neither.

(a)



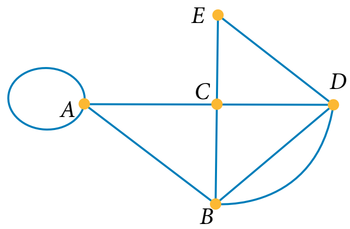
THINKING	WORKING
1 Determine the degree of each vertex.	*A*: degree of 1 *B*: degree of 8 *C*: degree of 3
2 Determine the start and end points if applicable.	Two odd vertices indicate that the graph is traceable, starting and finishing at *A* and *C*.
3 Interpret the result.	This graph is a semi-Eulerian graph.

(b)



1 Determine the degree of each vertex.	*A*: degree of 1 *B*: degree of 1 *C*: degree of 5 *D*: degree of 4 *E*: degree of 5
2 Determine the start and end points if applicable.	Four odd vertices, so the graph is not traceable.
3 Interpret the result.	This is neither an Eulerian graph nor a semi-Eulerian graph.

(c)

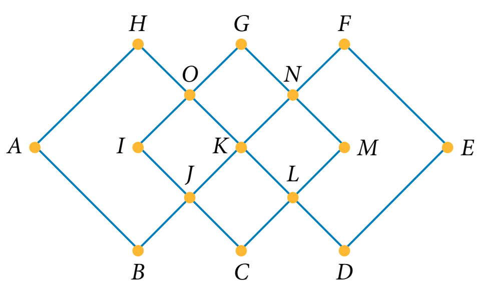


1 Determine the degree of each vertex.	*A*: degree of 2 *B*: degree of 4 *C*: degree of 4 *D*: degree of 4 *E*: degree of 2
2 Determine the start and end points if applicable.	All vertices are of even degree and therefore the graph is traceable from any vertex (because each has an entry and exit edge).
3 Interpret the result.	This is an Eulerian graph.

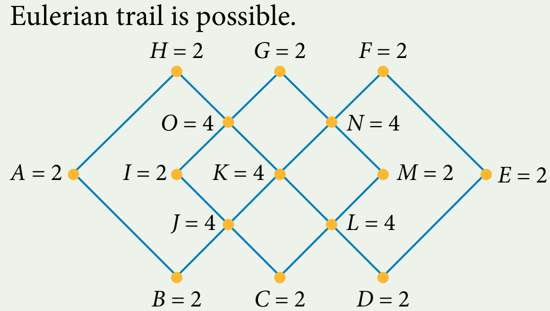
15 Eulerian and semi-Eulerian graphs

For each graph, describe an Eulerian trail or a semi-Eulerian trail.

(a)



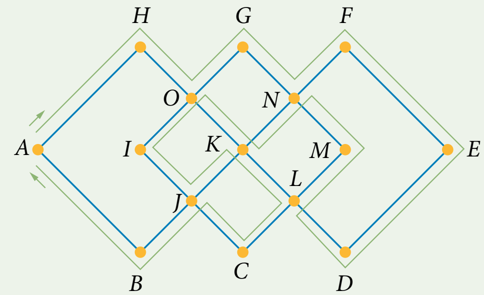
THINKING	WORKING
1 Determine the number of vertices of odd degree in the graph and identify the type of graph.	All vertices are of even degree. The graph is an example of an Eulerian graph, so an Eulerian trail is possible.



2 Trace a trail that traverses each edge only once.

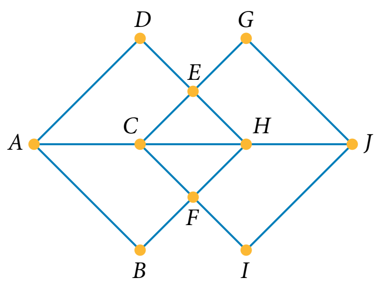
Describe an Eulerian trail by listing its vertices in order.

An Eulerian graph means you can create a closed trail starting from any vertex. One possibility is shown here, starting and finishing at vertex *A*.

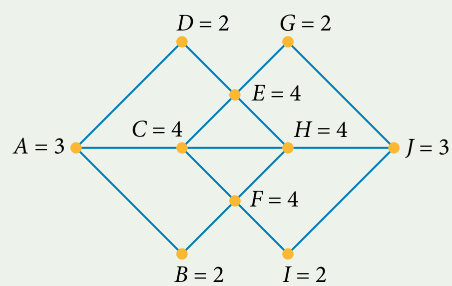


A-H-O-G-N-F-E-D-L-M-N-K-O-I-J-K-L-C-J-B-A

(b)



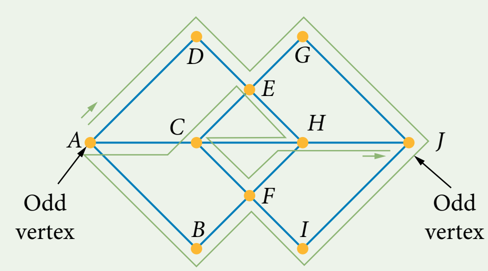
1 Determine the number of vertices of odd degree in the graph.



Vertices *A* and *J* are of odd degree. There are two vertices of odd degree. The graph is an example of a semi-Eulerian graph and contains a semi-Eulerian trail.

2 Describe a semi-Eulerian trail by listing its vertices in order.

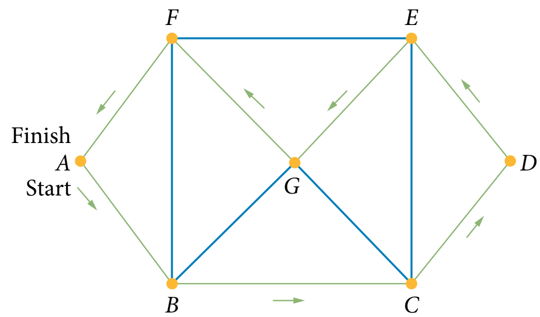
Start at vertex *A* and trace a trail that visits each edge only once and ends at the other odd degree vertex *J*. One possibility is shown here.



A-D-E-G-J-I-F-B-A-C-E-H-C-F-H-J

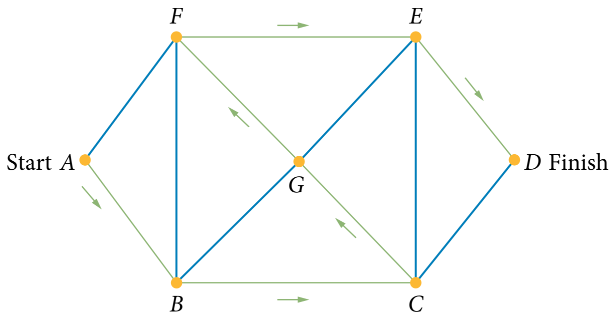
Hamiltonian graphs and semi-Hamiltonian graphs

A Hamiltonian graph contains a Hamiltonian cycle, which is *a closed path* that starts and ends at the same vertex and visits every other vertex exactly once. The Hamiltonian cycle shown here in green is *A-B-C-D-E-G-F-A*.



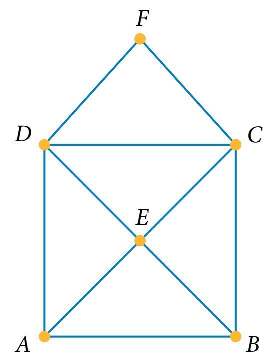
A semi-Hamiltonian graph contains a Hamiltonian path, which is an open path where each vertex is visited exactly once. The Hamiltonian path shown here in green is *A-B-C-G-F-E-D*.

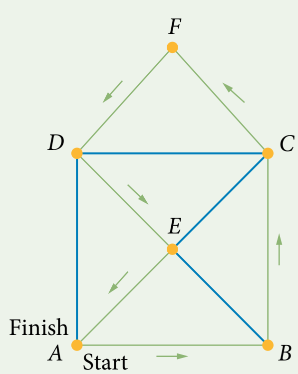
You can find Hamiltonian paths and cycles using trial and error.



16 Hamiltonian cycles and paths

For this network, describe a Hamiltonian cycle or Hamiltonian path (if possible).



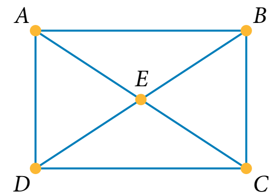
THINKING	WORKING
1 Using trial and error, determine a path that starts and ends at the same vertex and visits each vertex exactly once.	
2 Describe the graph and give the path by listing its vertices in order.	This graph is an example of a Hamiltonian graph. It contains a closed path, called a Hamiltonian cycle. *A-B-C-F-D-E-A*

EXERCISE 7.4 Walks, trails, paths and cycles

Worked Example

13

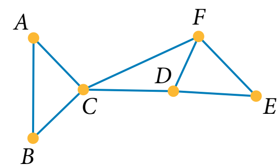
1 For the graph shown, describe each of the following as a walk, a trail, a path or a cycle.



(a) *A-B-E-D-A-E* (b) *A-E-D-A*

(c) *E-C-D-E-C-B* (d) *E-B-C-D*

2 For the graph shown, describe each of the following as a walk, a trail, a path or a cycle.



(a) *F-E-D-C-F* (b) *F-E-D-F-C-D-F*

(c) *F-D-C-B-A* (d) *F-E-D-F-C-B-A*

3 Which of the following statements is false?

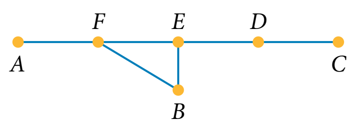
A In a walk, edges and vertices can be repeated.

B In a path, neither edges nor vertices can be repeated.

C In a trail, edges can be repeated.

D In a trail, vertices can be repeated.

4 Consider the network shown.



(a) Which of the following describes a path in the network?

A *A-F-E-B-A* B *A-F-B-E-B* C *A-F-B-E-D* D *A-F-B-E-F*

(b) Describe the common error made by a student who chose the first incorrect option in part (a).

(c) Which term best describes the route *A-F-E-B-F*?

A loop B trail C path D cycle

(d) Which term best describes the route *A-F-E-D-E* through the network?

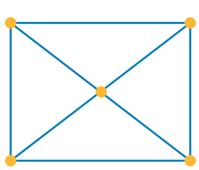
A walk B trail C path D cycle

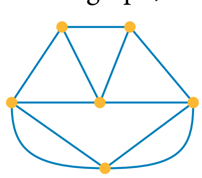
(e) Which of the following walks is a cycle?

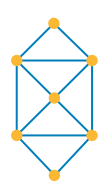
A *A-F-E-D-C* B *A-F-E-F-A* C *B-E-D-C-B* D *B-E-F-B*

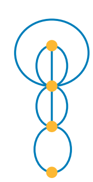
14

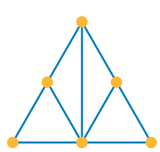
5 Describe each graph as an Eulerian graph, a semi-Eulerian graph or neither.

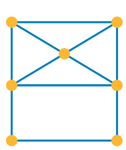
(a)


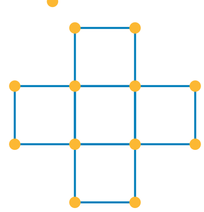
(b)


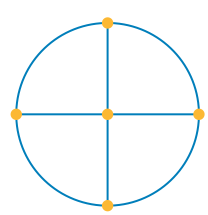
(c)


(d)


(e)


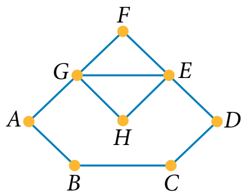
(f)


(g)


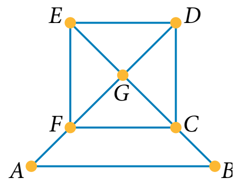
(h)


6 For each graph, state whether a Hamiltonian cycle, Hamiltonian paths only, or neither, can be found.

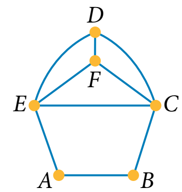
(a)



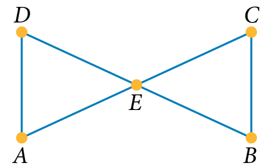
(b)



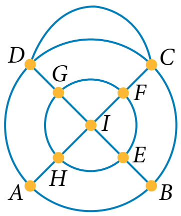
(c)



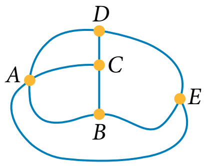
(d)



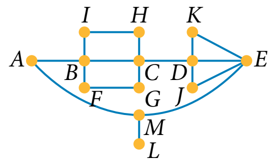
(e)



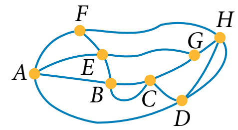
(f)



(g)



(h)



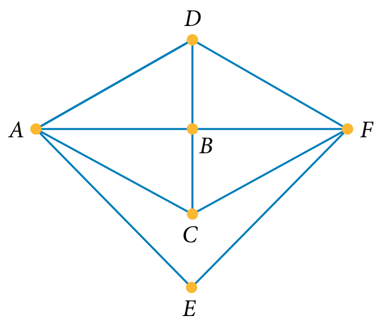
7 Consider the graphs in the previous question.

(a) Which of the graphs are Eulerian graphs?

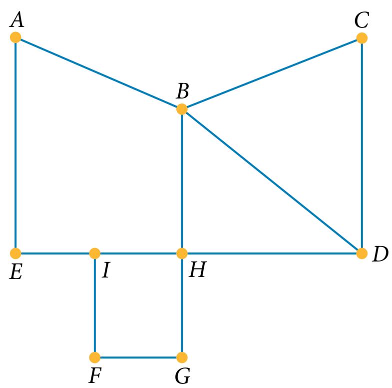
(b) Which of the graphs are semi-Eulerian graphs?

Worked Example 15

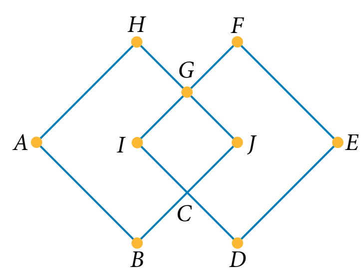
8 Describe an Eulerian trail or a semi-Eulerian trail for the graph.



9 A graph of the layout of a large retail shop is shown here. The shop contains several aisles, with scanners at each intersection point, labelled A to I. Identify a route that includes every aisle exactly once. The route may start and end at any scanner.



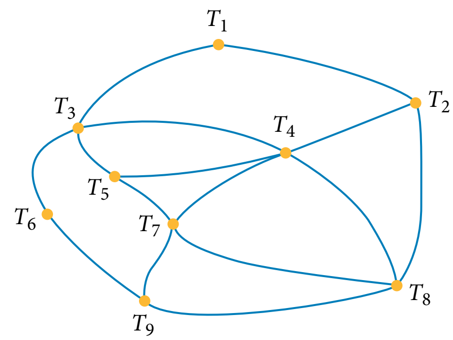
10 Describe an Eulerian trail or a semi-Eulerian trail for the graph.



Worked Example

16

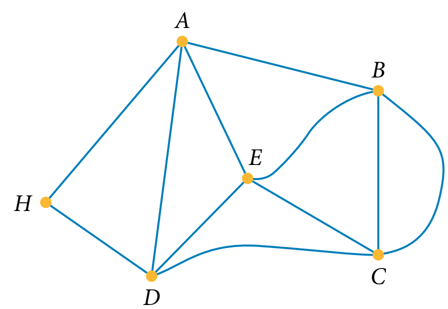
11 A section of a national park contains nine very old trees, labelled T_1 to T_9 in the graph. The trees are connected by paths as shown.



(a) Identify a route that visits each tree exactly once, starting at T_1 and ending at T_5.

(b) Is it possible to walk along each path exactly once in a complete tour of the park?

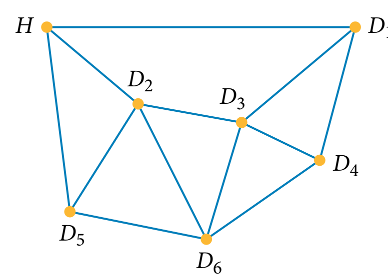
12 Cibatta is a keen cyclist who wishes to visit several shops during a morning ride along several main roads. The shops are labelled from A to E on the graph. The edges represent roads. Cibatta leaves from her home H and returns there at the end of her ride.



(a) Describe a route that takes Cibatta to every shop exactly once.

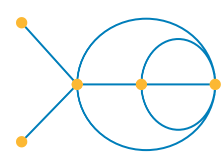
(b) Describe a route that takes Cibatta along each of the roads exactly once.

13 Marita is a medical sales representative and wishes to visit several doctors to promote her company's services. She makes a sketch showing the location of her home H and the locations of the medical clinics D_1 to D_6.



Identify a route that takes Marita from her home to each clinic exactly once and then home again.

14 Make alterations to the graph to achieve the required effect.



(a) Add one edge to transform the network to a semi-Eulerian graph.

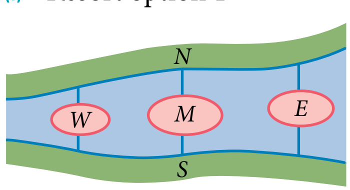
(b) Show three other solutions for part (a) that are not isomorphic to your first solution, or to each other.

(c) Add two edges to transform the original network to an Eulerian trail. Show two solutions that are not isomorphic.

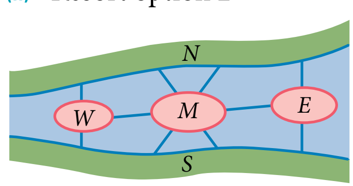
15 A wealthy oil sheik has been presented with three options for a resort, based on three islands: West (W), Middle (M) and East (E), over a river and two sides, North bank (N) and South bank (S). The three options are shown below.

(a) For each option, redraw the map as a network, labelling each vertex. Then describe, if possible, an example of a semi-Eulerian trail or an Eulerian trail from your graph.

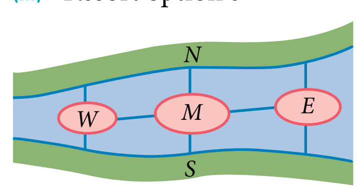
(i) Resort option 1



(ii) Resort option 2



(iii) Resort option 3



(b) Which option should the sheik choose if he wants the maximum number of bridges, on the condition that his guest can travel across each bridge exactly once?

Weighted graphs

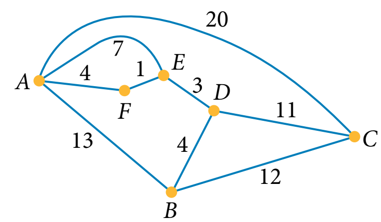
7.5

A map can be drawn as a graph with the distances between locations written along the edges representing roads. Such a graph is called a *weighted graph*.

You can use weighted graphs to solve many problems, such as finding the shortest route from one place to another.

17 Shortest distance between two points

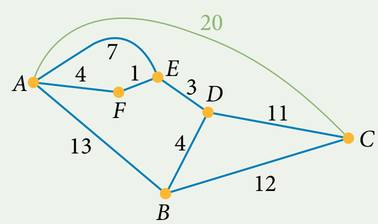
Determine the shortest route between *A* and *C* on the weighted graph shown. All distances are in kilometres.



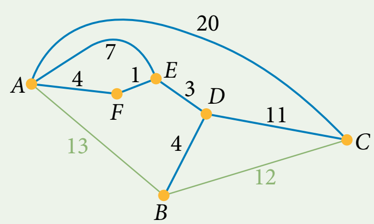
THINKING

1 Consider several routes from the starting point to the end point.

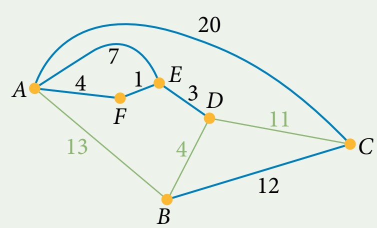
WORKING



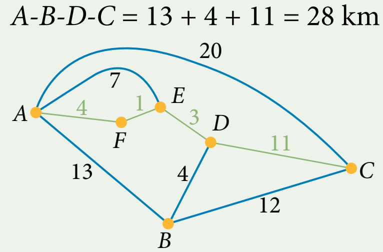
A-*C* = 20 km



A-*B*-*C* = 13 + 12 = 25 km



A-*B*-*D*-*C* = 13 + 4 + 11 = 28 km



A-*F*-*E*-*D*-*C* = 4 + 1 + 3 + 11 = 19 km

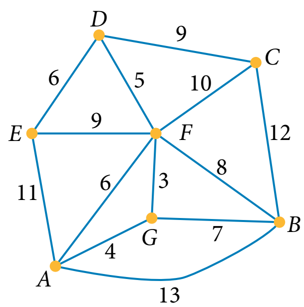
2 Identify the shortest route.

The shortest path is route *A*-*F*-*E*-*D*-*C*.

The classic Travelling Salesperson Problem asks: 'Given a list of cities that are connected by roads, what is the shortest possible route a salesperson must cover if each town must be visited exactly once and he or she must return to the starting point?' The solution requires a Hamiltonian cycle.

18 Shortest Hamiltonian cycle

Determine the shortest Hamiltonian cycle in the graph.

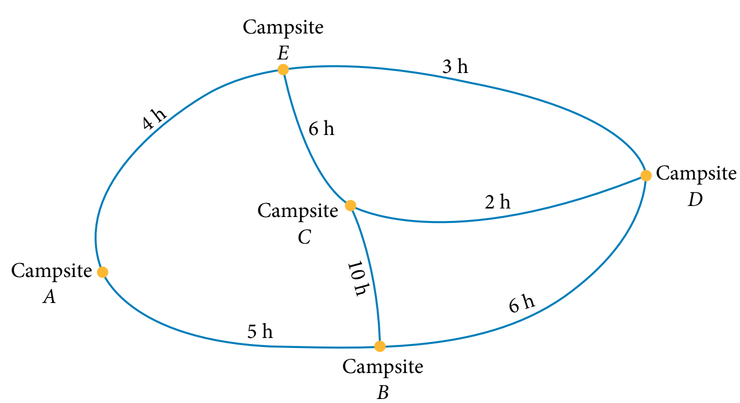


THINKING	WORKING
1 To identify a Hamiltonian cycle, visit each vertex exactly once and return to your starting point. Use trial and error to determine the shortest cycle.	Possible cycles include: *A-G-F-B-C-D-E-A* = 4 + 3 + 8 + 12 + 9 + 6 + 11 = 53 km *A-B-C-D-E-F-G-A* = 13 + 12 + 9 + 6 + 9 + 3 + 4 = 56 km *A-G-B-F-C-D-E-A* = 4 + 7 + 8 + 10 + 9 + 6 + 11 = 55 km *A-G-B-C-D-E-F-A* = 4 + 7 + 12 + 9 + 6 + 9 + 6 = 53 km *A-F-G-B-C-D-E-A* = 6 + 3 + 7 + 12 + 9 + 6 + 11 = 54 km *A-G-B-C-D-F-E-A* = 4 + 7 + 12 + 9 + 5 + 9 + 11 = 57 km *A-G-B-C-F-D-E-A* = 4 + 7 + 12 + 10 + 5 + 6 + 11 = 55 km
2 Identify the shortest Hamiltonian cycles.	The shortest route is 53 km. *A-G-F-B-C-D-E-A* and its reverse route *A-E-D-C-B-F-G-A*, are each 53 km.

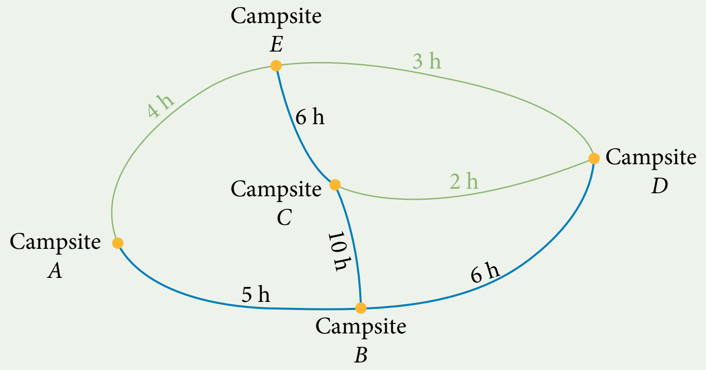
Although the weighted graphs look like maps, they are effective for solving problems involving shortest time.

19 Hiking times

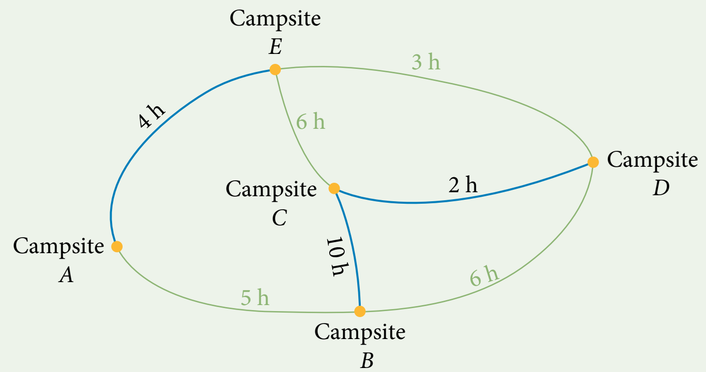
The weighted graph shows estimated walking times in hours along the walking paths that join campsites *A*, *B*, *C*, *D* and *E*.



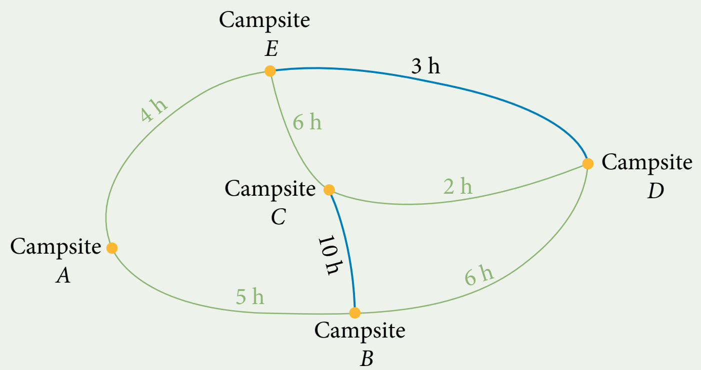
(a) Determine the quickest route from campsite A to campsite C.

THINKING	WORKING
1 Use trial and error to determine the quickest route between A and C.	 $A\text{-}E\text{-}D\text{-}C = 4 + 3 + 2 = 9\,\text{h}$
2 Interpret the result.	The quickest route is A-E-D-C, which takes 9 hours.

(b) Determine the quickest route from campsite A to campsite C that passes through every campsite.

1 Use trial and error to determine the shortest route from A to C that passes through each vertex.	 $A\text{-}B\text{-}D\text{-}E\text{-}C = 5 + 6 + 3 + 6 = 20\,\text{h}$
2 Interpret the result.	The quickest route is A-B-D-E-C, which takes 20 hours.

(c) Determine the quickest route from campsite A that visits each campsite and returns to campsite A.

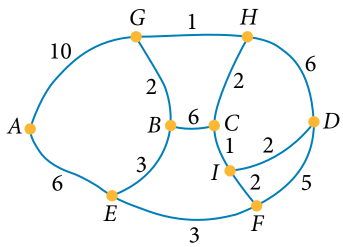
1 Use trial and error to determine the Hamiltonian cycle beginning and ending at A.	 $A\text{-}B\text{-}D\text{-}C\text{-}E\text{-}A = 5 + 6 + 2 + 6 + 4 = 23\,\text{h}$
2 Interpret the result.	The quickest route is A-B-D-C-E-A, or the reverse, A-E-C-D-B-A, which take 23 hours.

EXERCISE 7.5 Weighted graphs

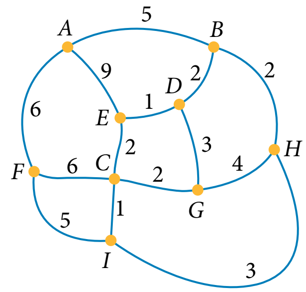
Worked Example 17

1 Determine the shortest routes from A to C in the following graphs. Distances are given in metres.

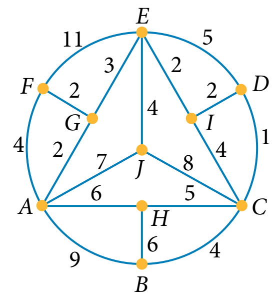
(a)



(b)



(c)



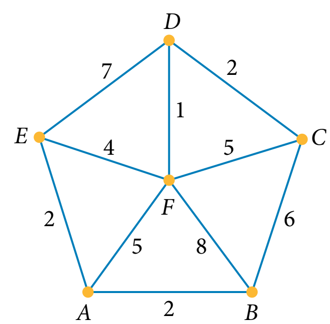
2 The graph gives road distances in kilometres between some locations within an outer suburb.

(a) The shortest routes from A to C are:

A A-B-C and A-F-C B A-B-C and A-E-D-C

C A-B-C and A-F-D-C D A-B-C and A-E-F-C

(b) Explain the common error made by a student who chose the first incorrect option.



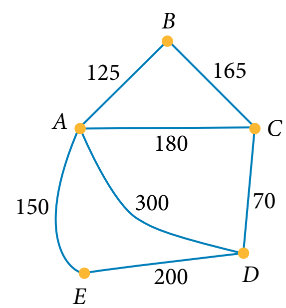
3 The graph gives distances in metres between some campsites in a state forest.

(a) The length of the semi-Eulerian path from D to C is:

A 70 m B 595 m

C 640 m D 1190 m

(b) Explain the common error made by a student who chose the third incorrect option.

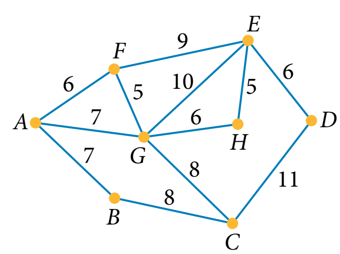


4 Distances in kilometres between various towns labelled A to H are shown in the weighted graph.

Determine the shortest route and distance in kilometres between the following pairs of towns:

(a) G and D

(b) A and D

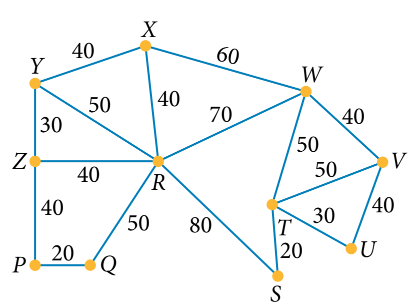


5 Distances in metres along paths between various exotic plants in a botanical garden are shown in the weighted graph.

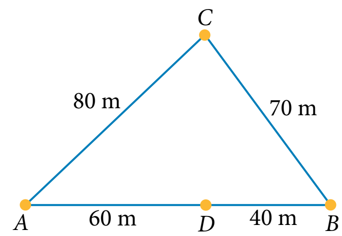
Determine the shortest route between the given pairs of plants and calculate this distance.

(a) P and U

(b) Z and V



6 A football coach places four witches' hats A, B, C and D as shown in the weighted graph. The hats are in fixed positions and the distances between them are given in metres. The coach instructs her players to start at any hat and run to a different hat along a course that includes every other hat.



Determine the paths that players can take, and their minimum distances if they choose the given two hats as their start and end points.

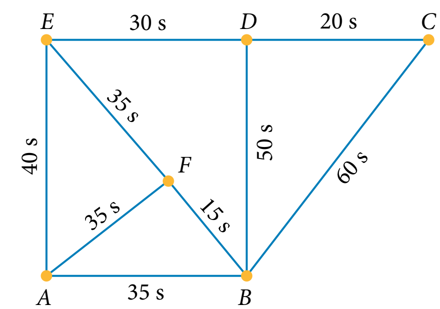
(a) A and B (b) A and C (c) A and D
(d) B and C (e) B and D (f) C and D

Worked Example 19

7 An office worker notes the time it takes him to walk between particular city buildings during lunchtime. The times in seconds are shown on the weighted graph.

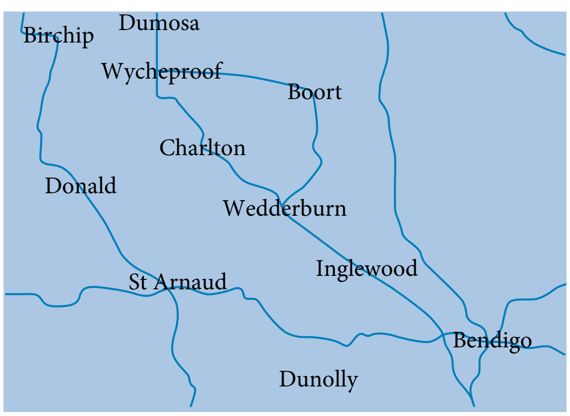
(a) What is the minimum route and time in seconds that it takes the worker to walk from A to C?

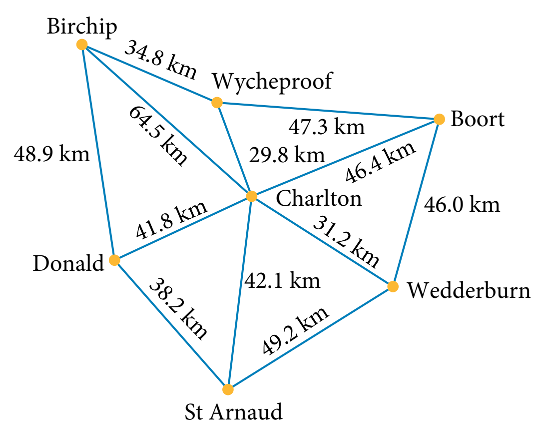
(b) What is the minimum route and time in seconds for a route that visits every building only once, if the walker starts and ends at vertex A?



8 A tour of seven central Victorian towns is to start and finish in Charlton. The towns are shown on the map below and are also represented on a weighted graph.

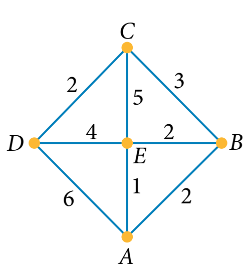
List the towns in order to show the shortest route that the tour can take. What is the distance for this route?



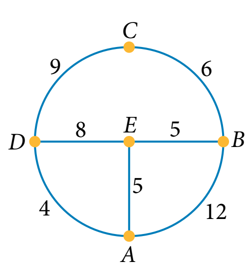


9 Determine the shortest Hamiltonian cycle, where possible, in each of the following weighted graphs. All distances are given in kilometres. 18

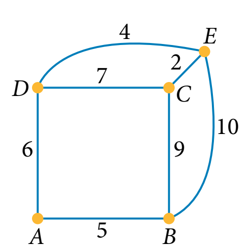
(a)



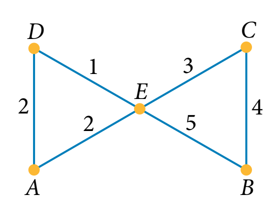
(b)



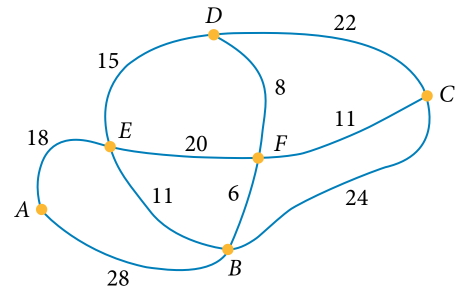
(c)



(d)



10 The distances in kilometres between local attractions labelled A to F are shown on the weighted graph.

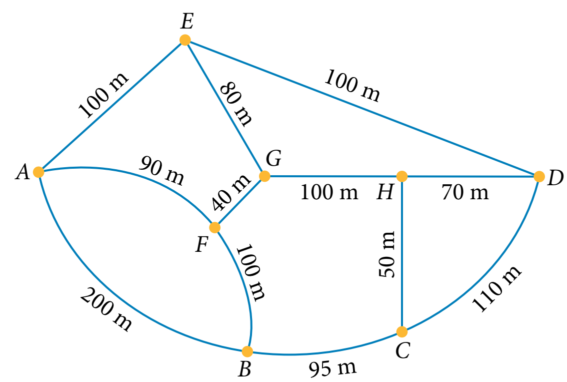


(a) Determine the shortest route from A to C and calculate its length in kilometres.

(b) A tourist wishes to visit every local attraction once. What is the shortest route in kilometres from A to C that visits all the local attractions?

(c) A sales representative from a company that prints brochures visits all local attractions, starting and ending at A. What is the route that gives the minimum distance and what is the distance in kilometres she must travel? (She may visit a local attraction more than once.)

11 A developer draws a weighted graph of a new estate to assist a real estate agent in drawing up accurate plans. Points A to H on the graph represent survey markers and distances are given in metres.

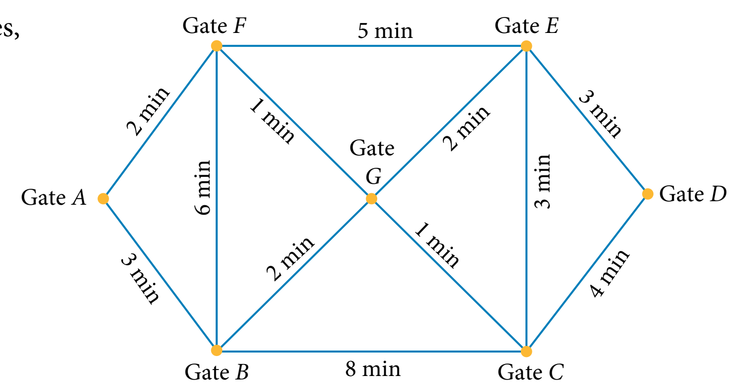


(a) The agent wishes to walk from A to D to check five markers, including A and D, in as short a distance as possible. What route can he take and how long in metres is the route?

(b) The developer wishes to check the positions of all survey markers starting at A. What route should she take to visit each marker and how far must she travel in metres?

12 The weighted graph shows walking times, in minutes, along special walkways between departure gates at an airport.

Security staff regularly travel various routes between gates, and the airport manager wants to know the answers to the following questions. Provide these answers for her.



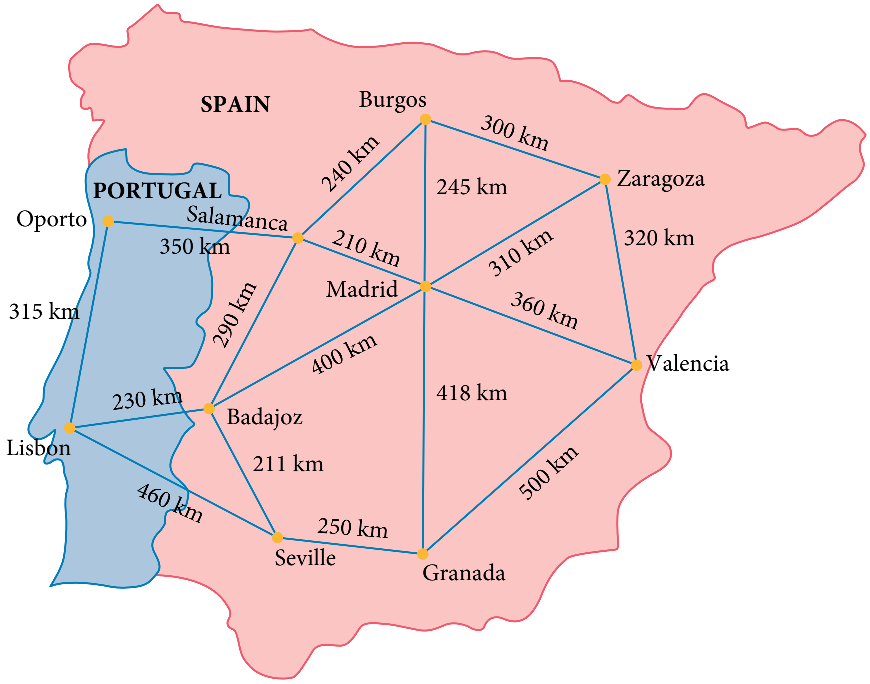
(a) If it is *not* necessary to visit every gate, determine the quickest route a staff member can take to go from gates A to D and state how long it will it take.

(b) If it is necessary to visit every gate, which path gives the quickest route between gates A and D? (A gate may be visited more than once.)

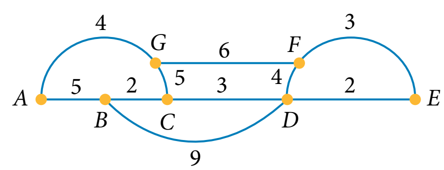
(c) Determine the quickest cycle that visits each gate exactly once, starting at gate A, and state how long it will take.

(d) Starting and ending at gate A, determine the quickest route travelling along every special walkway and state how long it will take.

13 Juan is planning a tour of Spain and Portugal. He wants to visit the cities shown on the map, starting and ending in Madrid. List the order of the cities he can visit so that the distance he travels is the shortest, and then calculate that distance in kilometres.



14 Determine the shortest Hamiltonian cycle in the graph. All distances are given in kilometres.



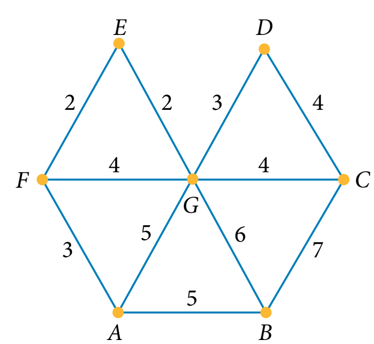
15 The graph gives distances between locations in kilometres.

A new road is to be built between *E* and *D* of just 1 km. Describe how this will affect the shortest path described by each of the following.

(a) *F* to *C*

(b) *E* to *B*

(c) the Hamiltonian cycle starting at *A*



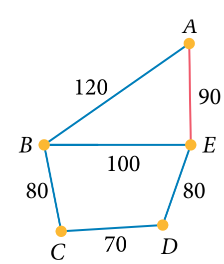
16 The graph gives distances between locations in kilometres.

A new road is proposed between *A* and *E* with a length of 90 km.

(a) Explain how the new road will affect the shortest distance between *A* and *D*.

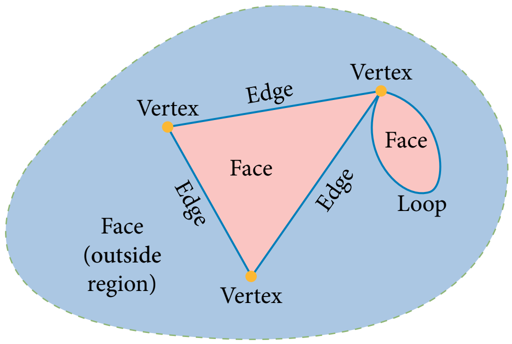
(b) The speed limit for *A*-*B*, *B*-*C* and *E*-*D* is 100 km/h, and for *B*-*E* and *C*-*D* it is 80 km/h. The proposed road will have a speed limit of 60 km/h.

How will the new road affect the quickest journey between *A* and *D*?



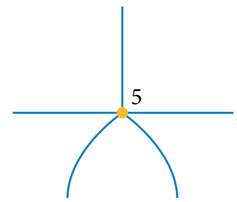
7 Summary

Graph terminology



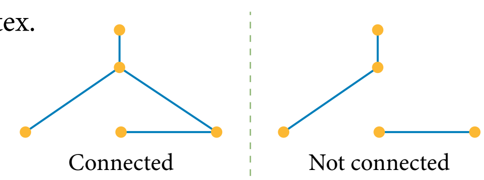
Degree of a vertex

The degree of a vertex is the number of entries/exits extending from the vertex.
A loop counts as 1 edge but adds 2 to the degree of a vertex.



Connected graph

A graph is connected if there is a path connecting every vertex.



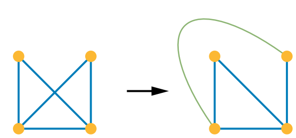
Bridge

A bridge is an edge in a connected graph that, if removed, leaves the graph disconnected.

In the disconnected graph shown on the right, a bridge is the edge that was removed from the connected graph shown on the left.

Planar graph

A graph is planar if it can be drawn so that no edges cross.



The same planar graph drawn two ways

Euler's formula

Euler's formula expresses the relationship between the numbers of vertices (v), edges (e) and faces (f) in a *connected planar* graph: $v + f - e = 2$.

Subgraph

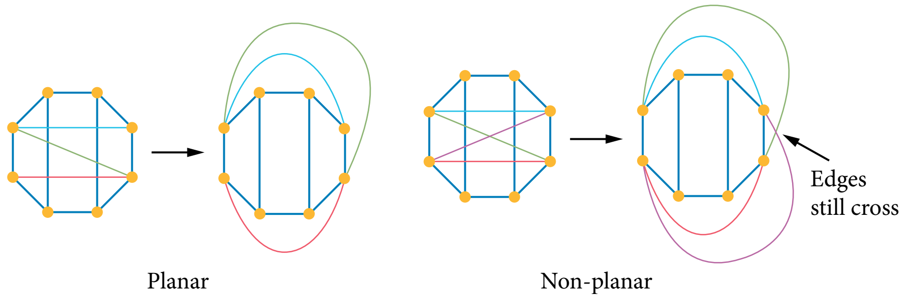
A subgraph contains parts of another graph, but may be missing some vertices or edges.

Simple graph

A simple graph contains no loops or multiple edges between vertices.

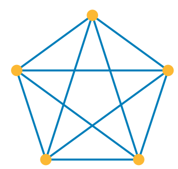
Isomorphic graph

An isomorphic graph is an equivalent graph having the same connections but not drawn identically.



Complete graph

In a complete graph, every vertex is connected to each of the other vertices with one edge between each pair of vertices.

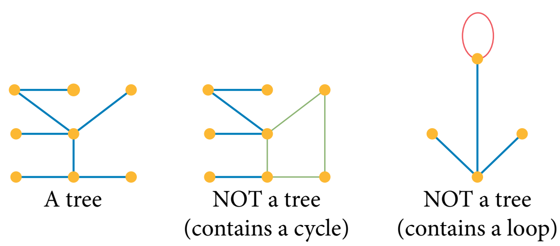


If v is the number of vertices and e is the number of edges in a complete graph, then $e = \frac{v(v-1)}{2}$.

Tree

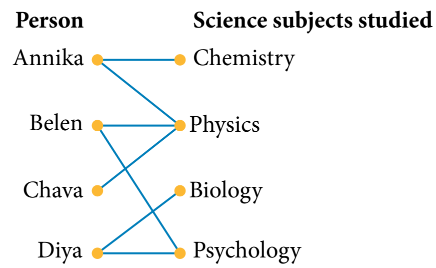
A tree is a connected graph that contains no loops or cycles.

A tree with v vertices contains $v - 1$ edges.



Bipartite graph

A bipartite graph is one whose vertices can be split into two sets so that each edge of the graph joins a vertex from the first set to a vertex in the second set.



Directed graph (digraph)

A directed graph contains vertices joined by directed edges, or arcs.

Practical networks

Use edges to represent:

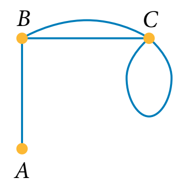
- relationships between people or objects.
- paths between vertices.

Use an arc (directed edge) to point:

- from a winner to a loser, 'defeats'.
- from a producer to a consumer, 'is consumed by'.

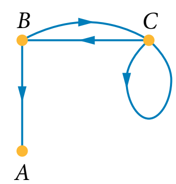
Adjacency matrices

An adjacency matrix lists the number of edges joining each pair of vertices in a non-directed graph.



	A	B	C
A	0	1	0
B	1	0	2
C	0	2	1

In a directed graph, an adjacency matrix lists the number of ways to get from one vertex to another by following a single arrow.



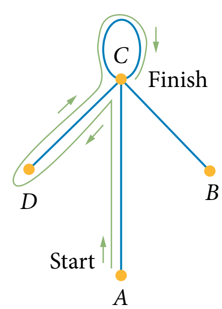
From \ To	A	B	C
A	0	0	0
B	1	0	1
C	0	1	1

Walk

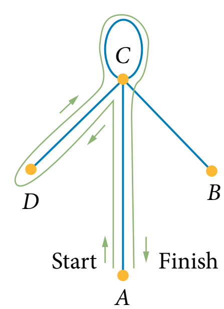
A walk is any connected sequence of vertices.

An open walk starts and ends at different vertices.

A closed walk starts and ends at the same vertex.



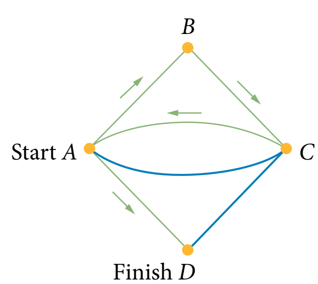
An open walk: *A-C-D-C-C*



A closed walk: *A-C-D-C-C-A*

Trail

A trail is a walk with no repeated edges.

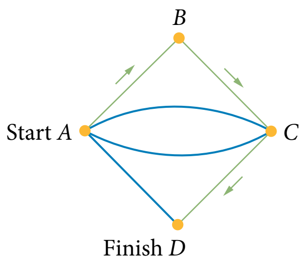


A trail *A-B-C-A-D*

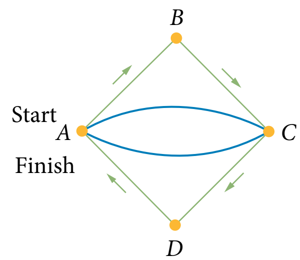
Path

A path is a walk with no repeated vertices or edges.

A cycle is a closed path.



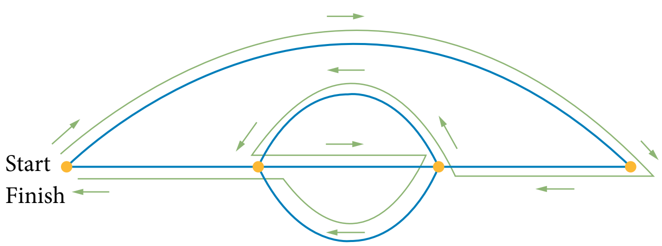
An open path: *A-B-C-D*



A closed path, or cycle: *A-B-C-D-A*

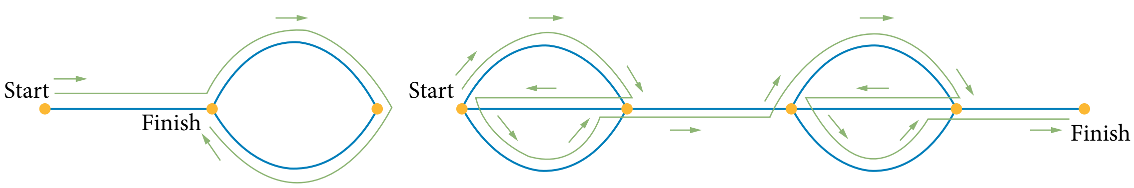
Eulerian graph and Eulerian trail

An Eulerian graph contains a closed trail, called an Eulerian trail, that includes every edge exactly once and that starts and ends at the same vertex. The degree of every vertex is even.



Semi-Eulerian graph and semi-Eulerian trail

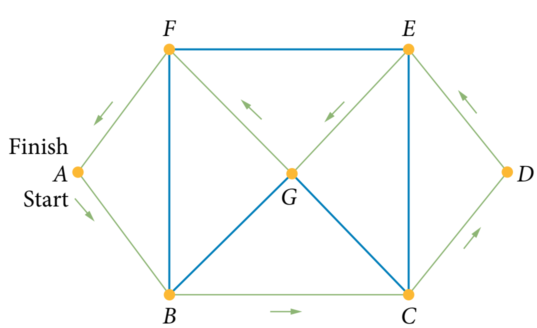
A semi-Eulerian graph contains an open trail, called a semi-Eulerian trail, that includes every edge exactly once but starts and ends at different vertices. The degrees of exactly two vertices are odd.



Semi-Eulerian graphs

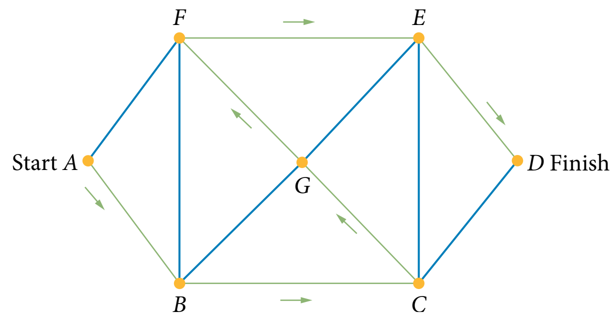
Hamiltonian graph and Hamiltonian cycle

A Hamiltonian graph contains a Hamiltonian cycle (a closed path), which is a path that starts and ends at the same vertex and visits every other vertex exactly once.



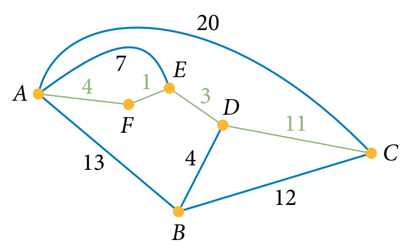
Semi-Hamiltonian graph and Hamiltonian path

A semi-Hamiltonian graph contains a Hamiltonian path, which is an open path that visits every vertex in a network exactly once, starting and ending at different vertices.

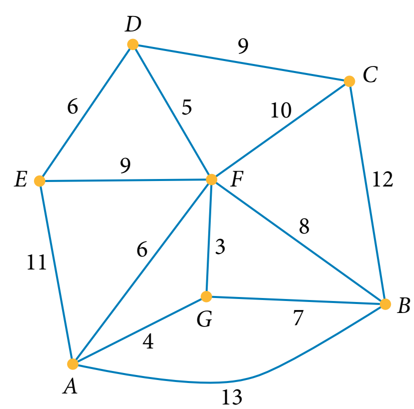


Weighted graphs

Weighted graphs contain distances, costs or times for each edge. These numbers are called lengths or weights. Find the shortest path by trial and error.



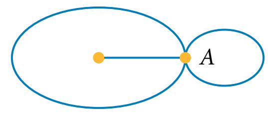
Try various Hamiltonian cycles to find the shortest circuit that visits each vertex only once (apart from the start/end vertex).



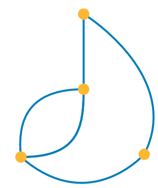
$A\text{-}G\text{-}F\text{-}B\text{-}C\text{-}D\text{-}E\text{-}A = 4 + 3 + 8 + 12 + 9 + 6 + 11 = 53$

Chapter review 7

1 Determine the degree of vertex *A* in the graph. Exercise 7.1

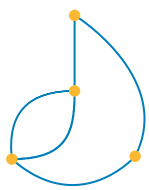


2 Determine the number of faces, edges and vertices in the graph. Exercise 7.1

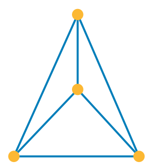


3 Which of the following graphs are isomorphic to the graph shown at right? Exercise 7.1

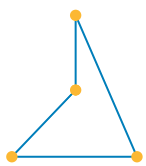
A



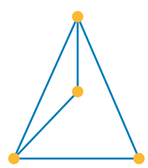
B

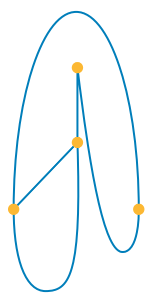


C



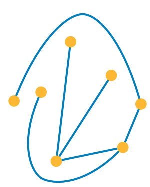
D



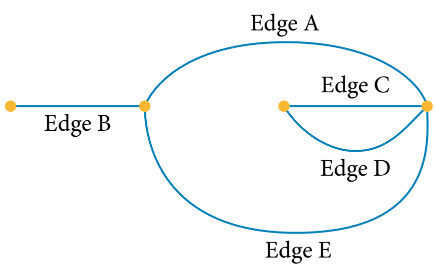


4 Which of the following terms do *not* describe the graph? Exercise 7.1

planar, connected, simple, directed, tree



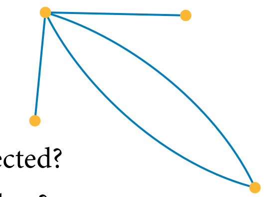
5 Which edge in the graph is a bridge? Exercise 7.1



6 A connected planar graph contains 18 edges and 17 vertices. If an extra edge is placed between two different vertices, without crossing other edges, how many faces will the new graph contain? Exercise 7.1

Exercise 7.1

7 Consider the graph shown at right.



(a) State the number of vertices, edges and faces that the graph contains.

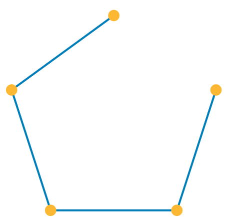
(b) Answer each of the following yes/no questions.

- **(i)** Is the graph planar?
- **(ii)** Is the graph connected?
- **(iii)** Is the graph simple?
- **(iv)** Is the graph complete?
- **(v)** Is the graph a tree?
- **(vi)** Does the graph contain a bridge?
- **(vii)** Does the graph contain a loop?

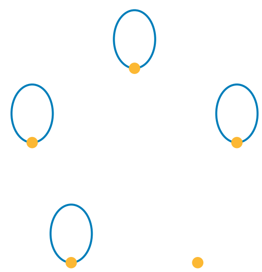
Exercise 7.1

8 The following graphs have 5 vertices and 4 edges. State the number of faces for each graph.

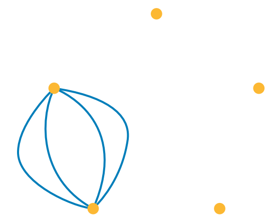
(a)



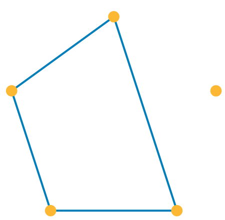
(b)



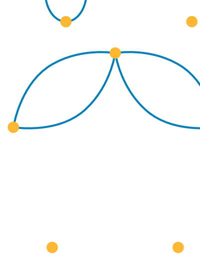
(c)



(d)



(e)



Exercise 7.1

9 Complete the table for connected planar graphs.

Number of vertices (v)	Number of edges (e)	Number of faces (f)
15	20	
9		11
	21	20
35	55	
	1	1
601		401

Exercise 7.1

10 How many edges are in a complete graph with the following properties?

(a) a graph with 1 vertex

(b) a graph with 12 vertices

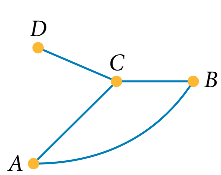
(c) a graph with 100 vertices

(d) a graph with 1000 vertices

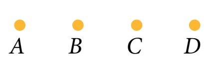
Exercise 7.1

11 Draw an isomorphic graph for each of the following graphs on the template given.

(a)

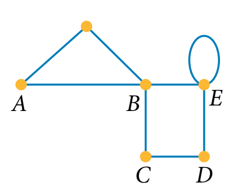


Graph

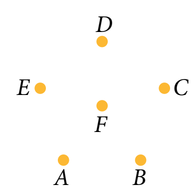


Isomorphic graph

(b)



Graph



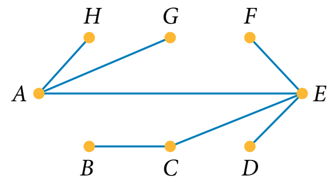
Isomorphic graph

12 The connected graph contains 8 vertices and 7 edges. Exercise 7.1

Draw subgraphs that contain each of the following:

(a) vertices A, E, G and H and 3 edges

(b) vertices A, B, C, D and E and 4 edges.



13 A food web for ocean organisms is shown here. The arrows point towards the consumer in each case. Exercise 7.2

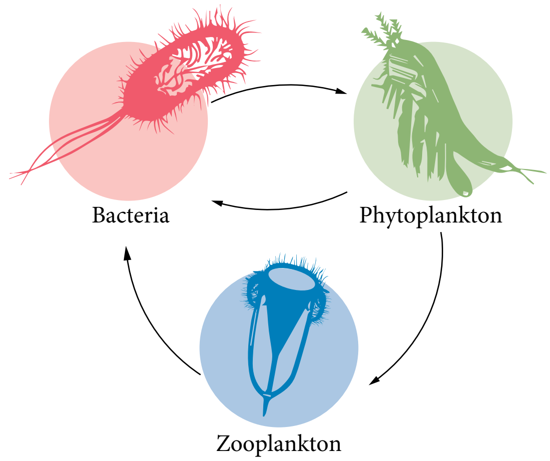
Which statement about this food web is correct?

A Bacteria consume phytoplankton and zooplankton, and are consumed by zooplankton.

B Phytoplankton consume bacteria and zooplankton, and are consumed by zooplankton.

C Phytoplankton consume bacteria, and are consumed by zooplankton and bacteria.

D Zooplankton consume only bacteria.



14 Each connection in the social network shown here indicates a fictional friendship between members of the network. Exercise 7.2

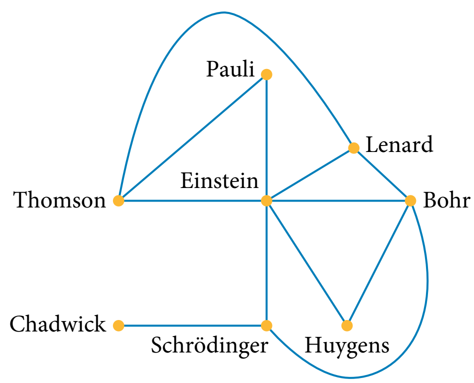
Which of the following statements is true?

A Einstein is friends with everybody in the network.

B Schrödinger has four friends in the network.

C Bohr is friends with Huygens and Pauli.

D Thomson is friends with Lenard and Pauli but not with Chadwick.



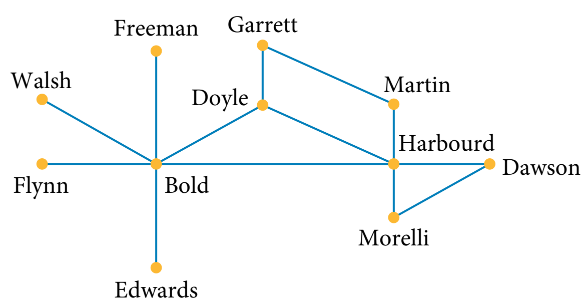
15 In his novel, an author created several characters who ultimately meet each other. The graph represents the relationships among the characters. Exercise 7.2

(a) Based on the graph, who is the main character in the novel?

(b) Minor characters in this novel meet only one other character. Who are the minor characters?

(c) Who are the characters that Doyle meets?

(d) One chapter in the novel involves a sequence of meetings that begins with Walsh and ends with Dawson. What is the minimum number of characters involved in these meetings and what is the order in which they occur?



(e) In the novel, Dawson, Freeman, Harbourd, Morelli and Walsh are 'good guys'. Flynn is 'neutral' and the others are 'bad guys'. Draw a bipartite graph with the good guys and Flynn on the left, and the other guys on the right, and use lines to indicate meetings between good guys and bad guys.

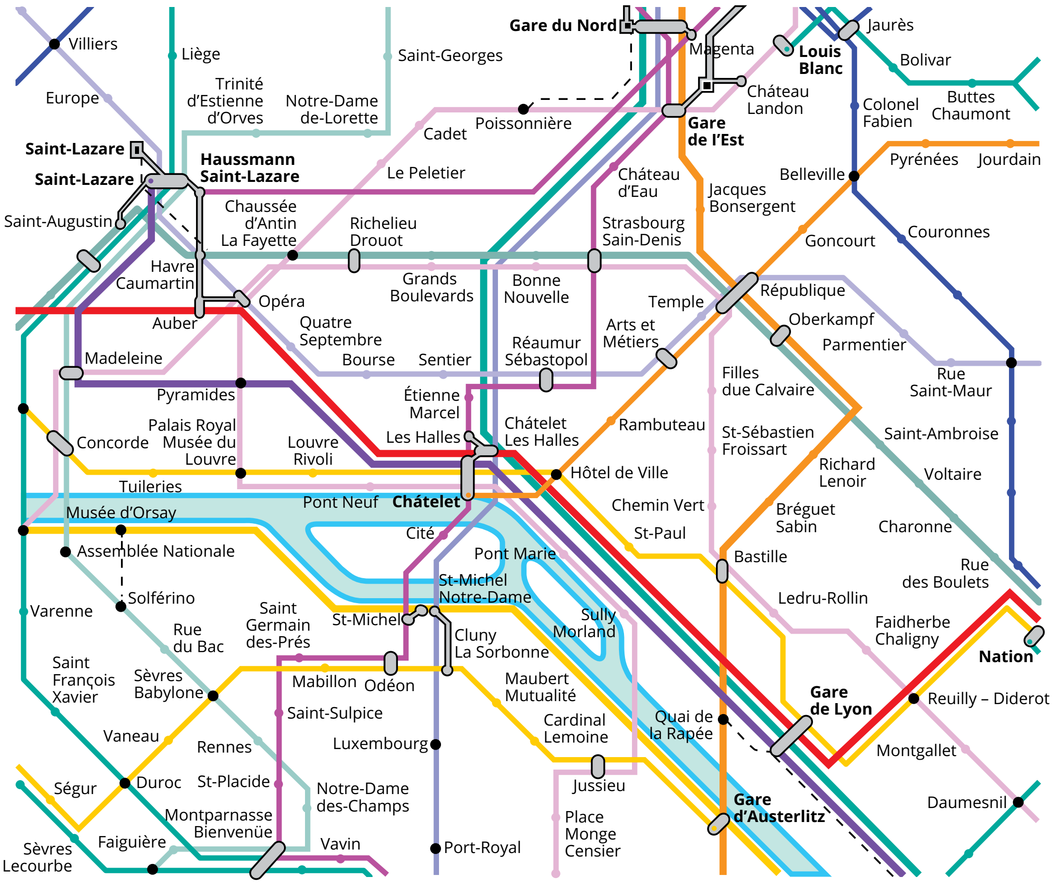
Exercise 7.2

16 A partial map of the Paris Metro, a public transportation system, appears below. Interchanges between different lines are shown as small grey circles or ovals.

(a) What is the number of lines that are connected by the interchange at Gare de Lyon, located near the bottom right corner of the map on the red line?

(b) What is the station north of Luxembourg on the mauve line starting from the bottom centre of the map?

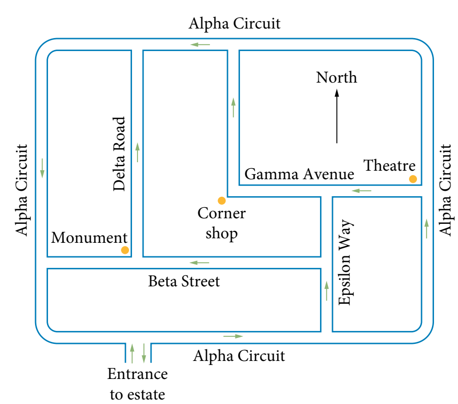
(c) Determine the stations, in order, on the direct route from Saint François Xavier at the bottom left corner of the map to Châtelet at the centre of the map along the river.



Exercise 7.2

17 A town planner designs a new estate with one-way travel restrictions.

Which of the trips (by car) listed below is possible?



A Drive north from the corner shop along Gamma Avenue, turn right onto Alpha Circuit and travel to the theatre.

B Drive north from the monument along Delta Road, turn right onto Alpha Circuit and travel to the theatre.

C Drive east from the entrance along Alpha Circuit, turn left onto Epsilon Way, then left onto Gamma Avenue and travel to the corner shop.

D Drive east from the monument along Beta Street, turn left onto Epsilon Way, then turn right on Gamma Avenue and travel to the theatre.

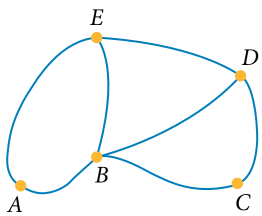
18 A network with vertices A, B, C and D has the following connections: Exercise 7.3

- A is connected to B by two different edges and to C by one edge.
- C is connected to D by two different edges.
- D is connected to itself by a loop.

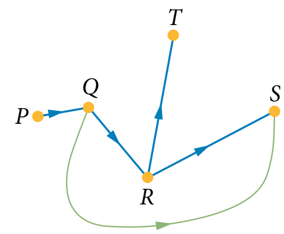
Construct the adjacency matrix for this network.

19 Construct an adjacency matrix for each of the following networks. Exercise 7.3

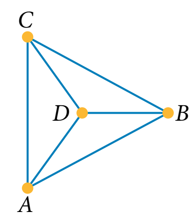
(a)



(b)

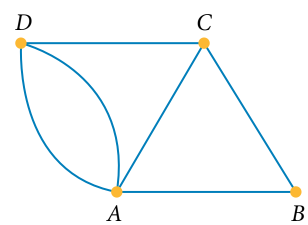


20 Which does the sequence A-B-C-D-A-C in the graph describe: a cycle, a path, a trail or a walk? Exercise 7.4



21 Describe the features of an open walk in terms of where it starts and finishes, and whether repeats of vertices and edges are allowed. Exercise 7.4

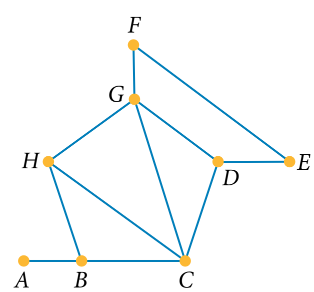
22 A semi-Eulerian trail can be traversed on the network shown here. Which vertices must be used to start and finish? Exercise 7.4



23 For the graph shown in Question 22, describe all Hamiltonian paths that start at A. Exercise 7.4

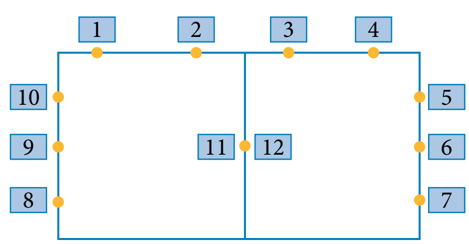
24 Consider the network shown here. Exercise 7.4

Between which two vertices could an additional edge be added to allow a semi-Eulerian trail within this graph? List all possible pairs.



25 A school campsite consists of cabins connected by narrow paths, as shown. The teachers stay in cabins 11 and 12. Exercise 7.4

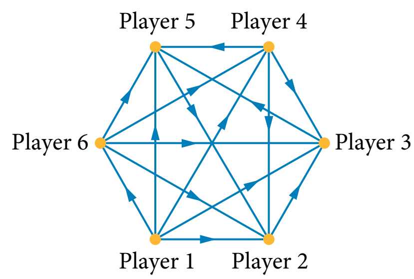
A teacher plans to start from cabin 12 and stop at each cabin once to wake up students at 6 am for a day of fun-filled activities. After he visits each student cabin, he returns to his own cabin. Describe the route he must take if the first cabin he stops at is cabin 2.



Exercise 7.2

26 The graph describes the results of a six-person squash tournament, where the arrow reads 'defeats'.

Which list shows the top three players, based on number of games won?



A Players 1, 4 and 5 **B** Players 1, 4 and 6

C Players 2, 3 and 5 **D** Players 2, 3 and 6

Exercise 7.2

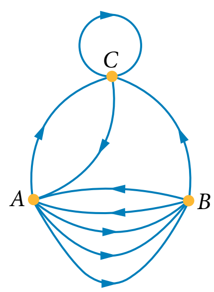
27 The organiser of a tennis round robin competition is deciding whether to hold a 10-player competition or a 12-player competition. How many more matches would a 12-player competition require than a 10-player competition?

Exercise 7.3

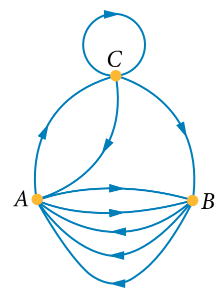
28 An adjacency matrix for a digraph is shown.
Which of the graphs below corresponds to this matrix?

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 2 & 1 \\ 3 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

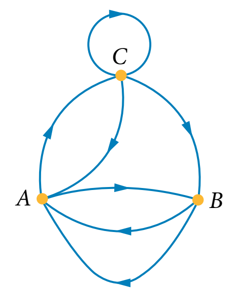
A



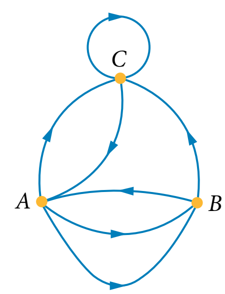
B



C



D



Exercise 7.3

29 Draw a possible graph that corresponds to the adjacency matrix.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Exercise 7.3

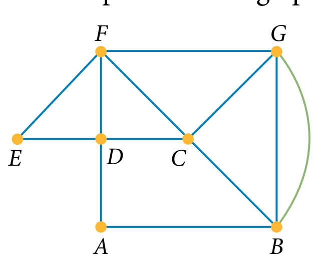
30 Draw a possible digraph that corresponds to the adjacency matrix.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

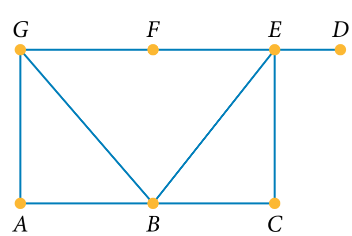
Exercise 7.4

31 Give an example of an Eulerian trail or a semi-Eulerian trail and a Hamiltonian path for each graph.

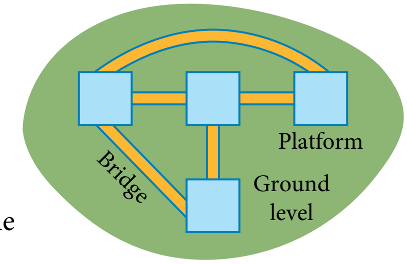
(a)



(b)



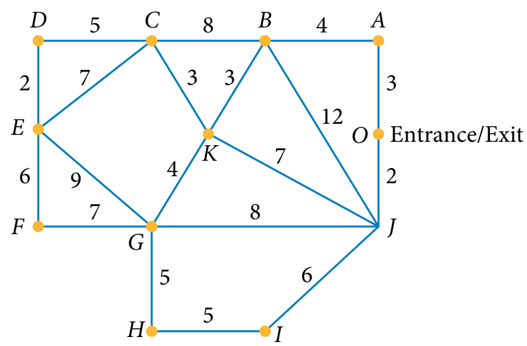
32 The graph shown here represents a children's playground, consisting of four square platforms connected by wobbly chain bridges. Exercise 7.4



(a) Represent the playground with a graph, writing the degree of each vertex next to the vertex.

(b) Is it possible for a child to run between the platforms on the playground and cross each bridge exactly once? Explain your answer.

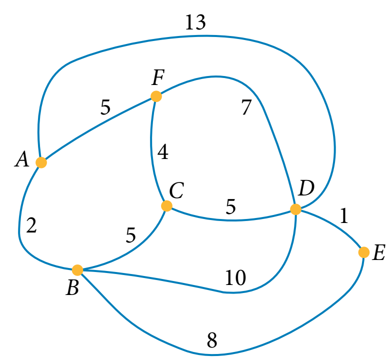
33 A museum arranges several displays and provides visitors with a plan showing walking times (in minutes) between key points. Exercise 7.5



(a) Jake is hungry and wants to have a very quick look through the museum from the entrance at O to the cafeteria at E. State the route Jake will take from the entrance as well as the time in minutes.

(b) Serena wishes to start at the museum entrance and visit every display only once before exiting. What is the route that Serena will take after entering the museum and how long will it take to walk?

34 Refer to the graph to answer the following questions. Exercise 7.5



(a) What is the length of the shortest route from A to D? Give an example.

(b) What is the length of the shortest route from A to E that includes all vertices? Give an example.

(c) What is the length of the shortest cycle that includes all vertices? Give an example.

(d) Is it possible to trace every edge of this graph without lifting your pencil?

35 A series of boardwalks link several small islands in a wetlands reserve, as shown here. Exercise 7.4

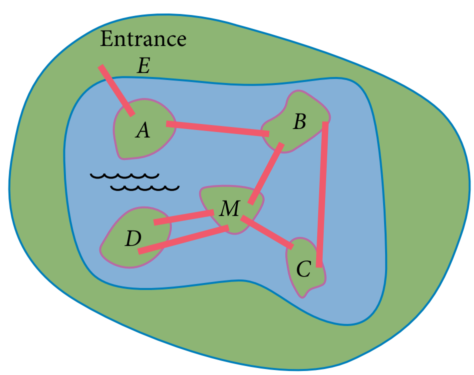
(a) Redraw the illustration as a graph with vertices and edges.

(b) Is it possible to draw a Hamiltonian cycle?

(c) Is an Eulerian trail possible?

(d) Is it possible to have a semi-Eulerian trail that starts at the entrance?

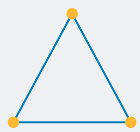
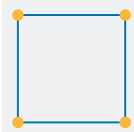
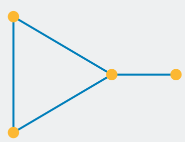
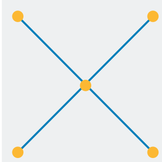
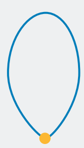
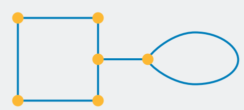
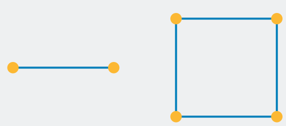
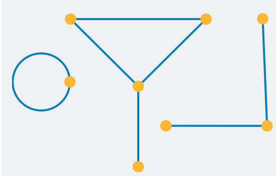
(e) It is decided to add another mainland link to an island that does not have one. Which island should be chosen if an Eulerian cycle will be possible in the new arrangement?



Exercise 7.2

36 Several graphs are shown below.

(a) Determine the sum of the degrees of the vertices and the number of edges in each graph to complete the table.

	Graph	Sum of degrees	Number of edges		Graph	Sum of degrees	Number of edges
(i)				(ii)			
(iii)				(iv)			
(v)				(vi)			
(vii)				(viii)			
(ix)				(x)			
(xi)				(xii)			

(b) Describe the relationship between the sum of the degrees and the number of edges for these examples, and explain why this will always be true.

Exercise 7.2

37 There is an extension of the Rock–Paper–Scissors game that adds Spock and Lizard to create a five-option game.

Spock

Lizard

The relationships between the five options are listed below. The winning option is the first option listed in each pair.

Rock blunts Scissors.
Lizard poisons Spock.
Paper wraps up Rock.
Scissors scare Lizard.
Scissors cut Paper.
Rock crushes Lizard.
Spock vaporises Rock.
Lizard eats Paper.
Paper disproves Spock.
Spock smashes Scissors.

(a) Draw a digraph to illustrate all the game options.

(b) Two friends play six rounds of the game. The options they choose in each game are listed below. What is the final score of the game?

Game	1	2	3	4	5	6
Player A	Rock	Paper	Scissors	Spock	Lizard	Spock
Player B	Spock	Rock	Scissors	Lizard	Paper	Paper

(c) Another variation is a four-option game called Rock–Paper–Scissors–Well, which adds the following rules to the rules of the original game:

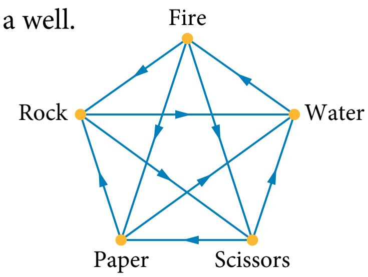
Well defeats Scissors and Rock, because both can fall into a well.

Paper defeats Well by covering it.

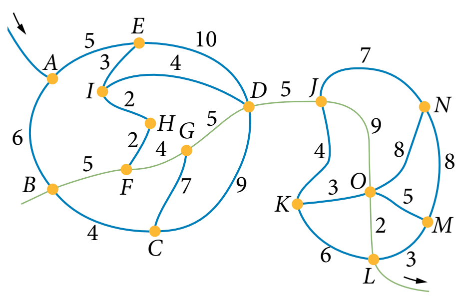
(i) Draw a graph to illustrate this game.

(ii) Why is this called an 'unbalanced' game?

(d) Is the game shown at right balanced or unbalanced? Justify your answer by giving your reasoning.



38 A travel agent is researching travel options for her customers. The weighted graph represents distances in kilometres between places of interest on the map. Blue lines represent bus routes and the green line represents a train route. Exercise 7.5



(a) Determine the minimum distance in kilometres between each of the given vertices, including bus and train routes.

(i) H and O (ii) B and N

(b) Determine the sequence of points in each example that covers the given distance:

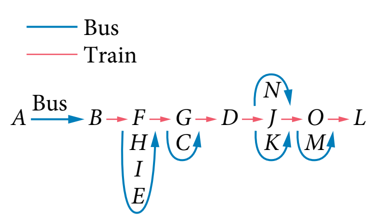
(i) A to B, distance = 27 km

(ii) H to E, distance = 30 km

(iii) H to O, distance = 43 km

(c) Determine the shortest route from A to L, and the cost of a trip on this route, if the bus fare is \$2 per kilometre and the train fare is \$1.20 per kilometre.

(d) A train runs along the red path shown in the graph and costs travellers \$1.20 per kilometre. A tourist decides to travel by train but stops at various points to take short bus trips to places not on the train line. The cost of each bus ride is \$2 per kilometre. Her trip is outlined in the diagram. Use the distances in the weighted graph above to find the total cost of this tourist's trip.



8

Networks and decision mathematics

8 Recall

Solve linear equations

1 Solve each equation for x.

(a) $2x - 1 = 9$

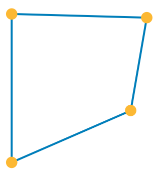
(b) $\frac{x}{4} = -5$

(c) $\frac{3x-7}{10} = 2$

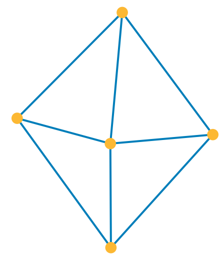
Subgraphs that are trees

2 Remove edges from each graph until the subgraph is a tree. Retain all the vertices of the original graph. There will be more than one solution in each case.

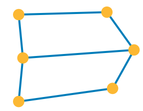
(a)



(b)



(c)



Constructing a matrix from a table

3 Convert each table to a matrix, maintaining the same number of rows and columns.

(a) average maximum temperatures (°C) in two cities

	September	October	November
Paris	21	16	11
London	19	14	10

(b) wickets taken and runs made by three all-rounders in the first innings of a cricket match

	Wickets	Runs
Fleser	2	33
Noros	5	23
Milderwuth	0	24

Complementary percentages

4 Determine the complement of each percentage by subtracting each from 100%.

(a) 30%

(b) 95%

(c) 12.5%

Minimum spanning trees

8.1

Spanning trees

A spanning tree is usually a subgraph of a given graph that connects all vertices with the minimum number of edges and, like all trees, it has no loops or cycles.

A *spanning tree* is a connected subgraph connecting v vertices with $(v - 1)$ edges.

A *minimum spanning tree* is a spanning tree that has the shortest possible length or total weight of the edges.

Applications of minimum spanning trees include constructing electrical networks that connect isolated settlements in order to keep the total length of cable to be used at a minimum.

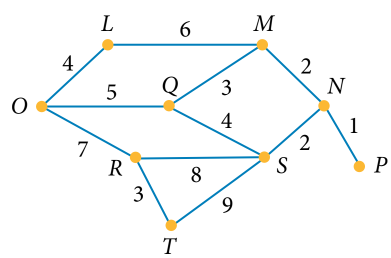
The trial and error method

A minimum spanning tree can be identified by trial and error.

The trial and error method involves inspection. Do not rely on the physical lengths of the edges, as the weighting can represent a variety of factors, for example, time, cost or mass, not just length.

1 Identify a minimum spanning tree by inspection

Identify the minimum spanning tree for this network and calculate its length.



THINKING

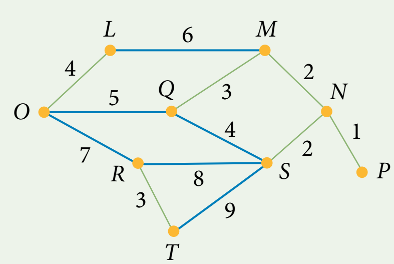
1 Identify the number of vertices, and use this to state the number of edges within the minimum spanning tree.

2 Highlight edges with the smallest weights, avoiding loops and cycles and connecting as many vertices as you can.

Determine the sum of the individual weights so far.

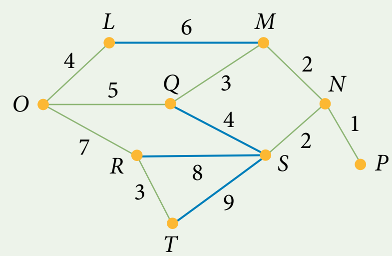
WORKING

The graph contains 9 vertices, so a spanning tree will have 8 edges.



$3 + 2 + 2 + 1 + 3 + 4 = 15$

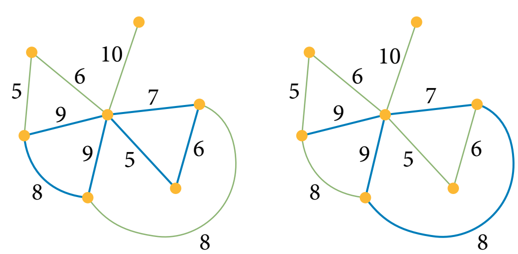
The edge connecting Q-S will not be included in the spanning tree, as point Q is already connected to the graph through Q-$M = 3$ and would form a cycle.

3 Analyse the disconnected graph.	The edge *O-L* is not connected to the larger subgraph. There are three ways to connect the points to the subgraph: *O-Q* = 5, *L-M* = 6 or *O-R* = 7. The edge *R-T* is not connected to the larger subgraph. There are three ways to connect the points to the sub-graph: *T-S* = 9, *R-S* = 8 or *R-O* = 7. To create a minimum spanning the two smallest weights, *O-Q* = 5 and *R-O* = 7 are used to connect the final vertices.
4 Complete the minimum spanning tree, ensuring no loops, and calculate its value.	 Value of the minimum spanning tree: 15 + 5 + 7 = 27 units

Minimum spanning trees can be used to determine the best way of connecting vertices in a network in many different contexts. Usually, you are asked to minimise the total distance needed to connect all points, but the process can also find the minimum cost linking all points in a network. In general, the *weight* of an edge can represent distance, cost or time.

If a network has n vertices, then its minimum spanning tree has $(n-1)$ edges. The remaining edges in the network are unnecessary and are not included in a minimum spanning tree.

Note, however, that a minimum spanning tree may not be unique. Both the minimum spanning trees highlighted in green below have a length of 40 units.



Prim's algorithm

While trial and error is an acceptable method for some networks, for more complicated networks you need to follow a fixed sequence of steps to be sure you have found a correct solution.

In 1958 Robert C. Prim, an electrical engineer and computer scientist from the USA, developed a strategy for finding a minimum spanning tree.

You can use *Prim's algorithm* to build a spanning tree by starting at any vertex and traversing one edge at a time until every vertex is included.

Steps for carrying out Prim's algorithm:

STEP 1: Choose any vertex in the network.

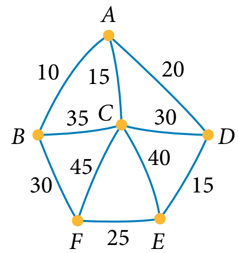
STEP 2: Look at all the edges connected to that vertex, choose the one with the lowest weight and add it to the tree.

STEP 3: Look at all the vertices connected so far and choose the edge with the lowest weight that connects the subgraph to another vertex, provided it does not form a cycle. Add it to the tree.

STEP 4: Repeat step 3 until you have connected all the vertices in the tree.

2 Using Prim's algorithm

Use Prim's algorithm to determine the minimum spanning tree for this network. Calculate the length of the minimum spanning tree. Weights are given in kilometres.

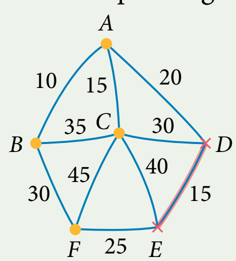


THINKING

1 Select a starting vertex and identify the lowest weighted edge from this vertex to any other vertex in the network.

WORKING

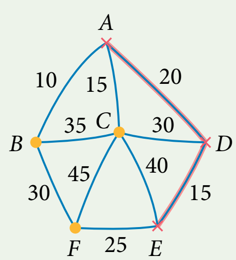
Starting with E, the edges show $E\text{-}C = 40$, $E\text{-}D = 15$ and $E\text{-}F = 25$. $E\text{-}D$ is the edge with the minimum weight that will connect vertex E to the spanning tree.



2 From the two connected vertices, identify the lowest weighted edge to connect another vertex to the subgraph.

Edges to consider: $E\text{-}F = 25$, $E\text{-}C = 40$, $D\text{-}C = 30$ and $D\text{-}A = 20$.

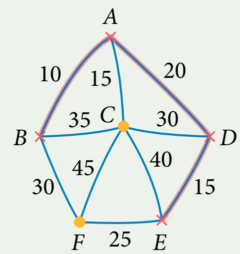
The path with the minimum weight, connecting another vertex to the subgraph, is $D\text{-}A$.



3 From the connected vertices, identify the lowest weighted edge to connect another vertex to the subgraph.

Edges to consider: $E\text{-}F = 25$, $E\text{-}C = 40$, $D\text{-}C = 30$, $A\text{-}C = 15$ and $A\text{-}B = 10$.

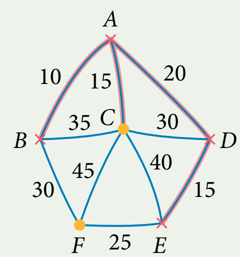
Connect vertex A to B.



4 From the connected vertices, identify the lowest weighted edge to connect another vertex to the subgraph without forming a cycle.

Edges to consider: $A\text{-}C = 15$, $B\text{-}C = 35$, $D\text{-}C = 30$, $E\text{-}C = 40$, $E\text{-}F = 25$ and $B\text{-}F = 30$.

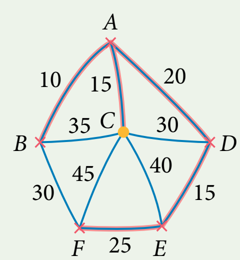
Connect vertex A to C.



5 From the connected vertices, identify the lowest weighted edge to connect another vertex to the subgraph without forming a cycle.

The only edges left to consider without making a cycle are: $E\text{-}F = 25$, $B\text{-}F = 30$ and $C\text{-}F = 45$.

Connect E to F.



6 Calculate the weight of this minimum spanning tree.

Weight of the minimum spanning tree:

$15 + 20 + 10 + 15 + 25 = 85$ km

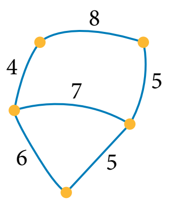
Remember, it does not matter which vertex you choose first. If you had started somewhere other than vertex E, the minimum spanning tree would still have the same length.

EXERCISE 8.1 Minimum spanning trees

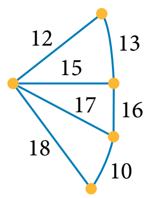
Worked Example

1 Identify the minimum spanning tree for each network and calculate its weight. 1

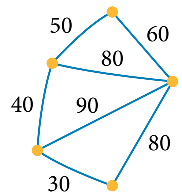
(a)



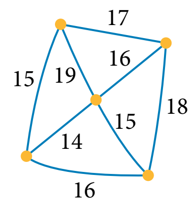
(b)



(c)



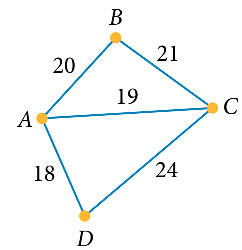
(d)



2 Consider the network shown.

(a) Name the edges that make up the minimum spanning tree in this network.

(b) Determine the weight of the minimum spanning tree.



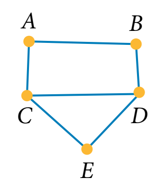
3 Consider the network shown.

(a) Which of the following options gives the edges that could be removed to leave a spanning tree?

A *C-D* only **B** *A-B* and *A-C*

C *D-E* and *D-B* **D** *E-C* and *E-D*

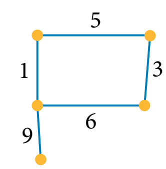
(b) Give reasons for dismissing the first incorrect option.



4 Consider the network shown.

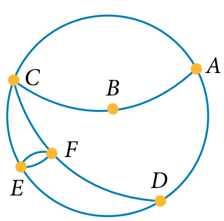
(a) Give a diagram and the weight of each possible spanning tree.

(b) Determine the weight of the minimum spanning tree.

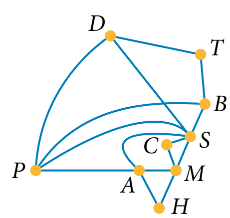


5 Determine the number of edges to be removed to obtain a spanning tree from each of the following networks.

(a)



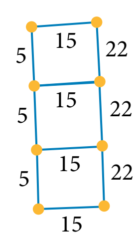
(b)



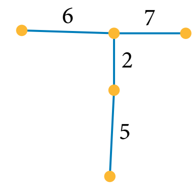
6 Consider the network shown.

(a) How many different minimum spanning trees can be drawn from the network?

(b) Determine the weight of the minimum spanning tree.



7 From the minimum spanning tree shown, create a possible graph from which it came, for each of the following cases. Your network will not be unique in each case.



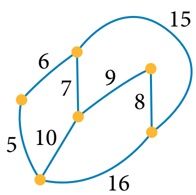
(a) The network has one cycle.

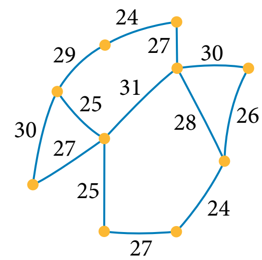
(b) The network has two cycles.

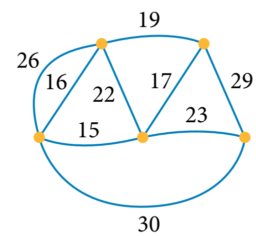
(c) The network has three cycles.

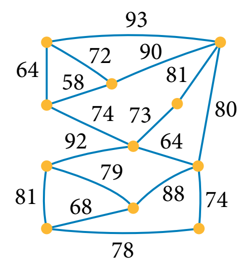
Worked Example 2

8 Use Prim's algorithm to construct a minimum spanning tree, and determine its weight for each of the graphs shown.

(a)


(b)


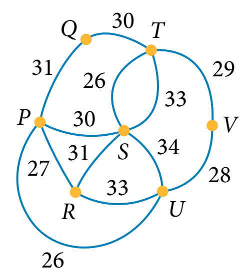
(c)


(d)


9 Which of these edges is in the minimum spanning tree for this network?

A *P-Q* **B** *R-S*

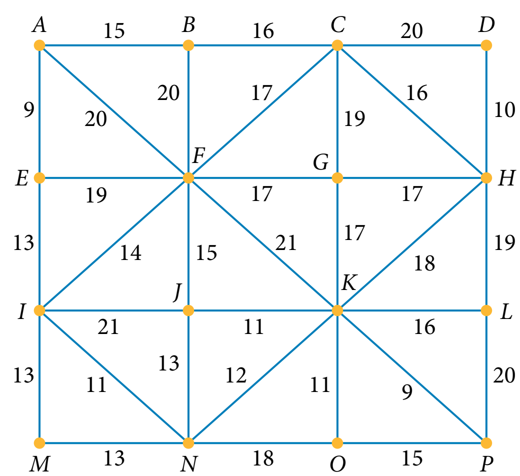
C *Q-T* **D** *P-S*



10 Consider the network shown.

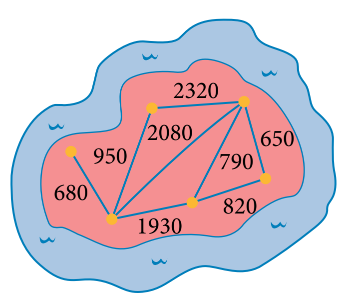
(a) Draw the minimum spanning tree.

(b) Determine the weight of the minimum spanning tree.



11 This map of a small island shows the locations of six mobile phone towers and possible ways of connecting the towers using underground cables. The distances between the towers are given in metres.

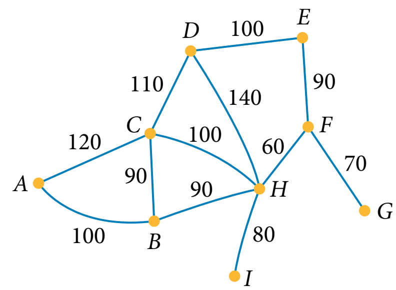
Determine the minimum length of cable needed to connect the six towers.



12 The weighted graph shows the locations and lengths, in metres, of pipes that connect several water troughs on a farm.

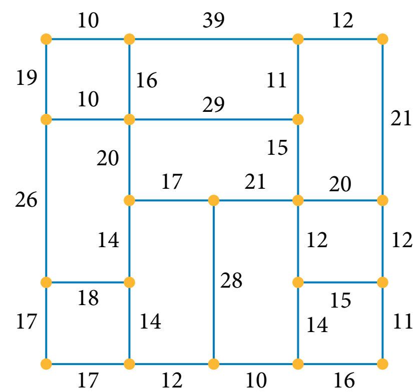
(a) Which pipes can be removed so that all troughs still have a water supply and the total length of the pipes is as small as possible?

(b) Calculate the total length of the minimum spanning tree.



13 The Central Business District has traffic lights at 20 intersections, as shown in the simplified weighted graph. The council plans to use underground cables to link the sensors at these intersections. The construction work will also involve paying business owners compensation for disrupting their businesses along the affected routes. The numbers in the graph represent the estimated compensation costs in thousands of dollars between each pair of intersections.

Determine the minimum amount the council will have to pay in compensation if the sensors at all 20 intersections are linked.



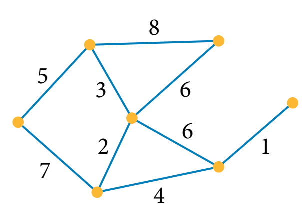
14 The table shows the estimated costs to repair sewer pipes between five pumping stations *A*, *B*, *C*, *D* and *E* after an earthquake. The red cell in the table indicates that no cost estimate is available. Initially, the repair is to be done in a way that keeps the cost to a minimum. Each pumping station can be connected to at least one other station but no unnecessary connections are to be repaired.

Station *A*				
$60 000	**Station *B***			
$90 000	$100 000	**Station *C***		
$80 000	$90 000	$130 000	**Station *D***	
	$80 000	$70 000	$50 000	**Station *E***

(a) Determine the minimum estimated cost of repairing the sewer pipes so that all five pumping stations are linked.

(b) Determine the minimum estimated cost if pumping stations *C* and *D* have to be connected directly.

15 In the network shown, the highest length and lowest length have been placed the wrong way around. What difference, if any, will this make to the length of the minimum spanning tree?



8.2 Network flow

Maximum flow

Applications of directed graphs include traffic and water flow, where the rate of flow of vehicles on a road or the rate of flow of water in a pipe is written on each arc of a weighted network.

A typical problem may ask for the maximum possible flow through a network.

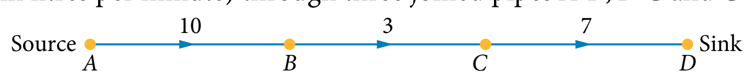
> In a maximum-flow problem, the start point and the finish point are often referred to as the *source* and the *sink*, respectively.

Single-line networks

The simplest networks have no deviations.

3 Maximum flow for a single line

The water flow rate (in litres per minute) through three joined pipes *A*-*B*, *B*-*C* and *C*-*D* is shown below.



Maximum pipe flow rates
(litres per minute)

Determine the maximum flow from source *A* to sink *D*.

THINKING	WORKING
1 Determine the minimum value on the single line.	The smallest of 10, 3 and 7 is 3.
2 The maximum flow is the minimum flow for any section of a single line.	The pipe *B*-*C* restricts the flow through the network to a maximum flow of 3 litres per minute.

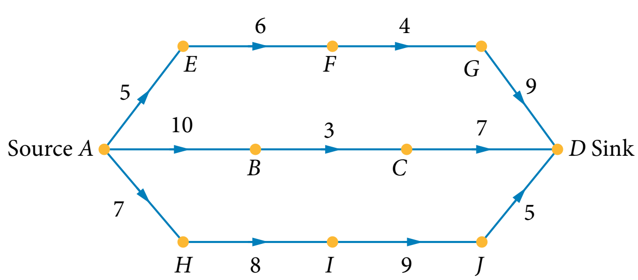
Multiple lines

If there is more than one line, the maximum flow rate is the *total* of the maximum rate for each line.

4 Maximum flow for multiple lines by inspection

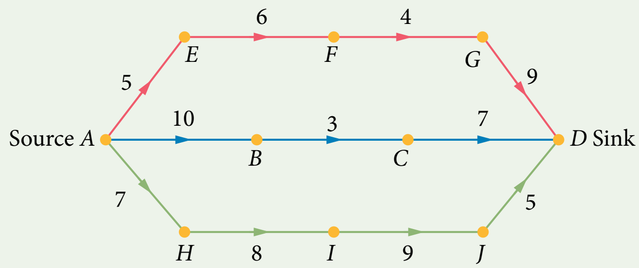
The water flow through a system of pipes is shown in the diagram. Determine the maximum flow (in litres per minute) from source *A* to sink *D*.

(a)



THINKING	WORKING

1 Identify all paths from the source to the sink.



From the source to the sink, there are three paths:

A-E-F-G-D, *A-B-C-D*, *A-H-I-J-D*

2 Determine the minimum value on each line.

The maximum flow in the top path *A-E-F-G-D* is 4 L/min.

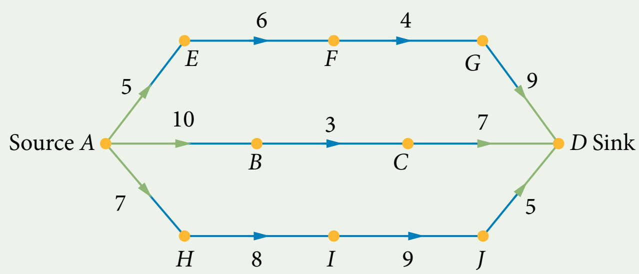
The maximum flow in the middle path *A-B-C-D* is 3 L/min.

The maximum flow in the bottom path *A-H-I-J-D* is 5 L/min.

3 Interpret the result.

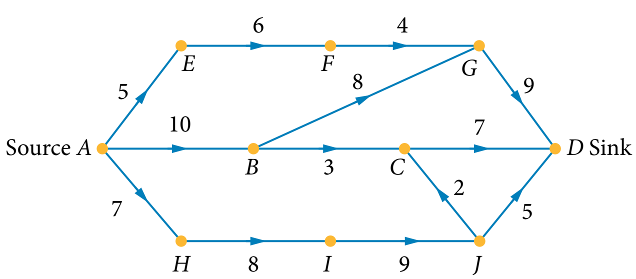
The maximum flow from source *A* to sink *D* is $4 + 3 + 5 = 12$ litres per minute.

4 Evaluate the reasonableness of your result.

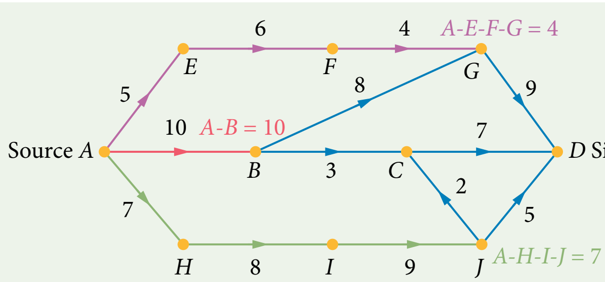


The maximum flow leaving the source is $5 + 10 + 7 = 22$ and the maximum flow into the sink is $9 + 7 + 5 = 21$. Therefore the maximum flow must be less than or equal to 21 litres per minute.

(b)



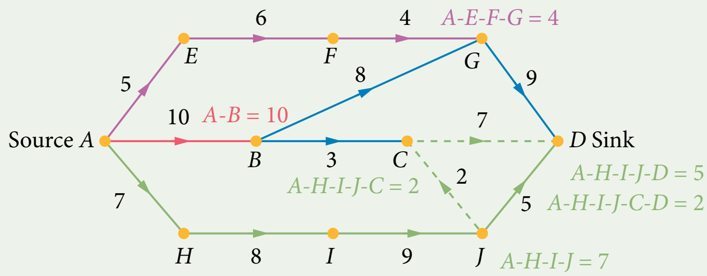
1 Analyse the maximum flow through each path until two or more arcs connect to a vertex.



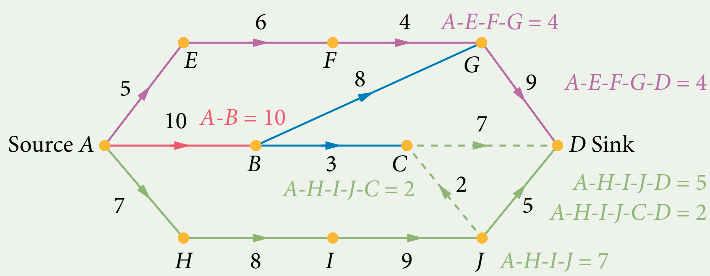
From the source there are three lines:

A-E-F-G = 4, *A-B* = 10, *A-H-I-J* = 7

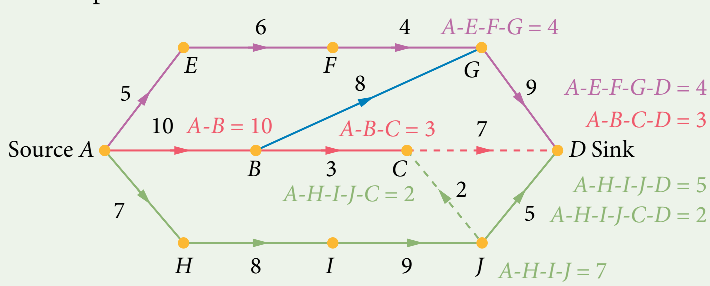
2 First consider the maximum flow through the network directly to the sink. Then determine the maximum flow through alternative arcs, by considering the restrictions on the arcs leading into the sink.



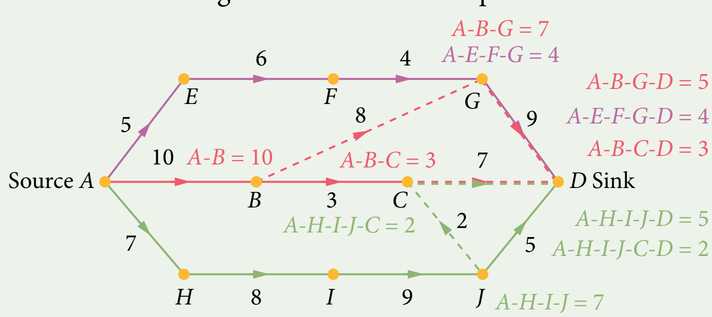
Beginning with the bottom path, the flow through *C-D* is not at maximum capacity, currently taking only 2 litres per minute from a possible 7.



The top path shows that the flow through *G-D* is not at maximum capacity, currently taking only 4 litres per minute from a possible 9.



The path *A-B-C-D* has a maximum flow of 3 litres per minute. Note that the flow through *C-D* sums to 5 litres per minute.



The final flow path needs to be planned cautiously.

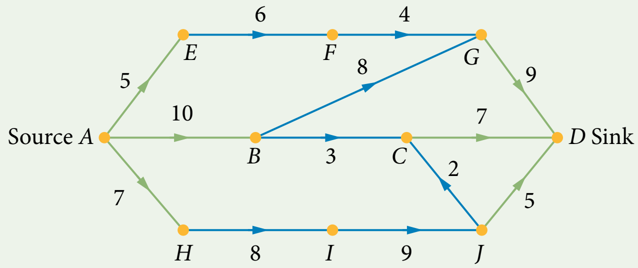
Recall that from *B*, 3 litres per minute is directed to *C*, leaving 7 of the 10 litres per minute to be directed to *G*.

G-D has a maximum capacity of 9 litres per minute, partially occupied by the 4 litres per minute from the path *A-E-F-G*, so the maximum flow through *A-B-G-D* is 5 litres per minute.

3 Interpret the result.

The maximum flow from source A to sink D is $5 + 4 + 3 + 5 + 2 = 19$ litres per minute.

4 Evaluate the reasonableness of your result.



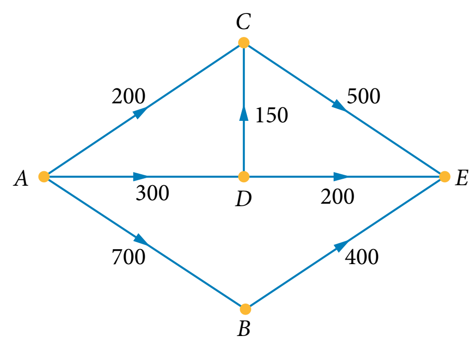
The maximum flow leaving the source is $5 + 10 + 7 = 22$ and the maximum flow into the sink is $9 + 7 + 5 = 21$. Therefore the maximum flow must be less than or equal to 21 litres per minute.

Many paths

Another method for determining the exact value of the maximum flow from source to sink is the many paths method.

5 Maximum flow using the many paths method

The maximum number of vehicles that can travel along particular roads during the busiest 15 minute period of the day is shown in the network diagram.



Determine the maximum total flow from source A to sink E for the network, using the many paths method.

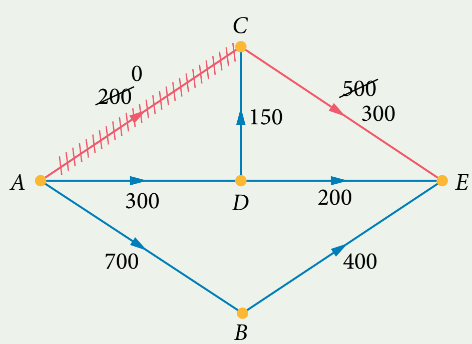
THINKING

WORKING

1 First consider the path across the top A-C-E.

Note the smallest flow on this path in a table.

Subtract this value from each value on the path. Write the remaining flows, crossing out the arc with zero remaining flow.

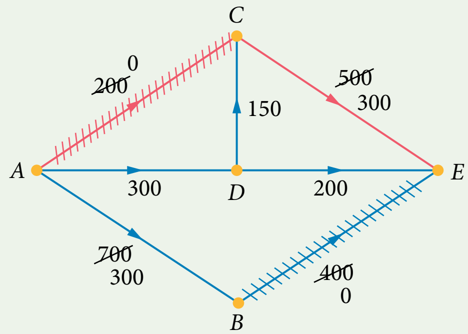


Maximum flow rate subtotal
200
Total (max. flow)

2 Consider the path across the bottom *A-B-E*.

Note the smallest flow on this path in the table.

Subtract this value from each value on the path. Write the remaining flows, crossing out the arc with zero remaining flow.

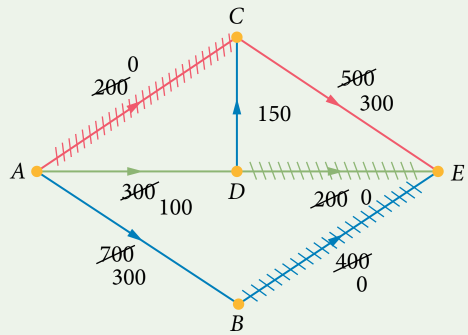


Maximum flow rate subtotal
200
400
Total (max. flow)

3 Consider the path through the middle *A-D-E*.

Note the smallest flow on this path in the table.

Subtract this value from each value on the path. Write the remaining flows, crossing out the arc with zero remaining flow.

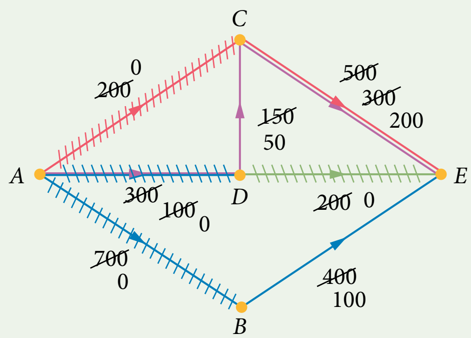


Maximum flow rate subtotal
200
400
200
Total (max. flow)

4 Consider path *A-D-C-E*.

Note the smallest flow on this path in the table.

Subtract this value from each value on the path. Write the remaining flows, crossing out the arc with zero remaining flow.



Maximum flow rate subtotal
200
400
200
100
900
Total (max. flow)

5 Interpret the answer.

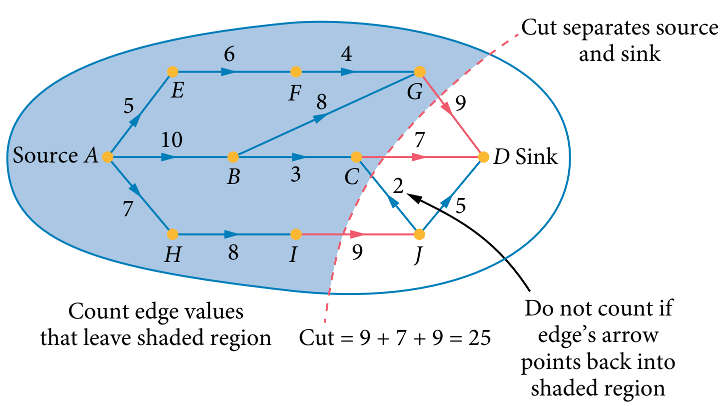
The total maximum flow is 900 vehicles in 15 minutes.

The minimum-cut method

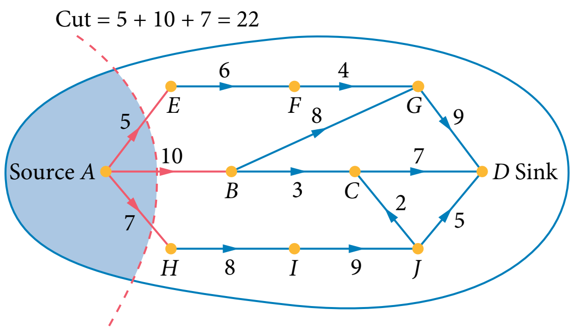
Another method for determining the exact value of the maximum flow from source to sink is the minimum-cut method.

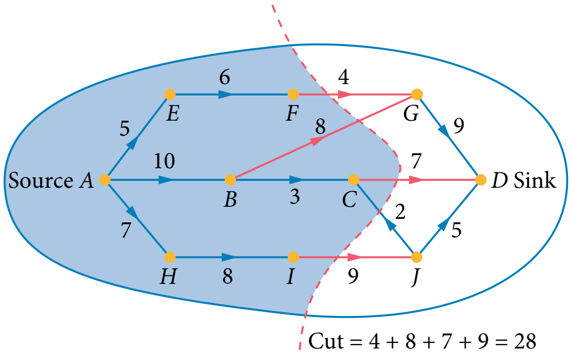
A *cut* is a curve drawn across the arcs of a digraph that separates the source and sink in a weighted directed network.

The value of the cut is the sum of the weights (or flows) of arcs whose directions point *away* from the source side of the cut. It can be helpful to shade the source side of a cut before calculating a cut's value.

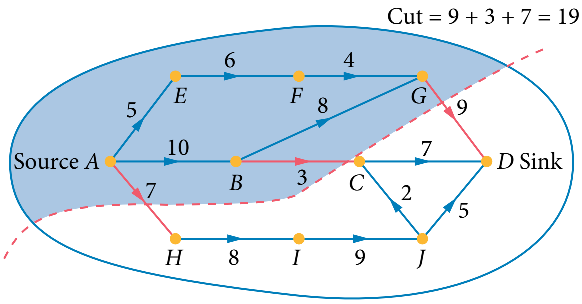


Some other possible cuts for this network are shown below.





The *minimum cut* for this network is 19, as shown below. This is the *maximum total flow* value from the previous method. The minimum cut passes through the restricting edge in each path from the source to the sink and, hence, gives the maximum possible flow.

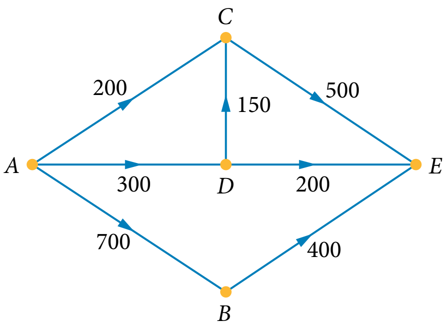


In a weighted digraph, such as one showing network flows, the minimum cut gives the maximum total flow from the source to the sink. For this method to be effective, it is important to identify and determine the capacity of all cuts.

6 Maximum flow using the minimum-cut method

The maximum number of vehicles that can travel along particular roads during the busiest 15 minute period of the day is shown in the network diagram.

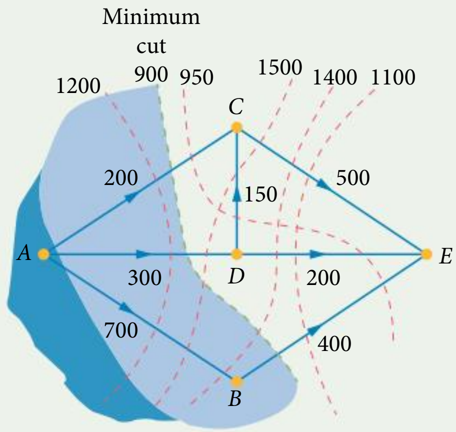
Determine the maximum total flow from source A to sink E for the network, using the minimum-cut method.



THINKING

1 Make several cuts and determine the minimum cut.

WORKING



The minimum cut is 900.

2 Interpret the result.

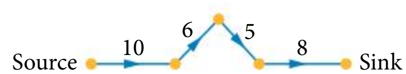
The maximum flow is 900 vehicles in the busiest 15 minute period.

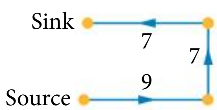
EXERCISE 8.2 Network flow

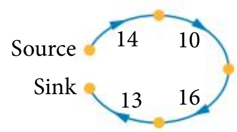
Worked Example 3

1 Determine the maximum flow from source to sink for each of the following networks.

(a)

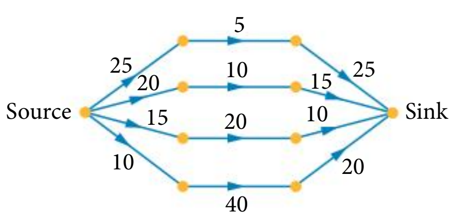

(b)


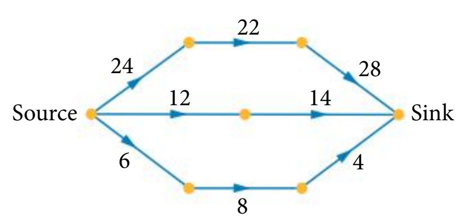
(c)


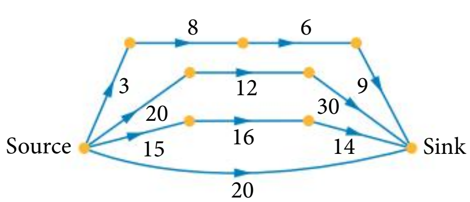
(d)


Worked Example 4

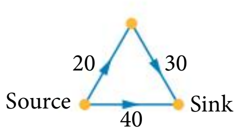
2 Determine the maximum flow from source to sink for each of the following networks.

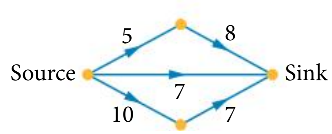
(a)


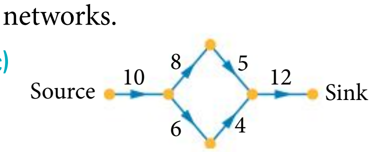
(b)


(c)


3 Determine the maximum flow from source to sink in the following networks.

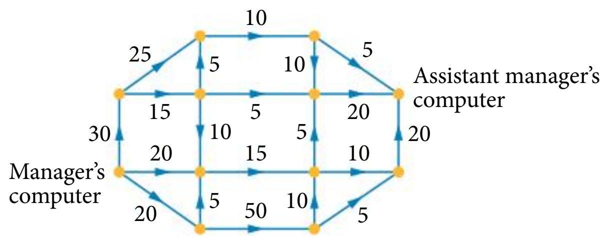
(a)


(b)


(c)


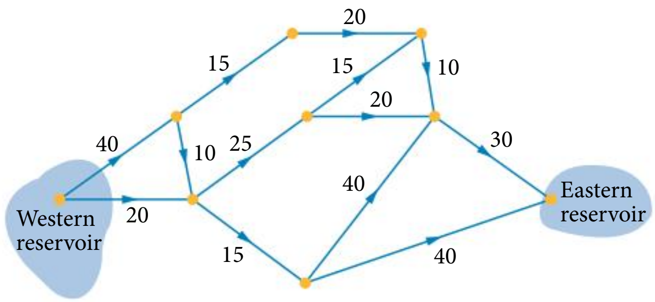
4 The number of megabytes (MB) per second of information that can pass between computers in an office network is shown at right. Worked Example 6

Using the minimum-cut method, determine the maximum amount of data that can flow every second from the manager's computer to the assistant manager's computer in this network.



5 The number of megalitres (ML) of water that can travel along various sections of an irrigation channel network per day is shown in the diagram.

Use the minimum-cut method to determine the maximum daily flow from the western reservoir to the eastern reservoir.



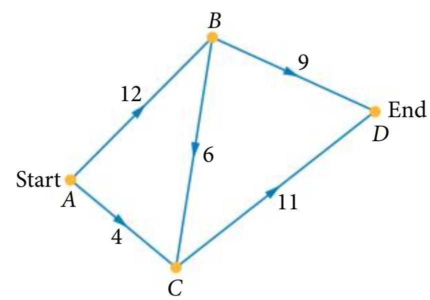
6 The network shows the maximum flow in several pipes, from vertex A to vertex D.

(a) Determine the maximum flow in the network.

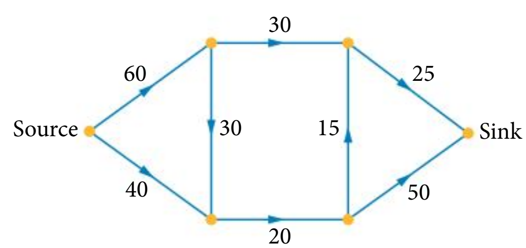
A 13 units B 21 units

C 16 units D 20 units

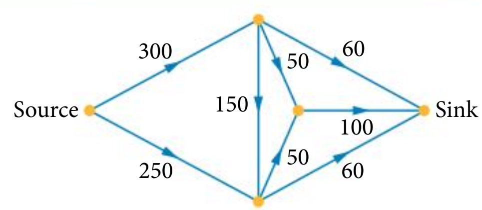
(b) Which options could have been eliminated simply by considering the flow from the pipes leaving directly from the start?



7 Determine the maximum flow from source to sink for the network shown here.



8 Use the many paths method to determine the maximum total flow from source to sink in the network. 5



9 The diagram shows maximum flow rates in individual pipes in Smallville's stormwater network. Several cuts are also shown.

(a) What are the values of cut 1, cut 2 and cut 3, respectively?

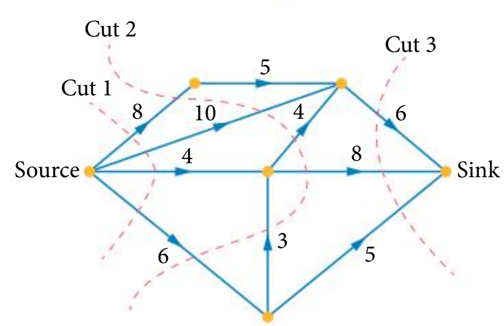
A 4, 4 and 5 B 28, 36 and 19

C 28, 33 and 19 D 28, 39 and 19

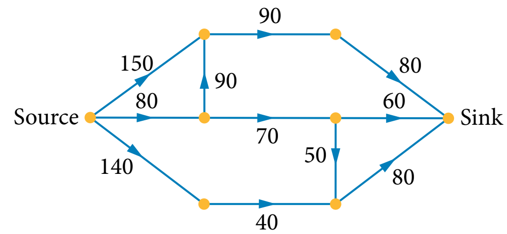
(b) What is the maximum possible flow, through the network?

A 16 B 17

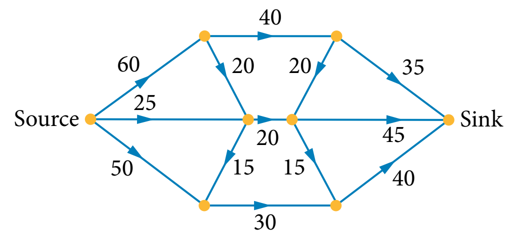
C 18 D 19



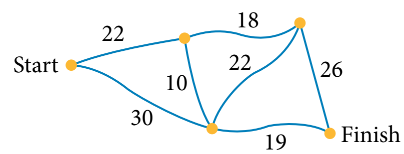
10 Determine the maximum total flow from the source to the sink for the network.



11 Determine the maximum total flow from the source to the sink in the network.



12 The diagram represents a network of water pipes.



(a) Determine the maximum flow through the network in litres per minute.

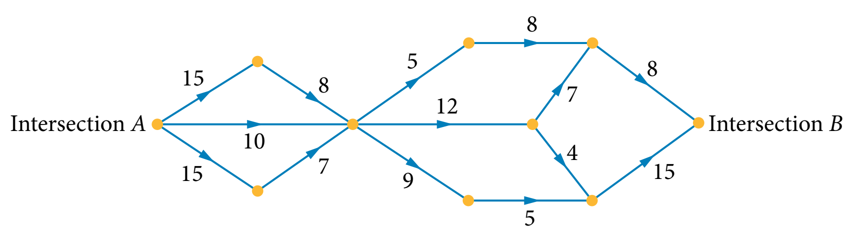
(b) The city engineer has decided that only *one* pipe can be upgraded to take more water. Which pipe should be chosen in order to increase the maximum flow as much as possible?

(c) By how much should the capacity of this pipe be changed?

(d) Determine the new maximum flow after the change has been made.

13 The number of vehicles (measured in 1000s) per hour that can travel along major roads between two intersections *A* and *B* in a city during peak times are shown in the traffic network diagram.

Currently 10 000 vehicles per hour travel between these points during peak times.



(a) The number of vehicles using these roads is expected to grow. Theoretically, how many more vehicles could travel between intersections *A* and *B* than currently do at peak times?

(b) Suggest an upgrade to one particular section of the road that will allow the maximum traffic flow to increase to 20 000 at peak times.

8.3 Allocation problems

Allocation problems

Allocation problems involve assigning or allocating several jobs to several people (with each person being assigned one job), in a way that minimises or maximises a quantity of interest.

In simple cases, it may be possible to allocate jobs 'by inspection', as in the following example.

Matrix form

A matrix is a 'cut down' version of a table containing rows (and columns) of values. Matrices can be used to save time when dealing with tables of values. For example, the table at right shows the amount charged, in dollars, by three people to do three jobs in the process of manufacturing a set of toys.

	Illustrations	Jigsaws	Kites
Amy	9	10	20
Brad	10	15	25
Cam	12	12	15

This table may be written in matrix form as: $\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} I & J & K \end{array} \\ \begin{bmatrix} 9 & 10 & 20 \\ 10 & 15 & 25 \\ 12 & 12 & 15 \end{bmatrix} \end{array}$ or even more simply as: $\begin{bmatrix} 9 & 10 & 20 \\ 10 & 15 & 25 \\ 12 & 12 & 15 \end{bmatrix}$.

In this situation, the aim is to determine the minimum (or maximum) amount that can be charged, depending on how the tasks are allocated.

Solving by inspection

The minimum cost allocation can be calculated by inspection.

7 Determine the minimum cost allocation by inspection

The table lists the costs in dollars of three people doing three jobs in the process of manufacturing a set of toys.

Calculate the total (minimum) cost by the inspection method.

	Illustrations	Jigsaws	Kites
Amy	9	10	20
Brad	10	15	25
Cam	12	12	15

THINKING

1 Write the table in matrix form and identify the minimum cost for each job.

WORKING

$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} I & J & K \end{array} \\ \begin{bmatrix} \textcircled{9} & \textcircled{10} & 20 \\ 10 & 15 & 25 \\ 12 & 12 & \textcircled{15} \end{bmatrix} \end{array}$

2 Make sure only one job is assigned to each person.	Amy cannot do two jobs, so either Illustrations or Jigsaws must be assigned to Brad. $\begin{matrix} & I & J & K \\ A & \textcircled{9} & \textcircled{10} & 20 \\ B & \textcircled{10} & 15 & 25 \\ C & 12 & 12 & \textcircled{15} \end{matrix}$ or $\begin{matrix} & I & J & K \\ A & \textcircled{9} & \textcircled{10} & 20 \\ B & 10 & \textcircled{15} & 25 \\ C & 12 & 12 & \textcircled{15} \end{matrix}$
3 Compare the options.	The first option would increase costs by \$1, while the second option would increase costs by \$5, so choose the first option.
4 Write the allocation.	Illustrations by Brad will cost \$10. Jigsaws by Amy will cost \$10. Kites by Cam will cost \$15.
5 Add the costs to find the total (minimum) cost and interpret the result.	\$10 + \$10 + \$15 = \$35 The minimum cost for the set is \$35 and can be achieved by assigning Kites to Cam, Illustrations to Brad and Jigsaws to Amy.

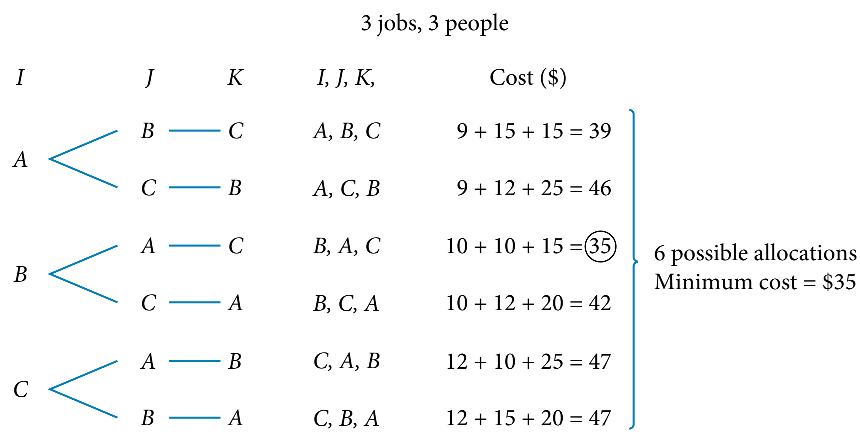
Determine the minimum cost allocation by testing all options

Another way to determine the allocation that results in the minimum total cost is to list all possible allocations, and add the costs for each case. For three jobs and three people there are six possible allocations.

A way of systematically listing the possibilities is to use a tree diagram. All possible allocations of jobs from the previous example are listed in the following tree diagram.

You can see that minimum total cost is attained when jobs are assigned as shown here.

	Illustrations	Jigsaws	Kites
A	9	[10]	20
B	[10]	15	25
C	12	12	[15]



Allocation using the Hungarian algorithm

The Hungarian algorithm is a systematic method of solving more complex allocation problems.

8 The Hungarian algorithm

Use the Hungarian algorithm to determine an allocation of jobs that minimises the total cost of producing the toy set.

	Illustrations	Jigsaws	Kites
Amy	9	10	20
Brad	10	15	25
Cam	12	12	15

THINKING | WORKING

1 Write the information in a matrix.

$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} I & J & K \end{array} \\ \begin{bmatrix} 9 & 10 & 20 \\ 10 & 15 & 25 \\ 12 & 12 & 15 \end{bmatrix} \end{array}$$

2 Perform a *row reduction*, where the minimum value in each row is subtracted from each value in the row.

$$\begin{bmatrix} 9 & 10 & 20 \\ 10 & 15 & 25 \\ 12 & 12 & 15 \end{bmatrix} \begin{array}{c} -9 \\ -10 \\ -12 \end{array} \rightarrow \begin{bmatrix} 0 & 1 & 11 \\ 0 & 5 & 15 \\ 0 & 0 & 3 \end{bmatrix}$$

3 Perform a *column reduction*, where the minimum value in each column is subtracted from each value in the column.

$$\begin{bmatrix} 0 & 1 & 11 \\ 0 & 5 & 15 \\ 0 & 0 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 8 \\ 0 & 5 & 12 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-0 \quad -0 \quad -3$$

4 Determine the minimum number of horizontal and vertical lines required to cover the zeros. If you can cover the zeros with fewer lines than there are columns, circle the smallest uncovered value.

$$\begin{bmatrix} 0 & \textcircled{1} & 8 \\ 0 & 5 & 12 \\ 0 & 0 & 0 \end{bmatrix}$$

5 Subtract the minimum uncovered number from all uncovered values and add it to the intersection of the lines.

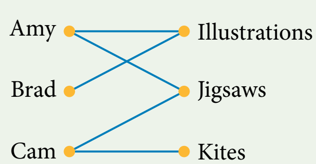
$$\begin{bmatrix} 0 & 0 & 7 \\ 0 & 4 & 11 \\ 1 & 0 & 0 \end{bmatrix}$$

6 Determine the minimum number of horizontal and vertical lines required to cover the zeros.

$$\begin{bmatrix} 0 & 0 & 7 \\ 0 & 4 & 11 \\ 1 & 0 & 0 \end{bmatrix}$$

Given that 3 lines are required to cover the zeros in the 3×3 matrix, the allocation can begin.

7 Draw a bipartite graph demonstrating the allocation of zeros.



8 Allocate the tasks either by identifying the rows/columns with only a single 0, or starting with a vertex with degree 1 in the bipartite graph.

$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} I & J & K \end{array} \\ \begin{bmatrix} 9 & \boxed{10} & 20 \\ \boxed{10} & 15 & 25 \\ 12 & 12 & \boxed{15} \end{bmatrix} \end{array}$$

9 Interpret the answer.	The minimum cost to produce the set of toys is (10 + 10 + 15) = $35 and can be achieved by assigning Kites to Cam, Illustrations to Brad and Jigsaws to Amy.

Consider the following example of a more complex allocation problem that involves four jobs and four people.

Amando, Borat, Chrissie and Digby are interior designers who have been hired to renovate one room each in a home. They each quote the following costs, in thousands of dollars, for refurbishing the four rooms. In order to finish the renovation in time, the home owner must assign one room to each renovator.

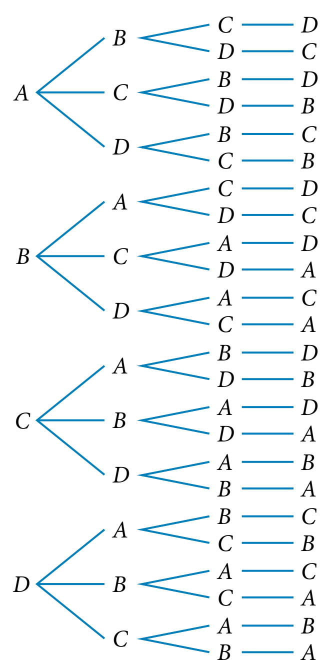
	Indoor spa	Junior's bedroom	Kitchen	Living room
Amando	15	16	21	16
Borat	16	20	15	25
Chrissie	25	22	16	22
Digby	15	23	18	17

If the home owner wishes to minimise costs, which person should be assigned to each room?

The 24 possible listings are shown below.

4 jobs, 4 people

Job *I* Job *J* Job *K* Job *L*



Jobs *I, J, K, L*	Cost ($'000)
A, B, C, D	15 + 20 + 16 + 17 = 68
A, B, D, C	15 + 20 + 18 + 22 = 75
A, C, B, D	15 + 22 + 15 + 17 = 69
A, C, D, B	15 + 22 + 18 + 25 = 80
A, D, B, C	15 + 23 + 15 + 22 = 75
A, D, C, B	15 + 23 + 16 + 25 = 79
B, A, C, D	16 + 16 + 16 + 17 = (65)
B, A, D, C	16 + 16 + 18 + 22 = 72
B, C, A, D	16 + 22 + 21 + 17 = 76
B, C, D, A	16 + 22 + 18 + 16 = 72
B, D, A, C	16 + 23 + 21 + 22 = 82
B, D, C, A	16 + 23 + 16 + 16 = 71
C, A, B, D	25 + 16 + 15 + 17 = 73
C, A, D, B	25 + 16 + 18 + 25 = 84
C, B, A, D	25 + 20 + 21 + 17 = 83
C, B, D, A	25 + 20 + 18 + 16 = 79
C, D, A, B	25 + 23 + 21 + 25 = 94
C, D, B, A	25 + 23 + 15 + 16 = 79
D, A, B, C	15 + 16 + 15 + 22 = 68
D, A, C, B	15 + 16 + 16 + 25 = 72
D, B, A, C	15 + 20 + 21 + 22 = 78
D, B, C, A	15 + 20 + 16 + 16 = 67
D, C, A, B	15 + 22 + 21 + 25 = 83
D, C, B, A	15 + 22 + 15 + 16 = 68

24 possible allocations
Minimum cost = $65 000

You can see that minimum total cost is attained when jobs are assigned as highlighted here.

	Indoor spa	Junior's bedroom	Kitchen	Living room
Amando	15	[16]	21	16
Borat	[16]	20	15	25
Chrissie	25	22	[16]	22
Digby	15	23	18	[17]

9 Allocation to minimise

Amando, Borat, Chrissie and Digby are interior designers who have been hired to renovate one room each in a home. They each quote the following costs, in thousands of dollars, for refurbishing the four rooms. The home owner must assign one room to each renovator.

	Indoor spa	Junior's bedroom	Kitchen	Living room
Amando	15	16	21	16
Borat	16	20	15	25
Chrissie	25	22	16	22
Digby	15	23	18	17

Use the Hungarian algorithm to determine an allocation of jobs that minimises the total cost of the renovation. State the minimum cost of the renovation.

THINKING / WORKING

1 Display the data as a matrix with appropriate headings.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} I & J & K & L \end{array} \\ \begin{bmatrix} 15 & 16 & 21 & 16 \\ 16 & 20 & 15 & 25 \\ 25 & 22 & 16 & 22 \\ 15 & 23 & 18 & 17 \end{bmatrix} \end{array}$$

2 Perform a row reduction, by subtracting the minimum value in each row from each value in the row.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} I & J & K & L \end{array} \\ \begin{bmatrix} 15 & 16 & 21 & 16 \\ 16 & 20 & 15 & 25 \\ 25 & 22 & 16 & 22 \\ 15 & 23 & 18 & 17 \end{bmatrix} \end{array}\begin{array}{c} -15 \\ -15 \\ -16 \\ -15 \end{array} \rightarrow \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} I & J & K & L \end{array} \\ \begin{bmatrix} 0 & 1 & 6 & 1 \\ 1 & 5 & 0 & 10 \\ 9 & 6 & 0 & 6 \\ 0 & 8 & 3 & 2 \end{bmatrix} \end{array}$$

3 Perform a column reduction, by subtracting the minimum value in each column from each value in the column.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} I & J & K & L \end{array} \\ \begin{bmatrix} 0 & 1 & 6 & 1 \\ 1 & 5 & 0 & 10 \\ 9 & 6 & 0 & 6 \\ 0 & 8 & 3 & 2 \end{bmatrix} \\ \begin{array}{cccc} -0 & -1 & -0 & -1 \end{array} \end{array} \rightarrow \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} I & J & K & L \end{array} \\ \begin{bmatrix} 0 & 0 & 6 & 0 \\ 1 & 4 & 0 & 9 \\ 9 & 5 & 0 & 5 \\ 0 & 7 & 3 & 1 \end{bmatrix} \end{array}$$

4 Determine the minimum number of horizontal and vertical lines required to cover the zeros. If you can cover all zeros with fewer lines than there are rows, circle the smallest uncovered number.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} I & J & K & L \end{array} \\ \begin{bmatrix} 0 & 0 & 6 & 0 \\ \textcircled{1} & 4 & 0 & 9 \\ 9 & 5 & 0 & 5 \\ 0 & 7 & 3 & 1 \end{bmatrix} \end{array}$$

5 Subtract the minimum uncovered number from each uncovered number and add it to each number where the lines intersect.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} I & J & K & L \end{array} \\ \begin{bmatrix} 0 & 0 & 7 & 0 \\ 0 & 3 & 0 & 8 \\ 8 & 4 & 0 & 4 \\ 0 & 7 & 4 & 1 \end{bmatrix} \end{array}$$

6 Determine the minimum number of horizontal and vertical lines required to cover the zeros. If you can cover all zeros with fewer lines than there are rows, circle the smallest uncovered number.

	I	*J*	*K*	*L*
A	0	0	7	0
B	0	3	0	8
C	8	4	0	4
D	0	7	4	(1)

7 Subtract the minimum uncovered number from each uncovered number and add it to each number where the lines intersect.

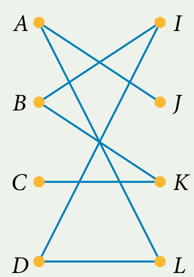
	I	*J*	*K*	*L*
A	1	0	8	0
B	0	2	0	7
C	8	3	0	3
D	0	6	4	0

8 Determine the minimum number of horizontal and vertical lines required to cover the zeros. If the minimum number of lines is equal to the number of rows, then you can start allocating.

	I	*J*	*K*	*L*
A	1	0	8	0
B	0	2	0	7
C	8	3	0	3
D	0	6	4	0

Given that 4 lines are required to cover the zeros in the 4×4 matrix, the allocation can begin.

9 Create a bipartite graph connecting people with jobs, based on the zeros.



10 Allocate jobs based on zeros. You may use a bipartite graph and then a table, or just a table.

Start with any row containing just one zero:

	I	*J*	*K*	*L*
A	1	**0**	8	**0**
B	**0**	2	**0**	7
C	8	3	[**0**]	3
D	**0**	6	4	**0**

Chrissie must do the kitchen.

	I	*J*	*K*	*L*
A	1	**0**	8	**0**
B	[**0**]	2	**0**	7
C	8	3	[**0**]	3
D	**0**	6	4	**0**

Borat cannot do the kitchen so he must do the indoor spa, leaving Digby to do the lounge and Amando to do Junior's bedroom.

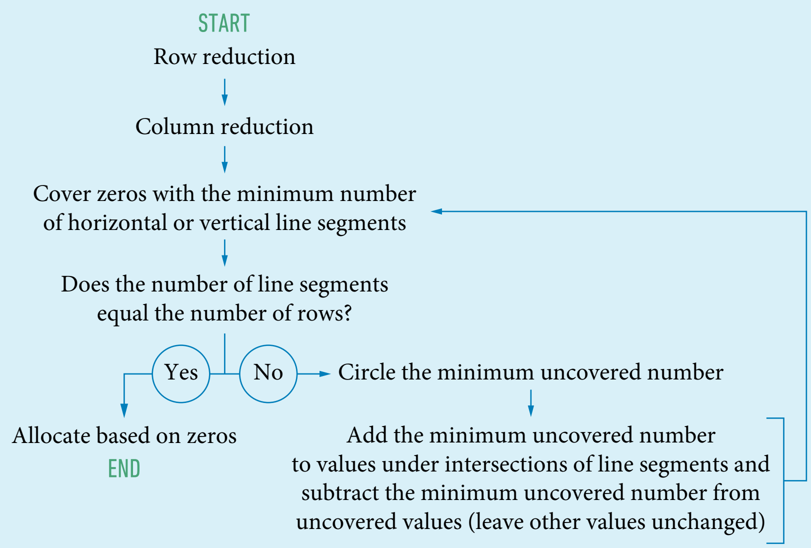
	Task *I*	Task *J*	Task *K*	Task *L*	Total cost
Person	*B*	*A*	*C*	*D*	
Cost ($'000s)	16	16	16	17	**$65 000**

The Hungarian algorithm may only be used on *square* matrices, or tables with equal numbers of rows and columns.

The flowchart outlines the steps in solving allocation problems using the Hungarian algorithm.

In some cases, there may be two choices of person for a particular job, and either person may be chosen, as both choices will produce the same minimum cost.

Hungarian algorithm summary



Allocating to maximise

If required to determine an allocation that *maximises* an overall quantity (e.g. a score or income), you must create a complementary matrix before continuing with the Hungarian algorithm.

For a complementary matrix, replace each element in the matrix with 'Largest value in matrix *minus* the element value'.

10 Constructing a complementary matrix

For the matrix, construct a complementary matrix in terms of the maximum element in the matrix.

$$\begin{bmatrix} 15 & 16 & 21 & 16 \\ 16 & 20 & 15 & 25 \\ 25 & 22 & 16 & 22 \\ 15 & 23 & 18 & 17 \end{bmatrix}$$

THINKING	WORKING
1 Identify the maximum element in the matrix.	$\begin{bmatrix} 15 & 16 & 21 & 16 \\ 16 & 20 & 15 & \mathbf{25} \\ \mathbf{25} & 22 & 16 & 22 \\ 15 & 23 & 18 & 17 \end{bmatrix}$ The maximum value in the matrix is 25.

2 Subtract each element from the maximum element.

Complementary matrix:

$$\begin{bmatrix} \mathbf{25}-15 & \mathbf{25}-16 & \mathbf{25}-21 & \mathbf{25}-16 \\ \mathbf{25}-16 & \mathbf{25}-20 & \mathbf{25}-15 & \mathbf{25}-25 \\ \mathbf{25}-25 & \mathbf{25}-22 & \mathbf{25}-16 & \mathbf{25}-22 \\ \mathbf{25}-15 & \mathbf{25}-23 & \mathbf{25}-18 & \mathbf{25}-17 \end{bmatrix} = \begin{bmatrix} 10 & 9 & 4 & 9 \\ 9 & 5 & 10 & 0 \\ 0 & 3 & 9 & 3 \\ 10 & 2 & 7 & 8 \end{bmatrix}$$

3 Use the Hungarian algorithm to make the assignments.

Row reduction:

$$\begin{bmatrix} 10 & 9 & 4 & 9 \\ 9 & 5 & 10 & 0 \\ 0 & 3 & 9 & 3 \\ 10 & 2 & 7 & 8 \end{bmatrix} \begin{matrix} -4 \\ -0 \\ -0 \\ -2 \end{matrix} \rightarrow \begin{bmatrix} 6 & 5 & 0 & 5 \\ 9 & 5 & 10 & 0 \\ 0 & 3 & 9 & 3 \\ 8 & 0 & 5 & 6 \end{bmatrix}$$

From the row reduction there is already a 0 in each column.

4 Assign the tasks in the complementary matrix.

$$\begin{bmatrix} 6 & 5 & \boxed{0} & 5 \\ 9 & 5 & 10 & \boxed{0} \\ \boxed{0} & 3 & 9 & 3 \\ 8 & \boxed{0} & 5 & 6 \end{bmatrix}$$

5 Transfer the allocations to the original matrix and determine the maximum value.

$$\begin{bmatrix} 15 & 16 & \boxed{21} & 16 \\ 16 & 20 & 15 & \boxed{25} \\ \boxed{25} & 22 & 16 & 22 \\ 15 & \boxed{23} & 18 & 17 \end{bmatrix}$$

The maximum value of the allocation within the matrix is $25 + 23 + 21 + 25 = 94$.

EXERCISE 8.3 Allocation problems

Worked Example 7

1 For each of the following 'jobs/people' tables (numbers represent cost in dollars):
- Write the given information in matrix form.
- Assign one job to each person so that the total cost is minimised.

(a)

	Job *I*	Job *J*	Job *K*
Ava	8	11	16
Blair	9	8	19
Colt	7	10	17

(b)

	Job *I*	Job *J*	Job *K*
Ana	80	35	20
Briar	50	40	15
Che	70	32	18

(c)

	Job *P*	Job *Q*	Job *R*
Xia	19	51	38
Yuri	16	55	40
Zac	21	59	33

(d)

	Job *I*	Job *J*	Job *K*	Job *L*
Art	30	50	65	85
Bart	25	45	80	80
Chiara	40	55	70	95
Drake	20	60	75	90

(e)

	Job I	Job J	Job K	Job L
Arun	85	43	100	225
Bodhi	53	49	143	257
Cate	58	48	121	209
Dee	89	59	149	245

(f)

	Job S	Job T	Job U	Job V
Phil	13	62	70	79
Quinton	44	74	93	64
Rob	19	67	88	76
Stuart	25	56	61	75

2 By calculating the cost of all possible allocations, determine the minimum cost allocation of jobs I, J and K for persons A, B and C, given their cost per hour (in dollars) in the matrix, for each case.

(a)
$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} I & J & K \end{array} \\ \begin{bmatrix} 7 & 12 & 14 \\ 10 & 9 & 19 \\ 8 & 11 & 17 \end{bmatrix} \end{array}$$

(b)
$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} I & J & K \end{array} \\ \begin{bmatrix} 20 & 15 & 19 \\ 18 & 14 & 17 \\ 16 & 10 & 16 \end{bmatrix} \end{array}$$

3 Determine the resultant matrix after a row reduction and column reduction.

(a) $\begin{bmatrix} 5 & 18 & 3 \\ 6 & 2 & 4 \\ 10 & 9 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 22 & 34 & 12 \\ 25 & 23 & 18 \\ 22 & 27 & 15 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & 6 & 12 & 23 \\ 0 & 5 & 17 & 25 \\ 4 & 7 & 19 & 21 \\ 8 & 2 & 11 & 20 \end{bmatrix}$ (d) $\begin{bmatrix} 44 & 71 & 25 & 14 \\ 32 & 72 & 25 & 18 \\ 65 & 80 & 25 & 13 \\ 25 & 83 & 25 & 15 \end{bmatrix}$

4 For each of the following matrices:

- Cover rows and columns containing zeros with as few horizontal and vertical lines as possible.
- Subtract the minimum uncovered number from every uncovered number.
- Add the minimum uncovered number to numbers that are covered twice.
- Write the resulting matrix as your answer.

(a) $\begin{bmatrix} 5 & 2 & 0 \\ 0 & 0 & 4 \\ 8 & 7 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 25 & 22 \\ 36 & 0 & 0 \\ 0 & 16 & 18 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 10 & 7 & 0 \\ 0 & 16 & 3 & 7 \\ 0 & 12 & 5 & 0 \\ 11 & 0 & 0 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & 6 & 6 & 0 \\ 2 & 0 & 0 & 0 \\ 8 & 8 & 1 & 0 \\ 4 & 3 & 1 & 0 \end{bmatrix}$

5 The Hungarian algorithm results in the matrices shown below.

Create a bipartite graph connecting people (M, N, O, P) with jobs (W, X, Y, Z), based on zeros.

(a)
$$\begin{array}{c} \\ M \\ N \\ O \\ P \end{array} \begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{bmatrix} 4 & 0 & 0 & 4 \\ 0 & 9 & 16 & 0 \\ 0 & 0 & 12 & 13 \\ 10 & 0 & 27 & 6 \end{bmatrix} \end{array}$$

(b)
$$\begin{array}{c} \\ M \\ N \\ O \\ P \end{array} \begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{bmatrix} 120 & 200 & 0 & 70 \\ 312 & 0 & 30 & 0 \\ 180 & 90 & 0 & 0 \\ 0 & 0 & 250 & 210 \end{bmatrix} \end{array}$$

(c)
$$\begin{array}{c} \\ M \\ N \\ O \\ P \end{array} \begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{bmatrix} 8 & 0 & 5 & 0 \\ 6 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 5 & 7 & 2 \end{bmatrix} \end{array}$$

(d)
$$\begin{array}{c} \\ M \\ N \\ O \\ P \end{array} \begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{bmatrix} 0 & 3 & 0 & 6 \\ 4 & 0 & 2 & 3 \\ 0 & 0 & 9 & 2 \\ 2 & 7 & 1 & 0 \end{bmatrix} \end{array}$$

6 The Hungarian algorithm has been applied to a particular matrix showing the cost of persons A, B, C and D performing tasks I, J, K and L. The matrix shown at right is obtained.

Based on this matrix and using the Hungarian algorithm method, which assignment produces the lowest total cost?

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} I & J & K & L \end{array} \\ \begin{bmatrix} 8 & 6 & 0 & 5 \\ 2 & 0 & 0 & 7 \\ 0 & 3 & 4 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

A

Task	*I*	*J*	*K*	*L*
Person	*A*	*D*	*B*	*C*

B

Task	*I*	*J*	*K*	*L*
Person	*B*	*C*	*D*	*A*

C

Task	*I*	*J*	*K*	*L*
Person	*A*	*D*	*C*	*B*

D

Task	*I*	*J*	*K*	*L*
Person	*C*	*B*	*A*	*D*

7 The Overloch surf club enters a team of three in a beach triathlon involving three stages: running, swimming and paddle boarding. Each team member must compete in one stage only. The table shows the team members' average placing in previous competitions.

Team member	Run	Swim	Paddle boarding
Gus	8	11	4
Hong	10	15	11
Isaac	24	21	22

Allocate each team member to a stage of the event most likely to produce the best total placing, based on previous results.

Worked Example 10

8 The following matrices represent allocation problems where maximum profit is being determined. Because the processes being used produce minimum values, construct complementary matrices based on the maximum element in each matrix.

(a) $\begin{bmatrix} 15 & 13 & 8 \\ 14 & 12 & 4 \\ 15 & 14 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 1 & 1 \\ 7 & 10 & 5 \\ 4 & 6 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 21 & 12 & 3 & 16 \\ 3 & 4 & 11 & 13 \\ 5 & 6 & 12 & 14 \\ 7 & 6 & 21 & 13 \end{bmatrix}$

8

9 An allocation problem involves three people A, B and C who must each be assigned one of three jobs I, J and K. The initial matrix for this problem shows, for example, that person B charges $19 to complete job J.

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} I & J & K \end{array} \\ \begin{bmatrix} 11 & 24 & 6 \\ 15 & 19 & 8 \\ 15 & 17 & 14 \end{bmatrix} \end{array}$$

(a) How many zeros will the matrix contain after row reduction and column reduction?

A 2 **B** 3 **C** 4 **D** 5

(b) Calculate the total minimum cost.

9

10 A parent is seeking tutors for a weekly session in four different subjects for her son, who is studying Year 12. She checks out the hourly rate of four tutoring businesses for the four subjects. The rates, in dollars, are listed in the table. The parent wishes to share the tutoring jobs among the four businesses.

	Maths	Physics	Chemistry	Biology
A Plus	49	41	58	81
Best Results	43	50	55	67
Clever Kids	43	59	65	92
Directed Study	49	43	71	82

Use the Hungarian algorithm method to find the minimum total weekly cost.

11 Josiah has four jobs he needs done, and gets quotes from four home maintenance contractors for each of these jobs. The jobs and prices quoted, in dollars, are listed in the table.

Use the Hungarian algorithm to determine which contractor should be awarded which job if Josiah is to minimise his total costs. Determine the total cost.

	Mowing	Paving	Roof repair	Skylight installation
Ace Home Maintenance	50	145	70	350
Better Home Services	45	120	60	425
Cheaper Home Repairs	65	130	65	320
Dane's Home Help	55	150	80	395

12 Allison, Brian, Cody and Dirk are painters who have been hired to paint the interior of one unit each in a block of units. They quote the costs, in dollars, shown in the table. In order to get the job done in time, the builder must assign one unit to each painter. Use the Hungarian algorithm to determine the answer.

If the builder wishes to minimise his costs, which painter should he assign to each unit? Determine the total cost.

	Unit 1	Unit 2	Unit 3	Unit 4
Allison	980	800	1190	740
Brian	1080	940	1100	930
Cody	860	830	900	1040
Dirk	960	830	910	1120

13 Dim employs four tradies (Achilles, Bacchus, Cali and Daman) to travel to four locations ($L1$, $L2$, $L3$ and $L4$). The distance, in kilometres, that each of Dim's tradies must travel to each location is given in the table.

Using the Hungarian algorithm method, determine which tradie Dim should assign to each location if the total distance travelled by the tradies is to be minimised. State the total distance.

	$L1$	$L2$	$L3$	$L4$
Achilles	20	18	18	18
Bacchus	21	19	22	22
Cali	23	18	21	22
Daman	20	19	23	20

14 A taxi dispatcher must send one of four taxis ($T1$, $T2$, $T3$ and $T4$) to each of four suburbs (W, X, Y and Z). The estimated time, in minutes, it will take each taxi to reach each location is given in the table.

Which taxi should be sent to each location if the waiting time experienced by customers is to be minimised? Use the Hungarian algorithm method and state the total waiting time.

	W	X	Y	Z
$T1$	29	33	29	29
$T2$	28	31	31	32
$T3$	29	29	31	30
$T4$	29	32	33	33

15 The times, in minutes, taken by four workers (Adair, Beth, Catriona and Dalia) to complete four jobs are listed in the table. Using the Hungarian algorithm method, assess which worker should be assigned to each job if the total time taken to complete the four jobs is to be minimised. State the minimum total time.

	Job 1	Job 2	Job 3	Job 4
Adair	73	72	75	87
Beth	89	86	79	89
Catriona	76	78	70	83
Dalia	77	77	79	79

16 Mr Holder must decide the order in which his relay team members will run in an inter-school competition. He records the times run by the team members in various positions during practice. The average times, in seconds, for each member are shown in the table.

Based on practice times and using the Hungarian algorithm method, in which order should Mr Holder place his relay team?

	Runners			
	1	2	3	4
Manny	32	33	35	33
Norman	40	30	33	39
Oscar	34	39	38	39
Percy	33	36	38	31

17 A concert promoter must organise one concert for each of four bands, in one of four Australian cities: Melbourne (M), Sydney (S), Brisbane (B) or Perth (P). Each city is to host one concert only. The estimated cost, in thousands of dollars, of accommodation for each band at their preferred hotel in each city is given in the table.

Using the Hungarian algorithm method, assess which band should play in each city, if accommodation costs are to be minimised. Determine the total cost of accommodation for the bands.

	M	S	B	P
Yabba Dabba	48	66	54	27
Pink Kinks	48	61	58	26
Bowling Pebbles	39	72	43	29
Stratus Crow	42	58	41	26

18 A manufacturing company sells and ships its products to five countries. The company has five shipping containers, each filled with a different type of product. The company wishes to send one container to each country to clear its inventory as quickly as possible. The amount the company can sell each shipment for depends on the country. The amounts for which each product can be sold are given in the table.

Which country should the company sell each container to if it is to maximise its sales revenue, and what is the maximum total sales revenue?

	Country				
	A	*B*	*C*	*D*	*E*
Container *U*	2	3	3	5	4
Container *V*	4	5	7	4	6
Container *W*	6	5	4	5	6
Container *X*	5	7	8	6	6
Container *Y*	4	6	5	3	5

19 Hija is a freelance political journalist who works for four national newspapers: *Unheralded News* (UN), *Age Old News* (AON), *Daily Telepath* (DT) and *Sunday Mews* (SM). These newspapers insist on exclusive articles, so he only ever sells each article to one newspaper. He has written four articles on the topics: Refugees, Gay marriage, Climate change and Education reform. The four newspapers offer him the amounts, in dollars, listed in the table.

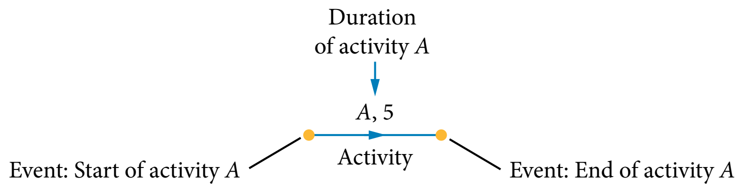
	UN	AON	DT	SM
Refugees	950	600	900	1500
Gay marriage	850	650	900	1450
Climate change	800	900	900	1350
Education reform	800	600	850	1350

Which article should Hija sell to each newspaper if he is to maximise his earnings for the four articles, and what payment will he receive for each article? What is the total payment?

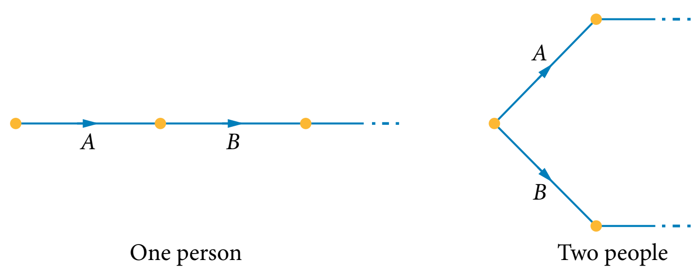
8.4 Project networks

Projects

A manufacturing process, a building project and planning a concert may each be described in terms of a directed graph. Such 'digraphs' are called project networks, and are drawn so each arc (directed edge) represents a particular activity (usually labelled with a capital letter). Vertices are called events, and represent the start or end of an activity. Activities that point towards other activities are called predecessors, and must be completed before the activity they point to may be started. The duration of each activity is usually written with its label in the middle of the edge representing the activity.



The appearance of a project network may depend on how many people are available to work on its various activities. For example, one person may not be able to work on activities simultaneously, but two people may be able to do so.



The starting time for any activity must allow for all predecessor activities to finish.

For example, the concrete around the base of a fence post must have enough time to set before rails are hammered onto it. A cake must have enough time to bake and cool before it is iced.

11 Constructing a project network diagram

Each work morning, Dave gets dressed (5 minutes), cooks breakfast (12 minutes) and eats breakfast (6 minutes) while listening to 15 minutes of music before he brushes his teeth (2 minutes).

(a) Tabulate the activities, giving times (minutes) and immediate predecessors for each.

THINKING

1 List and label each activity.

List the label of the immediate predecessor(s), if any.

Write the duration of each activity.

WORKING

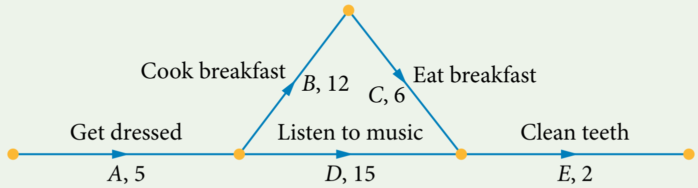
Activity	Immediate predecessors	Duration (minutes)
A: get dressed	–	5
B: cook breakfast	*A*	12
C: eat breakfast	*B*	6
D: listen to music	*A*	15
E: clean teeth	*C*, *D*	2

(b) Construct a project network diagram to describe Dave's morning routine.

1 Place the activities in the order in which they occur.

Branch the network if activities occur simultaneously.

2 Write the labels, or descriptions, on each arc, followed by the duration in minutes.



Earliest starting time (EST) and latest starting time (LST)

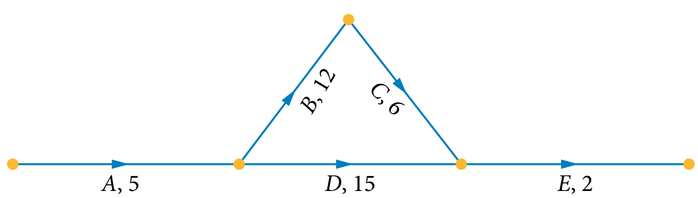
The earliest starting time, or EST, is the earliest time an activity can begin, after all predecessors have been completed. The earliest starting time for the beginning of a project is usually denoted as 0.

The latest starting time, or LST, is the latest time an activity can begin without delaying the project (i.e. without delaying the earliest completion time).

The earliest starting time (EST) and latest starting time (LST) for each activity may be found from the network graph or table.

12 EST, LST and minimum completion time

Use the project network diagram to determine the earliest starting time and latest starting time for each activity. Summarise your results in a table.

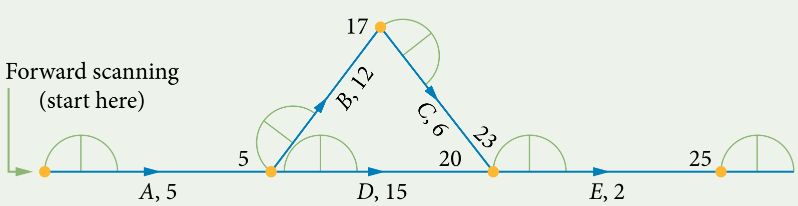


THINKING

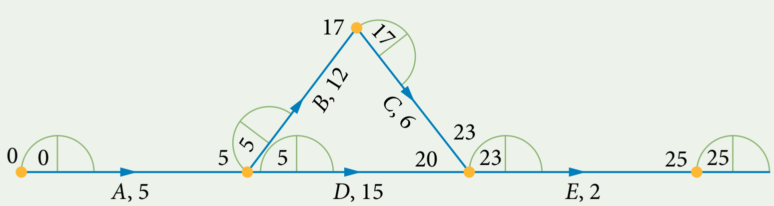
1 Draw curved 'double boxes' at the start of each activity, and at the end.

Determine the time to reach each vertex. Write these values at each vertex.

WORKING



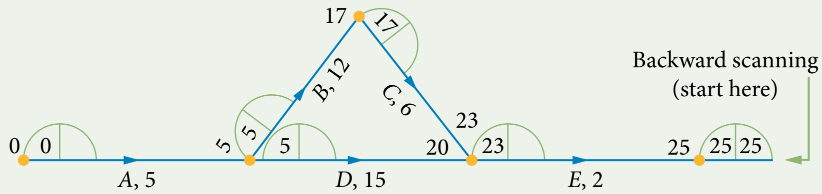
2 Write the largest of the vertex values in the left of each curved box. These are the EST values.



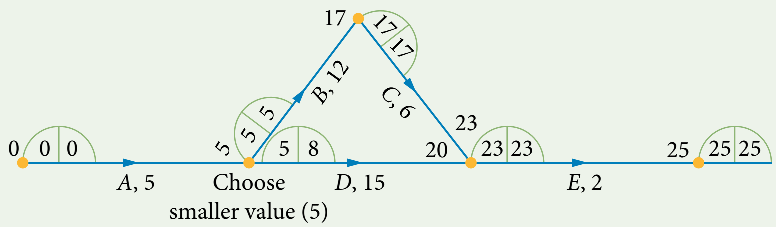
3 Interpret the values of the forward scan.

Activity *A* (getting dressed) has no predecessor and hence has an EST (earliest starting time) of 0. Dave cannot clean his teeth (activity E) until the following paths of activities are complete: *A-B-C* (dressed, cooked and eaten breakfast) = 23 minutes, *A-D* (dressed and listened to music) = 20 minutes after the project started. Therefore the EST for activity *E* is 23 minutes after Dave starts getting dressed (project commencement).

4 Copy the final EST value to the right side of the last box.



5 Complete a backward scan by subtracting the activity duration times along each path, writing the reduced value in each right box as you go. When working back from a vertex that has two right box values, use the smaller value to compute the next one.



6 Interpret the values of the backward scan.

The project can be completed in 25 minutes. Activity *E* takes 2 minutes, meaning that in order for the project to be completed (Dave to be ready) in 25 minutes, he must start brushing his teeth (activity *E*) 23 minutes after the project commences (i.e. when he starts getting ready).

From above, *B-C* = 18 minutes and *D* = 15 minutes, meaning that it takes Dave 3 minutes longer to cook and eat his breakfast than it does to listen to 15 minutes of music. Therefore the LST to start listening to music (activity *D*) is 3 minutes after the commencement of activity *B* (cooking his breakfast).

7 Tabulate the EST and LST values for each activity.

Activity	EST	LST
A	0	0
B	5	5
C	17	17
D	5	8
E	23	23

The earliest starting time for the activities indicate the earliest time that Dave can begin the activity to allow him to achieve what he needs to in a logical order.

The latest starting time indicates the latest possible starting time for Dave to start the task and still finish the project in the required time.

Critical activities

The *critical* activities for which EST = LST are shown in red in the table.

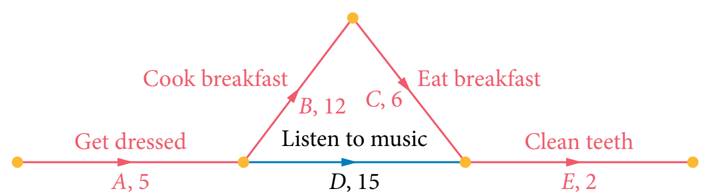
Activity	EST	LST
A Get dressed	0	0
B Cook breakfast	5	5
C Eat breakfast	17	17
D Listen to music	5	8
E Clean teeth	23	23

If an activity has the same earliest starting time and latest starting time, it is said to be *critical*.

The *critical path* in a network is the *longest* path and includes all critical activities.

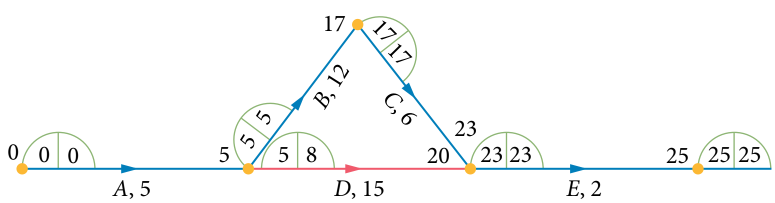
The critical path for Dave's morning ritual is shown in red.

Critical path analysis is important as it is helpful in deciding which activities to delay, or which activities to complete more quickly, when presented with various options.



Critical path and float times

The critical path for Dave's routine is shown in blue.



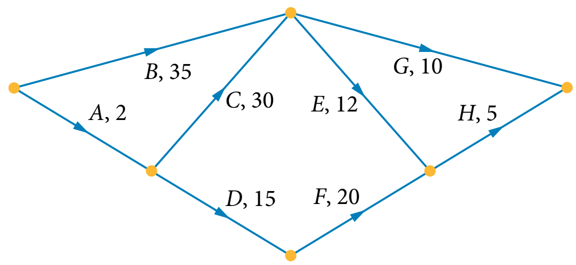
The only non-critical activity is 'Listen to music', with a float time of 8 − 5 = 3 minutes. The start of the activity 'listen to music' could be delayed by up to 3 minutes without affecting the finishing time of the routine.

Float time = LST − EST

Critical activities have no float time.

13 Critical path using forward and backward scanning

A project network involving activities A to H is shown. The duration of each activity is given in minutes.

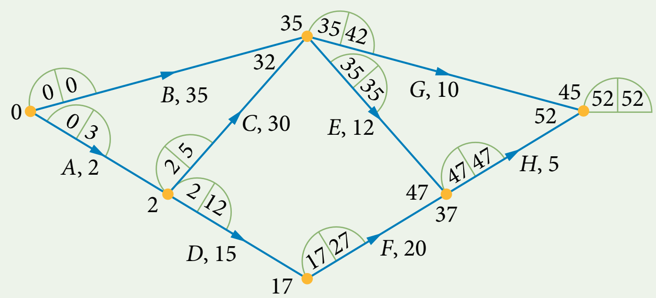


Using forward and backward scanning, identify the critical path and determine the minimum time in which the project may be completed. Summarise the findings in a table showing activity, duration, predecessors, EST, LST and float times.

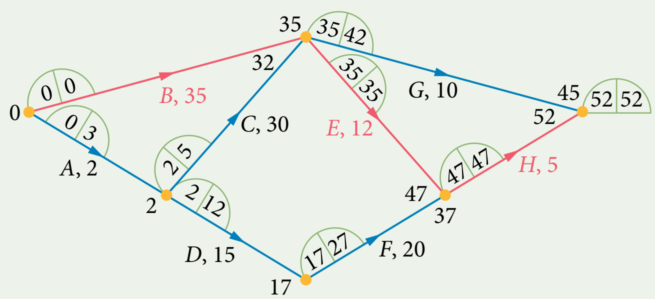
THINKING

WORKING

1 Use forward and backward scanning.



2 Activities that have equal EST and LST values are critical. Highlight the edges containing these activities.



3 List the critical activities, in order, that make up the critical path.

Critical path: *B-E-H*

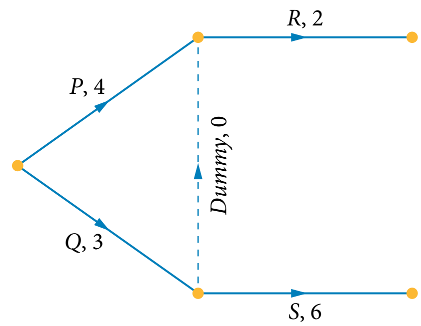
The minimum completion time is 52 minutes.

4 Use the values from the network diagram to complete the table.

Activity	Immediate predecessors	Duration (minutes)	EST	LST	Float
A	–	2	0	3	3
B	–	35	0	0	0
C	*A*	30	2	5	3
D	*A*	15	2	12	10
E	*B, C*	12	35	35	0
F	*D*	20	17	27	10
G	*B, C*	10	35	42	7
H	*E, F*	5	47	47	0

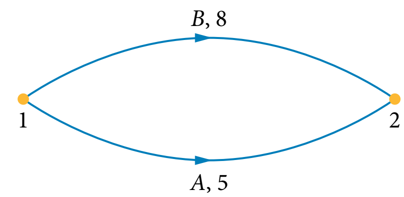
Dummy activities

A *dummy activity* is an activity that has zero duration. There are two main reasons for using dummy activities.



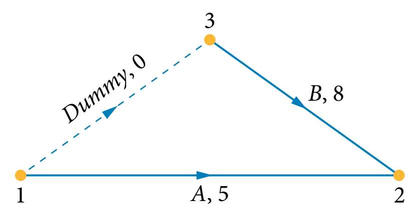
Reason 1: If one activity (R) has two immediate predecessors (P and Q), while another activity (S) has just the one immediate predecessor (Q), you can show all predecessors correctly if you use a 'dummy activity' of zero duration. Dummy activities are drawn with dashed lines, and are not usually listed when describing a critical path.

Notice how Q precedes R by no time, so it is an immediate predecessor of R, as required.



Reason 2: Multiple edges between two vertices are to be avoided in project networks. This is not a problem when you label activities using letters written near the middle of edges, but it causes problems when computer programs are used to describe activities, as these programs often identify an activity by the vertices at each end of an edge. If there are two edges drawn between two vertices, such a program cannot uniquely identify either of the activities.

For example, a program may label two vertices '1' and '2' respectively.

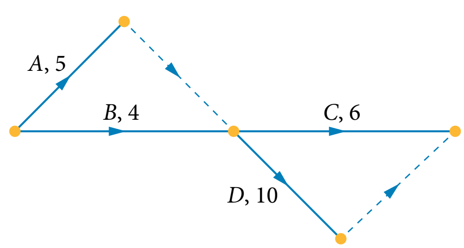


You can easily distinguish between the two activities by labelling them 'A' and 'B', but a computer program using only vertex labels would not be able to distinguish between the edges, as it would denote both as '1-2'. Using a dummy activity solves this problem.

Now a computer program using only vertex labels can distinguish between the edges, and denote edge A as '1-2', and edge B as '3-2'.

14 Network diagram containing a dummy activity

The dashed line segments in the project network shown here are dummy activities of zero duration. Activity durations are given in minutes.



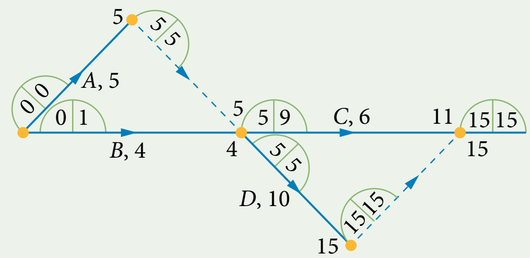
Determine the critical path of the network and state the minimum completion time. Summarise the findings in a table showing activity, duration, predecessors, EST, LST and float times.

THINKING

WORKING

1 Use forward and backward scanning. List the activities, in order, that have equal EST and LST values, ignoring the dummy activities.

Write the minimum completion time.



Critical path: *A-D*

The minimum completion time is 15 minutes.

2 Complete the summary table from the diagram. Note that *A* is an immediate predecessor of *D*, as the dummy activity connecting it to *D* has zero duration.

Activity	Immediate predecessors	Duration (minutes)	EST	LST	Float
A	–	5	0	0	0
B	–	4	0	1	1
C	*A*, *B*	6	5	9	4
D	*A*, *B*	10	5	5	0

15 Constructing a network diagram from a table

The following table describes a particular project network.

Activity	Duration (minutes)	Immediate predecessors
A	3	–
B	5	–
C	6	*A*, *B*
D	4	*B*
E	2	*C*
F	3	*C*, *D*
G	5	*E*, *F*
H	3	*E*, *F*

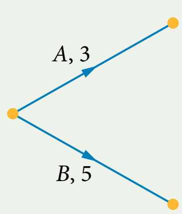
Draw the network and determine the critical path and minimum time for completion.

THINKING

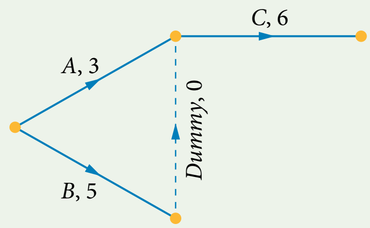
WORKING

1 Begin at the top of the table and work down.

2 Activities *A* and *B* have no immediate predecessors, so start these from a common point.

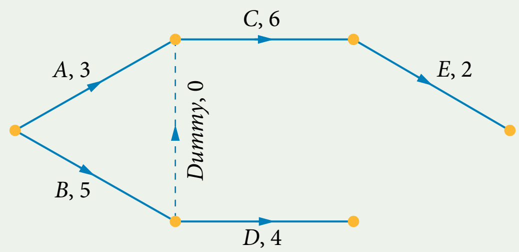


3 *C* follows *A* and *B*, but cannot connect directly to both, so join *C* to *A* and use a dummy activity to connect *B* to *C*.

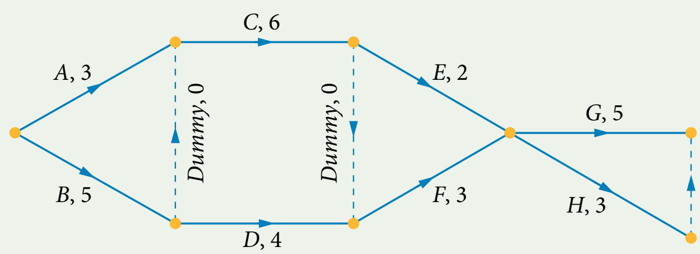


4 *D* follows *B*.

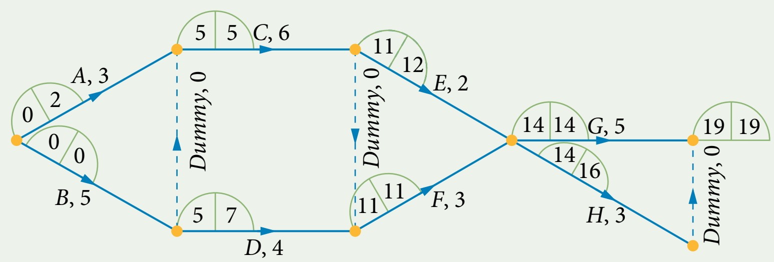
E follows *C*.



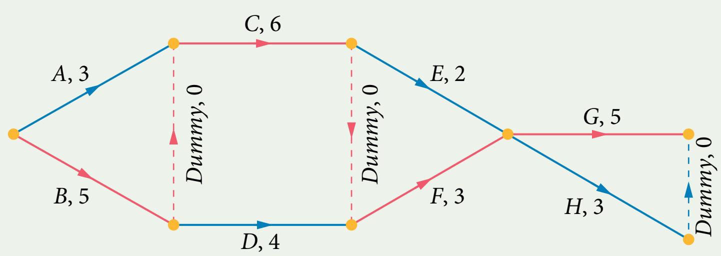
5 *F* follows *C* and *D* (requiring a dummy activity), and with *E* (see last two rows of table), leads to *G* and *H*.



6 Complete forward and backward scanning.



7 Highlight the critical path, which is the longest path through the network.



Critical path: *B-C-F-G*

8 Determine the shortest time in which the project could be completed.

$5 + 6 + 3 + 5 = 19$ minutes

The critical activities in the project are *B-C-F-G*. The minimum time to complete the project is 19 minutes.

EXERCISE 8.4 Project networks

Worked Example

1 For each of the following procedures, tabulate the activities, giving times (minutes) and immediate predecessors for each. Then construct a project network diagram. 11

(a) After-school routine: leisure activity (30 minutes); homework part A (90 minutes); eat dinner (15 minutes); homework part B (30 minutes); shower (10 minutes) while listening to 20 minutes of music.

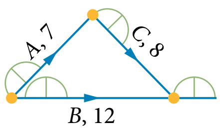
(b) Building a model: cut out and assemble the pieces (20 minutes); glue the edges (15 minutes); glue dries (60 minutes); prepare stickers (10 minutes); add stickers (10 minutes).

(c) After-school swimming lesson: walk to car (3 minutes); drive to pool (15 minutes) while eating snacks (10 minutes); change into togs (3 minutes); lesson (30 minutes); shower and change (10 minutes).

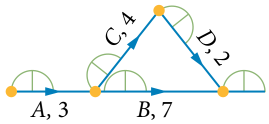
(d) Making dinner: set table (3 minutes); boil carrots (20 minutes); microwave potatoes (5 minutes); steam broccoli (5 minutes); prepare salmon (4 minutes); grill salmon (8 minutes); serve to plates (3 minutes).

2 Use the project network diagram to determine the earliest starting time and latest starting time for each activity. 12

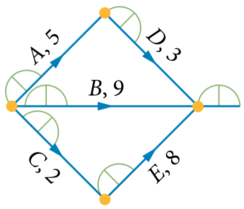
(a)



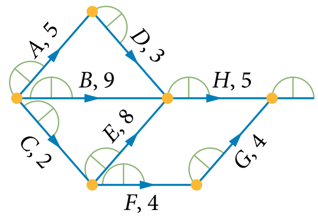
(b)



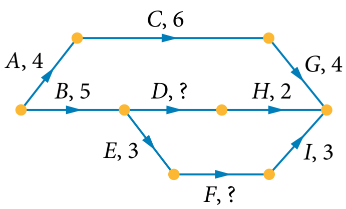
(c)



(d)



3 Consider the project network shown at right.



(a) Determine the earliest starting time for activity *G*.

A 2 B 4 C 6 D 10

(b) Determine the earliest finishing time for activity *G*.

A 4 B 6 C 10 D 14

(c) Which one of the following statements is correct?

A The immediate predecessors of *D* are *A* and *B*.

B The immediate predecessor of *D* is *B*.

C The immediate predecessor of *D* is *E*.

D The immediate predecessors of *D* are *B* and *E*.

(d) Which durations for activities *D* and *F* would result in three critical paths?

A *D*, 6 and *F*, 3 B *D*, 6 and *F*, 6 C *D*, 7 and *F*, 3 D *D*, 7 and *F*, 4

4 A particular activity in a project network has a latest starting time of 15 hours and a float time of 6 hours.

(a) What is the earliest starting time for this activity?

A 6 hours B 9 hours C 15 hours D 21 hours

(b) Explain the common error made by a student who chose the third incorrect option.

5 The table describes a stairs construction project.

(a) Sketch a possible network, use forward and backward scanning, and highlight the critical path.

(b) List the activities, in order, that form the critical path.

(c) Determine the minimum completion time for the project.

(d) Determine the float time for each non-critical activity.

Activity	Description	Duration (days)	Immediate predecessors
A	Obtain materials	1	–
B	Dig post holes	1	–
C	Assemble stairs	2	*A*
D	Cement posts in holes	3	*B*
E	Install stairs	1	*C*, *D*
F	Build hand rails	1	*E*
G	Paint stairs and rails	1	*F*

6 The table describes a window repair project.

(a) Sketch a possible network, use forward and backward scanning, and highlight the critical path.

(b) List the activities, in order, that form the critical path.

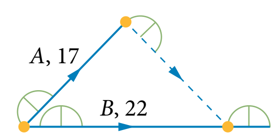
(c) Determine the minimum completion time for the project.

(d) Determine the float time for each non-critical activity.

Activity	Description	Duration (minutes)	Immediate predecessors
A	Order glass	5	–
B	Collect tools, buy putty	30	–
C	Collect glass	40	*A*
D	Remove broken window	15	*B*, *C*
E	Putty inside	8	*D*
F	Install new glass	10	*E*
G	Putty outside	8	*F*
H	Sweep up broken glass	5	*D*

Worked Example 14

7 The dashed line segment in this project network is a dummy activity of zero duration. Activity durations are given in days.



(a) Use forward and backward scanning to fill in the empty boxes in the diagram provided.

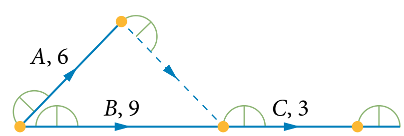
(b) Write the activities, in order, that form the critical path.

(c) Determine the minimum time in which the project may be completed.

(d) Summarise the key features of the project under the following headings:

Activity	Duration (days)	Immediate predecessor	EST	LST	Float (days)

8 The dashed line segments in this project network are dummy activities of zero duration. Activity durations are given in days.



(a) Use forward and backward scanning to fill in the empty boxes on the diagram provided.

(b) Write the activities, in order, that form the critical path.

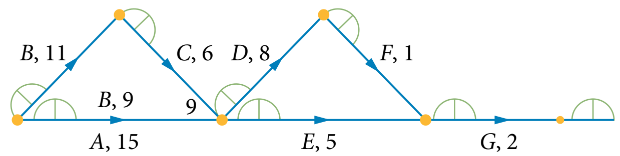
(c) Determine the minimum time in which the project can be completed.

(d) Summarise the key features of the project under the following headings:

Activity	Duration (days)	Immediate predecessor	EST	LST	Float (days)

Worked Example 13

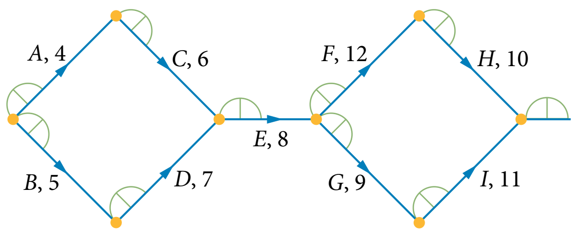
9 For the following project network, the duration of each activity is given in days.



(a) Use forward and backward scanning to fill in the empty boxes on the diagram provided.
(b) Write the activities, in order, that form the critical path.
(c) Determine the minimum time in which the project can be completed.
(d) Summarise the key features of the project under the given headings:

Activity	Duration (days)	Immediate predecessor	EST	LST	Float (days)

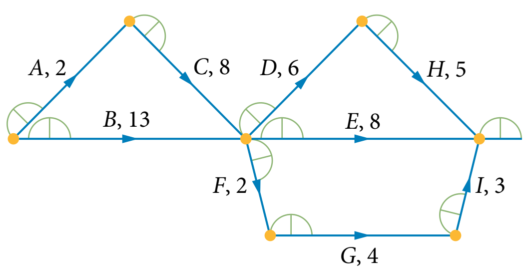
10 For the following project network, the duration of each activity is given in days.



(a) Use forward and backward scanning to determine the EST and LST for each activity.
(b) Write the activities, in order, that form the critical path.
(c) Determine the minimum time in which the project can be completed.
(d) Summarise the key features of the project under the given headings:

Activity	Duration (days)	Immediate predecessor	EST	LST	Float (days)

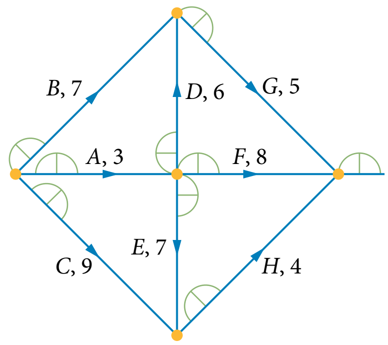
11 For the following project network, the duration of each activity is given in days.



(a) Use forward and backward scanning to determine the EST and LST for each activity.
(b) Write the activities, in order, that form the critical path.
(c) Determine the minimum time in which the project can be completed.
(d) Summarise the key features of the project under the following headings:

Activity	Duration (days)	Immediate predecessor	EST	LST	Float (days)

12 For the following project network, the duration of each activity is given in days.



(a) Use forward and backward scanning to fill in the empty boxes on the diagram provided.

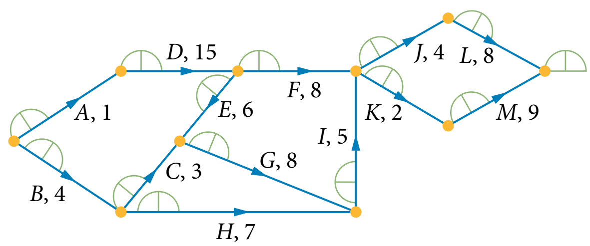
(b) Write the activities, in order, that form the critical path.

(c) Determine the minimum time in which the project may be completed.

(d) Summarise the key features of the project under the following headings:

Activity	Duration (days)	Immediate predecessor	EST	LST	Float (days)

13 For the following project network, the duration of each activity is given in days.



(a) Use forward and backward scanning to fill in the empty boxes on the diagram provided.

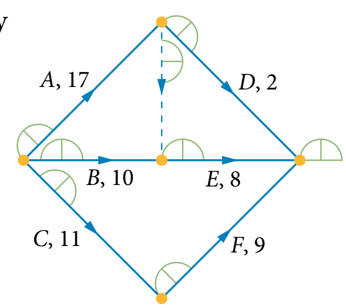
(b) Write the activities, in order, that form the critical path.

(c) Determine the minimum time in which the project can be completed.

(d) Summarise the key features of the project under the following headings:

Activity	Duration (days)	Immediate predecessor	EST	LST	Float (days)

14 The dashed line segments in the project network shown here are dummy activities of zero duration. Activity durations are given in days.



(a) Use forward and backward scanning to determine the EST and LST for each activity.

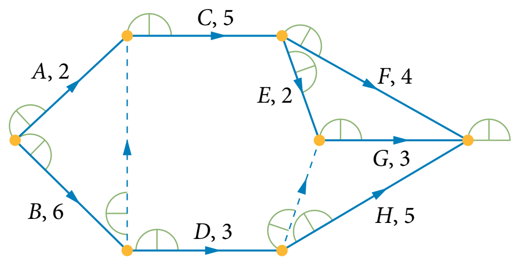
(b) Write the activities, in order, that form the critical path.

(c) Determine the minimum time in which the project can be completed.

(d) Summarise the key features of the project under the following headings:

Activity	Duration (days)	Immediate predecessor	EST	LST	Float (days)

15 The dashed line segments in this project network are dummy activities of zero duration. Activity durations are given in days.



(a) Use forward and backward scanning to determine the EST and LST for each activity.

(b) Write the activities, in order, that form the critical path.

(c) Determine the minimum time in which the project may be completed.

(d) Summarise the key features of the project under the following headings:

Activity	Duration (days)	Immediate predecessor	EST	LST	Float (days)

Worked Example 15

16 The table describes a bread-making process.

Activity	Description	Duration (minutes)	Immediate predecessors
A	Grease tins	1	–
B	Measure out ingredients	2	–
C	Pre-heat oven	5	–
D	Mix ingredients	3	*B*
E	Allow dough to rise	20	*D*
F	Knead dough, place in tins	5	*A*, *E*
G	Bake	20	*C*, *F*

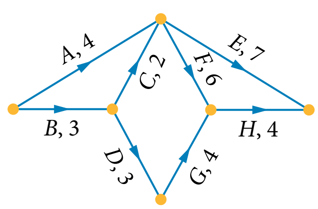
(a) Sketch a possible network, use forward and backward scanning, and highlight the critical path.

(b) List the activities, in order, that form the critical path.

(c) Determine the minimum completion time for the project.

(d) Determine the float time for each non-critical activity.

17 The duration of each activity in the project network shown here is given in hours.



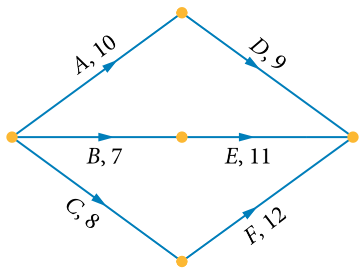
(a) Determine the critical path for this project network and the earliest completion time.

(b) Activity *H* is reduced in duration by 2 hours. What is the critical path and the earliest completion time for the new conditions?

(c) Activity *F* is reduced in duration by 4 hours on the original project. What is the critical path and the earliest completion time for the new conditions?

(d) Activity *A* is reduced in duration by 2 hours on the original project network. What is the critical path and the earliest completion time for the new conditions?

18 The network for a construction project is shown here, with activity durations in days.



(a) What is the critical path for this construction project and the earliest completion time?

The project manager has $5000 available to spend on speeding up completion. There are three activities that may have their duration reduced: activities *A*, *B* and *C*. It costs $1000 per day to reduce the duration of each of these activities.

(b) What is the minimum possible duration of the project if the manager uses the $5000 to reduce the overall completion time?

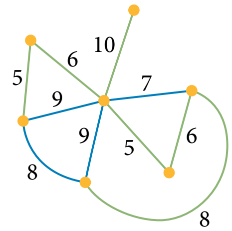
Summary

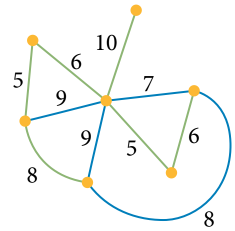
Minimum spanning trees

A spanning tree is a collection of edges in a network through which all vertices are joined. It is a subgraph of the original connected network (with n vertices) and has $(n - 1)$ edges.

The minimum spanning tree is the one with the shortest possible length.

A minimum spanning tree may not be unique.





Prim's algorithm for determining a minimum spanning tree:

STEP 1: Choose any vertex in the network.

STEP 2: Identify the lowest-weighted edge attached to the vertex and add it to the subgraph.

STEP 3: Identify the lowest-weighted edge attached to the subgraph that does not form a cycle and add it to the subgraph.

STEP 4: Repeat step 3 until you have connected all the vertices to the subgraph and formed the tree.

Maximum flow

Network diagrams often represent pipes or roads.

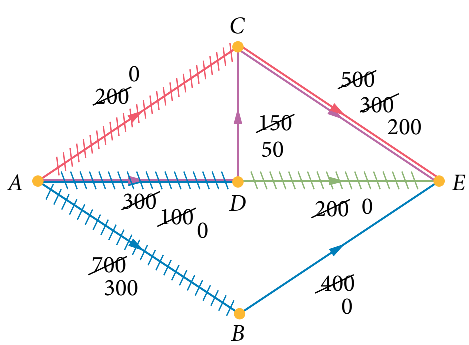
In a maximum-flow problem, the start point and the finish point are often referred to as the *source* and the *sink*, respectively.

The maximum flow of a single line is the minimum flow for any section of that line.

If there is more than one line, the maximum flow rate is the *total* of the maximum rate for each line.

Many paths method

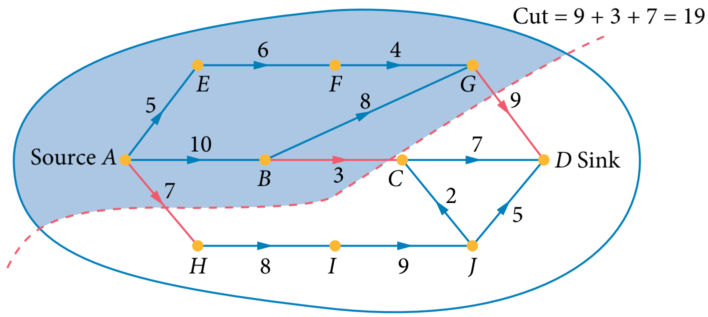
For complex systems, the many paths method will find the maximum flow.



Maximum flow rate subtotal
200
400
200
100
900
Total (max. flow)

Minimum-cut method

Another method for determining the exact value of the maximum flow from source to sink is the minimum-cut method.



Allocation problems

Allocation problems involve assigning (or allocating) several jobs to several people (with each person assigned one job), in a way that minimises the total cost (or other quantity).

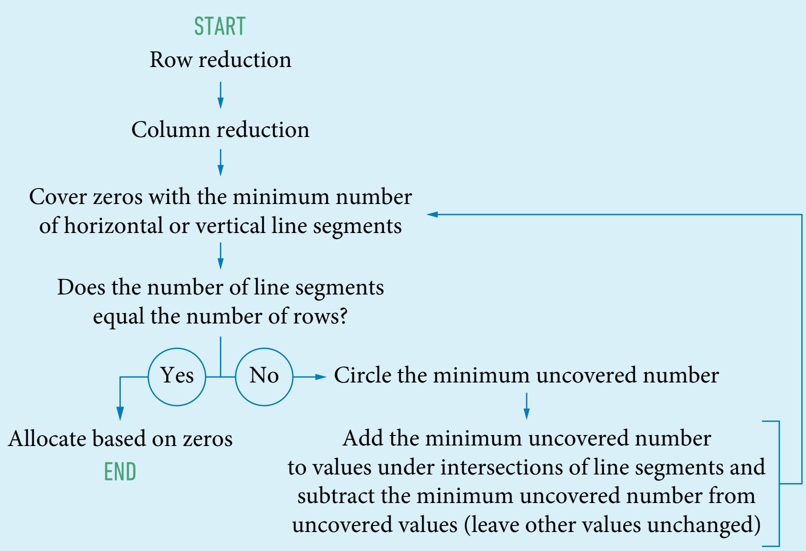
The possibilities can be displayed in a square matrix, with the people listed in rows and the jobs in columns. Each cell shows the cost for a particular person to do a particular job.

Inspection is a 'trial and error' method. Start by looking for cells with low values.

Trying all possibilities involves calculating the cost of every combination and selecting the minimum total, which usually involves displaying the combinations using a tree diagram.

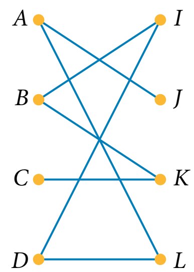
The Hungarian algorithm uses matrix operations to obtain the minimal cost allocation.

Hungarian algorithm summary



A bipartite graph can connect people with jobs, based on zeros resulting from the Hungarian algorithm.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} I & J & K & L \end{array} \\ \begin{bmatrix} 1 & \mathbf{0} & 8 & \mathbf{0} \\ \mathbf{0} & 2 & \mathbf{0} & 7 \\ 8 & 3 & \mathbf{0} & 3 \\ \mathbf{0} & 6 & 4 & \mathbf{0} \end{bmatrix} \end{array}$$



Allocate *J* to *A* and *K* to *C* and so on until each person has one job.

If required to determine an allocation that *maximises* an overall quantity, you must create a complementary matrix before continuing with the Hungarian algorithm.

For a complementary matrix, replace each element in the matrix with 'Largest value in matrix *minus* the element value'.

Project networks

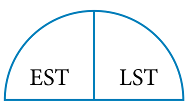
The earliest completion time for a project is calculated by adding the times on the longest possible path from start to finish. This path is called the critical path.

The earliest starting time, EST, is the earliest time an activity can begin after all predecessors have been completed.

The latest starting time, LST, is the latest time an activity can begin without delaying the project (i.e. without delaying the earliest completion time).

Forward scanning:

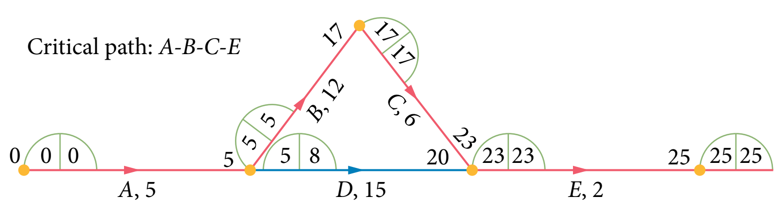
- Write zero at the first vertex, then add times to find the total 'length' of each path taken to reach each vertex. If there is more than one value, use the largest value.
- Write these values at each vertex and at the left of each curved box (ESTs).



Backward scanning:

- Copy the final EST value into the right side of the last box.
- Working backwards through the network, subtract activity duration times along each path, writing the reduced value in each right box as you go. If a vertex has more than one right box value, use the smallest value to compute the next one.

The critical path is one that includes critical activities, where EST = LST.



Float time for non-critical activities is the time an activity may be delayed without delaying the overall project.

Float time = LST − EST

A dummy activity is an activity that has zero duration.

Dummy activities are used to show predecessors correctly. They are not usually listed when describing a critical path.

Chapter review 8

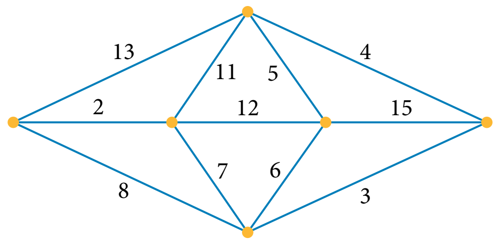
Exercise 8.1

1 Prim's algorithm is applied to this network, starting at the leftmost vertex.

Which weights are included in the first three edges?

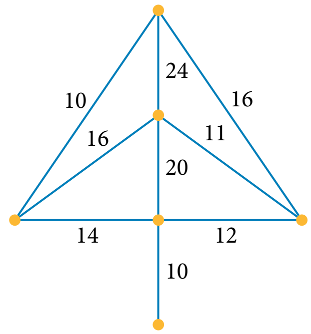
A 2, 3 and 4 **B** 2, 3 and 7

C 2, 7 and 8 **D** 2, 7 and 11



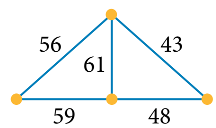
Exercise 8.1

2 Determine the total weight of the minimum spanning tree for the network shown.

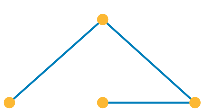


Exercise 8.1

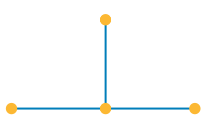
3 Which of the networks below is a minimum spanning tree for the first network shown here?



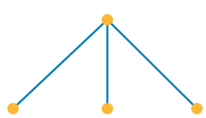
A



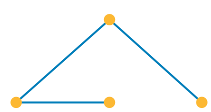
B



C

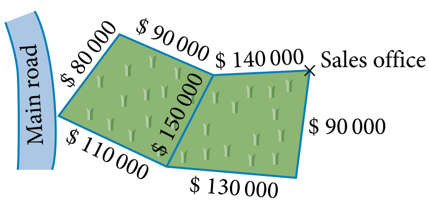


D



Exercise 8.1

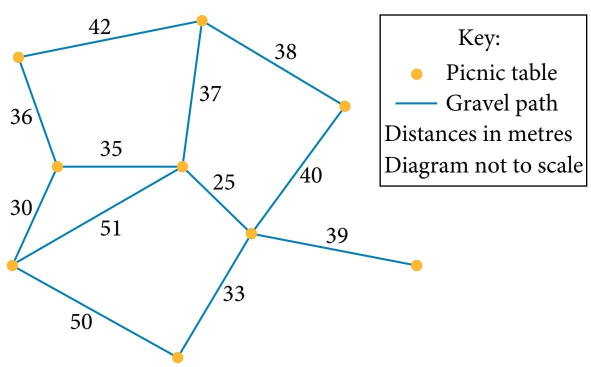
4 A developer has used a bulldozer to create roads in a subdivision. Now the developer has to tar-seal a route from the main road to the sales office. The cost for each section is shown in the network diagram. Determine the minimum cost.



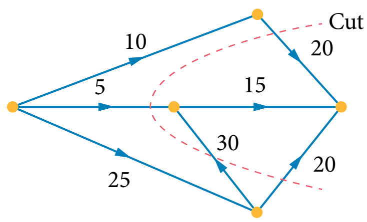
Exercise 8.1

5 Picnic tables in a nature park are connected by gravel paths as shown. The park manager wishes to pave some of these paths so that each table has wheelchair access via a paved path. Each path is 1.5 m wide.

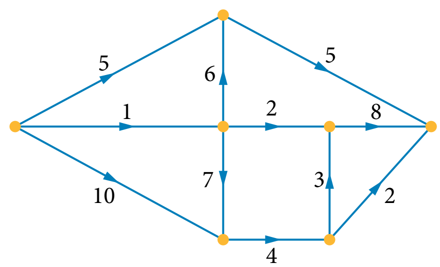
Determine the minimum area of pavers required.



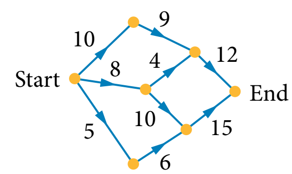
6 Determine the value of the cut. Exercise 8.2



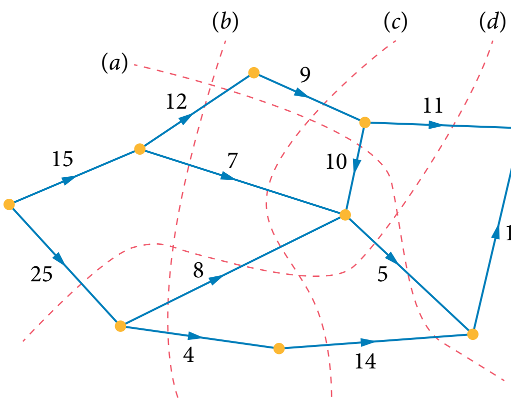
7 Determine the minimum cut for the network. Exercise 8.2



8 Determine the maximum flow through this network from start to end. Exercise 8.2

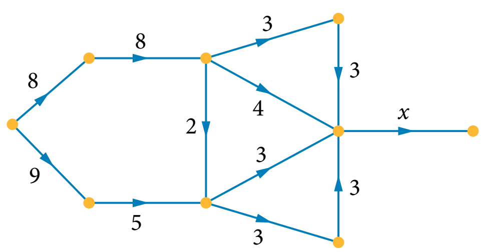


9 Determine the value of each cut. Exercise 8.2



10 The maximum flow is shown for each edge of a network, except for the final edge connecting to the sink. Exercise 8.2

Determine the smallest value of x that maximises the flow through the network.



11 After row reduction, what does this matrix become? Select the correct option. Exercise 8.3

12	17	5	15
17	22	6	18
15	20	7	13
11	18	8	14

A

1	0	0	2
16	5	1	5
14	3	2	0
0	1	3	1

B

7	12	0	10
11	16	0	12
8	13	0	6
3	10	0	6

C

7	12	0	10
12	17	1	13
10	15	2	8
6	13	3	9

D

10	5	17	7
5	0	16	4
7	2	15	9
11	4	14	8

Exercise 8.3

12 After column reduction, what does this matrix become?

$$\begin{bmatrix} 10 & 7 & 0 & 9 \\ 5 & 0 & 11 & 6 \\ 0 & 13 & 5 & 8 \\ 0 & 6 & 12 & 3 \end{bmatrix}$$

Exercise 8.3

13 Three children quote the following amounts (in dollars) to do three tasks for their parents. Each child will be paid for one task.

Which child should the parents pay for each task, if the parents wish to do the following?

Child	Cut grass	Do dishes	Vacuum the lounge
Jacob	5	2	2
Krystal	9	3	5
Louis	7	4	1

(a) Assign one job to each child so that the total cost is minimised.

(b) Assign one job to each child so that the total cost is maximised.

Exercise 8.3

14 The Hungarian algorithm results in the matrix shown.

(a) Create a bipartite graph connecting people (M, N, O, P) with jobs (W, X, Y, Z), based on zeros.

(b) Use the bipartite graph to allocate a job to each person.

$$\begin{array}{c} \\ M \\ N \\ O \\ P \end{array} \begin{array}{c} \begin{array}{cccc} W & X & Y & Z \end{array} \\ \begin{bmatrix} 0 & 15 & 6 & 0 \\ 85 & 0 & 0 & 33 \\ 0 & 12 & 0 & 25 \\ 95 & 11 & 0 & 25 \end{bmatrix} \end{array}$$

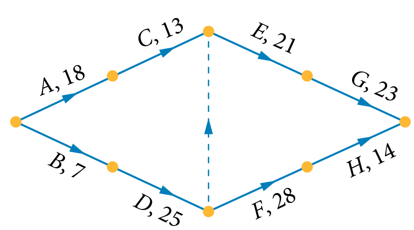
Exercise 8.3

15 Construct a complementary matrix based on the maximum element in this matrix.

$$\begin{bmatrix} 4 & 9 & 16 \\ 11 & 2 & 7 \\ 8 & 5 & 9 \end{bmatrix}$$

Exercise 8.4

16 The project network lists activity durations in hours. A dummy activity is shown as a dashed line segment.



(a) What is the minimum completion time for the project?

(b) What is the earliest starting time for activity E?

(c) What are the immediate predecessors of activity E?

(d) What is the maximum time by which activity H may be delayed, without delaying completion of the project?

(e) If the duration of activity F is increased by 4 hours to 32 hours, for how long will the project be delayed?

Exercise 8.4

17 The table describes Project X.

Activity	Duration (hours)	Immediate predecessors
A	2	–
B	7	A
C	1	B, E
D	5	–
E	6	D
F	3	D
G	3	B, E
H	4	F
I	10	G, H

(a) What is the earliest starting time for activity H?

A 4 hours

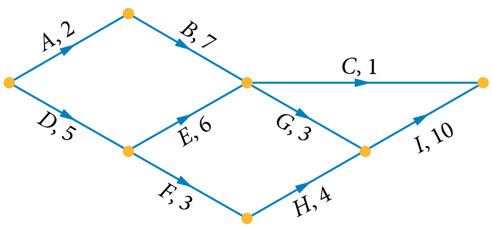
B 5 hours

C 7 hours

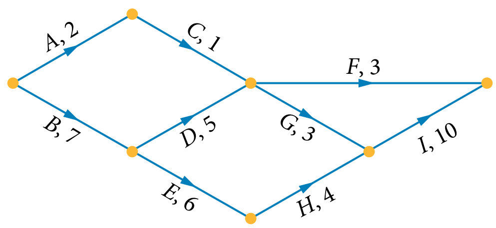
D 8 hours

(b) Which of the graphs below best illustrates Project X?

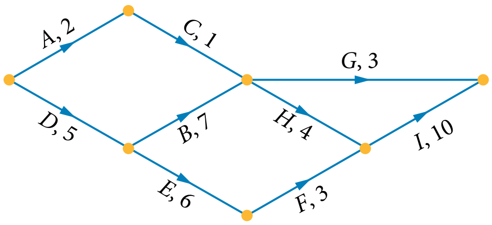
A



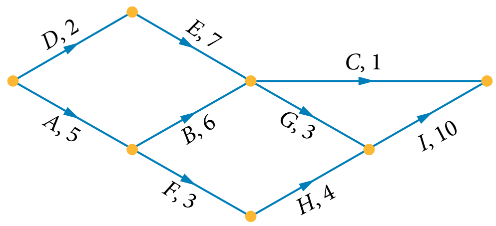
B



C



D



18 Draw a project network diagram for Project Z, corresponding to the table.

Exercise 8.4

Activity	Duration (hours)	Immediate predecessors
A	7	–
B	25	–
C	13	A
D	18	C, B
E	11	C, B
F	9	C, B
G	6	F

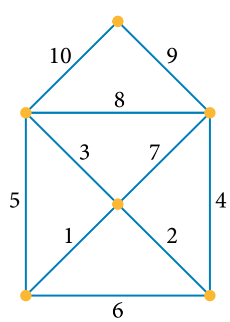
19 An alternative to Prim's algorithm for finding a minimum spanning tree is Kruskal's algorithm, which states:

Exercise 8.1

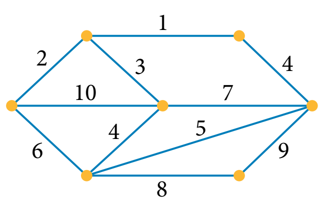
Highlight edges in order of size (whether connected to previous edges or not), ignoring any edge that produces a circuit, until all vertices are included.

Use Kruskal's algorithm to determine the total weight of the minimum spanning tree for each of the following networks.

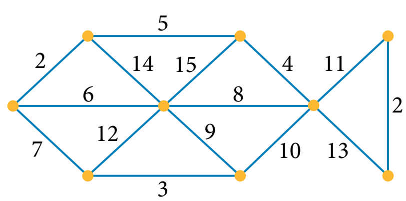
(a)



(b)



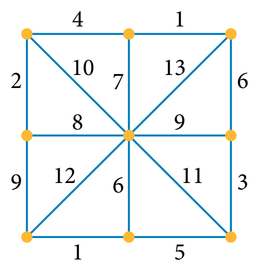
(c)



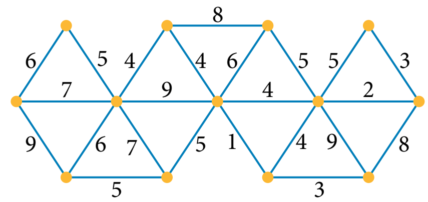
Exercise 8.1

20 Highlight the minimum spanning tree for each of the following networks and determine the total weight of the minimum spanning tree.

(a)

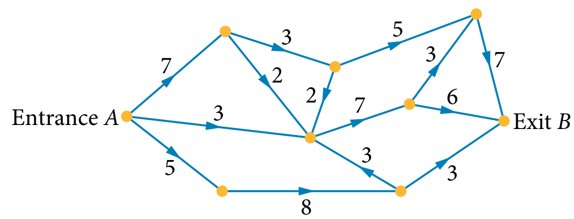


(b)



Exercise 8.2

21 The number of ants (in thousands per minute) that can travel along various tunnels in a nest is given in the network diagram. Sentry ants ensure that the ants travel only in the directions shown. Ants enter the nest at *A* and exit at *B*.

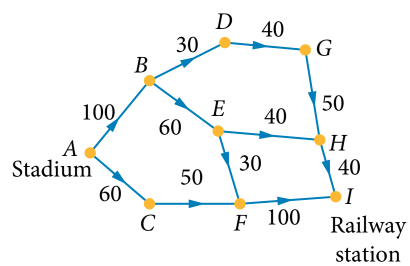


What is the maximum number of ants (in thousands per minute) that can travel through the nest?

A 13 **B** 14 **C** 15 **D** 16

Exercise 8.2

22 The diagram shows a network of footpaths starting at a stadium and ending at a nearby railway station. The numbers alongside each footpath show the maximum capacity of pedestrians per minute.



(a) Determine the maximum flow of pedestrians per minute from the stadium to the railway station.

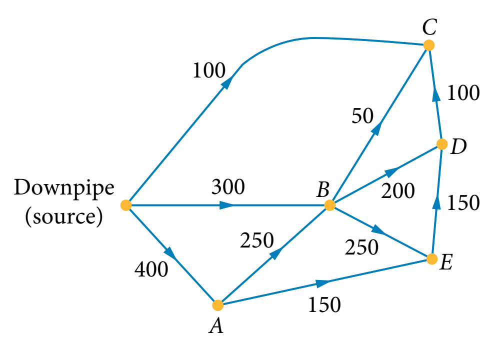
(b) One of the footpaths is to be upgraded.

(i) Determine which footpath should be chosen in order to increase the maximum flow as much as possible.

(ii) By how much could the flow of pedestrians per minute increase?

Exercise 8.2

23 A stormwater system is made up of a network of pipes that connects a downpipe from a roof to an open drain. The pipes are already in position, as shown, and the locations of possible drains are marked *A*, *B*, *C*, *D* and *E*. The capacity of each pipe in litres per minute is shown. The arrows show the direction of flow. The system is gravity fed, meaning that the arrows represent downhill flow of water.



(a) List the drain locations in order from highest to lowest.

(b) Determine the maximum flow, in litres per minute, through the network if all pipes are connected and the drain is placed at *C*.

(c) Where should the drain be placed to remove as much water as possible from the system? *Note*: In this case, one or more pipes may not be connected.

24 What does this matrix become after applying the Hungarian algorithm? Select the correct option. Exercise 8.3

$$\begin{bmatrix} 16 & 18 & 16 & 20 \\ 14 & 18 & 11 & 24 \\ 19 & 13 & 10 & 22 \\ 18 & 15 & 19 & 22 \end{bmatrix}$$

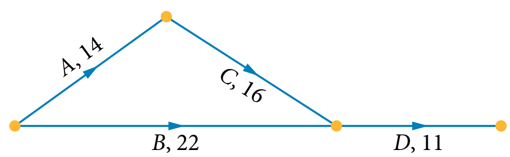
A $\begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 4 & 0 & 6 \\ 6 & 0 & 0 & 5 \\ 3 & 0 & 4 & 3 \end{bmatrix}$ **B** $\begin{bmatrix} 0 & 2 & 0 & 4 \\ 3 & 7 & 0 & 13 \\ 9 & 3 & 0 & 12 \\ 3 & 0 & 4 & 7 \end{bmatrix}$ **C** $\begin{bmatrix} 0 & 2 & 3 & 0 \\ 0 & 4 & 0 & 6 \\ 6 & 0 & 0 & 5 \\ 3 & 0 & 7 & 3 \end{bmatrix}$ **D** $\begin{bmatrix} 3 & 5 & 3 & 3 \\ 3 & 7 & 3 & 9 \\ 9 & 3 & 3 & 8 \\ 3 & 0 & 7 & 3 \end{bmatrix}$

25 The table shows the personal best times (rounded to the nearest second) of four swimmers who have the same coach, for four events. Exercise 8.3

The coach enters these swimmers in a competition and wishes to allocate each to an event based on personal best times. Which allocation minimises the total of the personal best times for the four events?

Swimmer	Time (s)			
	50 m	100 m	200 m	400 m
Wendel	26	55	154	235
Xavier	27	55	149	243
Yertle	28	51	146	239
Zac	28	54	150	243

26 A simple project is described by the network diagram shown, with each duration given in minutes. Exercise 8.4



(a) Determine the activities, in order, in the critical path.

(b) State the minimum time in which the project can be completed.

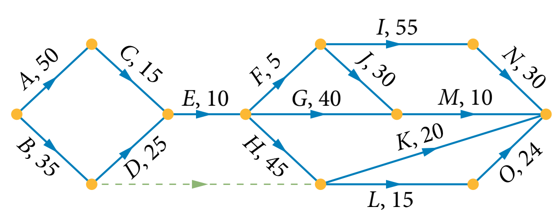
(c) Which activity can be delayed without lengthening the project completion time?

(d) By how much would activity *B* have to be delayed to cause an overall delay of 10 minutes?

(e) If activity *A* is reduced in duration by 5 minutes, what will be the reduction in the minimum project completion time?

(f) If activity *A* is reduced in duration by 10 minutes, what will be the reduction in the minimum project completion time?

27 Times in this project network diagram are given in hours. Exercise 8.4



(a) Determine the earliest starting time for:

(i) activity *C* **(ii)** activity *E*.

(b) Which activities are the immediate predecessors of activity *L*?

(c) Determine the minimum duration of the project.

(d) What is the latest starting time of activity *N*?

(e) What is the latest starting time of activity *M*?

Exercise 8.4

28 The table describes a widget manufacturing process.

(a) Draw a possible network diagram, apply forward and backward scanning, and highlight the critical path.

(b) Determine the activities that make up the critical path.

(c) What is the minimum completion time for the project?

(d) Determine how long each non-critical activity can be delayed (assuming no other activity is delayed) without delaying completion of the project.

Activity	Duration (hours)	Immediate predecessors
A	3	–
B	8	–
C	5	*A*
D	10	*A*
E	2	*A*
F	7	*C*
G	6	*B*, *E*
H	4	*D*, *F*, *G*
I	9	*B*, *E*

Exercise 8.4

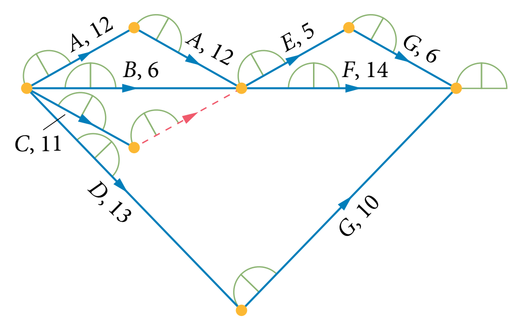
29 The duration of each activity in the following project network is given in days.

(a) Apply forward and backward scanning to fill in the empty boxes on the diagram provided.

(b) Determine the activities that form the critical path.

(c) State the minimum time in which the project can be completed.

(d) Complete a table showing Activity, Duration, Immediate predecessors, EST, LST and Float times.



Exercise 8.4

30 Describe the effect on the minimum completion time for the project described in the previous question in each of the following situations.

(a) Activity *B* is delayed by 3 days.

(b) Activity *A* is delayed by 4 days.

(c) Activity *F* is delayed by 2 days.

(d) Activity *G* is delayed by 5 days.

Exercise 8.1

31 Another alternative to Prim's algorithm for finding a minimum spanning tree is the Reverse-delete algorithm.

Here are the steps for the Reverse-delete algorithm:

STEP 1: Identify the longest edge. Remove this edge, provided it is not a bridge (i.e. taking it away doesn't isolate (disconnect) a vertex or vertices).

STEP 2: Look for the next longest edge. Remove it, provided taking it away doesn't isolate a vertex or vertices.

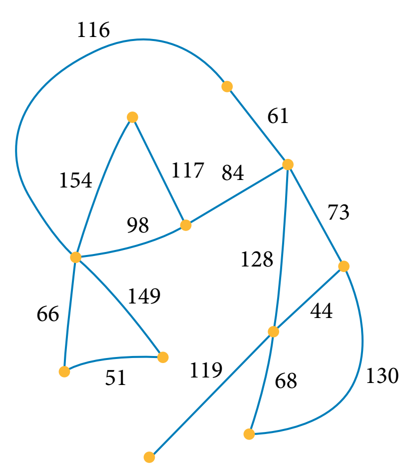
STEP 3: Repeat step 2 until no more edges can be removed.

Consider the network shown here.

(a) Determine, in the order in which they are removed, the lengths of the edges that are *not* part of the final minimum spanning tree.

(b) Draw the minimum spanning tree.

(c) Determine the total length of the minimum spanning tree.



32 Donato wishes to buy four computer components (I, J, K, L), and checks the prices at five nearby stores (A, B, C, D, E). The cost, in dollars, of each component at each store is given in the table. Donato has vouchers at each of the stores and would like to buy one component from each store, and to minimise the total cost.

Exercise 8.3

Store	Component I	Component J	Component K	Component L
A	75	85	90	62
B	80	89	92	65
C	77	91	95	62
D	82	93	87	60
E	76	90	93	64

In order to use the Hungarian algorithm (and determine the minimum total cost), there must be an equal number of stores and components, so imagine that Donato requires an imaginary fifth component (M) that is free from each store, and apply the Hungarian algorithm.

(a) From which store should Donato buy each component?

(b) What is the total cost of the components?

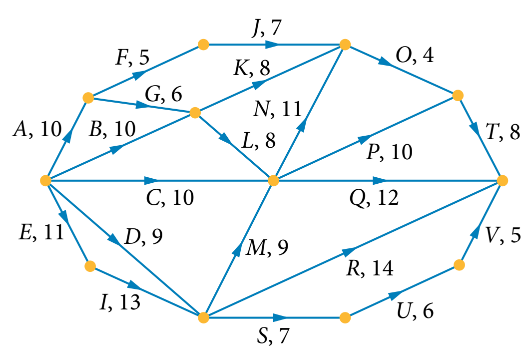
(c) Donato's friend Val suggests that instead of applying the Hungarian algorithm, the minimum cost can be found by simply choosing the cheapest store for component I, then the cheapest (but different) store for component J, and so on.

What would be the total cost, using this method?

(d) Is Val's method valid?

33 A complex project network is shown. The activity durations are given in days.

Exercise 8.4



(a) Determine the critical path for this project network and calculate the minimum project duration.

(b) Which activities have float times of 15 days or more?

(c) Determine the effect on the duration of the project in each of the following cases.

(i) The duration of activity S is increased by 10 days.

(ii) The duration of activity T is increased by 3 days.

(iii) The duration of activity Q is increased by 14 days.

(iv) The durations of activities L and N are reduced by 6 days each.

Exam review UNIT 4

Paper 1: Simple familiar

Exercise 5.1

1 The table shows the growth of an investment account n quarters after it started, where A_0 represents the amount initially invested. Values have been rounded to the nearest cent.

Quarter n	Balance, A_n ($)
0	7700.00
1	7807.80
2	7917.11
3	8027.95

(a) Determine the quarterly interest rate, to 1 decimal place.

(b) Calculate the annual (nominal) interest rate.

(c) Determine the account balance after one year.

(d) Write a recurrence relation that models the growth in the quarterly investment account balance. Write the relation in the form $A_0 = a,\ A_{n+1} = kA_n$.

Exercise 5.2

2 Stephanie borrowed $11 500 for her working holiday overseas. She will repay the loan plus all added interest when she returns after two years. Compound interest will be charged monthly at a rate of 6% p.a.

(a) Write a recurrence relation to model this loan.

(b) Determine the loan balance after one year.

(c) How much interest will have been added to the loan account by the time Stephanie returns after two years?

Exercise 5.2

3 The interest earned on Sam's savings account is 5.25% p.a. calculated monthly. After five years, the account balance is $5697.36. Determine the principal amount used to start this account.

Exercise 5.3

4 A family has $12 000 to invest for up to 10 years. They need to decide between two investment options, each with compound interest calculated annually.

Option 1: 5% p.a. for the first 5 years and 4.5% p.a. every year after that

Option 2: 4.5% p.a. for the first 5 years and 5% p.a. every year after that

(a) Determine the account balances of both options after 5 years.

(b) Determine the account balances of both options after 10 years.

(c) Explain why both account balances should be essentially the same after 10 years.

(d) If the rates for the 10th year continue for the 11th year, explain why Option 2 will show a greater account balance after 11 years.

Exercise 5.4

5 A loan of $5000 for 5 years has interest added monthly at the rate of 15% p.a.

(a) Use the formula $i_{\text{effective}} = r^n - 1$ to calculate the effective annual rate of interest, to 2 decimal places.

(b) Use the formula $i_{\text{effective}} = \left(1 + \frac{i}{n}\right)^n - 1$ to determine the effective annual rate of interest, to 2 decimal places.

6 A swimming club wants to borrow \$120 000 to add two rooms to its clubhouse. They can borrow this amount at 4.1% p.a. with interest calculated monthly. If they are able to repay no more than \$300 each month, how long will it take to repay this loan? Exercise 6.2

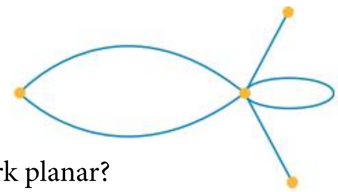
7 A perpetuity is purchased for \$145 000 and pays \$406.00 every month. Exercise 6.4

(a) Write a recurrence relation that models this perpetuity.

(b) What is the annual interest rate applied to this perpetuity?

8 A savings account is opened with a deposit of \$3800 and earns interest of 4.68% p.a. compounding monthly. If \$100 is added to the account each month, how long will it take for the account to increase to at least \$5000? Exercise 6.5

9 Examine the network shown here. Exercise 7.1



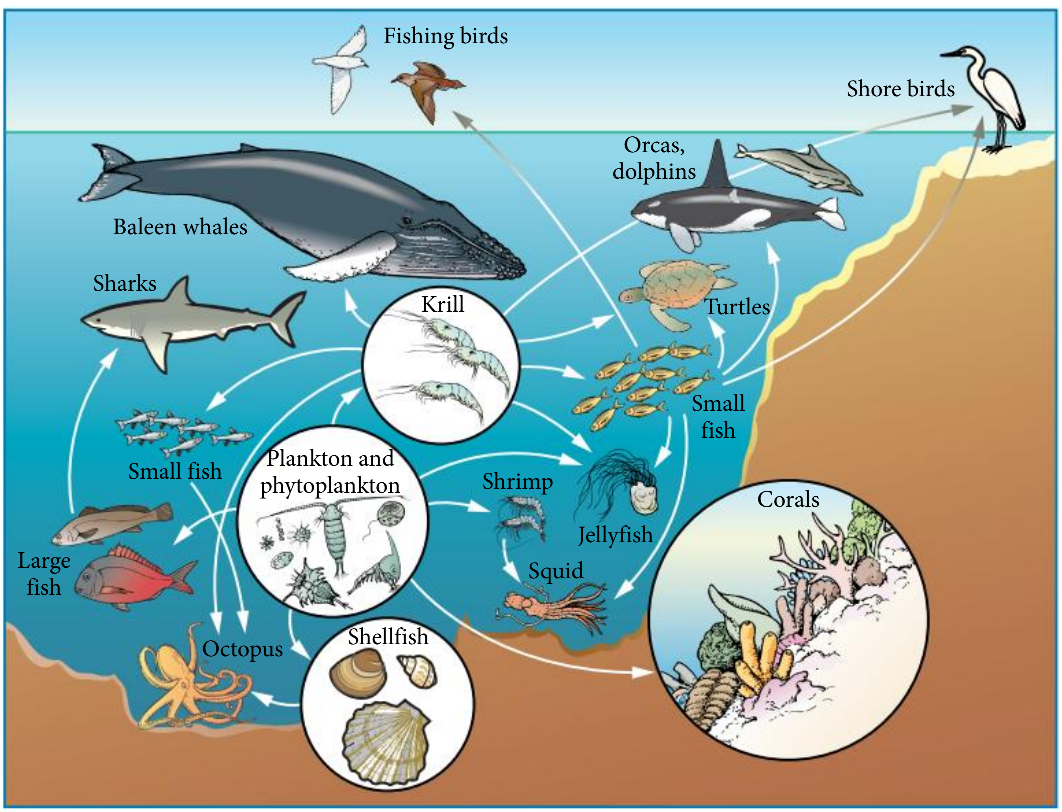
(a) How many edges, vertices and faces are in this network?

(b) Answer either yes or no to each of the following questions.

(i) Is the network simple?
(ii) Is the network planar?
(iii) Is the network complete?
(iv) Is the network directed?
(v) Is the network connected?
(vi) Is the network a tree?

10 How many vertices are there in a connected planar graph that has 47 edges and 15 faces? Exercise 7.1

11 An ocean food web is shown here. Exercise 7.2



(a) Based on the food web, what does an octopus eat?

(b) Which animals are at the bottom of this food web?

(c) Based on the food web, which animals eat large fish?

Exercise 7.3

12 Based on the adjacency matrix for an undirected network shown here, determine whether each statement about the network is true or false.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

(a) The network contains 2 loops.

(b) The network has 4 vertices.

(c) Only 1 vertex connects to each of the other vertices.

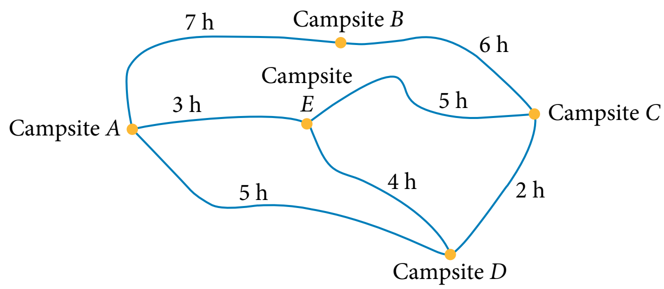
(d) There are no vertices with degrees greater than 1.

(e) The network is connected.

(f) The network contains no vertices of odd degree.

Exercise 7.5

13 The weighted graph shows the estimated walking times in hours along the walking paths that join campsites *A*, *B*, *C*, *D* and *E*.



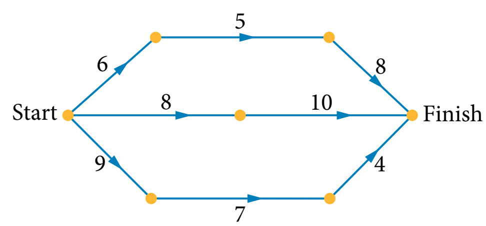
(a) Determine the quickest route from campsite *A* to campsite *C*.

(b) Determine the quickest route from campsite *A* to campsite *B* that passes through every campsite.

(c) Determine the quickest route from campsite *A* that visits each campsite and returns to campsite *A*.

Exercise 8.2

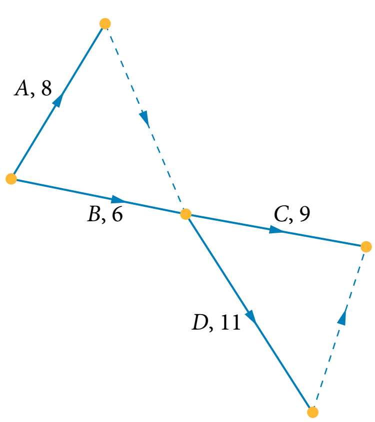
14 Determine the maximum possible flow through the network from the start to the finish.



Exercise 8.4

15 The dashed line segments in the project network shown below are dummy activities of zero duration. Activity durations are given in minutes.

Determine the critical path of the network and calculate the completion time.



Paper 2: Complex familiar and complex unfamiliar

1 After three years, Karen's loan account balance is $17 605.74. This is $1066.76 more than it was a year earlier. Exercise 5.1

(a) If the interest is compounded annually, write a recurrence relation to model Karen's loan.

(b) If Karen pays off this loan plus added interest at the end of the fourth year, how much will she have to pay?

2 To pay for an overseas holiday, Eliza takes out a short-term personal loan of $12 500, with interest charged at 19.7% p.a. compounding daily. The full amount is to be paid when she returns in 63 days. How much interest will Eliza be charged while on holidays? Exercise 5.1

3 Determine the interest earned on an investment of $580 over 2 years, given that an investment of $1240 earning the same interest rate earned $31.32 in 6 months. Exercise 5.2

4 An investment of $2680 has grown to $2948.91 in 3 years after compound interest was added each quarter. Determine the annual effective rate of interest, to 2 decimal places. State any assumptions you made and justify your working mathematically.

5 A finance company charged monthly interest at 7.42% p.a. on a loan of $12 000. No repayments have been made on the loan, and the balance is now $13 162.49. How long ago was the money borrowed? Exercise 5.2

6 Determine the value of an annuity investment of $10 500 with an interest rate of 4% p.a. calculated monthly, given that $500 is added to the investment each month for 30 years. Give your answer to the nearest thousand dollars. Exercise 6.6

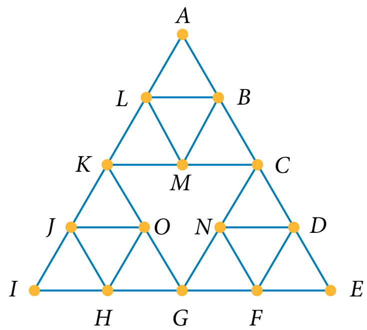
7 Construct a graph showing the following connections between 10 LED lights on a circuit board: Exercise 7.2

- a connection between LED 1 and LED 2,
- a connection between LED 3 and LED 5,
- connections from LED 3 to LED 2 and LED 7,
- connections from LED 4 to LED 3, LED 5 and LED 6,
- a connection from LED 5 to LED 9 through LED 8, and
- a connection between LED 10 and LED 8.

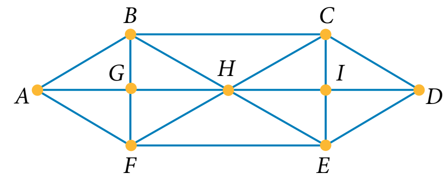
Exercise 7.4

8 For each graph shown, describe the Eulerian trail or semi-Eulerian trail contained, if possible.

(a)

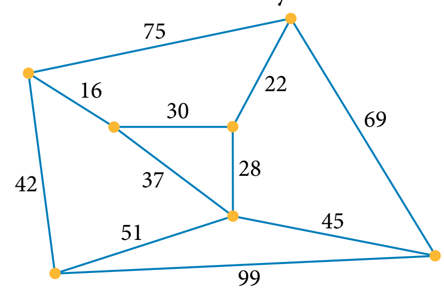


(b)



Exercise 8.1

9 A fibre optic communication system must connect each of the towns in the network shown below. The edges in the network show roads linking the towns, with distances in kilometres. The cable must be laid alongside roads so excavation machinery has access.



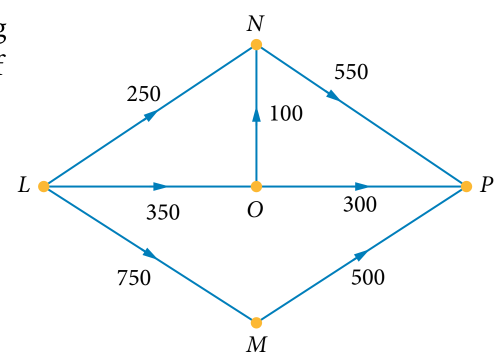
(a) Highlight roads to show how the cable should be laid if the length of cable is to be minimised.

(b) Determine the minimum length of cable required to link the towns.

Exercise 8.2

10 The maximum number of vehicles that can travel along particular roads during the busiest 15 minute period of the day is shown in the network diagram.

Determine the maximum total flow from source L to sink P for the network, using the many paths method.



Exercise 8.3

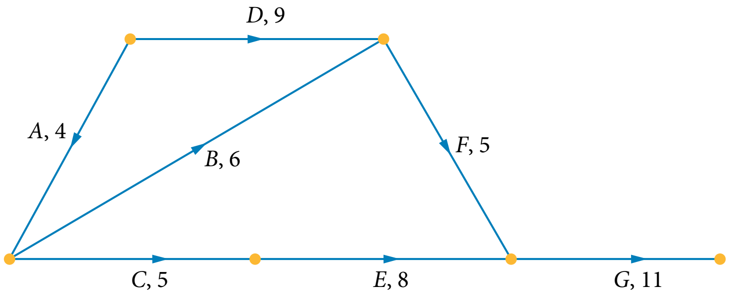
11 Alana, Bjorn, Cameron and Dana are interior designers who have been hired to renovate one room each in a home. They each quote the following costs, in thousands of dollars, for refurbishing the four rooms.

	Indoor spa	Junior's bedroom	Kitchen	Living room
Alana	13	15	20	15
Bjorn	15	18	14	23
Cameron	24	21	15	20
Dana	14	22	17	16

Use the Hungarian algorithm to determine an allocation of jobs that minimises the total cost of the renovation. State the minimum cost of the renovation.

12 A $170 000 loan is to be fully repaid in 10 years with equal monthly payments in whole dollars. The interest rate on the loan is 6% p.a. compounding monthly. Use a spreadsheet to determine how much of the $170 000 loan will have been repaid after three years. Exercise 6.1

13 The network for a construction project is shown here, with activity durations in days. The project manager has $15 000 available to spend on speeding up completion. There are two activities that may be reduced in duration: activities D and E. It costs $3000 per day to reduce the duration of activity D, and $2000 per day to reduce the duration of activity E. Exercise 8.4



What is the minimum possible duration of the project if the manager uses the $15 000 to reduce the overall completion time?

Exam review UNITS 3 & 4

Paper 1: Simple familiar

Exercise 1.4

1 Determine the y-intercept of the least-squares line for the following data.

x	1.2	2.8	3.7	4.6	5.1	7.4	8.3	9
y	51	46	47	32	19	21	7	4

A -6.3 **B** 9.8 **C** 56.3 **D** 61.7

Exercise 2.3

2 For the data set, calculate and interpret the seasonal index for spring.

Season	Summer	Autumn	Winter	Spring
Sales	\$1850	\$3420	\$9875	\$2430

A 14% below the average season **B** 45% below the average season
C 55% below the average season **D** 86% below the average season

Exercise 3.1

3 For the sequence -2, 4, 10, 16... determine the value of the 25th term in the sequence.

A 142 **B** 146 **C** 148 **D** 152

Exercise 4.2

4 Calculate the distance (in kilometres) between the locations with coordinates 28°N, 71°E and 28°N, 23°W.

A 3114 **B** 4713 **C** 9229 **D** 10 453

Exercise 5.3

5 Determine the nominal interest rate for an investment to double in value in 6.5 years, compounding monthly.

A 8.92% p.a. **B** 10.71% p.a. **C** 11.08% p.a. **D** 11.25% p.a.

Exercise 6.6

6 Determine the amount of each monthly payment made over 12 years if \$480 000 is invested in an annuity earning 5.85% p.a., compounding monthly.

A \$2306.91 **B** \$3333.33 **C** \$4646.91 **D** \$56 783.09

Exercise 7.1

7 Determine the number of edges in a complete graph with 9 vertices.

A $7 + f$ **B** 8 **C** 28 **D** 36

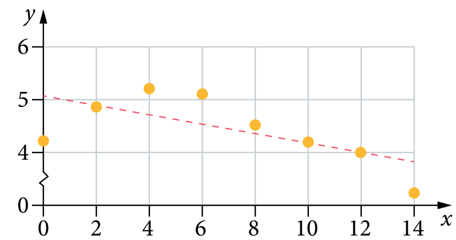
Exercise 1.2

8 Brad and Angelina both keep statistics on the health of children in war-torn countries. Brad compares the health of a child with the average income of the family, while Angelina compares the health of a child with the number of children in the family. If a larger number indicates better health, estimate the respective values of r that Brad and Angelina are likely to find. Assume both associations are moderate.

9 Consider the scatter plot and data table shown. Exercise 1.5

x	0	2	4	6	8	10	12	14
y	4.25	4.90	5.23	5.10	4.68	4.27	3.96	3.25

Without doing any further calculations, construct a table showing whether the residual value is positive or negative, and use this to comment on the assumption of linearity.



10 For each association **(a)** and **(b)** listed below: Exercise 1.3

- where causation is observed, complete the statement: '__% of the variation in variable 2 can be explained by the variation in variable 1. __% of the change is due to the variation in other factors.'
- where causation cannot be verified, explain other factors that might lead to an association.

(a) Distance travelled versus fuel consumed: $r = 0.98$ and $R^2 = 0.96$

(b) Number of fast food outlets versus the number of public toilets: $r = 0.6$ and $R^2 = 0.36$

11 What type of trend (random, increasing, decreasing, seasonal, cyclic) would you expect to see in a time series for each of the following? Explain your answer in each case. Exercise 2.1

(a) The number of marine species living on the Great Barrier Reef over the past 150 years.

(b) The value of Australia's ASX Index (the share prices of Australia's listed companies) over the past 50 years.

(c) The frequency of wars around the world.

12 Calculate the seasonal indices for each 3 month period over the year. Exercise 2.3

Period	Dec–Feb	Mar–May	Jun–Aug	Sep–Nov
Sales ($)	86	43	59	72

13 Determine the 9th term of each sequence. Exercise 3.1, 3.3

(a) a geometric sequence with the 1st term 24 and the 6th term 768

(b) an arithmetic sequence with the 1st term 18 and the 10th term 49.5

14 In 2015, Garth's food expenses increased by $12 each month. In January 2016, his food expenses were $900. Exercise 3.2

(a) If Garth's monthly food expenses continued to increase by the same amount, how much would his food expenses be in December 2016?

(b) What would Garth's total food expenses be in 2016?

15 Use a map of Australia to locate towns or cities with the given coordinates. Exercise 4.1

(a) 26°S,139°E **(b)** 38°S,144°E **(c)** 35°S,118°E

16 Determine the local time for viewers in the given cities when the Big Bash cricket match starts in Brisbane at 4:10 pm in January. Exercise 4.3

(a) Hobart, Tas **(b)** Alice Springs, NT **(c)** Adelaide, SA **(d)** Perth, WA

Exercise 5.1, 5.2

17 Gudrun's part-time job has enabled her to deposit \$1250 in a credit union account. Interest at 4.08% p.a. is added to the account every three months.

Let A_n be the account balance at the end of quarter number n.

(a) Write a recurrence relation to model the balance in this account.

(b) Determine the account balance after one year.

(c) Determine the account balance at the start of the third year.

Exercise 5.4

18 An investment of \$2000 has compound interest of 7% p.a. calculated and added daily.

(a) Determine the investment account balance after one year.

(b) Use the account balance from part (a) to determine the effective annual rate of interest.

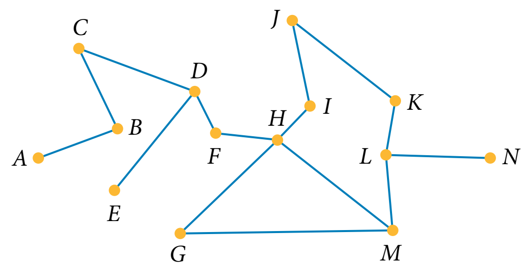
Exercise 6.1

19 A house loan of \$135 000 is taken out at 6.9% p.a. compounded monthly. If a repayment of \$1150 is made each month, determine the amount owing after each of the following periods.

(a) 6 months (b) 1 year

Exercise 7.2

20 A graph of an underground sewer network is shown here. The vertices labelled from A to N represent access points, and edges represent pipes connecting the points.



(a) Can this network be described as a tree? Explain why or why not.

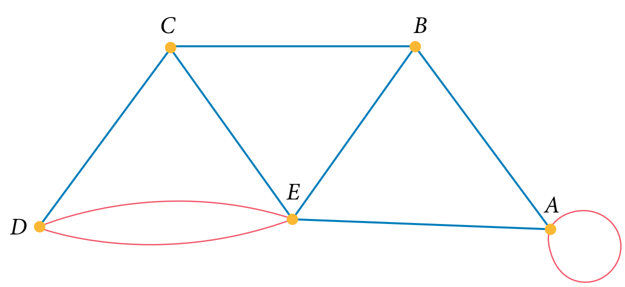
(b) How many access points are there in the network?

(c) Identify a path from access point E to access point N that contains exactly 9 access points.

(d) Using only vertical and horizontal line segments, draw a subnetwork of the sewer system containing access points A, B, C, D, E, F and H, and any connections between them.

Exercise 7.3

21 Construct the adjacency matrix for the graph shown here.



Exercise 8.3

22 The Hungarian algorithm results in the following matrix.

(a) Create a bipartite graph connecting the people (M, N, O, P) with jobs (W, X, Y, Z), based on zeros.

(b) Use the bipartite graph to allocate a job to each person.

	W	X	Y	Z
M	12	7	0	0
N	0	2	5	0
O	8	0	0	16
P	11	4	3	0

Paper 2: Complex familiar and complex unfamiliar

1 Ceu conducts a series of experiments to find an equation that describes the relationship between the rate of a chemical reaction and the temperature at which the reaction takes place. The results are shown in the table. Exercise 1.6

Temperature (K)	300	350	400	450	500	550
Rate of reaction (mol/min)	16.4	21.2	26.8	32.0	37.1	41.9

(a) Determine which of the variables is the explanatory variable and draw the scatter plot. Comment on the linearity of the trend.

(b) Calculate the correlation coefficient. Give your answer to 2 decimal places.

(c) Use linear regression to calculate the equation of the line of best fit. Give your answer to 1 decimal place.

(d) Use the regression equation to predict the rate of reaction at 370 K. Comment on your confidence in this prediction.

(e) Use the regression equation to predict the rate of reaction at 150 K. Comment on your confidence in this prediction.

2 For each of the following rainfall data sets, construct a 4-point moving mean (to the nearest whole number) and then compare the information by plotting both sets of data on the same set of axes. Exercise 2.2

Season	Sum	Aut	Win	Spr	Sum	Aut	Win	Spr	Sum	Aut	Win	Spr
Rainfall (mm)	400	500	50	60	300	600	20	80	200	500	30	50

3 A full water tank with a total capacity of 20 000 litres begins to leak at a rate of 8% per hour. Exercise 3.4

(a) Write a rule for the volume of water in the tank (V_n) after n hours.

(b) How much water is in the tank after 6 hours? Give your answer to the nearest litre.

(c) After how many hours will there be less than 2000 litres of water remaining? Give your answer as a whole number.

4 Monique, who comes from Riga, Latvia, is going on holiday to Amsterdam, Netherlands, via Copenhagen, Denmark. Her itinerary is as follows: Exercise 4.4

Dep Riga 7:30 am, 20/01; arr Copenhagen 8:10 am, 20/01; dep Copenhagen 3:00 pm, 20/01; arr Amsterdam 4:25 pm, 20/01

(a) Determine the time in Riga, UTC + 2, when Monique arrives in Amsterdam, UTC + 1.

(b) Calculate the total flying time, in hours and minutes.

(c) Determine the individual flight times, given that Copenhagen is UTC + 1.

Exercise 5.2

5 Alistair borrows $560 from a payday loan company for 10 days. He must repay a total of $580 to clear the debt.

(a) Determine the annual (nominal) rate of compound interest that the company used to calculate the interest for Alistair's 10 day loan, to 2 decimal places.

(b) If Alistair does not repay the loan plus interest in 10 days, interest will continue to be charged at the annual interest determined in part **(a)**. How many days will it take for the loan account balance to reach $1000?

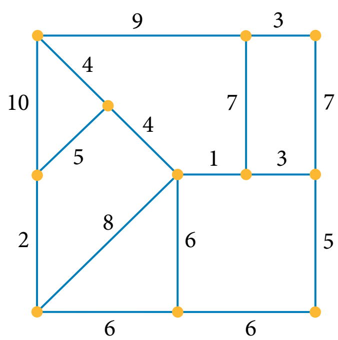
Exercise 7.1

6 If a complete graph has 45 edges, how many vertices does it have?

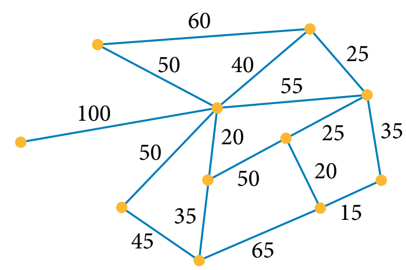
Exercise 8.1

7 For each of the following networks, highlight a minimum spanning tree and then determine its length.

(a)



(b)



Exercise 2.4

8 The unemployment rate until the end of 2017 (as a percentage) is shown in the graph.

(a) Starting from the peak of unemployment during the Global Financial Crisis (during 2009), write three linear equations to model the unemployment rates from that point until the end of 2017.

(b) Use each equation to predict the unemployment rate at the end of 2020. Comment on the suitability of using each model, and the reasonableness of each value.

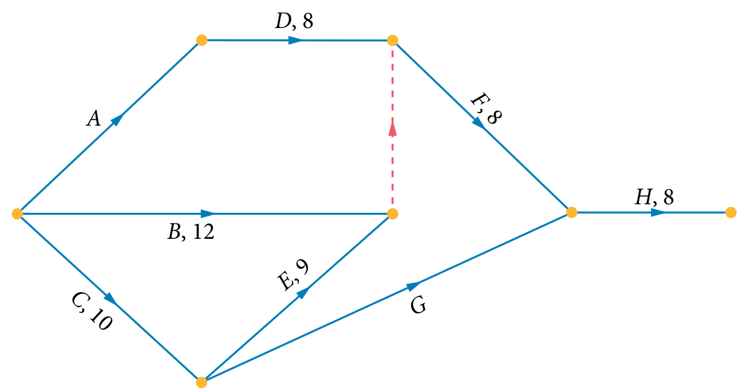
9 A manufacturing company needs to construct a new facility to keep up with production expectations. The company needs to borrow \$549 000 for the construction project. The bank offers the company a rate of 7.1% p.a. compounded monthly over 18 years. Exercise 6.3

(a) Use a spreadsheet to determine the monthly repayments, rounded up to the nearest dollar.

(b) The company has record sales for the first three years of the loan, and decides to refinance the loan. How much does the company owe after three years?

(c) The company's accountant believes they can afford to repay the loan much faster. The remainder of the loan, after the third year, is to be paid in another 10 years. Assume the interest rate stays the same. How much does the company now have to pay per month?

10 Pierre discovers the secret recipe for his great grandfather's special barbeque sauce, and records it as a project table and a network diagram. Unfortunately, silverfish have eaten parts of these. Exercise 8.4



(a) Pierre learns that the critical path through the network is C-E-F-H. Construct a project table with the given headings, including the maximum possible durations for activities A and G.

Activity	Duration (minutes)	Immediate predecessor	EST (minutes)	LST (minutes)

(b) What is the minimum time for preparation of the sauce?

Answers

Chapter 1: Bivariate data analysis

RECALL

1

x	15	16	17	18	19
f	2	9	10	12	8
Relative frequency	5%	22%	24%	29%	20%

2 (a) 2.24 (b) 0.85 (c) 13.50

3 (a) 2 (b) $\frac{8}{5}$ (c) $-\frac{6}{5}$

4 (a) $\frac{8}{7}$ (b) $\frac{2}{3}$ (c) $-\frac{5}{3}$

5 (a) $(0, 7)$ (b) $(0, 5)$ (c) $(0, -4)$

6 (a) $b = 3, a = 5$ (b) $b = 1, a = -6$ (c) $b = -3, a = 4$

7 (a) 6 (b) 9 (c) 3.225

EXERCISE 1.1

1 (a)

	Year 12	Year 7	Total
Popcorn	35	40	75
Fries	21	26	47
Choc top	11	50	61
Total	67	116	183

(b)

	Year 12	Year 7	Total
Popcorn	19%	22%	41%
Fries	11%	14%	26%
Choc top	6%	27%	33%
Total	37%	63%	100%

2 (a)

	Children	Teenagers	Adults	Totals
Chips	20	31	10	61
Vegetables	2	1	28	31
Salad	5	20	63	88
Totals	27	52	101	180

(b) (i) 34% (ii) 29% (iii) 38%

(iv) 64% more of the children

3 **D**

4 (a)

	Lex Luthor	Goldfinger	The Penguin	The Joker	KAOS	Total
Males	24%	5%	43%	21%	6%	100%
Females	34%	9%	24%	22%	11%	100%

(b) Females: Lex Luthor; males: The Penguin

5 (a)

	Blue	Brown	Yellow	White
Peanut	59%	46%	37%	51%
Sultana	41%	54%	63%	49%
Total	100%	100%	100%	100%

(b) Blue

6 (a)

	Vanilla	Other flavour	Total
Cone	44%	56%	100%
Cup	36%	64%	100%

(b) 8% more customers eating from a cone eat vanilla ice-cream than those eating from a cup.

(c) 58% more customers ate vanilla ice-cream from a cone than from a cup.

7 (a) **D**

(b) You must be able to compare with the total for the categories that are thought to influence the relative frequencies of another variable.

8 (a)

	Year 7	Year 12
< 30 minutes	83%	29%
≥ 30 minutes	17%	71%
Total	100%	100%

(b) 17% of Year 7 students travel for 30 or more minutes, compared to 71% of Year 12 students, a percentage difference of 54%.

9 (a) 65–74 years

(b) No, only 6% of 18–24 year olds have high blood pressure.

10 (a)

	Cold	No cold	Total
Sugar tablet	23%	77%	100%
Vitamin C tablet	19%	81%	100%

(b) For the sample, 4% fewer students taking vitamin C caught a cold than those taking the placebo.

11 (a)

	Males %	Females %
Agree	36	68
Disagree	64	32
Total	100	100

(b) 32% more of the women agreed with the proposal than the men.

12 (a)

	Overweight to obese					
	Males		Females		Total	
	Number	%	Number	%	Number	%
Diabetic	14	14	8	7	22	10
Pre-diabetic	9	9	7	6	16	7
Healthy range	77	77	105	88	182	83
Total	100	100	120	100	220	100

(b)

	Normal to underweight					
	Males		Females		Total	
	Number	%	Number	%	Number	%
Diabetic	20	7	10	3	30	5
Pre-diabetic	13	4	15	5	28	5
Healthy range	267	89	265	91	532	90
Total	300	100	290	100	590	100

(c) The risk of diabetes increases by 7% for those in the overweight to obese category. For males in this category, the risk of diabetes increases by 12%, and for females in this category it increases by 5%.

13 (a)

	0 aces	1 ace	2 aces	3 aces	4 aces	Total
Slow serve	7	8	8	4	0	27
Fast serve	1	8	14	30	20	73
Total	8	16	22	34	20	100

(b) 34

(c) Slow servers: 15% fast servers: 68%

(d) 53% more of the fast servers scored more than two aces, compared with the slow servers.

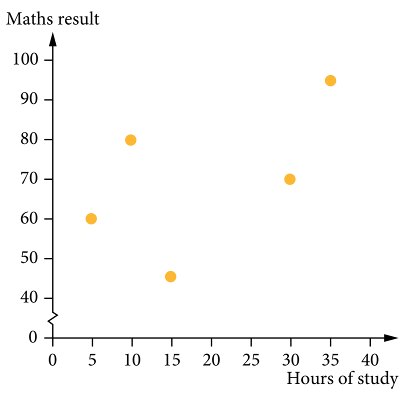
14 According to this data it appears that men earned more than women by about $250 per week. Median salaries could be explored to account for the possibilities of outliers and the average time fractions worked by each gender would be a useful figure. The rule would be limited to the sample taken and would be expected to vary over time.

15 (a) Although the South Island has a larger area than the North Island, it has four fewer rivers longer than 100 km and three fewer rivers longer than 150 km. It has one more river longer than 200 km.

(b) The data does not suggest that the larger the island, the longer the rivers it has.

EXERCISE 1.2

1 The maths results show a moderate positive linear association.



2 (a) Years of employment

(b) Rainfall

(c) Temperature

(d) Cans of soft drink consumed

3 (a) No association

(b) Strong positive linear

(c) Moderate negative linear

(d) Strong positive non-linear

4 Moderate positive non-linear

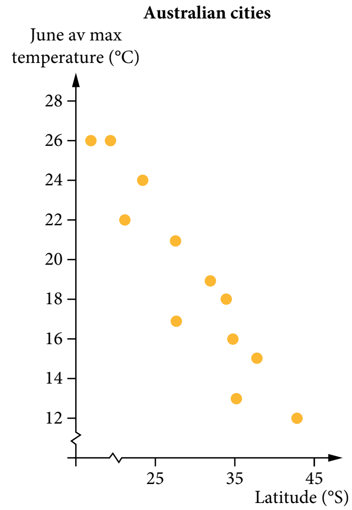
5 (a) **D**

(b) Sometimes there is no obvious explanatory variable.

6 (a) Latitude

(b) Negative linear association

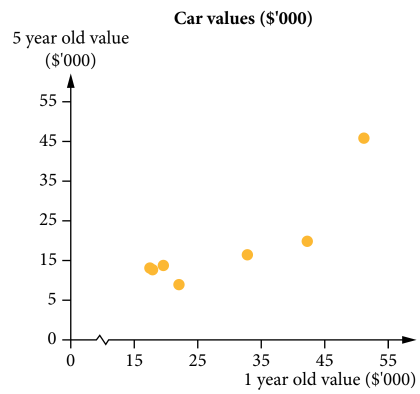
(c) Moderate negative linear association



7 (a) 1-year-old value

(b) Positive linear association

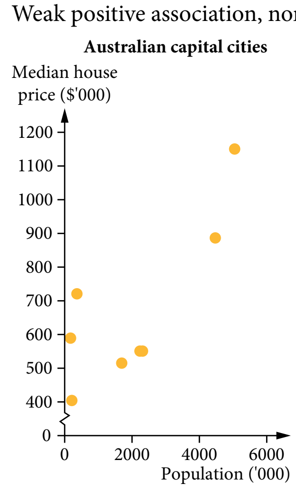
(c) Moderate positive non-linear association



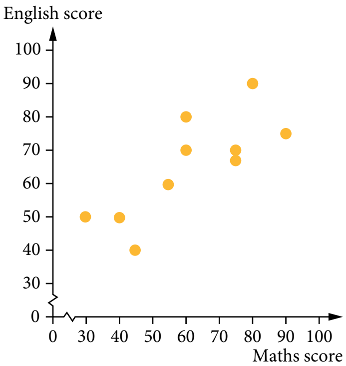
8 (a) Population

(b) Positive linear association

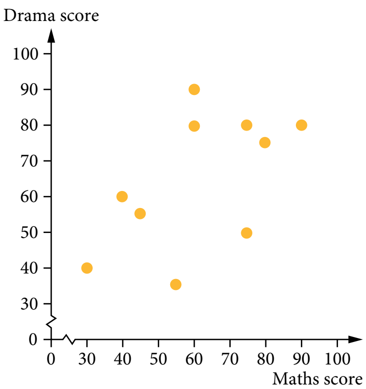
(c) Weak positive association, non-linear



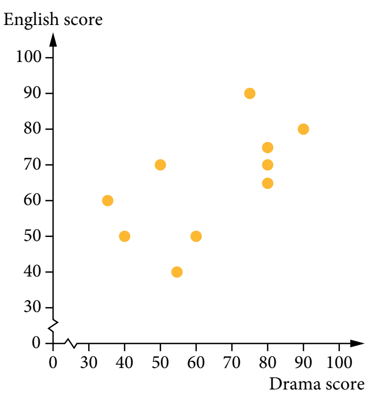
9 **(a)**



(b)



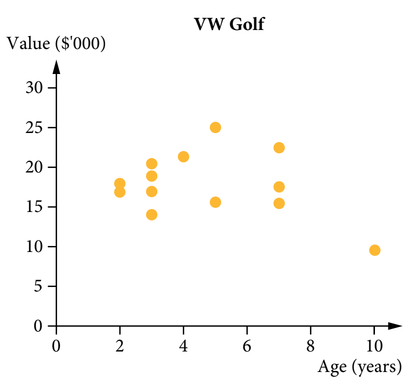
(c)

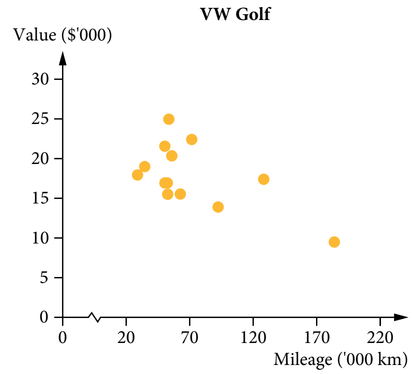


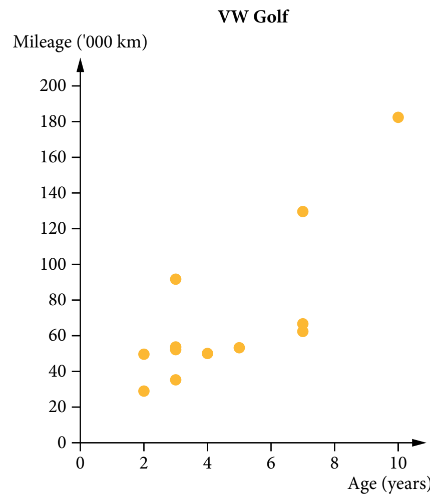
(d) Maths and English appear to have the strongest association.

10 **(a)** Value

(b)



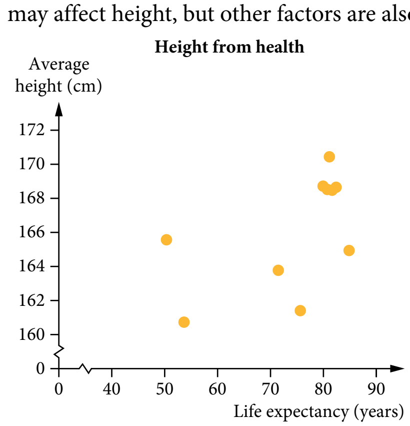




(c) Value and age: weak negative (non-linear). Value and mileage: moderate negative linear. Mileage and age: moderate positive linear. Mileage is only moderately associated with age. This could be partly responsible for the weak connection between age and value. All models are restricted by the lifespan of the car.

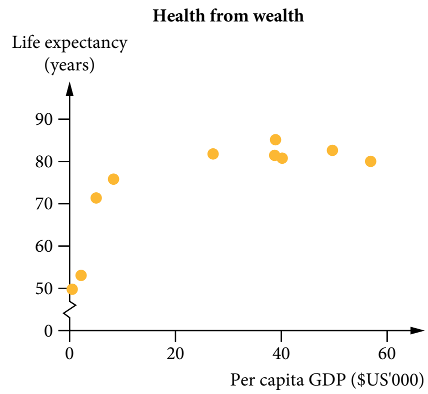
11 **(a)** **(i)** Life expectancy (explanatory variable) and average adult height

(ii) Weak positive non-linear association. Good health may affect height, but other factors are also involved.

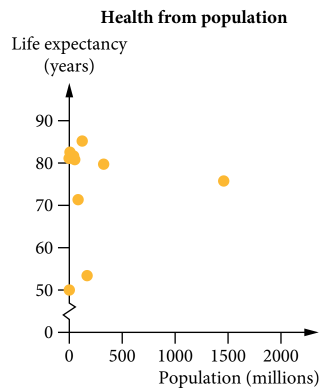


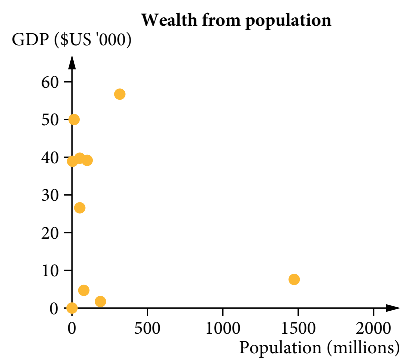
(b) **(i)** Per capita GDP (explanatory variable) and life expectancy

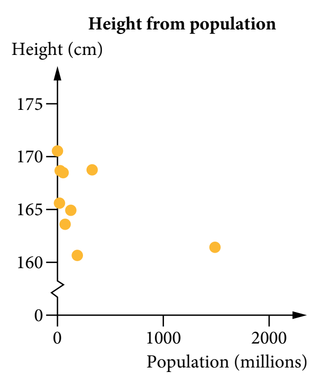
(ii) Moderate positive non-linear association. National wealth appears to be associated with better national health up to, say, $US30 000, after which there is no increased benefit.



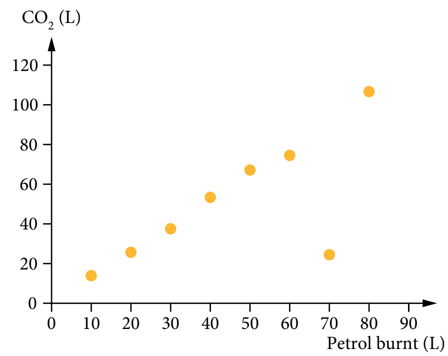
12 Even with the outlier (China) removed, there is no association between population size and any of the other three variables.

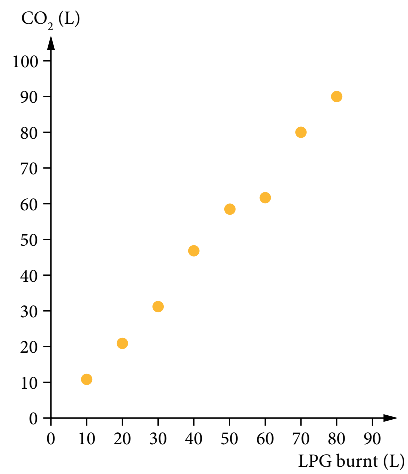






13 **(a)**

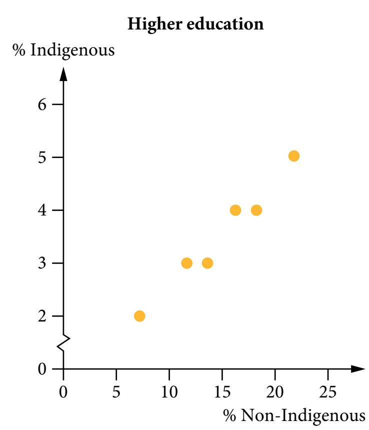




(b) Strong positive linear for both. Petrol burnt has an outlier at (70, 25).

(c) LPG. The gradient is roughly 1.1 L of CO_2 per litre of fuel. For petrol it is closer to 1.3 L of CO_2 per litre of fuel.

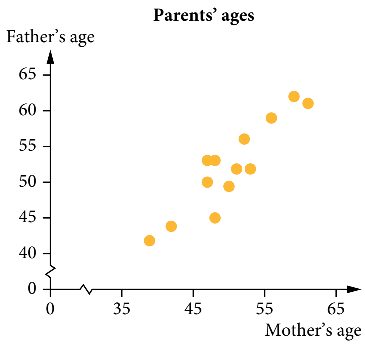
14 **(a)**



(b) Strong, positive and linear. The data shows that, as the percentage of non-Indigenous attendance increases, the percentage of Indigenous attendance also increases.

(c) Indigenous: factor of 2.5, non-Indigenous: factor of 3.1. Non-Indigenous percentage increase is more rapid.

15 (a)



Moderate positive linear association. Fathers are generally around 2 years older than mothers.

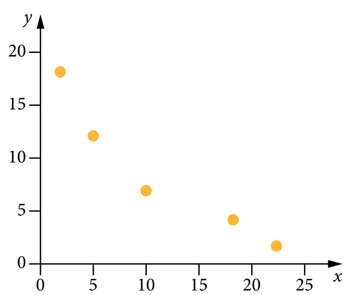
(b) If same class members and same day of the year, the pattern of dots would look the same, but the values on the axes would be 4 years less on both axes. The verbal analysis would be identical. If the day of the year was different or the class did not have the same members, you would still expect the same association as in (a).

(c) The pattern of dots would be similar, but the values on the axes would be 18 years less on both axes. The verbal analysis would be identical to the answer for (a).

EXERCISE 1.3

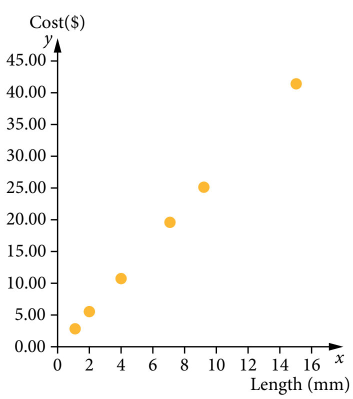
1 (a) $r = 0.98$ (b) $r = 0.94$ (c) $r = -0.13$ (d) $r = 0.90$

2 (a)



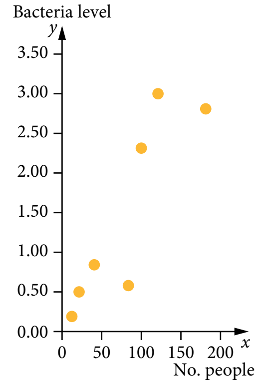
$r = -0.95$; strong negative linear

(b)



$r = 1.00$; perfect positive linear

(c)

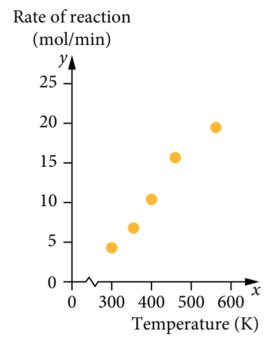


$r = 0.89$; strong positive linear

3 (a) $r = 0.46$ (b) (500, 182)

(c) A. Outliers are generally excluded from the data set, so calculations are an accurate representation of the study.

(d) $r = 0.99$



(e) r changes from 0.46 to 0.99.

(f) The number 182 (mol/min) was most likely meant to be 18.2 (mol/min).

4 (a) Set B (b) Set A (c) Set C (d) Set A

5 (a) (i) $r = 0.92$ (ii) $r = 0.92$ (iii) $r = 0.92$

(b) The correlation coefficients are the same for each set.

The second set is the same as the first with 10 subtracted from each y-value.

The third is set is the first with each value doubled.

The points would cluster around the line of best fit in exactly the same way.

6 (a) **A**

(b) It is impossible to tell whether the association is linear without seeing the scatter plot.

7 (a) 64% of the variation in the quality of performance can be explained by the variation in hours of rehearsal; 36% of the variation is due to other factors.

(b) The number of fast food outlets and the number of hospitals in a town depend on the population.

(c) 36% of variation in the number of registered motorbikes can be explained by the variation in number of registered cars; 64% of the variation is due to other factors.

(d) The results in Maths and Physics exams depend on other factors such as interest, hours of study and intelligence.

(e) 96% of variation in the number of contestants remaining in a reality TV series can be explained by the variation in episode number; 4% of the variation is due to other factors.

8 (a) $r = 0.95$ (b) $r = -0.85$ (c) $r = 0.80$ (d) $r = -0.74$

9 $r_{xz} = -0.99$

10 Possible solution: $(15.2, 240)$; increase in r from 0.5041 to 0.5578, increase in s_{xy} from 28.56 to 31.34.

EXERCISE 1.4

1 (a) \$860 (b) \$150

(c) $\text{cost}(\$) = 860 + 150 \times (\text{number of teams})$ (d) \$4460

2 (a) \$10 (b) \$42

(c) $\text{cost}(\$) = 10 + 42 \times (\text{number of hours})$ (d) \$304

3 (a) \$120 (b) \$360

(c) $\text{cost}(\$) = 120 \times (\text{number of nights}) + 360$ (d) \$2040

4 (a) $y = 1.5 + 0.5x$ (b) $y = 11 - 2x$

(c) $y = 6x - 68$ (d) $y = 50 - 0.25x$

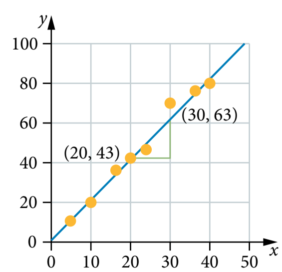
5 (a) $y = 6.28 + 1.56x$; $r = 0.98$, strong positive linear association

(b) $y = 50.22 - 6.52x$; $r = -1.00$, strong negative linear association

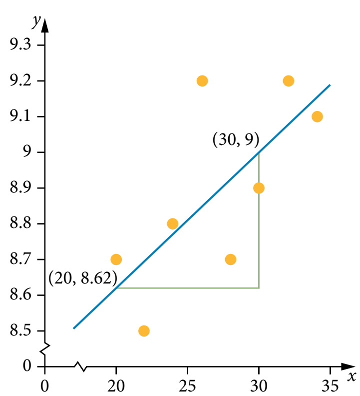
(c) $y = 0.37x - 1.5$; $r = 0.99$, strong positive linear association

(d) $y = 2.42 + 19.81x$; $r = 0.95$, strong positive linear association

6 y-intercept $(0, 3)$, gradient 2, equation $y = 3 + 2x$



7 (a) y-intercept $(0, 7.9)$, gradient 2, equation $y = 7.9 + 0.038x$

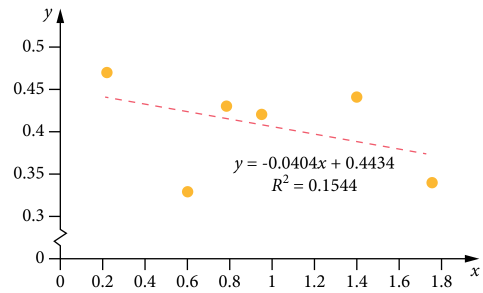


(b) The student has failed to recognise that the x-axis does not start at 0.

8 (a) D (b) B

9 B

10 (a)



$r = -0.39$, $\text{speed (m/s)} = -0.04 \times \text{depth (m)} + 0.44$

(b) There is a weak association, $r = -0.39$, linking an increase in water depth to slowing of the speed.

(c) The equation suggests a speed of 0.44 m/s at the surface, with a decrease of 0.04 m/s in the flow of the river for each metre of depth.

(d) 15% of the change in water flow rate responds to, or can be explained by, the change in the depth of water. 85% of the variation in water flow rate is in response to other factors.

11 (a) $r = -1$, $\text{temperature (°C)} = -6.5 \times \text{altitude (km)} + 15$

(b) The altitude is the explanatory variable and the temperature is the response variable. There is a strong negative linear association between the variables. For every increase in altitude of one kilometre, the temperature decreases by 6.5°C.

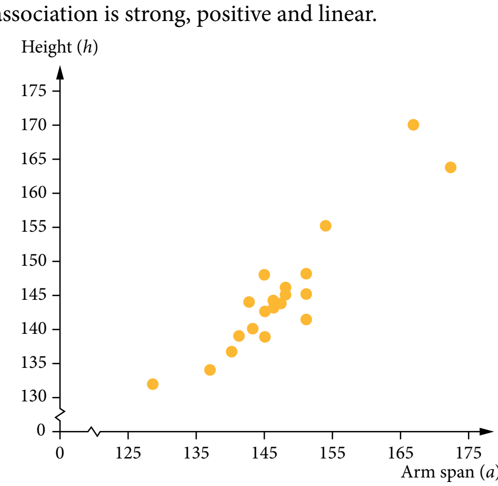
12 (a) $r = 0.99$, wife's age = $2.7 + 0.91 \times$ husband's age

(b) $r = 0.72$, wife's age = $11.31 + 0.62 \times$ husband's age

(c) Set A fits the regression line better, with an r-value of 0.99, which is closer to 1 than that of Set B. Set B has an outlier at $(70, 32)$, which significantly reduces the value of r.

13 (a) Arm span

(b) The association is strong, positive and linear.



(c) $r = 0.92$

(d) $h = 10.42 + 0.91a$ or
height (cm) = 10.42 + 0.91 × arm span (cm)

(e) A correlation coefficient of 0.92 shows a strong correlation, meaning that arm span is a good predictor of height. The data is collected from people ranging in height from 128 cm to 172 cm. You would expect the model to be reliable for young adults, but the results may vary for young children.

14 (a) *Note*: Answers for the correlation coefficient estimate will vary but should be similar to those in the table.

Association	Explanatory variable	Strength and direction of association	Correlation coefficient estimate
Area and population	area	weak, positive	0.2
Area and motorcar thefts	area	weak, positive	0.2
Area and average weekly earnings	area	none	0
Population and motorcar thefts	population	moderate, positive	0.7
Population and average weekly earnings	population	none	0
Motor car thefts and average weekly earnings	earnings	weak, negative	-0.2

(b) Population versus area: $r = 0.08$

Motor car thefts versus area: $r = 0.10$

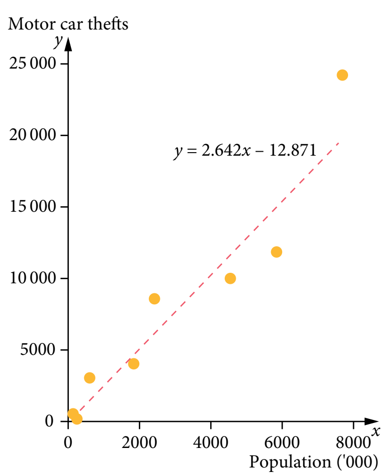
Earnings versus area: $r = 0.41$

Motor car thefts versus population: $r = 0.96$

Earnings versus population: $r = -0.17$

Motor car thefts versus earnings: $r = -0.16$

(c) Motor car thefts versus population has correlation coefficient close to 1 and the graph shows a strong linear association.



(d) number of motor car thefts = 2.64 × population ('000) − 12.87

15 (a) Only the set D scatter plot suggests a linear association that is strong enough for further analysis.

(b) $y = -0.38x + 21.70$

When $x = 0$, $y = 21.70$, except that the data may not be valid below $x = 20$.

For each increase of 1 in the value of x there is a corresponding decrease of 0.38 in the value of y.

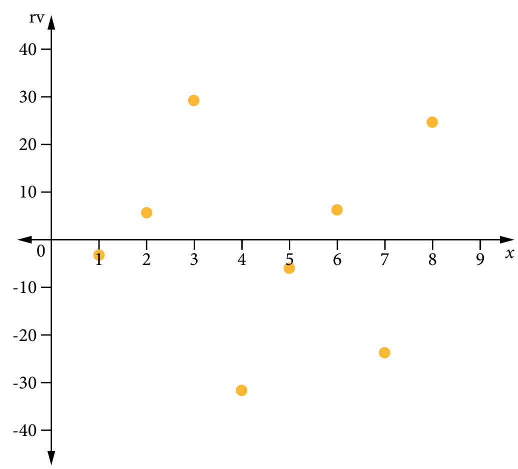
(c) $y = -0.38(x - 20) + 14.1$

For the smallest value in the domain, $x = 20$, the model predicts $y = 14.1$. For each increase of 1 in the value of x there is a corresponding decrease of 0.38 in the value of y.

EXERCISE 1.5

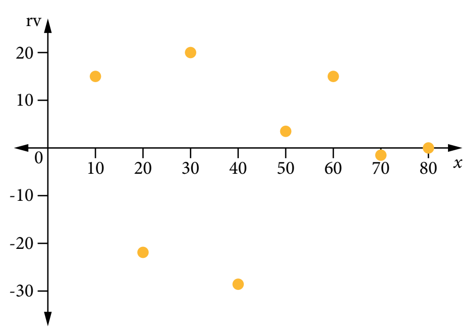
1 (a)

x	1	2	3	4	5	6	7	8
rv	-3.3	5.7	29.7	-31.3	-7.3	6.7	-24.3	24.7



(b)

x	10	20	30	40	50	60	70	80
rv	16	-22	20	-28	4	16	-2	0



2 (a) Yes (b) Yes (c) No (d) No

3 (a)

x	2	4	6	8	10	12	14
rv	0	–	+	–	–	+	0

The assumption of linearity is supported.

(b)

x	5	10	15	20	25	30	35	40
rv	0	–	0	+	+	0	–	–

The assumption of linearity is not supported.

(c)

x	-4	-2	0	2	4	6	8
rv	–	+	+	–	+	–	0

The assumption of linearity is supported.

4 **(a)** Scatter plot I: D; Scatter plot II: A; Scatter plot III: B; Scatter plot IV: C

(b) **A**

5 **(a)** **A**

(b) The student has given the predicted value instead of the actual value.

6 **(a)**

x	15	20	25	30
y	39.9	51.4	62.9	74.4

(b)

x	15	20	25	30
y	195.65	192.6	189.55	186.5

(c)

x	15	20	25	30
y	718.9	967.4	1215.9	1464.4

(d)

x	15	20	25	30
y	365.8	344.8	323.8	302.8

7 **(a)**

y_a	22	41	40	81	85
y_p	31.2	35.1	41.1	68.8	92.6
$y_a - y_p$	-9.8	5.9	-1.1	12.2	-7.6

(b)

y_a	17.8	20.0	21.1	23.4	23.9
y_p	18.1	19.7	21.2	22.8	24.4
$y_a - y_p$	-0.3	0.3	-0.1	0.6	-0.5

(c)

y_a	46.1	47.2	49.8	50.9	53.4
y_p	45.9	47.8	49.6	51.5	53.3
$y_a - y_p$	0.2	-0.6	0.2	-0.6	0.1

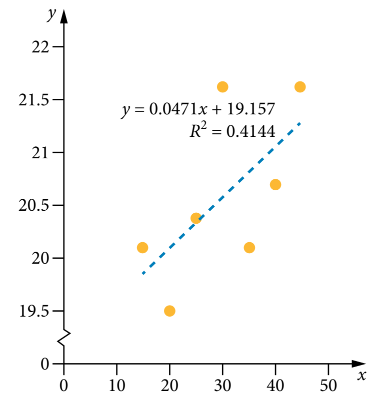
(d)

y_a	50.8	43.5	37.5	32.7	23.6
y_p	50.7	44.1	37.6	31.1	24.6
$y_a - y_p$	0.1	-0.6	-0.1	1.6	-1.0

8 **(a)** $y = 4.2 + x$

(b) $y = 73.6 - 4.7x$

9 **(a)**

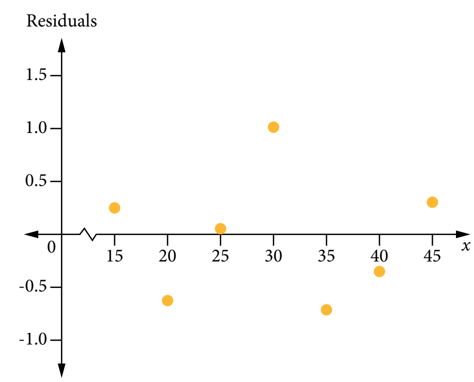


$y = 19.157 + 0.0471x$

(b)

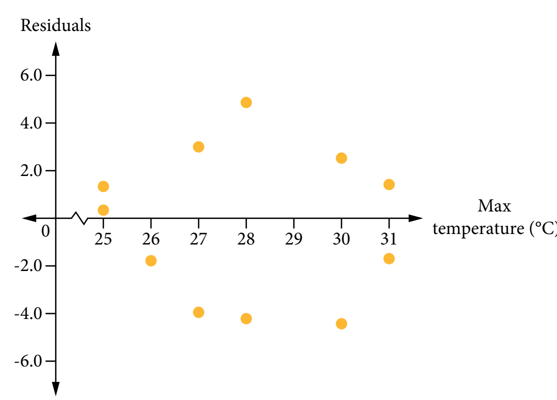
	A	B	C	D
1	x	y	y predicted	residuals
2	15	20.1	19.9	0.2
3	20	19.5	20.1	-0.6
4	25	20.4	20.3	0.1
5	30	21.6	20.6	1.0
6	35	20.1	20.8	-0.7
7	40	20.7	21.0	-0.3
8	45	21.6	21.3	0.3

(c)



The random order of the points confirms linearity.

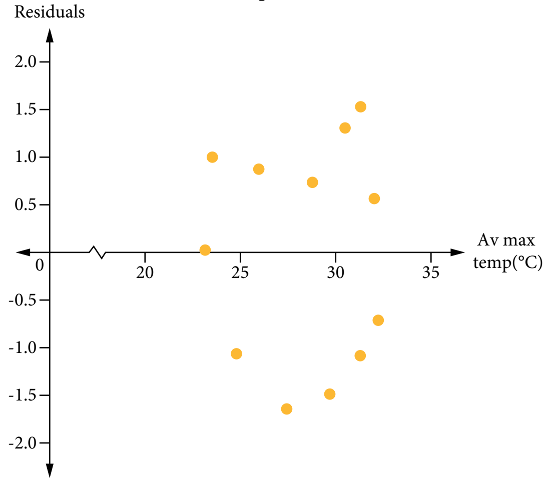
10



Linearity is supported and there is moderate correlation, $r = 0.63$.

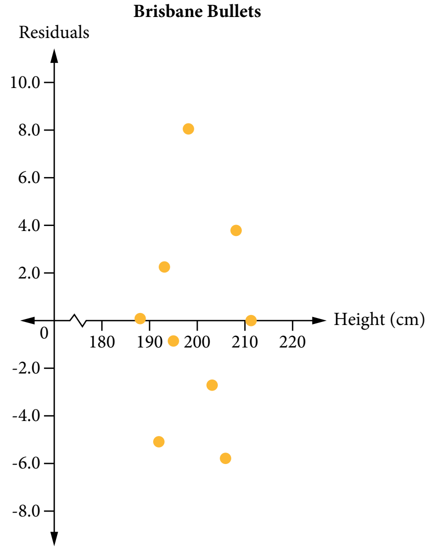
For any month in Cairns:
Number of days of rain = 1.2 × max temp(°C) – 18.4

11 **Rockhampton**



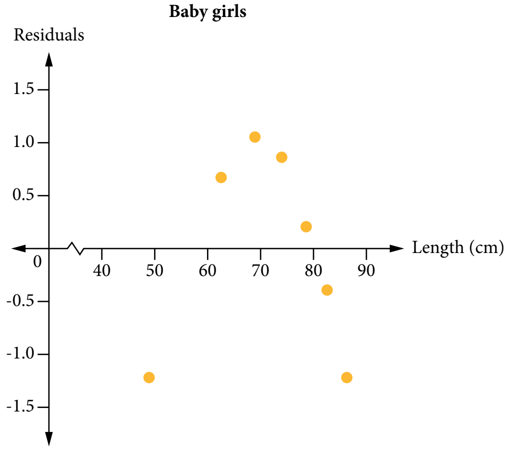
Linearity is supported with strong correlation, $r = 0.97$.
For Rockhampton monthly average temperatures (°C):
Minimum = 1.4 × maximum − 22.2

12 **Brisbane Bullets**



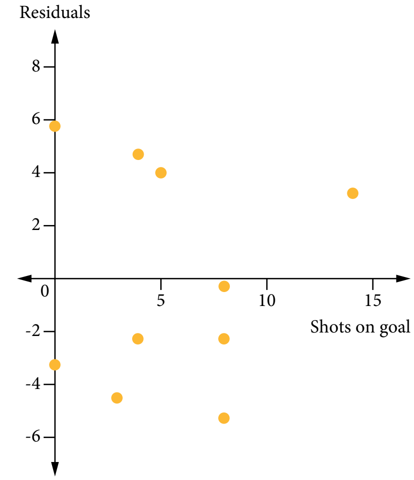
Linearity is supported, but the association is weak, $r = 0.29$.
Points per game = 0.17 × height (cm) − 24.2

13 **Baby girls**



Linearity is not supported, even though the correlation was strong, $r = 0.98$.

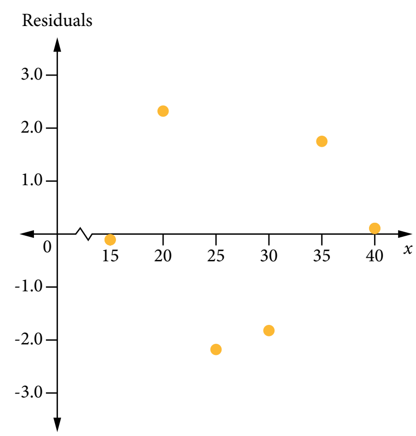
14 **Brisbane Roar FC**



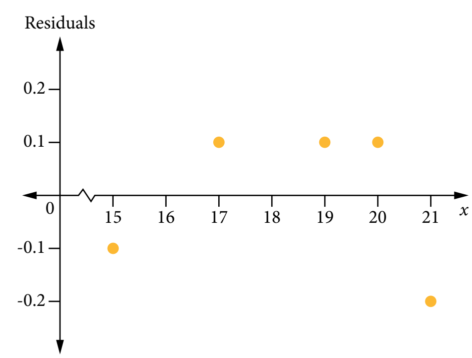
Linearity is supported, but the correlation is weak, $r = -0.25$.
Chances created = -0.25 × shots on goal + 7.25

15 $n = 18$

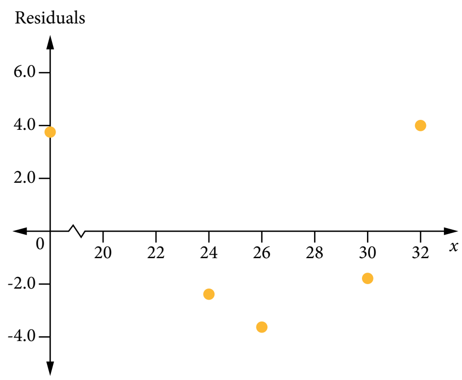
16 **(a)** Suggested points: (20, 22) and (35, 26) give $r = 0.85$.



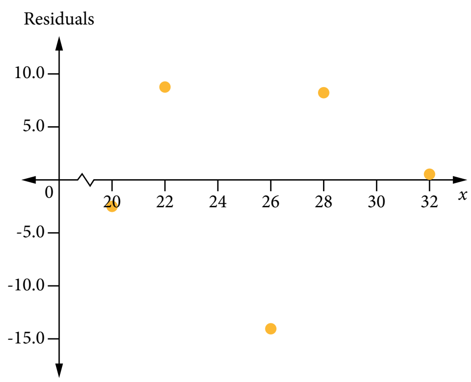
(b) Removed points: (16, 7.7) and (18, 7.3)



17 (a) Suggested points: (24, 47), (26, 50) and (30, 60) give $r = 0.94$

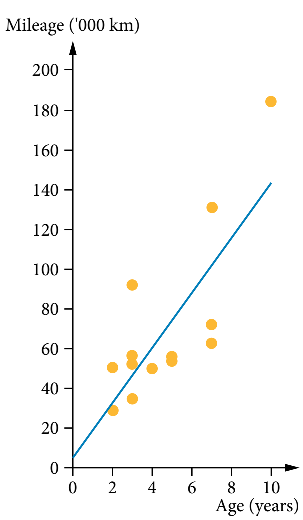


(b) Suggested points: (22, 60), (26, 45) and (28, 70) give $r = 0.69$



EXERCISE 1.6

1 (a)

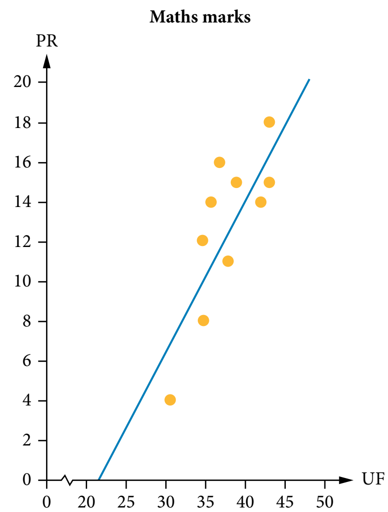


(b) (i) 130 000 km, interpolation

(ii) 5–6 years old, interpolation

(iii) 170 000 km, extrapolation

2 (a)



(b) (i) 14 or 15, interpolation

(ii) 46, extrapolation, reasonable because the point is close to the data points and the value is less than 50.

(iii) 35, interpolation

3 (a) **B**

(b) The student has substituted $x = 12$ instead of $y = 12$. It doesn't make sense that a one-year-old Mazda 3 would be less expensive than a five-year-old Mazda 3.

(c) The student has substituted $x = 12\,000$ or $y = 12\,000$ instead of $y = 12$.

4

Diameter (mm)	Length (m)	Interpolation or extrapolation?
1	0.9	interpolation
3	2.3	interpolation
14	10	extrapolation

5

Height (cm)	Mass (kg)	Interpolation or extrapolation?
150	73.6	interpolation
142.8	70	interpolation
32.8	15	extrapolation

6 (a) 13 200

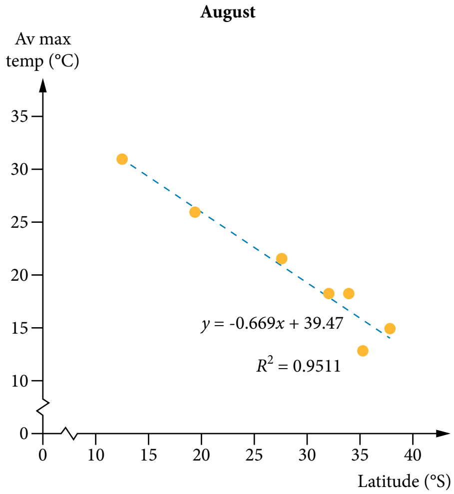
(b) 1400

7 **D**

8 (a) 17.2°C

(b) 26.2°C

9 (a)



August av. max temp (°C) = -0.67 × latitude (°S) + 39.47, $r = -0.98$

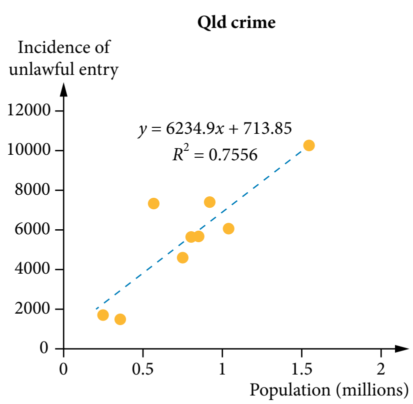
(b) There is a strong negative linear association between the variables. 95% of the variation in the maximum August temperature responds to, or can be explained by, the variation in latitude.

(c) **(i)** Adelaide

(ii) Adelaide 16.1°C; Hobart 10.8°C

(iii) As expected, the value for Hobart was predicted with less accuracy because extrapolation beyond the given data was used.

10 (a)



No. unlawful entries = 6235 × population (millions) + 714, $r = 0.87$

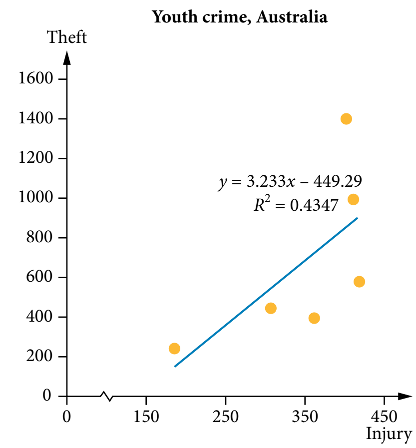
(b) There is a strong positive linear association between the variables.

76% of the variation in the incidence of unlawful entry responds to, or can be explained by, the variation in population; 24% responds to variation in other factors, such as the economy.

(c) 1855

(d) Mackay has a much lower incidence of unlawful entry than you would expect for its population.

11 (a)



Theft rate = 3.23 × injury rate − 449.29, $r = 0.66$

(b) There is a moderate positive linear association between the variables.

43% of the variation in theft rate responds to, or can be explained by, the variation in injury; 57% of the variation in theft can be explained by other factors including the economy and local laws.

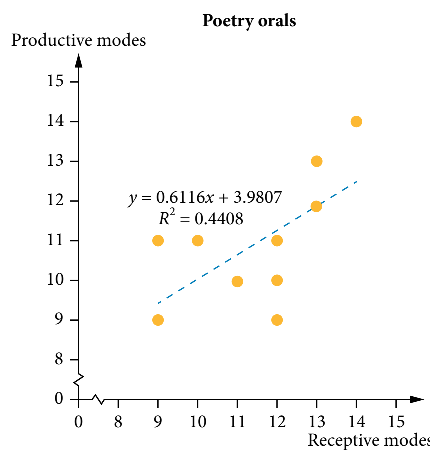
(c) Queensland had fewer youth thefts than expected, considering the number of crimes causing injury.

(d) Northern Territory had a much higher youth crime-causing-injury rate than expected, considering the theft rate.

12 (a)

I	11	14	13	12	13	12	13	9	13	12	10	9
II	10	14	12	9	12	10	13	9	12	11	11	11

(b)



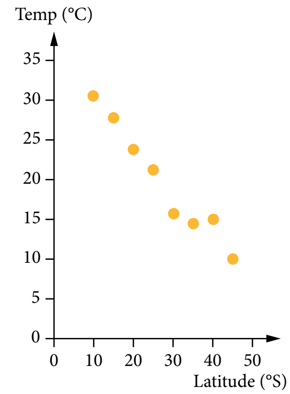
Productive mode result = 0.61 × receptive mode result + 3.98, $r = 0.66$

(c) There is a moderate positive association between the variables.

(d) C*

(e) D*

13 **(a)** **(i)** Explanatory variable: latitude

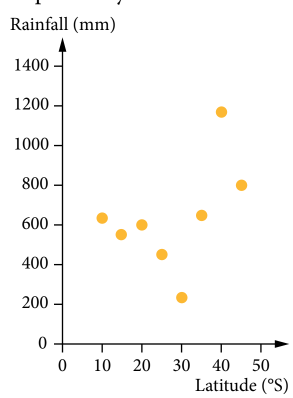


(ii) There is a strong negative linear association. As the latitude increases (going further south), the average maximum temperature decreases.

(iii) $r = -0.98$

(iv) Av. max temp $(°C) = -0.6 \times$ latitude $(°S) + 36.0$

(b) **(i)** Explanatory variable: latitude



(ii) There is a moderate positive linear association. As the latitude increases (going further south), the average rainfall increases.

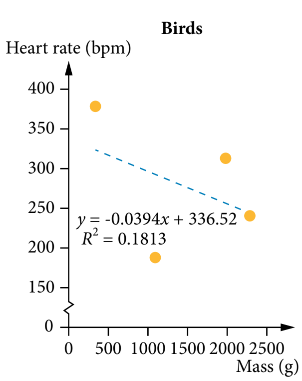
(iii) $r = 0.46$

(iv) Annual rainfall $(\text{mm}) = 10.1 \times$ latitude $(°S) + 360$

(c) The stronger association exists between latitude and temperature. This is evident because the r value is furthest from 0.

(d) 10°S to 45°S

14 **(a)**



Heart rate (bpm) $= -0.040 \times$ mass$(g) + 336.52$, $r = -0.43$

(b) Canary: predicted heart rate 336 bpm; actual heart rate 1000 bpm

Turkey: predicted heart rate -8 bpm; actual heart rate 193 bpm

15 **(a)** Possible set: (3, 88), (4, 104), (6, 88), (7, 80), (8, 88)

(b) $y = -2.14x + 101.58$, $r = -0.51$

(c) For $x = 5$, the actual value was 9 above the value predicted by the model.

(d) For $x = 9$, the actual value was 42 above the value predicted by the model.

(e) Residual value is greater by 33 for the prediction requiring extrapolation.

From the curve you can see that any actual values outside the domain used for the data set will be very different from the model values, with the difference increasing as the x-values become further from the chosen data.

CHAPTER REVIEW 1

1 **(a)** **B** **(b)** **C**

2 **(a)** Age **(b)** Hours spent studying

(c) Intensity of exercise **(d)** Hours of sunlight

(e) Distance travelled

3 **(a)** **D** **(b)** **C** **(c)** **A**

4 Alex: $0.75 \le r \le 1.00$; Sam: $-0.2 < r < 0.2$

5 $-0.6 < r < -1$

6 **(a)** Causal. The time taken to complete a journey depends solely on the speed travelled. A negative non-linear relationship.

(b) Non-causal. A positive relationship. Common response: access to health care.

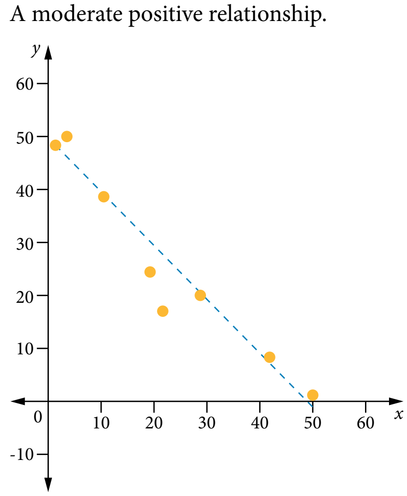
(c) Non-causal. A negative linear relationship. Common response: season.

(d) Non-causal. Confounding variables: age, gender, genetics.

(e) Causal. The cost of petrol depends on the number of litres purchased. Strong positive relationship.

(f) Causal. Job satisfaction depends on the level of education. A moderate positive relationship.

7 **(a)**

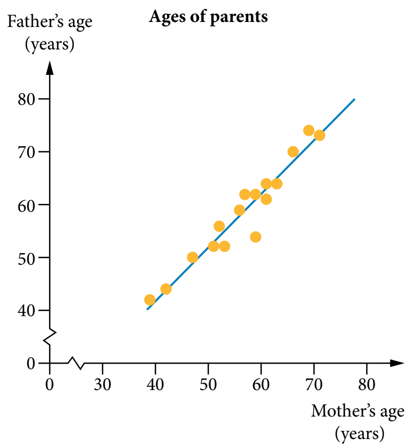


(b) $m = -1$ **(c)** $(0, 48)$ **(d)** $y = -x + 48$

8 -78

9 31 ≤ UF mark ≤ 43 and 4 ≤ PR mark ≤ 18

10 (a)



(b) (i) 52, interpolation

(ii) 58, interpolation

(iii) 78, extrapolation

11 (a) 19.2°C (b) 29.4°C

12 As the mass of food remaining decreases, the number of ants at the scene increases. The linear association is weak.

13 (a) Maximum daily temperature

(b) Positive

(c) People need ice for camping, picnics and parties. More of these occur in summer, so there is a common variable effect. For camping, more ice is needed in hotter weather as it melts faster, so this is a causal effect. For parties, more ice is needed in hotter weather as more cold drinks are needed, so this is a causal effect.

14 $r = \pm 0.8$

15 36%

16 (a) $y = -36.12 + 1.2x$

(b) $y = 2.35 - 0.06x$

17 (a)

Age	Enjoy	No opinion	Dislike	Total
10–20	3%	12%	1%	16%
20–30	3%	4%	3%	11%
30–40	2%	13%	18%	33%
>40	3%	11%	27%	40%
Total	11%	40%	49%	100%

(b)

Age	Enjoy	No opinion	Dislike	Total
10–20	17%	75%	8%	100%
20–30	31%	38%	31%	100%
30–40	6%	40%	54%	100%
>40	7%	27%	67%	100%

(c) In general, the youngest group mostly had no opinion (and might not have heard of hip-hop).

After that, the older the group, the more hip-hop is disliked.

18

	≤ 20	> 20	Total
Females	50%	50%	100%
Males	50%	50%	100%

50% of females and 50% of males had good marks.

From this data there is no association between gender and good marks in Physics.

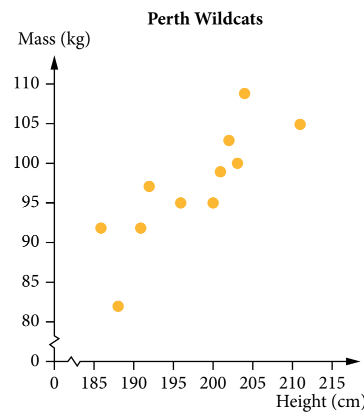
19 0.8621 (4 d.p.), a strong, linear, positive association

20 (a) **C** (b) **B**

21 **D**

22 (a) Mass

(b) Moderate positive linear association



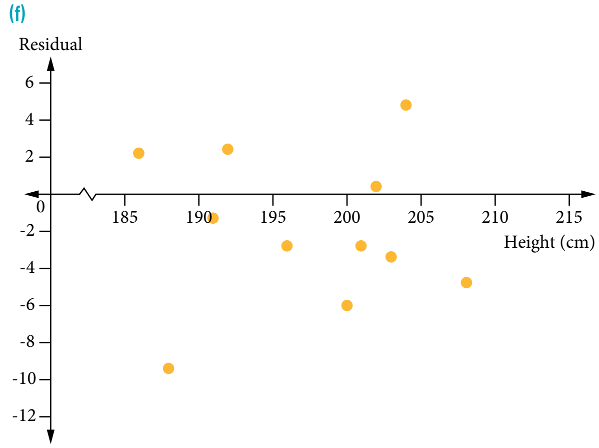
(c) $r = 0.83$

(d) $\text{mass (kg)} = 0.8 \times \text{height (cm)} - 59.0$

(e)

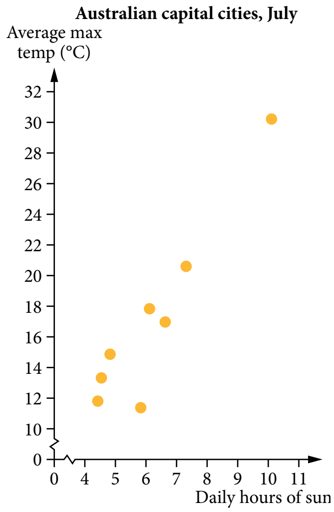
Name	JB	GH	JE	MK	TJ	DU	MM	JW	EB	SR	DM
Mass (kg)	92	99	95	109	105	97	95	100	82	103	92
Predicted mass (kg)	93.8	101.8	101	104.2	109.8	94.6	97.8	103.4	91.4	102.6	89.8
Residual	-1.8	-2.8	-6	4.8	-4.8	2.4	-2.8	-3.4	-9.4	0.4	2.2

(f)



The points seem to be distributed randomly, and therefore the data is assumed to be linear.

23 **(a)**



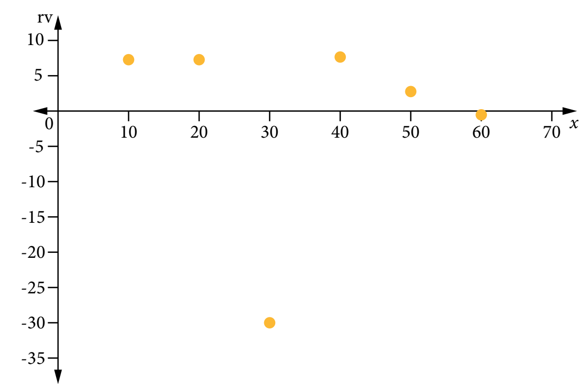
The association appears to be linear.

(b) av. max. temp $(°C) = 3.11 \times$ hours of sun $- 2.13$

(c) **(i)** 6.8 **(ii)** 24.0°C

24 **(a)**

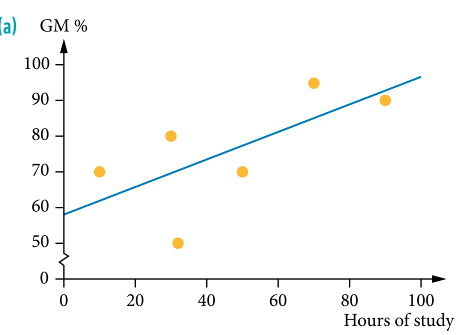
x	10	20	30	40	50	60
y	58	70	45	95	102	110
Predicted value	51	63	75	87	99	111
Residual	7	7	-30	8	3	-1



(b) Unless the third point is an outlier, the points seem random, so the data is assumed to be linear.

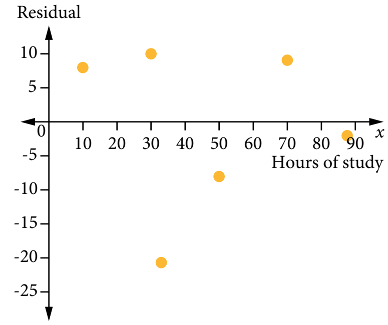
(c) Assuming the association is linear and the variables are not confounding or responding to a common response, the variation in x is responsible for 69% of the variation in y.

25 **(a)**



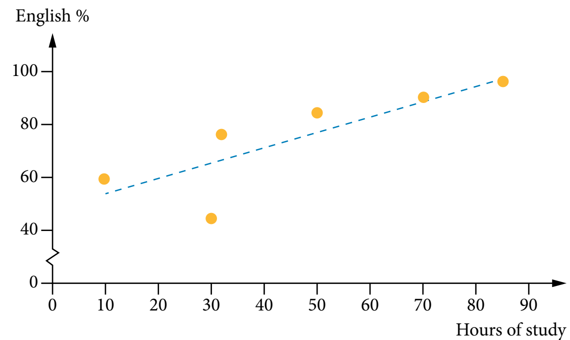
The association is moderate, positive and linear.

GM % $= 0.4 \times$ hours of study $+ 58.0$, with $r = 0.66$



The random pattern of the residuals supports the assumption of linearity.

(b)



The association is strong, positive and linear.
English % $= 0.6 \times$ hours of study $+ 48.4$, with $r = 0.83$

(c) English %: for 60 hours of study 84.4%; for 120 hours of study 120.4%. The first prediction involves interpolation, so it has a reasonable chance of accuracy. Other variables such as ability could drastically affect the accuracy.

The second prediction involves extrapolation and has yielded an impossible value.

(d) There is a clear association, with a higher correlation for English than General Maths. The association will have some confounding variables that are not controlled, such as the effectiveness of the study time and the hours spent on each individual subject (only the total is given). Each student's ability in each subject may have been fairly similar for this group. With a greater range of abilities, a positive association would be expected, but with a weaker correlation. Clearly, extrapolation from the data is likely to give meaningless results.

Chapter 2: Time series analysis

RECALL

1 **(a)** 42 **(b)** 0.2 and 7.2 **(c)** no outliers

2 **(a)** $\bar{x} = 15$ **(b)** $\bar{x} = 3.2$ **(c)** $\bar{x} = 17$

3 **(a)**

x	Relative frequency
12	0.22
13	0.63
14	0.16

(b)

x	83	84	85	86	87
Relative frequency	0.07	0.31	0.37	0.23	0.03

4 (a) $y = 8x - 14$ (b) $y = 201.9 - 3.16x$

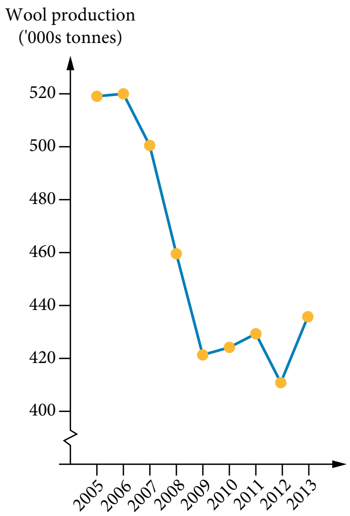
5 (a) $y = 22.7$ (b) $x = 3.8$

6 (a) $m = 40$; the car is travelling away from home at a speed of 40 km/h

(b) $c = 12$; the car was 12 km from home at the start

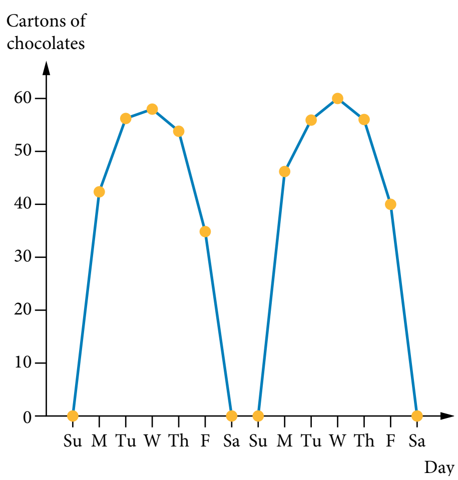
EXERCISE 2.1

1



Negative trend from 2006 to 2009, with little change after that.

2 (a)



(b) The pattern can be described as seasonal with a repeating weekly cycle. Production is greater in the middle of the week. There is no production or sales on the weekends. There does not appear to be any positive or negative trend.

(c) There does not appear to be any positive or negative trend.

3 (a) C

(b) Graph A appears to be cyclical, but there are a different number of points for each cycle.

(c) **B:** negative linear; **D:** positive linear

4 (a) 6 (b) 4 (c) 3 (d) 5

5 Random, irregular time series. Calculate the mean (or median) of the existing values to predict future values.

6 (a) The barley yields are irregular, producing peaks above the median and troughs below the median value.

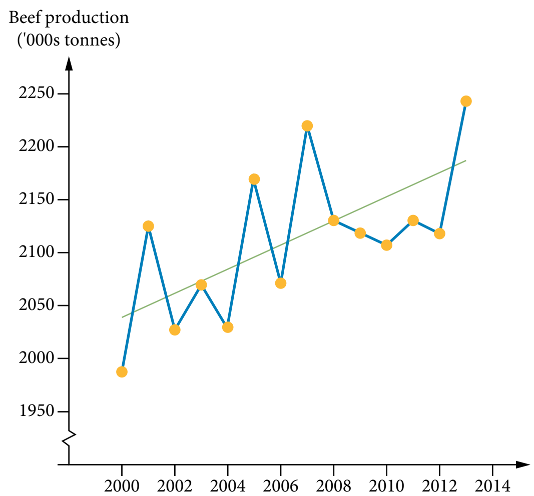
(b) The winning time has a negative linear trend.

(c) Seasonal data. There are approximately 10 peaks per 10 year period, indicating a yearly peak of mumps outbreak. Although the peak still exists each year, the number of cases during the peak decreases significantly from the mid-1950s.

7 (a) **C** (b) **A** (c) **A**

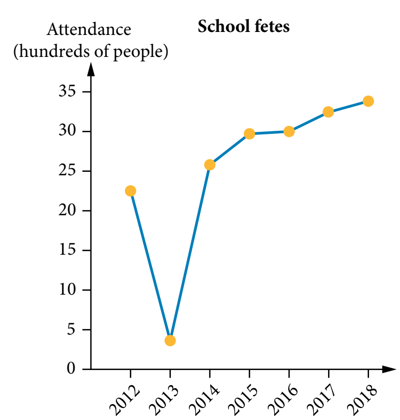
(d) **A** (e) **D**

8



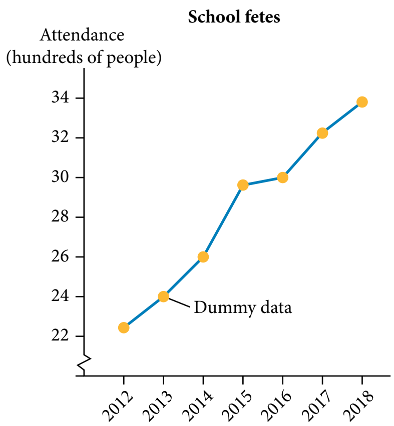
Positive trend

9 (a)



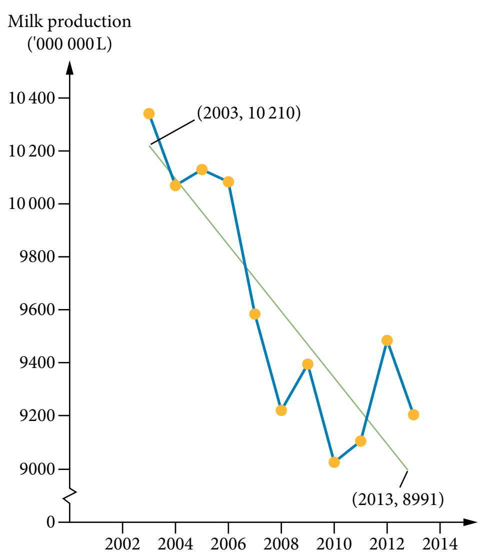
Weather event: 2013

(b) Dummy data: (2013, 24.1)



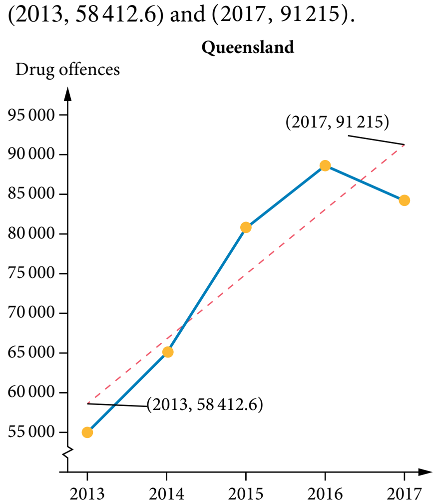
Apart from 2013 when there was a weather event, there is a fairly regular increase each year.

10 **(a)** $y = -122.11x + 254\,796.6$, endpoints of trendline: (2003, 10 210) and (2013, 8991).



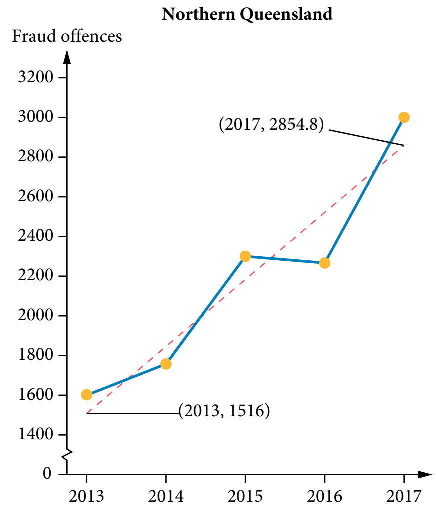
(b) The trendline correctly shows the negative trend; linearity is a poor description for points past, say, 2011.

11 **(a)** $y = 8200.6x - 16\,449\,395.2$; endpoints of trendline: (2013, 58 412.6) and (2017, 91 215).



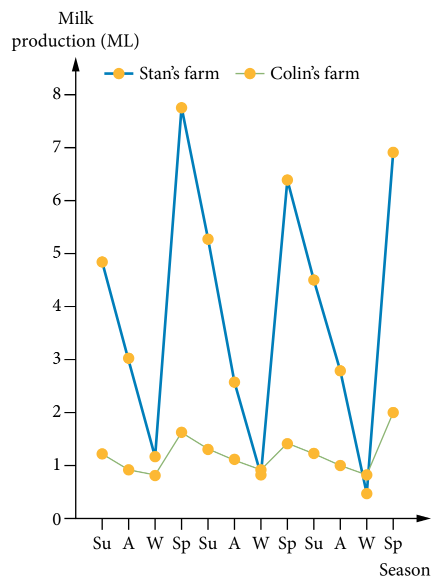
(b) Positive linear trend represented by the line.

12 **(a)** $y = 334.7x - 672\,235.1$; endpoints of trendline: (2013, 1516) and (2017, 2854.8)



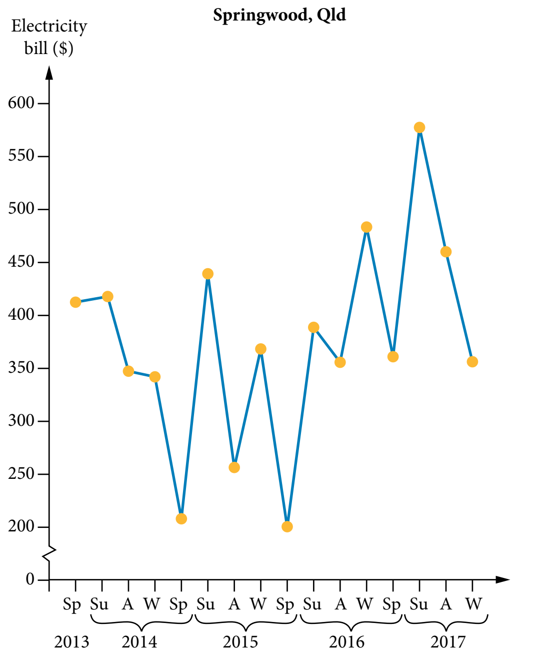
(b) Positive linear trend represented by the line.

13 **(a)**



(b) The seasonal variation for Stan's farm in Victoria is much greater than for Colin's farm in Queensland. Both experience maximum production in spring and minimum production in winter.

14 (a)



Summer and winter are more costly due to air conditioning for cooling in summer, and heaters in winter.

(b) Summer and winter of 2014. Possible dummy data: summer 2014: \$719; winter 2014: \$449.

(c) Summer 2017; possible estimate: \$510

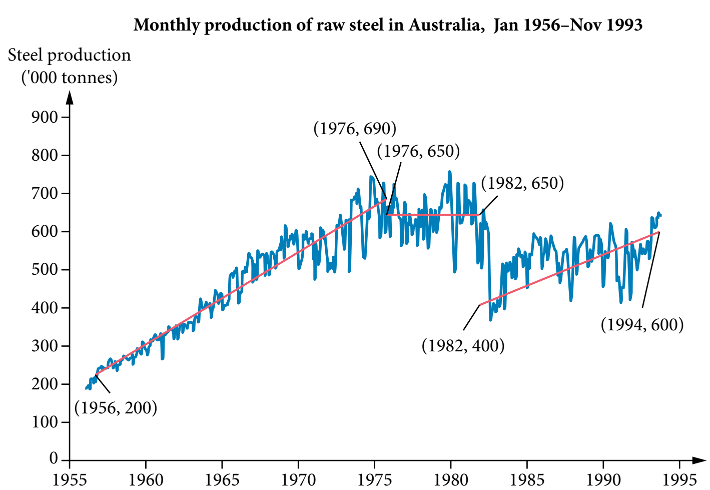
(d) The next point was for autumn/winter, and the last one was for winter/spring, so the usual peak for winter was missing. In future, the peaks for summer and winter and the troughs for autumn and spring will be smoothed out.

(e) August/September 2014 for the solar panels. March/April 2016 for the increased number of people.

15 (a) Production has positive linear trend from 1956 to 1976. It is stable for a few years (linear with a gradient of zero), with a dramatic fall in 1982 followed by a gradual increase (positive linear trend).

(b)

$$y = \begin{cases} 24.5x - 47722, & 1956 \le x < 1976 \\ 650, & 1976 \le x < 1982 \\ 16.7x - 32699.4, & 1982 \le x < 1994 \end{cases}$$



(c) (i) 6 663 000 tonnes **(ii)** 7 800 000 tonnes

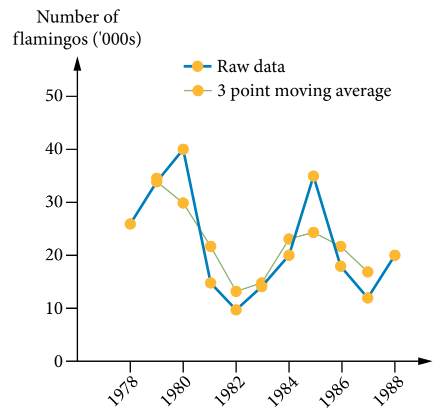
(iii) 6 704 000 tonnes

EXERCISE 2.2

1 (a)

Year	1978	1979	1980	1981	1982	1983	1984	1985	1986	1987	1988
Migration ('000)	26	34	40	15	10	14	20	35	18	12	20
3-point moving mean		33.3	29.7	21.7	13.0	14.7	23.0	24.3	21.7	16.7	

(b)



(c) **D**

(d) The first and final data values are removed as a result of the data being smoothed, but that is not the purpose of smoothing.

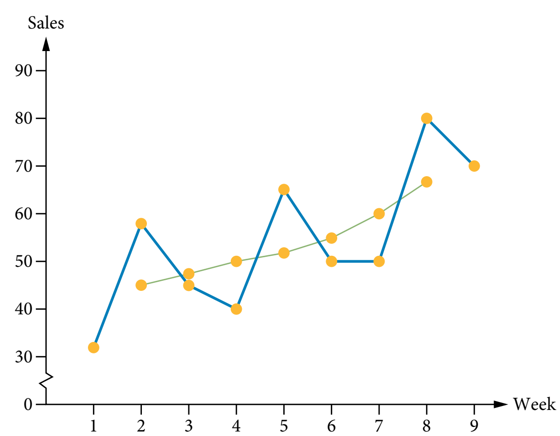
2 **A**

3 15 m

4 Wednesday: \$1.68; Thursday: \$1.70

5

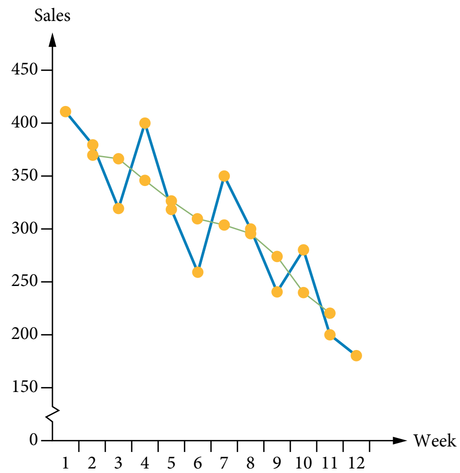
Week	1	2	3	4	5	6	7	8	9
Sales	32	58	45	40	65	50	50	80	70
3-point moving mean		45	47.7	50	51.7	55	60	66.7	



There is a positive trend in the data. The raw data indicated a possible 3 week cycle, which would make a seasonal pattern.

6

Week	1	2	3	4	5	6	7	8	9	10	11	12
Sales	410	380	320	400	320	260	350	300	240	280	200	180
3-point moving mean		370	366.7	346.7	326.7	310	303.3	296.7	273.3	240	220	

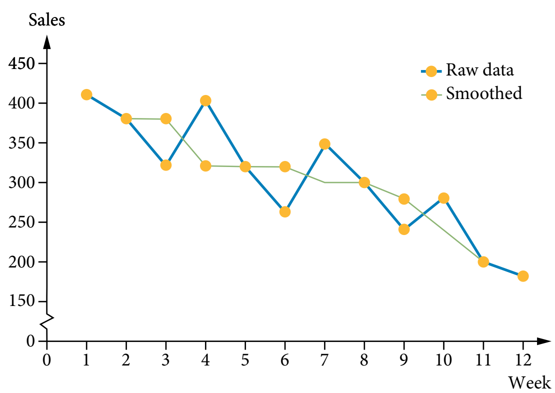


There is a negative trend in the data. The raw data indicated a possible 3 week cycle, which would make a seasonal pattern.

7

Week	1	2	3	4	5	6	7	8	9	10	11	12
Sales	410	380	320	400	320	260	350	300	240	280	200	180
		380	380	320	320	320	300	300	280	240	200	

8



9 28 950

10 131.9 mm

11 (a) 32°C

(b) 31.9°C

(c) 32.1°C

(d) 32.22°C

12

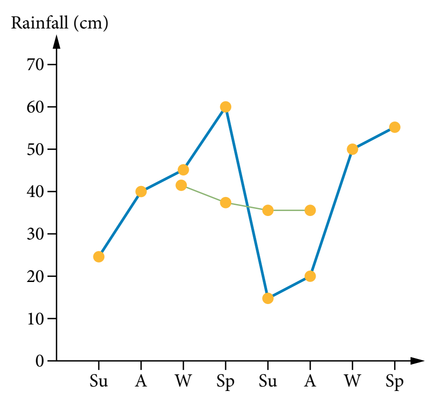
Year	Quarter	Sales	
2012	1	600	
	2	200	
	3	700	475
	4	300	525
2013	1	800	575
	2	400	612.5
	3	900	587.5
	4	400	537.5
2014	1	500	537.5
	2	300	575
	3	1000	
	4	600	

13

Year	Quarter	Sales	
2012	1	600	
	2	200	
	3	700	475
	4	300	525
2013	1	800	575
	2	400	600
	3	900	525
	4	400	450
2014	1	500	450
	2	300	500
	3	1 000	
	4	600	

14 (a)

Season	Sum	Aut	Win	Spr	Sum	Aut	Win	Spr
Rainfall (cm)	25	40	45	60	15	20	50	55
			41	38	36	36		

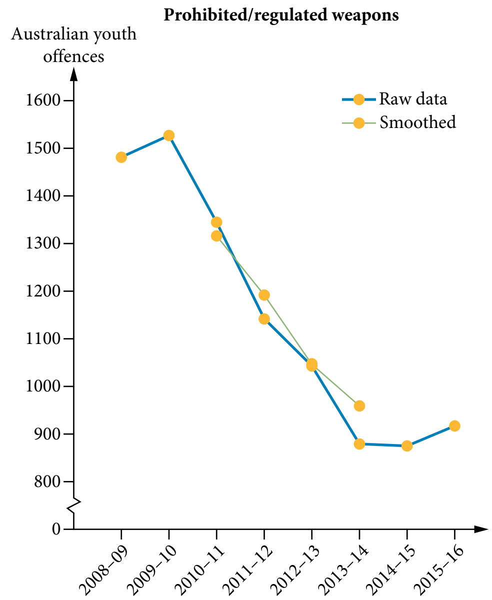


(b) To smooth out the seasonal variation.

15 (a)

2008–09	2009–10	2010–11	2011–12	2012–13	2013–14	2014–15	2015–16
1480	1529	1351	1145	1046	881	878	917
		1322	1187	1047	959		

(b) The smoothed data shows the negative trend. The raw data gives better detail about what happened at the beginning and end of the period.



16 (a)

Data	45	60	50	20	50	75	45	23	60	80	50	25	45	60	70	24
3-point moving mean		51.7	43.3	40	48.3	56.7	47.7	42.7	54.3	63.3	51.7	40	43.3	58.3	51.3	

(b)

Data	45	60	50	20	50	75	45	23	60	80	50	25	45	60	70	24
4-point moving mean			44.4	46.9	48.1	47.9	49.5	51.4	52.6	53.5	51.9	47.5	47.5	44.4		

(c)

Data	45	60	50	20	50	75	45	23	60	80	50	25	45	60	70	24
5-point moving mean			45	51	48	42.6	50.6	56.6	51.6	47.6	52	52	50	44.8		

(d)

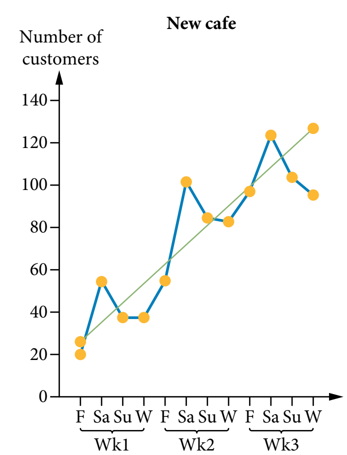
Data	45	60	50	20	50	75	45	23	60	80	50	25	45	60	70	24
12-point moving mean							48.6	48.6	49.5	50.5						

(e) As the data is collected quarterly, the 4-point moving mean is the most appropriate.

17 (a) For the regression equation, number the days, with the opening day being 1 and the final day of data being 12.

x	3	4	5	6	7	8	9	10
y	42.25	52.375	64.25	75.875	86.625	94.5	99.625	103.625

The regression equation is: $y = 9.135x + 18.01$



(b) The trendline allows you to see the overall positive change. Without the raw data you would not be aware of the regular peak values on Saturdays.

18 (a)

710	782	886	847	889	972	993	1005
		822.8	875.2	917.4	941.2		

(b)

1324	1545	1584	1751	2108	2301	2671	2546
		1649	1841.5	2071.875	2307.125		

EXERCISE 2.3

1 $SI_{Dec\text{-}Feb} = 0.67$, $SI_{Mar\text{-}May} = 0.97$, $SI_{Jun\text{-}Aug} = 1.03$, $SI_{Sep\text{-}Nov} = 1.33$

Dec–Feb: 33% below, Mar–May: 3% below, Jun-Aug: 3% above, Sep–Nov: 33% above average season

2 (a) $SI_{Jan} = 1.01$, $SI_{Feb} = 1.16$, $SI_{Mar} = 1.29$, $SI_{Apr} = 1.27$, $SI_{May} = 1.17$, $SI_{Jun} = 1.32$, $SI_{Jul} = 0.96$, $SI_{Aug} = 0.80$, $SI_{Sep} = 0.67$, $SI_{Oct} = 0.76$, $SI_{Nov} = 0.83$, $SI_{Dec} = 0.77$

(b) Compared to the average monthly rainfall of 101.15 mm, Sydney sees 33% less rain in September and 32% more rain in June.

3 (a) $SI_{Jan} = 1.67$, $SI_{Feb} = 1.66$, $SI_{Mar} = 1.47$, $SI_{Apr} = 0.97$, $SI_{May} = 0.77$, $SI_{Jun} = 0.71$, $SI_{Jul} = 0.59$, $SI_{Aug} = 0.48$, $SI_{Sep} = 0.48$, $SI_{Oct} = 0.79$, $SI_{Nov} = 1.02$, $SI_{Dec} = 1.40$

(b) $SI_{Su} = 1.57$, $SI_{Au} = 1.07$, $SI_{Wi} = 0.59$, $SI_{Sp} = 0.76$

4 **C**

5 (a) $x = 1.2$

(b)

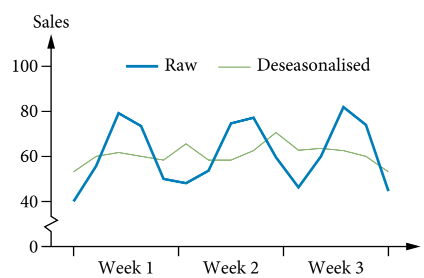
2-month period	Jan–Feb	Mar–Apr	May–Jun	Jul–Aug	Sep–Oct	Nov–Dec
Seasonal index	1.00	0.33	0.67	1.67	1.00	1.33

6 $x = 1.34$. Performance on Friday is 34% higher than the average weekday.

7

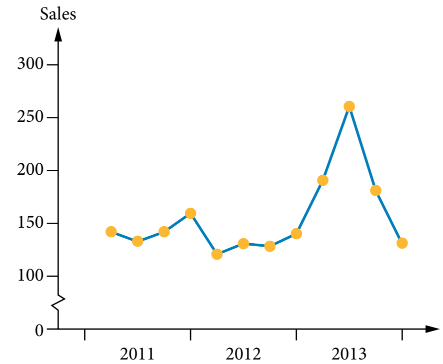
	Summer	Autumn	Winter	Spring
Rainfall (mm)	297.4	378.1	310.4	227.9
Seasonal index	0.980	1.246	1.023	0.751

8



The raw data shows a cyclical pattern, with sales building to a peak on Wednesdays and Thursdays and falling on Fridays and Mondays. The deseasonalised data shows a constant trend of approximately 60 units per day, with better than average sales occurring on the Monday and Friday of week 2.

9 (a)



(b) The data is not seasonal or cyclic. There is an increase in sales for most of 2013.

(c) As the time to turn off the analogue signal approached, those with an analogue set decided to purchase a newer TV so they could receive the digital signal.

(d) There are no apparent recurring patterns visible in the graph to indicate that the data is seasonal. Also, the data contains outliers, so would not be suitable as a means of forecasting sales for the following years.

10 $SI_{summer} = 0.60$, $SI_{autumn} = 1.20$, $SI_{winter} = 1.17$, $SI_{spring} = 1.04$

11 (a) **D**

(b) The student has used the total for each cycle, rather than the average, to determine each index.

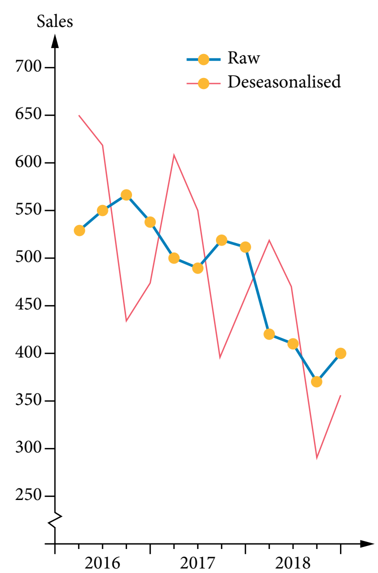
(c) **B**

12 (a)

Quarter 1	Quarter 2	Quarter 3	Quarter 4
1.234	1.134	0.755	0.877

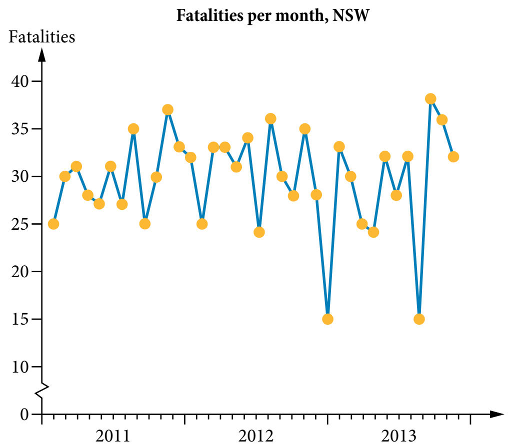
(b)

Seasonal index	Quarter			
	1	2	3	4
2016	527	547	570	536
2017	494	485	517	513
2018	421	414	371	399



(c) There is a negative trend evident. The reason for this cannot be determined.

13 **(a)**

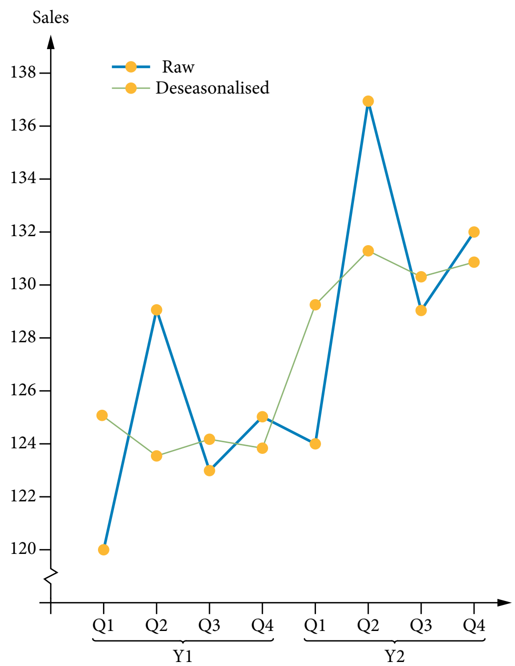


(b) The pattern is irregular, although there are regular peaks in August and November.

(c) $SI_{January} = 0.802$, $SI_{March} = 1.056$, $SI_{May} = 0.919$, $SI_{August} = 1.157$, $SI_{November} = 1.215$, $SI_{December} = 1.048$

(d) A variety of factors are possible, such as wet weather or holidays. NSW does not have a clear wet season, so there is no distinct effect. Holidays are sometimes spread over two months, and some vary (Easter and Labour day), hence the effects are blurred and there is no dramatic difference in the indices.

14



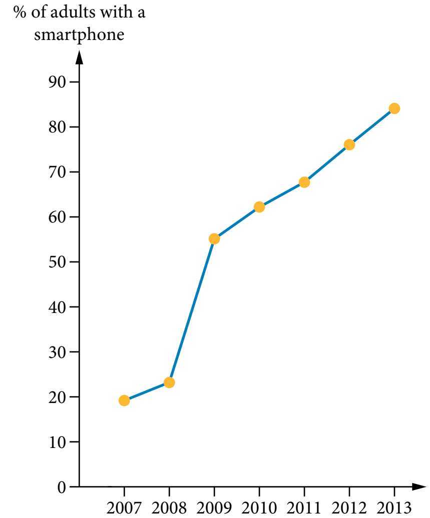
15 **(a)** February, April, July, August, September

(b) March: 226% above average

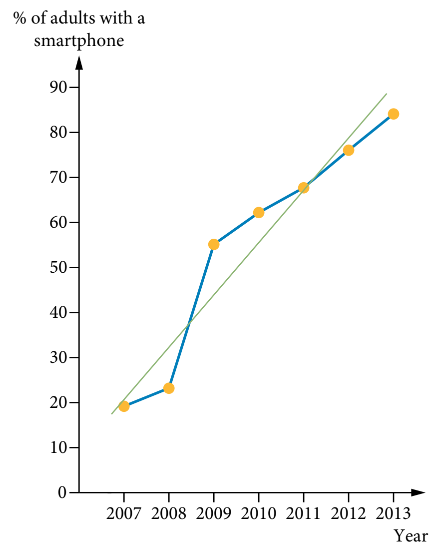
(c) Below average: 246.5 mm

EXERCISE 2.4

1 **(a)**



(b)

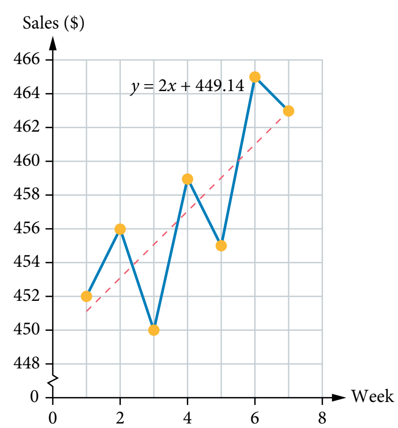


% adults with a smartphone = 11.2143 × year − 22 485.4

(c) 111%

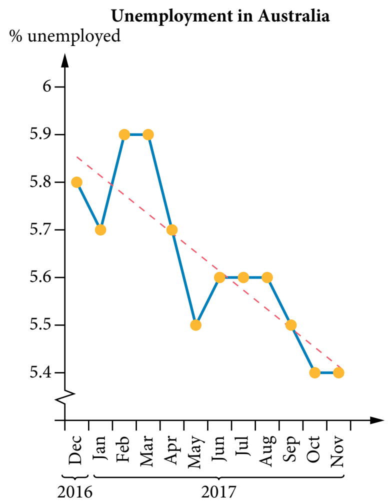
(d) This is an example of extrapolation giving an answer that is not reasonable.

2



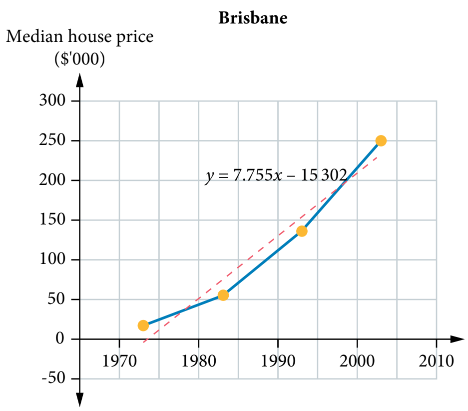
Sales ($) = 2 × week number + 449.14; $489

3



5.4% in December 2017 and 3.9% in December 2020

4 (a)



Median price $(\$'000) = 7.755 \times \text{year} - 15302$

The trendline is not suitable, because the points suggest a curve rather than a straight line.

(b) 1973–1983: \$3800/year; 1983–1993: \$8100/year; 1993–2003: \$11 250/year. Linear model: constant rate of \$7755/year.

5 $x = 43$

6 Monday, week 10

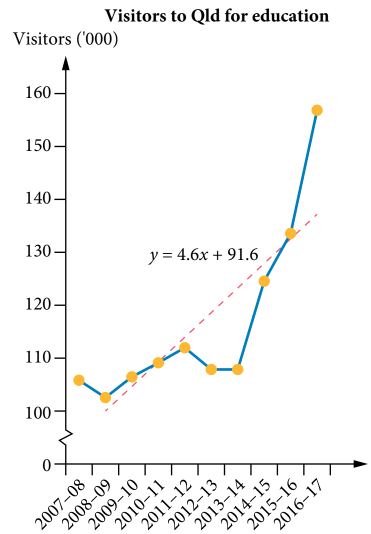
7 **B**

8 $x = 118$

9 (a) **D**

(b) The student has used the time elapsed, instead of counting the initial value as $x = 1$.

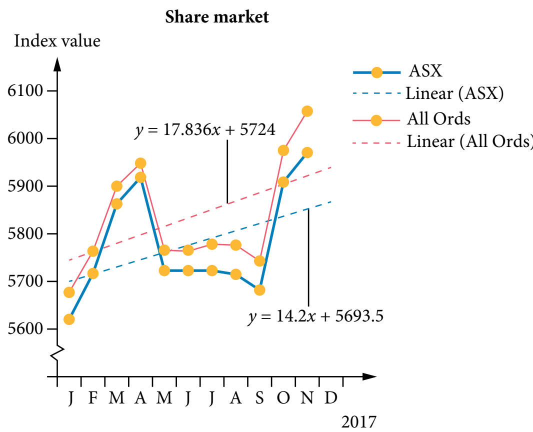
10



The linear model is inappropriate, as the graphed points suggest a curve.

The model predicts 133 000 visitors, which is unreasonable.

11 (a)

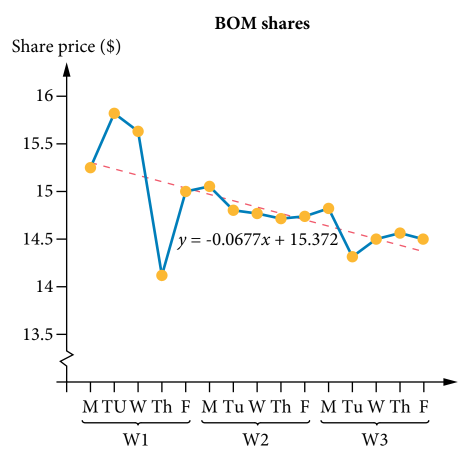


Both graphs show a positive trend.

(b) ASX: 5864; All Ords: 5938. More reasonable values would be numbers closer to the November 2017 figures.

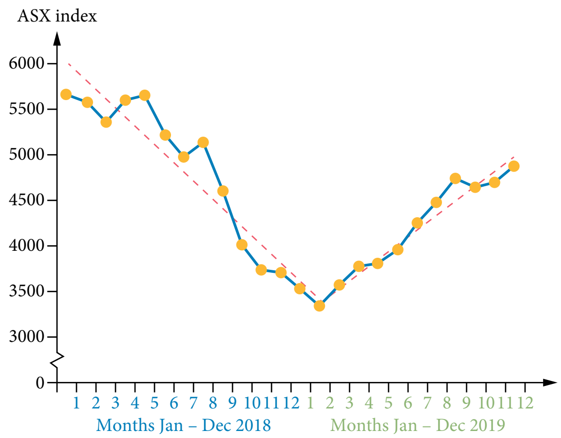
(c) ASX: 6034; All Ords: 6152. The values seem to be reasonable, as the linear model appears to be as appropriate as any other from the data available.

12 (a) Endpoints: (W1 M, 15.304) and (W3 F, 14.357)



(b) \$12.12

13 **(a)**

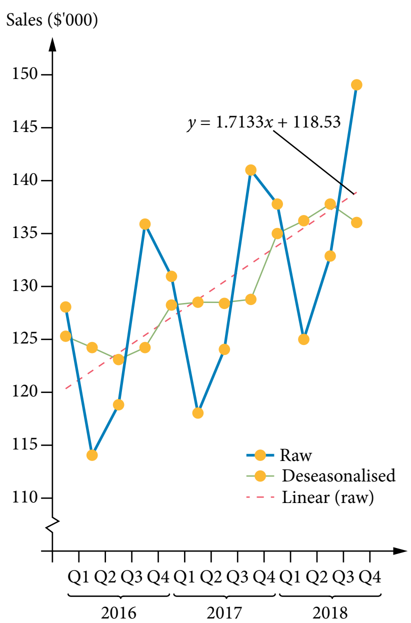


Endpoints of the least-squares regression lines:
(Jan 08, 6006.7) and (Feb 09, 3439.8), then
(Feb 09, 3420.9) and (Dec 09, 4972.5)

(b) March 2010

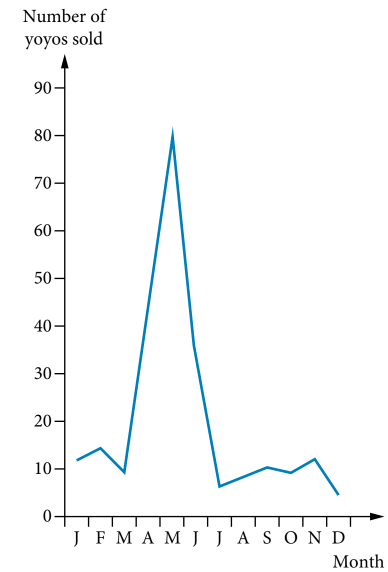
(c) May 2010

14 **(a)** Endpoints: (2016 Q1, 120.2) and (2018 Q4, 139.1)



(b) \$144 000, \$131 000, \$139 000, \$160 000

15 **(a)** $y = -1.84x + 32.39$



(b) -12; the model is clearly not suitable

(c) $y = -0.33x + 11$, 3 yoyos. The prediction is reasonable, but the model predicts a steady decline in numbers, so would give negative values for most of the following year. The other problem is the failure to predict the big sales figure if there is another promotion, as probably happened during this year.

CHAPTER REVIEW 2

1 Random, irregular time series. Calculate the mean (or median) of the existing values to predict future values.

2 **C**

3 Random

4 Seasonal, positive trend

5 600

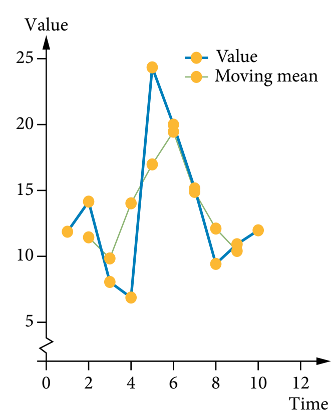
6 27.4

7 $a = 333$, $b = 500$

8 **A**

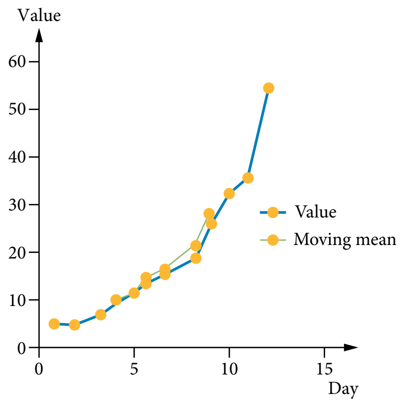
9

Time	1	2	3	4	5	6	7	8	9	10
Value	12	14	8	7	24	20	15	9	11	12
Moving mean		11.3	9.7	13	17	19.7	14.7	11.7	10.7	



10

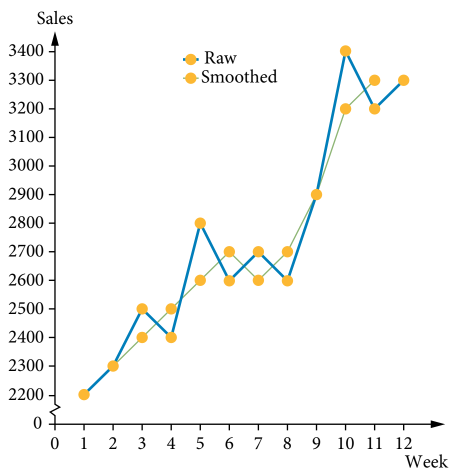
Day	1	2	3	4	5	6	7	8	9	10	11	12
Value	5	5	8	10	12	14	16	19	26	32	35	54
Moving mean				10	12	15	18	22	28			



11

Time	1	2	3	4	5	6	7	8	9	10
Value	12	14	8	7	24	20	15	9	11	12
		12	8	8	20	20	15	11	11	

12



13 $m = 1.08$, $n = 1.52$

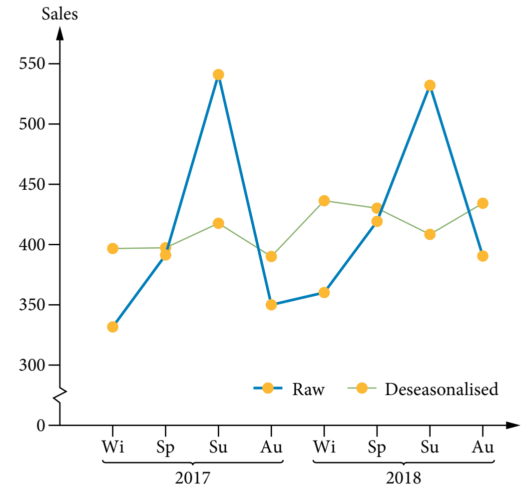
14 $x = 4111$, $y = 3897$

15 1056 jeans

16 (a) $a = 0.90$

(b)

	2017				2018			
Season	Win	Spr	Sum	Aut	Win	Spr	Sum	Au
Raw	330	390	540	350	360	420	530	390
Seasonalised	398	398	419	389	434	429	411	433



(c) Winter, 2018

17 $SI_{Q1} = 0.85$, $SI_{Q2} = 1.09$, $SI_{Q3} = 1.12$, $SI_{Q4} = 0.95$

18

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
2016	390	370	352	380
2017	434	388	372	373
2018	445	405	384	405

19

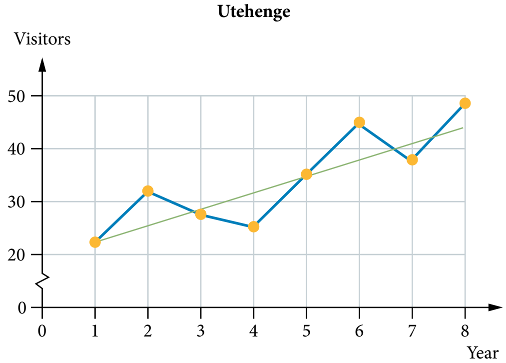
	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday	Sunday
Actual takings ($)	4305	2361	4794	4196	6886	8578	6470

20 $y = 28x + 84$, sales = 28 × week number + 84

21 $y = 254 - 33x$, number of birds = 254 − 33 × week number

22 (a)

Utehenge

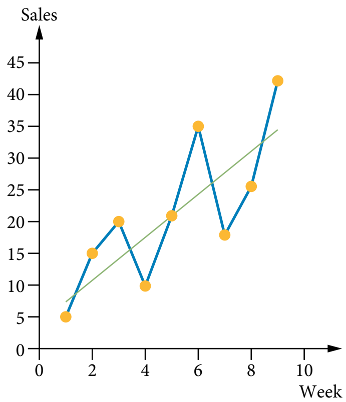


(b) visitors = 19.61 + 3.31 × year

(c) 53

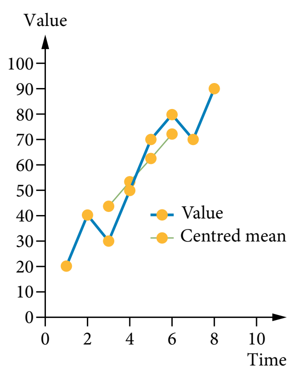
23 sales = 3.37 × week number + 4.5

Endpoints: (1, 7.87) and (9, 34.83)



24

Time	1	2	3	4	5	6	7	8
Value	20	40	30	50	70	80	70	90
Centred moving mean			41.25	52.5	62.5	72.5		



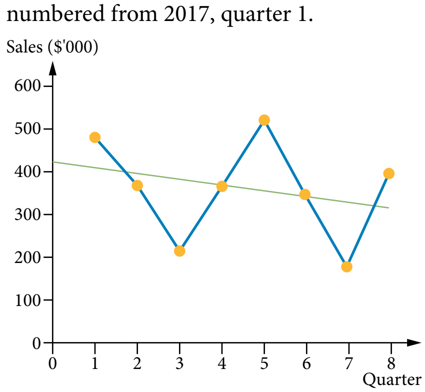
25

Time	1	2	3	4	5	6	7	8
Value	20	40	30	50	70	80	70	90
Centred moving median			40	52.5	65	72.5		

26 D

27 A

28 (a) sales ($'000) = 406 − 11 × quarter where quarters are numbered from 2017, quarter 1.

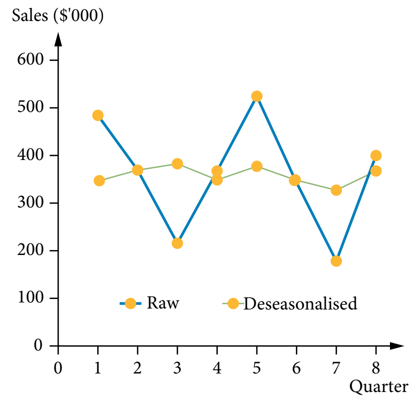


(b)

	1st quarter	2nd quarter	3rd quarter	4th quarter
Seasonal index	1.393 30	0.989 80	0.544 05	1.072 85

(c)

Deseasonalised	1st quarter	2nd quarter	3rd quarter	4th quarter
2017	344.51	363.71	385.99	344.88
2018	373.22	353.61	330.85	372.84

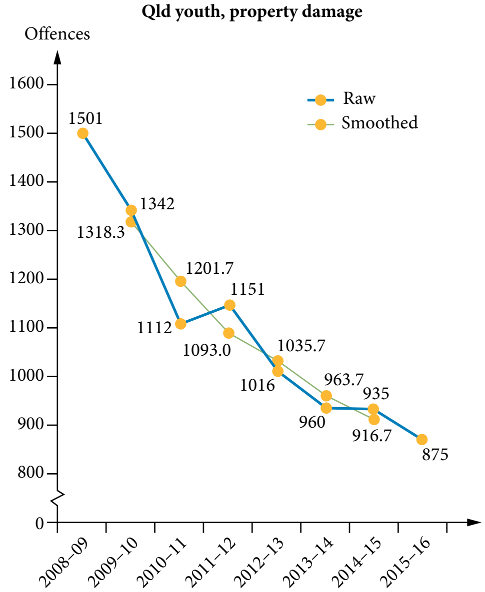


29 Dummy value: 69. A linear trendline appears to be suitable: $y = 4.458x + 56.606$. If the trend continues, the model predicts sales of 132 for May in the following year.

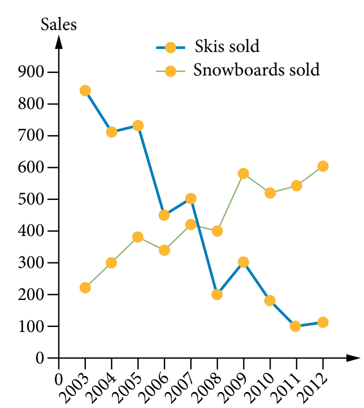
30

2008–09	2009–10	2010–11	2011–12	2012–13	2013–14	2014–15	2015–16
1501	1 342	1 112	1 151	1 016	940	935	875
	1 318.3	1 201.7	1 093	1 035.7	963.7	916.7	

Qld youth, property damage



31 (a)

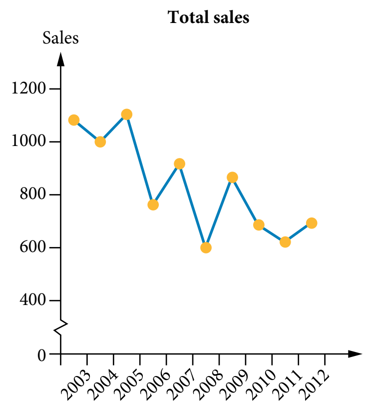


(b) Ski sales = 174 882 – 86.91 × year,
snowboard sales = 39.39 × year – 78 653.33

(c) Skis: -155; snowboards: 678

The predictions are extrapolations. The ski sales result is unreasonable, as the sales of skis is a positive quantity or zero and cannot be negative. The snowboard sales result is reasonable.

(d)



(e) Total sales = 96 288.67 – 47.52 × year

(f) From 2027 onwards

(g) No extrapolation is possible, because the first year gives negative values for ski sales.

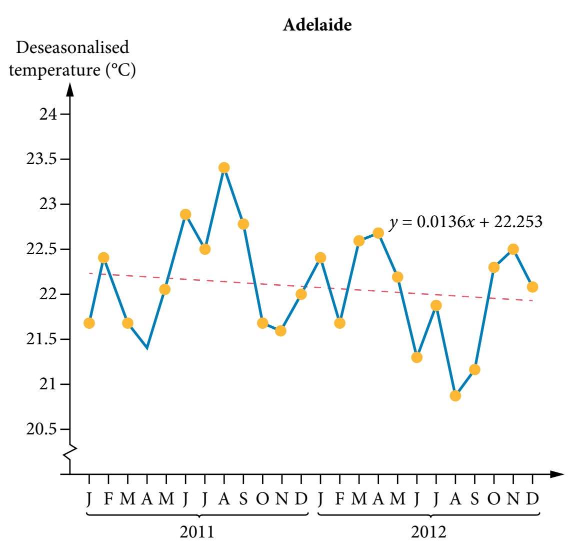
32 (a)

	Jan	Feb	Mar	Apr	May	June	July	Aug	Sep	Oct	Nov	Dec
SI	1.34	1.23	1.12	1.04	0.81	0.72	0.69	0.74	0.90	0.98	1.18	1.22

(b) Deseasonalised data:

	Jan	Feb	Mar	Apr	May	June	July	Aug	Sep	Oct	Nov	Dec
2011	21.7	22.4	21.7	21.4	22.1	22.9	22.5	23.4	22.8	21.7	21.6	22.0
2012	22.4	21.7	22.6	22.7	22.2	21.3	21.9	20.9	21.2	22.3	22.5	22.1

(c)



Deseasonalised temperature
= -0.0136 × month number + 22.253

(d) 29.4°C , 26.9°C , 24.7°C

Chapter 3: Growth and decay in sequences

RECALL

1 (a) 65, 73 **(b)** 47, 40

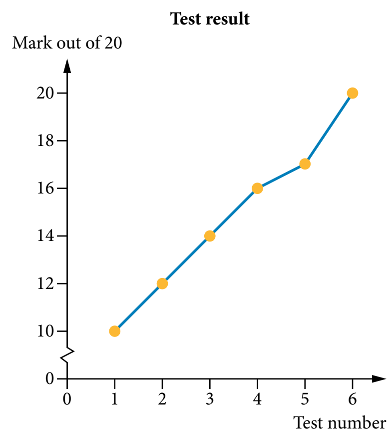
2 (a) 1134, 3402 **(b)** 3, 1

3 (a) $b + (h - 1)e = 85$ **(b)** $ar^{n-1} = 144$

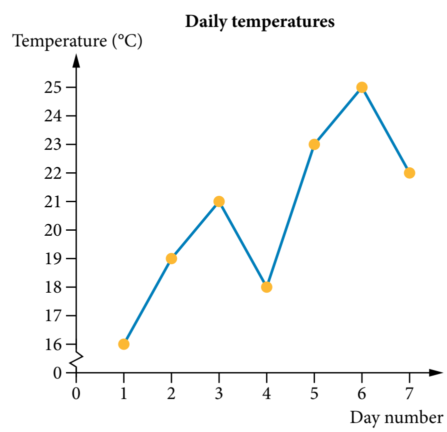
(c) $\frac{u\left(100 - s^3\right)}{10 - s} = 48$

4 (a) 708.5 **(b)** 42.78 **(c)** 20125

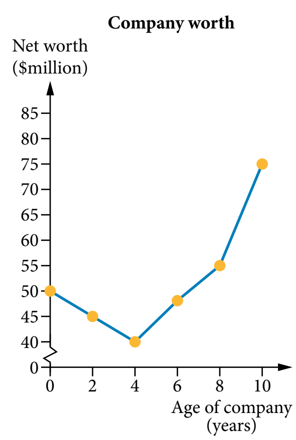
5 (a)



(b)



(c)



6 (a) $x = 5, y = 6$

(b) $x = 57, y = -9$

EXERCISE 3.1

1 (a) 3, 5, 7, 9 (b) 45, 40, 35, 30
(c) 2, 10, 18, 26 (d) 1, 12, 23, 34
(e) 4, -2, -8, -14 (f) 3, 1, -1, -3
(g) 5, 20, 35, 50 (h) -10, 21, 52, 83
(i) 6, -7, -20, -33 (j) 3.5, 3.75, 4, 4.25

2 $d = 36$

3 (a) 6, 8, 10, 12 (b) 8, 11, 14, 17
(c) 12, 10, 8, 6 (d) -9, -5, -1, 3
(e) 21, 18, 15, 12 (f) 64, 70, 76, 82
(g) 33, 28, 23, 18 (h) -52, -32, -12, 8
(i) 5000, 5350, 5700, 6050 (j) 1, 0.667, 0.334, 0.001

4 (a) The sequence is arithmetic. $t_{100} = 497$
(b) The sequence is arithmetic. $t_{30} = 64$
(c) The sequence is not arithmetic.
(d) The sequence is arithmetic. $t_{16} = 106$
(e) The sequence is arithmetic. $t_{51} = 59$
(f) The sequence is arithmetic. $t_{18} = -132$
(g) The sequence is arithmetic. $t_{26} = -63$
(h) The sequence is not arithmetic.
(i) The sequence is arithmetic. $t_{200} = 398$
(j) The sequence is arithmetic. $t_{33} = -184$
(k) The sequence is arithmetic. $t_{73} = -90$
(l) The sequence is arithmetic. $t_{15} = -72$
(m) The sequence is arithmetic. $t_{150} = 2295$
(n) The sequence is arithmetic. $t_{66} = 30.6$
(o) The sequence is arithmetic. $t_9 = 66.25$

5 (a) **B**
(b) There are only 5 (not 6) lots of the common difference separating t_1 and t_6.

6 (a) $t_{20} = 294$ (b) $t_{20} = 193$
(c) $t_{20} = 121.5$ (d) $t_{20} = -0.2$

7 (a) 499 (b) 20

8 (a) 73 (b) 14 (c) -34

9 (a) 8, 14, 20 (b) 4, 24, 44 (c) 19, 28, 37
(d) -11, 11, 33 (e) 5, 14.5, 24 (f) -60, -49.5, -39

10 (a) 130 (b) 86.5 (c) -1780 (d) -3

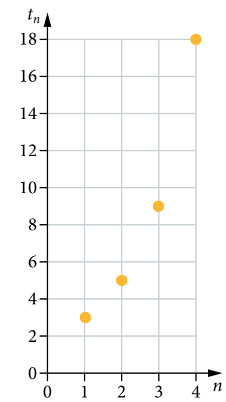
11 **D**

12 (a) $t_n = -6n + 50$; $t_{30} = -130$
(b) $t_n = 38n - 53$; $t_{17} = 593$
(c) $t_n = -11n + 13$; $t_{24} = -251$
(d) $t_n = 120n + 280$; $t_{50} = 6280$

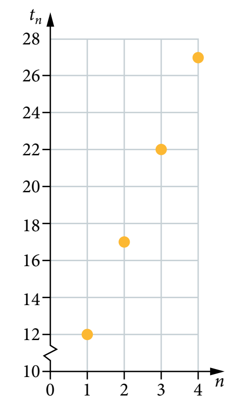
13 (a) t_{17} (b) t_{53} (c) t_{19} (d) t_{26}

14 (a) t_{11} (b) t_{22} (c) t_{151} (d) t_{220}

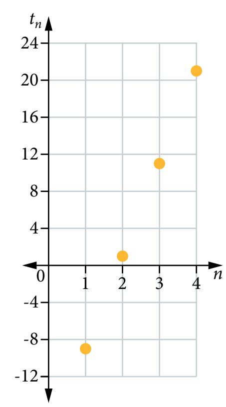
15 (a) -52, -39, -26, -13 (b) 8, 15, 22, 29
(c) -32, -27, -22, -17 (d) 49, 42, 35, 28

16 (a)


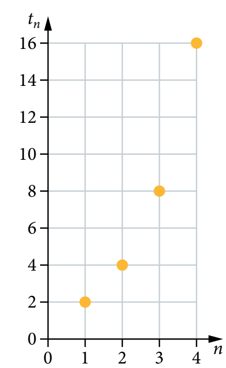
The sequence is not arithmetic.

(b)


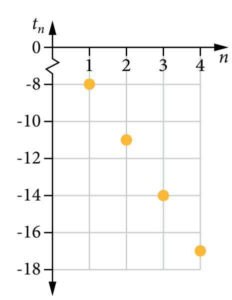
The sequence is arithmetic.

(c)


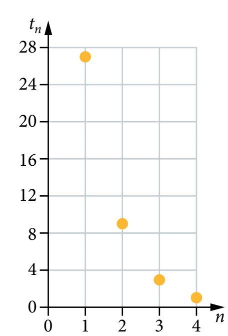
The sequence is arithmetic.

(d)


The sequence is not arithmetic.

(e)


The sequence is arithmetic.

(f)


The sequence is not arithmetic.

17 -33

18 (a) $t_{34} = 1005$ (b) $t_{168} = 5025$

19 $t_1 = 25, t_2 = 22, t_8 = 4, t_9 = 1, t_{11} = -5, t_{17} = -23, t_{18} = -26$.
The sequence has 18 terms.

20 $t_n = 9n - 75$

EXERCISE 3.2

1 \$1.90

2 42 km

3 (a) 185 (b) Saturday, day 13

4 (a) 28 (b) 16

5 (a) **D**

(b) The initial value has $n = 0$ for the number of days (time elapsed), not $n = 1$.

6 (a) \$180 (b) 91 km

7 (a) \$3.15 (b) \$5.00

8 (a) \$111 600 (b) \$5266.25

(c) \$16 392 000 (d) \$259 000

9 (a) \$14 million (b) \$71.7 million

10 (a) \$4130 (b) \$4720 (c) 7 years

11 (a) \$720 (b) 13 years

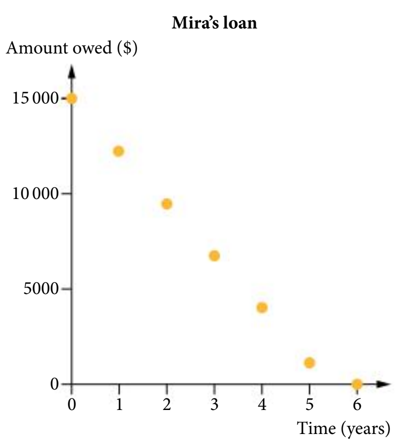
12

Age (years)	Value
0	\$42 000
1	\$35 700
2	\$29 400
3	\$23 100
4	\$16 800
5	\$10 500

13 (a) **A**

(b) **C**

(c)



14 (a) \$1800

(b) \$1560

(c) Assuming that the rate of depreciation continues at a constant rate, 2028

15 (a) \$50 000

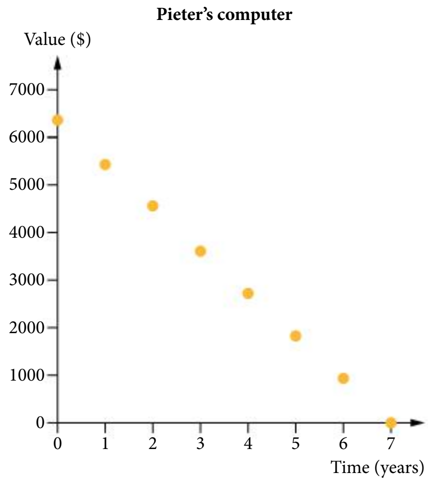
(b) 110 000

16 8 years

17 18.5% p.a.

18 (a) 14.29% p.a.

(b)



EXERCISE 3.3

1 (a) 9, 18, 36, 72 (b) 4, 12, 36, 108

(c) 2, 8, 32, 128 (d) 1, 5, 25, 125

(e) 3, -6, 12, -24 (f) -5, 15, -45, 135

(g) 16, 104, 676, 4394 (h) 200, 220, 242, 266.2

(i) 500, 1100, 2420, 5324 (j) 100, 90, 81, 72.9

(k) 360, 180, 90, 45 (l) 4000, -1000, 250, -62.5

2 (a) $a = 2, r = 200$ (b) $a = 50, r = 3.9$

(c) $a = 2.05, r = 0.88$ (d) $a = 0.003, r = -17$

(e) $a = -4, r = 4$ (f) $a = 25, r = 50$

(g) $a = 200, r = -1.08$

3 (a) **A**

(b) This would be the second term of an arithmetic sequence with $d = -4$.

(c) **D**

4 (a) 6, 12, 24, 48

(b) 1, 10, 100, 1000

(c) 20, 60, 180, 540

(d) -2, -6, -18, -54

(e) -5, 10, -20, 40

(f) -6, 15, -37.5, 93.75

(g) 243, 81, 27, 9

(h) 40, 5, 0.625, 0.078 125

(i) 6000, 6120, 6242.4, 6367.248

(j) 5000, 4950, 4900.5, 4851.495

(k) -0.75, -1.5, -3, -6

(l) 10 000, 10 500, 11 025, 11 576.25

5 (a) This is a geometric sequence. $t_{10} = 98\,415$

(b) This is a geometric sequence. $t_8 = 7.8125$

(c) This is a geometric sequence. $t_{21} = 3\,145\,728$

(d) This is a non-geometric sequence.

(e) This is a geometric sequence. $t_{13} = -4096$

(f) This is a geometric sequence. $t_9 = 0.000154$

(g) This is a geometric sequence. $t_5 = 0.06$

(h) This is a geometric sequence. $t_7 = 531\,441$

6 (a) This is a geometric sequence. $t_{11} = 3814.697$

(b) This is a geometric sequence. $t_{15} = 68.355$

(c) This is a non-geometric sequence.

(d) This is a non-geometric sequence.

(e) This is a geometric sequence. $t_{50} = 185\,800.857$

(f) This is a geometric sequence. $t_{10} = 1551.328$

(g) This is a geometric sequence. $t_6 = 411.523$

(h) This is a geometric sequence. $t_{14} = 0.809$

7 (a) $t_n = 13(3)^{n-1}$ (b) $t_n = 6(-6)^{n-1}$

(c) $t_n = -1000(2)^{n-1}$ (d) $t_n = 552(0.65)^{n-1}$

(e) $t_n = \frac{8}{3}(3.2)^{n-1}$

8 (a) $a = 77$ (b) $a = 12$ (c) $a = 105$ (d) $a = -40$

9 (a) $r = 2.5$ (b) $r = \pm 0.9$ (c) $r = \pm 2.2$ (d) $r = 7$

10 **B**

11 (a) $t_n = 2(10)^{n-1}$, $t_6 = 200\,000$

(b) $t_n = 15(3)^{n-1}$, $t_5 = 1215$

(c) $t_n = 1.5(2)^{n-1}$, $t_7 = 96$

(d) $t_n = 100(0.5)^{n-1}$, $t_6 = 3.125$

(e) $t_n = 6000(0.8)^{n-1}$, $t_4 = 3072$

(f) $t_n = -23(-7)^{n-1}$, $t_5 = -55\,223$

(g) $t_n = 35(-6)^{n-1}$, $t_8 = -9\,797\,760$

(h) $t_n = 0.11(7.9)^{n-1}$, $t_4 = 54.234\,29$

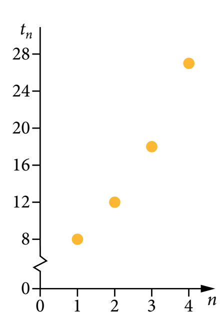
12 (a) 12th (b) 10th (c) 6th (d) 17th

13 (a) $n = 6$ (b) $n = 4$ (c) $n = 3$ (d) $n = 2$

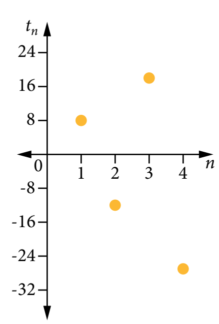
14 (a) 2, 6, 18, 54 (b) 4, 16, 64, 256

(c) -1.3, -6.5, -32.5, -162.5 (d) 6, -6, 6, -6

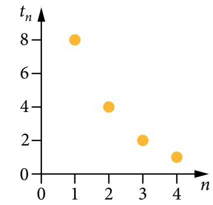
15 (a)



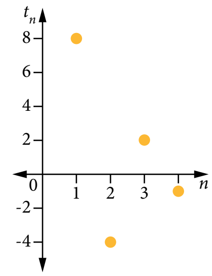
(b)



(c)



(d)



16 (a) $t_9 = 2\,343\,750$ (b) $t_{24} = 0.000\,000\,443\,32$

17 $t_1 = 2000$, $t_2 = -600$, $t_3 = 180$, $t_8 = -0.4374$ or $t_1 = -2000$, $t_2 = 600$, $t_3 = -180$, $t_8 = 0.4374$

The sequence has 8 terms.

EXERCISE 3.4

1 87 mm

2 $320

3 1.2 billion

4 6561 drops/min

5 $6144

6 $12 290

7 (a) **C**

(b) 0.35 has 2 decimal places, but 3.5% = 0.035.

8 **D**

9 (a) $109 399 (b) $15 055 243

(c) $190 180 (d) $274 676

10 (a) $859 (b) $1403 (c) $544

11 $2.62

12 $48.4 million

13 (a) 819 200 m^2 (b) 13 days

14 (a) 315 worms (b) 2022

15 (a) 31 505 (b) 2024 (c) 3229

16 32 days

17 7 episodes

18 (a) 6.05 m (b) after 7 bounces

19 (a)

Year	Value
0	$4 600
1	$3 680
2	$2 944
3	$2 355
4	$1 884
5	$1 507

(b)

Year	Claim
0	
1	$920
2	$736
3	$589
4	$471
5	$377
Total	$3 093

20 (a)

Year	Value	Depreciation
0	$2 750 000	
1	$2 310 000	$440 000
2	$1 940 400	$369 600
3	$1 629 936	$310 464

(b) 14 years

21 (a)

Year	Balance
0	$500.00
1	$537.50
2	$577.81
3	$621.15
4	$667.73
5	$717.81

(b) $217.81

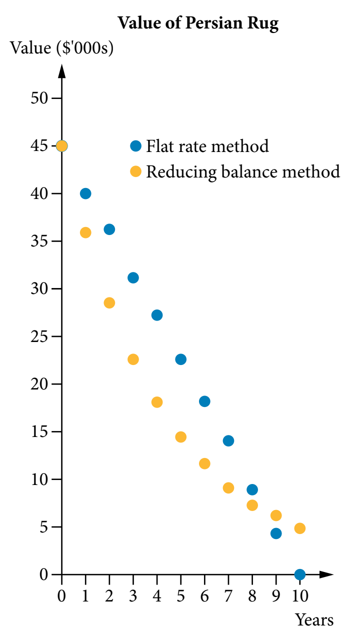
22 12 years

23 (a) 10 years (b) $51 723.70

24 28 days

25 (a)

Year	Flat rate value	Reducing balance value
2015	$45 000	$45 000
2016	$40 500	$36 000
2017	$36 000	$28 800
2018	$31 500	$23 040
2019	$27 000	$18 432
2020	$22 500	$14 746
2021	$18 000	$11 796
2022	$13 500	$9437
2023	$9000	$7550
2024	$4500	$6040
2025	$0	$4832



(b) Rupert should choose the reducing balance method, as each deduction ($9000, $7200 and $5760) would be more than the constant $4500 of the flat rate method.

(c) From 2020 to 2025

CHAPTER REVIEW 3

1 $t_1 = 3, t_2 = 20$

2 12

3 **C**

4 $m = 42$

5 -522, -387, -252, -117

6 $365.25

7 (a) $140 (b) 10 hours

8 (a) 101 lengths (b) 50 sections

9 $420

10 (a) Arithmetic sequence, $d = 7$

(b) Geometric sequence, $r = 5$

(c) Arithmetic sequence, $d = -2$

11 (a) -3, -12, -48, -192, -768 (b) -2, 13, 28, 43, 58

(c) 100, 77, 54, 31, 8

12 (a) $t_9 = 112$, sum of terms: 984

(b) $t_9 = 102.515625$, sum of terms: 1029.97

13 (a) 51, 48, 45 (b) 1215, 405, 135

(c) 102, 120, 138 (d) 22, -88, 352

14 (a) $t_n = 1875\left(\frac{1}{5}\right)^{n-1}$ (b) $t_n = 1966 - 91n$

15 $t_{n+1} = 3t_n$ with $t_1 = 19$

16 258

17 **D**

18 $682\frac{5}{8}$

19 61

20 375 tonnes

21 $14 409

22 $1212.01

23 (a) 64, 256, 1024, 4096, 16 384 (b) 268 435 456

24 **D**

25 **B**

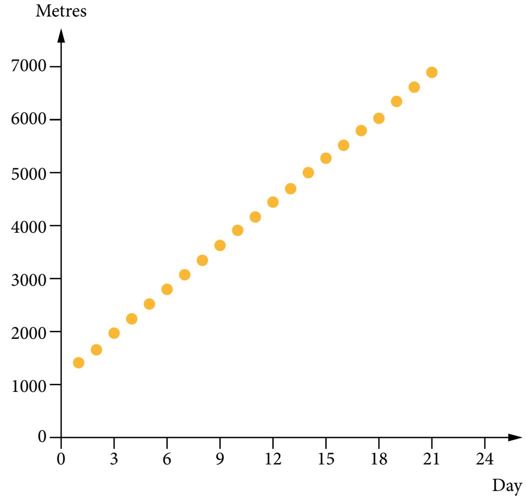
26 (a) $115 960 (b) 12th year

27 (a) $1360 (b) 2018

28 (a) 8 (b) 11

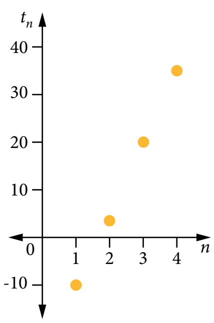
29 (a) 6600 m

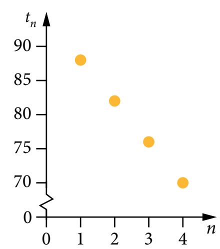
(b)



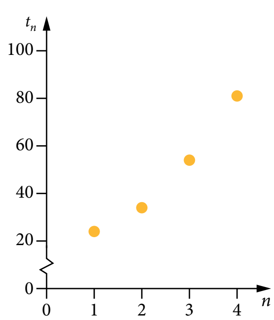
30 (a) -60, 73, 206 (b) 11, 33, 99

31 (a) -10, 5, 20, 35 (b) 88, 82, 76, 70





(c) 24, 36, 54, 81



32 9, -18, 36, -72

33 5 years

34 $2248

35 (a) $6 893 749.49 (b) 7 years

36 (a) $1079 (b) 7 years

37 (a) 9.3 cm (b) 6th (c) 198 cm

38 9.75%

39 (a) (i) 0.2 mm (ii) 3.2 mm (iii) 10.2 cm (iv) 104.9 m

(b) 24 times

(c) 42 times

40 (a) $N = 6^n$

(b)

n	N	Total
1	6	6
2	36	42
3	216	258
4	1 296	1 554
5	7 776	9 330
6	4 665	55 986
7	279 936	335 922

(c) Layer 13

Chapter 4: Earth geometry

RECALL

1 (a) 0001 (b) 1450 (c) 10:10 pm

2 (a) 53.7° (b) 125.25° (c) 2.67°

3 (a) 0.8660 (b) 0.9759 (c) 0.0872

4 (a) 145 cm (b) 1335 m

5 (a) 24 m (b) 148 mm (c) 539 km

6 (a) 8 h 25 min (b) 17 h 50 min

7 28.9 m

8 21.9 cm

EXERCISE 4.1

1 (a) Jerusalem (b) Rio de Janiero

(c) Mombasa (d) Dublin

2 (a) 30°N, 60°E (b) 0°, 40°W

(c) 20°S, 50°W (d) 25°S, 150°E

3 (a) **B**

(b) All of Australia is in the eastern hemisphere, including Western Australia.

4 **A**

5 (a) Caracas, Venezuela

(b) Ho Chi Minh City (Saigon), Vietnam

(c) Dunedin, New Zealand

(d) Salvador, Brazil

6 (a) Cloncurry, Queensland (b) Ballarat, Victoria

(c) Birdsville, Queensland (d) Parkes, NSW

7 (a) Osaka, Japan (b) Seoul, South Korea

(c) Singapore (d) Phnom Penh, Cambodia

8 (a) 36°N, 4°E (b) 9°N, 39°E

(c) 15°N, 17°W (d) 4°S, 22°E

9 (a) 27°28′S, 153°2′E (b) 33°52′S, 151°13′E
(c) 35°17′S, 149°8′E (d) 37°49′S, 144°58′E

10 (a) 42.88°S, 147.33°E (b) 34.93°S, 138.6°E
(c) 31.95°S, 115.85°E (d) 12.47°S, 130.85°E

11 (a) 28°2′S, 153°26′E (b) 27°56′S, 153°20′E
(c) 28°2′S, 153°17′E (d) 28°5′S, 153°22′E

12 (a) 11°9′S, 142°40′E (b) 10°54′S, 142°23′E
(c) 10°5′S, 142°13′E (d) 10°48′S, 142°33′E

13 (a) 23°10′S, 150°29′E (b) 23°8′S, 150°44′E
(c) 23°22′S, 150°32′E (d) 23°20′S, 150°46′E

14 (a) San Antonio, Texas: 29.8°N, 98.4°W
(b) Santa Fe, New Mexico: 35.4°N, 106.1°W
(c) Minneapolis, Minnesota: 44.6°N, 93.3°W

15 26°29′S, 147°57′E

EXERCISE 4.2

1 (a) 36° (b) 19° (c) 59° (d) 24°

2 (a) 101°43′ (b) 7°25′ (c) 8°53′ (d) 62.4°

3 (a) **C**
(b) Angular distances cannot be more than 180°, so use the minor sector angle.

4 (a) 7006 km (b) 6016 km (c) 2513 km (d) 1668 km

5 (a) **A** (b) **D** (c) **B**

6 (a) 28°48′ (b) 150°58′ (c) 174° (d) 8°

7 (a) 1171 km (b) 4796 km (c) 226 km (d) 18 151 km

8 10°

9 (a) 4000 km
(b) 3800 km
(c) (i) 1400 km; 1390 km (ii) 700 km; 734 km
(iii) 1600 km; 1603 km (iv) 2400 km; 2509 km

10 (a) 13 650 km (b) 76.41 km (c) 8091 km (d) 16 000 km

11 (a) 2417 km (b) 1680 km (c) 403 km (d) 181 km

12 (a) 9913 km (b) 11 332 km (c) 9722 km (d) 1992 km

13 (a) 485 km (b) 1918 km (c) 328 km (d) 540 km

14 (a) 180°; 20 016 km
(b) (i) 25°N, 30°W (ii) 41°22′S, 156°44′E

15 (a) 475 km; averaged the east–west amounts for the different parallel of latitude because they were different. Assumed the east–west distances were close enough to the shortest distance between points.
(b) 72 km; same assumptions as part (a)

EXERCISE 4.3

1 (a) *B* is ahead of *A* by 5 h 12 min.
(b) *B* is ahead of *A* by 1 h 12 min.
(c) *A* is ahead of *B* by 2 h 12 min.
(d) *B* is ahead of *A* by 6 h 40 min.

2 (a) *B* is ahead of *A* by 6 h 40 min.
(b) *A* is ahead of *B* by 11 h 4 min.
(c) *B* is ahead of *A* by 9 h 12 min.
(d) *A* is ahead of *B* by 6 h 12 min.

3 (a) 7:45 am (b) 9:15 am (c) 10:45 am (d) 10:15 am

4 (a) **A** (b) **D** (c) **A** (d) **B**

5 (a) 9:25 am in Brisbane, 10:25 am in Sydney, 10:25 am in Canberra, 10:25 am in Hobart, 9:55 am in Adelaide, 8:55 am in Darwin, 7:25 am in Perth
(b) noon in Brisbane: 1:00 pm in Melbourne; noon in Sydney, Canberra, Hobart: noon in Melbourne; noon in Adelaide: 12:30 pm in Melbourne; noon in Darwin: 1:30 pm in Melbourne; noon in Perth, 3:00 pm in Melbourne

6 (a) 9:00 pm New Year's Eve (b) 11:00 pm New Year's Eve
(c) 10:00 am New Year's Day (d) 3:00 pm New Year's Day

7 (a) UTC + 5 (b) UTC – 10 (c) UTC + 11 (d) UTC – 1

8 (a) 9:00 pm (b) 7:00 pm (c) 8:00 pm (d) 6:00 pm

9 (a) **D**
(b) The 11 h difference involves crossing the IDL, so this must be taken into account.

10 (a) 3:50 pm on 06/02 (b) 8:20 am on 06/02
(c) 3:50 am on 06/02 (d) 9:50 pm on 05/02

11 (a) 4:15 am, same day (b) 9:15 am, same day
(c) 3:15 pm, same day (d) 7:15 pm, same day

12 The office hours do not overlap. The Melbourne contact could stay at work until 6 pm to talk to the New York colleague as soon as the New York office opens. Alternatively, the New York colleague could open the office before 6 am to talk to the Melbourne colleague as the Melbourne office finishes work.

13 (a) 10 am to 5 pm (b) 9 am to 4 pm (c) 9 am to 2:30 pm

14 (a) 8 am to 3 pm on Sunday
(b) Reese: 6 am to 1 pm on Sunday; Hana: 2 pm to 9 pm on Saturday
(c) Minh in Qld: 8 am to 2 pm on Sunday; Reese: 6 am to 12 noon on Sunday; Hana: 3 pm to 9 pm on Saturday

15 6:10 am

EXERCISE 4.4

1 (a) 11:20 am Tuesday (b) 5:20 am Tuesday

2 (a) 7:20 pm Wednesday (b) 2:50 pm Wednesday

3 (a) 2:45 pm (b) 5:45 pm

4 (a) 6:30 pm (b) 1:30 pm

5 (a) **B** (b) **D**

6 (a) 770 km/h (b) 845 km/h

7 (a) 469 km/h (b) 493 km/h

8 (a) 607 km/h (b) 740 km/h

9 (a) **C** (b) The 2 h time difference has been ignored.

10 (a) 5:49 am on 04/11

(b) 17 h 4 min

(c) Sydney to Honolulu: 10 h 5 min

Honolulu to Los Angeles: 5 h 35 min

Los Angeles to Las Vegas: 1 h 24 min

11 (a) 9:40 pm on 15/02

(b) 21 h 15 min

(c) Brisbane to Dubai: 14 h 35 min

Dubai to Rome: 6 h 40 min

12 (a) 1:15 pm → 4:30 pm on Sunday; 19 h 10 min

(b) Edinburgh to Munich: 2 h 15 min

Munich to St Petersburg: 2 h 45 min

13 (a) 9:25 am

Assumed a constant ~20 min tarmac time, i.e. the flying time and flight times are not proportional.

Assumed no tail-wind.

Assumed the same average speed.

Assumed the flat map gave distances proportional to great circle curved distances.

(b) 6:15 pm

Assumed a constant ~20 min tarmac time, i.e. the flying time and flight times are not proportional.

Assumed no tail-wind.

Assumed the same average speed.

Assumed the flat map gave distances proportional to great circle curved distances.

14 07:35 or 7.35 am

This calculation assumes the same average speed for the original journeys and the future journey. The short flight may have had a smaller, slower aircraft, and it would not have cruised for very long at top speed. Also, take-off and landing will only happen once each with the single flight, so the actual ETA would be a lot earlier than the calculation suggests.

15 7 h 35 min

CHAPTER REVIEW 4

1 (a) 12°N, 105°E (b) 39°N, 85°W (c) 50°N, 14°E

2 (a) Australia (b) South America (c) Europe

3 (a) 0°, 52°W (b) 22°S, 116°E

4 **B**

5 34°34′N, 69°13′E

6 (a) 58° (b) 7°

7 8°52′

8 **C**

9 (a) 65° (b) 27°10′

10 4003 km

11 13 840 km

12 *B* is ahead of *A* by 3 h

13 *A* is ahead of *B* by 5 h 48 min. If the IDL were taken into account, *B* would be ahead of *A* by 17 h 12 min.

14 (a) 3:10 pm (b) 3:40 pm (c) 1:40 pm

15 (a) UTC – 3 (b) UTC + 4

16 (a) 1:15 pm on 27/12 (b) 10:15 am on 27/12

17 (a) 8:50 am Tuesday (b) 5:50 am Tuesday

18 (a) 9:35 pm (b) 10:05 pm

19 (a) 730 km/h (b) 618 km/h

20 (a) 23°10′S, 150°46′E (b) 23°16′S, 150°36′E

21 116 km

22 14 500 km

23 4838 km

24 **C**

25 **B**

26 north: 64°42′N, 10°5′E

south: 55°42′N, 10°5′E

east: 60°12′N, 19°43′E

west: 60°12′N, 0°27′E

27 1333 km. Assumptions: averaged the east–west amounts for the different parallel of latitude, because they were different. Assumed the east–west distances were close enough to the shortest distance between points.

28 Brisbane breakfast ↔ Los Angeles lunch

Brisbane lunch ↔ Beijing breakfast and Los Angeles dinner

Brisbane dinner ↔ Beijing lunch

Wellington breakfast ↔ London dinner

Wellington dinner ↔ London breakfast

Beijing dinner ↔ Los Angeles breakfast

New York breakfast ↔ Paris lunch

New York lunch ↔ Paris dinner

Perth lunch ↔ Tel Aviv breakfast

Perth dinner ↔ Tel Aviv lunch

29 9 h 15 min

Assume the same tail-wind advantage. Assume similar portion of take-off and landing times.

Exam review: Unit 3

PAPER 1

1 (a)

	Boys	Girls	Total
Both semesters	48	39	87
One semester only	11	27	38
Neither semester	82	95	177
Total	141	161	302

(b)

	Boys	Girls	Total
Both semesters	16%	13%	29%
One semester only	4%	9%	13%
Neither semester	27%	31%	59%
Total	47%	53%	100%

2 (a) There is a strong positive linear association.

(b) There is a moderate negative non-linear association.

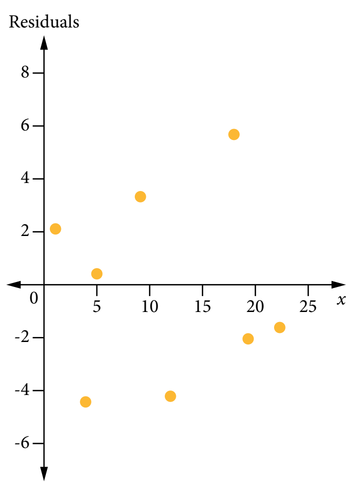
(c) There is no association.

3 (a) $80

(b) $73/h

4

x	1	4	5	9	12	18	19	22
rv	2.15	-4.4	0.75	3.35	-4.2	5.7	-2.15	-1.7

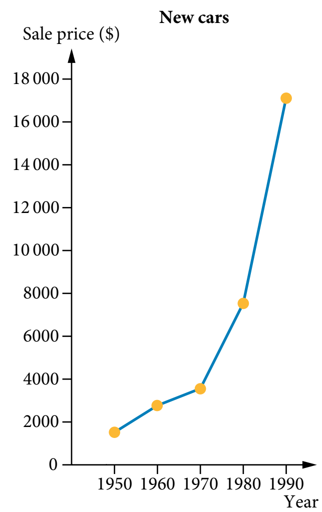


5 (a) Negative trend. The rate of decrease slows down after 1986.

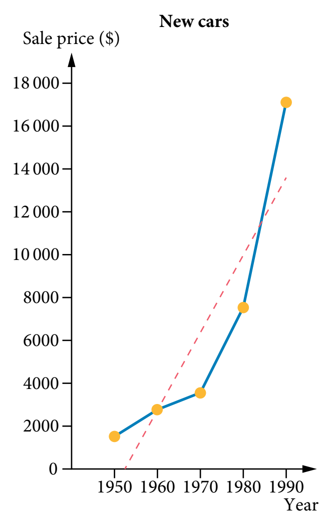
(b) Cyclic trend

(c) The trend is random.

6 (a)



(b)

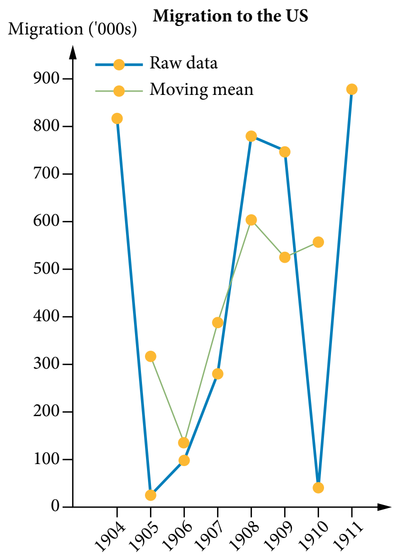


There is a positive trend that is non-linear.

7 (a)

Year	1904	1905	1906	1907	1908	1909	1910	1911
Migration ('000s)	812.9	26.5	100.5	285.3	782.9	751.8	41.6	878.6
Smoother data		313.3	137.4	389.6	606.7	525.4	557.3	

(b)



The extreme value(s) have been smoothed out.

8 (a) $x = 1.52$

(b)

	Jan–Feb	Mar–Apr	May–Jun	Jul–Aug	Sep–Oct	Nov–Dec
SI	0.48	1.44	2.4	0.48	0.24	0.96

9 (a) $d = 4$, $t_5 = 21$ (b) $r = -2$, $t_5 = -48$

10 (a) $t_7 = 14$ (b) $t_7 = -34\,816$

(c) $t_7 = 11764.9$

11 4050

12 (a) Helsinki, Finland

(b) Lima, Peru

(c) Oklahoma City, Oklahoma

13 (a) 1001 km

(b) 2913 km

14 (a) 1:35 am Thursday (b) 6:35 am Wednesday

PAPER 2

1 $r = 0.80$, $y = 3.15 + 1.11x$

2

Year	Quarter	Sales	4-point means
2016	1	300	
	2	250	
	3	480	412.5
	4	630	412.5
2017	1	280	417.5
	2	270	422.5
	3	500	426.25
	4	650	431.25
2018	1	290	437.5
	2	300	445
	3	520	
	4	690	

3 $SI_{\text{Dec-Feb}} = 1.05$, $SI_{\text{Mar-May}} = 0.89$, $SI_{\text{Jun-Aug}} = 0.75$, $SI_{\text{Sep-Nov}} = 1.31$

4 (a) $t_n = 1701\left(\frac{1}{3}\right)^{n-1}$, $t_8 = \frac{7}{9}$ (b) $t_n = 79n - 8$, $t_{13} = 1019$

5 (a) $32 085.27 (b) 15 years

6 (a) 5:40 am, next day (b) 4:40 pm, same day

(c) 12:40 pm, same day

7 Possible value: $y = 55$

8 mid-October 2023

9 19°4′S, 118°51′E

Chapter 5: Compound interest loans and investments

RECALL

1 (a) 80% (b) 107.5% (c) 3.75%

2 (a) 0.09 (b) 0.152 (c) 0.025

3 (a) 25% (b) 10% (c) 200%

4 (a) 32 (b) 849.6 (d) $1400

5 (a) 86.4 (b) 41.9175 (c) 532.5

6 (a) $4 < x < 5$ (b) $14 < t < 15$ (c) $R = 1.06$

7 (a) $t_1 = 200$, $t_2 = 300$, $t_3 = 450$, $t_4 = 675$

(b) $t_1 = 5000$, $t_2 = 4000$, $t_3 = 3200$, $t_4 = 2560$

EXERCISE 5.1

1 (a) $L_0 = 15\,000$, $L_{n+1} = 1.0565 \times L_n$

(b) $Q_0 = 11\,000$, $Q_{n+1} = 1.006\,875 \times Q_n$

(c) $P_0 = 1000$, $P_{n+1} = 1.02 \times P_n$

(d) $S_0 = 467.45$, $S_{n+1} = 1.036 \times S_n$

(e) $M_0 = 5600$, $M_{n+1} = 1.007 \times M_n$

(f) $L_0 = 17\,950$, $L_{n+1} = 1.006 \times L_n$

2 (a) **A**

(b) The student has not changed the percentage per annum to a percentage per period of 6 months.

3 (a) 9.2% p.a. (b) 4.5% p.a. (c) 10.7% p.a. (d) 15.5% p.a.

4 (a) 0.8%

(b) $A_0 = 5000$, $A_{n+1} = 1.008 \times A_n$

(c) $5203.23

5 (a) $7490, $8014.30, $8575.30 (b) $1575.30

6 (a) $L_0 = 2000$, $L_{n+1} = 1.005 \times L_n$ (b) $20.05

7 **B**

8 $3535.14

9 (a) $750 (b) $754.50

10 (a)

Half-year, n	Balance at end of half-year n, A_{n-1} ($)
0	60 000.00
1	62 040.00
2	64 149.36

(b) $64 149.36 (c) $4149.36 (d) $2181.08

11 (a) $A_0 = 2834.75,\ A_{n+1} = 1.011 \times A_n$

(b) $2865.93

(c) $2865.93

(d) Interest is only added every 3 months. The second amount of interest will only be added at the end of June. The balances at the end of March, April and May will all be the same. The balance at the end of June will be the next month to show additional interest.

(e) $2929.33

(f) $94.58

12 (a) B_1 represents the loan account balance at the end of June.

(b) $B_0 = 780,\ B_{n+1} = 1.006 \times B_n$

(c) $k = 4$

(d) $789.39

13 $4739.28

14 (a) 10.4% p.a. (b) 18% p.a. (c) 6.24% p.a.

(d) 1% p.a. (e) 6% p.a.

15 5.6% p.a.

16 (a) 7.2% p.a.

(b) $43.42 should be $41.42; interest earned in the fourth month

17 (a) $A_0 = 6380,\ A_{n+1} = 1.0035 \times A_n$ (b) $6938.05

18 $L_0 = 1200,\ L_{n+1} = 1.0185 L_n$; $1731.42

19 36 months

20 (a) $62 539.48 (b) $62 751.54 (c) $62 896.59

21 14 months

22 (a) $11 357.29 (b) After 14 months

23 (a) After 2 months, the first account is ahead for the first time temporarily

(b) After 73 months, the first account is ahead permanently

EXERCISE 5.2

1 (a) $50 723.19 (b) $46 246.94 (c) $44 005.08

2 $13 976.73

3 **D**

4 $1167.76

5 $10 100

6 $6310

7 $2614

8 $12 098.52

9 $656.24

10 Teri's investment, $1.28

11 6% p.a.

12 $73.69 greater if interest is calculated weekly.

13 (a) **C**

(b) The student has recognised the time frame but has not determined the number of compounding periods within that time frame.

14 10.41%

15 $26 200

16 17 quarters or 4 years 3 months

17 155 months

18 333 days

19 7.04% p.a.

20 122 months

21 23.33% p.a.

22 3.35% p.a.

23 7.61% p.a.

24 (a) $2844.42

(b) 18 quarters, or 4 years 6 months

25 9.2% p.a.

26 4.29% p.a.

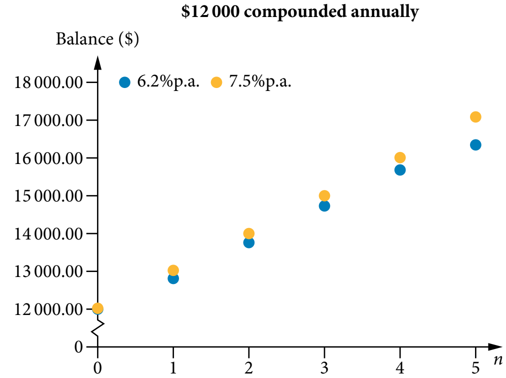
27 785%

EXERCISE 5.3

1 (a)

	A	B	C
1	n	6.2%p.a.	7.5%p.a.
2	0	12000.00	12000.00
3	1	12744.00	12900.00
4	2	13534.13	13867.50
5	3	14373.24	14907.56
6	4	15264.39	16025.63
7	5	16210.78	17227.55

$12 000 compounded annually

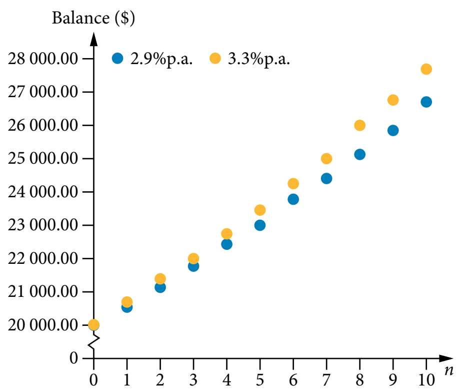


(b) The higher interest rate yielded $1016.77 more over 5 years.

2 (a)

	A	B	C
1	n	2.9%p.a.	3.3%p.a.
2	0	20000.00	20000.00
3	1	20580.00	20660.00
4	2	21176.82	21341.78
5	3	21790.95	22046.06
6	4	22422.89	22773.58
7	5	23073.15	23525.11
8	6	23742.27	24301.44
9	7	24430.80	25103.38
10	8	25139.29	25931.79
11	9	25868.33	26787.54
12	10	26618.51	27671.53

$20 000 compounded annually

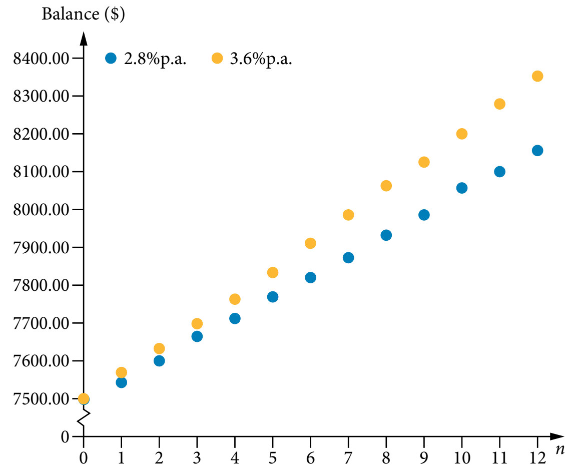


(b) The higher interest rate yielded $1053.02 more over 10 years.

3 (a)

	A	B	C
1	n	2.8%p.a.	3.6%p.a.
2	0	7500.00	7500.00
3	1	7552.50	7567.50
4	2	7605.37	7635.61
5	3	7658.61	7704.33
6	4	7712.22	7773.67
7	5	7766.20	7843.63
8	6	7820.56	7914.22
9	7	7875.31	7985.45
10	8	7930.44	8057.32
11	9	7985.95	8129.84
12	10	8041.85	8203.00
13	11	8098.14	8276.83
14	12	8154.83	8351.32

$7500 compounded quarterly

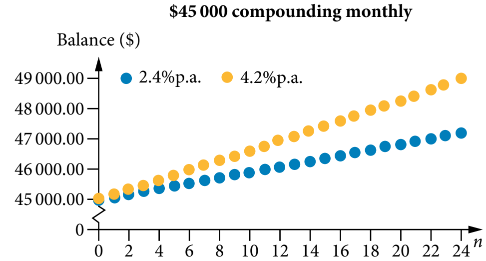


(b) The higher interest rate yielded $196.49 more over 3 years.

4 (a)

	A	B	C
1	n	2.4%p.a.	4.2%p.a.
2	0	45000.00	45000.00
3	1	45090.00	45157.50
4	2	45180.18	45315.55
5	3	45270.54	45474.16
24	22	47022.14	48595.36
25	23	47116.18	48765.44
26	24	47210.42	48936.12

$45 000 compounding monthly



(b) The higher interest rate yielded $1725.70 more over 2 years.

5 (a) $8124.49 **(b)** $8181.24 **(c)** Option 2, $56.75

6 Compounding monthly is $295.87 more expensive.

7 $7968.56

8 (a) A

(b) The student has forgotten to divide by 100.

9 Account 1, $3.65

10 (a) False; compound interest increases each period

(b) False; compound interest increases each period

11 (a) 4.5% p.a. **(b)** 3 years **(c)** 4% p.a.

12 $3.21 better if she transfers.

13 **D**

14 $B_0 = 12\,856.36,\ B_{n+1} = 1.0035 \times B_n$

15 **(a)** \$9646.81 **(b)** \$318.10

16 **(a)** $x = 4715.53$

(b) The annual rate of interest increased by 1.55% p.a.

(c) 9 years

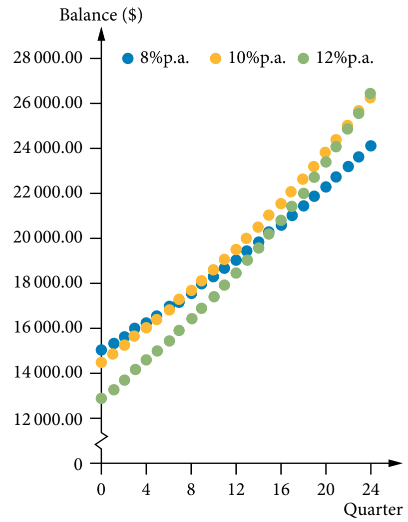
17 **(a)** \$5463.74 **(b)** \$5708.14 **(c)** 26.85%

18 \$22 308.29

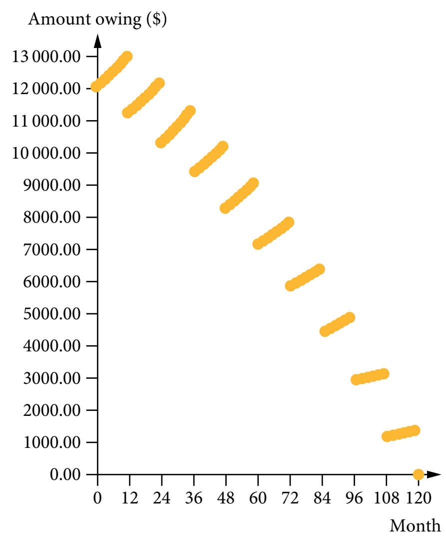
19 For up to and including 6 quarters: 8% p.a. with no fee

For 7–22 quarters: 10% p.a. with \$500 fee

For 23 quarters or more: 12% p.a. with \$2000 fee.



20 10 years; final payment: \$1294.22



EXERCISE 5.4

1 **(a)** \$4333.11

(b) 8.33%

2 9.87%

3 **(a)** **B**

(b) The student has forgotten to halve the nominal rate to match the compounding period.

(c) The nominal rate and annual effective rates are always fairly close.

4 **(a)** Company A: 9.5% p.a.; Company B: 9.4% p.a.

(b) Company A: 9.84%; Company B: 9.82%

(c) Company B

(d) No impact

5 **(a)** Compounding monthly: 7.98%; compounding annually: 7.92%

(b) Compounding annually: \$4.68

6 Account B, 3.81% p.a. > 3.80% p.a.

7 **D**

8 4.94% p.a.

9 16.15% p.a.

10 3.30% p.a.

11 8.90% p.a.

12 6.78% p.a.

13 10.60%

14 **(a)** 4.02% p.a. **(b)** 4.08%

15 Account A: 7.9%, monthly Account B: 6.4%, daily

Account C: 5.2%, quarterly

16 **(a)** 4.40% p.a. **(b)** 4.49%

CHAPTER REVIEW 5

1 **D**

2 \$68 499

3 \$4254

4 \$8640.00

5 **C**

6 \$2333

7 $L_0 = 7900,\ L_{n+1} = 1.019L_n$

8 9.6%

9 \$8161.47

10 \$1521

11 6.26%

12 \$67 541

13 **(a)** Account balance at the end of July.

(b) $J_0 = 1450,\ J_{n+1} = 1.0045J_n$

(c) Account balance at the end of August.

(d) \$1530.27

(e) Account balance at the end of December 2018.

(f) \$1482.92

14 \$1063

15 \$22 515

16 \$859.21

17 (a) \$8388.61 (b) \$8796.09

18 \$10 820

19 4.91%

20 (a) True (b) True

21 (a) \$16 639 (b) \$17 447 (c) \$17 522

22 (a) Option 1: \$13 677.10; Option 2: \$13 690.29

(b) Option 1: \$19 603.36; Option 2: \$19 698.03

23 \$4.13

24 5.72%

25 8.77%

26 (a) Loan: $L_0 = 20\,000$, $L_{n+1} = 1.0135L_n$

Investment: $A_0 = 20\,000$, $A_{n+1} = 1.0045A_n$

(b) The effective annual rate of interest of 5.51% for the loan is less than the effective annual rate of interest of 5.54% for the investment. The interest charged will be less than the interest earned.

27 (a) 2.89%

(b) \$262 965.43

(c) 9.57%

(d) \$541 000

(e) \$578 874

28 12 years

29 **D**

30 26 quarters or 6.5 years

31 8.13%

32 8.31%

33 36 months or 3 years

34 155 weeks

35 **C**

36 The effective interest rate for Option 2, 5.74%, is slightly higher than that for Option 1, 5.72%.

37 \$199.71

38 4 months

39 (a) \$19.87

(b) 1 month

(c) \$36 049.63

(d) The account balance from part (c) has \$501.16 more than that for Option 1.

(e) 4.33% p.a.

Chapter 6: Annuities and perpetuities

RECALL

1 (a) 8.6 (b) 0.064 (c) 0.0053

2 (a) 0.25 (b) 0.042 (c) 0.008

3 (a) $A_1 = 264$, $A_2 = 241$ (b) $A_1 = 392$, $A_2 = 439.04$

4 (a) $r = 1.2$ (b) $r = 1.06$ (c) $r = 1.0025$

5 (a) \$832.48 (b) 31 months

(c) \$803.39 (d) 3.90%

6 (a) 17 months (b) \$9571.30 (c) \$1669.67

EXERCISE 6.1

1 (a) $L_0 = 6000$, $L_{n+1} = 1.013 \times L_n - 300$

(b) \$78

(c) \$222

2 (a) Interest component \$35, principal component \$365

(b) Interest component \$40, principal component \$60

(c) Interest component \$162, principal component \$38

3 (a) \$7689.60 (b) \$23 132 (c) \$75 966.67

4 (a) $L_0 = 15\,000$, $L_{n+1} = 1.0451 \times L_n - 1200$

(b) $L_0 = 7684.50$, $L_{n+1} = 1.0075 \times L_n - 320$

(c) $L_0 = 23\,000$, $L_{n+1} = 1.0315 \times L_n - 1000$

5 **A**

6 (a) \$748.08 (b) \$12 920 (c) \$12 171.92

7

Period, n	Value of loan, L_n (\$)	Equal monthly repayments, R (\$)	Interest (\$)	Reduction in principal (\$)	Next principal, L_{n+1} (\$)
0	22 500.00	997.20	112.50	884.70	21 615.30
1	21 615.30	997.20	108.08	889.12	20 726.18
2	20 726.18	997.20	103.63	893.57	19 832.61

8

Period, n	Value of loan, L_n (\$)	Equal monthly repayments, R (\$)	Interest (\$)	Reduction in principal (\$)	Next principal, L_{n+1} (\$)
0	1 380.00	481.32	31.74	449.58	930.42
1	930.42	481.32	21.40	459.92	470.50
2	470.50	481.32	10.82	470.50	0.00
3	0.00	Loan has been repaid			

9 (a) $L_0 = 11\,000$, $L_{n+1} = 1.0305 \times L_n - 750$

(b) \$16.44

(c) \$3805.41

10 \$406.52

11 (a) C

(b) The student has automatically put the smaller value over the larger, whereas in this case the interest is larger than the repayment.

12

Period, n	Value of loan, L_n (\$)	Interest added at the end of the next period (\$)	Balance after interest added (\$)	Repayment (\$)	Account balance reduced by (\$)	Next principal, L_{n+1} (\$)
5	1238.13	49.53	1287.66	1287.66	1238.13	0.00

13 The balance remains at \$5000, because the interest took up all the repayments.

14 (a) The annual rate of interest for the loan is 7.2%.

(b) \$225

(c) \$2685

(d) \$2054.57

(e) $L_4 = 1.006 \times 2054.57 - 225$

(f) \$1627.95

15 (a) 6 months

(b) \$82.12

(c) 1.97%

(d) Tasha increased the repayment for the third month by \$700. This reduced the term to five months. In total this change saved \$13.31.

16 (a) \$31.67 (b) 7.60%

17 \$5188.42

18 (a) $L_0 = 3875$, $L_{n+1} = 1.0195 \times L_n - 850$

(b) 5 quarters

(c) \$820.63

(d) \$698.82

19 $L_0 = 6440$, $L_{n+1} = 10135 \times L_n - 1664.70$

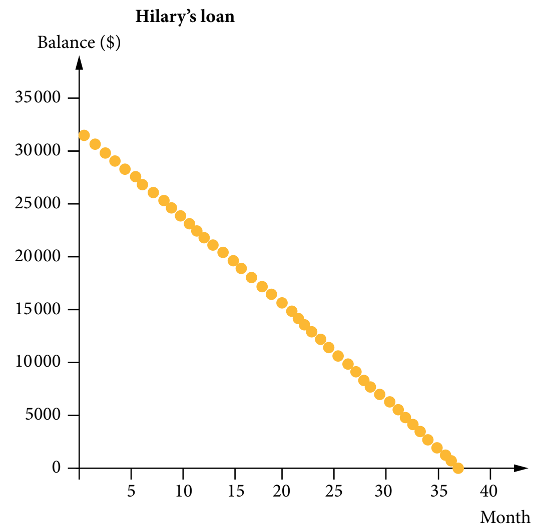
Period, n	Value of loan, L_n (\$)	Payment at the end of the next period (\$)	Interest portion of the payment (\$)	Principal reduced by (\$)	Next principal, L_{n+1} (\$)
0	6440.00	1664.70	86.94	1577.76	4862.24
1	4862.24	1664.70	65.64	1599.06	3263.18
2	3263.18	1664.70	44.05	1620.65	1642.53
3	1642.53	1664.70	22.17	1642.53	0

20 (a) $L_0 = 31600$, $L_{n+1} = 1.0039 \times L_n - 900$

(b) 38 months

(c) \$750.32

(d)



Because the principal reductions increase, the graph gets steeper to the right.

(e) \$2450.32

(f) \$2.91

21 (a) $L_0 = 7820$, $L_{n+1} = 1.0155 \times L_n - 520$

(b) 10 months

(c) \$651.09

(d) \$1151.09

22 \$4119.48

23 (a) \$6192.76 (b) \$32.09 (c) \$687.19

24 (a) \$19 874.90 (b) 6 quarters or 18 months

(c) 3 years (d) \$342.68

(e) 17.85%

25 \$19 500.00

26 \$40 097.56

EXERCISE 6.2

1 \$472.70

2 (a) \$10 (b) \$15 (c) \$60 (d) \$90

3 (a) \$600

(b) The periodic repayment is \$200 less than the \$600 interest charged in the first month. The account balance will increase by \$200 at the end of the first month. This is an increasing balance loan. Interest charged each month will increase geometrically, as will the loan account balance.

(c) \$610

4 (a) \$90

(b) Interest charged for the first period is less than the periodic repayment.

(c) \$11 640

5 (a) $30.40 (b) $5700

6 **B**

7 (a) $18 (b) $3398

8 (a) $8842.50

(b) 110.53%; the loan account balance will increase each quarter.

(c) $547 139.20

(d) $538 665.40

9 (a) $L_0 = 4000, L_{n+1} = 1.02 \times L_n - 200$

(b) $2590.55

(c) $L_0 = 2500, L_{n+1} = 1.025 \times L_n - 200$

(d) 16 months; $35.22

(e) $3035.32

10 (a) **C**

(b) The student has not converted the annual percentage to a quarterly percentage.

11 (a) 0.8%

(b) 9.6% p.a.

(c) $L_0 = 480, L_{n+1} = 1.008 \times L_n - 60$

(d) $136.23

(e) 9 months

(f) $18.08

(g) $18.08

(h) **C**

12 (a) 36 months or 3 years (b) Reduced by 1 month

13 The loan balance will increase from $342 000.

14 Reduced by 20 months

15 (a) 3 quarters or 9 months (b) $176.31

(c) 6 quarters or 18 months (d) $138.35

16 (a) $L_0 = 4000, L_{n+1} = 1.02 \times L_n - 65$

(b) $4015; the repayment would be less than the interest.

(c) $4266.18

(d) $981.18

17 (a) Interest for the first period is $536.25, which is $36.25 more than the repayment.

(b) The loan balance after four repayments is greater than the initial loan. The interest charge for the fifth period will be more than $536.25.

(c) $542.00

18 $340

19 (a) $4900

(b) 71 quarters or 17 years 9 months

20 (a) The principal is not reduced. It is increased by the amount that follows the negative sign.

(b) 1.85%

(c) $81.40

(d) 43 months or 3 years 7 months

(e) $887.65

(f) 32 months

(g) $221.05

21 $17 360

22 (a) $6825.23

(b) $111.93

(c) 17 quarters or 4 years 3 months

(d) $416.90

(e) $805

EXERCISE 6.3

1 $8240.03

2 (a) $L_0 = 4000, L_{n+1} = 1.02 \times L_n - 500$ for $0 \le n \le 2$

(b) $3580

(c) $L_0 = 3151.60, L_{n+1} = 1.02 \times L_n - 1000$

(d) $1258.92

3 (a) The outstanding balance reduces after every repayment.

(b) 1%

(c) $80

(d) The monthly rate of interest increased from 1% to 1.2%, so the nominal rate changed from 12% p.a. to 14.4% p.a.

(e) Payments increased to $250 from the end of the fourth month.

4 (a) **B**

(b) The student has not realised that the first column gives the initial balance and does not involve a payment.

5 **B**

6 A missed repayment means the interest charged for the period will increase the loan account balance with no usual deduction for a repayment. The earlier this happens, the greater the number of remaining periods for the increase to be compounded. Depending on the parameters of the loan, the term of the loan may need to increase.

7 (a) 120 months or 10 years (b) 2 months

8 441 months

9 27 months

10 $7974.25

11 (a) $297

(b) It is an increasing balance loan.

(c) $16 610.95

(d) The loan becomes an interest-only loan.

12 (a) 2 years 6 months (b) $2524.92

13 $10 106.64

14 $13 921.78

15 9 months, $1332.50

16 (a) $1910.97 (b) 27 months

17 (a) 98 months or 8 years 2 months (b) $404

18 (a) 16 months (b) $224.58 (c) $514.45

EXERCISE 6.4

1 $286 350

2 **A**

3 $645.75

4 5.6% p.a.

5 $444 445

6 (a) Interest $1125, principal reduction $375

(b) Interest $281.25, principal reduction $218.75

(c) Interest $500, principal reduction $100

(d) Interest $78, principal reduction $72

7 $A_1 = \$37\,720$, $A_2 = \$35\,427.46$, $A_3 = \$33\,122.31$

8 (a) $225 680.35 (b) $225 064.26

9 (a) $15 000

(b) $4500

(c) $3840.54

(d) 5.8% p.a.

(e) $A_0 = 15\,000$, $A_{n+1} = 1.058 \times A_n - 4500$

10 (a) $A_0 = 60\,000$, $A_{n+1} = 1.0065 \times A_n - 900$

(b)

Period, n	Balance at end of period n, A_n	Payment withdrawn at the end of the next period ($)	Interest portion of payment ($)	Rest of payment ($)	Balance at end of period $n+1$ ($)
0	60 000.00	900.00	390.00	510.00	59 490.00
1	59 490.00	900.00	386.69	513.11	58 976.69
2	58 976.69				

(c) $352.33

11 9.82% p.a.

12 (a) $2322.81 (b) $220 449.23

(c) 1.6% (d) $2359.98

13 (a) $88 381.77

(b) $19 203.42

(c) 80 months; $1051.15

14 (a) 140 months (b) $24 512.81

15 (a) $700 option, shorter by 108 months. The higher payment gives a shorter term, as expected.

(b) $600 option, more interest by $39 696.33

16 The annuity with a 3.25% p.a. interest rate would end after 23 years, paying a total of $344 242.05. At 6.5% p.a., the annuity will continue paying $15 000 every year indefinitely.

17 (a) Annual $117 520, quarterly $117 476.89

(b) The quarterly option claimed interest three times during the year, so the principal was reduced before the end of the year. Even though the capital is reduced, the payments were accessed much earlier.

(c) $3485.11

EXERCISE 6.5

1 (a) $A_1 = \$120\,150$, $A_2 = \$120\,300.56$, $A_3 = \$120\,451.69$

(b) $A_1 = \$300\,400$, $A_2 = \$300\,801.20$, $A_3 = \$301\,203.60$

(c) $A_1 = \$80\,100$, $A_2 = \$80\,200.63$, $A_3 = \$80\,301.88$

(d) $A_1 = \$225\,212.50$, $A_2 = \$225\,425.96$, $A_3 = \$225\,640.38$

2 (a) **D** (b) **B**

3 (a) **B**

(b) The student has forgotten to convert the annual interest rate to a quarterly rate.

4 (a)

Annuity balance at end of period n, A_n ($)	Add interest ($)	Add periodic contribution ($)	Amount added to annuity ($)	Next annuity balance, A_{n+1} ($)
4800.00	96.00	100.00	196.00	4996.0

(b)

Annuity balance at end of period n, A_n ($)	Add interest ($)	Add periodic contribution ($)	Amount added to annuity ($)	Next annuity balance, A_{n+1} ($)
12 000.00	60.00	400.00	460.00	12 460.00

(c)

Annuity balance at end of period n, A_n ($)	Add interest ($)	Add periodic contribution ($)	Amount added to annuity ($)	Next annuity balance, A_{n+1} ($)
6800.00	32.64	239.56	272.20	7072.20

(d)

Annuity balance at end of period n, A_n ($)	Add interest ($)	Add periodic contribution ($)	Amount added to annuity ($)	Next annuity balance, A_{n+1} ($)
19 813.43	148.60	345.0	493.60	20 307.03

5 (a) 3.6% (b) $6190.62 (c) $500.00 (d) $8107.28

6 (a) $A_1 = \$4038.51$, $A_2 = \$4252.64$, $A_3 = \$4467.52$

(b) $A_1 = \$12\,351$, $A_2 = \$12\,703.49$, $A_3 = \$13\,057.48$

(c) $A_1 = \$18\,416.73$, $A_2 = \$18\,634.26$, $A_3 = \$18\,852.59$

(d) $A_1 = \$8152.67$, $A_2 = \$8305.96$, $A_3 = \$8459.88$

7 (a) $83 049.47 (b) $1445.06 (c) $54 494.53

8 (a) $666.37 (b) $46 506.37 (c) 19 quarters

9 Option 1, because the interest on half the contributions is calculated over a longer time.

10 (a) $15 3450 < $153 997.59 The annuity has kept up with inflation.

(b) $255 750 > $255 612.38 The annuity has not kept up with inflation.

(c) $61 380 < $63 774.91 The annuity has kept up with inflation.

11 (a) $A_0 = 743.51$, $A_{n+1} = 1.0032 \times A_n + 100$

(b) 26 months

12 (a) $8277.37 (b) 9 quarters (c) $12 700 (d) $2765.12

13 (a) $25 679.92

(b) $A_{12} = 25\,679.92$, $A_{n+1} = 1.0035 \times A_n + 240$

(c) 39 months

14 (a) $A_0 = 37\,780.29$, $A_{n+1} = 1.052 \times A_n + 5000$

(b) $44 744.87

(c) End of 2023

(d) $379 000

15 $279 330.77

16 $355

17 $127 875

EXERCISE 6.6

1 (a) $6932.90 (b) $6919.56

(c) $4260.02 (d) $30 092.42

2 (a) $88.25 (b) $133.38

(c) $316.62 (d) $93.84

3 (a) $40 385.46 (b) $63115.64

(c) $24 473.99 (d) $18 336.30

4 (a) $103.40 (b) $1768.27

(c) $650.79 (d) $381.74

5 (a) $12 649.80

(b) (i) 17 years (ii) $241 929.57

6 $16181.19

7 (a) $6697.63 (b) $10 962.11

8 (a) **C**

(b) The student has used the percentage per month as if it was the percentage per annum.

9 25%

10 (a) $206.25 (b) $20 202.85

11 $100152.08

12 33 quarters

13 $81.63, assuming interest rates on the investment account remain constant for 75 years.

14 (a) Based on 15 hours per week and investing 20% of his wage, the account will be worth $34 533 by the time he finishes his education. Initial deposits of $69 will be worth $98.80 due to compound interest, whereas later contributions will be worth a little over $69.

(b) By contributing an extra 15% of his wage (or $51.75) per week, Dakari will have saved an extra $25 899 over the 9 year period, allowing him to save a total of $60 432.72 while studying.

15 Ignoring inflation, Olivia is at least $78.70 per week better off if she keeps the property. Assuming inflation of 2% p.a., if Olivia keeps the property she will be approximately $400 per week better off, and still own a property with an anticipated value of $498 000.

CHAPTER REVIEW 6

1 $100; 12% p.a.

2 $4630

3 $4159.22

4

Period, n	Balance at end of period n, L_n ($)	Interest added at the end of the next period ($)	Balance after interest added ($)	Repay ($)	Account balance reduced by ($)	Balance at end of period $n+1$, L_{n+1} ($)
0	$10 000	$450.00	$10 450	$3 637.73	$3 187.73	$6 812.27
1	$6 812.27	= 4.5% × 6812.27	$7 118.82	$3 637.73	$3 331.18	= 6812.27 – 3331.18
2	$3 481.09	$156.65	$3 637.74	$3 637.74	$3 481.09	$0.00

5 (a) $3083.16 (b) $2654.52

6 $701.80

7 increasing balance

8 6

9 $75 900

10 30 months

11 6%

12 **A**

13 (a) Interest-only loan (b) $252.75 (c) $112.33

14 (a)

Period, n	Principal at end of period n, L_n ($)	Repay ($)	Interest added at the end of the next period ($)	Principal reduced by ($)	Principal at end of period $n+1$, L_{n+1} ($)
0	1500.00	200.00	22.50	177.50	1322.50
1	1322.50	200.00	19.84	180.16	1142.34
2	1142.34	400.00	17.14	382.86	759.48
3	759.48	400.00	11.39	388.61	370.87
4	370.87	376.43	5.56	394.44	-23.57

(b) 1 year 3 months

15 (a) **C** (b) **A** (c) **D**

16 (a) The principal increased by \$292.00 due to the interest.

(b) \$89.11

17 $A_0 = 40\,000,\ A_{n+1} = 1.0302A_n - 2400$

18 **A**

19 4.86%

20 **B**

21 (a) $A_0 = 210\,000,\ A_{n+1} = 1.0108 \times A_n - 6500$

(b) 40 quarters or 10 years

(c) \$6165.05

22 \$26 315.79

23 \$248 000

24 **B**

25 **D**

26 \$12 278.86

27 \$3134.82

28 \$7526.35

29 (a) \$294.67 (b) \$53 000 (c) \$229.67

(d) \$8000; 5 years (e) \$9880

30 (a) \$8283.42 (b) 54 months (c) \$171.99

(d) \$3283.42 (e) \$3671.99

31 \$5500

32 (a) \$13 130.88 (b) \$69.11

33 (a) \$350 000 at 5.9% and \$20 000 at 11.8%

(b) 13 months; \$373 042.82

(c) \$371 917.55

(d) The interest for the first month will be \$1828.59, which exceeds what he is able to repay. Therefore, the loan balance will continue to increase despite each repayment.

34 (a) \$236.84 (b) \$856.84

35 (a) \$7570 (b) \$7850

36 (a) (i) \$104 119.65 (ii) \$91 145.95

(b) 16 years 9 months

Chapter 7: Graphs and networks

RECALL

1 (a) 10 (b) 16 (c) 3

2 (a) $b = 3$ (b) $p = 8$ (c) $e = 10$

3 (a) $x = 18$ (b) $x = 8$ (c) $x = \frac{3}{4}$

4 (a) 8 (b) 4 (c) 9

5 (a) $n = 8$ (b) $n = 10$ (c) $n = 5$

6 (a) faces 6; edges 12; vertices 8

(b) faces 7; edges 12; vertices 7

(c) faces 6; edges 12; vertices 8

EXERCISE 7.1

1 (a) $v = 5, e = 6, f = 3$ $v + f - e = 2$; connected

(b) $v = 4, e = 6, f = 4$ $v + f - e = 2$; connected

(c) $v = 5, e = 5, f = 2$ $v + f - e = 2$; connected

(d) $v = 3, e = 4, f = 3$ $v + f - e = 2$; connected

(e) $v = 5, e = 8, f = 5$ $v + f - e = 2$; connected

(f) $v = 3, e = 5, f = 4$ $v + f - e = 2$; connected

(g) $v = 5, e = 2, f = 1$ $v + f - e = 4$; not connected

(h) $v = 9, e = 8, f = 1$ $v + f - e = 2$; connected

2 (a) $e = 24$ (b) $v = 10$ (c) $f = 5$ (d) $e = 18$

(e) $e = 27$ (f) $v = 92$ (g) $f = 36$ (h) $v + f = 50$

3 (a) connected planar graph

(b) connected planar graph

(c) not a connected planar graph

(d) not a connected planar graph

(e) not a connected planar network

(f) connected planar graph

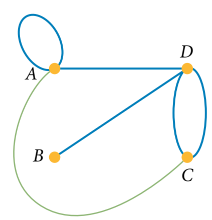
4 (a) connected, planar, simple (b) connected, planar

(c) connected, planar, simple (d) connected, planar, directed

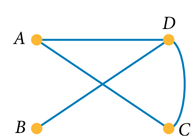
(e) connected, planar, simple (f) connected, planar

(g) planar, simple (h) connected, planar, simple, bipartite

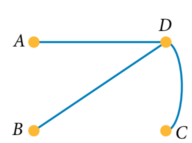
5 (a) Sample graph



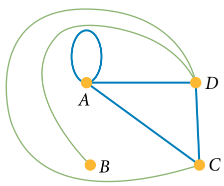
(b) Sample graph



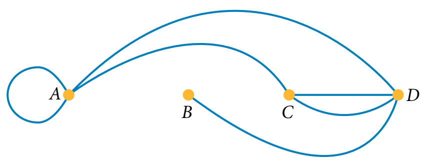
(c) Sample graph



(d) **(i)** Sample graph



(ii) Sample graph



6 **(a)** *A*: 3; *B*: 3; *C*: 2; *D*: 4; *E*: 0

(b) *P*: 3; *Q*: 5

7 **(a)** True **(b)** False **(c)** True **(d)** False

(e) False **(f)** True **(g)** False **(h)** False

8 edge 3, edge 5

9 **(a)** $e = 21$ **(b)** $e = 45$

10 **(a)** $e = 66$ **(b)** $e = 190$

11 **(a)** Isomorphic **(b)** Not isomorphic **(c)** Isomorphic

(d) Not isomorphic **(e)** Isomorphic **(f)** Not isomorphic

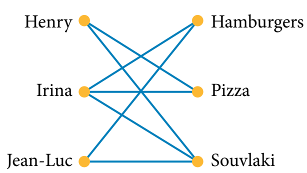
12 **(a)** Tree **(b)** Tree **(c)** Tree

(d) Not a tree **(e)** Not a tree **(f)** Not a tree

13 **C**

14 **(a)** False **(b)** True **(c)** True **(d)** True

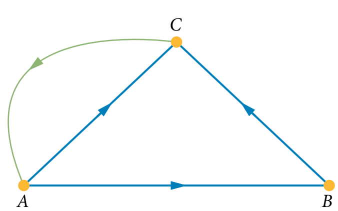
15



16 **(a)** False **(b)** True **(c)** True

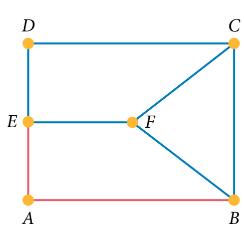
(d) True **(e)** True **(f)** False

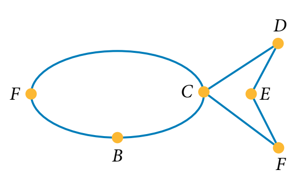
17



18 **(a)** **A**

(b)

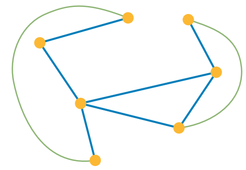




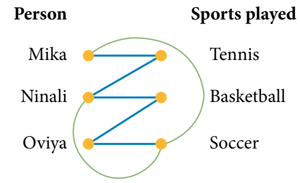
F appears twice, so this is not a subgraph.

19 The number of edges that must be removed is 3. If any more are removed the graph will become disconnected.

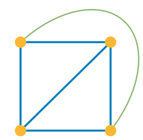
20 **(a)** $v = 7, e = 9, f = 4$



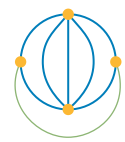
(b) $v = 6, e = 7, f = 3$



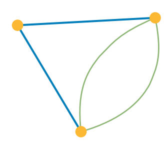
(c) $v = 4, e = 6, f = 4$



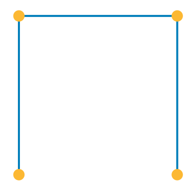
(d) $v = 4, e = 8, f = 6$



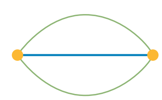
21 **(a)** Planar and connected. Answers will vary.



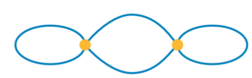
(b) Planar and connected. Answers will vary.



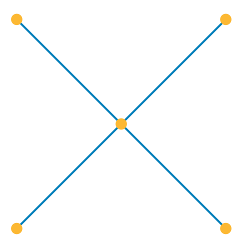
(c) Planar and connected. Answers will vary.



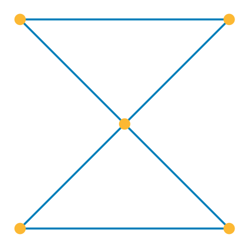
(d) Planar and connected. Answers will vary.



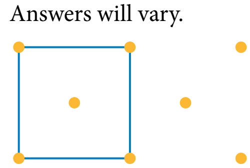
(e) Planar and connected. Answers will vary.



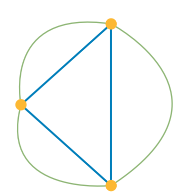
(f) Planar and connected. Answers will vary.



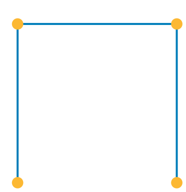
(g) The graph is not connected. Answers will vary.



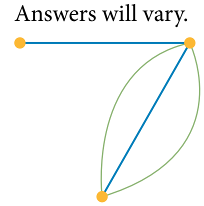
(h) Planar and connected. Answers will vary.



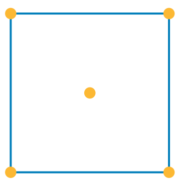
22 **(a)**



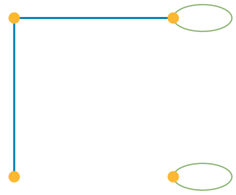
(b) Answers will vary.



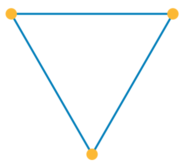
(c) Answers will vary.



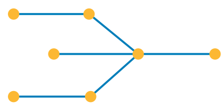
(d) Answers will vary.



(e)

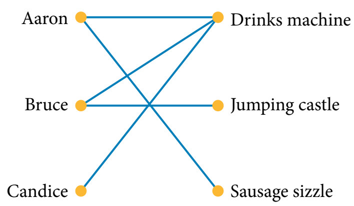


(f) Answers will vary.



23 (a) $v = 6$ (b) $v = 9$ (c) $v = 4$ (d) $v = 8$

24 (a)



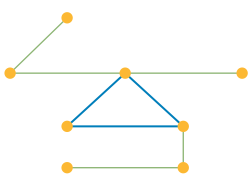
(b) Bruce the jumping castle; Aaron the drinks machine; Candice the sausage sizzle

Bruce the jumping castle; Aaron the sausage sizzle; Candice the drinks machine

(c) The most popular job is the drinks machine and the least popular is the jumping castle.

(d) Bruce

25 The possible bridges are marked in green.



EXERCISE 7.2

1 (a) Taurus, Cancer, Scorpius, Aries, Aquarius

(b) 14

(c) Taurus, Leo, Libra

(d) Cancer

2 (a) Louise and James

(b) Princess Anne and Mark Phillips

(c) Married into the Royal Family

(d) 3

(e) Queen Elizabeth; Prince Charles; Prince William; Prince George

3 (a) True (b) True (c) False

(d) True (e) True (f) False

4 (a) **B**

(b) The arrow reads ‘defeats’ and points away from the winning choice in each case. The student may have thought the arrow pointed to the winning choice.

5 (a) Yes

(b) SE

(c) Yes

(d) Duane Street, Church Street, Thomas Street and West Broadway

(e) A: Greenwich and Chambers

B: Chambers and Church

C: Church and Thomas

D: Church and Leonard

E: Thomas and West Broadway

F: West Broadway and Chambers

6 (a) (i) Nematodes, bacteria

(ii) Bacteria, protozoa, fungi

(iii) Humans, predatory birds

(b) (i) False (ii) True (iii) False

(iv) False (v) True

(c) A spider is eaten by a mouse or a songbird, which is then eaten by an eagle.

7 (a)

Team	Number of wins
Aphids	3
Beetles	0
Caterpillars	1
Dragonflies	2
Earwigs	4
Fruit flies	3

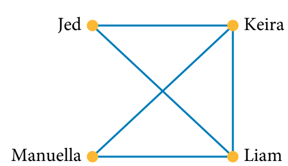
(b) Earwigs

(c) Earwigs and Caterpillars; Caterpillars and Fruit flies

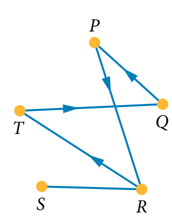
8 (a) True (b) False (c) True

(d) False (e) False (f) True

9



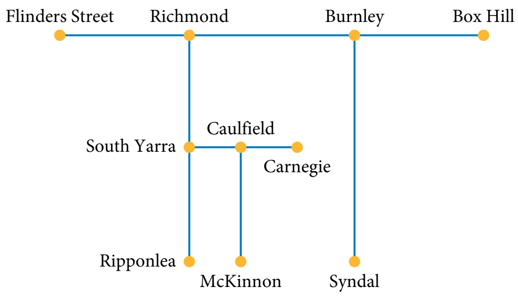
10



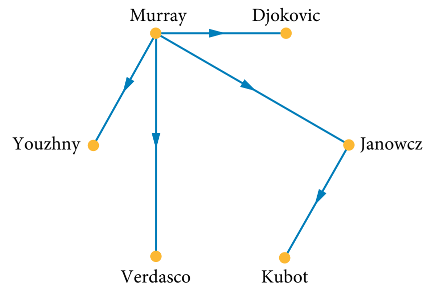
11 (a) 12 (b) The Frankston line

(c) 12 (d) South Yarra station

(e)

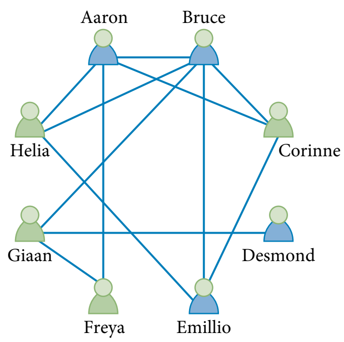


(f)



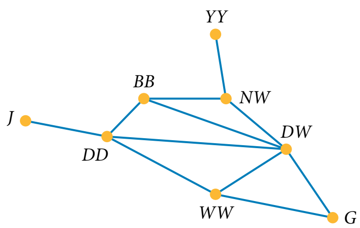
12 A

13 (a)

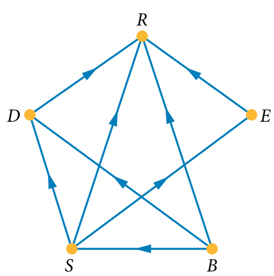


(b) Giaan

14



15



16 (a) A minimum of two flights is required.

(b) Melbourne to Perth; Melbourne to Brisbane; Melbourne to Cairns

(c) Yes, it is possible to add direct flights.

(d) Melbourne, Hobart, Melbourne, Perth, Adelaide, Melbourne, Alice Springs, Melbourne, Cairns, Brisbane, Sydney, Melbourne

17 (a) J. del Potro

(b) Three

(c) One

(d) A. Murray with 18 games lost

(e) A. Murray beat M. Youzhny, F. Verdasco, J. Janowcz, N. Djokovic

EXERCISE 7.3

1
$$\begin{matrix} & \begin{matrix} A & B & C \end{matrix} \\ \begin{matrix} A\\B\\C \end{matrix} & \begin{bmatrix} 0&1&1\\1&0&1\\1&1&1 \end{bmatrix} \end{matrix}$$

2 (a)
$$\begin{matrix} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} A\\B\\C\\D \end{matrix} & \begin{bmatrix} 0&1&1&0\\1&0&1&0\\1&1&0&1\\0&0&1&0 \end{bmatrix} \end{matrix}$$

(b)
$$\begin{matrix} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} A\\B\\C\\D \end{matrix} & \begin{bmatrix} 0&1&0&0\\1&0&1&1\\0&1&0&0\\0&1&0&1 \end{bmatrix} \end{matrix}$$

(c)
$$\begin{matrix} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} A\\B\\C\\D \end{matrix} & \begin{bmatrix} 0&1&0&0\\1&0&1&0\\0&1&0&0\\0&0&0&0 \end{bmatrix} \end{matrix}$$

(d)
$$\begin{matrix} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} A\\B\\C\\D \end{matrix} & \begin{bmatrix} 0&1&1&1\\1&0&1&0\\1&1&0&1\\1&0&1&0 \end{bmatrix} \end{matrix}$$

(e)
$$\begin{matrix} & \begin{matrix} A & B & C \end{matrix} \\ \begin{matrix} A\\B\\C \end{matrix} & \begin{bmatrix} 0&2&0\\2&0&2\\0&2&0 \end{bmatrix} \end{matrix}$$

(f)
$$\begin{matrix} & \begin{matrix} A & B & C & D \end{matrix} \\ \begin{matrix} A\\B\\C\\D \end{matrix} & \begin{bmatrix} 1&1&0&1\\1&1&1&0\\0&1&1&1\\1&0&1&1 \end{bmatrix} \end{matrix}$$

(g)
$$\begin{matrix} & \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{matrix} A\\B\\C\\D\\E \end{matrix} & \begin{bmatrix} 0&1&0&0&0\\1&0&1&0&1\\0&1&0&1&2\\0&0&1&0&1\\0&1&2&1&0 \end{bmatrix} \end{matrix}$$

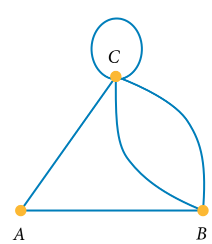
(h)
$$\begin{matrix} & \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{matrix} A\\B\\C\\D\\E \end{matrix} & \begin{bmatrix} 0&1&0&1&1\\1&0&1&1&1\\0&1&0&1&1\\1&1&1&0&0\\1&1&1&0&0 \end{bmatrix} \end{matrix}$$

3 (a) C

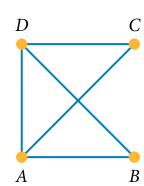
(b) The student has counted the loop at D as two connections instead of one.

4 D

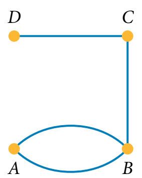
5 (a)



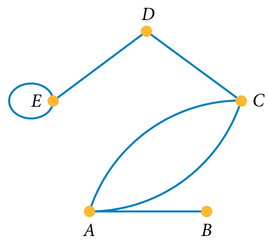
(b)



(c)



(d)



6

	A	B	C	D
A	0	0	0	1
B	1	0	0	0
C	1	0	0	0
D	0	0	0	0

7 (a)

	A	B	C
A	0	2	0
B	0	0	1
C	0	0	0

(b)

	A	B	C
A	0	1	1
B	0	0	1
C	0	0	0

(c)

	A	B	C	D
A	0	1	0	1
B	0	1	0	0
C	1	1	0	0
D	0	0	1	0

(d)

	A	B	C	D
A	0	1	1	0
B	1	0	1	1
C	0	0	0	0
D	0	0	0	0

(e)

	A	B	C	D
A	0	1	1	1
B	0	0	2	0
C	0	0	0	0
D	0	1	0	0

(f)

	A	B
A	2	1
B	0	1

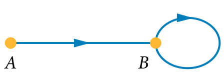
(g)

	A	B	C	D	E
A	0	1	1	1	0
B	0	0	1	0	0
C	0	0	0	1	0
D	0	0	0	0	1
E	1	0	0	0	0

(h)

	A	B	C	D	E
A	0	1	0	0	0
B	0	0	1	1	1
C	0	0	0	1	0
D	0	0	0	0	0
E	0	0	0	1	0

8



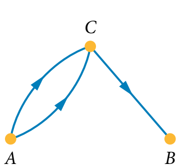
9

	R	P	S
R	0	0	1
P	1	0	0
S	0	1	0

10

	A	B	C	D	E	F
A	0	1	1	1	0	0
B	0	0	0	0	0	0
C	0	1	0	0	0	0
D	0	1	1	0	0	0
E	1	1	0	1	0	1
F	1	1	0	1	0	0

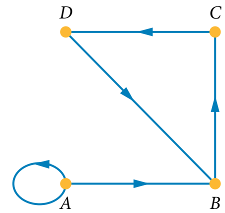
11



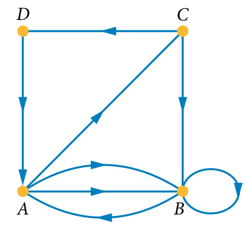
12

	B	R	M	I	T	C
B	0	1	0	0	0	0
R	1	0	1	1	0	0
M	0	1	0	0	1	0
I	0	1	0	0	0	0
T	0	0	1	0	0	1
C	0	0	0	0	1	0

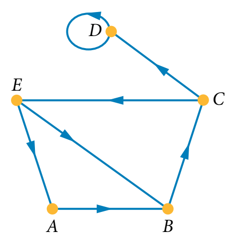
13 (a)



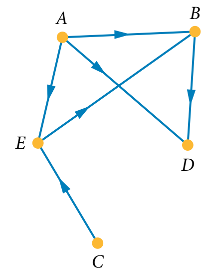
(b)



14 (a)



(b)



15

	A	B	C	D	E	F
A	0	3	0	0	0	0
B	3	0	1	1	1	0
C	0	1	0	0	2	0
D	0	1	0	0	0	0
E	0	1	2	0	1	0
F	0	0	0	0	0	2

16 (a) Two of many possibilities. The rows are interchangeable.

0	1	0	1	1
1	0	1	1	1
0	1	0	0	0
1	1	0	0	0
1	1	0	0	0

and

0	1	1	1	1
1	0	0	1	1
1	0	0	0	0
1	1	0	0	0
1	1	0	0	0

(b) Two of many possibilities. The rows are interchangeable.

0	1	0	0	0
0	0	1	0	0
0	0	0	0	0
0	1	0	0	1
1	0	0	0	0

and

0	0	0	0	0
0	0	1	0	0
0	0	0	1	0
1	0	0	0	0
0	1	0	1	0

17 (a) The fourth column, instead of being all zeros, will now have the first three rows as 1s.

(b) The second row and the second column will be all zeros.

Change 2 to 0 in the first position of the second row and second column.

Change 1 to 0 in the third position of the second row and second column.

18 Added: a loop at *C*; a connection between *A* and *C*.

Removed: one of the two loops at *D*; the connection between *A* and *B*; one of the two connections between *A* and *D*.

EXERCISE 7.4

1 **(a)** Trail **(b)** Cycle **(c)** Walk **(d)** Open path

2 **(a)** Cycle **(b)** Walk

(c) Open path **(d)** Trail (open)

3 **C**

4 **(a)** **C**

(b) There is no edge from *B* to *A*.

(c) **B**

(d) **A**

(e) **D**

5 **(a)** Neither **(b)** Semi-Eulerian graph

(c) Eulerian graph **(d)** Semi-Eulerian graph

(e) Neither **(f)** Semi-Eulerian graph

(g) Eulerian graph **(h)** Neither

6 **(a)** Hamiltonian paths only **(b)** A Hamiltonian cycle

(c) A Hamiltonian cycle **(d)** Hamiltonian paths only

(e) A Hamiltonian cycle **(f)** A Hamiltonian cycle

(g) Neither **(h)** A Hamiltonian cycle

7 **(a)** (a) and (d) **(b)** (b), (c), (e), (g) and (h)

8 One of many semi-Eulerian trails: *D-B-F-D-A-B-C-F-E-A-C*

9 *D-C-B-A-E-I-F-G-H-D-B-H-I*, *D-C-B-A-E-I-F-G-H-B-D-H-I*, *D-C-B-D-H-B-A-E-I-H-G-F-I*, *D-C-B-D-H-B-A-E-I-F-G-H-I*, *D-C-B-H-G-F-I-E-A-B-D-H-I*, *D-C-B-H-I-F-G-H-D-B-A-E-I*.

There are many others, including the reverse of each of these.

10 One of many Eulerian trails: *A-H-G-J-C-I-G-F-E-D-C-B-A*

11 **(a)** One possible path is $T_1 - T_2 - T_8 - T_9 - T_6 - T_3 - T_4 - T_7 - T_5$.

(b) No, it is not possible.

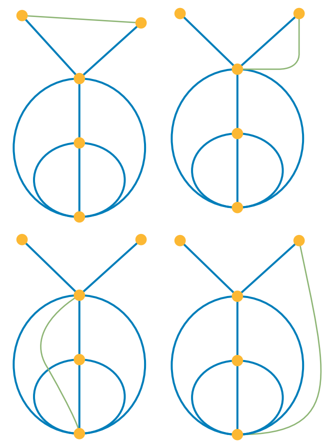
12 **(a)** Possible answers include: *H-A-B-C-E-D-H*, *H-D-E-C-B-A-H*, *H-A-E-B-C-D-H*, *H-D-C-B-E-A-H*, *H-D-C-E-B-A-H*.

(b) *H-A-B-C-B-E-C-D-E-A-D-H*, *H-A-B-C-B-E-D-C-E-A-D-H*, *H-A-B-C-B-E-A-D-E-C-D-H*, *H-A-B-C-D-E-C-B-E-A-D-H*, *H-A-B-C-D-E-B-C-E-A-D-H*, *H-A-B-C-D-A-E-B-C-E-D-H*, *H-A-B-C-D-A-E-C-B-E-D-H*, as well as other trails, including the reverse of each of these.

13 $H - D_1 - D_4 - D_3 - D_2 - D_6 - D_5 - H$,
$H - D_1 - D_4 - D_3 - D_6 - D_2 - D_5 - H$,
$H - D_1 - D_4 - D_6 - D_3 - D_2 - D_5 - H$,
$H - D_1 - D_3 - D_4 - D_6 - D_2 - D_5 - H$,
$H - D_1 - D_3 - D_4 - D_6 - D_5 - D_2 - H$

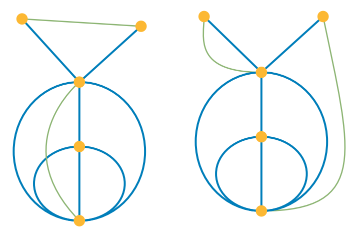
Other solutions include the reverse of each of these.

14 **(a)**

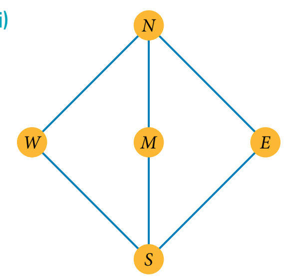


(b) Answers given with **(a)**.

(c)

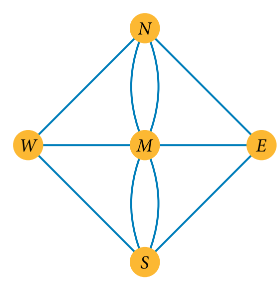


15 **(a)** **(i)**



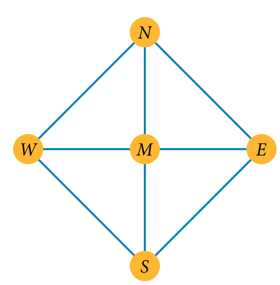
Possible options: *N-W-S-E-N-M-S*, *N-E-S-W-N-M-S*

(ii)



W-N-E-S-W-M-N-M-S-M-E

(iii) No Eulerian trails or semi-Eulerian trails are possible.



(b) Option 2

EXERCISE 7.5

1 (a) The shortest route is *A-E-F-I-C*. Length = 12 m

(b) The shortest route is *A-B-D-E-C*. Length = 10 m

(c) The shortest route is *A-G-E-I-D-C*. Length = 10 m

2 (a) **C**

(b) Paths with fewer vertices are not always shorter.

3 (a) **D**

(b) The student has found the shortest Hamiltonian path: *D-E-A-B-C*.

4 (a) *G-E-D*; 16 km (b) *A-F-E-D*; 21 km

5 (a) *P-Q-R-S-T-U*; 200 m (b) *Z-R-W-V*; 150 m

6 (a) 230 m (b) 170 m (c) 190 m

(d) 180 m (e) 210 m (f) 220 m

7 (a) *A-E-D-C*; 90 s

(b) *A-E-D-C-B-F-A* or *A-F-B-C-D-E-A*; 200 s

8 Charlton, St Arnaud, Donald, Birchip, Wycheproof, Boort, Wedderburn, Charlton; 288.5 km

9 (a) Possible cycle: *A-B-C-D-E-A*; 12 km

(b) Possible cycle: *A-D-C-B-E-A*; 29 km

(c) Possible cycle: *A-D-E-C-B-A*; 26 km

(d) None exists.

10 (a) *A-B-F-C*; 45 km

(b) The shortest route is *A-E-B-F-D-C*; 65 km

(c) *A-E-D-F-C-F-B-A*; 97 km

11 (a) *A-F-G-H-D*; 300 m (b) *A-F-G-E-D-H-C-B*; 525 m

12 (a) *A-F-G-E-D* or *A-F-G-C-D*; 8 min

(b) *A-B-G-F-G-C-E-D*; 14 min

(c) *A-B-G-C-D-E-F-A* or *A-F-E-D-C-G-B-A*; 20 min

(d) Many possible trails including *A-B-C-D-E-C-G-E-F-G-B-F-A*; 40 min

13 Madrid, Salamanca, Oporto, Lisbon, Badajoz, Seville, Granada, Valencia, Zaragoza, Burgos, Madrid; 2931 km

14 Possible cycle: *A-G-F-E-D-C-B-A*; 25 km

15 (a) 1 km shorter (b) No effect (c) 1 km shorter

16 (a) The new shortest route *A-E-D* at 170 km cuts 100 km off the previous shortest route *A-B-C-D* at 270 km.

(b) The new quickest route *A-E-D* at 2 h 18 min cuts a theoretical 34.5 minutes off the previous quickest route *A-B-C-D* at 2 h 52.5 min.

CHAPTER REVIEW 7

1 5

2 4 vertices, 5 edges, 3 faces

3 **D**

4 Directed

5 Edge *B*

6 4

7 (a) 4 vertices, 4 edges and 2 faces

(b) (i) Yes (ii) Yes (iii) No (iv) No

(v) No (vi) Yes (vii) No

8 (a) $f = 1$ (b) $f = 5$ (c) $f = 4$

(d) $f = 2$ (e) $f = 3$

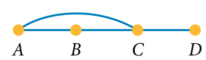
9

Number of vertices (v)	Number of edges (e)	Number of faces (f)
15	20	7
9	18	11
3	21	20
35	55	22
2	1	1
601	1000	401

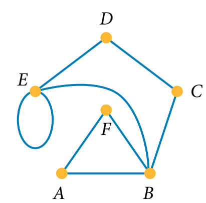
10 (a) $e = 0$ (b) $e = 66$

(c) $e = 4950$ (d) $e = 499\,500$

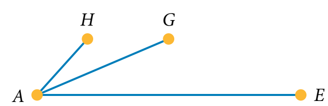
11 (a)



(b)

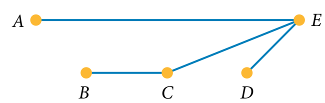


12 (a)



or any isomorphic equivalent

(b)



or any isomorphic equivalent

13 **C**

14 **D**

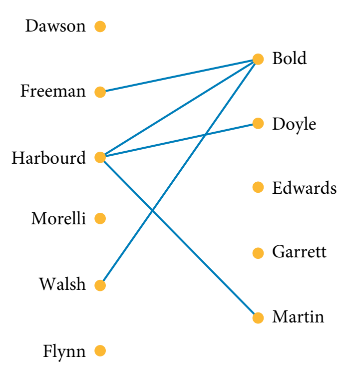
15 (a) Bold is the main character.

(b) Walsh, Flynn, Freeman and Edwards

(c) Garrett, Bold and Harbourd

(d) The characters involved in these meetings, in order, are Walsh, Bold, Harbourd and Dawson.

(e)



There are five meetings that take place between a good guy and a bad guy. There are two good guys who never meet a bad guy.

16 **(a)** There are four lines that are connected by the interchange.

(b) St-Michel Notre Dame station

(c) Saint François Xavier, Duroc, Vaneau, Sevres Babylone, Mabillon, Odéon, St-Michel, Cité, Châtelet

17 **C**

18

	A	*B*	*C*	*D*
A	0	2	1	0
B	2	0	0	0
C	1	0	0	2
D	0	0	2	1

19 **(a)**

	A	*B*	*C*	*D*	*E*
A	0	1	0	0	1
B	1	0	1	1	1
C	0	1	0	1	0
D	0	1	1	0	1
E	1	1	0	1	0

(b)

	P	*Q*	*R*	*S*	*T*
P	0	1	0	0	0
Q	0	0	1	1	0
R	0	0	0	1	1
S	0	0	0	0	0
T	0	0	0	0	0

20 Open trail

21 Starts and ends at different vertices, can repeat edges and vertices

22 *C* and *D*

23 *A-B-C-D*, *A-D-C-B* (two of these)

24 *A* and *B*, *A* and *D*, *A* and *H*, *B* and *D*, *B* and *H*, *D* and *H*

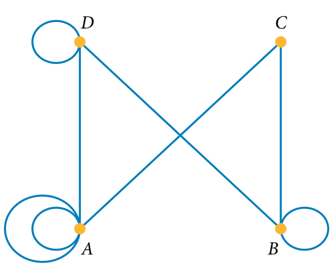
25 12, 2, 1, 10, 9, 8, 7, 6, 5, 4, 3, 12

26 **B**

27 21

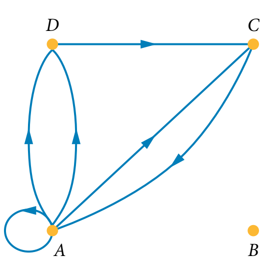
28 **B**

29



or any isomorphic equivalent

30

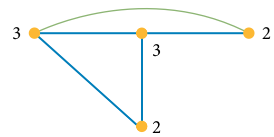


or any isomorphic equivalent

31 **(a)** Eulerian trail: *A-B-G-F-E-D-C-B-G-C-F-D-A*
Hamiltonian path: *A-B-G-F-C-D-E*

(b) Semi-Eulerian trail: *D-E-F-G-A-B-C-E-B-G*
Hamiltonian path: *D-E-F-G-A-B-C*

32 **(a)**



(b) It is possible because it has exactly 2 odd vertices.

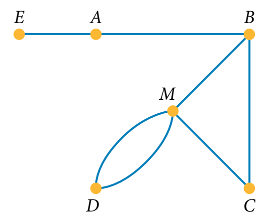
33 **(a)** *O-J-G-E*; 19 minutes

(b) *O-A-B-K-C-D-E-F-G-H-I-J-O* or the reverse; 51 minutes

34 **(a)** 11; *A-B-E-D* **(b)** 19; *A-B-C-F-D-E*

(c) 25; *A-F-C-D-E-B-A* **(d)** No

35 **(a)** Possible solution. Isomorphic graphs are also correct.



(b) No, it is not possible to draw a Hamiltonian cycle that starts at the entrance and visits each island exactly once.

(c) No, there is no Eulerian cycle that starts at the entrance.

(d) Yes, it is possible to have a semi-Eulerian trail that starts at the entrance.

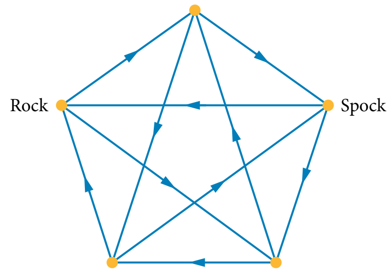
(e) Island B

36 **(a)**

Graph	Sum of degrees	Number of edges
(i)	0	0
(ii)	2	1
(iii)	4	2
(iv)	6	3
(v)	6	3
(vi)	8	4
(vii)	8	4
(viii)	8	4
(ix)	2	1
(x)	14	7
(xi)	10	5
(xii)	14	7

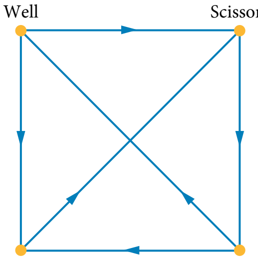
(b) The sum of the degrees is double the number of edges. Any edge ends in two different vertices unless it is a loop. In either case, the addition of an edge adds one to the degree of each of two vertices, or two to the degree of a single vertex.

37 **(a)**



(b) Player B: 3; Player A: 2

(c) **(i)**



(ii) Paper and Well can each defeat 2 other options. Rock and Scissors can defeat only 1 other option.

(d) The game is unbalanced because Fire defeats 3 other options. Paper, Rock and Scissors each defeat 2 options. Water defeats only 1 option.

38 **(a)** **(i)** $H\text{-}I\text{-}D\text{-}J\text{-}K\text{-}O = 18\,\text{km}$

(ii) $B\text{-}C\text{-}D\text{-}J\text{-}N = 25\,\text{km}$ and $B\text{-}F\text{-}H\text{-}I\text{-}D\text{-}J\text{-}N = 25\,\text{km}$

(b) **(i)** *A-E-I-H-F-G-C-B*

(ii) *H-I-D-C-B-A-E*

(iii) *H-F-G-D-J-N-M-L-K-O*

(c) $A\text{-}E\text{-}I\text{-}D\text{-}J\text{-}K\text{-}O\text{-}L = 26\,\text{km}$; *A-E-I-D* (on bus) *D-J* (on train) *J-K-O* (on bus) *O-L* (on train); $46.40

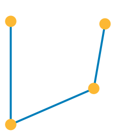
(d) $168

Chapter 8: Networks and decision maths

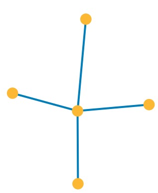
RECALL

1 **(a)** $x = 5$ **(b)** $x = -20$ **(c)** $x = 9$

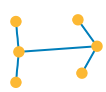
2 **(a)**



(b)



(c)



3 **(a)**

	S	*O*	*N*
Paris	21	16	11
London	19	14	10

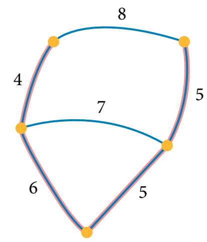
(b)

	W	*R*
Fleser	2	33
Noros	5	23
Milderwuth	0	24

4 **(a)** 70% **(b)** 5% **(c)** 87.5%

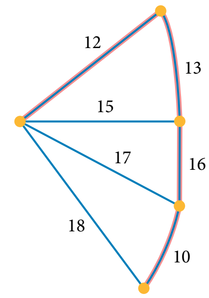
EXERCISE 8.1

1 **(a)**



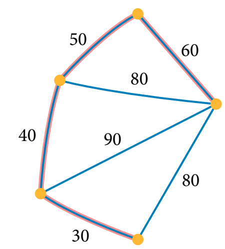
Weight: 20

(b)



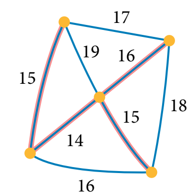
Weight: 51

(c)



Weight: 180

(d)



Weight: 60

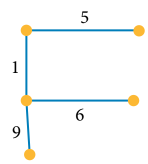
2 **(a)** *A-D*, *A-C*, *A-B* **(b)** 57

3 **(a)** C

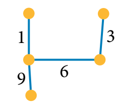
(b) The subgraph contains a cycle.

The graph has 5 vertices, the minimum spanning tree has 4 edges, so 2 edges must be removed.

4 **(a)**



Weight: 21



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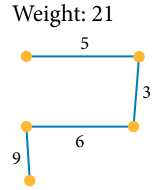
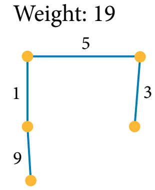


Weight: 23



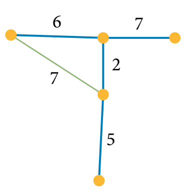
Weight: 18

(b) 18

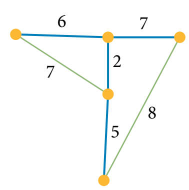
5 **(a)** 5 **(b)** 7

6 **(a)** 1 **(b)** 75

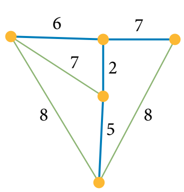
7 **(a)**



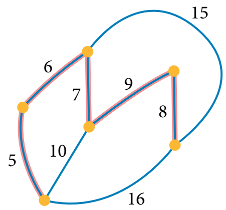
(b)



(c)

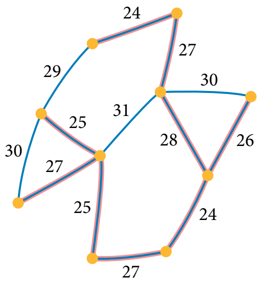


8 (a)



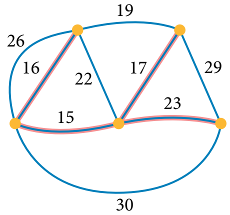
Weight: 35

(b)



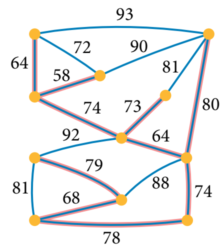
Weight: 233

(c)



Weight: 71

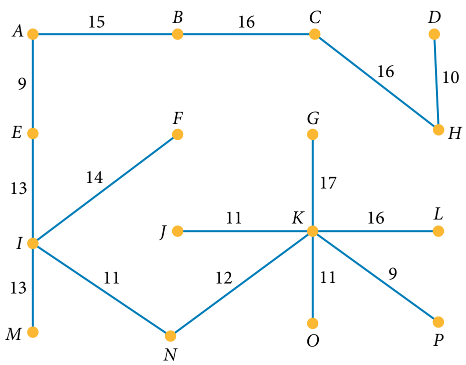
(d)



Weight: 712

9 **C**

10 (a) This is a sample answer; there are several correct trees.



(b) 193

11 5000 m

12 (a) *A-C*, *C-D*, *C-H* and *D-H* (b) 680 m

13 $259 000

14 (a) $260 000 (b) $320 000

15 The actual minimum spanning tree will be longer by 2.

EXERCISE 8.2

1 (a) 10 (b) 5 (c) 7 (d) 10

2 (a) 35 (b) 38 (c) 49

3 (a) 60 (b) 19 (c) 9

4 30 MB per second

5 40 ML per day

6 (a) **C** (b) **B** and **D**

7 45

8 220

9 (a) **B** (b) **A**

10 190

11 90

12 (a) 45 litres per minute

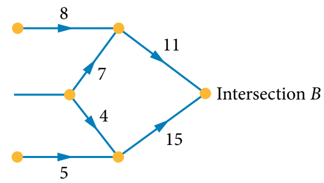
(b) Either of the pipes leading to the finish.

(c) 7 litres per minute

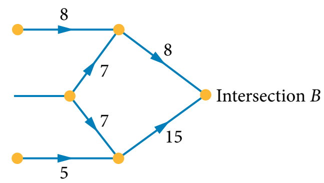
(d) 52 litres per minute

13 (a) 7000 more vehicles could travel between intersections *A* and *B*.

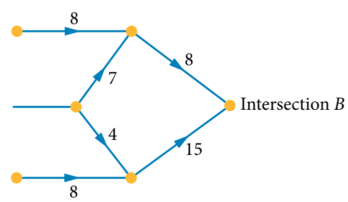
(b)



Or



Or



EXERCISE 8.3

1 (a)

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} I & J & K \end{array} \\ \begin{bmatrix} 8 & 11 & 16 \\ 9 & 8 & 19 \\ 7 & 10 & 17 \end{bmatrix} \end{array}$$

Job *I*	Job *J*	Job *K*
Colt	Blair	Ava

Cost: $31

(b)

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} I & J & K \end{array} \\ \begin{bmatrix} 80 & 35 & 20 \\ 50 & 40 & 15 \\ 70 & 32 & 18 \end{bmatrix} \end{array}$$

Job *I*	Job *J*	Job *K*
Briar	Che	Ana

Cost: $102

(c)

	P	Q	R
X	19	51	38
Y	16	55	40
Z	21	59	33

Job *P*	Job *Q*	Job *R*
Yuri	Xia	Zac

Cost: $100

(d)

	I	J	K	L
A	30	50	65	85
B	25	45	80	80
C	40	55	70	95
D	20	60	75	90

Three alternatives:

Job *I*	Job *J*	Job *K*	Job *L*
Drake	Chiara	Art	Bart
Drake	Art	Chiara	Bart
Drake	Bart	Chiara	Art

Cost: $220

(e)

	I	J	K	L
A	85	43	100	225
B	53	49	143	257
C	58	48	121	209
D	89	59	149	245

Job *I*	Job *J*	Job *K*	Job *L*
Bodhi	Dee	Arun	Cate

Cost: $421

(f)

	S	T	U	V
P	13	62	70	79
Q	44	74	93	64
R	19	67	88	76
S	25	56	61	75

Job *S*	Job *T*	Job *U*	Job *V*
Phil	Rob	Stuart	Quinton

Cost: $205

2 (a)

Job *I*	Job *J*	Job *K*
C	*B*	*A*

(b) Two alternatives:

Job *I*	Job *J*	Job *K*
A	*C*	*B*
B	*C*	*A*

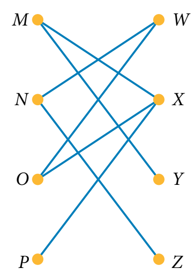
3 (a) $\begin{bmatrix} 0 & 15 & 0 \\ 2 & 0 & 2 \\ 7 & 8 & 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} 3 & 17 & 0 \\ 0 & 0 & 0 \\ 0 & 7 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 3 & 0 & 3 \\ 0 & 5 & 8 & 8 \\ 0 & 3 & 6 & 0 \\ 6 & 0 & 0 & 1 \end{bmatrix}$ **(d)** $\begin{bmatrix} 20 & 3 & 4 & 0 \\ 4 & 0 & 0 & 0 \\ 42 & 13 & 5 & 0 \\ 0 & 14 & 3 & 0 \end{bmatrix}$

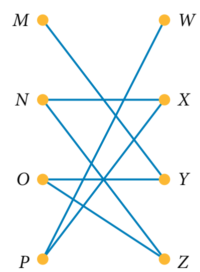
4 (a) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 6 \\ 6 & 5 & 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} 0 & 9 & 6 \\ 52 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 7 & 4 & 0 \\ 0 & 13 & 0 & 7 \\ 0 & 9 & 2 & 0 \\ 14 & 0 & 0 & 7 \end{bmatrix}$ **(d)** $\begin{bmatrix} 4 & 5 & 5 & 0 \\ 2 & 0 & 0 & 1 \\ 7 & 7 & 0 & 0 \\ 3 & 2 & 0 & 0 \end{bmatrix}$

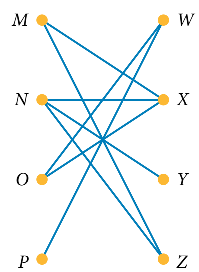
5 (a)



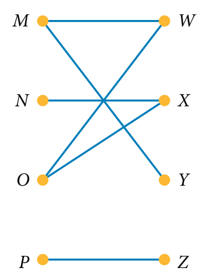
(b)



(c)



(d)



6 **D**

7

Run	Swim	Paddle boarding
Hong	Isaac	Gus

8 (a) $\begin{bmatrix} 0 & 2 & 7 \\ 1 & 3 & 11 \\ 0 & 1 & 9 \end{bmatrix}$ **(b)** $\begin{bmatrix} 7 & 9 & 9 \\ 3 & 0 & 5 \\ 6 & 4 & 8 \end{bmatrix}$ **(c)** $\begin{bmatrix} 0 & 9 & 18 & 5 \\ 18 & 17 & 10 & 8 \\ 16 & 15 & 9 & 7 \\ 14 & 15 & 0 & 8 \end{bmatrix}$

9 (a) **D**

(b) The minimum total cost is $36.

10

	Maths	Physics	Chemistry	Biology
Tutor	Clever kids	Directed Study	A Plus	Best results

Cost: $211

11

	Mowing	Paving	Roof repair	Skylight
Contractor	Dane's	Better	Ace	Cheaper

Cost: $565

12

Unit 1	Unit 2	Unit 3	Unit 4
Cody	Brian	Dirk	Allison

Cost: $3450

13

L1	L2	L3	L4
Bacchus	Cali	Achilles	Daman

77 km

14

W	X	Y	Z
T4	T3	T2	T1

118 minutes

15

Job 1	Job 2	Job 3	Job 4
Catriona	Adair	Beth	Dalia

306 minutes

16 Oscar, Norman, Manny, Percy

17

Melbourne	Sydney	Brisbane	Perth
Bowling Pebbles	Pink Kinks	Stratus Crow	Yabba Dabba

$168 000

18

Country	A	B	C	D	E
Container	W	Y	X	U	V

$31 million

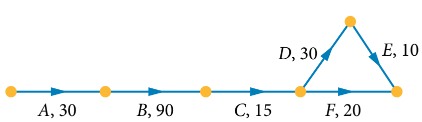
19 $4150

Unheralded News	Age Old News	Daily Telepath	Sunday Mews
Refugees	Climate change	Education reform	Gay marriage

EXERCISE 8.4

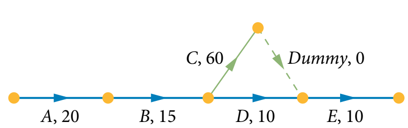
1 (a)

Activity	Immediate predecessors	Duration (minutes)
A: leisure activity	–	30
B: homework part A	*A*	90
C: eat dinner	*B*	15
D: homework part B	*C*	30
E: shower	*D*	10
F: listen to music	*C*	20



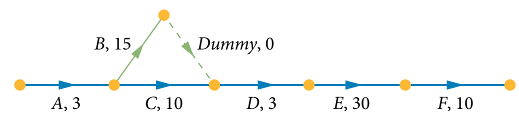
(b)

Activity	Immediate predecessors	Duration (minutes)
A: cut out and assemble pieces	–	20
B: glue the edges	*A*	15
C: glue dries	*B*	60
D: prepare stickers	*B*	10
E: add stickers	*C*, *D*	10



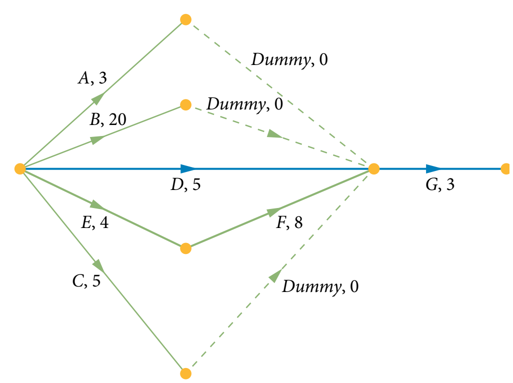
(c)

Activity	Immediate predecessors	Duration (minutes)
A: walk to car	–	3
B: drive to pool	*A*	15
C: eat snacks	*A*	10
D: change into togs	*B*, *C*	3
E: lesson	*D*	30
F: shower and change	*E*	10



(d)

Activity	Immediate predecessors	Duration (minutes)
A: set table	–	3
B: boil carrots	–	20
C: microwave potatoes	–	5
D: steam broccoli	–	5
E: prepare salmon	–	4
F: grill salmon	*E*	8
G: serve to plates	*A*, *B*, *C*, *D*, *F*	3



2 (a)

Activity	EST	LST
A	0	0
B	0	3
C	7	7

(b)

Activity	EST	LST
A	0	0
B	3	3
C	3	4
D	7	8

(c)

Activity	EST	LST
A	0	2
B	0	1
C	0	0
D	5	7
E	2	2

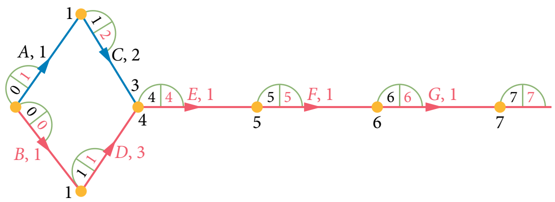
(d)

Activity	EST	LST
A	0	2
B	0	1
C	0	0
D	5	7
E	2	2
F	2	7
G	6	11
H	10	10

3 (a) **D** (b) **D** (c) **B** (d) **C**

4 (a) **B** (b) The student has confused EST with LST.

5 (a)

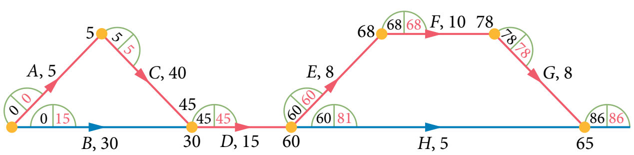


(b) *B-D-E-F-G*

(c) 7 days

(d) A: 1 hour; C: 1 hour

6 (a)

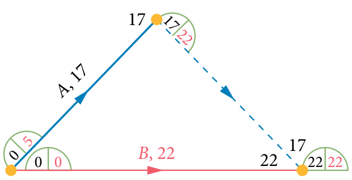


(b) *A-C-D-E-F-G*

(c) 86 minutes

(d) *B*: 15 minutes; *H*: 21 minutes

7 (a)



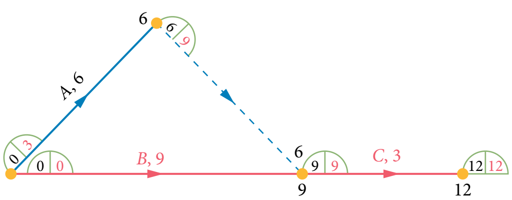
(b) *B*

(c) 22 days

(d)

Activity	Duration (days)	Immediate predecessor	EST	LST	Float (days)
A	17	–	0	5	5
B	22	–	0	0	0

8 (a)



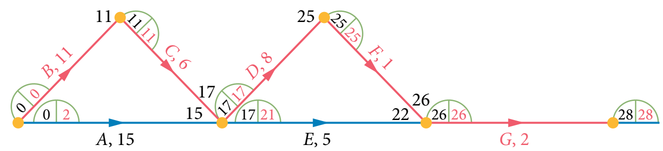
(b) *B-C*

(c) 12 days

(d)

Activity	Duration (days)	Immediate predecessor	EST	LST	Float (days)
A	6	–	0	3	3
B	9	–	0	0	0
C	3	*A, B*	9	9	0

9 (a)



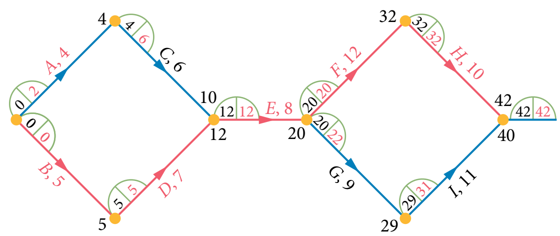
(b) *B-C-D-F-G*

(c) 28 days

(d)

Activity	Duration (days)	Immediate predecessor	EST	LST	Float (days)
A	15	–	0	2	2
B	11	–	0	0	0
C	6	*B*	11	11	0
D	8	*A, C*	17	17	0
E	5	*A, C*	17	21	4
F	1	*D*	25	25	0
G	2	*E, F*	26	26	0

10 (a)



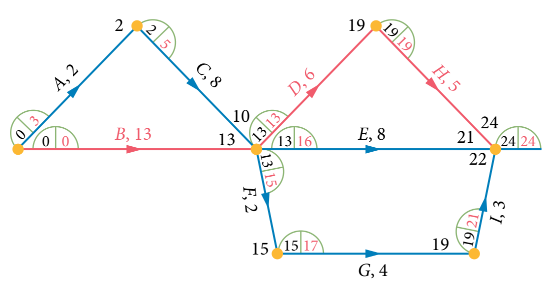
(b) *B-D-E-F-H*

(c) 42 days

(d)

Activity	Duration (days)	Immediate predecessor	EST	LST	Float (days)
A	4	–	0	2	2
B	5	–	0	0	0
C	6	*A*	4	6	2
D	7	*B*	5	5	0
E	8	*C, D*	12	12	0
F	12	*E*	20	20	0
G	9	*E*	20	22	2
H	10	*F*	32	32	0
I	11	*G*	29	31	2

11 (a)



(b) *B-D-H*

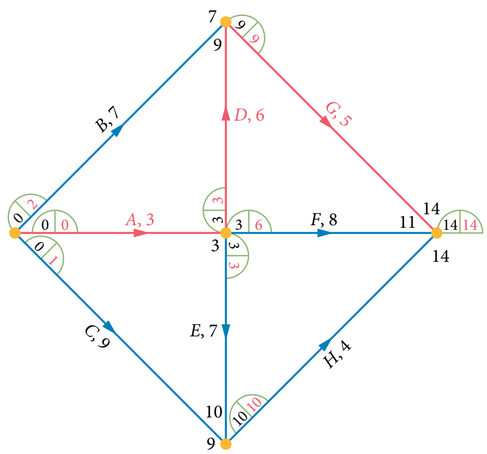
(c) 24 days

EXERCISE 8.4

(d)

Activity	Duration (days)	Immediate predecessor	EST	LST	Float (days)
A	2	–	0	3	3
B	13	–	0	0	0
C	8	*A*	2	5	3
D	6	*B*, *C*	13	13	0
E	8	*B*, *C*	13	16	3
F	2	*B*, *C*	13	15	2
G	4	*F*	15	17	2
H	5	*D*	19	19	0
I	3	*G*	19	21	2

12 (a)



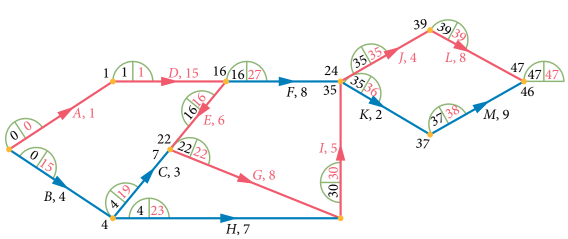
(b) *A*-*D*-*G* or *A*-*E*-*H*

(c) 14 days

(d)

Activity	Duration (days)	Immediate predecessor	EST	LST	Float (days)
A	3	–	0	0	0
B	7	–	0	2	2
C	9	–	0	1	1
D	6	*A*	3	3	0
E	7	*A*	3	3	0
F	8	*A*	3	6	3
G	5	*B*, *D*	9	9	0
H	4	*C*, *E*	10	10	0

13 (a)



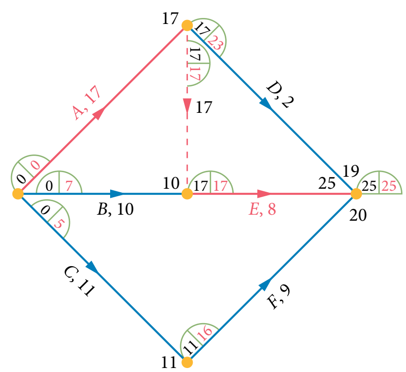
(b) *A*-*D*-*E*-*G*-*I*-*J*-*L*

(c) 47 days

(d)

Activity	Duration (days)	Immediate predecessor	EST	LST	Float (days)
A	1	–	0	0	0
B	4	–	0	15	15
C	3	*B*	4	19	15
D	15	*A*	1	1	0
E	6	*D*	16	16	0
F	8	*D*	16	27	11
G	8	*C*, *E*	22	22	0
H	7	*B*	4	23	19
I	5	*G*, *H*	30	30	0
J	4	*F*, *I*	35	35	0
K	2	*F*, *I*	35	36	1
L	8	*J*	39	39	0
M	9	*K*	37	38	1

14 (a)



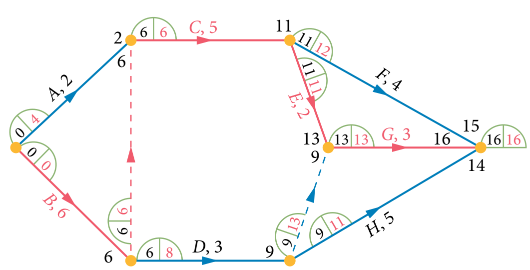
(b) *A*-*E*

(c) 25 days

(d)

Activity	Duration (days)	Immediate predecessor	EST	LST	Float (days)
A	17	–	0	0	0
B	10	–	0	7	7
C	11	–	0	5	5
D	2	*A*	17	23	6
E	8	*A*, *B*	17	17	0
F	9	*C*	11	16	5

15 (a)



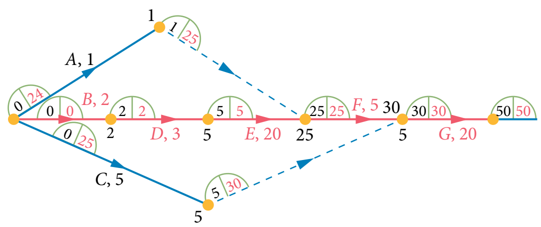
(b) *B-C-E-G*

(c) 16 days

(d)

Activity	Duration (days)	Immediate predecessor	EST	LST	Float (days)
A	2	–	0	4	4
B	6	–	0	0	0
C	5	*A*, *B*	6	6	0
D	3	*B*	6	8	2
E	2	*C*	11	11	0
F	4	*C*	11	12	1
G	3	*D*, *E*	13	13	0
H	5	*D*	9	11	2

16 (a)



(b) *B-D-E-F-G*

(c) 50 minutes

(d) *A*: 24 minutes; *C*: 25 minutes

17 (a) *B-C-F-H*; 15 hours

(b) *B-C-F-H*; 13 hours

(c) *B-D-G-H*; 14 hours

(d) *B-C-F-H*; 15 hours

18 (a) *C-F*; 20 days (b) 18 days

CHAPTER REVIEW 8

1 **B**

2 57

3 **A**

4 $310 000

5 409.5 m^2

6 75

7 10

8 22 units

9 Cut (a): 41, Cut (b): 31, Cut (c): 38, Cut (d): 49

10 $x = 13$

11 **B**

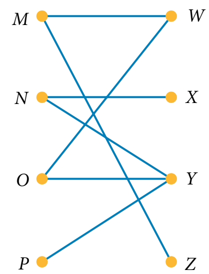
12

$$\begin{bmatrix} 10 & 7 & 0 & 6 \\ 5 & 0 & 11 & 3 \\ 0 & 13 & 5 & 5 \\ 0 & 6 & 12 & 0 \end{bmatrix}$$

13 (a) Cut grass: Jacob; Do dishes: Krystal; Vacuum the lounge: Louis

(b) Cut grass: Krysta; Do dishes: Louis; Vacuum the lounge: Jacob

14 (a)



(b)

Job	*W*	*X*	*Y*	*Z*
Person	*O*	*N*	*P*	*M*

15

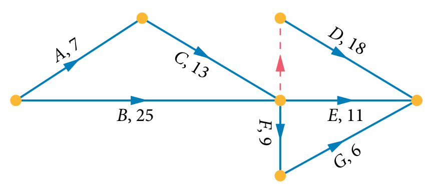
$$\begin{bmatrix} 12 & 7 & 0 \\ 5 & 14 & 9 \\ 8 & 11 & 7 \end{bmatrix}$$

16 (a) 76 hours (b) 32 hours (c) *C* and *D*

(d) 2 hours (e) 2 hours

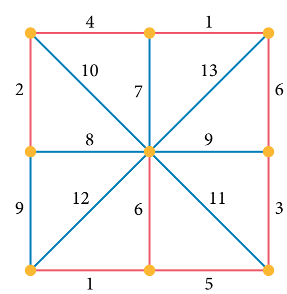
17 (a) **D** (b) **A**

18



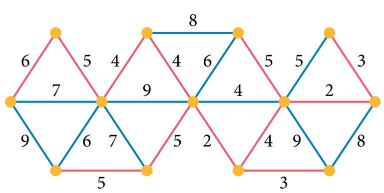
19 (a) 19 (b) 22 (c) 40

20 (a)



Total: 28

(b)



Total: 43

21 **A**

22 (a) 120 pedestrians per minute

(b) (i) *E-F* or *H-I* or both

(ii) 20 pedestrians per minute

23 (a) *A*, *B*, *E*, *D*, *C* (b) 250 L/min (c) Drain *B*

24 **C**

25 Xavier 50 m; Zac 100 m; Yertle 200 m; Wendel 400 m

26 (a) *A-C-D* (b) 41 minutes (c) Activity *B*

(d) 18 minutes (e) 5 minutes (f) 8 minutes

27 (a) (i) 50 hours (ii) 65 hours

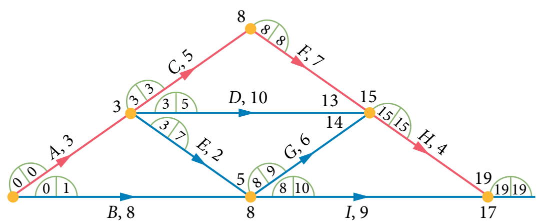
(b) Activities *B* and *H*

(c) 165 hours

(d) 135 hours

(e) 155 hours

28 (a)



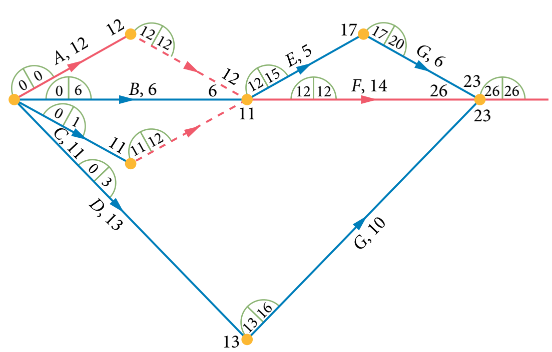
(b) *A-C-F-H*

(c) 19 hours

(d)

Non-critical activity	May be delayed by (hours)
B	1
D	2
E	4
G	1
I	2

29 (a)



(b) *A-F*

(c) 26 days

(d)

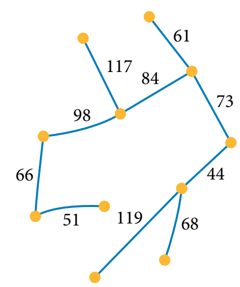
Activity	Duration (days)	Immediate predecessor	EST	LST	Float (days)
A	12	–	0	0	0
B	6	–	0	6	6
C	11	–	0	1	1
D	13	–	0	3	3
E	5	*A*, *B*, *C*	12	15	3
F	14	*A*, *B*, *C*	12	12	0
G	10	*D*	17	20	3

30 (a) No effect (b) 4 days

(c) 2 days (d) 2 days

31 (a) 154, 149, 130, 128, 116

(b)



(c) The total length of the minimum spanning tree is 781 units.

32 (a)

Component	*I*	*J*	*K*	*L*
Store	*E*	*A*	*D*	*C*

(b) $310

(c) $317

(d) No, Val's method is not valid for determining the minimum cost, because the Hungarian algorithm gives a lower cost.

33 (a) *A-G-L-N-O-T*; 47 days

(b) *R*, *S*, *U* and *V*

(c) (i) No effect (ii) Increased by 3 days

(iii) Increased by 3 days (iv) Decreased by 5 days

Exam review: Unit 4

PAPER 1

1 (a) 1.4% (b) 5.6%

(c) $8140.34 (d) $A_0 = 7700$, $A_{n+1} = 1.014 \times A_n$

2 (a) $L_0 = 11500$, $L_{n+1} = 1.005 \times L_n$

(b) $12 209.29

(c) $1462.34

3 $4384.50

4 (a) Option 1: \$15 315.38; Option 2: \$14 954.18

(b) \$19 084.75 and \$19 084.74 respectively.

(c) The original balance was multiplied by $(1.045)^5$ and $(1.05)^5$ for each.

(d) Option 2 will have had six years of 5% growth and five years of 4.5% growth, whereas Option 1 will have had one year less of the higher growth and one year more of the lower growth.

5 (a) 16.08% p.a. (b) 16.08% p.a.

6 They will never repay this loan, because it is an increasing balance loan.

7 (a) $A_0 = 145\,000$, $A_{n+1} = 1.0028 \times A_n - 406$

(b) 3.36% p.a.

8 11 months

9 (a) $e = 5$, $v = 4$, $f = 3$

(b) (i) No (ii) Yes (iii) No

(iv) No (v) Yes (vi) No

10 $v = 34$

11 (a) Shellfish, small fish, krill

(b) Plankton and phytoplankton

(c) Sharks

12 (a) True (b) True (c) True

(d) False (e) True (f) False

13 (a) *A-D-C*; 7 h

(b) *A-E-D-C-B*; 15 h

(c) *A-E-D-C-B-A* or its reverse *A-B-C-D-E-A*: 22 h

14 17 units

15 *AD*; 19 minutes

PAPER 2

1 (a) $L_0 = 14\,595.45$, $L_{n+1} = 1.0645 \times L_n$

(b) \$18 741.31

2 \$432.22

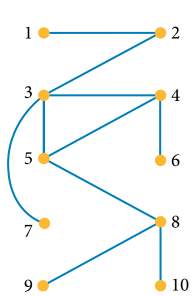
3 \$60.87

4 3.24% p.a.

5 15 months

6 \$382 000

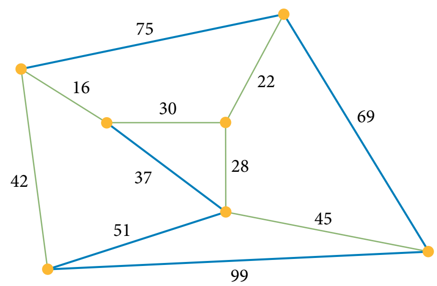
7



8 (a) Possible Eulerian trail:
A-L-K-J-I-H-G-F-E-D-C-B-L-M-K-O-J-H-O-G-N-F-D-N-C-M-B-A

(b) Possible semi-Eulerian trail:
A-F-G-H-F-E-H-I-E-D-I-C-H-B-G-A-B-C-D

9 (a)



(b) 183 km

10 1100 vehicles per 15 minute period

11

Task	Indoor spa	Junior's room	Kitchen	Living room
Person	Bjorn	Alana	Cameron	Dana

Cost: \$61 000

12 \$40 830.88

13 25 days

Exam review: Units 3 & 4

PAPER 1

1 **D**

2 **B**

3 **A**

4 **C**

5 **B**

6 **C**

7 **D**

8 Brad: $0.5 < r < 0.75$, Angelina: $-0.75 < r < -0.5$

9 There is a clear pattern, so linearity is not confirmed.

10 (a) 96% of the change in fuel consumed can be explained by the change in distance travelled. 4% of the change is due to the variation other factors.

(b) The numbers of fast food outlets and public toilets both respond to population.

11 (a) Decreasing trend. Global warming and human factors such as overfishing are having a negative impact on the survival of marine species on the Great Barrier Reef, causing some to become extinct.

(b) Cyclic trend. Long-term share market performances fluctuate. This performance is unpredictable, so the values have a cyclic, not seasonal, trend.

(c) Random trend. Unfortunately, there are wars around the world at any time, but their occurrence over the long term cannot be predicted.

12 $SI_{\text{Dec-Feb}} = 1.32$, $SI_{\text{Mar-May}} = 0.66$, $SI_{\text{Jun-Aug}} = 0.92$, $SI_{\text{Sep-Nov}} = 1.11$

13 (a) $t_9 = 6144$ (b) $t_9 = 46$

14 (a) \$1032 (b) \$11592

15 (a) Birdsville, Queensland

(b) Geelong, Victoria

(c) Albany, Western Australia

16 (a) 5:10 pm (b) 3:40 pm

(c) 4:40 pm (d) 2:10 pm

17 (a) $A_0 = 1250$, $A_{n+1} = 1.0102 \times A_n$

(b) \$1301.78

(c) \$1355.72

18 (a) \$2145.00 (b) 7.25%

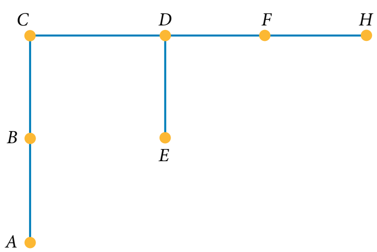
19 (a) \$132 725.02 (b) \$130 370.41

20 (a) No; it contains a cycle.

(b) 14

(c) *E-D-F-H-I-J-K-L-N*

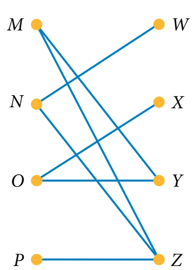
(d)



21

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{matrix} A & B & C & D & E \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 1 & 1 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

22 (a)

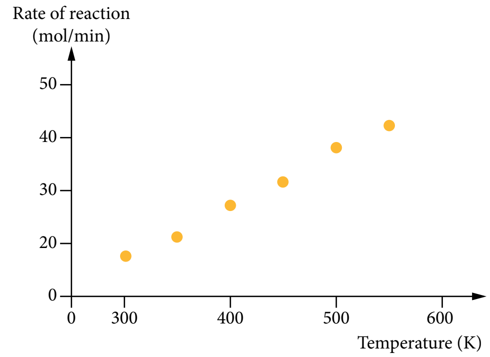


(b)

Job	*W*	*X*	*Y*	*Z*
Person	*N*	*O*	*M*	*P*

PAPER 2

1 (a) Temperature is the explanatory variable.



The graph shows a very strong positive linear trend.

(b) $r = 1.00$

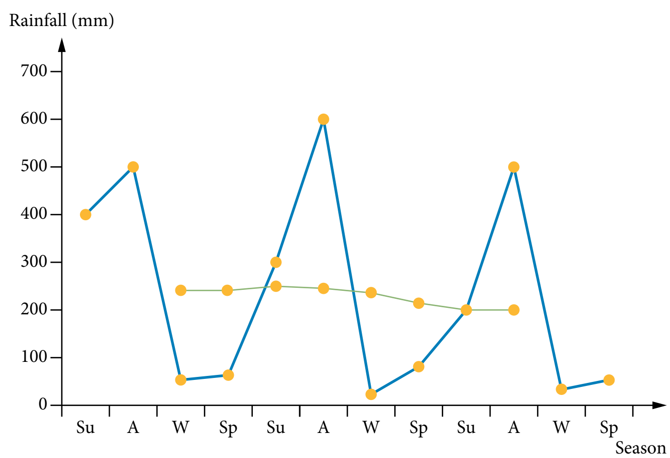
(c) Rate of reaction (mol/min) = 0.1 × temperature (K) − 14.6

(d) 22.4 mol/min. With interpolation and very high association, this prediction is likely to be accurate.

(e) 0.4 mol/min. With extrapolation, confidence in the prediction is reduced.

2

Season	Sum	Aut	Win	Spr	Sum	Aut	Win	Spr	Sum	Aut	Win	Spr
Rainfall (mm)	400	500	50	60	300	600	20	80	200	500	30	50
			253	228	253	245	250	225	200	203	195	
			241	241	249	248	238	213	202	199		



3 (a) $V_n = 20\,000(0.92)^n$ (b) 12127 L (c) 28 hours

4 (a) 5:25 pm

(b) 3 h 5 min

(c) Riga to Copenhagen: 1 h 40 min

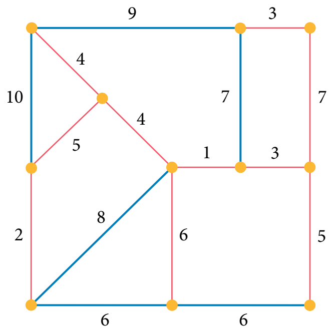
Copenhagen to Amsterdam: 1 h 25 min

5 **(a)** 128.31% p.a.

(b) 166 days

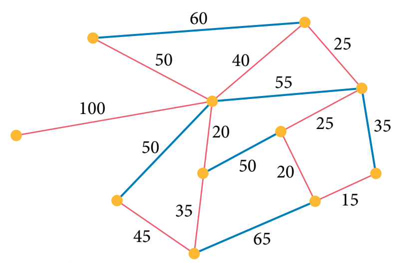
6 $v = 10$

7 **(a)**



Length: 48

(b)



Length: 375

8 **(a)** Let x represent the start of each year, so $x = 2010$ gives the start of 2010, and y is the unemployment rate as a percentage.

From near the start of 2009 to early 2011:
$y = -0.4x + 809.5$

From early 2011to early 2015: $y = 0.35x - 699$

From early 2015 to the end of 2017: $y = -0.275x + 560.4$

(b) 1.1%, 8.4% and 4.2% respectively.

All involve extrapolation, so they assume the trend continues. The first two trends are clearly no longer relevant. For the third trend to continue for three more years is very unlikely, given the cyclic nature of the unemployment figures.

9 **(a)** \$4510 **(b)** \$498 542.46 **(c)** \$5815

10 **(a)** Barbeque sauce recipe:

Activity	Duration (minutes)	Immediate predecessor	EST (minutes)	LST (minutes)
A	10	–	0	1
B	12	–	0	7
C	10	–	0	0
D	8	*A*	10	11
E	9	*C*	10	10
F	8	*B*, *D*, *E*	19	19
G	16	*C*	10	11
H	8	*F*, *G*	27	27

(b) 35 minutes