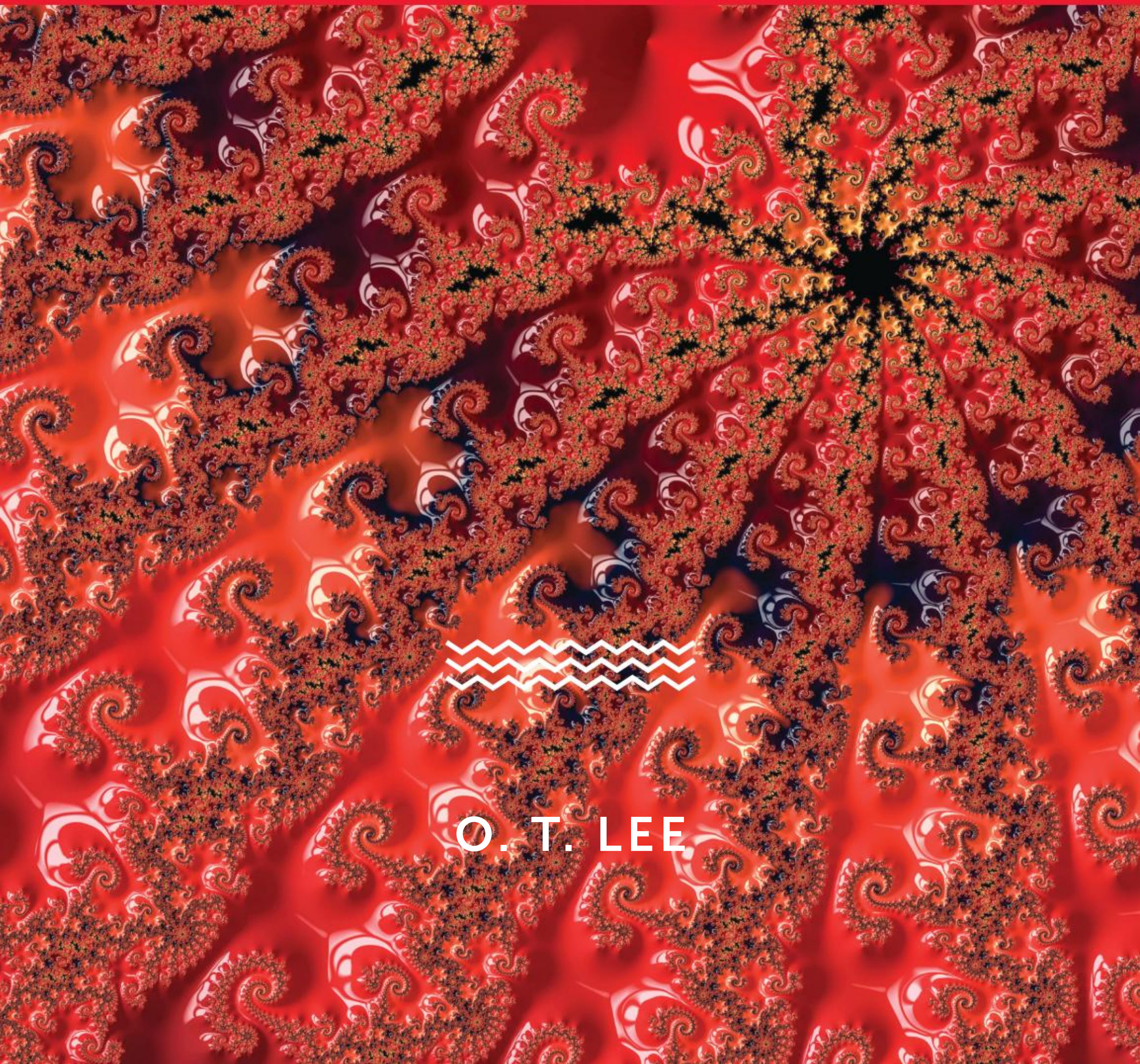
YEAR 12 ATAR COURSE REVISED EDITION

ACADEMIC TASK FORCE

REVISION SERIES

MATHEMATICS APPLICATIONS

UNITS 3 & 4



O. T. LEE

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ACADEMIC GROUP
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Achieve Success at School

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About the Author
Dr O. T. Lee is an author of many books which are used extensively in WA schools. Dr Lee is an exceptional, insightful teacher with wide-ranging experience as a WACE marker.

Mathematics Applications Revision Series Units 3 & 4

Contents

Notes

Mathematics Applications Revision Series Units 3 & 4

- The Mathematics Applications Revision Series Units 3 & 4 provides a comprehensive set of revision/review questions for the new year 12 Mathematics Applications Units 3 & 4 course.

- The review questions are written at test/examination level for both the Calculator Free and Calculator Assumed Sections and presented in a write-on format in topical order.

- This book exposes students to questions and problems at test/examination level.

- These questions are suitable for end-of-topic reviews and pre-test and pre-examination reviews.

- It is accompanied by a set of fully worked solutions with which students can measure their solutions. These solutions are often not the only solutions but they provide a model for students to work with. Students, interrogate your solutions to understand your errors and your successes. It may sometimes to possible to achieve a correct numerical answer with faulty reasoning!

- Do not memorise solutions. Understand the techniques and processes used in relation to the questions asked.

Notes

Data Handling

- Measures of Location: *mean*, *median* and *mode.*
- Measures of Spread: *range*, *inter-quartile range* (IQR), *standard deviation s* (from calculator)
  - For large sample sizes:
    Min ≈ $\bar{x} - 3s$ and Max ≈ $\bar{x} + 3s$
- *Outliers* (univariate statistics):
  - Scores that are less than Q1 – 1. 5 × IQR or more than Q3 + 1.5 × IQR
  - Scores that are less than $\bar{x} - 3s$ or more than $\bar{x} + 3s$.
- An *explanatory variable* is a variable that is used to explain changes in a *response variable*.
- *Row percentages* are calculated using row totals, while *column percentages* are calculated using column totals.

Bivariate Statistics

- *Line of best fit* or the *least squares regression line*. The best line that can be drawn through the points on a scatter-graph
  - Outliers are data points that go against the trend indicated by the line of best fit.
- The *coefficient of linear correlation r*, indicates how well the points on a scatter-graph fit a straight line pattern.
- If two variables are correlated, it does *not* imply that that a cause and effect relationship exists between the variables.
- $-1 \le r \le 1$.
  The fit gets stronger as r approaches –1 or 1.
  When $r = 1$: the fit is perfect and positive (the fitted line has a positive gradient).
  When $r = -1$: the fit is perfect and negative.
  When $r = 0$, no linear relationship.
- The coefficient of determination r^2
  - If $r = 0.9$, $r^2 = 0.81$; 81% of the variation in the response variable can be explained by the relationship between the explanatory and response variables
- When the line of best fit $y = ax + b$ is used to predict the value of y:

r value	*x*-value	Reliability
strong to perfect	Inside data range	Good
near perfect	Just outside data range	Good
not important	Outside data range	Poor

- A linear model is considered inadequate if the plot of its residuals has a pattern.

Time Series

- The period counts the number of time intervals between consecutive peaks or troughs.
- Moving averages are used to smooth out the fluctuations present in a time series to make the trend clearer. Centred moving averages are used when the period is an even number.
- The line of best fit through the moving averages is called the trend line. The trend is increasing if the gradient of this line is positive, decreasing if the gradient is negative.
- Average percentage method
  - Seasonal Index for Quarter 1
    = Average of all "quarter 1 variables expressed as % of quarterly mean"
  - Sum of seasonal indices = 1

Period	Q1	Q2	Q3	Q4
Variable	a	b	c	d
Variable as % of *quarterly mean*	$\frac{a}{q}$	$\frac{b}{q}$	$\frac{c}{q}$	$\frac{d}{q}$
Variable	e	f	g	h
Variable as % of *quarterly mean*	$\frac{e}{r}$	$\frac{f}{r}$	$\frac{g}{r}$	$\frac{h}{r}$

Quarterly mean $q = \dfrac{a+b+c+d}{4}$

Quarterly mean $r = \dfrac{e+f+g+h}{4}$

Seasonal Index for Q1 = $\dfrac{\frac{a}{q}+\frac{e}{r}}{2}$

- If a time series has a period of n then there are n seasons and a seasonal index for each season.
- Seasonally Adjusted Time Series
  $= \dfrac{\text{Time Series}}{\text{Seasonal Index}}$
- Predicted Time Series
  = Predicted Seasonally Adjusted Time Series × Seasonal Index

Linear Recurrence Relation

$T_{n+1} = bT_n + c \quad T_1 = a$

Arithmetic Sequences

- $a, a + d, a + 2d, a + 3d, \ldots$
  There is a constant difference d between two consecutive terms.
- Recursive equation: $T_{n+1} = T_n + d \quad T_1 = a$
- Explicit rule for the nth term: $T_n = a + (n-1)d$

Geometric Progression

- $a, ar, ar^2, ar^3, \ldots$
  There is a common multiplier (ratio) r between two consecutive terms
- Recursive equation: $T_{n+1} = T_n \times r$ $T_1 = a$
- Explicit rule for nth term: $T_n = a \times r^{n-1}$

Exponential Growth and Decay

- $P(t + 1) = P(t) \times r$ where $P(0)$ = initial value
- $P(t) = P(0)\, r^t$

Simple Interest

- Simple interest earned = P × R × T
  P : principal (amount invested)
  R : interest rate per annum in decimal form
  T : time in years

Compound Interest:

- Recursive formula:
  $b(n)$: balance at the end of period n
  r: interest rate per period, in decimal form
  $b(n + 1) = b(n) \times (1 + r)$ where $b(0) = P$
- Explicit formula:
  $$\text{Final Amount} = P \times (1 + \frac{r}{n})^{nt}$$
  P: principal (amount invested)
  r : interest rate per annum in decimal form
  n : number of times compounded per year
  t : number of years

Effective Rate of Interest

$$\text{Effective rate} = (1 + \frac{I}{n})^n - 1$$

I: interest rate per annum in decimal form
n : number of times compounded per year

Reducible Balance Loan

- $b(n)$: Balance at the end of period n.
  r : Interest rate per period, in decimal form
  P : Regular repayment
  L : Loan amount
  $b(n + 1) = b(n) \times (1 + r) - P$ $b(0) = L$

Inflation

- $v(n)$: value at end of period n
  r : Rate of inflation in decimal form
  $v(n + 1) = v(n) \times (1 + r)$ $v(0) = A$
  $v(n) = A \times (1 + r)^n$

*Depreciation: Diminishing value**/Reducing Balance***

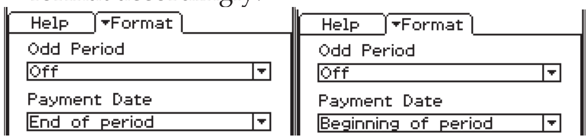
- $v(n)$: value at end of period n
  r : Rate of depreciation in decimal form
  $v(n + 1) = v(n) \times (1 - r)$ $v(0) = A$
  $v(n) = A \times (1 - r)^n$

Depreciation: Prime Cost/Flat Rate

- $v(n)$: value at end of period n
  r : Rate of depreciation in decimal form
  $v(n + 1) = v(n) - r\,A$ $v(0) = A$
  $v(n) = A - nrA$

Using CAS Finance Wizard

- Check when payments are made and set format accordingly.



Graph Theory

- *Adjacency matrix*: a_{ij} = number of edges connecting vertex i to vertex j.
- *Bipartite graph*:
  Vertices separable into two distinct groups.
  No edges between vertices within the set.
  Edges connect vertices from one set to vertices of second set.
- *Bridge*: An edge that connects two graphs. Without this edge, the two graphs are not connected to each other.
- *Complete Graph*: Each vertex is connected to every other vertex.
- *Connected Graph*: All vertices are connected to each other.
- *Cycle*: Closed walk which starts and ends at the same vertex. No repeat Edges.
  No repeat vertices except start/end
- *Degree of a vertex*:
  Number of edges connected to a vertex.
  Loops are counted twice.
- *Digraph*: Graphs with edges that are directed.
- *Eulerian Circuit (Eulerian)*:
  Covers every edge exactly once.
  Starts and ends at same vertex.
  All vertices are even.
- *Euler's Formula/Rule*: $V + F - E = 2$
  V: Number of Vertices
  F: Number of Faces/Regions
  E: Number of Edges
- *Eulerian Trial (semi-Eulerian)*:
  Covers each edge exactly once.
  Start and End vertices different.
  Exactly two odd vertices.
- *Hamiltonian Cycle:*
  Includes every vertex exactly once.
  Starts and Finishes at same vertex.
- *Hamiltonian Path:*
  Includes every vertex exactly once.
  Start and End vertices different.

- *K_n graph*: A complete graph with *n* vertices.
- *Length of walk* between two vertices:
  Number of edges between the two vertices.
- *Multiple edges*: Two or more edges connecting a pair of vertices.
- *Path*: No repeat edges and vertices
- *Planar Graph*:
  No edges intersect outside of vertices
- *Simple Graph*: No multiple edges or loops
- *Sub-graph*: Part of a graph
- *Trail*: No repeat edges
- *Traversable Network*:
  - Exactly two odd vertices or no odd vertices
- *Walk*:
  A sequence of vertices.
  Vertices & edges can be repeated

Shortest path

- From the start point, work out all possible routes to each point that is connected directly to the start point. These points are now "live". Select and mark out the shortest route.
- Work out all possible routes from the start point to all points that are connected to the "live" points". These points are now "live". Select and mark out the shortest route to each of these points.
- Continue the above step until the destination point is reached. Retrace from the destination point to the start point using the marked shortest routes. Shortest route will consist of the marked routes from the start point to the destination point.

Minimum Spanning Tree

Connects all nodes with *no loops*. Use Prim's algorithm either on a diagram or on a table:

- Start at any node (N1) and connect to the nearest unconnected node (N2)
- From either of the "live nodes" N1 or N2 , connect to the next nearest unconnected node N3. N3 now becomes "live".
- Connect from all "live" nodes until all nodes are connected without forming any loops.

Maximum flow

- Source: No incoming links.
- Sink: No outgoing links
- Start by identifying a possible path from *source* to *sink*. Find the flow of this path. This is the largest amount that can flow along this path.
- The capacity of each used link is now reduced by the amount that has passed through it. The capacities of the unused links remain unchanged.
- Making use of the remaining capacities in the used links and the unused links, systematically identify another path from source to sink and note down the flow. The capacities of all used links are reduced by the amounts that have passed through them.
- Repeat the above until there are no more possible flow paths from the source to the sink

Project Networks

- A project network is a directed graph consisting of a sequence of connected activities.
- The next activity cannot start until all its immediate predecessor activities have been completed.
- The minimum completion time is the shortest time taken to complete each and every activity of the project in the correct sequence.
- The path that gives rise to the minimum completion time is called the critical path (CP).
- Delays on activities on the critical path will delay the completion of the project.
- Delays on activities outside the critical path may/may not delay the project.
- To determine the minimum number of workers required to complete project in minimum time: assign one worker to the CP and as many workers as is required so that all activities outside critical path are taken care of without delaying the minimum completion time.

Hungarian Algorithm for a $n \times n$ matrix

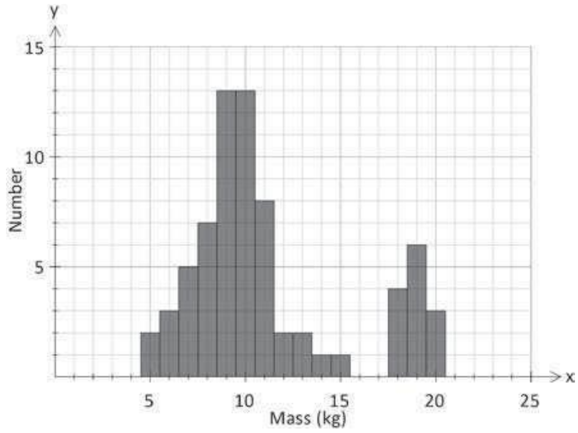
- Minimum cost.
  Step 1: Subtract the smallest entry in each row from all the entries in that row.
  Step 2: Subtract the smallest entry in each column from all the entries in that column.
  Step 3: Cross out *all* the zero entries with a *minimum* number of horizontal and vertical lines
  Step 4: If the number of "zero-lines" is *n*, go to Step 7. If the number of "zero-lines" is less then *n*, go to Step 5.
  Step 5: Determine *x*, the smallest non-zero entry not crossed out by the lines. Subtract this entry from each uncrossed out entry. Add *x* to each entry located at the intersection of zero lines.
  Step 6: Uncross all entries. Repeat Steps 3, 4 and 5 until *n* zero lines are obtained.
  Step 7: Assign worker to task according to location of the final zero entries.
- Maximum Returns
  Subtract each entry from the largest entry in the matrix. Execute Steps 1 to 7.

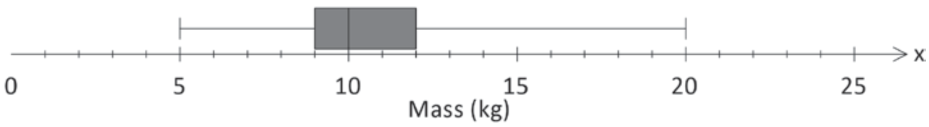
01 Statistical Investigation Process

Calculator Free

1. [8 marks: 1, 3, 2, 2]

Gilbert owns a small hobby farm where he is experimenting with a new seedless watermelon hybrid. The accompanying diagrams show the distribution of the mass of the seventy watermelons harvested as a histogram and a box-plot.





(a) State one feature of the distribution of the mass of watermelons that is conveyed in the histogram but not conveyed in the box-plot.

(b) The box-plot indicates the presence of outliers.

(i) How many of the watermelons can be classified as outliers in terms of their mass?

(ii) Should the outliers be removed when calculating the mean mass of the watermelons harvested? Why?

(iii) Suggest two questions that should be asked regarding the "outliers".

Calculator Free

2. [8 marks: 2, 2, 2, 2]

The accompanying diagram shows the stem and leaf plot of the marks obtained by two groups of 25 students each in a common mathematics test.

Class A		Class B
9 8 2	0	2 3 4 6
9 9 9 9 6 6	1	
5 4	2	5 6 8
6 4 1	3	8 9
1	4	1 1
3 2	5	4
9 8 6 6	6	1 1 4 7 9
3	7	
4 1	8	3 8 8
7	9	3 5 5 9 9

(a) Verify that the median mark for Class B is higher than the median for class A..

(b) Based on the higher median mark obtained by the students in class B, it was suggested that the teacher of Class B was a *better* teacher than the teacher of Class A. Identify two issues that could be used to challenge this suggestion.

(c) The two classes were taught by the same teacher and were not streamed according to their mathematical ability. Suggest two factors that could account for the differences in the marks.

(d) Both classes were taught by the same teacher and were streamed according to their mathematical ability. Comment on the statement "Class B was more successful in this test".

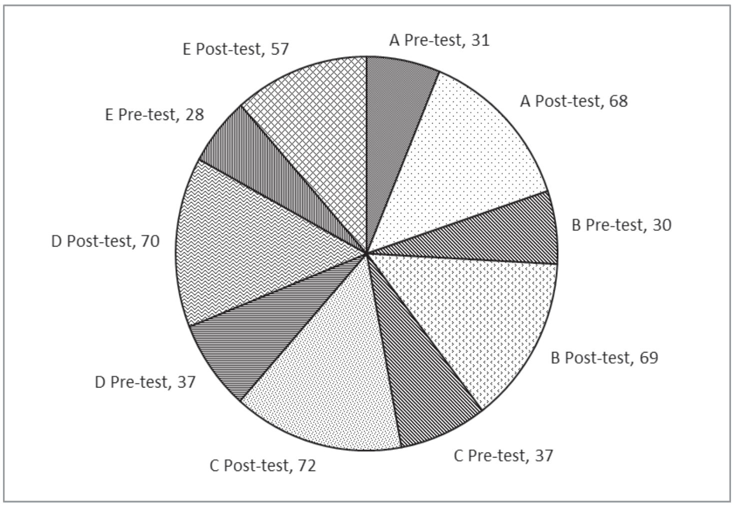
Calculator Free

3. [3 marks]

Five groups of students taking the same mathematics unit were tested just before a topic was taught (pre-test) and immediately after the topic was taught (post-test). The table below shows the results for these tests for the five groups of students.

Group	Pre-test	Post-test
A	31	68
B	30	69
C	37	72
D	37	70
E	28	57

Kim wanted to show the changes between the results of the pre-test and that of the post-test and drew a pie graph shown below.



Discuss whether a pie graph is suitable for making the comparisons intended.

Calculator Assumed

4. [10 marks: 2, 2, 1, 3, 2]

Hannah wishes to determine the percentage of university graduates that have found jobs that have some relevance with their university education and qualifications.

(a) Write one question she could ask in a survey of university graduates that will provide her with the information she needs.

(b) Hannah interviewed 100 graduates working in several office blocks in the city where she lived. Suggest two ways she could improve the quality of information she will obtain.

The following table shows the results of a survey that Hannah conducted on 100 university graduates.

Age	Relevant to current job	Not relevant to current job	Total
20 – 24	4	7	11
25 – 29	8	13	21
30 – 34	12	10	22
35 – 39	13	8	21
40 – 44	10	2	12
45 – 49	10	3	13

(c) What type of graph would be most suitable for displaying the information above?

Calculator Assumed

4. (d) Use the graph type you chose in (c) to display the given information.

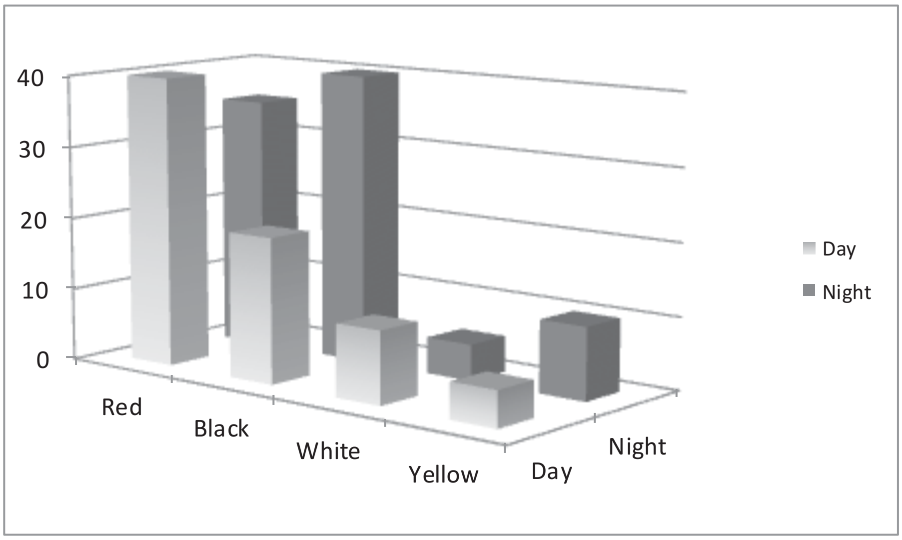
(e) From the information in the table above, determine with reasons if there is a relationship between the age of the graduate interviewed and the relevance of the graduate's university qualifications with the graduate's current job.

02 Associations between Categorical variables

Calculator Free

1. [5 marks: 1, 1, 3]

The diagram below shows the number of vehicle accidents described by the colour of the vehicles and the time of the accidents; day-time or night-time.



(a) A student noted that fewer white coloured cars were involved in accidents at night-time and concluded that it is safer to drive white coloured cars at night-time. Give one reason why this need not necessarily be true.

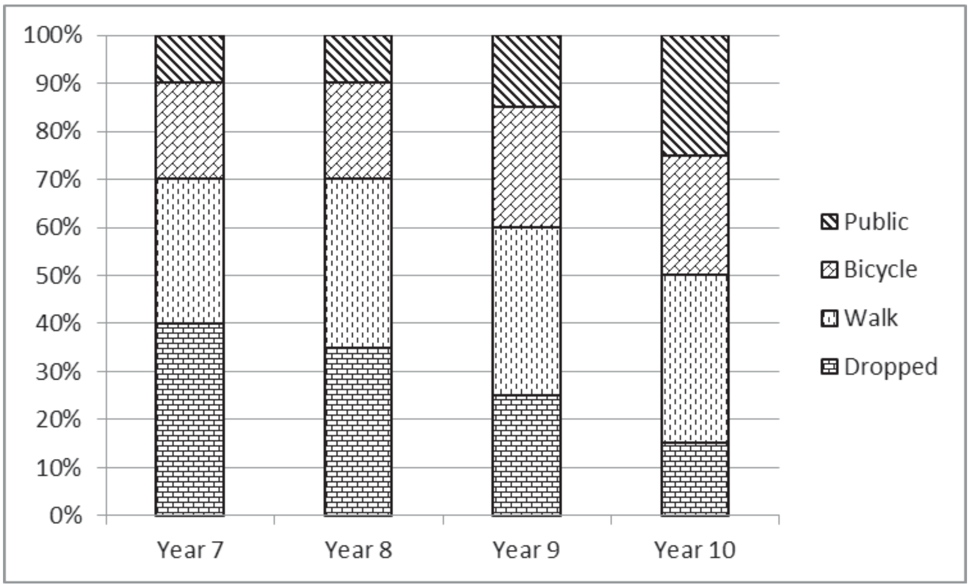
(b) Describe another potential association between colour of vehicle, the number of vehicle accidents and the time of the accidents.

(c) State three questions you would ask to ascertain the credibitity of the association you described in (a).

Calculator Free

2. [6 marks: 3, 3]

The diagram below shows the percentage of students in the listed school year groups and their mode of travel to school.



(a) Identify a possible negative association between school year and the mode of travel to school. Describe this association and state the response variable and the explanatory variable.

(b) Identify a possible positive association between school year and the mode of travel to school. Describe this association and state the response variable and the explanatory variable.

Calculator Assumed

3. [8 marks: 2, 2, 4]

The accompanying diagram shows the different makes of cars parked at three different suburban shopping centres on a school-day morning. The shopping centres A, B and C are located respectively at high, middle and low income suburbs.

	A	B	C
Australian	70	60	40
German	150	80	20
Korean	40	130	120
Japanese	90	140	110
Others	60	70	50

(a) Complete the table below showing the row percentages.

	A	B	C
Australian	41	35	24
German			
Korean	14	45	41
Japanese	26	41	
Others	33	39	28

(b) Complete the table below showing the column percentages.

	A	B	C
Australian	17	13	12
German		17	6
Korean		27	
Japanese	22	29	
Others	15	15	15

(c) Determine with reasons if there is a relationship between the make of cars parked and the level of income of the suburb. Clearly identify the response and explanatory variables. State any assumptions you made.

Calculator Assumed

4. [9 marks: 1, 2, 2, 4]

A sample of 400 voters were asked to respond YES or NO to the question, "Should uranium mining be permitted in Western Australia (WA)?". The results are tabulated in the first table below.

	YES	No
WA born	10%	55%
Not WA born	30%	5%

The second table shows the gender distribution of those in the sample who responded YES.

	WA born	Not WA born
Female	5%	35%
Male	15%	45%

(a) How many in the sample responded YES?

(b) What percentage of respondents who voted YES was male?

(c) What percentage of those who voted YES was WA born?

(d) Discuss the relationship between birth state and sex and the way the vote was cast? Give reasons for your answer.

Calculator Assumed

5. [10 marks: 2, 2, 2, 2, 2]

The table below shows the distribution of adults by age groups (in years) who choose not to have their medical prescriptions filled and their accompanying reasons.

Reason	18-24	25-35	35-45	46-50	51-60	61-70	70+	Total
Cost of medication is too high	28	18	15	17	18	24	30	150
Wait to see if they get better without medication	39	25	19	20	24	35	36	198
Fear that they might become dependent on the medication	5	12	18	14	13	10	10	82
Do not trust their doctor's diagnosis	22	18	17	17	18	22	24	138
Total	94	73	69	68	73	91	100	568

(a) Complete the table below showing column percentages (to the nearest %). Some entries have been adjusted to match the percentage total.

Reason	18-24	25-35	35-45	46-50	51-60	61-70	70+
Cost of medication is too high	30		22	25	25	26	30
Wait to see if they get better without medication	42		27	29	32	39	36
Fear that they might become dependent on the medication	5	16	26	21		11	10
Do not trust their doctor's diagnosis	23	25	25	25		24	24

(b) Complete the table below showing row percentages (to the nearest %). Some entries have been adjusted to match the percentage total.

Reason	18-24	25-35	35-45	46-50	51-60	61-70	70+
Cost of medication is too high	19	12	10	11	12	16	20
Wait to see if they get better without medication			10	10	13	17	18
Fear that they might become dependent on the medication	6	15	22	17	16	12	12
Do not trust their doctor's diagnosis	17	13	12	12	13		

(c) Comment on the distribution of adults in the study who choose not to have their prescriptions filled because they do not trust their doctor's diagnosis.

Calculator Assumed

5. (d) Comment on the distribution of adults in the study who choose not to have their prescriptions filled because they fear becoming dependent on the medication.

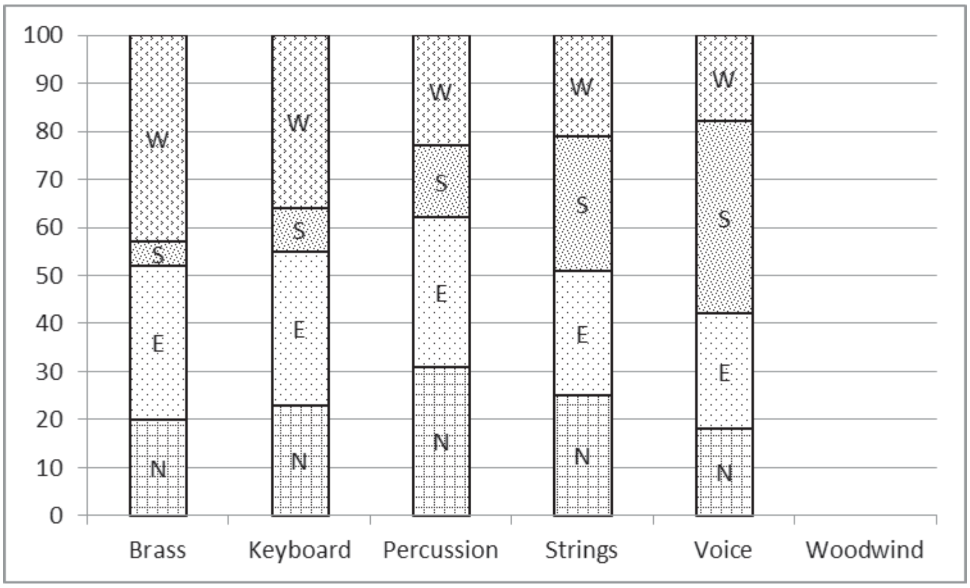
(e) Which age group form the largest group among those who gave “the high cost of medication” as the reason? Justify your answer.

Calculator Assumed

6. [15 marks: 3, 2, 2, 2, 1, 2, 2, 1]

Macliver College is a specialist co-educational music college catering for students from years 7 to 12. The table below shows the distribution of a *sample* of students from Macliver College according to their musical instrument category and the geographical area of their homes (North, East, South or West of the College). Also included in the table is the gender distribution of students in each instrument category. The accompanying graph is a percentage stacked column graph of some of the information displayed in the table.

	North	East	South	West	Females	Males
Brass	22	35	5	46	36	72
Keyboard	24	34	10	38	54	52
Percussion	33	33	16	24	24	82
Strings	21	22	24	18	72	13
Voice	15	20	33	15	68	15
Woodwind	6	10	26	8	29	21



(a) Complete the percentage stacked column graph above.

(b) Comment on the composition of students in this sample with Brass as their instrument category.

Calculator Assumed

6. (c) In this sample, what proportion of students from the South are in the Voice category?

(d) Comment on the instrument selection of students from the South in this sample.

(e) In which category/categories are the female students in the majority?

(f) Is it appropriate to conclude that there are more female than male students in this College? Give reasons for your answer.

(g) Does this sample provide evidence that instrument category is related to geographical location of the students' homes? Justify your answer, stating any assumptions you make.

(h) Suggest a possible reason for the high proportion of students in Woodwind coming from the South of the College.

Calculator Assumed

7. [14 marks: 2, 2, 2, 2, 3, 3]

Table 1 shows the results of a survey conducted in country X which describes the distribution of adults with tertiary and post-graduate qualifications and their region of birth.

Table 1

	Tertiary	Post-graduate	Total
A	45	28	73
B	47	36	83
C	43	31	74
D	57	44	101
E	62	25	87
Total	254	164	418

Table 2 shows the distribution of those with tertiary qualifications according to gender.

Table 2

Tertiary	Males	Females	Total
A	22	23	45
B	31	16	47
C	24	19	43
D	32	25	57
E	42	20	62
Total	151	103	254

Table 3 shows the distribution of those with post-graduate qualifications according to gender.

Table 3

Post-graduate	Males	Females	Total
A	19	9	28
B	25	11	36
C	18	13	31
D	24	20	44
E	15	10	25
Total	101	63	164

(a) What proportion of those surveyed were born in A and have post-graduate qualifications?

(b) What proportion of those surveyed are females with post-graduate qualifications and are born in A?

Calculator Assumed

7. (c) What proportion of those with tertiary and post-graduate qualifications are females.

(d) Identify with reasons, the region of birth of those with post-graduate qualifications with roughly equal representation from both sexes .

(e) Comment on the distribution of those with post-graduate qualifications in terms of their birth region and gender.

(f) Country X has a population of 25 million and 30% of the population have tertiary or post-graduate qualifications. Use the results of this survey to estimate how many in country X are females with post-graduate qualifications and are born in region A.

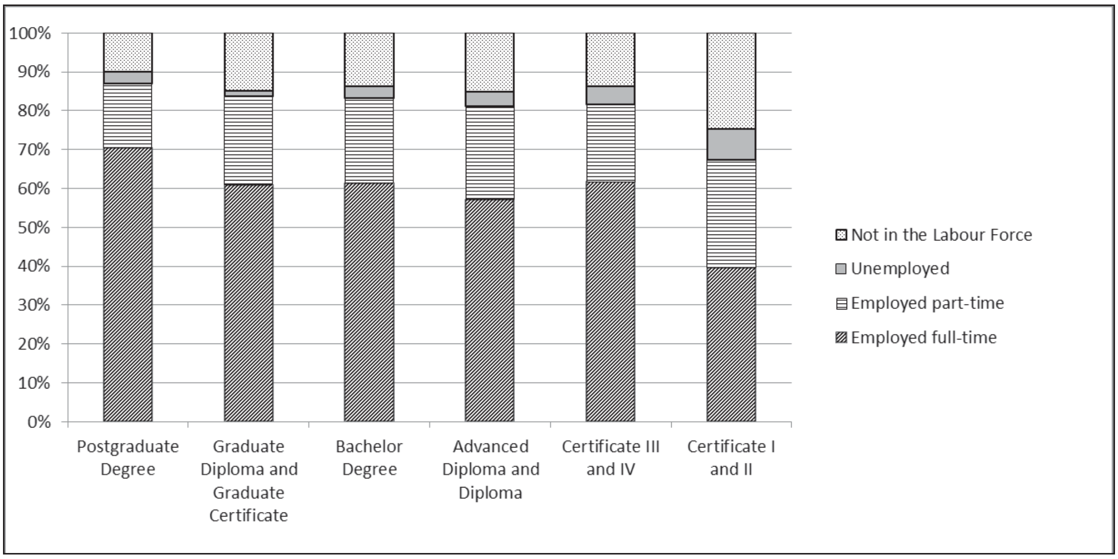
Calculator Assumed

8. [9 marks: 2, 1, 1, 1, 1, 3]

The table below shows the estimated number of persons (thousands) employed full-time, part-time, unemployed (but looking for work) and not in the labour force and their post-school qualifications.

	Employed full-time	Employed part-time	Unemployed	Not in the Labour Force	Total
Postgraduate Degree	700	165	30	100	995
Graduate Diploma and Graduate Certificate	400	150	10	98	658
Bachelor Degree	1,600	570	81	360	2,611
Advanced Diploma and Diploma	870	368	58	230	1,526
Certificate III and IV	1,790	580	135	400	2,905
Certificate I and II	184	130	37	115	466
Total	5,544	1,963	351	1,303	9,161

The chart given below is drawn using the information in the table above.



(a) Explain what calculations were required to construct the chart shown.

(b) Determine the association between post-school qualifications and full-time employment status.

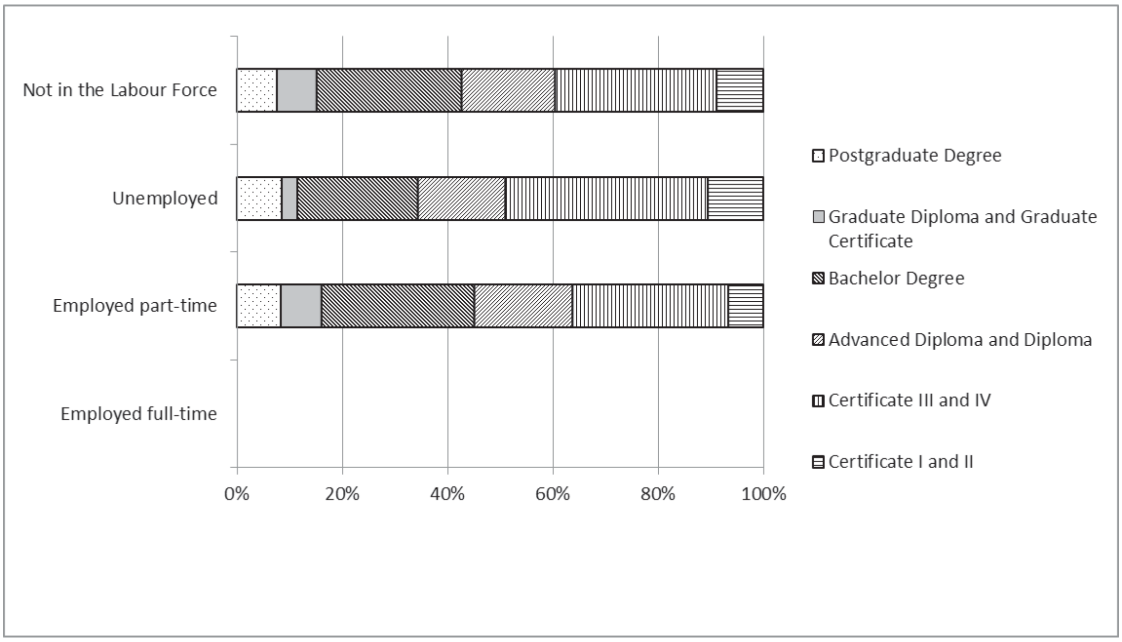
Calculator Assumed

8. (c) Determine the association between post-school qualifications and those not in the labour force.

(d) Calculate the percentage of those with post-graduate qualifications not in the labour force.

(e) Determine the proportion of those with certificates in part-time employment.

(f) Complete the chart drawn below.



03 Associations between Numerical Variables

Calculator Free

1. [2 marks]

The linear relationship between two variables, x and y, is described as negative.
The least squares regression line has equation $y = a + bx$.
Determine with reasons which of the following statement(s) *must* be true.

A. Both a and b must be positive.
B. Both a and b must be negative.
C. a must be positive.
D. a must be negative.
E. b must be negative.

---

2. [2 marks]

The least squares regression line between Q and T is given by $Q = –48.4 – 1.2t$.
Which of the following statement(s) *must* be true?

A. There could be a negative linear relationship between Q and t
B. As t increases, Q decreases.
C. As t increases, Q increases.
D. The coefficient of linear correlation between Q and t is negative.
E. The coefficient of linear correlation between Q and t is positive.

Calculator Free

3. [2 marks]

The coefficient of linear correlation between two variables, x and y, is 0.95.
Which of the following statement(s) *must* be true?

A. As x increases, y increases.
B. An increase in x causes an increase in y.
C. 95% of the data points lie on the line of best fit between x and y.
D. The relationship between x and y may actually be non-linear.
E. The line of best fit between x and y must have a positive gradient and a positive vertical intercept.

---

4. [2 marks]

The coefficient of determination between the variables x and y is 0.81 and the line of regression of y on x is $y = -0.4x + 3.2$.
Which of the following statements must be true?

A. The coefficient of linear correlation between x and y is 0.9.
B. The coefficient of linear correlation between x and y is –0.9.
C. The coefficient of linear correlation between x and y is $-(0.81^2)$.
D. The regression line passes through 81% of the points.
E. The response variable is y and the explanatory variable is x.

---

5. [2 marks]

The line of regression of b on f is $b = 2.54f + 122.5$ with a coefficient of linear correlation of 0.8. Which of the following statements must be true?

A. The coefficient of determination is 0.64
B. The response variable is f and the explanatory variable is b.
C. 64% of the variation in b can be explained by the relationship between b and f.
D. As f decreases b decreases.
E. Increases in b are caused by increases in f.

Calculator Free

6. [5 marks: 2, 1, 2]

The least squares regression line between N and t is given by $N = 0.05t + 2.51$.

(a) Find the average increase in N corresponding to an increase of 20 units in t.

(b) Predict the value of N when $t = 50$.

(c) The actual value of N when $t = 50$ is 4.95. Find the residual associated with the prediction in (b).

---

7. [5 marks: 2, 1, 2]

The least squares regression line of Physics Marks p on Mathematics Marks m is $p = 1.5m + 5.0$.

(a) Explain clearly why the correlation coefficient between p and m cannot be –0.8.

(b) Calculate the predicted increase in y when x increases by 10 units.

(c) The actual value of y when $x = 20$ is 37.
Use the regression line given to predict the value of y when $x = 20$.
Hence, determine the associated residual.

Calculator Free

8. [8 marks: 1, 1, 2, 2, 2]

For a sample of 100 students, the coefficient of linear correlation between the length of a student's foot and the number of errors in a spelling test is –0.9.

(a) Determine the nature of the data for "the length of a student's foot".

(b) Determine the nature of the data for "the number of spelling errors in a spelling test".

(c) Determine the percentage of variation in the number of errors in the spelling test that is explained by the length of a student's foot.

(d) Use the information given to determine with reasons if increasing the length of a student's foot will reduce the number of errors in the spelling test.

(e) The average foot length is 15.4 cm and the average number of spelling errors is 8.4. A student with a foot length of 22.2 cm had 15 spelling errors in the test. Determine the possible effect on the original correlation coefficient if this student's data were included. Give a reason for your answer.

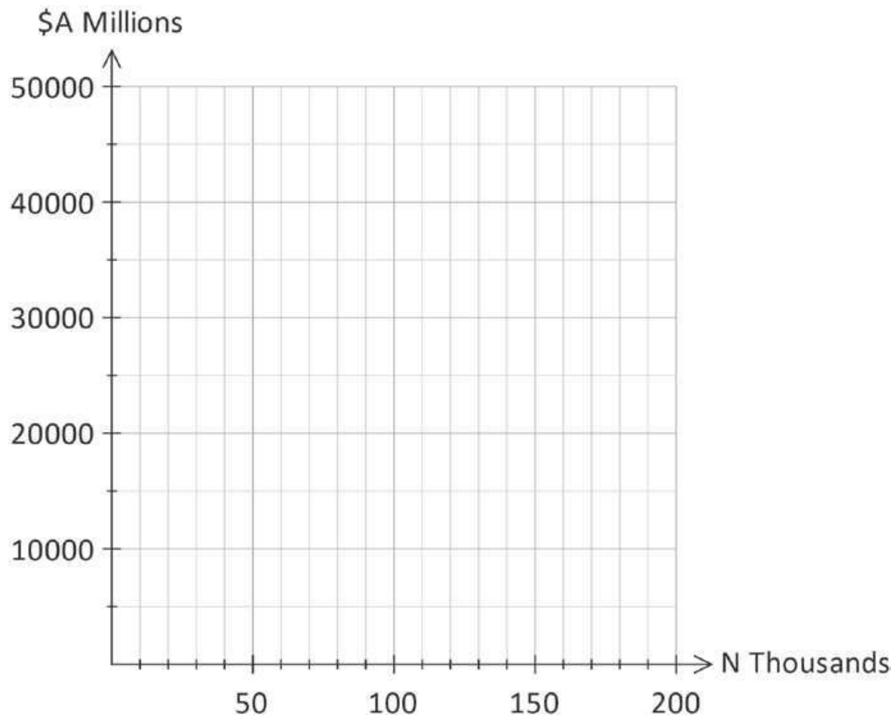
Calculator Assumed

9. [14 marks: 2, 2, 1, 1, 2, 3, 1, 2]

The following table displays the Number Employed (thousands) and the corresponding Annual Turnover (in $million) for several types of industry.

Type of Industry	Number Employed ('000), *N*	Annual Turnover $m, *A*
Food, beverages	160	43 000
Textiles	75	10 000
Wood and paper	60	11 000
Printing	90	14 000
Energy Products	90	30 000
Non-metal mineral products	40	9 000
Metal products	150	37 000
Machinery	200	40 000

(a) Draw a scatter-graph for this data.



(b) Calculate the coefficient of linear correlation and comment on the nature of the relationship between *N* and *A*

Calculator Assumed

9. (c) Find the gradient of the least squares regression line of A on N.

(d) Find the vertical intercept of the least squares regression line of A on N.

(e) Determine the increase in turnover for every 10 000 increase in the number employed.

(f) Use the least squares regression line to predict the annual turnover for a business that employs 5 000 people.
Comment on the reliability of your prediction.

(g) An American reporter converts the turnover to US$ using the conversion AUD$1 = US$0.70. Calculate the correlation coefficient between the Number employed and the Annual Turnover in US$.

(h) Comment on the statement made by a politician that increasing the number of people employed will create a higher annual turnover.

Calculator Assumed

10. [13 marks: 2, 2, 2, 2, 2, 2, 1]

The table below shows the heights of 20 students (in inches) and their self-esteem rating (out of 10, the higher the score, the higher the self-esteem).

Height, h	Self-esteem rating, s
68	4.1
71	4.6
62	3.8
75	4.4
58	3.2
60	3.1
67	3.8
68	4.1
71	4.3
69	3.7
68	3.5
67	3.2
63	3.7
62	3.3
60	3.4
63	4.0
65	4.1
67	3.8
63	3.4
61	3.6

(a) Find the mean height and the associated standard deviation.

(b) Find the mean and median self-esteem rating.

Calculator Assumed

10. (c) Calculate the coefficient of linear correlation between s and h.
Comment on the type of relationship between the s and h.

(d) Calculate the least squares regression line that will allow you to predict the self-esteem of a student given the student's height.

(e) Use your regression line to determine the increase/decrease in the self-esteem rating for every inch drop in height.

(f) Ubiuty is 50 inches tall. Use your regression line in (e) to predict Ubiuty's self-esteem rating. Comment on the reliability of your prediction.

(g) The heights were converted to centimetres (1 inch = 2.54 cm) for Australian readers. What is the new coefficient of linear correlation between the height of students and their self-esteem ratings?

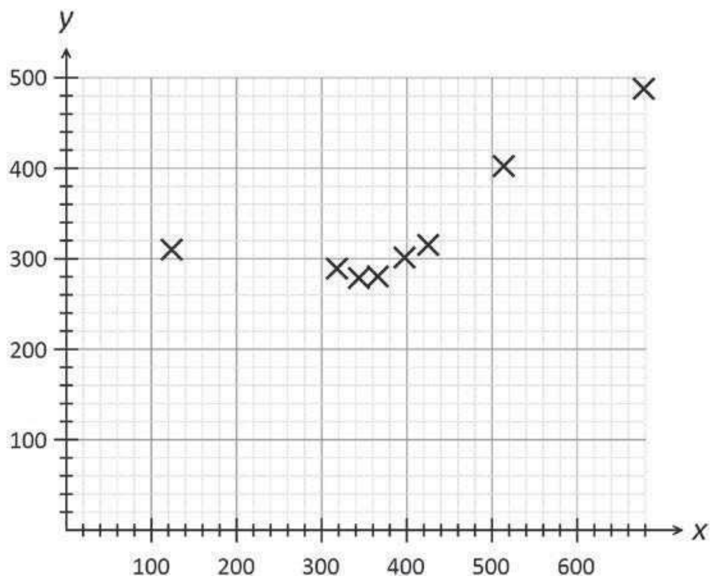
Calculator Assumed

11. [14 marks: 1, 1, 1, 2, 3, 2, 2, 2]

The table below shows the recorded number of births and deaths in several suburbs over a period of one year.

Suburbs	No. of births x	No. of deaths, y
A	345	278
B	425	315
C	125	310
D	319	289
E	679	487
F	514	401
G	367	280
H	398	301

(a) The scatter-graph for this data is drawn below. Draw the line of best fit.



(b) Mark with a circle the outlier on the scatter-graph in (a).

(c) Remove the identified outlier. Draw the line of best fit for the remaining data.

(d) For the given data points, discuss the effect of the outlier on the vertical intercept and the gradient of the line of best fit on the data points.

Calculator Assumed

11. (e) Give the most reliable prediction for the number of deaths for a suburb with 500 births? Describe how you obtained your answer.

.

(f) Estimate the increase/decrease in deaths for every increase of 50 births.

(g) George argues that since there is a good correlation between the number of births and deaths, he should move to a suburb with lower births, so that he has a smaller chance of dying. Comment mathematically on his statement.

(h) Suggest a reason why an increase in the number of births is accompanied by an increase in the number of deaths.

Calculator Assumed

12. [10 marks: 2, 2, 4, 4]

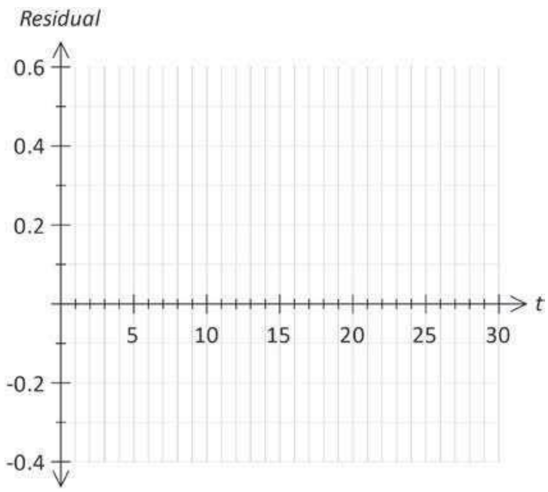
The table below shows pairs of readings of Q and t obtained from an experiment.

t	1	5	8	11	13	14	17	19	20	24
Q	2.0	2.6	3.4	4.1	4.9	5.0	6.2	6.6	7.0	8.6
Residual										

(a) Find the least squares regression line of Q on t and the corresponding coefficient of linear correlation between Q and t.

(b) Complete the row of linear residuals above.

(c) On the axes provided below, draw graph of the residuals for Q. Comment on the appropriateness of a linear relationship between Q and t.



(d) Use your answer in (a) to predict the value of Q when t = 12 and comment on the reliability of your result.

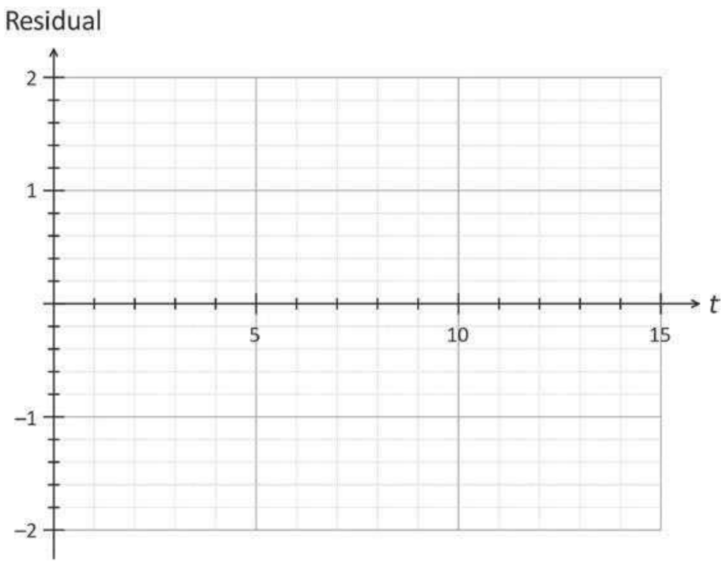
Calculator Assumed

13. [9 marks: 2, 2, 2, 3]

The least squares regression line of Q on t is given by $Q = 0.86t - 2.07$.
The coefficient of linear correlation between Q and t is 0.9191.
The table below shows the residuals associated with this regression line.

t	1	2	3	4	5	6	7	8	9	10
Residual	0.08	−1.28	1.77	−0.49	−0.24	1.2	−0.05	−2.01	0.64	0.38

(a) Sketch on axes provided below the graph of the residuals against t.



(b) Determine with reasons if fitting a linear model to the values of t and Q is appropriate.

(c) Predict the value of Q for $t = 11$. Give your answer correct to 1 decimal place

(d) Comment on the reliability of your prediction in (c).

04 Data Investigation Process

Calculator Free

1. [10 marks: 2, 2, 3, 3]

Mrs Mazzart teaches music and mathematics. She wishes to investigate the claim that mathematical competence and musical competence are related. Together with some of her students, they design a statistical investigation to study this claim.

(a) State a possible response and explanatory variable for the investigation.

(b) Describe the data that need to be collected and how the data is to be collected.

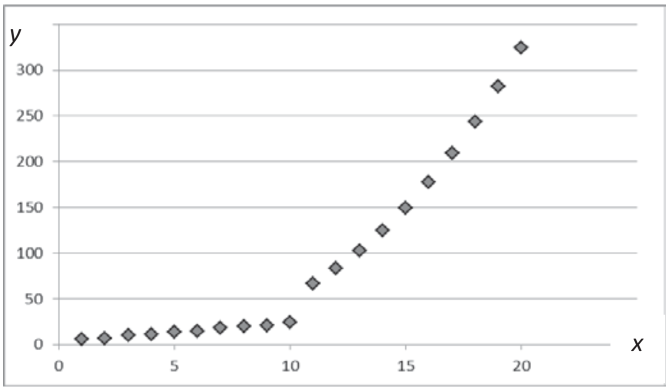
(c) Describe how you would display and analyse the data.

(d) Describe how you would interpret the data you analysed.

Calculator Free

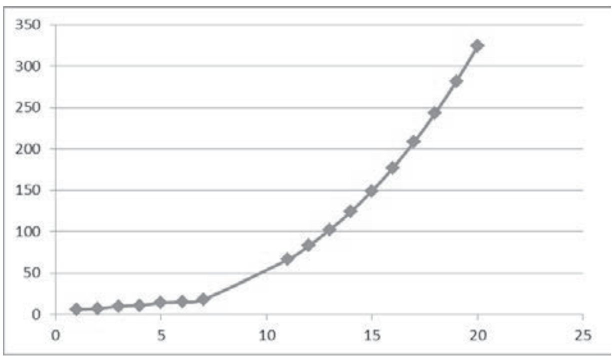
2. [6 marks: 4, 2]

The diagram below shows the scatter-graph between the variables x and y.



(a) Discuss the nature of the association between the variables x and y.

(b) A researcher decided that the y-values associated with $x = 8$, $x = 9$ and $x = 10$ were incorrectly measured/recorded. The researcher removed these scores and obtained the resulting curve.



Was the researcher justified in doing so? Give your reasons.

Calculator Assumed

3. [12 marks: 2, 1, 5, 4]

The table below shows the age (in months) and their heights (in cm) for a group of 20 students.

Age, x	Height, y
120	98
119	100
120	99
118	102
120	105
117	96
116	95
119	101
122	103
118	97
192	170
190	175
186	168
185	165
189	172
185	165
190	173
188	167
187	166
191	172

(a) Plot a scatter-graph of the data in your CAS calculator and describe the scatter-plot.

(b) Find the coefficient of linear correlation between x and y.
Give your answer to 4 decimal places.

Calculator Assumed

3. (c) Predict the height of a student aged 150 months. Explain clearly how you obtained your answer and comment on the reliability of your prediction.

(d) Use the most appropriate part of the table to predict the height of a student aged 121 months. Explain your choice of data and show clearly how you obtained your answer. Comment on the reliability of your prediction.

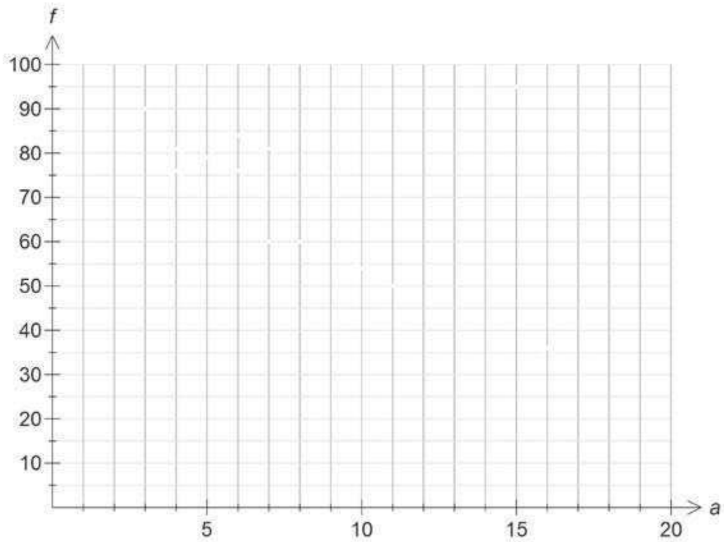
Calculator Assumed

4. [12 marks: 2, 1, 3, 2, 2, 2]

The Head of a Mathematics Department produced the following table to show the link between lesson attendance and final mark.

No of absences for the year, *a*	Final Mark, *f*
6	84
7	81
8	60
11	50
4	81
15	95
5	79
7	60
6	76
3	90
16	36
10	54
4	76

(a) Draw a scatter graph for this table.



(b) Identify the outlier in the data set.

Calculator Assumed

4. (c) Discuss the effect the outlier has on the coefficient of linear correlation between a and f.

(d) Use an appropriate selection of the given data and a suitable statistical calculation to estimate the final mark for Jon who was absent for 18 lessons. Comment on the reliability of your answer.

(f) Determine the average increase/decrease in the final mark for every five absences.

(g) From the calculated coefficient of linear correlation, the Head of Department concluded that poor attendance is the cause of poor final marks. Discuss mathematically the truth or falsity of this statement.

Calculator Assumed

5. [10 marks]

The table below shows the mass (m kg), height (h cm) and the time taken to run a 400 m race for 10 athletes.

Athlete	Mass, m (kg)	Height, h (cm)	Time, t (sec)
A	67	178	45.1
B	65	195	41.0
C	72	187	46.6
D	66	188	44.1
E	69	171	48.2
F	71	189	42.4
G	73	173	49.4
H	67	187	43.3
I	70	175	47.6
J	75	183	47.5

Athlete K weighs 68 kg and is 176 cm tall.
Use an appropriate statistical method to determine two possible predictions for the time K will take to run the 400 m race. Determine with reasons, which is the more reliable prediction.

Calculator Assumed

6. [12 marks]

In a science experiment, students were required to measure and determine the algebraic relationship between the variables x and y. Tables 1 and 2, shows respectively the results obtained by Ariel and Bernie. Chang combines the results obtained by Ariel and Bernie to form a larger collection of 16 pairs of measurements of x and y.

Table 1	(Ariel)
x	y
2	2.4
5	2.8
8	3
1	2
12	3.8
10	3.4
17	4.6
6	2.8

Table 2	(Bernie)
x	y
3	2.3
4	2.4
7	2.8
9	3.1
11	3.4
13	3.7
14	4
20	5.3

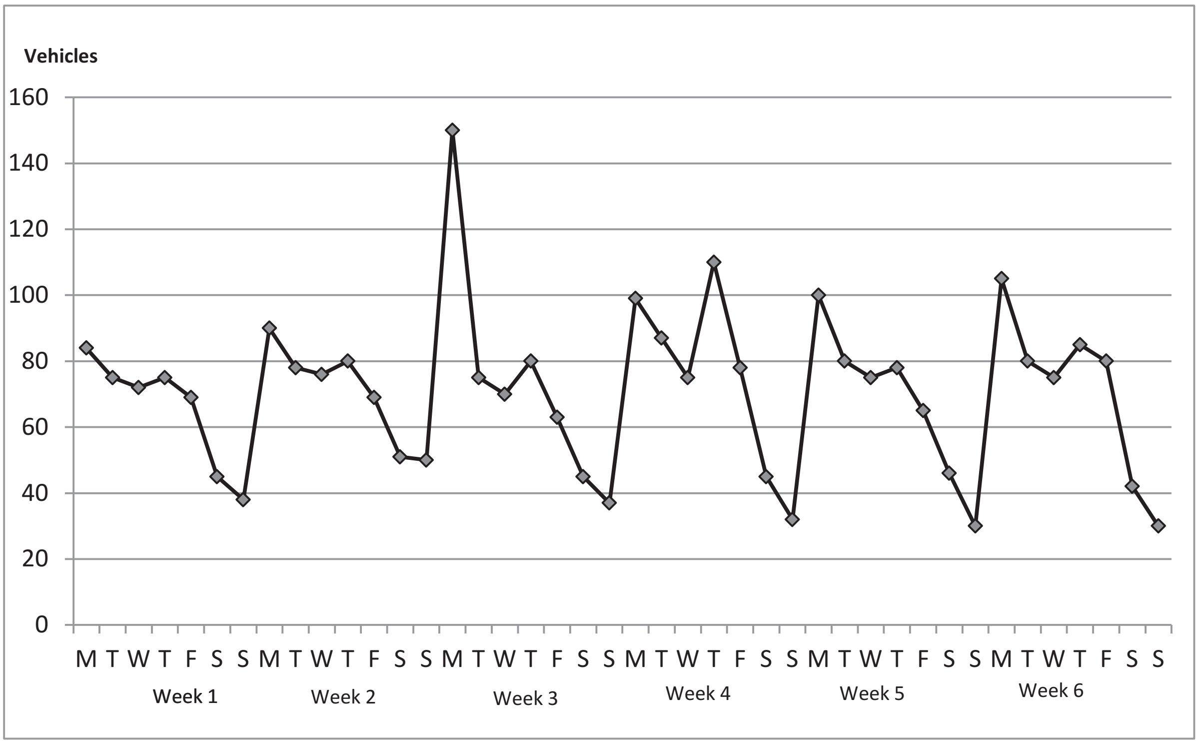
Discuss using statistical methods whether it is appropriate for Chang to combine the results obtained by Ariel and Bernie.
Investigate the effects of combining the results obtained by Ariel and Bernie.

05 Time Series

Calculator Free

1. [6 marks: 4, 2]

The accompanying diagram shows the number of vehicles passing through a road junction between 8.00 am and 8.30 am each day.



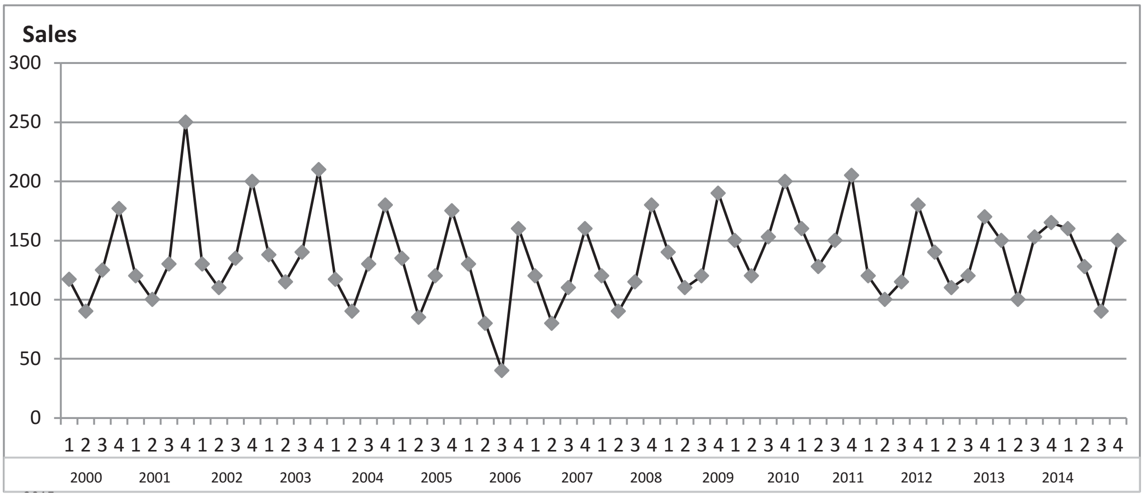
(a) Describe the time series in terms of its period, trend, seasonal and unseasonal fluctuations.

(b) Suggest an appropriate method to smooth the fluctuations.

Calculator Free

2. [9 marks: 2, 2, 3, 2]

The accompanying diagram shows the sales of vehicles in a certain city for the various quarters between 2000 and 2015 inclusive.



(a) Determine and describe the period for the time series.

(b) Describe any long term fluctuations (cycles) in this time series.

(c) Describe any unusual fluctuations in this time series.

(d) The overall trend in the time series is masked by the fluctuations.
Suggest a procedure to smooth these fluctuations to expose the trend.

Calculator Free

3. [5 marks: 2, 3]

The accompanying table shows a time series data with period of 4 and some of the associated 4 point centred moving averages.

Time	Data	4 pt. cma
1	4	
2	8	
3	16	***a***
4	16	14
5	12	16
6	***b***	18
7	24	20.125
8	24	22.375
9	21	24.625
10	25	27
11	33	
12	34	

(a) Calculate the value of a.
Show clearly how you obtained your answer.

(b) Calculate the value of b.
Show clearly how you obtained your answer.

Calculator Assumed

4. [8 marks: 2, 4, 2]

The table below shows the number of vehicles, N, entering a national park each month. A set of six point centred moving averages, M, is calculated and displayed in column 4.

Year	Month	No. of vehicles, N	Moving Average, M
2013	Jan	950	
	Feb	850	
	Mar	650	
	Apr	700	750.8
	May	620	753.3
	June	720	754.2
	July	**A**	755.8
	Aug	850	767.5
	Sept	660	783.3
	Oct	710	791.7
	Nov	750	806.7
	Dec	780	821.7
2014	Jan	1020	825.8
	Feb	990	**B**
	Mar	700	815.0
	Apr	720	
	May	690	
	June	760	

(a) The six point moving average for March 2014 is 815. Show how this was calculated.

(b) Find the values of **A** and **B**.

(c) Determine with reasons if it would be appropriate to use a set of five point moving averages to smooth the time series.

Calculator Assumed

5. [15 marks: 2, 4, 2, 2, 2, 3]

The accompanying table shows the number of vehicles caught speeding along a freeway over 20 consecutive working days (Mondays to Fridays inclusive) and the corresponding five point moving averages.

Day t	Vehicles, v	5 ma m
1	66	
2	31	
3	32	41.8
4	***a***	48.2
5	52	56.2
6	98	58.4
7	71	70.6
8	43	77.6
9	89	80.0
10	87	84.4
11	110	91.4
12	93	***b***
13	78	94.0
14	72	96.2
15	117	98.0
16	121	104.2
17	102	110.6
18	109	108.8
19	104	
20	108	

(a) Explain clearly why a five point moving average is the most appropriate type of moving average to mask any fluctuations.

(b) Calculate the values of ***a*** and ***b***.

The line of best fit through the moving averages has equation $\hat{m} = 4.4012t + 35.5627$ with a coefficient of determination of 95.5%.

(c) Determine with reasons the secular trend of the number of vehicles caught speeding along this freeway.

Calculator Assumed

5. (d) Predict the moving average for t = 19.

(e) Use your answer in (d) to predict the number of cars that will be caught speeding for t = 21.

(f) *Discuss* the reliability of your prediction in (e).

Calculator Assumed

6. [9 marks: 1, 4, 4]

The accompanying table shows the average price of commodity X over a period of 26 weeks and a set of calculated moving averages.

Week	Price (cents)	Moving Average
1	129.7	
2	158.9	
3	152.1	153.225
4	148.3	165.300
5	177.5	177.250
6	207.7	189.075
7	198.9	201.275
8	196.1	***a***
9	227.3	224.925
10	253.5	237.125
11	246.7	249.325
12	245.9	261.150
13	275.1	273.100
14	300.3	284.738
15	295.5	296.625
16	***b***	309.200
17	325.9	322.150
18	350.1	335.163
19	349.3	347.675
20	340.5	360.000
21	375.7	372.200
22	398.9	384.400
23	398.1	396.475
24	389.3	409.050
25	423.5	
26	451.7	

(a) Determine with reasons the best set of moving averages to smooth the time series.

(b) Find the values of ***a***, and ***b***.

(c) Calculate the equation of the line of best fit through the moving averages. Hence, determine with reasons the trend of the time series.

Calculator Assumed

7. [4 marks: 2, 2]

(a) The table below shows the quarterly seasonal indices for a time series.

Quarter 1	Quarter 2	Quarter 3	Quarter 4
1.09	1.05	0.98	x

Use the average percentage method to determine the value of x.

(b) The table below shows the bi-monthly seasonal indices for a time series.

Jan-Feb	Mar-Apr	May-June	July-Aug	Sept-Oct	Nov-Dec
0.99	1.01	1.02	1.01	x	0.98

Use the average percentage method to determine the value of x.

8. [8 marks: 2, 2, 4]

The monthly seasonal index for visitor numbers to a certain city for June is 1.06.

(a) Comment on the visitor numbers in June in relation to the average monthly number of visitors.

(b) The number of visitors to this city in June 2015 was 150 000. Determine the seasonally adjusted number of visitors to this city in June 2015.

(c) The line of best fit through the seasonally adjusted figures for monthly visitor numbers for 2010 to 2015 is $s = 1200m + 85\ 000$ where January 2010 is $m = 1$. Predict the visitor numbers for June 2016.

Calculator Assumed

9. [8 marks: 2 marks each]

A store operates six days a week. Table 1 below shows the net income (to the nearest $100) from this store over four weeks.

Week	Mon	Tues	Wed	Thurs	Fri	Sat	Daily Average
1	4 500	5 000	7 000	12 000	11 000	15 000	9 083.33
2	4 600	5 000	7 200	11 900	10 900	15 300	***a***
3	4 500	***b***	7 100	13 000	12 000	16 000	9 583.33
4	4 400	4 800	7 000	12 100	12 000	15 800	9 350.00

Table 2 below shows the daily sales as a percentage of the weekly daily average and the seasonal indices for the data in the table above. Some of the cells have been deliberately left blank.

Week	Mon	Tues	Wed	Thurs	Fri	Sat
1	0.495 41	0.550 46	***c***	1.321 10	1.211 01	1.651 38
2	0.502 73	0.546 45	0.786 89	1.300 55	1.191 26	1.672 13
3	0.469 57	0.511 30	0.740 87	1.356 52	1.252 17	1.669 57
4	0.470 59	0.513 37	0.748 66	1.294 12	1.283 42	1.689 84
Index	0.484 6	0.5304		1.3181	***d***	1.670 7

Calculate the values of $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$.

Calculator Assumed

10. [10 marks: 6, 2, 2]

The table below shows the quarterly sales figures ($m) for a company. The seasonal quarterly index (using the average percentage method) for Q4 is 1.1146.

Year	Quarter	Sales ($m)	Quarterly Mean, Q	Sales as % of Q
2012	Q1	700	698.25	1.0025
	Q2	621		*a*
	Q3	683		0.9782
	Q4	789		1.13
2013	Q1	841	839	1.0024
	Q2	*b*		0.8939
	Q3			0.9821
	Q4	941		1.1216
2014	Q1	974	971.5	1.0026
	Q2	*c*		
	Q3	951		0.9789
	Q4	1077		1.1086
2015	Q1	1124	1141	0.9851
	Q2	1049		0.9194
	Q3	1138		0.9974
	Q4	1253		*d*

(a) Determine the values of a, b, c and d.

(b) Calculate the seasonally adjusted sales figure for Q4 of 2015.

(c) The predicted seasonally adjusted sales figure for Q4 of 2016 is $1200 million. What is the predicted sales figure for Q4 of 2016?

Calculator Assumed

11. [18 marks: 2, 2, 2, 2, 4, 4, 2]

The table below shows the number of passengers travelling on a certain bus route and the corresponding seasonally adjusted daily number of passengers. The seasonal daily index (using the average percentage method) for Wednesdays is 1. 0291.

Week	Day	Passengers	Seasonally adjusted, s
1	Mon	1134	771
	Tue	1013	918
	Wed	973	945
	Thu	1013	842
	Fri	933	952
	Sat	611	931
	Sun	517	928
2	Mon	1214	826
	Tue	1053	954
	Wed	1026	
	Thu	1080	897
	Fri	933	952
	Sat	691	1053
	Sun	678	1217
3	Mon	2018	1373
	Tue	1013	918
	Wed		919
	Thu	1080	897
	Fri	852	870
	Sat	611	931
	Sun	504	905
4	Mon	1335	908
	Tue	1174	1064
	Wed	1013	984
	Thu	1482	1231
	Fri	1053	1075
	Sat	611	931
	Sun	437	785

(a) Determine the daily mean number of passengers for week 1.

Calculator Assumed

11. (b) Calculate the proportion of passengers travelling on Wednesday of the first week in relation to the mean daily number of passengers in the first week.

(c) Calculate the seasonally adjusted number of passengers for Wednesday of the second week.

(d) Calculate the actual number of passengers for Wednesday of the third week.

(e) Determine the seasonal daily index for Sundays and explain what this index means.

(f) Using Monday of week one as $t = 1$, the line of best fit of the seasonally adjusted numbers against time t is $s = 3.735t + 909.198$ with a coefficient of correlation value of 0.232.

(i) Predict the number of passengers on this route on Wednesday of week 5.

(ii) Comment on the reliability of your prediction.

Calculator Assumed

12. [13 marks: 2, 2, 2, 4, 3]

Data is collected between 2003 to 2019 inclusive for a time series variable with a seasonal pattern of four quarters. The line of best fit through the deseasonalised data is $s = -0.5t + 200$, where $t = 1$ represents quarter 1 of 2003. The line of best fit has a linear correlation coefficient of -0.94. The seasonal indices for quarters 1, 2, 3 and 4 are respectively 120, 110, x and 80.

(a) Calculate the value of x.

(b) Determine with reasons, which seasons had time series values below the seasonal mean.

(c) Predict the seasonally adjusted value for Quarter 4 of 2022.

(d) Predict the time series value when $t = 110$.

(e) *Discuss* the reliability of the prediction in (d).

Calculator Assumed

13. [9 marks: 1, 1, 7]

The table below shows the quarterly sales of passenger cars for a car company and the accompanying seasonally adjusted sales numbers. Some of the cells in the table have been deliberately left blank. The seasonal index for the March quarter is 0.970 65.

Year	Quarter	Sales	Seasonally Adjusted Sales
2013	Mar	515	530.5723
	June		533.8706
	Sept	540	534.3757
	Dec	550	536.0232
2014	Mar	545	561.4794
	June		554.0166
	Sept	560	554.1674
	Dec	570	555.5149
2015	Mar	555	571.7818
	June		574.1627
	Sept	585	578.9070
	Dec	590	575.0067
2016	Mar		597.5377
	June	595	599.3453
	Sept	600	593.7508
	Dec	610	594.4985

(a) The seasonally adjusted sales figures for the March quarter of 2014 is 561.4794. Show how this value is calculated.

(b) Calculate the actual sales numbers for the March quarter of 2016.

(c) Using the information given above and an appropriate statistical method, predict the profit for the March quarter of 2017. Show all working.

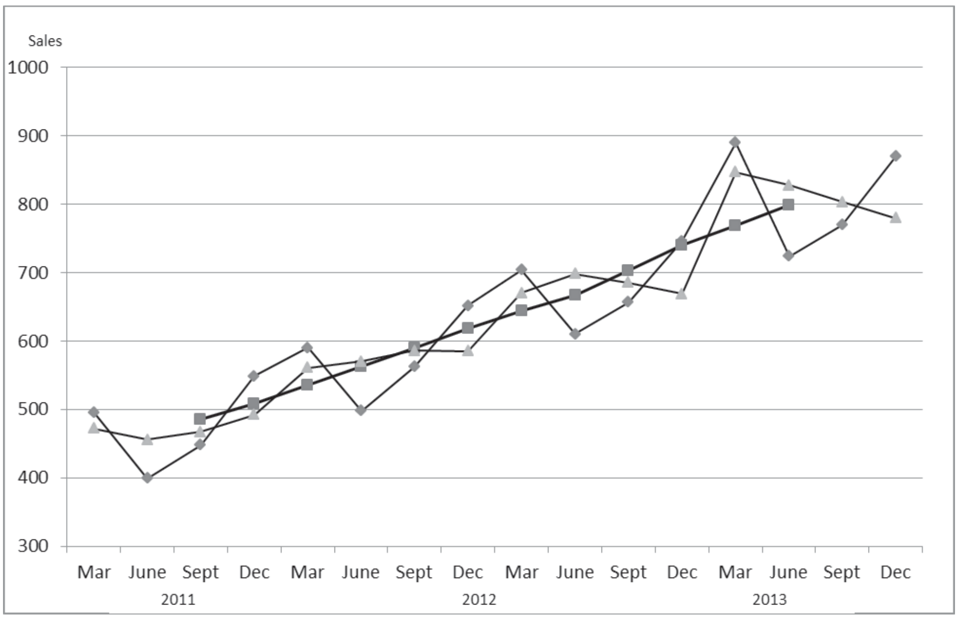
Calculator Assumed

14. [16 marks: 7, 2, 1, 4, 2]

The table below shows the quarterly number of cars sold by a car-yard. Included in the table are the corresponding 4 point centred moving averages and deseasonalised data. The seasonal quarterly index (using the average percentage method) for March is 1.0511.

Year	Quarter	Sales	4 pt. cma	Deseasonalised
2011	Mar	496		*a*
	June	399		456
	Sept	448	*b*	467
	Dec	549	508.88	492
2012	Mar	590	535.50	561
	June	498	562.63	570
	Sept	562	589.75	586
	Dec	652	618.00	585
2013	Mar	*c*	643.88	670
	June	610	667.50	698
	Sept	657	702.50	685
	Dec	746	740.00	669
2014	Mar	890	768.38	847
	June	*d*	798.00	828
	Sept	770		803
	Dec	870		780

The diagram below shows the plot of the time series, the four point centred moving averages and the deseasonalised data.



Calculator Assumed

14. (a) Calculate the values of a, b, c and d.

(b) Determine with reasons which of moving-averages or deseasonalised data perform a better job in smoothing the time series data.

(c) State one disadvantage of using moving averages to smooth the time series data.

Using the March Quarter of 2011 as $t = 1$, the regression line of moving averages on time t is $m = 28.57t + 392.13$ and the regression line of deseasonalised data on time t is $s = 26.54t + 409.95$.

(d) Use each of these two lines to predict the sales for the March quarter of 2015.

(e) Is the difference in predictions in (d) significant? Why?

06 Arithmetic Sequences

Calculator Free

1. [5 marks: 2, 2, 1]

Consider the sequence: 18 25 32 39 46….

(a) Find the recursive rule for this sequence.

(b) Find the rule for the n^{th} term of this sequence.

(c) Find the 50^{th} term of this sequence.

---

2. [6 marks: 2, 2, 2]

The n^{th} term of an arithmetic sequence is described by the rule $T(n) = 100 - 5n$, where $n = 1, 2, 3, 4, 5, \ldots$.

(a) Find the first 5 terms of the sequence.

(b) State the recursive rule for this sequence.

(c) One of the terms of this sequence is –895. Find the term that is three terms after this term.

Calculator Free

3. [5 marks: 2, 3]

The n^{th} term of an arithmetic sequence is described by the rule $\mathrm{T}(n) = 20 + 2n$, where the $n = 0, 1, 2, 3, \ldots$.

(a) Find the recursive rule of this sequence.

(b) Find which term equals 1000.

---

4. [6 marks: 2, 2, 2]

A sequence is defined by the recursive rule $a(n + 1) = a(n) + 0.5$ where $a(1) = 5$

(a) Find the first 5 terms of the sequence.

(b) Find the rule for n^{th} term in this sequence.

(c) Which term equals 10?

Calculator Free

5. [5 marks: 1, 2, 2]

The n^{th} term of an arithmetic sequence is described by the rule $T(n) = 10 + 3n$, where the first term is 13.

(a) Find the 20th term of this sequence.

(b) Find which term equals 355.

(c) State the recursive rule for this sequence using the terms a_n, a_{n-1} and a_0.

---

6. [5 marks: 3, 2]

The terms of a sequence are defined by $T_{n+1} - T_n + 20 = 0$ with $T_1 = 100$

(a) Show that this sequence is an arithmetic sequence.

(b) How many positive terms are there in this sequence?

Calculator Free

7. [6 marks: 3, 3]

The 8th term and 13th term of an arithmetic sequence are 50 and 80 respectively.

(a) Determine with reasons the 3rd term of this sequence.

(b) Determine with reasons the common difference and first term of this sequence.

8. [7 marks: 3, 4]

The difference between the tenth term and fifth term of an arithmetic sequence is 25.

(a) Determine with reasons, the difference between the thirtieth term and the fortieth term of this sequence.

(b) Determine the first term of the sequence if the fifth term of the sequence is 30.

Calculator Free

9. [7 marks: 3, 4]

The sixth term of an arithmetic sequence is double its fourth term. The first term of the sequence is 20 and the common difference is d.

(a) Show that $T_4 = 2 \times d$.

(b) Hence, find the recursive rule for the sequence.

---

10. [7 marks: 2, 3, 2]

An arithmetic sequence has first term a and common difference d.
The difference between the seventh term and the third term of an arithmetic sequence is equal to the third term.

(a) Write t_3 and t_7 respectively the third and seventh term in this sequence, in terms of a and d.

(b) Show clearly that $a = 2d$.

(c) Provide one possible arithmetic sequence of eight terms with the property that "*the difference between the seventh term and the third term is equal to the third term*".

Calculator Assumed

11. [3 marks: 1, 2]

A sequence is defined by the recursive rule $u(n + 1) = u(n) + 8$ where $u(1) = -104$.

(a) Find the 10th term.

(b) How many negative terms are there? Justify your answer.

---

12. [6 marks: 3, 1, 2]

Consider the sequence: 3 7 11 15 19 …..

(a) Find the 20th term.

(b) Which term equals 319?

(c) How many terms are there with values less than 500? Justify your answer.

Calculator Assumed

13. [4 marks: 2, 2]

A new road 85 km long is being laid. At the end of the Stage 1, 35 km of road had been laid. It took 45 days to complete Stage 1. For Stage 2, covering the remaining 50 km, each day an extra 600 m of new road is completed.
Let $b(n)$ be the length of completed road at the start of day n into Stage 2.

(a) Write a recursive equation for the length of completed road at the start of day n.

(b) Find how long it would take for entire road to be laid.

---

14. [5 marks: 3, 2]

Fiona is paid \$500 per week. After every 26 weeks, she receives a \$50 per week pay rise. Assume that the pattern of the pay rise is valid for 10 years and assume that there are 52 weeks in a year.
Let $a(n)$: Fiona's weekly pay at the start of half year number n.

(a) Write a recursive formula for Fiona's weekly pay.
Hence, find Fiona's weekly pay after 3 years?

(b) When will Fiona's pay be \$1000 per week?

Calculator Assumed

15. [8 marks: 2, 1, 2, 3]

The speed of a car is 8 ms^{-1} when it starts to accelerate. Its speed increases by 0.2 ms^{-1} each second so that its speed t seconds after it starts accelerating, $v(t)$ is as displayed in the table below.

t	1	2	3	4
$v(t)$ ms^{-1}	8.2	8.4	8.6	8.8

(a) Write a recursive equation for the speed of the car.

(b) Write an equation for the speed of the car in terms of t, where $t \geq 0$.

(c) The maximum permissible speed for the car along this stretch of road is 30 ms^{-1}. Find how long it takes the car to reach its maximum permissible speed. Justify your answer.

(d) As the car approaches its destination it decelerates so that it loses speed at a rate of 0.35 ms^{-1} each second. Use an appropriate recursive rule to find the time it takes the car to come to a stop from its maximum permissible speed.

Calculator Assumed

16. [9 marks: 2, 2, 3, 1, 1]

A special filter is installed to remove airborne particles from a dust-free room. The filter has to be changed every 10 weeks. In the first five weeks of operation, it removes 10 000 particles each week and thereafter its filtering capacity reduces by 500 particles each week.

(a) Write a recursive equation that describes the filtering capacity of the filter for the first five weeks.

(b) Write a recursive equation that describes the filtering capacity of the filter from the sixth week to the tenth week inclusive.

(c) Write in terms of k, an equation that describes the number of particles filtered in week k, if $6 \leq k \leq 10$.

(d) Find the total number of particles filtered in the first 5 weeks.

(e) Find the total number of particles filtered by the end of the 7th week.

07 Geometric Sequences

Calculator Free

1. [4 marks: 2, 2]

The n^{th} term of a sequence is given by $T(n) = 4 \times 2^n$, where $n = 1, 2, 3, 4, 5, \ldots\ldots$.

(a) Find the first 5 terms of the sequence.

(b) State the recursive rule for this sequence.

---

2. [4 marks: 2, 2]

Consider the sequence: 8 16 32 64 128 ….

(a) Find the recursive rule for this sequence.

(b) Given the n^{th} term is $T(n) = a \times b^n$, for $n = 1, 2, 3, \ldots$, find a and b.

---

3. [2 marks]

A sequence is defined by the recursive rule $a(n + 1) = a(n) \times 2$ where $a(1) = 3$.
Given the n^{th} term is $T(n) = a \times b^{n-1}$, for $n = 1, 2, 3, \ldots$, find a and b.

Calculator Free

4. [5 marks: 2, 3]

The n^{th} term of a sequence is given by the rule $T(n) = 2^{n+2}$ for $n = 1, 2, 3, \ldots$.

(a) Find the recursive rule of this sequence.

(b) Find the term number of the first term that exceeds 100.

---

5. [7 marks: 1, 2, 2, 2]

A sequence is given by the recursion equation $t_{n+1} = 5t_n$, where the $t_1 = 4$.

(a) Complete the table below listing the first five terms in this sequence.

t_1	t_2	t_3	t_4	t_5

(b) State the n^{th} term (general term) rule for this sequence.

(c) Complete the table below that lists the difference between two consecutive terms.

	$t_2 - t_1$	$t_3 - t_2$	$t_4 - t_3$	$t_5 - t_4$
Difference				

(d) The difference between two consecutive terms is 50 000.
State with reasons, these two consecutive terms.

Calculator Free

6. [4 marks]

   The difference between the fourth term and first term of a geometric sequence is 52. Calculate the common ratio of the sequence if the first term is 2.

---

7. [5 marks]

   4, a, b and 108 are consecutive terms of a geometric sequence.
   Determine the recursive rule for this sequence. Show all working.

---

8. [6 marks: 3, 3]

   Consider the five consecutive terms of a sequence: 2, a, b, c, 162

   (a) Explain why if $a = 40$, then the sequence cannot be an arithmetic sequence.

   (b) Explain why if $c = 54$, then the sequence can be a geometric sequence.

Calculator Assumed

9. [6 marks: 2, 2, 2]

Consider the following sequence of six terms.

10, –5, 2.5, –1.25, 0.625, –0.3125, 0.15650

(a) Explain clearly why the following sequence is not a geometric sequence.

(b) Change one of the terms in the given sequence to turn the sequence into a geometric sequence.

(c) State the recursive rule for the sequence you created in part (b).

---

10. [3 marks: 1, 2]

A sequence is defined by the recursive rule $u(n + 1) = u(n) \times 1.5$ with $u(1) = 2$.

(a) Find the 10th term to 4 significant figures.

(b) Which term first exceeds 1 000 000. Justify your answer.

Calculator Assumed

11. [6 marks: 3, 1, 2]

Consider the sequence: 3 –6 12 –24 48 …..

(a) Find the 15th term.

(b) Which term equals –6144?

(c) How many positive terms are there with values less than 1 000 000?
Justify your answer.

---

12. [6 marks: 4, 2]

A sequence is described by the rule $T_n = 1500(-1.04)^n$, where $n = 1, 2, 3, ..$.

(a) Show that the sequence is a geometric sequence. State the recursive rule.

(b) Find the first term that exceeds 2 000.

Calculator Assumed

13. [4 marks: 2, 2]

In 2005, a recycling centre processed 2000 tonnes of plastic containers. The amount of plastic containers processed each year, increased by 8 per cent per year. Let $a(n)$ be the amount of plastic containers processed in year n.

(a) Write a recursive rule for the amount of plastic containers recycled.

(b) In which year will the recycling centre for the first time recycle more than 5 000 tonnes of plastic containers? Assume that the recycling centre is able to handle the growth in the amount of containers recycled.

---

14. [5 marks: 2, 1, 2]

An observation balloon is released from a height of 50 metres and allowed to float vertically upwards. The height increase in the first minute is 80 metres. Thereafter, the height increase during each subsequent minute is 85% of the height increase during the previous minute. Ignore air and wind resistance. Let $h(n)$ be the height increase in the nth minute.

(a) Write a recursive equation for the height increase.

(b) Find the height increase (nearest cm) in the 5th minute.

(c) During which minute did the height increase first drop below 1 metre?

Calculator Assumed

15. [10 marks: 2, 2, 3, 3]

A ball is dropped from a height of 3 metres. The first time it hits the ground it bounces up to 2.7 m. The height reached after each bounce is 90% of the height reached in the previous bounce. Let $h(n)$ be the height of the ball reached after the nth bounce, where $n = 1, 2, 3, \ldots$.

(a) Write a recursive equation for the height reached by the ball after the nth bounce.

(b) Write a non-recursive equation for the height reached by the ball after the nth bounce.

(c) Calculate the difference in heights reached between the fifth bounce and the second bounce. Give your answer to the nearest cm.

(d) When does the difference in heights reached between consecutive bounces first differ by less than 0.1 m? Justify your answer.

Calculator Assumed

16. [9 marks: 2, 2, 2, 3]

An investment account pays 12% interest compounded annually. That is, each year the account grows by 12%. Julia invests $20 000 in this account for 20 years. No new money was added to and no withdrawals were made from the investment account. Let $b(n)$ be the balance of the account at the start of year n.

(a) Write a recursive equation for the balance of the investment account.

(b) Write an explicit equation for the balance of the investment account.

(c) Find the account balance after 8 years.

(d) Find the minimum number of years required for the balance to exceed $1 000 000.

08 Linear Recurrence Relations

Calculator Free

1. [7 marks: 1, 2, 2, 2]

A sequence is defined by the general term rule $t_n = 4 + 2n$ for $n = 1, 2, 3, \ldots$.

(a) List the first 5 terms of the sequence.

(b) State the recursive rule for this sequence.

(c) Consider the first 100 terms in this sequence. How many terms are there that are multiples of 5?

(d) How many terms are there that are less than 200?

---

2. [4 marks]

Consider the sequence: 3 –2 8 –12 28 ….

This sequence follows the rule $t_{n+1} = a\,t_n + b$. Determine the values of a and b.

Calculator Free

3. [7 marks: 2, 2, 3]

A sequence is defined by the recursive rule $a(n + 1) = a(n) - 2$ where $a(1) = -50$.

(a) Find the first 5 terms of the sequence.

(b) Find the rule for n^{th} term in this sequence.

(c) Find the term number of the first term that is greater than –100?

---

4. [4 marks]

Consider the sequence: 5 13 29 61 125 ….

The recursive rule of this sequence is of the form $T_{n+1} = aT_n + b$ with $T_1 = 5$.
Determine the values of a and b.

Calculator Assumed

5. [6 marks]

A sequence is defined by the rule $h_{n+1} = 2 - 3h_n$ where $n = 1, 2, 3, \ldots.$ for $h_1 = k$.
Given that k is a constant, find k if $h_4 = -598$.

---

6. [9 marks: 3, 6]

A sequence is defined by the recurrence relation

$$T_{n+2} = a\,T_{n+1} + b\,T_n \quad \text{where } T_1 = x \text{ and } T_2 = y.$$

(a) Find the first 5 terms of the sequence if $a = -1$, $b = 1$, $x = 2$, $y = 3$.

(b) The sequence 10, 20, 70, 200, 610 obeys the recurrence relation given.
Find the values of a and b.

Calculator Assumed

7. [8 marks: 2, 3, 3]

Consider the sequence: 2, x, 10.

(a) Find the value(s) of x if the sequence follows the rule $u_n - u_{n-1} = 4$.

(b) Find the value(s) of x if the sequence follows the rule $f(n + 1) = k\,f(n)$ where k is a constant.

(c) Find the value(s) of x if the sequence follows the rule $u(n + 2) = u(n + 1) - \{u(n)\}^2$.

---

8. [4 marks]

Describe in words the meaning of the following recursive equation.
List the first 5 terms of this sequence.
$C_{i+3} = C_{i+2} + C_{i+1} + C_i$ for $i = 1, 2, 3, \ldots$, where $C_1 = 0$, $C_2 = 0$ and $C_3 = -1$

Calculator Assumed

9. [8 marks: 2, 2, 2, 2]

Consider the following sequence of coordinates (x, y):

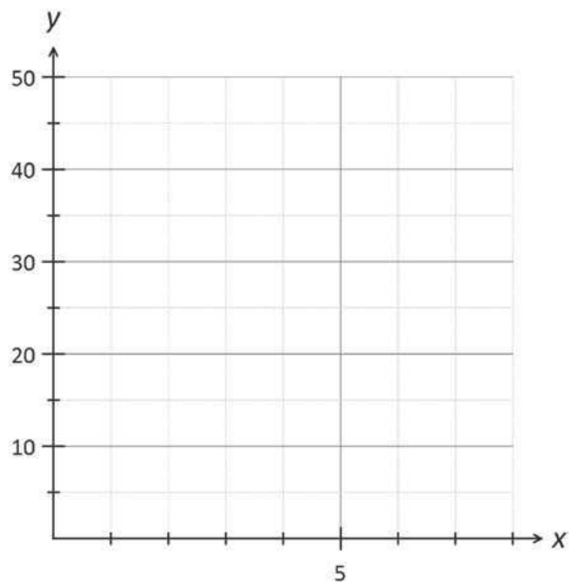
$$(3, 3), (4, 6), (5, 12), (6, 24), (7, 48), \ldots.$$

(a) Find the next set of coordinates in this sequence.

(b) Find the recursive rule for the x-coordinate and the recursive rule for the y-coordinate.

(c) Find the 20th set of coordinates in this sequence.

(d) Plot the coordinates and determine the shape of the curve passing through these points.



Calculator Assumed

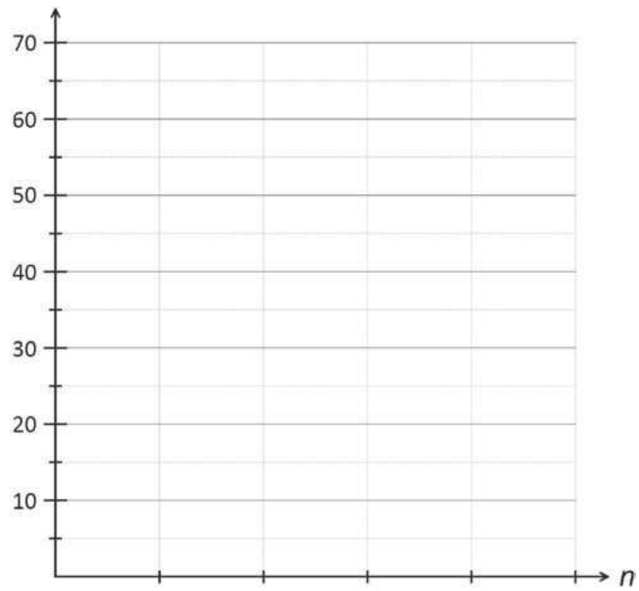
10. [12 marks: 3, 3, 3, 3]

Consider the sequences A, B and C defined by the recurrence relations.

A: $a(n + 1) = a(n) + 3,$ $a(0) = 1$
B: $b(n + 1) = 2\,b(n)$ $b(0) = 1$
C: $c(n + 1) = 2\,c(n) + 3$ $c(0) = 1$

(a) List the first six terms of each of the above sequences.

(b) In the axes provided below, plot the terms of each of the given sequences with respect to n. Fit as many points into the diagram as is possible.



(c) Comment on the shape of the plot of each of these recurrence relations.

(d) Discuss the rate of growth between the terms in each of the given sequences.

Calculator Assumed

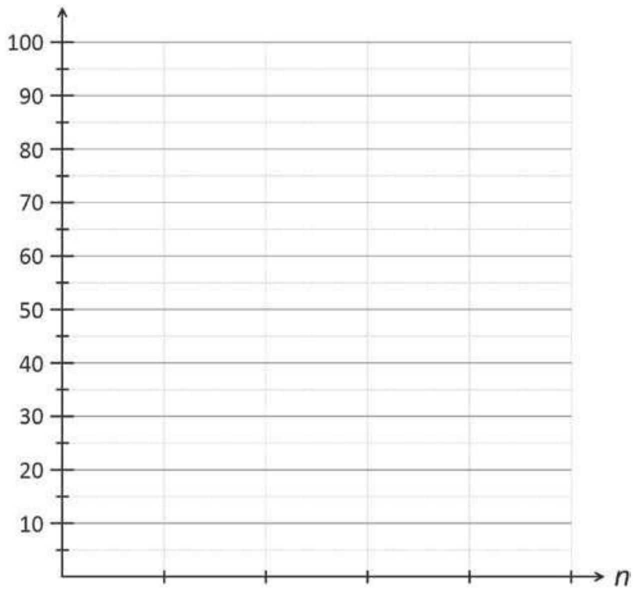
11. [12 marks: 3, 3, 3, 3]

Consider the sequences A, B and C defined by the recurrence relations.

A: $a(n+1) = a(n) - 2,$ $a(0) = 100$
B: $b(n+1) = 0.8\,b(n)$ $b(0) = 100$
C: $c(n+1) = 0.8\,c(n) - 2$ $c(0) = 100$

(a) List the first six terms of each of the above sequences.

(b) In the axes provided below, plot the terms of each of the given sequences with respect to n. Fit as many points into the diagram as is possible.



(c) Comment on the shape of the plot of each of these recurrence relations.

(d) Discuss the rate of decay between the terms in each of the given sequences.

Calculator Assumed

12. [16 marks: 4, 12]

Consider the sequences A, B, C and D defined by the recurrence relations.

A: $a(n + 1) = 0.05\, a(n)$, $a(0) = 1$
B: $b(n + 1) = 0.05\, b(n) + 1$ $b(0) = 1$
C: $c(n + 1) = -0.05\, c(n) + 1$ $c(0) = 1$
D: $d(n + 1) = -0.05\, d(n)$ $d(0) = 1$

(a) Complete the table given below which lists some terms in these sequences.

n	$a(n)$	$b(n)$	$c(n)$	$d(n)$
0	1	1	1	1
1	0.05	1.05	0.95	–0.05
2		1.0525		0.0025
3	0.000125		0.952375	
4	0.00000625	1.05263125	0.95238125	0.00000625
5	3.125×10^{-7}	1.052631563	0.952380938	-3.125×10^{-7}

(b) Describe with reasons, the terms in each of these sequences using the words "positive", "negative", "increasing", "decreasing" and "steady-state".
You may need to provide further terms to support your descriptions.

Calculator Assumed

13. [9 marks: 3, 6]

The table 1 below shows 3 recursion formulae.

	Recursion Formula	
A	$a_{n+1} = a_n + 10$	$a_1 = 50$
B	$b_{n+1} = 1.1\ b_n$	$b_1 = 50$
C	$c_{n+1} = 1.1\ c_n - 4$	$c_1 = 50$

(a) Which of the three recursion formulae with reach 100 fastest?
Give reasons for your answer.

Table 2 shows 4 general term formulae.

	General Term Formula for n = 1, 2, 3, …
I	$t_n = \dfrac{500 \times 1.1^n}{11}$
II	$t_n = 50 \times 1.1^n$
III	$t_n = 10n + 40$
IV	$t_n = 40 + 10 \times 1.1^{n-1}$

(b) Match each of the recursion formulae in the Table 1 with a general term formula in Table 2. Explain how you arrived at your answer.

Calculator Assumed

14. [8 marks: 2, 2, 4]

Consider the recursion formula $a(n+1) = 0.85 \times a(n) + 10$ $a(1) = 80$.

(a) Determine to three significant figures, the ninth term of this sequence

(b) Determine the term number of the first term that is less than 67.
Justify your answer.

(c) Describe with reasons, the behaviour of this sequence.

15. [6 marks: 4, 2]

Consider the recursion formula $A(n+1) = 0.9 \times A(n) + 5$ $\text{A}(1) = 100$

(a) Describe with reasons, the behaviour of this sequence.

(b) The rule $A(n+1) = 0.9 \times A(n) + k$ with $A(1) = 100$
gives a sequence of constant terms (all the terms have the same value).
Determine the value of k.

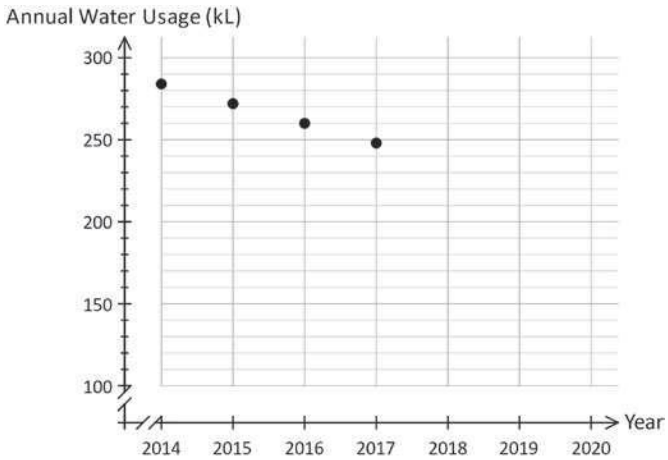
09 Growth & Decay Models

Calculator Assumed

1. [9 marks: 2, 2, 3, 2]

The table and the accompanying graph below show the amount of water (kL) used by a household in a given suburb each year for the years 2014 to 2017.

Year	2014	2015	2016	2017
Annual Water Usage (kL)	284	272	260	248



(a) Assuming that the water usage pattern continues, plot on the diagram above the predicted water usage in 2018, 2019 and 2020.

(b) Write a rule in its simplest form for the expected annual water usage n years after 2014.

(c) Using this pattern, determine when the annual water usage first drops below 10 kL.

(d) Determine with reasons if the rule in (b) may be used to make long term predictions of water usage for households in this suburb.

Calculator Assumed

2. [8 marks: 1, 2, 2, 3]

   Q, the number of organisms (*in hundreds*) in a laboratory culture is related to time t (days) by the formula $Q(t + 1) = 1.075 \times Q(t)$ where $Q(0) = 16$.

   (a) How many organisms were there at the start?

   (b) What is the growth/decay rate? *State clearly the units used.*

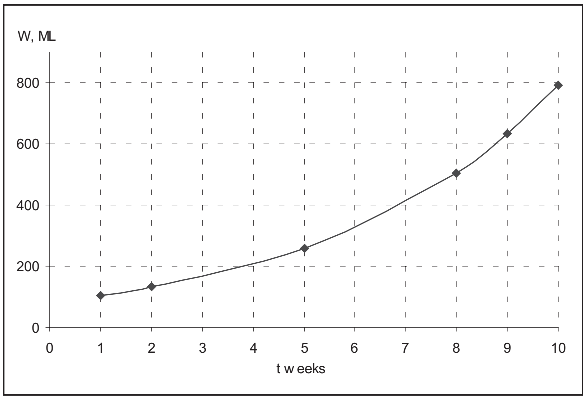
   (c) Find the number of organisms after one week.

   (d) How long will it take for the population to reach 2 000?
   Give your answer to the nearest day.

Calculator Assumed

3. [9 marks: 3, 5, 1]

The amount of water, W MegaLitres, in a newly constructed dam at time t weeks is shown in the graph below.



Three models were suggested for this data:

Equation I	$W(t) = W(t-1) \times 1.25$ where $W(0) = 80$,
Equation II	$W(t+1) = W(t) \times 1.26$ where $W(0) = 100$
Equation III	$W(t+1) = W(t) \times 1.26$ where $W(0) = 50$

(a) Which of these three models best represents the data given? Why?

(b) Use your chosen model to:
  (i) estimate the amount of water in the dam after 20 weeks.
  (ii) find when the amount of water in the dam will first exceed 3 000 ML.

(c) What is the most important assumption underlying the model you chose in part (a)?

Calculator Assumed

4. [10 marks: 2, 2, 3, 3]

A study of the population of a rare marsupial found the population growth rate was 9.5% per annum. At the commencement of the study (at the start of 1997) the population was 2 000.

(a) Write a recursive formula for predicting the population, P, t years after 1997.

(b) Predict the population at the start of 2007 (to nearest whole number).

(c) Show clearly that the population t years after 1997 can also be written as $P(t) = A \times b^t$ where A and b are constants. State the values of A and b.

(d) Predict when the population first exceeds 10 000. Show clearly how you obtained your answer.

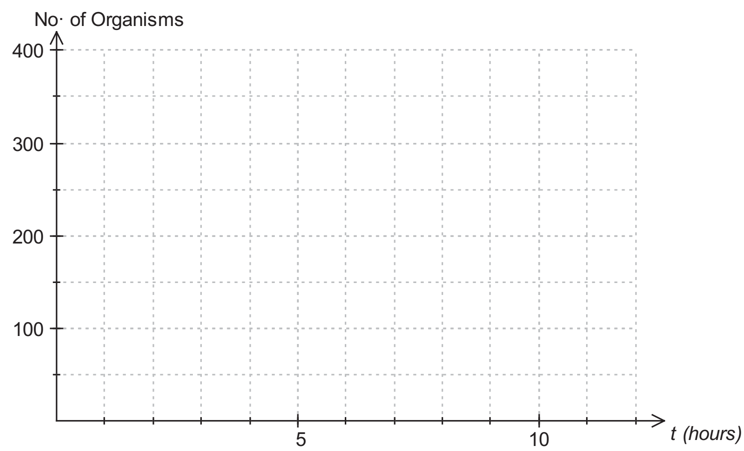
Calculator Assumed

5. [7 marks: 2, 2, 3]

In a laboratory experiment, two types of organisms were placed in a sealed biosphere. The number of organism A in the biosphere is modeled by $N = 200 \times 1.04^t$ where t is time in hours.

(a) Find the recursive formula that describes the number of organisms at time t hours.

(b) Sketch the graph of the population of A against t for $0 \le t \le 12$ in the axes provided below.



The number of organism B at time t hours is given in the table below.

t	1	2	3	4	5	6	7	8	9	10	11	12
B	400	368	338	310	285	262	241	221	203	186	171	157

(c) Plot the population of B against t onto the axes given in part b. Hence determine during which hour the population of A exceeds the population of B.

Calculator Assumed

6. [9 marks: 2, 1, 3, 3]

A farm machine is worth \$400 000 new. Let $V(n)$ be the value of the machine at the end of year n. $V(n)$ is calculated using two different methods.

(a) Method A:
Each year, the value of the machine is decreased by \$50 000.
(i) Write a recursive equation for the value of the machine.

(ii) Calculate the value of the machine after 5years.

(b) Method B
At end of each year, the value of the machine is decreased by 20% of the previous year's value. Calculate the value of the machine after 5 years. Show clearly how you arrived at your answer.

(c) When does the Method B give a higher value for the machine than Method A?

Calculator Assumed

7. [10 marks: 3, 1, 1, 2, 3]

The maximum capacity of a grain silo is 80 tonnes. Initially there was 50 tonnes of grain in the silo. At the start of each day, 20% of the grain in the silo is trucked out. At the end of each day 15 tonnes of grain is trucked in to be stored in the silo. Let $C(n)$ represent the amount of grain in the silo at the end of day n.

(a) Write a linear recursion relation for the amount of grain in the silo at the end of day n.

(b) Determine the amount of grain in the silo at the end of the eighth day.

(c) Determine the amount of grain trucked out on the morning of the ninth day.

(d) Find the difference in the amount of grain in the silo between the end of the second day and the end of the eighth day inclusive?

(e) Determine with reasons if the capacity of the grain silo will be exceeded.

Calculator Assumed

8. [10 marks: 3, 1, 3, 3]

The maximum capacity of a swimming pool is 2 500 kL.
Initially there was 2 000 kL of water in the pool. Each day the pool loses 5% of the amount of water present at the start of the day, due to evaporation. At the end of each day 125 kL of water is added to the pool.

Let $W(n)$ represent the amount of water in the pool at the end of day n.

(a) Write a linear recursion relation for the amount of water in the pool at the end of day n.

(b) How much water is added into the pool between the third day and the sixth day inclusive?

(c) Determine the increase in the amount of water in the pool between the start of the third day and the end of the sixth day .

(d) Describe with reasons, the amount of water in the pool in the long term.

Calculator Assumed

9. [9 marks: 4, 3, 2]

The population of the City of Tam at the start of 2000 is 90 000. Each year, the population grows naturally (through births) at a rate of 1.5% of the previous year's population. Each year 300 persons move into the city and 1 000 persons leave the city. Each year, the natural death rate of the population is 0.5% of the previous year's population. Let P_n be the population of the city (in thousands) n years after 2000.

(a) Explain clearly why $P_n = a\,P_{n-1} + b$ where $P_0 = 90$, $n = 1, 2, 3\ldots$.
State the values of a and b.

(b) Determine the values of P_1, P_2 and P_3.

(c) The mayor of a neighbouring city described the City of Tam as "a city with a decaying population".
(i) How would the mayor support this statement?

(ii) How would you contradict this statement?

10 Financial Mathematics I

(Simple Interest, Compound Interest and Depreciation)

Calculator Assumed

1. [10 marks: 2, 2, 1, 2, 3]

(a) Calculate the effective annual interest rate on an account that pays interest at a rate of 3% per annum compounded daily.

(b) A savings account pays interest at a rate of $100a$% per annum compounded monthly. Calculate a if the effective annual rate of interest is 4%.

(c) A savings account pays interest at a rate of 4.5% per annum compounded n times per year. Calculate n if:

(i) the effective annual rate of interest is 4.5%.

(ii) the effective annual rate of interest is 4.60075%.

(d) An account pays interest at a rate of $100a$% per annum compounded n times per year. The effective annual rate of interest is 3.66%.
Calculate the interest earned after 5 years on an initial deposit of \$10 000.

Calculator Assumed

2. [6 marks: 3, 3]

Account X pays interest at a rate of 6.0 % compounded *monthly*.
Account Y pays interest at a rate of $100r$ % compounded *daily*.

(a) Calculate the effective rate of interest for Account X.

(b) Calculate the minimum value for r if Account Y pays more interest than account X.

3. [5 marks]

$100 000 is to be invested for several years.
Account P offers interest at 6% per annum compounded annually.
Account Q offers a flat rate interest of 6% per annum.

Explain with ***reasons*** why Account P pays better interest compared to the Account Q.

Calculator Assumed

4. [14 marks: 3, 2, 2, 3, 4]

Zoe invests \$200 000 in an account that pays simple interest at a rate of 6% per year. The interest is paid at the end of each year and is not added to the principal. Let B(n) be the account balance at the end of n years.

(a) Determine a recurrence relation for the account balance after n years.

(b) Determine a general rule for the account balance after n years.

(c) Find n when the account balance is \$296 000.

(d) Find the minimum number of years required for the balance to first exceed \$500 000.

(e) How much would Zoe have to deposit into the account if she wishes the account balance to exceed \$500 000 after a minimum of 10 years? Assume that the interest rate remains at 6% per annum.

Calculator Assumed

5. [10 marks: 1, 4, 2, 3]

$10 000 is invested at a rate of 8% per year with interest compounded monthly.

(a) Calculate the monthly interest rate.

(b) Use a recurrence relation to calculate the value of the investment after 5 years and the total interest earned.

(c) How much more interest is earned after 5 years compared to if only simple interest is paid at a rate of 8% per year?

(d) Calculate the annual simple interest rate that will yield the same interest as the interest earned after 5 years with interest compounded monthly.

Calculator Assumed

6. [10 marks: 1, 4, 2, 3]

$100 000 is invested at a rate of 4.75% per annum with interest compounded monthly.

(a) Calculate the monthly interest rate.

(b) Use a recurrence relation to calculate value of the investment after 10 years and the total interest earned.

(c) Find the time taken for the investment to double its initial value.

(d) If the interest rate per annum were to be doubled, determine the time taken for the initial investment to double. Assume that the interest is compounded monthly.

Calculator Assumed

7. [10 marks: 2, 2, 4, 2]

$10 000 is invested in account A that pays simple interest *monthly*
at a rate of 3% pa.
$10 000 is invested in another account B that pays interest at a rate of 2.9% pa compounded *monthly*.

(a) Write a recursion equation for the account balance in A after *n* months.

(b) Write a recursion equation for the account balance in B after *n* months.

(c) Discuss with reasons which account is more profitable in the short term and in the long term.

(d) Determine with reasons if your answer in (c) will change if the amount invested in each account was $1 000 000?

Calculator Assumed

8. [8 marks: 1, 1, 1, 2, 3]

Garrett is depreciating his home office equipment at 15% each year using the flat rate method (prime cost method). Under the flat rate method, an item loses the same amount of value each year. It cost Garrett \$10 000 to purchase and install his home office equipment.

(a) What is the annual depreciation?

(b) What would the office system be worth after 2 years?

(c) When will the office system have a paper value of \$0.

(d) An alternative method of depreciation is to use the diminishing value method (reducing balance method), where each year the asset loses a constant percentage of its previous year's value. Use the diminishing value method with a depreciation rate of 15% per year to find:

(i) the value of the office system after 2 years.

(ii) the minimum number of years required for the value of the office system to be worth less than \$1000.

Calculator Assumed

9. [8 marks: 2, 2, 2, 2]

Jack and Jill bought a block of land for $500 000 and built a $300 000 house on the block. The value of the land appreciates (increases) at a constant rate of 8% per year while the value of the house depreciates (decreases) at a constant rate of 4% per year.

(a) Calculate the value of the block of land after 2 years.

(b) Calculate the value of the house after 2 years.

(c) Write a mathematical expression for the total value of the block of land and the value of the house after n years.

(d) Calculate when the total value of the land and the house first exceeds $1 000 000.

11 Financial Mathematics II (Loans)

Calculator Assumed

1. [9 marks: 2, 4, 3]

Henry borrowed $10 000 to buy his first car. Interest is charged on the opening balance each month at a rate of 15% per annum. Henry repays $800 each month for the first 10 months and thereafter he repays $900 per month (except for the final payment). All payments are made at the end of each month. The final payment cannot exceed the regular payments.

(a) Why would Henry take more than 12 months to repay this loan?

The table below shows the state of Henry's account over the life of the loan.

Month	Opening Balance	Interest	Repayment	Closing Balance
1	10000.00	125.00	800.00	9325
2	9325.00	116.56	800.00	8641.56
3	8641.56	108.02	800.00	7949.58
4	7949.58	99.37	800.00	7248.95
5	7248.95	90.61	800.00	6539.56
6	6539.56	81.74	800.00	5821.3
7	5821.3	72.77	800.00	5094.07
8	5094.07	63.68	800.00	4357.75
9	4357.75	54.47	800.00	3612.22
10	3612.22	45.15	800.00	2857.37

(b) Complete the table above to find how long Henry will take to repay the loan.

(c) How much interest would Henry have paid for the whole loan?

Calculator Assumed

2. [11 marks: 2, 2, 2, 3, 2]

Nick borrows $10 000 to buy a new car. Interest is charged at a reducible rate of R% per annum calculated monthly on the previous month's closing balance (assume that this rate is fixed for the life of the loan). Nick pays back $$B$ at the end of every month except for the last payment. Each repayment is made after the interest for the month has been added onto the amount owing at the start of the month. The table below shows the state of Nick's car loan account.

Start of Month	Amount Owing $	Interest $	Amount Owing after Interest and Repayment $
1	10 000	125	8 625
2	$ 8,625.00	$ 107.81	$7,232.81
3	$ 7,232.81	$ 90.41	$5,823.22
4	$ 5,823.22	$ 72.79	$4,396.01
5	$ 4,396.01	$ 54.95	$2,950.96

(a) Find the value of B.

(b) Find the value of R.

(c) Write a recursive rule to determine the closing balance at the end of each month.

(d) How long will Nick take to repay the loan and state the final payment.

(e) How much interest did Nick pay over the life of the loan?

Calculator Assumed

3. [9 marks: 2, 2, 2, 3]

Adam took a loan worth \$200 000 to purchase a property. Interest is charged on the opening balance each month. Adam's monthly payments are made at the end of each month.

(a) The table below is an account statement of Adam's loan for the first 6 months.

Month	Opening Balance	Interest charged	Repayment	Closing Balance
1	\$200,000.00	\$948.33	\$948.33	\$200,000.00
2	\$200,000.00	\$948.33	\$948.33	\$200,000.00
3	\$200,000.00	\$948.33	\$948.33	\$200,000.00
4	\$200,000.00	\$948.33	\$948.33	\$200,000.00
5	\$200,000.00	\$948.33	\$948.33	\$200,000.00
6	\$200,000.00	\$948.33	\$948.33	\$200,000.00

(i) What was the rate of interest (per annum)?

(ii) Describe the main feature of Adam's loan.

(c) In another scenario, Adam plans to pay off his entire loan in 25 years with the same rate of interest charged as in (a).

(i) What would his monthly payments be (nearest \$)?

(ii) Calculate the interest paid over the life of the loan and hence the effective annual rate of interest.

Calculator Assumed

4. [14 marks: 1, 3, 2, 3, 2, 3]

Jamie borrowed $10 000 to buy her car. Interest is charged on the opening balance each month at a rate of 9% per annum. At the end of each month, Jamie repays $500 except for the final payment. The final payment cannot exceed the regular payments. The table below shows, b_n, the opening balance at the start of each month and b_{n+1}, the closing balance at the end of each month.

Month	Opening Balance b_n	Interest	Repayment	Closing Balance b_{n+1}
1				
2				
3				

(a) Find the monthly interest rate.

(b) Complete the table above and state the amount Jamie owed at the end of the third month.

(c) Write a recursive rule to determine the closing balance at the end of each month.

(d) How long will Jamie take to repay her loan and how much is the final payment?

(e) What is the total interest paid?

(f) Will Jamie pay off her loan in half the time taken in (d) if she doubled her monthly repayments? Justify your answer.

Calculator Assumed

5. [14 marks: 2, 2, 3, 1, 3, 3]

Olivia has a mortgage of $600 000 on her house. Interest is calculated at 5.7% per annum on the month's opening balance and credited at the end of each month. Monthly repayments are made at the end of each month. The loan and its accrued interest have to be repaid in 30 years.

For the first 5 years, Olivia is permitted by the conditions of her loan to repay only the interest part of the loan. To pay off the loan within the remaining 25 years, Olivia has to make a minimum monthly payment of $3 756.54.

(a) Calculate her monthly repayments for the first 5 years.

(b) If Olivia makes the minimum monthly payment for the next 25 years, what would be the total interest she would have paid for the house? Do not include the interest paid in the first 5 years.

(c) Olivia makes the minimum monthly payments from the start of year six to the end of year ten

(i) Write a recursive rule to determine the amount owing at the end of each month.

(ii) Determine the amount left owing at the end year ten.

Calculator Assumed

5. (c) (iii) Starting from the eleventh year, Olivia pays double the minimum monthly payment each month. How many more months will Olivia need to repay the entire loan?

(iv) Calculate the final payment in Question 5 (c) (iii).

Calculator Assumed

6. [12 marks: 2, 2, 2, 3, 3]

Sky has a mortgage of $600 000 on her luxury apartment. The table given below shows the state of her reducible balance mortgage account for month 117 of her loan. Assume that the interest rate remains unchanged throughout the life of the loan. Sky repays $6 000 each month.

Month	Starting Amount	Interest	Repayment	Amount Still Owing
117	$316,516.02	$2,347.49	$6,000.00	$312,863.52

(a) Find the annual interest rate charged.

(b) How much is the principal reduced by over the 118th month?

(c) Write a recursive rule to determine the closing balance at the end of each month, starting from month 117.

(d) Determine how long Sky would take to reduce the amount owing to half the original amount borrowed.

(e) From the start of the 120th month onwards, Sky converts her loan to an "interest only" non-reducible loan. That is, her monthly repayments are just sufficient to cover the interest charged for that month. What is the amount of this monthly repayment?

Calculator Assumed

7. [10 marks: 2, 2, 6]

Jackie pays $4000 monthly towards her monthly reducible-balance mortgage of $500 000. Interest is charged at 8% per annum.

(a) Complete the table below to determine the balance at the end of the second month.

Month	Starting Amount	Interest	Repayment	Amount Still Owing
1	$500 000.00	$3333.33	$4000.00	$499 333.33
2				

Her sister Joan suggests that she should make weekly payments instead. Assume that Jackie's Bank agrees to convert her mortgage to a weekly reducible-balance mortgage and Jackie would now pay $1000 per week with interest remaining at 8% per annum. (Assume 1 year = 52 weeks)

(b) Write a recursive rule to determine the closing balance at the end of each week.

(c) At the end of the first year, determine which of the two payment methods is better, and by how much. Explain clearly how you arrived at your answer.

Calculator Assumed

8. [12 marks: 2, 2, 2, 2, 4]

A couple needs \$100 000 to renovate their house.
Option A is a reducible balance loan with interest compounded monthly at 4% per annum. The loan is to be repaid in equal monthly instalments in 4 years.
Option B is an *interest only* loan over 4 years with a flat rate interest of 3.9% per annum. The borrower only pays the interest charged during the term of the loan. At the end of the loan, the borrower pays the full amount borrowed. The interest charged is repaid in equal monthly installments.

(a) Calculate the monthly instalment for Option A.

(b) Calculate the total cost for Option A.

(c) Calculate the monthly payments for Option B.

(d) To pay the \$100 000 borrowed in Option B, the couple opens a savings account that pays interest at 3.3% per annum compounded monthly. The couple deposits a fixed amount into the account at the start of each month. Interest is paid at the end of each month on the opening balance at the start of each month after the monthly deposit is made. What should the monthly deposit be if the account balance at the end of 4 years is \$100 000?

Calculator Assumed

8. (e) Determine with reasons, which option should the couple take to borrow the \$100 000 needed to fund their home renovations.

Calculator Assumed

9. [11 marks: 3, 3, 5]

Maxine has a mortgage of \$750 000 on her house. Interest is calculated on the monthly opening balance at 6% per annum. Maxine pays \$ 1200 at the end of each week.

(a) Show that the loan has been reduced by \$37 130 (to the nearest \$) by the end of the second year.

(b) Determine how long it will take Maxine to repay the entire loan.

(c) What is the total interest paid by Maxine?

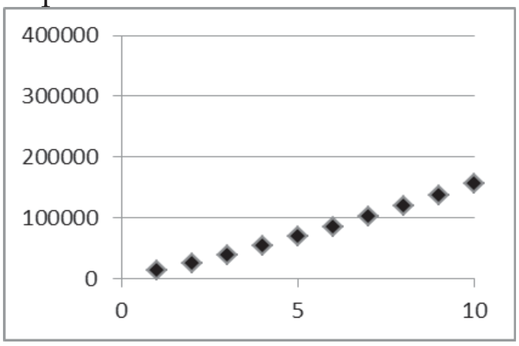
12 Financial Mathematics III (Annuities)

Calculator Free

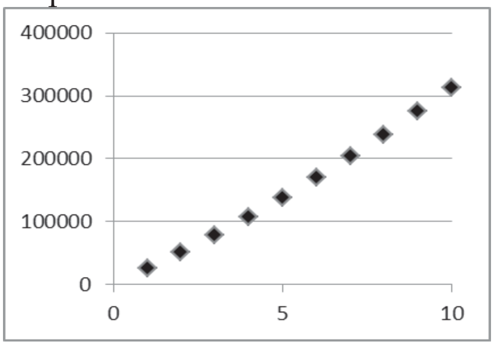
1. [8 marks: 2, 2, 2, 2]

The four graphs given below show the closing annual balances of several investment accounts over time in years. Each of these accounts has the same initial balance of \$0. Match each of the given graphs with one of the given recursion relations.

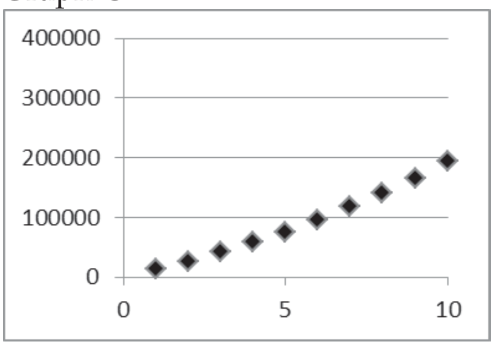
Graph A



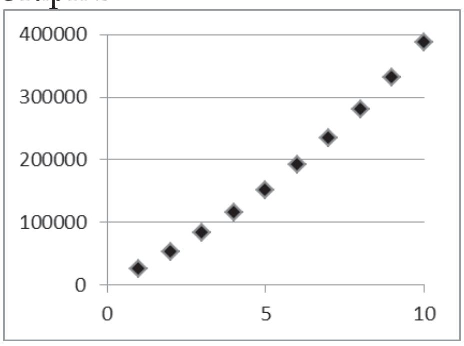
Graph B



Graph C



Graph D



Recursion Relations:

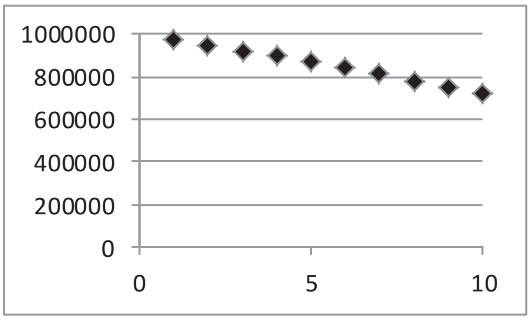
I	$b(n+1) = [b(n) + 12\,000 \times 1.05^{n-1}] \times 1.048$	$b(0) = 0$
II	$b(n+1) = [b(n) + 12\,000 \times 1.05^{n-1}] \times 1.048$	$b(1) = 0$
III	$b(n+1) = [b(n) + 12\,000] \times 1.048$	$b(1) = 0$
IV	$b(n+1) = [b(n) + 24\,000] \times 1.048$	$b(1) = 0$
V	$b(n+1) = [\mathrm{b}(n) + 24\,000 \times 1.05^{n-1}] \times 1.048$	$b(1) = 0$
VI	$b(n+1) = [b(n) + 12\,000] \times 1.048$	$b(1) = 0$

Calculator Free

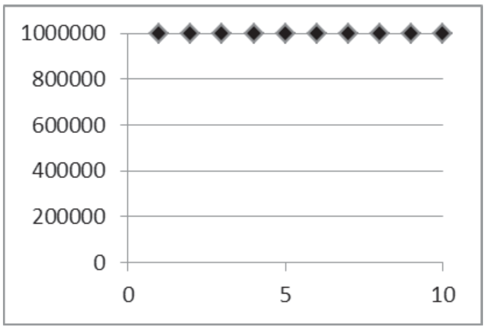
2. [8 marks: 2, 2, 2, 2]

The four graphs given below show the closing annual balances of several annuities over time in years. Each of these accounts has the same initial balance of \$1 000 000. Match each of the given graphs with one of the given recursion relations.

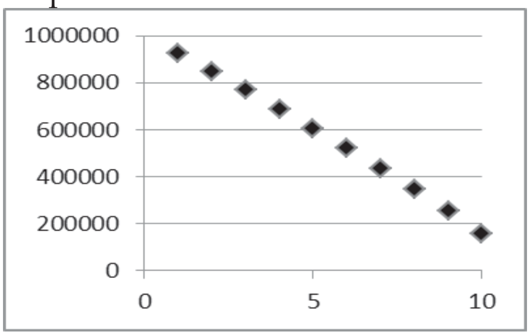
Graph A



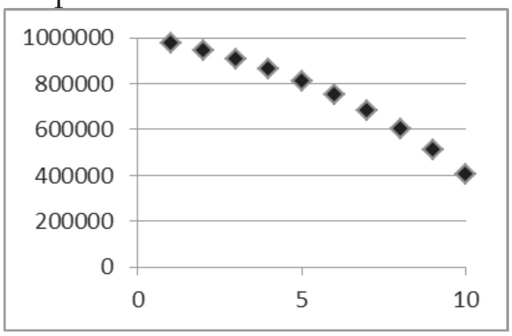
Graph B



Graph C



Graph D



Recursion Relations:

I	$b(n+1) = 1.025\, b(n) - 50\,000 \times 1.1^{n-1}$	$b(1) = 1\,000\,000$
II	$b(n+1) = 1.03\, b(n) - 25\,000$	$b(1) = 1\,000\,000$
III	$b(n+1) = 1.025 b(n) - 25\,000$	$b(1) = 1\,000\,000$
IV	$b(n+1) = 1.025\, b(n) - 100\,000$	$b(1) = 1\,000\,000$
V	$b(n+1) = 1.025\, b(n) - 100\,000$	$b(0) = 1\,000\,000$
VI	$b(n+1) = 1.025 b(n) - 50\,000$	$b(1) = 1\,000\,000$

Calculator Assumed

3. [10 marks: 2, 2, 2, 2, 2]

Mark owns a superannuation account. The account was opened on July 1st 2008, with an opening deposit of $1000. On the first of each month thereafter, Mark deposits $1000 into the account. The superannuation company claims that the average long term growth rate is 12% per annum on the opening balance each month and paid at the end of each month.

Month	Opening Balance $	Balance after monthly deposit ($)	Growth for the month ($)	Balance at the end of the month ($)
1	0	1000		
2				

(a) Complete the table above to find the account balance after 2 months.

(b) Write a recursive rule to determine the closing balance at the end of each month.

(c) Find the account balance at the end of 10 years.

(d) Find the total interest earned after 10 years.

(e) Find the average annual percentage return after 10 years.

Calculator Assumed

4. [10 marks: 2, 2, 2, 2, 2,]

May & Sydney operate an investment account for their children's education with an initial deposit of $5000. They make regular monthly deposits of $2000 each at the end of each month for 10 years. The table below shows the first few months of the account. Assume that the account pays a fixed rate of interest of 6% per annum. The monthly growth is calculated on the opening monthly balance and credited to the account at the end of each month.

Month	Opening Balance $	Growth for the month ($)	Monthly Deposit ($)	Balance at the end of the month ($)
1	5 000	25	2 000	7 025
2	7 025	35.125	2 000	9 060.125
3				

(a) Complete the table above to find the account balance at the end of the third month.

(b) Write a recursive rule to determine the account balance at the end of each month.

(c) Determine the account balance at the end of the 10 year period.

(d) Find the total interest earned after 10 years.

(e) Find the average annual percentage return after 10 years.

Calculator Assumed

5. [9 marks: 2, 2, 2, 1, 2]

Penny operates a superannuation account with the investment division of a bank. On January 1st 2001, the account balance was \$40 000.
On the first of each month, Penny deposits \$500 into the account. Assume that her account will grow by an average of 9% per annum over a period of 30 years. Each month's growth is calculated on the opening monthly balance and credited to her account at the end of each month.

Month	Opening Balance \$	Balance after monthly deposit (\$)	Growth for the month (\$)	Balance at the end of the month (\$)
1	40 000	40 500		
2				

(a) Complete the table above to find Penny's balance after 2 months.

(b) Write a recursive rule to determine the account balance at the end of each month.

(c) Use the recursive formula to find her account balance after 30 years.

The explicit formula for the account balance, b_n, at the end of each month:

$$b_n = (40\,000 \times r^n) + 500 \times r \times \frac{(1-r^n)}{1-r} \text{ where } n = 1, 2, 3\,.$$

(d) What should the value of r be?

(e) Use the above formula to calculate her account balance after **30 years**.

Calculator Assumed

6. [15 marks: 2, 3, 1, 3, 3, 3]

William starts work on 1st July 2016 with an annual salary of $60 000.
His annual salary increases by 3% every year. His employer contributes 9.5% of his annual salary in four equal quarterly amounts into a superannuation fund held in his name. The employer's contributions are transferred into his fund at the end of each quarter. The superannuation fund pays interest at 6% per annum compounded quarterly.

(a) Calculate the employer's quarterly contribution into his fund in his first year of work?

(b) The recursion formula to describe the fund balance at the end of quarter n of his *first* year of work is given by $\mathrm{B}(n) = \mathrm{B}(n-1) \times (1+a) + b \quad \mathrm{B}(0) = c$.
Determine the values of a, b and c.

(c) Calculate the fund balance at the end of the *first* year.

(d) Calculate the employer's quarterly contribution into his fund in his *second* year of work?

Calculator Assumed

6. (e) Calculate the fund balance at the end of the *second* year. Show clearly how you obtained your answer.

(f) The fund balance at the end of the 29th year is \$ 607 354.96.
Calculate the fund balance at the end of the 30th year.
Show clearly how you obtained your answer.

Calculator Assumed

7. [10 marks: 2, 2, 3, 3]

Sheldon opens an investment account with an initial deposit of \$10 000. The account pays interest on the opening monthly balance at a rate of 6% per annum and credited into the account at the end of each month. Each fortnight thereafter Sheldon deposits \$400 into this account.

(a) Calculate the account balance after 30 years.

(b) How much interest after 30 years?

(c) How much extra each fortnight should he contribute for the account balance to exceed \$1 000 000 after 30 years?

(d) Would contributing \$200 each week instead of \$400 each fortnight improve the fund balance at the end of 30 years? Justify your answer.

Calculator Assumed

8. [13 marks: 2, 3, 2, 2, 2, 2]

Leonard has $500 000 in an account that pays interest on the opening monthly balance at the end of each month. Interest is paid at 0.25% per month. Leonard withdraws $1000 at the end of each month after the interest has been paid. The table below shows the monthly account balances.

Month	Opening Balance	Interest for the month	Monthly withdrawal	Closing Balance
1	$500 000.00	$1 250. 00	$1 000	$500 250.00
2	$500 250.00	$1 250.63	$1 000	$500 500.63
3	$500 500.63	$1 251.25	$1 000	$500 751.88
4	$500 751.88	$1 251.88	$1 000	$501 003.76
5	$501 003.76	$1 252.51	$1 000	$501 256.27
6				

(a) Complete the table above for month 6.

(b) Write a recursion relation for the opening balance for each month.

(c) Find the account balance at the start of month 12.

(d) Find the account balance at the end of month 24.

(e) Without further calculations, describe with reasons, the state of the account balances if Leonard were to withdraw $1000 at the start of each month beginning on the start of month 2.

(f) Without further calculations, describe the account balances if Leonard were to withdraw $1250 at the end of each month.

Calculator Assumed

9. [13 marks: 2, 3, 2, 3, 1, 2]

Amy has $800 000 in an annuity that pays interest at 3% on the opening annual balance. Interest is paid at the end of each year. From the end of the first year on, Amy withdraws $40 000 at the end of each year after the interest has been paid. Assume that Amy had sufficient funds for the first year. The table below shows the annual account balances for the first few years.

Year	Opening Balance	Interest for the year	Annual withdrawal	Closing Balance
1	$800 000	$24 000	$40 000	$784 000
2	$784 000	$23 520	$40 000	$767 520
3	$767 520	$23 025.60	$40 000	$750 545.60
4	$750 545.60	$22 516.37	$40 000	$733 061.97
5				

(a) Complete the table above for year 5.

(b) Write a recursion relation for the closing balance for each year.

(c) Find the account balance at the end of 20 years.

(d) How many years will the funds last? Justify your answer.

(e) How much should Amy withdraw each year (after the interest has been paid) if she wishes the account balance to remain constant?

(f) If Amy chooses the option in part (e), what would the funds left in the account be worth after 30 years if the annual inflation rate is 2%.

Calculator Assumed

10. [15 marks: 2, 4, 2, 7]

Raj has $1 500 000 in an annuity that pays interest at 3% on the opening annual balance. Interest is paid at the end of each year. At the end of the each year after the interest is paid, Raj withdraws a certain sum for his living expenses for the following year. At the end of the first year he withdraws $60 000. From then on, the amount he withdraws increases by 5% each year. The table below shows his account balance over several years.

Year	Opening Balance	Interest for the year	Annual withdrawal	Closing Balance
1	$1 500 000	$45 000	$60000	$1 485 000
2	$1 485 000	$44 550	$63000	$1 466 550
3	$1 466 550	$43 996.50	$66150	$1 444 396.50
4	$1 444 396.50	$43331.90	$69457.50	$1 418 270.90
5				

(a) Complete the table below for year 5.

(b) Write a recursion rule to describe the closing balance in for each year.

(c) Determine when the account balance first drops below $500 000.

(d) How much longer would his funds last if he withdrew a constant $60 000 at the end of each year? Show how you obtained your answer.

Calculator Assumed

11. [7 marks: 2, 5]

Howard expects to live a further 25 years after he retires. Consider an annuity that pays interest on the opening balance at a rate of 3.6% per annum with interest paid at the end of the year. Withdrawals are made after the interest is paid. Howard and his wife Bernadette estimate that they require about $60 000 each year.

(a) How much (nearest $10 000) would the couple need to start this annuity, for the funds to last at least 25 years?

(b) Bernadette expects to live a further 10 years after Howard expires. Bernadette estimates that she would require about $40 000 per year. How much would the couple need to start this annuity in order to fund them over their estimated lifespans? Explain how you obtained your answer.

Calculator Assumed

12. [6 marks: 2, 4]

Shane is 25 and has just started a full time job. He expects to work until 65 and live until 85. In his retirement he estimates that he will need about \$80 000 per year. Shane is considering the following financial policies.

- A superannuation policy which pays interest on the opening annual balance at a rate of 5.6% per annum and credited into the account at the end of each year.
- An annuity which pays interest on the opening balance at a rate of 4% per annum with interest paid at the end of the year. Withdrawals are made after the interest is paid.

(a) How much would Shane require to start the annuity to fund him over his expected lifespan? (Give your answer to the nearest \$10 000)

(b) How much would Shane need to contribute annually into the superannuation policy to fund the retirement he requires? Explain how you obtained your answer.

Calculator Assumed

13. [6 marks: 2, 4]

Alexa is 25 and has just started a full time job. She expects to work until 65 and live until 90. In her retirement she estimates that she will need about \$70 000 per year. Alexa is considering the following financial policies.

- An income stream that pays interest on the opening balance at a rate of 3.8% per annum with interest paid at the end of the year. Withdrawals are made after the interest is paid.
- A superannuation policy that pays interest on the opening annual balance at a rate of 6.5% per annum and credited into the account at the end of each year.

(a) How much would Alexa require to start the income stream to fund her over her expected lifespan and yet leave \$1 000 000 to her family? (Give your answer to the nearest \$10 000)

(b) How much would Alexa need to contribute annually into the superannuation policy to fund the retirement she requires and leave \$1 000 000 to her family? Explain how you obtained your answer.

Calculator Assumed

14. [7 marks: 2, 1, 2, 2]

When Wendy retired at 65, she sold her share in a business for \$1 500 000.
At the time of her retirement, women were expected to live a further 25 years.

(a) Option A.
She invests the sum received in a perpetuity which pays interest at a rate of 7.2% per annum calculated monthly. The perpetuity pays her a monthly amount.

(i) Determine the monthly amount she receives.

(ii) How much is left in the perpetuity account after the 300^{th} payment?

(b) Option B
She invests the sum in an annuity that pays interest on the monthly opening balance at a rate of 7.8 % calculated monthly and withdraws \$10 000 each month after the interest has been credited. The annuity is shut down when the balance goes below \$10 000.

(i) How many full \$10 000 withdrawals can Wendy make?

(ii) What would be the balance in the annuity account when it was shut down?

13 Graph Theory I

(Basic terminology, planar graphs, paths and trails)

Calculator Free

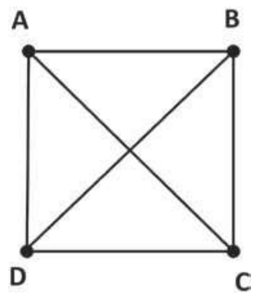
1. [3 marks]

   Complete the table below for the following connected planar graphs.

Number of Vertices, V	Number of Faces, F	Number of Edges, E
7	7	
8		11
	6	12

2. [7 marks: 1, 2, 4]

   The accompanying diagram shows a drawing of a graph with four vertices and six edges.



   (a) Explain clearly why the drawing is non-planar.

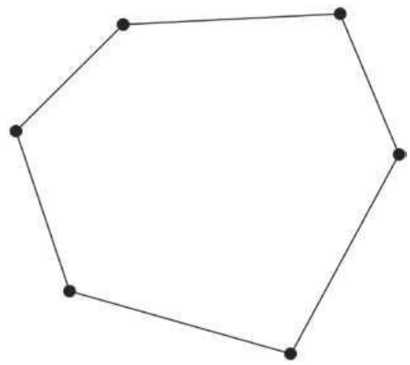
   (b) In the space provided below, provide a planar representation of this graph.

   (c) How many *complete planar* sub-graphs are possible? Name (or draw) the complete planar sub-graphs.

Calculator Free

3. [4 marks: 2, 2]

   The diagram below shows a graph with six vertices and six edges.



   (a) Explain what the term "complete graph" means.

   (b) Add edges to the graph so that the graph becomes a complete graph.

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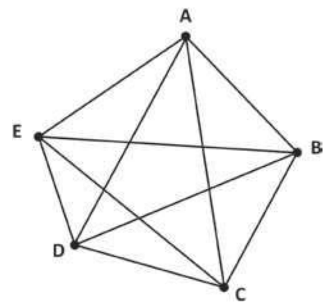
4. [6 marks: 1, 5]

   The accompanying diagram shows a K_n graph.



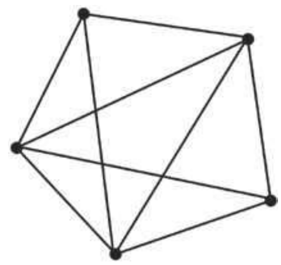
   (a) Determine the value of n.

   (b) Explain with an example, why this K_n graph is not a planar graph.

Calculator Free

5. [6 marks: 1, 2, 3]

(a) Explain why adding an additional edge to the accompanying diagram would make the graph non-planar.



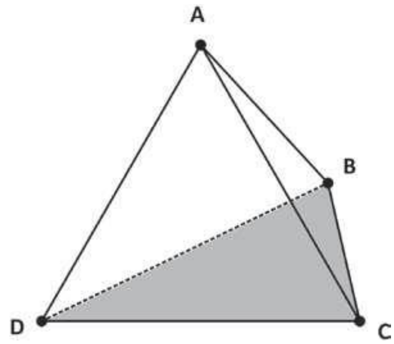
(b) Draw a graph with 5 vertices such that the sum of the degrees of all the vertices is 20.

(c) Draw a planar graph which contains no loops with 5 vertices and 15 edges.

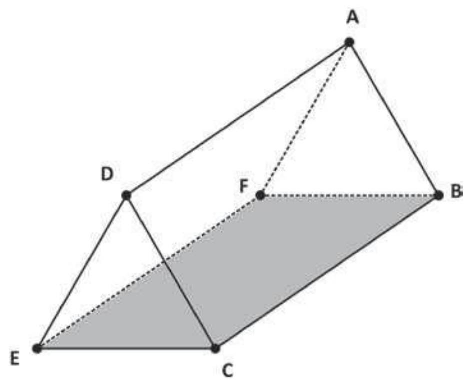
Calculator Free

6. [6 marks: 3, 3]

   (a) ABCD is a triangular pyramid with ΔBCD as its base. Represent the pyramid ABCD as a planar graph.



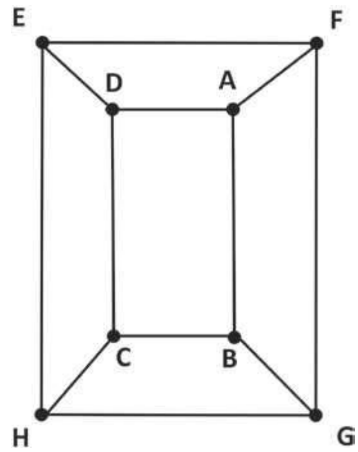
   (b) ABCDEF is a triangular prism with BCEF as its base. Represent the prism ABCDEF as a planar graph.



Calculator Free

7. [13 marks: 3, 2, 2, 3, 3]

The accompanying diagram shows a planar graph.

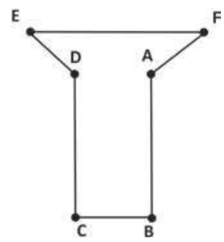


(a) Sketch a three dimensional object which shares this planar graph.

(b) Identify an open Hamiltonian path (semi-Hamiltonian) for this planar graph.

(c) Identify a Hamiltonian cycle (closed Hamiltonian) for this planar graph

(d) Identify and draw a sub-graph consisting of six vertices which forms an Eulerian circuit.



(e) Identify and draw a sub-graph consisting of eight vertices which forms a semi-Eulerian trail.

Calculator Free

8. [11 marks: 1, 2, 2, 3, 3]

(a) Explain what a connected planar graph is.

(b) A connected planar graph has a total of 12 vertices and faces.
How many edges are there in this graph?

(c) The number of edges less the number of vertices in a connected planar graph is 4. How many faces are there in this graph?

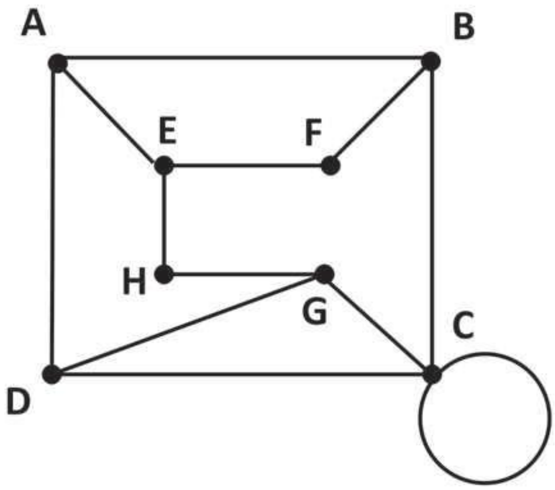
(d) Is it possible to have a connected planar graph with 6 vertices and 4 edges?
Justify your answer.

(e) Draw a connected planar graph with 6 vertices and 8 faces.

Calculator Free

9. [12 marks: 1, 2, 2, 3, 2, 2]

The diagram below shows a graph.



(a) State the degree of vertex C.

(b) Determine with reasons if this graph is a simple graph.

(c) Verify that Euler's rule works for this graph.

(d) Determine with reasons if this graph is traversable.

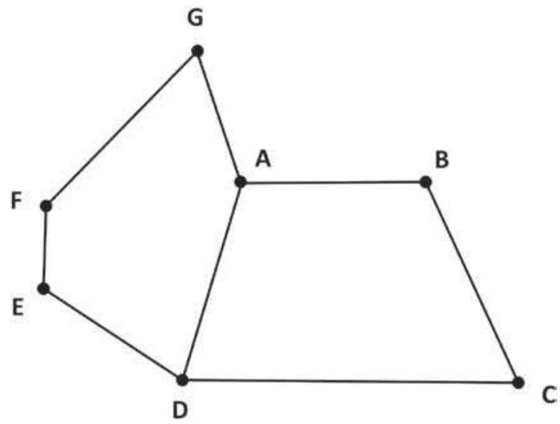
(e) Identify a Hamiltonian cycle for the graph given.

(f) Identify and state the length of the longest trail from C to A.

Calculator Free

10. [10 marks: 2, 3, 3, 2]

The accompanying diagram shows a graph.



(a) Determine with reasons if the edge EF is a bridge.

(b) Determine with reasons if this graph forms a semi-Eulerian trail. If it is, state the semi-Eulerian trail.

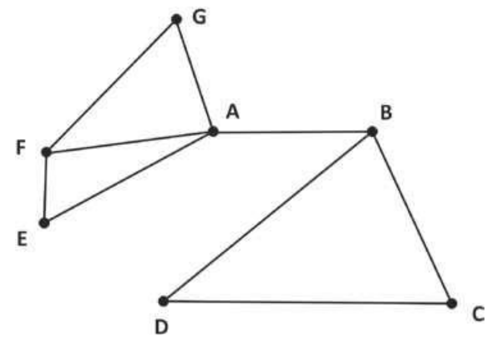
(c) Determine with reasons which edge you would remove so that this graph forms an Eulerian Circuit? Give a possible Eulerian circuit.

(d) In the space provided below, draw the sub-graph A-B-C-D-A. Add edges to this sub-graph so that it becomes a complete planar graph.

Calculator Free

11. [16 marks: 3, 3, 2, 2, 2, 4]

The accompanying diagram shows a graph.



(a) Determine with reasons if the graph is simple.

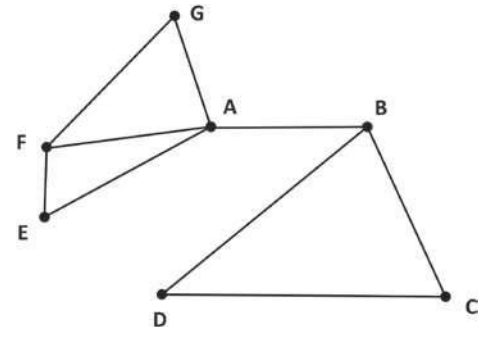
(b) Determine with reasons if the edge AB is a bridge.

(c) This graph does not have a Hamiltonian path. Add an extra edge to this graph so that a Hamiltonian path may be formed. State this path.

(d) Verify that Euler's rule works for this graph.

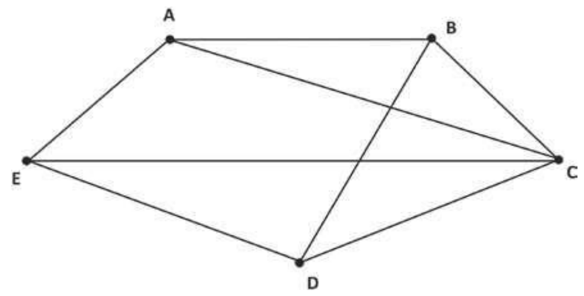
(e) Identify the semi-Eulerian trail in this graph.

(f) Add an additional edge to the graph so that an Eulerian circuit may be formed. Explain the reason for your choice. State this circuit.



Calculator Free

12. [10 marks: 3, 3, 2, 2]



(a) For the graph given above, determine with reasons if the walk ABCAECD is semi-Eulerian (forms an Eulerian Trail).

(b) Explain why the graph given above is neither Eulerian or semi-Eulerian.

(c) State with reasons, an edge that can be removed so that the graph becomes semi-Eulerian.

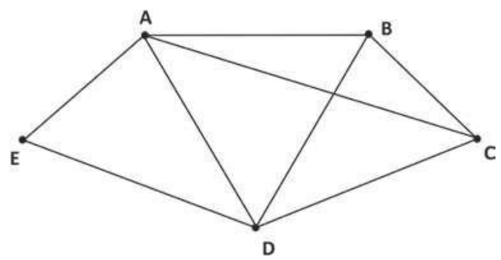
(d) Another vertex F is added to the graph. State with reasons how many additional edges need to be added to make the new graph Eulerian.

Calculator Free

13. [6 marks: 2, 2, 2]

(a) Compare and contrast between a trail and a path.

(b) Consider the accompanying graph.



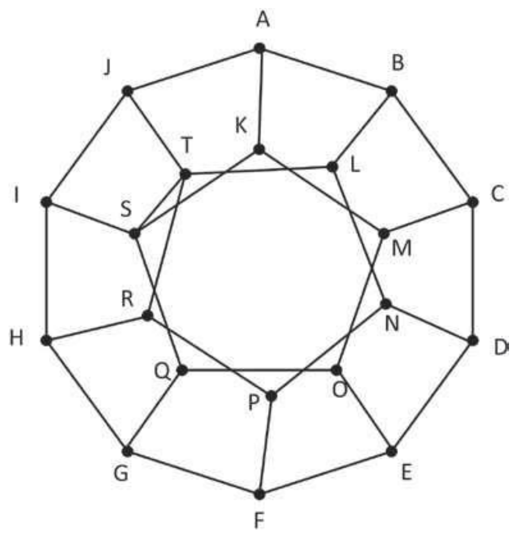
(i) Determine with reasons if the walk ABCDA is a Hamiltonian Cycle.

(ii) Determine with reasons if the walk ABCAEDCA is an Eulerian Trail.

14. [7 marks: 3, 4]

(a) Compare and contrast the features of a cycle and a Hamiltonian cycle.

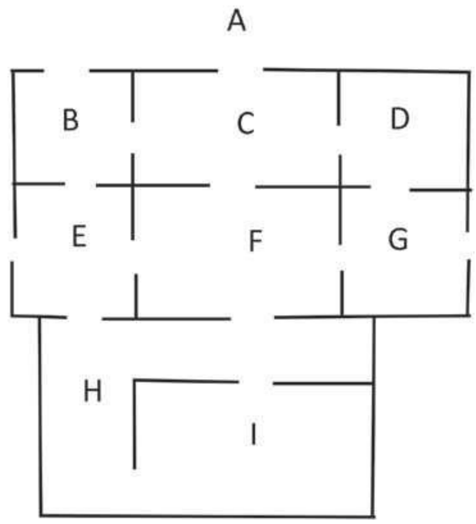
(b) Describe a Hamiltonian cycle for the accompanying graph.



Calculator Free

15. [10 marks: 5, 2, 3]

The accompanying diagram shows a maze divided into regions marked with the letters A to I. The entries into the various regions are drawn as gaps.



(a) Draw a graph to represent this maze, with the entries represented by edges.

(b) Determine with reasons if it is possible to start from A going through each entry exactly once and returning to A.

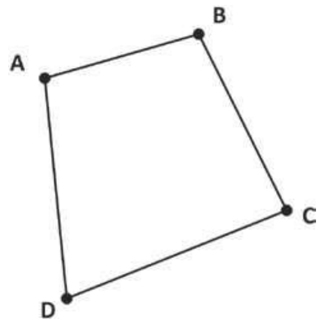
(c) Is it possible to create a trail that passes through each entry exactly once? Give a reason if it is not possible and give the trail if it possible.

14 Graph Theory II (Bipartite graphs & Adjacency matrices)

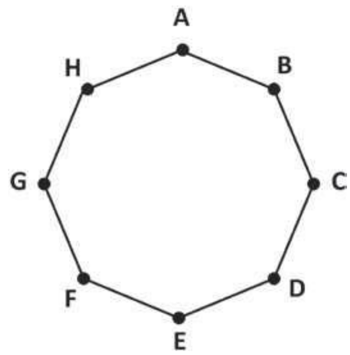
Calculator Free

1. [10 marks: 3, 4, 3]

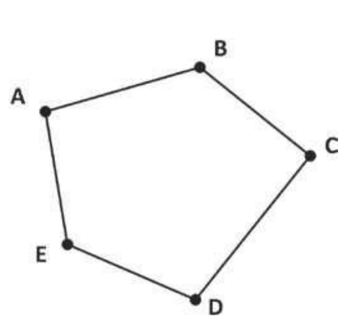
(a) Redraw the accompanying diagram as a bipartite graph.



(b) Redraw the accompanying diagram as a bipartite graph.



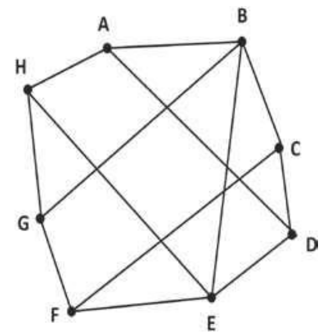
(c) Explain why the graph in the accompanying diagram is not bipartite.



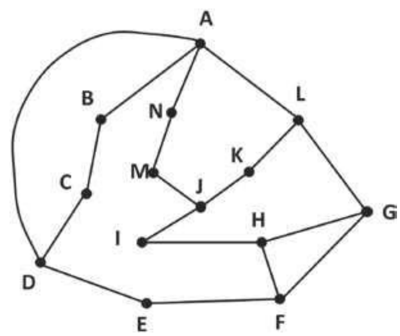
Calculator Free

2. [6 marks: 3, 3]

(a) For the graph drawn in the accompanying diagram, determine if the graph is bipartite or otherwise.
If the graph is not bipartite, explain why.
If the graph is bipartite, redraw the graph identifying the bipartite sets.



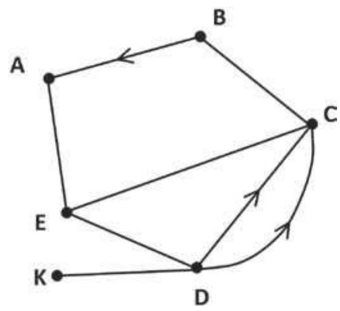
(b) Explain why the graph drawn in the accompanying diagram is not bipartite.
Suggest one way of making this graph bipartite without changing the number of links and vertices.



Calculator Assumed

3. [11 marks: 2, 2, 3, 2, 1, 1]

The accompanying diagram shows a mixed graph.



(a) Identify and state an Eulerian trail.

(b) Identify and state a Hamiltonian path.

(c) State the adjacency matrix **M** for this mixed graph.

(d) Calculate $\mathbf{M}^2$.

(e) Use your answer in (d) to determine:

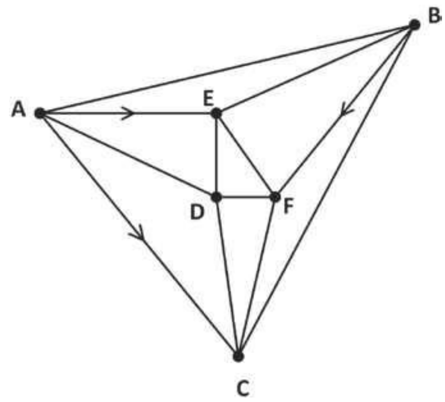
(i) the number of connections between the vertices K and C through another vertex.

(ii) the most number of walks of length 2 between any two vertices.

Calculator Assumed

4. [12 marks: 2, 3, 3, 2, 1, 1]

The accompanying diagram shows a mixed graph.



(a) Identify and state a Hamiltonian cycle.

(b) Identify and state an Eulerian circuit.

(c) State the adjacency matrix **M** for this mixed graph.

(d) Calculate $\mathbf{M}^2$.

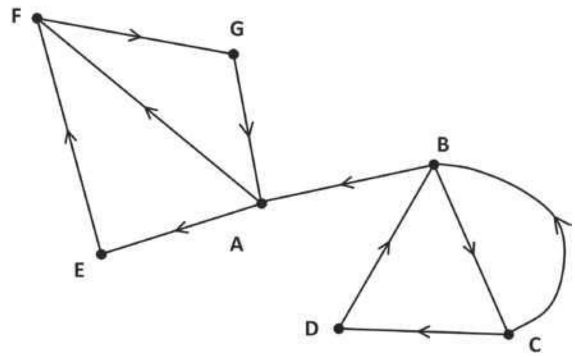
(e) Use your answer in (d) to determine:
(i) the number of ways B can be reached from D by two stage walks

(ii) vertices that *cannot* reach A with a two stage walk

Calculator Assumed

5. [12 marks: 2, 1, 2, 2, 2, 3]

The accompanying diagram shows a directed graph.



(a) Determine with reasons if this graph is a simple directed graph.

(b) Identify and state a Hamiltonian path.

(c) Identify and state a Eulerian trail.

(d) Show that Euler's Formula works for this directed graph.

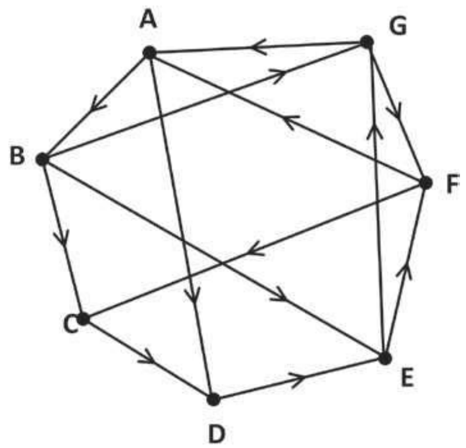
(e) State the adjacency matrix **M** for this directed graph.

(f) Which vertex/vertices can be reached from E by a three-stage walk? Explain how you obtained your answer.

Calculator Assumed

6. [14 marks: 3, 2, 4, 3, 2]

The accompanying diagram shows a directed graph.



(a) Describe all walks from A to C that are of length 4.

(b) Identify and state a Hamiltonian cycle.

(c) An Eulerian trial may be formed if the direction of one of the edges is reversed. Identify this edge and state the Eulerian trial.

(d) State the adjacency matrix M for this digraph.

(e) Which vertex/vertices cannot be reached from vertex C by four-stage walks?

Calculator Assumed

7. [9 marks: 3, 3, 3]

The adjacency matrix of a planar graph is given by

$$M = \text{From} \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \text{To} \\ \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

(a) Draw the planar graph.

(b) Draw a three dimensional solid that is represented by this planar graph.

(c) How many more walks of length four than walks of length three are there between A and F. Explain how you obtained your answer.

Calculator Assumed

8. [8 marks: 3, 3, 2]

The adjacency matrix of a graph is given by

$$M = \text{From} \quad \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \text{To} \\ \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

(a) Draw this graph.

(b) Represent this graph in bipartite form.

(c) Without any calculations, explain why there are no walks of lengths more than one?

Calculator Assumed

9. [9 marks: 2, 2, 2, 3]

Consider the following adjacency matrix M =
$$\begin{array}{cc} & \text{To} \\ & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \text{From} \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{pmatrix} 0 & 2 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}.$$

For this adjacency matrix, a loop counts as one edge.

(a) Determine with reasons if M represents a simple graph.

(b) Determine with reasons if M represents a directed or an undirected graph.

(c) Determine with reasons if M represents a connected graph.

(d) For an undirected graph:

$2 \times$ Number of Edges = Sum of elements in M + Number of loops.

For a directed graph:

Number of Edges = Sum of elements in M

Determine the number of edges and hence the number of faces in the graph represented by this matrix, given that the graph is planar.

Calculator Assumed

10. [10 marks: 2, 3, 2, 3]

Consider the following adjacency matrix.

$$M = \text{From} \quad \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \\ J \\ K \end{array} \begin{array}{c} \text{To} \\ \begin{array}{cccccccccc} A & B & C & D & E & F & G & H & J & K \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

(a) Determine with reasons if the graph it represents is a simple graph.

(b) Determine with reasons if the graph is directed or undirected.

(c) Identify all the odd vertices (in-degree and out-degree) in this graph.

(d) For an undirected graph:

$2 \times$ Number of Edges = Sum of elements in M + Number of loops.

For a directed graph:

Number of Edges = Sum of elements in M

Determine the number of edges and hence the number of faces in the graph represented by this matrix, given that the graph is connected planar graph.

Calculator Assumed

11. [17 marks: 4, 3, 3, 4, 3]

Teams A, B, C, D and E are in Group 1 of a soccer competition where they play each other exactly once. The results of the games are:

- A defeated B, C and E
- B defeated D
- C defeated B and E
- D defeated A and C
- E defeated B and D.

(a) Draw a digraph that represents the results of these matches where a directed edge points from the winning team to the losing team.

(b) The digraph in (b) can be expressed as an adjacency matrix **M**. The element "1" is to indicate a win and the element "0" to indicate a loss. Complete the matrix M

$$\begin{array}{cc} & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 & \square \\ 0 & 0 & \square & \square & \square \\ 0 & \square & 0 & \square & \square \\ \square & \square & \square & 0 & \square \\ 0 & \square & \square & \square & 0 \end{pmatrix} \end{array}$$

Calculator Assumed

11. (c) The elements in $\mathbf{M}^2$ represents the number of "two-stage wins" between the teams. An incomplete $\mathbf{M}^2$ is shown below. The element in row 1 column 2 indicates that A has 2 "two-stage wins" against B: A beat E who beat B and A beat C who beat B. Complete the matrix $\mathbf{M}^2$ below.

$$\mathbf{M}^2 = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} 0 & 2 & 0 & 2 & \square \\ 1 & 0 & \square & \square & \square \\ 0 & \square & 0 & \square & \square \\ \square & \square & \square & 0 & \square \\ 1 & \square & \square & \square & 0 \end{pmatrix} \end{array}$$

(d) Calculate $\mathbf{M} + \mathbf{M}^2$. Then multiply your answer with the column matrix $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

The resulting matrix gives the total number of direct wins and "two-stage wins" for each team. Use this answer to rank the teams.

(e) Calculate $\mathbf{M} + \mathbf{M}^2 + \mathbf{M}^3$. Then multiply your answer with $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

Use this result to rank the teams.

Calculator Assumed

12. [8 marks: 3, 3, 2]

A, B, C, D, E and F meet at a party and each shakes the other's hands.

(a) Draw a graph that represents the "handshakes" between these six people. [An edge connecting two vertices represents two persons sharing a handshake.]

(b) Due to prior disagreements, A, C and E refuse to shake hands with each other. Similarly B, D and F refuse to shake hands with each other. Draw a bipartite graph to represent the handshakes between these six persons.

(c) Write an adjacency matrix for the graph you drew in (b).

Calculator Assumed

13. [13 marks: 3, 4, 1, 3, 2]

Bus route P runs from A to G to F to E and back to A.
Bus route Q runs from A to B to G to D to E and back to A.
Bus route R runs from B to G to F to C to B.
Bus route S runs from B to D to C and back to B.

(a) Construct a digraph that conveys these routes.

(b) Identify and state a Hamiltonian cycle. Describe the sequence of bus routes required to achieve the Hamiltonian cycle.

(c) The digraph is not Eulerian as it has more than two odd vertices. Identify all the odd vertices in this diagraph.

(d) Which edge would you remove to make this digraph Eulerian? State the Eulerian circuit.

(e) How would you reorganise the bus routes to achieve your answer in (d).

15 Shortest Paths

Calculator Assumed

1. [9 marks: 3, 3, 3]

The table below shows the distances (km) between the stated towns.

	A	B	C	D
A	–	20	15	50
B	20	–	10	20
C	15	10	–	15
D	50	20	15	–

(a) Represent the data shown in the table above as a clearly labelled weighted planar graph.

(b) Describe the shortest Hamiltonian cycle in this graph and state its length.

(c) Which link would you remove so that the remaining graph contains the shortest possible semi-Eulerian trail? Describe this trail and state its length.

Calculator Assumed

2. [8 marks: 4, 1, 3]

The table below shows the distances (km) between the stated towns.

	A	B	C	D	E	F
A	–	15	25	40	16	30
B	15	–	–	50	12	24
C	25	–	–	30	8	–
D	40	50	30	–	17	–
E	16	12	8	17	–	–
F	30	24	–	–	–	–

(a) Draw a clearly labelled weighted graph to represent the data shown in the above table.

(b) Which town is the most "isolated" town? Why?

(c) Identify the shortest Hamiltonian path and state its length.

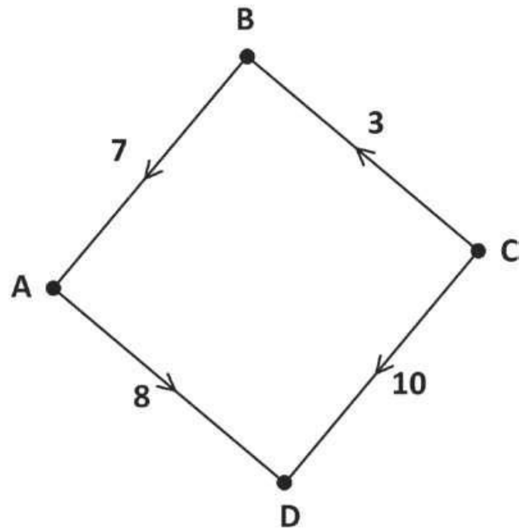
Calculator Assumed

3. [6 marks: 3, 1, 2]

The following table show the times taken to travel between the buildings A, B, C and D in *minutes*. Note the times taken are not symmetrical due to road conditions and one-way streets. For example, the time taken to travel from A to B is 6 minutes but the time taken to travel from B to A is 7 minutes.

		TO			
		A	B	C	D
FROM	A	–	6	3	8
	B	7	–	4	–
	C	1	3	–	10
	D	6	–	4	–

(a) Complete the weighted directed graph below to show the travelling times between these buildings. Label your diagram carefully.



(b) State the mathematical term used to describe a walk that starts and finishes at the same building visiting each building exactly once except for the start/finish building.

(c) State clearly the quickest walk starting from A visiting all the three other buildings and finishing back at A. State the time taken.

Calculator Assumed

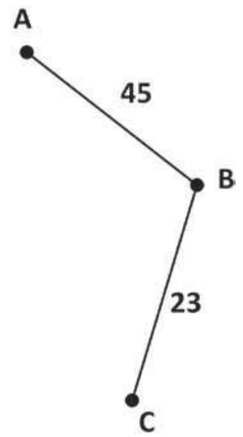
4. [8 marks: 2, 4, 2]

Towns A, B, C, D and E lie along the same stretch of a long highway in this order. The following table shows the *cumulative* distances in km between these towns. For example the shaded cell indicates that the distance between A and D passing through B and C in turn is 85 km.

A				
45	B			
68	23	C		
85	40	17	D	
94	49	26	k	E

(a) Calculate the value of k.

(b) Complete the weighted graph below, which shows the road connections between these town.

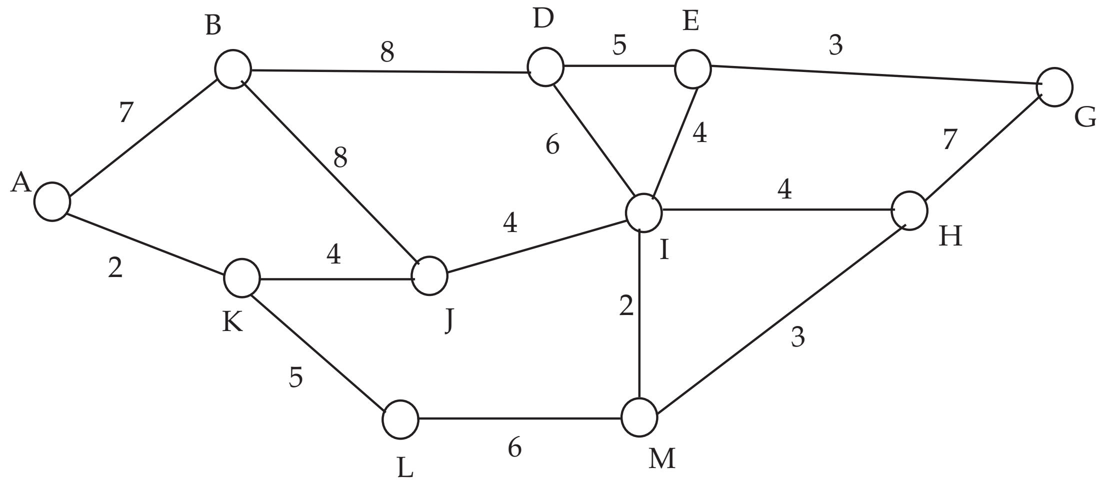


(c) If it was possible to join E to C by one new road, providing it was shorter than the current distance between E and C, what should its length be?

Calculator Assumed

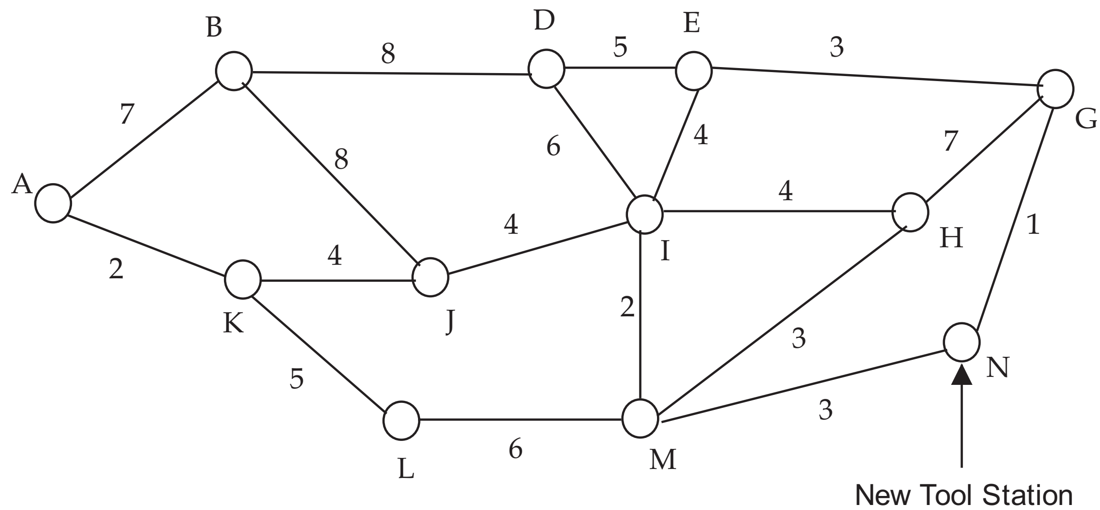
5. [6 marks: 3, 3]

The weighted graph below represents the passage-ways and their lengths (metres) between several work stations. No other passage-ways connect these work stations.



(a) Find the shortest walk between A and G. Give the length of this walk.

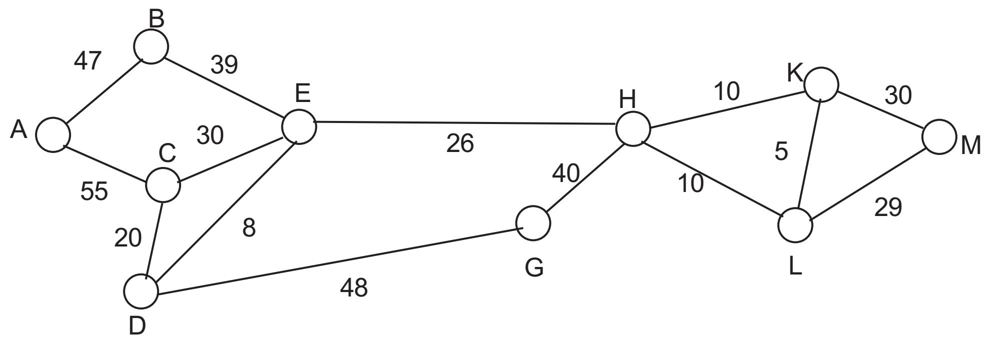
(b) A new tool station is located as shown. Discuss the impact of this new tool station on the minimum distance between A and G.



Calculator Assumed

6. [10 marks: 3, 2, 2, 3]

The weighted graph below shows a road network between several towns. Distances are shown alongside the edges and are in km.



(a) Find the shortest path from A to M. Give this distance.

(b) Find the shortest path from A to M via E <u>and</u> K. Give this distance.

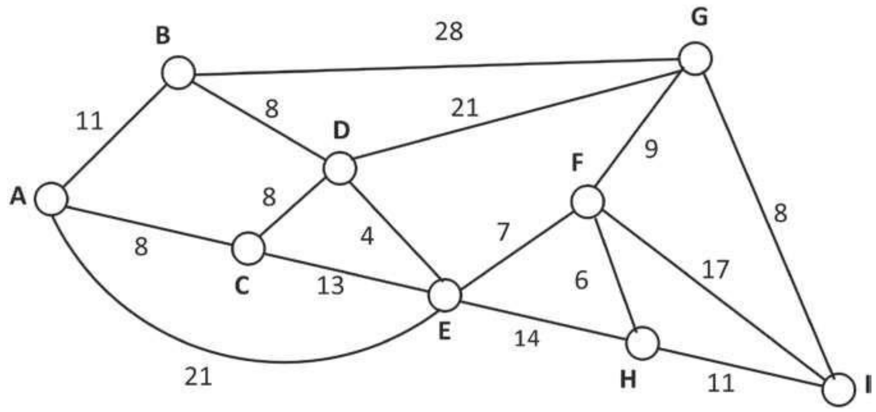
(c) Find the shortest path from A to M via E <u>or</u> K. Give this distance.

(d) Find the shortest path starting from A and finishing at A visiting each town at least once. Give this distance.

Calculator Assumed

7. [11 marks: 4, 3, 4]

In the graph given below, the weights located on the edges indicate the road distances in km between the towns listed.



(a) Determine all possible shortest routes between A and I and state the distance.

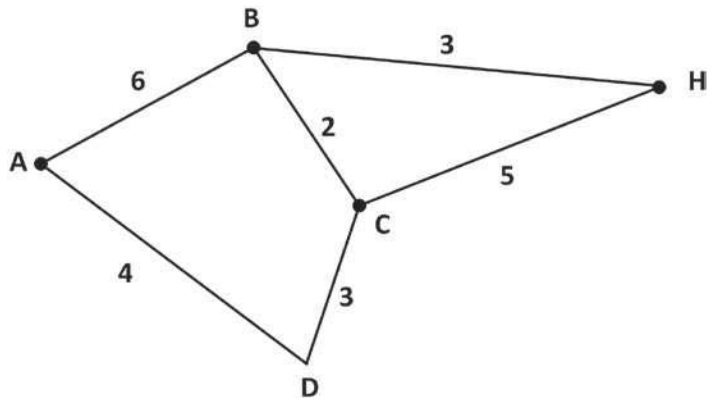
(b) Town F is flooded and all roads in and out of the town are closed to traffic. Determine the impact this has on the shortest route between A and I.

(c) A new road with distance d km is created between D and H. What should the distance of this road be if it is to have any impact on the shortest route between A and I? Justify your answer.

Calculator Assumed

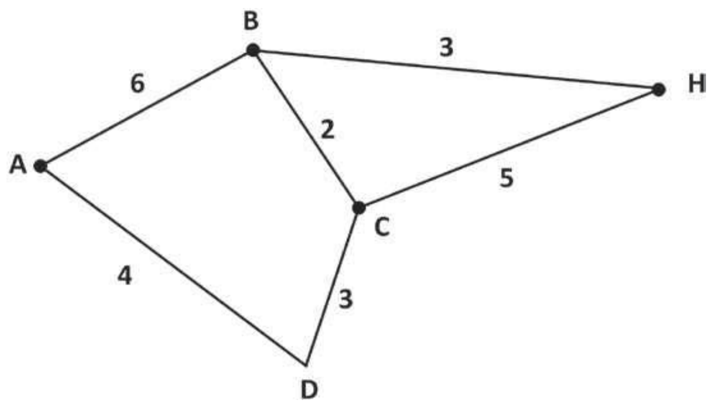
8. [8 marks: 4, 4]

The diagram below shows the gravel tracks and distances (in km) between the paddock gates A, B, C, D and H.



(a) Each evening Kaylah needs to visit every *gate* (at least once) to make sure that each gate is shut for the night.
Describe the *shortest* route Kaylah should take if she starts from and finishes at gate H. Give the distance of this route.

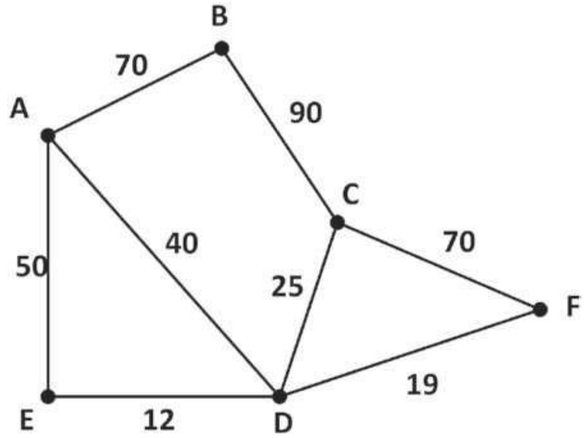
(b) Each morning Jasmin needs to travel along each gravel track in this network (at least once) to check that the fences are in order.
Describe the *shortest* route Jasmin should take if she starts from and finishes at H. Give the distance of this route.



Calculator Assumed

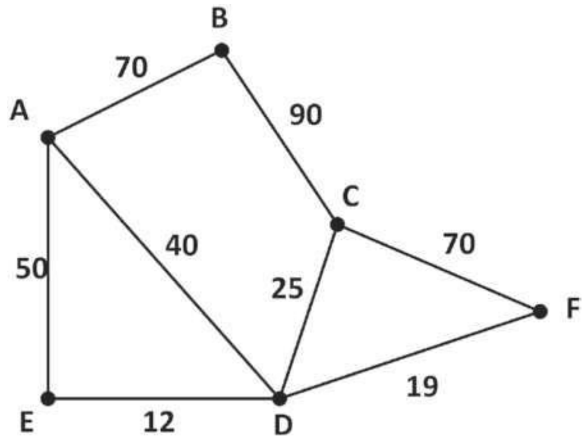
9. [10 marks: 2, 4, 4]

The diagram below shows the links and distances (in km) between the towns A, B, C, D, E and F.



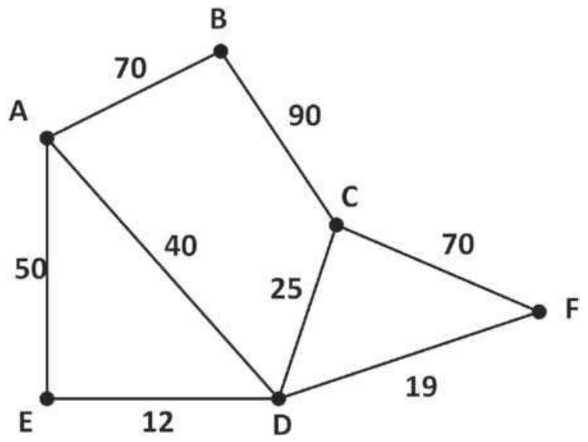
(a) State the shortest path from A to F and give this shortest distance.

(b) John needs to visit every *town* in this network (at least once). Describe the *shortest* route John should take if he starts from and finishes at A.
Give the distance of this route.



Calculator Assumed

9. (c) Kim needs to travel along every edge in this network (at least once). Describe the *shortest* route Kim should take if he starts from and finishes at A.
Give the distance of this route.

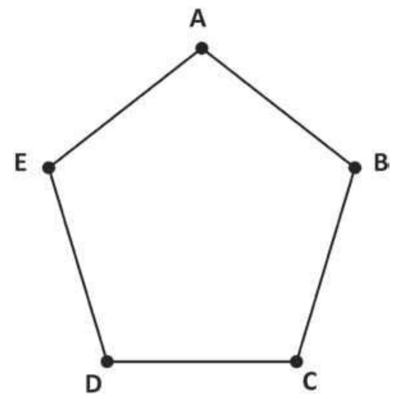


16 Trees

Calculator Free

1. [7 marks: 2, 3, 2]

The accompanying diagram shows a graph formed by the edges of a pentagon.



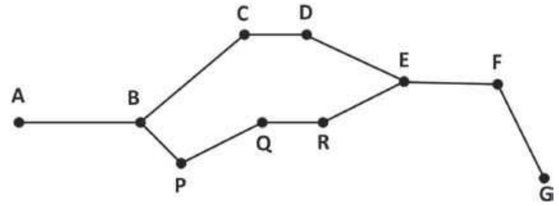
(a) How many sub-graphs that are trees of length two are there within this graph? Name these trees.

(b) How many sub-graphs that are trees of length of more than two are there within this graph? Name these trees.

(c) How many sub-graphs that are trees of length at least 2 are there in a graph formed by the edges of a decagon (10-sided polygon)?

2. [4 marks]

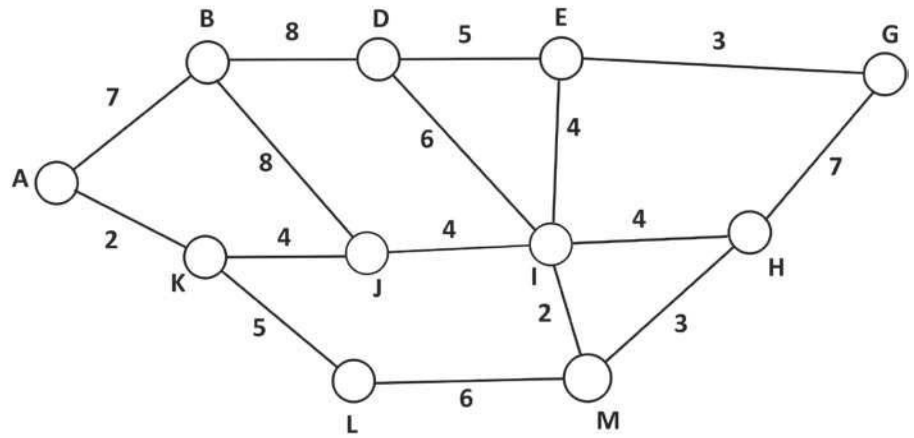
Explain why the graph shown in the accompanying diagram is not a tree. Explain how you would turn this graph into a tree. Draw the graph of the tree in the space below.



Calculator Assumed

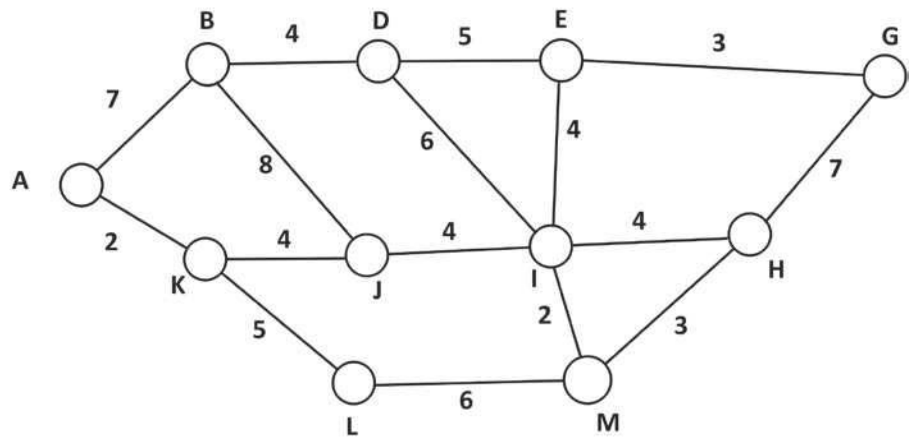
3. [6 marks: 3, 3]

The network below shows the distances (metres) between the various tool stations in a workshop.



(a) Draw in the minimum spanning tree for this network in the diagram above. State the minimum distance.

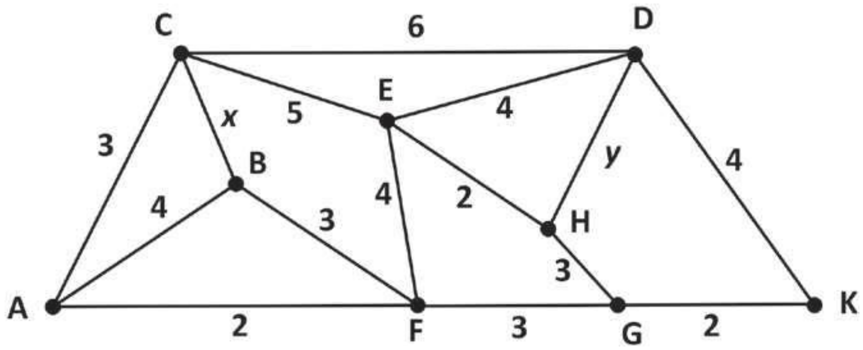
(b) A new walk-way of length 4 metres is built between B and D. Determine the effect this new walk-way will have on the minimum spanning tree and the minimum spanning distance.



Calculator Assumed

4. [9 marks: 4, 2, 3]

The diagram below shows the cost (in thousands of dollars) of connecting several laboratories in a research facility with new high speed data transmission cables.



(a) For $x = y = 2$, indicate on the given diagram, the cheapest way of connecting all the laboratories in the research facility. State this cost.

(b) For $x = 2$, find the range of values for y if there is to be only one possible way to connect these laboratories with minimum cost.

(c) For $y = 2$, find the value(s) of x if there is to be more than one possible way to connect these laboratories with minimum cost. State the various possibilities.

Calculator Assumed

5. [9 marks: 2, 3, 4]

A Peer-to-Peer computer network is planned for Carl's home. The table below shows the cost (in $) of installing the data cables between 5 terminals in Carl's home.

	A	B	C	D	E
A	–	20	14	x	24
B	20	–	18	16	8
C	14	18	–	32	28
D	x	16	32	–	19
E	24	8	28	19	–

(a) Display this information as a weighted graph that displays the terminals and the cost of each connection.

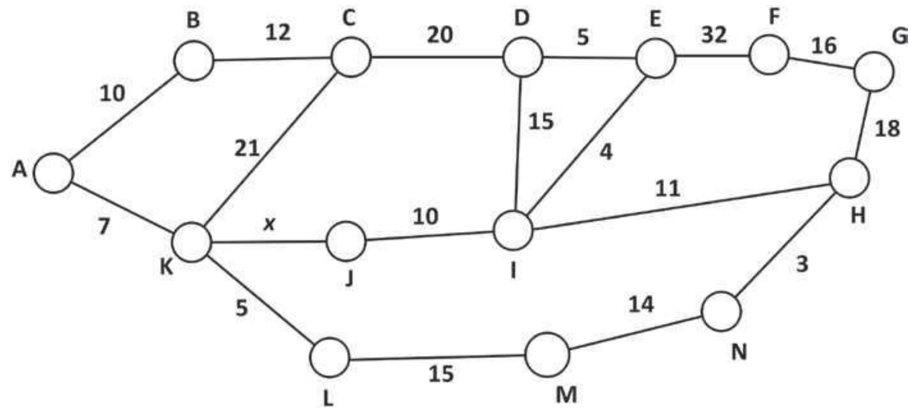
(b) For x = 40, identify the minimal spanning tree for this graph and state the minimum cost.

(c) Find the largest possible value of x if AD is to be part of the minimal spanning tree. State the minimum spanning tree and the minimum cost

Calculator Assumed

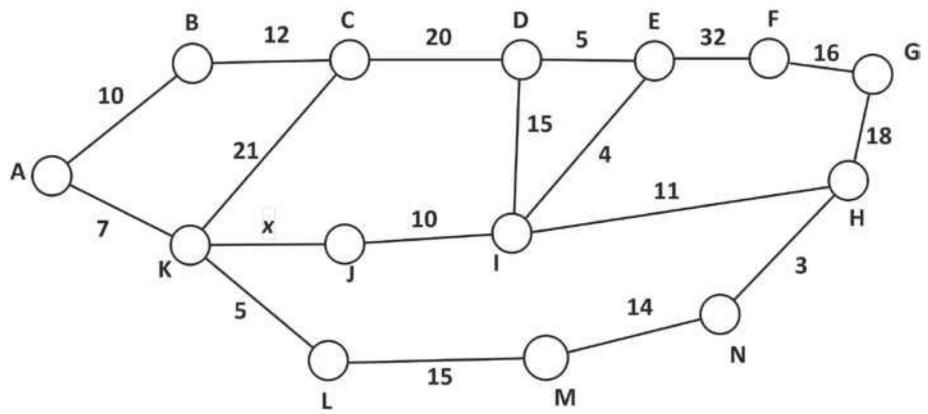
6. [12 marks: 4, 4, 4,]

The network below shows distances (in km) between the various towns in the north east of a state.



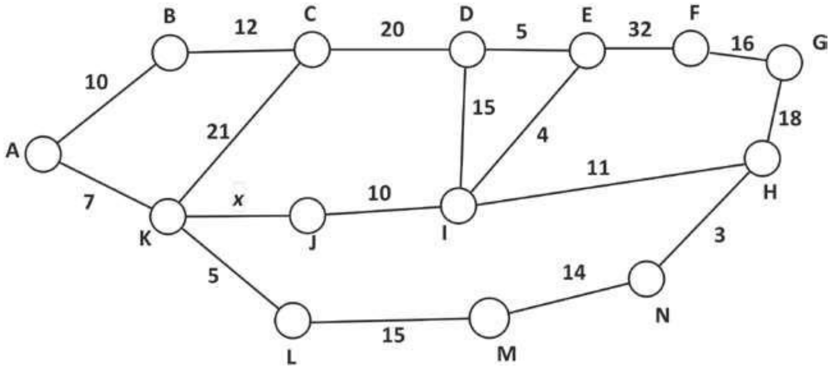
(a) Given that $x = 10$, draw in the minimum spanning tree for this network in the diagram above. Explain what the minimum spanning tree means in the context of this network.

(b) For $x = 10$, find the shortest route between A and G *passing through* I. Give the length of this route. Explain what this shortest route means in the context of this question.



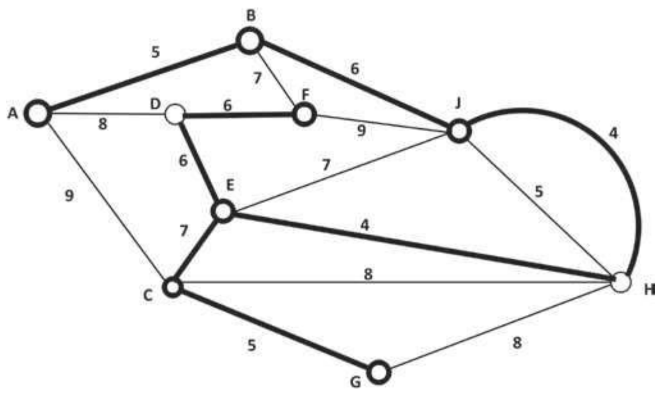
Calculator Assumed

6. (c) If the shortest path between A and G is AKJIHG, find the value(s) of x.



7. [6 marks: 2, 4]

The accompanying diagram shows the minimum spanning tree drawn (drawn in bold) onto a network showing the distances in hundreds of metres between several buildings. Prim's algorithm was used to determine the minimum spanning tree starting at building E.



(a) Explain with reasons, which building is the next to be connected to E.

(b) Determine with reasons, which building is the last in the sequence of buildings to be connected?

Calculator Assumed

8. [7 marks: 2, 3, 2]

A high-speed fibre optic network links 6 computer terminals within an office. The distances between the 6 terminals (in metres), A, B, C, D, E, and F are shown in the table provided below.

	A	B	C	D	E	F
A	–	13	14	9	6	11
B	13	–	9	7	16	13
C	14	9	–	10	14	9
D	9	7	10	–	15	8
E	6	16	14	15	–	8
F	11	13	9	8	8	–

(a) Find the shortest possible amount of cable which must be used to link these terminals.

(b) An electrician requires a diagram of the shortest network which is planned. Draw such a diagram.

(c) If it were necessary to increase the distance from D to B by 3 metres in order to run the cable around Alana's new reception desk, what would be the shortest amount of cable needed to span this network?

Calculator Assumed

9. [9 marks: 2, 4, 3]

The table below shows the distances (km) between several cities.

	A	B	C	D	E	F	G	H
A	–	110	150	70	20	300	220	510
B	110	–	30	25	80	260	120	450
C	150	30	–	40	60	80	200	300
D	70	25	40	–	10	250	140	210
E	20	80	60	10	–	310	150	420
F	300	260	80	250	310	–	400	200
G	220	120	200	140	150	400	–	300
H	510	450	300	210	420	200	300	–

(a) Using Prim's Algorithm to determine the minimum spanning tree, starting from town A, what town should be connected to A? Why?

(b) Use Prim's Algorithm to determine the minimum spanning tree for this network. State the minimum distance.

(c) State the minimum spanning tree and the minimum distance if the edge DG must be within this tree.

Calculator Assumed

10. [9 marks: 4, 1, 4]

The table below shows the distances (km) between several cities.

	A	B	C	D	E	F	G	H
A		75	33	23	92	96	50	33
B	75		32	99	56	30	61	20
C	33	32		55	56	35	98	93
D	23	99	55		69	41	68	57
E	92	56	56	69		92	38	98
F	96	30	35	41	92		71	21
G	50	61	98	68	38	71		67
H	33	20	93	57	98	21	67	

(a) Use Prim's algorithm to determine the minimum spanning tree and state its length.

(b) Using Prim's algorithm, starting at vertex H, what is the last town to be connected to the tree.

(c) State the minimum spanning tree and the minimum distance if the edge BH cannot be within this tree.

Calculator Assumed

11. [9 marks: 4, 3, 2]

The distances (in kilometres) between the various parks A, B, C, D, E, and F are shown in the table provided below.

	A	B	C	D	E	F	G	H
A	–	8	6	14	2	5	7	6
B	8	–	11	5	9	18	7	3
C	6	11	–	7	5	12	10	8
D	14	5	7	–	6	4	15	12
E	2	9	5	6	–	7	6	9
F	5	18	12	4	7	–	8	6
G	7	7	10	15	6	8	–	11
H	6	3	8	12	9	6	11	–

(a) State the minimum spanning tree and state its length.

(b) If town F is removed from the network, state with reasons, the length of the new minimum spanning tree.

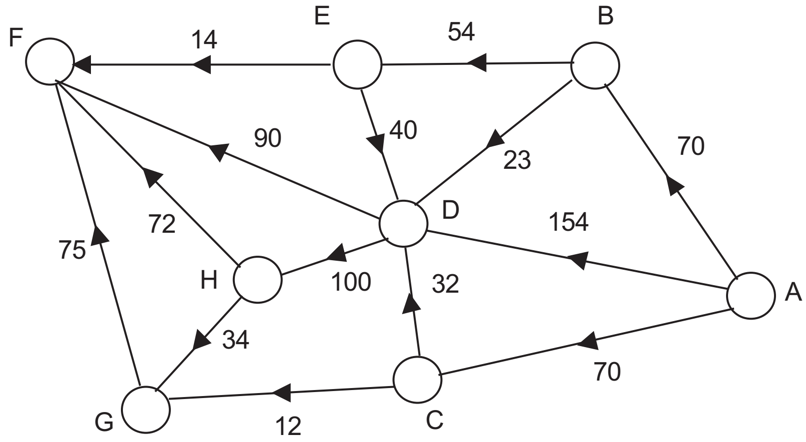
(c) Referring back to the original network. A new park is built at N.
The distances (km) to N from A, B, C, D, E, F, G and H are respectively
14, 8, 7, 16, 10, 9, 11 and 9.
State with reasons the length of the new minimum spanning tree.

17 Flow Networks

Calculator Assumed

1. [10 marks: 1, 4, 2, 3]

The following diagram shows a system of roads in a busy commercial business district with the corresponding capacities (in vehicles per 5 minutes).



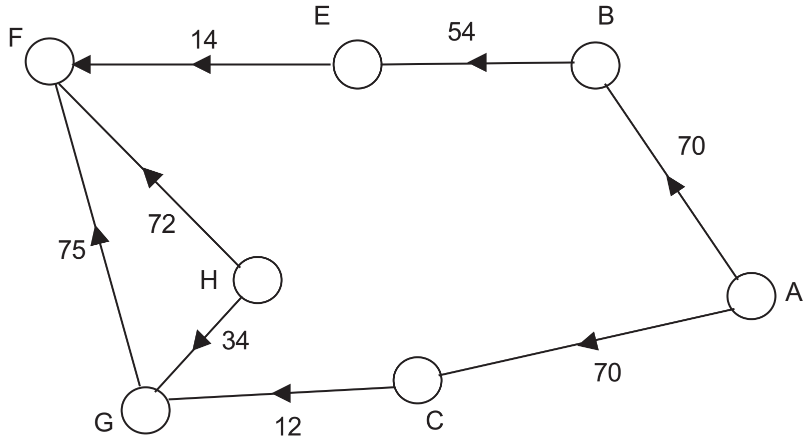
(a) State the source and sink.

(b) Calculate the maximum flow for this network. Show sufficient working for your answer to be checked.

(c) In the diagram above, draw the cut that corresponds to the maximum flow.

Calculator Assumed

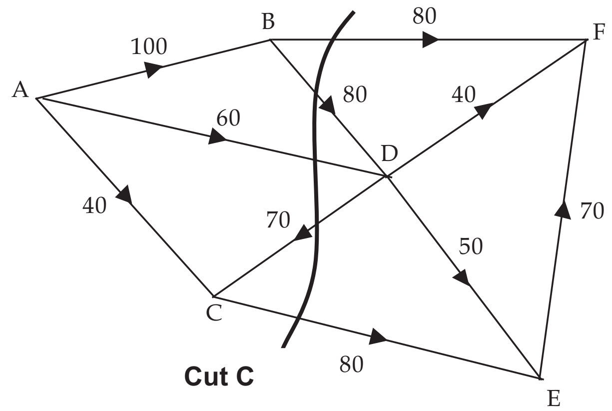
1. (d) The traffic network is heavily dependent on node D. If node D were to be made non-functional, then there would be severe traffic disruptions. Assume that there was a severe storm and node D was destroyed by the storm. Assume that all other nodes sustained minimal damage. Suggest a new link that could be created linking two other existing nodes (other than the source and the sink) that would best relieve the traffic flow. What should the maximum capacity of this link be?



Calculator Assumed

2. [9 marks: 1, 1, 3, 2, 2]

The map below shows 6 buildings A, B, C, D. E and F which are connected by one-way streets. The arrows show the direction of the flow of traffic. The number on each link refers to the capacity of the link (in number of vehicles per minute).



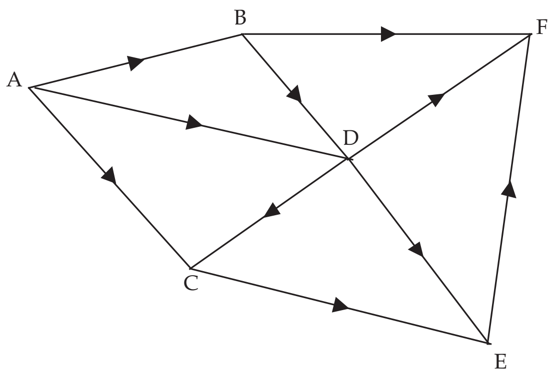
(a) Determine the "source" and the "sink" for this network.

(b) Determine the capacity of the cut C drawn in the diagram above.

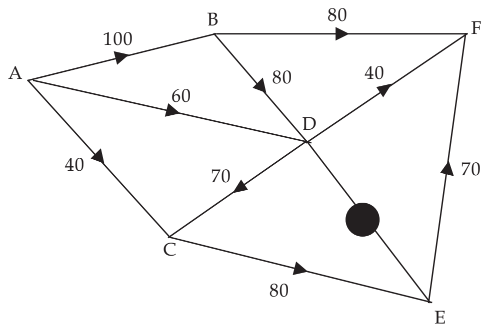
(c) Determine the maximum flow for this traffic network. Justify your answer.

Calculator Assumed

2. (d) Indicate the traffic flow that corresponds to the maximum flow in the diagram given below.



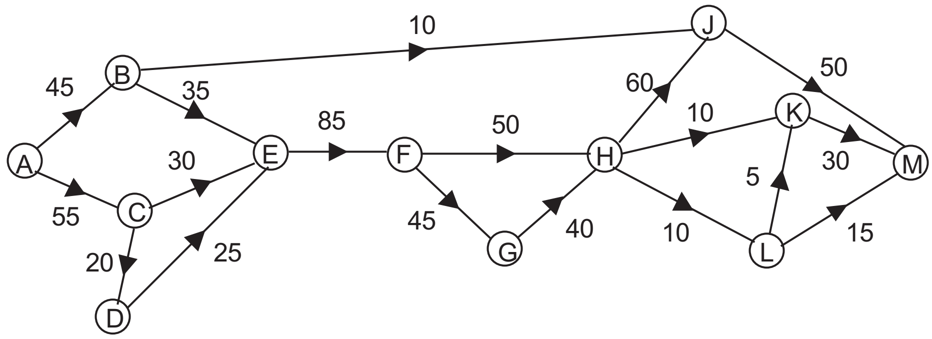
(e) An accident occurs on the road between D and E and is temporarily closed to traffic. By how much would the maximum traffic flow change?



Calculator Assumed

3. [12 marks: 1, 2, 3, 3, 3]

The diagram below shows a communications network. The number accompanying each link represents the capacity of the link in Gigabits per second.



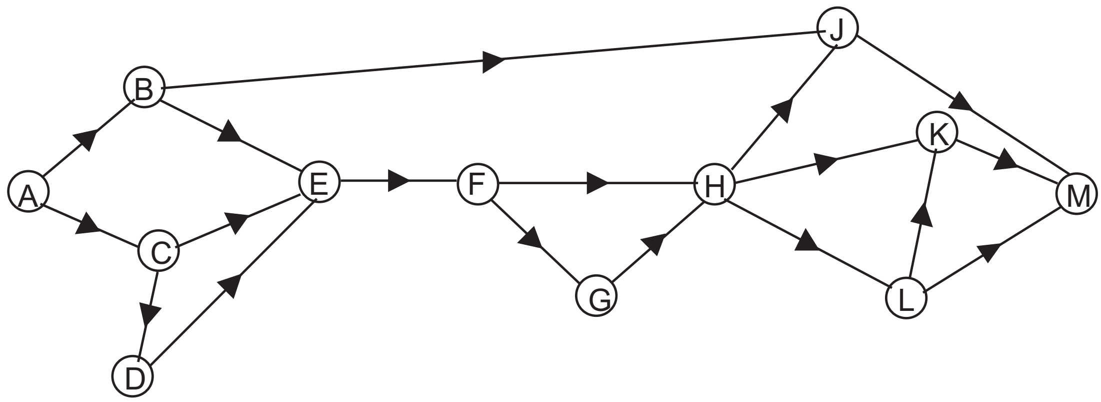
(a) Explain clearly why E is not the sink for this network.

(b) Determine with reasons the maximum possible capacity for the path ABEFHJM.

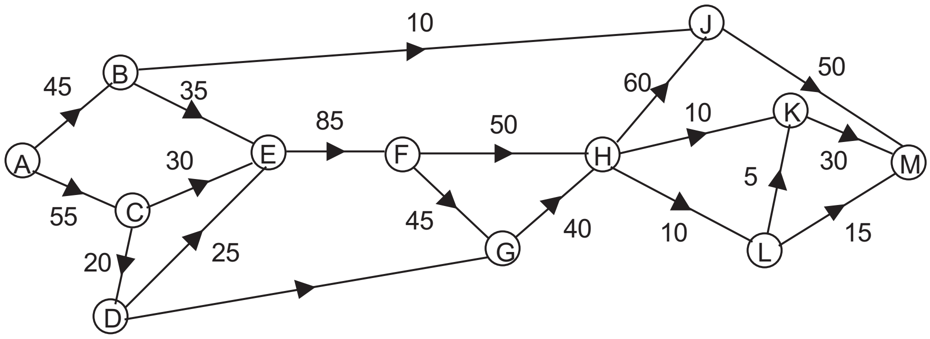
(c) Find the maximum flow between A and M.
To obtain full marks you need to state the flows for the relevant paths.

Calculator Assumed

3. (d) In the diagram below, indicate clearly how the maximum flow between A and M as in question 3(c) can be achieved.



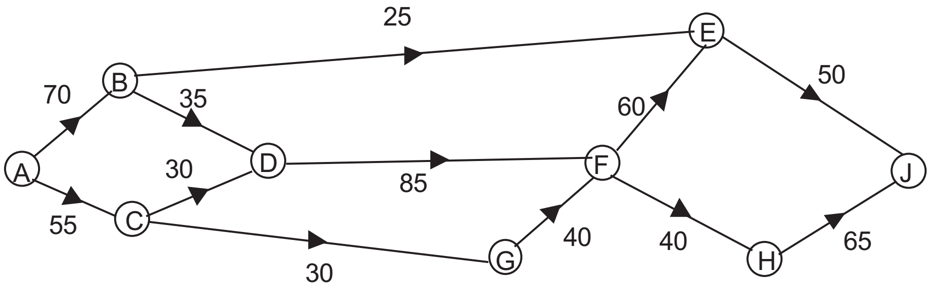
(e) A new direct link between D and G is developed.
Describe clearly how this will affect the maximum flow between A and M.



Calculator Assumed

4. [12 marks: 1, 3, 3, 2, 1, 2]

The diagram below shows the maximum capacity (vehicles per hour) of a road network.



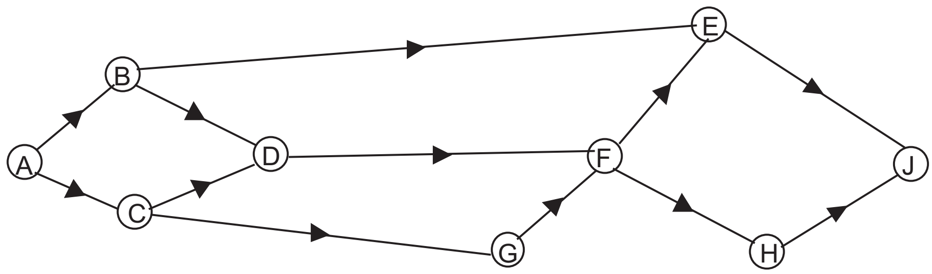
(a) Determine the maximum number of vehicles that the road network is capable of conveying (per hour) from A to D.

(b) Will the maximum number of vehicles coming into D from A be able to travel from D to F without being held-up? Explain.

(c) Find the maximum flow of this network.
State clearly the flow in each possible path.

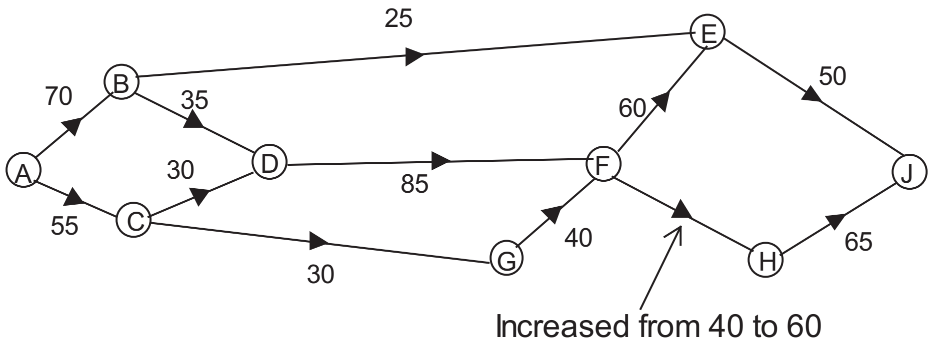
Calculator Assumed

4. (d) Indicate in the diagram below, the unused capacity when maximum flow is achieved.



(e) Which link(s) has the most waste in terms of carrying capacity?

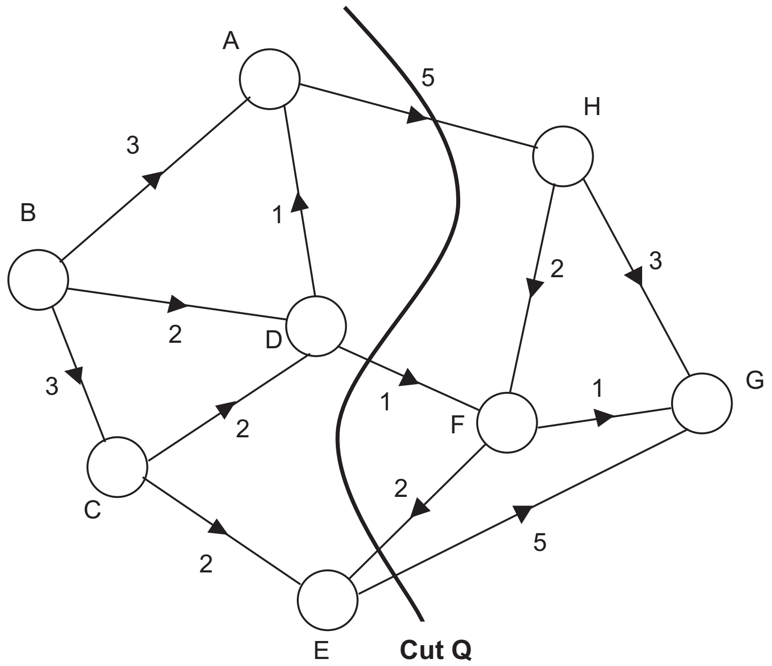
(f) State the increase in the maximum flow from A to J if the flow capacity between F and H were increased from 40 to 60. Justify your answer.



Calculator Assumed

5. [8 marks: 1, 1, 3, 3]

The diagram below shows the traffic system of the Central Business District of a City. The nodes represent signal lights at the corresponding junctions. The numbers on the links represent the maximum number of cars (× 10) capable of passing though each junction each minute.



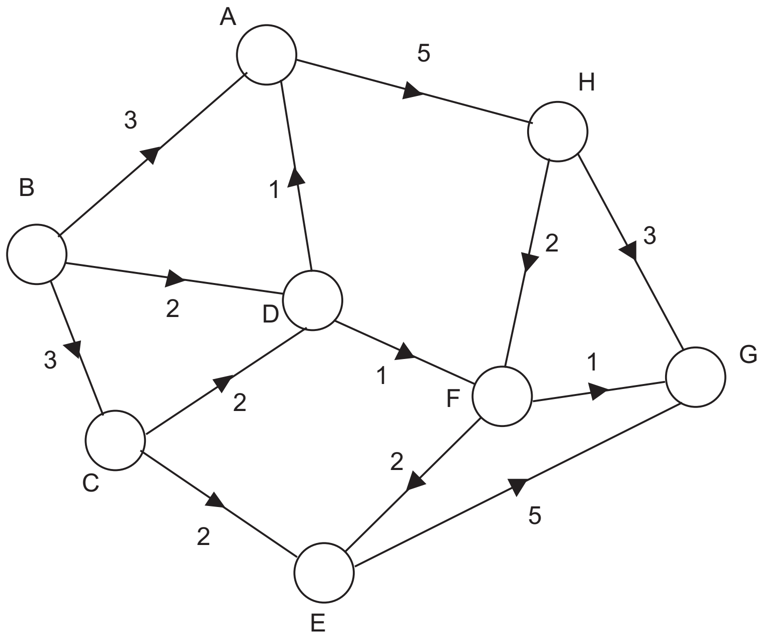
(a) Identify the source and the sink.

(b) State the capacity of the cut Q drawn in the diagram above.

(c) What is the maximum flow of traffic through this system?
Show clearly how you arrived at your answer.

Calculator Assumed

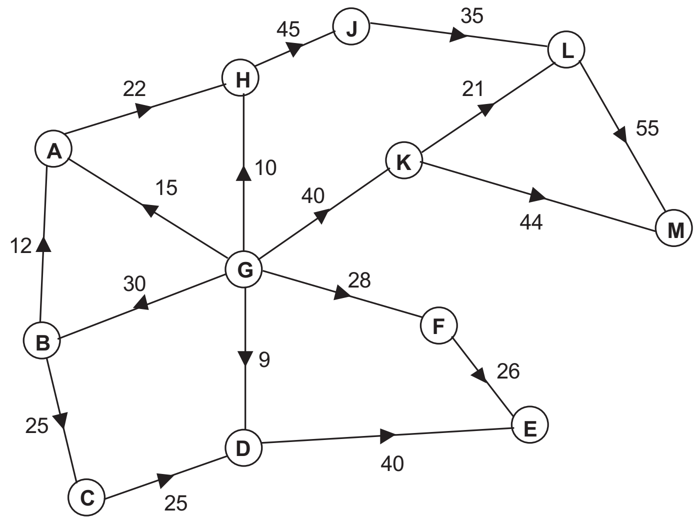
5. (d) A breakdown of traffic lights occurs at one of the junctions (not the source junction or the sink junction). Assume that the traffic lights operate independently from each other. Determine which set of the lights will cause the most reduction in traffic flow. Identify the junction at which this set of lights is located. By how much is the maximum flow of the network reduced?



Calculator Assumed

6. [8 marks: 2, 3, 3]

The diagram below shows the maximum number of vehicles that can travel along a network each minute.

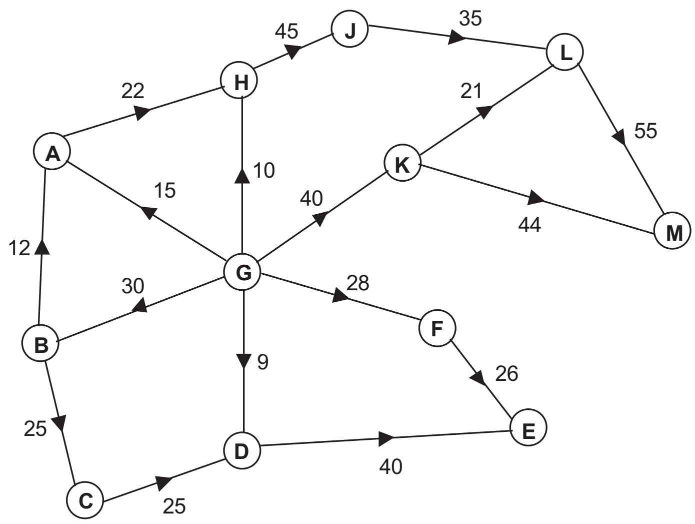


(a) Consider only those paths that lead from G to E. Find the maximum number of vehicles that can travel from G to E each minute.

(b) Consider only those paths that lead from G to M. Find the maximum number of vehicles that can travel from G to M each minute.

Calculator Assumed

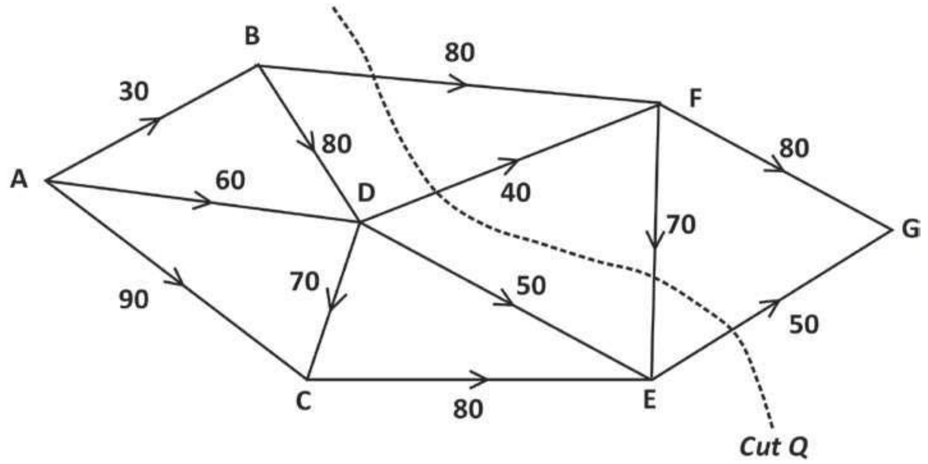
6. (c) Considering the network as a whole, find the maximum number of vehicles that can flow from G to E and M.



Calculator Assumed

7. [13 marks: 1, 2, 3, 2, 2, 3]

The map below shows 6 buildings A, B, C, D. E and F which are connected by one-way streets. The arrows show the direction of flow of traffic. The capacity of each street (in number of vehicles per minute) is given in the numbers alongside the edges.



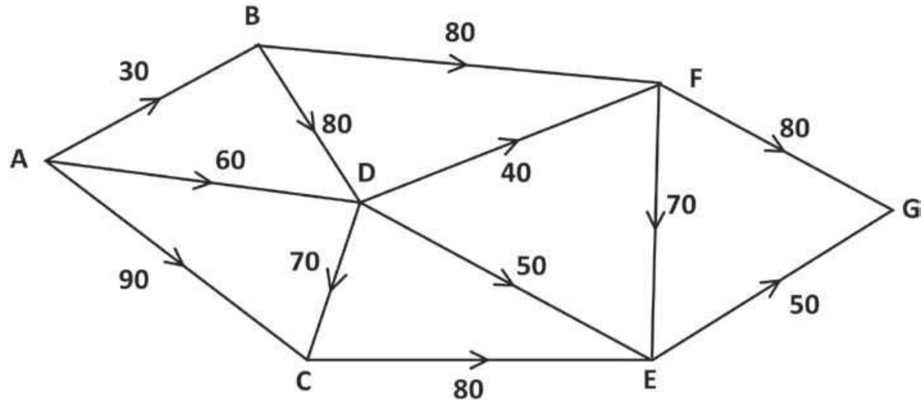
(a) Determine the capacity of the cut Q drawn in the diagram above.

(b) In the diagram above, draw a cut labelled R with capacity of 300 vehicles per minute.

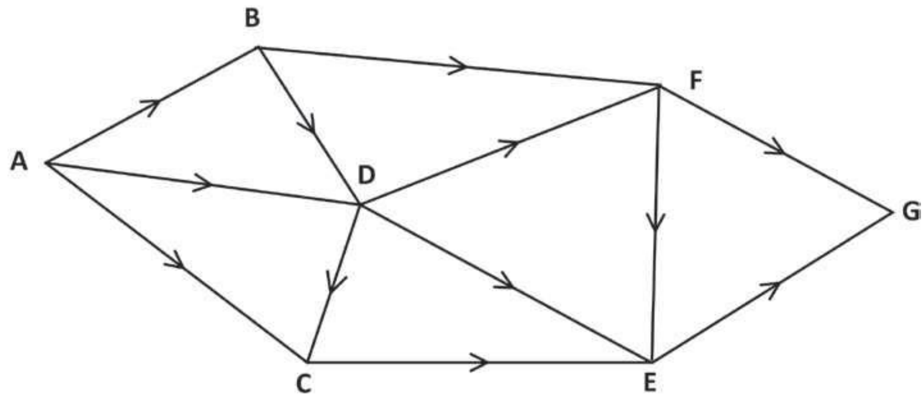
(c) Determine the maximum flow for this traffic network.
Show clearly how you obtained your answer.

Calculator Assumed

7. (d) In the diagram below, draw the cut that corresponds to the maximum flow.



(e) In the diagram below, indicate the unused capacity when maximum flow is achieved.

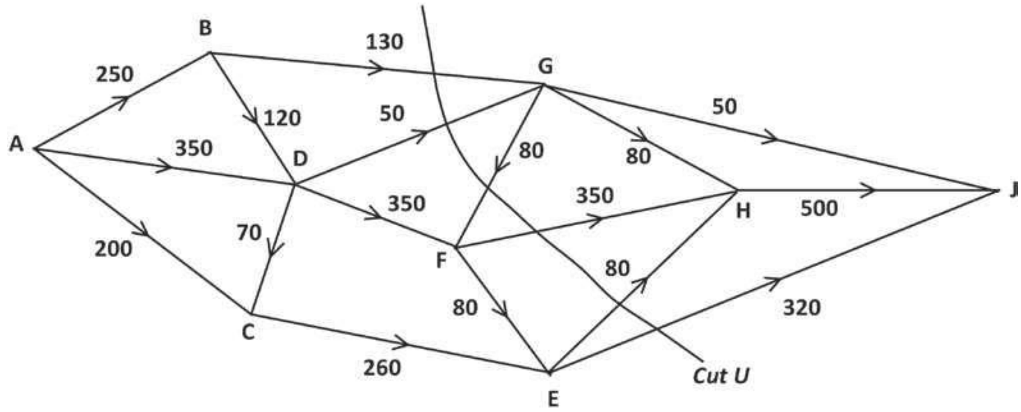


(f) The Mayor of the city wishes to improve the maximum flow so that it matches the flow from the source. How would you achieve this if you were allowed to change the traffic flow of one of these streets and improve the capacity of one of these streets?

Calculator Assumed

8. [12 marks: 1, 1, 3, 1, 2, 2, 2]

The diagram below shows the capacities of a storm water drainage system with flows described in kL per hour.



(a) State the source and sink for this network.

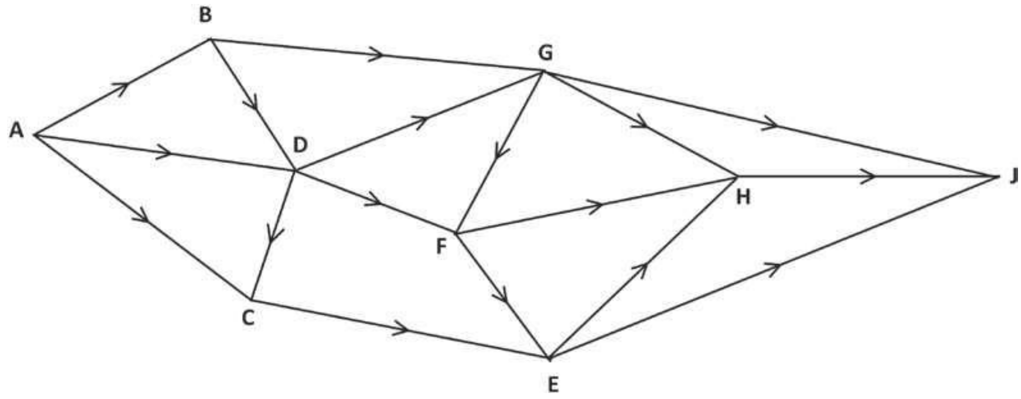
(b) State the value of the Cut U.

(c) Determine the maximum flow for this network. Show clearly how you obtained your answer.

(d) In the diagram above, draw and label the cut that corresponds to the maximum flow.

Calculator Assumed

8. (e) In the diagram below, indicate the unused capacity when maximum flow is achieved.



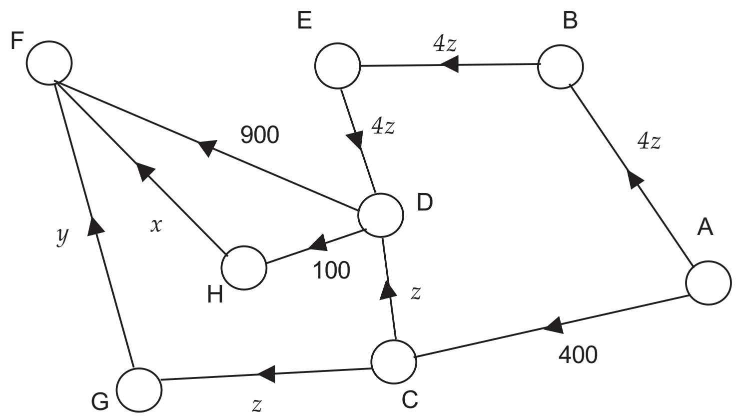
(f) Discuss if it is possible to allow all the flow from the source to exit through the sink without any delay by upgrading the capacity of just one of the pipes.

(g) Discuss the impact on the maximum flow if the capacity of the pipe EH is reduced from 80 to 50 kL per hour.

Calculator Assumed

9. [9 marks: 1, 6, 2]

The following network diagram shows the flow in a systems of pipes that achieve the maximum flow for the system (in Litres per minute).



(a) State the source and sink.

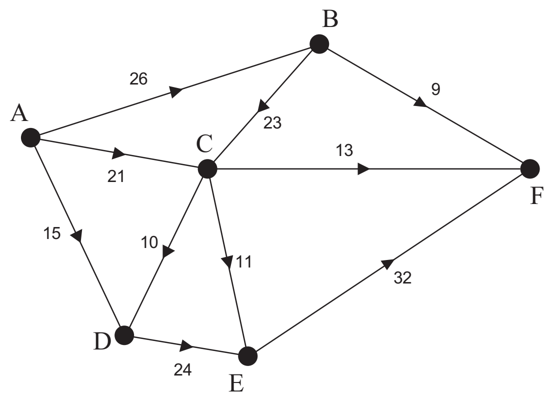
(b) Calculate x, y and z. Justify your answers.

(c) Calculate the maximum flow.

Calculator Assumed

10. [10 marks: 3, 4, 3]

An airline has flights that connect airports at A, B, C, D, E, and F. The network drawn below shows the maximum number of passengers, in hundreds, that can be carried at a certain peak time of day.



(a) Find the maximum number of passengers that can be carried from A to F. Show clearly how you obtained your answer.

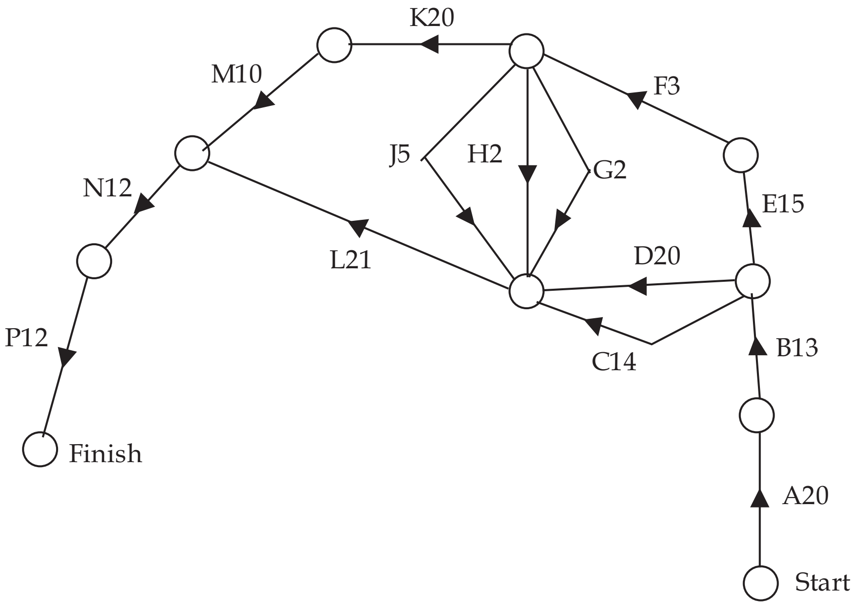
(b) If 95% of the passengers are adults and if the cost for a child fare from A to F is 60% of the adult fare of $1 200 (irrespective of the number of connecting flights), how much could the airline expect to earn during this peak period?

(c) An AFL match is scheduled to be held at an oval located at F between the local team of C and the local team of F. The airline intends to put on extra flights on the E to F route during this peak period to maximise capacity. How many extra passengers should such flights cater for? Justify your answer.

18 Project Networks

1. [10 marks: 2, 2, 2, 4,]

The diagram below shows a project network with time measured in minutes.



(a) State the critical path.

(b) If the project was started at 8.00 am, what is the earliest time the project could be completed?

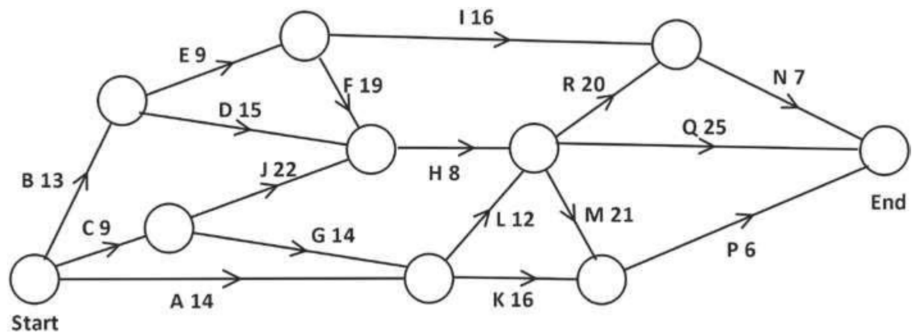
(c) If the project was started at 8.00 am, find the latest time activity L could start without delaying the completion time? Justify your answer.

(d) Activity D is delayed by 8 minutes. Discuss with reasons, the effect this will have on the completion time of the project.

Calculator Assumed

2. [9 marks: 2, 2, 1, 3, 1]

The diagram below shows the tasks required to complete a project. The project starts at 9.00 am. The times for the required tasks are in minutes. Each task requires a workers full attention.



(a) Determine the critical path(s) for this project.

(b) State the earliest time the project can be completed.

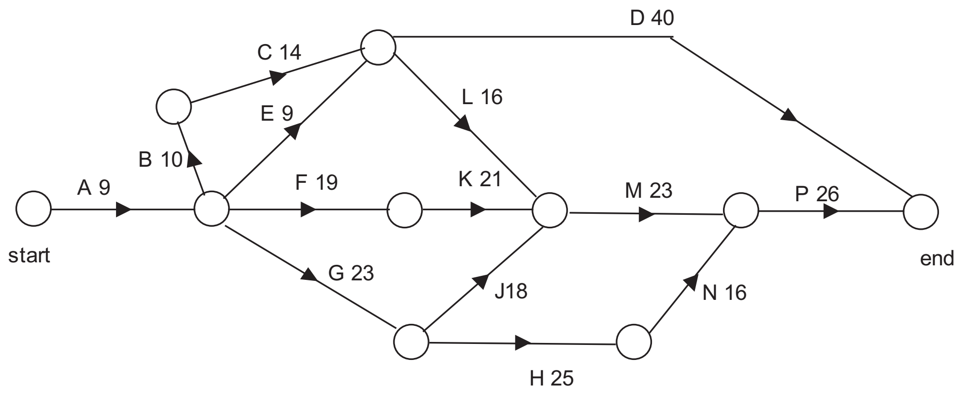
(c) What is the Earliest Starting Time for activity G if the minimum completion time (MCT) is to be maintained?

(d) State with reasons the Latest Starting Time for activity G if the MCT is to be maintained.

(e) State all the activities that have float (slack) times of 0 minutes.

Calculator Assumed

3. [11 marks: 4, 1, 3, 3]



For the project network above, the minimum times required to complete the various activities are recorded in days.

(a) Find the minimum completion time and the corresponding critical path(s).

(b) By how many days can activity E be delayed without affecting the minimum completion time?

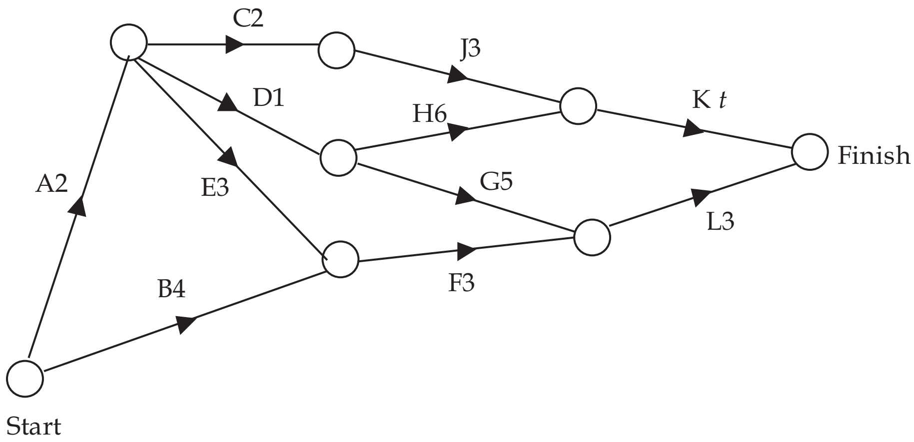
(c) Activity H can now be completed in 22 days. Determine the effect this will have on the minimum completion time and the critical path.

(d) After some reorganisation, it is now possible to commence activity C immediately after the completion of activity A. Discuss the effect of this reorganisation on the minimum completion time and the critical path.

Calculator Assumed

4. [7 marks: 4, 3]

The diagram below shows a project network, with time measured in days.



(a) Complete the Activity Table to describe the project network.

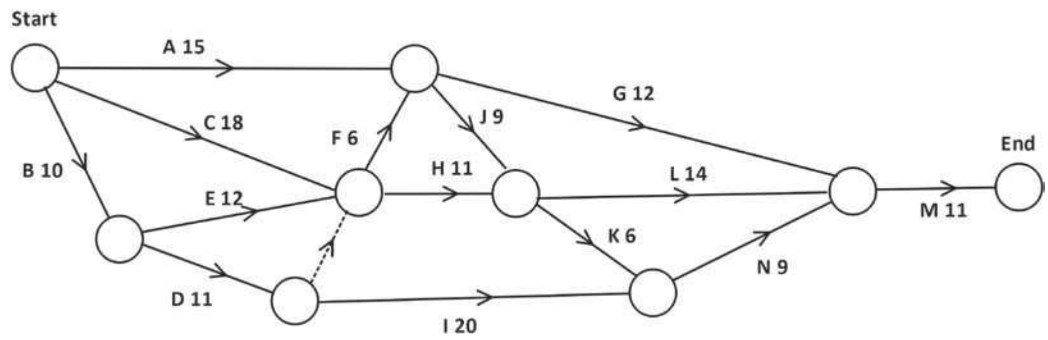
Activity	Time Taken (days)	Immediate Predecessor(s)

(b) Find the value of t if the minimum completion time is 13 days.
State the corresponding critical path.

Calculator Assumed

5. [10 marks: 1, 2, 2, 2, 3]

The diagram below shows a project network. The times for the required tasks are in hours. Each task requires a workers full attention.



(a) State the immediate predecessors of Task I.

(b) State the immediate predecessors of Task F.

(c) Explain the purpose of the dummy link represented by the dotted line in the diagram above.

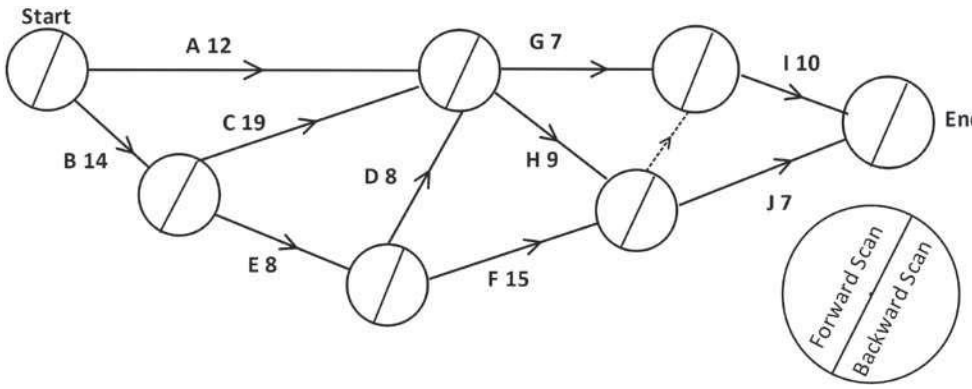
(d) State the task that *will always* be on the critical path. Explain.

(e) Determine the critical path and minimum completion time for this network.

Calculator Assumed

6. [13 marks: 4, 3, 2, 4]

The project network shown below has time for the various tasks recorded in days.



(a) Complete the forward and backward scans for this network.
Explain the purpose of the forward and backward scans.

(b) Use your answer in (a) to explain why Task D is not on the critical path.

(c) Identify the critical path and the minimum completion time for this network.

(d) The minimum completion time may be reduced by hiring more sophisticated equipment for Task C. It will cost $5 000 per day to reduce the completion time for task C by one day. Determine with reasons, how the original minimum completion time may be reduced and by how many days and the extra cost incurred.

Calculator Assumed

7. [10 marks: 4, 2, 2, 2]

The table below shows the tasks required for the building of a factory and the time required in days. The project was started on the 1st of October 2019.

Task	Task Description	Duration	Prerequisites
A	Clear land	42	–
B	Lay foundations	28	A
C	Build walls	42	B
D	Electrical wiring	21	C
E	Plastering	21	D, H
F	Landscaping	28	B
G	Interior work	35	E
H	Roof	14	C

(a) Draw a project network to represent the above information.

(b) When will be the earliest completion date?

(c) If, because of an industrial dispute, the bricklayers went on strike for 5 days, determine the earliest completion date.

(d) If, because of poor weather, the roof job took an extra 10 days, determine the earliest completion date.

Calculator Assumed

8. [11 marks: 4, 2, 5]

The table below describes the various activities involved in assembling a computer.

Activity	Description of Activity	Immediate Predecessor	Time (min)
A	Install mother-board	–	4
B	Test hard drive	A	30
C	Install hard drive	B, E	4
D	Install I/O ports	A	5
E	Install DVD-RW	D	5
F	Test DVD-RW	E	t
G	Install operating system	C, F	15
H	Test assembled computer	G	12

(a) Construct a project network for this table of activities.

(b) The minimum completion time for this project is 65 minutes.
State a possible critical path.

(c) Given that F is not on the critical path, find the possible values of t.
Justify your answer.

Calculator Assumed

9. [10 marks: 4, 3, 3]

The table below shows the activities and their corresponding predecessors and times for a project network.

Activities	Immediate Predecessors	Required time (minutes)
A	–	10
B	–	7
C	A	6
D	A	8
E	B	10
F	B, C, D	16
G	E	6
H	E	4
I	F, G, H	12
J	H	6

(a) Draw the graph for this network.

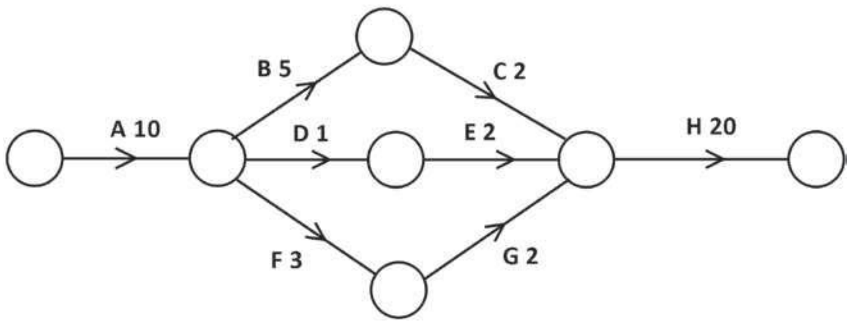
(b) Determine with reasons if this project can be completed within 45 minutes.

(c) If each activity requires a worker's full attention, what is the minimum number of workers that are required for this project to be completed in minimum time? Justify your answer.

Calculator Assumed

10. [8 marks: 2, 3, 3]

The diagram below shows a project network for a job being handled by Bianca. Each activity can <u>only</u> be done by one person and time is recorded in minutes.



(a) What is the minimum time required by Bianca, working by herself, to complete all the tasks?

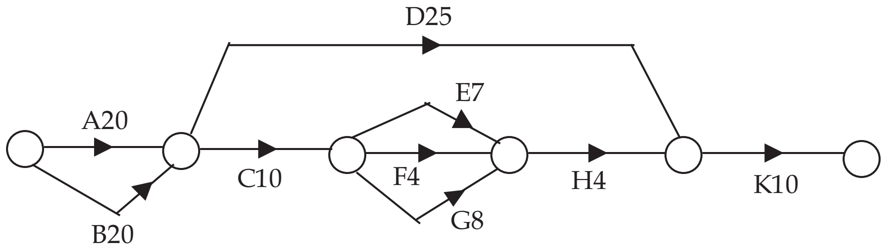
(b) What is the minimum time required by Bianca, if she is assisted by Teneka, to complete all the tasks?

(c) Would it save time, if Leesa came along to give Bianca and Teneka some assistance? Justify your answer.

Calculator Assumed

11. [9 marks: 2, 3, 4]

Seb is involved in setting up a hall for a birthday party. The diagram below shows the project network for this task. Each activity requires a person's full attention and the associated time is measured in minutes.



(a) What is the minimum time required by Seb, working alone, to complete the job? Justify your answer.

(b) What would the minimum completion time be if Stefan gave Seb a hand right from the start? Justify your answer.

(c) By how much would the minimum completion time in part (b) change if Thomas and Stefan were available to help Seb from the start? Justify your answer.

19 Assignment Problems

Calculator Free

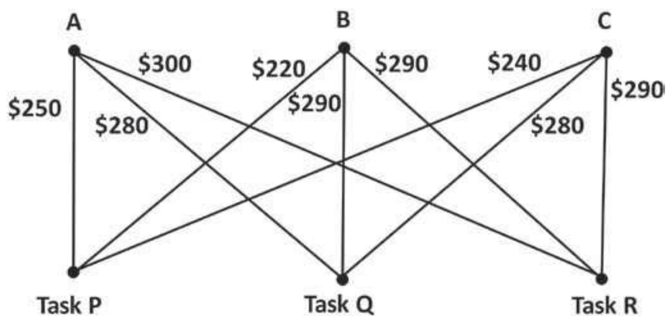
1. [2 marks]

The accompanying table shows the different mathematics courses taken by several students. Display this information as a bipartite graph.

Student	Mathematics Course
Penny	Specialist, Methods
Sheldon	Applications, Essentials
Leonard	Methods, Applications
Amy	Methods

2. [4 marks: 1, 3]

The accompanying bipartite graph shows the labour costs associated with tasks P, Q and R performed by workers A, B and C. One worker is to be assigned to one task with no two workers assigned to the same task.



(a) Find the cost of assigning A to task P, B to task R and C to task Q.

(c) Find the lowest cost associated with assigning one worker to one task. State how this is achieved.

Calculator Free

3. [6 marks: 2, 1, 2, 1]

The accompanying table details the preferences students A, B and C allocated to courses P, Q and R. "1" indicates the student's first preference. It is intended that each student is matched to one course.

	Course P	Course Q	Course R
Student A	1	3	2
Student B	2	1	3
Student C	1	2	3

(a) Represent the information in the table above in the form of a bipartite graph.

(b) Calculate the sum of the preferences if A, B and C are allocated respectively to courses P, Q and R.

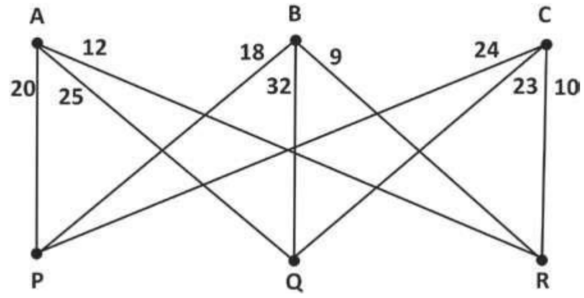
(c) Determine how the courses should be allocated to A, B and C so that the sum of the preferences is minimised.

(d) Explain what it means when the sum of preferences is minimised.

Calculator Assumed

4. [7 marks: 2, 1, 2, 2]

The diagram below is a bipartite graph describing the number of components produced by workers A, B and C at workstations P, Q and R over a period of one hour.



(a) Complete the matrix representation of the bipartite graph.

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} P & Q & R \end{array} \\ \begin{pmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix} \end{array}$$

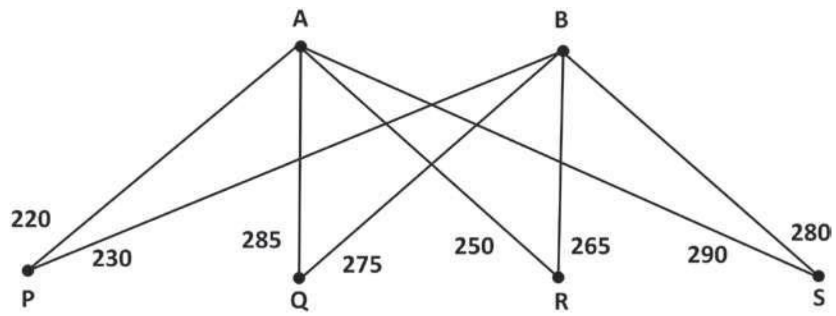
(b) Calculate the total number of components produced when A, B and C are respectively assigned to R, P and Q.

(c) Which assignment produced a total of 58 components?

(d) Determine the optimum assignment(s) so that the total number of components is maximised. State this maximum.

Calculator Assumed

5. [9 marks: 3, 4, 2]



(a) The bipartite graph above shows the daily labour costs ($) associated with tasks P, Q, R and S performed by workers A and B. One worker is to be assigned to exactly one task with the constraint that the total cost is to be minimised. State the minimum cost and the solution(s) to the assignment problem.

(b) The bipartite graph above shows the hourly profit ($) generated through tasks P, Q, R and S performed by workers A and B. One worker is to be assigned to exactly one task with the constraint that the total profit is to be maximised. State the maximum profit and the solution(s) to the assignment problem.

(c) The bipartite graph above shows the time (minutes) taken to complete tasks A and B by workers P, Q, R and S. One worker is to be assigned to exactly one task with the constraint that the total time taken is to be minimised. State the minimum time taken and the solution(s) to the assignment problem.

Calculator Free

6. [12 marks: 3, 3, 3, 3]

Matrices A, B, C and D are reduced row and reduced column cost matrices (opportunity cost matrices) obtained through the Hungarian algorithm. Determine with reasons if these matrices:

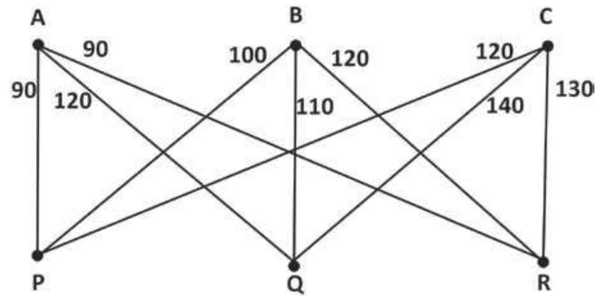
- are unable to assign an optimum solution without further steps
- readily provide a unique optimum solution
- readily provide multiple optimum solutions.

$$A = \begin{pmatrix} 1 & 0 & 5 & 7 \\ 2 & 5 & 5 & 0 \\ 1 & 5 & 0 & 8 \\ 0 & 6 & 2 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 0 & 1 & 6 \\ 0 & 0 & 2 & 0 \\ 2 & 5 & 0 & 3 \\ 0 & 6 & 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 2 & 0 & 5 \\ 3 & 8 & 0 \\ 5 & 0 & 6 \end{pmatrix} \quad D = \begin{pmatrix} 5 & 1 & 0 & 9 \\ 0 & 2 & 0 & 0 \\ 1 & 3 & 0 & 8 \\ 0 & 4 & 1 & 0 \end{pmatrix}$$

Calculator Assumed

7. [6 marks: 3, 2, 1]

The accompanying diagram shows a bipartite graph describing the hourly costs ($) associated with workers A, B and C performing tasks P, Q and R.



(a) Complete the tables below showing the use of the Hungarian algorithm to assign one worker to exactly one task while minimising the total average cost. [Use as many "blank tables" as you need. Indicate the order in which the tables were used.]

	P	Q	R
A			
B			
C			

	P	Q	R
A			
B			
C			

	P	Q	R
A			
B			
C			

	P	Q	R
A			
B			
C			

	P	Q	R
A			
B			
C			

	P	Q	R
A			
B			
C			

(b) State the optimum assignment and the total hourly cost.

(c) What is the least average cost per worker?

Calculator Assumed

8. [7 marks: 5, 2]

The accompanying table shows the running times (seconds) for sprinters A, B, C and D for each of the four legs of a 4 × 100 m relay race.

Time	Leg 1	Leg 2	Leg 3	Leg 4
Sprinter A	9.9	9.4	9.7	9.3
Sprinter B	10.0	9.3	9.9	9.1
Sprinter C	9.8	9.6	9.7	9.5
Sprinter D	9.7	9.2	9.5	9.1

(a) Complete the tables below showing the use of the Hungarian algorithm to assign one sprinter to exactly one leg of the race while minimising the total time taken for the race. [Use as many "blank tables" as you need. Indicate the order in which the tables were used.]

(b) State the optimum assignment of sprinters to the legs of the race. State the minimum time achievable.

Calculator Assumed

9. [9 marks: 6, 3]

The accompanying table shows the times (minutes) taken by workers A, B, C and D to complete Tasks 1, 2, 3 and 4.

Time	Task 1	Task 2	Task 3	Task 4
Worker A	23	32	41	16
Worker B	24	35	39	17
Worker C	23	31	40	18
Worker D	25	31	40	16

(a) Use the Hungarian algorithm to assign one worker to exactly one task with the constraint that the total time taken to complete the four tasks is a minimum. State all the optimum assignments and the corresponding time taken.

(b) If the time for B to complete Task 3 was 38 minutes instead of 39 minutes discuss how this would affect the optimum assignment in (a).

Calculator Assumed

10. [9 marks: 6, 3]

The accompanying matrix shows the number of new customers signed up per day by salespersons A, B, C and D at outlets located at shopping centres P, Q, R and S.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} P & Q & R & S \end{array} \\ \begin{pmatrix} 15 & 12 & 18 & 6 \\ 16 & 14 & 20 & 6 \\ 14 & 13 & 19 & 7 \\ 17 & 15 & 20 & 7 \end{pmatrix} \end{array}$$

(a) Use the Hungarian algorithm to assign one salesperson to exactly one shopping centre maximising the total number of new customers signed up. State all the optimum assignments and the corresponding sales made. Show each step of the process.

(b) If the number of new customers signed up by B at shopping centre R was 22 instead of 20, discuss how this would affect the optimum assignment in (a).

Calculator Assumed

11. [10 marks, 7, 3]

The accompanying table shows the times (hours) taken by workers A, B, C and D to complete Tasks 1, 2, 3 and 4.

Time	Task 1	Task 2	Task 3	Task 4
Worker A	124	154	256	23
Worker B	132	165	248	21
Worker C	128	154	248	24
Worker D	124	158	255	21

(a) Show the use of the Hungarian algorithm to assign one worker to exactly one task with the constraint that the total time taken to complete the four tasks is a minimum. State the optimum assignment(s) and the corresponding time taken.

(b) If because of health reasons, Worker A must be assigned to Task 4, discuss the impact of this requirement on the total time to complete the four tasks.

Calculator Assumed

12. [9 marks]

The accompanying table shows the times (minutes) taken by workers A, B, and C to complete Tasks 1, 2, 3 and 4.

Time	Task 1	Task 2	Task 3	Task 4
Worker A	21	28	26	12
Worker B	22	26	26	10
Worker C	23	27	28	10

Use the Hungarian algorithm to assign one worker to exactly one task with the constraint that the total time taken to complete the three chosen tasks is a minimum. State all the optimum assignments and the corresponding time taken. Show each step of the process.

Calculator Assumed

13. [6 marks]

The accompanying table shows the average daily costs for workers A, B, C, D and E performing Tasks 1, 2, 3 and 4.

Cost	Task 1	Task 2	Task 3	Task 4
A	500	580	620	550
B	480	600	610	540
C	490	590	600	550
D	510	580	610	560
E	490	590	590	570

Use the Hungarian algorithm to assign one worker to exactly one task while minimizing the total cost. State two possible optimum assignments and the corresponding cost. Show each step of the process.

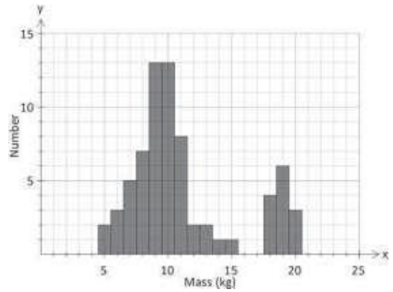
Fully Worked Solutions

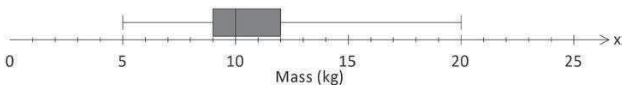
01 Statistical Investigation Process

Calculator Free

1. [8 marks: 1, 3, 2, 2]

Gilbert owns a small hobby farm where he is experimenting with a new seedless watermelon hybrid. The accompanying diagrams show the distribution of the mass of the seventy watermelons harvested as a histogram and a box-plot.





(a) State one feature of the distribution of the mass of watermelons that is conveyed in the histogram but not conveyed in the box-plot.

The histogram clearly indicates that there are two clusters. ✓

(b) The box-plot indicates the presence of outliers.

(i) How many of the watermelons can be classified as outliers in terms of their mass?

IQR = 3 ✓
Outliers have mass that exceed $12 + 3 \times 1.5 = 16.5$ ✓
Hence, number of outliers $= 4 + 6 + 3 = 13$ ✓

(ii) Should the outliers be removed when calculating the mean mass of the watermelons harvested? Why?

No! The "outliers" make up about 20% of the water-melons harvested. ✓✓
OR
Yes! The two clusters indicate that they should be separated into 2 groups. ✓✓

(iii) Suggest two questions that should be asked regarding the "outliers".

- Did these watermelons come from the same vine?
- Did these watermelons grow under the same conditions?
- Could these watermelons be a new mutation?

✓✓

Calculator Free

2. [8 marks: 2, 2, 2, 2]

The accompanying diagram shows the stem and leaf plot of the marks obtained by two groups of 25 students each in a common mathematics test.

Class A		Class B
9 8 2	0	2 3 4 6
9 9 9 9 6 6	1	
5 4	2	5 6 8
6 4 1	3	8 9
1	4	1 1
3 2	5	4
9 8 6 6	6	1 1 4 7 9
3	7	
4 1	8	3 8 8
7	9	3 5 5 9 9

(a) Verify that the median mark for Class B is higher than the median for class A..

Median (class A) = 34 ✓

Median (class B) = 61 ✓

(b) Based on the higher median mark obtained by the students in class B, it was suggested that the teacher of Class B was a *better* teacher than the teacher of Class A. Identify two issues that could be used to challenge this suggestion.

Among them: ✓✓
- Which teacher wrote the test?
- Were the classes streamed according to ability?
- Did both classes have the same amount of time to prepare for the test?
- Were both classes prepared in the same way for the test?

(c) The two classes were taught by the same teacher and were not streamed according to their mathematical ability. Suggest two factors that could account for the differences in the marks.

Among them: ✓✓
- Were the tests taken at the same time of the day and under similar conditions?
- Did both classes have the same amount of time to prepare for the test?
- Were both classes prepared in the same way for the test?

(d) Both classes were taught by the same teacher and were streamed according to their mathematical ability. Comment on the statement "Class B was more successful in this test".

Class B was more successful in terms of "raw marks".
However, given that the classes were streamed, it could be possible that Class A students had better performance in terms of how much they had improved from previous tests. ✓✓

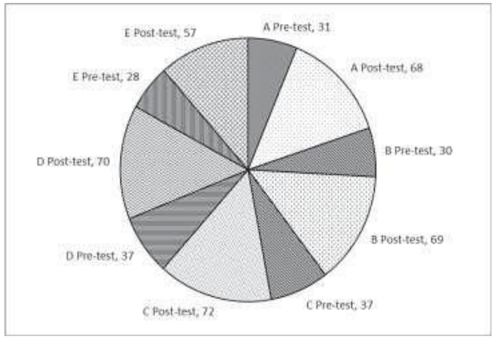
Calculator Free

3. [3 marks]

Five groups of students taking the same mathematics unit were tested just before a topic was taught (pre-test) and immediately after the topic was taught (post-test). The table below shows the results for these tests for the five groups of students.

Group	Pre-test	Post-test
A	31	68
B	30	69
C	37	72
D	37	70
E	28	57

Kim wanted to show the changes between the results of the pre-test and that of the post-test and drew a pie graph shown below.



Discuss whether a pie graph is suitable for making the comparisons intended.

- ✓ The increase in the size of the slices do show the changes between the marks for the pre-test and post-test.
- ✓ However, the sizes of the slices shown are percentages of the whole which in this instance is meaningless.
- ✓ A more suitable graph would be a side by side bar graph.

Calculator Assumed

4. [10 marks: 2, 2, 1, 3, 2]

Hannah wishes to determine the percentage of university graduates that have found jobs that have some relevance with their university education and qualifications.

(a) Write one question she could ask in a survey of university graduates that will provide her with the information she needs.

Any reasonable question. ✓
A question that offers options describing varying levels of relevance. ✓
Eg. Tick box that applies. A. Very relevant.
Mark on a scale (Highly relevant to no relevance)

(b) Hannah interviewed 100 graduates working in several office blocks in the city where she lived. Suggest two ways she could improve the quality of information she will obtain.

Increase size of sample. ✓
Survey graduates in other parts of the city, or other cities. ✓

The following table shows the results of a survey that Hannah conducted on 100 university graduates.

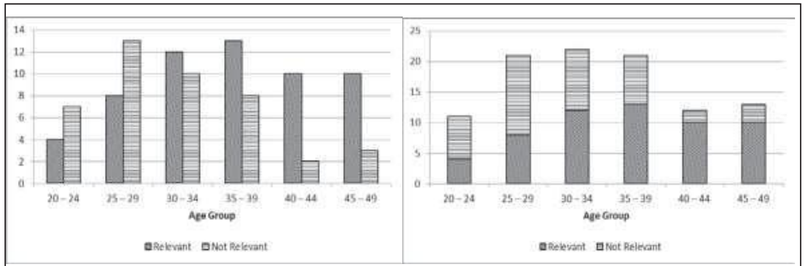
Age	Relevant to current job	Not relevant to current job	Total
20 – 24	4	7	11
25 – 29	8	13	21
30 – 34	12	10	22
35 – 39	13	8	21
40 – 44	10	2	12
45 – 49	10	3	13

(c) What type of graph would be most suitable for displaying the information above?

Side by side column graph or stacked column graph ✓

Calculator Assumed

4. (d) Use the graph type you chose in (c) to display the given information.



✓ Correct graph drawn ✓ Accurate columns
✓ Labels correctly applied

(e) From the information in the table above, determine with reasons if there is a relationship between the age of the graduate interviewed and the relevance of the graduate's university qualifications with the graduate's current job.

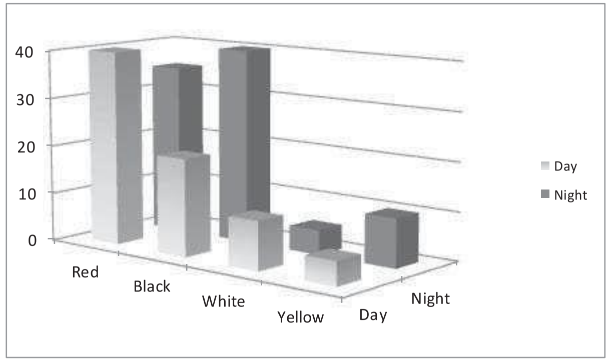
✓ Proportion of those in relevant jobs increases with age group.
✓ Evidence provided.
From 4/11 for those in 20-24 age group to 10/13 for those in 45-49 age group.

02 Associations between Categorical variables

Calculator Free

1. [5 marks: 1, 1, 3]

The diagram below shows the number of vehicle accidents described by the colour of the vehicles and the time of the accidents; day-time or night-time.



(a) A student noted that fewer white coloured cars were involved in accidents at night-time and concluded that it is safer to drive white coloured cars at night-time. Give one reason why this need not necessarily be true.

Need to compare *percentages* of the different coloured cars on the road involved in accidents. ✓
(There could be fewer white cars on the road at night or overall.)

(b) Describe another potential association between colour of vehicle, the number of vehicle accidents and the time of the accidents.

Red coloured cars appear to be involved in most accidents! ✓

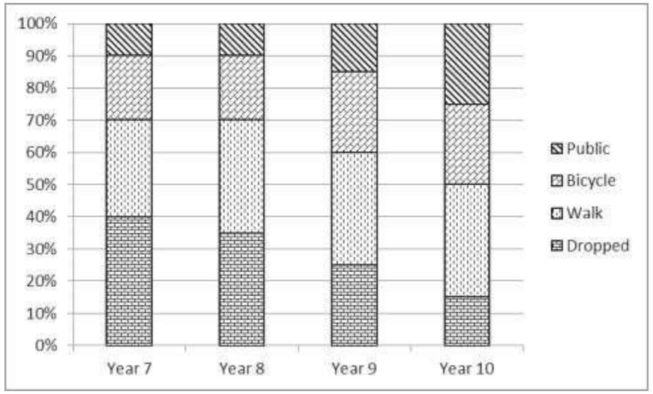
(c) State three questions you would ask to ascertain the credibility of the association you described in (a).

- Method of data collection. ✓
  (Eg. When, where and how was the data collected.)
- Comparison of percentages rather than absolute numbers. ✓
  (Eg. What is the total number of red-coloured vehicles on the road?)
- Explore other explanatory variables. ✓
  (Eg. What are the ages of the drivers involved in *all* the accidents?)

Calculator Free

2. [6 marks: 3, 3]

The diagram below shows the percentage of students in the listed school year groups and their mode of travel to school.



(a) Identify a possible negative association between school year and the mode of travel to school. Describe this association and state the response variable and the explanatory variable.

Response Variable: % of students in year group being dropped in school. ✓
Explanatory Variable: Year group level (age of student) ✓
Conjecture: % of students in year group dropped in school appears to decline with the age of students (as identified by year level). ✓

(b) Identify a possible positive association between school year and the mode of travel to school. Describe this association and state the response variable and the explanatory variable.

Response Variable: % of students in year group travelling by public transport. ✓
Explanatory Variable: Year group level (age of student) ✓
Conjecture: % of students in year group travelling by public transport appears to increase with the age of students (as identified by year level). ✓

Calculator Assumed

3. [8 marks: 2, 2, 4]

The accompanying diagram shows the different makes of cars parked at three different suburban shopping centres on a school-day morning. The shopping centres A, B and C are located respectively at high, middle and low income suburbs.

	A	B	C
Australian	70	60	40
German	150	80	20
Korean	40	130	120
Japanese	90	140	110
Others	60	70	50

(a) Complete the table below showing the row percentages.

	A	B	C
Australian	41	35	24
German	60	32	8
Korean	14	45	41
Japanese	26	41	32
Others	33	39	28

✓✓

(b) Complete the table below showing the column percentages.

	A	B	C
Australian	17	13	12
German	36	17	6
Korean	10	27	35
Japanese	22	29	32
Others	15	15	15

✓✓

(c) Determine with reasons if there is a relationship between the make of cars parked and the level of income of the suburb. Clearly identify the response and explanatory variables. State any assumptions you made.

- Assumption: Being a school-day morning, car owners are local residents with respect to the shopping centres. (That is, there are few cross suburb shoppers.)
- 60% of German made cars were parked at the high income shopping centre.
- The most popular make of cars at the high income shopping centre were of German make (36%).
- Hence, shoppers at the high income suburb favour German cars.
  Response variable: Make of car
  Explanatory variable: Income level of suburb

Assumption ✓
Relationship ✓
Variables ✓
Reason(s) ✓

Calculator Assumed

4. [9 marks: 1, 2, 2, 4]

A sample of 400 voters were asked to respond YES or NO to the question, "Should uranium mining be permitted in Western Australia (WA)?". The results are tabulated in the first table below.

	YES	No
WA born	10%	55%
Not WA born	30%	5%

The second table shows the gender distribution of those in the sample who responded YES.

	WA born	Not WA born
Female	5%	35%
Male	15%	45%

(a) How many in the sample responded YES?

No. of YES = 400 × 40% = 160 ✓

(b) What percentage of respondents who voted YES was male?

% of YES who were Male = 15 + 45 ✓
= 60 ✓

(c) What percentage of those who voted YES was WA born?

% of YES who were WA born = $\frac{10}{40}\times 100$ ✓
= 25 ✓

(d) Discuss the relationship between birth state and sex and the way the vote was cast? Give reasons for your answer.

- 25% of those who voted YES were WA born.
Hence respondents who were WA born tended to vote No.
- 60% of those who voted YES were males. 40% of those who voted YES were females.

Hence, respondents who were female tended to vote No irrespective of birth state.

Birth State and Vote with reasons. ✓✓
Gender and Vote with reasons. ✓✓

Calculator Assumed

5. [10 marks: 2, 2, 2, 2, 2]

The table below shows the distribution of adults by age groups (in years) who choose not to have their medical prescriptions filled and their accompanying reasons.

Reason	18-24	25-35	35-45	46-50	51-60	61-70	70+	Total
Cost of medication is too high	28	18	15	17	18	24	30	150
Wait to see if they get better without medication	39	25	19	20	24	35	36	198
Fear that they might become dependent on the medication	5	12	18	14	13	10	10	82
Do not trust their doctor's diagnosis	22	18	17	17	18	22	24	138
Total	94	73	69	68	73	91	100	568

(a) Complete the table below showing column percentages (to the nearest %). Some entries have been adjusted to match the percentage total.

Reason	18-24	25-35	35-45	46-50	51-60	61-70	70+
Cost of medication is too high	30	25	22	25	25	26	30
Wait to see if they get better without medication	42	34	27	29	32	39	36
Fear that they might become dependent on the medication	5	16	26	21	18	11	10
Do not trust their doctor's diagnosis	23	25	25	25	25	24	24

(b) Complete the table below showing row percentages (to the nearest %). Some entries have been adjusted to match the percentage total.

Reason	18-24	25-35	35-45	46-50	51-60	61-70	70+
Cost of medication is too high	19	12	10	11	12	16	20
Wait to see if they get better without medication	20	13	10	10	13	17	18
Fear that they might become dependent on the medication	6	15	22	17	16	12	12
Do not trust their doctor's diagnosis	17	13	12	12	13	16	17

(c) Comment on the distribution of adults in the study who choose not to have their prescriptions filled because they do not trust their doctor's diagnosis.

A constant percentage of about 25% of each age group do not trust their doctor's diagnosis. ✓✓

Calculator Assumed

5. (d) Comment on the distribution of adults in the study who choose not to have their prescriptions filled because they fear becoming dependent on the medication.

The percentage in each age group grows from 5% for 18-24 year olds to peak at 26% of 35-45 year olds and then starts to decline to 10% for the those aged 70+. ✓✓

(e) Which age group form the largest group among those who gave "the high cost of medication" as the reason? Justify your answer.

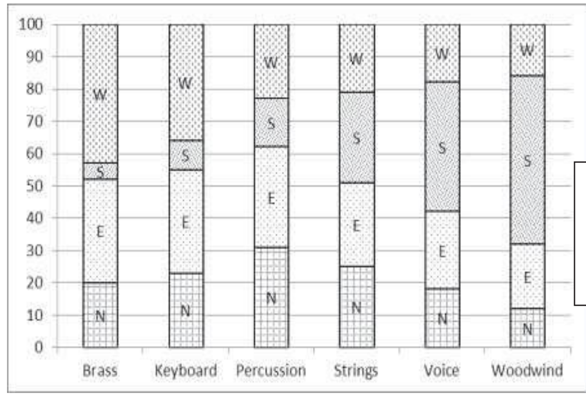
The 70+ group with 20%. ✓✓

Calculator Assumed

6. [15 marks: 3, 2, 2, 2, 1, 2, 2, 1]

Macliver College is a specialist co-educational music college catering for students from years 7 to 12. The table below shows the distribution of a *sample* of students from Macliver College according to their musical instrument category and the geographical area of their homes (North, East, South or West of the College). Also included in the table is the gender distribution of students in each instrument category. The accompanying graph is a percentage stacked column graph of some of the information displayed in the table.

	North	East	South	West	Females	Males
Brass	22	35	5	46	36	72
Keyboard	24	34	10	38	54	52
Percussion	33	33	16	24	24	82
Strings	21	22	24	18	72	13
Voice	15	20	33	15	68	15
Woodwind	6	10	26	8	29	21



Column for Woodwind to be completed. Correct % ✓✓✓

(a) Complete the percentage stacked column graph above.

(b) Comment on the composition of students in this sample with Brass as their instrument category.

- Dominated by students from the West (≈ 42%) and East (≈ 32%). Together they form ≈ 75% of the Brass students. ✓
- Minority from the South (≈ 5%). ✓

Calculator Assumed

6. (c) In this sample, what proportion of students from the South are in the Voice category?

> Proportion = $\frac{33}{114} \times 100 \approx 29\%$ ✓✓

(d) Comment on the instrument selection of students from the South in this sample.

> - In order of "popularity":
>   Voice, Woodwind, Strings, Percussion, Keyboard, Brass. ✓✓

(e) In which category/categories are the female students in the majority?

> - Keyboard, Strings, Voice and Woodwind. ✓

(f) Is it appropriate to conclude that there are more female than male students in this College? Give reasons for your answer.

> Conclusion only appropriate if the sample is representative of the student population. ✓✓

(g) Does this sample provide evidence that instrument category is related to geographical location of the students' homes? Justify your answer, stating any assumptions you make.

> Assuming that the sample is representative of the student population, the percentage stacked column graph shows that within each instrument category there is a distinct variation in the proportion of students from the different geographical zones.
> Hence, YES. ✓✓
>
> [Not likely for sample bias to produce such variation.]

(h) Suggest a possible reason for the high proportion of students in Woodwind coming from the South of the College.

> - Sample could be biased.
> - Feeder schools in the South have better Woodwind Programmes. ✓

Calculator Assumed

7. [14 marks: 2, 2, 2, 2, 3, 3]

Table 1 shows the results of a survey conducted in country X which describes the distribution of adults with tertiary and post-graduate qualifications and their region of birth.

Table 1

	Tertiary	Post-graduate	Total
A	45	28	73
B	47	36	83
C	43	31	74
D	57	44	101
E	62	25	87
Total	254	164	418

Table 2 shows the distribution of those with tertiary qualifications according to gender.

Table 2

Tertiary	Males	Females	Total
A	22	23	45
B	31	16	47
C	24	19	43
D	32	25	57
E	42	20	62
Total	151	103	254

Table 3 shows the distribution of those with post-graduate qualifications according to gender.

Table 3

Post-graduate	Males	Females	Total
A	19	9	28
B	25	11	36
C	18	13	31
D	24	20	44
E	15	10	25
Total	101	63	164

(a) What proportion of those surveyed were born in A and have post-graduate qualifications?

> Proportion = $\frac{28}{418} \times 100 \approx 6.7\%$ ✓✓

(b) What proportion of those surveyed are females with post-graduate qualifications and are born in A?

> Proportion = $\frac{9}{418} \times 100 \approx 2.2\%$ ✓✓

Calculator Assumed

7. (c) What proportion of those with tertiary and post-graduate qualifications are females.

Proportion = $\frac{63+103}{418} \times 100 \approx 39.7\%$ ✓✓

(d) Identify with reasons, the region of birth of those with post-graduate qualifications with roughly equal representation from both sexes .

Region D. ✓
Proportion from D with Post-graduate qualifications who are males
$= \frac{24}{44} \times 100 \approx 55\%$
Proportion from D with Post-graduate qualifications who are females ✓
$\approx 45\%$

(e) Comment on the distribution of those with post-graduate qualifications in terms of their birth region and gender.

- Region D is the birth region for the greatest number with post-graduate qualifications (i) overall and (ii) among the females.
  Men from D with post-graduate qualifications are ranked second in total numbers.
- Region B is the birth region for the most number of males with post-graduate qualifications but is ranked second overall.
- Region E is the birth region of the least number with post-graduate qualifications (i) overall and (ii) among the males.

Comment on region with greatest number overall and in terms of gender.
Comment on region with least number overall and in terms of gender. ✓✓✓

(f) Country X has a population of 25 million and 30% of the population have tertiary or post-graduate qualifications. Use the results of this survey to estimate how many in country X are females with post-graduate qualifications and are born in region A.

Number $\approx \frac{9}{418} \times 0.3 \times 25$ million ✓✓
≈ 0.1615 million

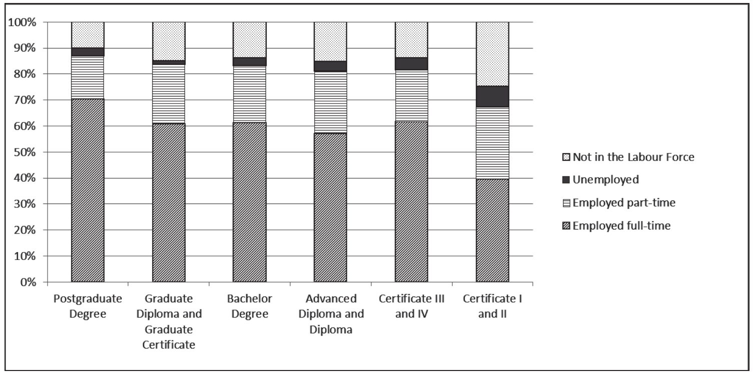
Calculator Assumed

8. [9 marks: 2, 1, 1, 1, 1, 3]

The table below shows the estimated number of persons (thousands) employed full-time, part-time, unemployed (but looking for work) and not in the labour force and their post-school qualifications.

	Employed full-time	Employed part-time	Unemployed	Not in the Labour Force	Total
Postgraduate Degree	700	165	30	100	995
Graduate Diploma and Graduate Certificate	400	150	10	98	658
Bachelor Degree	1,600	570	81	360	2,611
Advanced Diploma and Diploma	870	368	58	230	1,526
Certificate III and IV	1,790	580	135	400	2,905
Certificate I and II	184	130	37	115	466
Total	5,544	1,963	351	1,303	9,161

The chart given below is drawn using the information in the table above.



(a) Explain what calculations were required to construct the chart shown.

Row percentages. ✓✓

(b) Determine the association between post-school qualifications and full-time employment status.

The level of full-time employment declines with the decrease in the level of post-school qualifications. ✓

Calculator Assumed

8. (c) Determine the association between post-school qualifications and those not in the labour force.

> The proportion of those not in the labour force increases with the decrease in the level of post-school qualifications. ✓

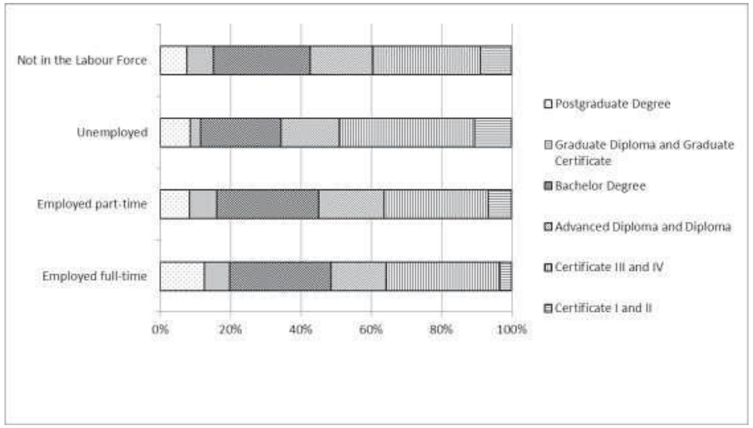
(d) Calculate the percentage of those with post-graduate qualifications not in the labour force.

> Required percentage = $\frac{100}{995}\times 100 \approx 10\%$ ✓

(e) Determine the proportion of those with certificates in part-time employment.

> Required percentage = $\frac{710}{1963}\times 100 \approx 36\%$ ✓

(f) Complete the chart drawn below.



> Correct % for employed full-time. ✓✓
> Labels ✓

03 Associations between Numerical Variables

Calculator Free

1. [2 marks]

The linear relationship between two variables, x and y, is described as negative.
The least squares regression line has equation $y = a + bx$.
Determine with reasons which of the following statement(s) *must* be true.

A. Both a and b must be positive.
B. Both a and b must be negative.
C. a must be positive.
D. a must be negative.
E. b must be negative.

> Statement E is the only correct statement. ✓
> As the relationship between x and y is negative,
> the least squares regression line must have a negative gradient.
> Hence b must be negative. ✓

2. [2 marks]

The least squares regression line between Q and T is given by $Q = -48.4 - 1.2t$.
Which of the following statement(s) *must* be true?

A. There could be a negative linear relationship between Q and t
B. As t increases, Q decreases.
C. As t increases, Q increases.
D. The coefficient of linear correlation between Q and t is negative.
E. The coefficient of linear correlation between Q and t is positive.

> Statements A, B and D must be true. ✓✓

Calculator Free

3. [2 marks]

The coefficient of linear correlation between two variables, x and y, is 0.95.
Which of the following statement(s) *must* be true?

A. As x increases, y increases.
B. An increase in x causes an increase in y.
C. 95% of the data points lie on the line of best fit between x and y.
D. The relationship between x and y may actually be non-linear.
E. The line of best fit between x and y must have a positive gradient and a positive vertical intercept.

> A and D are true. ✓✓

---

4. [2 marks]

The coefficient of determination between the variables x and y is 0.81 and the line of regression of y on x is $y = -0.4x + 3.2$.
Which of the following statements must be true?

A. The coefficient of linear correlation between x and y is 0.9.
B. The coefficient of linear correlation between x and y is –0.9.
C. The coefficient of linear correlation between x and y is $-(0.81^2)$.
D. The regression line passes through 81% of the points.
E. The response variable is y and the explanatory variable is x.

> B and E are true. ✓✓

---

5. [2 marks]

The line of regression of b on f is $b = 2.54f + 122.5$ with a coefficient of linear correlation of 0.8. Which of the following statements must be true?

A. The coefficient of determination is 0.64
B. The response variable is f and the explanatory variable is b.
C. 64% of the variation in b can be explained by the relationship between b and f.
D. As f decreases b decreases.
E. Increases in b are caused by increases in f.

> A, C and D are true. ✓✓

Calculator Free

6. [5 marks: 2, 1, 2]

The least squares regression line between N and t is given by $N = 0.05t + 2.51$.
(a) Find the average increase in N corresponding to an increase of 20 units in t.

> Average increase in N per unit time = gradient of line
> = 0.05 ✓
> Hence, average increase in $N = 0.05 \times 20 = 1$ ✓

(b) Predict the value of N when $t = 50$.

> Predicted value of $N = 0.05 \times 50 + 2.51$
> = 5.01 ✓

(c) The actual value of N when $t = 50$ is 4.95. Find the residual associated with the prediction in (b).

> Residual = Actual – Predicted ✓
> = 4.95 – 5.01
> = –0.06 ✓

---

7. [5 marks: 2, 1, 2]

The least squares regression line of Physics Marks p on Mathematics Marks m is $p = 1.5m + 5.0$.
(a) Explain clearly why the correlation coefficient between p and m cannot be –0.8.

> Regression line has a positive gradient ✓ $\Rightarrow$ r cannot be negative. ✓

(b) Calculate the predicted increase in y when x increases by 10 units.

> Increase in $y = 1.5 \times 10 = 15$ units ✓

(c) The actual value of y when $x = 20$ is 37.
Use the regression line given to predict the value of y when $x = 20$.
Hence, determine the associated residual.

> $\hat{y} = 1.5 \times 20 + 5 = 35$ ✓
> Residual = 37 – 35 = 2 ✓

Calculator Free

8. [8 marks: 1, 1, 2, 2, 2]

For a sample of 100 students, the coefficient of linear correlation between the length of a student's foot and the number of errors in a spelling test is –0.9.

(a) Determine the nature of the data for "the length of a student's foot".

numerical and continuous ✓

(b) Determine the nature of the data for "the number of spelling errors in a spelling test".

numerical and discrete ✓

(c) Determine the percentage of variation in the number of errors in the spelling test that is explained by the length of a student's foot.

$r^2 = 0.9^2 = 0.81$ ✓
Hence 81% ✓

(d) Use the information given to determine with reasons if increasing the length of a student's foot will reduce the number of errors in the spelling test.

Statement is incorrect. ✓
Strong correlation does not imply a causal relationship exists between the variables. ✓

(e) The average foot length is 15.4 cm and the average number of spelling errors is 8.4. A student with a foot length of 22.2 cm had 15 spelling errors in the test. Determine the possible effect on the original correlation coefficient if this student's data were included. Give a reason for your answer.

Correlation coefficient would be less negative. ✓

New data point is an outlier. ✓

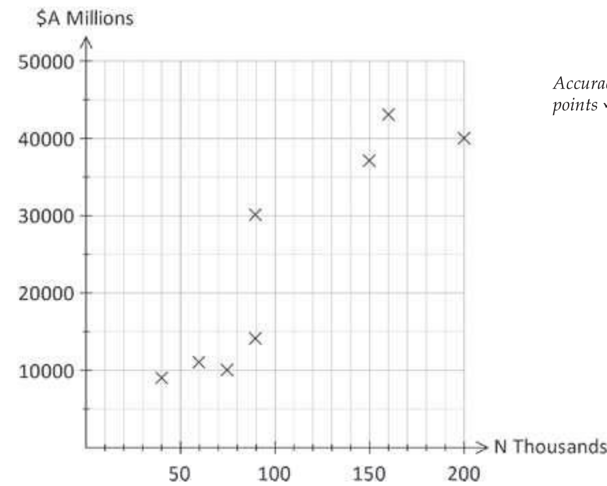
Calculator Assumed

9. [14 marks: 2, 2, 1, 1, 2, 3, 1, 2]

The following table displays the Number Employed (thousands) and the corresponding Annual Turnover (in $million) for several types of industry.

Type of Industry	Number Employed ('000), N	Annual Turnover $m, A
Food, beverages	160	43 000
Textiles	75	10 000
Wood and paper	60	11 000
Printing	90	14 000
Energy Products	90	30 000
Non-metal mineral products	40	9 000
Metal products	150	37 000
Machinery	200	40 000

(a) Draw a scatter-graph for this data.



Accuracy of points ✓✓

(b) Calculate the coefficient of linear correlation and comment on the nature of the relationship between N and A

$r = 0.9073$ ✓
The relationship between N and A is a strong and positive linear relationship. ✓

Calculator Assumed

9. (c) Find the gradient of the least squares regression line of A on N.

Gradient = 240.02 ✓

(d) Find the vertical intercept of the least squares regression line of A on N.

Vertical intercept = –1702.50 ✓

(e) Determine the increase in turnover for every 10 000 increase in the number employed.

Increase in turnover = 240.02 × 10 ✓
≈ \$2 400 million ✓

(f) Use the least squares regression line to predict the annual turnover for a business that employs 5 000 people.
Comment on the reliability of your prediction.

When N = 5 thousand, ✓
$A = 240.02 \times 5 - 1702.50$
≈ –\$502 million ✓

Prediction is unreliable as an extreme extrapolation is involved. ✓

(g) An American reporter converts the turnover to US\$ using the conversion AUD\$1 = US\$0.70. Calculate the correlation coefficient between the Number employed and the Annual Turnover in US\$.

r = 0.9073 (unchanged) ✓

(h) Comment on the statement made by a politician that increasing the number of people employed will create a higher annual turnover.

Statement is inaccurate as there is not necessarily a cause and effect relationship between the number employed and the annual turnover. ✓✓

Calculator Assumed

10. [13 marks: 2, 2, 2, 2, 2, 2, 1]

The table below shows the heights of 20 students (in inches) and their self-esteem rating (out of 10, the higher the score, the higher the self-esteem).

Height, h	Self-esteem rating, s
68	4.1
71	4.6
62	3.8
75	4.4
58	3.2
60	3.1
67	3.8
68	4.1
71	4.3
69	3.7
68	3.5
67	3.2
63	3.7
62	3.3
60	3.4
63	4.0
65	4.1
67	3.8
63	3.4
61	3.6

(a) Find the mean height and the associated standard deviation.

Mean height = 65.4 cm ✓

Std. dev. for height = 4.2942 ✓

(b) Find the mean and median self-esteem rating.

Mean self-esteem rating = 3.755 cm ✓

Median self-esteem rating = 3.75 ✓

Calculator Assumed

10. (c) Calculate the coefficient of linear correlation between s and h. Comment on the type of relationship between the s and h.

$r = 0.7306$ ✓

There is a moderately strong positive linear relationship between s and h. ✓

(d) Calculate the least squares regression line that will allow you to predict the self-esteem of a student given the student's height.

$s = 0.0707h - 0.8663$ ✓✓

(e) Use your regression line to determine the increase/decrease in the self-esteem rating for every inch drop in height.

Gradient of regression line = 0.0707. ✓
Hence, for every 1 inch drop in height, self-esteem rating will drop by ≈ 0.07 points. ✓

(f) Ubiuty is 50 inches tall. Use your regression line in (e) to predict Ubiuty's self-esteem rating. Comment on the reliability of your prediction.

$s = 0.0707 \times 50 - 0.8663$
$= 2.7$ ✓

Prediction is not reliable as an extreme extrapolation is involved. ✓

(g) The heights were converted to centimetres (1 inch = 2.54 cm) for Australian readers. What is the new coefficient of linear correlation between the height of students and their self-esteem ratings?

$r = 0.7306$
r remains unchanged as both variables had multipliers with the same sign (h was multiplied by 2.54 and s was multiplied by 1). ✓

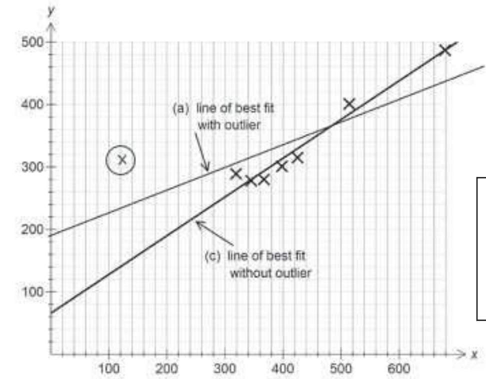
Calculator Assumed

11. [14 marks: 1, 1, 1, 2, 3, 2, 2, 2]

The table below shows the recorded number of births and deaths in several suburbs over a period of one year.

Suburbs	No. of births x	No. of deaths, y
A	345	278
B	425	315
C	125	310
D	319	289
E	679	487
F	514	401
G	367	280
H	398	301

(a) The scatter-graph for this data is drawn below. Draw the line of best fit.



(a) Line of best fit ✓
(b) Outlier ✓
(c) Line of best fit ✓

(b) Mark with a circle the outlier on the scatter-graph in (a).

(c) Remove the identified outlier. Draw the line of best fit for the remaining data.

(d) For the given data points, discuss the effect of the outlier on the vertical intercept and the gradient of the line of best fit on the data points.

The outlier increases the vertical intercept and decreases the gradient of the line. ✓✓

Calculator Assumed

11. (e) Give the most reliable prediction for the number of deaths for a suburb with 500 births? Describe how you obtained your answer.

.

Prediction is most reliable with the outlier (125, 310) removed: ✓
line of best fit has equation $y = 65.79 + 0.62x$. ✓
When $x = 500$, $y \approx 376$ ✓

(f) Estimate the increase/decrease in deaths for every increase of 50 births.

Gradient of regression line = 0.62 ✓

Hence, for every increase of 50 births,
number of deaths will increase by $0.62 \times 50 \approx 31$. ✓

(g) George argues that since there is a good correlation between the number of births and deaths, he should move to a suburb with lower births, so that he has a smaller chance of dying. Comment mathematically on his statement.

Conclusion is incorrect as it assumes that strong correlation implies a cause and effect relationship. ✓✓

(h) Suggest a reason why an increase in the number of births is accompanied by an increase in the number of deaths.

Larger population. ✓
Need to consider births as % of population and deaths as % of population. ✓

Calculator Assumed

12. [10 marks: 2, 2, 4, 4]

The table below shows pairs of readings of Q and t obtained from an experiment.

t	1	5	8	11	13	14	17	19	20	24
Q	2.0	2.6	3.4	4.1	4.9	5.0	6.2	6.6	7.0	8.6
Residual	0.49	−0.07	−0.14	−0.30	−0.08	−0.27	0.06	−0.12	−0.01	0.44

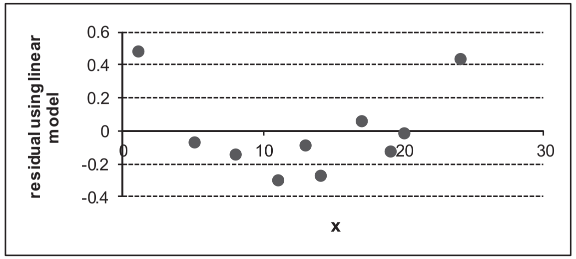
✓✓

(a) Find the least squares regression line of Q on t and the corresponding coefficient of linear correlation between Q and t.

$\hat{Q} = 0.2892t + 1.2225$ ✓
$r = 0.9917$ ✓

(b) Complete the row of linear residuals above.

(c) On the axes provided below, draw graph of the residuals for Q. Comment on the appropriateness of a linear relationship between Q and t.



Residual plot. ✓✓
Residual plot has a pattern. ✓
Hence, a linear relationship is not appropriate. ✓

(d) Use your answer in (a) to predict the value of Q when $t = 12$ and comment on the reliability of your result.

$t = 12$, $\hat{Q} = 4.7$ ✓✓
Although the coefficient of linear correlation is very strong, the residual plot indicates that a linear model is not appropriate. ✓
Hence, the prediction must be treated with great caution. ✓

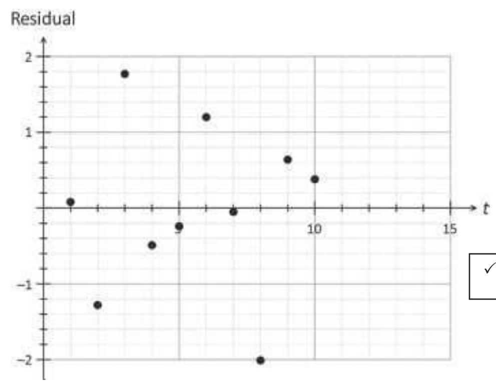
Calculator Assumed

13. [9 marks: 2, 2, 2, 3]

The least squares regression line of Q on t is given by $Q = 0.86t - 2.07$.
The coefficient of linear correlation between Q and t is 0.9191.
The table below shows the residuals associated with this regression line.

t	1	2	3	4	5	6	7	8	9	10
Residual	0.08	−1.28	1.77	−0.49	−0.24	1.2	−0.05	−2.01	0.64	0.38

(a) Sketch on axes provided below the graph of the residuals against t.



✓✓

(b) Determine with reasons if fitting a linear model to the values of t and Q is appropriate.

Residual plot is random. ✓
Hence, fitting the linear model is appropriate. ✓

(c) Predict the value of Q for $t = 11$. Give your answer correct to 1 decimal place

$Q = 0.86 \times 11 - 2.07$ ✓
$= 7.39 \approx 7.4$ ✓

(d) Comment on the reliability of your prediction in (c).

Linear fit is appropriate and
correlation is very high, and the base value
for making the prediction is just outside ✓✓
the domain of the data set, prediction is reliable. ✓

04 Data Investigation Process

Calculator Free

1. [10 marks: 2, 2, 3, 3]

Mrs Mazzart teaches music and mathematics. She wishes to investigate the claim that mathematical competence and musical competence are related. Together with some of her students, they design a statistical investigation to study this claim.

(a) State a possible response and explanatory variable for the investigation.

Response variable: Performance in Mathematics Test ✓
Explanatory variable: Performance in Music Test ✓
OR
Response variable: Performance in Music Test
Explanatory variable: Performance in Mathematics Test

(b) Describe the data that need to be collected and how the data is to be collected.

- Identify a group of students that are enrolled in the same music course and same mathematics course. ✓
- For each student, the marks for the music course and the marks for the mathematics course is retrieved from the teacher's marks book. ✓

(c) Describe how you would display and analyse the data.

- Display collected data on a scatter-graph. ✓
- If the scatter graph shows a possible linear relationship between the variables, calculate the coefficient of linear correlation between the variables. ✓
- Plot a graph of the calculated residuals. ✓

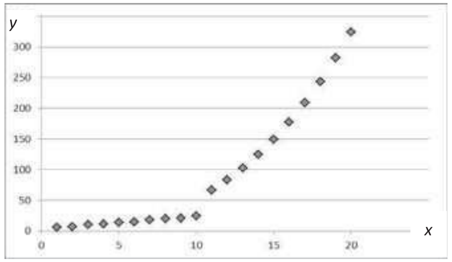
(d) Describe how you would interpret the data you analysed.

- Use the value of the coefficient of linear correlation r to describe the strength and type of relationship between the variables ✓
- Use the residual plot to confirm the suitability of using a linear model to describe the relationship between the variables. ✓
- State that the relationship between variables is that of an association and not of causation. ✓

Calculator Free

2. [6 marks: 4, 2]

The diagram below shows the scatter-graph between the variables x and y.

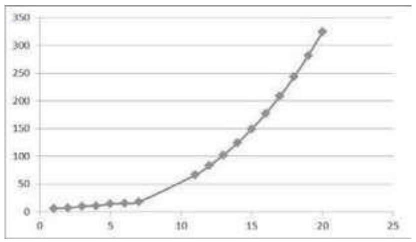


(a) Discuss the nature of the association between the variables x and y.

For $1 \le x \le 10$, ✓
the association appears to be a positive linear association. ✓

For $11 \le x \le 20$, ✓
the association appears to be non-linear positive association. ✓

(b) A researcher decided that the y-values associated with $x = 8$, $x = 9$ and $x = 10$ were incorrectly measured/recorded. The researcher removed these scores and obtained the resulting curve.



Was the researcher justified in doing so? Give your reasons.

No.
The scatter-plot indicates clearly that there are two sets of relationships between x and y. ✓✓

Calculator Assumed

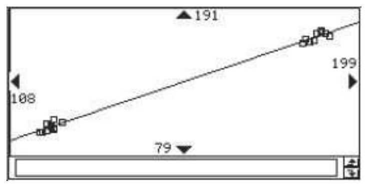
3. [12 marks: 2, 1, 5, 4]

The table below shows the age (in months) and their heights (in cm) for a group of 20 students.

Age, x	Height, y
120	98
119	100
120	99
118	102
120	105
117	96
116	95
119	101
122	103
118	97
192	170
190	175
186	168
185	165
189	172
185	165
190	173
188	167
187	166
191	172

(a) Plot a scatter-graph of the data in your CAS calculator and describe the scatter-plot.

The plotted points are located within two separate clusters which are widely separated. ✓
Within each cluster, the points appear to be linearly related. ✓



(b) Find the coefficient of linear correlation between x and y.
Give your answer to 4 decimal places.

$r = 0.9981$ ✓

Calculator Assumed

3. (c) Predict the height of a student aged 150 months. Explain clearly how you obtained your answer and comment on the reliability of your prediction.

Equation of line of best fit on the entire data set obtained from calculator: ✓

$y = 1.0050x - 19.9115$ ✓

For $x = 150$, from line of best fit: $\hat{y} = 130.8 = 131$ ✓

The point corresponding to $x = 150$ is in between the two widely separated clusters. ✓
Hence, although the correlation is extremely strong, the predicted value must be treated with caution. ✓

Or
Equation of line of best fit on the first cluster obtained from calculator: ✓

$y = 1.2862x - 53.3346$ ✓

For $x = 150$, from line of best fit: $\hat{y} = 139.6 = 140$ ✓
Associated correlation is $r = 0.6940$.
Hence, the predicted value is an extrapolated value and is hence not reliable. ✓✓

(d) Use the most appropriate part of the table to predict the height of a student aged 121 months. Explain your choice of data and show clearly how you obtained your answer. Comment on the reliability of your prediction.

$x = 121$ lies within the first cluster. ✓
Equation of line of best fit on the first cluster obtained from calculator:

$y = 1.2862x - 53.3346$ ✓

For $x = 121$, from line of best fit: $\hat{y} = 102.3 = 102$ ✓
Associated correlation is $r = 0.6940$.
The scatter-graph shows a linear relationship.
The predicted value is an interpolated value with a moderately strong linear correlation.
Hence, the predicted value is fairly reliable. ✓

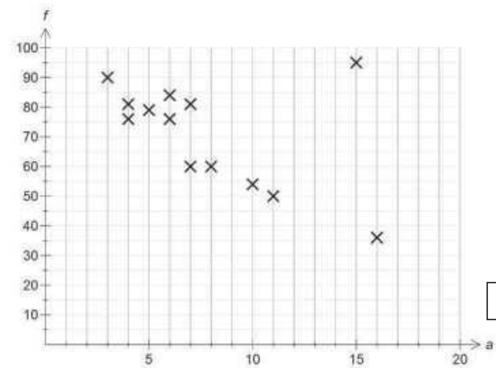
Calculator Assumed

4. [12 marks: 2, 1, 3, 2, 2, 2]

The Head of a Mathematics Department produced the following table to show the link between lesson attendance and final mark.

No of absences for the year, a	Final Mark, f
6	84
7	81
8	60
11	50
4	81
15	95
5	79
7	60
6	76
3	90
16	36
10	54
4	76

(a) Draw a scatter graph for this table.



✓✓

(b) Identify the outlier in the data set.

Outlier is (15, 95). ✓

Calculator Assumed

4. (c) Discuss the effect the outlier has on the coefficient of linear correlation between a and f.

> With the outlier, $r_{af} = -0.4954$. ✓
> Without the outlier, $r_{af} = -0.9222$ ✓
> Hence, the outlier reduces the strength of the coefficient of linear correlation. ✓

(d) Use an appropriate selection of the given data and a suitable statistical calculation to estimate the final mark for Jon who was absent for 18 lessons. Comment on the reliability of your answer.

> Without the outlier, equation of the line of best fit:
> $f = 99.14 - 4.17a$. ✓
>
> When $a = 18$, $f \approx 24$
> The scatter-graph without the outlier shows that a linear relationship exists.
> The estimate is based on a single a-value that is just outside the known data set. This is however accompanied with a very strong linear correlation. Hence, estimate may be considered reliable. ✓

(f) Determine the average increase/decrease in the final mark for every five absences.

> Without the outlier, equation of the line of best fit:
> $f = 99.14 - 4.17a$.
>
> Gradient = –4.17 ✓
> Hence, for every 5 absences Final mark will decrease by $4.17 \times 5 \approx 21$ marks. ✓

(g) From the calculated coefficient of linear correlation, the Head of Department concluded that poor attendance is the cause of poor final marks. Discuss mathematically the truth or falsity of this statement.

> Conclusion is incorrect as it assumes that strong correlation implies a cause and effect relationship. ✓✓

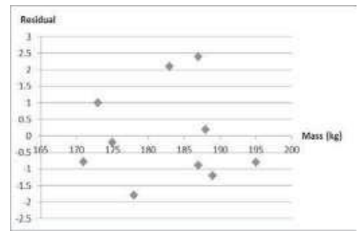
Calculator Assumed

5. [10 marks]

The table below shows the mass (m kg), height (h cm) and the time taken to run a 400 m race for 10 athletes.

Athlete	Mass, m (kg)	Height, h (cm)	Time, t (sec)
A	67	178	45.1
B	65	195	41.0
C	72	187	46.6
D	66	188	44.1
E	69	171	48.2
F	71	189	42.4
G	73	173	49.4
H	67	187	43.3
I	70	175	47.6
J	75	183	47.5

Athlete K weighs 68 kg and is 176 cm tall.
Use an appropriate statistical method to determine two possible predictions for the time K will take to run the 400 m race. Determine with reasons, which is the more reliable prediction.

> Correlation between m and t, $r_{mt} = 0.6633$ ✓
> Regression line t on m: $\hat{a} = 0.5606t + 6.5568$ ✓
> When $m = 68$ $\hat{a} = 44.7$ sec ✓
>
> Correlation between h and t, $r_{ht} = -0.8594$ ✓
> Regression line t on h: $\hat{a} = -0.2993t + 100.1715$ ✓
> When $h = 176$ $\hat{a} = 47.5$ ✓
>
> Since $|r_{ht}| > |r_{mt}|$, $\hat{a} = 47.5$ is a possibly a better predictor for t. ✓
>
> 
>
> ✓
>
> The residual plot is random and hence the linear model used for h against t is appropriate. ✓
> This confirms that $\hat{a} = 47.5$ is a better predictor for t. ✓

Calculator Assumed

6. [12 marks]

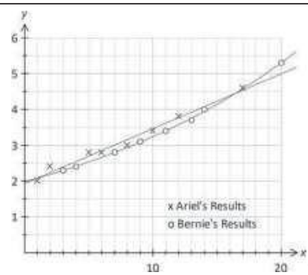
In a science experiment, students were required to measure and determine the algebraic relationship between the variables x and y. Tables 1 and 2, shows respectively the results obtained by Ariel and Bernie. Chang combines the results obtained by Ariel and Bernie to form a larger collection of 16 pairs of measurements of x and y.

Table 1	(Ariel)
x	y
2	2.4
5	2.8
8	3
1	2
12	3.8
10	3.4
17	4.6
6	2.8

Table 2	(Bernie)
x	y
3	2.3
4	2.4
7	2.8
9	3.1
11	3.4
13	3.7
14	4
20	5.3

Discuss using statistical methods whether it is appropriate for Chang to combine the results obtained by Ariel and Bernie.
Investigate the effects of combining the results obtained by Ariel and Bernie.

From scatter graph drawn, Ariel's results seem to fit a linear model ✓
whereas Bernie's result fits a non-linear model. ✓
Hence, it would not be appropriate for Chang to combine the two sets of results. ✓

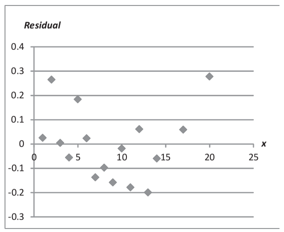


✓✓

For Ariel: correlation between x and y, $r_{xy} = 0.9915$ ✓
For Bernie: correlation between x and y, $r_{xy} = 0.9900$ ✓
For Chang: correlation between x and y, $r_{xy} = 0.9862$ ✓

The relationship between x and y cannot be both linear and non-linear at the same time. Hence, either Ariel is incorrect or Bernie is incorrect. ✓

If Ariel is correct, then to combine her results with Bernie's would reduce the coefficient of linear correlation from 0.9915 to 0.9862.
Also, the residual plot has a clear a pattern and hence, indicates that the linear model would be incorrect. That is, if the relationship between x and y is linear, to include Bernie's results would make the relationship non-linear. ✓✓



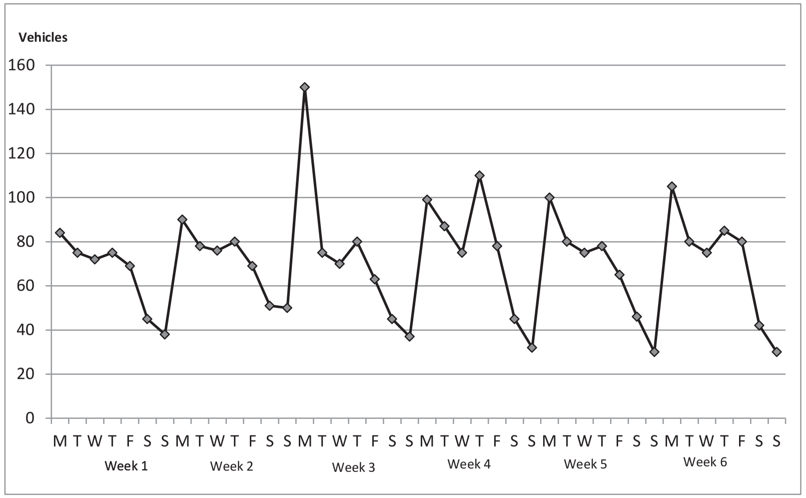
✓

05 Time Series

Calculator Free

1. [6 marks: 4, 2]

The accompanying diagram shows the number of vehicles passing through a road junction between 8.00 am and 8.30 am each day.



(a) Describe the time series in terms of its period, trend, seasonal and unseasonal fluctuations.

- Time series has a period of 7 days; it peaks each Monday and bottoms out each Sunday. ✓
- The trend is somewhat ambiguous. ✓
  There is a slight increase in the peak traffic and a slight decrease in the minimum traffic over time.
- There is unusually high peak on Monday of the 3rd week. ✓
- There is an unseasonal high fluctuation on Thursday of week 4. ✓

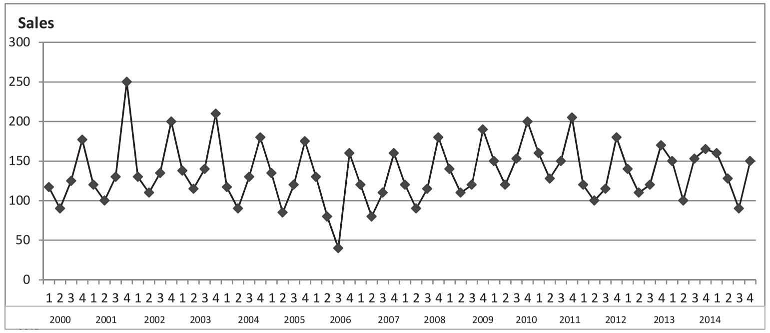
(b) Suggest an appropriate method to smooth the fluctuations.

Calculate and plot seven point moving averages. ✓✓

Calculator Free

2. [9 marks: 2, 2, 3, 2]

The accompanying diagram shows the sales of vehicles in a certain city for the various quarters between 2000 and 2015 inclusive.



(a) Determine and describe the period for the time series.

- Time series has a period of 4 quarters. ✓
- Peaks in Q4 and bottoms out in Q2 ✓

(b) Describe any long term fluctuations (cycles) in this time series.

- There appears to be a long term cycle of 8 years. ✓✓

(c) Describe any unusual fluctuations in this time series.

- The peak is unusually high in Q4 of 2001. ✓
- There is an unseasonal downturn in Q3 of 2006 and 2015. (Q3 sales usually trend upwards from Q2, but in these instances they were lower than Q2) ✓✓

(d) The overall trend in the time series is masked by the fluctuations. Suggest a procedure to smooth these fluctuations to expose the trend.

- Calculate and plot four point centred moving averages. ✓✓

Calculator Free

3. [5 marks: 2, 3]

The accompanying table shows a time series data with period of 4 and some of the associated 4 point centred moving averages.

Time	Data	4 pt. cma
1	4	
2	8	
3	16	***a***
4	16	14
5	12	16
6	***b***	18
7	24	20.125
8	24	22.375
9	21	24.625
10	25	27
11	33	
12	34	

(a) Calculate the value of ***a***. Show clearly how you obtained your answer.

$\boldsymbol{a} = \dfrac{0.5(4)+8+16+16+0.5(12)}{4}$ ✓

$= 12$ ✓

(b) Calculate the value of ***b***. Show clearly how you obtained your answer.

$\dfrac{0.5(16)+12+\boldsymbol{b}+24+0.5(24)}{4} = 18$ ✓

$8 + 12 + \boldsymbol{b} + 24 + 12 = 72$ ✓

$\boldsymbol{b} = 16$ ✓

Calculator Assumed

4. [8 marks: 2, 4, 2]

The table below shows the number of vehicles, N, entering a national park each month. A set of six point centred moving averages, M, is calculated and displayed in column 4.

Year	Month	No. of vehicles, N	Moving Average, M
2013	Jan	950	
	Feb	850	
	Mar	650	
	Apr	700	750.8
	May	620	753.3
	June	720	754.2
	July	**A**	755.8
	Aug	850	767.5
	Sept	660	783.3
	Oct	710	791.7
	Nov	750	806.7
	Dec	780	821.7
2014	Jan	1020	825.8
	Feb	990	**B**
	Mar	700	815.0
	Apr	720	
	May	690	
	June	760	

(a) The six point moving average for March 2014 is 815. Show how this was calculated.

$$\frac{0.5\times780+1020+990+700+720+690+0.5\times760}{6}$$ ✓✓

(b) Find the values of **A** and **B**.

A = 755.8 × 6 – (700 × 0.5 + 620 + 720 + 850 + 660 + 710 × 0.5) ✓
= 979.8
= 980 ✓

$$\mathbf{B} = \frac{0.5\times750+780+1020+990+700+720+0.5\times690}{6}$$ ✓
= 821.7 ✓

(c) Determine with reasons if it would be appropriate to use a set of five point moving averages to smooth the time series.

No. The period of the time series is six. Hence, it will not smooth the time series as adequately as a six point centred moving average. ✓✓

Calculator Assumed

5. [15 marks: 2, 4, 2, 2, 2, 3]

The accompanying table shows the number of vehicles caught speeding along a freeway over 20 consecutive working days (Mondays to Fridays inclusive) and the corresponding five point moving averages.

Day t	Vehicles, v	5 ma m
1	66	
2	31	
3	32	41.8
4	***a***	48.2
5	52	56.2
6	98	58.4
7	71	70.6
8	43	77.6
9	89	80.0
10	87	84.4
11	110	91.4
12	93	***b***
13	78	94.0
14	72	96.2
15	117	98.0
16	121	104.2
17	102	110.6
18	109	108.8
19	104	
20	108	

(a) Explain clearly why a five point moving average is the most appropriate type of moving average to mask any fluctuations.

The time series peaks at t = 1, 6, 11, 17 ✓
Hence, the period of the time series is five. ✓

(b) Calculate the values of ***a*** and ***b***.

a = 48.2 × 5 – 31 – 32 – 52 – 98 ✓
= 28 ✓

$$b = \frac{87+110+93+78+72}{5}$$ ✓
= 88.0 ✓

The line of best fit through the moving averages has equation $\hat{m} = 4.4012t + 35.5627$ with a coefficient of determination of 95.5%.

(c) Determine with reasons the secular trend of the number of vehicles caught speeding along this freeway.

Secular trend is increasing. ✓
Reason:
The gradient of the trend line which is 4.4012 is positive. ✓

Calculator Assumed

5. (d) Predict the moving average for $t = 19$.

$\hat{m} = 4.4012 \times 19 + 35.5627$ ✓
$= 119.1855 \approx 119.2$ ✓

(e) Use your answer in (d) to predict the number of cars that will be caught speeding for $t = 21$.

$119.2 = \dfrac{102+109+104+108+x}{5}$ ✓
$x = 173$ ✓

(f) *Discuss* the reliability of your prediction in (e).

Prediction must be treated with great caution. ✓

Reason:

Although the coefficient of determination is very high, the predicted value of 173 is much too high. ✓

Based on a period of 5, there should be a peak at $t = 21$.
However, the increase from the previous "peak" to the "peak" $t = 1$ to $t = 6$ is 32; from $t = 6$ to $t = 11$ is 12 and from $t = 11$ to $t = 16$ is 11.
This gives an average peak to peak increase of about 20 vehicles.

But for $t = 16$ to $t = 21$, the increase is predicted to be 52 which is way above the average peak to peak increase. ✓

Calculator Assumed

6. [9 marks: 1, 4, 4]

The accompanying table shows the average price of commodity X over a period of 26 weeks and a set of calculated moving averages.

Week	Price (cents)	Moving Average
1	129.7	
2	158.9	
3	152.1	153.225
4	148.3	165.300
5	177.5	177.250
6	207.7	189.075
7	198.9	201.275
8	196.1	***a***
9	227.3	224.925
10	253.5	237.125
11	246.7	249.325
12	245.9	261.150
13	275.1	273.100
14	300.3	284.738
15	295.5	296.625
16	***b***	309.200
17	325.9	322.150
18	350.1	335.163
19	349.3	347.675
20	340.5	360.000
21	375.7	372.200
22	398.9	384.400
23	398.1	396.475
24	389.3	409.050
25	423.5	
26	451.7	

(a) Determine with reasons the best set of moving averages to smooth the time series.

Four point centred moving averages, as period is 4 weeks ✓

(b) Find the values of $\boldsymbol{a}$, and $\boldsymbol{b}$.

$\boldsymbol{a} = \dfrac{(207.7\times0.5 + 198.9 + 196.1 + 227.3 + 253.5\times0.5)}{4}$ ✓
$= 213.225$ ✓

$\dfrac{(295.5\times0.5 + \boldsymbol{b} + 325.9 + 350.1 + 349.3\times0.5)}{4} = 322.150$ ✓
$\boldsymbol{b} = 290.2$ ✓

(c) Calculate the equation of the line of best fit through the moving averages. Hence, determine with reasons the trend of the time series.

Moving Average = 12.1853 × time + 115.6186 ✓✓

Trend is increasing as the trend line has a positive gradient. ✓✓

Calculator Assumed

7. [4 marks: 2, 2]

(a) The table below shows the quarterly seasonal indices for a time series.

Quarter 1	Quarter 2	Quarter 3	Quarter 4
1.09	1.05	0.98	x

Use the average percentage method to determine the value of x.

$x = 4 - 1.09 - 1.05 - 0.98$
$= 0.88$ ✓✓

(b) The table below shows the bi-monthly seasonal indices for a time series.

Jan-Feb	Mar-Apr	May-June	July-Aug	Sept-Oct	Nov-Dec
0.99	1.01	1.02	1.01	x	0.98

Use the average percentage method to determine the value of x.

$x = 6 - 0.99 - 1.01 - 1.02 - 1.01 - 0.98$
$= 0.99$ ✓✓

8. [8 marks: 2, 2, 4]

The monthly seasonal index for visitor numbers to a certain city for June is 1.06.

(a) Comment on the visitor numbers in June in relation to the average monthly number of visitors.

Visitor number in June is on average 6% higher than the monthly figures. ✓✓

(b) The number of visitors to this city in June 2015 was 150 000. Determine the seasonally adjusted number of visitors to this city in June 2015.

Seasonally adjusted numbers $= \dfrac{150000}{1.06}$ ✓
$= 141\,509$ ✓

(c) The line of best fit through the seasonally adjusted figures for monthly visitor numbers for 2010 to 2015 is $s = 1200m + 85\,000$ where January 2010 is $m = 1$. Predict the visitor numbers for June 2016.

$\hat{s} = 1200 \times 78 + 85\,000 = 178\,600$ ✓✓
Predicted numbers $= 178\,600 \times 1.06 \approx 189\,316$ ✓✓

Calculator Assumed

9. [8 marks: 2 marks each]

A store operates six days a week. Table 1 below shows the net income (to the nearest $100) from this store over four weeks.

Week	Mon	Tues	Wed	Thurs	Fri	Sat	Daily Average
1	4 500	5 000	7 000	12 000	11 000	15 000	9 083.33
2	4 600	5 000	7 200	11 900	10 900	15 300	***a***
3	4 500	***b***	7 100	13 000	12 000	16 000	9 583.33
4	4 400	4 800	7 000	12 100	12 000	15 800	9 350.00

Table 2 below shows the daily sales as a percentage of the weekly daily average and the seasonal indices for the data in the table above. Some of the cells have been deliberately left blank.

Week	Mon	Tues	Wed	Thurs	Fri	Sat
1	0.495 41	0.550 46	***c***	1.321 10	1.211 01	1.651 38
2	0.502 73	0.546 45	0.786 89	1.300 55	1.191 26	1.672 13
3	0.469 57	0.511 30	0.740 87	1.356 52	1.252 17	1.669 57
4	0.470 59	0.513 37	0.748 66	1.294 12	1.283 42	1.689 84
Index	0.484 6	0.5304		1.3181	***d***	1.670 7

Calculate the values of ***a***, ***b***, ***c*** and ***d***.

$\boldsymbol{a} = \dfrac{4600+5000+7200+11900+10900+15300}{6}$ ✓
$= 9150.00$ ✓

$\boldsymbol{b} = 6 \times 9583.33 - 4500 - 7100 - 13000 - 12000 - 16000$ ✓
$= 4899.98 \approx 4900$ ✓

$\boldsymbol{c} = \dfrac{7000}{9083.33}$ ✓
$\approx 0.770\,64$ ✓

$\boldsymbol{d} = \dfrac{1.21101+1.19126+1.25217+1.28342}{4}$ ✓
≈ 1.2345 ✓

Calculator Assumed

10. [10 marks: 6, 2, 2]

The table below shows the quarterly sales figures ($m) for a company. The seasonal quarterly index (using the average percentage method) for Q4 is 1.1146.

Year	Quarter	Sales ($m)	Quarterly Mean, Q	Sales as % of Q
2012	Q1	700	698.25	1.0025
	Q2	621		*a*
	Q3	683		0.9782
	Q4	789		1.13
2013	Q1	841	839	1.0024
	Q2	*b*		0.8939
	Q3			0.9821
	Q4	941		1.1216
2014	Q1	974	971.5	1.0026
	Q2	*c*		
	Q3	951		0.9789
	Q4	1077		1.1086
2015	Q1	1124	1141	0.9851
	Q2	1049		0.9194
	Q3	1138		0.9974
	Q4	1253		*d*

(a) Determine the values of a, b, c and d.

$a = \dfrac{621}{698.25} = 0.8894$ ✓

$b = 839 \times 0.8939 \approx 750$ ✓

$c = 971.5 \times 4 - 974 - 951 - 1077$ ✓
$= 884$ ✓

$d = 1.1146 \times 4 - 1.13 - 1.1216 - 1.1086$ ✓
$= 1.0982$ ✓

(b) Calculate the seasonally adjusted sales figure for Q4 of 2015.

Seasonally adjusted $= \dfrac{1253}{1.1146}$ ✓
$\approx$ \$1 124 million ✓

(c) The predicted seasonally adjusted sales figure for Q4 of 2016 is $1200 million. What is the predicted sales figure for Q4 of 2016?

Predicted Sales $= 1200 \times 1.1146$ ✓
$\approx$ \$1 338 million ✓

Calculator Assumed

11. [18 marks: 2, 2, 2, 2, 4, 4, 2]

The table below shows the number of passengers travelling on a certain bus route and the corresponding seasonally adjusted daily number of passengers. The seasonal daily index (using the average percentage method) for Wednesdays is 1. 0291.

Week	Day	Passengers	Seasonally adjusted, *s*
1	Mon	1134	771
	Tue	1013	918
	Wed	973	945
	Thu	1013	842
	Fri	933	952
	Sat	611	931
	Sun	517	928
2	Mon	1214	826
	Tue	1053	954
	Wed	1026	
	Thu	1080	897
	Fri	933	952
	Sat	691	1053
	Sun	678	1217
3	Mon	2018	1373
	Tue	1013	918
	Wed		919
	Thu	1080	897
	Fri	852	870
	Sat	611	931
	Sun	504	905
4	Mon	1335	908
	Tue	1174	1064
	Wed	1013	984
	Thu	1482	1231
	Fri	1053	1075
	Sat	611	931
	Sun	437	785

(a) Determine the daily mean number of passengers for week 1.

Daily mean for week 1
$= \dfrac{1134+1013+973+1013+933+611+517}{7}$ ✓
≈ 884.8571 ✓

Calculator Assumed

11. (b) Calculate the proportion of passengers travelling on Wednesday of the first week in relation to the mean daily number of passengers in the first week.

> Proportion = $\dfrac{973}{884.8571} \approx 1.0996$ ✓✓

(c) Calculate the seasonally adjusted number of passengers for Wednesday of the second week.

> Seasonally adjusted = $\dfrac{1026}{1.0291} \approx 997$ ✓✓

(d) Calculate the actual number of passengers for Wednesday of the third week.

> Actual numbers = $919 \times 1.0291 \approx 946$ ✓✓

(e) Determine the seasonal daily index for Sundays and explain what this index means.

> Seasonal Index for Sundays = $\dfrac{517}{928}$ ✓
> ≈ 0.5571 ✓
>
> Passenger numbers on Sundays are 44.3% lower than daily mean passenger numbers. ✓✓

(f) Using Monday of week one as $t = 1$, the line of best fit of the seasonally adjusted numbers against time t is $s = 3.735t + 909.198$ with a coefficient of correlation value of 0.232.

(i) Predict the number of passengers on this route on Wednesday of week 5.

> $\hat{s} = 3.735 \times 31 + 909.198 = 1024.98$ ✓✓
> Predicted numbers = $1024.98 \times 1.0291 \approx 1055$ ✓✓

(ii) Comment on the reliability of your prediction.

> Prediction needs to be treated with great caution as coefficient of correlation is low indicating that the seasonally adjusted figures may not actually fit a linear model. ✓✓

Calculator Assumed

12. [13 marks: 2, 2, 2, 4, 3]

Data is collected between 2003 to 2019 inclusive for a time series variable with a seasonal pattern of four quarters. The line of best fit through the deseasonalised data is $s = -0.5t + 200$, where $t = 1$ represents quarter 1 of 2003. The line of best fit has a linear correlation coefficient of -0.94. The seasonal indices for quarters 1, 2, 3 and 4 are respectively 120, 110, x and 80.

(a) Calculate the value of x.

> $120 + 110 + x + 80 = 400$ ✓
> $x = 90$ ✓

(b) Determine with reasons, which seasons had time series values below the seasonal mean.

> Q3 and Q4 are below the seasonal mean. ✓
> Reason: Seasonal index is less than 100. ✓

(c) Predict the seasonally adjusted value for Quarter 4 of 2022.

> Quarter 4 of 2022 $\Rightarrow$ $t = 80$ ✓
> $s = -0.5 \times 80 + 200$
> $= 160$ ✓

(d) Predict the time series value when $t = 110$.

> $s = -0.5 \times 110 + 200$
> $= 145$ ✓
> $t = 110 \Rightarrow$ Quarter 2 ✓
> Predicted time series = 145×1.1 ✓
> $= 159.5$ ✓

(e) *Discuss* the reliability of the prediction in (d).

> The available data covers only 68 quarters.
> $t = 110$ corresponds to 42 quarters beyond the available data. ✓
> Hence, although correlation is very strong, this is an extreme extrapolation. ✓
> Hence, the prediction is not reliable. ✓

Calculator Assumed

13. [9 marks: 1, 1, 7]

The table below shows the quarterly sales of passenger cars for a car company and the accompanying seasonally adjusted sales numbers. Some of the cells in the table have been deliberately left blank. The seasonal index for the March quarter is 0.970 65.

Year	Quarter	Sales	Seasonally Adjusted Sales
2013	Mar	515	530.5723
	June		533.8706
	Sept	540	534.3757
	Dec	550	536.0232
2014	Mar	545	561.4794
	June		554.0166
	Sept	560	554.1674
	Dec	570	555.5149
2015	Mar	555	571.7818
	June		574.1627
	Sept	585	578.9070
	Dec	590	575.0067
2016	Mar		597.5377
	June	595	599.3453
	Sept	600	593.7508
	Dec	610	594.4985

(a) The seasonally adjusted sales figures for the March quarter of 2014 is 561.4794. Show how this value is calculated.

$\dfrac{545}{0.97065} = 561.4794$ ✓

(b) Calculate the actual sales numbers for the March quarter of 2016.

$x = 597.5377 \times 0.97065$
$= 580$ ✓

(c) Using the information given above and an appropriate statistical method, predict the profit for the March quarter of 2017. Show all working.

Let March quarter of 2013 be $t = 1$. ✓
Line of best fit between seasonally adjusted data and time is $\hat{s} = 4.8545t + 524.0503$ ✓✓
March quarter of 2017 is $t = 17$. ✓
Predicted seasonally adjusted for $t = 17$ is
$\hat{s} = 4.8545 \times 17 + 524.0503$
$= 606.5768$ ✓
Hence, predicted profit for $t = 17$ is
606.5768×0.97065 ✓
≈ 589 ✓

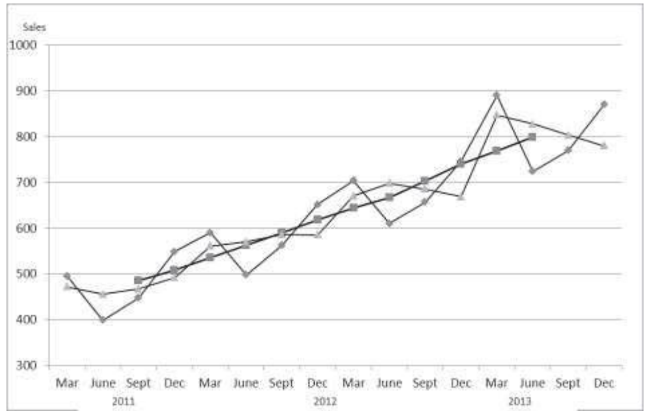
Calculator Assumed

14. [16 marks: 7, 2, 1, 4, 2]

The table below shows the quarterly number of cars sold by a car-yard. Included in the table are the corresponding 4 point centred moving averages and deseasonalised data. The seasonal quarterly index (using the average percentage method) for March is 1.0511.

Year	Quarter	Sales	4 pt. cma	Deseasonalised
2011	Mar	496		*a*
	June	399		456
	Sept	448	*b*	467
	Dec	549	508.88	492
2012	Mar	590	535.50	561
	June	498	562.63	570
	Sept	562	589.75	586
	Dec	652	618.00	585
2013	Mar	*c*	643.88	670
	June	610	667.50	698
	Sept	657	702.50	685
	Dec	746	740.00	669
2014	Mar	890	768.38	847
	June	*d*	798.00	828
	Sept	770		803
	Dec	870		780

The diagram below shows the plot of the time series, the four point centred moving averages and the deseasonalised data.



Calculator Assumed

14. (a) Calculate the values of a, b, c and d.

$a = \dfrac{496}{1.0511} \approx 472$ ✓✓

$b = \dfrac{(496\times0.5 + 399 + 448 + 549 + 590\times0.5)}{4}$ ✓

≈ 484.75 ✓

$c = 670 \times 1.0511 \approx 704$ ✓

$d = 4 \times 798 - 0.5(746) - 890 - 770 - 0.5(870)$ ✓

≈ 724 ✓

(b) Determine with reasons which of moving-averages or deseasonalised data perform a better job in smoothing the time series data.

From the graph drawn, moving averages does a better job as the points are a closer fit to a straight line. ✓✓

(c) State one disadvantage of using moving averages to smooth the time series data.

Absence of moving averages for the first two and last two quarters. ✓

Using the March Quarter of 2011 as $t = 1$, the regression line of moving averages on time t is $m = 28.57t + 392.13$ and the regression line of deseasonalised data on time t is $s = 26.54t + 409.95$.

(d) Use each of these two lines to predict the sales for the March quarter of 2015.

$\hat{m} = 28.57 \times 15 + 392.13 = 820.68$ ✓

$\dfrac{(890\times0.5 + 724 + 770 + 870 + x\times0.5)}{4} = 820.68$

Predicted numbers = 947.44 ≈ 947 ✓

$\hat{s} = 26.54 \times 17 + 409.95 = 861.13$ ✓

Predicted numbers = 861.13 × 1.0511 ≈ 905 ✓

(e) Is the difference in predictions in (d) significant? Why?

Difference between the two predictions = 42 cars.
Which is approximately a 4.5% difference. ✓
Hence, the difference is significant. ✓

06 Arithmetic Sequences

Calculator Free

1. [5 marks: 2, 2, 1]

Consider the sequence: 18 25 32 39 46….

(a) Find the recursive rule for this sequence.

Recursive rule is $T_{n+1} = T_n + 7$ ✓ with $T_1 = 18$ ✓

(b) Find the rule for the n^{th} term of this sequence.

$T(n) = 7n + 11$, $n = 1, 2, 3, \ldots$ ✓✓

(c) Find the 50^{th} term of this sequence.

$T(50) = 7 \times 50 + 11 = 361$ ✓

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2. [6 marks: 2, 2, 2]

The n^{th} term of an arithmetic sequence is described by the rule $T(n) = 100 - 5n$, where $n = 1, 2, 3, 4, 5, \ldots$.

(a) Find the first 5 terms of the sequence.

$T(1) = 100 - 5 \times 1 = 95$
$T(2) = 100 - 5 \times 2 = 90$
$T(3) = 100 - 5 \times 3 = 85$
$T(4) = 100 - 5 \times 4 = 80$
$T(5) = 100 - 5 \times 5 = 75$ [–½ per error, round down ✓✓]

(b) State the recursive rule for this sequence.

Recursive rule is $T_{n+1} = T_n - 5$ ✓ with $T_1 = 95$ ✓

(c) One of the terms of this sequence is –895. Find the term that is three terms after this term.

Term is –895 – 5 – 5 – 5 = –910 ✓✓

Calculator Free

3. [5 marks: 2, 3]

The n^{th} term of an arithmetic sequence is described by the rule $T(n) = 20 + 2n$, where the $n = 0, 1, 2, 3, \ldots$.

(a) Find the recursive rule of this sequence.

$T(n+1) = T(n) + 2 \qquad T(0) = 20$ ✓✓

(b) Find which term equals 1000.

$20 + 2n = 1000$ ✓
$2n = 980$
$n = 490$ ✓
Hence, the 491st term. ✓

4. [6 marks: 2, 2, 2]

A sequence is defined by the recursive rule $a(n+1) = a(n) + 0.5$ where $a(1) = 5$

(a) Find the first 5 terms of the sequence.

$a(1) = 5$
$a(2) = a(1) + 0.5 = 5 + 0.5 = 5.5$
$a(3) = a(2) + 0.5 = 5.5 + 0.5 = 6$
$a(4) = a(3) + 0.5 = 6 + 0.5 = 6.5$
$a(5) = a(4) + 0.5 = 6.5 + 0.5 = 7$ [−½ per error, round down ✓✓]

(b) Find the rule for n^{th} term in this sequence.

$a(n) = 0.5n + 4.5$, $n = 1, 2, 3, \ldots$ ✓✓

(c) Which term equals 10?

$0.5n + 4.5 = 10$ ✓
$0.5n = 5.5$
$n = 11$
Hence, 11th term. ✓

Calculator Free

5. [5 marks: 1, 2, 2]

The n^{th} term of an arithmetic sequence is described by the rule $T(n) = 10 + 3n$, where the first term is 13.

(a) Find the 20th term of this sequence.

$T(20) = 10 + 3 \times 20 = 70$ ✓

(b) Find which term equals 355.

$T(n) = 355 \Rightarrow 10 + 3n = 355$ ✓
$3n = 345$
$n = 115$ ✓
Hence, the 115th term.

(c) State the recursive rule for this sequence using the terms a_n, a_{n-1} and a_0.

$a_n = a_{n-1} + 3, \qquad a_0 = 13$ ✓✓

6. [5 marks: 3, 2]

The terms of a sequence are defined by $T_{n+1} - T_n + 20 = 0$ with $T_1 = 100$

(a) Show that this sequence is an arithmetic sequence.

Rewrite recursive rule as $T_{n+1} - T_n = -20$ with $T_1 = 100$ ✓
Hence, difference between any two consecutive terms is constant. ✓
Hence, sequence is an arithmetic sequence. ✓

(b) How many positive terms are there in this sequence?

Terms are 100, 80, 60, 40, 20, 0, −20 …..
Hence, there are 5 positive terms. ✓✓

Calculator Free

7. [6 marks: 3, 3]

The 8th term and 13th term of an arithmetic sequence are 50 and 80 respectively.

(a) Determine with reasons the 3rd term of this sequence.

> The 3rd term is as "far" from the 8th term
> as the 8th term is from the 13th term. ✓
> Difference between 8th & 13th term = 80 – 50 = 30 ✓
> Hence, 3rd term = 50 – 30 = 20 ✓

(b) Determine with reasons the common difference and first term of this sequence.

> Difference between terms = 30.
> There are 5 "terms" in between these two terms. ✓
> Hence, common difference = $\frac{30}{5} = 6$ ✓
> First term = 20 – 12 = 8 ✓

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8. [7 marks: 3, 4]

The difference between the tenth term and fifth term of an arithmetic sequence is 25.

(a) Determine with reasons, the difference between the thirtieth term and the fortieth term of this sequence.

> As sequence is an Arithmetic sequence,
> difference between terms is a constant. ✓
> Difference between any 5 consecutive terms is always a constant 25. ✓
> Hence difference between 30^{th} term and 40^{th} term is $-25 \times 2 = -50$. ✓

(b) Determine the first term of the sequence if the fifth term of the sequence is 30.

> There are 5 "terms" between T(5) and T(10).
> Hence, common difference = $\frac{25}{5} = 5$ ✓✓
> First term = 30 – 4 × 5 = 10 ✓✓

Calculator Free

9. [7 marks: 3, 4]

The sixth term of an arithmetic sequence is double its fourth term. The first term of the sequence is 20 and the common difference is *d*.

(a) Show that $T_4 = 2 \times d$.

> Given $T_6 = 2 \times T_4$:
> $T_6 = T_4 + 2 \times d$ ✓
> $\Rightarrow 2 \times T_4 = T_4 + 2 \times d$ ✓
> $T_4 = 2 \times d$ ✓

(b) Hence, find the recursive rule for the sequence.

> $T_4 = 2 \times d$
> But $T_4 = 20 + 3 \times d$ ✓
> $2 \times d = 20 + 3 \times d$ ✓
> $d = -20$ ✓
> Hence, $T_{n+1} = T_n - 20$ with $T_1 = 20$ ✓

---

10. [7 marks: 2, 3, 2]

An arithmetic sequence has first term *a* and common difference *d*.
The difference between the seventh term and the third term of an arithmetic sequence is equal to the third term.

(a) Write t_3 and t_7 respectively the third and seventh term in this sequence, in terms of *a* and *d*.

> $t_3 = a + 2d$ ✓
> $t_7 = a + 6d$ ✓

(b) Show clearly that $a = 2d$.

> $t_7 - t_3 = t_3$ ✓
> $(a + 6d) - (a + 2d) = a + 2d$ ✓
> $\Rightarrow a = 2d$ ✓

(c) Provide one possible arithmetic sequence of eight terms with the property that "*the difference between the seventh term and the third term is equal to the third term*".

> $a = 2d$
> Let $d = 1$. ✓
> Hence: 2, 3, 4, 5, 6, 7, 8, 9 ✓

Calculator Assumed

11. [3 marks: 1, 2]

A sequence is defined by the recursive rule $u(n+1) = u(n) + 8$ where $u(1) = -104$.

(a) Find the 10th term.

> Use CAS Calculator:
> $u(10) = -32$ ✓

(b) How many negative terms are there? Justify your answer.

> Use CAS Calculator:
> $u(13) = -8$
> $u(14) = 0$ ✓
> Hence, there are 13 negative terms. ✓

---

12. [6 marks: 3, 1, 2]

Consider the sequence: 3 7 11 15 19 …..

(a) Find the 20th term.

> Recursive rule is $T_{n+1} = T_n + 4$ with $T_1 = 3$ ✓ ✓
> From Calculator:
> $T_{20} = 79$ ✓

(b) Which term equals 319?

> Use CAS Calculator:
> $u(80) = 319$
> Hence, 80th term. ✓

(c) How many terms are there with values less than 500? Justify your answer.

> Use CAS Calculator:
> $u(125) = 499$
> $u(126) = 503$ ✓
> Hence, there are 125 terms. ✓

Calculator Assumed

13. [4 marks: 2, 2]

A new road 85 km long is being laid. At the end of the Stage 1, 35 km of road had been laid. It took 45 days to complete Stage 1. For Stage 2, covering the remaining 50 km, each day an extra 600 m of new road is completed.
Let $b(n)$ be the length of completed road at the start of day n into Stage 2.

(a) Write a recursive equation for the length of completed road at the start of day n.

> $b(n+1) = b(n) + 0.6$ where $b(1) = 35$. ✓✓

(b) Find how long it would take for entire road to be laid.

> From Calculator:
> $b(85) = 85.4$ ✓
> Hence, 84 days to complete Stage 2
> Hence, 129 days to complete laying the road. ✓

---

14. [5 marks: 3, 2]

Fiona is paid \$500 per week. After every 26 weeks, she receives a \$50 per week pay rise. Assume that the pattern of the pay rise is valid for 10 years and assume that there are 52 weeks in a year.
Let $a(n)$: Fiona's weekly pay at the start of half year number n.

(a) Write a recursive formula for Fiona's weekly pay.
Hence, find Fiona's weekly pay after 3 years?

> Then, $a(n+1) = a(n) + 50$ where $a(1) = 500$. ✓✓
> After 3 years: $n = 6$
> $\Rightarrow a(6) = 750$ ✓

(b) When will Fiona's pay be \$1000 per week?

> From Calculator:
> $a(11) = 1000$ ✓
> $n = 11 \Rightarrow$ After the start of the 6th year. ✓

Calculator Assumed

15. [8 marks: 2, 1, 2, 3]

The speed of a car is 8 ms^{-1} when it starts to accelerate. Its speed increases by 0.2 ms^{-1} each second so that its speed t seconds after it starts accelerating, $v(t)$ is as displayed in the table below.

t	1	2	3	4
$v(t)$ ms^{-1}	8.2	8.4	8.6	8.8

(a) Write a recursive equation for the speed of the car.

$v(t+1) = v(t) + 0.2$ where $v(0) = 8$. ✓✓

(b) Write an equation for the speed of the car in terms of t, where $t \geq 0$.

First Difference = 0.2
Hence, $v(t) = 8 + 0.2t$, $t \geq 0$ ✓

(c) The maximum permissible speed for the car along this stretch of road is 30 ms^{-1}. Find how long it takes the car to reach its maximum permissible speed. Justify your answer.

Max. speed = 30.
Hence, $8 + 0.2t = 30$ ✓
$t = 110$ seconds ✓

(d) As the car approaches its destination it decelerates so that it loses speed at a rate of 0.35 ms^{-1} each second. Use an appropriate recursive rule to find the time it takes the car to come to a stop from its maximum permissible speed.

$v(t+1) = v(t) - 0.35$ where $v(0) = 30$. ✓✓

From Calculator:
$v(85) = 0.25$
$v(86) = -0.1$
Hence, in the 86th second . ✓

Calculator Assumed

16. [9 marks: 2, 2, 3, 1, 1]

A special filter is installed to remove airborne particles from a dust-free room. The filter has to be changed every 10 weeks. In the first five weeks of operation, it removes 10 000 particles each week and thereafter its filtering capacity reduces by 500 particles each week.

(a) Write a recursive equation that describes the filtering capacity of the filter for the first five weeks.

$T_{n+1} = T_n$ ✓ where $T_1 = 10\,000$ ✓ $(1 \leq n \leq 5)$ *(optional)*

(b) Write a recursive equation that describes the filtering capacity of the filter from the sixth week to the tenth week inclusive.

$T_{n+1} = T_n - 500$ ✓ where $T_1 = 9\,500$ ✓

(c) Write in terms of k, an equation that describes the number of particles filtered in week k, if $6 \leq k \leq 10$.

$P = 10\,000 - 500(k-5)$ ✓✓
$= 12\,500 - 500k$ ✓

(d) Find the total number of particles filtered in the first 5 weeks.

Total = 10 000 × 5 = 50 000 ✓

(e) Find the total number of particles filtered by the end of the 7th week.

Total = 50 000 + 9 500 + 9 000 = 68 500 ✓

07 Geometric Sequences

Calculator Free

1. [4 marks: 2, 2]

The n^{th} term of a sequence is given by $T(n) = 4 \times 2^n$, where $n = 1, 2, 3, 4, 5, \ldots$.

(a) Find the first 5 terms of the sequence.

$T(1) = 4 \times 2 = 8$ $T(2) = 4 \times 2^2 = 16$
$T(3) = 4 \times 2^3 = 32$ $T(4) = 4 \times 2^4 = 64$
$T(5) = 4 \times 2^5 = 128$ [−½ per error, round down ✓✓]

(b) State the recursive rule for this sequence.

Recursive rule is $T_{n+1} = T_n \times 2$ ✓ with $T_1 = 8$ ✓

2. [4 marks: 2, 2]

Consider the sequence: 8 16 32 64 128 ….

(a) Find the recursive rule for this sequence.

Recursive rule is $T_{n+1} = T_n \times 2$ ✓ with $T_1 = 8$ ✓

(b) Given the n^{th} term is $T(n) = a \times b^n$, for $n = 1, 2, 3, \ldots$, find a and b.

Hence, $T(n) = 8 \times 2^{n-1}$ ✓
$= 4 \times 2^n$
Hence, $a = 4, b = 2$ ✓

3. [2 marks]

A sequence is defined by the recursive rule $a(n + 1) = a(n) \times 2$ where $a(1) = 3$.
Given the n^{th} term is $T(n) = a \times b^{n-1}$, for $n = 1, 2, 3, \ldots$, find a and b.

$T(n) = 3 \times 2^{n-1}$ ✓✓

Calculator Free

4. [5 marks: 2, 3]

The n^{th} term of a sequence is given by the rule $T(n) = 2^{n+2}$ for $n = 1, 2, 3, \ldots$.

(a) Find the recursive rule of this sequence.

$T(n + 1) = T(n) \times 2$ $T(1) = 8$ ✓✓

(b) Find the term number of the first term that exceeds 100.

$2^6 = 64$ ✓
$2^7 = 128$ ✓
Hence, $n = 5$ ✓

5. [7 marks: 1, 2, 2, 2]

A sequence is given by the recursion equation $t_{n+1} = 5t_n$, where the $t_1 = 4$.

(a) Complete the table below listing the first five terms in this sequence.

t_1	t_2	t_3	t_4	t_5
4	20	100	500	2 500

(b) State the n^{th} term (general term) rule for this sequence.

$t_n = 4 \times 5^{n-1}$ ✓✓

(c) Complete the table below that lists the difference between two consecutive terms.

	$t_2 - t_1$	$t_3 - t_2$	$t_4 - t_3$	$t_5 - t_4$
Difference	16	80	400	2 000

✓✓

(d) The difference between two consecutive terms is 50 000.
State with reasons, these two consecutive terms.

Differences form a GP with first term 16 and common ratio 5.
50 000 is the 6th term in the difference table. ✓
Hence, the two terms are t_7 & t_6 . ✓

Calculator Free

6. [4 marks]

The difference between the fourth term and first term of a geometric sequence is 52. Calculate the common ratio of the sequence if the first term is 2.

> Let common ratio be r.
> $T(4) = 2r^3$ ✓
> Hence: $2r^3 - 2 = 52$ ✓
> $r^3 = 27$ ✓
> $r = 3$ ✓

7. [5 marks]

4, a, b and 108 are consecutive terms of a geometric sequence.
Determine the recursive rule for this sequence. Show all working.

> Let common ratio be r.
> $4r^3 = 108$ ✓
> $r^3 = 27$ ✓
> $r = 3$ ✓
> Hence, recursive rule is
> $T(n+1) = T(n) \times 3$ ✓ $T(1) = 4$ ✓

8. [6 marks: 3, 3]

Consider the five consecutive terms of a sequence: 2, a, b, c, 162

(a) Explain why if $a = 40$, then the sequence cannot be an arithmetic sequence.

> If sequence is arithmetic, then common difference = $a - 2 = 38$. ✓
> Then $b = 40 + 38 = 78$ and $c = 78 + 38 = 116$ ✓
> The fifth term would be $116 + 38 = \neq 162$ ✓
> Hence, sequence cannot be arithmetic.

(b) Explain why if $c = 54$, then the sequence can be a geometric sequence.

> If sequence is geometric, then common ratio = $\frac{162}{54} = 3$. ✓
> Then $b = \frac{54}{3} = 18$ and $a = \frac{18}{3} = 6$. ✓
> The first term would be $\frac{6}{3} = 2$ which matches the first term of the sequence. ✓
> Hence, sequence can be geometric.

Calculator Assumed

9. [6 marks: 2, 2, 2]

Consider the following sequence of six terms.
10, –5, 2.5, –1.25, 0.625, –0.3125, 0.15650

(a) Explain clearly why the following sequence is not a geometric sequence.

> $\frac{T_2}{T_1} = -0.5$, $\frac{T_3}{T_2} = -0.5$, $\frac{T_4}{T_3} = -0.5$, $\frac{T_5}{T_4} = -0.5$, but $\frac{T_6}{T_5} = -0.5008 \neq -\frac{1}{2}$ ✓
> Hence, the sequence is not a GP as there is no common ratio. ✓

(b) Change one of the terms in the given sequence to turn the sequence into a geometric sequence.

> Change T_6 to 0.15625. ✓✓
> (so that $\frac{T_6}{T_5} = -0.5$)

(c) State the recursive rule for the sequence you created in part (b).

> Recursive rule is $T_{n+1} = T_n \times (-0.5)$ ✓ where $T_1 = 10$ ✓

10. [3 marks: 1, 2]

A sequence is defined by the recursive rule $u(n+1) = u(n) \times 1.5$ with $u(1) = 2$.

(a) Find the 10th term to 4 significant figures.

> Use CAS Calculator:
> $u(10) = 76.89$ ✓

(b) Which term first exceeds 1 000 000. Justify your answer.

> Use CAS Calculator:
> $u(33) = 862\,879.7665492$
> $u(34) = 1\,294\,319.649824$ ✓
> Hence, the 34th term. ✓

Calculator Assumed

11. [6 marks: 3, 1, 2]

Consider the sequence: 3 –6 12 –24 48 …..

(a) Find the 15th term.

Recursive rule is	$T_{n+1} = T_n \times (-2)$ with $T_1 = 3$ ✓ ✓
From Calculator:	$T_{15} = 49\ 152$ ✓

(b) Which term equals –6144?

Use CAS Calculator:
$u(12) = -6144$
Hence, 12th term. ✓

(c) How many positive terms are there with values less than 1 000 000? Justify your answer.

Use CAS Calculator:
$u(19) = 786\ 432$
$u(20) = -1\ 572\ 864$
$u(21) = 3\ 145\ 728$ ✓
Hence, there are 10 positive terms. ✓

12. [6 marks: 4, 2]

A sequence is described by the rule $T_n = 1500(-1.04)^n$, where $n = 1, 2, 3, ..$.

(a) Show that the sequence is a geometric sequence. State the recursive rule.

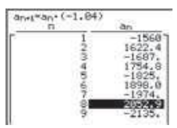
When $n = 1$, $T_1 = 1500 \times (-1.04)^1 = -1560$
$n = 2$, $T_2 = 1500 \times (-1.04)^2 = -1560 \times (-1.04) = T_1 \times (-1.04)$
$n = 3$, $T_3 = 1500 \times (-1.04)^3 = T_2 \times (-1.04)$ ✓
and so on …..
Clearly, the ratio of consecutive terms is constant (= –1.04) ✓

Hence recursive rule is $T_{n+1} = T_n \times (-1.04)$ where $T_1 = -1560$ ✓✓

(b) Find the first term that exceeds 2 000.

Use $T_{n+1} = T_n \times (-1.04)$ where $T_1 = -1560$

$T_6 = 1897.98$, $T_8 = 2052.85$ ✓
Hence, the 8th term. ✓



Calculator Assumed

13. [4 marks: 2, 2]

In 2005, a recycling centre processed 2000 tonnes of plastic containers. The amount of plastic containers processed each year, increased by 8 per cent per year. Let $a(n)$ be the amount of plastic containers processed in year n.

(a) Write a recursive rule for the amount of plastic containers recycled.

Recursive rule is:
$a(n + 1) = a(n) \times 1.08$ with $a(1) = 2000$ ✓ ✓

(b) In which year will the recycling centre for the first time recycle more than 5 000 tonnes of plastic containers? Assume that the recycling centre is able to handle the growth in the amount of containers recycled.

From Calculator:
$a(12) = 4663.3$
$a(13) = 5036.3$ ✓
Hence, in the 13th year (ie year 2017) ✓

14. [5 marks: 2, 1, 2]

An observation balloon is released from a height of 50 metres and allowed to float vertically upwards. The height increase in the first minute is 80 metres. Thereafter, the height increase during each subsequent minute is 85% of the height increase during the previous minute. Ignore air and wind resistance. Let $h(n)$ be the height increase in the nth minute.

(a) Write a recursive equation for the height increase.

Recursive rule is:
$h(n + 1) = h(n) \times 0.85$ with $h(1) = 80$ ✓ ✓

(b) Find the height increase (nearest cm) in the 5th minute.

From Calculator:
$h(5) = 41.7605 = 41.76$ m ✓

(c) During which minute did the height increase first drop below 1 metre?

From Calculator:
$h(27) = 1.1695$ $h(28) = 0.9940$ ✓
Hence, in the 28th minute ✓

Calculator Assumed

15. [10 marks: 2, 2, 3, 3]

A ball is dropped from a height of 3 metres. The first time it hits the ground it bounces up to 2.7 m. The height reached after each bounce is 90% of the height reached in the previous bounce. Let $h(n)$ be the height of the ball reached after the nth bounce, where $n = 1, 2, 3, \ldots$.

(a) Write a recursive equation for the height reached by the ball after the nth bounce.

$h(n+1) = h(n) \times 0.9$ where $h(1) = 2.7$ ✓ ✓

(b) Write a non-recursive equation for the height reached by the ball after the nth bounce.

$h(n) = 2.7 \times 0.9^{n-1}$ ✓✓
or $h(n) = 3 \times 0.9^{n}$

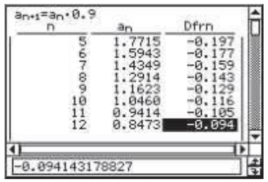
(c) Calculate the difference in heights reached between the fifth bounce and the second bounce. Give your answer to the nearest cm.

$h(5) = 1.77147$ ✓
$h(2) = 2.43$ ✓
Difference $= 0.65853$
≈ 0.66 m (or 66 cm) ✓

(d) When does the difference in heights reached between consecutive bounces first differ by less than 0.1 m? Justify your answer.

$h(10) = 1.0460$ $h(11) = 0.9414$
$h(12) = 0.8473$

$h(11) - h(10) = -0.105$ ✓
$h(12) - h(11) = -0.094$ ✓
$\Rightarrow$ Between the 11th and 12th bounce. ✓



Calculator Assumed

16. [9 marks: 2, 2, 2, 3]

An investment account pays 12% interest compounded annually. That is, each year the account grows by 12%. Julia invests $20 000 in this account for 20 years. No new money was added to and no withdrawals were made from the investment account. Let $b(n)$ be the balance of the account at the start of year n.

(a) Write a recursive equation for the balance of the investment account.

Recursive rule is:
$b(n+1) = b(n) \times 1.12$ with $b(1) = 20\,000$ ✓ ✓

(b) Write an explicit equation for the balance of the investment account.

n	1	2	3	4
$b(n)$	20 000	$20\,000 \times 1.12$	$20\,000 \times 1.12^2$	$20\,000 \times 1.12^3$
Ratio	1.12	1.12	1.12	

Hence, $b(n) = 20\,000 \times 1.12^{n-1}$, $n = 1, 2, 3, \ldots$ ✓✓

(c) Find the account balance after 8 years.

Balance after 8 years $= b(9)$. ✓
From Calculator:
$b(9) = \$49\,519.26$ ✓

(d) Find the minimum number of years required for the balance to exceed $1 000 000.

Need to solve $20\,000 \times 1.12^{n-1} = 1\,000\,000$ ✓
Use "solver"
$n = 35.5$ ✓
Hence need 36 years. ✓

08 Linear Recurrence Relations

Calculator Free

1. [7 marks: 1, 2, 2, 2]

A sequence is defined by the general term rule $t_n = 4 + 2n$ for $n = 1, 2, 3, \dots$.

(a) List the first 5 terms of the sequence.

6, 8, 10, 12, 14 ✓

(b) State the recursive rule for this sequence.

$a_{n+1} = a_n + 2 \quad a_1 = 6$ ✓✓

(c) Consider the first 100 terms in this sequence. How many terms are there that are multiples of 5?

10, 20, 30, 40, 50, 60, 70, 80, 90, 100 ✓
Hence 10 terms. ✓

(d) How many terms are there that are less than 200?

$4 + 2n < 200$ ✓
$2n < 196$
$n < 98$
Hence, 97 terms. ✓

2. [4 marks]

Consider the sequence: 3 –2 8 –12 28 ….

This sequence follows the rule $t_{n+1} = a\,t_n + b$. Determine the values of a and b.

For $n = 1$: $t_2 = a\,t_1 + b$
$-2 = 3a + b$ (1) ✓

For $n = 2$: $t_3 = a\,t_2 + b$
$8 = -2a + b$ (2) ✓

(1) – (2): $-10 = 5a \Rightarrow a = -2$ ✓
$b = 4$ ✓

Calculator Free

3. [7 marks: 2, 2, 3]

A sequence is defined by the recursive rule $a(n + 1) = a(n) - 2$ where $a(1) = -50$.

(a) Find the first 5 terms of the sequence.

–50, –52, –54, –56, –58
[–½ per error, round down ✓✓]

(b) Find the rule for n^{th} term in this sequence.

$a(n) = -50 + (n - 1) \times (-2)$ ✓
$= -48 - 2n$ ✓

(c) Find the term number of the first term that is greater than –100?

$-48 - 2n > -100$ ✓
$2n < 52$ ✓
Hence 25th term. ✓

4. [4 marks]

Consider the sequence: 5 13 29 61 125 ….

The recursive rule of this sequence is of the form $T_{n+1} = aT_n + b$ with $T_1 = 5$. Determine the values of a and b.

For $n = 1$: $T_2 = aT_1 + b$
$13 = 5a + b$ (1) ✓

For $n = 2$: $T_3 = aT_2 + b$
$29 = 13a + b$ (2) ✓

(2) – (1): $16 = 8a \Rightarrow a = 2$ ✓
$b = 3$ ✓

Calculator Assumed

5. [6 marks]

A sequence is defined by the rule $h_{n+1} = 2 - 3h_n$ where $n = 1, 2, 3, \ldots$ for $h_1 = k$.
Given that k is a constant, find k if $h_4 = -598$.

When $n = 3$, $h_4 = 2 - 3h_3 \Rightarrow -598 = 2 - 3h_3$ ✓
$\Rightarrow h_3 = 200$ ✓
When $n = 2$, $h_3 = 2 - 3h_2 \Rightarrow 200 = 2 - 3h_2$ ✓
$\Rightarrow h_2 = -66$ ✓
When $n = 1$, $h_2 = 2 - 3h_1 \Rightarrow -66 = 2 - 3h_1$ ✓
$\Rightarrow h_1 = \dfrac{68}{3}$
Hence, $k = \dfrac{68}{3}$ ✓

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6. [9 marks: 3, 6]

A sequence is defined by the recurrence relation
$$T_{n+2} = a\,T_{n+1} + b\,T_n \text{ where } T_1 = x \text{ and } T_2 = y.$$

(a) Find the first 5 terms of the sequence if $a = -1$, $b = 1$, $x = 2$, $y = 3$.

Rule is $T_{n+2} = -T_{n+1} + T_n$ with $T_1 = 2$ and $T_2 = 3$

When $n = 1$, $T_3 = -T_2 + T_1 = -(3) + 2 = -1$ ✓
When $n = 2$, $T_4 = -T_3 + T_2 = -(-1) + 3 = 4$ ✓
When $n = 3$, $T_5 = -T_4 + T_3 = -(4) + (-1) = -5$ ✓

Hence, first five terms are 2, 3, –1, 4 and –5.

(b) The sequence 10, 20, 70, 200, 610 obeys the recurrence relation given.
Find the values of a and b.

When $n = 1$, $T_3 = aT_2 + bT_1$
But $T_1 = 10$, $T_2 = 20$ and $T_3 = 70 \Rightarrow 70 = 20a + 10b$ ✓✓

When $n = 2$, $T_4 = aT_3 + bT_2$
But $T_2 = 20$, $T_3 = 70$ and $T_4 = 200 \Rightarrow 200 = 70a + 20b$ ✓✓

Solve these two equations simultaneously using an appropriate routine:
$a = 2$ and $b = 3$ ✓✓

Calculator Assumed

7. [8 marks: 2, 3, 3]

Consider the sequence: 2, x, 10.

(a) Find the value(s) of x if the sequence follows the rule $u_n - u_{n-1} = 4$.

Rule can be interpreted in words as:
"the difference between two consecutive terms is always 4".
Hence, $10 - x = 4 \Rightarrow x = 6$ ✓✓

(b) Find the value(s) of x if the sequence follows the rule $f(n+1) = k\,f(n)$ where k is a constant.

Rule can be rewritten as $\dfrac{f(n+1)}{f(n)} = k$.
That is, the ratio of two consecutive terms is a constant (= k).
Hence, $\dfrac{x}{2} = \dfrac{10}{x}$ ✓
$\Rightarrow x^2 = 20$ ✓
Hence, $x = \pm\sqrt{20}$ ✓

(c) Find the value(s) of x if the sequence follows the rule
$u(n+2) = u(n+1) - \{u(n)\}^2$.

When $n = 1$, $u(3) = u(2) - \{u(1)\}^2$
Hence, $10 = x - 2^2$ ✓✓
$\Rightarrow x = 14$ ✓

---

8. [4 marks]

Describe in words the meaning of the following recursive equation.
List the first 5 terms of this sequence.
$C_{i+3} = C_{i+2} + C_{i+1} + C_i$ for $i = 1, 2, 3, \ldots$, where $C_1 = 0$, $C_2 = 0$ and $C_3 = -1$

Current Term = Sum of the previous 3 terms ✓✓

When $n = 1$, $C_4 = C_3 + C_2 + C_1 = (-1) + 0 + 0 = -1$ ✓
When $n = 2$, $C_5 = C_4 + C_3 + C_2 = (-1) + (-1) + 0 = -2$ ✓

Hence, first 5 terms are 0, 0, –1, –1, –2 .

Calculator Assumed

9. [8 marks: 2, 2, 2, 2]

Consider the following sequence of coordinates (x, y):

$(3, 3), (4, 6), (5, 12), (6, 24), (7, 48), \ldots$

(a) Find the next set of coordinates in this sequence.

> First term of each ordered pair follows the sequence 3, 4, 5, 6, 7, …. .
> This is an arithmetic sequence with $a = 3$ and $d = 1$.
> Second term of each ordered pair follows the sequence 3, 6, 12, 24, 48, …. .
> This is a geometric sequence with $a = 3$ and $r = 2$.
>
> Hence, next ordered pair is (8, 96). ✓✓

(b) Find the recursive rule for the x-coordinate and the recursive rule for the y-coordinate.

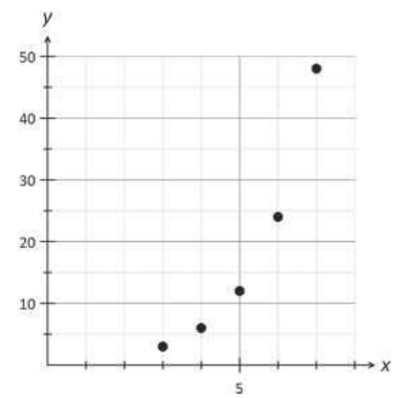
> x-coordinate has recursive equation: $x(n + 1) = x(n) + 1$, $x(1) = 3$ ✓
>
> y-coordinate has recursive equation: $y(n + 1) = y(n) \times 2$, $y(1) = 3$ ✓

(c) Find the 20th set of coordinates in this sequence.

> From Calculator:
> $x(20) = 22$ $y(20) = 1\,572\,864$
> Hence, 20th set is (22, 1 572 864). ✓✓

(d) Plot the coordinates and determine the shape of the curve passing through these points.

> Plot of points. ✓
>
> The points lie on an exponential curve. ✓



Calculator Assumed

10. [12 marks: 3, 3, 3, 3]

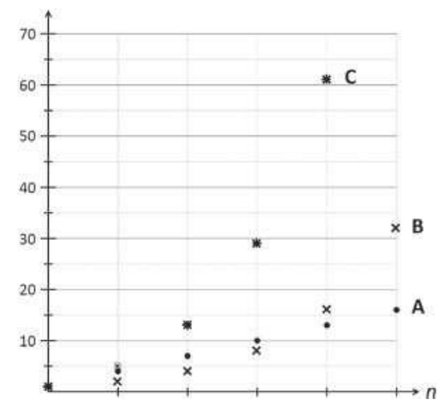
Consider the sequences A, B and C defined by the recurrence relations.

A: $a(n + 1) = a(n) + 3$, $a(0) = 1$
B: $b(n + 1) = 2\,b(n)$ $b(0) = 1$
C: $c(n + 1) = 2\,c(n) + 3$ $c(0) = 1$

(a) List the first six terms of each of the above sequences.

Sequence A:	1, 4, 7, 10, 13, 16	✓
Sequence B:	1, 2, 4, 8, 16, 32	✓
Sequence C:	1, 5, 13, 29, 61, 125	✓

(b) In the axes provided below, plot the terms of each of the given sequences with respect to n. Fit as many points into the diagram as is possible.



> Plots ✓✓✓

(c) Comment on the shape of the plot of each of these recurrence relations.

> Sequence A: Linear plot ✓
> Sequence B: Exponential plot ✓
> Sequence C: Exponential plot ✓

(d) Discuss the rate of growth between the terms in each of the given sequences.

> In sequence A, the growth rate is constant. ✓
> In sequence B, the growth rate increases at a constant rate. ✓
> In sequence C, the growth rate increases at an increasing rate. ✓

Calculator Assumed

11. [12 marks: 3, 3, 3, 3]

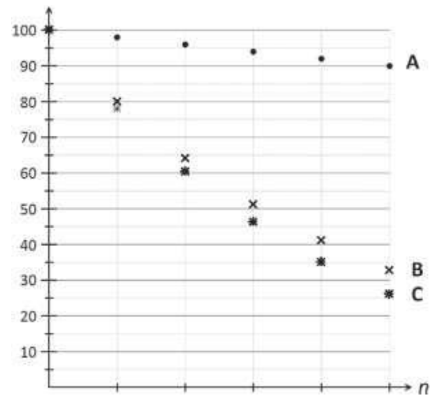
Consider the sequences A, B and C defined by the recurrence relations.

A: $a(n+1) = a(n) - 2$, $a(0) = 100$
B: $b(n+1) = 0.8\,b(n)$ $b(0) = 100$
C: $c(n+1) = 0.8\,c(n) - 2$ $c(0) = 100$

(a) List the first six terms of each of the above sequences.

Sequence A:	100, 98, 96, 94, 92, 90	✓
Sequence B:	100, 80, 64, 51.2, 40.96, 32.768	✓
Sequence C:	100, 78, 60.4, 46.32, 35.056, 26.0448	✓

(b) In the axes provided below, plot the terms of each of the given sequences with respect to n. Fit as many points into the diagram as is possible.



Plots ✓✓✓

(c) Comment on the shape of the plot of each of these recurrence relations.

Sequence A: Linear plot ✓
Sequence B: Exponential plot ✓
Sequence C: Exponential plot ✓

(d) Discuss the rate of decay between the terms in each of the given sequences.

In sequence A, the decay rate is constant. ✓
In sequence B, the decay rate increases at a constant rate. ✓
In sequence C, the decay rate increases at an increasing rate. ✓

Calculator Assumed

12. [16 marks: 4, 12]

Consider the sequences A, B, C and D defined by the recurrence relations.

A: $a(n+1) = 0.05\,a(n)$, $a(0) = 1$
B: $b(n+1) = 0.05\,b(n) + 1$ $b(0) = 1$
C: $c(n+1) = -0.05\,c(n) + 1$ $c(0) = 1$
D: $d(n+1) = -0.05\,d(n)$ $d(0) = 1$

(a) Complete the table given below which lists some terms in these sequences.

n	$a(n)$	$b(n)$	$c(n)$	$d(n)$
0	1	1	1	1
1	0.05	1.05	0.95	–0.05
2	0.0025	1.0525	0.9525	0.0025
3	0.000125	1.052625	0.952375	–0.000125
4	0.00000625	1.05263125	0.95238125	0.00000625
5	3.125×10^{-7}	1.052631563	0.952380938	-3.125×10^{-7}

(b) Describe with reasons, the terms in each of these sequences using the words "positive", "negative", "increasing", "decreasing" and "steady-state". You may need to provide further terms to support your descriptions.

Sequence A: $A(20) = 9.5367\times 10^{-27}$ $A(21) = 4.76837\times 10^{-28}$ ✓
Hence, terms are all positive and decreasing. ✓✓

Sequence B: $B(11) = B(12) = 1.052\,631\,578\,947\,37$ ✓
Hence, terms are all positive and in a steady-state but larger than the first term. ✓✓

Sequence C: $C(11) = C(12) = 0.952\,380\,952\,380\,952$ ✓
Hence, terms are all positive and in a steady-state but smaller than the first term. ✓✓

Sequence D: $D(11) = -4.882\,812\,5 \times 10^{-15}$
$D(12) = 2.44\,140\,625 \times 10^{-16}$ ✓
Hence, terms decreasing but oscillating between positive and negative terms. ✓✓

Calculator Assumed

13. [9 marks: 3, 6]

The table 1 below shows 3 recursion formulae.

	Recursion Formula	
A	$a_{n+1} = a_n + 10$	$a_1 = 50$
B	$b_{n+1} = 1.1\ b_n$	$b_1 = 50$
C	$c_{n+1} = 1.1\ c_n - 4$	$c_1 = 50$

(a) Which of the three recursion formulae with reach 100 fastest? Give reasons for your answer.

> $a_6 = 100$ ✓
> $b_9 = 107.18$ $b_8 = 97.44$
> $c_{20} = 101.16$ $c_{19} = 95.60$ ✓
> Hence, formula A. ✓

Table 2 shows 4 general term formulae.

	General Term Formula for $n = 1, 2, 3, \ldots$
I	$t_n = \dfrac{500 \times 1.1^n}{11}$
II	$t_n = 50 \times 1.1^n$
III	$t_n = 10n + 40$
IV	$t_n = 40 + 10 \times 1.1^{n-1}$

(b) Match each of the recursion formulae in the Table 1 with a general term formula in Table 2. Explain how you arrived at your answer.

> Recursion Formula A is linear.
> General Term Formula III is the only linear term formula. ✓
> Hence Formula A is matched with III ✓
>
> Recursion Formula B is geometric.
> General Term Formulae I & II are geometric.
> But Formula II gives $b_1 = 55$. ✓
> Hence B is with I. ✓
>
> Formula C is neither linear nor geometric. ✓
> Formula II is geometric.
> Recursion Formula C is with IV. ✓

Calculator Assumed

14. [8 marks: 2, 2, 4]

Consider the recursion formula $a(n+1) = 0.85 \times a(n) + 10$ $a(1) = 80$.

(a) Determine to three significant figures, the ninth term of this sequence

> $a(9) = 70.2999 \approx 70.3$ ✓✓

(b) Determine the term number of the first term that is less than 67. Justify your answer.

> $a(23) = 67.04$
> $a(24) = 66.98$ ✓
> Hence, the 24th term. ✓

(c) Describe with reasons, the behaviour of this sequence.

> $a(60) \approx 66.6676$ $a(70) \approx 66.6668$
> Term values are decreasing. ✓
>
> $a(100) \approx 66.6667$ $a(101) \approx 66.6667$
> That is: $a(100) \approx a(101)$. ✓
> Hence, sequence is a decreasing sequence ✓
> with a steady state solution of $66.\overline{6}$. ✓

---

15. [6 marks: 4, 2]

Consider the recursion formula $A(n+1) = 0.9 \times A(n) + 5$ A(1) = 100

(a) Describe with reasons, the behaviour of this sequence.

> $A(1) = 100$
> $A(199) = 50.0000000435501$
> $A(200) = 50.0000000391951$ ✓
> Hence, sequence is decreasing ✓
> to a long term steady state solution of 50 ✓✓

(b) The rule $A(n+1) = 0.9 \times A(n) + k$ with $A(1) = 100$ gives a sequence of constant terms (all the terms have the same value). Determine the value of k.

> If $k = 10$,
> sequence is 100, 100, 100, …. ✓✓

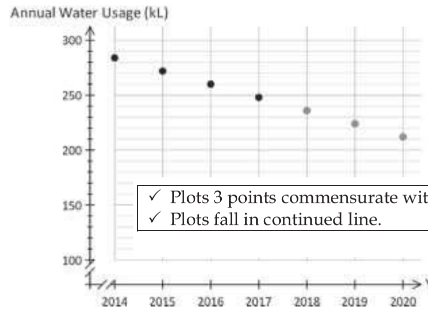
09 Growth & Decay Models

Calculator Assumed

1. [9 marks: 2, 2, 3, 2]

The table and the accompanying graph below show the amount of water (kL) used by a household in a given suburb each year for the years 2014 to 2017.

Year	2014	2015	2016	2017
Annual Water Usage (kL)	284	272	260	248



✓ Plots 3 points commensurate with given years.
✓ Plots fall in continued line.

(a) Assuming that the water usage pattern continues, plot on the diagram above the predicted water usage in 2018, 2019 and 2020.

(b) Write a rule in its simplest form for the expected annual water usage n years after 2014.

$U(n) = 284 + (n - 1) \times (-12)$ ✓
$= 296 - 12n$ ✓

(c) Using this pattern, determine when the annual water usage first drops below 10 kL.

$296 - 12n < 10$ ✓
$n > 23.8$ ✓
23.8 years after 2014 $\Rightarrow$ 2038 ✓

(d) Determine with reasons if the rule in (b) may be used to make long term predictions of water usage for households in this suburb.

NO! ✓
As water usage becomes negative after 25 years. ✓

Calculator Assumed

2. [8 marks: 1, 2, 2, 3]

Q, the number of organisms (*in hundreds*) in a laboratory culture is related to time t (days) by the formula $Q(t + 1) = 1.075 \times Q(t)$ where $Q(0) = 16$.

(a) How many organisms were there at the start?

$Q(0) = 16$
Hence, no. of organisms = $16 \times 100 = 1600$ ✓

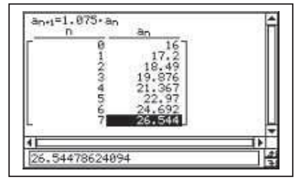
(b) What is the growth/decay rate? *State clearly the units used.*

Growth rate = 7.5% per day.
✓ ✓

(c) Find the number of organisms after one week.

Using CAS Calculator
generated table: $Q(7) = 26.5448$. ✓

Hence, no. of organisms after 1 week
$= 26.5448 \times 100 = 2654$ ✓



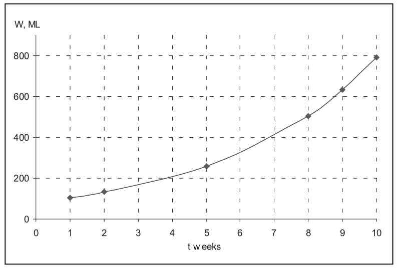
(d) How long will it take for the population to reach 2 000? Give your answer to the nearest day.

When population = 2000, Q = 20 ✓
From Table in (c), $t = 4$ ✓
Hence, 4 days. ✓

Calculator Assumed

3. [9 marks: 3, 5, 1]

The amount of water, W MegaLitres, in a newly constructed dam at time t weeks is shown in the graph below.



Three models were suggested for this data:

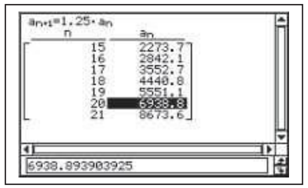
Equation I	$W(t) = W(t-1) \times 1.25$ where $W(0) = 80$,
Equation II	$W(t+1) = W(t) \times 1.26$ where $W(0) = 100$
Equation III	$W(t+1) = W(t) \times 1.26$ where $W(0) = 50$

(a) Which of these three models best represents the data given? Why?

> From graph, when $t = 1$, $W \approx 100$ ✓
> Eqn I. $W(1) = 100$
> Eqn II. $W(1) = 126$
> Eqn III. $W(1) = 63$ ✓
> Hence, Eqn. I best fits the curve. $W = 80 \times 1.25^t$ ✓

(b) Use your chosen model to:
 (i) estimate the amount of water in the dam after 20 weeks.
 (ii) find when the amount of water in the dam will first exceed 3 000 ML.

> (i) From CAS Calculator generated table, $W(20) = 6938.9$ ML ✓✓✓
> (ii) From CAS Calculator generated table, $W > 3000$ when $t = 17$ weeks. ✓✓



(c) What is the most important assumption underlying the model you chose in part (a)?

> The rate of increase remains constant throughout. ✓

Calculator Assumed

4. [10 marks: 2, 2, 3, 3]

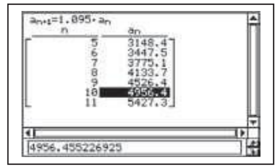
A study of the population of a rare marsupial found the population growth rate was 9.5% per annum. At the commencement of the study (at the start of 1997) the population was 2 000.

(a) Write a recursive formula for predicting the population, P, t years after 1997.

> $P(t) = P(t-1) \times 1.095$ where $P(0) = 2\,000$ ✓✓

(b) Predict the population at the start of 2007 (to nearest whole number).

> Start of 2007 is $t = 10$. ✓
> Hence $P(10) = 4956$ ✓

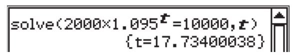


(c) Show clearly that the population t years after 1997 can also be written as $P(t) = A \times b^t$ where A and b are constants. State the values of A and b.

> $P(1) = P(0) \times 1.095 = 2\,000 \times 1.095$
> $P(2) = P(1) \times 1.095 = (2\,000 \times 1.095) \times 1.095 = 2\,000 \times 1.095^2$ ✓
> $P(3) = P(2) \times 1.095 = (2\,000 \times 1.095^2) \times 1.095 = 2\,000 \times 1.095^3$
> Therefore $P(t) = 2\,000 \times 1.095^t$ ✓✓

(d) Predict when the population first exceeds 10 000. Show clearly how you obtained your answer.

> $2\,000 \times 1.095^t = 10\,000$ ✓
> Use CAS Solve command: $t = 17.73$ ✓
> Hence in the year 2014.



Calculator Assumed

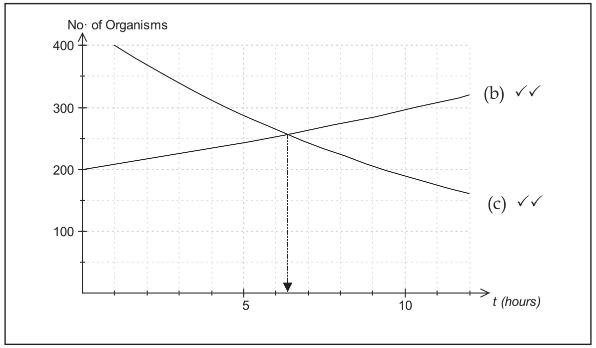
5. [7 marks: 2, 2, 3]

In a laboratory experiment, two types of organisms were placed in a sealed biosphere. The number of organism A in the biosphere is modeled by $N = 200 \times 1.04^t$ where t is time in hours.

(a) Find the recursive formula that describes the number of organisms at time t hours.

$N(0) = 200 \times 1.04^0 = 200$
Hence, recursive formula is $N(t) = N(t-1) \times 1.04$ where $N(0) = 200$ ✓✓

(b) Sketch the graph of the population of A against t for $0 \le t \le 12$ in the axes provided below.



The number of organism B at time t hours is given in the table below.

t	1	2	3	4	5	6	7	8	9	10	11	12
B	400	368	338	310	285	262	241	221	203	186	171	157

(c) Plot the population of B against t onto the axes given in part b. Hence determine during which hour the population of A exceeds the population of B.

From the graph, during the 7th hour. ✓

Calculator Assumed

6. [9 marks: 2, 1, 3, 3]

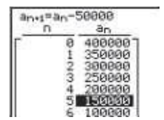
A farm machine is worth $400 000 new. Let $V(n)$ be the value of the machine at the end of year n. $V(n)$ is calculated using two different methods.

(a) Method A:
Each year, the value of the machine is decreased by $50 000.
(i) Write a recursive equation for the value of the machine.

$V(n+1) = V(n) - 50\,000$ ✓ $V(0) = 400\,000$ ✓

(ii) Calculate the value of the machine after 5years.

$V(5) = \$150\,000$ ✓



(b) Method B
At end of each year, the value of the machine is decreased by 20% of the previous year's value. Calculate the value of the machine after 5 years. Show clearly how you arrived at your answer.

$V(n+1) = V(n) \times 0.8$ ✓ $V(0) = 400\,000$ ✓

OR $V(n) = 400000 \times 0.8^n$ ✓✓

$V(5) = \$131\,072$ ✓

(c) When does the Method B give a higher value for the machine than Method A?

For Method B: $V(5) = \$131\,072$
$V(6) = \$104\,857.60$ ✓
For Method A: $V(5) = \$150\,000$
$V(6) = \$100\,000$ ✓
Hence, after 6 years. ✓

Calculator Assumed

7. [10 marks: 3, 1, 1, 2, 3]

The maximum capacity of a grain silo is 80 tonnes. Initially there was 50 tonnes of grain in the silo. At the start of each day, 20% of the grain in the silo is trucked out. At the end of each day 15 tonnes of grain is trucked in to be stored in the silo. Let $C(n)$ represent the amount of grain in the silo at the end of day n.

(a) Write a linear recursion relation for the amount of grain in the silo at the end of day n.

$C(n+1) = C(n) \times 0.8 + 15$ where $W(0) = 50$ ✓ ✓ ✓

(b) Determine the amount of grain in the silo at the end of the eighth day.

$C(8) = 70.8057$ ✓

(c) Determine the amount of grain trucked out on the morning of the ninth day.

Amount trucked out = 70.8057 × 0.2
= 14.16 tonnes ✓

(d) Find the difference in the amount of grain in the silo between the end of the second day and the end of the eighth day inclusive?

$C(8) = 70.8057$ $C(2) = 59$
Hence, difference = additional 11.8057 tonnes ✓ ✓

(e) Determine with reasons if the capacity of the grain silo will be exceeded.

$C(99) = 74.99999$
$C(100) = 74.99999$ ✓
The long term state is 75 tonnes. ✓
Hence, the amount of grain will never exceed 75 tonnes. ✓

Calculator Assumed

8. [10 marks: 3, 1, 3, 3]

The maximum capacity of a swimming pool is 2 500 kL.
Initially there was 2 000 kL of water in the pool. Each day the pool loses 5% of the amount of water present at the start of the day, due to evaporation. At the end of each day 125 kL of water is added to the pool.

Let $W(n)$ represent the amount of water in the pool at the end of day n.

(a) Write a linear recursion relation for the amount of water in the pool at the end of day n.

$W(n+1) = W(n) \times 0.95 + 125$ where $W(0) = 2\,000$ ✓ ✓ ✓

(b) How much water is added into the pool between the third day and the sixth day inclusive?

Amount added = 125 × 4 = 500 kL ✓

(c) Determine the increase in the amount of water in the pool between the start of the third day and the end of the sixth day .

Amount of water at start of day 3 = W(2) = 2048.8 kL ✓
Amount of water at end of day 6 = W(6) = 2132.4 ✓
Increase = 83.6 kL ✓

(d) Describe with reasons, the amount of water in the pool in the long term.

W(0) = 2000
W(299) = 2499.99999
W(300) = 2499.99999 ✓

Hence, the amount of water in the pool increases ✓
from 2000 kL to a maximum not exceeding 2500 kL ✓

Calculator Assumed

9. [9 marks: 4, 3, 2]

The population of the City of Tam at the start of 2000 is 90 000. Each year, the population grows naturally (through births) at a rate of 1.5% of the previous year's population. Each year 300 persons move into the city and 1 000 persons leave the city. Each year, the natural death rate of the population is 0.5% of the previous year's population. Let P_n be the population of the city (in thousands) n years after 2000.

(a) Explain clearly why $P_n = a\,P_{n-1} + b$ where $P_0 = 90$, $n = 1, 2, 3\ldots$.
State the values of a and b.

$a = 1.015 - 0.005 = 1.01$ ✓ $b = 0.3 - 1.0 = -0.7$ ✓

Population n years after 2000 = Population $(n-1)$ years after 2000
× Growth/Decay Rate, a
+ Net migration, b ✓✓

(b) Determine the values of P_1, P_2 and P_3.

$P_n = 1.01 \times P_{n-1} + (-0.7)$ $P_0 = 90$

When $n = 1$, $P_1 = 1.01 \times 90 - 0.7 = 90.2$ ✓
When $n = 1$, $P_2 = 1.01 \times 90.2 - 0.7 = 90.402$ ✓
When $n = 1$, $P_3 = 1.01 \times 90.402 - 0.7 = 90.6060$ ✓

(c) The mayor of a neighbouring city described the City of Tam as "a city with a decaying population".
(i) How would the mayor support this statement?

There are more people (1000) leaving the city than there are people moving in (300). ✓

(ii) How would you contradict this statement?

Since $P_3 > P_2 > P_1 > P_0$,
there is a net population growth from one year to the next. ✓

10 Financial Mathematics I

(Simple Interest, Compound Interest and Depreciation)

Calculator Assumed

1. [10 marks: 2, 2, 1, 2, 3]

(a) Calculate the effective annual interest rate on an account that pays interest at a rate of 3% per annum compounded daily.

Effective annual rate $= \left(1 + \frac{0.03}{365}\right)^{365} - 1$ ✓
$= 0.03045$
$\approx 3.045\%$ ✓

(b) A savings account pays interest at a rate of 100a% per annum compounded monthly. Calculate a if the effective annual rate of interest is 4%.

$\left(1 + \frac{a}{12}\right)^{12} - 1 = 0.04$ ✓
$a = 0.039285$
≈ 0.03929 ✓

solve((1+x/12)^12−1=0.04, x)|x≥0
{x=0.039285}

(c) A savings account pays interest at a rate of 4.5% per annum compounded n times per year. Calculate n if:
(i) the effective annual rate of interest is 4.5%.

$n = 1$ ✓

(ii) the effective annual rate of interest is 4.60075%.

$\left(1 + \frac{0.045}{n}\right)^{n} - 1 = 0.0460075$ ✓
$n \approx 52$ ✓

solve((1+0.045/x)^x−1=0.0460075, x)
{x=51.98855044, x=51.98855118}

(d) An account pays interest at a rate of 100a% per annum compounded n times per year. The effective annual rate of interest is 3.66%.
Calculate the interest earned after 5 years on an initial deposit of \$10 000.

Interest earned = 10 000 × 0.0366 × 5 ✓✓
= \$1830 ✓

Calculator Assumed

2. [6 marks: 3, 3]

Account X pays interest at a rate of 6.0 % compounded *monthly*.
Account Y pays interest at a rate of $100r$ % compounded *daily*.

(a) Calculate the effective rate of interest for Account X.

Effective rate of interest for X = $\left(1+\frac{0.06}{12}\right)^{12} - 1$ ✓✓
$= 0.0616778$
Hence, effective rate is 6.17%. ✓

(b) Calculate the minimum value for r if Account Y pays more interest than account X.

Effective rate of interest for Y $= \left(1+\frac{r}{365}\right)^{365} - 1$ ✓
Effective rate of interest for Y > Effective rate of interest for X
$\left(1+\frac{r}{365}\right)^{365} - 1 > 0.0616778$ ✓
$r > 0.05986$ ✓

solve((1+x/365)^365>1.0616778
{x>0.05985539419}

3. [5 marks]

$100 000 is to be invested for several years.
Account P offers interest at 6% per annum compounded annually.
Account Q offers a flat rate interest of 6% per annum.

Explain with ***reasons*** why Account P pays better interest compared to the Account Q.

The interest earned in the first year for both accounts is $6 000. ✓

In the Account P, each year the principal grows by the amount of interest earned the previous year. ✓
Hence, interest earned in subsequent years will exceed $6 000. ✓

For the Account Q, the principal remains constant. ✓
Hence, the interest earned each year remains at $6 000. ✓

Calculator Assumed

4. [14 marks: 3, 2, 2, 3, 4]

Zoe invests $200 000 in an account that pays simple interest at a rate of 6% per year. The interest is paid at the end of each year and is not added to the principal. Let B(n) be the account balance at the end of n years.

(a) Determine a recurrence relation for the account balance after n years.

Simple interest per year = 200 000 × 0.06 = $12 000. ✓
The yearly account balances form a sequence:
200 000, (200 000 + 12 000), (200 000 + 2 × 12 000), ….

Hence: $B(n) = B(n-1) + 12\,000$, $B(0) = 200\,000$. ✓✓

(b) Determine a general rule for the account balance after n years.

General rule $B(n) = 200\,000 + 12\,000n$ for $n = 0, 1, 2, 3, \ldots$ ✓✓

(c) Find n when the account balance is $296 000.

$200\,000 + 12\,000n = 296\,000$ ✓
$\Rightarrow n = 8$ ✓

(d) Find the minimum number of years required for the balance to first exceed $500 000.

$200\,000 + 12\,000n > 500\,000$ ✓
$n > 25$ ✓
Hence, at least 26 years. ✓

(e) How much would Zoe have to deposit into the account if she wishes the account balance to exceed $500 000 after a minimum of 10 years? Assume that the interest rate remains at 6% per annum.

Let deposit be $$x$.
Interest each year = $0.06x$ ✓
Account balance after 10 years = $x + 0.06x \times 10$ ✓
Hence, $x + 0.06x \times 10 = 500\,000$ ✓
$x = 312500$
Hence, initial deposit = $312 500 ✓

Calculator Assumed

5. [10 marks: 1, 4, 2, 3]

$10 000 is invested at a rate of 8% per year with interest compounded monthly.

(a) Calculate the monthly interest rate.

> Monthly rate = $\frac{8}{12} = \frac{2}{3}$% (or $0.\dot{6}$ or 0.67) ✓

(b) Use a recurrence relation to calculate the value of the investment after 5 years and the total interest earned.

> Recurrence relation: $a(n+1) = a(n) \times (1 + \frac{(2/3)}{100})$, $a(0) = 10\,000$ ✓✓
> From Calculator: $a(60) = \$14\,898.46$ ✓
> Hence, interest earned = 14 898.46 – 10 000 = $4 898.46 ✓

(c) How much more interest is earned after 5 years compared to if only simple interest is paid at a rate of 8% per year?

> For simple interest:
> Interest earned each year = 10 000 × 0.08 = 800
> Interest earned in 5 years = 800 × 5 = 4 000 ✓
> Hence, extra interest earned = $4 898.46 – $4 000 = $898.46 ✓

(d) Calculate the annual simple interest rate that will yield the same interest as the interest earned after 5 years with interest compounded monthly.

> Simple Interest Rate = $\frac{4898.46}{10000} \div 5$ ✓✓
> = 0.09797
> That is ≈ 9.8% simple interest pa. ✓

Calculator Assumed

6. [10 marks: 1, 4, 2, 3]

$100 000 is invested at a rate of 4.75% per annum with interest compounded monthly.

(a) Calculate the monthly interest rate.

> Monthly rate = 4.75/12 = 0.3958% ✓

(b) Use a recurrence relation to calculate value of the investment after 10 years and the total interest earned.

> Recurrence relation: $a(n+1) = a(n) \times (1 + \frac{0.0475}{12})$, $a(0) = 100\,000$ ✓✓
> From Calculator: $a(120) = \$160\,650.72$ ✓
> Hence, interest earned = 160 650.72– 100 000 = $60 650.72 ✓

(c) Find the time taken for the investment to double its initial value.

> From Calculator: $a(175) = \$199\,639.08$
> $a(176) = \$200429.32$ ✓
> Hence, 176 months or 14 years, 8 months. ✓

(d) If the interest rate per annum were to be doubled, determine the time taken for the initial investment to double. Assume that the interest is compounded monthly.

> Recurrence relation: $a(n+1) = a(n) \times (1 + \frac{0.0475 \times 2}{12})$, $a(0) = 100\,000$ ✓
> From Calculator: $a(87) = \$198583.21$
> $a(88) = \$200155.32$ ✓
> Hence, value is doubled after 88 months ✓

Calculator Assumed

7. [10 marks: 2, 2, 4, 2]

$10 000 is invested in account A that pays simple interest *monthly* at a rate of 3% pa.
$10 000 is invested in another account B that pays interest at a rate of 2.9% pa compounded *monthly*.

(a) Write a recursion equation for the account balance in A after n months.

$B(n+1) = B(n) + 25$ ✓ $B(0) = 10\,000$ ✓

(b) Write a recursion equation for the account balance in B after n months.

$B(n+1) = B(n) \times \left(1+\frac{0.029}{12}\right)$ ✓ $B(0) = 10\,000$ ✓

(c) Discuss with reasons which account is more profitable in the short term and in the long term.

Account A is more profitable if the investment term is 28 months or less. ✓
Account B is more profitable if the investment term is 29 months or more. ✓

Evidence:
A(28) = $ 10 700 B(28) = $ 10 699.21 ✓
A(29) = $ 10 725 B(29) = $ 10 725.07 ✓

(d) Determine with reasons if your answer in (c) will change if the amount invested in each account was $1 000 000?

There will be no change. ✓

The balances in each account would be multiplied by the same factor of 100. ✓

Calculator Assumed

8. [8 marks: 1, 1, 1, 2, 3]

Garrett is depreciating his home office equipment at 15% each year using the flat rate method (prime cost method). Under the flat rate method, an item loses the same amount of value each year. It cost Garrett $10 000 to purchase and install his home office equipment.

(a) What is the annual depreciation?

Annual depreciation = 10 000 × 0.15
= $1500 ✓

(b) What would the office system be worth after 2 years?

Value after 2 years = 10 000 – 2 × 1500
= $7 000 ✓

(c) When will the office system have a paper value of $0.

No. of years = 10 000/1500 = 6.7 years
That is after 7 years. ✓

(d) An alternative method of depreciation is to use the diminishing value method (reducing balance method), where each year the asset loses a constant percentage of its previous year's value. Use the diminishing value method with a depreciation rate of 15% per year to find:

(i) the value of the office system after 2 years.

Value after 2 years = $10\,000 \times 0.85^2$ ✓
= $7 225 ✓

(ii) the minimum number of years required for the value of the office system to be worth less than $1000.

$10\,000 \times 0.85^n < 1000$ ✓
$n > 14.168$ ✓
That is, at least 15 years. ✓

```
solve(10000×0.85^x<1000,x)
{x>14.16810399}
```

Calculator Assumed

9. [8 marks: 2, 2, 2, 2]

Jack and Jill bought a block of land for \$500 000 and built a \$300 000 house on the block. The value of the land appreciates (increases) at a constant rate of 8% per year while the value of the house depreciates (decreases) at a constant rate of 4% per year.

(a) Calculate the value of the block of land after 2 years.

Value = $500\,000 \times 1.08^2$ ✓
= \$583 200 ✓

(b) Calculate the value of the house after 2 years.

Value = $300\,000 \times 0.96^2$ ✓
= \$276 480 ✓

(c) Write a mathematical expression for the total value of the block of land and the value of the house after n years.

Total Value = $500\,000 \times 1.08^n + 300\,000 \times 0.96^n$
✓ ✓

(d) Calculate when the total value of the land and the house first exceeds \$1 000 000.

$500\,000 \times 1.08^n + 300\,000 \times 0.96^n = 1\,000\,000$ ✓
$n = 5.435$
Hence, after 6 years. ✓

```
solve(500000×1.08^x+300000×0.96^x=1000000
            {x=-28.03470806,x=5.43537139}
```


11 Financial Mathematics II (Loans)

Calculator Assumed

1. [9 marks: 2, 4, 3]

Henry borrowed \$10 000 to buy his first car. Interest is charged on the opening balance each month at a rate of 15% per annum. Henry repays \$800 each month for the first 10 months and thereafter he repays \$900 per month (except for the final payment). All payments are made at the end of each month. The final payment cannot exceed the regular payments.

(a) Why would Henry take more than 12 months to repay this loan?

Total paid in 12 months = (800 × 10) + (900 × 2) = 9800 ✓
This is less than amount borrowed (\$10 000). ✓
Hence, Henry will need more than 12 months.

The table below shows the state of Henry's account over the life of the loan.

Month	Opening Balance	Interest	Repayment	Closing Balance
1	10000.00	125.00	800.00	9325
2	9325.00	116.56	800.00	8641.56
3	8641.56	108.02	800.00	7949.58
4	7949.58	99.37	800.00	7248.95
5	7248.95	90.61	800.00	6539.56
6	6539.56	81.74	800.00	5821.3
7	5821.3	72.77	800.00	5094.07
8	5094.07	63.68	800.00	4357.75
9	4357.75	54.47	800.00	3612.22
10	3612.22	45.15	800.00	2857.37
11	2857.37	35.72	900	1993.09
12	1993.09	24.91	900	1118.00
13	1118.00	13.98	900	231.98
14	231.98	2.90	234.88	0
				–1 per error ✓✓✓

(b) Complete the table above to find how long Henry will take to repay the loan.

Harry will take 14 months. ✓

(c) How much interest would Henry have paid for the whole loan?

Total amount paid = 800 × 10 + 900 × 3 + 234.88 = \$10 934.88 ✓
Hence, interest paid = 10 934.88 – 10 000 = \$934.88 ✓✓

Calculator Assumed

2. [11 marks: 2, 2, 2, 3, 2]

Nick borrows $10 000 to buy a new car. Interest is charged at a reducible rate of R% per annum calculated monthly on the previous month's closing balance (assume that this rate is fixed for the life of the loan). Nick pays back $$B$ at the end of every month except for the last payment. Each repayment is made after the interest for the month has been added onto the amount owing at the start of the month. The table below shows the state of Nick's car loan account.

Start of Month	Amount Owing $	Interest $	Amount Owing after Interest and Repayment $
1	10 000	125	8 625
2	$ 8,625.00	$ 107.81	$7,232.81
3	$ 7,232.81	$ 90.41	$5,823.22
4	$ 5,823.22	$ 72.79	$4,396.01
5	$ 4,396.01	$ 54.95	$2,950.96

(a) Find the value of B.

$10000 + 125 - B = 8625$ ✓
Hence, B = $1500 ✓

(b) Find the value of R.

Monthly = $\frac{125}{10000} \times 100 = 1.25\%$ ✓
Hence, annual rate $R = 1.25 \times 12 = 15\%$ ✓

(c) Write a recursive rule to determine the closing balance at the end of each month.

$b_{n+1} = b_n \times 1.0125 - 1500, \quad b_0 = 10\,000$ ✓ ✓

(d) How long will Nick take to repay the loan and state the final payment.

Use $b_{n+1} = b_n \times 1.0125 - 1500, \quad b_0 = 10\,000$
$b_7 = 6.45 \quad b_8 = -1493.47$ ✓
Hence, Nick will take 8 months. ✓
Final payment = –1493.47 +1500 = $6.53 ✓

(e) How much interest did Nick pay over the life of the loan?

Total interest paid = 1500 × 7 + 6.53 – 10000 ✓
= $506.53 ✓

Calculator Assumed

3. [9 marks: 2, 2, 2, 3]

Adam took a loan worth $200 000 to purchase a property. Interest is charged on the opening balance each month. Adam's monthly payments are made at the end of each month.

(a) The table below is an account statement of Adam's loan for the first 6 months.

Month	Opening Balance	Interest charged	Repayment	Closing Balance
1	$200,000.00	$948.33	$948.33	$200,000.00
2	$200,000.00	$948.33	$948.33	$200,000.00
3	$200,000.00	$948.33	$948.33	$200,000.00
4	$200,000.00	$948.33	$948.33	$200,000.00
5	$200,000.00	$948.33	$948.33	$200,000.00
6	$200,000.00	$948.33	$948.33	$200,000.00

(i) What was the rate of interest (per annum)?

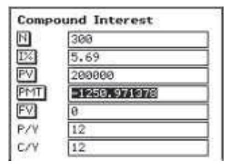
Rate of interest $= \frac{948.33}{200\,000}$ per month ✓
$= \frac{948.33}{200\,000} \times 12$
$= 0.0569 = 5.69\%$ per year ✓

(ii) Describe the main feature of Adam's loan.

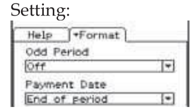
The repayments were sufficient only to cover the interest for the month and the principal remains unchanged. ✓✓

(c) In another scenario, Adam plans to pay off his entire loan in 25 years with the same rate of interest charged as in (a).
(i) What would his monthly payments be (nearest $)?

Use CAS Finance App (TVM Calculator):
Payment date: End of period.
N = 300, I = 5.69, PV = 200 000
FV = 0, P/Y=C/Y = 12 ✓
Monthly payments PMT = $1251 ✓



CAS Finance App Setting:



(ii) Calculate the interest paid over the life of the loan and hence the effective annual rate of interest.

Interest Paid = 1251 × 300 – 200 000 = 175 300 ✓
Effective annual interest $= \left(\frac{175300}{200000} \div 25\right) \times 100$ ✓
$\approx 3.51\%$ ✓

Calculator Assumed

4. [14 marks: 1, 3, 2, 3, 2, 3]

Jamie borrowed \$10 000 to buy her car. Interest is charged on the opening balance each month at a rate of 9% per annum. At the end of each month, Jamie repays \$500 except for the final payment. The final payment cannot exceed the regular payments. The table below shows, b_n, the opening balance at the start of each month and b_{n+1}, the closing balance at the end of each month.

Month	Opening Balance b_n	Interest	Repayment	Closing Balance b_{n+1}
1	\$10 000	\$75	\$500	\$9575
2	\$9575	\$71.81	\$500	\$9146.81
3	\$9146.81	\$68.60	\$500	\$8715.41

(a) Find the monthly interest rate.

Monthly rate = 9/12 = 0.75% ✓

(b) Complete the table above and state the amount Jamie owed at the end of the third month.

Table completed ✓✓
Amount owed at end of 3 months = \$8715.41 ✓

(c) Write a recursive rule to determine the closing balance at the end of each month.

$b_{n+1} = b_n \times 1.0075 - 500$ ✓ $b_0 = 10000$ ✓

(d) How long will Jamie take to repay her loan and how much is the final payment?

Use $b_{n+1} = b_n \times 1.0075 - 500$ $b_0 = 10000$
$b_{21} = 372.73$ $b_{22} = -124.48$ ✓
Hence, Jamie will take 22 months to pay off the loan. ✓
Final payment = –124.48 + 500 = \$375.52 ✓

(e) What is the total interest paid?

Total Interest Paid = 21 × \$500 + \$375.52 – \$10 000
= \$875.52 ✓✓

(f) Will Jamie pay off her loan in half the time taken in (d) if she doubled her monthly repayments? Justify your answer.

If repayments doubled, recursive formula would be
$b_{n+1} = b_n \times 1.0075 - 1000$ $b_0 = 10000$ ✓✓
Using this formula, Jamie needs 11 months to pay off the loan. Hence, yes. ✓

Calculator Assumed

5. [14 marks: 2, 2, 3, 1, 3, 3]

Olivia has a mortgage of \$600 000 on her house. Interest is calculated at 5.7% per annum on the month's opening balance and credited at the end of each month. Monthly repayments are made at the end of each month. The loan and its accrued interest have to be repaid in 30 years.

For the first 5 years, Olivia is permitted by the conditions of her loan to repay only the interest part of the loan. To pay off the loan within the remaining 25 years, Olivia has to make a minimum monthly payment of \$3 756.54.

(a) Calculate her monthly repayments for the first 5 years.

Monthly rate = $\frac{0.057}{12}$ = 0.00475 ✓
Monthly payment = 600 000 × 0.00475 = \$2 850 ✓

(b) If Olivia makes the minimum monthly payment for the next 25 years, what would be the total interest she would have paid for the house? Do not include the interest paid in the first 5 years.

Total payments = 3756.54 × 25 × 12 = \$ 1 126 962 ✓
Interest paid = 1 126 962 – 600 000 = \$ 526 962 ✓

(c) Olivia makes the minimum monthly payments from the start of year six to the end of year ten

(i) Write a recursive rule to determine the amount owing at the end of each month.

$B(n+1) = B(n) \times (1 + 0.00475) - 3756.54$ ✓✓ $B(0) = 600\,000$ ✓

(ii) Determine the amount left owing at the end year ten.

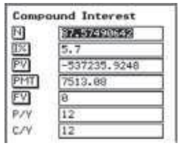
B(60) = \$ 537 235.92 ✓

Calculator Assumed

5. (c) (iii) Starting from the eleventh year, Olivia pays double the minimum monthly payment each month. How many more months will Olivia need to repay the entire loan?

Use CAS Finance App (TVM Calculator):
Payment date: End of period.
I = 5.7, PV = –537 235.9248 , FV = 0, P/Y=C/Y=12 ✓
Number of months N = 88.57 ✓

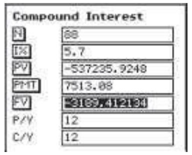
Olivia needs another 88 months to repay the rest of the loan. ✓



(iv) Calculate the final payment in Question 5 (c) (iii).

Use CAS Finance App (TVM Calculator):
Payment date: End of period.
N = 88, I = 5.7, PV = –537 235.9248 , PMT = 7 513.08, P/Y=C/Y=12 ✓
Final Balance FV = –3 189.41 ✓

Excess payment with 88th payment = \$ 3 189.41
Hence, final payment = 7 513.08 – 3 189.41 = \$ 4 323.67 ✓



Calculator Assumed

6. [12 marks: 2, 2, 2, 3, 3]

Sky has a mortgage of \$600 000 on her luxury apartment. The table given below shows the state of her reducible balance mortgage account for month 117 of her loan. Assume that the interest rate remains unchanged throughout the life of the loan. Sky repays \$6 000 each month.

Month	Starting Amount	Interest	Repayment	Amount Still Owing
117	\$316,516.02	\$2,347.49	\$6,000.00	\$312,863.52

(a) Find the annual interest rate charged.

Interest Rate = 2 347.49/316 516.02 × 12 × 100
= 8.9 % per annum ✓✓

(b) How much is the principal reduced by over the 118th month?

Reduction = 6000 – (312 863.52 × 0.089/12) ✓
= \$3 679.60 ✓

(c) Write a recursive rule to determine the closing balance at the end of each month, starting from month 117.

Let b_n : account balance at the end of n months after the start of month 117
$b_{n+1} = b_n \times 1.00741\overline{6} - 6000$ ✓ $b_0 = 316\,516.02$ ✓

(d) Determine how long Sky would take to reduce the amount owing to half the original amount borrowed.

Use $b_{n+1} = b_n \times 1.00741\overline{6} - 6000$ $b_0 = 316\,516.02$
$b_4 = 301742.65$, $b_5 = 297980.58$ ✓
Hence, at the end of 5 months after the start of month 117. ✓
Hence, end of 121 months. ✓

(e) From the start of the 120th month onwards, Sky converts her loan to an "interest only" non-reducible loan. That is, her monthly repayments are just sufficient to cover the interest charged for that month. What is the amount of this monthly repayment?

Use $b_{n+1} = b_n \times 1.00741\overline{6} - 6000$ $b_0 = 316\,516.02$
Balance at start of month 120 = Balance at the end of month 119
$b_3 = 305\,477.03$ ✓
Monthly interest on \$305 477.03 = 305 477.03 × $0.00741\overline{6}$ ✓
= \$2 265.62 ✓

Calculator Assumed

7. [10 marks: 2, 2, 6]

Jackie pays $4000 monthly towards her monthly reducible-balance mortgage of $500 000. Interest is charged at 8% per annum.

(a) Complete the table below to determine the balance at the end of the second month.

Month	Starting Amount	Interest	Repayment	Amount Still Owing
1	$500 000.00	$3333.33	$4000.00	$499 333.33
2	$499 333.33	$3328.89	$4000.00	$498 662.22

✓✓

Her sister Joan suggests that she should make weekly payments instead. Assume that Jackie's Bank agrees to convert her mortgage to a weekly reducible-balance mortgage and Jackie would now pay $1000 per week with interest remaining at 8% per annum. (Assume 1 year = 52 weeks)

(b) Write a recursive rule to determine the closing balance at the end of each week.

Let b_n : account balance at the end of the nth week
$b_{n+1} = b_n \times 1.00153846 - 1000$ ✓ $\quad b_0 = 500\ 000$ ✓

(c) At the end of the first year, determine which of the two payment methods is better, and by how much. Explain clearly how you arrived at your answer.

Let a_n : account balance at the end of the nth month
$a_{n+1} = a_n \times 1.00\overline{6} - 4000 \quad a_0 = 500\ 000$ ✓✓
$a_{12} = \$491\ 700.05$ ✓

Let b_n : account balance at the end of the nth week
$b_{n+1} = b_n \times 1.00153846 - 1000 \quad b_0 = 500\ 000$
$b_{52} = \$487\ 516.89$ ✓

Hence, the weekly payment is better. ✓
At the end of one year, the weekly payment method has reduced the amount owing by an extra $4183.16. ✓

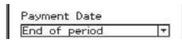
Calculator Assumed

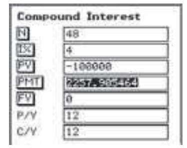
8. [12 marks: 2, 2, 2, 2, 4]

A couple needs $100 000 to renovate their house.
Option A is a reducible balance loan with interest compounded monthly at 4% per annum. The loan is to be repaid in equal monthly instalments in 4 years.
Option B is an <u>*interest only*</u> loan over 4 years with a flat rate interest of 3.9% per annum. The borrower only pays the interest charged during the term of the loan. At the end of the loan, the borrower pays the full amount borrowed. The interest charged is repaid in equal monthly installments.

(a) Calculate the monthly instalment for Option A.

Use CAS Finance App (TVM Calculator):
Payment date: End of period.
N = 28, I = 4, PV = – 100 000, FV = 0, P/Y=C/Y=12 ✓
Monthly instalment PMT = $ 2 257.91 ✓





(b) Calculate the total cost for Option A.

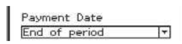
Total Payments = 2257.91 × 48 ✓
= $108 379.68 ✓

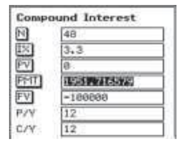
(c) Calculate the monthly payments for Option B.

Interest for the year = 100 000 × 0.039 = $3 900 ✓
Monthly payments = 3 900 ÷ 12 = $325 ✓

(d) To pay the $100 000 borrowed in Option B, the couple opens a savings account that pays interest at 3.3% per annum compounded monthly. The couple deposits a fixed amount into the account at the start of each month. Interest is paid at the end of each month on the opening balance at the start of each month after the monthly deposit is made. What should the monthly deposit be if the account balance at the end of 4 years is $100 000?

Use CAS Finance App (TVM Calculator):
Payment date: End of period.
N = 28, I = 3.3, PV = 0, FV = –100000, P/Y=C/Y=12 ✓
Monthly instalment = $ 1 951.72 ✓





Calculator Assumed

8. (e) Determine with reasons, which option should the couple take to borrow the $100 000 needed to fund their home renovations.

Monthly Cost for Option A = $ 2 257.91 ✓

Monthly Cost for Option B = 325 + 1951.72
= $2 276.72 ✓

Hence, monthly cost for Option A is cheaper. ✓

Hence, couple should choose Option A. ✓

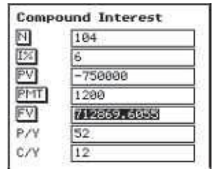
Calculator Assumed

9. [11 marks: 3, 3, 5]

Maxine has a mortgage of $750 000 on her house. Interest is calculated on the monthly opening balance at 6% per annum. Maxine pays $ 1200 at the end of each week.

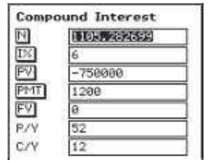
(a) Show that the loan has been reduced by $37 130 (to the nearest $) by the end of the second year.

Use CAS Finance App (TVM Calculator):
Payment date: End of period.
N = 104, I = 6, PV = – 750 000, PMT = 1 200,
P/Y= 52, C/Y=12 ✓
Final Balance FV = 712 870 ✓
Hence, loan reduced by 750 000 – 712 870 = $37 130
✓



(b) Determine how long it will take Maxine to repay the entire loan.

Use CAS Finance App (TVM Calculator):
Payment date: End of period.
N = 104, I = 6, PV = –750 000, PMT = 1 200, FV = 0
P/Y= 52, C/Y=12 ✓
Number of Instalments N = 1 105.28 ✓
Hence, loan will be repaid in 1 106 weeks ✓



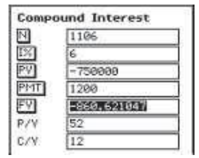
(c) What is the total interest paid by Maxine?

Use CAS Finance App (TVM Calculator):
Payment date: End of period.
N = 1106, I = 6, PV = –750 000 , PMT = 1 200,
P/Y=52, C/Y=12 ✓
Final Balance FV = –860.62 ✓

Excess payment with 1 106th payment = $ 860.62
Hence, final payment = 1 200 – 860.62 = $ 339.38 ✓

Total payments = 1 200 × 1105 + 339.38 = $1 326 339.38 ✓

Hence, total interest paid = 1 326 339.38 – 750 000
= $576 339.38 ✓



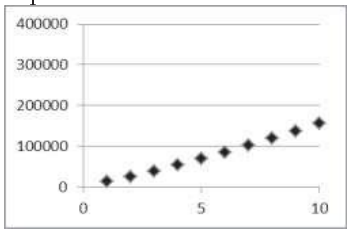
12 Financial Mathematics III (Annuities)

Calculator Free

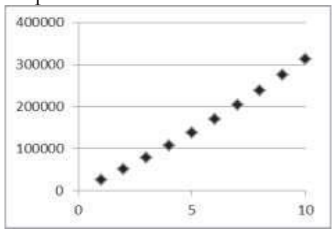
1. [8 marks: 2, 2, 2, 2]

The four graphs given below show the closing annual balances of several investment accounts over time in years. Each of these accounts has the same initial balance of $0. Match each of the given graphs with one of the given recursion relations.

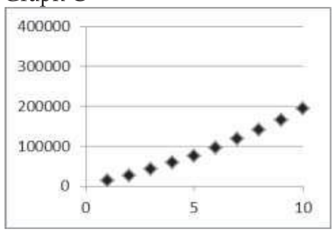
Graph A



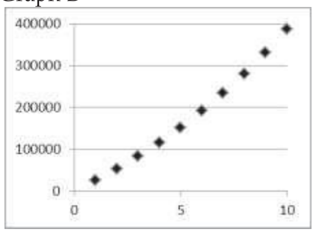
Graph B



Graph C



Graph D



Recursion Relations:

I	$b(n+1) = [b(n) + 12\,000 \times 1.05^{n-1}] \times 1.048$	$b(0) = 0$
II	$b(n+1) = [b(n) + 12\,000 \times 1.05^{n-1}] \times 1.048$	$b(1) = 0$
III	$b(n+1) = [b(n) + 12\,000] \times 1.048$	$b(1) = 0$
IV	$b(n+1) = [b(n) + 24\,000] \times 1.048$	$b(1) = 0$
V	$b(n+1) = [b(n) + 24\,000 \times 1.05^{n-1}] \times 1.048$	$b(1) = 0$
VI	$b(n+1) = [b(n) + 12\,000] \times 1.048$	$b(1) = 0$

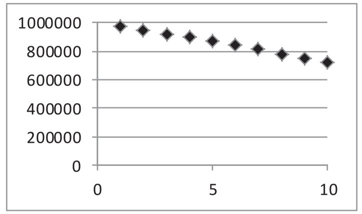
Graph A:	III	Graph B:	IV
Graph C:	II	Graph D:	V

Calculator Free

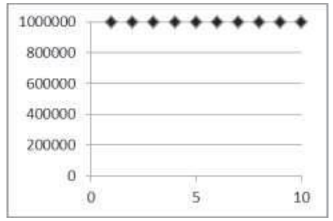
2. [8 marks: 2, 2, 2, 2]

The four graphs given below show the closing annual balances of several annuities over time in years. Each of these accounts has the same initial balance of $1 000 000. Match each of the given graphs with one of the given recursion relations.

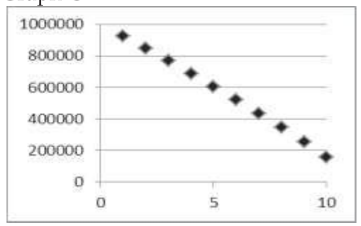
Graph A



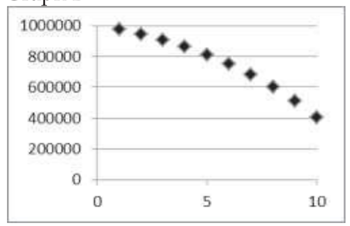
Graph B



Graph C



Graph D



Recursion Relations:

I	$b(n+1) = 1.025\,b(n) - 50\,000 \times 1.1^{n-1}$	$b(1) = 1\,000\,000$
II	$b(n+1) = 1.03\,b(n) - 25\,000$	$b(1) = 1\,000\,000$
III	$b(n+1) = 1.025b(n) - 25\,000$	$b(1) = 1\,000\,000$
IV	$b(n+1) = 1.025\,b(n) - 100\,000$	$b(1) = 1\,000\,000$
V	$b(n+1) = 1.025\,b(n) - 100\,000$	$b(0) = 1\,000\,000$
VI	$b(n+1) = 1.025b(n) - 50\,000$	$b(1) = 1\,000\,000$

Graph A:	VI	Graph B:	III
Graph C:	IV	Graph D:	I

Calculator Assumed

3. [10 marks: 2, 2, 2, 2, 2]

Mark owns a superannuation account. The account was opened on July 1st 2008, with an opening deposit of \$1000. On the first of each month thereafter, Mark deposits \$1000 into the account. The superannuation company claims that the average long term growth rate is 12% per annum on the opening balance each month and paid at the end of each month.

Month	Opening Balance \$	Balance after monthly deposit (\$)	Growth for the month (\$)	Balance at the end of the month (\$)
1	0	1000	10	1010
2	1010	2010	20.10	2030.10

–1 per error ✓✓

(a) Complete the table above to find the account balance after 2 months.

(b) Write a recursive rule to determine the closing balance at the end of each month.

Let b_n : account balance at the end of the nth month
$b_{n+1} = (b_n + 1000) \times 1.01$ ✓ $b_0 = 0$ ✓

(c) Find the account balance at the end of 10 years.

Use $b_{n+1} = (b_n + 1000) \times 1.01$ $b_0 = 0$
$b_{120} = \$232\,339.08$ ✓✓

(d) Find the total interest earned after 10 years.

Total interest earned $= b_{120} - 1\,000 \times 120$
$= \$232\,339.08 - 120\,000$ ✓
$= \$112\,339.08$ ✓

(e) Find the average annual percentage return after 10 years.

Average annual return $= \dfrac{112\,339.80}{120\,000} \times 100 \div 10$ ✓
$= 9.36\%$ ✓

Calculator Assumed

4. [10 marks: 2, 2, 2, 2, 2,]

May & Sydney operate an investment account for their children's education with an initial deposit of \$5000. They make regular monthly deposits of \$2000 each at the end of each month for 10 years. The table below shows the first few months of the account. Assume that the account pays a fixed rate of interest of 6% per annum. The monthly growth is calculated on the opening monthly balance and credited to the account at the end of each month.

Month	Opening Balance \$	Growth for the month (\$)	Monthly Deposit (\$)	Balance at the end of the month (\$)
1	5 000	25	2 000	7 025
2	7 025	35.125	2 000	9 060.125
3	9 060.125	45.30	2 000	11 105.43

–1 per error ✓✓

(a) Complete the table above to find the account balance at the end of the third month.

(b) Write a recursive rule to determine the account balance at the end of each month.

Let b_n : account balance at the end of the nth month
$b_{n+1} = b_n \times 1.005 + 2000$ ✓ $b_0 = 5000$ ✓

(c) Determine the account balance at the end of the 10 year period.

Use $b_{n+1} = b_n \times 1.005 + 2000$ $b_0 = 5000$
$b_{120} = \$336\,855.68$ ✓✓

(d) Find the total interest earned after 10 years.

Total interest earned $= b_{120} - (5\,000 + 2\,000 \times 120)$
$= \$336\,855.68 - 245\,000$ ✓
$= \$91\,855.68$ ✓

(e) Find the average annual percentage return after 10 years.

Average annual return $= \dfrac{91\,855.68}{245\,000} \times 100 \div 10$ ✓
$= 3.75\%$ ✓

Calculator Assumed

5. [9 marks: 2, 2, 2, 1, 2]

Penny operates a superannuation account with the investment division of a bank. On January 1st 2001, the account balance was $40 000.
On the first of each month, Penny deposits $500 into the account. Assume that her account will grow by an average of 9% per annum over a period of 30 years. Each month's growth is calculated on the opening monthly balance and credited to her account at the end of each month.

Month	Opening Balance $	Balance after monthly deposit ($)	Growth for the month ($)	Balance at the end of the month ($)
1	40 000	40 500	303.75	40 803.75
2	40 803.75	41 303.75	309.78	41 613.53

–1 per error ✓✓

(a) Complete the table above to find Penny's balance after 2 months

(b) Write a recursive rule to determine the account balance at the end of each month.

Let b_n : account balance at the end of the nth month
$b_{n+1} = (b_n + 500) \times 1.0075$ ✓ $b_0 = 40\,000$ ✓

(c) Use the recursive formula to find her account balance after 30 years.

Use $b_{n+1} = (b_n + 500) \times 1.0075$ with $B_0 = 40\,000$
$b_{360} = 1\,511\,460.07$ ✓
Hence, account balance after 30 years = $1 511 460.07 ✓

The explicit formula for the account balance, b_n, at the end of each month:

$$b_n = (40\,000 \times r^n) + 500 \times r \times \frac{(1-r^n)}{1-r} \quad \text{where } n = 1, 2, 3\,.$$

(d) What should the value of r be?

$r = 1.0075$ ✓

(e) Use the above formula to calculate her account balance after **30 years**.

Hence, $b_n = (40\,000 \times 1.0075^{360}) + 500 \times 1.0075 \times \frac{(1 - 1.0075^{360})}{1 - 1.0075}$ ✓
= $1 511 460.07 ✓

Calculator Assumed

6. [15 marks: 2, 3, 1, 3, 3, 3]

William starts work on 1st July 2016 with an annual salary of $60 000.
His annual salary increases by 3% every year. His employer contributes 9.5% of his annual salary in four equal quarterly amounts into a superannuation fund held in his name. The employer's contributions are transferred into his fund at the end of each quarter. The superannuation fund pays interest at 6% per annum compounded quarterly.

(a) Calculate the employer's quarterly contribution into his fund in his first year of work?

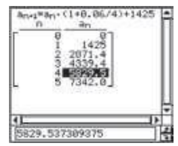
Annual contribution = 60 000 × 0.095 ✓
= $5 700
Quarterly contribution = $1425 ✓

(b) The recursion formula to describe the fund balance at the end of quarter n of his *first* year of work is given by $B(n) = B(n-1) \times (1+a) + b \quad B(0) = c$.
Determine the values of a, b and c.

$B(n) = B(n-1) \times (1 + 0.015) + 1\,425 \quad B(0) = 0$
$a = \frac{0.06}{4} = 0.015$ ✓ $b = 1425$ ✓ $c = 0$ ✓

(c) Calculate the fund balance at the end of the *first* year.

B(4) = $ 5 829.54 ✓



(d) Calculate the employer's quarterly contribution into his fund in his *second* year of work?

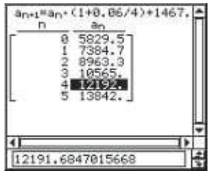
Annual Salary in year 2 = 60 000 × 1.03 ✓
Quarterly contribution in year 2
$= 60\,000 \times 1.03 \times \frac{0.095}{4}$ ✓
= 1425 × 1.03 = $1467.75 ✓

Calculator Assumed

6. (e) Calculate the fund balance at the end of the *second* year. Show clearly how you obtained your answer.

$B(n) = B(n-1) \times \left(1+\frac{0.06}{4}\right) + 1467.75 \quad B(0) = 5\,829.54$ ✓✓

Balance at end of year 2, B(4) = \$ 12 191.68 ✓



(f) The fund balance at the end of the 29th year is \$ 607 354.96.
Calculate the fund balance at the end of the 30th year.
Show clearly how you obtained your answer.

Annual salary for year 30 = $60\,000 \times 1.03^{29}$
Quarterly contribution in year 30

$= 60\,000 \times 1.03^{29} \times \frac{0.095}{4}$ ✓

$= 3358.11$

$B(n) = B(n-1) \times \left(1+\frac{0.06}{4}\right) + 3358.11 \quad B(0) = 607\,354.96$ ✓

Balance at end of year 30, B(30) = \$ 658 362.10 ✓

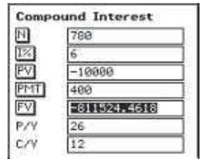
Calculator Assumed

7. [10 marks: 2, 2, 3, 3]

Sheldon opens an investment account with an initial deposit of \$10 000. The account pays interest on the opening monthly balance at a rate of 6% per annum and credited into the account at the end of each month. Each fortnight thereafter Sheldon deposits \$400 into this account.

(a) Calculate the account balance after 30 years.

Use CAS Finance App (TVM Calculator):
Payment date: End of period.
N = 780, I = 6, PV = – 10 000, PMT = 400,
P/Y= 26, C/Y=12 ✓
Final Balance FV = \$811 524.46 ✓



(b) How much interest after 30 years?

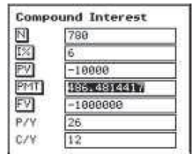
Total Contributions = 10 000 + (400 × 26 × 30) = \$322 000 ✓

Hence, interest earned = 811 524.46 – 322 000 = \$489 524.46 ✓

(c) How much extra each fortnight should he contribute for the account balance to exceed \$1 000 000 after 30 years?

Use CAS Finance App (TVM Calculator):
Payment date: End of period.
N = 780, I = 6, PV = – 10 000, FV = – 1 000 000,
P/Y= 26, C/Y=12 ✓
Instalment PMT = \$486.48 ✓

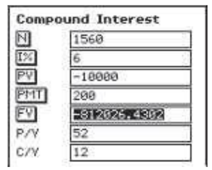
Hence an extra \$86.48 per fortnight. ✓



(d) Would contributing \$200 each week instead of \$400 each fortnight improve the fund balance at the end of 30 years? Justify your answer.

Use CAS Finance App (TVM Calculator):
Payment date: End of period.
N = 1560, I = 6, PV = – 10 000, PMT = 400,
P/Y= 26, C/Y=12 ✓
Final Balance FV = \$812 026.53 ✓

Hence, yes, by an extra \$502.07. ✓



Calculator Assumed

8. [13 marks: 2, 3, 2, 2, 2, 2]

Leonard has $500 000 in an account that pays interest on the opening monthly balance at the end of each month. Interest is paid at 0.25% per month. Leonard withdraws $1000 at the end of each month after the interest has been paid. The table below shows the monthly account balances.

Month	Opening Balance	Interest for the month	Monthly withdrawal	Closing Balance
1	$500 000.00	$1 250. 00	$1 000	$500 250.00
2	$500 250.00	$1 250.63	$1 000	$500 500.63
3	$500 500.63	$1 251.25	$1 000	$500 751.88
4	$500 751.88	$1 251.88	$1 000	$501 003.76
5	$501 003.76	$1 252.51	$1 000	$501 256.27
6	$501 256.27	$1 253.14	$1 000	$501 509.41

(a) Complete the table above for month 6.

(b) Write a recursion relation for the opening balance for each month.

$b(n+1) = b(n) \times 1.0025 - 1000 \quad b(1) = 500\,000$
✓✓ ✓

(c) Find the account balance at the start of month 12.

$b(12) = \$502\,784.63$ ✓✓

(d) Find the account balance at the end of month 24.

$b(25) = \$\,506\,175.70$ ✓✓

(e) Without further calculations, describe with reasons, the state of the account balances if Leonard were to withdraw $1000 at the start of each month beginning on the start of month 2.

Whether the withdrawal is made at the end of the month or the start of the next month, the opening monthly balance remains the same Hence, no change from the previous arrangement. ✓✓

(f) Without further calculations, describe the account balances if Leonard were to withdraw $1250 at the end of each month.

Monthly withdrawal of $1250 is equal to the interest earned over the first month. Hence, account balances would remain at $500 000 at the start of each month. ✓✓

Calculator Assumed

9. [13 marks: 2, 3, 2, 3, 1, 2]

Amy has $800 000 in an annuity that pays interest at 3% on the opening annual balance. Interest is paid at the end of each year. From the end of the first year on, Amy withdraws $40 000 at the end of each year after the interest has been paid. Assume that Amy had sufficient funds for the first year. The table below shows the annual account balances for the first few years.

Year	Opening Balance	Interest for the year	Annual withdrawal	Closing Balance
1	$800 000	$24 000	$40 000	$784 000
2	$784 000	$23 520	$40 000	$767 520
3	$767 520	$23 025.60	$40 000	$750 545.60
4	$750 545.60	$22 516.37	$40 000	$733 061.97
5	$733 061.97	$21 991.86	$40 000	$715 053.83

(a) Complete the table above for year 5.

(b) Write a recursion relation for the closing balance for each year.

$b(n+1) = b(n) \times 1.03 - 40\,000 \quad b(0) = 800\,000$
✓✓ ✓

(c) Find the account balance at the end of 20 years.

$b(20) = \$370\,074$ ✓✓

(d) How many years will the funds last? Justify your answer.

$b(30) \approx \$38\,793.35$ ✓
$b(31) \approx -\$42.85$
Balance at end of year 31 before annual withdrawal
$\approx 40\,000 - 42.85 = \$39\,957.15$ ✓
Hence, the funds will last no more than 32 years. ✓

(e) How much should Amy withdraw each year (after the interest has been paid) if she wishes the account balance to remain constant?

Withdrawal = $24 000 ✓

(f) If Amy chooses the option in part (e), what would the funds left in the account be worth after 30 years if the annual inflation rate is 2%.

Funds worth $= \dfrac{800\,000}{1.02^{30}} \approx \$441\,657$ ✓✓

Calculator Assumed

10. [15 marks: 2, 4, 2, 7]

Raj has $1 500 000 in an annuity that pays interest at 3% on the opening annual balance. Interest is paid at the end of each year. At the end of the each year after the interest is paid, Raj withdraws a certain sum for his living expenses for the following year. At the end of the first year he withdraws $60 000. From then on, the amount he withdraws increases by 5% each year. The table below shows his account balance over several years.

Year	Opening Balance	Interest for the year	Annual withdrawal	Closing Balance
1	$1 500 000	$45 000	$60000	$1 485 000
2	$1 485 000	$44 550	$63000	$1 466 550
3	$1 466 550	$43 996.50	$66150	$1 444 396.50
4	$1 444 396.50	$43331.90	$69457.50	$1 418 270.90
5	$1 418 270.90	$42 548.13	$72930.38	$1 387 888.65

(a) Complete the table below for year 5.

(b) Write a recursion rule to describe the closing balance in for each year.

Recursion Rule for balance at the end of each year is:
$b(n+1) = 1.03\,b(n) - 60\,000 \times 1.05^n$ ✓ ✓ ✓ $\quad b(0) = 1\,500\,000$ ✓

(c) Determine when the account balance first drops below $500 000.

b(17) = $561 759.39
b(18) = $441 091.07 ✓
Hence, at the end of the 18th year. ✓

(d) How much longer would his funds last if he withdrew a constant $60 000 at the end of each year? Show how you obtained your answer.

For increasing annual withdrawals:
$b(21) \approx \$13\,438$ ✓
Hence, funds will last no longer than 22 years. ✓

For constant annual withdrawals:
$b(n+1) = 1.03\,b(n) - 60\,000 \quad b(0) = 1\,500\,000$ ✓✓
$b(46) \approx \$52478 \quad b(47) = -\$5948$ ✓
Hence, funds will last no longer than 48 years. ✓

Therefore, with constant withdrawals, his funds would last him an extra 26 years! ✓

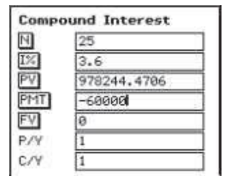
Calculator Assumed

11. [7 marks: 2, 5]

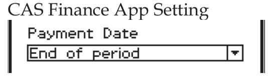
Howard expects to live a further 25 years after he retires. Consider an annuity that pays interest on the opening balance at a rate of 3.6% per annum with interest paid at the end of the year. Withdrawals are made after the interest is paid. Howard and his wife Bernadette estimate that they require about $60 000 each year.

(a) How much (nearest $10 000) would the couple need to start this annuity, for the funds to last at least 25 years?

Use CAS Finance App (TVM Calculator):
Payment date: End of period.
N = 25, I = 3.6, PMT = –60000, FV = 0
P/Y= 1, C/Y = 1 ✓
Initial Balance PV ≈ $978 244
≈ $980 000 ✓



CAS Finance App Setting

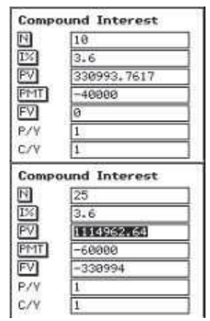


(b) Bernadette expects to live a further 10 years after Howard expires. Bernadette estimates that she would require about $40 000 per year. How much would the couple need to start this annuity in order to fund them over their estimated lifespans? Explain how you obtained your answer.

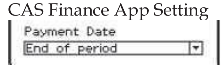
Use CAS Finance App (TVM Calculator):
Payment date: End of period.
N = 10, I = 3.6, PMT = –40000, FV = 0
P/Y= 1, C/Y = 1 ✓
Present Value PV = 330 993.76
Bernadette requires a further $330 994. ✓
Hence, balance after 25 years should be ≈ $330 994. ✓

Use CAS Finance App (TVM Calculator):
Payment date: End of period.
N = 25, I = 3.6, PMT = –60 000, FV = –330 994
P/Y= 1, C/Y = 1 ✓
Present Value PV = 1 114 962.64

Initial balance should be
≈ $1 114 962.64 ≈ $1 115 000 ✓



CAS Finance App Setting



Calculator Assumed

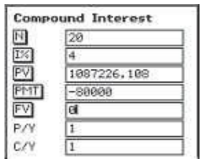
12. [6 marks: 2, 4]

Shane is 25 and has just started a full time job. He expects to work until 65 and live until 85. In his retirement he estimates that he will need about \$80 000 per year. Shane is considering the following financial policies.

- A superannuation policy which pays interest on the opening annual balance at a rate of 5.6% per annum and credited into the account at the end of each year.
- An annuity which pays interest on the opening balance at a rate of 4% per annum with interest paid at the end of the year. Withdrawals are made after the interest is paid.

(a) How much would Shane require to start the annuity to fund him over his expected lifespan? (Give your answer to the nearest \$10 000)

Use CAS Finance App (TVM Calculator):
Payment date: End of period.
N = 20, I = 4, PMT = –80 000, FV = 0
P/Y= 1, C/Y = 1 ✓
Present Value PV = 1087226.11
Amount required = \$1 087 226.11
≈ \$1 088 000 ✓

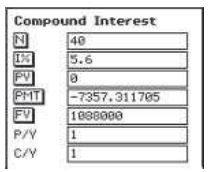


CAS Finance App Setting

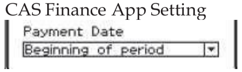
(b) How much would Shane need to contribute annually into the superannuation policy to fund the retirement he requires? Explain how you obtained your answer.

Shane requires \$1 088 000 ✓
in his superannuation account after 40 years. ✓

Use CAS Finance App (TVM Calculator):
Payment date: Beginning of period.
N = 40, I = 5.6, PV = 0, FV = 1 088 000
P/Y= 1, C/Y = 1 ✓
PMT = –7357.31
Annual contribution is ≈ \$7360 per annum. ✓



CAS Finance App Setting



Calculator Assumed

13. [6 marks: 2, 4]

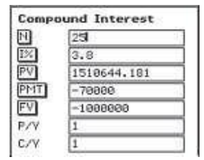
Alexa is 25 and has just started a full time job. She expects to work until 65 and live until 90. In her retirement she estimates that she will need about \$70 000 per year. Alexa is considering the following financial policies.

- An income stream that pays interest on the opening balance at a rate of 3.8% per annum with interest paid at the end of the year. Withdrawals are made after the interest is paid.
- A superannuation policy that pays interest on the opening annual balance at a rate of 6.5% per annum and credited into the account at the end of each year.

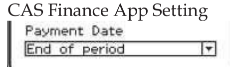
(a) How much would Alexa require to start the income stream to fund her over her expected lifespan and yet leave \$1 000 000 to her family? (Give your answer to the nearest \$10 000)

Use CAS Finance App (TVM Calculator):
Payment date: End of period.
N = 25, I = 3.8, PMT = –70 000, FV = –1 000 000
P/Y= 1, C/Y = 1 ✓
Present Value PV = 1 510 644.18

Amount required ≈ \$1 510 644.18 ≈ \$1 511 000 ✓✓



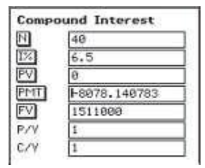
CAS Finance App Setting



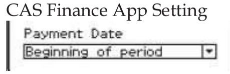
(b) How much would Alexa need to contribute annually into the superannuation policy to fund the retirement she requires and leave \$1 000 000 to her family? Explain how you obtained your answer.

Alexa requires \$1 511 000 ✓
in her superannuation account after 40 years. ✓

Use CAS Finance App (TVM Calculator):
Payment date: Beginning of period.
N = 40, I = 6.5, PV = 0, FV = 1 511 000
P/Y= 1, C/Y = 1 ✓
PMT = –8078.14
Annual contribution is ≈ \$8080 per annum. ✓



CAS Finance App Setting



Calculator Assumed

14. [7 marks: 2, 1, 2, 2]

When Wendy retired at 65, she sold her share in a business for $1 500 000.
At the time of her retirement, women were expected to live a further 25 years.

(a) Option A.
She invests the sum received in a perpetuity which pays interest at a rate of 7.2% per annum calculated monthly. The perpetuity pays her a monthly amount.

(i) Determine the monthly amount she receives.

Monthly rate = $\frac{0.072}{12}$ = 0.006 ✓
Monthly payment = 1 500 000 × 0.006 = $9 000 ✓

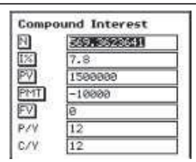
(ii) How much is left in the perpetuity account after the 300th payment?

Balance = $1 500 000 ✓

(b) Option B
She invests the sum in an annuity that pays interest on the monthly opening balance at a rate of 7.8 % calculated monthly and withdraws $10 000 each month after the interest has been credited. The annuity is shut down when the balance goes below $10 000.

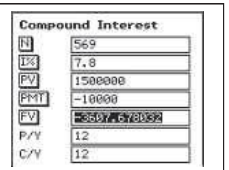
(i) How many full $10 000 withdrawals can Wendy make?

Use CAS Finance App (TVM Calculator):
Payment date: End of period.
I = 7.8, PV = 1 500 000, PMT = –10 000, FV = 0
P/Y= 12, C/Y=12 ✓
Number of Instalments N = 569.4
Wendy can make 569 full withdrawals. ✓



(ii) What would be the balance in the annuity account when it was shut down?

Use CAS Finance App (TVM Calculator):
Payment date: End of period.
N = 569 I = 7.8, PV = 1 500 000, PMT = –10 000,
P/Y= 12, C/Y=12, ✓
FV = –3607.68
Balance after 569th withdrawal = $3 607.68 ✓



13 Graph Theory I

(Basic terminology, planar graphs, paths and trails)

Calculator Free

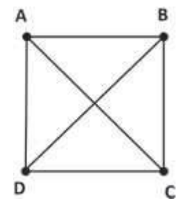
1. [3 marks]

Complete the table below for the following connected planar graphs.

Number of Vertices, V	Number of Faces, F	Number of Edges, E
7	7	12
8	5	11
8	6	12

2. [7 marks: 1, 2, 4]

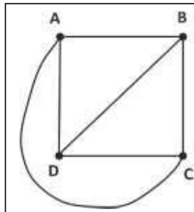
The accompanying diagram shows a drawing of a graph with four vertices and six edges.



(a) Explain clearly why the drawing is non-planar.

The edges AC and BD intersect outside of the vertices. ✓

(b) In the space provided below, provide a planar representation of this graph.



Redraw either AC or BD. ✓✓

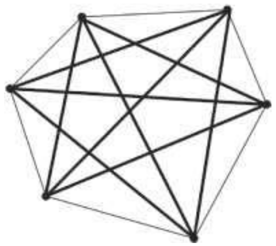
(c) How many *complete planar* sub-graphs are possible?
Name (or draw) the complete planar sub-graphs.

4 vertices & 6 edges: A-B-C-D-A-C | B ✓
3 vertices & 3 edges: A-B-C-A, A-B-D-A, B-D-C-B, A-D-C-A ✓✓
Total = 5 sub-graphs. ✓

Calculator Free

3. [4 marks: 2, 2]

The diagram below shows a graph with six vertices and six edges.



All vertices connected to each other. ✓✓

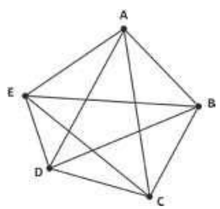
(a) Explain what the term "complete graph" means.

A complete graph is a simple graph where all the vertices are connected to each other by an edge. ✓✓

(b) Add edges to the graph so that the graph becomes a complete graph.

4. [6 marks: 1, 5]

The accompanying diagram shows a K_n graph.

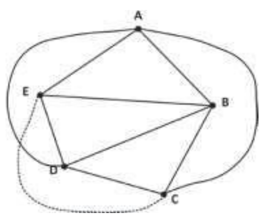


(a) Determine the value of n.

$n = 5$ as there are 5 vertices. ✓

(b) Explain with an example, why this K_n graph is not a planar graph.

- There are five intersecting internal diagonals AD, AC, EB, EC and BD. ✓
- Three of the diagonals have to be redrawn on the outside so that they do not intersect. ✓
- But the third redrawn diagonal will always intersect with another of the redrawn diagonal. ✓
- For example: AD and AC are redrawn on the outside. EC cannot remain inside as it intersects BD. But EC drawn on the outside will intersect AD. Hence, EC must intersect with one of the diagonals. ✓ ✓
- Hence, this graph cannot be drawn as a planar graph.



Calculator Free

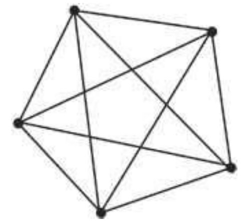
5. [6 marks: 1, 2, 3]

(a) Explain why adding an additional edge to the accompanying diagram would make the graph non-planar.

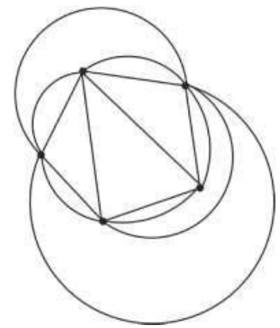
Drawing an extra edge would make the graph a K5 graph which is non-planar. ✓

(b) Draw a graph with 5 vertices such that the sum of the degrees of all the vertices is 20.

✓ Has 5 vertices .
✓ Total no of edges = 20



(c) Draw a planar graph which contains no loops with 5 vertices and 15 edges.

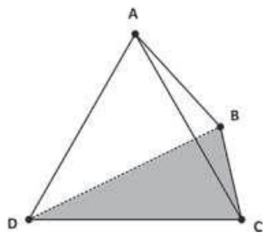


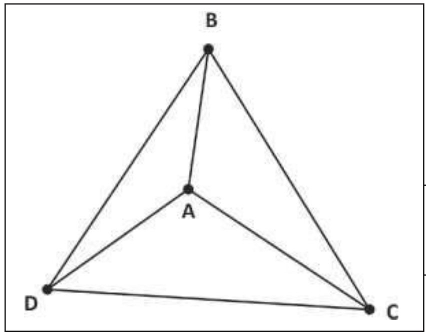
✓ Has 5 vertices
✓ Has 15 edges.
✓ Graph is planar.

Calculator Free

6. [6 marks: 3, 3]

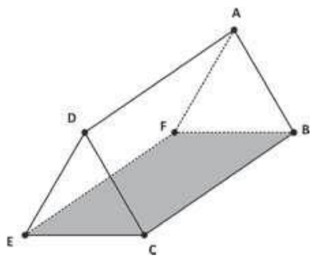
(a) ABCD is a triangular pyramid with ΔBCD as its base. Represent the pyramid ABCD as a planar graph.

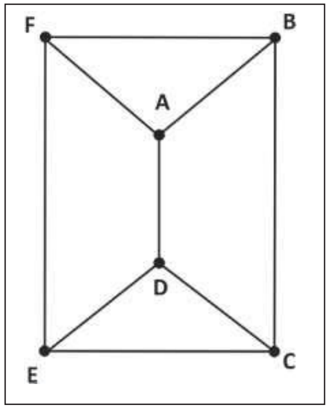




Many possible solutions.
4 vertices, 6 edges & 4 faces ✓
Planar ✓✓

(b) ABCDEF is a triangular prism with BCEF as its base. Represent the prism ABCDEF as a planar graph.



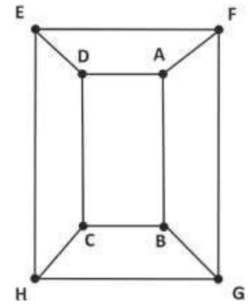


Many possible solutions.
6 vertices, 9 edges & 5 faces ✓
Planar ✓✓

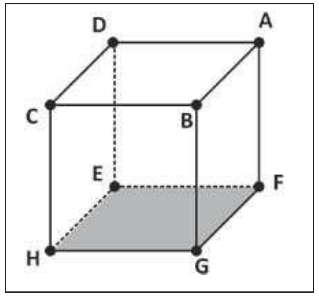
Calculator Free

7. [13 marks: 3, 2, 2, 3, 3]

The accompanying diagram shows a planar graph.



(a) Sketch a three dimensional object which shares this planar graph.



Many possible solutions.
8 vertices, 12 edges & 6 faces ✓
Trapezoidal prism ✓✓

(b) Identify an open Hamiltonian path (semi-Hamiltonian) for this planar graph.

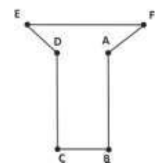
Many possibilities. A-F-G-B-C-D-E-H ✓✓

(c) Identify a Hamiltonian cycle (closed Hamiltonian) for this planar graph

Many possibilities. A-F-G-B-C-H-E-D-A ✓✓

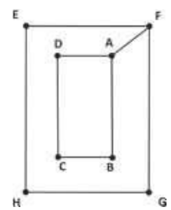
(d) Identify and draw a sub-graph consisting of six vertices which forms an Eulerian circuit.

Many possibilities.
A-F-E-D-C-B-A
✓✓✓



(e) Identify and draw a sub-graph consisting of eight vertices which forms a semi-Eulerian trail.

Many possibilities.
A-B-C-D-A-F-G-H-E-F
✓✓✓



Calculator Free

8. [11 marks: 1, 2, 2, 3, 3]

(a) Explain what a connected planar graph is.

A planar graph is a connected graph where the edges intersect only at the vertices. ✓

(b) A connected planar graph has a total of 12 vertices and faces. How many edges are there in this graph?

$V + F = E + 2$
$12 = E + 2$ ✓
$E = 10$ ✓

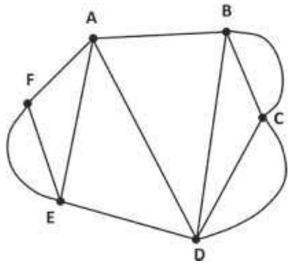
(c) The number of edges less the number of vertices in a connected planar graph is 4. How many faces are there in this graph?

$V + F = E + 2$
$F - 2 = E - V$ ✓
$= 4$
$F = 6$ ✓

(d) Is it possible to have a connected planar graph with 6 vertices and 4 edges? Justify your answer.

For a planar graph:
$V + F = E + 2$
If $V = 6$ and $E = 4$, then $F = 0$. ✓
But a connected graph has at least 1 face. ✓
Hence, not possible. ✓

(e) Draw a connected planar graph with 6 vertices and 8 faces.

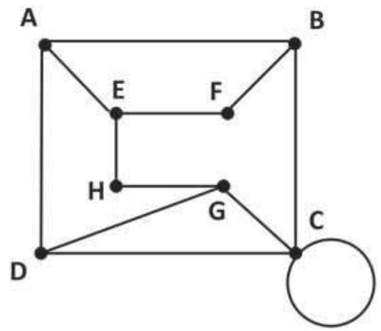


$V = 6$, $F = 8$, $E = 12$ ✓✓✓

Calculator Free

9. [12 marks: 1, 2, 2, 3, 2, 2]

The diagram below shows a graph.



(a) State the degree of vertex C.

Degree of vertex C = 5 ✓

(b) Determine with reasons if this graph is a simple graph.

There is a loop at C. ✓
Hence, not a simple graph. ✓

(c) Verify that Euler's rule works for this graph.

$V = 8$ $F = 6$ $E = 12$ ✓
$V + F - E = 8 + 6 - 12 = 2$ ✓
Hence, Euler's rule works.

(d) Determine with reasons if this graph is traversable.

Vertices A, B, C, E and G are odd. ✓
Hence, graph is not traversable as it has more than two odd vertices. ✓✓

(e) Identify a Hamiltonian cycle for the graph given.

ABFEHGCDA or reverse ✓✓

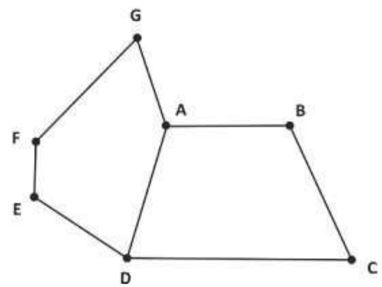
(f) Identify and state the length of the longest trail from C to A.

C-C-B-F-E-H-G-C-D-A ✓
Length = 9 ✓

Calculator Free

10. [10 marks: 2, 3, 3, 2]

The accompanying diagram shows a graph.



(a) Determine with reasons if the edge EF is a bridge.

EF is not a bridge.	✓
Removing EF still leaves the graph connected.	✓

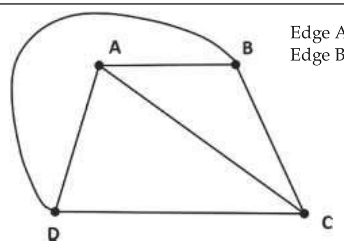
(b) Determine with reasons if this graph forms a semi-Eulerian trail. If it is, state the semi-Eulerian trail.

Graph is semi-Eulerian.	✓
Graph has exactly two odd vertices at A and D.	✓
semi-Eulerian trail: ABCDAGFED or equivalent	✓

(c) Determine with reasons which edge you would remove so that this graph forms an Eulerian Circuit? Give a possible Eulerian circuit.

Removing AD would make all vertices even.	✓✓
This would create an Eulerian circuit ABCDEFGA.	✓

(d) In the space provided below, draw the sub-graph A-B-C-D-A. Add edges to this sub-graph so that it becomes a complete planar graph.

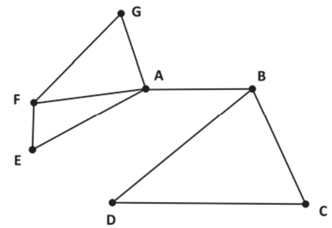


Edge AC ✓
Edge BD ✓

Calculator Free

11. [16 marks: 3, 3, 2, 2, 2, 4]

The accompanying diagram shows a graph.



(a) Determine with reasons if the graph is simple.

The graph is simple.	✓
There are no loops	✓
and no adjacent vertices are connected by multiple edges.	✓

(b) Determine with reasons if the edge AB is a bridge.

AB is a bridge.	✓
If it is removed, the graph becomes two disconnected subgraphs	✓
consisting of the vertices BCD and AEFG respectively.	✓

(c) This graph does not have a Hamiltonian cycle. Add an extra edge to this graph so that a Hamiltonian cycle may be formed. State this path.

Add DE.	✓
Hamiltonian cycle: ABCDEFGA	✓

(d) Verify that Euler's rule works for this graph.

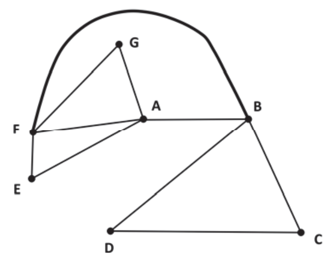
V = 7 F = 4 E = 9	
V + F – E = 7 + 4 – 9 = 2	✓✓
Hence, Euler's rule works.	

(e) Identify the semi-Eulerian trail in this graph.

Semi-Eulerian Trail: BCDBAGFEAF or equivalent.	✓✓

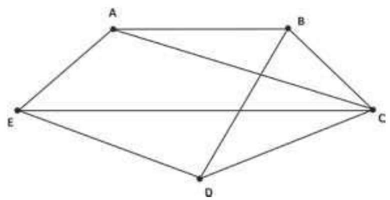
(f) Add an additional edge to the graph so that an Eulerian circuit may be formed. Explain the reason for your choice. State this circuit.

Add BF.	✓
Reason: To make all vertices even.	✓
Eulerian circuit: BCDBAFEAGFB or equivalent.	✓✓



Calculator Free

12. [10 marks: 3, 3, 2, 2]



(a) For the graph given above, determine with reasons if the walk ABCAECD is semi-Eulerian (forms an Eulerian Trail).

Not semi-Eulerian. ✓
Reason: It does not cover all edges: ✓
DE , BD excluded. ✓

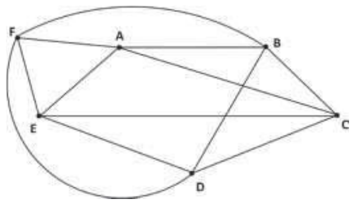
(b) Explain why the graph given above is neither Eulerian or semi-Eulerian.

Graph has 4 odd vertices; A, B, D and E. ✓
It cannot be Eulerian as Eulerian graphs have no odd vertices. ✓
It cannot be semi-Eulerian as semi-Eulerian graphs have exactly 2 odd vertices. ✓

(c) State with reasons, an edge that can be removed so that the graph becomes semi-Eulerian.

Graph has 4 odd vertices; A, B, D and E. ✓
Removing a link between any two of the odd vertices (eg BD),
leaves two remaining odd vertices, ✓
which makes the graph semi-Eulerian.

(d) Another vertex F is added to the graph. State with reasons how many additional edges need to be added to make the new graph Eulerian.



Four additional edges, AF, BF, EF
and DF. ✓
This makes all vertices even, ✓
and hence graph becomes Eulerian.

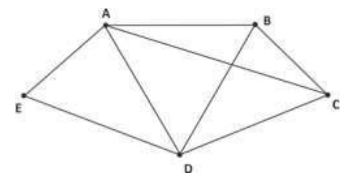
Calculator Free

13. [6 marks: 2, 2, 2]

(a) Compare and contrast between a trail and a path.

A trail and a path have no repeated edges. ✓
However in a path, vertices cannot be repeated and
in a trail, vertices can be repeated. ✓

(b) Consider the accompanying graph.



(i) Determine with reasons if the walk ABCDA is a Hamiltonian Cycle.

Not a Hamiltonian cycle ✓
as it does not include vertex E. ✓

(ii) Determine with reasons if the walk ABCAEDCA is an Eulerian Trail.

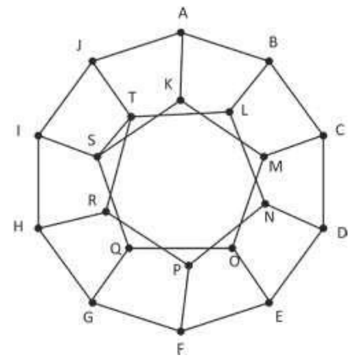
Not an Eulerian Trial as it does not include edge AD.
✓ ✓

14. [7 marks: 3, 4]

(a) Compare and contrast the features of a cycle and a Hamiltonian cycle.

Both are closed walks with no repeat edges and no repeat vertices. ✓
A Hamiltonian cycle visits all vertices in the graph ✓
whereas in a cycle, not all vertices need be visited. ✓

(b) Describe a Hamiltonian cycle for the accompanying graph.

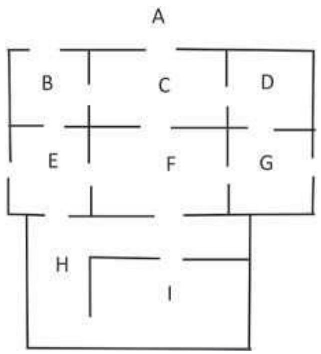


✓ Starts and finishes path at the same vertex.
✓ Connects all vertices.
✓ Vertices are not repeated.
✓ Cycle is correct
(eg AJIHGFEDCBLNPRTSQOMKA)

Calculator Free

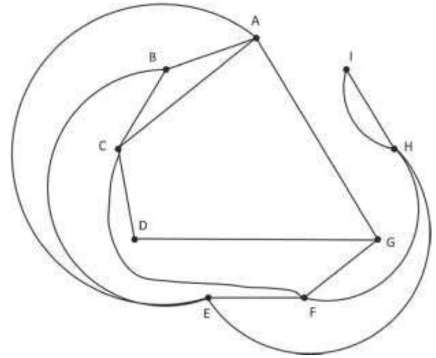
15. [10 marks: 5, 2, 3]

The accompanying diagram shows a maze divided into regions marked with the letters A to I. The entries into the various regions are drawn as gaps.



(a) Draw a graph to represent this maze, with the entries represented by edges.

✓ 9 vertices
✓ 15 edges
✓ Odd vertices: B and G
✓ All other vertices even.
✓ All correct.



(b) Determine with reasons if it is possible to start from A going through each entry exactly once and returning to A.

No. ✓
Graph is not Eulerian as there B and G are odd vertices. ✓

(c) Is it possible to create a trail that passes through each entry exactly once? Give a reason if it is not possible and give the trail if it possible.

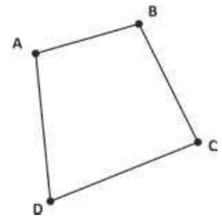
Yes it is possible. ✓
One possible trail:
B-C-F-E-H-I-H-F-G-D-C-A-B-E-A-G ✓✓

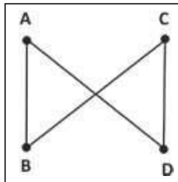
14 Graph Theory II (Bipartite graphs & Adjacency matrices)

Calculator Free

1. [10 marks: 3, 4, 3]

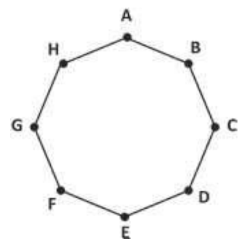
(a) Redraw the accompanying diagram as a bipartite graph.

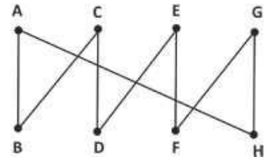




set {A, C} and set {B, D} ✓
Correct edges. ✓✓

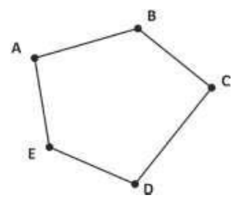
(b) Redraw the accompanying diagram as a bipartite graph.

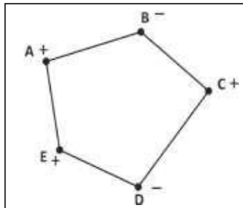




{A, C, E, G} ✓
{B, D, F, H} ✓
Correct edges. ✓✓

(c) Explain why the graph in the accompanying diagram is not bipartite.





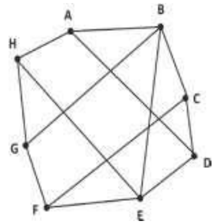
Graph showing + / – ✓

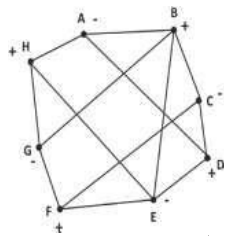
When the connected vertices are alternately assigned into groups + or –, we end up with connected vertices A and E in the same group. Hence, graph cannot be bipartite. ✓✓

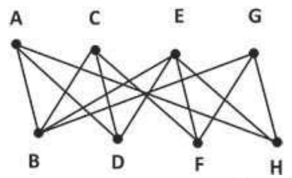
Calculator Free

2. [6 marks: 3, 3]

(a) For the graph drawn in the accompanying diagram, determine if the graph is bipartite or otherwise.
If the graph is not bipartite, explain why.
If the graph is bipartite, redraw the graph identifying the bipartite sets.

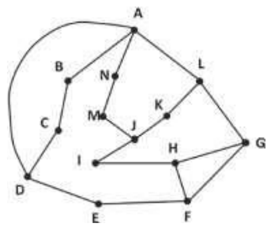


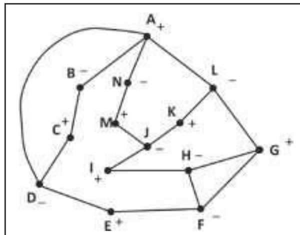


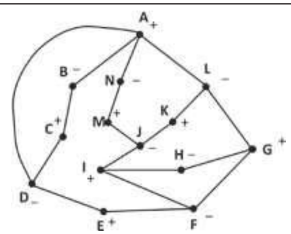


Graph is bipartite. ✓　　✓✓

(b) Explain why the graph drawn in the accompanying diagram is not bipartite. Suggest one way of making this graph bipartite without changing the number of links and vertices.





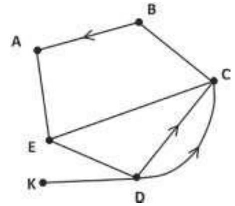


F, G and H form a triangle with two vertices from the same set. ✓
Which means that there are vertices within the same set connected to each other, making the graph non-bipartite. ✓
To make graph bipartite, change link from FH to FI. ✓

Calculator Assumed

3. [11 marks: 2, 2, 3, 2, 1, 1]

The accompanying diagram shows a mixed graph.



(a) Identify and state an Eulerian trail.

K – D → C – E – D → C – B → A – E ✓✓

(b) Identify and state a Hamiltonian path.

`

K – D → C – B → A – E ✓✓

(c) State the adjacency matrix **M** for this mixed graph.

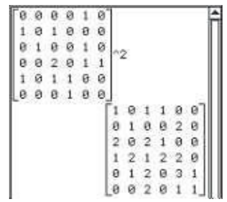
$$\mathbf{M} = \text{From}\ \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ K \end{array}\!\!\begin{array}{c} \text{To} \\ \begin{array}{cccccc} A & B & C & D & E & K \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

✓✓✓

(d) Calculate $\mathbf{M}^2$.

$$\mathbf{M}^2 = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 2 & 0 & 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 2 & 2 & 0 \\ 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 2 & 0 & 1 & 1 \end{pmatrix}$$

✓✓



(e) Use your answer in (d) to determine:

(i) the number of connections between the vertices K and C through another vertex.

No. of connections = 2 ✓

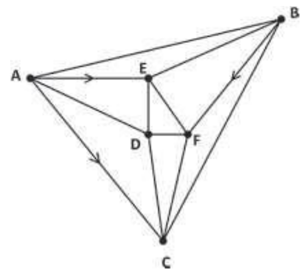
(ii) the most number of walks of length 2 between any two vertices.

Most number of walks of length 2 = 3 ✓

Calculator Assumed

4. [12 marks: 2, 3, 3, 2, 1, 1]

The accompanying diagram shows a mixed graph.



(a) Identify and state a Hamiltonian cycle.

A → E – D – F – C – B – A ✓✓

(b) Identify and state an Eulerian circuit.

A → E – B → F – E – D – A → C – D – F – C – B – A ✓✓✓

(c) State the adjacency matrix **M** for this mixed graph.

$$\mathbf{M} = \text{From}\ \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \text{To} \\ \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$ ✓✓✓

(d) Calculate $\mathbf{M}^2$.

$$\mathbf{M}^2 = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 & 4 \\ 0 & 3 & 2 & 4 & 2 & 2 \\ 2 & 0 & 3 & 1 & 3 & 2 \\ 0 & 3 & 2 & 4 & 2 & 2 \\ 2 & 0 & 3 & 1 & 3 & 2 \\ 1 & 2 & 1 & 2 & 1 & 3 \end{pmatrix}$$ ✓✓

(e) Use your answer in (d) to determine:

(i) the number of ways B can be reached from D by two stage walks

3 ways ✓

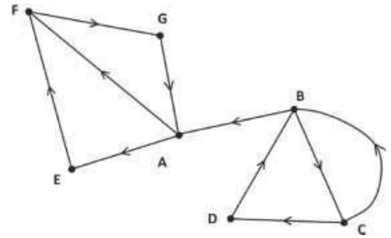
(ii) vertices that *cannot* reach A with a two stage walk

B and D. ✓

Calculator Assumed

5. [12 marks: 2, 1, 2, 2, 2, 3]

The accompanying diagram shows a directed graph.



(a) Determine with reasons if this graph is a simple directed graph.

Graph is not simple as BC is connected with two edges. ✓✓

(b) Identify and state a Hamiltonian path.

C – D – B – A – E – F – G or equivalent. ✓

(c) Identify and state a Eulerian trail.

C – D – B – C – B – A – E – F – G – A – F or equivalent. ✓✓

(d) Show that Euler's Formula works for this directed graph.

V = 7 F = 5 E = 10 ✓
V + F – E = 7 + 5 – 10 = 2 ✓
Hence, Euler's Formula works.

(e) State the adjacency matrix **M** for this directed graph.

$$\mathbf{M} = \text{From}\ \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \\ G \end{array} \begin{array}{c} \text{To} \\ \begin{array}{ccccccc} A & B & C & D & E & F & G \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$ ✓✓

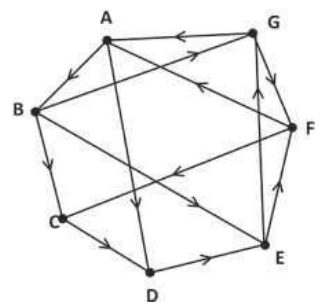
(f) Which vertex/vertices can be reached from E by a three-stage walk? Explain how you obtained your answer.

From $\mathbf{T} = \mathbf{M}^3$: $t_{51} = 1$, $t_{52} = t_{53} = t_{54} = t_{55} = t_{56} = t_{57} = 0$ ✓✓
Hence, only A can be reached from E by a 3-stage walk. ✓

Calculator Assumed

6. [14 marks: 3, 2, 4, 3, 2]

The accompanying diagram shows a directed graph.



(a) Describe all walks from A to C that are of length 4.

A – D – E – F – C	✓
A – B – E – F – C	✓
A – B – G – F – C	✓

(b) Identify and state a Hamiltonian cycle.

A – B – C – D – E – G – F – A ✓✓

(c) An Eulerian trial may be formed if the direction of one of the edges is reversed. Identify this edge and state the Eulerian trial.

Reverse the direction in edge BC. ✓
Eulerian trial:
C-D-E-F-C-B-E-G-A-B-G-F-A-D ✓✓✓

(d) State the adjacency matrix M for this digraph.

$$M = \text{From}\ \begin{array}{c} \\ A\\B\\C\\D\\E\\F\\G \end{array}\!\!\begin{array}{c} \text{To} \\ \begin{array}{ccccccc} A & B & C & D & E & F & G \end{array} \\ \begin{pmatrix} 0&1&0&1&0&0&0\\ 0&0&1&0&1&0&1\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&1\\ 1&0&1&0&0&0&0\\ 1&0&0&0&0&1&0 \end{pmatrix}\end{array} \quad ✓✓✓$$

(e) Which vertex/vertices cannot be reached from vertex C by four-stage walks?

From $\mathbf{M}^4$: B, D, E, G ✓✓

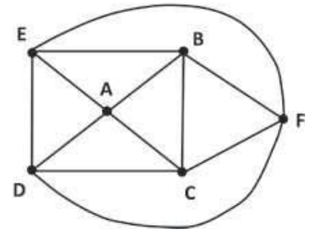
Calculator Assumed

7. [9 marks: 3, 3, 3]

The adjacency matrix of a planar graph is given by

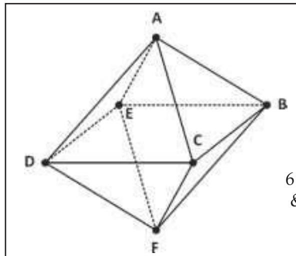
$$M = \text{From}\ \begin{array}{c} \\ A\\B\\C\\D\\E\\F \end{array}\!\!\begin{array}{c} \text{To} \\ \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 0&1&1&1&1&0\\ 1&0&1&0&1&1\\ 1&1&0&1&0&1\\ 1&0&1&0&1&1\\ 1&1&0&1&0&1\\ 0&1&1&1&1&0 \end{pmatrix}\end{array}$$

(a) Draw the planar graph.



6 vertices & 12 edges
& 8 faces ✓
No edges crossing ✓✓

(b) Draw a three dimensional solid that is represented by this planar graph.



6 vertices & 12 edges
& 8 faces ✓✓✓

(c) How many more walks of length four than walks of length three are there between A and F. Explain how you obtained your answer.

From $\mathbf{S} = \mathbf{M}^3$: $s_{16} = 8$ ✓
From $\mathbf{T} = \mathbf{M}^4$: $t_{16} = 48$ ✓
Hence, $48 - 8 = 40$. ✓

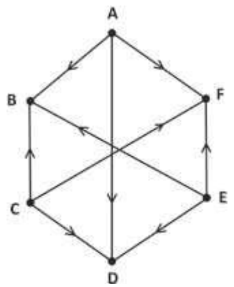
Calculator Assumed

8. [8 marks: 3, 3, 2]

The adjacency matrix of a graph is given by

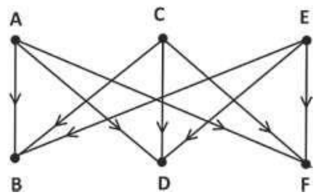
$$M = \text{From}\ \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \text{To} \\ \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

(a) Draw this graph.



6 vertices & 9 edges ✓✓✓

(b) Represent this graph in bipartite form.



Sets {A, C, E } and {B, D, F } ✓
Correct connections and directions ✓✓

(c) Without any calculations, explain why there are no walks of lengths more than one?

Vertices A, C and E have no incoming connections while vertices B, D and F have no outgoing connections. Hence, all walks terminate at B, D or F. ✓✓

Calculator Assumed

9. [9 marks: 2, 2, 2, 3]

Consider the following adjacency matrix $M = \text{From}\ \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \text{To} \\ \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} 0 & 2 & 1 & 1 & 1 \\ 2 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$.

For this adjacency matrix, a loop counts as one edge.

(a) Determine with reasons if M represents a simple graph.

Graph is not a simple graph. ✓
$m_{1,2} = 2 \Rightarrow$ 2 edges between A and B. ✓

(b) Determine with reasons if M represents a directed or an undirected graph.

Graph is not directed. ✓
M is symmetrical about its main diagonal. ✓

(c) Determine with reasons if M represents a connected graph.

Graph is connected. ✓
There are no rows/columns of zeros. ✓

(d) For an undirected graph:
$2 \times$ Number of Edges = Sum of elements in M + Number of loops.
For a directed graph:
Number of Edges = Sum of elements in M
Determine the number of edges and hence the number of faces in the graph represented by this matrix, given that the graph is planar.

Graph is not directed.
Hence, $2 \times$ number of edges $= 22 + 0$. ✓
$\Rightarrow$ No. of edges $E = 11$ ✓
Number of Faces $F = E + 2 - V = 11 + 2 - 5 = 8$ ✓

Calculator Assumed

10. [10 marks: 2, 3, 2, 3]

Consider the following adjacency matrix.

$$M = \begin{array}{c} \\ \text{From} \end{array} \begin{array}{c|cccccccccc|} & \multicolumn{10}{c}{\text{To}} \\ & A & B & C & D & E & F & G & H & J & K \\ A & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ B & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ C & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ D & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ E & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ F & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ G & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ H & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ J & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ K & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}$$

(a) Determine with reasons if the graph it represents is a simple graph.

$m_{5,5} = 1$ indicating the existence of a loop. ✓
Hence, graph is not simple. ✓

(b) Determine with reasons if the graph is directed or undirected.

$m_{1,7} = 1$ and $m_{7,1} = 0$. ✓
Hence, M is not symmetrical about its main diagonal. ✓
Hence, graph is directed. ✓

(c) Identify all the odd vertices (in-degree and out-degree) in this graph.

A, E, G, H, and J ✓✓

(d) For an undirected graph:
$2 \times$ Number of Edges = Sum of elements in M + Number of loops.
For a directed graph:
Number of Edges = Sum of elements in M
Determine the number of edges and hence the number of faces in the graph represented by this matrix, given that the graph is connected planar graph.

Graph is directed.
Hence, number of edges = 34. ✓
Number of Faces = E + 2 – V = 34 + 2 – 10 = 26 ✓✓

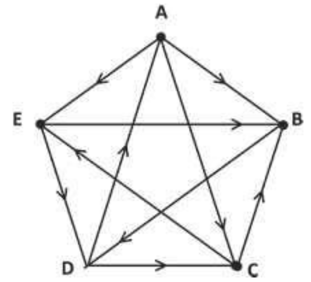
Calculator Assumed

11. [17 marks: 4, 3, 3, 4, 3]

Teams A, B, C, D and E are in Group 1 of a soccer competition where they play each other exactly once. The results of the games are:

- A defeated B, C and E
- C defeated B and E
- E defeated B and D.
- B defeated D
- D defeated A and C

(a) Draw a digraph that represents the results of these matches where a directed edge points from the winning team to the losing team.



5 vertices, 10 edges ✓
Directions correct ✓✓✓

(b) The digraph in (b) can be expressed as an adjacency matrix **M**. The element "1" is to indicate a win and the element "0" to indicate a loss. Complete the matrix M

$$\begin{array}{c|ccccc|} & A & B & C & D & E \\ A & 0 & 1 & 1 & 0 & \boxed{1} \\ B & 0 & 0 & \boxed{0} & \boxed{1} & \boxed{0} \\ C & 0 & \boxed{1} & 0 & \boxed{0} & \boxed{1} \\ D & \boxed{1} & \boxed{0} & \boxed{1} & 0 & \boxed{0} \\ E & 0 & \boxed{1} & \boxed{0} & \boxed{1} & 0 \end{array}$$

✓✓✓

Calculator Assumed

11. (c) The elements in $\mathbf{M}^2$ represents the number of "two-stage wins" between the teams. An incomplete $\mathbf{M}^2$ is shown below. The element in row 1 column 2 indicates that A has 2 "two-stage wins" against B: A beat E who beat B and A beat C who beat B. Complete the matrix $\mathbf{M}^2$ below.

$$\mathbf{M}^2 = \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} 0 & 2 & 0 & 2 & \boxed{1} \\ 1 & 0 & \boxed{1} & \boxed{0} & \boxed{0} \\ 0 & \boxed{1} & 0 & \boxed{2} & \boxed{0} \\ \boxed{0} & \boxed{2} & \boxed{1} & 0 & \boxed{2} \\ 1 & \boxed{0} & \boxed{1} & \boxed{1} & 0 \end{pmatrix} \end{array}$$

✓✓✓

(d) Calculate $\mathbf{M} + \mathbf{M}^2$. Then multiply your answer with the column matrix $\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}$.

The resulting matrix gives the total number of direct wins and "two-stage wins" for each team. Use this answer to rank the teams.

$$(\mathbf{M}+\mathbf{M}^2)\times\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix} = \begin{pmatrix} 0 & 3 & 1 & 2 & 2 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 2 & 1 \\ 1 & 2 & 2 & 0 & 2 \\ 1 & 1 & 1 & 2 & 0 \end{pmatrix} \times \begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}8\\3\\5\\7\\5\end{pmatrix}$$

✓✓ ✓

Hence: A, D, C=E, B ✓

(e) Calculate $\mathbf{M} + \mathbf{M}^2 + \mathbf{M}^3$. Then multiply your answer with $\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}$.

Use this result to rank the teams.

$$(\mathbf{M}+\mathbf{M}^2+\mathbf{M}^3\times\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix} = \begin{pmatrix} 2 & 4 & 3 & 5 & 2 \\ 1 & 2 & 2 & 1 & 2 \\ 2 & 2 & 2 & 3 & 1 \\ 1 & 5 & 2 & 4 & 3 \\ 2 & 3 & 3 & 2 & 2 \end{pmatrix} \times \begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}16\\8\\10\\15\\12\end{pmatrix}$$

✓✓

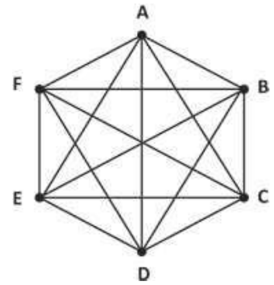
Hence: A, D, E, C, B ✓

Calculator Assumed

12. [8 marks: 3, 3, 2]

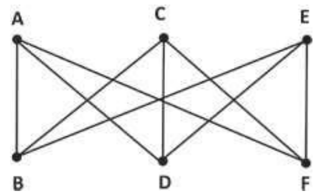
A, B, C, D, E and F meet at a party and each shakes the other's hands.

(a) Draw a graph that represents the "handshakes" between these six people. [An edge connecting two vertices represents two persons sharing a handshake.]



6 vertices ✓
15 edges ✓
Correct connections ✓

(b) Due to prior disagreements, A, C and E refuse to shake hands with each other. Similarly B, D and F refuse to shake hands with each other. Draw a bipartite graph to represent the handshakes between these six persons.



Sets {A, C, E} and {B, D, E} ✓ ✓
9 connections ✓

(c) Write an adjacency matrix for the graph you drew in (b).

$$\text{From}\quad \begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \text{To} \\ \begin{array}{cccccc} A & B & C & D & E & F \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

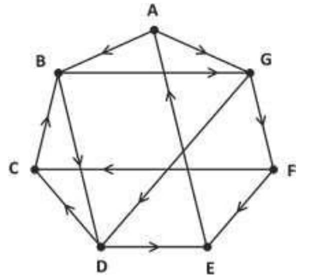
✓✓

Calculator Assumed

13. [13 marks: 3, 4, 1, 3, 2]

Bus route P runs from A to G to F to E and back to A.
Bus route Q runs from A to B to G to D to E and back to A.
Bus route R runs from B to G to F to C to B.
Bus route S runs from B to D to C and back to B.

(a) Construct a digraph that conveys these routes.



6 vertices, 12 edges ✓
Directions correct ✓✓

(b) Identify and state a Hamiltonian cycle. Describe the sequence of bus routes required to achieve the Hamiltonian cycle.

A – G – F – C – B – D – E – A ✓✓

Route P: A to G
Route R: G to F to C to B
Route S: B to D
Route Q: D to E to A. ✓✓

(c) The digraph is not traversable as it has more than two odd vertices. Identify all the odd vertices in this diagraph.

A, E, F and C ✓

(d) Which edge would you remove to make this digraph traversable? State the traversable path.

Remove edge CF: ✓
Eulerian circuit: A – G – F – E – A – B – G – D – C – B – D – E ✓✓

(e) How would you reorganise the bus routes to achieve your answer in (d).

Bus routes P, Q and S remain unchanged. ✓
Remove route R, thereby removing edge CF ✓

Route P: A to G to F to E to A
Route Q: A to B to G to D
Route S: D to C to B to D
Route Q: D to E ✓✓

15 Shortest Paths

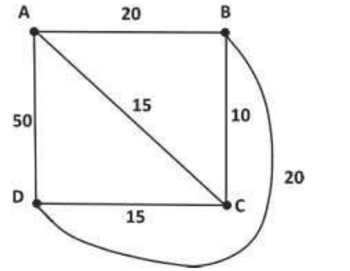
Calculator Assumed

1. [9 marks: 3, 3, 3]

The table below shows the distances (km) between the stated towns.

	A	B	C	D
A	–	20	15	50
B	20	–	10	20
C	15	10	–	15
D	50	20	15	–

(a) Represent the data shown in the table above as a clearly labelled weighted planar graph.



4 nodes ✓
6 links ✓
Correct weights ✓

(b) Describe the shortest Hamiltonian cycle in this graph and state its length.

B – D – C – A – B ✓✓

Length = 20 + 15 + 15 + 20 = 70 km. ✓

(c) Which link would you remove so that the remaining graph contains the shortest possible semi-Eulerian trail? Describe this trail and state its length.

Remove AD. ✓
Trail: B – D – C – A – B – C ✓
Length: 80 km ✓

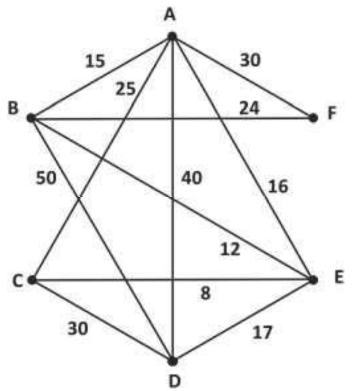
Calculator Assumed

2. [8 marks: 4, 1, 3]

The table below shows the distances (km) between the stated towns.

	A	B	C	D	E	F
A	–	15	25	40	16	30
B	15	–	–	50	12	24
C	25	–	–	30	8	–
D	40	50	30	–	17	–
E	16	12	8	17	–	–
F	30	24	–	–	–	–

(a) Draw a clearly labelled weighted graph to represent the data shown in the above table.



6 nodes ✓
11 links ✓✓
weights ✓

(b) Which town is the most "isolated" town? Why?

Town F as it is only linked by two roads. ✓

(c) Identify the shortest Hamiltonian path and state its length.

Hamiltonian path: F – B – A – E – C – D ✓✓

Length: 89 km. ✓

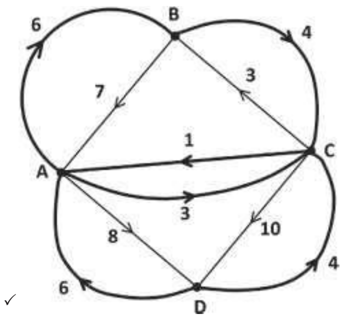
Calculator Assumed

3. [6 marks: 3, 1, 2]

The following table show the times taken to travel between the buildings A, B, C and D in *minutes*. Note the times taken are not symmetrical due to road conditions and one-way streets. For example, the time taken to travel from A to B is 6 minutes but the time taken to travel from B to A is 7 minutes.

		TO			
		A	B	C	D
FROM	A	–	6	3	8
	B	7	–	4	–
	C	1	3	–	10
	D	6	–	4	–

(a) Complete the weighted directed graph below to show the travelling times between these buildings. Label your diagram carefully.



4 vertices ✓
10 edges ✓
Correct directions ✓

(b) State the mathematical term used to describe a walk that starts and finishes at the same building visiting each building exactly once except for the start/finish building.

Hamiltonian cycle. ✓

(c) State clearly the quickest walk starting from A visiting all the three other buildings and finishing back at A. State the time taken.

Quickest walk is:
$A \rightarrow D \rightarrow C \rightarrow B \rightarrow A$ ✓
Time taken = 22 minutes. ✓

Calculator Assumed

4. [8 marks: 2, 4, 2]

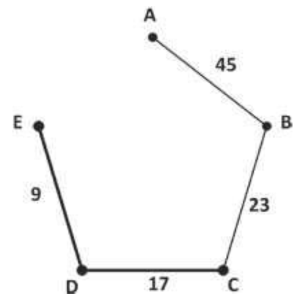
Towns A, B, C, D and E lie along the same stretch of a long highway in this order. The following table shows the *cumulative* distances in km between these towns. For example the shaded cell indicates that the distance between A and D passing through B and C in turn is 85 km.

A				
45	B			
68	23	C		
85	40	17	D	
94	49	26	*k*	E

(a) Calculate the value of *k*.

$k = 94 - (45 + 23 + 17) = 9$ km. ✓✓
or $k = 94 - 85 = 9$

(b) Complete the weighted graph below, which shows the road connections between these town.



Connections
CD and DE drawn. ✓
Distances correct ✓✓
No connections between AC, AD etc. ✓

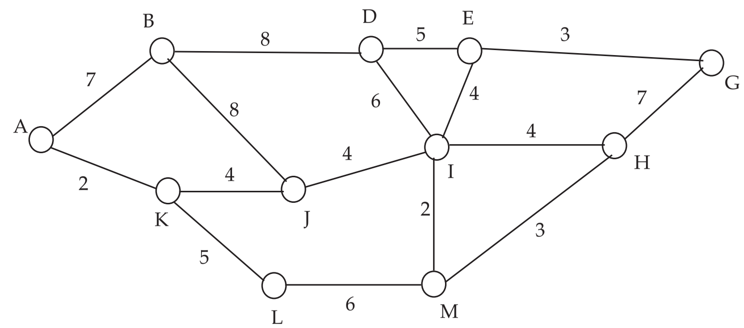
(c) If it was possible to join E to C by one new road, providing it was shorter than the current distance between E and C, what should its length be?

$0 < \text{length} < 17 + 9$
That is $0 < \text{length} < 26$ km ✓✓

Calculator Assumed

5. [6 marks: 3, 3]

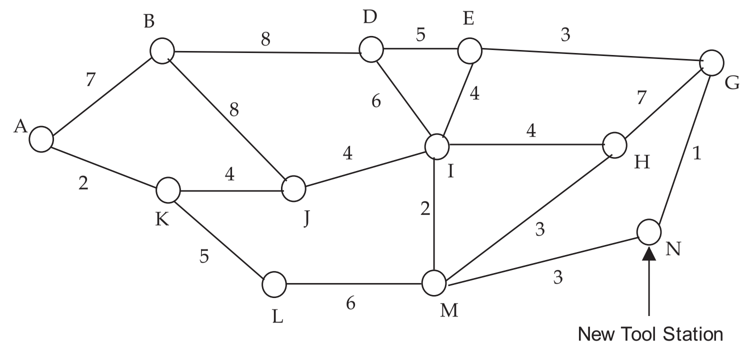
The weighted graph below represents the passage-ways and their lengths (metres) between several work stations. No other passage-ways connect these work stations.



(a) Find the shortest walk between A and G. Give the length of this walk.

Shortest walk = A → K → J → I → E → G ✓✓
Shortest distance = 17 m. ✓

(b) A new tool station is located as shown. Discuss the impact of this new tool station on the minimum distance between A and G.

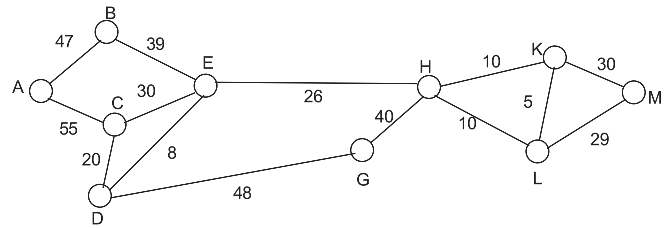


New Shortest walk = A → K→ J → I → M → N → G ✓✓
Shortest distance = 16 m. ✓

Calculator Assumed

6. [10 marks: 3, 2, 2, 3]

The weighted graph below shows a road network between several towns. Distances are shown alongside the edges and are in km.



(a) Find the shortest path from A to M. Give this distance.

Shortest path is ACDEHLM.	✓✓
Distance = 148 km	✓

(b) Find the shortest path from A to M via E <u>and</u> K. Give this distance.

Shortest path is ACDEHKM.	✓
Distance = 149 km	✓

(c) Find the shortest path from A to M via E <u>or</u> K. Give this distance.

Shortest path is ACDEHLM.	✓
Distance = 148 km	✓

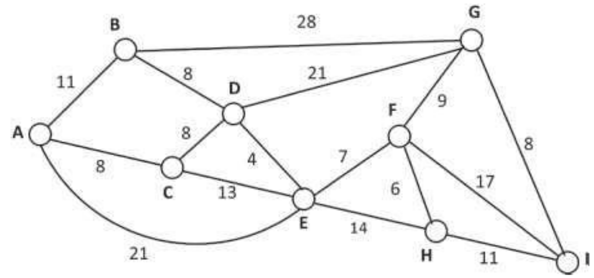
(d) Find the shortest path starting from A and finishing at A visiting each town at least once. Give this distance.

Shortest path is ACDEHLM → KHGDEBA	✓✓
Distance = 148 + 222 = 370km	✓

Calculator Assumed

7. [11 marks: 4, 3, 4]

In the graph given below, the weights located on the edges indicate the road distances in km between the towns listed.



(a) Determine all possible shortest routes between A and I and state the distance.

Shortest distance = 44 km	✓
ACDEFGI	✓
ACDEFI	✓
ACDEFHI	✓

(b) Town F is flooded and all roads in and out of the town are closed to traffic. Determine the impact this has on the shortest route between A and I.

New shortest distance = 45 km		✓
New Paths:	ADGI	✓
	ADEHI	✓

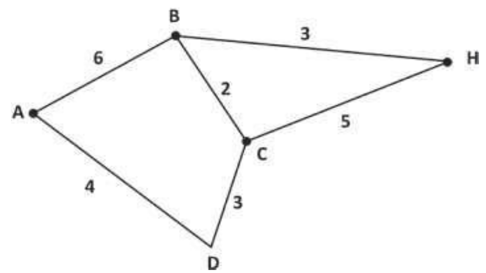
(c) A new road with distance d km is created between D and H. What should the distance of this road be if it is to have any impact on the shortest route between A and I? Justify your answer.

Shortest distance from A to I = 44 km	✓
Shortest distance from A to D = 16 km	
Distance H to I = 11 km	
Total distance (A to D + H to I) = 16 + 11 = 27	✓
Hence, DH must be less than 44 – 27 = 17 km.	✓
Hence, $0 < d < 17$ km	✓

Calculator Assumed

8. [8 marks: 4, 4]

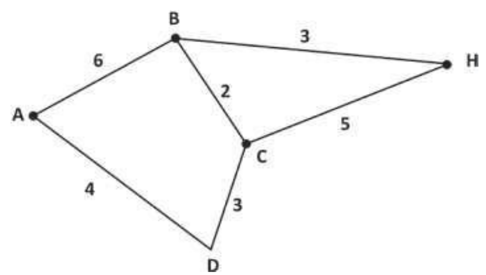
The diagram below shows the gravel tracks and distances (in km) between the paddock gates A, B, C, D and H.



(a) Each evening Kaylah needs to visit every *gate* (at least once) to make sure that each gate is shut for the night.
Describe the *shortest* route Kaylah should take if she starts from and finishes at gate H. Give the distance of this route.

> Shortest route is HBCDABH with a distance of 21 km.
> ✓ ✓ ✓ ✓

(b) Each morning Jasmin needs to travel along each gravel track in this network (at least once) to check that the fences are in order.
Describe the *shortest* route Jasmin should take if she starts from and finishes at H. Give the distance of this route.

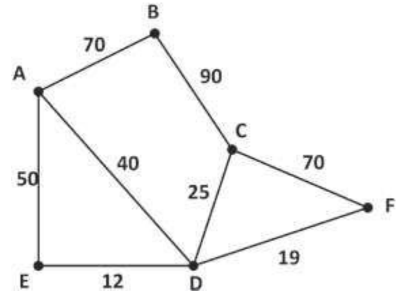


> Shortest route is HCBCDABH with a distance of 25 km.
> ✓ ✓ ✓ ✓
> or HBCBADCH with a distance of 25 km.
> ✓ ✓ ✓ ✓
> or HCBADCBH with a distance of 25 km.
> ✓ ✓ ✓ ✓

Calculator Assumed

9. [10 marks: 2, 4, 4]

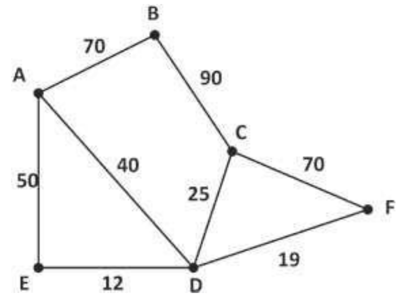
The diagram below shows the links and distances (in km) between the towns A, B, C, D, E and F.



(a) State the shortest path from A to F and give this shortest distance.

> Shortest path is A – D – F with a distance of 59 km.
> ✓ ✓

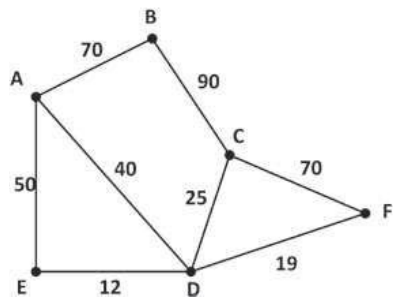
(b) John needs to visit every *town* in this network (at least once). Describe the *shortest* route John should take if he starts from and finishes at A.
Give the distance of this route.



> Shortest route is:
> A → E → D → F → D → C → B → A ✓✓✓
> Distance = 285 km. ✓

Calculator Assumed

9. (c) Kim needs to travel along every edge in this network (at least once). Describe the *shortest* route Kim should take if he starts from and finishes at A. Give the distance of this route.



Shortest route is:
A → B → C → F → D → C → D → E → A → D → A ✓✓✓
Distance = 441 km. ✓

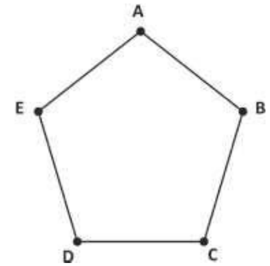
(Alternative routes with distance 441 km exist.)

16 Trees

Calculator Free

1. [7 marks: 2, 3, 2]

The accompanying diagram shows a graph formed by the edges of a pentagon.



(a) How many sub-graphs that are trees of length two are there within this graph? Name these trees.

5 trees. ✓
EAB, ABC, BCD, CDE, DEA ✓

(b) How many sub-graphs that are trees of length of more than two are there within this graph? Name these trees.

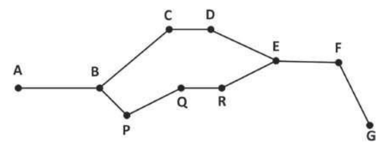
Length 3: ABCD, BCDE, CDEA, DEAB, EABC ✓
Length 4: ABCDE, BCDEA, CDEAB, DEABC, EABCD ✓
Total no. of trees = 10 ✓

(c) How many sub-graphs that are trees of length at least 2 are there in a graph formed by the edges of a decagon (10-sided polygon)?

Total no. of trees = trees of length 2 + trees of length 3 + …
+ trees of length 9
= 10 × 8 = 80. ✓✓

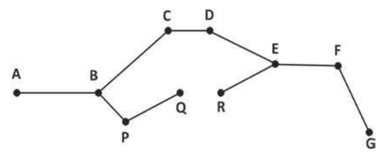
2. [4 marks]

Explain why the graph shown in the accompanying diagram is not a tree. Explain how you would turn this graph into a tree. Draw the graph of the tree in the space below.



Graph is not a tree because of the presence of a loop BCDERQPB. ✓

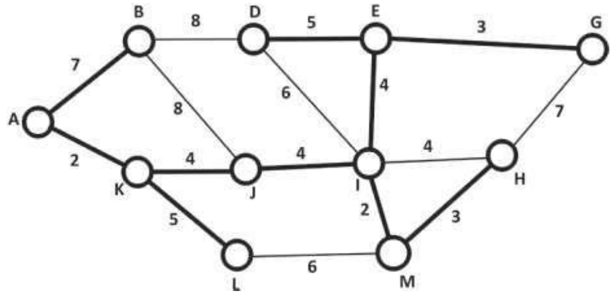
Sever the connection between any of the adjacent vertices in the loop to turn the graph into a tree. ✓
✓✓



Calculator Assumed

3. [6 marks: 3, 3]

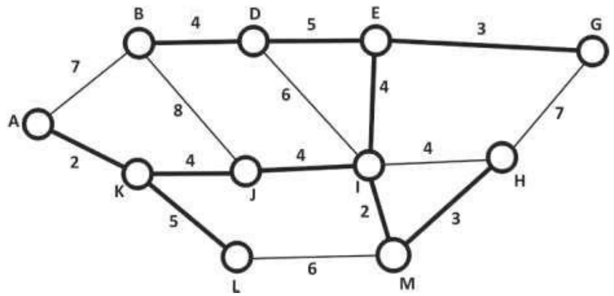
The network below shows the distances (metres) between the various tool stations in a workshop.



(a) Draw in the minimum spanning tree for this network in the diagram above. State the minimum distance.

Minimum spanning tree (presence of loop gets 0)	✓✓
Minimum distance = 39 metres.	✓

(b) A new walk-way of length 4 metres is built between B and D. Determine the effect this new walk-way will have on the minimum spanning tree and the minimum spanning distance.

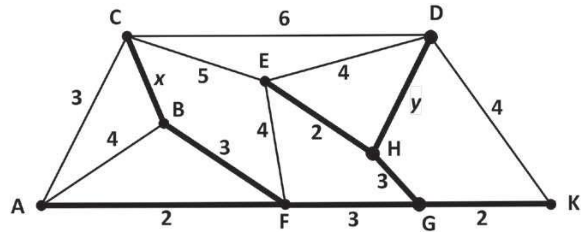


Minimum spanning distance = 39 – 7 + 4 = 36 metres.	✓
Minimum spanning tree: Connection from A to B replaced with connection from D to B.	✓✓

Calculator Assumed

4. [9 marks: 4, 2, 3]

The diagram below shows the cost (in thousands of dollars) of connecting several laboratories in a research facility with new high speed data transmission cables.



(a) For $x = y = 2$, indicate on the given diagram, the cheapest way of connecting all the laboratories in the research facility. <u>State this cost.</u>

Minimum spanning tree.	✓✓
Minimum cost = $19 000	✓✓

(b) For x= 2, find the range of values for y if there is to be only one possible way to connect these laboratories with minimum cost.

$0 < y < 4$	✓✓

(c) For $y = 2$, find the value(s) of x if there is to be more than one possible way to connect these laboratories with minimum cost. State the various possibilities.

$x = 3$ ✓

```
    C
    |
    B               C   B
    |               |   |
A – F – G – K       A – F – G – K
        |                   |
    E – H – D   or      E – H – D   ✓✓
```

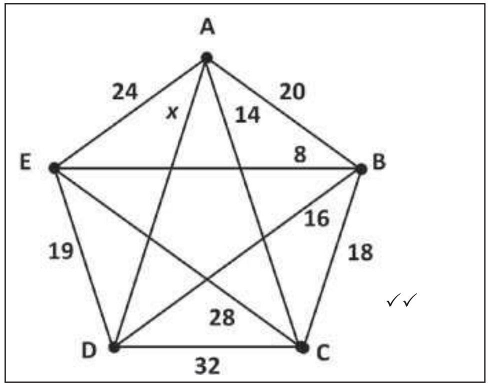
Calculator Assumed

5. [9 marks: 2, 3, 4]

A Peer-to-Peer computer network is planned for Carl's home. The table below shows the cost (in $) of installing the data cables between 5 terminals in Carl's home.

	A	B	C	D	E
A	–	20	14	x	24
B	20	–	18	16	8
C	14	18	–	32	28
D	x	16	32	–	19
E	24	8	28	19	–

(a) Display this information as a weighted graph that displays the terminals and the cost of each connection.



✓✓

(b) For $x = 40$, identify the minimal spanning tree for this graph and state the minimum cost.

Minimal spanning tree:
D
|
E – B – C – A ✓✓
Minimum cost: $56 ✓

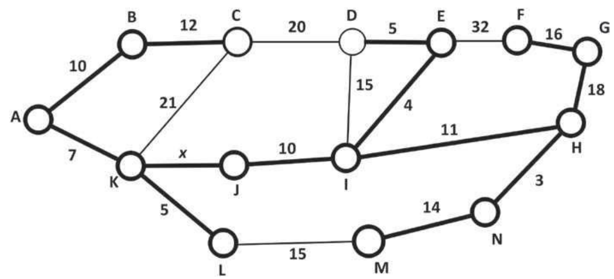
(c) Find the largest possible value of x if AD is to be part of the minimal spanning tree. State the minimum spanning tree and the minimum cost

$0 < x < 18$ ✓✓
Minimal spanning tree:
E – B – D – A – C ✓
Minimum cost = $38 + x$ ✓

Calculator Assumed

6. [12 marks: 4, 4, 4,]

The network below shows distances (in km) between the various towns in the north east of a state.

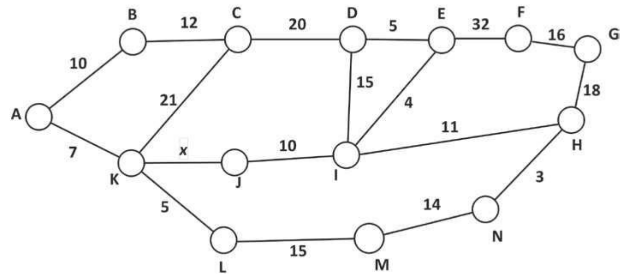


(a) Given that $x = 10$, draw in the minimum spanning tree for this network in the diagram above. Explain what the minimum spanning tree means in the context of this network.

Minimum Spanning Tree ✓✓

A tree which connects *all vertices* where the total length of the connecting edges a *minimum*. ✓✓

(b) For $x = 10$, find the shortest route between A and G *passing through* I. Give the length of this route. Explain what this shortest route means in the context of this question.

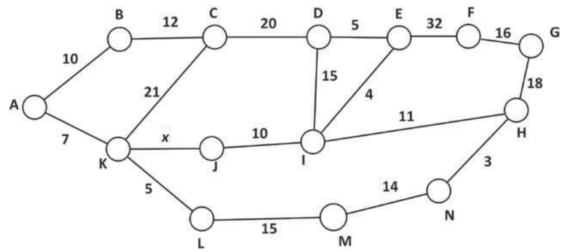


Shortest route: AKJIHG ✓✓
Length of route: 56 km ✓

The shortest route is a tree that connects A to G with the total length of the connections a minimum ✓

Calculator Assumed

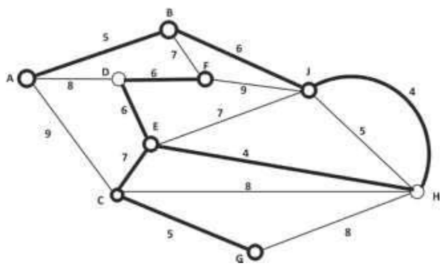
6. (c) If the shortest path between A and G is AKJIHG, find the value(s) of x.



Length of AKJIHG = $46 + x$ ✓
Next shortest route is AKLMNHG with length 62. ✓
Hence: $46 + x < 62$
$0 < x < 16$ ✓✓

7. [6 marks: 2, 4]

The accompanying diagram shows the minimum spanning tree drawn (drawn in bold) onto a network showing the distances in hundreds of metres between several buildings. Prim's algorithm was used to determine the minimum spanning tree starting at building E.



(a) Explain with reasons, which building is the next to be connected to E.

Building H. ✓
It is the closest to E. ✓

(b) Determine with reasons, which building is the last in the sequence of buildings to be connected?

Sequence of sub-trees.
E → H → J → B → A ✓
Followed by E → D → F ✓
Finally E → C → G. ✓
Hence, last building in sequence is G. ✓

Calculator Assumed

8. [7 marks: 2, 3, 2]

A high-speed fibre optic network links 6 computer terminals within an office. The distances between the 6 terminals (in metres), A, B, C, D, E, and F are shown in the table provided below.

	A	B	C	D	E	F
A	–	13	14	9	6	11
B	13	–	9	7	16	13
C	14	9	–	10	14	9
D	9	7	10	–	15	8
E	6	16	14	15	–	8
F	11	13	9	8	8	–

(a) Find the shortest possible amount of cable which must be used to link these terminals.

Using Prim's algorithm, the minimum spanning distance = 38 m. ✓✓

(b) An electrician requires a diagram of the shortest network which is planned. Draw such a diagram.

Using Prim's Algorithm, minimum spanning tree is:

$A \overset{6}{\text{——}} E \overset{8}{\text{——}} F \overset{8}{\text{——}} D \overset{7}{\text{——}} B \overset{9}{\text{——}} C$

–1 per error ✓✓✓

OR

$A \overset{6}{\text{——}} E \overset{8}{\text{——}} F \overset{8}{\text{——}} D \overset{7}{\text{——}} B$, with $F \overset{9}{\text{——}} C$

(c) If it were necessary to increase the distance from D to B by 3 metres in order to run the cable around Alana's new reception desk, what would be the shortest amount of cable needed to span this network?

New minimum spanning tree is:

$A \overset{6}{\text{——}} E \overset{8}{\text{——}} F \overset{8}{\text{——}} D$, with $F \overset{9}{\text{——}} C \overset{9}{\text{——}} B$

–1 per error ✓
Hence, minimum spanning distance = 40 m. ✓

Calculator Assumed

9. [9 marks: 2, 4, 3]

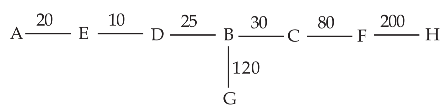
The table below shows the distances (km) between several cities.

	A	B	C	D	E	F	G	H
A	–	110	150	70	20	300	220	510
B	110	–	30	25	80	260	120	450
C	150	30	–	40	60	80	200	300
D	70	25	40	–	10	250	140	210
E	20	80	60	10	–	310	150	420
F	300	260	80	250	310	–	400	200
G	220	120	200	140	150	400	–	300
H	510	450	300	210	420	200	300	–

(a) Using Prim's Algorithm to determine the minimum spanning tree, starting from town A, what town should be connected to A? Why?

Next town should be E. ✓
It is the town closest to A. ✓

(b) Use Prim's Algorithm to determine the minimum spanning tree for this network. State the minimum distance.

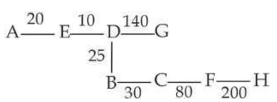


–1 per error ✓✓✓

Minimum spanning distance = 485 km. ✓

(c) State the minimum spanning tree and the minimum distance if the edge DG must be within this tree.

Minimum spanning tree is now:



–1 per error ✓✓

Minimum spanning distance = 505 km. ✓

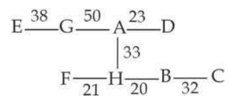
Calculator Assumed

10. [9 marks: 4, 1, 4]

The table below shows the distances (km) between several cities.

	A	B	C	D	E	F	G	H
A		75	33	23	92	96	50	33
B	75		32	99	56	30	61	20
C	33	32		55	56	35	98	93
D	23	99	55		69	41	68	57
E	92	56	56	69		92	38	98
F	96	30	35	41	92		71	21
G	50	61	98	68	38	71		67
H	33	20	93	57	98	21	67	

(a) Use Prim's algorithm to determine the minimum spanning tree and state its length.



or equivalent

–1 per error ✓✓✓

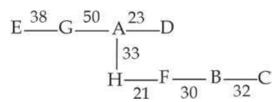
Minimum spanning distance = 217km. ✓

(b) Using Prim's algorithm, starting at vertex H, what is the last town to be connected to the tree.

Town E ✓

(c) State the minimum spanning tree and the minimum distance if the edge BH cannot be within this tree.

Using Prim's Algorithm, minimum spanning tree is:



–1 per error ✓✓✓

Minimum spanning distance = 227km. ✓

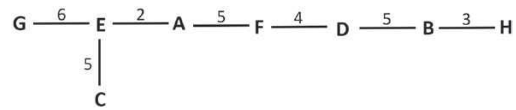
Calculator Assumed

11. [9 marks: 4, 3, 2]

The distances (in kilometres) between the various parks A, B, C, D, E, and F are shown in the table provided below.

	A	B	C	D	E	F	G	H
A	–	8	6	14	2	5	7	6
B	8	–	11	5	9	18	7	3
C	6	11	–	7	5	12	10	8
D	14	5	7	–	6	4	15	12
E	2	9	5	6	–	7	6	9
F	5	18	12	4	7	–	8	6
G	7	7	10	15	6	8	–	11
H	6	3	8	12	9	6	11	–

(a) State the minimum spanning tree and state its length.



–1 per error ✓✓✓

Minimum Length = 30 km ✓

(b) If town F is removed from the network, state with reasons, the length of the new minimum spanning tree.

In the original minimum spanning tree, if town F is removed, then the network splits into two disconnected sub-trees. ✓

The shortest links reconnecting the two sub-trees are DE and HA both of lengths 6 km. ✓
Hence, new minimum length = 30 – 9 + 6 = 27 ✓

(c) Referring back to the original network. A new park is built at N. The distances (km) to N from A, B, C, D, E, F, G and H are respectively 14, 8, 7, 16, 10, 9, 11 and 9.
State with reasons the length of the new minimum spanning tree.

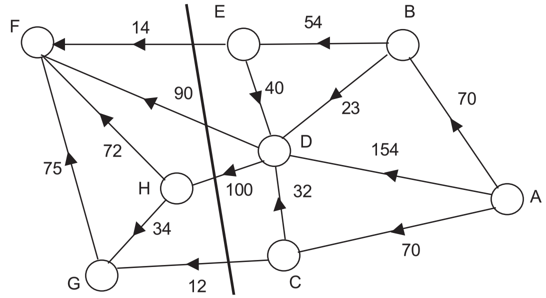
N is closest to C. ✓
Hence, add the distance N to C to the minimum length from (a).
New minimum length = 30 + 7 = 37 km. ✓

17 Flow Networks

Calculator Assumed

1. [10 marks: 1, 4, 2, 3]

The following diagram shows a system of roads in a busy commercial business district with the corresponding capacities (in vehicles per 5 minutes).



Cut for Maximum Flow

(a) State the source and sink.

Source = A Sink = F ✓

(b) Calculate the maximum flow for this network. Show sufficient working for your answer to be checked.

Path	Flow
ABEF	14
ADF	90
ACGF	12
ACDHF	32
ADHF	40
ADHGF	24
ABDHGF	4
Max. Flow	216

–1 per error ✓✓✓

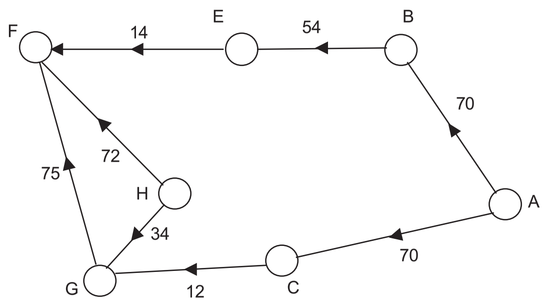
Hence, maximum flow = 216 vehicles per 5 minutes ✓

(c) In the diagram above, draw the cut that corresponds to the maximum flow.

Cut correctly drawn. ✓✓

Calculator Assumed

1. (d) The traffic network is heavily dependent on node D. If node D were to be made non-functional, then there would be severe traffic disruptions. Assume that there was a severe storm and node D was destroyed by the storm. Assume that all other nodes sustained minimal damage. Suggest a new link that could be created linking two other existing nodes (other than the source and the sink) that would best relieve the traffic flow. What should the maximum capacity of this link be?

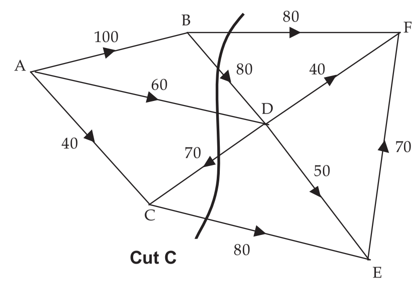


The maximum flow is:
ABEF 14
ACGF 12
Total 26
This leaves a slack (unused capacity) of:
56 along AB
58 along AC ✓
Hence to increase maximise flow, create a Link from C to H ✓
with maximum capacity = 58 vehicles per 5 minutes ✓

Calculator Assumed

2. [9 marks: 1, 1, 3, 2, 2]

The map below shows 6 buildings A, B, C, D. E and F which are connected by one-way streets. The arrows show the direction of the flow of traffic. The number on each link refers to the capacity of the link (in number of vehicles per minute).



(a) Determine the "source" and the "sink" for this network.

Source = A Sink = F ✓

(b) Determine the capacity of the cut C drawn in the diagram above.

Capacity of cut = 80 + 80 + 60 + 80 = 300 ✓

(c) Determine the maximum flow for this traffic network. Justify your answer.

Path	Flow
ABF	80
ABDF	20
ADF	20
ADEF	40
ACEF	30
Max. Flow	190

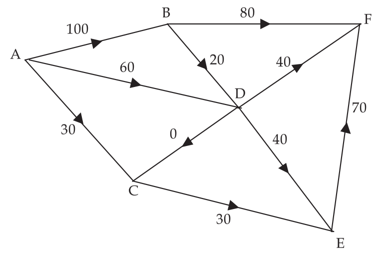
–1 per error ✓✓

Hence, maximum flow = 190 vehicles per minute. ✓

Note:
Other flows are possible but the maximum remains at 190.

Calculator Assumed

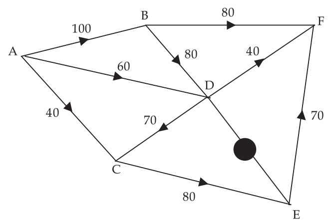
2. (d) Indicate the traffic flow that corresponds to the maximum flow in the diagram given below.



–1 per error ✓✓

Note:
Other flows are possible, commensurate with the flow paths listed in part (b).

(e) An accident occurs on the road between D and E and is temporarily closed to traffic. By how much would the maximum traffic flow change?



Path	Flow
ABF	80
ABDF	20
ADF	20
ADCEF	40
ACEF	30
Max. Flow	190

✓

Hence, maximum flow remains unchanged at 190 vehicles per minute. ✓

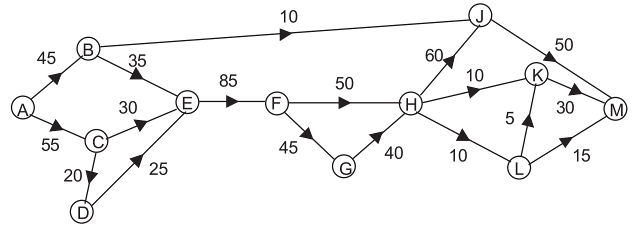
Or

The flow of 40 from D to E could be diverted through C and E.
Hence, maximum flow remains unchanged.

Calculator Assumed

3. [12 marks: 1, 2, 3, 3, 3]

The diagram below shows a communications network. The number accompanying each link represents the capacity of the link in Gigabits per second.



(a) Explain clearly why E is not the sink for this network.

There is an outgoing link EF. ✓

(b) Determine with reasons the maximum possible capacity for the path ABEFHJM.

Maximum capacity is 35 as BE has a capacity of only 35. ✓✓

(c) Find the maximum flow between A and M.
To obtain full marks you need to state the flows for the relevant paths.

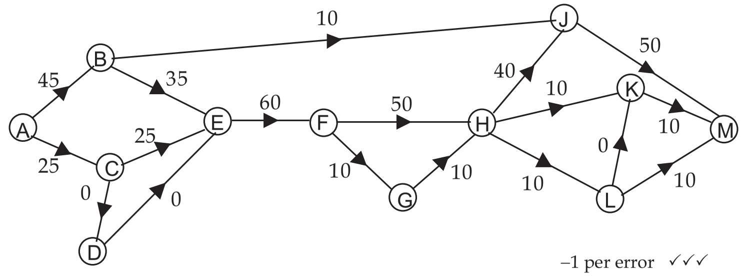
Path	Flow
ABJM	10
ABEFHJM	35
ACEFHJM	5
ACEFHKM	10
ACEFGHLM	10
Max. Flow	70

–1 per error ✓✓

Hence, maximum flow is 70 Gigabits per second. ✓

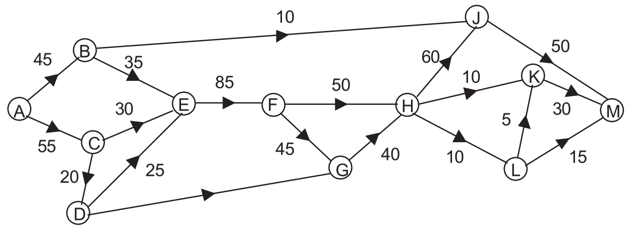
Calculator Assumed

3. (d) In the diagram below, indicate clearly how the maximum flow between A and M as in question 3(c) can be achieved.



–1 per error ✓✓✓

(e) A new direct link between D and G is developed.
Describe clearly how this will affect the maximum flow between A and M.

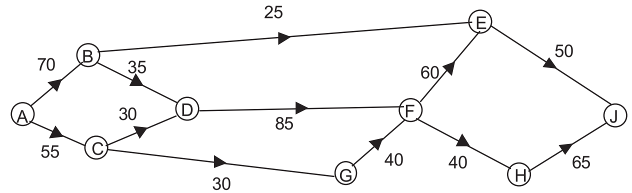


No effect as the "full capacity links" are HK, HL and JM.
✓ ✓ ✓

Calculator Assumed

4. [12 marks: 1, 3, 3, 2, 1, 2]

The diagram below shows the maximum capacity (vehicles per hour) of a road network.



(a) Determine the maximum number of vehicles that the road network is capable of conveying (per hour) from A to D.

Maximum capacity = 30 + 35 = 65 ✓

(b) Will the maximum number of vehicles coming into D from A be able to travel from D to F without being held-up? Explain.

Will not be held up . ✓
Capacity of DF is 85 which is greater than the maximum capacity from A to D (which is 65). ✓✓

(c) Find the maximum flow of this network.
State clearly the flow in each possible path.

Path	Flow
ABEJ	25
ABDFEJ	25
ABDFHJ	10
ACDFHJ	30
Max. Flow	90

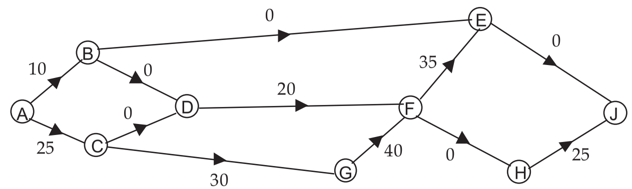
–1 per error ✓✓

Hence, maximum flow is 90 vehicles per hour. ✓

Note:
Other flows are possible but maximum remains at 90.

Calculator Assumed

4. (d) Indicate in the diagram below, the unused capacity when maximum flow is achieved.



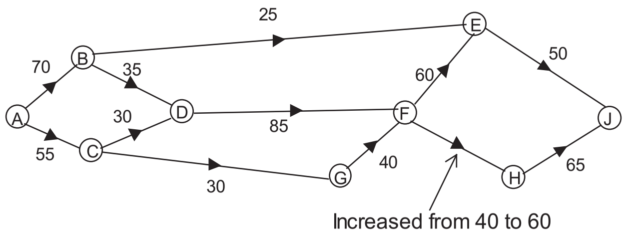
−1 per error ✓✓
Depends on flow in part (c).

(e) Which link(s) has the most waste in terms of carrying capacity?

From the flow diagram above, the unused links GF (40) and FE (35),. ✓

Note:
Answer is dependent on flow diagram in (d).

(f) State the increase in the maximum flow from A to J if the flow capacity between F and H were increased from 40 to 60. Justify your answer.



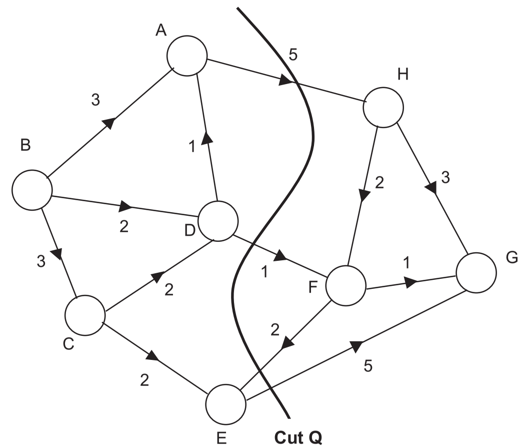
There would be an additional path ACGFHJ that could carry an extra 20 vehicles per hour. ✓

Hence, the maximum flow would be increased by 20 vehicles per hour. ✓

Calculator Assumed

5. [8 marks: 1, 1, 3, 3]

The diagram below shows the traffic system of the Central Business District of a City. The nodes represent signal lights at the corresponding junctions. The numbers on the links represent the maximum number of cars (× 10) capable of passing though each junction each minute.



(a) Identify the source and the sink.

Source = B Sink = G ✓

(b) State the capacity of the cut Q drawn in the diagram above.

Capacity of cut = 5 + 1 + 5 = 11 ✓

(c) What is the maximum flow of traffic through this system? Show clearly how you arrived at your answer.

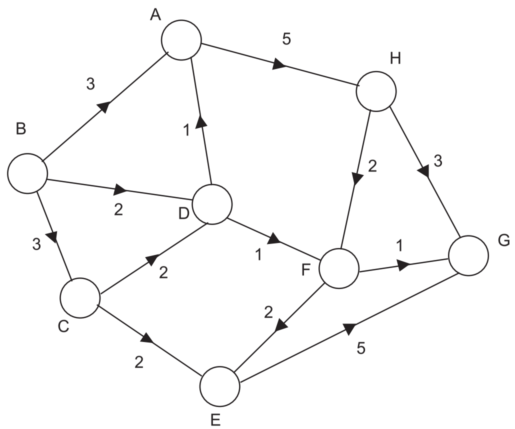
Path	Flow
BAHG	3
BDFG	1
BDAHFEG	1
BCEG	2
Max. Flow	7

−1 per error ✓✓

Hence, maximum flow is 7 × 10 = 70 vehicles per minute. ✓
[Other flows possible with the same maximum flow.]

Calculator Assumed

5. (d) A breakdown of traffic lights occurs at one of the junctions (not the source junction or the sink junction). Assume that the traffic lights operate independently from each other. Determine which set of the lights will cause the most reduction in traffic flow. Identify the junction at which this set of lights is located. By how much is the maximum flow of the network reduced?



Junction H. ✓
Flow is now along:
BCEG 2
BDFG 1 ✓
Hence, maximum flow is now 3 × 10 = 30 cars per minute.
There is a reduction of 40 cars per minute. ✓

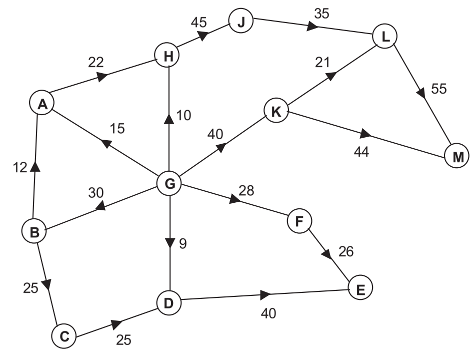
OR

Junction A. ✓
Flow is now along:
BDFG 1
BCEG 2 ✓
Hence, maximum flow is now 3 × 10 = 30 cars per minute.
There is a reduction of 40 cars per minute. ✓

Calculator Assumed

6. [8 marks: 2, 3, 3]

The diagram below shows the maximum number of vehicles that can travel along a network each minute.



(a) Consider only those paths that lead from G to E. Find the maximum number of vehicles that can travel from G to E each minute.

Path	Flow
GFE	26
GDE	9
GBCDE	25
Max. Flow	60

Justification ✓

Hence, maximum flow is 60 vehicles per minute.✓

(b) Consider only those paths that lead from G to M. Find the maximum number of vehicles that can travel from G to M each minute.

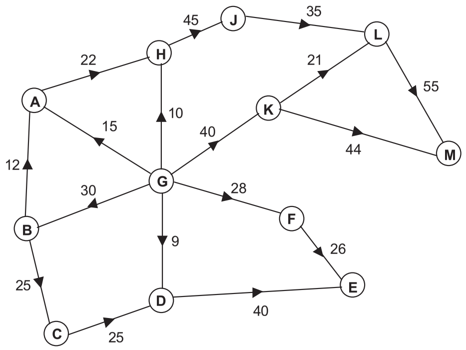
Path	Flow
GKM	40
GHJLM	10
GAHJLM	15
GBAHJLM	7
Max. Flow	72

–1 per error ✓✓

Hence, maximum flow is 72 vehicles per minute. ✓

Calculator Assumed

6. (c) Considering the network as a whole, find the maximum number of vehicles that can flow from G to E and M.

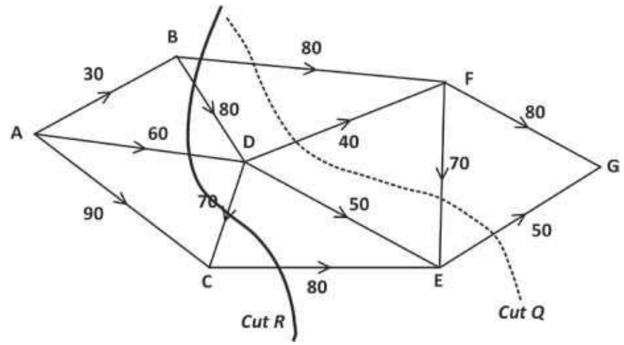


From the flows in (a) and (b),
a total of 25 + 7 = 32 flows through GB.
But GB has a capacity of only 30.
Hence, Total Flow = 60 + 72 – 2 = 130
✓ ✓ ✓

Calculator Assumed

7. [13 marks: 1, 2, 3, 2, 2, 3]

The map below shows 6 buildings A, B, C, D. E and F which are connected by one-way streets. The arrows show the direction of flow of traffic. The capacity of each street (in number of vehicles per minute) is given in the numbers alongside the edges.



(a) Determine the capacity of the cut Q drawn in the diagram above.

Capacity of cut = 80 + 40 + 50 = 170 vehicles/min ✓

(b) In the diagram above, draw a cut labelled R with capacity of 300 vehicles per minute.

Cut correctly drawn. ✓✓

(c) Determine the maximum flow for this traffic network. Show clearly how you obtained your answer.

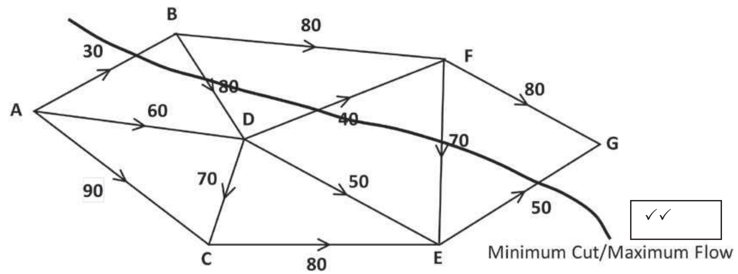
Path	Flow
ABFG	30
ADFG	40
ADEG	20
ACEG	30
Max. Flow	120

✓✓

Hence, maximum flow is 120 vehicles per minute. ✓

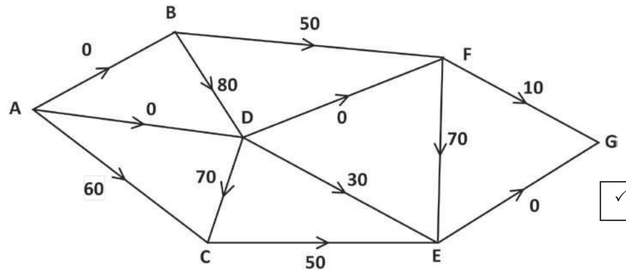
Calculator Assumed

7. (d) In the diagram below, draw the cut that corresponds to the maximum flow.



✓✓

(e) In the diagram below, indicate the unused capacity when maximum flow is achieved.



✓✓

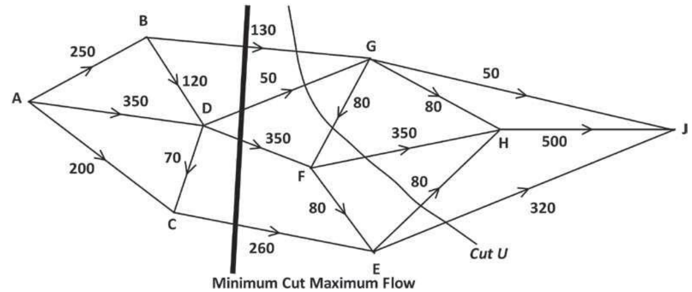
(f) The Mayor of the city wishes to improve the maximum flow so that it matches the flow from the source. How would you achieve this if you were allowed to change the traffic flow of one of these streets and improve the capacity of one of these streets?

> Increase capacity of EG by 60 vehicles per minute ✓
> and reverse direction of traffic flow from C to D. ✓
>
> This will use up all the unused capacity in AC, and part of the unused capacities of CE and DE. Maximum flow will increase to 180. ✓

Calculator Assumed

8. [12 marks: 1, 1, 3, 1, 2, 2, 2]

The diagram below shows the capacities of a storm water drainage system with flows described in kL per hour.



(a) State the source and sink for this network.

> Source: A Sink: J ✓

(b) State the value of the Cut U.

> Value = 130+50+350+80+320 = 930 ✓

(c) Determine the maximum flow for this network. Show clearly how you obtained your answer.

One possible solution.

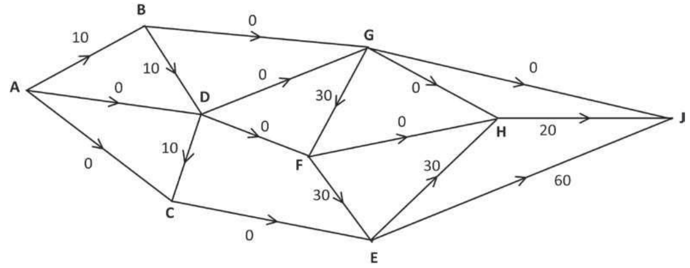
Path	Flow	
ABGJ	50	
ADEHJ	350	
ACEJ	200	✓
ABGHJ	80	
ABDGFEHJ	50	
ABDCEJ	50	✓
Max. Flow	790 kL per hour.	✓

(d) In the diagram above, draw and label the cut that corresponds to the maximum flow.

> ✓ Cut correctly drawn and labelled.

Calculator Assumed

8. (e) In the diagram below, indicate the unused capacity when maximum flow is achieved.



✓	Total unused capacity from Source = 10 and Total unused capacity into sink = 80.
✓	All other unused capacities correct.

(f) Discuss if it is possible to allow all the flow from the source to exit through the sink without any delay by upgrading the capacity of just one of the pipes.

> Upgrading CE by an extra capacity of ✓
> 10 kL per hour will allow all inflow to exit through the sink. ✓

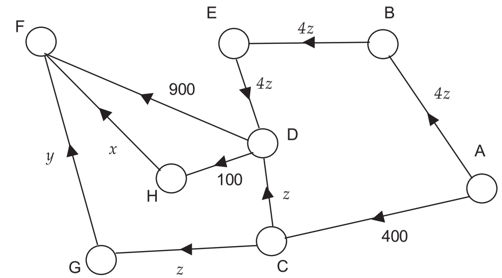
(g) Discuss the impact on the maximum flow if the capacity of the pipe EH is reduced from 80 to 50 kL per hour.

> No Impact. ✓
> Flow of 30 kL per hour through EH can be diverted through EJ. ✓

Calculator Assumed

9. [9 marks: 1, 6, 2]

The following network diagram shows the flow in a systems of pipes that achieve the maximum flow for the system (in Litres per minute).



(a) State the source and sink.

> Source = A Sink = F ✓

(b) Calculate x, y and z. Justify your answers.

> Flow along DH = Flow along HF $\Rightarrow x = 100$ ✓✓
>
> Flow along AC = Flow along CG + Flow along CD
> $\Rightarrow 400 = z + z \quad \Rightarrow z = 200$ ✓✓
>
> Flow along CG = Flow along GF
> $z = y \quad \Rightarrow y = 200$ ✓✓

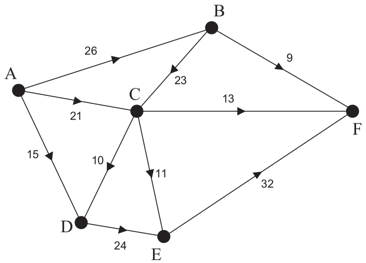
(c) Calculate the maximum flow.

> Diagram shows the flow that achieves maximum flow.
> Hence, maximum flow = $x + y + 900$
> = 1 200 Litres per minute. ✓✓

Calculator Assumed

10. [10 marks: 3, 4, 3]

An airline has flights that connect airports at A, B, C, D, E, and F. The network drawn below shows the maximum number of passengers, in hundreds, that can be carried at a certain peak time of day.



(a) Find the maximum number of passengers that can be carried from A to F. Show clearly how you obtained your answer.

Path	Flow
ABF	9
ACF	13
ADEF	15
ACDEF	8
ABCEF	9
Max. Flow	54

–1 per error ✓✓

Hence, maximum flow is 54 × 100 = 5400 passengers. ✓

(b) If 95% of the passengers are adults and if the cost for a child fare from A to F is 60% of the adult fare of \$1 200 (irrespective of the number of connecting flights), how much could the airline expect to earn during this peak period?

Number of adults = 5400 × 0.95 = 5130 ✓
Number of children = 5400 – 5130 = 270 ✓

Revenue = 5130 × \$1200 + 270 × (0.6 × 1200) = \$ 6 350 400 ✓✓

(c) An AFL match is scheduled to be held at an oval located at F between the local team of C and the local team of F. The airline intends to put on extra flights on the E to F route during this peak period to maximise capacity. How many extra passengers should such flights cater for? Justify your answer.

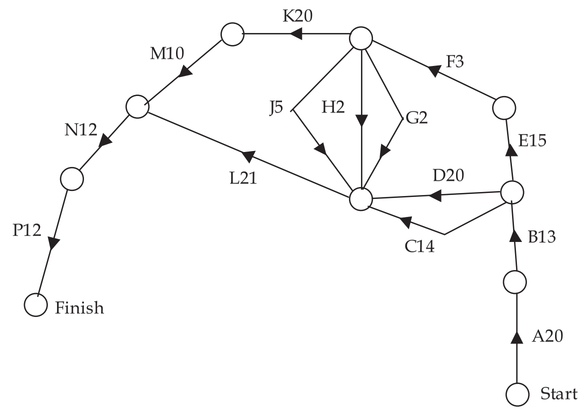
Path CDEF allows an extra 100 passengers. ✓
Path CEF allows an extra 200 passengers. ✓

Hence, the flights can cater for an extra 200 + 100 = 300 passengers. ✓

18 Project Networks

1. [10 marks: 2, 2, 2, 4,]

The diagram below shows a project network with time measured in minutes.



(a) State the critical path.

Start – ABEFKMNP – Finish ✓✓

(b) If the project was started at 8.00 am, what is the earliest time the project could be completed?

Minimum completion time = 105 minutes ✓
Hence, earliest completion time = 8.00 + 105 min = 9.45 am ✓

(c) If the project was started at 8.00 am, find the latest time activity L could start without delaying the completion time? Justify your answer.

There is a slack of 81 – (56 + 21) = 4 minutes in starting L. ✓
Hence, latest starting time for L = 8.00 + 56 min + 4 min = 9.00 am ✓

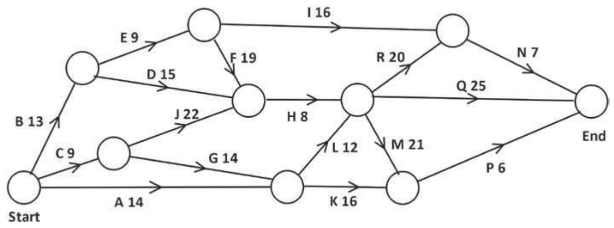
(d) Activity D is delayed by 8 minutes. Discuss with reasons, the effect this will have on the completion time of the project.

There will be a new critical path of
Start – ABDLNP – Finish ✓✓
The completion time will be delayed by 1 minute. ✓✓

Calculator Assumed

2. [9 marks: 2, 2, 1, 3, 1]

The diagram below shows the tasks required to complete a project. The project starts at 9.00 am. The times for the required tasks are in minutes. Each task requires a workers full attention.



(a) Determine the critical path(s) for this project.

Critical Path: BEFHRN and BEFHMP ✓✓

(b) State the earliest time the project can be completed.

MCT = 76 minutes. ✓ Hence: 10:16 am ✓

(c) What is the Earliest Starting Time for activity G if the minimum completion time (MCT) is to be maintained?

9.09 am ✓

(d) State with reasons the Latest Starting Time for activity G if the MCT is to be maintained.

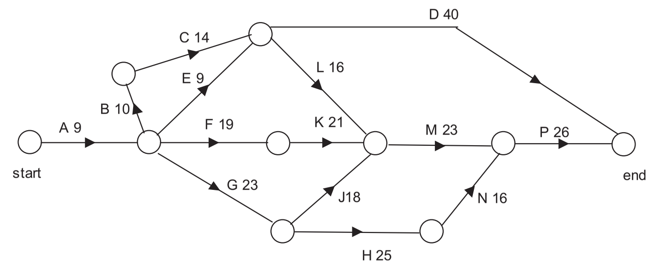
LCT for L = 49 LST for L = 49 – 12 = 37 Hence, LCT for G = 37 ✓ LST for G = 37 – 14 = 23 ✓ Hence: 9.23 am ✓

(e) State all the activities that have float (slack) times of 0 minutes.

B, E, F, H, R, N, M, P ✓

Calculator Assumed

3. [11 marks: 4, 1, 3, 3]



For the project network above, the minimum times required to complete the various activities are recorded in days.

(a) Find the minimum completion time and the corresponding critical path(s).

Critical paths are: Start – A – G – J – M – P – Finish ✓✓ Start – A – G – H – N – P – Finish. ✓ Minimum completion time = 99 days ✓

(b) By how many days can activity E be delayed without affecting the minimum completion time.

E can be delayed 40 – 16 – 9 = 15 days. ✓

(c) Activity H can now be completed in 22 days. Determine the effect this will have on the minimum completion time and the critical path.

Start – A – G – H – N – P – Finish is no longer a critical path. ✓ But Start – A – G – J – M – P – Finish remains a critical path. ✓ Hence, no change on the minimum completion time. ✓

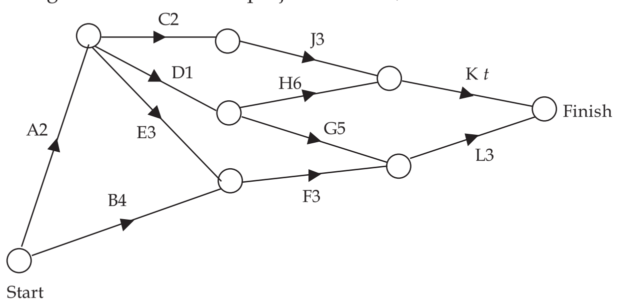
(d) After some reorganisation, it is now possible to commence activity C immediately after the completion of activity A. Discuss the effect of this reorganisation on the minimum completion time and the critical path.

The longest path A – C – L requires only 39 days which is less than that the 50 days required for A – G – J along the critical path. ✓✓ Hence, there is no effect on the critical path and hence the minimum completion time. ✓

Calculator Assumed

4. [7 marks: 4, 3]

The diagram below shows a project network, with time measured in days.



(a) Complete the Activity Table to describe the project network.

Activity	Time Taken (days)	Immediate Predecessor(s)
A	2	–
B	4	–
C	2	A
D	1	A
E	3	A
F	3	B, E
G	5	D
H	6	D
J	3	C
K	t	J, H
L	3	F, G

–1 per error ✓✓✓✓

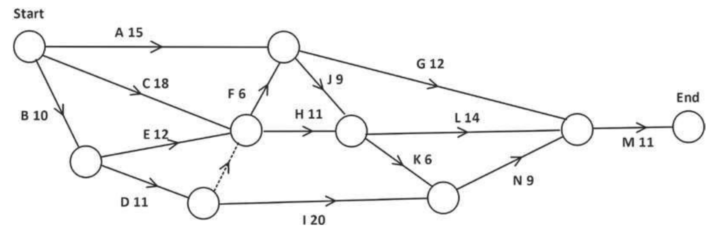
(b) Find the value of *t* if the minimum completion time is 13 days.
State the corresponding critical path.

For a minimum completion time of 13 days, critical path must be Start – ADHK – Finish ✓✓
with $t = 4$ days. ✓

Calculator Assumed

5. [10 marks: 1, 2, 2, 2, 3]

The diagram below shows a project network. The times for the required tasks are in hours. Each task requires a workers full attention.



(a) State the immediate predecessors of Task I.

D ✓

(b) State the immediate predecessors of Task F.

C, E ✓
D ✓

(c) Explain the purpose of the dummy link represented by the dotted line in the diagram above.

To indicate that I and F share a common immediate predecessor ✓
but not all immediate predecessors. ✓

(d) State the task that <u>*will always*</u> be on the critical path. Explain.

Task M. ✓
It is the unique final task. ✓

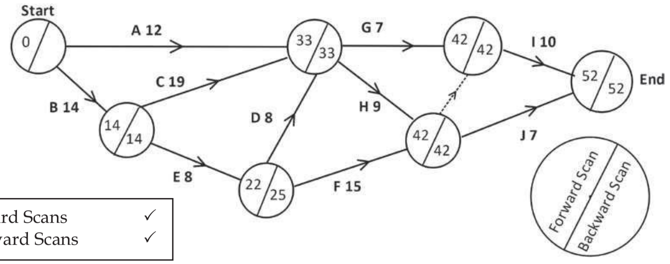
(e) Determine the critical path and minimum completion time for this network.

Critical Path: BEFJLM ✓✓
Minimum Completion Time = 62 hours ✓

Calculator Assumed

6. [13 marks: 4, 3, 2, 4]

The project network shown below has time for the various tasks recorded in days.



Forward Scans	✓
Backward Scans	✓

(a) Complete the forward and backward scans for this network. Explain the purpose of the forward and backward scans.

> Forward scans to indicate Earliest Starting Times of Tasks. ✓
> Backwards scans to indicate Latest Starting Times of Tasks. ✓

(b) Use your answer in (a) to explain why Task D is not on the critical path.

> Tasks on the critical path share the same time for EST and LST. ✓
> But the EST for D is 22 minutes which is different from its LST which is 25 minutes. ✓✓
> Hence, D is not on the critical path.

(c) Identify the critical path and the minimum completion time for this network.

> Critical Path: BCHI ✓
> Minimum Completion Time = 52 days ✓

(d) The minimum completion time may be reduced by hiring more sophisticated equipment for Task C. It will cost \$5 000 per day to reduce the completion time for task C by one day. Determine with reasons, how the original minimum completion time may be reduced and by how many days and the extra cost incurred.

> Spend an extra \$15 000 to reduce the time for task C by 3 days ✓
> reducing the project time from 52 days to 49 days. ✓
>
> Latest completion time for Task D is 30 days. ✓
> Hence, if Task C is reduced to 16 days from 19 days,
> the latest completion time for Task C will also be 30 days. ✓
> G can now start 3 days earlier, reducing the project time by 3 days.

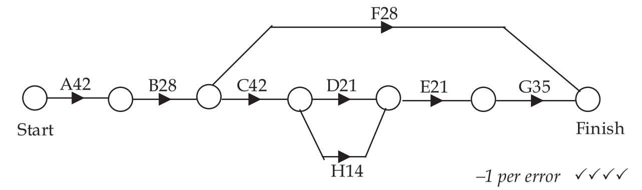
Calculator Assumed

7. [10 marks: 4, 2, 2, 2]

The table below shows the tasks required for the building of a factory and the time required in days. The project was started on the 1st of October 2019.

Task	Task Description	Duration	Prerequisites
A	Clear land	42	–
B	Lay foundations	28	A
C	Build walls	42	B
D	Electrical wiring	21	C
E	Plastering	21	D, H
F	Landscaping	28	B
G	Interior work	35	E
H	Roof	14	C

(a) Draw a project network to represent the above information.



–1 per error ✓✓✓✓

(b) When will be the earliest completion date?

> Minimum completion time = 189 days. ✓
> Hence, earliest completion date = 6th April the 2016. ✓

(c) If, because of an industrial dispute, the bricklayers went on strike for 5 days, determine the earliest completion date.

> Since, C is part of the critical path, the completion date will now be 5 days later, 11th April 2016. ✓✓

(d) If, because of poor weather, the roof job took an extra 10 days, determine the earliest completion date.

> New critical path ABCHEG with minimum completion time of 192 days.
> Hence, completion date will be 9th April 2016. ✓✓

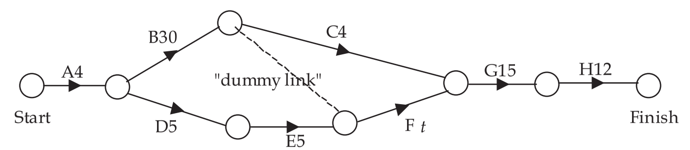
Calculator Assumed

8. [11 marks: 4, 2, 5]

The table below describes the various activities involved in assembling a computer.

Activity	Description of Activity	Immediate Predecessor	Time (min)
A	Install mother-board	–	4
B	Test hard drive	A	30
C	Install hard drive	B, E	4
D	Install I/O ports	A	5
E	Install DVD-RW	D	5
F	Test DVD-RW	E	t
G	Install operating system	C, F	15
H	Test assembled computer	G	12

(a) Construct a project network for this table of activities.



"dummy link" ✓
–1 per error for rest ✓✓✓

(b) The minimum completion time for this project is 65 minutes. State a possible critical path.

Possible critical path = Start – A – B – C –G – H – Finish. ✓✓

(c) Given that F is not on the critical path, find the possible values of t. Justify your answer.

Critical path must be: Start – A – B – C –G – H – Finish.
Path A – B – C takes 4 + 30 + 4 = 38 minutes. ✓
Path A – D – E – F takes $4 + 5 + 5 + t = t + 14$ minutes. ✓

For F not to be on the critical path, $t + 14 < 38$ ✓
$\Rightarrow \quad t < 24$.
Hence, $0 < t < 24$ minutes. ✓✓

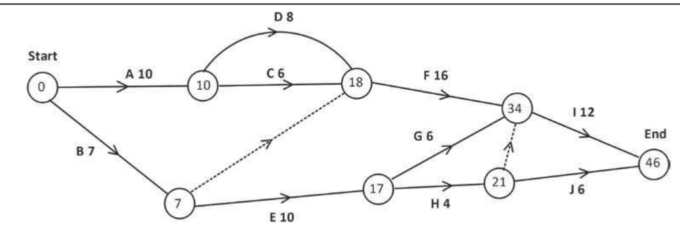
Calculator Assumed

9. [10 marks: 4, 3, 3]

The table below shows the activities and their corresponding predecessors and times for a project network.

Activities	Immediate Predecessors	Required time (minutes)
A	–	10
B	–	7
C	A	6
D	A	8
E	B	10
F	B, C, D	16
G	E	6
H	E	4
I	F, G, H	12
J	H	6

(a) Draw the graph for this network.



✓ All activities included. ✓ First dummy link.
✓ Second dummy link. ✓ All other links correct.

(b) Determine with reasons if this project can be completed within 45 minutes.

Critical Path is ADFI. ✓
Minimum Completion Time is 46 minutes. ✓
Hence, no! ✓

(c) If each activity requires a worker's full attention, what is the minimum number of workers that are required for this project to be completed in minimum time? Justify your answer.

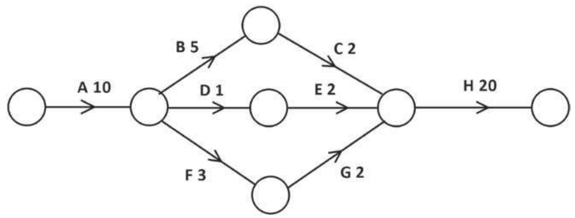
Worker 1 completes all tasks on critical path. ✓
Worker 2: B, C, G and J. ✓
Worker 3: H ✓

Hence, 3 workers are needed.

Calculator Assumed

10. [8 marks: 2, 3, 3]

The diagram below shows a project network for a job being handled by Bianca. Each activity can <u>only</u> be done by one person and time is recorded in minutes.



(a) What is the minimum time required by Bianca, working by herself, to complete all the tasks?

> Minimum time = 10+ 5 + 2 + 1 + 2 + 3 + 2 + 20 ✓
> = 45 minutes ✓

(b) What is the minimum time required by Bianca, if she is assisted by Teneka, to complete all the tasks?

> Critical Path is ABCH with minimum completion time of 37 minutes. ✓
> Bianca completes activities on critical path.
> Remaining activities D, E, F and G which can be completed by Teneka require 8 minutes but must be completed within a 7 minute window ✓ to be in sync with the critical path.
> Hence, minimum completion time is 37 + 1 = 38 minutes. ✓

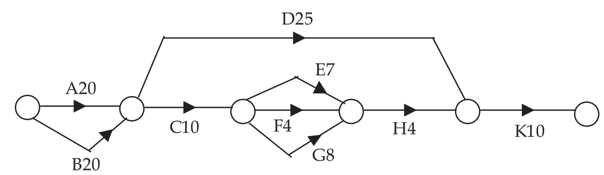
(c) Would it save time, if Leesa came along to give Bianca and Teneka some assistance? Justify your answer.

> Yes.
> Bianca could complete the activities on the critical path which will take 37 minutes. ✓
> Teneka could handle D and E.
> Leesa could handle F and G. ✓
> The project could be completed in 37 minutes. ✓

Calculator Assumed

11. [9 marks: 2, 3, 4]

Seb is involved in setting up a hall for a birthday party. The diagram below shows the project network for this task. Each activity requires a person's full attention and the associated time is measured in minutes.



(a) What is the minimum time required by Seb, working alone, to complete the job? Justify your answer.

> Minimum time = 20 + 20 + 10 + 25 + 7 + 4 + 8 + 4 + 10 ✓
> = 108 minutes ✓

(b) What would the minimum completion time be if Stefan gave Seb a hand right from the start? Justify your answer.

> Critical Path is ADK with minimum completion time of 55 minutes.
>
> Seb handles all the tasks in the critical path and can complete A and D in 45 minutes.
> Stefan handles rest of the tasks.
> Total amount of time for Stefan to complete B, C, E, F, G and H is 53 minutes.
> Hence, earliest start time for K is 53 minutes.
> Hence, time taken = 53 + 10 = 63 minutes ✓✓✓

(c) By how much would the minimum completion time in part (b) change if Thomas and Stefan were available to help Seb from the start? Justify your answer.

	Simultaneous Tasks			
Seb	A	D	K	Total Time Taken
Stefan	B	C, G		
Thomas		E, F, H		
Time Taken	20	25	10	55 min

Justification ✓✓

Hence, time taken = 55 minutes ✓
Hence, time saved = 63 – 55 = 8 min, ✓

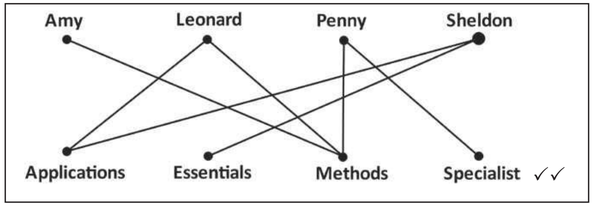
19 Assignment Problems

Calculator Free

1. [2 marks]

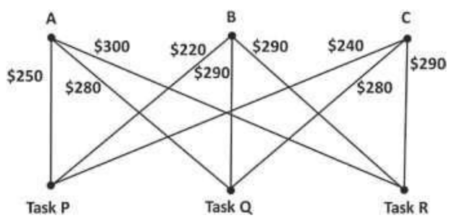
The accompanying table shows the different mathematics courses taken by several students. Display this information as a bipartite graph.

Student	Mathematics Course
Penny	Specialist, Methods
Sheldon	Applications, Essentials
Leonard	Methods, Applications
Amy	Methods



2. [4 marks: 1, 3]

The accompanying bipartite graph shows the labour costs associated with tasks P, Q and R performed by workers A, B and C. One worker is to be assigned to one task with no two workers assigned to the same task.



(a) Find the cost of assigning A to task P, B to task R and C to task Q.

Cost = $250 + $290 + $280 = $820 ✓

(c) Find the lowest cost associated with assigning one worker to one task. State how this is achieved.

Assign A to Task Q, B to Task P and C to Task R. ✓✓
Cost = $280 + $220 + $290 = $790 ✓

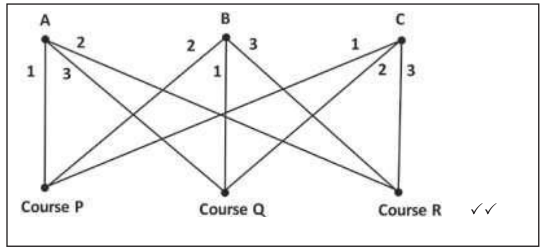
Calculator Free

3. [6 marks: 2, 1, 2, 1]

The accompanying table details the preferences students A, B and C allocated to courses P, Q and R. "1" indicates the student's first preference. It is intended that each student is matched to one course.

	Course P	Course Q	Course R
Student A	1	3	2
Student B	2	1	3
Student C	1	2	3

(a) Represent the information in the table above in the form of a bipartite graph.



(b) Calculate the sum of the preferences if A, B and C are allocated respectively to courses P, Q and R.

Sum of preferences = 1 + 1 + 3 = 5 ✓

(c) Determine how the courses should be allocated to A, B and C so that the sum of the preferences is minimised.

Allocate A to course R, B to course Q and C to course P. ✓
Sum of preferences = 4 ✓

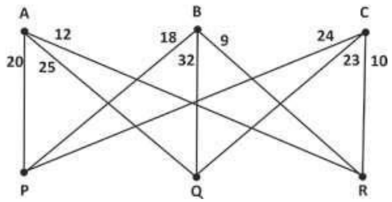
(d) Explain what it means when the sum of preferences is minimised.

As many students as is possible are allocated their initial preferences. ✓

Calculator Assumed

4. [7 marks: 2, 1, 2, 2]

The diagram below is a bipartite graph describing the number of components produced by workers A, B and C at workstations P, Q and R over a period of one hour.



(a) Complete the matrix representation of the bipartite graph.

$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} P & Q & R \end{array} \\ \begin{pmatrix} \boxed{20} & \boxed{25} & \boxed{12} \\ \boxed{18} & \boxed{32} & \boxed{9} \\ \boxed{24} & \boxed{23} & \boxed{10} \end{pmatrix} \end{array}$$

✓✓

(b) Calculate the total number of components produced when A, B and C are respectively assigned to R, P and Q.

Total number of components = 12 + 18 + 23 = 53 ✓

(c) Which assignment produced a total of 58 components?

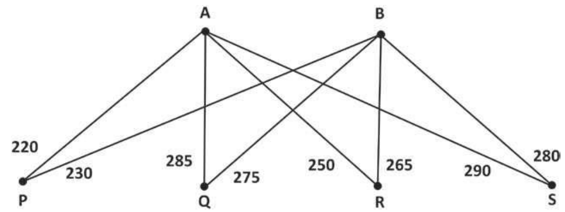
Allocate A to Q, B to R and C to P. ✓✓

(d) Determine the optimum assignment(s) so that the total number of components is maximised. State this maximum.

Allocate A to R, B to Q and C to P. ✓
Maximum = 68 components ✓

Calculator Assumed

5. [9 marks: 3, 4, 2]



(a) The bipartite graph above shows the daily labour costs ($) associated with tasks P, Q, R and S performed by workers A and B. One worker is to be assigned to exactly one task with the constraint that the total cost is to be minimised. State the minimum cost and the solution(s) to the assignment problem.

Assign A to task R and B to Task P. ✓✓
Total cost = $230 + $250 = $480 ✓

(b) The bipartite graph above shows the hourly profit ($) generated through tasks P, Q, R and S performed by workers A and B. One worker is to be assigned to exactly one task with the constraint that the total profit is to be maximised. State the maximum profit and the solution(s) to the assignment problem.

Assign A to Task Q and B to Task S. ✓
Total cost = $285 + $280 = $565 ✓
or assign A to Task S and B to Task Q ✓
Total cost = $290 + $275 = $565 ✓

(c) The bipartite graph above shows the time (minutes) taken to complete tasks A and B by workers P, Q, R and S. One worker is to be assigned to exactly one task with the constraint that the total time taken is to be minimised. State the minimum time taken and the solution(s) to the assignment problem.

Assign P to task B and A to Task R. ✓
Total time = 230 + 250 = 480 minutes ✓

Calculator Free

6. [12 marks: 3, 3, 3, 3]

Matrices A, B, C and D are reduced row and reduced column cost matrices (opportunity cost matrices) obtained through the Hungarian algorithm. Determine with reasons if these matrices:

- are unable to assign an optimum solution without further steps
- readily provide a unique optimum solution
- readily provide multiple optimum solutions.

$$A = \begin{pmatrix} 1 & 0 & 5 & 7 \\ 2 & 5 & 5 & 0 \\ 1 & 5 & 0 & 8 \\ 0 & 6 & 2 & 4 \end{pmatrix} \quad B = \begin{pmatrix} 3 & 0 & 1 & 6 \\ 0 & 0 & 2 & 0 \\ 2 & 5 & 0 & 3 \\ 0 & 6 & 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 2 & 0 & 5 \\ 3 & 8 & 0 \\ 5 & 0 & 6 \end{pmatrix} \quad D = \begin{pmatrix} 5 & 1 & 0 & 9 \\ 0 & 2 & 0 & 0 \\ 1 & 3 & 0 & 8 \\ 0 & 4 & 1 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 5 & 7 \\ 2 & 5 & 5 & 0 \\ 1 & 5 & 0 & 8 \\ 0 & 6 & 2 & 4 \end{pmatrix}$$

- There are four "zero lines" which equals the number of rows. ✓
- There is exactly one zero in each column and row. ✓
- Hence, an optimum solution is available. ✓

$$B = \begin{pmatrix} 3 & 0 & 1 & 6 \\ 0 & 0 & 2 & 0 \\ 2 & 5 & 0 & 3 \\ 0 & 6 & 0 & 0 \end{pmatrix}$$

- There are four "zero lines" which equals the number of rows. ✓
- There is at least one zero in each column and row. Further, rows 2 & 4 have 3 zeros each. ✓
- Hence, multiple optimum solutions are available. ✓

$$C = \begin{pmatrix} 2 & 0 & 5 \\ 3 & 8 & 0 \\ 5 & 0 & 6 \end{pmatrix}$$

- There are only two "zero lines" which is less than the number of rows (three rows). ✓✓
- Hence, no optimum solution is available at this stage. ✓

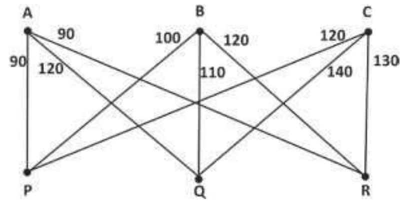
$$D = \begin{pmatrix} 5 & 1 & 0 & 9 \\ 0 & 2 & 0 & 0 \\ 1 & 3 & 0 & 8 \\ 0 & 4 & 1 & 0 \end{pmatrix}$$

- There are only three "zero lines" which is less than the number of rows (four rows). ✓✓
- Hence, no optimum solution is available at this stage. ✓

Calculator Assumed

7. [6 marks: 3, 2, 1]

The accompanying diagram shows a bipartite graph describing the hourly costs ($) associated with workers A, B and C performing tasks P, Q and R.



(a) Complete the tables below showing the use of the Hungarian algorithm to assign one worker to exactly one task while minimising the total average cost. [Use as many "blank tables" as you need. Indicate the order in which the tables were used.]

	P	Q	R
A	90	120	90
B	100	110	120
C	120	140	130

→

	P	Q	R
A	0	30	0
B	0	10	20
C	0	20	10

✓

	P	Q	R
A	0	20	0
B	0	0	20
C	0	10	10

✓✓

	P	Q	R
A			
B			
C			

	P	Q	R
A			
B			
C			

	P	Q	R
A			
B			
C			

(b) State the optimum assignment and the total hourly cost.

Assign A to task R, B to task Q and C to task P. ✓
Total hourly cost = $320 ✓

(c) What is the least average cost per worker?

Average cost per worker = 320/3 ≈ $107 ✓

Calculator Assumed

8. [7 marks: 5, 2]

The accompanying table shows the running times (seconds) for sprinters A, B, C and D for each of the four legs of a 4 × 100 m relay race.

Time	Leg 1	Leg 2	Leg 3	Leg 4
Sprinter A	9.9	9.4	9.7	9.3
Sprinter B	10.0	9.3	9.9	9.1
Sprinter C	9.8	9.6	9.7	9.5
Sprinter D	9.7	9.2	9.5	9.1

(a) Complete the tables below showing the use of the Hungarian algorithm to assign one sprinter to exactly one leg of the race while minimising the total time taken for the race. [Use as many "blank tables" as you need. Indicate the order in which the tables were used.]

9.9	9.4	9.7	9.3
10.0	9.3	9.9	9.1
9.8	9.6	9.7	9.5
9.7	9.2	9.5	9.1

→

0.6	0.1	0.4	0
0.9	0.2	0.8	0
0.3	0.1	0.2	0
0.6	0.1	0.4	0

✓

0.3	0	0.2	0
0.6	0.1	0.6	0
0	0	0	0
0.3	0	0.2	0

✓✓

→

0.1	0	0	0
0.4	0.1	0.4	0
0	0.2	0	0.2
0.1	0	0	0

✓✓

(b) State the optimum assignment of sprinters to the legs of the race. State the minimum time achievable.

Assign A to leg 3, B to leg 4, C to leg 1 and D to leg 2. ✓
Or Assign A to leg 2, B to leg 4, C to leg 1 and D to leg 3.
Minimum time = 37.8 seconds. ✓

Calculator Assumed

9. [9 marks: 6, 3]

The accompanying table shows the times (minutes) taken by workers A, B, C and D to complete Tasks 1, 2, 3 and 4.

Time	Task 1	Task 2	Task 3	Task 4
Worker A	23	32	41	16
Worker B	24	35	39	17
Worker C	23	31	40	18
Worker D	25	31	40	16

(a) Use the Hungarian algorithm to assign one worker to exactly one task with the constraint that the total time taken to complete the four tasks is a minimum. State all the optimum assignments and the corresponding time taken.

23	32	41	16
24	35	39	17
23	31	40	18
25	31	40	16

→

7	16	25	0
7	18	22	0
5	13	22	0
9	15	24	0

2	3	3	0
2	5	0	0
0	0	0	0
4	2	2	0

→

0	1	1	0
2	5	0	2
0	0	0	2
2	0	0	0

Row reduction. ✓
Column reduction & zero lines ✓
Revision & zero lines ✓

Optimum assignments:
- Assign A to task 1, B to task 3, C to task 2 and D to task 4. ✓
- Assign A to task 4, B to task 3, C to task 1 and D to task 2. ✓

Minimum time = 109 minutes ✓

(b) If the time for B to complete Task 3 was 38 minutes instead of 39 minutes discuss how this would affect the optimum assignment in (a).

The assignments would remain unchanged. ✓
In the optimum assignments to reduce total time, B is already assigned to Task 3. ✓
Hence, if B takes 1 minute less, the total minimum time would reduce by 1 minute. ✓

Calculator Assumed

10. [9 marks: 6, 3]

The accompanying matrix shows the number of new customers signed up per day by salespersons A, B, C and D at outlets located at shopping centres P, Q, R and S.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} P & Q & R & S \end{array} \\ \begin{pmatrix} 15 & 12 & 18 & 6 \\ 16 & 14 & 20 & 6 \\ 14 & 13 & 19 & 7 \\ 17 & 15 & 20 & 7 \end{pmatrix} \end{array}$$

(a) Use the Hungarian algorithm to assign one salesperson to exactly one shopping centre maximising the total number of new customers signed up. State all the optimum assignments and the corresponding sales made. Show each step of the process.

$$\begin{pmatrix} 15 & 12 & 18 & 6 \\ 16 & 14 & 20 & 6 \\ 14 & 13 & 19 & 7 \\ 17 & 15 & 20 & 7 \end{pmatrix} \longrightarrow \begin{pmatrix} 5 & 8 & 2 & 14 \\ 4 & 6 & 0 & 14 \\ 6 & 7 & 1 & 13 \\ 3 & 5 & 0 & 13 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 6 & 0 & 12 \\ 4 & 6 & 0 & 14 \\ 5 & 6 & 0 & 12 \\ 3 & 5 & 0 & 13 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 2 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Entries subtracted from largest entry ✓
Row reduction ✓
Column reduction & zero lines ✓✓

Assign A to P, B to R, C to S and D to Q. ✓
Total customers signed up = 57 ✓

(b) If the number of new customers signed up by B at shopping centre R was 22 instead of 20, discuss how this would affect the optimum assignment in (a).

The assignments would remain unchanged. ✓
In the optimum assignment to maximise the number of customers signed up, B is already assigned to R. ✓
Hence, if B's sign-ups increase by 2, the total number of sign-ups would increase by 2 to 59. ✓

Calculator Assumed

11. [10 marks, 7, 3]

The accompanying table shows the times (hours) taken by workers A, B, C and D to complete Tasks 1, 2, 3 and 4.

Time	Task 1	Task 2	Task 3	Task 4
Worker A	124	154	256	23
Worker B	132	165	248	21
Worker C	128	154	248	24
Worker D	124	158	255	21

(a) Show the use of the Hungarian algorithm to assign one worker to exactly one task with the constraint that the total time taken to complete the four tasks is a minimum. State the optimum assignment(s) and the corresponding time taken.

101	131	233	0
111	144	227	0
104	130	224	0
103	137	234	0

✓

0	1	9	0
10	14	3	0
3	0	0	0
2	7	10	0

✓

0	1	9	2
8	12	1	0
3	0	0	2
0	5	8	0

✓

0	0	8	2
8	11	0	0
4	0	0	3
0	4	7	0

✓

For minimum total time: Assign A(2), B(4), C(3) and D(1) ✓
or A(1), B(3), C(2) and D(4)] ✓
Minimum total time = 547 man-hours ✓

(b) If because of health reasons, Worker A must be assigned to Task 4, discuss the impact of this requirement on the total time to complete the four tasks.

A is assigned to Task 4.
By inspection,
Assign D to Task 1, C to Task 3, B to Task 2. ✓✓
Total time is now 549 man-hours (an increase of 2 man-hours) ✓

Calculator Assumed

12. [9 marks]

The accompanying table shows the times (minutes) taken by workers A, B, and C to complete Tasks 1, 2, 3 and 4.

Time	Task 1	Task 2	Task 3	Task 4
Worker A	21	28	26	12
Worker B	22	26	26	10
Worker C	23	27	28	10

Use the Hungarian algorithm to assign one worker to exactly one task with the constraint that the total time taken to complete the three chosen tasks is a minimum. State all the optimum assignments and the corresponding time taken. Show each step of the process.

21	28	26	12
22	26	26	10
23	27	28	10
0	0	0	0

→

9	16	14	0
12	16	16	0
13	17	18	0
0	0	0	0

0	7	5	0
3	7	7	0
4	8	9	0
0	0	0	9

→

0	7	5	3
0	4	4	0
1	5	6	0
0	0	0	12

0	3	1	3
0	0	0	0
1	1	2	0
4	0	0	16

Insert dummy rows ✓
Row reduction & lines ✓✓
First revision. ✓
Second revision. ✓
Final revision ✓

Assign A to task 1, B to task 2 and C to task 4. ✓
or Assign A to task 1, B to task 3 and C to task 4 ✓
Minimum time = 57 minutes ✓

Calculator Assumed

13. [6 marks]

The accompanying table shows the average daily costs for workers A, B, C, D and E performing Tasks 1, 2, 3 and 4.

Cost	Task 1	Task 2	Task 3	Task 4
A	500	580	620	550
B	480	600	610	540
C	490	590	600	550
D	510	580	610	560
E	490	590	590	570

Use the Hungarian algorithm to assign one worker to exactly one task while minimizing the total cost. State two possible optimum assignments and the corresponding cost. Show each step of the process.

500	580	620	550	0
480	600	610	540	0
490	590	600	550	0
510	580	610	560	0
490	590	590	570	0

→

20	0	30	10	0
0	20	20	0	0
10	10	10	10	0
30	0	20	20	0
10	10	0	30	0

10	0	20	0	0
0	30	20	0	10
0	10	0	0	0
20	0	10	10	0
10	20	0	30	10

Dummy columns ✓
Column reduction & zero lines ✓✓

Assign A to task 2, B to task 1, C to task 4 and E to task 3. ✓
or Assign B to task 4, C to task 1, D to task 2 and E to task 3 ✓
Minimum cost = $2200 ✓

Other optimum solutions with minimum cost of $2200 exist.

ACADEMIC TASK FORCE
ATAR COURSE BOOKS

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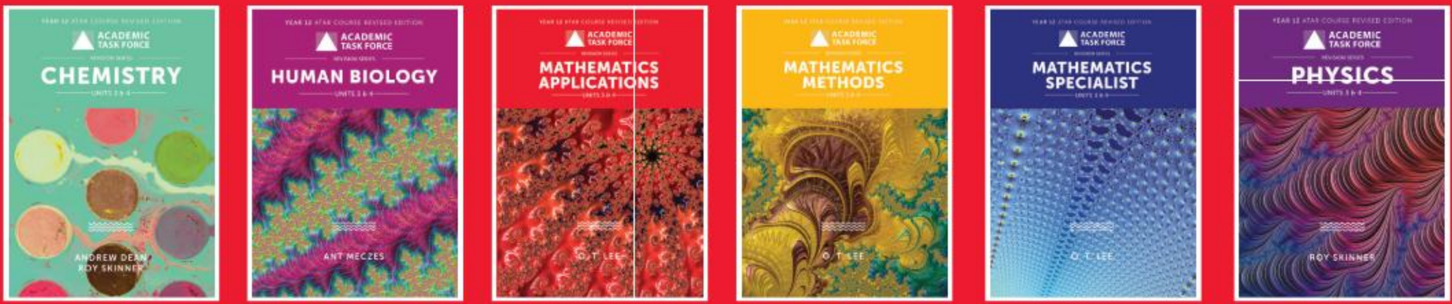

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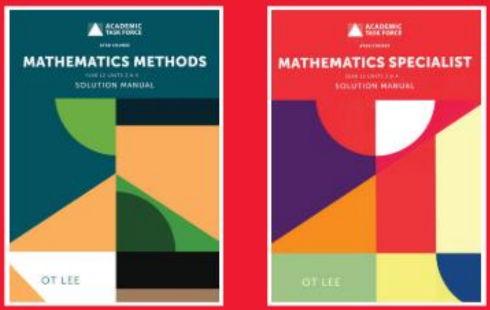

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